MAT 201
Larson/Edwards – Section 1.4
Continuity and One-Sided Limits

When one says that "a function is continuous at $x = c$" or "a function is continuous on an open interval", what conclusions can we draw about the function?

<u>Definition of Continuity</u>:

Continuity at a Point: A function f is continuous at c if the following three conditions are met:

1. $f(c)$ is defined
2. $\lim\limits_{x \to c} f(x)$ exists
3. $\lim\limits_{x \to c} f(x) = f(c)$

Continuity on an Open Interval: A function is continuous on an open interval (a, b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

Note: Polynomial functions are continuous at every point x. Rational functions are continuous at every point x where the denominator does not equal 0.

<u>Discontinuity</u>: If a function f is defined on an open interval containing a real number c, and f is not continuous at c, then f is said to have a discontinuity at c. Discontinuities come in two categories:

1. ***Removable*:** A discontinuity at $x = c$ is called removable if f can be made continuous by appropriately defining (or redefining) $f(c)$.

When one says that “a function is continuous at $x = c$” or
“a function is continuous on an open interval”, what conclusions can we draw about the function?

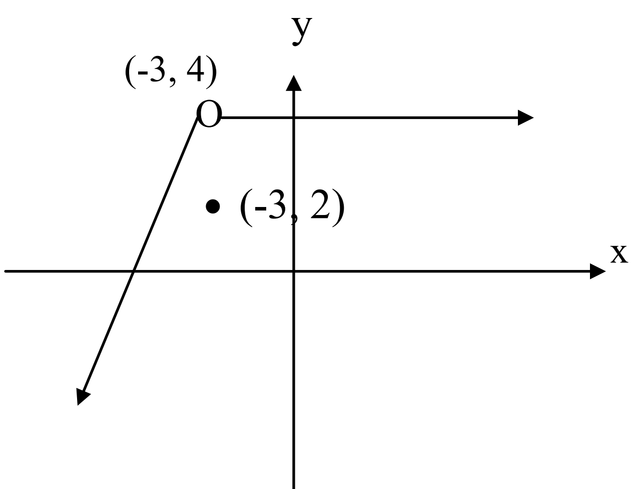
You can draw the function on
that open interval without
lifting your pencil.

a. ***Ex:*** The function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at $x = 2$ (or the point (2, 4)), but it is removable since we can reduce or redefine the function to $f(x) = x + 2$

2. ***Nonremovable*:** A discontinuity at $x = c$ is called nonremovable if the function cannot be made continuous at $x = c$ by defining (or redefining) $f(c)$.

   a. Ex: The function $f(x) = \dfrac{x - 2}{x + 3}$ is discontinuous at $x = -3$ and it is nonremovable since we cannot redefine the function by reducing.

Ex: Consider the following function that is defined on all real numbers:

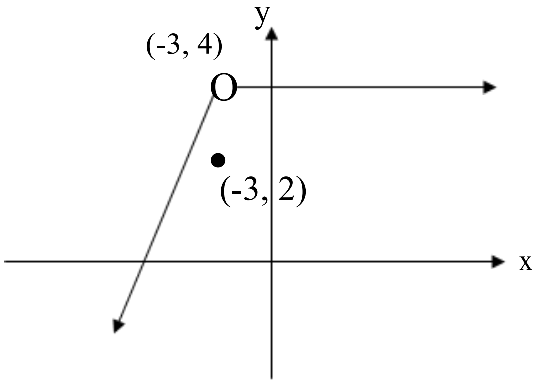


Use the definition of continuity to show why the function is discontinuous at x = -3.

<u>Properties of Continuity</u>: If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Scalar Multiple: bf
2. Sum or Difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, if $g(c) \neq 0$

Ex: Consider the following function that is defined on all real numbers:



Use the definition of continuity to show why the function is discontinuous at x = -3.

$c = -3$

✓1) $f(-3)$ is defined $= 2$

✓2) $\lim_{x \to -3} f(x) = 4$

X3) $\lim_{x \to -3} f(x) \stackrel{?}{=} f(-3)$

$4 \neq 2$

<u>One sided limits</u>:

1. The function *f* has the ***right-hand limit*** *L* as *x* approaches *c* from the right, written $\lim_{x \to c^+} f(x) = L$ if the values *f(x)* can be made as close to *L* as we please by taking *x* sufficiently close to (but not equal to) *c* and greater than *c*.
2. The function f has the ***left-hand limit*** *L* as *x* approaches *c* from the left, written $\lim_{x \to c^-} f(x) = L$ if the values *f(x)* can be made as close to *L* as we please by taking *x* sufficiently close to (but not equal to) *c* and less than *c*.

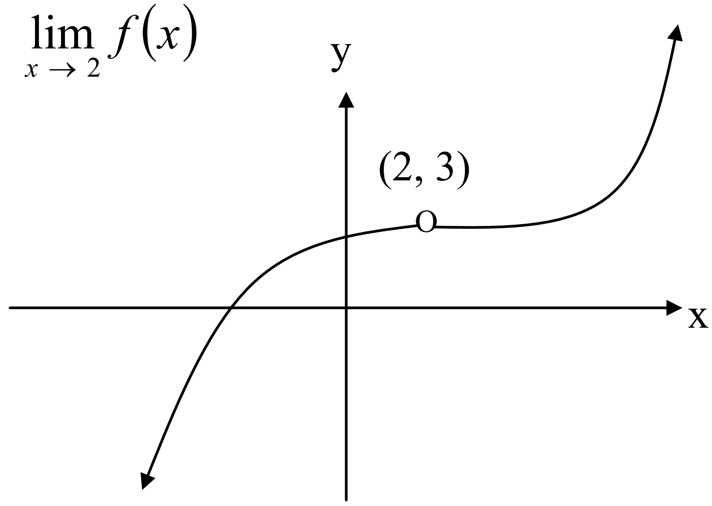
<u>Existence of a Limit</u>: Let *f* be a function and let *c* and *L* be real numbers. The limit of *f(x)* as *x* approaches *c* is *L* if and only if

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

What does this say?

Ex: Use the graph and the definition for the existence of a limit to determine if the limit exists, and if it exists find the limit:

a) $\lim_{x \to 2} f(x)$



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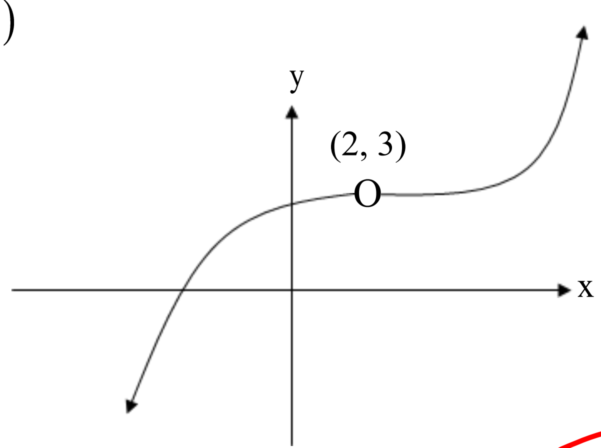


$\lim_{x \to 2^+} f(x) = 3$

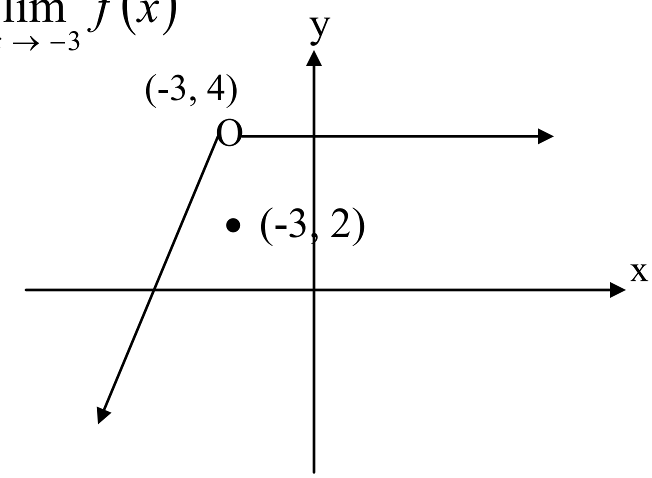
//

$\lim_{x \to 2^-} f(x) = 3$

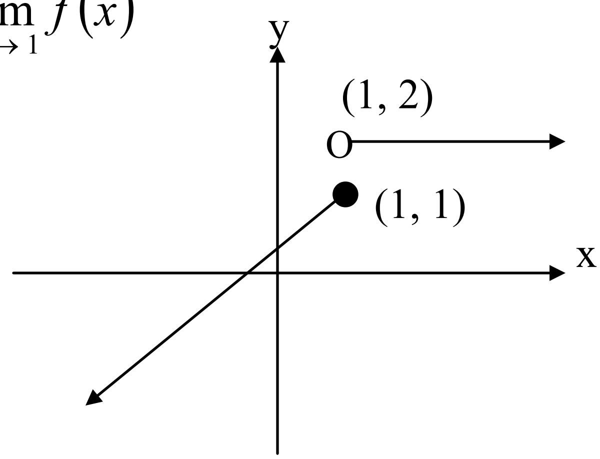
$\therefore$ Limit exists ;

$\lim_{x \to 2} f(x) = 3$

b) $\lim_{x \to -3} f(x)$



c) $\lim_{x \to 1} f(x)$

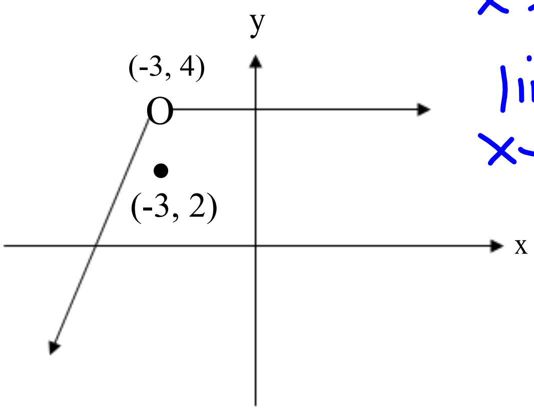


Ex: Find the limit, if it exists.

a) $\lim_{x \to 4^+} \dfrac{4 - x}{x^2 - 16}$

b) $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 - 3x, & x \le 1 \\ x^2 + 2x - 5, & x > 1 \end{cases}$

b) $\lim_{x \to -3} f(x)$

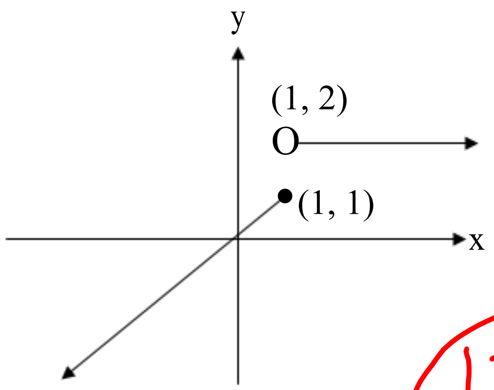


$\lim_{x \to -3^+} f(x) = 4$

$\lim_{x \to -3^-} f(x) = 4$

$$\boxed{\begin{gathered}\lim_{x \to -3} f(x) = 4 \\ \text{(Exists)}\end{gathered}}$$

c) $\lim\limits_{x \to 1} f(x)$



$\lim\limits_{x \to 1^+} f(x) = 2$

$\neq$

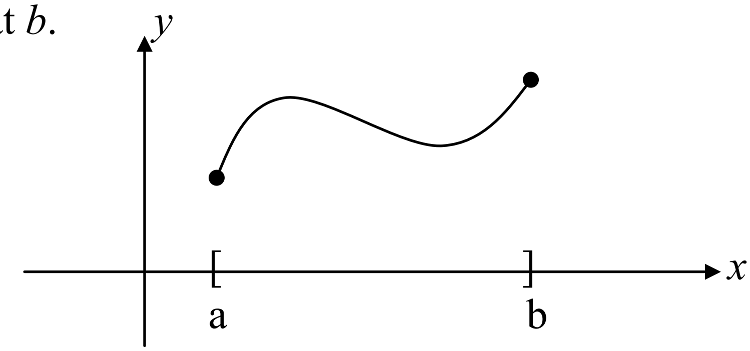
$\lim\limits_{x \to 1^-} f(x) = 1$

Limit D.N.E.

Definition of Continuity on a Closed Interval:
A function f is continuous on the closed interval [a, b] if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b).$$

The function f is continuous from the right at a and continuous from the left at b.



Intermediate Value Theorem:
If f is continuous on the closed interval [a, b], $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in [a, b] such that $f(c) = k$.

What does the Intermediate Value Theorem (IVT) mean?

Ex: Explain why the function $f(x) = x^2 - 2 - \cos x$ has a zero in the interval $[0, \pi]$.

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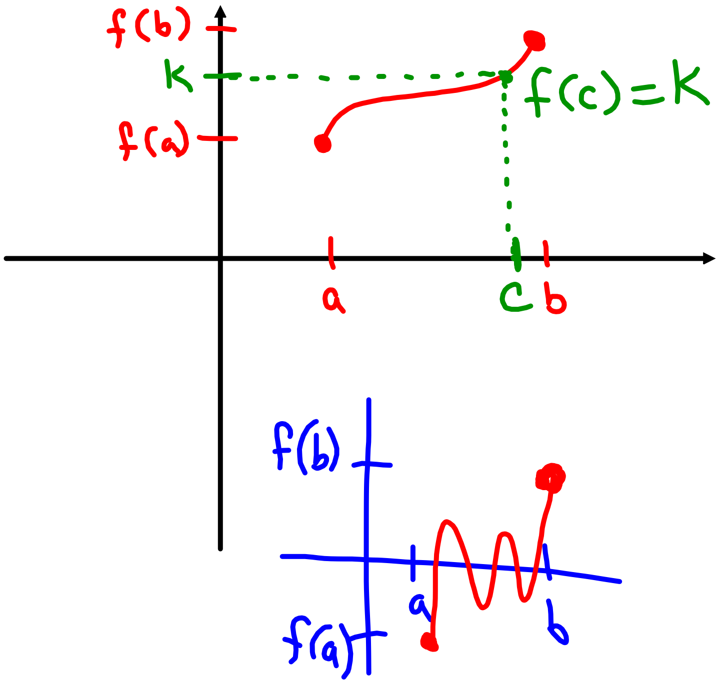
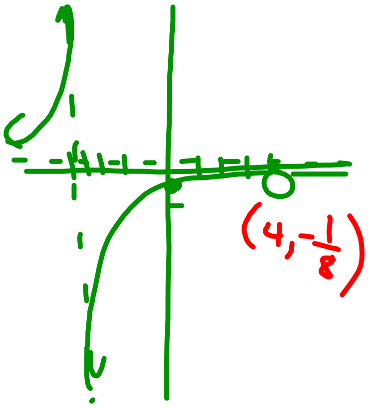


Ex: Find the limit, if it exists.

$$\lim_{x \to 4^+} \frac{4 - x}{x^2 - 16} = \lim_{x \to 4^+} \frac{\overset{-1}{\cancel{4-x}}}{(x+4)\cancel{(x-4)}}$$

Hole @ $\left(4, -\frac{1}{8}\right)$



$$= \lim_{x \to 4^+} \frac{-1}{x+4}$$

$$= -\frac{1}{8}$$

a) $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 - 3x, & x \le 1 \\ x^2 + 2x - 5, & x > 1 \end{cases}$

$$\lim_{x \to 1^-} f(x) = (1)^3 - 3(1) = -2$$

$$\lim_{x \to 1^+} f(x) = (1)^2 + 2(1) - 5 = -2$$

$$\lim_{x \to 1} f(x) = -2$$

Ex: Explain why the function $f(x) = \underset{\text{cont.}}{x^2} - \underset{\text{cont.}}{2} - \cos x$ cont.
has a zero in the interval [0, π].

Is $f(x)$ continuous on $[0, \pi]$? Continuous since all the parent functions listed being subtracted are continuous on $[0, \pi]$.

Does $f(a) \neq f(b)$?

$f(0) = -3$

$f(\pi) = \pi^2 - 1 \approx 8.86$

$f(0) \neq f(\pi)$

Is 0 between -3 & 8.86? Yes!

(By I.V.T)

$\therefore$ There exists at least one value $x = c$ in $[0, \pi]$ s.t. $f(c) = 0$.