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PHYS 311
HW 1

Problem 1

Prove $\vec{\alpha}\times(\vec{\beta}+\vec{\gamma}) = (\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma})$

$\vec{\alpha} = (\alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z})$

$\vec{\beta} = (\beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z})$

$\vec{\gamma} = (\gamma_x\hat{x} + \gamma_y\hat{y} + \gamma_z\hat{z})$

Cross Product

$a\times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$

$\vec{\alpha}\times(\vec{\beta}+\vec{\gamma})$

$\vec{\beta}+\vec{\gamma} = (\beta_x+\gamma_x)\hat{x} + (\beta_y+\gamma_y)\hat{y} + (\beta_z+\gamma_z)\hat{z}$

$\vec{\alpha}\times(\vec{\beta}+\vec{\gamma}) = (\alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z})\times((\beta_x+\gamma_x)\hat{x} + (\beta_y+\gamma_y)\hat{y} + (\beta_z+\gamma_z)\hat{z})$

$$\vec{\alpha}\times(\vec{\beta}+\vec{\gamma}) : \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \alpha_x & \alpha_y & \alpha_z \\ \beta_x+\gamma_x & \beta_y+\gamma_y & \beta_z+\gamma_z \end{vmatrix} = \langle \alpha_y(\beta_z+\gamma_z) - \alpha_z(\beta_y+\gamma_y), \alpha_z(\beta_x+\gamma_x) - \alpha_x(\beta_z+\gamma_z), \alpha_x(\beta_y+\gamma_y) - \alpha_y(\beta_x+\gamma_x)\rangle$$

$$\vec{\alpha}\times(\vec{\beta}+\vec{\gamma}) = \langle \alpha_y(\beta_z+\gamma_z) - \alpha_z(\beta_y+\gamma_y), \alpha_z(\beta_x+\gamma_x) - \alpha_x(\beta_z+\gamma_z), \alpha_x(\beta_y+\gamma_y) - \alpha_y(\beta_x+\gamma_x)\rangle$$

$(\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma})$

$(\vec{\alpha}\times\vec{\beta})$

$(\alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z})\times(\beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z})$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \end{vmatrix} = \langle \alpha_y\beta_z - \alpha_z\beta_y, \alpha_z\beta_x - \alpha_x\beta_z, \alpha_x\beta_y - \alpha_y\beta_x\rangle$$

$(\vec{\alpha}\times\vec{\gamma})$

$(\alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z})\times(\gamma_x\hat{x} + \gamma_y\hat{y} + \gamma_z\hat{z})$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \alpha_x & \alpha_y & \alpha_z \\ \gamma_x & \gamma_y & \gamma_z \end{vmatrix} = \langle \alpha_y\gamma_z - \alpha_z\gamma_y, \alpha_z\gamma_x - \alpha_x\gamma_z, \alpha_x\gamma_y - \alpha_y\gamma_x\rangle$$

$(\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma})$: $\langle \alpha_y\beta_z - \alpha_z\beta_y, \alpha_z\beta_x - \alpha_x\beta_z, \alpha_x\beta_y - \alpha_y\beta_x\rangle$

$+$

$\langle \alpha_y\gamma_z - \alpha_z\gamma_y, \alpha_z\gamma_x - \alpha_x\gamma_z, \alpha_x\gamma_y - \alpha_y\gamma_x\rangle$

$$\langle \alpha_y\beta_z - \alpha_z\beta_y + \alpha_y\gamma_z - \alpha_z\gamma_y, \alpha_z\beta_x - \alpha_x\beta_z + \alpha_z\gamma_x - \alpha_x\gamma_z, \alpha_x\beta_y - \alpha_y\beta_x + \alpha_x\gamma_y - \alpha_y\gamma_x\rangle$$

$$(\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma}) = \langle \alpha_y(\beta_z+\gamma_z) - \alpha_z(\beta_y+\gamma_y), \alpha_z(\beta_x+\gamma_x) - \alpha_x(\beta_z+\gamma_z), \alpha_x(\beta_y+\gamma_y) - \alpha_y(\beta_x+\gamma_x)\rangle$$

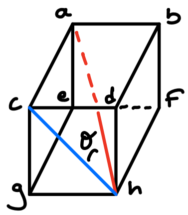
$$\vec{\alpha}\times(\vec{\beta}+\vec{\gamma}) = \langle \alpha_y(\beta_z+\gamma_z) - \alpha_z(\beta_y+\gamma_y), \alpha_z(\beta_x+\gamma_x) - \alpha_x(\beta_z+\gamma_z), \alpha_x(\beta_y+\gamma_y) - \alpha_y(\beta_x+\gamma_x)\rangle$$

$$(\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma}) = \langle \alpha_y(\beta_z+\gamma_z) - \alpha_z(\beta_y+\gamma_y), \alpha_z(\beta_x+\gamma_x) - \alpha_x(\beta_z+\gamma_z), \alpha_x(\beta_y+\gamma_y) - \alpha_y(\beta_x+\gamma_x)\rangle$$

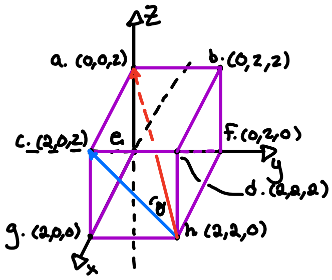
From the above pink and purple equations, we notice that $[\vec{\alpha}\times(\vec{\beta}+\vec{\gamma})] = [(\vec{\alpha}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\gamma})]$. Therefore we can say that the cross product is distributive.

Problem 2

Body and Face Diagonals



Face Diagonal
Body Diagonal



Face Diagonal : $\vec{hc} \equiv \vec{FD} = \langle 0,-2,2 \rangle$

Body Diagonal : $\vec{ha} = \vec{BD} = \langle -2,-2,2 \rangle$

$\vec{FD} \cdot \vec{BD} = \langle 0,-2,2 \rangle \cdot \langle -2,-2,2 \rangle$

$0(-2) - 2(-2) + 2(2) = 4+4 = 8$

$\vec{FD} \cdot \vec{BD} = 8$

$|FD| = 2\sqrt{2}$

$|BD| = 2\sqrt{3}$

$\vec{FD} \cdot \vec{BD} = |FD| \cdot |BD| \cdot \cos\theta$

$8 = 2\sqrt{2} \cdot 2\sqrt{3} \cdot \cos\theta$

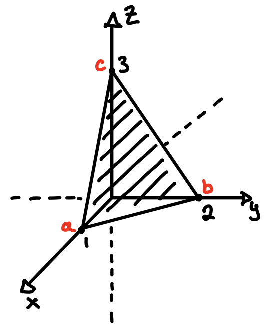
$8 = 4\sqrt{6} \cdot \cos\theta$

$\frac{\sqrt{6}}{3} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right) = 35.26°$

$$\boxed{\begin{array}{l}\theta = 35.26° \\ \theta = 0.6155 \text{ rads}\end{array}}$$

Problem 3



$\vec{a} = \overline{ca}$ $\quad a = (1,0,0) \quad \vec{a} = \langle 1,0,-3 \rangle$ $\qquad \vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

$\vec{b} = \overline{cb}$ $\quad b = (0,2,0) \quad \vec{b} = \langle 0,2,-3 \rangle$

$c = (0,0,3)$

$$(\vec{a}\times\vec{b}) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 2 & -3 \end{vmatrix} = \langle 0(-3) - (-3)(2), -3(0) - 1(-3), 1(2) - 0(0) \rangle$$
$$= \langle 6, 3, 2 \rangle$$

$$\vec{a}\times\vec{b} = \langle 6,3,2 \rangle$$

unit vector: $|\vec{a}\times\vec{b}| = \sqrt{6^2 + 3^2 + 2^2} = 7$

$$\boxed{\begin{aligned} \hat{n} &= \langle 6/7, 3/7, 2/7 \rangle \\ \hat{n} &= 6/7\,\hat{x} + 3/7\,\hat{y} + 2/7\,\hat{z} \end{aligned}}$$

Problem 4

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$$

$$\vec{A}=(A_x\hat{x}+A_y\hat{y}+A_z\hat{z})$$
$$\vec{B}=(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$
$$\vec{C}=(C_x\hat{x}+C_y\hat{y}+C_z\hat{z})$$

$$\vec{a}\times\vec{b}=\langle a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1\rangle$$

$\vec{A}\times(\vec{B}\times\vec{C})$:

$$\vec{B}\times\vec{C} : \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \langle B_yC_z-B_zC_y,\ B_zC_x-B_xC_z,\ B_xC_y-B_yC_x\rangle$$

$$(\vec{B}\times\vec{C})=\langle B_yC_z-B_zC_y,\ B_zC_x-B_xC_z,\ B_xC_y-B_yC_x\rangle$$

$$\vec{A}\times(\vec{B}\times\vec{C})=\langle A_x, A_y, A_z\rangle\times\langle B_yC_z-B_zC_y,\ B_zC_x-B_xC_z,\ B_xC_y-B_yC_x\rangle$$

$$\vec{A}\times(\vec{B}\times\vec{C}) : \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z-B_zC_y & B_zC_x-B_xC_z & B_xC_y-B_yC_x \end{vmatrix}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \langle A_y(B_xC_y-B_yC_x)-A_z(B_zC_x-B_xC_z),\ A_z(B_yC_z-B_zC_y)-A_x(B_xC_y-B_yC_x),\ A_x(B_zC_x-B_xC_z)-A_y(B_yC_z-B_zC_y)\rangle$$

$\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$:

$$\vec{A}\cdot\vec{C} : (A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\cdot(C_x\hat{x}+C_y\hat{y}+C_z\hat{z})$$
$$\vec{A}\cdot\vec{C}=A_xC_x+A_yC_y+A_zC_z$$
$$\vec{B}(\vec{A}\cdot\vec{C})=(A_xC_x+A_yC_y+A_zC_z)(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$
$$\vec{B}(\vec{A}\cdot\vec{C})=\langle (A_xC_x+A_yC_y+A_zC_z)B_x,\ (A_xC_x+A_yC_y+A_zC_z)B_y,\ (A_xC_x+A_yC_y+A_zC_z)B_z\rangle$$
$$\vec{B}(\vec{A}\cdot\vec{C})=\langle A_xB_xC_x+A_yB_xC_y+A_zB_xC_z,\ A_xB_yC_x+A_yB_yC_y+A_zB_yC_z,\ A_xB_zC_x+A_yB_zC_y+A_zB_zC_z\rangle$$

$$\vec{A}\cdot\vec{B} : (A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\cdot(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$
$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$$
$$\vec{C}(\vec{A}\cdot\vec{B})=(A_xB_x+A_yB_y+A_zB_z)(C_x\hat{x}+C_y\hat{y}+C_z\hat{z})$$
$$\vec{C}(\vec{A}\cdot\vec{B})=\langle (A_xB_x+A_yB_y+A_zB_z)C_x,\ (A_xB_x+A_yB_y+A_zB_z)C_y,\ (A_xB_x+A_yB_y+A_zB_z)C_z\rangle$$
$$\vec{C}(\vec{A}\cdot\vec{B})=\langle A_xB_xC_x+A_yB_yC_x+A_zB_zC_x,\ A_xB_xC_y+A_yB_yC_y+A_zB_zC_y,\ A_xB_xC_z+A_yB_yC_z+A_zB_zC_z\rangle$$

$$\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) : \langle A_xB_xC_x+A_yB_xC_y+A_zB_xC_z,\ A_xB_yC_x+A_yB_yC_y+A_zB_yC_z,\ A_xB_zC_x+A_yB_zC_y+A_zB_zC_z\rangle$$
$$-\ \langle A_xB_xC_x+A_yB_yC_x+A_zB_zC_x,\ A_xB_xC_y+A_yB_yC_y+A_zB_zC_y,\ A_xB_xC_z+A_yB_yC_z+A_zB_zC_z\rangle$$
$$\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})=\langle A_yB_xC_y+A_zB_xC_z-A_yB_yC_x-A_zB_zC_x,\ A_xB_yC_x+A_zB_yC_z-A_xB_xC_y-A_zB_zC_y,\ A_xB_zC_x+A_yB_zC_y-A_xB_xC_z-A_yB_yC_z\rangle$$

$$\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) = \langle A_y(B_xC_y-B_yC_x)-A_z(B_zC_x-B_xC_z),\ A_z(B_yC_z-B_zC_y)-A_x(B_xC_y-B_yC_z),\ A_x(B_zC_x-B_xC_z)-A_y(B_yC_z-B_zC_y)\rangle$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \langle A_y(B_xC_y-B_yC_x)-A_z(B_zC_x-B_xC_z),\ A_z(B_yC_z-B_zC_y)-A_x(B_xC_y-B_yC_x),\ A_x(B_zC_x-B_xC_z)-A_y(B_yC_z-B_zC_y)\rangle$$
$$\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B}) = \langle A_y(B_xC_y-B_yC_x)-A_z(B_zC_x-B_xC_z),\ A_z(B_yC_z-B_zC_y)-A_x(B_xC_y-B_yC_z),\ A_x(B_zC_x-B_xC_z)-A_y(B_yC_z-B_zC_y)\rangle$$

The pink and purple equations are the same. Therefore we can say
$$\boxed{\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})}$$

Problem 5

$$\vec{\mathscr{r}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

Source - $(1,4,5)$ - x', y', z'

Field - $(1,2,3)$ - x, y, z

$$\mathscr{r} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$\mathscr{r} = (1-1)\hat{x} + (2-4)\hat{y} + (3-5)\hat{z}$$

$$\mathscr{r} = 0\hat{x} - 2\hat{y} - 2\hat{z}$$

$$|\mathscr{r}| = \sqrt{0^2+2^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\hat{\mathscr{r}} = \frac{0\hat{x}}{2\sqrt{2}} - \frac{\hat{y}}{\sqrt{2}} - \frac{\hat{z}}{\sqrt{2}}$$

$$\boxed{\begin{aligned} |\vec{\mathscr{r}}| &= 2\sqrt{2} \\ \vec{\mathscr{r}} &= 0\hat{x} - 2\hat{y} - 2\hat{z} \\ \hat{\mathscr{r}} &= 0\hat{x} - \frac{\hat{y}}{\sqrt{2}} - \frac{\hat{z}}{\sqrt{2}} \end{aligned}}$$

Problem 6

a.) $f(x,y,z) = x^2 + y^5 + z^3$

$$\nabla f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right>$$

$$\frac{\partial f}{\partial x} = 2x \quad : \quad \frac{\partial f}{\partial y} = 5y^4 \quad : \quad \frac{\partial f}{\partial z} = 3z^2$$

$$\boxed{\begin{array}{l} \nabla f(x,y,z) = \left< 2x, 5y^4, 3z^2 \right> \\ \text{or } \; 2x\hat{x} + 5y^4\hat{y} + 3z^2\hat{z} \end{array}}$$

b.) $f(x,y,z) = x^2 y^5 z^3$

$$\nabla f(x,y,z) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right>$$

$$\frac{\partial f}{\partial x} = 2xy^5z^3 \quad : \quad \frac{\partial f}{\partial y} = 5x^2y^4z^3 \quad : \quad \frac{\partial f}{\partial z} = 3x^2y^5z^2$$

$$\boxed{\begin{array}{l} \nabla f(x,y,z) = \left< 2xy^5z^3, 5x^2y^4z^3, 3x^2y^5z^2 \right> \\ \text{or } \; 2xy^5z^3\hat{x} + 5x^2y^4z^3\hat{y} + 3x^2y^5z^2\hat{z} \end{array}}$$

c.) $f(x,y,z) = \ln(2x)e^z\sin(3y)$

$$\nabla f(x,y,z) = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right>$$

$$\frac{\partial f}{\partial x} = \frac{e^z\sin(3y)}{x} \quad : \quad \frac{\partial f}{\partial y} = 3\ln(2x)e^z\cos(3y) \quad : \quad \frac{\partial f}{\partial z} = \ln(2x)e^z\sin(3y)$$

$\frac{d}{dx}\ln(u) = \frac{u'}{u}$ $\qquad$ $\frac{d}{du}\sin(u) = u' \cdot \cos(u)$ $\qquad$ $\frac{d}{du}e^u = e^u$

$\frac{d}{dx}\ln(2x) = \frac{2}{2x} = \frac{1}{x}$ $\qquad$ $\frac{d}{dy}\sin(3y) = 3\cos(3y)$ $\qquad$ $\frac{d}{dz}e^z = e^z$

$$\boxed{\begin{array}{l} \nabla f(x,y,z) = \left< x^{-1}e^z\sin(3y), 3\ln(2x)e^z\cos(3y), \ln(2x)e^z\sin(3y) \right> \\ \text{or } \; \frac{e^z\sin(3y)}{x}\hat{x} + 3\ln(2x)e^z\cos(3y)\hat{y} + \ln(2x)e^z\sin(3y) \end{array}}$$

Problem 7

$h(x,y) = 3(2xy - 3x^2 - 4y^2 - 8x + 8y + 5)$

$h(x,y) = 6xy - 9x^2 - 12y^2 - 24x + 24y + 15$

$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 \rightarrow$ in our case $h(x,y)$ iff $D > 0$ & $h_{xx}(a,b) < 0$, (a,b) is a local max

a.)

$h_x = 6y - 18x - 24$

$h_y = 6x - 24y + 24$

$h_x = 0: \; 0 = 6y - 18x - 24$

$-18x + 6y = 24$

$h_y = 0: \; 0 = 6x - 24y + 24$

$6x - 24y = -24$

$h_{xx} = -18 \qquad h_{xy} = 6$

$h_{yy} = -24$

$6x - 24y = -24$

$-18x + 6y = 24$

$-18(-24) - (6)^2 = 396$

$$\text{rref}\left(\begin{vmatrix} 6 & -24 & -24 \\ -18 & 6 & 24 \end{vmatrix}\right) = \begin{vmatrix} 1 & 0 & -\frac{12}{11} \\ 0 & 1 & \frac{8}{11} \end{vmatrix}$$

$\boxed{\text{8/11 of a mile north, 12/11 of a mile west, of Topeka Kansas}}$

b.) $h(x,y) = 6xy - 9x^2 - 12y^2 - 24x + 24y + 15$

$y = 8/11$ of a mile

$x = \frac{-12}{11}$ of a mile

$h(-12/11, 8/11) = 6(\frac{-12}{11})(\frac{8}{11}) - 9(\frac{-12}{11})^2 - 12(\frac{8}{11})^2 - 24(\frac{-12}{11}) + 24(\frac{8}{11}) + 15 = 36.82$ Feet

$\boxed{\text{The hill is 36.8 Feet tall}}$

c.) $h(x,y) = 6xy - 9x^2 - 12y^2 - 24x + 24y + 15$

$\nabla h(x,y): \; \frac{\partial h}{\partial x} = 6y - 18x - 24$ ⋮

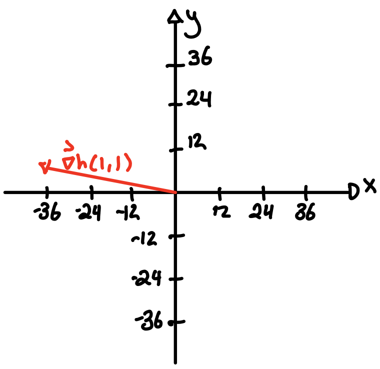
$\frac{\partial h}{\partial y} = 6x - 24y + 24$ ⋮

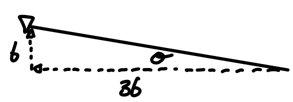
$\nabla h(x,y) = \langle 6y - 18x - 24, 6x - 24y + 24 \rangle$

$\nabla h(1,1) = \langle 6(1) - 18(1) - 24, 6(1) - 24(1) + 24 \rangle$

$\nabla h(1,1) = \langle -36, 6 \rangle$

$|\nabla(1,1)| = \sqrt{(-36)^2 + (6)^2} = 6\sqrt{37} \simeq 36.5$ Ft/mile





$\tan\theta = \frac{6}{36} = \frac{1}{6}$

$\theta = \tan^{-1}(\frac{1}{6}) = 9.5°$

$\boxed{\text{The slope is 36.5 Ft/mile steep at 9.5° North of West of Topeka Kansas}}$