Exercise 1

Calculate $\hat{U}_{prep}$ for a 3 particle System

$$A = \frac{1}{2}\left(\hat{I}\otimes\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{I}\otimes\hat{\sigma}_y + \hat{I}\otimes\hat{\sigma}_y\otimes\hat{I} - \hat{I}\otimes\hat{\sigma}_y\otimes\hat{\sigma}_y \right)$$

$$B = \frac{1}{2}\left(\hat{I}\otimes\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_y\otimes\hat{I} + \hat{\sigma}_y\otimes\hat{I}\otimes\hat{I} - \hat{\sigma}_y\otimes\hat{\sigma}_y\otimes\hat{I} \right)$$

$$C = \frac{1}{2}\left(\hat{I}\otimes\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{I}\otimes\hat{\sigma}_y + \hat{\sigma}_y\otimes\hat{I}\otimes\hat{I} - \hat{\sigma}_y\otimes\hat{I}\otimes\hat{\sigma}_y \right)$$

$$\hat{U}_{prep} = A\cdot B\cdot C$$

$$A = \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 & 0 & 0 & 0 & 0 \\ i & 1 & -1 & -i & 0 & 0 & 0 & 0 \\ i & -1 & 1 & -i & 0 & 0 & 0 & 0 \\ 1 & i & i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & -i & 1 \\ 0 & 0 & 0 & 0 & i & 1 & -1 & -i \\ 0 & 0 & 0 & 0 & i & -1 & 1 & -i \\ 0 & 0 & 0 & 0 & 1 & i & i & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 & -i & 0 & 1 & 0 \\ 0 & 1 & 0 & -i & 0 & -i & 0 & 1 \\ i & 0 & 1 & 0 & -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 & 0 & -1 & 0 & -i \\ i & 0 & -1 & 0 & 1 & 0 & -i & 0 \\ 0 & i & 0 & -1 & 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 & i & 0 & 1 & 0 \\ 0 & 1 & 0 & i & 0 & i & 0 & 1 \end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix} 1 & -i & 0 & 0 & -i & 1 & 0 & 0 \\ i & 1 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & -i & 1 \\ 0 & 0 & i & 1 & 0 & 0 & -1 & -i \\ i & -1 & 0 & 0 & 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & -1 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i & 0 & 0 & i & 1 \end{pmatrix}$$

$$A\cdot B\cdot C = \frac{1}{8}\begin{pmatrix} 1 & -i & -i & 1 & 0 & 0 & 0 & 0 \\ i & 1 & -1 & -i & 0 & 0 & 0 & 0 \\ i & -1 & 1 & -i & 0 & 0 & 0 & 0 \\ 1 & i & i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -i & -i & 1 \\ 0 & 0 & 0 & 0 & i & 1 & -1 & -i \\ 0 & 0 & 0 & 0 & i & -1 & 1 & -i \\ 0 & 0 & 0 & 0 & 1 & i & i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -i & 0 & -i & 0 & 1 & 0 \\ 0 & 1 & 0 & -i & 0 & -i & 0 & 1 \\ i & 0 & 1 & 0 & -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 & 0 & -1 & 0 & -i \\ i & 0 & -1 & 0 & 1 & 0 & -i & 0 \\ 0 & i & 0 & -1 & 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 & i & 0 & 1 & 0 \\ 0 & 1 & 0 & i & 0 & i & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i & 0 & 0 & -i & 1 & 0 & 0 \\ i & 1 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 & -i & 1 \\ 0 & 0 & i & 1 & 0 & 0 & -1 & -i \\ i & -1 & 0 & 0 & 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & -1 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i & 0 & 0 & i & 1 \end{pmatrix}$$

$$\boxed{\hat{U}_{prep} = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & -i & 0 & -i & i & 0 \\ i & 0 & 0 & -i & 0 & i & -i & 0 \\ 0 & i & i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & -i & -i & 0 \\ 0 & i & -i & 0 & i & 0 & 0 & -i \\ 0 & -i & i & 0 & i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & i & i & 0 \end{pmatrix}}$$

Exercise 2

Calculate $\hat{\rho}_{prep}$ for 3 particle system

$$\hat{\rho}_{prep} = \hat{\rho}_i \otimes \hat{\rho}_i \otimes \hat{\rho}_i \quad : \quad \hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{8}\begin{pmatrix} 1 & r & r & r^2 & r & r^2 & r^2 & r^3 \\ r & 1 & r^2 & r & r^2 & r & r^3 & r^2 \\ r & r^2 & 1 & r & r^2 & r^3 & r & r^2 \\ r^2 & r & r & 1 & r^3 & r^2 & r^2 & r \\ r & r^2 & r^2 & r^3 & 1 & r & r & r^2 \\ r^2 & r & r^3 & r^2 & r & 1 & r^2 & r \\ r^2 & r^3 & r & r^2 & r & r^2 & 1 & r \\ r^3 & r^2 & r^2 & r & r^2 & r & r & 1 \end{pmatrix}}$$

Exercise 3

Calculate $\hat{\rho}_{prep} = \hat{u}_{prep}\,\hat{\rho}_{prep}\,\hat{u}^{\dagger}_{prep}$

$$\hat{u}_{prep} = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & -i & 0 & -i & i & 0 \\ i & 0 & 0 & -i & 0 & i & -i & 0 \\ 0 & i & i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & -i & -i & 0 \\ 0 & i & -i & 0 & i & 0 & 0 & -i \\ 0 & -i & i & 0 & i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & i & i & 0 \end{pmatrix}, \quad \hat{\rho}_{prep} = \frac{1}{8}\begin{pmatrix} 1 & r & r & r^2 & r & r^2 & r^2 & r^3 \\ r & 1 & r^2 & r & r^2 & r & r^3 & r^2 \\ r & r^2 & 1 & r & r^2 & r^3 & r & r^2 \\ r^2 & r & r & 1 & r^3 & r^2 & r^2 & r \\ r & r^2 & r^2 & r^3 & 1 & r & r & r^2 \\ r^2 & r & r^3 & r^2 & r & 1 & r^2 & r \\ r^2 & r^3 & r & r^2 & r & r^2 & 1 & r \\ r^3 & r^2 & r^2 & r & r^2 & r & r & 1 \end{pmatrix}$$

$$\hat{u}^{\dagger}_{prep} = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & -i & 0 & -i & i & 0 \\ i & 0 & 0 & -i & 0 & i & -i & 0 \\ 0 & i & i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & -i & -i & 0 \\ 0 & i & -i & 0 & i & 0 & 0 & -i \\ 0 & -i & i & 0 & i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & i & i & 0 \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{32}\begin{pmatrix} 0 & -i & -i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & -i & 0 & -i & i & 0 \\ i & 0 & 0 & -i & 0 & i & -i & 0 \\ 0 & i & i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & -i & -i & 0 \\ 0 & i & -i & 0 & i & 0 & 0 & -i \\ 0 & -i & i & 0 & i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & i & i & 0 \end{pmatrix}\begin{pmatrix} 1 & r & r & r^2 & r & r^2 & r^2 & r^3 \\ r & 1 & r^2 & r & r^2 & r & r^3 & r^2 \\ r & r^2 & 1 & r & r^2 & r^3 & r & r^2 \\ r^2 & r & r & 1 & r^3 & r^2 & r^2 & r \\ r & r^2 & r^2 & r^3 & 1 & r & r & r^2 \\ r^2 & r & r^3 & r^2 & r & 1 & r^2 & r \\ r^2 & r^3 & r & r^2 & r & r^2 & 1 & r \\ r^3 & r^2 & r^2 & r & r^2 & r & r & 1 \end{pmatrix}\begin{pmatrix} 0 & -i & -i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & -i & 0 & -i & i & 0 \\ i & 0 & 0 & -i & 0 & i & -i & 0 \\ 0 & i & i & 0 & -i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & -i & -i & 0 \\ 0 & i & -i & 0 & i & 0 & 0 & -i \\ 0 & -i & i & 0 & i & 0 & 0 & -i \\ i & 0 & 0 & i & 0 & i & i & 0 \end{pmatrix}$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{8}\begin{pmatrix} 3r^2+1 & 0 & 0 & 0 & 0 & 0 & 0 & -r(r^2+3) \\ 0 & 1-r^2 & 0 & 0 & 0 & 0 & r(1-r^2) & 0 \\ 0 & 0 & 1-r^2 & 0 & 0 & r(1-r^2) & 0 & 0 \\ 0 & 0 & 0 & 1-r^2 & r(1-r^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & r(1-r^2) & 1-r^2 & 0 & 0 & 0 \\ 0 & 0 & r(1-r^2) & 0 & 0 & 1-r^2 & 0 & 0 \\ 0 & r(1-r^2) & 0 & 0 & 0 & 0 & 1-r^2 & 0 \\ -r(r^2+3) & 0 & 0 & 0 & 0 & 0 & 0 & 3r^2+1 \end{pmatrix}}$$

Exercise 4

Write $\hat{P}_{prep}$ in tensor product form

$$
\begin{aligned}
[1,1] &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| & [5,4] &= |1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\
[1,8] &= |0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| & [5,5] &= |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\
[2,2] &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| & [6,3] &= |1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
[2,7] &= |0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| & [6,6] &= |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\
[3,3] &= |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| & [7,2] &= |1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| \\
[3,6] &= |0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| & [7,7] &= |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
[4,4] &= |0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| & [8,1] &= |1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| \\
[4,5] &= |0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| & [8,8] &= |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1|
\end{aligned}
$$

$$
\begin{aligned}
\hat{P}_{prep} = \frac{1}{8}\Big[& (3r^2+1)|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| - r(r^2+3)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| + (1-r^2)|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\
& + r(1-r^2)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| + (1-r^2)|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| + r(1-r^2)|0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| \\
& + (1-r^2)|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| + r(1-r^2)|0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| + r(1-r^2)|1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\
& + (1-r^2)|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + r(1-r^2)|1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| + (1-r^2)|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\
& + r(1-r^2)|1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| + (1-r^2)|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| - r(r^2+3)|1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| \\
& + (3r^2+1)|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big]
\end{aligned}
$$

$\hat{\rho}_i = |0\rangle\langle 0| : \hat{\rho}_F = |0\rangle\langle 0|$, $\hat{\rho}_i = |1\rangle\langle 1| : \hat{\rho}_F = |1\rangle\langle 1|$, $\hat{\rho}_i = |0\rangle\langle 1| : \hat{\rho}_F = (1-2\xi)|0\rangle\langle 1|$, $\hat{\rho}_i = |1\rangle\langle 0| : \hat{\rho}_F = (1-2\xi)|1\rangle\langle 0|$

$$
\begin{aligned}
\hat{P}_{prep} = \frac{1}{8}\Big[& (3r^2+1)|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| - r(r^2+3)(1-2\lambda)(1-2\alpha)(1-2\beta)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| + (1-r^2)|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\
& + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|0\rangle\langle 1| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| + (1-r^2)|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
& + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| + (1-r^2)|0\rangle\langle 0| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
& + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|0\rangle\langle 1| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\
& + (1-r^2)|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|1\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
& + (1-r^2)|1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |1\rangle\langle 1| + r(1-r^2)(1-2\lambda)(1-2\alpha)(1-2\beta)|1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |0\rangle\langle 1| + (1-r^2)|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
& - r(r^2+3)(1-2\lambda)(1-2\alpha)(1-2\beta)|1\rangle\langle 0| \otimes |1\rangle\langle 0| \otimes |1\rangle\langle 0| + (3r^2+1)|1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big]
\end{aligned}
$$

$\gamma = (1-2\lambda)(1-2\alpha)(1-2\beta)$

$$
\boxed{\hat{P}_{prep} = \frac{1}{8}\begin{pmatrix}
(3r^2+1) & 0 & 0 & 0 & 0 & 0 & 0 & -r(r^2+3)\gamma \\
0 & (1-r^2) & 0 & 0 & 0 & 0 & \gamma r(1-r^2) & 0 \\
0 & 0 & (1-r^2) & 0 & 0 & \gamma r(1-r^2) & 0 & 0 \\
0 & 0 & 0 & (1-r^2) & \gamma r(1-r^2) & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma r(1-r^2) & (1-r^2) & 0 & 0 & 0 \\
0 & 0 & \gamma r(1-r^2) & 0 & 0 & (1-r^2) & 0 & 0 \\
0 & \gamma r(1-r^2) & 0 & 0 & 0 & 0 & (1-r^2) & 0 \\
-r(r^2+3)\gamma & 0 & 0 & 0 & 0 & 0 & 0 & (3r^2+1)
\end{pmatrix}}
$$

Exercise 5

Calculate QFI

$$H_i = \sum_m \frac{1}{\eta_m}\left(\frac{\partial \eta_m}{\partial \lambda}\right)^2 + 2\sum_{n=m} \frac{\eta_n-\eta_m}{\eta_n+\eta_m}\left|\langle \varphi_m | \frac{\partial|\varphi_n\rangle}{\partial\lambda}\right|^2 \quad : \quad \begin{matrix}\eta \equiv \text{eigenvalue} \\ \mu \equiv \text{eigenstate}\end{matrix} \quad : \quad \eta = \det(\hat{A} - \eta\hat{I}) = 0$$

$$\hat{A}_1 = \frac{1}{8}\begin{pmatrix} (3r^2+1) & -r(r^2+3)\gamma \\ -r(r^2+3)\gamma & (3r^2+1)\end{pmatrix}, \quad \hat{A}_2 = \frac{1}{8}\begin{pmatrix}(1-r^2) & \gamma r(1-r^2) \\ \gamma r(1-r^2) & (1-r^2)\end{pmatrix}, \quad \hat{A}_3 = \frac{1}{8}\begin{pmatrix}(1-r^2) & \gamma r(1-r^2) \\ \gamma r(1-r^2) & (1-r^2)\end{pmatrix}, \quad \hat{A}_4 = \frac{1}{8}\begin{pmatrix}(1-r^2) & \gamma r(1-r^2) \\ \gamma r(1-r^2) & (1-r^2)\end{pmatrix}$$

i.) Calculate H_1:

$$\det(\hat{A}_1 - \eta\hat{I}) = \frac{1}{8}\begin{pmatrix} (3r^2+1) & -r(r^2+3)\gamma \\ -r(r^2+3)\gamma & (3r^2+1)\end{pmatrix} - \frac{1}{8}\begin{pmatrix} 8\eta & 0 \\ 0 & 8\eta\end{pmatrix} = \frac{1}{8}\begin{pmatrix} (3r^2+1)-8\eta & -r(r^2+3)\gamma \\ -r(r^2+3)\gamma & (3r^2+1)-8\eta\end{pmatrix}$$

$$\left|\frac{1}{8}\begin{pmatrix} (3r^2+1)-8\eta & -r(r^2+3)\gamma \\ -r(r^2+3)\gamma & (3r^2+1)-8\eta\end{pmatrix}\right| = \frac{1}{8}\left(\left((3r^2+1)-8\eta\right)^2 - r^2(r^2+3)^2\gamma^2\right)$$

$$\frac{1}{8}\left(\left((3r^2+1)-8\eta\right)^2 - r^2(r^2+3)^2\gamma^2\right) = 0 \quad : \quad \left((3r^2+1)-8\eta\right)^2 = r^2(r^2+3)^2\gamma^2$$

$$(3r^2+1) - 8\eta = \pm r(r^2+3)\gamma \quad \therefore \quad \eta = \frac{1}{8}\left((3r^2+1) \pm r(r^2+3)\gamma\right) \quad : \quad \gamma = (1-2\lambda)(1-2\alpha)(1-2\beta)$$

$$\eta_1 = \frac{1}{8}\left((3r^2+1) + r(r^2+3)\gamma\right), \quad \eta_2 = \frac{1}{8}\left((3r^2+1) - r(r^2+3)\gamma\right)$$

$H(\eta_1)$:

$$\left(\frac{\partial\eta_1}{\partial\lambda}\right) = -\frac{1}{8}\cdot 2r(r^2+3)(1-2\alpha)(1-2\beta) = \frac{-1}{4}r(r^2+3)(1-2\alpha)(1-2\beta)$$

$$\left(\frac{\partial\eta_1}{\partial\lambda}\right)^2 = \frac{1}{16}r^2(r^2+3)^2(1-2\alpha)^2(1-2\beta)^2 \quad : \quad \frac{1}{\eta_1} = \frac{8}{(3r^2+1)+r(r^2+3)\gamma}$$

$$\frac{1}{\eta_1}\cdot\left(\frac{\partial\eta_1}{\partial\lambda}\right)^2 = \frac{8}{(3r^2+1)+r(r^2+3)\gamma}\cdot\frac{1}{16}r^2(r^2+3)^2(1-2\alpha)^2(1-2\beta)^2$$

$$H(\eta_1) = \frac{1}{2}\cdot\frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{(3r^2+1)+r(r^2+3)\gamma}$$

$H(\eta_2)$:

$$\left(\frac{\partial\eta_2}{\partial\lambda}\right) = \frac{1}{8}\cdot 2r(r^2+3)(1-2\alpha)(1-2\beta) = \frac{1}{4}r(r^2+3)(1-2\alpha)(1-2\beta)$$

$$\left(\frac{\partial\eta_2}{\partial\lambda}\right)^2 = \frac{1}{16}r^2(r^2+3)^2(1-2\alpha)^2(1-2\beta)^2 \quad : \quad \frac{1}{\eta_2} = \frac{8}{(3r^2+1)-r(r^2+3)\gamma}$$

$$\frac{1}{\eta_2}\cdot\left(\frac{\partial\eta_2}{\partial\lambda}\right)^2 = \frac{8}{(3r^2+1)-r(r^2+3)\gamma}\cdot\frac{1}{16}r^2(r^2+3)^2(1-2\alpha)^2(1-2\beta)^2$$

$$H(\eta_2) = \frac{1}{2}\cdot\frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{(3r^2+1)-r(r^2+3)\gamma}$$

Exercise 5 Continued

$$H_1 = \frac{1}{2}\cdot\frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{(3r^2+1)+r(r^2+3)\gamma} + \frac{1}{2}\cdot\frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{(3r^2+1)-r(r^2+3)\gamma}$$

$$= \frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{2}\left(\frac{1}{(3r^2+1)+r(r^2+3)\gamma} + \frac{1}{(3r^2+1)-r(r^2+3)\gamma}\right)$$

$$= \frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2}{2}\left(\frac{(3r^2+1)-\cancel{r(r^2+3)\gamma}+(3r^2+1)+\cancel{r(r^2+3)\gamma}}{(3r^2+1)^2-r^2(r^2+3)^2\gamma^2}\right)$$

$$= \frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2(3r^2+1)}{(3r^2+1)^2-r^2(r+3)^2\gamma^2}$$

$$\boxed{H_1 = \frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2(3r^2+1)}{(3r^2+1)^2-r^2(r+3)^2\gamma^2}}$$

ii.) Calculate H_2:

$$\det(\hat{A}_1-\eta\hat{I}) = \frac{1}{8}\begin{pmatrix}(1-r^2) & \gamma r(1-r^2)\\ \gamma r(1-r^2) & (1-r^2)\end{pmatrix} - \frac{1}{8}\begin{pmatrix}8\eta & 0\\ 0 & 8\eta\end{pmatrix} = \frac{1}{8}\begin{pmatrix}(1-r^2)-8\eta & \gamma r(1-r^2)\\ \gamma r(1-r^2) & (1-r^2)-8\eta\end{pmatrix}$$

$$\left|\frac{1}{8}\begin{pmatrix}(1-r^2)-8\eta & \gamma r(1-r^2)\\ \gamma r(1-r^2) & (1-r^2)-8\eta\end{pmatrix}\right| = \frac{1}{8}\left(\left((1-r^2)-8\eta\right)^2-\gamma^2r^2(1-r^2)^2\right) = 0$$

$$\left((1-r^2)-8\eta\right)^2 = \gamma^2r^2(1-r^2)^2 \quad \therefore \quad \eta = \frac{1}{8}\left((1-r^2)\pm\gamma r(1-r^2)\right) : \quad \gamma=(1-2\lambda)(1-2\alpha)(1-2\beta)$$

$$\eta_1 = \frac{1}{8}\left((1-r^2)+\gamma r(1-r^2)\right), \quad \eta_2 = \frac{1}{8}\left((1-r^2)-\gamma r(1-r^2)\right)$$

$H(\eta_1)$: $\left(\frac{\partial\eta_1}{\partial\lambda}\right) = \frac{1}{8}\cdot -2r(1-r^2)(1-2\alpha)(1-2\beta)$: $\left(\frac{\partial\eta_1}{\partial\lambda}\right) = -\frac{1}{4}\cdot r(1-r^2)(1-2\alpha)(1-2\beta)$

$$\left(\frac{\partial\eta_1}{\partial\lambda}\right)^2 = \frac{1}{16}r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2 \quad : \quad \frac{1}{\eta_1} = \frac{8}{(1-r^2)+\gamma r(1-r^2)}$$

$$\frac{1}{\eta_1}\cdot\left(\frac{\partial\eta_1}{\partial\lambda}\right)^2 = \frac{8}{(1-r^2)+\gamma r(1-r^2)}\cdot\frac{1}{16}r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2 = \frac{1}{2}\,\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)+\gamma r(1-r^2)}$$

$$H(\eta_1) = \frac{1}{2}\,\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)+\gamma r(1-r^2)}$$

$H(\eta_2)$: $\left(\frac{\partial\eta_2}{\partial\lambda}\right) = \frac{1}{8}\,2r(1-r^2)(1-2\alpha)(1-2\beta)$: $\left(\frac{\partial\eta_2}{\partial\lambda}\right) = \frac{1}{4}r(1-r^2)(1-2\alpha)(1-2\beta)$

$$\left(\frac{\partial\eta_2}{\partial\lambda}\right)^2 = \frac{1}{16}r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2 \quad : \quad \frac{1}{\eta_2} = \frac{8}{(1-r^2)-\gamma r(1-r^2)}$$

Exercise 5 Continued

$$\frac{1}{\eta_2}\cdot\left(\frac{\partial \eta_2}{\partial \lambda}\right)^2 = \frac{8}{(1-r^2)-\gamma r(1-r^2)}\cdot\frac{1}{16}\,r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2 = \frac{1}{2}\cdot\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)-\gamma r(1-r^2)}$$

$$H(\eta_2) = \frac{1}{2}\,\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)-\gamma r(1-r^2)}$$

$$H_2 = \frac{1}{2}\,\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)+\gamma r(1-r^2)} + \frac{1}{2}\,\frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)-\gamma r(1-r^2)}$$

$$= \frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{2}\left(\frac{1}{(1-r^2)+\gamma r(1-r^2)} + \frac{1}{(1-r^2)-\gamma r(1-r^2)}\right)$$

$$= \frac{r^2(1-r^2)^2(1-2\alpha)^2(1-2\beta)^2}{2}\left(\frac{(1-r^2)-\gamma r\cancel{(1-r^2)} + (1-r^2)+\gamma r\cancel{(1-r^2)}}{(1-r^2)^2-\gamma^2 r^2(1-r^2)^2}\right)$$

$$= \frac{r^2(1-r^2)^3(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)^2-\gamma^2 r^2(1-r^2)^2}$$

$$\boxed{H_2 = \frac{r^2(1-r^2)^3(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)^2-\gamma^2 r^2(1-r^2)^2}}$$

iii.) Calculate H : $H = H_1 + 3H_2$

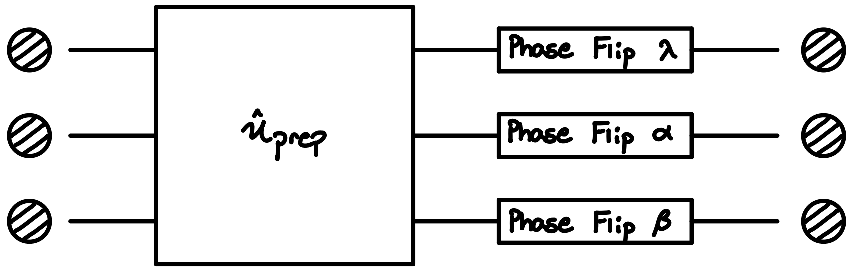
$$H = \frac{r^2(r+3)^2(1-2\alpha)^2(1-2\beta)^2(3r^2+1)}{(3r^2+1)^2 - r^2(r+3)^2\gamma^2} + 3\,\frac{r^2(1-r^2)^3(1-2\alpha)^2(1-2\beta)^2}{(1-r^2)^2-\gamma^2 r^2(1-r^2)^2}$$

$$= r^2(1-2\alpha)^2(1-2\beta)^2\left(\frac{(r+3)^2(3r^2+1)}{(3r^2+1)^2 - r^2(r+3)^2\gamma^2} + \frac{3(1-r^2)^3}{(1-r^2)^2-\gamma^2 r^2(1-r^2)^2}\right)$$

$$= r^2(1-2\alpha)^2(1-2\beta)^2\left(\frac{(r+3)^2(3r^2+1)\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right) + 3(1-r^2)^3\left((3r^2+1)^2 - r^2(r+3)^2\gamma^2\right)}{\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)\left((3r^2+1)^2 - r^2(r+3)^2\gamma^2\right)}\right)$$

$$\boxed{H = r^2(1-2\alpha)^2(1-2\beta)^2\left(\frac{(r+3)^2(3r^2+1)\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right) + 3(1-r^2)^3\left((3r^2+1)^2 - r^2(r+3)^2\gamma^2\right)}{\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)\left((3r^2+1)^2 - r^2(r+3)^2\gamma^2\right)}\right)}$$

Exercise 6

Determine Gain For  over 

$$H_2 = \frac{4r^2}{1-(1-2\lambda)^2 r^2}$$

$$H_1 = r^2(1-2\alpha)^2(1-2\beta)^2\left(\frac{(r+3)^2(3r^2+1)\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)+3(1-r^2)^3\left((3r^2+1)^2-r^2(r+3)^2\gamma^2\right)}{\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)\left((3r^2+1)^2-r^2(r+3)^2\gamma^2\right)}\right)$$

$$\gamma = (1-2\lambda)(1-2\alpha)(1-2\beta)$$

$$G = \frac{H_1}{H_2}$$

$$\boxed{G = \frac{(1-(1-2\lambda)^2 r^2)(1-2\alpha)^2(1-2\beta)^2}{4}\left(\frac{(r+3)^2(3r^2+1)\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)+3(1-r^2)^3\left((3r^2+1)^2-r^2(r+3)^2\gamma^2\right)}{\left((1-r^2)^2-\gamma^2 r^2(1-r^2)^2\right)\left((3r^2+1)^2-r^2(r+3)^2\gamma^2\right)}\right)}$$