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MATH 360
CP Ch. 5.2

Legendre's differential equation

(1) $$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (n \text{ constant})$$

is one of the most important ODE's in Physics. It arises in numerous problems, particularly in boundary value problems for spheres (take a quick look at Example 1 in Sec. 12.10).

The equation involves a parameter n, whose value depends on the physical or engineering problem. So (1) is actually a whole family of ODE'S. For $n=1$ we solved it in Example 3 of Sec. 5.1 (look back at it). Any solution of (1) is called a Legendre Function. The study of these and other "higher" functions not occurring in calculus is called the theory of special functions. Further special functions will occur in the next sections.

Dividing (1) by $1-x^2$, we obtain the standard form needed in Theorem 1 of Sec. 5.1 and we see that the coefficients $-2x(1-x^2)$ and $n(n+1)/(1-x^2)$ of the new equation are analytic at $x=0$, so that we may apply the power series method. Substituting

(2) $$y = \sum_{m=0}^{\infty} a_m x^m$$

and its derivatives into (1), and denoting the constant $n(n+1)$ simply by K, we obtain

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + K\sum_{m=0}^{\infty} a_m x^m = 0.$$

By writing the first expression as two separate series we have the equation

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} K a_m x^m = 0.$$

It may help you to write out the first few terms of each series explicitly, as in Example 3 of Sec. 5.1; or you may continue as follows. To obtain the same general power x^s in all four series, set $m-2=s$ (thus $m=s+2$) in the first series and simply write s instead of m in the other three series. This gives

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2}x^s - \sum_{s=2}^{\infty} s(s-1)a_s x^s - \sum_{s=1}^{\infty} 2s a_s x^s + \sum_{s=0}^{\infty} K a_s x^s = 0.$$

(Note that in the first series the summation begins with $s=0$.) Since this equation with the right side 0 must be an identity in x if (2) is to be a solution of (1), the sum of the coefficients of each power of x on the left must be zero. Now x^0 occurs in the first and fourth series only, and gives [remember that $K=n(n+1)$]

(3a) $$2\cdot 1 a_2 + n(n+1)a_0 = 0$$

x^1 occurs in the first, third, and fourth series and gives

(3b) $$3\cdot 2 a_3 + [-2+n(n+1)]a_1 = 0.$$

The higher powers $x^2, x^3, \ldots$ occur in all four series and give

(3c) $$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0.$$

The expression in the brackets [...] can be written $(n-s)(n+s+1)$, as you may readily verify. Solving (3a) for a_2 and (3b) for a_3 as well as (3c) for a_{s+2}, we obtain the general formula

(4) $$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \qquad (s = 0, 1, \ldots).$$

This is called a recurrence relation or recursion formula. (Its derivation you may verify with your CAS). It gives each coefficient in terms of the second one preceding it, except for a_0 and a_1, which are left as arbitrary constants. We find succesively

$$a_2 = -\frac{n(n+1)}{2!}a_0$$

$$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_5 = -\frac{(n-3)(n+4)}{5\cdot 4}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

and so on. By inserting these expressions for the coefficients into (2) we obtain

(5) $$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where

(6) $$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$

(7) $$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots$$

These series converge for $|x| < 1$ (see Prob. 4; or they may terminate, see below). Since (6) contains even powers of x only, while (7) contains odd powers of x only, the ratio y_1/y_2 is not a constant, so that y_1 and y_2 are not proportional and are thus linearly independent solutions. Hence (5) is a general solution of (1) on the interval $-1 < x < 1$.

Note that $x = \pm 1$ are the points at which $1 - x^2 = 0$, so that the coefficients of the standardized ODE are no longer analytic. So it should not surprie you that we do not get a longer convergence interval of (6) and (7), unless these series terminate after finitely many powers. In that case, the series become polynomials.