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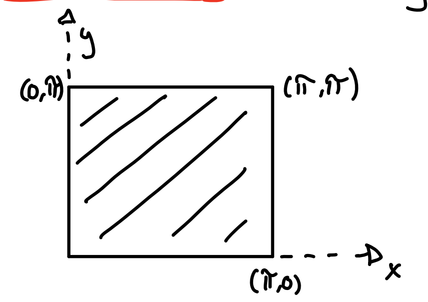
MATH 360
HW ch.12.9

12.9 # 13

Problem 12.9.13

$0.1xy(\pi-x)(\pi-y) \quad c^2=1 \ , \ V_0=0$



$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t\right)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dx\,dy$$

$$B_{mn}^* = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)dx\,dy$$

$f(x,y) = xy \ , \ g(x,y)=0 \ , \ a=\pi, \ b=\pi \ , \ c=1$

$$\lambda_{mn} = \pi\sqrt{\frac{m^2}{\pi^2}+\frac{n^2}{\pi^2}} = \sqrt{m^2+n^2} \quad , \quad \lambda_{mn} = \sqrt{m^2+n^2}$$

$B_{mn}^* = 0$ since $g(x,y)=0$

$$B_{mn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\sin(mx)\sin(ny)\,dx\,dy \rightarrow \frac{4}{\pi^2}\left[\int_0^\pi y\sin(ny)\underset{\underset{\textcircled{2}}{\uparrow}}{\,dy}\int_0^\pi x\sin(mx)\underset{\underset{\textcircled{1}}{\uparrow}}{\,dx}\right]$$

① $\int_0^\pi x\sin(nx)\,dx \qquad u=x \ , \ du=dx \qquad$ Parts: $uv - \int v\,du$

$v' = \sin(nx)\,dx \ , \ v = \frac{-\cos(nx)}{n}$

$$-\frac{x\cos(nx)}{n}\Big|_0^\pi + \int_0^\pi \cos(nx)\,dx = \frac{-\pi}{n}\cos(n\pi) + \left[\overset{0}{\cancel{\frac{\sin(nx)}{n}}}\right]_0^\pi = \frac{-\pi}{n}(-1)^n$$

$$\frac{\pi}{n}(-1)^1(-1)^n = \frac{\pi}{n}(-1)^{n+1} = \textcircled{1}$$

② $\int_0^\pi y\sin(my)\,dy \qquad u=y \ , \ du=dy$

$v' = \sin(my)\,dy = \frac{-\cos(my)}{m}$

$$-\frac{y\cos(my)}{m}\Big|_0^\pi + \int_0^\pi \cos(my)\,dy = \frac{-\pi}{m}\cos(m\pi) + \left[\frac{\sin(my)}{m}\right]_0^\pi \quad \frac{-\pi}{m}(-1)^m$$

$$\frac{\pi}{m}(-1)^1(-1)^m = \frac{\pi}{m}(-1)^{m+1} = \textcircled{2}$$

$$B_{mn} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\sin(mx)\sin(ny)\,dx\,dy = \frac{4}{\pi^2}\left[\frac{\pi^2}{mn}(-1)^{m+1}(-1)^{n+1}\right] = \frac{4}{mn}(-1)^{m+n+2}$$

Problem 12.9.13 | Continued

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos \lambda_{mn} t + \cancelto{0}{B^*_{mn}\sin} \lambda_{mn}t\, t\right)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

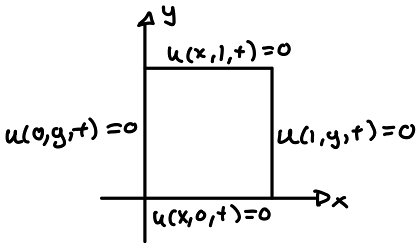
$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{4}{mn}(-1)^{m+n+2}\cos\left(\sqrt{m^2+n^2}\,t\right)\sin(mx)\sin(ny)}$$

$$u(x,y,t) = 4\Big[\cos\sqrt{2}t\sin(x)\sin(y) + \frac{1}{4}\cos(2\sqrt{2}t)\sin(2x)\sin(2y) + \frac{1}{9}\cos(3\sqrt{2}t)\sin(3x)\sin(3y)$$

$$+ \frac{1}{16}\cos(4\sqrt{2}t)\sin(4x)\sin(4y) + \frac{1}{26}\cos(5\sqrt{2}t)\sin(5x)\sin(5y) + \frac{1}{36}\cos(6\sqrt{2}t)\sin(6x)\sin(6y)\Big]$$

Example

Find $u(x,y,t)$ of the figure with $c^2=1$ and initial velocity is 0.



BVP of the form: $u_{tt} = u_{xx} + u_{yy}$

$a=b=1$, $f(x,y)=y$

$c=1$,

Solution: $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\cos(\lambda_{mn}t) + B^*_{mn}\sin(\lambda_{mn}t)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$, $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[B_{mn}\cos(\lambda_{mn}t)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) , B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$B_{mn} = \frac{4}{1}\int_0^1\int_0^1 y\sin(m\pi x)\sin(n\pi y)\,dx\,dy = 4\int_0^1 y\sin(n\pi y)\,dy\cdot\left[-\frac{\cos(m\pi x)}{m\pi}\right]_0^1$$

$$= 4\left[-\frac{\cos(m\pi)}{m\pi} + \frac{\cos(0)}{m\pi}\right]\int_0^1 y\sin(n\pi y)\,dy$$

$u=y$ $du=dy$

$v' = \sin(n\pi y)\,dy$ $v = \frac{-\cos(n\pi y)}{(n\pi)}$

$$= \frac{4}{m\pi}\left[1-(-1)^m\right]\cdot\left[-y\frac{\cos(n\pi y)}{(n\pi)}\Big|_0^1 + \frac{1}{(n\pi)}\int_0^1\cos(n\pi y)\,dy\right]$$

$$= \frac{4}{m\pi}\left[1-(-1)^m\right]\cdot\left[-\frac{\cos(n\pi)}{(n\pi)} + \cancel{\frac{1}{(n\pi)^2}\cdot\sin(n\pi y)}\Big|_0^1\right] = \frac{4}{mn(\pi)^2}\cdot\left[(1)-(-1)^m\right]\left((-1)^n\right)$$

$$B_{mn} = \frac{8\cdot(-1)^n}{mn(\pi)^2} \text{ iff } m=2k+1$$

$$0 \text{ iff } m=2k$$