# Physics 396

## Homework Set 3

1. Consider one inertial reference frame with coordinates $(t', x', y', z')$ moving at a constant speed $v$ relative to another inertial reference frame with coordinates $(t, x, y, z)$. The coordinate transformations relating the coordinates of these two frames are of the form

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and are known as *Lorentz boosts.*

a) By first calculating differentials of Eqs. (1), explicitly show that these Lorentz boosts leave the line element unchanged.

In PHYS 230, you first learned three rules before arriving at the general Lorentz boosts. Explicitly show that the Lorentz boosts reduce to the following rules for special cases...

b) *Moving clocks run slow* by the factor $\sqrt{1 - v^2/c^2}$.

c) *Moving objects are contracted* by the factor $\sqrt{1 - v^2/c^2}$.

d) Two clocks synchronized in their own frame are NOT synchronized in other frames. The front (*leading*) clock reads an earlier time (*lags*) the chasing clock by

$$\Delta t' = vD/c^2 \tag{2}$$

where $D$ is the *rest distance* between them.

2. Maxwell's equations constitute a set of coupled, first-order, linear partial differential equations that classically describe $\vec{E}$ and $\vec{B}$ fields as arising from charges and currents. These equations can be decoupled by taking the curl of two of them and then employing the remaining two. In regions of space where there are no charges or currents, these

equations take the form

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0 \tag{3}$$

$$\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = 0, \tag{4}$$

where $\vec{E} = \vec{E}(x, y, z, t)$, $\vec{B} = \vec{B}(x, y, z, t)$, and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

It is noted that Eqs. (3) and (4) correspond to wave equations in 3D and predict the existence of electromagnetic waves that move at a constant speed $c$.

Consider one inertial reference frame with coordinates $(t', x', y', z')$ moving at a constant speed $v$ relative to another inertial reference frame with coordinates $(t, x, y, z)$. Show that Eqs. (3) and (4) are *invariant* under the *Lorentz boosts* given in Eq. (1). This means that both the $S$ and $S'$ observers will have equivalent wave equations in their own frame of reference and that both observers will predict that the electromagnetic waves propagate at the same speed $c$.

3. a) Construct a spacetime diagram at rest relative to you with coordinates $(ct, x)$. Indicate the positions of two events ($A$, $B$) by dots. The line connecting the two events should be labeled $\ell$ and forming an angle $\theta$ with the $x$ axis.

   b) Now construct the line element $\Delta s^2$ in terms of $\ell$ and $\theta$.

   c) Setting $\ell = 1$, plot $\Delta s^2$ for angles $0 < \theta < \pi/2$ in Excel or an equivalent (or better) plotting software.

   d) What is the *range* of values for this $\Delta s^2$? What angle has the *smallest magnitude* for this distance between points that defines this Minkowski spacetime? For what angles do we find the *maximum* and *minimum* values of $\Delta s^2$?

4. Two spaceships $A$ and $B$ are moving in opposite directions, each measures the speed of the other to be $(12/13)c$. Relative to our frame of reference, spaceship $A$ moves to the right and spaceship $B$ moves to the left. Each spaceship contains two clocks with one at the nose and one at the tail, synchronized with one another in each ship's frame. For spaceship $A$, label the clocks as $N_A$ and $T_A$ corresponding to the nose and tail clock of $A$, respectively. Likewise, for spaceship $B$ label the nose and tail clocks as $N_B$ and $T_B$. The rest length of spaceship $A$ is 598 m and the rest length of spaceship $B$ is 299 m. Just as the nose of $B$ reaches the nose of $A$, both ships set their nose clocks to read $t = 0$. Analyze this problem from the rest frame of spaceship $A$.
   a) Sketch both spaceships from $A$'s point of view at the moment when the noses meet. Use a ruler to make sure that your drawn lengths are consistent with your calculated values.
   b) At this moment described in a), what are the readings of all four ship clocks?
   c) How long does it take the nose of $B$ to reach the tail of $A$?
   d) When the nose of $B$ reaches the tail of $A$, again sketch both spaceships from $A$'s point of view.
   e) At this moment described in d), what are the readings of all four ship clocks?

5. Spaceship $A$ leaves Earth for a distant star and travels at a constant speed of $v_A = (5/13)c$. Assume that the Earth and the star are mutually at rest and that their clocks have been previously synchronized. Both spaceship $A$'s clock and the Earth's clock read $t = 0$ at the start of this journey. After 7.00 years have passed on Earth's clock, spaceship $B$ departs Earth in route for the same distant star traveling at a constant speed of $v_B = (12/13)c$. Remarkably, both spaceship $A$ and $B$ arrive at the star at precisely the same time, according to the star's clock.
   a) Draw a set of three pictures in the rest frame of the Earth and star for the three important events, that is, at the time of spaceship $A$'s departure, at the time of spaceship $B$'s departure, and at the time of arrival of spaceships $A$ and $B$.
   b) According to Earth's clock, how long does it take the spaceships to arrive at this star?

c) What is this rest length distance from Earth to the star?

d) At the moment when spaceship $B$ leaves Earth, what does the spaceship $A$'s clock read?

e) At the moment when the spaceships arrive at the star, how much time passed by on spaceship $A$'s clock?

f) How long did spaceship $B$'s journey take to reach this star, according to spaceship $B$'s clock?

g) What is the distance from the Earth to the star, according to spaceship $A$?

h) What is the distance from the Earth to the star, according to spaceship $B$?

6. Reconsider Problem 5 that featured two spaceships, $A$ and $B$, that both departed Earth for a distant star traveling at speeds $(5/13)c$ and $(12/13)c$, respectively. Spaceship $B$ departed Earth 7.00 years after Spaceship $A$, according to Earth's clocks. Let the Earth rest frame be the $S$ frame, Spaceship $A$'s rest frame be the $S'$, and Spaceship $B$'s rest frame be the $S''$ frame.

a) Draw a spacetime diagram in Earth's frame. This spacetime diagram should take up most of a full sheet of paper with a scale of 1 $c{\cdot}$yr : 2 cm. Be sure to label each axis and include a proper scale.

b) Draw the world lines of Spaceship $A$ and Spaceship $B$ on this spacetime diagram, labeling them on the diagram as such.

c) Draw the $(x', ct')$ and $(x'', ct'')$ axes on this spacetime diagram, labeling them as such.

d) Indicate the locations of the three important events on this spacetime diagram with a dot, labeling them as $E1$, $E2$, and $E3$.

e) Draw the world line of the distant star as a dashed line.