MAT 202
Larson – Section 8.5
Partial Fractions

In Precalculus we discussed a process called ***partial fraction decomposition***. This process involves decomposing a rational function into a sum/difference of simpler rational functions. The benefit of knowing this process becomes very clear in the following example:

$$\int \frac{1}{x^2 - 5x + 6}\,dx$$

To evaluate this integral without partial fractions, we would need to complete the square in the denominator, then apply trigonometric substitution. Eventually (see page 542 of text), after much work, we would arrive at our solution of:

$$\int \frac{1}{x^2 - 5x + 6}\,dx = \ln|x - 3| - \ln|x - 2| + C$$

If we had recognized that the rational expression $\dfrac{1}{x^2 - 5x + 6}$

can be written as $\dfrac{1}{x - 3} - \dfrac{1}{x - 2}$, we would have been able to integrate each fraction separately and as a result the amount of work needed would have been cut in half. We will now learn how to decompose rational functions into partial fractions.

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{x-3} - \frac{1}{x-2}\,dx$$

$$= \int \frac{1}{x-3}\,dx - \int \frac{1}{x-2}\,dx$$

$$\ln|x-3| - \ln|x-2| + C$$

$$\int \frac{1}{\left(x^2-5x+\frac{25}{4}\right)+6-\frac{25}{4}}\,dx$$

$$\int \frac{1}{\left(x-\frac{5}{2}\right)^2-\frac{1}{4}}\,dx$$

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$-\int \frac{1}{\left(\frac{1}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}\,dx$$

$$-\left[\frac{1}{2\left(\frac{1}{2}\right)}\ln\left|\frac{\frac{1}{2}+x-\frac{5}{2}}{\frac{1}{2}-x+\frac{5}{2}}\right|\right] + C$$

$$-\ln\left|\frac{x-2}{3-x}\right| + C$$

$$-\left(\ln|x-2| - \ln|3-x|\right) + C$$

$$-\ln|x-2| + \ln|3-x| + C$$

$$|-1(x-3)|$$

$$|-1|\,|x-3|$$

Decomposition of $\frac{N(x)}{D(x)}$ into Partial Fractions:

1. **Divide (using long division) if improper.** If the degree of *N(x)* is greater than the degree of *D(x)*, then divide *D(x)* into *N(x)*.

$$\frac{N(x)}{D(x)} = P(x) + \frac{N_1(x)}{D(x)}$$

Apply steps 2, 3, and 4 (below) to the proper rational expression $\frac{N_1(x)}{D(x)}$.

2. **Factor denominator.** Completely factor the denominator into factors of the form $(px + q)^m$ (linear factors) and $(ax^2 + bx + c)^n$ (quadratic factors) where $(ax^2 + bx + c)$ is non-factorable over the reals.
3. **Linear Factors.** For *each* factor of the form $(px + q)^m$ the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. **Quadratic Factors.** For *each* factor of the form $(ax^2 + bx + c)^n$ the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{(ax^2 + bx + c)} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Ex: Use partial fractions to integrate the following:

a) $\int \frac{3x + 13}{x^2 + 5x + 6}\,dx$

Ex: Use partial fractions to integrate the following:

a) $\int \frac{3x+13}{x^2+5x+6}dx = \int \frac{3x+13}{(x+3)(x+2)}dx$

$$\frac{A(x+2)}{(x+3)(x+2)} + \frac{B(x+3)}{(x+2)(x+3)} = \frac{3x+13}{(x+3)(x+2)}$$

$$\frac{Ax+2A+Bx+3B}{(x+3)(x+2)} = \frac{3x+13}{(x+3)(x+2)}$$

$$Ax + \underline{\underline{2A}} + Bx + \underline{\underline{3B}} = 3x+13$$

$$(A+B)x = 3x$$

$$\begin{array}{r} 2A + 3B = 13 \\ -2(A+B=3) \\ \hline B = 7 \\ A = -4 \end{array}$$

$$\int \frac{-4}{x+3} + \frac{7}{x+2}\,dx$$

$$= \boxed{-4\ln|x+3| + 7\ln|x+2| + C}$$

b) $\int \frac{3x^2 + 7x - 1}{x^3 + 2x^2 + x} dx$

c) $\int \frac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2} dx$

b) $$\int \frac{3x^2+7x-1}{x^3+2x^2+x}\,dx = \int \frac{3x^2+7x-1}{x(x^2+2x+1)}\,dx$$

$$= \int \frac{3x^2+7x-1}{x(x+1)^2}\,dx$$

$$\frac{A(x+1)^2}{x(x+1)^2} + \frac{Bx(x+1)}{x(x+1)(x+1)} + \frac{Cx}{(x+1)^2x} = \frac{3x^2+7x-1}{x(x+1)^2}$$

$$A(x^2+2x+1) + Bx^2 + Bx + Cx = 3x^2+7x-1$$

$$\underline{\underline{Ax^2}} + \boxed{2Ax} + \boxed{A} + \underline{\underline{Bx^2}} + \boxed{Bx} + \boxed{Cx} = 3x^2 + 7x\boxed{-1}$$

$A+B=3$ $\quad -1+B=3$ $\quad B=4$

$2A+B+C=7$ $\quad 2(-1)+4+C=7$

$A=-1$ $\quad 2+C=7$

$C=5$

$$\int \frac{-1}{x}\,dx + \int \frac{4}{x+1}\,dx + \int \frac{5}{(x+1)^2}\,dx$$

$u=x+1$

$du=dx$

$5\int \frac{1}{(x+1)^2}\,dx$

$5\int u^{-2}\,du$

$$-\ln|x| + 4\ln|x+1| + {}^{-}5\cdot\frac{1}{x+1} + C$$

$$\boxed{-\ln|x| + 4\ln|x+1| - \frac{5}{x+1} + C}$$

c) $\int \frac{2x^3 - x^2 + x + 5}{x^2 + 3x + 2}dx = \int 2x - 7 + \frac{18x+19}{x^2+3x+2}\,dx$

$$\begin{array}{r}
2x - 7 \\
x^2 + 3x + 2 \;\big)\; \overline{\cancel{2x^3} - x^2 + x + 5} \\
\underline{-\cancel{2x^3} \mp 6x^2 \mp 4x} \quad \downarrow \\
-\cancel{7x^2} - 3x + 5 \\
\underline{+\cancel{7x^2} + 21x + 14} \\
18x + 19
\end{array}$$

$$\int 2x\,dx - \int 7\,dx + \int \frac{18x+19}{(x+2)(x+1)}\,dx$$

$$\int \frac{18x+19}{(x+1)(x+2)}\,dx \Rightarrow \frac{A(x+2)}{(x+1)(x+2)} + \frac{B(x+1)}{(x+2)(x+1)} = \frac{18x+19}{(x+1)(x+2)}$$

$$Ax + 2A + Bx + B = 18x + 19$$

$$\begin{array}{l}
-1(A + B = 18) \\
\underline{2A + B = 19} \\
A \quad = 1 \\
B = 17
\end{array}$$

$$\int \frac{1}{x+1}dx + \int \frac{17}{x+2}\,dx$$

$$\int 2x\,dx - \int 7\,dx + \int \frac{1}{x+1}\,dx + \int \frac{17}{x+2}\,dx$$

$$\boxed{x^2 - 7x + \ln|x+1| + 17\ln|x+2| + C}$$

d) $\int \frac{3x^4 + 24x^2 - 2x + 16}{x^5 + 8x^3 + 16x} dx$

d) $\displaystyle\int \frac{3x^4+24x^2-2x+16}{x^5+8x^3+16x}\,dx$

$$\int \frac{3x^4+24x^2-2x+16}{x(x^4+8x^2+16)}\,dx$$

$$\int \frac{3x^4+24x^2-2x+16}{x(x^2+4)^2}\,dx$$

$$\frac{A(x^2+4)^2}{x(x^2+4)^2}+\frac{x(Bx+C)(x^2+4)}{x(x^2+4)^2}+\frac{x(Dx+E)}{x(x^2+4)^2}$$

$$A(x^4+8x^2+16)+(Bx^2+Cx)(x^2+4)+Dx^2+Ex$$

$$Ax^4+8Ax^2+16A+Bx^4+4Bx^2+Cx^3+4Cx+Dx^2+Ex$$

$$=3x^4+24x^2-2x+16$$

$A+B=3 \rightarrow B=2$

$C=0$

$8A+4B+D=24 \rightarrow 8+8+D=24 \quad D=8$

$4C+E=-2 \rightarrow E=-2$

$16A=16 \rightarrow A=1$

$$\int \frac{1}{x}\,dx+\int\frac{2x}{x^2+4}\,dx+\int\frac{8x-2}{(x^2+4)^2}\,dx$$

$$\int \frac{1}{x}\,dx+\int\frac{2x}{x^2+4}\,dx+4\int\frac{2x}{(x^2+4)^2}\,dx-2\int\frac{1\,dx}{(x^2+4)^2}$$

easy $\quad u=x^2+4 \quad du=2x\,dx \quad$ u-sub $\quad 4\int u^{-2}\,du \quad$ Trig sub.

$$\ln|x|+\ln|x^2+4|-\frac{4}{x^2+4}-2\left(\ ?\ \right)$$

$x=2\tan\theta \quad dx=2\sec^2\theta\,d\theta$

$$\int \frac{1}{\left((2\tan\theta)^2+4\right)^2}\cdot 2\sec^2\theta\,d\theta$$

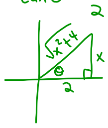
$(4\tan^2\theta+4)^2$

$(4(\tan^2\theta+1))^2$

$$\int\frac{2\sec^2\theta}{(4\sec^2\theta)^2}\,d\theta=\int\frac{\cancel{2}\,\cancel{\sec^2\theta}}{\cancel{16}\,\sec^{\cancel{4}}\theta}\,d\theta$$

$\tan\theta=\frac{x}{2}$

$$=\frac{1}{8}\int\cos^2\theta\,d\theta$$



$$=\frac{1}{8}\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$=\frac{1}{8}\left[\int\frac{1}{2}\,d\theta+\frac{1}{2}\int\cos 2\theta\,d\theta\right]$$

$u=2\theta \quad du=2\,d\theta \quad \frac{1}{2}du=d\theta$

$\theta=\arctan\frac{x}{2}$

$$=\frac{1}{8}\left[\frac{1}{2}\theta+\frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right]$$

$\sin\theta=\frac{x}{\sqrt{x^2+4}}$

$\cos\theta=\frac{2}{\sqrt{x^2+4}}$

$$=\frac{1}{16}\theta+\frac{1}{32}(2\sin\theta\cos\theta)$$

$$=\frac{1}{16}\arctan\frac{x}{2}+\frac{1}{\cancel{16}\,8}\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{\cancel{2}\,1}{\sqrt{x^2+4}}$$

$$=\frac{1}{16}\arctan\frac{x}{2}+\frac{x}{8(x^2+4)}+C$$

$$\ln|x|+\ln(x^2+4)-\frac{4}{x^2+4}-2\left(\frac{1}{16}\arctan\frac{x}{2}+\frac{x}{8(x^2+4)}\right)$$

$$\boxed{\ln|x|+\ln(x^2+4)-\frac{4}{x^2+4}-\frac{1}{8}\arctan\frac{x}{2}-\frac{x}{4(x^2+4)}+C}$$