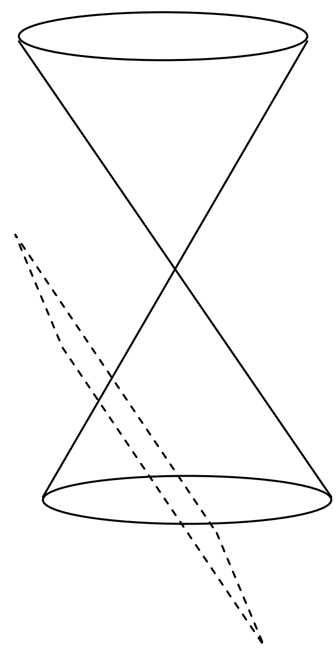
MAT 202
Larson – Section 10.1
Conics & Calculus

A conic section (or just simply "conic") is the intersection of a plane and a double-napped cone. The resulting "cut", depending on where you have sliced the cone will be one of the four conic sections: ***Circle***, ***Parabola***, ***Ellipse*** or ***Hyperbola***.



How would you have to slice the cone with a plane to get a circle? How would you have to slice the cone with a plane to get a parabola?

Note: When the plane passes through the vertex, the resulting figure is a ***degenerate conic***, that is, one that does not resemble a circle, parabola, ellipse, or hyperbola. A degenerate conic will either be a point, a line or two intersecting lines.

Circle

***Circle*:** A circle of radius r with center at the point (h, k) is the collection of all points (x, y) in a plane equidistant from the center (h, k).

<u>Equation of a Circle in Standard Form</u>:

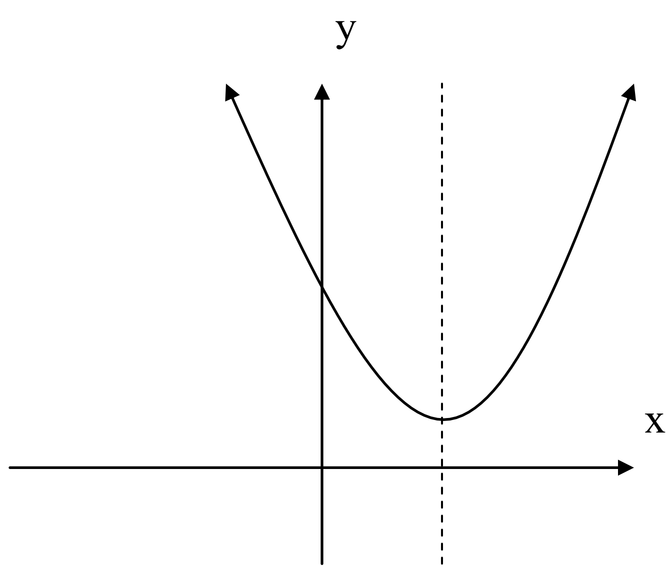
$$(x-h)^2+(y-k)^2=r^2$$

where (h, k) is the center and r is the radius.

What is the standard form for an equation of a circle with center (0, 0)?

Parabola

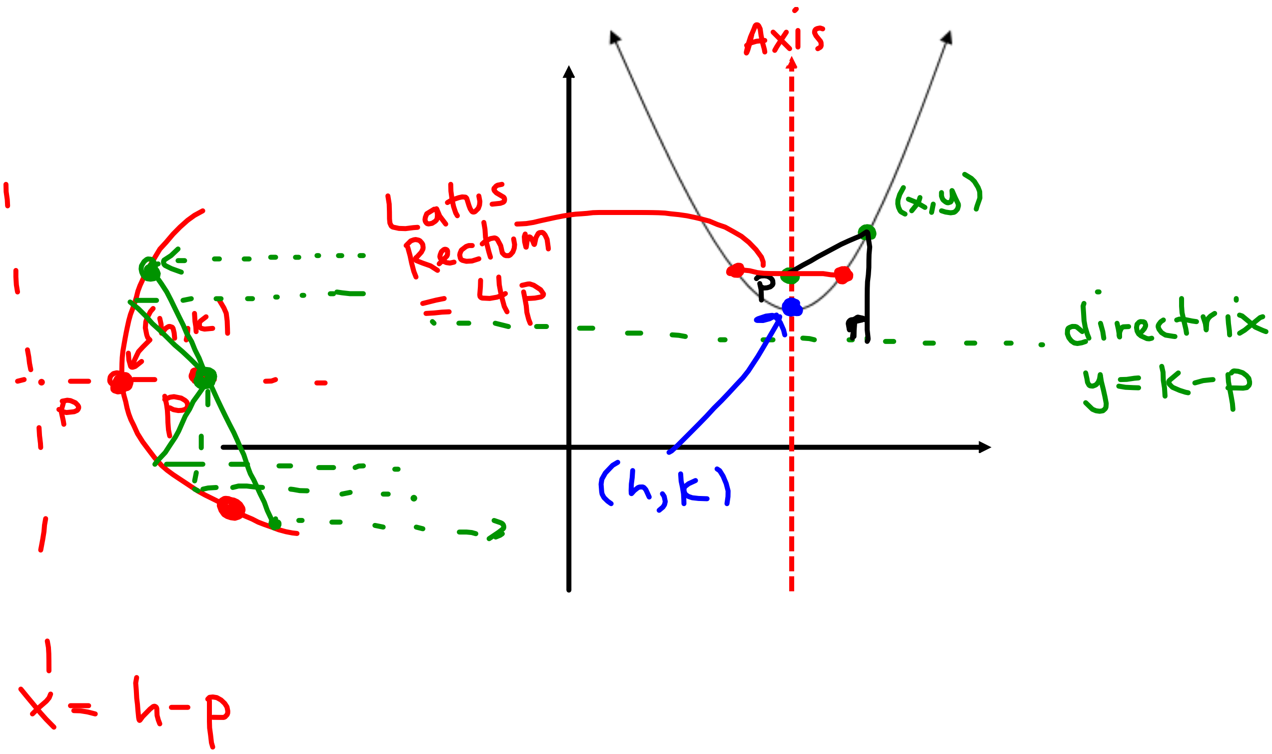
<u>Parabola</u>: A parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line, the ***directrix***, and a fixed point, the ***focus***, not on the line. The midpoint between the focus and the directrix is the ***vertex***, and the line through the focus and vertex is the ***axis*** of the parabola.



What is the standard form for an equation of a circle with center (0, 0)?

$$(x-0)^2+(y-0)^2=r^2$$

$$\boxed{x^2+y^2=r^2}$$


Axis
Latus Rectum = 4p
(x,y)
p
directrix
y=k-p
(h,k)
(h,k)
p
p
x= h-p

<u>Equation of a Parabola in Standard Form</u>:

1. $(x-h)^2 = 4p(y-k)$ Vertical axis and directrix: $y = k - p$
2. $(y-k)^2 = 4p(x-h)$ Horizontal axis and directrix: $x = h - p$

where (h, k) is the vertex and p is the directed distance from the vertex to the focus. That is, the focus lies on the axis, p units from the vertex. A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a ***focal chord***. What is the significance of "4p"?

What are the possible equations of a parabola in standard form if the vertex is (0, 0)?

Ex: Find the vertex, focus, and directrix of the parabola and sketch its graph:
$(x+5)+(y-1)^2 = 0$

Ex: Find the standard form of the equation of the parabola with vertex: (-1, 2), directrix: y = 4

What is the significance of "4p"?

Length of the $\underbrace{\text{Latus Rectum}}_{\text{"Straight Side"}}$ is 4p.

What are the possible equations of a parabola in standard form if the vertex is (0, 0)?

1) $x^2 = 4py$

2) $y^2 = 4px$

Ex: Find the vertex, focus, and directrix of the parabola

and sketch its graph: $(x+5)+(y-1)^2=0$

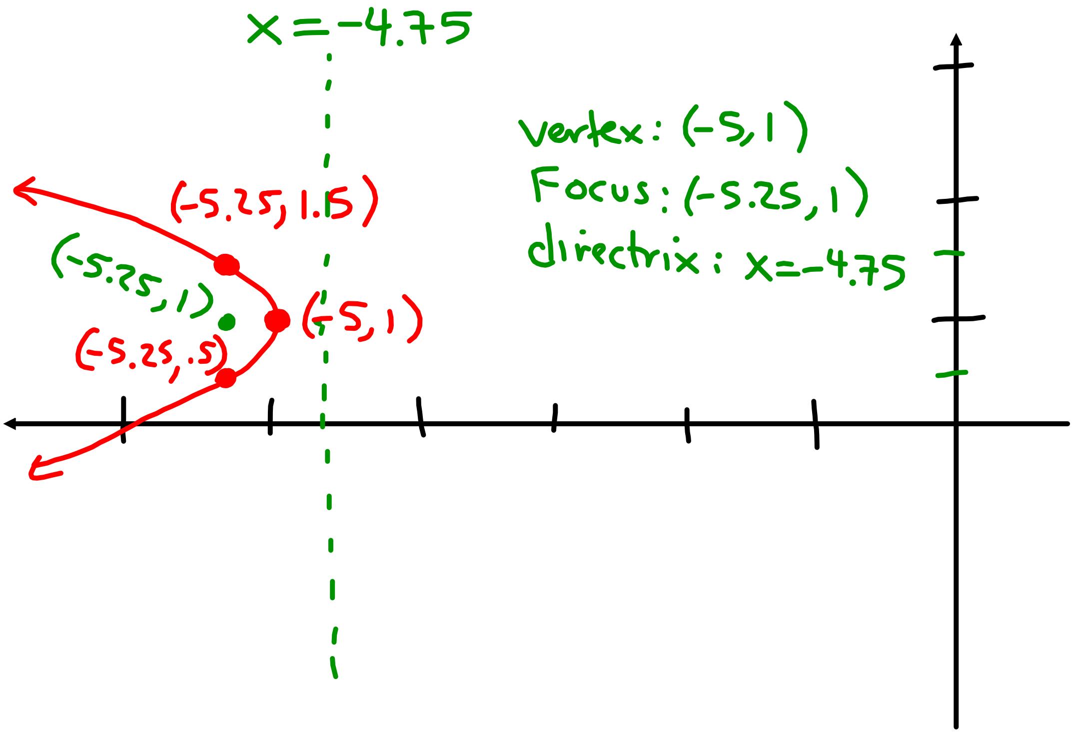
$-(x+5)$ $\qquad$ $-(x+5)$

$(y-1)^2=-(x+5)$

$(y-1)^2=4\left(-\frac{1}{4}\right)(x+5)$

$p=-1/4$

Parabola opening to the left.

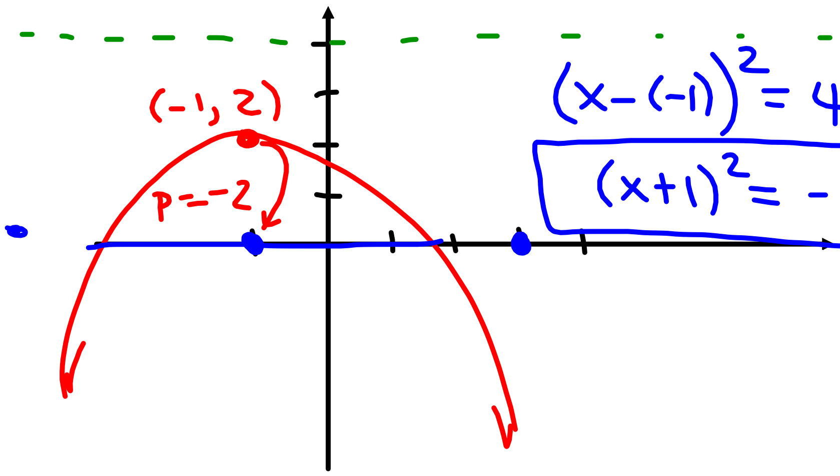


Vertex: (-5, 1)
Focus: (-5.25, 1)
directrix: x=-4.75

Ex: Find the standard form of the equation of the parabola with:

vertex: (-1, 2), directrix: y = 4

Axis is vertical: $(x-h)^2 = 4p(y-k)$

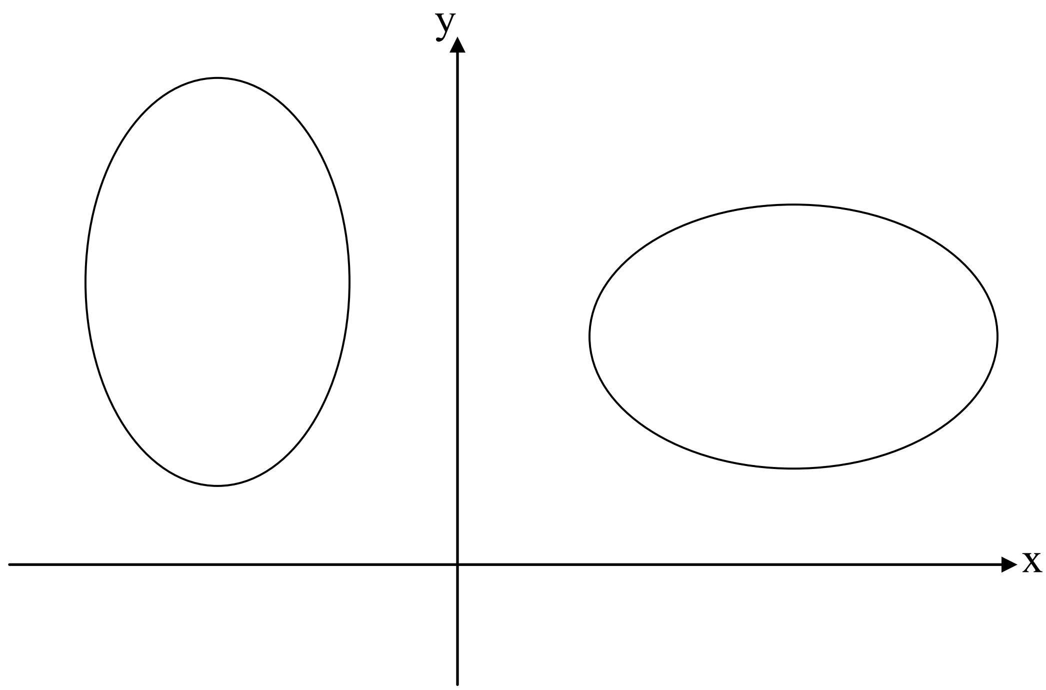


$(x-(-1))^2 = 4(-2)(y-2)$

$\boxed{(x+1)^2 = -8(y-2)}$

Ellipse

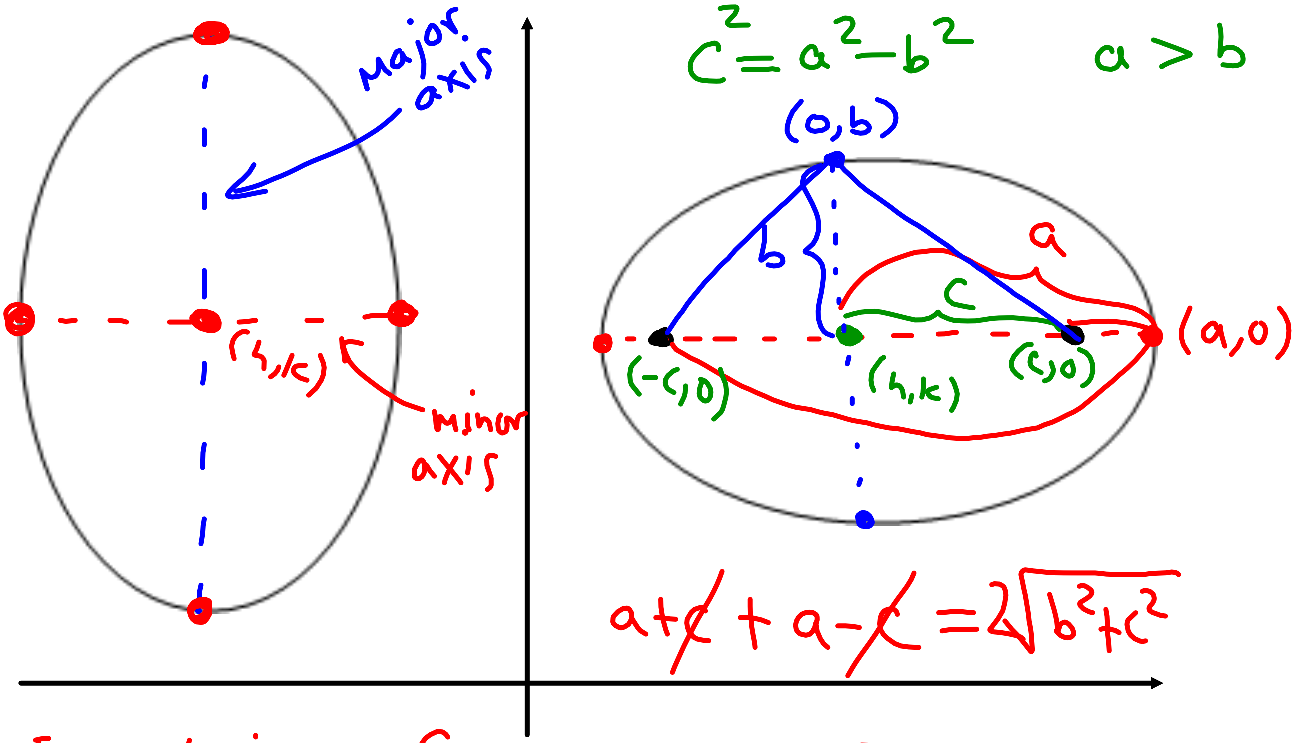
Ellipse*:** An ellipse is the set of all points (x, y) the sum of whose distances from two distinct points (foci***) is constant. The line through the foci intersects the ellipse at two points called the ***vertices***. Its midpoint is the ***center*** of the ellipse. The chord joining the vertices is the ***major axis*** and the chord perpendicular to the major axis at the center is the ***minor axis*** of the ellipse.



Equation of an Ellipse in Standard Form: with center (h, k) and major and minor axes of lengths $2a$ and $2b$ respectively, where $0 < b < a$, is:

1. $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ Major axis is horizontal
2. $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ Major axis is vertical

The foci lie on the major axis c units from the center, with $c^2 = a^2 - b^2$. How do we derive this?



$$c^2 = a^2 - b^2 \qquad a > b$$

$$a + \cancel{c} + a - \cancel{c} = 2\sqrt{b^2 + c^2}$$

Eccentricity: $\frac{c}{a}$

$$2a = 2\sqrt{b^2 + c^2}$$

$$a = \sqrt{b^2 + c^2}$$

$$a^2 = b^2 + c^2$$

$$a^2 - b^2 = c^2$$

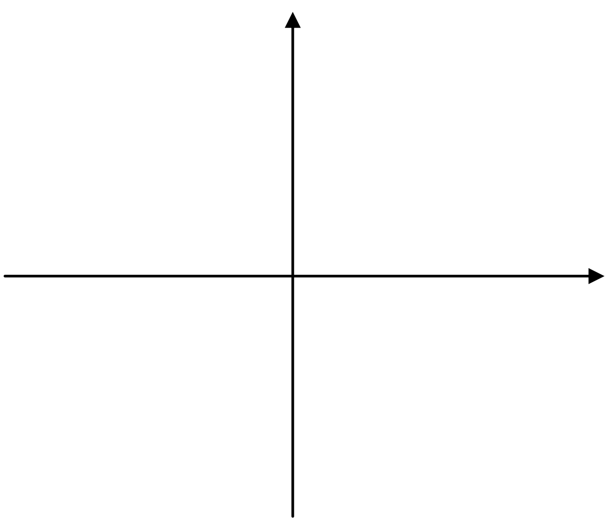
The foci lie on the major axis c units from the center, with $c^2 = a^2 - b^2$. How do we derive this?

Derived on the previous slide.

The ***eccentricity*** (i.e. “ovalness” or “off-centerness”) e of an ellipse is given by the formula: $e = \frac{c}{a}$. Note: the closer that e gets to 0, the more the ellipse looks like a circle and the closer that e gets to 1, the more “narrow” the ellipse gets.

If the center of the ellipse is located at the origin, what will the equations look like?

Ex: Find the center, vertices, foci, and eccentricity of the given ellipse and sketch its graph: $3x^2 + y^2 + 18x - 2y - 8 = 0$



Ex: Find the standard form of the equation of the ellipse with: Foci: (0, 0), (4, 0); Major axis length 8

If the center of the hyperbola is located at the origin, what will the equations look like?

1) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2) $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Ex: Find the center, vertices, foci, and eccentricity of the given ellipse and sketch its graph:

$3x^2 + y^2 + 18x - 2y - 8 = 0$

$3x^2 + 18x \quad + y^2 - 2y \quad = 8$

$3(x^2 + 6x + \underline{9}) + (y^2 - 2y + \underline{1}) = 8 + \underline{27} + \underline{1}$

$\dfrac{3(x+3)^2}{36} + \dfrac{(y-1)^2}{36} = \dfrac{36}{36}$

$\dfrac{(x+3)^2}{12} + \dfrac{(y-1)^2}{36} = 1$

Vertical Major axis

$a = 6 \quad b = 2\sqrt{3}$

$c^2 = 36 - 12$

$c^2 = 24$

$c = 2\sqrt{6}$

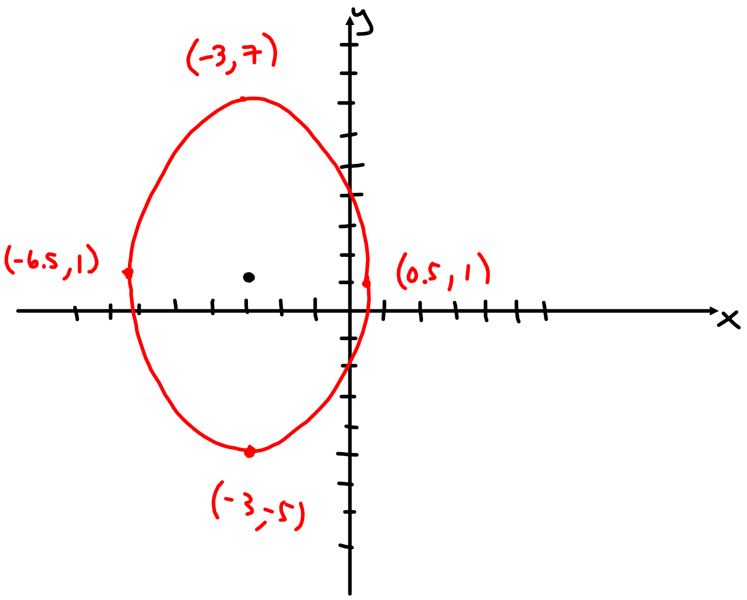
Center: $\boxed{(-3, 1)}$

Vertices: $(-3, 1 \pm 6) \rightarrow \boxed{(-3, 7)\ (-3, -5)}$

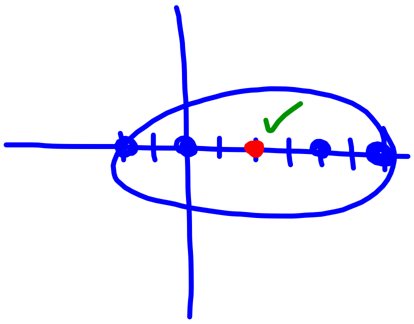
Foci: $\boxed{(-3, 1 \pm 2\sqrt{6})}$

Along minor axis: $(-3 \pm 2\sqrt{3}, 1)$

Eccentricity: $\dfrac{c}{a} = \dfrac{2\sqrt{6}}{6} = \dfrac{\sqrt{6}}{3} \approx \boxed{0.81}$



Ex: Find the standard form of the equation of the ellipse with:
Foci: (0, 0), (4, 0); Major axis length 8 ← $2a=8 = a=4$



Major axis horizontal

center @ (2,0)

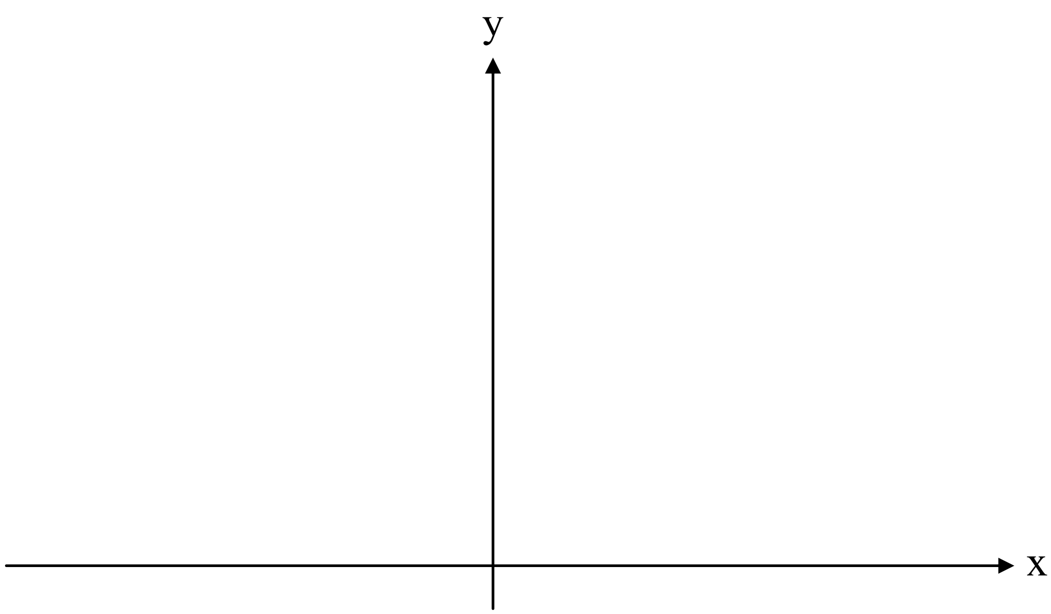
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$c^2 = a^2 - b^2$$
$$b^2 = a^2 - c^2$$
$$b^2 = 16 - 4$$
$$b^2 = 12$$

$$\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$$

Hyperbola

Hyperbola*:** A hyperbola is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points (foci***) is constant. In a hyperbola the two disconnected "curves" are ***branches***. The line through the two foci intersects the hyperbola at its two ***vertices***. The line segment connecting the vertices is called the ***transverse axis***, and the midpoint of the transverse axis is called the ***center*** of the hyperbola. The line passing through the center of the hyperbola perpendicular to the transverse axis is the ***conjugate axis***.



Equation of a Hyperbola in Standard Form: with center (h, k)

1. $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ Transverse axis is horizontal

   Asymptotes: $y = k \pm \dfrac{b}{a}(x-h)$

2. $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ Transverse axis is vertical

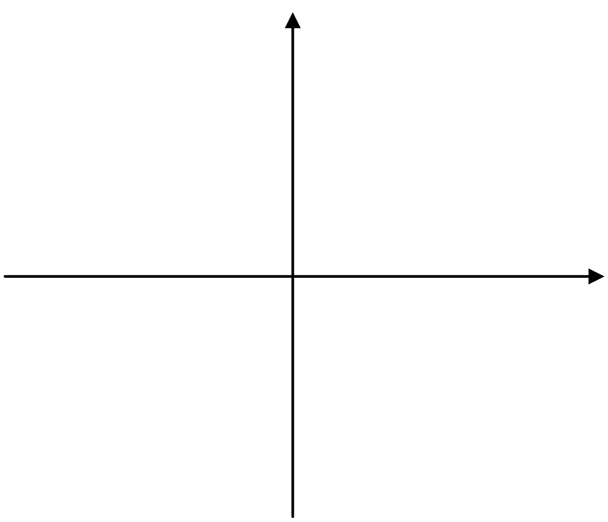
   Asymptotes: $y = k \pm \dfrac{a}{b}(x-h)$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$.

If the center of the hyperbola is located at the origin, what will the equations look like?

Note: As with ellipses, the eccentricity of a hyperbola is $e = \frac{c}{a}$. Because $c > a$, then $e > 1$. When the ***eccentricity*** is "large", the branches are nearly flat, and when e is close to 1, the branches are more pointed.

Ex: Find the center, vertices, foci and the asymptotes of the hyperbola and sketch its graph: $x^2 - 9y^2 + 36y - 72 = 0$



Ex: Find the standard form of the equation of the hyperbola with: Vertices: (2, 3), (2, -3); Passing through the point (5, 4)

Ex: Find the center, vertices, foci and the asymptotes of the hyperbola and sketch its graph:

$$x^2 - 9y^2 + 36y - 72 = 0$$

$$x^2 - 9(y^2 - 4y + \underline{4}) = 72 - \underline{36}$$

$$\frac{x^2}{36} - \frac{9(y-2)^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{36} - \frac{(y-2)^2}{4} = 1$$

Transverse Axis Horizontal

$a = 6$, $b = 2$

$c = \sqrt{40} = 2\sqrt{10}$

$y - k = \pm\frac{b}{a}(x - h)$

Center: $(0,2)$

Vertices: $(-6,2), (6,2)$

Foci: $(-2\sqrt{10}, 2), (2\sqrt{10}, 2)$

Asymptotes: $y - 2 = \pm\frac{1}{3}x$

Ex: Find the standard form of the equation of the hyperbola with: Vertices: (2, 3), (2, -3); Passing through the point (5, 4)

center @ (2, 0)

Transverse Axis is vertical :

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$a = 3$

$$\frac{(4-0)^2}{9} - \frac{(5-2)^2}{b^2} = 1$$

$$\boxed{\frac{y^2}{9} - \frac{(x-2)^2}{81/7} = 1}$$

$$9b^2\left(\frac{16}{9} - \frac{9}{b^2} = 1\right)$$

$$16b^2 - 81 = 9b^2$$

$$7b^2 = 81$$

$$b^2 = 81/7$$

Classifying Conics:

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a:

1. Circle if A = C
2. Parabola if AC = 0
3. Ellipse if AC > 0
4. Hyperbola if AC < 0

In this section, you will be asked to find the arc length. Recall the formula for arc length in chapter 7:

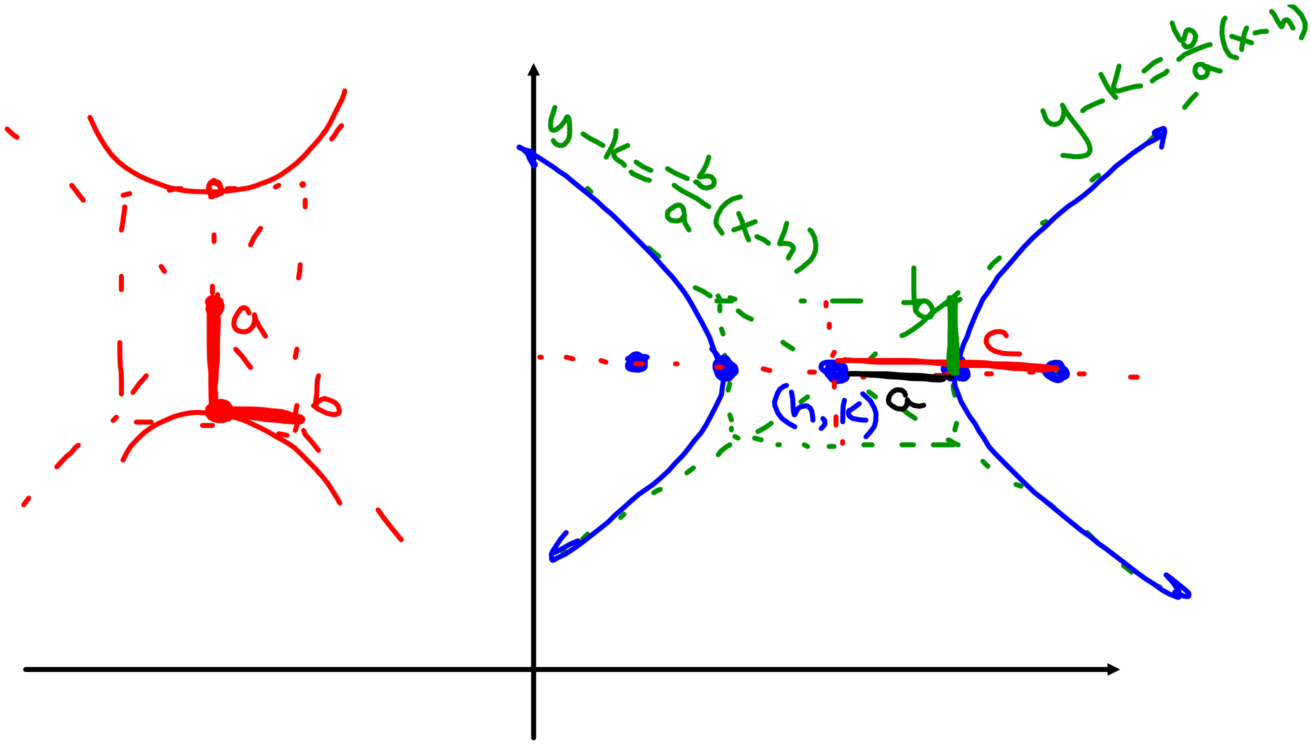
Definition of Arc Length:

Let the function given by *y = f(x)* represent a smooth curve on the interval [*a*, *b*]. The arc length of *f* between *a* and *b* is:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve given by *x = g(y)*, the arc length of *g* between *c* and *d* is:

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$


a
b
y-k = -b/a (x-h)
y-k = b/a (x-h)
b
c
(h, k)
a