Phys 396 2018

Lecture 9

Today: HW by 5

Thurs: * Seminar
* New HW out

Fri:

Raffel 3.2 4.1→4.2, 4.4

Multiple qubit states

Consider two distinct qubits, labeled A and B. In order to accommodate all possible measurement outcomes we will have to represent the states of the system using tensor products. The standard basis vectors are

$$|0\rangle_A |0\rangle_B \equiv |0\rangle_A \otimes |0\rangle_B \equiv |00\rangle$$

$$|0\rangle_A |1\rangle_B \equiv |0\rangle_A \otimes |1\rangle_B \equiv |01\rangle$$

$$|1\rangle_A |0\rangle_B \equiv |1\rangle_A \otimes |0\rangle_B \equiv |10\rangle$$

$$|1\rangle_A |1\rangle_B \equiv |1\rangle_A \otimes |1\rangle_B \equiv |11\rangle$$

The general state of such a two qubit system can be represented via:

$$|\Psi\rangle = a_0|0\rangle|0\rangle + a_1|0\rangle|1\rangle + a_2|1\rangle|0\rangle + a_3|1\rangle|1\rangle.$$

We need to interpret this in terms of measurements. We want to use the same basic rules:

> Given an outcome of a measurement and associated state $|\varphi\rangle$, the probability of the outcome is $|\langle\varphi|\Psi\rangle|^2$.

In order for this to make sense we need rules for inner products + bras. Suppose $|\Psi\rangle$ and $|\Phi\rangle$ are of the form

$$|\Psi\rangle = |\psi_A\rangle|\psi_B\rangle$$

$$|\Phi\rangle = |\phi_A\rangle|\psi_B\rangle$$

then

$$\langle\Phi| \equiv |\Phi\rangle^{+} \equiv \langle\phi_A|\langle\phi_B|$$

and

$$\langle\Phi|\Psi\rangle = \langle\phi_A|\psi_A\rangle\langle\phi_B|\psi_B\rangle$$

We can extend this by requiring linearity on the right element and

$$|\Phi\rangle = \alpha_1|\Phi_1\rangle + \alpha_2|\Phi_2\rangle \Rightarrow \langle\Phi| = \alpha_1^*\langle\Phi_1| + \alpha_2^*\langle\Phi_2|$$

Using this we can show that the basis $\{|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle\}$ is orthonormal. Then we can see that for the generic state

$$|\Psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$$

the measwement in this basis yields outcomes with probabilities

$$\text{Prob}(00) = |\langle 00|\Psi\rangle|^2 = |a_0|^2$$

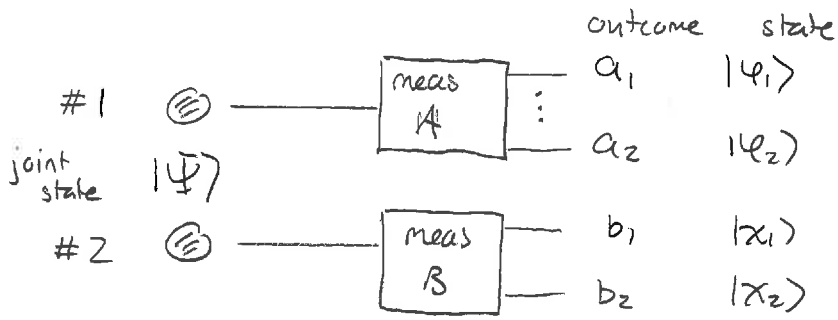
$$\text{Prob}(01) = |\langle 01|\Psi\rangle|^2 = |a_1|^2$$

$$\vdots$$

These must add to 1 so we get $|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1$

Measurements on multiple qubits

Consider a process (relabel the particles



We can calculate outcome probabilities via

$$\text{Prob}\,(a_1 \text{ AND } b_1) = |\langle\varphi_1|\langle\chi_1|\,|\Psi\rangle|^2$$

$$\text{Prob}\,(a_1 \text{ AND } b_2) = |\langle\varphi_1|\langle\chi_2|\,|\Psi\rangle|^2$$

$$\vdots$$

$$\text{Prob}\,(a_2 \text{ AND } b_2) = |\langle\varphi_2|\langle\chi_2|\Psi\rangle|^2$$

We can eventually combine these probabilities into probabilities of outcomes of measurements on one particle regardless of the other.

For example:

$$\text{Prob}\,(a_1 \text{ regardless of } b) = \text{Prob}\,(a_1 \text{ AND } b_1) + \text{Prob}\,(a_1 \text{ AND } b_2)$$

$$= |\langle\varphi_1|\langle\chi_1|\Psi\rangle|^2 + |\langle\varphi_1|\langle\chi_2|\Psi\rangle|^2$$

etc,...

1 Measurements on two qubits

Consider two qubits ("1" on the left and "2" on the right) in the state

$$|\Psi\rangle = \frac{1}{2}\left(|00\rangle + i\,|01\rangle + |10\rangle + i\,|11\rangle\right).$$

In the following a measurement is performed on "1" and a measurement on "2."

a) Determine the probabilities of all four outcomes if the measurement basis for "1" is $\{|0\rangle, |1\rangle\}$ and the measurement basis "2" is $\{|0\rangle, |1\rangle\}$.

b) Determine the probabilities of all four outcomes if the measurement basis for "1" is

$$\left\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}$$

and the measurement basis "2" is

$$\left\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}.$$

c) Determine the probabilities of all four outcomes if the measurement basis for "1" is

$$\left\{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}$$

and the measurement basis "2" is

$$\left\{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle, \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\right\}.$$

Ans: a) We need to compute

$$\text{Prob}(00) = \left|\langle 0|\langle 0|\Psi\rangle\right|^2 = \left|\langle 00|\Psi\rangle\right|^2$$

$$\langle 00|\Psi\rangle = \frac{1}{2}\underbrace{\langle 00|00\rangle}_{\langle 0|0\rangle\langle 0|0\rangle} + \frac{i}{2}\underbrace{\langle 00|01\rangle}_{\langle 0|0\rangle\cancel{\langle 0|1\rangle}\,0} + \frac{1}{2}\underbrace{\langle 00|10\rangle}_{\cancel{\langle 0|1\rangle}\,0\,\langle 0|0\rangle} + \frac{i}{2}\underbrace{\langle 00|11\rangle}_{\cancel{\langle 0|1\rangle\langle 0|1\rangle}\,0\;0}$$

$$= \frac{1}{2}$$

$$\text{Prob}(00) = \frac{1}{4}$$

$$\text{Prob}(01) = |\langle 01|\Psi\rangle|^2 = \left|\frac{i}{2}\right|^2 = 1/4$$

$$\text{Prob}(10) = |\langle 10|\Psi\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$\text{Prob}(11) = |\langle 11|\Psi\rangle|^2 = \left|\frac{i}{2}\right|^2 = \frac{1}{4}.$$

b) $$\text{Prob}(++) = |\langle\varphi_1|\langle\chi_1|\Psi\rangle|^2$$

$$\langle\varphi_1| = \frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|$$

$$\langle\chi_1| = \frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|$$

$$\langle\varphi_1|\langle\chi_1| = \frac{1}{2}\left\{\langle 00| + \langle 01| + \langle 10| + \langle 11|\right\}$$

$$\text{Prob}(++) = \left|\frac{1}{4}\left(1+i+1+i\right)\right|^2 = \left|\frac{1+i}{2}\right|^2 = \frac{(1+i)(1-i)}{4} = \frac{1}{2}$$

$$\text{Prob}(++) = 1/2$$

For $(+-)$ we need

$$\langle\varphi_1|\langle\chi_2| = \frac{1}{2}\left(\langle 00| - \langle 01| + \langle 10| - \langle 11|\right)$$

$$\text{Prob}(+-) = \left|\frac{1}{4}\left(1-i+1-i\right)\right|^2 = \frac{1}{2}$$

$$\text{Prob}(+-) = 1/2$$

For $(-+)$ we need

$$\langle\varphi_2|\langle\chi_1| = \frac{1}{2}\left(\langle 00| + \langle 01| - \langle 10| - \langle 11|\right)$$

$$\text{Prob}(-+) = \left|\frac{1}{4}\left(1+i-1-i\right)\right| = 0$$

$$\text{Prob}(-+) = 0$$

For $--$ we need $\langle \varphi_2 | \langle \chi_2 | = \frac{1}{2}\left(\langle 00| - \langle 01| - \langle 10| + \langle 11| \right)$

$$\text{Prob}(--) = \left| \frac{1}{4}(1 - i - 1 + i) \right|^2 = 0 \qquad \text{Prob}(--) = 0$$

c) List the outcomes + states

"1"	"2"	
+	+	$\left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) = \frac{1}{2}\left(\langle 00| - i\langle 01| + \langle 10| - i\langle 11|\right)$
+	−	..... $\frac{1}{2}\left(\langle 00| + i\langle 01| + \langle 10| + i\langle 11|\right)$
−	+	..... $\frac{1}{2}\left(\langle 00| - i\langle 01| - \langle 10| + i\langle 11|\right)$
−	−	..... $\frac{1}{2}\left(\langle 00| + i\langle 01| - \langle 10| - i\langle 11|\right)$

$$\text{Prob}(++) = \left| \frac{1}{4}(1+1+1+1) \right| = 1$$

$$\text{Prob}(+-) = \left| \frac{1}{4}(1-1+1-1) \right| = 0$$

$$\text{Prob}(-+) = \left| \frac{1}{4}(1+1-1-1) \right| = 0$$

$$\text{Prob}(--) = \left| \frac{1}{4}(1-1-1+1) \right| = 0$$

Part b) of the example illustrates: If we are only interested in the outcome of measurement on "1" regardless of that on "2", then

$$\text{Prob}(+ \text{ regardless of "2"}) = \text{Prob}(++) + \text{Prob}(+-) = 1$$

$$\text{Prob}(- \quad \text{"} \quad \text{"} \quad \text{"2"}) = 0$$

Thus if we measure $\left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$ on "1" we will be assured of the "+" outcome regardless of the outcome of the same on "2".

The measurement of part c) indicates that we will definitely attain the outcomes associated with

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad \text{on "1"}$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \quad \text{on "2"}$$

This suggests that the state of the combined system is

$$|\Psi\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)}_{\text{state "1"}} \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle+i|1\rangle\right)}_{\text{state of "2"}}$$

and multiplication reveals that this is indeed true.

General measurements

Suppose that we only measure on system "1" and want statistics regardless of any measurement on system "2". Thus suppose that we measure A with the given associated states. How can we get the probabilities of outcomes?

One way is to express the state as

some state, not necessarily normalized.

$$|\Psi\rangle = |\varphi_1\rangle|\phi_1\rangle + |\varphi_2\rangle|\phi_2\rangle$$

where $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ are the states associated with measurement A.

Then:

$$\text{Prob}(a_1) = \langle\phi_1|\phi_1\rangle$$

$$\text{Prob}(a_2) = \langle\phi_2|\phi_2\rangle$$

If one cannot do this, then an alternative is :

* Given one wants to measure on "1" with a measurement with states $|\varphi_i\rangle$ and outcomes a_i:

1) Express

$$|\Psi\rangle = |\phi_0\rangle|0\rangle + |\phi_1\rangle|1\rangle$$

where $\{|0\rangle, |1\rangle\}$ are a measurement basis for "2". Here $\{|\phi_0\rangle, |\phi_1\rangle\}$ will not necessarily be orthonormal.

2) Then

$$\mathrm{Prob}(a_i) = |\langle\varphi_i|\phi_0\rangle|^2 + |\langle\varphi_i|\phi_1\rangle|^2$$

One can see that this is plausible if we measure:

* In basis $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ for "1"
* In basis $\{|0\rangle, |1\rangle\}$ for "2"
* Add each probability for "1" over outcomes for "2".

Example: Suppose that

$$|\Psi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle \right)$$

Determine probabilities for the following measurements on the left qubit:

a) $\{ |0\rangle, |1\rangle \}$

b) $\left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\}$

Answer: a)

$$|\Psi\rangle = \underbrace{\frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle\right)}_{|\phi_0\rangle} |0\rangle + \underbrace{\frac{1}{\sqrt{3}} |0\rangle}_{|\phi_1\rangle} |1\rangle$$

$$\text{Prob}(0) = |\langle 0|\phi_0\rangle|^2 + |\langle 0|\phi_1\rangle|^2$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

$$\text{Prob}(1) = |\langle 1|\phi_0\rangle|^2 + |\langle 1|\phi_1\rangle|^2$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2 + 0 = \frac{1}{3}$$

b)

$$\text{Prob}(+) = \left| \left(\frac{1}{\sqrt{2}} \langle 0| + \langle 1|\right)\left(\frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle\right)\right) \right|^2$$

$$+ \left| \frac{1}{\sqrt{2}} \left(\langle 0| + \langle 1|\right) \frac{1}{\sqrt{3}} |0\rangle \right|^2$$

$$= \frac{1}{6}\, 2^2 + \frac{1}{6} = \frac{5}{6}$$

$$\text{Prob}(-) = \left| \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right)\left(\frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle\right)\right) \right|^2 + \left| \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right) \frac{1}{\sqrt{3}} |1\rangle \right|^2$$

$$= \frac{1}{6}\, 0 + \frac{1}{6} = \frac{1}{6}$$

Entangled states

We have seen an example where a superposition can be written as a product. So

$$|\Psi\rangle = \frac{1}{2}\left(|00\rangle + i|01\rangle + |10\rangle + i|11\rangle\right)$$

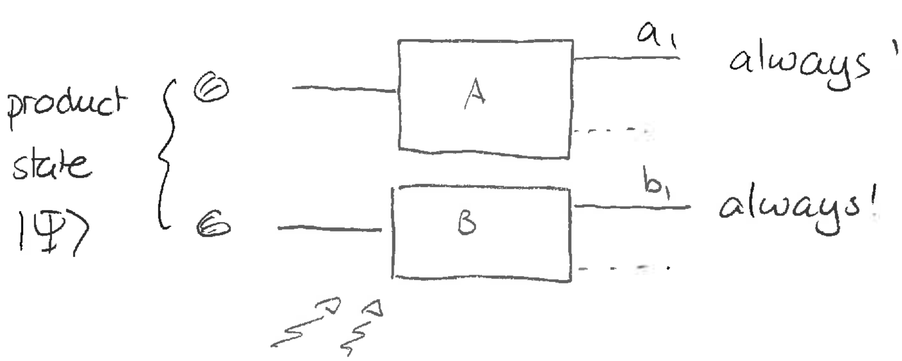
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

and we saw that, for such a state there is a pair of measuwements, one on each qubit so that the pair give one outcome with certainty.

This is an example of the following:

> A state $|\Psi\rangle$ is a <u>product state</u> if there exist states $|\varphi_A\rangle$ and $|\chi_B\rangle$ for the two subsystems such that
>
> $$|\Psi\rangle = |\varphi_A\rangle|\chi_B\rangle$$

The physical interpretation of this is:



there is some choice of measurement, one on each subsystem so that the combination always gives one particular pair of outcomes.

There is a sense of independence between the two systems. Specifically one can show that

If the two systems are in a product state then the statistics of any measurement on a single system are independent of the choice of measurement on the other system

But is it always possible to express states in this form? The following exercise illustrates this

Exercise: Let

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|01\rangle - |10\rangle\}$$

a) Consider a measurement in $\{|0\rangle, |1\rangle\}$. List possible outcomes + probabilities. Is there a correlation between outcomes on the two systems? Is either outcome on one system definite?

b) Using

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

and

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

rewrite $|\Psi\rangle$ in terms of $\{|+\rangle, |-\rangle\}$. Use the result to list probabilities of outcomes of measurements in basis $\{|+\rangle, |-\rangle\}$. Is there a correlation between outcomes?

c) By considering arbitrary general single qubit states

$$|\varphi\rangle = a_0|0\rangle + a_1|1\rangle$$

$$|\chi\rangle = b_0|0\rangle + b_1|1\rangle$$

Can one find a, b s.t. $|\Psi\rangle = |\varphi\rangle|\chi\rangle$.

Answer: a)

outcomes		state	prob
0	0	$\lvert 00\rangle$	$\lvert\langle 00\lvert\Psi\rangle\rvert^2 = 0$
0	1	$\lvert 01\rangle$	$\frac{1}{2}$
1	0	$\lvert 10\rangle$	$\frac{1}{2}$
1	1	$\lvert 11\rangle$	0

Yes one will always get opposites "0" for one and "1" for other

No, cannot say which

b) $$\Psi = \frac{1}{\sqrt{2}}\left[\frac{1}{2}(|+\rangle+|-\rangle)(|+\rangle-|-\rangle) - \frac{1}{2}(|+\rangle-|-\rangle)(|+\rangle+|-\rangle)\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\cancel{|++\rangle} - |+-\rangle + |-+\rangle - \cancel{|--\rangle} - \cancel{|++\rangle} + |-+\rangle - |+-\rangle - \cancel{|--\rangle}\right]$$

$$= \frac{1}{\sqrt{2}}\left[|-+\rangle - |+-\rangle\right]$$

Similar to a)

$\mathrm{Prob}(++) = 0$

$\left.\begin{array}{l}\mathrm{Prob}(-+) = \frac{1}{2} \\ \mathrm{Prob}(+-) = \frac{1}{2}\end{array}\right\}$ always opposite

$\mathrm{Prob}(--) = 0$

c) If true $|\Psi\rangle = a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle$.

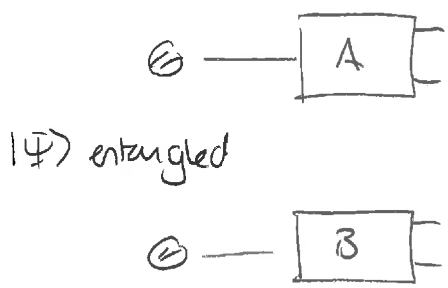
Then $a_0b_0 = 0$, $a_1b_1 = 0$; $\left.\begin{array}{l}a_0b_1 = 1/\sqrt{2} \\ a_1b_0 = 1/\sqrt{2}\end{array}\right\}$ neither $a_0 = 0$ nor $b_0 = 0$

either $a_0 = 0$ or $b_0 = 0$ contradict.

This particular state cannot be written as a product state. Then:

> An *entangled* state is one which cannot be written as a product state.

In terms of measurements



No choice of single qubit measurements that yield one pair with certainty.