2.5 Nonlinear Models: Logistic Equation

13, 17, 35

2.5.13

a.)

$y' = (r - ay)y$
$L = r/a$
$y' = (1 - \frac{a}{r}y)y$
$y' = r(1 - y/L)y$

$y' = (-t + y)y$
$L = -t/a$

$y' = -r(1 - y/L)y$

$y' = r(1 - y/L)y$

$y(t) = \dfrac{L}{1 + (\frac{L}{y_0} - 1)e^{-rt}}$

$\frac{dy}{dt} = -r(1 - y/L)y$

$\frac{dy}{(1 - y/L)y} = -r\,dt$

$\int \frac{1}{y}\,dy + \int \frac{1}{1 - y/L}\,dy = -r\,dt$

$\ln y - \ln(y/L - 1) = -rt + C$

$\ln\left(\frac{y}{y/L - 1}\right) = -rt + C$

$\frac{y}{y/L - 1} = Ke^{-rt}$

$K = \frac{y}{y/L - 1}$

$y' = r(1 - y/L)y$

$y' = r(1 - \frac{y}{L})y$

$\frac{1}{(1 - \frac{y}{L})y}\,dy = r\,dt$

$\frac{1}{(1 - \frac{y}{L})} + \frac{1}{y}\,dy = r\,dt$

$\ln|y| + 1/L(-L\ln|1 - \frac{y}{L}|) = rt + C$

$\ln(y) - \ln(1 - \frac{y}{L}) = rt + C$

$\ln\left(\frac{y}{1 - \frac{y}{L}}\right) = rt + C$

$\left(\frac{y}{1 - \frac{y}{L}}\right) = Ke^{rt}$ $\quad K = \frac{y_0}{1 - y_0/L}$

a.) Same equation

b.)

$y' = r(1 - \frac{y}{L})y \quad y_0 = L$
$y' = r(1 - \frac{L}{L})L$
$= r(1 - 1)L$
$= 0$

$y' = y - \sqrt{y}$
$y' = L - \sqrt{L}$
$= L - L$
$= 0$

b. Same result

C.) $0 < y_0 < L$, $\quad y_0 = L$, $\quad y_0 > L$

$y = \dfrac{L}{1 + (\frac{L}{y_0} - 1)e^{-rt}}$

The denominator is never 0.
∴ The denominator will approach 1 as $t \to \infty$. Which is what is on the graph.

$y = \dfrac{L}{1 + (\frac{L}{y_0} - 1)e^{-rt}}$

The denominator goes to 1, which is what we see on the graph.
$y = L$, equilibria

$y = \dfrac{L}{1 + (\frac{L}{y_0} - 1)e^{-rt}}$

The denominator is never 1, this is what we see on the graph.

C. ✓

D.) $0 < y_0 < L$

$t^* = \frac{1}{r}\ln\left(\frac{L}{y_0} - 1\right)$

$\frac{d^2y}{dt^2} = dt\left[r(1 - y/l)y\right]$

$\frac{d^2y}{dt^2} = r(1 - y/l)\frac{dy}{dt} - \frac{ry}{l}\frac{dy}{dt}$

$= \left((r - y/l) - \frac{ry}{l}\right)\frac{dy}{dt}$

$0 = r - 2ry/l$

$= (r - 2ry/l)(r(1 - y/l)y)$ to eventually get

$y = \dfrac{L}{1 + (L/y_0 - 1)e^{-rt}}$ ⇒ Sub in for y to

$\frac{dy}{dt} = r(1 - y/l)y$

$r\left(\frac{L}{y_0} - 1\right)e^{-rt} - r = 0$

$\left(\frac{L}{y_0} - 1\right)e^{-rt} = 1$

$e^{-rt} = \dfrac{1}{\frac{L}{y_0} - 1}$

$e^{rt} = L/y_0 - 1$

$rt = \ln(L/y_0 - 1)$

$t = \dfrac{\ln(L/y_0 - 1)}{r}$

i.) $t^* = \dfrac{\ln(L/y_0 - 1)}{r}$
ii.) $L/2$
iii.) $\frac{rL}{4}$

$r - \frac{2ry}{L} = 0$

$\frac{2ry}{L} = r$

$2ry = Lr$

$2y = L$

$y = \frac{L}{2}$

$y' = r(1 - y/l)y$
$= r(1 - \frac{L/2}{l})\frac{L}{2}$
$= r(1 - \frac{1}{2})\frac{L}{2}$
$= r(\frac{1}{2})\frac{L}{2}$
$= \frac{rL}{4}$

2.5.17

$x_0 = 1{,}000$

$L = 80{,}000$

1,000 originally heard

one day 10,000

$$\frac{dx}{dt} = 80{,}000\,K\,(80{,}000 - x)$$

$$\frac{dx}{dt} = 80{,}000\,K\left(1 - \frac{x}{80{,}000}\right)x$$

$$x = \frac{L}{1 + \left(\frac{L}{x_0} - 1\right)e^{-rt}}$$

$$= \frac{80{,}000}{1 + \left(\frac{80{,}000}{1{,}000} - 1\right)e^{-rt}}$$

$$x = \frac{80{,}000}{1 + 79e^{-rt}}$$

$$10{,}000 = \frac{80{,}000}{1 + 79e^{-r(1)}}$$

$t = 1$

$x = 10{,}000$

$$10{,}000(1 + 79e^{-r}) = 80{,}000$$

$$1 + 79e^{-r} = 8$$

$$79e^{-r} = 7$$

$$e^{-r} = \frac{7}{79}$$

$$-r = \ln(7/79)$$

$$r = 2.42$$

$$y = \frac{L}{1 + (y_0/L - 1)e^{-rt}}$$

$$\frac{dx}{dt} = Kx(L - x)$$

$$X(t) = \frac{80{,}000}{1 + 79e^{-2.42t}}$$

2.5.35

$$\frac{dy}{dt} = \alpha y - y^3$$

$$0 = \alpha y - y^3$$

$$= y(\alpha - y^2)$$

$y = 0$

$y = \pm\sqrt{\alpha}$

a.) when $\alpha \le 0$, $y = 0$ or imaginary number

$y = 0$ is only Solution

when $y = 0$

$$0 = y(\alpha - y^2)$$

$$= y(0 - y^2)$$

$$0 = -y^3$$

$$y = 0$$

$\alpha \le 0$

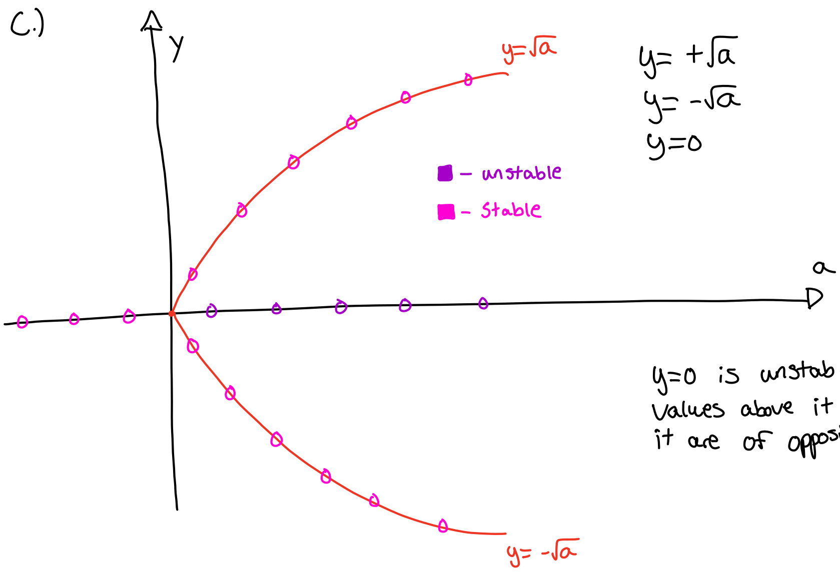
$y = 0$ is eq point

b.) when $\alpha \ge 0$

E.Q's : $y = 0$, $y = +\sqrt{\alpha}$, $y = -\sqrt{\alpha}$

$\therefore$ There are three E.Q's

c.)



$y = +\sqrt{a}$

$y = -\sqrt{a}$

$y = 0$

Bifurcation points occur where there are a change of stability on the graph

Bifurcation at $\alpha = 0$

$y = 0$ is unstabe since values above it and below it are of opposite sign.