Phys 311 Homework Set 1 Solutions

1. Let $\vec{\alpha} = \alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z}$

$\vec{\beta} = \beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z}$

$\vec{\gamma} = \gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z}$

now

$$\vec{\alpha} \times (\vec{\beta} + \vec{\gamma}) = (\alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z}) \times (\beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z} + \gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z})$$

$$= (\alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z}) \times (\beta_x \hat{x} + \beta_y \hat{y} + \beta_z \hat{z})$$

$$+ (\alpha_x \hat{x} + \alpha_y \hat{y} + \alpha_z \hat{z}) \times (\gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z}) = (\vec{\alpha} \times \vec{\beta}) + (\vec{\alpha} \times \vec{\gamma})$$

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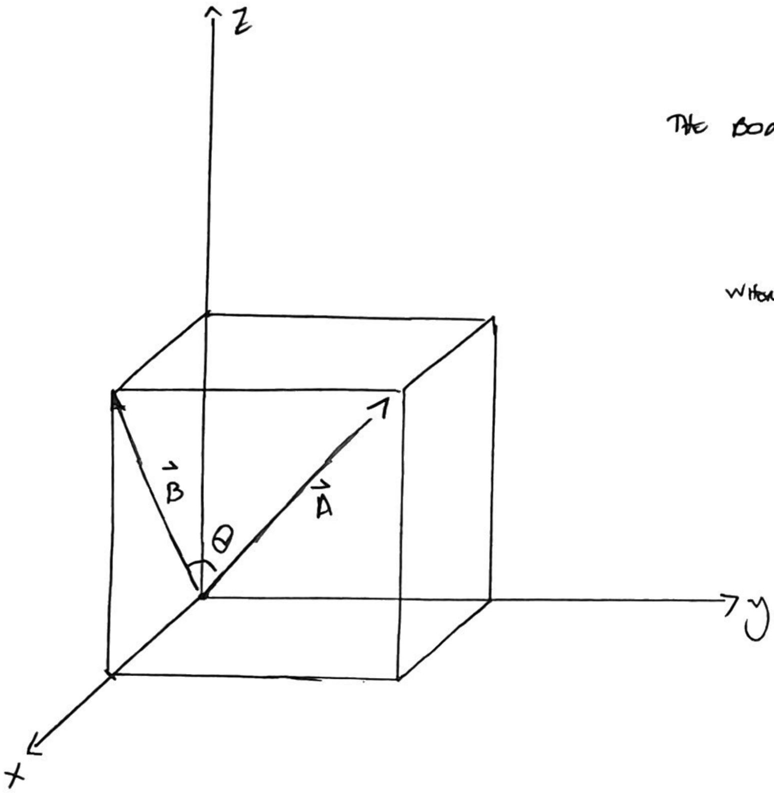


THE BODY DIAGONAL IS...

$$\vec{A} = \hat{x} + \hat{y} + \hat{z}$$

WHEREAS A FACE DIAGONAL IS...

$$\vec{B} = \hat{x} + \hat{z}$$

NOTICE THAT:

$$\vec{A}\cdot\vec{A} = A^2 = (\hat{x}+\hat{y}+\hat{z})\cdot(\hat{x}+\hat{y}+\hat{z}) = 1+1+1 = 3 \qquad \text{so} \quad A = \sqrt{3}$$

$$\vec{B}\cdot\vec{B} = B^2 = (\hat{x}+\hat{z})\cdot(\hat{x}+\hat{z}) = 1+1 = 2 \qquad \text{so} \quad B = \sqrt{2}$$

NOW

$$\vec{A}\cdot\vec{B} = (\hat{x}+\hat{y}+\hat{z})\cdot(\hat{x}+\hat{z}) = 2$$

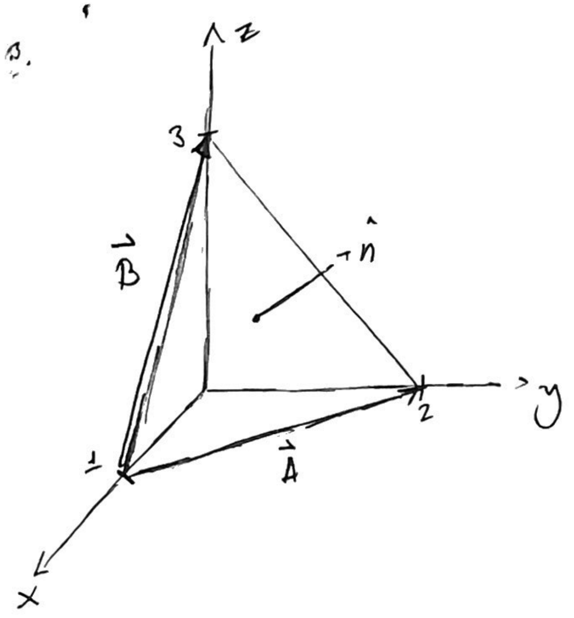
BUT ALSO

$$\vec{A}\cdot\vec{B} = AB\cos\theta = \sqrt{3}\sqrt{2}\cos\theta = \sqrt{6}\cos\theta$$

so

$$2 = \sqrt{6}\cos\theta \quad \therefore \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.3^\circ$$

3.



SINCE $\hat{n}$ IS PERPENDICULAR TO THE SHADED PLANE, I NEED TO CONSTRUCT TWO VECTORS THAT LIE ON THE SHADED PLANE....

Let

$$\vec{A} = -\hat{x} + 2\hat{y}$$

$$\vec{B} = -\hat{x} + 3\hat{z}$$

NOW

$\vec{A} \times \vec{B}$ IS PERPENDICULAR TO THE PLANE $\therefore$ PARALLEL TO $\hat{n}$ BY THE RIGHT HAND RULE

So

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \hat{x}(6-0) + \hat{y}(0\cdot -1 - 3\cdot -1) + \hat{z}(0-(-2))$$

$$= 6\hat{x} + 3\hat{y} + 2\hat{z}$$

NOW

$$(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = (6\hat{x}+3\hat{y}+2\hat{z})\cdot(6\hat{x}+3\hat{y}+2\hat{z}) = 36+9+4 = 49$$

So

$$|\vec{A}\times\vec{B}| = 7$$

TO GET $\hat{n}$, DIVIDE $\vec{A}\times\vec{B}$ BY ITS MAGNITUDE...

$$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$$

$$\boxed{\hat{n} = \frac{6}{7}\hat{x} + \frac{3}{7}\hat{y} + \frac{2}{7}\hat{z}}$$

4. Let $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$

$\vec{C} = C_x \hat{x} + C_y \hat{y} + C_z \hat{z}$

NOW

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \hat{x}(B_y C_z - B_z C_y) + \hat{y}(B_z C_x - B_x C_z) + \hat{z}(B_x C_y - B_y C_x)$$

SO

(*)

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix}$$

$$= \hat{x}\left(A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)\right)$$

$$+ \hat{y}\left(A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_x)\right)$$

$$+ \hat{z}\left(A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)\right)$$

WHEREAS

$$\vec{B}(\vec{A} \cdot \vec{C}) = (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})(A_x C_x + A_y C_y + A_z C_z)$$

$$= B_x (A_x C_x + A_y C_y + A_z C_z)\hat{x} + B_y (A_x C_x + A_y C_y + A_z C_z)\hat{y} + B_z (A_x C_x + A_y C_y + A_z C_z)\hat{z}$$

AND

$$\vec{C}(\vec{A} \cdot \vec{B}) = (C_x \hat{x} + C_y \hat{y} + C_z \hat{z})(A_x B_x + A_y B_y + A_z B_z)$$

$$= C_x (A_x B_x + A_y B_y + A_z B_z)\hat{x} + C_y (A_x B_x + A_y B_y + A_z B_z)\hat{y} + C_z (A_x B_x + A_y B_y + A_z B_z)\hat{z}$$

SO

$$
\begin{aligned}
\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) = &\left[B_x(\cancel{A_x C_x} + A_y C_y + A_z C_z) - C_x(\cancel{A_x B_x} + A_y B_y + A_z B_z)\right]\hat{x} \\
&+ \left[B_y(A_x C_x + \cancel{A_y C_y} + A_z C_z) - C_y(A_x B_x + \cancel{A_y B_y} + A_z B_z)\right]\hat{y} \\
&+ \left[B_z(A_x C_x + A_y C_y + \cancel{A_z C_z}) - C_z(A_x B_x + A_y B_y + \cancel{A_z B_z})\right]\hat{z}
\end{aligned}
$$

WHERE I GROUPED BY $\hat{x}, \hat{y}, \hat{z}$ AND CANCELED LIKE TERMS.

NOW REARRANGING...

$$
\begin{aligned}
\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) = &\left[A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)\right]\hat{x} \\
&+ \left[A_z(B_y C_z - B_z C_y) - A_x(B_x C_y - B_y C_x)\right]\hat{y} \\
&+ \left[A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)\right]\hat{z}
\end{aligned}
$$

WHICH PRECISELY MATCHES (*)!

$$5.\ \vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (\hat{x} + 2\hat{y} + 3\hat{z})m - (\hat{x} + 4\hat{y} + 5\hat{z})m$$

$$= (-2\hat{y} - 2\hat{z})m = -2(\hat{y} + \hat{z})m$$

$$\therefore \left[\vec{\mathscr{r}} = -2(\hat{y} + \hat{z})m \right]$$

now

$$\vec{\mathscr{r}} \cdot \vec{\mathscr{r}} = \mathscr{r}^2 = -2(\hat{y} + \hat{z})m \cdot -2(\hat{y} + \hat{z})m = 4(1 + 1)m^2 = 8m^2$$

$$\therefore \quad \mathscr{r} = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}}$$

$$\left[\mathscr{r} = \sqrt{8}\,m = 2\sqrt{2}\,m \right]$$

Lastly,

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{-2(\hat{y} + \hat{z})m}{2\sqrt{2}\,m} = -\frac{1}{\sqrt{2}}(\hat{y} + \hat{z})m$$

$$\left[\hat{\mathscr{r}} = -\frac{1}{\sqrt{2}}(\hat{y} + \hat{z})m \right]$$

6. $\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$

a) NOW $f(x,y,z) = x^2 + y^5 + z^3$

so $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 5y^4$, $\frac{\partial f}{\partial z} = 3z^2$

so $\left[\vec{\nabla} f = (2x)\hat{x} + (5y^4)\hat{y} + (3z^2)\hat{z}\right]$

b) HERE $f(x,y,z) = x^2 y^5 z^3$

so $\frac{\partial f}{\partial x} = 2xy^5z^3$, $\frac{\partial f}{\partial y} = 5x^2y^4z^3$, $\frac{\partial f}{\partial z} = 3x^2y^5z^2$

so

$\vec{\nabla} f = (2xy^5z^3)\hat{x} + (5x^2y^4z^3)\hat{y} + (3x^2y^5z^2)\hat{z}$

$\left[\vec{\nabla} f = xy^4z^2\left((2yz)\hat{x} + (5xz)\hat{y} + (3xy)\hat{z}\right)\right]$

c) LASTLY $f(x,y,z) = \ln(2x)e^z\sin(3y)$

so $\frac{\partial f}{\partial x} = e^z\sin(3y)\frac{\partial}{\partial x}\ln(2x) = e^z\sin(3y)\cdot\frac{1}{2x}\cdot 2 = \frac{1}{x}e^z\sin(3y)$

SINCE $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$\frac{\partial f}{\partial y} = \ln(2x)e^z\frac{\partial}{\partial y}\sin(3y) = 3\ln(2x)e^z\cos(3y)$

$\frac{\partial f}{\partial z} = \ln(2x)e^z\sin(3y)$

so

$\vec{\nabla} f = \frac{1}{x}e^z\sin(3y)\hat{x} + 3\ln(2x)e^z\cos(3y)\hat{y} + \ln(2x)e^z\sin(3y)\hat{z}$

$\left[\vec{\nabla} f = e^z\left(\frac{1}{x}\sin(3y)\hat{x} + 3\ln(2x)\cos(3y)\hat{y} + \ln(2x)\sin(3y)\hat{z}\right)\right]$

7. TO FIND THE TOP OF THE HILL OF HEIGHT

$$h(x,y) = 3(2xy - 3x^2 - 4y^2 - 8x + 8y + 5),$$

WE NEED TO SET THE GRADIENT EQUAL TO ZERO.

NOW

$$\vec{\nabla} h = \hat{x}\frac{\partial h}{\partial x} + \hat{y}\frac{\partial h}{\partial y} + \hat{z}\frac{\partial h}{\partial z}$$

NOW

$$\frac{\partial h}{\partial x} = 3(2y - 6x - 8)$$

$$\frac{\partial h}{\partial y} = 3(2x - 8y + 8)$$

SO

$$\vec{\nabla} h = 3(2y - 6x - 8)\hat{x} + 3(2x - 8y + 8)\hat{y}$$

THIS IS EQUAL TO ZERO WHEN...

1) $2y - 6x - 8 = 0$

2) $2x - 8y + 8 = 0$

ADDING YIELDS...

$$-4x - 6y = 0$$

$$x = -\frac{3}{2}y$$

PLUGGING THIS BACK INTO 2) YIELDS...

$$-3y - 8y + 8 = 0 \quad \text{or} \quad 8 = 11y \quad \therefore \boxed{y = \frac{8}{11}} \quad \therefore x = -\frac{3}{2}\cdot\frac{8}{11} = \frac{-12}{11}$$

SO THE TOP OF THE HILL IS LOCATED AT

a) $\boxed{(x,y) = \left(\frac{-12}{11}, \frac{8}{11}\right) \text{MILES}}$

b) THE HEIGHT OF THE HILL @ $(x,y) = \left(\frac{-12}{11}, \frac{8}{11}\right)$ MILES

$$h\left(x=\tfrac{-12}{11}, y=\tfrac{8}{11}\right) = 3\left[2\left(-\tfrac{12}{11}\right)\left(\tfrac{8}{11}\right) - 3\left(\tfrac{-12}{11}\right)^2 - 4\left(\tfrac{8}{11}\right)^2 - 8\left(-\tfrac{12}{11}\right) + 8\left(\tfrac{8}{11}\right) + 5\right] \text{ft}$$

$$= 3\left[-\frac{192}{11^2} - \frac{432}{11^2} - \frac{256}{11^2} + \frac{96}{11} + \frac{64}{11} + 5\right] \text{ft}$$

$$= 3\left[-\frac{880}{11^2} + \frac{160}{11} + 5\right] \text{ft} = \frac{3}{11^2}\left[-880 + 160(11) + 5(11)^2\right]$$

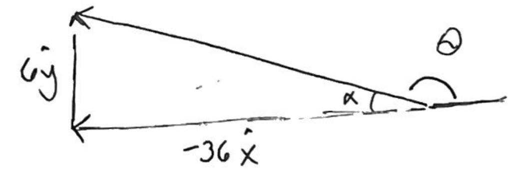
$$= \frac{3}{11^2}\left[-880 + 1760 + 605\right] \text{ft} = \frac{4{,}455}{121} \text{ ft}$$

$$\therefore \left[h\left(x = \tfrac{-12}{11}, y = \tfrac{8}{11}\right) = 36.8 \text{ ft}\right]$$

c) NOW

$$\vec{\nabla} h(x = 1\text{MILE}, y = 1\text{MILE}) = 3(2y - 6x - 8)\hat{x} + 3(2x - 8y + 8)\hat{y}\Big|_{\substack{x=1\\y=1}}$$

$$= 3(2-6-8)\hat{x} + 3(2-8+8)\hat{y}$$



so

$$\vec{\nabla} h(x = 1\text{MILE}, y = 1\text{MILE}) = \left(-36\hat{x} + 6\hat{y}\right)\frac{\text{ft}}{\text{MILE}}$$

NOTICE:

) THE MAGNITUDE $|\vec{\nabla} h|$ GIVES THE SLOPE (RATE OF INCREASE) ALONG THE MAXIMUM INCREASE

$$\vec{\nabla} h \cdot \vec{\nabla} h = (36)^2 \frac{\text{ft}^2}{\text{MILE}^2} + 6^2 \frac{\text{ft}^2}{\text{MILE}^2} = 1332 \text{ ft}^2/\text{MILE}^2$$

$$\text{so } \left[|\vec{\nabla} h| = \sqrt{\vec{\nabla} h \cdot \vec{\nabla} h} = 36.5 \text{ ft/MILE}\right]$$

c) NOW $\tan\alpha = \frac{6}{36}$ $\therefore$ [$\alpha = 9.5°$ NORTH OF WEST]