Integrating Factor Method $y'+P(t)y=f(t)$

① Find the integrating factor $\mu(t)=e^{\int P(t)\,dt}$

② Multiply each side of the differential equation by the integrating factor to get

$$e^{\int P(t)\,dt}[y'+P(t)y]=f(t)e^{\int P(t)\,dt}$$

Which will always reduce to

$$\frac{d}{dt}\left[e^{\int P(t)\,dt}y(t)\right]=f(t)e^{\int P(t)\,dt}$$

③ Find the anti-derivative in 2 to get

$$e^{\int P(t)\,dt}y(t)=\int f(t)e^{\int P(t)\,dt}\,dt+C$$

④ Solve the final equation of 3 algebraically for y to get the general solution

$$y=e^{-\int P(t)\,dt}\int f(t)e^{\int P(t)\,dt}\,dt+Ce^{-\int P(t)\,dt}$$

Euler-Langrange Method for Linear ODE's

$y'+P(t)y=f(t)$

① Solve $y'+P(t)y=0$ to obtain:

$$y_h=Ce^{-\int P(t)\,dt}$$

② Solve $v'(t)e^{-\int P(t)\,dt}=f(t)$ for $v(t)$ to obtain $y_p=v(t)e^{-\int P(t)\,dt}$

③ Combine steps 1 & 2 to form the general solution $y(t)=y_h+y_p$

④ Solve for IVP if applicable

$$y=y_h+y_p$$

Homogeneous Solutions $ay''+by'+cy$

$\Delta=b^2-4ac$

Case 1 $\Delta>0$	Real unequal roots: $r_1,r_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Overdamped motion: $y_h=C_1e^{r_1t}+C_2e^{r_2t}$
Case 2 $\Delta=0$	Real repeated root: $r=-\frac{b}{2a}$	Critically damped motion: $y_h=C_1e^{rt}+C_2te^{rt}$
Case 3 $\Delta<0$	Complex conjugate roots: $r_1,r_2=\alpha\pm\beta i$ $\alpha=-\frac{b}{2a}$, $\beta=\frac{\sqrt{4ac-b^2}}{2a}$	underdamped motion: $y_h=e^{\alpha t}(C_1\cos\beta t+C_2\sin\beta t)$

Undetermined Coefficients

Table 4.4.1 Predicting Forms of Particular Solutions

Forcing Function $f(t)$	⟹ Particular Solution $y_p(t)$
(i) K	A_0
(ii) $P_n(t)$	$A_n(t)$
(iii) Ce^{kt}	A_0e^{kt}
(iv) $C\cos\omega t+D\sin\omega t$	$A_0\cos\omega t+B_0\sin\omega t$
(v) $P_n(t)e^{kt}$	$A_n(t)e^{kt}$
(vi) $P_n(t)\cos\omega t+Q_n(t)\sin\omega t$	$A_n(t)\cos\omega t+B_n(t)\sin\omega t$
(vii) $Ce^{kt}\cos\omega t+De^{kt}\sin\omega t$	$A_0e^{kt}\cos\omega t+B_0e^{kt}\sin\omega t$
(viii) $P_n(t)e^{kt}\cos\omega t+Q_n(t)e^{kt}\sin\omega t$	$A_n(t)e^{kt}\cos\omega t+B_n(t)e^{kt}\sin\omega t$

- $P_n(t)$, $Q_n(t)$, $A_n(t)$, $B_n(t)\in\mathbb{P}_n$ (hence $A_0,B_0\in\mathbb{P}_0=\mathbb{R}$) and $K,\omega,C,$ and D are real constants
- In (iv)-(viii), both terms must be included in y_p, even if only one of the terms is present in $f(t)$

If any term or terms of y_p are found in y_h (i.e., if such terms are solutions of $ay''+by'+cy=0$), multiply the expression for y_p by t (or, if necessary, by t^2) to eliminate the duplication.

Variation of Parameters

Method of Variation of Parameters for Determining a Particular Solution y_p for $L(y)=y''+p(t)y'+q(t)y=f(t)$

Step 1: Determine two linearly independent solutions y_1 and y_2 of the corresponding homogeneous equation $L(y)=0$

Step 2: Solve for v_1' and v_2' the system

$$y_1v_1'+y_2v_2'=0$$
$$y_1'v_1'+y_2'v_2'=f$$

or determine v_1' and v_2' from Cramers Rule

$$v_1'=\frac{\begin{vmatrix}0 & y_2\\ f & y_2'\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{-y_2f}{W(y_1,y_2)}\qquad v_2'=\frac{\begin{vmatrix}y_1 & 0\\ y_1' & f\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{y_1f}{W(y_1,y_2)}$$

where $W(y_1,y_2)=y_1y_2'-y_1'y_2$ is the Wronskian

Step 3: Integrate the results of Step 2 to find v_1 & v_2

Step 4: Compute $y_p=v_1y_1+v_2y_2$

Note: The derivation is for an operator L having coefficient 1 for y'', so equations having a leading coefficient different from 1 must be divided by that coefficient in order to determine the standard form for $f(t)$.