MAT 202
Larson – Section 5.2
The Natural Logarithmic Function: Integration

Recall the differentiation rules for the natural logarithmic function:
u is a differentiable function of x such that $u \neq 0$.

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x} \quad \textbf{and} \quad \frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$$

From these two rules the antiderivative is as follows.

<u>Log Rule for Integration</u>: Let u be a differentiable function of x such that $u \neq 0$.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Alternatively, since $du = u'dx$, formula #2 above can be written as:

3. $\displaystyle\int \frac{u'}{u}\,dx = \ln|u| + C$

In this chapter, we will begin by using the log rule with various integration techniques, such as, ***change of variables***, ***pattern recognition*** (i.e., recognizing quotient forms), ***using long division***, and ***u-substitution***.

Ex: Use ***change of variables*** and the log rule to integrate: $\int \frac{x^2}{5 - x^3}\, dx$

Ex: Use ***long division*** and the log rule to integrate: $\int \frac{x^3 - 6x - 20}{x + 5}\, dx$

Ex: Use ***change of variables*** and the log rule to integrate: $\int \frac{3x}{(x + 4)^2}\, dx$

Ex: Use ***change of variables*** and the log rule to integrate:

$$\int \frac{x^2}{5-x^3}\,dx \quad -\tfrac{1}{3}du$$

u

$u = 5 - x^3$

$du = -3x^2\,dx$

$-\frac{1}{3}du = x^2\,dx$

$$-\frac{1}{3}\int \frac{1}{u}\,du$$

$$= -\frac{1}{3}\ln|u| + C$$

$$= \boxed{-\frac{1}{3}\ln|5-x^3| + C}$$

Ex: Use ***long division*** and the log rule to integrate: $-\int \frac{115}{x+5}\,dx$

$$\int \frac{x^3-6x-20}{x+5}\,dx = \int x^2-5x+19-\frac{115}{x+5}\,dx$$

$$\begin{array}{r|l}
 & x^2-5x+19 \\
\hline
x+5 & x^3+0x^2-6x-20 \\
 & \cancel{-x^3} \mp 5x^2 \\
\hline
 & -\cancel{5x^2}-6x \\
 & +\cancel{5x^2} \pm 25x \\
\hline
 & \cancel{19x}-20 \\
 & -\cancel{19x} \mp 95 \\
\hline
 & -115
\end{array}$$

$$= \boxed{\frac{x^3}{3}-\frac{5x^2}{2}+19x-115\ln|x+5|+C}$$

Ex: Use ***change of variables*** and the log rule to integrate:

$$\int \frac{3x}{(x+4)^2}\,dx = \int \frac{3(u-4)}{u^2}\,du$$

$u = x+4$

$du = dx$

$x = u-4$

$$= \int \frac{3}{u} - \frac{12}{u^2}\,du$$

$$= 3\ln|u| - \frac{12u^{-1}}{-1} + C$$

$$= \boxed{3\ln|x+4| + \frac{12}{x+4} + C}$$

Ex: Use ***u-substitution*** and the log rule to integrate: $\displaystyle\int \frac{1}{1+\sqrt{3x}}\,dx$

Ex: What do you notice about the quotient in the following integral? How would you proceed to integrate?

$$\int \frac{x^2+4x}{x^3+6x^2+5}\,dx$$

Ex: Solve the differential equation. The solution must pass through the given point: $\dfrac{dy}{dx} = \dfrac{2x}{x^2-9x}$, $(0, 4)$

Ex: Use ***u-substitution*** and the log rule to integrate:

$$\int \frac{1}{1+\sqrt{3x}}\,dx \longrightarrow \int \frac{1}{u} \cdot \frac{2}{3}\underbrace{(3x)^{1/2}}_{u-1}\,du$$

$u = 1 + \sqrt{3x} \rightarrow \sqrt{3x} = u - 1$

$du = \frac{1}{2}(3x)^{-1/2} \cdot 3\,dx$

$du = \frac{3}{2}(3x)^{-1/2}\,dx$

$du = \frac{3}{2(3x)^{1/2}}\,dx$

$dx = \frac{2}{3}(3x)^{1/2}\,du$

$$\frac{2}{3}\int \frac{u-1}{u}\,du$$

$$\frac{2}{3}\int 1 - \frac{1}{u}\,du$$

$$= \frac{2}{3}\left(u - \ln|u|\right) + C$$

$$= \boxed{\frac{2}{3}(1+\sqrt{3x}) - \frac{2}{3}\ln\left|1+\sqrt{3x}\right| + C}$$

Ex: What do you notice about the quotient in the following integral? How would you proceed to integrate?

$$\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5}\,dx = \frac{1}{3}\int \frac{1}{u}\,du$$

$$u = x^3 + 6x^2 + 5$$

$$du = (3x^2 + 12x)\,dx$$

$$du = 3(x^2 + 4x)\,dx$$

$$\frac{1}{3}du = (x^2 + 4x)\,dx$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \boxed{\frac{1}{3}\ln|x^3 + 6x^2 + 5| + C}$$

Ex: Solve the differential equation. The solution must pass through the given point: $\frac{dy}{dx} = \frac{2x}{x^2 - 9x}, (0, 4)$

$y = \int \frac{2x}{x^2 - 9x} dx$

$y = \int \frac{2\cancel{x}}{\cancel{x}(x-9)} dx = \int \frac{2}{x-9} dx$

$y = 2 \ln|x-9| + C$

$4 = 2 \ln|0-9| + C$

$4 = 2 \ln 9 + C$

$C = 4 - 2 \ln 9$

$\boxed{y = 2 \ln|x-9| + 4 - 2 \ln 9}$

Ex: Find the area of the region bounded by the graphs of the equations

$$y = \frac{5x}{x^2 + 2}, \quad x = 1, \quad x = 5, \quad y = 0.$$

Up to this point, we know how to integrate $f(x) = \sin x$ and $f(x) = \cos x$. We will now develop integration rules for the remaining four trigonometric functions.

<u>Integrals of the Six Basic Trigonometric Functions</u>:

$$\int \sin u \, du = -\cos u + C \qquad \int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C \qquad \int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

Ex: Prove that $\int \cot u \, du = \ln|\sin u| + C$

Ex: Prove that $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

Ex: Find the area of the region bounded by the graphs of the equations

$$y = \frac{5x}{x^2 + 2}, \quad x = 1, \quad x = 5, \quad y = 0$$

Area: $A = \int_1^5 \frac{5x}{x^2+2}\,dx$

$u = x^2 + 2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$= \frac{5}{2}\int_1^5 \frac{1}{u}\,du$

$= \frac{5}{2}\ln|x^2+2|\Big]_1^5$

$= \frac{5}{2}\ln|27| - \frac{5}{2}\ln|3|$

$= \boxed{\frac{5}{2}\ln 9}$

Ex: Prove that $\int \cot u \, du = \ln|\sin u| + C$

$$\int \frac{\cos u}{\sin u} \, du \qquad x = \sin u$$

$$dx = \cos u \, du$$

$$= \int \frac{1}{x} \, dx$$

$$= \ln|x| + C = \ln|\sin u| + C \quad \blacksquare$$

Ex: Prove that

$$\int \csc u \, du = -\ln\left|\csc u + \cot u\right| + C$$

$$\int \boxed{\frac{\csc u\,(\csc u + \cot u)}{\csc u + \cot u}\, du}$$

$$x = \csc u + \cot u$$

$$dx = (-\csc u \cot u - \csc^2 u)\, du$$

$$dx = -\boxed{\csc u\,(\cot u + \csc u)\, du}$$

$$= -\int \frac{1}{x}\, dx$$

$$= -\ln|x| + C$$

$$= -\ln\left|\csc u + \cot u\right| + C \quad \blacksquare$$