Exercise 1

a.) using

i.)

$|0\rangle|0\rangle \rightsquigarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |0\rangle|1\rangle \rightsquigarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |1\rangle|0\rangle \rightsquigarrow \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |1\rangle|1\rangle \rightsquigarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

find the 4x4 matrix for CNOT. Show that it is unitary.

$|0\rangle|0\rangle \to |0\rangle|0\rangle \; : \; |0\rangle|1\rangle \to |0\rangle|1\rangle \; : \; |1\rangle|0\rangle \to |1\rangle|1\rangle \; : \; |1\rangle|1\rangle \to |1\rangle|0\rangle$

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$\therefore$

$$\boxed{U_{CNOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}}$$

ii.)

$$U_{CNOT}^{\dagger} \times U_{CNOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \times \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$\boxed{U_{CNOT} \text{ is unitary}}$

b.) Consider $|\psi_0\rangle = |0\rangle \cdot \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |0\rangle|1\rangle\right)$.

i.) Find $|\psi\rangle = U_{CNOT}|\psi_0\rangle$

$$|\psi\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |0\rangle|1\rangle\right)}$$

ii.) Is this state entangled

$\boxed{\text{This state is a product}}$

Exercise 1 Continued

c.) Consider $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$

i.) Find $|\psi\rangle = U_{CNOT}|\psi_0\rangle$

$$|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \therefore \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)}$$

ii.) Is this state entangled

$\boxed{\text{This state is entangled.}}$

d.) Consider $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

i.) Find $|\psi\rangle = U_{CNOT}|\psi_0\rangle$

$$|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \therefore \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)}$$

ii.) Is this state entangled

$\boxed{\text{This state is a product}}$

e.) Consider $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$

i.) Find $|\psi\rangle = U_{CNOT}|\psi_0\rangle$

$$|\psi\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \therefore \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|1\rangle)}$$

ii.) Is this state entangled

$\boxed{\text{This state is a product}}$

Exercise 2

Show that $U_{CNOT} = \frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} + \frac{1}{2}(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x = \frac{1}{2}\left[\hat{I}\otimes\hat{I} + \hat{\sigma}_z\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x - \hat{\sigma}_z\otimes\hat{\sigma}_x\right]$

$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$U_{CNOT} = \frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} + \frac{1}{2}(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x$$

$$\hat{I}+\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \; : \; \hat{I}-\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; : \; \frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \; : \; \frac{1}{2}(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} + \frac{1}{2}(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

Exercise 3

One important class of unitaries has form: $\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_n + \hat{\sigma}_n\otimes\hat{I} - \hat{\sigma}_n\otimes\hat{\sigma}_n)$

Where n is some direction.

a.) Determine the matrix for $\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_z + \hat{\sigma}_z\otimes\hat{I} - \hat{\sigma}_z\otimes\hat{\sigma}_z)$

i.) $\hat{I}\otimes\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; $\hat{I}\otimes\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\hat{\sigma}_z\otimes\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$; $\hat{\sigma}_z\otimes\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_z + \hat{\sigma}_z\otimes\hat{I} - \hat{\sigma}_z\otimes\hat{\sigma}_z) = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_z + \hat{\sigma}_z\otimes\hat{I} - \hat{\sigma}_z\otimes\hat{\sigma}_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}$$

ii.) Determine action on $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\cdot\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)$

$$|\psi\rangle = \hat{U}|\psi_0\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \therefore \quad |\psi\rangle = (|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)$$

iii.) Entangled or not?

This is an entangled state

Exercise 3 Continued

b.) Determine the matrix for $\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x + \hat{\sigma}_x\otimes\hat{I} - \hat{\sigma}_x\otimes\hat{\sigma}_x)$

i.) $\hat{I}\otimes\hat{I} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$: $\hat{I}\otimes\hat{\sigma}_x = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$

$\hat{\sigma}_x\otimes\hat{I} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$: $\hat{\sigma}_x\otimes\hat{\sigma}_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$

$$\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x + \hat{\sigma}_x\otimes\hat{I} - \hat{\sigma}_x\otimes\hat{\sigma}_x) = \left[\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} + \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} - \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\right]$$

$$\boxed{\frac{1}{2}(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x + \hat{\sigma}_x\otimes\hat{I} - \hat{\sigma}_x\otimes\hat{\sigma}_x) = \begin{pmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&1\\-1&1&1&1\end{pmatrix}}$$

ii.) Determine action on $|\psi_0\rangle = |0\rangle|0\rangle$

$$|\psi\rangle = \hat{U}|\psi_0\rangle = \frac{1}{2}\begin{pmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&1\\-1&1&1&1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)}$$

iii.) Entangled or not ?

$\boxed{\text{This is an entangled State}}$

Exercise 4

Suppose single qubit has density operator: $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$ where $0 \le r \le 1$. Determine probabilities of outcomes.

a.) SG $\hat{z}$

i.) SG$+\hat{z}$: $\text{Prob}(+\hat{z}) = \langle\langle 0 | \hat{\rho} | 0 \rangle\rangle$

$$\hat{\rho}|0\rangle = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+r \\ 0 \end{pmatrix} : \langle 0|\hat{\rho}|0\rangle = (1\ 0)\cdot\frac{1}{2}\begin{pmatrix} 1+r \\ 0 \end{pmatrix} = \frac{1+r}{2}$$

$$\boxed{\text{prob}(+\hat{z}) = \frac{1+r}{2}}$$

ii.) SG$-\hat{z}$: $\text{Prob}(-\hat{z}) = \langle\langle 1 | \hat{\rho} | 1 \rangle\rangle$

$$\hat{\rho}|1\rangle = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1-r \end{pmatrix} : \langle 1|\hat{\rho}|1\rangle = (0\ 1)\cdot\frac{1}{2}\begin{pmatrix} 0 \\ 1-r \end{pmatrix} = \frac{1-r}{2}$$

$$\boxed{\text{prob}(-\hat{z}) = \frac{1-r}{2}}$$

b.) SG $\hat{x}$

i.) SG$+\hat{x}$: $\text{Prob}(+\hat{x}) = \langle\langle +\hat{x} | \hat{\rho} | +\hat{x} \rangle\rangle$

$$\hat{\rho}|+\hat{x}\rangle = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ 1-r \end{pmatrix} : \langle +\hat{x}|\hat{\rho}|+\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ 1)\cdot\frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ 1-r \end{pmatrix} = \frac{1}{4}\cdot 2$$

$$\boxed{\text{prob}(+\hat{x}) = 1/2}$$

ii.) SG$-\hat{x}$: $\text{Prob}(-\hat{x}) = \langle\langle -\hat{x} | \hat{\rho} | -\hat{x} \rangle\rangle$

$$\hat{\rho}|-\hat{x}\rangle = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ r-1 \end{pmatrix} : \langle -\hat{x}|\hat{\rho}|-\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ -1)\cdot\frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ r-1 \end{pmatrix} = \frac{1}{4}\cdot 2$$

$$\boxed{\text{prob}(-\hat{x}) = 1/2}$$

Exercise 5

Suppose single qubit has density operator: $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ where $0 \le r \le 1$. Determine probabilities of outcomes.

a.) SG $\hat{z}$

i.) SG$+\hat{z}$: $\text{Prob}(+\hat{z}) = \langle\langle 0|\hat{\rho}|0\rangle\rangle$

$$\hat{\rho}|0\rangle = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ r \end{pmatrix} : \langle 0|\hat{\rho}|0\rangle = (1\ 0)\cdot\frac{1}{2}\begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{1}{2}$$

$$\boxed{\text{prob}(+\hat{z}) = 1/2}$$

ii.) SG$+\hat{z}$: $\text{Prob}(+\hat{z}) = \langle\langle 1|\hat{\rho}|1\rangle\rangle$

$$\hat{\rho}|1\rangle = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} r \\ 1 \end{pmatrix} : \langle 1|\hat{\rho}|1\rangle = (0\ 1)\cdot\frac{1}{2}\begin{pmatrix} r \\ 1 \end{pmatrix} = \frac{1}{2}$$

$$\boxed{\text{prob}(-\hat{z}) = 1/2}$$

b.) SG $\hat{x}$

i.) SG$+\hat{x}$: $\text{Prob}(+\hat{x}) = \langle\langle +\hat{x}|\hat{\rho}|+\hat{x}\rangle\rangle$

$$\hat{\rho}|+\hat{x}\rangle = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ r+1 \end{pmatrix} : \langle +\hat{x}|\hat{\rho}|+\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ 1)\cdot\frac{1}{2\sqrt{2}}\begin{pmatrix} 1+r \\ 1+r \end{pmatrix} = \frac{1+r}{2}$$

$$\boxed{\text{prob}(+\hat{x}) = \frac{1+r}{2}}$$

ii.) SG$-\hat{x}$: $\text{Prob}(-\hat{x}) = \langle\langle -\hat{x}|\hat{\rho}|-\hat{x}\rangle\rangle$

$$\hat{\rho}|-\hat{x}\rangle = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1-r \\ r-1 \end{pmatrix} : \langle -\hat{x}|\hat{\rho}|-\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ -1)\cdot\frac{1}{2\sqrt{2}}\begin{pmatrix} 1-r \\ r-1 \end{pmatrix} = \frac{1-r}{2}$$

$$\boxed{\text{prob}(-\hat{x}) = \frac{1-r}{2}}$$

Exercise 6

For the following States + Probs construct density operators matrix

a.) $|0\rangle$ with prob q, $|1\rangle$ with prob $1-q$

$$q|0\rangle\langle 0| + (1-q)|1\rangle\langle 1| = q\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + (1-q)\begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = q\begin{pmatrix}1&0\\0&0\end{pmatrix} + (1-q)\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\therefore \quad \boxed{\hat{\rho} = \begin{pmatrix}q&0\\0&1-q\end{pmatrix}}$$

b.) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ with prob q, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ with prob $1-q$

$$q|{+\hat{x}}\rangle\langle{+\hat{x}}| + (1-q)|{-\hat{x}}\rangle\langle{-\hat{x}}| = q\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}(1\ 1) + (1-q)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}(1\ -1)$$

$$= \frac{q}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{(1-q)}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}q&q\\q&q\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1-q&q-1\\q-1&1-q\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&2q-1\\2q-1&1\end{pmatrix}$$

$$\therefore \quad \boxed{\hat{\rho} = \frac{1}{2}\begin{pmatrix}1&2q-1\\2q-1&1\end{pmatrix}}$$

c.) Show that

i.) $\hat{\rho} = \frac{1}{2}\begin{pmatrix}1+r&0\\0&1-r\end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_z)$

$$\hat{I} = \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \hat{\sigma}_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} : \ \hat{I} + r\hat{\sigma}_z = \begin{pmatrix}1&0\\0&1\end{pmatrix} + r\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}r&0\\0&-r\end{pmatrix} = \begin{pmatrix}1+r&0\\0&1-r\end{pmatrix}$$

$$\frac{1}{2}(\hat{I} + r\hat{\sigma}_z) = \frac{1}{2}\begin{pmatrix}1+r&0\\0&1-r\end{pmatrix} = \hat{\rho}$$

$$\therefore \quad \boxed{\hat{\rho} = \frac{1}{2}\begin{pmatrix}1+r&0\\0&1-r\end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_z)}$$

ii.) $\hat{\rho} = \frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)$

$$\hat{I} = \begin{pmatrix}1&0\\0&1\end{pmatrix},\ \hat{\sigma}_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} : \ \hat{I} + r\hat{\sigma}_x = \begin{pmatrix}1&0\\0&1\end{pmatrix} + r\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&r\\r&0\end{pmatrix} = \begin{pmatrix}1&r\\r&1\end{pmatrix}$$

$$\frac{1}{2}(\hat{I} + r\hat{\sigma}_x) = \frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix} = \hat{\rho}$$

$$\therefore \quad \boxed{\hat{\rho} = \frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)}$$

Exercise 7

$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$ use $Tr[\hat{\rho}\hat{P}_0]$ to get probabilities for

a.) SG $\hat{x}$ $\quad |+\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\langle +\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$, $\langle -\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$

i.)

$$\hat{P}_+ = |+\hat{x}\rangle\langle +\hat{x}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r & 1+r \\ 1-r & 1-r \end{pmatrix} : Tr(\hat{\rho}\cdot\hat{P}_+) = \frac{1}{4}(1+r+1-r) = \frac{1}{2}$$

$$\boxed{prob(+\hat{x}) = \frac{1}{2}}$$

ii.)

$$\hat{P}_- = |-\hat{x}\rangle\langle -\hat{x}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} : \hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_- = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r & -r-1 \\ r-1 & 1-r \end{pmatrix} : Tr(\hat{\rho}\cdot\hat{P}_+) = \frac{1}{4}(1+r+1-r) = \frac{1}{2}$$

$$\boxed{prob(-\hat{x}) = \frac{1}{2}}$$

b.) SG $\hat{y}$ $\quad |+\hat{y}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-\hat{y}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$\langle +\hat{y}| = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)$, $\langle -\hat{y}| = \frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)$

i.)

$$\hat{P}_+ = |\hat{y}\rangle\langle\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} : \hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r & -i(1+r) \\ i(1-r) & 1-r \end{pmatrix} : Tr(\hat{\rho}\cdot\hat{P}_+) = \frac{1}{4}(1+r+1-r) = \frac{1}{2}$$

$$\boxed{prob(+\hat{y}) = \frac{1}{2}}$$

Exercise 7 Continued

ii.)

$$\hat{\rho}_- = |-\hat{y}\rangle\langle -\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad : \quad \hat{\rho}_- = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\hat{\rho} \cdot \hat{\rho}_- = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r & i(1+r) \\ -i(1-r) & 1-r \end{pmatrix} \quad : \quad Tr(\hat{\rho} \cdot \hat{\rho}_-) = \frac{1}{4}(1+r+1-r) = \frac{1}{2}$$

$$\boxed{prob(-\hat{y}) = \frac{1}{2}}$$

Exercise 8

$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$ use $Tr[\hat{\rho}\hat{P}_o]$ to get probabilities for

a.) SG $\hat{x}$ $\quad |+\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\langle +\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$, $\langle -\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$

i.)

$$\hat{P}_+ = |+\hat{x}\rangle\langle +\hat{x}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \therefore \hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r & 1+r \\ 1+r & 1+r \end{pmatrix} \therefore Tr(\hat{\rho}\cdot\hat{P}_+) = \frac{1}{4}(1+r+1+r) = \frac{1}{2}(1+r)$$

$$\boxed{\text{prob}(+\hat{x}) = \frac{1}{2}(1+r)}$$

ii.)

$$\hat{P}_- = |-\hat{x}\rangle\langle -\hat{x}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \therefore \hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1-r & r-1 \\ r-1 & 1-r \end{pmatrix} \therefore Tr(\hat{\rho}\cdot\hat{P}_-) = \frac{1}{4}(1-r+1-r) = \frac{1}{2}(1-r)$$

$$\boxed{\text{prob}(-\hat{x}) = \frac{1}{2}(1-r)}$$

b.) SG $\hat{y}$ $\quad |+\hat{y}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-\hat{y}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$\langle +\hat{y}| = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)$, $\langle -\hat{y}| = \frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)$

i.)

$$\hat{P}_+ = |\hat{y}\rangle\langle \hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \therefore \hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+ir & r-i \\ r+i & 1-ir \end{pmatrix} \therefore Tr(\hat{\rho}\cdot\hat{P}_+) = \frac{1}{4}(1+ir+1-ir) = \frac{1}{2}$$

$$\boxed{\text{prob}(+\hat{y}) = \frac{1}{2}}$$

Exercise 8 Continued

ii.)

$$\hat{\rho}_- = |-\hat{Y}\rangle\langle -\hat{Y}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad : \quad \hat{\rho}_- = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\hat{\rho}\cdot\hat{\rho}_- = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1-ir & i+r \\ r-i & 1+ir \end{pmatrix} \quad : \quad Tr(\hat{\rho}\cdot\hat{\rho}_-) = \frac{1}{4}(1-ir+1+ir) = \frac{1}{2}$$

$$\boxed{prob(-\hat{Y}) = \frac{1}{2}}$$

Exercise 9

a.) $\hat{\rho} = \frac{1}{2}(\hat{I} + \hat{\sigma}_z)$ can find $|\psi\rangle$ s.t. $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad \hat{\rho} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\rho} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad : \quad |\psi\rangle = |0\rangle \quad \therefore \quad |\psi\rangle\langle\psi| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

$$\boxed{|\psi\rangle = |0\rangle}$$

b.) $\hat{\rho} = \frac{1}{2}(\hat{I} + \hat{\sigma}_x)$ can find $|\psi\rangle$ s.t. $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad : \quad \hat{\rho} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad : \quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \therefore \quad |\psi\rangle\langle\psi| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \checkmark$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}$$

Exercise 10

$\hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)$, $\hat{u}(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

using $\hat{\rho}(\alpha) = \hat{u}(\alpha)\hat{\rho}_0\hat{u}^{\dagger}(\alpha)$, find $\hat{\rho}(\alpha)$

$$\hat{\rho}(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\alpha/2} & re^{-i\alpha/2} \\ re^{i\alpha/2} & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix}$$

$\therefore$

$$\boxed{\hat{\rho}(\alpha) = \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix}}$$

Exercise 11

For just one particle

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \quad \hat{\theta} \longrightarrow \boxed{U(\alpha)} \longrightarrow \text{measure SG } \hat{x}$$

$U(\alpha) \longrightarrow \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

measure SG $\hat{x}$ $\longrightarrow$ Fisher information

$$\hat{\rho}(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\alpha/2} & re^{-i\alpha/2} \\ re^{i\alpha/2} & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix} \quad : \quad \hat{\rho}(\alpha) = \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix}$$

$$|\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad \langle +\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|), \quad \langle -\hat{x}| = \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)$$

Measure $+\hat{x}$: $\langle +\hat{x}|\hat{\rho}(\alpha)|+\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ 1) \cdot \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{4}(1\ 1)\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \frac{1}{4}(1\ 1)\begin{pmatrix} 1 + re^{-i\alpha} \\ 1 + re^{i\alpha} \end{pmatrix} = \frac{1}{4}\left(1 + re^{-i\alpha} + 1 + re^{i\alpha}\right)$$

$$\langle +\hat{x}|\hat{\rho}(\alpha)|+\hat{x}\rangle = \frac{1}{4}(2 + re^{-i\alpha} + re^{i\alpha}) = \frac{1}{4}\left(2 + r\cos(\alpha) - \cancel{ir\sin(\alpha)} + r\cos(\alpha) + \cancel{ir\sin(\alpha)}\right) = \frac{1}{2}(1 + r\cos(\alpha))$$

$$\langle +\hat{x}|\hat{\rho}(\alpha)|+\hat{x}\rangle = \frac{1}{2}(1 + r\cos(\alpha)) \longrightarrow P_+$$

Measure $-\hat{x}$: $\langle -\hat{x}|\hat{\rho}(\alpha)|-\hat{x}\rangle = \frac{1}{\sqrt{2}}(1\ -1) \cdot \frac{1}{2}\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{4}(1\ -1)\begin{pmatrix} 1 & re^{-i\alpha} \\ re^{i\alpha} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$= \frac{1}{4}(1\ -1)\begin{pmatrix} 1 - re^{-i\alpha} \\ re^{i\alpha} - 1 \end{pmatrix} = \frac{1}{4}\left(1 - re^{-i\alpha} + 1 - re^{i\alpha}\right)$$

$$\langle -\hat{x}|\hat{\rho}(\alpha)|-\hat{x}\rangle = \frac{1}{4}(2 - re^{-i\alpha} - re^{i\alpha}) = \frac{1}{4}\left(2 - r\cos(\alpha) + \cancel{ir\sin(\alpha)} - r\cos(\alpha) - \cancel{ir\sin(\alpha)}\right) = \frac{1}{2}(1 - r\cos(\alpha))$$

$$\langle -\hat{x}|\hat{\rho}(\alpha)|-\hat{x}\rangle = \frac{1}{2}(1 - r\cos(\alpha)) \longrightarrow P_-$$

$$F = F_+ + F_-$$

$$F_+ : P_+ = \frac{1}{2}(1 + r\cos(\alpha)) \quad : \quad \frac{\partial P_+}{\partial \alpha} = -\frac{r}{2}\sin(\alpha) \quad : \quad \left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{r^2}{4}\sin^2(\alpha)$$

$$\frac{2}{1 + r\cos(\alpha)} \cdot \frac{r^2}{4}\sin^2(\alpha) = \frac{r^2\sin^2(\alpha)}{2(1 + r\cos(\alpha))} = F_+$$

Exercise 11 continued

$$F_- : P_- = \frac{1}{2}(1-r\cos(\alpha)) : \frac{\partial P_-}{\partial \alpha} = \frac{r}{2}\sin(\alpha) : \left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{r^2}{4}\sin^2(\alpha)$$

$$\frac{2}{1-r\cos(\alpha)} \cdot \frac{r^2}{4}\sin^2(\alpha) = \frac{r^2\sin^2(\alpha)}{2(1-r\cos(\alpha))} = F_-$$

$$F = F_+ + F_- = \frac{r^2\sin^2(\alpha)}{2(1+r\cos(\alpha))} + \frac{r^2\sin^2(\alpha)}{2(1-r\cos(\alpha))} = \frac{r^2}{2}\left(\frac{\sin^2(\alpha)}{1+r\cos(\alpha)} + \frac{\sin^2(\alpha)}{1-r\cos(\alpha)}\right)$$

$$= \frac{r^2}{2}\left(\frac{(1-r\cos(\alpha))\sin^2(\alpha) + (1+r\cos(\alpha))\sin^2(\alpha)}{1-r^2\cos^2(\alpha)}\right) = \frac{r^2}{2}\left(\frac{2\cdot\sin^2(\alpha)}{1-r^2\cos^2(\alpha)}\right)$$

$$= \frac{r^2\cdot\sin^2(\alpha)}{1-r^2\cos^2(\alpha)} =$$

$$\boxed{F = \frac{r^2\cdot\sin^2(\alpha)}{1-r^2\cos^2(\alpha)}}$$