MAT 201
Larson/Edwards – Section 4.4
The Fundamental Theorem of Calculus

<u>The Fundamental Theorem of Calculus</u>: If a function *f* is continuous on the closed interval [*a*, *b*] and *F* is an antiderivative of *f* on the interval [*a*, *b*], then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus states that the derivative and definite integral have an inverse relationship much the same as multiplication and division are inverse operations. The limit processes used to define the derivative and the definite integral preserve this relationship.

<u>Guidelines for Using the FTC</u>:

1. The Fundamental Theorem of Calculus describes a way of evaluating a definite integral, not a procedure for finding antiderivatives.
2. Provided you can find the antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
3. In applying the FTC, it is helpful to use the notation:
   $$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$
4. The constant of integration *C* can be dropped. Why?

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Our book uses this notation

$$\Big[F(x)\Big]_a^b$$

4. The constant of integration C can be dropped. Why?

$$\boxed{F(x)\Big|_a^b}$$

My notation

$$\int_a^b f(x)\,dx = F(x)+C\Big]_a^b$$

$$= \Big[F(b)+C\Big] - \Big[F(a)+C\Big]$$

$$= F(b)\cancel{+C} - F(a)\cancel{-C}$$

Ex: Evaluate the definite integral:

a) $\int_{-1}^{0} (2x - 1)\,dx$

b) $\int_{0}^{1} \frac{x - \sqrt{x}}{3}\,dx$

Ex: Evaluate the definite integral:

a) $\int_{-1}^{0}(2x-1)\,dx = \overbrace{x^2 - x}^{F(x)}\Big|_{-1}^{0}$

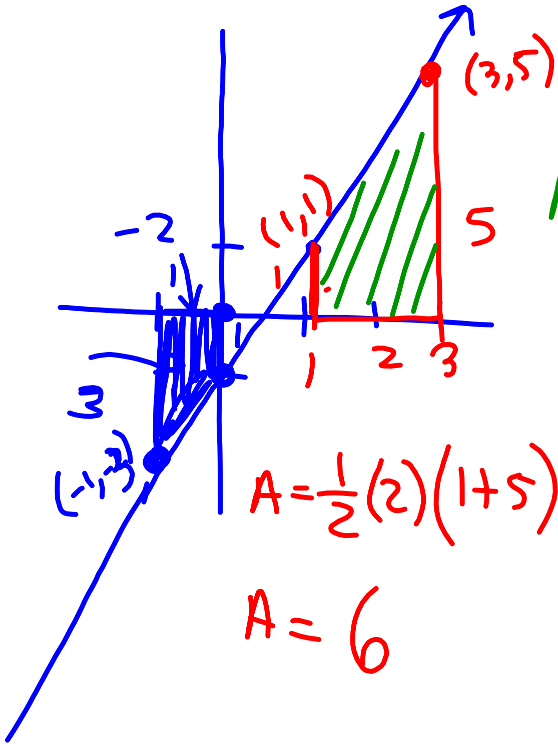
$a = -1$, $b = 0$

$F(b) - F(a)$

$= (0^2 - 0) - ((-1)^2 - (-1))$

$= 0 - 2$

$= -2$



$A = \frac{1}{2}(2)(1+5)$

$A = 6$

$A = \int_{1}^{3}(2x-1)\,dx$

$= x^2 - x\Big|_{1}^{3}$

$= (9-3) - (1-1)$

$= 6$ sq. units

b) $\int_0^1 \frac{x - \sqrt{x}}{3}\,dx$

$$= \int_0^1 \frac{1}{3}x - \frac{1}{3}x^{1/2}\,dx$$

$$= \frac{1}{3}\cdot\frac{x^2}{2} - \frac{1}{3}\frac{x^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{x^2}{6} - \frac{2}{9}x^{3/2}\Big|_0^1$$

$$= \left(\frac{(1)^2}{6} - \frac{2(1)^{3/2}}{9}\right) - (0)$$

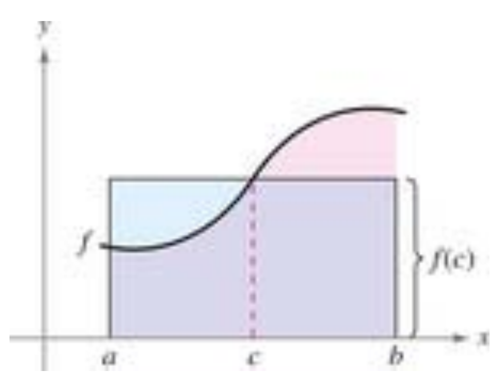
$$= \frac{1\cdot 3}{6\cdot 3} - \frac{2\cdot 2}{9\cdot 2} = \frac{-1}{18}$$

Ex: Find the area of the region bounded by the graphs of the equations $y = 1 + \sqrt[3]{x}$, $x = 0$, $x = 2$, $y = 0$.

<u>Mean Value Theorem for Integrals (Theorem 4.10)</u>:
If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

What is this saying?



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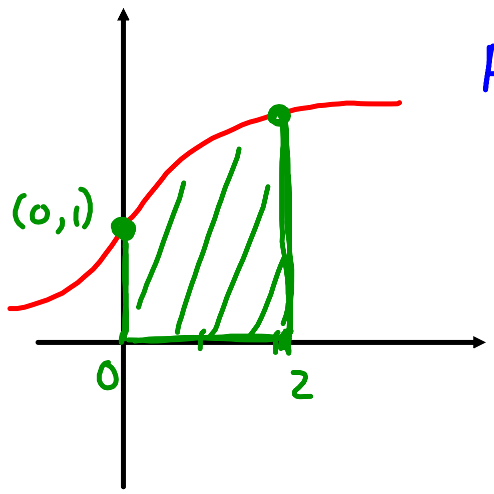


$$A = \int_0^2 1+\sqrt[3]{x}\, dx$$

$$= \int_0^2 1 + x^{1/3}\, dx$$

$$= x + \frac{x^{4/3}}{4/3}\Big|_0^2$$

$$= x + \frac{3x^{4/3}}{4}\Big|_0^2$$

$$= \left(2 + \frac{3(2)^{4/3}}{4}\right) - (0)$$

$$\approx 3.89 \text{ sq. units}$$

<u>Average Value of a Function on an Interval</u>:
If f is integrable on the closed interval $[a, b]$, then the ***average value*** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Note: The value *f(c)* given in the Mean Value Theorem for Integrals is the average value! Why???

Ex: Find the average value(s) of the function over the given interval and all values of x in the interval for which the function equals its average value. $f(x) = x^3,\ [0, 1]$

Average Value of a Function on an Interval:
If f is integrable on the closed interval $[a, b]$, then the ***average value*** of f on the interval is $= f(c)$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Note: The value *f(c)* given in the Mean Value Theorem for Integrals is the average value!
Why???

$$\left(\int_a^b f(x)\,dx = f(c)\,(b-a)\right)\frac{1}{(b-a)}$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$$

Ex: Find the $\overbrace{\text{average value(s)}}^{f(c)}$ of the function over the given interval and all $\underbrace{\text{values of } x}_{c}$ in the interval for which the function equals its average value. $f(x) = x^3, [\overset{a}{0}, \overset{b}{1}]$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{1-0}\int_0^1 x^3\,dx$$

$$= \int_0^1 x^3\,dx$$

$$= \frac{x^4}{4}\Big|_0^1 = \frac{1}{4} = f(c)$$

b) $f(c) = c^3$

$$\sqrt[3]{\frac{1}{4}} = \sqrt[3]{c^3}$$

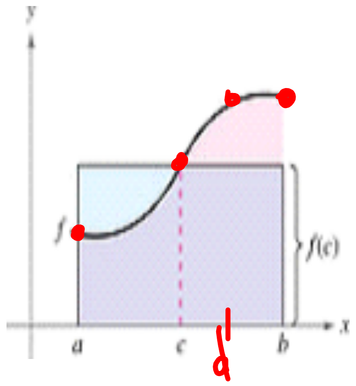
$$\frac{1}{\sqrt[3]{4}} = c$$

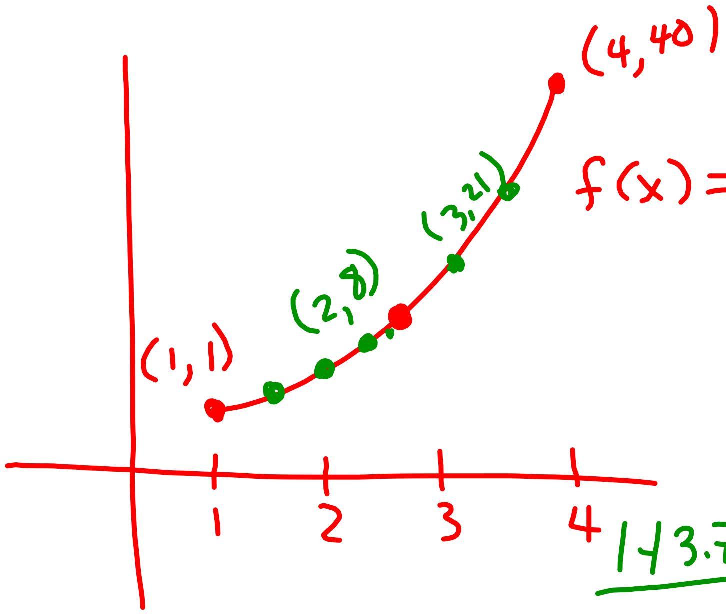
$$\boxed{c = \frac{\sqrt[3]{2}}{2} \approx 0.63}$$

<u>Mean Value Theorem for Integrals (Theorem 4.10)</u>:
If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = \underbrace{f(c)}_{\text{height of rectangle}}\underbrace{(b-a)}_{\text{width of the rectangle}}$$

What is this saying?





$f(x) = 3x^2 - 2x$

$$\frac{1 + 8 + 21 + 40}{4} = 17.5$$

$$\frac{1 + 3.75 + 8 + 13.75 + 21 + 29.75 + 40}{7}$$

$$\approx 16.75$$