Tues: HW by 5pm

Thurs: Read. 12.3.

Relativity and Electromagnetic Theory

We have seen that electromagnetic theory is invariant under Lorentz transformations via the scheme:

sources described in terms of point charges
$\rho, \vec{J}$ $\longrightarrow$ charge densities transform in such a way as to respect Lorentz contraction

$\downarrow$

Maxwell's equations give fields:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$\longrightarrow$ How do fields transform?

$\downarrow$

Lorentz force law gives relativisity force

$$\vec{F} = Q(\vec{E} + \vec{u}\times\vec{B})$$

where

$$F_x = \frac{dp^1}{dt} \qquad F_y = \frac{dp^2}{dt}$$

$\longrightarrow$ components of 4 momentum

$$p^1 = \frac{1}{\sqrt{1-u^2/c^2}}\,p_x \qquad p^2 = \frac{1}{\sqrt{1-u^2/c^2}}\,p_y$$

We would like a formalism that

1) transforms source charge and current densities between inertial frames

2) eventually gives field transformations between inertial frames.

Lorentz invariant formulation of electromagnetism

We will use the potential formulation of electromagnetic theory:

Potentials

$V \sim$ scalar

$\vec{A} \sim$ vector

$\longrightarrow$

Fields

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\downarrow$

Lorentz gauge

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t}$$

$\downarrow$

Potentials are related to sources by:

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

continuity equation:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

We need to

1) rewrite these in terms of x^0, x^1, x^2, x^3

2) show invariance of various constituents of the formalism.

Equations in terms of relativistic co-ordinates

Suppose that we aim to compute potentials from sources. This requires operations of the form:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] f = g.$$

Now with

$$x^0 = ct \quad \Rightarrow \quad t = x^0/c$$
$$x^1 = x$$
$$x^2 = y$$
$$x^3 = z$$

we get:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}$$

Thus we define the d'Alembertian operator:

$$\boxed{\Box^2 := \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3} - \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}}$$

We can the immediately express the equations for the potentials as

Will show $\Box^2$ is invariant

$$\boxed{\begin{aligned} \Box^2 V &= -\rho/\epsilon_0 \\ \Box^2 \vec{A} &= -\mu_0 \vec{J} \end{aligned}}$$

Next consider the sources. These must satisfy the continuity equation:

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial}{\partial x^1} J_x + \frac{\partial}{\partial x^2} J_y + \frac{\partial}{\partial x^3} J_z + c\frac{\partial \rho}{\partial x^0} = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x^1} J_x + \frac{\partial}{\partial x^2} J_y + \frac{\partial}{\partial x^3} J_z + \frac{\partial}{\partial x^0}(c\rho) = 0}$$

Now consider the Lorentz gauge condition:

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z + \frac{\partial}{\partial x^0}\left(\frac{V}{c}\right) = 0}$$

These motivate the definitions of

The current four-vector is

$$\{J^\mu\} = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

and the potential four-vector is

$$\{A^\mu\} = \begin{pmatrix} V/c \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

With these definitions the continuity equation becomes:

$$\boxed{\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0}$$

Will show these are invariant.

The Lorentz gauge condition becomes:

$$\boxed{\frac{\partial A^0}{\partial x^0} + \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} = 0}$$

Finally the current four vector generates the potential four-vector via:

$$\boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

Proof:
$$\Box^2 A^0 = \Box^2 V/c = \frac{1}{c}\Box^2 V = -\frac{1}{c}\rho/\epsilon_0$$

$$= -\frac{1}{c^2}\frac{c\rho}{\epsilon_0}$$

$$= -\frac{\mu_0\epsilon_0}{\epsilon_0} J^0 = -\mu_0 J^0$$

The others follow in a straightforward way. ∎

1 Electric fields in relativistic coordinates

Electric fields are generated via

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.$$

a) Determine expressions for the components of the electric field~~s~~ in terms of derivatives involving relativistic coordinates.

Magnetic fields are generated via

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

b) Determine expressions for the components of the magnetic field~~s~~ in terms of derivatives involving relativistic coordinates.

Answer: a) $$E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}$$

$$= -\frac{\partial V}{\partial x^1} - c\,\frac{1}{c}\frac{\partial A_x}{\partial t} \qquad = -\frac{\partial (cA^0)}{\partial x^1} - c\,\frac{\partial A^1}{\partial x^0}$$

$$\Rightarrow E_x = -c\left[\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right]$$

Similarly

$$E_y = -c\left[\frac{\partial A^0}{\partial x^2} + \frac{\partial A^2}{\partial x^0}\right]$$

$$E_z = -c\left[\frac{\partial A^0}{\partial x^3} + \frac{\partial A^3}{\partial x^0}\right]$$

b) $$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}$$

similarly $$B_y = \frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}$$

$$B_z = \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}.$$

Collecting these gives:

$$E_x = -c\left[\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right] \qquad B_x = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}$$

$$E_y = -c\left[\frac{\partial A^0}{\partial x^2} + \frac{\partial A^2}{\partial x^0}\right] \qquad B_y = \frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}$$

$$E_z = -c\left[\frac{\partial A^0}{\partial x^3} + \frac{\partial A^3}{\partial x^0}\right] \qquad B_z = \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}$$

So we get a relativistic formulation of electromagnetic theory.

Current four vector describes sources

$$\{J^\mu\} = \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix}$$

Continuity equation

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0$$

Compute potential four vector via

$$A^\mu = \begin{pmatrix} V/c \\ A_x \\ A_y \\ A_z \end{pmatrix} \qquad \Box^2 A^\mu = -\mu_0 J^\mu$$

where

$$\Box^2 = -\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}$$

satisfies Lorentz gauge $\quad \dfrac{\partial A^0}{\partial x^0} + \dfrac{\partial A^1}{\partial x^1} + \dfrac{\partial A^2}{\partial x^2} + \dfrac{\partial A^3}{\partial x^3} = 0$

Fields via

$$E_x = -c\left[\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right] \qquad B_x = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}$$

$$E_y = -c\left[\frac{\partial A^0}{\partial x^2} + \frac{\partial A^2}{\partial x^0}\right] \qquad B_y = \frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}$$

$$E_z = -c\left[\frac{\partial A^0}{\partial x^3} + \frac{\partial A^3}{\partial x^0}\right] \qquad B_z = \frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}$$

Invariance of the formalism under Lorentz transformations

Suppose that the S' frame is related to the S frame in the usual way. Specifically

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

The transformation will require:

1) a new source current four vector $\{J'^\mu\}$
2) a new d'Alembertian $\Box'$
3) a new potential vector $\{A'^\mu\}$.

Consider the source current four vector. Suppose that in frame S the sources are a rest. Then $J^0 = \rho c$ and $J^1 = J^2 = J^3 = 0$. We know that in the primed frame the current density becomes

$$\rho' = \gamma\rho \quad \Rightarrow \quad J'^0 = \gamma J^0$$

Now in the primed frame these sources move with velocity $-v\hat{x}$ and thus the current density is

$$J'_x = -\rho' v = -\gamma\rho v = -\gamma J^0 \frac{v}{c} = -\gamma\beta J^0$$
$$J'_y = 0$$
$$J'_z = 0$$

Thus these obey
$$J'^0 = \gamma(J^0 - \beta J^1)$$
$$J'^1 = \gamma(J^1 - \beta J^0)$$
$$J'^2 = J^2$$
$$J'^3 = J^3$$

We could use the relativistic velocity transformations to verify the transformation of a current in the S frame. The results are:

The source current 4-vector transforms as:

$$J'^0 = \gamma(J^0 - \beta J^1)$$
$$J'^1 = \gamma(J^1 - \beta J^0)$$
$$J'^2 = J^2$$
$$J'^3 = J^3$$

Now consider the continuity equation:

$$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0$$

Then
$$\frac{\partial}{\partial x^0} = \frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} + \frac{\partial x'^1}{\partial x^0}\frac{\partial}{\partial x'^1} + \frac{\partial x'^2}{\partial x^0}\frac{\partial}{\partial x'^2} + \frac{\partial x'^3}{\partial x^0}\frac{\partial}{\partial x'^3}$$

$$= \gamma\frac{\partial}{\partial x'^0} - \gamma\beta\frac{\partial}{\partial x'^1}$$

$$\frac{\partial}{\partial x^1} = -\gamma\beta\frac{\partial}{\partial x'^0} + \gamma\frac{\partial}{\partial x'^1}$$

$$\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x'^2}$$

$$\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x'^3}$$

So
$$\frac{\partial J^0}{\partial x^0} = \gamma\left[\frac{\partial}{\partial x'^0} - \beta\frac{\partial}{\partial x'^1}\right]\gamma\left(J'^0 + \beta J'^1\right) = \gamma^2\left[\frac{\partial J'^0}{\partial x'^0} - \beta\frac{\partial J'^0}{\partial x'^1} + \beta\frac{\partial J'^1}{\partial x'^0} - \beta^2\frac{\partial J'^1}{\partial x'^1}\right]$$

$$\frac{\partial J^1}{\partial x^1} = \gamma\left[\frac{\partial}{\partial x'^1} - \beta\frac{\partial}{\partial x'^0}\right]\gamma\left(J'^1 + \beta J'^0\right) = \gamma^2\left[\frac{\partial J'^1}{\partial x'^1} - \beta\frac{\partial J'^1}{\partial x'^0} + \beta\frac{\partial J'^0}{\partial x'^1} - \beta^2\frac{\partial J'^0}{\partial x'^0}\right]$$

So $$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} = \gamma^2\left[\frac{\partial J'^0}{\partial x'^0}(1-\beta^2) + \frac{\partial J'^1}{\partial x'^1}(1-\beta^2)\right]$$

$$= \underbrace{(1-\beta^2)\gamma^2}_{=1.}\left[\frac{\partial J'^0}{\partial x'^0} + \frac{\partial J'^1}{\partial x'^1}\right]$$

Thus $$\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = \frac{\partial J'^0}{\partial x'^0} + \frac{\partial J'^1}{\partial x'^1} + \frac{\partial J'^2}{\partial x'^2} + \frac{\partial J'^3}{\partial x'^3}$$

This means that all inertial observers agree on the continuity equation. So

> If the current four-vector transforms as the co-ordinates under a Lorentz transformation then all inertial observers will agree that the continuity equation is satisfied.

Now consider the d'Alembertian operator. In the primed frame:

$$\Box'^2 = -\frac{\partial}{\partial x'^0}\frac{\partial}{\partial x'^0} + \frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^1} + \frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^2} + \frac{\partial}{\partial x'^3}\frac{\partial}{\partial x'^3}$$

Now using

$$x^0 = \gamma(x'^0 + \beta x'^1) \qquad \Rightarrow \qquad \frac{\partial}{\partial x'^0} = \frac{\partial x^0}{\partial x'^0}\frac{\partial}{\partial x^0} + \frac{\partial x^1}{\partial x'^0}\frac{\partial}{\partial x^1}$$
$$x^1 = \gamma(x'^1 + \beta x'^0)$$

$$= \gamma\frac{\partial}{\partial x^0} + \gamma\beta\frac{\partial}{\partial x^1} = \gamma\left(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}\right)$$

$$\frac{\partial}{\partial x'^1} = \gamma\beta\frac{\partial}{\partial x^0} + \gamma\frac{\partial}{\partial x^1} = \gamma\left(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$$

$$\Rightarrow \Box'^2 = -\gamma^2\left(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}\right)\left(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}\right) + \gamma^2\left(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)\left(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}$$

$$= -\underbrace{\gamma^2(1-\beta^2)}_{1}\left(\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0}\right) + \underbrace{\gamma^2(1-\beta^2)}_{1}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3}$$

$$= \Box^2$$

Thus

The form of the d'Alembertian is invariant between inertial observers

We then get that if electromagnetic theory is to be invariant under Lorentz transformations

$$\Box'^2 A'^\mu = -\mu_0 J'^\mu$$

$$\Box^2 A'^\mu = -\mu_0 J'^\mu$$

So if $\{A^\mu\}$ transforms in the same way that $\{J^\mu\}$ does then this will work. We thus require:

The potential four vectors transform as:

$$A'^0 = \gamma(A^0 - \beta A^1)$$

$$A'^1 = \gamma(A^1 - \beta A^0)$$

$$A'^2 = A^2$$

$$A'^3 = A^3$$

Transformation of fields

We can determine how the fields transform via the relationship between potential and fields. Then

$$E'_x = -c\left[\frac{\partial A'^0}{\partial x'^1} + \frac{\partial A'^1}{\partial x'^0}\right] \qquad B'_x = \frac{\partial A'^3}{\partial x'^2} - \frac{\partial A'^2}{\partial x'^3}$$

$$E'_y = -c\left[\frac{\partial A'^0}{\partial x'^2} + \frac{\partial A'^2}{\partial x'^0}\right] \qquad B'_y = \frac{\partial A'^1}{\partial x'^3} - \frac{\partial A'^3}{\partial x'^1}$$

$$\vdots \qquad\qquad \vdots$$

2 Field transformations

a) Determine an expression for E'_x in terms of field components in the S' frame.

b) Determine an expression for E'_y in terms of field components in the S' frame.

Answer: a)

$$E'_x = -c \frac{\partial}{\partial x'^1} \gamma(A^0 - \beta A^1)$$
$$- c \frac{\partial}{\partial x'^0} \gamma(A^1 - \beta A^0)$$

Then
$$\frac{\partial}{\partial x'^1} = \gamma\left(\beta \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$$
$$\frac{\partial}{\partial x'^0} = \gamma\left(\frac{\partial}{\partial x^0} + \beta \frac{\partial}{\partial x^1}\right)$$

$$\Rightarrow \quad E'_x = -c\gamma^2\left\{\left(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)(A^0 - \beta A^1) + \left(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}\right)(A^1 - \beta A^0)\right\}$$

$$= -c\gamma^2\left\{\cancel{\frac{\partial A^0}{\partial x^0}(\beta - \beta)} + \frac{\partial A^0}{\partial x^1}(1-\beta^2) + \frac{\partial A^1}{\partial x^0}(1-\beta^2) + \cancel{\frac{\partial A^1}{\partial x^1}(-\beta + \beta)}\right\}$$

$$= -c\underbrace{\gamma^2(1-\beta^2)}_{1}\left\{\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right\} = -c\left(\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right)$$
$$= E_x$$

$$\Rightarrow \quad E'_x = E_x$$

b) $$E_y' = -c\left[\frac{\partial A'^0}{\partial x'^2} + \frac{\partial A'^2}{\partial x'^0}\right]$$

$$= -c\,\frac{\partial}{\partial x^2}\left\{\gamma\left(A^0 - \beta A^1\right)\right\} \quad -c\gamma\left(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}\right)\left\{A^2\right\}.$$

$$= -c\gamma\left\{\frac{\partial A^0}{\partial x^2} + \frac{\partial A^2}{\partial x^0} + \beta\left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)\right\}.$$

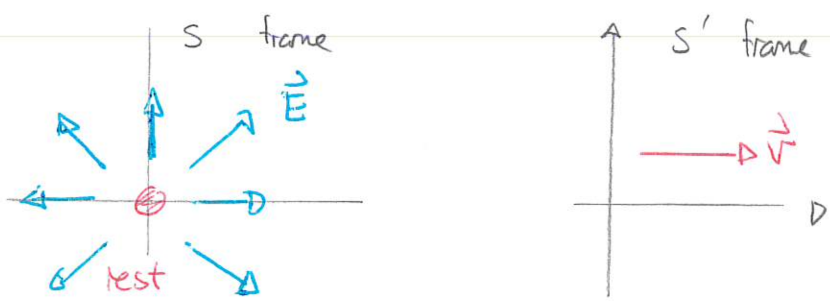
$$= -c\gamma\underbrace{\left(\frac{\partial A^0}{\partial x^2} + \frac{\partial A^2}{\partial x^0}\right)}_{-E_y/c} - c\gamma\beta\underbrace{\left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)}_{B_z}$$

$$\Rightarrow \quad E_y' = \gamma E_y - c\gamma\frac{v}{c}B_z \qquad \Rightarrow \quad E_y' = \gamma\left(E_y - vB_z\right)$$

We can continue for all the components

> If the S' frame travels with velocity $\vec{v} = v\hat{x}$ with respect to the S frame then the fields in the frames transform as:
>
> $$E_x' = E_x \qquad\qquad B_x' = B_x$$
>
> $$E_y' = \gamma(E_y - vB_z) \qquad\qquad B_y' = \gamma(B_y + \tfrac{v}{c^2} E_z)$$
>
> $$E_z' = \gamma(E_z + vB_y) \qquad\qquad B_z' = \gamma(B_z - \tfrac{v}{c^2} E_y)$$

We see that the electric and magnetic fields are mixed under the Lorentz transformations. For example



only produces $\vec{E}$ fields