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MATH 360
CP 10.1

Line Integrals

This requires that we represent the curve C by a parametric representation.

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k} \qquad (a \leq t \leq b)$$

A line integral of a vector function $\vec{F}(\vec{r})$ over a curve $C: \vec{r}(t)$ dt

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt \qquad \vec{r}\,' = \frac{d\vec{r}}{dt}$$

And another way of writing the previous statement,

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt$$

Closed Loop Line Integral

when the loop is closed we use,

$$\oint_C \text{ instead of } \int_C$$

Examples 1 & 2

Ex 1: Evaluation of a Line Integral in the Plane

Find the value of the line integral (3) when $\vec{F}(\vec{r}) = [-y, -xy] = -y\hat{\imath} - xy\hat{\jmath}$ and C is the circular arc in Fig. 220 from A to B.

Solution. We may represent C by $\vec{r}(t) = [\cos(t), \sin(t)] = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$, where $0 \leq t \leq \pi/2$. Then $x(t) = \cos(t)$, $y(t) = \sin(t)$ and

$$\vec{F}(\vec{r}(t)) = -y(t)\hat{\imath} - x(t)y(t)\hat{\jmath} = [-\sin(t), -\cos(t)\sin(t)] = -\sin(t)\hat{\imath} - \cos(t)\sin(t)\hat{\jmath}.$$

By differentiation, $\vec{r}\,'(t) = [-\sin(t), \cos(t)] = -\sin(t)\hat{\imath} + \cos(t)\hat{\jmath}$, so that by (3) [use (10) in App. 3.1; set $\cos(t) = u$ in the second term]

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{\pi/2} [-\sin(t), -\cos(t)\sin(t)] \cdot [-\sin(t), \cos(t)]\ dt$$
$$= \int_0^{\pi/2} (\sin^2(t) - \cos^2(t)\sin(t))\,dt$$
$$= \int_0^{\pi/2} \tfrac{1}{2}(1 - \cos(2t))\,dt - \int_1^0 u^2(-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521$$

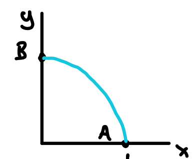

Fig 220.

Ex 2: Line Integral in Space

The evaluation of line integrals in Space is practically the same as it is in the plane. To see this, Find the value of (3) when $\vec{F}(\vec{r}) = [x,y,z] = z\hat{\imath} + x\hat{\jmath} + y\hat{k}$ and C is the helix (Fig. 221)

$$\vec{r}(t) = [\cos(t), \sin(t), 3t] = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + 3t\hat{k} \qquad 0 \le t \le 2\pi \tag{4}$$

Solution. From (4) we have $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = 3t$. Thus

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = (3t\hat{\imath} + \cos(t)\hat{\jmath} + \sin(t)\hat{k}) \cdot (-\sin(t)\hat{\imath} + \cos(t)\hat{\jmath} + 3\hat{k})$$

The dot product is $3t(-\sin(t)) + \cos^2(t) + 3\sin(t)$. Hence (3) gives

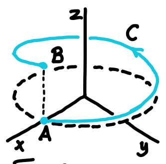

Fig 221.

$$\int_C \vec{F}(r) \cdot d\vec{r} = \int_0^{2\pi} (-3t\sin(t) + \cos^2(t) + 3\sin(t))\, dt = 6\pi + \pi = 7\pi \approx 21.99$$

Ex 5 A Line Integral of the Form (8)

Integrate $\vec{F}(\vec{r}) = [xy, yz, z]$ along the helix in Example 2.

Solution. $\vec{F}(\vec{r}(t)) = [\cos(t)\sin(t), 3t\sin(t), 3t]$ integrated with respect to t from 0 to 2π gives

$$\int_0^{2\pi} \vec{F}(\vec{r}(t))\, dt = \left[-\tfrac{1}{2}\cos^2(t), 3\sin(t) - 3t\cos(t), \tfrac{3}{2}t^2\right]\Big|_0^{2\pi} = [0, -6\pi, 6\pi^2]$$