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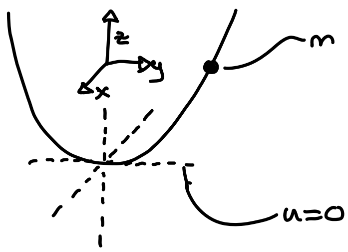
Dr. Workman
PHYS 342
HW 6 Ch. 7

Problem 1



$az = x^2+y^2=r^2$ $\quad L = T-u \quad u = mgh = mgz$

$az = r^2 \quad az = x^2+y^2$

$x^2+y^2=r^2$

$T = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2) \quad u = mgz$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{K=1}^{\infty} \lambda_K \frac{\partial F_K}{\partial q_i} = 0 \qquad F = az - r^2$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$L = \frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2) - mgz \qquad mr^2\dot{\theta} = \frac{\partial L}{\partial \dot{\theta}}, \quad \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \sum_{K=1}^{\infty} \lambda_K \frac{\partial F}{\partial r} = 0 : \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}, \quad \frac{\partial F}{\partial r} = (-2r)$$

$$mr\dot{\theta}^2 - m\ddot{r} - 2\lambda r = 0$$

$$\underline{mr\dot{\theta}^2 - m\ddot{r} - 2\lambda r = 0}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \sum_{K=1}^{\infty} \lambda_K \frac{\partial F}{\partial \theta} = 0 : \quad \frac{\partial L}{\partial \theta} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}, \quad \frac{\partial F}{\partial \theta} = 0$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$$

$$\underline{2\dot{r}\dot{\theta} + r\ddot{\theta} = 0} \qquad \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} + \sum_{K=1}^{\infty} \lambda_K \frac{\partial F}{\partial z} = 0 : \quad \frac{\partial L}{\partial z} = -mg, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z}, \quad \frac{\partial F}{\partial z} = a$$

$$-mg - m\ddot{z} + \lambda a = 0$$

$$\underline{-mg - m\ddot{z} + \lambda a = 0}$$

$$\boxed{\ddot{r} = r\dot{\theta}^2 - \frac{2r\lambda}{m} \quad \ddot{z} = \frac{\lambda a}{m} - g \quad \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} \quad f = az - r^2}$$

Problem 2

a.) $\dot{\theta} = \omega = \sqrt{2g/a}$ $\qquad \ddot{r} = 0 \qquad \ddot{z} = 0$

$$r \cdot \frac{2g}{a} - \frac{2r\lambda}{m} = 0 \qquad 0 = \frac{\lambda a}{m} - g \;, \quad \lambda = \frac{mg}{a}$$

$$r\frac{2g}{a} - \frac{2r}{m}\left(\frac{mg}{a}\right) = 0$$

$$\frac{2gr}{a} - \frac{2gr}{a} = 0 \quad ✓$$

$$\boxed{\text{with } \lambda = \frac{mg}{a}}$$

b.) $\omega = \frac{2\pi}{P}$, $P = \frac{1}{f}$

$$\ddot{r} = r\dot{\theta}^2 - \frac{2r\lambda}{m} : \lambda = \frac{mg}{a} \qquad r = r_0 + u \qquad \frac{\cancel{\partial L}}{\cancel{\partial \theta}}^{\,0} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 : -\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$\ddot{u} = (r_0+u)\dot{\theta}^2 - \frac{2(r_0+u)\lambda}{m} \qquad \dot{r} = \dot{u} \qquad \ddot{r} = \ddot{u}$$

must be = to constant

$m r^2 \dot{\theta} = \alpha$

$\dot{\theta} = \frac{\alpha}{mr^2}$: we say $\dot{\theta} = \omega$: $\omega = \frac{\alpha}{mr^2} = \frac{\alpha}{mah}$: $\alpha = \omega \cdot mah$

comes from radius a, height h

$\dot{\theta} = \frac{wah}{r^2}$: $\dot{\theta}^2 = \frac{\omega^2 a^2 h^2}{r^4}$

now,

$$\ddot{r} = -\frac{2r}{m}\left(\frac{mg}{a}\right) + r\left(\frac{wah}{r^2}\right)^2 = -\frac{2rg}{a} + \frac{\omega^2 a^2 h^2}{r^3} \quad : \quad \ddot{r} = \frac{\omega^2 a^2 h^2}{r^3} - \frac{2gr}{a}$$

$$r \to r_0 + u \quad : \quad \ddot{r} = \frac{\omega^2 a^2 h^2}{(r_0+u)^3} - \frac{2g(r_0+u)}{a}$$

Smallest oscillation : $(r_0+u)^3 = r_0(1+u/r_0)^3$ $\qquad \sum_{n=0}^{\infty} \frac{F^{(n)}(x-a)^{(n)}}{n!}$

Taylor Expand: $F(u) = r_0^{-1}(1+u/r_0)^{-3}$

$$F(0) = r_0^{-1}(1+0/r_0)^{-3} = r_0^{-1}(1)^{-3} = r_0^{-1} \quad : \quad F^{(0)} = \frac{1}{r_0}$$

$$F'(u) = -3r_0^{-1}(1+u/r_0)^{-4} \cdot r^{-1} = -3r_0^{-2}(1+u/r_0)^{-4}$$

$$F'(0) = -3r_0^{-2}(1+0/r_0)^{-4} = -3r_0^{-2}(1) \quad : \quad F^{(1)} = \frac{-3}{r_0^2}$$

$$\text{Taylor} = \frac{1}{r_0} - \frac{3}{r_0^2}u = 1 - \frac{3}{r_0}u \quad : \quad \left[1 - \frac{3}{r_0}u\right]$$

Problem 3 | Continued

$ah=r_0^2 \Leftrightarrow az=r^2,\quad z=h,\quad r=r_0$

$$\ddot{u} = \frac{\omega^2 a^2 h^2}{r_0^3}\left(1-\frac{3}{r_0}u\right) - \frac{2g(r_0+u)}{a}$$

$$= \frac{\omega^2 r_0^4}{r_0^3}\left(1-\frac{3}{r_0}u\right) - \frac{2gr_0}{a} - \frac{2gu}{a} = \frac{\omega^2 r_0^{\cancel{4}}}{\cancel{r_0^3}} - \frac{3\omega^2 \cancel{r_0^4}}{\cancel{r_0^4}}u - \frac{2gr_0}{a} - \frac{2gu}{a}$$

$$= \omega^2 r_0 - \frac{2g}{a}r_0 - 3\omega^2 u - \frac{2g}{a}u \quad : \quad \text{Let } \omega = \sqrt{\frac{2g}{a}}$$

$$= \cancel{\frac{2g}{a}r_0} - \cancel{\frac{2g}{a}r_0} - 3\cdot\frac{2g}{a}u - \frac{2g}{a}u = -\frac{8g}{a}u \qquad \therefore \quad \ddot{u} = -\frac{8g}{a}u$$

$\Delta = b^2-4ac \quad : \quad \Delta < 0 \quad \therefore \quad u = A\cos(\beta t) + B\sin(\beta t) \quad : \quad \beta = -\frac{8g}{a}$

$$\boxed{u(t) = A\cos\left(-\frac{8g}{a}t\right) + B\sin\left(-\frac{8g}{a}t\right) \qquad f = \frac{1}{\pi}\sqrt{\frac{2g}{a}},\quad T = \pi\sqrt{\frac{2g}{a}}}$$

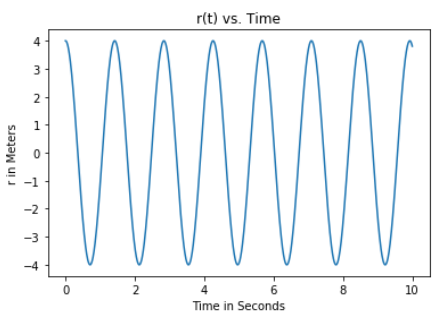
$\omega^2 = \frac{8g}{a}$: $\omega = \frac{2\pi}{T}$ $\quad \frac{2\pi}{T} = \sqrt{\frac{8g}{a}}$: $\frac{1}{T} = f$

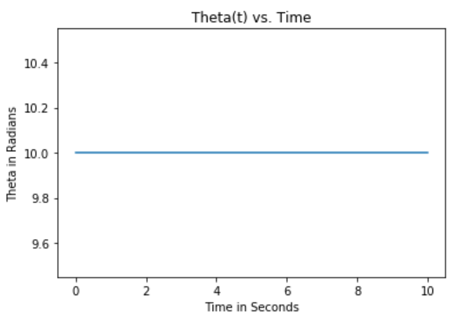
$f\cdot 2\pi = 2\sqrt{\frac{2g}{a}}$ $\quad\therefore\quad f = \frac{1}{\pi}\sqrt{\frac{2g}{a}}$: $T = f^{-1}$

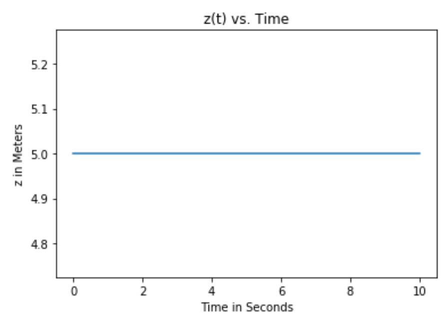
$T = \frac{1}{\frac{1}{\pi}\sqrt{2g/a}} = \pi\sqrt{\frac{a}{2g}}$: $T = \pi\sqrt{\frac{a}{2g}}$

c.) Plot Solutions to (1) $\quad m = 1.0$ kg, $a = 1.0$ m, $g = 9.8$ m/s²

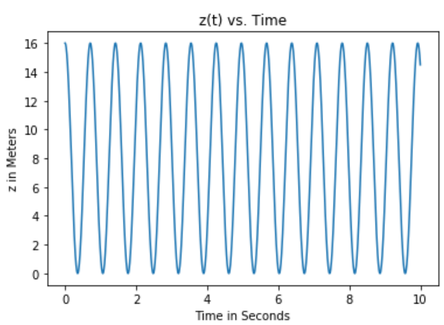
$\ddot{r} = r\dot{\theta}^2 - \frac{2rg}{a}$, Plot for $r(t)$ $\qquad \ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}$, Plot for $\theta(t)$ $\qquad \ddot{z} = \frac{\lambda a}{m} - g$: $\lambda = \frac{mg}{a}$, Plot of $z(t)$







Initial Conditions : $R_0 = 4$, $\dot{R}_0 = 0$, $z_0 = 5$, $\dot{z}_0 = 0$, $\theta_0 = 10$, $\dot{\theta}_0 = 0$



Problem 3

$$F(x,t) = \frac{K}{x^2} e^{-(t/\tau)} \qquad L = T - U \ , \quad H = T + U \quad , \quad U = -\int F(r)\, dr$$

$L:$ $T = \frac{1}{2} m v^2 \cong \frac{1}{2} m \dot{x}^2$

$$U: \quad U(x) = -\int_i^f \frac{ke^{-(t/\tau)}}{x^2} dx = -ke^{-(t/\tau)} \int_i^f \frac{1}{x^2} dx = ke^{-(t/\tau)} \left. \frac{1}{x} \right|_i^f = \frac{ke^{-(t/\tau)}}{\Delta x} \qquad \Delta x \to x$$

$$U = \frac{ke^{-(t/\tau)}}{x}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{ke^{-(t/\tau)}}{x}$$

$H:$ $T = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$

$$U: \quad U(x) = -\int_i^f \frac{ke^{-(t/\tau)}}{x^2} dx = -ke^{-(t/\tau)} \int_i^f \frac{1}{x^2} dx = ke^{-(t/\tau)} \left. \frac{1}{x} \right|_i^f = \frac{ke^{-(t/\tau)}}{\Delta x} \qquad \Delta x \to x$$

$$U = \frac{ke^{-(t/\tau)}}{x}$$

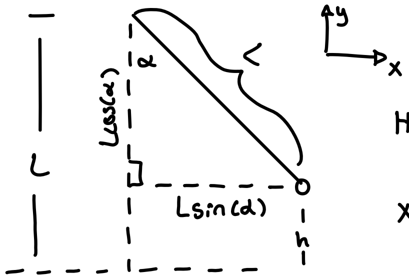
$$H = \frac{p^2}{2m} + \frac{ke^{-(t/\tau)}}{x}$$

$$\boxed{L = \frac{1}{2} m \dot{x}^2 - \frac{K}{x} e^{-(t/\tau)} \ , \quad H = \frac{p^2}{2m} + \frac{K}{x} e^{-(t/\tau)}}$$

There are no dissipative forces that are being applied to the particle in this problem. Because of this we can say that energy is conserved in this system. Because energy is conserved in this system, we can say the Hamiltonian represents the energy.

Problem 4)

a.)



$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$H = L\cos(\alpha) + y \quad \therefore \quad y = H - L\cos(\alpha)$

$x = L\sin(\alpha)\,,\ y = L\cos(\alpha)\,,\ \dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = 0$

$L = T - U \qquad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,,\ U = mgh\,,\ h = L(1-\cos(\alpha))$

constant Length!

$T = \frac{1}{2}m(\dot{L}^2 + L^2\dot{\alpha}^2)\,,\ U = mgL(1-\cos(\alpha)) \qquad L = \frac{1}{2}m(\cancel{\dot{L}^2}^{\,0} + L^2\dot{\alpha}^2) - mgL(1-\cos(\alpha))$

$L = \frac{1}{2}mL^2\dot{\alpha}^2 - mgL(1-\cos(\alpha))$

$\dfrac{\partial L}{\partial \alpha} = -mgL\sin(\alpha)\,,\ \dfrac{\partial L}{\partial \dot{\alpha}} = mL^2\dot{\alpha}\,,\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\alpha}} = mL^2\ddot{\alpha} \quad \therefore \quad mL^2\ddot{\alpha} = -mgL\sin(\alpha) \qquad P_\alpha = m\ell^2\dot{\alpha}$

$\ddot{\alpha} = -\dfrac{g}{L}\sin(\alpha)$

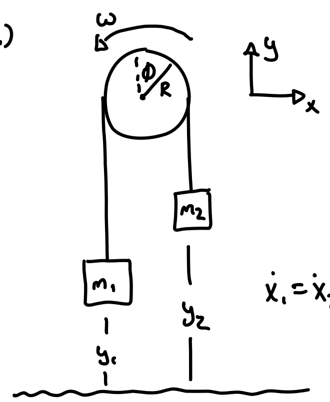
$H = \sum_i P_i\dot{q}_i - L = P_\alpha\dot{\alpha} - \frac{1}{2}mL^2\dot{\alpha}^2 + mgL(1-\cos(\alpha))$

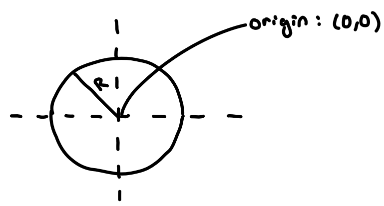
$= mL^2\dot{\alpha}\cdot\dot{\alpha} - \frac{1}{2}mL^2\dot{\alpha}^2 + mgL(1-\cos(\alpha)) = mL^2\dot{\alpha}^2 - \frac{1}{2}mL^2\dot{\alpha}^2 + mgL(1-\cos(\alpha))$

$H = \dfrac{mL^2\dot{\alpha}^2}{2} + mgL(1-\cos(\alpha)) \qquad \dot{\alpha} = \dfrac{\partial H}{\partial P_\alpha} = \dfrac{P_\alpha}{m\ell^2}\,,\ \dot{P}_\alpha = -mgL\sin(\alpha)$

$$\boxed{H = \frac{P_\alpha^2}{2mL^2} + mgL(1-\cos(\alpha))}$$

b.)





$L = T - U$

$H = T + U$

$\dot{x}_1 = \dot{x}_2 \qquad \ell = y_1 + y_2 \qquad \dot{y}_1 = -\dot{y}_2 \qquad I = \frac{1}{2}MR^2 \qquad \omega = \dfrac{v}{R} = \dfrac{\dot{y}}{R}$

$T:\ \frac{1}{2}m_1\dot{y}_1^2\,,\ \frac{1}{2}m_2\dot{y}_2^2\,,\ \frac{1}{2}I\omega^2 \qquad U:\ m_1gy_1\,,\ m_2gy_2$

$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}(I)\dfrac{\dot{y}^2}{R^2} \qquad \dot{y}_1 \to \dot{y}_2\,,\ y_1 \to y$

$U = -m_1gy_1 - m_2g(\ell - y_1) = -m_1gy - m_2g(\ell - y) \qquad m_1gy + m_2g\ell - m_2gy = y(m_1g - m_2g) + m_2g\ell$

$y_1 \to y$

$L = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dfrac{\dot{y}^2}{R^2} + m_1gy + m_2g(\ell - y) \qquad H = T + U = \sum_i P_i\dot{q}_i - L$

$\dfrac{\partial L}{\partial y} = g(m_1 - m_2) \quad \dfrac{\partial L}{\partial \dot{y}} = m_1\dot{y} + m_2\dot{y} + \dfrac{I\dot{y}}{R^2} \qquad H = \dot{y}(m_1 + m_2 + I/R^2)\dot{y} - \frac{1}{2}\dot{y}^2(m_1 + m_2 + I/R^2) - m_1gy - m_2g(\ell - y)$

Problem 4) Continued

$$H = \frac{1}{2}\dot{y}^2(m_1 + m_2 + I/R^2) - m_1 g y - m_2 g(\ell - y)$$

$$\boxed{\begin{aligned} H &= \frac{P_y^2}{2(m_1 + m_2 + I/R^2)} - m_1 g y - m_2 g(\ell - y) \\ P_y &= (m_1 + m_2 + I/R^2)\dot{y} \\ \dot{P}_y &= g(m_1 - m_2) \end{aligned}}$$

Problem 5

$F = k/r^2 \quad , \quad U = -\int_i^f F(r)\,dr \qquad L = T - U \, , \quad H = T + U \qquad T = \frac{1}{2}mv^2$

$U = -\int_i^f \frac{k}{r^2}\,dr = \frac{k}{r}\Big|_i^f = \frac{k}{\Delta r} \to \frac{k}{r} \qquad U = \frac{k}{r} \qquad \dot{r}^2 + r^2\dot{\theta}^2$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r} \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad H = T + U = \sum_i p_i \dot{q}_i - L$

r: $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad H = m\dot{r}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$

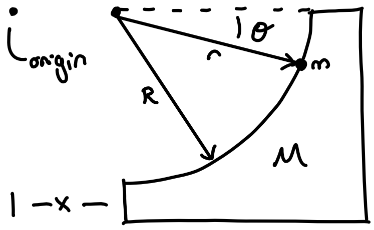
$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + \frac{k}{r^2} \, , \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r} \, , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} \qquad P_r = m\dot{r} \, , \quad \dot{P}_r = mr\dot{\theta}^2 + \frac{k}{r^2}$

θ: $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \qquad H = mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$

$\frac{\partial L}{\partial \theta} = 0 \, , \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \, , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} \qquad P_\theta = mr^2\dot{\theta} \, , \quad \dot{P}_\theta = 0$

$$\boxed{\begin{array}{c|c} H = \frac{P_r^2}{m} - \frac{P_r^2}{2m} - \frac{P_\theta^2}{2mr^2} + \frac{k}{r} & H = \frac{P_\theta^2}{mr^2} - \frac{P_r^2}{2m} - \frac{P_\theta^2}{2mr^2} + \frac{k}{r} \\ P_r = m\dot{r} \, , \quad \dot{P}_r = mr\dot{\theta}^2 + \frac{k}{r^2} & P_\theta = mr^2\dot{\theta} \, , \quad \dot{P}_\theta = 0 \end{array}}$$

Problem 6)



y, x

$L = T - U$

$X_M = x \qquad X_m = x + r\cos\theta \qquad \dot{x}_m = \dot{x} - r\cdot\dot{\theta}\sin\theta\ , \quad \dot{x}_M = \dot{x}$

$Y_M = 0 \qquad Y_m = -r\sin\theta \qquad \dot{y}_m = -r\cdot\dot{\theta}\cos\theta\ ,$

$$T = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) + \frac{1}{2}M(\dot{x}_M)^2\ , \quad U = mgy$$

$$L = \frac{1}{2}m(\dot{x}_m^2+\dot{y}_m^2) + \frac{1}{2}M(\dot{x}_M)^2 + mgr\sin\theta = \frac{1}{2}m([\dot{x} - r\cdot\dot{\theta}\sin\theta]^2 + [-r\cdot\dot{\theta}\cos\theta]^2) + \frac{1}{2}M(\dot{x})^2 + mgr\sin\theta$$

$$= \frac{1}{2}m[(\dot{x}^2 - 2r\cdot\dot{x}\cdot\dot{\theta}\sin\theta + r^2\cdot\dot{\theta}^2\cdot\sin^2\theta) + (r^2\cdot\dot{\theta}^2\cdot\cos^2\theta)] + \frac{1}{2}M(\dot{x}^2) + mgr\sin\theta$$

$$= \frac{1}{2}m[\dot{x}^2 - 2r\cdot\dot{x}\cdot\dot{\theta}\sin\theta + r^2\cdot\dot{\theta}^2] + \frac{1}{2}M\dot{x}^2 + mgr\sin\theta$$

$$L = \frac{1}{2}m(\dot{x}^2 - 2r\dot{x}\dot{\theta}\sin\theta + r^2\dot{\theta}^2) + \frac{1}{2}M(\dot{x}^2) + mgr\sin\theta$$

For x: $\quad \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

$$\frac{\partial L}{\partial x} = 0 \quad : \quad \frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m(2\dot{x} - 2r\dot{\theta}\sin\theta) + \frac{1}{2}M(2\dot{x}) = m(\dot{x} - r\dot{\theta}\sin\theta) + M(\dot{x})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m(\ddot{x} - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta) + M(\ddot{x})$$

$$= m\ddot{x} - mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + M\ddot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x}(M+m) - mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\ddot{x}(M+m) - mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) = 0$$

$$\boxed{\ddot{x} = \frac{mr(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)}{(M+m)}}$$

For θ: $\quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}m(-2r\dot{x}\dot{\theta}\cos\theta) + mgr\cos\theta\ , \quad \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m(-2r\dot{x}\sin\theta + 2r^2\dot{\theta}) = m(r^2\dot{\theta} - r\dot{x}\sin\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m(r^2\ddot{\theta} - r\ddot{x}\sin\theta - r\dot{\theta}\dot{x}\cos\theta)$$

$$-m(r\dot{x}\dot{\theta}\cos\theta) + mgr\cos\theta = mr^2\ddot{\theta} - mr(\ddot{x}\sin\theta + \dot{\theta}\dot{x}\cos\theta)$$

$$-(r\dot{x}\dot{\theta}\cos\theta) + gr\cos\theta + r(\ddot{x}\sin\theta + \dot{\theta}\dot{x}\cos\theta) = r^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{(\ddot{x}\sin\theta + \cancel{\dot{\theta}\dot{x}\cos\theta}) + g\cos\theta - \cancel{\dot{x}\dot{\theta}\cos\theta}}{r}$$

$$\boxed{\ddot{\theta} = \frac{\ddot{x}\sin\theta + g\cos\theta}{r}}$$