6.4 Stability and Linear Classification

1-6, 11-14, 9

6.4.1

$\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \bar{x}$ (Saddle point)

Saddle point
unstable

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} \right| = 0$

$(1-\lambda)(-2-\lambda) - 1(4)$

$-2 + \lambda + \lambda^2 - 4$

$\lambda^2 + \lambda - 6$

$(\lambda+2)(\lambda-3)$

$\lambda = -2, 3$

$(A - \lambda I)\bar{v}_1$

$\lambda = 2$

$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x = y$ $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3$

$(A - \lambda I)\bar{v}_1$

$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x = 0$, $y = 0$ $\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 < 0 < \lambda_2 \therefore$ unstable

6.4.3

$\bar{x}' = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \bar{x}$ (star node)

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{bmatrix} \right| = 0$

$(-2-\lambda)(-2-\lambda)$

$4 + 4\lambda + \lambda^2$

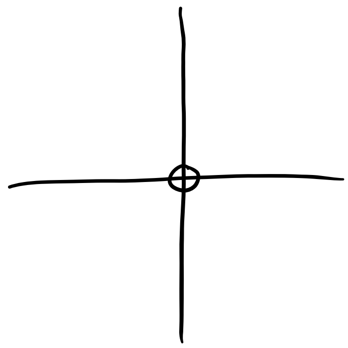
$-\lambda - 2 = 0$

$\lambda = -2, -2$

$(A - \lambda I)\bar{v}_1$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ rref $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\lambda_1, \lambda_2 < 0$



Stable star node

6.4.5

$\bar{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \bar{x}$ (node)

$|A - \lambda I| = 0$

$\begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix} = 0$

$(2-\lambda)(4-\lambda) - 1(3)$

$8 - 6\lambda + \lambda^2 - 3$

$\lambda^2 - 6\lambda + 5$

$(\lambda-5)(\lambda-1)$

$\lambda = 1, 5$

$\lambda_1, \lambda_2 > 0$

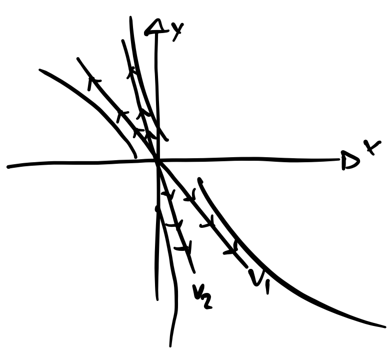
$\lambda = 1$

$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$ rref $= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x = -y$ $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 5$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ rref $= \begin{bmatrix} 1 & -\frac{1}{3}y \\ 0 & 0 \end{bmatrix}$ $x = \frac{1}{3}y$ $v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

unstable node



6.4.2)

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$ (center)

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \right| = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$\lambda = -i$

$(A - \lambda I)\vec{v}$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ -1 & i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad x = iy \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda = i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad x = -iy \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Imaginary ∴ Neutrally Stable

6.4.4)

$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}$ (degenerate node) Eigen values repeated

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -2-\lambda & 1 \\ 0 & -2-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)(-2-\lambda) - 1(0)$$

$$4 + 4\lambda + \lambda^2$$

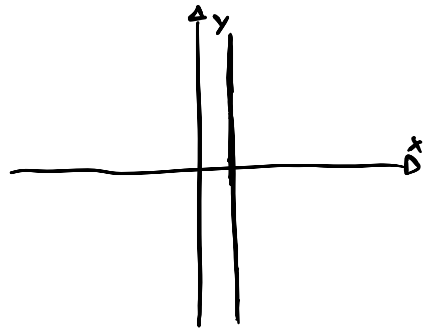
$$(\lambda+2)(\lambda+2) = 0$$

$$\lambda = -2$$

$(A - \lambda I)\vec{v}$

$\lambda = -2$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{v} \to \begin{matrix} y = 0 \\ x = \text{free} \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



degenerate node
neutrally stable

6.4.6)

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \vec{x}$

$$\Delta = b^2 - 4ac = 1 - 4(1)(1) = -3$$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} = 0$$

$$\alpha = \frac{-b}{2a} = \frac{-1}{2}$$

$$\lambda + \lambda^2 - 1(-1)$$

$$\lambda^2 + \lambda + 1$$

$$\beta = \frac{\sqrt{4-1}}{2(1)} \quad \beta = \frac{\sqrt{3}}{2}$$

$$\lambda = -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}$$

$$\left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)$$

$$\tfrac{1}{4} - \tfrac{3}{4}i^2 = \tfrac{1}{4} + \tfrac{3}{4} = 1$$

$\lambda = -\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}$

$$\begin{bmatrix} \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} & 1 \\ -1 & -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} \end{bmatrix} \to \begin{bmatrix} 1 & \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \\ -1 & -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \end{bmatrix} \to \begin{bmatrix} 1 & \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \\ 0 & 0 \end{bmatrix}$$

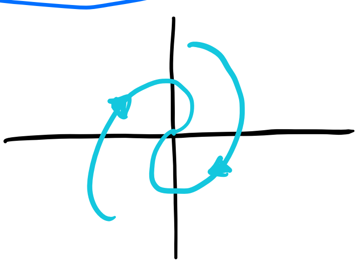
$$x = \left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)y \quad v_1 = \begin{bmatrix} \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \\ 1 \end{bmatrix}$$

$\lambda = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$

$$\begin{bmatrix} \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i & 1 \\ -1 & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \end{bmatrix} \to \begin{bmatrix} 1 & \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \\ -1 & -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \end{bmatrix} \to \begin{bmatrix} 1 & \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i \\ 0 & 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i \\ 1 \end{bmatrix} \quad x = \left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)y$$

Since both eigenvalues are imaginary, $\alpha < 0$, the solution spirals to the origin. Aysmptotically stable.



6.4.P.11) $\bar{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}$

$\left| \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \right| = 0 :$ $(-\lambda)(-\lambda) - 1(0)$

$\lambda^2 = 0$

$\lambda = 0, 0$

$\lambda = 0$

$(A - \lambda I) v_1$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_2 = 0$, $x_1 = \text{free}$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left[t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right]$

$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left[\begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$

$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{rref} = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\quad y = 1$

$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

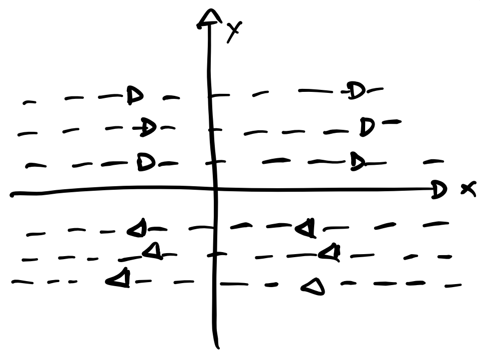
$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\bar{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$ $\quad x_1' = +x_2$, $\quad x_2' = 0$

$0 = x_2 \; ; \; y = 0$

$0 = 0$



6.4.13) $\bar{x}' = \begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix} \bar{x}$

$|A - \lambda I| = 0$

$\begin{bmatrix} 3-\lambda & -9 \\ 1 & -3-\lambda \end{bmatrix} = 0$

$(3-\lambda)(-3-\lambda) + 9(1)$

$-9 + 0 + \lambda^2 + 9 = 0$

$\lambda^2 = 0$

$\lambda = 0$

$(A - \lambda I) v_1$

$\lambda = 0$

$\begin{bmatrix} 3 & -9 \\ 1 & -3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\quad x = 3y \quad \bar{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \left[t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right]$

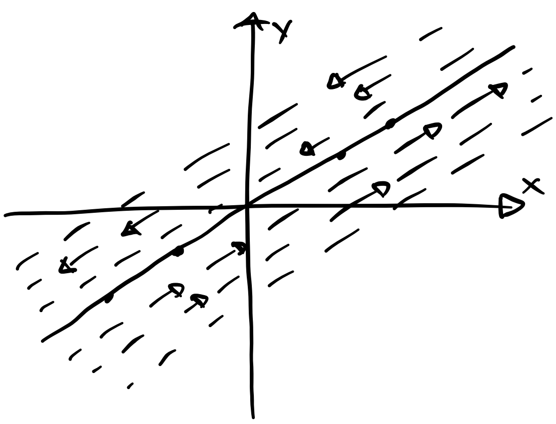
$\left[\begin{array}{cc|c} 3 & -9 & 3 \\ 1 & -3 & 1 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x - 3y = 1$, $\quad x = 1 + 3y$

$x(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \left[t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right]$

$\bar{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$3x + y = 0$

$y = -3x$



6.4.12)

$$\bar{x}' = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \bar{x}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ -4 & -2-\lambda \end{bmatrix}$$

$(2-\lambda)(-2-\lambda)+4$

$-4+\lambda^2+4$

$\lambda^2=0$

$\lambda=0$

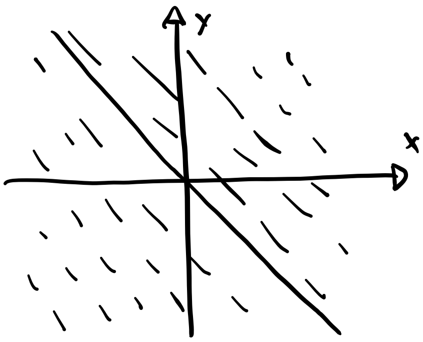
$v_1: (A-\lambda I)\bar{v}_1=0$

$\lambda=0$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x=-\tfrac{1}{2}y \quad v_1=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$v_2: (A-\lambda I)\bar{v}_2=\bar{v}_1$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \quad x=\tfrac{1}{2}-\tfrac{1}{2}y \quad \bar{v}_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\bar{x}(t) = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\left[t\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$= c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\left[\begin{bmatrix} t \\ 1-2t \end{bmatrix}\right]$$

$$\boxed{\bar{x}(t) = c_1\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2\begin{bmatrix} t \\ -2t+1 \end{bmatrix}}$$

$2x+y=0 : y=-2x$

$-4x-2y=0 : y=-2x$

6.4.14)

$$\bar{x}' = \begin{bmatrix} -4 & 2 \\ -8 & 4 \end{bmatrix} x$$

$|A-\lambda I|=0$

$\lambda=0$

$(A-\lambda I)\bar{v}_1=0$

$x=\frac{1}{2}y$

$$\begin{bmatrix} -4 & 2 \\ -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \quad \bar{v}_1=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$(A-\lambda I)\bar{v}_2=\bar{v}_1$

$$\left[\begin{array}{cc|c} -4 & 2 & 1 \\ -8 & 4 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 \end{array}\right] \quad \bar{v}_2=\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$x=\frac{1}{2}y-\frac{1}{4}$

$$\bar{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}\right)$$

$-4x+2y=0 \qquad 2y=4x$

$-8x+4y=0 \qquad y=2x$

$4y=8x$

$y=2x$

$$\boxed{\bar{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} t \\ 2t+\frac{1}{2} \end{bmatrix}}$$