5.1 Power Series Method

The **power series method** is the method for solving linear ODEs with ***variable*** coefficients.
It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions. In this section we begin by explaining the idea of the power series method.

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

Theorem 1

$$y'' + p(x)y' + q(x)y = r(x)$$

Existence of Power Series Solutions

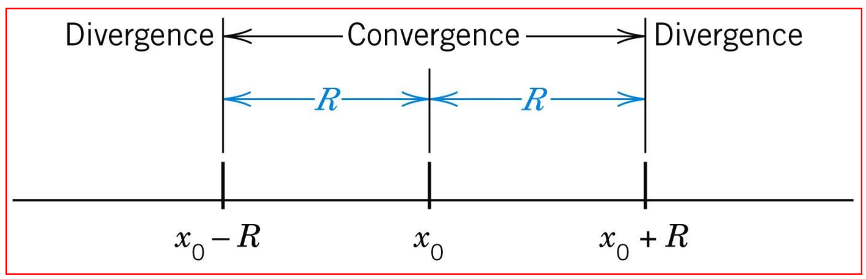
If p, q, and r in (12) *are analytic at $x = x_0$, then every solution of* (12) *is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.*

$$y'' + p(x)y' + q(x)y = r(x)$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

$$\text{(a)}\quad R = \frac{1}{\lim\limits_{m\to\infty} \sqrt[m]{|a_m|}}, \quad \text{(b)}\quad R = \frac{1}{\lim\limits_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$



Idea and Technique of the Power Series Method

For a given ODE

(4) $$y'' + p(x)y' + q(x)y = 0$$

we first represent $p(x)$ and $q(x)$ by power series in powers of x (or of $x - x_0$ if solutions in powers of $x - x_0$ are wanted).

(2) $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

(3) $$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

(5) $$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Further Theory: Operations on Power Series (continued)

1. Termwise Differentiation.

A power series may be differentiated term by term.

More precisely: if

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

converges for $|x - x_0| < R$ where $R > 0$, then the series obtained by differentiating term by term also converges for those x and represents the derivative y' of y for those x:

$$y'(x) = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1} \qquad (|x - x_0| < R).$$

Further Theory: Operations on Power Series (continued)

2. Termwise Addition.

Two power series may be added term by term. More precisely: if the series

(13) $$\sum_{m=0}^{\infty} a_m (x - x_0)^m \quad \text{and} \quad \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then the series

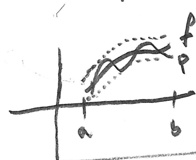
$$\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m$$

converges and represents $f(x) + g(x)$ for each x that lies in the interior of the convergence interval common to each of the two given series.

Ch 5 Series Solution of ODEs. Special Functions.

Stone - Weierstrass Theorem

Every continuous function $f(x)$ on $[a,b]$ can be uniformly approximated by a polynomial $p(x)$:



$$|f(x) - p(x)| < \varepsilon, \quad \forall x \in [a,b]$$

The importance of using a polynomial $p(x)$ to represent $f(x)$ is that polynomials are easy to work with & compute, requiring only addition/subtraction & multiplication:

$$f(x) \cong a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Taylor series & Sturm-Liouville orthogonal eigenpolynomials (Legendre & others) provide examples of polynomial expansions

1) $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$, $\quad \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$ (Taylor)

2) $f(x) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi)$ (Legendre, see Ch 12.11, p. 596)

$$A_n = \frac{2n+1}{2R^n} \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

When solving an ODE for a solution function $y = f(x)$, the power series method assumes y has the form $y = \sum_{m=0}^{\infty} a_m x^m$. We use the ODE to solve for the coeffs a_m, rather than by integration as for the PDEs in Chapter 12. One reason for this is because the expansion functions $y_m = a_m x^m$ do not form an orthogonal set on $[a,b]$.

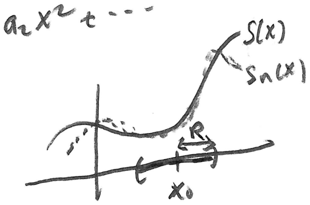
Certain ODEs important in applications have important solution functions $y(x) = \sum_{m=0}^{\infty} a_m x^m$, and these are called special functions. Bessel functions, Legendre polynomials, Chebyshev polynomials, Hermite & Jacobi polynomials, & gamma functions are all examples.

Ch 5.1 Power Series Method

Power Series

$$s(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$ (powerseries about x_0)

$$s(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$ (powerseries about $x_0 = 0$)



n-th partial sum

$$S_n(x) = \sum_{m=0}^{n} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$$ = finite sum = polynomial

$$R_n(x) = \text{remainder} = a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots$$ = infinite series

---

Note that $\sum_{m=0}^{\infty} a_m (x-x_0)^m = \lim_{n\to\infty} \sum_{m=0}^{n} a_m (x-x_0)^m$

Thus $y(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$ is a limit, and a limit either exists (converges) or does not exist (diverges). A power series will always converge for $x = x_0$. If the power series converges for other x, then the set of all x for which it converges will be an interval centered at x_0 & with radius R = radius of convergence

---

Radius given by $R = \dfrac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}} = \dfrac{1}{\lim_{m\to\infty} \left|\dfrac{a_{m+1}}{a_m}\right|}$

Needed for convergence of power series

To see this, recall ① ratio & ② root test:

① $\lim_{n\to\infty} \left| \dfrac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} |x-x_0| = L \cdot |x-x_0| < 1$

$\Rightarrow |x-x_0| < \frac{1}{L} \Rightarrow -\frac{1}{L} < x-x_0 < \frac{1}{L}$

$\Rightarrow x_0 - \frac{1}{L} < x < x_0 + \frac{1}{L}$

Similarly, ② $\lim_{n\to\infty} \sqrt[m]{|a_m(x-x_m)^m|} = \lim_{m\to\infty} \sqrt[m]{|a_m|} \cdot |x-x_0|$

$= L|x-x_0| < 1 \Rightarrow |x-x_0| < \frac{1}{L} \Rightarrow -\frac{1}{L} < x-x_0 < \frac{1}{L}$

$x_0 - \frac{1}{L} < x < x_0 + \frac{1}{L}$

In a 200-level differential equations class, solutions to a linear 2nd order homogeneous ODE with constant coefficient

$$y'' + p y' + q y = 0$$

are typically found by assuming $y = e^{rx}$.

If the coefficient are variable, then ODE has form

$$y'' + p(x) y' + q(x) y = 0.$$

<u>Theorem</u> The ODE above will have a power series solution if $p(x)$ & $q(x)$ have power series representations with radius of convergence $R > 0$.

Ex $\quad y'' + y = 0$

Solution Assume $y = \sum_{m=0}^{\infty} a_m x^m$. Then

$$y' = \sum_{m=1}^{\infty} a_m \cdot m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} a_m \cdot m(m-1) x^{m-2}$$

Then substitute these into ODE:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} + a_m \right] x^m = 0$$

$$\therefore \; (m+2)(m+1) a_{m+2} + a_m = 0, \quad m = 0, 1, 2, \ldots$$

$$\boxed{a_{m+2} = \frac{-a_m}{(m+2)(m+1)}, \quad m = 0, 1, 2, \ldots}$$

Recursion relation

$a_0 = \text{arb}$

$a_1 = \text{arb}$

$a_2 = \dfrac{-a_0}{(2)(1)}$	$a_3 = \dfrac{-a_1}{(3)(2)}$
$a_4 = \dfrac{-a_2}{(4)(3)} = \dfrac{a_0}{4!}$	$a_5 = \dfrac{-a_3}{(5)(4)} = \dfrac{a_1}{5!}$
$\vdots$	$\vdots$
$a_{2k} = \dfrac{(-1)^k a_0}{(2k)!}$	$a_{2k+1} = \dfrac{(-1)^k a_1}{(2k+1)!}$

$$\therefore \; y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\Rightarrow \quad \boxed{y(x) = a_0 \cos x + a_1 \sin x}$$

Ex (Airy's Eqn) $y'' - xy = 0, \; -\infty < x < \infty$

Solution Assume $y = \sum_{m=0}^{\infty} a_m x^m$. Then

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Substitute into ODE:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - x \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$(2)(1) a_2 + \sum_{m=1}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$2a_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} - a_{m-1} \right] x^m = 0$$

$2a_2 = 0$, $\quad (m+2)(m+1) a_{m+2} - a_{m-1} = 0$

$\boxed{a_2 = 0}$ $\quad a_{m+2} = \dfrac{a_{m-1}}{(m+2)(m+1)}$

Shift index

$$\boxed{a_m = \frac{a_{m-3}}{m(m-1)}}$$

$a_0 = \text{arb}$

$a_1 = \text{arb}$

$a_2 = 0$

$a_3 = \dfrac{a_0}{(3)(2)}$

$a_4 = \dfrac{a_1}{(4)(3)}$

$a_5 = 0$

$a_6 = \dfrac{a_3}{(6)(5)} = \dfrac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}$

$a_7 = \dfrac{a_4}{7 \cdot 6} = \dfrac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$

$a_{3k} = \dfrac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1)(3k)}$

$a_{3k+1} = \dfrac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3k)(3k+1)}$

$a_{3k+2} = 0$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + \sum_{m=3}^{\infty} a_m x^m$$

$$\therefore y(x) = a_0 \left[1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1)(3k)} \right] + a_1 \left[\sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3k)(3k+1)} \right]$$

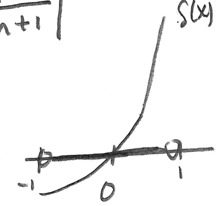
Ch 5.1 In-Class Examples

① Find radius of convergence

(a) $\sum_{m=0}^{\infty} m x^m$

$$R = \lim_{m\to\infty} \frac{1}{\left|\frac{a_{m+1}}{a_m}\right|} = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty} \left|\frac{m}{m+1}\right|$$

$$= \lim_{m\to\infty} \left|\frac{1}{1+\underbrace{\frac{1}{m}}_{\to 0}}\right| = 1 \qquad \boxed{\therefore R = 1}$$



$x_0 = 0,\ R = 1$

(b) $\sum_{m=0}^{\infty} m x^{2m} = \sum_{m=0}^{\infty} m (x^2)^m = \sum_{m=0}^{\infty} m t^m$, where $t = x^2$

$$R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty} \frac{m}{m+1} = \lim_{m\to\infty} \frac{1}{1+\frac{1}{m}} = 1 \Rightarrow R^* = 1$$

$\Rightarrow |t| < 1$

$\Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$

$\Rightarrow R = 1$

(c) $\sum_{m=0}^{\infty} \frac{(-1)^m}{5^m} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{5^m} t^m$, $t = x^2$

$$\lim_{m\to\infty} \frac{1}{\sqrt[m]{|a_m|}} = \lim_{m\to\infty} \frac{1}{\sqrt[m]{\left|\frac{(-1)}{5}\right|^m}} = \lim_{m\to\infty} \frac{1}{1/5} = 5$$

$\therefore R^* = 5 \Rightarrow |t| < 5 \Rightarrow x^2 < 5 \Rightarrow -\sqrt{5} < x < \sqrt{5} \Rightarrow \boxed{R = \sqrt{5}}$

② $y'' + y' - x^2 y = 0$

Solution Assume $y = \sum_{m=0}^{\infty} a_m x^m$. Then $y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$, $y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

Substitute y, y' & y'' into ODE:

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^{m-1} - x^2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\overset{①}{\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m} + \overset{②}{\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m} - \overset{③}{\sum_{m=0}^{\infty} a_m x^{m+2}} = 0 \qquad ③ = \sum_{m=2}^{\infty} a_{m-2} x^m$$

$$\overbrace{(2)(1) a_2 + (3)(2) a_3 x}^{①} + \overbrace{a_1 + 2 a_2 x}^{②} + \sum_{m=2}^{\infty} \Big[\overbrace{(m+2)(m+1) a_{m+2}}^{①} + \overbrace{(m+1) a_{m+1}}^{②} - \overbrace{a_{m-2}}^{③} \Big] x^m = 0$$

(*) (**)

$$\boxed{a_{m+2} = \frac{-(m+1) a_{m+1} + a_{m-2}}{(m+2)(m+1)}, \quad m = 2, 3, 4, \ldots}$$

$a_0 = \text{arb}$

$a_1 = \text{arb}$

(*) $2a_2 + a_1 = 0 \Rightarrow a_2 = -\frac{1}{2} a_1$

(**) $6a_3 + 2a_2 = 0 \Rightarrow a_3 = -\frac{1}{3} a_2 = \frac{1}{3 \cdot 2} a_1$

$(m=2)$ $a_4 = \dfrac{-3a_3 + a_0}{4 \cdot 3} = -\dfrac{a_3}{4} + \dfrac{a_0}{4 \cdot 3} = \dfrac{-a_1}{4 \cdot 3 \cdot 2} + \dfrac{a_0}{4 \cdot 3}$

$(m=3)$ $a_5 = \dfrac{-4a_4 + a_1}{5 \cdot 4} = -\dfrac{a_4}{5} + \dfrac{a_1}{5 \cdot 4} = \dfrac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} - \dfrac{a_0}{5 \cdot 4 \cdot 3} + \dfrac{a_1}{5 \cdot 4} \cdot \dfrac{6}{6}$

$$= \frac{7a_1}{5 \cdot 4 \cdot 3 \cdot 2} - \frac{a_0}{5 \cdot 4 \cdot 3}$$

$$\therefore \boxed{y(x) = a_0 \left[1 + \frac{1}{12} x^4 - \frac{1}{60} x^5 + \cdots \right] + a_1 \left[x - \frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{7}{120} x^5 - \cdots \right]}$$