Statistical and Thermal Physics: Homework 15

Due: 10 April 2020

1 Spin system microstates

Consider an isolated system of spins in a magnetic field. A spin up particle has energy $-\mu B$ and a spin down particle μB. If there are five particles in the ensemble consider the following microstates:

State A: $\uparrow\uparrow\uparrow\downarrow\downarrow$
State B: $\downarrow\uparrow\uparrow\downarrow\downarrow$
State C: $\downarrow\downarrow\uparrow\uparrow\uparrow$
State D: $\downarrow\downarrow\downarrow\uparrow\uparrow$

According to the fundamental postulates of statistical physics which of these states are guaranteed to occur with the same probability as each other? For which will the probabilities probably be different? Explain your answer

2 Toy system microstates

A "toy" system consists of two interacting subsystems, A and B. Subsystem A has two particles (labeled a_1 and a_2) and subsystem B has three particles (with individual particles labeled similarly). Particles in either system can only have discrete energies. The state of each particle is labeled by the number of energy units that it has. Macrostates for the pair of systems are describe by the particle numbers and total energies for the systems. The following are various microstates of the system.

	Subsystem A		Subsystem B		
Particle	a_1	a_2	b_1	b_1	b_3
Microstate 1	4	1	1	1	1
Microstate 2	2	2	1	2	1
Microstate 3	3	2	1	1	2
Microstate 4	3	2	1	0	2
Microstate 5	0	5	1	1	1

a) Which microstates represent the same macrostate?

b) For which of these microstates is it possible to state with certainty that they will occur with the same probability?

3 Stirling's approximation

Consider the standard form of Stirling's log approximation

$$\ln n! \approx n \ln n - n + \frac{1}{2} \ln (2\pi n).$$

A weaker version is

$$\ln n! \approx n \ln n - n.$$

Check both versions of Stirling's log approximation for $n = 15$ and $n = 150$. Comment on their accuracy.

4 Interacting Einstein solids: analytical approximations

a) Use Stirling's approximation to show that for an Einstein solid

$$\begin{aligned}\Omega(N,q) &\approx \left(1 + \frac{q}{N-1}\right)^{N-1} \left(1 + \frac{N-1}{q}\right)^{q} \sqrt{\frac{N+q-1}{2\pi(N-1)q}} \\ &\approx \left(1 + \frac{q}{N}\right)^{N} \left(1 + \frac{N}{q}\right)^{q} \sqrt{\frac{N+q}{2\pi Nq}}\end{aligned}$$

whenever $q \gg 1$ and $N \gg 1$.

b) Assume that $q \gg N$. Using the Taylor series approximation,

$$\left(1 + \frac{x}{n}\right)^{n} \approx e^{x}$$

whenever $n \gg 1$, show that

$$\Omega(N,q) \approx \left(\frac{eq}{N}\right)^{N} \frac{1}{\sqrt{2\pi N}}.$$

c) Suppose that two Einstein solids, labeled A and B, interact. Show that the multiplicity for the combined system is

$$\Omega = \kappa \, q_A^{N_A} q_B^{N_B}$$

where κ is a term that does not depend on q_A or q_B.

d) Determine an expression for the entropy for the combined system.

e) If the system is isolated, then $q = q_A + q_B$ is fixed. Use this to rewrite the entropy in terms of q_A only. Show that the entropy attains a maximum when

$$\frac{q_A}{q_B} = \frac{N_A}{N_B}.$$

Explain in words what this means for the way in which energy units are shared between the two subsystems when they are in thermal equilibrium.

5 Interacting spin systems

Consider two spin systems, A and B, that interact. For each the magnetic field is such that $\mu B = 1$. Let E_A be the energy of A, N_A be the number of particles in A with similar variables representing B. Suppose that N_A and N_B are fixed and the total energy $E = E_A + E_B$ is also fixed.

a) Let N_{A+} be the number of particles in subsystem A with spin up. Show that

$$E_A = N_A - 2N_{A+}.$$

b) Noting that the total energy for the system is fixed, explain how you would describe a macrostate of the system. Which variables are necessary to describe the macrostate? Are any of these variables redundant?

c) Show that the multiplicity of the macrostate is

$$\Omega = \binom{N_A}{\frac{N_A - E_A}{2}} \binom{N_B}{\frac{N_B - E_A + E}{2}}.$$

Now suppose that $N_A = 4$ and $N_B = 12$ and initially $E_A = 4$ and $E_B = 4$. The systems are allowed to interact.

d) List all possible macrostates for the composite system and the probabilities with which each occurs.

e) Determine the equilibrium macrostate. Determine the energy per particle in each subsystem in this macrostate. In which direction did the energy flow as the systems reached equilibrium?

f) Determine the mean energy for each subsystem.

6 Counting states: electron in a box

a) Consider an electron in a one-dimensional box with length L. Determine $\Gamma(E)$ if $E = 110\frac{h^2}{8mL^2}$ where m is the mass of the electron.

b) Consider an electron in a three-dimensional cubic box whose sides have length L. Determine $\Gamma(E)$ if $E = 15\frac{h^2}{8mL^2}$ where m is the mass of the electron.

7 Gould and Tobochnik, *Statistical and Thermal Physics*, 4.12, page 198. The OSP program name is Particle in a Box. The points represent the actual number of states; the line plots the expression of Eq. (4.39); this is the asymptotic relation. The data can be obtained via Views → Data Table.

8 Volumes of hyperspheres

The volume of a sphere of radius R in n dimensions is given by:

$$V_n(R) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} R^n.$$

Verify that this is correct for $n = 1, 2,$ and 3.

If the states have the same energy then the probabilities with which they occur are the same For any microstate

Energy $= N_+ (-\mu B) + N_- (\mu B)$

Thus for state A

$$E_A = 3(-\mu B) + 2\mu B \quad \Rightarrow \quad E_A = -\mu B$$

Similarly

$$E_B = \mu B$$

$$E_C = -\mu B$$

$$E_D = \mu B$$

Thus A, C will occur with the same probability as each other

B, D " " " " " " " " "

but A, B are probably different.

a) The E_A, E_B combination must be the same

microstate	E_A	E_B	$E = E_A + E_B$
1	5	3	8
2	4	4	8
3	5	4	9
4	5	3	8
5	5	3	8

So microstates 1, 4, 5 represent the same macrostate.

The other two belong to distinct macrostates.

b) Only microstates with the same total energy. Thus

1, 2, 4, 5

occur with the same probability

n	Exact $\ln n!$	Stirling $n\ln n - n + \frac{1}{2}\ln 2\pi n$	% diff	$n\ln n - n$	% diff
15	27.899	27.894	0.02%	25.621	8.2%
150	605.020	605.010	0.00%	601.595	0.6 %

accurate to within 0.1%

less accurate
reasonable for
n large
enough

a) $N! = N^N e^{-N} \sqrt{2\pi N}$

$$\Omega(N,q) = \binom{N+q-1}{q} = \frac{(N+q-1)!}{q!(N-1)!}$$

$$\cong \frac{(N+q-1)^{N+q-1} \cancel{e^{-(N+q-1)}} \sqrt{2\pi(N+q-1)}}{q^q \cancel{e^{-q}} \sqrt{2\pi q}\,(N-1)^{N-1} \cancel{e^{-(N-1)}} \sqrt{2\pi(N-1)}}$$

$$= \left(\frac{N+q-1}{q}\right)^q \left(\frac{N+q-1}{N-1}\right)^{N-1} \sqrt{\frac{N+q-1}{2\pi q(N-1)}}$$

$$= \left(1+\frac{q}{N-1}\right)^{N-1} \left(1+\frac{N-1}{q}\right)^q \sqrt{\frac{N+q-1}{2\pi q(N-1)}}$$

$$\approx \left(1+\frac{q}{N}\right)^N \left(1+\frac{N}{q}\right)^q \sqrt{\frac{N+q}{2\pi q N}}$$

b) $\left(1+\frac{q}{N}\right)^N \approx \left(\frac{q}{N}\right)^N$ $\qquad N+q \approx q.$

$\left(1+\frac{N}{q}\right)^q \approx e^N$

So $\Omega(N,q) \approx \left(\frac{q}{N}\right)^N e^N \frac{1}{\sqrt{2\pi N}} = \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$

c) $\Omega = \Omega_A \Omega_B$

Then $\Omega_A = \left(\frac{e q_A}{N_A}\right)^{N_A} \frac{1}{\sqrt{2\pi N_A}}$

$\Omega_B = \left(\frac{e q_B}{N_B}\right)^{N_B} \frac{1}{\sqrt{2\pi N_B}}$

So

$$\Omega = \frac{1}{2\pi\sqrt{N_A N_B}} \left(\frac{e}{N_A}\right)^{N_A} q_A^{N_A} \left(\frac{e}{N_B}\right)^{N_B} q_B^{N_B}$$

$$= \underbrace{\frac{1}{2\pi\sqrt{N_A N_B}} \left(\frac{e}{N_A}\right)^{N_A} \left(\frac{e}{N_B}\right)^{N_B}}_{\text{Let this be } K} q_A^{N_A} q_B^{N_B}$$

So $\Omega = K q_A^{N_A} q_B^{N_B}$.

d) $S = k\ln\Omega = k\left[\ln K + N_A \ln q_A + N_B \ln q_B\right]$

e) $S = k\ln K + kN_A \ln q_A + kN_B \ln(q - q_A)$

$$\frac{\partial S}{\partial q_A} = \frac{kN_A}{q_A} - \frac{kN_B}{q - q_A} = 0$$

$$\Rightarrow \frac{N_A}{q_A} = \frac{N_B}{q - q_B} \Rightarrow \frac{N_A}{q_A} = \frac{N_B}{q_B} \Rightarrow \frac{q_A}{q_B} = \frac{N_A}{N_B}$$

The ratio of energies in A vs B is the same as the ratio of oscillator numbers

a) $E_A = -\mu B N_{A+} + \mu B N_{A-}$

$= -N_{A+} + N_{A-}$

Then $N_{A-} = N_A - N_{A+} \Rightarrow E_A = N_A - 2N_{A+}$

b) We need E_A, E_B or N_{A+}, N_{B+}. But since $E = E_A + E_B$ is fixed we could just use:

EITHER E_A (or N_{A+})

OR E_B (or N_{B+})

c) Multiplicity of A is

$$\Omega_A = \binom{N_A}{N_{A+}}$$

Multiplicity of B is

$$\Omega_B = \binom{N_B}{N_{B+}}$$

Multiplicity is

$$\Omega = \Omega_A \Omega_B = \binom{N_A}{N_{A+}}\binom{N_B}{N_{B+}}$$

Now $N_{A+} = \frac{N_A - E_A}{2}$ $\quad N_{B+} = \frac{N_B - E_B}{2} = \frac{N_B - (E - E_A)}{2}$

gives:

$$\Omega = \binom{N_A}{\frac{N_A - E_A}{2}}\binom{N_B}{\frac{N_B - E + E_A}{2}}$$

We list these by N_{A+} and N_{B+}. Note that

$$E = E_A + E_B \Rightarrow 8 = N_A - 2N_{A+} + N_B - 2N_{B+}$$

$$\Rightarrow 8 = 4 + 12 - 2(N_{A+} + N_{B+})$$

$$\Rightarrow -8 = -2(N_{A+} + N_{B+})$$

$$\Rightarrow 4 = N_{A+} + N_{B+}$$

N_{A+}	N_{B+}	$E_A = N_A - 2N_{A+} = 4 - 2N_{A+}$	$E_B = N_B - 2N_{B+} = 12 - 2N_{B+}$	Ω_A	Ω_B	Ω	Prob
0	4	4	4	1	495	495	0.272
1	3	2	6	4	220	880	0.484
2	2	0	8	6	66	396	0.218
3	1	-2	10	4	12	48	0.026
4	0	-4	12	1	1	1	0.0005
						$\sum_i = 1820$	

e) $N_{A+} = 1 \Rightarrow E_A = 2 \rightarrow$ energy per particle $= 2/4 = 0.5$

$N_{B+} = 3$ $E_B = 6 \rightarrow$ energy " " $= 6/12 = 0.5$

f) $\bar{E}_A = \sum E_A \, prob(E_A) = 2$

$\bar{E}_B = \sum E_B \, prob(E_B) = 6$

a) The states are labeled by $n = 1, 2, 3 \ldots$ with energy $E_n = \frac{h^2}{8mL^2} n^2$

So $n = 1, 2, 3, 4, \ldots, 10$ all have $E \leq 100 \frac{h^2}{8mL^2}$

Thus $\Gamma(E) = 10$

b) The states are labeled by $n_x, n_y, n_z = 1, 2, \ldots$ with energy

$$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

States are

n_x	n_y	n_z	E in multiples of $h^2/8mL^2$
1	1	1	3
1	2	1	6
2	1	1	6
1	1	2	6
1	1	3	11
1	3	1	11
3	1	1	11
1	1	4	18
1	4	1	18
4	1	1	18
1	2	3	
1	3	2	
2	1	3	14
2	3	1	
3	1	2	
3	2	1	
2	2	2	12
2	2	3	17
2	3	2	17
3	2	2	17

$n_x n_y n_z$	E
1 2 2	9
2 2 1	9
2 1 2	9

$\Gamma = 17$

G+T Ch 4

Prob 4.12

Let N_1 = actual number

N_2 = asymptotic number.

Then we want $\frac{|N_2 - N_1|}{N_1} \times 100 < 1$.

Data indicates this occurs when $R = 117$ since here

$N_1 = 10859$

$N_2 = 10751$

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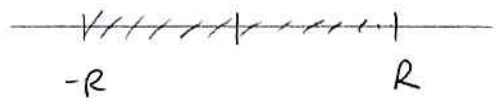
$\underline{n=1}$ $\quad V_1 = \dfrac{2\pi^{1/2}}{1\,\Gamma(1/2)} R^1$

$\Gamma(1/2) = \pi^{1/2} \Rightarrow V_1 = 2R$

This is length of segment



$\underline{n=2}$ $\quad V_2 = \dfrac{2\pi}{2\,\underbrace{\Gamma(1)}_{=1}} R^2 = \pi R^2$ = area circle

$\underline{n=3}$ $\quad V_3 = \dfrac{2\pi^{3/2}}{3\,\Gamma(3/2)} R^3$

$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2$

$\Rightarrow V_3 = \frac{4}{3}\pi R^3$ $\Rightarrow$ Volume sphere