Navier-Stokes Equations

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1 Introduction to Fluid Dynamics and the Navier-Stokes Equations

Fluid dynamics, a fundamental branch of physics and applied mathematics, is concerned with the behavior of liquids and gases in motion. This field is not only pivotal in understanding a vast array of natural phenomena but also plays a crucial role in a myriad of engineering applications. From the intricate patterns formed by smoke to the powerful currents of the ocean, fluid dynamics offers insights into both the elegantly simple and the complexly chaotic aspects of the natural world.

The study of fluid motion dates back centuries, but it was in the 19th century that significant strides were made in formalizing these concepts into mathematical models. The Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, stand as a monumental achievement in this regard. These equations were developed independently by Navier in 1822 and later refined by Stokes in 1845. They synthesized Newton's second law of motion with the principles governing the movement of fluids, providing a comprehensive mathematical framework to describe the flow of incompressible fluids.

At their core, the Navier-Stokes equations are a set of nonlinear partial differential equations that describe the motion of fluid substances. These equations encapsulate various factors such as fluid velocity, pressure, density, and viscosity, offering a window into understanding how these elements interact within a fluid flow. The beauty of the Navier-Stokes equations lies in their generality; they are applicable to a wide range of scenarios, from the slow flow of oil in a pipeline to the turbulent currents of air around a speeding aircraft.

In this exploration of the Navier-Stokes equations, we will delve into their mathematical formulation, unravel their physical interpretations, and appreciate their wide-ranging applications. We will also confront the challenges they pose, not just in terms of their complex solutions but also in the broader context of their significance in modern mathematics and physics. As we embark on this journey, we stand at the intersection of mathematical elegance and the intricate dance of fluid particles, ready to unravel the mysteries of fluid dynamics through the lens of the Navier-Stokes equations. The main references are [Chorin and Marsden, 1993], [Panton, 2013] and [Batchelor, 2000].

2 Navier-Stokes Equations

The Navier-Stokes equations describe the motion of fluid substances and are a set of nonlinear partial differential equations. They can be written in different forms based on the assumptions about the fluid, such as incompressibility and compressibility.

2.1 Incompressible Fluid

For an incompressible fluid, the Navier-Stokes equations consist of the continuity equation and the momentum equation.

- **Continuity Equation:** This equation states that the divergence of the velocity field is zero, representing the conservation of mass in the fluid.

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

where $\mathbf{u}$ is the velocity field of the fluid.

- **Momentum Equation:** This equation represents the conservation of momentum.

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \tag{2}$$

where ρ is the density of the fluid, p is the pressure field, μ is the dynamic viscosity, and $\mathbf{f}$ represents external forces such as gravity.

2.2 Compressible Fluid

For compressible flows, the equations are modified to account for the variation in density.

- **Continuity Equation for Compressible Fluid:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3}$$

- **Momentum Equation for Compressible Fluid:**

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \tau + \mathbf{f} \tag{4}$$

where τ is the stress tensor, which includes viscous stresses.

3 The Navier-Stokes Problem

The Navier–Stokes existence and smoothness problem concerns the mathematical properties of solutions to the Navier–Stokes equations, a system of partial differential equations that describe the motion of a fluid in space. Solutions to the Navier–Stokes equations are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the Navier–Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering. Even more basic (and seemingly intuitive) properties of the solutions to Navier–Stokes have never been proven. For the three-dimensional system of equations, and given some initial conditions, mathematicians have neither proved that smooth solutions always exist, nor found any counter-examples. This is called the Navier–Stokes existence and smoothness problem.

References

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