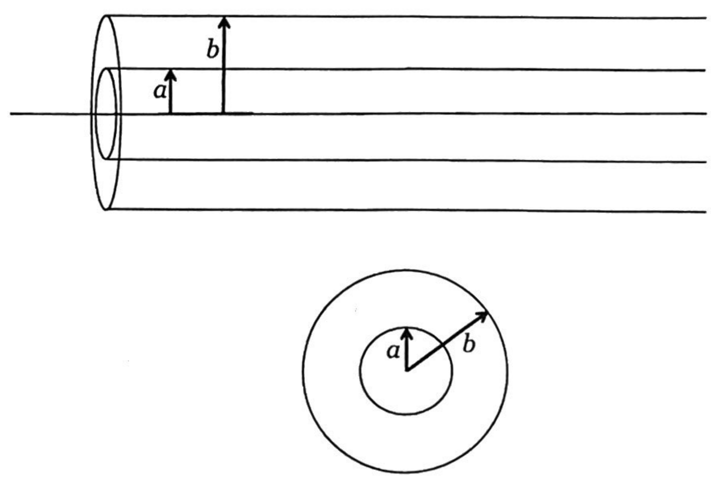
Physics 311

Exam 3

1. An infinitely long straight wire carries a uniform charge-per-unit-length, $\lambda = +Q/\ell$.[1] The wire is surrounded by an infinitely long, thick coaxial cylindrical shell (inner radius a, outer radius b). The thick cylindrical shell carries no net charge.

   a) Find the surface charge density σ at $s = a$ and at $s = b$ in terms of $\lambda, \ell, a,$ and b. For full credit, show all work.

   b) Find the potential difference between a point on the inner radius a and a point at $s = 2b$.



[1] For every length of wire ℓ, there is a charge $+Q$.

2. Two infinite metal plates lie parallel to the xz plane, one at $y = 0$ and one at $y = a$. [2] The top plate is grounded and the bottom plate is maintained at a constant potential $-V_0$.

   a) Using Laplace's equation, construct the *general solution* for the electric potential.

   b) By applying boundary conditions, construct the electric potential in terms of V_0, y, and a.

[2]These plates are truly infinite, where both x and z run from $-\infty$ to ∞.

3. A rectangular pipe, running parallel to the x-axis (from $-\infty$ to ∞), has three grounded metal sides at $y = 0$, $y = b$, and $z = 0$. The fourth side, at $z = a$, is maintained at a specific potential $V_0(y)$.

a) By choosing a product solution and using the method of *Separation of Variables*, explicitly arrive at a set of ordinary differential equations (ODEs). (Show your work!)

b) By solving the above ODE's, arrive at the *appropriate* separable solution to Laplace's equation.

c) By applying the boundary conditions

$$V(y=0) = V(y=b) = V(z=0) = 0, \tag{1}$$

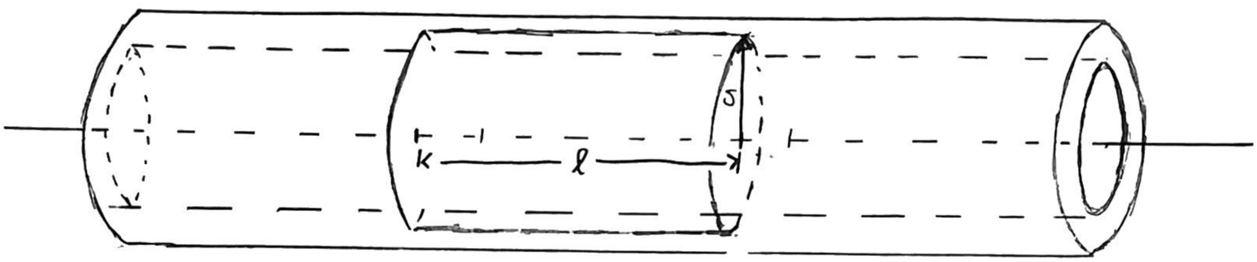
construct the *general solution* for the electric potential, which amounts to a linear combination of an *infinite set* of solutions.

d) For the case when $V_0(y) = -V_0$ (a negative constant), calculate the potential, expressing the final solution as a sum with no undetermined constants.

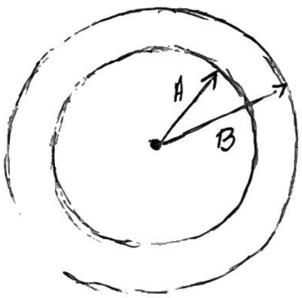
Hint: You may find the integral relation

$$\begin{aligned} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx &= 0 \quad \text{if} \quad n \neq m \\ &= \frac{a}{2} \quad \text{if} \quad n = m \end{aligned} \tag{2}$$

PHYS 311 EXAM 3 SOLUTIONS



a)



CONSIDER A LENGTH OF WIRE ℓ, WHICH HAS A CHARGE $+Q$ SO

$$\lambda = \frac{Q}{\ell}$$

· INSIDE OF THE THICK COAXIAL SHELL, THERE IS NO $\vec{E}$-FIELD.

· BY GAUSS' LAW...

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon}$$ THERE IS NO CHARGE ENCLOSED

CONSIDER: A COAXIAL CYLINDRICAL SURFACE OF RADIUS S...

$$Q_{enc} = \underbrace{\lambda \ell}_{+Q} + \underbrace{\sigma 2\pi A \ell}_{-Q} = 0 \quad \text{SO} \quad \sigma_A = -\frac{\lambda \ell}{2\pi A \ell} = \frac{-\lambda}{2\pi A} = \frac{-Q}{2\pi A \ell}$$

SINCE THE THICK COAXIAL SHELL IS NEUTRAL, THE SURFACE CHARGE DENSITY AT B IS..

$$\sigma_B = \frac{+Q}{2\pi B \ell} = \frac{\lambda \ell}{2\pi B \ell} = +\frac{\lambda}{2\pi B}$$

SO

$$\boxed{\begin{aligned} \sigma_A &= -\frac{\lambda}{2\pi A} \\ \sigma_B &= +\frac{\lambda}{2\pi B} \end{aligned}}$$

NOW CONSIDER COAXIAL CYLINDRICAL SURFACES OF RADIUS s FOR THREE CASES..

CI : $s<A$

CII : $A<s<B$

CIII : $s>B$

CASE I : $s<A$ ...

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{WHICH YIELDS} \quad E2\pi s\ell = \frac{\lambda\ell}{\varepsilon_0} \quad \text{so} \quad E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s} \quad \text{so} \quad \left[\vec{E}_I = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}\hat{s}\right]$$

CASE II : $A<s<B$...

GAUSS' LAW YIELDS $\quad E2\pi s\ell = 0 \quad$ so $\quad \left[\vec{E}_{II} = 0\,\hat{s}\right]$

CASE III : $s>B$...

GAUSS' LAW YIELDS $\quad E2\pi s\ell = \dfrac{\sigma_0 2\pi B\ell}{\varepsilon_0} = \dfrac{\lambda\ell}{\varepsilon_0} \quad$ so $\quad \left[\vec{E}_{III} = \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{1}{s}\hat{s}\right]$

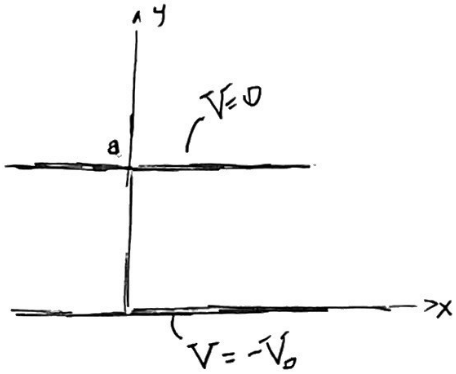
NOW

$$V(s=A) - V(s=2B) = -\int_{2B}^{A}\vec{E}\cdot d\vec{\ell} = -\int_{2B}^{B}\vec{E}_{III}\cdot d\vec{\ell} - \cancel{\int_{B}^{A}\vec{E}_{II}\cdot d\vec{\ell}}^{\,0} = -\frac{\lambda}{2\pi\varepsilon_0}\int_{2B}^{B}\frac{ds}{s} = \frac{\lambda}{2\pi\varepsilon_0}\ln s\Big|_{B}^{2B}$$

NOW $d\vec{\ell} = ds\,\hat{s}$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln(2B) - \ln(B)\right] = \frac{\lambda}{2\pi\varepsilon_0}\ln(2)$$

2.



a) LAPLACE'S EQN TAKES THE FORM..

$$\frac{\partial^2 V}{\partial y^2} = 0$$

SINCE $V = V(y)$

LAPLACE'S EQN BECOMES

$$\frac{d^2 V}{dy^2} = 0$$

WHICH HAS THE SOLUTION

$$\boxed{V(y) = my + b}$$

b) APPLYING BOUNDARY CONDITIONS..

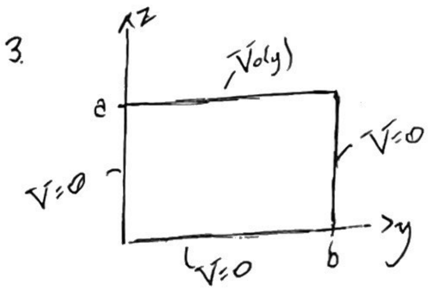
$$V(y=0) = -V_0 = b$$

$$V(y=a) = 0 = ma - V_0 \quad \therefore \quad m = \frac{V_0}{a}$$

$$V(y) = \frac{V_0}{a} y - V_0 = V_0\left(\frac{y}{a} - 1\right)$$

$$\boxed{V(y) = V_0\left(\frac{y}{a} - 1\right)}$$

3.



a) THE LAPLACE EQN TAKES THE FORM...

$$\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

CHOOSING A PRODUCT SOLUTION..

$$V(y,z) = Y(y) Z(z)$$

LAPLACE'S EQN BECOMES..

$$\frac{1}{YZ}\left[Z\frac{d^2Y}{dy^2} + Y\frac{d^2Z}{dz^2} = 0 \right]$$

yields

$$\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz}}_{k^2} = 0$$

so

$$\frac{d^2Y}{dy^2} = -k^2 Y$$

$$\frac{d^2Z}{dz^2} = k^2 Z$$

b) $Y(y) = A\sin(ky) + B\cos(ky)$ w/ $V(y,z) = Y(y)Z(z)$

$Z(z) = Ce^{kz} + De^{-kz}$

) $V(y=0) = B(Ce^{kz} + De^{-kz}) = 0 \quad \therefore \boxed{B=0}$

$V(y=b) = A\sin(kb)(Ce^{kz} + De^{-kz}) = 0 \quad \therefore kb = n\pi \quad \therefore \boxed{k_n = \frac{n\pi}{b}}$

$V(z=0) = A\sin(ky)(C+D) = 0 \quad \therefore \boxed{D=-C}$

so $V(y,z) = AC(e^{kz} - e^{-kz})\sin(ky) = 2AC\sinh(kz)\sin(ky) = C\sinh(kz)\sin(ky)$

let $2AC \to C$

SO THE GENERAL SOLUTION TAKES THE FORM...

$$\boxed{V(y,z) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}z\right)\sin\left(\frac{n\pi}{b}y\right)}$$

d) NOW

$$V(z=a) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = -V_0$$

MULTIPLYING BOTH SIDES BY $\sin\left(\frac{m\pi y}{b}\right) dy$: INTEGRATING,

$$\sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \underbrace{\int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) dy}_{\frac{b}{2}\delta_{nm}} = -V_0 \int_0^b \sin\left(\frac{m\pi y}{b}\right) dy$$

SO

$$c_m \frac{b}{2} \sinh\left(\frac{m\pi a}{b}\right) = -V_0 \frac{b}{m\pi} \cos\left(\frac{m\pi y}{b}\right)\Bigg|_b^0 = -V_0 \frac{b}{m\pi}\left(1 - \cos(m\pi)\right)$$

SO

$$c_m = -\frac{2V_0}{m\pi} \cdot \frac{1}{\sinh\left(\frac{m\pi a}{b}\right)} \left(1 - \cos(m\pi)\right)$$

$$\therefore \quad c_m = -\frac{4V_0}{m\pi} \cdot \frac{1}{\sinh\left(\frac{m\pi a}{b}\right)} \quad m \text{ ODD}$$

$$= 0 \quad \text{FOR } m \text{ EVEN}$$

$$\therefore \quad \boxed{V(y,z) = -\frac{4V_0}{\pi} \sum_{n \text{ ODD}} \frac{1}{n} \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi}{b} z\right) \sin\left(\frac{n\pi y}{b}\right)}$$

1a) FOR GAUSSIAN SURFACE OF RADIUS $a<s<b$. $E=0$ $\therefore Q_{enc}=0$ +1

$$Q_{enc} = \lambda l + \sigma 2\pi A l = 0 \quad +1$$

$$\sigma_A = -\frac{\lambda}{2\pi A} \quad +5$$

SINCE SHELL IS NEUTRAL, $+Q$ ON OUTER SURFACE +1

$$\sigma_B = +\frac{\lambda}{2\pi B} \quad +5$$

b) $$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad +3$$

$$\oint \vec{E}\cdot d\vec{a} = E\,2\pi s l \quad +3$$

$$\vec{E}_{I} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}\hat{s} \quad \text{for } s<a \quad +2$$

$$\vec{E}_{II} = 0 \quad \text{for } a<s<b \quad +2$$

$$\vec{E}_{III} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}\hat{s} \quad s>b \quad +2$$

$$V(s=a) - V(s=2b) = -\int_{2b}^{a} \vec{E}\cdot d\vec{l} \quad +3$$

$$= -\int_{2b}^{b} \vec{E}_{III}\cdot d\vec{l} \quad +2$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln(2) \quad +3$$

2 a) $$\frac{\partial^2 V}{\partial y^2} = 0 \quad +10 \quad (+5 \text{ POINTS IN 2D or 3D})$$

$$V(y) = my + b \quad +8$$

b) $$V(y=0) = b = -V_0 \quad +5$$

$$V(y=a) = ma - V_0 = 0 \quad +5$$

$$m = \frac{V_0}{a}$$

$$V(y) = V_0\left(\frac{y}{a} - 1\right) \quad +5$$