Physics 396

Homework Set 8

1. In geometrized units, the Schwarzschild line element takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where M is the total mass of the central object. Consider two massless shells concentric with the central object. The inner shell has a circumference of $6\pi M$ and the outer shell has a circumference of $20\pi M$.

a) Calculate the physical radial distance between these two shells.

b) Calculate the spatial volume between the two spherical shells.

2. In class, we arrived at an equation of motion describing the radial coordinate of a freely-falling particle about a spherically-symmetric massive object of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r), \tag{2}$$

where M is the mass of the central object, $\mathcal{E} \equiv (e^2 - 1)/2$, and

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}. \tag{3}$$

a) Setting $r \equiv Mx$ and $\ell \equiv M\tilde{\ell}$, show that Eq. (3) can be written in the form

$$V_{eff}(x) = -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3}. \tag{4}$$

b) Using your favorite plotting software (i.e. Maple, Matlab, Wolframalpha, etc.), plot $V_{eff}(x)$ vs x for $\tilde{\ell} = 3$, $\tilde{\ell} = \sqrt{12}$, $\tilde{\ell} = 4$, and $\tilde{\ell} = 4.5$. Your x-axis should have a range of $2 < x < 50$ to properly show the curves. These four curves should be positioned on the same plot.

3. Consider the spacetime geometry exterior to a spherically symmetric massive object, in the presence of a constant vacuum energy. The line element takes the form

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where M is the mass of the central object and Λ is the cosmological constant. It is noted that the Λ can take on both positive and negative values. Here, we're interested in obtaining the radial equation of motion for a *massive particle* freely-falling in this curved spacetime, which is analogous to Eq. (2).

a) The line element of Eq. (5) contains two symmetries associated with the t and ϕ coordinates. Construct the two *first integrals* associated with these symmetries of the form

$$-e \equiv \underset{\sim}{\xi} \cdot \underset{\sim}{u} \tag{6}$$

$$\ell \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u}, \tag{7}$$

where e and ℓ are constants, $\underset{\sim}{u}$ is the four-velocity of the massive particle, and ξ, η are the Killing vectors associated with the t, ϕ coordinates, respectively.

b) Another *first integral* can be constructed from the fact that freely-falling massive particles follow *timelike* four-velocities, namely,

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1. \tag{8}$$

Using Eq. (5), construct this first integral.

c) As the angular momentum of this freely-falling massive particle is conserved, one can choose the motion to occur in the $\theta = \pi/2$ plane. Using this fact and Eqs. (6) and (7), show that Eq. (8) can be written in the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r), \tag{9}$$

where $\mathcal{E} \equiv (e^2 - 1)/2$ and

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2) \tag{10}$$

4. Reconsider the spacetime geometry of the previous problem where the line element is given in Eq. (5).

   a) Construct the equation that determines the radii corresponding to the *extrema* of the effective potential. Write the expression as a polynomial and state its order.

   b) Construct the equation that determines the radii corresponding to the *turning points* of the motion.

   c) Setting $r \equiv Mx$, $\ell \equiv M\tilde{\ell}$, and $\Lambda \equiv \tilde{\Lambda}/M^2$, show that Eq. (10) can be written in the form

$$V_{eff}(x) = -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3} - \frac{\tilde{\Lambda}}{6}(\tilde{\ell}^2 + x^2). \tag{11}$$

   d) Using your favorite plotting software (i.e. Maple, Matlab, Wolframalpha, etc.), plot $V_{eff}(x)$ vs x for $\tilde{\ell} = 4.5$ and $\tilde{\Lambda} = +1.0 \times 10^{-5}$, $\tilde{\Lambda} = 0$, and $\tilde{\Lambda} = -1.0 \times 10^{-5}$. Your x-axis should have a range of $2 < x < 80$ to properly show the curves. These three curves should be positioned on the same plot.

5. We now wish to consider *light ray orbits* in the spacetime described by the line element of Eq. (5).

   a) Construct the analogous expressions to Eqs. (6) and (7) in terms of the affine parameter λ.

   b) Construct the first integral corresponding to the fact that light rays have *null* four-velocites, namely,

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = 0. \tag{12}$$

   c) One can choose the motion to occur in the $\theta = \pi/2$ plane. Using this fact and Eqs. (6) and (7), show that Eq. (12) can be written in the form

$$\frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r), \tag{13}$$

   where

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right). \tag{14}$$

   d) Calculate the radius or radii corresponding to the *extrema* of this effective potential. Calculate the value(s) of this effective potential at this radius (or these radii).

e) Construct the equation that determines the radii corresponding to the *turning points* of the motion.