MAT 201
Larson/Edwards – Section 1.3
Evaluating Limits Analytically

In section 1.2 we learned that the limit of *f(x)* as *x* approaches *c* does not depend on the value of *f* at $x = c$. In some cases, however, the limit may be precisely *f(c)*. In such cases, the limit can be evaluated by ***direct substitution***.

<u>Evaluating Limits by Direct Substitution</u>: In many cases (that is, when the function is continuous at $x = c$) $\lim_{x \to c} f(x)$ can be found by direct substitution: $\lim_{x \to c} f(x) = f(c)$.

Ex: Find the indicated limit by direct substitution, if possible:

a) $\lim_{x \to 5}(3x - 2)$

b) $\lim_{x \to 0} \frac{x^2 - 5}{x + 1}$

c) $\lim_{x \to -1}\left(\frac{2x^2 - 3x}{x}\right)$

Ex: Find the indicated limit by direct substitution, if possible:

a) $\lim_{x \to 5} (3x - 2) = 3(5) - 2$

$= 13$

b) $\lim_{x \to 0} \frac{x^2 - 5}{x + 1} = \frac{0^2 - 5}{0 + 1}$

$= \boxed{-5}$

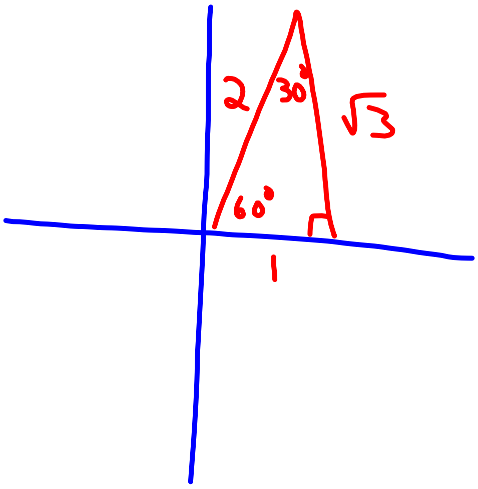
c) $\lim_{x \to -1} \left(\frac{2x^2 - 3x}{x} \right) = \frac{2(-1)^2 - 3(-1)}{(-1)}$

$= \frac{5}{-1}$

$= -5$

d) $\lim_{x \to \frac{\pi}{3}} (\sin x) = \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2}$



d) $\lim_{x \to \frac{\pi}{3}} (\sin x)$

e) $\lim_{x \to -2} \frac{x^2 - 4}{2x^2 + x^3}$

f) $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25}$

We see that in both examples (e) and (f), direct substitution alone doesn't work for us since $\frac{0}{0}$ is an indeterminate form, and therefore we cannot determine the limit. Next, we will learn ways to rewrite functions so that we can use direct substitution to find the limit.

e) $\lim_{x \to -2} \frac{x^2 - 4}{2x^2 + x^3} = \frac{(-2)^2 - 4}{2(-2)^2 + (-2)^3} = \frac{0}{0}$

indeterminate

f) $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = \frac{\sqrt{25} - 5}{25 - 25} = \frac{0}{0}$

Functions That Agree At All But One Point (Theorem 1.7):
Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).$$

This theorem states that if we can remove a discontinuity at $x = c$ by factoring and cancelling common factors, then we can find the limit.

Ex: Use the theorem above (1.7) to find the limit:

a) $\lim\limits_{x \to -2} \frac{x^2 - 4}{2x^2 + x^3}$

b) $\lim\limits_{x \to 25} \frac{\sqrt{x} - 5}{x - 25}$

Functions That Agree At All But One Point (Theorem 1.7): Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\boxed{\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c).}$$

Ex: Use the theorem above (1.7) to find the limit:

a) $$\lim_{x \to -2} \underbrace{\frac{x^2 - 4}{2x^2 + x^3}}_{f(x)} = \lim_{x \to -2} \frac{\cancel{(x+2)}(x-2)}{x^2\cancel{(2+x)}}$$

$$= \lim_{x \to -2} \underbrace{\frac{x-2}{x^2}}_{g(x)} = \frac{-2-2}{(-2)^2} = -1$$

g(c)

b) $\lim_{x \to 25} \frac{\sqrt{x} - 5}{x - 25} = \lim_{x \to 25} \frac{\cancel{\sqrt{x} - 5}^{\,1}}{(\sqrt{x}+5)\cancel{(\sqrt{x}-5)}}$

$= \lim_{x \to 25} \frac{1}{\sqrt{x}+5}$

$= \frac{1}{\sqrt{25}+5} = \frac{1}{10}$

$$\lim_{x \to 25} \frac{(\sqrt{x}-5)(\sqrt{x}+5)}{(x-25)(\sqrt{x}+5)}$$

$$\lim_{x \to 25} \frac{\cancel{x-25}^{\,1}}{\cancel{(x-25)}_{1}(\sqrt{x}+5)}$$

$$\lim_{x \to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}$$

Finding Limits By Rationalizing: In some cases when radicals are involved and direct substitution may not be used it is helpful to rationalize either the numerator or denominator then re-attempt direct substitution.

Ex: Find the limit, if it exists: $\lim\limits_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

Tools For Finding Limits:

1. Try to find the limit by direct substitution.
2. If the limit of *f(x)* as *x* approaches *c* cannot be evaluated by direct substitution, try dividing out or rationalizing techniques.
3. Use a graph or table to reinforce your conclusion.

Useful Theorems:

1.1: Let *b* and *c* be real numbers and let *n* be a positive integer.

a) $\lim\limits_{x \to c} b = b$

b) $\lim\limits_{x \to c} x = c$

c) $\lim\limits_{x \to c} x^n = c^n$

Ex: Find the limit, if it exists: $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\left(\sqrt{x+2} + \sqrt{2}\right)}{\left(\sqrt{x+2} + \sqrt{2}\right)}$$

$$\lim_{x \to 0} \frac{x + \cancel{2} + \cancel{\sqrt{2} \cdot \sqrt{x+2}} - \cancel{\sqrt{2} \cdot \sqrt{x+2}} - \cancel{2}}{x\left(\sqrt{x+2} + \sqrt{2}\right)}$$

$$\lim_{x \to 0} \frac{\overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}}\left(\sqrt{x+2} + \sqrt{2}\right)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}} \ \underline{\underline{\text{or}}} \ \frac{\sqrt{2}}{4}}$$

1.2: Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = K$

a) Scalar multiple: $\lim\limits_{x \to c}[b \cdot f(x)] = bL$

b) Sum or Difference: $\lim\limits_{x \to c}[f(x) \pm g(x)] = L \pm K$

c) Product: $\lim\limits_{x \to c}[f(x) \cdot g(x)] = L \cdot K$

d) Quotient: $\lim\limits_{x \to c}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{L}{K}$, $K \neq 0$

e) Power: $\lim\limits_{x \to c}[f(x)]^n = L^n$

1.3: a) If p is a polynomial function and c is a real number, then $\lim\limits_{x \to c} p(x) = p(c)$.

b) If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

1.4: Let n be a positive integer. The following limit is valid for all c if n is odd and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}.$$

1.5: If f and g are functions such that $\lim\limits_{x \to c} g(x) = L$ and $\lim\limits_{x \to L} f(x) = f(L)$, then $\lim\limits_{x \to c} f(g(x)) = f\left(\lim\limits_{x \to c} g(x)\right) = f(L)$.

1.6: Let c be a real number in the domain of the given trigonometric function.

a) $\lim\limits_{x \to c} \sin x = \sin c$

b) $\lim\limits_{x \to c} \cos x = \cos c$

c) $\lim_{x \to c} \tan x = \tan c$

d) $\lim_{x \to c} \cot x = \cot c$

e) $\lim_{x \to c} \sec x = \sec c$

f) $\lim_{x \to c} \csc x = \csc c$

1.7: WE ALREADY COVERED

1.8: The Squeeze Theorem: If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$ then $\lim_{x \to c} f(x)$ exists and is equal to L.

1.9: a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

b) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$