MAT 202
Larson – Section 5.5
Bases Other than *e* and Applications

Up to this point we've only considered differentiation & integration of base e logarithmic and exponential functions. We will now investigate the calculus of logarithmic & exponential functions with a base not equal to e.

Definition of Exponential Function to Base *a*:
If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}$$

If a = 1, then y = 1^x = 1 is a constant function

Recall from College Algebra the change of base formulas:

Change of Base Formula:
Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows:

1. Converting from base a to base b: $\log_a x = \dfrac{\log_b x}{\log_b a}$
2. Converting from base a to base 10: $\log_a x = \dfrac{\log x}{\log a}$
3. Converting from base a to base e: $\log_a x = \dfrac{\ln x}{\ln a}$

Ex: Evaluate the following with your calculator and round your answer to the nearest hundredth:

$$\log_7 \frac{2}{9}$$

Ex: Evaluate the following with your calculator and round your answer to the nearest hundredth:

$$\log_7 \frac{2}{9} = \frac{\ln 2/9}{\ln 7} \approx \boxed{-0.77}$$

Derivatives for Bases other than *e*:

Let a be a positive real number (a $\neq$ 1), and let *u* be a differentiable function of *x*.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}\left[\log_a x\right] = \left(\frac{1}{\ln a}\right)\frac{1}{x}$
4. $\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$

Ex: Find the derivative of the following:

a) $y = x\left(6^{-2x}\right)$

b) $h(t) = \log_5 (4 - t)^2$

Derivatives for Bases other than *e*:

Let a be a positive real number (a ≠ 1), and let u be a differentiable function of x.

1. $\frac{d}{dx}\left[a^{x}\right]=(\ln a)a^{x}$
2. $\frac{d}{dx}\left[a^{u}\right]=(\ln a)a^{u}\frac{du}{dx}$
3. $\frac{d}{dx}[\log_{a}x]=\left(\frac{1}{\ln a}\right)\frac{1}{x}$
4. $\frac{d}{dx}[\log_{a}u]=\left(\frac{1}{\ln a}\right)\left(\frac{1}{u}\right)\frac{du}{dx}$

$$\frac{d}{dx}\left[a^{x}\right]=\frac{d}{dx}\left[e^{(\ln a)x}\right]$$

$$=e^{(\ln a)x}\cdot\ln a$$

$$=a^{x}\cdot\ln a$$

$$=(\ln a)a^{x}$$

$$\frac{d}{dx}\left[\log_{a}x\right]=\frac{d}{dx}\left[\frac{\ln x}{\ln a}\right]$$

$$=\frac{1}{\ln a}\cdot\frac{1}{x}$$

$$\ln\left(\frac{x}{a}\right)=\ln x-\ln a$$

Ex: Find the derivative of the following:

a) $y = x\left(6^{-2x}\right)$

6^u $\quad \ln 6 \cdot 6^u \cdot u'$

$y' = 1 \cdot 6^{-2x} + x \cdot \ln 6 \cdot 6^{-2x} \cdot -2$

$y' = 6^{-2x} - 2x \ln 6 \cdot 6^{-2x}$

$\boxed{y' = 6^{-2x}\left(1 - 2x \ln 6\right)}$

b) $h(t) = \log_5 (4-t)^2 = 2\log_5 (4-t)$

$h'(t) = 2 \cdot \frac{1}{\ln 5} \cdot \frac{1}{4-t} \cdot -1$

$h'(t) = \frac{-2}{[\ln(5)](4-t)} = \frac{-2}{(4-t)\ln 5}$

$\overset{or}{=} \frac{2}{(t-4)\ln 5}$

c) $f(t) = t^{3/2} \log_2 \sqrt{t+1}$

Ex: Find the equation of the tangent line to the graph of $y = \log_3 x$, at the point (27, 3).

Ex: Use logarithmic differentiation to find $\dfrac{dy}{dx}$: $y = (x-2)^{x+1}$

Ex: Find the equation of the tangent line to the graph of the function $y = (\ln x)^{\cos x}$, at the point $(e, 1)$.

c) $f\left(t\right) = t^{3/2}\log_2\sqrt{t+1}$

$$f(t) = \frac{1}{2}t^{3/2}\log_2(t+1)$$

$$f'(t) = \frac{3}{4}t^{1/2}\cdot\log_2(t+1) + \frac{1}{\ln 2}\cdot\frac{1}{t+1}\cdot\frac{1}{2}t^{3/2}$$

$$\boxed{f'(t) = \frac{3}{4}\sqrt{t}\,\log_2(t+1) + \frac{1}{2(t+1)\ln 2}t^{3/2}}$$

Ex: Use logarithmic differentiation to find $\dfrac{dy}{dx}$:

$$\ln y \overset{\ln}{=} \left((x-2)^{x+1}\right) \qquad \log_{x-2} y = x+1$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[(x+1)\ \ln(x-2)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln(x-2) + \frac{1}{x-2} \cdot x+1$$

$$y\left(\frac{1}{y}\frac{dy}{dx} = \ln(x-2) + \frac{x+1}{x-2}\right)$$

$$\frac{dy}{dx} = y\left(\ln(x-2) + \frac{x+1}{x-2}\right)$$

$$\boxed{\frac{dy}{dx} = (x-2)^{x+1}\left(\ln(x-2) + \frac{x+1}{x-2}\right)}$$

Ex: Find the equation of the tangent line to the graph of the function $y = (\ln x)^{\cos x}$, at the point $(e, 1)$.

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\cos x \cdot \ln(\ln x)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\sin x \cdot \ln(\ln x) + \cos x \cdot \frac{1}{\ln x}\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = (\ln x)^{\cos x}\left(-\sin x \cdot \ln(\ln x) + \frac{\cos x}{x \ln x}\right)$$

$$m = (\ln e)^{\cos e}\left(-\sin e \cdot \ln(\ln e) + \frac{\cos e}{e \ln e}\right)$$

$$m = 1\left(0 + \frac{\cos e}{e}\right) = \frac{\cos e}{e}$$

$$1 = \frac{\cos e}{\cancel{e}}\cdot \cancel{e} + b$$

$$1 - \cos e = b$$

$$\boxed{y = \frac{\cos e}{e}x + 1 - \cos e}$$

Tools for Integrating Exponential Functions with a Base Other Than *e*:

1. Convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or
2. Integrate directly, using the integration formula:

$$\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$$

Ex: Find the indefinite integral: $\int (x + 4) 6^{(x + 4)^2} \, dx$

Ex: Evaluate the definite integral: $\int_0^1 \left(5^x - 3^x\right) dx$

Note: As a consequence of logarithmic differentiation we may prove the Power Rule from Calculus I to be extended to cover any real number *n* (not just *n* must be a rational number).

$$1.\ \frac{d}{dx}\left[x^n\right] = n \cdot x^{n-1} \qquad 2.\ \frac{d}{dx}\left[u^n\right] = n \cdot u^{n-1}$$

$n \in$ *any number, and u is a differentiable function of x.*

Ex: Find the indefinite integral: $\int (x+4)6^{(x+4)^2}\,dx$

$$\frac{1}{2}\int 6^u\,du$$

$u=(x+4)^2$

$du = 2(x+4)\,dx$

$\frac{1}{2}du = (x+4)\,dx$

$$= \frac{1}{2}\cdot\frac{1}{\ln 6}\cdot 6^u + C$$

$$= \boxed{\frac{1}{2\ln 6}\cdot 6^{(x+4)^2} + C}$$

Ex: Evaluate the definite integral: $\int_0^1 \left(5^x - 3^x\right) dx$

$$\int_0^1 5^x\,dx - \int_0^1 3^x\,dx$$

$$\frac{1}{\ln 5}\cdot 5^x \Big]_0^1 - \frac{1}{\ln 3}\cdot 3^x \Big]_0^1$$

$$\left(\frac{5}{\ln 5} - \frac{1}{\ln 5}\right) - \left(\frac{3}{\ln 3} - \frac{1}{\ln 3}\right)$$

$$\boxed{\frac{4}{\ln 5} - \frac{2}{\ln 3} \approx 0.66}$$