# MAT 202
# Larson – Section 10.3
# Parametric Equations & Calculus

Now that we can represent a graph in the plane by a set of parametric equations, how can we use calculus to study plane curves? In this section we will review (i) how to find the slope of the tangent line to a curve, (ii) how to find the arc length of a curve, and (iii) how to find the surface area of a solid of revolution of a set of parametric equations…WITHOUT changing them to rectangular form!

***<u>Parametric Form of the Derivative</u>*:** If a smooth curve $C$ is given by the equations $x = f(t)$ and $y = g(t)$ then the slope of $C$ at $(x, y)$ is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

***<u>Higher Order Derivatives</u>*:**

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left[\dfrac{dy}{dx}\right]}{\dfrac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

$$\frac{d^3y}{dx^3} = \frac{\dfrac{d}{dt}\left[\dfrac{d^2y}{dx^2}\right]}{\dfrac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

**Recall:** Finding the slope of a tangent line is associated with the first derivative, and finding concavity is associated with the second derivative.

***Higher Order Derivatives***:

$$\boxed{\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0}$$

$$\frac{d^3 y}{dx^3} = \frac{\frac{d}{dt}\left[\frac{d^2 y}{dx^2}\right]}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

Ex: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and find the slope and concavity (if possible) at the given value of the parameter. $x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad when\ t = 2$

Ex: Find the equations of the tangent lines at the point where the curve crosses itself. $x = 2\sin 2t, \quad y = 3\sin t$

***<u>Horizontal & Vertical Tangents</u>*****:** If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, then the curve represented by $x = f(t)$ and $y = g(t)$ has a **horizontal tangent** at $(f(t_0), g(t_0))$. Similarly, if $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, then the curve represented by $x = f(t)$ and $y = g(t)$ has a **vertical tangent** at $(f(t_0), g(t_0))$.

Ex: Find all points (if any) of horizontal and vertical tangency to the curve. $x = 3\cos\theta, \quad y = 3\sin\theta$

Ex: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and find the slope and concavity (if possible) at the given value of the parameter.

$$x = \sqrt{t}, \quad y = \sqrt{t-1}, \quad when\ t = 2$$

$$y = \sqrt{x^2-1} \qquad \frac{dy}{dx} = 2x \cdot \frac{1}{2}(x^2-1)^{-1/2}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{\sqrt{1}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cancel{\tfrac{1}{2}}(t-1)^{-1/2}}{\cancel{\tfrac{1}{2}}t^{-1/2}} = \frac{t^{1/2}}{(t-1)^{1/2}}$$

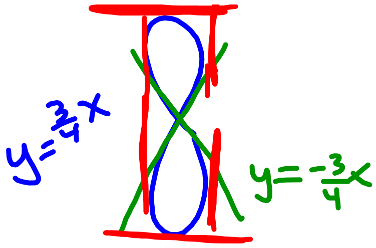
$$@\ t=2,\ m = \frac{2^{1/2}}{(2-1)^{1/2}} = \frac{\sqrt{2}}{1} = \boxed{\sqrt{2}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\left[\frac{\frac{1}{2}t^{-1/2}(t-1)^{1/2} - \frac{1}{2}t^{1/2}(t-1)^{-1/2}}{t-1}\right]}{\frac{1}{2}t^{-1/2}}$$

$$@\ t=2 \qquad \frac{\frac{\frac{1}{2\sqrt{2}}(1)^{1/2} - \frac{\sqrt{2}}{2}(1)^{-1/2}}{1}}{\frac{1}{2\sqrt{2}}} < 0$$

$\therefore$ $\boxed{\text{Concave down}}$

Ex: Find the equations of the $\boxed{\text{tangent lines}}$ at the point where the curve crosses itself. $x = 2\sin 2t, \quad y = 3\sin t$




Where does it cross itself?

| t | x | y |
|---|---|---|
| 0 | 0 | 0 |
| $\frac{\pi}{4}$ | 2 | $\frac{3\sqrt{2}}{2}$ |
| $\frac{\pi}{2}$ | 0 | 3 |
| $\frac{3\pi}{4}$ | -2 | $\frac{3\sqrt{2}}{2}$ |
| $\pi$ | 0 | 0 |

@ $t = 0, \pi$, they cross.

point: $(0,0)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{4\cos 2t}$$

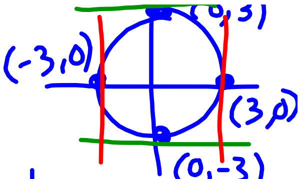
@ $t = 0$

$$m = \frac{3\cos 0}{4\cos 2\cdot 0} = \frac{3}{4} \qquad \boxed{y = \frac{3}{4}x}$$

@ $t = \pi$

$$m = \frac{3\cos\pi}{4\cos 2\pi} = \frac{-3}{4} \qquad \boxed{y = \frac{-3}{4}x}$$

Ex: Find all points (if any) of horizontal and vertical tangency to the curve. $x = 3\cos\theta, \quad y = 3\sin\theta$




For Horizontal tangency we want:

$$\frac{dy}{dx} = 0 \rightarrow 0 = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \rightarrow \frac{dy}{d\theta} = 0$$

For Vertical tangency we want:

$$\frac{dy}{dx} = \text{undefined} \rightarrow \text{und.} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \rightarrow \frac{dx}{d\theta} = 0$$

$$\frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dx}{d\theta} = -3\sin\theta$$

Horizontal tangency: $\frac{dy}{d\theta} = 0$ & $\frac{dx}{d\theta} \neq 0$

$3\cos\theta = 0$

$\cos\theta = 0$

$\theta = \frac{\pi}{2} + n\pi \rightarrow \frac{\pi}{2}, \frac{3\pi}{2}$

H.T. Points: $(0, 3), (0, -3)$

Vertical tangency: $\frac{dy}{d\theta} \neq 0$ & $\frac{dx}{d\theta} = 0$

$0 = -3\sin\theta$

$\sin\theta = 0$

$\theta = n\pi \rightarrow 0, \pi$

V.T. points: $(3,0), (-3,0)$

***Arc Length In Parametric Form***: If a smooth curve $C$ is given by the equations $x = f(t)$ and $y = g(t)$ such that $C$ does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of $C$ over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

***Area of a Surface of Revolution***: If a smooth curve $C$ is given by the equations $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area $S$ of the surface of revolution formed by revolving $C$ about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, Revolution about $x$-axis : $g(t) \ge 0$
2. $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, Revolution about $y$-axis : $f(t) \ge 0$