Ch 2.4 Error Analysis for Iterative Methods

Defn: Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ & α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

An iterative technique of the form $p_n = g(p_{n-1})$ is of order α if $\{p_n\}_{n=0}^{\infty}$ converges to $p = g(p)$ of order α.

High order convergence usually indicates faster convergence than low order.

Linear convergence : $\alpha = 1$

Quadratic convergence: $\alpha = 2$

<u>Example</u> Consider the two sequences $\{p_n\}_{n=0}^{\infty}$ & $\{\tilde{p}_n\}_{n=0}^{\infty}$ given by

$$p_n = \frac{1}{2^n}, \quad \tilde{p}_n = \left(\frac{1}{2}\right)^{2^n-1}$$

Then $p_n \to 0$ and $\tilde{p}_n \to 0$ with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = 0.5$$

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1} - p|}{|\tilde{p}_n - p|^2} = \lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = \frac{\left(\frac{1}{2}\right)^{2^{n+1}-1}}{\left(\frac{1}{2}\right)^{2^n-1}} = 0.5$$

Thus $\{p_n\}$ converges <u>linearly to zero</u> (of order 1) with <u>asymptotic error constant</u> 0.5 while $\{\tilde{p}_n\}$ converges <u>quadratically to zero</u> with the same <u>asymptotic error constant</u>.

See Table 2.7 for sequence values, ($p_0=1$)

Note that quadratic convergence is much faster than linear convergence.

Table 2.7

n	Linear Convergence Sequence $\{p_n\}_{n=0}^{\infty}$ $(0.5)^n$	Quadratic Convergence Sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ $(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

In this example,

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\hat{p}_n|^2} = 0.5$$

Thus

$$|p_{n+1}| \cong 0.5\,|p_n| \approx (0.5)^2\,|p_{n-1}| \approx \cdots \approx (0.5)^{n+1}\,|p_0|$$

$$|\tilde{p}_{n+1}| \cong 0.5\,|\tilde{p}_n|^2 \approx (0.5)^3\,|\tilde{p}_{n-1}|^4 \approx \cdots \approx (0.5)^{2^{n+1}-1}\,|\tilde{p}_0|^{2^{n+1}}$$

Recall $\lambda = 0.5$ is the asymptotic error constant, with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda > 0.$$

Compare this with results above.
Note that a good initial value gives a smaller error, and it is desirable to have $\lambda < 1$.

Math 361 Ch 2.4 Extra Notes

Key Definition

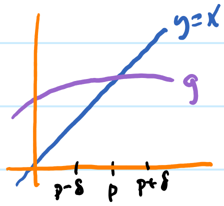
If $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$

then $p_n \to p$ with order α, and λ = asymptotic error constant want <1

Theorems Suppose we are looking for a fixed point p for g, that is, $g(p)=p$ and $p_{n+1}=g(p_n)$ is how we find p.

① If $g'(p) \neq 0$, then $\alpha = 1$ (linear convergence) ← proof uses Taylor Series & MVT

② If $g'(p)=0$ and $g''(x) \in C[a,b]$ with $|g''(x)| \le M$ on $[a,b]$, then $\exists\, \delta > 0$ s.t. if $p_0 \in [p-\delta, p+\delta]$ then $\alpha \ge 2$ (at least quadratic conv)



Theorem Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$.

Suppose in addition that $g' \in C(a,b)$ and $\exists\, k \in (0,1)$ s.t. $|g'(x)| \le k$ on (a,b).

If $g'(p) \neq 0$, then for any $p_0 \in [a,b]$, the sequence given by $p_n = g(p_{n-1})$, $n \ge 1$, converges only linearly to the unique fixed point $p \in [a,b]$.

Proof By Fixed Point Theorem, $p_n \to p$. Since $g' \in C(a,b)$, use MVT to obtain

$$p_{n+1} - p = g(p_n) - p = g'(c_n) \cdot (p_n - p)$$

where $c_n \in (p, p_n)^*$. Since $p_n \to p$, it follows that $c_n \to p$. Since $g' \in C(a,b)$, $\lim_{n\to\infty} g'(c_n) = g'(p) \in (0,1)$. Thus

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\overset{\alpha=1}{}}} = \lim_{n\to\infty} |g'(c_n)| = \underbrace{|g'(p)|}_{\text{Asymptotic error constant}} \in (0,1). \quad \blacksquare$$

$\alpha = 1$

Asymptotic error constant

By Previous Theorem, if $g'(p) \neq 0$, then fixed point iteration exhibits linear convergence with asymptotic error constant $|g'(p)|$.

Thus higher order convergence for fixed point methods can occur only when $g'(p) = 0$. The next theorem describes additional conditions that ensure ~~the~~ quadratic convergence.

<u>Theorem</u> Let p solve $g(x) = x$, $p \in (a,b)$. Suppose $g'(p) = 0$ and $g'' \in C(a,b)$ with $|g''(x)| \leq M$ on (a,b). Then $\exists\, \delta > 0$ s.t. for $p_0 \in [p-\delta, p+\delta]$ the sequence $p_n = g(p_{n-1})$ converges at least quadratically to p. Moreover, for sufficiently large n,

$$|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$$

Proof Choose $k \in (0,1)$ and $\delta > 0$ s.t.

$|g'(x)| \le k$ and g'' continuous,

for $x \in [p-\delta, p+\delta]$. (Recall $g'(p) = 0$).

Using a method given in Section 2.3, $p_n \in [p-\delta, p+\delta]$, $n = 0, 1, 2, \ldots$, where p_0 is in $[p-\delta, p+\delta]$.

Then, using Taylor Series about p,

$$g(x) = g(p) + g'(p)\cdot(x-p) + \frac{g''(\xi(x))}{2}(x-p)^2,$$

where $\xi(x)$ between x and p, and $x \in [p-\delta, p+\delta]$. Thus

$$g(x) = p + \tfrac{1}{2} g''(\xi(p)) \cdot (x-p)^2$$

and hence

$$p_{n+1} = g(p_n) = p + \tfrac{1}{2} g''(\xi(p_n)) \cdot (p_n - p)^2$$

Therefore

$$p_{n+1} - p = \tfrac{1}{2} g''(\xi(p_n)) \cdot (p_n - p)^2$$

Since $p_n \to p$, $\xi(p_n) \to p$, since $\xi(p_n)$ between p_n & p.

From here, we can obtain Theorem results (discuss). ▨

From the previous two theorems, note that quadratically convergent fixed pt. methods involve functions $g(x)$ for which $g'(p) = 0$.

When we seek a root p of $f(x)$, we can convert this to a fixed pt. procedure by defining

$$g(x) = x - \phi(x) f(x),$$

where $\phi(x)$ is a differiable function to be chosen later.

To obtain quadratic convergence, we need $g'(p) = 0$. We have

$$g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x)$$

Thus $g'(p) = 1 - \phi'(p)\cancelto{0}{f(p)} - \phi(p) f'(p)$,

so choose $\phi(p) = 1/f'(p)$, i.e., $\phi(x) = 1/f'(x)$.

Hence

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \text{(Newton's Method)}$$

In the previous discusion, we require that $f'(p) \neq 0$. Newton's Method and the Secant Method give problems when $f'(p) = 0$ when $f(p) = 0$, and when $f'(p_n) \to 0$ simultaneously with $f(p_n)$:

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

To examine these difficulties in more detail, we make the following definition.

<u>Definition</u> A solution p of $f(x) = 0$ is a <u>zero of multiplicity m</u> of f if for $x \neq p$, we can write

$$f(x) = (x-p)^m q(x)$$

where $\lim_{x \to p} q(x) \neq 0$.

Here, $q(x)$ represents the part of f that does not contribute to the zero of f.

A <u>simple zero</u> is one for which $m = 1$.

Theorem A function $f \in C'[a,b]$ has a simple zero at $p \in (a,b)$ iff $f(p) = 0$ and $f'(p) \neq 0$.

Proof Suppose f has a simple zero at p. Then $f(p) = 0$ and $f(x) = (x-p)q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Since $f \in C'[a,b]$,

$$f'(p) = \lim_{x \to p} f'(x) = \lim_{x \to p} \left[q(x) + (x-p)q'(x) \right]$$

$$= \lim_{x \to p} q(x) \neq 0.$$

Conversely, suppose $f(p) = 0$, and $f'(p) \neq 0$. Using a Taylor series about p, we have

$$f(x) = \cancel{f(p)}^{\,0} + f'(\xi(x)) \cdot (x-p),$$

where $\xi(x)$ between x and p. Since $f \in C'[a,b]$,

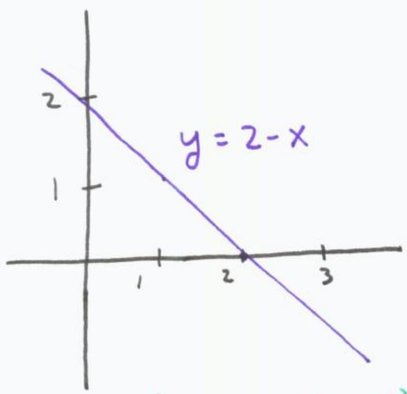
$$\lim_{x \to p} f'(\xi(x)) = f'\left(\lim_{x \to p} \xi(x) \right) = f'(p) \neq 0.$$

Let $q(x) = f'(\xi(x))$. Then $f(x) = (x-p)q(x)$ where $\lim_{x \to p} q(x) \neq 0$. ∎

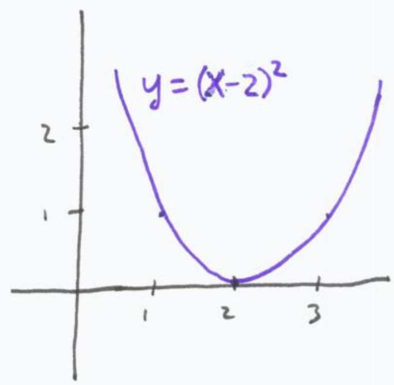
Theorem The function $f \in C^m[a,b]$ has a zero of multiplicity m at $p \in (a,b)$ iff

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p)$$

but $f^{(m)}(p) \neq 0$.



simple zero (m=1)



m = 2

Example $f(x) = e^x - x - 1$.

Note that $f(0) = 0$ and $f'(0) = 0$, while $f''(0) = 1$. Thus f has a zero of multiplicity 2 at $p = 0$.

See Figure 2.11 for a graph of f, and Table 2.8 for Newton's Method iterates. Convergence is not quadratic. $(p_0 = 1)$

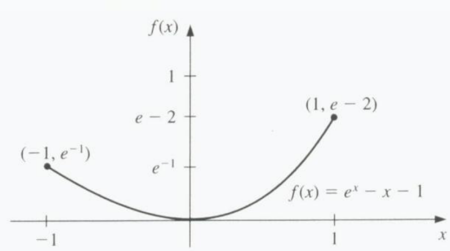


Table 2.8

n	p_n	n	p_n
0	1.0	9	2.7750×10^{-3}
1	0.58198	10	1.3881×10^{-3}
2	0.31906	11	6.9411×10^{-4}
3	0.16800	12	3.4703×10^{-4}
4	0.08635	13	1.7416×10^{-4}
5	0.04380	14	8.8041×10^{-5}
6	0.02206	15	4.2610×10^{-5}
7	0.01107	16	1.9142×10^{-6}
8	0.005545		

One method of handling multiple roots is to define

$$\mu(x) = \frac{f(x)}{f'(x)}.$$

If p is a zero of f of multiplicity m, and $f(x) = (x-p)^m q(x)$, then

$$\mu(x) = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p) q'(x)}$$

$$= (x-p) \cdot \frac{q(x)}{m q(x) + (x-p) q'(x)}$$

Thus $\mu(p) = 0$, but $q(p) \neq 0$ and hence

$$\frac{q(p)}{m q(p) + (p-p) q'(p)} = \frac{1}{m} \neq 0$$

Thus $\mu(x)$ has a simple zero at p.

Hence Newton's Method can be applied to $\mu(x)$.

We have $f(x) = (x-p)^m q(x)$ and

$$\mu(x) = \frac{f(x)}{f'(x)} = (x-p)\frac{q(x)}{mq(x) + (x-p)q'(x)}$$

To apply Newton's Method to $\mu(x)$, let

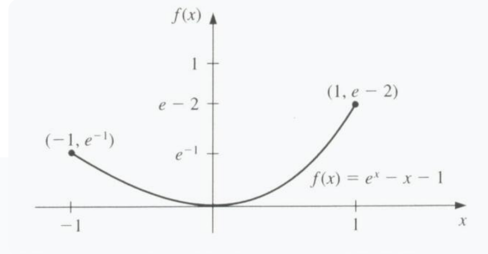
$$g(x) = x - \frac{\mu(x)}{\mu'(x)}$$

Taking derivatives & simplifying,

$$g(x) = x - \frac{f(x)\cdot f'(x)}{[f'(x)]^2 - f(x)\cdot f''(x)}$$

If g has the required continuity conditions, functional iteration (Newton's Method) applied to g will be quadratically convergent, regardless of the value of m, the multiplicity of the zero p of f.

Drawbacks: Need $f''(x)$, and formula for g complicated. Also, cancellation error in denominator of g.



Example $f(x) = e^x - x - 1$

Double zero at $p = 0$. Use

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

$$= x - \frac{(e^x - x - 1)(e^x - 1)}{(e^x - 1)^2 - (e^x - x - 1)(e^x)}$$

Thus

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{(e^{p_{n-1}} - p_{n-1} - 1)(e^{p_{n-1}} - 1)}{(e^{p_{n-1}} - 1)^2 - e^{p_{n-1}}(e^{p_{n-1}} - p_{n-1} - 1)}$$

Starting with $p_0 = 1$, the results of this method are given in Table 2.9. Quadratic convergence.

Results here are much faster than with Newton's Method applied to f (Table 2.8), where convergence was linear.

However, it turns out that both numerator and denominator of g approach zero, and no improvement occurs after p_5.

Table 2.8

n	p_n	n	p_n
0	1.0	9	2.7750×10^{-3}
1	0.58198	10	1.3881×10^{-3}
2	0.31906	11	6.9411×10^{-4}
3	0.16800	12	3.4703×10^{-4}
4	0.08635	13	1.7416×10^{-4}
5	0.04380	14	8.8041×10^{-5}
6	0.02206	15	4.2610×10^{-5}
7	0.01107	16	1.9142×10^{-6}
8	0.005545		

Table 2.9

n	p_n
1	$-2.3421061 \times 10^{-1}$
2	$-8.4582788 \times 10^{-3}$
3	$-1.1889524 \times 10^{-5}$
4	$-6.8638230 \times 10^{-6}$
5	$-2.8085217 \times 10^{-7}$

In our class notes for this section, one of the examples is to solve $f(x) = e^x - x - 1 = 0$. Since $f(0) = f'(0) = 0$ and $f''(0) \neq 0$, it follows that $p = 0$ is a zero of multiplicity two. See Figure 1. In the work below, we compare Newton's Method with the Modified Newton's Method. Note that in Figure 2 we see that $g_1'(0) \neq 0$ while $g_2'(0) = 0$.

$$f(x) = e^x - x - 1$$

$$f'(x) = e^x - 1$$

$$f''(x) = e^x$$

$$g_1(x) = x - \frac{f(x)}{f'(x)}$$

$$g_2(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

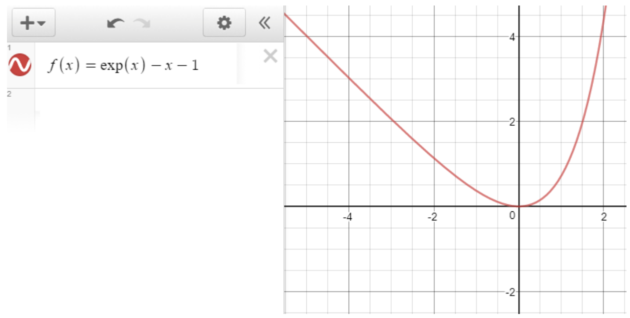


Figure 1

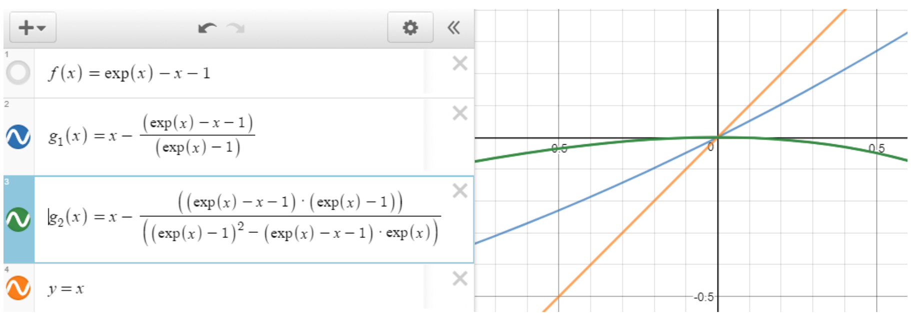


Figure 2

	Find Root	f(x) =exp(x)-x-1										
		f'(x) =exp(x)-1										
		f"(x) = exp(x)										
		g1(x) = x - f(x)/f'(x)					Roots	f(root)				
		g2(x) = x - f(x)f'(x)/((f'(x))^2-f(x)f"(x))					0	0				
n	f(x)	f'(x)	g1(x)	Abs Error	Rel Error	n	f(x)	f'(x)	f"(x)	g2(x)	Abs Error	Rel Error
0			1	1	#DIV/0!	0				1	1	#DIV/0!
1	0.71828183	1.71828183	0.581976707	0.581977	#DIV/0!	1	0.718282	1.718282	2.718282	-0.234210614	0.234211	#DIV/0!
2	0.20759569	0.7895724	0.319055041	0.319055	#DIV/0!	2	0.025406	-0.2088	0.791195	-0.00845828	0.008458	#DIV/0!
3	0.05677201	0.37582705	0.167996173	0.167996	#DIV/0!	3	3.57E-05	-0.00842	0.991577	-1.18902E-05	1.19E-05	#DIV/0!
4	0.01493591	0.18293208	0.086348874	0.086349	#DIV/0!	4	7.07E-11	-1.2E-05	0.999988	-4.21859E-11	4.22E-11	#DIV/0!
5	0.00383773	0.0901866	0.043795704	0.043796	#DIV/0!	5	0	-4.2E-11	1	-4.21859E-11	4.22E-11	#DIV/0!
6	0.00097319	0.04476889	0.022057685	0.022058	#DIV/0!							
7	0.00024507	0.02230275	0.011069387	0.011069	#DIV/0!							
8	6.1492E-05	0.01113088	0.005544905	0.005545	#DIV/0!							
9	1.5401E-05	0.00556031	0.002775014	0.002775	#DIV/0!							
10	3.8539E-06	0.00277887	0.001388149	0.001388	#DIV/0!							
11	9.6392E-07	0.00138911	0.000694235	0.000694	#DIV/0!							
12	2.4104E-07	0.00069448	0.000347158	0.000347	#DIV/0!							
13	6.0266E-08	0.00034722	0.000173589	0.000174	#DIV/0!							
14	1.5067E-08	0.0001736	8.6797E-05	8.68E-05	#DIV/0!							
15	3.767E-09	8.6801E-05	4.33991E-05	4.34E-05	#DIV/0!							
16	9.4175E-10	4.34E-05	2.16997E-05	2.17E-05	#DIV/0!							
17	2.3544E-10	2.17E-05	1.08499E-05	1.08E-05	#DIV/0!							
18	5.886E-11	1.085E-05	5.42495E-06	5.42E-06	#DIV/0!							

For the Modified Newton Method, begin by formatting your Excel sheet as follows.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		Find Root	f(x) =exp(x)-x-1										
2			f'(x) =exp(x)-1										
3			f''(x) = exp(x)										
4			g1(x) = x - f(x)/f'(x)					Roots	f(root)				
5			g2(x) = x - f(x)f'(x)/((f'(x))^2-f(x)f''(x))										
6													
7	*n*	f(x)	f'(x)	g1(x)	Abs Error	Rel Error	*n*	f(x)	f'(x)	f''(x)	g2(x)	Abs Error	Rel Error
8	0												
9	1												

Use the Excel formulas shown below for $g_1(x)$ and for $g_2(x)$ in the table above to get started, and then drag down to complete the table for as many iterations is required. Note that to compute the absolute and relative error you will need to perform the "What-If Analysis", unless the solution p is given or known.

n	f(x)	f'(x)	g1(x)	Abs Error	Rel Error
0			1	=ABS(H5-D8)	=ABS(H5-D8)/ABS(H5)
1	=EXP(D8)-D8-1	=EXP(D8)-1	=D8-B9/C9	=ABS(H5-D9)	=ABS(H5-D9)/ABS(H5)
2	=EXP(D9)-D9-1	=EXP(D9)-1	=D9-B10/C10	=ABS(H5-D10)	=ABS(H5-D10)/ABS(H5)

n	f(x)	f'(x)	f''(x)	g2(x)	Abs Error	Rel Error
0				1	=ABS(H5-K8)	=ABS(H5-K8)/ABS(H5)
1	=EXP(K8)-K8-1	=EXP(K8)-1	=EXP(K8)	=K8-H9*I9/(I9^2-H9*J9)	=ABS(H5-K9)	=ABS(H5-K9)/ABS(H5)
2	=EXP(K9)-K9-1	=EXP(K9)-1	=EXP(K9)	=K9-H10*I10/(I10^2-H10*J10)	=ABS(H5-K10)	=ABS(H5-K10)/ABS(H5)