Electromagnetic Theory II: Class Exam I

2 March 2021

Name: Solution **Total:** /50

Instructions

- There are 5 questions on 8 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

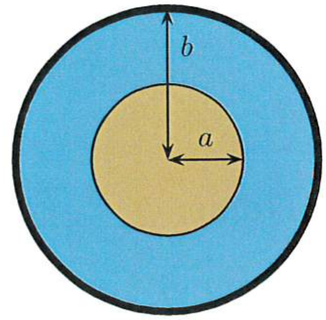
$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

Three infinitely long cylindrical materials are arranged as illustrated. The inner material, with radius a, is solid and carries charge with density, in cylindrical coordinates,



$$\rho(\mathbf{r}') = \alpha s'^2$$

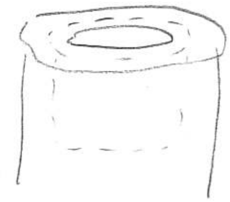
where $\alpha > 0$ is a constant. The middle material is a pipe with inner radius a and outer radius b and is a linear dielectric with permittivity ϵ. The outer material is a conducting cylindrical shell with radius b. It carries a uniform charge density such that the net charge per unit length along the axis of the entire arrangement is zero.

+11 a) Determine the electric field at all locations within and beyond the materials.

In all cases $\vec{E} = \epsilon \vec{D}$ and $\oint \vec{D} \cdot d\vec{a} = Q_{\text{free enc}}$. By symmetry

$\vec{D} = D_s(s)\hat{s}$

Gaussian surface - cylinder radius s, height h.



- on top/bottom $d\vec{a} = da\,\hat{z} \Rightarrow \vec{D} \cdot d\vec{a} = 0$

- on sides

$$d\vec{a} = s\,d\phi\,dz\,\hat{s}$$

$$\Rightarrow \quad \vec{D} \cdot d\vec{a} = D_s(s)\,s\,d\phi\,dz$$

So $$\oint \vec{D} \cdot d\vec{a} = D_s(s)\,s \int_0^{2\pi} d\phi \int_0^h dz = D_s(s)\,2\pi s h \Rightarrow D_s = \frac{Q_{\text{free enc}}}{2\pi s h}$$

For $s < a$ $$Q_{\text{free enc}} = \int \rho\, d\tau' = \int_0^s ds' \int_0^{2\pi} d\phi' \int_0^h dz'\, s'\, \alpha s'^2$$

$$\Rightarrow Q_{\text{free enc}} = \frac{2\pi\alpha h}{4} s^4$$

Question 1 continued ...

Thus $s < a$ $D_s = \frac{\alpha s^3}{4}$

For $a < s < b$ $Q_{\text{free enc}} = \frac{2\pi \alpha h}{4} a^4 \Rightarrow D_s = \frac{\alpha a^4}{4s}$

For $b < s$ $D_s = 0$

Thus

$$\vec{E} = \begin{cases} \frac{\alpha s^3}{4\epsilon_0}\hat{s} & s<a \\ \frac{\alpha a^4}{\epsilon 4 s}\hat{s} & a \leq s \leq b \\ 0 & b \leq s \end{cases}$$

b) Determine the bound surface charge densities on the dielectric. +2

$\sigma_b = \vec{P}\cdot\hat{n}$ $\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P} = (\epsilon - \epsilon_0)\vec{E}$

At $s = a$, $\hat{n} = -\hat{s}$

$\sigma_b = -\hat{s}\cdot(\epsilon-\epsilon_0)\vec{E}$

$= -\frac{(\epsilon-\epsilon_0)}{\epsilon 4}\alpha a^3$

At $s = b$ $\hat{n} = \hat{s}$

$\sigma_b = +\frac{(\epsilon-\epsilon_0)}{4\epsilon}\alpha\frac{a^4}{b}$

c) Is it possible that there is some arrangement of cylindrically symmetrical charge distributions on the inner or outer cylinders so that the *bound volume charge density* is non-zero? Explain your answer. +2

$\rho_b = -\vec{\nabla}\cdot\vec{P}$ here $\vec{P} = \frac{\beta}{s}\hat{s}$ where β is a constant.

Then $\vec{\nabla}\cdot\vec{P} = \frac{1}{s}\frac{\partial}{\partial s}(s P_s) = 0$. There will never be a bound ~~ch~~ volume charge.

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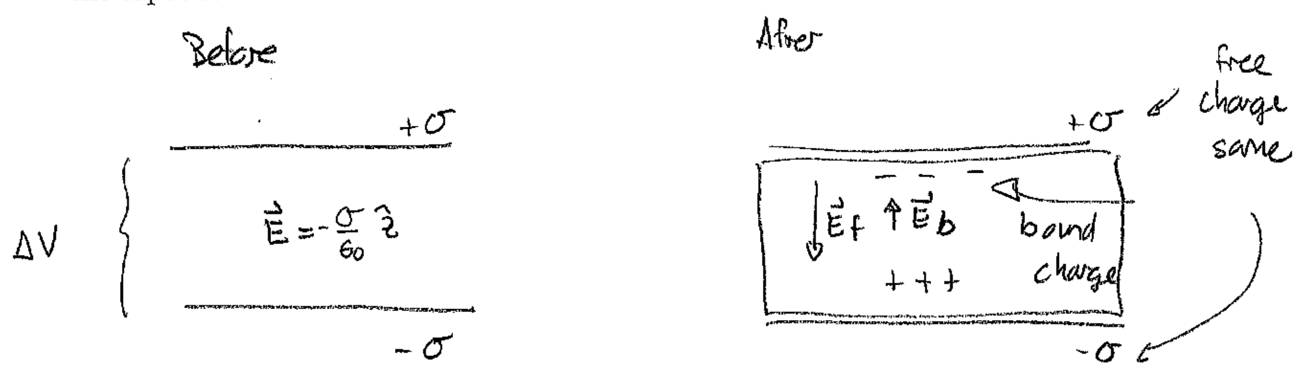
Question 2

A parallel plate capacitor consists of two flat plates that are so closely spaced that they can be assumed to be infinite. The capacitor is initially connected to a power supply which provides potential difference ΔV across gap between the plates. The power supply is disconnected after the capacitor has charged completely. Subsequently a linear dielectric is inserted between the capacitor plates. Explain qualitatively what effect inserting the dielectric has on: the electric field between the plates, the potential difference across the plates and the energy stored in the capacitor.



The bound charge reduces the field since $\vec{E} = \vec{E}_f + \vec{E}_b$

Then $\Delta V = -\int \vec{E}\cdot d\vec{l}$ will reduce in magnitude.

The energy is $U = \frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau$

is reduced since $\vec{E}$ is reduced.

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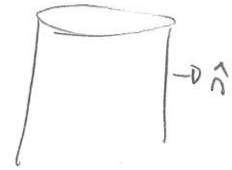
Question 3

An infinitely long solid cylinder with radius R has magnetization, given in cylindrical coordinates by,

$$\mathbf{M}(\mathbf{r}') = M_0 \frac{s'}{R}\,\hat{\mathbf{z}} \qquad = M_z \hat{z}$$

where $M_0 > 0$ is a constant. There are no free currents anywhere on the cylinder.



4 ~~4~~ 3 a) Determine the bound surface and volume current densities at all locations.

$\vec{K}_b = \hat{M} \times \hat{n}$, $\hat{n} = \hat{s}$, $s' = R$

$= M_0 \frac{R}{R} \underbrace{\hat{z} \times \hat{s}}_{\hat{\phi}}$

$\vec{K}_b = M_0 \hat{\phi}$

$\vec{J}_b = \vec{\nabla} \times \vec{M}$

$= \frac{1}{s} \cancel{\frac{\partial M_z}{\partial \phi}}^{0} \hat{s} - \frac{\partial M_z}{\partial s} \hat{\phi}$

$= -\frac{M_0}{R} \hat{\phi}$

6 ~~6~~ b) Determine the magnetic field at all locations.

$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$

Then $\oint \vec{H} \cdot d\vec{l} = I_{\text{free enc}}$ $\qquad \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = 0$

Now there are no free currents. So $\oint \vec{H} \cdot d\vec{l} = 0$ for any loop

$\Rightarrow \vec{H} = 0$.

$\Rightarrow \frac{1}{\mu_0} \vec{B} = M$

Question 3 continued …

Thus $\vec{B} = \mu_0 \vec{M} = \mu_0 M_0 \frac{s}{R} \hat{z}$ inside

Outside $\vec{M} = 0 \Rightarrow \vec{B} = 0$

So

$$\vec{B} = \begin{cases} \mu_0 M_0 \frac{s}{R} \hat{z} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

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Question 4

A spherical conductor has radius a and total charge Q. Outside the conductor

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}.$$

Show that this satisfies electric field boundary conditions at the conductor surface.

Inside $\vec{E}_{in} = \vec{0}$ We need $\vec{E}_{out}^{\parallel} = \vec{E}_{in}^{\parallel}$

Outside $\vec{E}_{out} = \frac{1}{4\pi\epsilon_0} \frac{Q}{a^2} \hat{r}$ $\quad 0 \quad 0$ ✓

Also $E_{out}^{\perp} - E_{in}^{\perp} = \vec{E}_{out} \cdot \hat{r} - \vec{E}_{in} \cdot \hat{r}$

$= \frac{1}{4\pi\epsilon_0} \frac{Q}{a^2}$

$= \frac{1}{\epsilon_0} \frac{Q}{4\pi a^2}$

$= \frac{1}{\epsilon_0} \sigma_f \Rightarrow E_{out}^{\perp} - E_{in}^{\perp} = \frac{\sigma_f}{\epsilon_0}$

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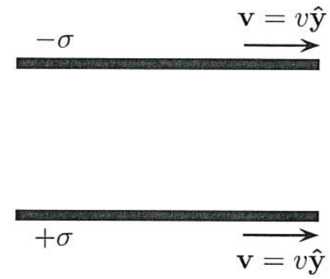
Question 5

Two parallel flat plates each have area A and are separated by distance d. The upper plate carries uniform surface charge density $-\sigma$ and the lower plate carries uniform surface charge density $+\sigma$, where $\sigma > 0$. The plates are dragged with constant speeds v in the same direction as illustrated. The electric and magnetic fields are zero everywhere except between the plates, where



depth w

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}} \qquad \text{and} \qquad \mathbf{B} = \mu_0 \sigma v\,\hat{\mathbf{x}}$$

Here $\hat{\mathbf{x}}$ points out of the page and $\hat{\mathbf{z}}$ up the page.

+6 a) Determine the total energy stored in the fields.

$$U = \frac{\epsilon_0}{2}\int_{vol} \vec{E}\cdot\vec{E}\,d\tau + \frac{1}{2\mu_0}\int_{vol} \vec{B}\cdot\vec{B}\,d\tau$$

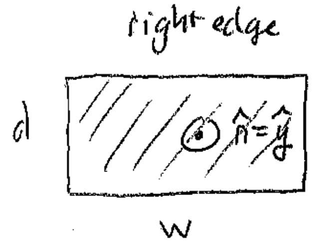
$$= \frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2}\int_{vol} d\tau + \frac{1}{2\mu_0}\mu_0^2\sigma^2 v^2 \underbrace{\int_{vol} d\tau}_{Ad}$$

$$= \frac{\sigma^2}{2\epsilon_0}\left[1 + \mu_0\epsilon_0 v^2\right] Ad$$

+5 b) Determine the direction of energy flow and the rate at which energy passes through the edge of the plate system on the right of the illustrated diagram.

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\frac{\sigma}{\epsilon_0}\mu_0 \sigma v \underbrace{\hat{z}\times\hat{x}}_{\hat{y}} = \frac{\sigma^2}{\epsilon_0} v\,\hat{y}$$

Question 5 continued ...



energy flow in direction of $\vec{S} \Rightarrow \hat{y}$

$$= \int \vec{S} \cdot d\vec{a}$$

$$= \frac{\sigma^2 v}{\epsilon_0} dw$$

c) Suppose that the arrangement of the plates was the same but that each had infinite area and the speed with which they were moving was not constant. Applying Gauss' law would yield that the electric field is $\mathbf{E} = \sigma/\epsilon_0 \, \hat{\mathbf{z}}$ and applying Ampère's law would yield a magnetic field, $\mathbf{B} = \mu_0 \sigma v \, \hat{\mathbf{x}}$. These two laws were both derived in static situations. Explain how you could verify whether these fields are correct or not when the speed with which the plates move is not constant.

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Substitute into Maxwell's eqns. $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow 0 = 0$ works

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow 0 = -\mu_0 \sigma \frac{dv}{dt} \hat{x}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow 0 = 0 \quad \text{works}$$

1 Does not hold!

$$\underbrace{\vec{\nabla} \times \vec{B}}_{0} = \mu_0 \underbrace{\vec{J}}_{0} + \mu_0 \epsilon_0 \underbrace{\frac{\partial \vec{E}}{\partial t}}_{0} \quad \text{works}$$

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