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PHYS 396
HW 9

✓ ✓ ✓ ✓ ✓ ✓
1, 2, 3, 4, 5, 6

Problem 1

$$ds^2 = -\left(1 - 2M/r\right)dt^2 + \left(1 - 2M/r\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \longrightarrow (1)$$

$r > 2M$: $\quad t = v - r - 2M \ln|r/2M - 1| \longrightarrow (2)$

$r < 2M$: $\quad t = v - r - 2M \ln|1 - r/2M| \longrightarrow (3)$

for (3) we take a differential w.r.t v and r

$$dt = dv - dr - 2M \cdot \frac{1}{1 - r/2M} \cdot \frac{-1}{2M}\, dr = dv - dr + (1 - r/2M)^{-1} dr$$

$$= dv - (1 + (1 - r/2M)^{-1})dr = dv - (r/2M / (r/2M - 1))dr$$

$$= dv - \frac{r/2M}{r/2M - 1} \cdot \frac{2m/r}{2m/r}\, dr = dv - \frac{dr}{1 - 2M/r} \quad : \quad dt = dv - \frac{dr}{(1 - 2M/r)} \longrightarrow (4)$$

putting (4) into (1)

$$ds^2 = -(1 - 2M/r)\left(dv - dr(1 - 2M/r)^{-1}\right)^2 + dr^2(1 - 2M/r)^{-1} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -(1 - 2M/r)(dv^2 - 2dvdr(1 - 2M/r)^{-1} + dr^2(1 - 2M/r)^{-2}) + dr^2(1 - 2M/r)^{-1} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -(1 - 2M/r)dv^2 + 2dvdr \cancel{- dr^2(1 - 2M/r)} \cancel{+ dr^2(1 - 2M/r)^{-1}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$= -(1 - 2M/r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \longrightarrow (5)$$

Our line element transforms to (6) when $r < 2M$ and thus t is in the form of (3).

$$\therefore \quad \boxed{ds^2 = -(1 - 2M/r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}$$

Problem 2

$$ds^2 = -(1-2m/r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \longrightarrow (6)$$

a.) The symmetries found in (6) are: (v, r, θ, φ)

$$\underset{\sim}{\xi} = (1,0,0,0) \; : \; \underset{\sim}{\eta} = (0,0,0,1) \longrightarrow (7)$$

Taking a dot product with:

$$\underset{\sim}{u}^0 \text{ and } \underset{\sim}{u}^3 \longrightarrow (8)$$

we get

$$\underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{v\beta}\xi^v u^\beta = g_{vr}\xi^v u^r + g_{vv}\xi^v u^v \longrightarrow (9)$$

(9) becomes: $\frac{dr}{d\tau} - (1-2m/r)\frac{dv}{d\tau}$

and thus e is,

$$\boxed{e = (1-2m/r)\frac{dv}{d\tau} - \frac{dr}{d\tau}} \longrightarrow (10)$$

$$\underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{\varphi\varphi}\eta^3 u^3 = \longrightarrow (11)$$

(11) becomes: $r^2\sin^2\theta\frac{d\varphi}{d\tau}$

and thus l is,

$$\boxed{l = r^2\sin^2\theta\frac{d\varphi}{d\tau}} \longrightarrow (12)$$

b.) using (6) we can construct $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1 \longrightarrow (13)$

(13) takes the form of: $-1 = g_{\alpha\beta}u^\alpha u^\beta$

Employing our metric from (6), (13) can be re-written as,

$$-1 = g_{vv}u^v u^v + g_{vr}u^v u^r + g_{\theta\theta}u^\theta u^\theta + g_{\varphi\varphi}u^\varphi u^\varphi \longrightarrow (14)$$

(14) is thus: $-1 = -(1-2m/r)\left(\frac{dv}{d\tau}\right)^2 + 2\left(\frac{dv}{d\tau}\right)\left(\frac{dr}{d\tau}\right) + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$

with $\underset{\sim}{u}\cdot\underset{\sim}{u}$ now,

$$\boxed{\underset{\sim}{u}\cdot\underset{\sim}{u} = -(1-2m/r)\left(\frac{dv}{d\tau}\right)^2 + 2\left(\frac{dv}{d\tau}\right)\left(\frac{dr}{d\tau}\right) + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2} \longrightarrow (15)$$

c.) Setting $\theta = \pi/2$, (15) will reduce to

$$-(1-2m/r)\left(\frac{dv}{d\tau}\right)^2 + \left(\frac{dv}{d\tau}\right)\left(\frac{dr}{d\tau}\right) + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \longrightarrow (16)$$

Problem 2| Continued

Re-organizing (10) & (12) we get:

$$\frac{dv}{d\tau} = (1-2m/r)^{-1}(e + dr/d\tau) \quad : \quad \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

$$-(1-2m/r)\left[(1-2m/r)^{-2}(e+dr/d\tau)^2\right] + \left[2(1-2m/r)^{-1}(e+dr/d\tau)\right]\frac{dr}{d\tau} + r^2 \cdot \frac{\ell^2}{r^4} = -1$$

$$-(1-2m/r)^{-1}(e+dr/d\tau)^2 + 2(1-2m/r)^{-1}(e \cdot dr/d\tau + (dr/d\tau)^2) + \frac{\ell^2}{r^2} = -1$$

$$(1-2m/r)^{-1}\left[-(e+dr/d\tau)^2 + (2e\, dr/d\tau + 2(dr/d\tau)^2)\right] = -1 - \frac{\ell^2}{r^2}$$

$$(1-2m/r)^{-1}\left[-e^2 - 2e(\cancel{dr}/d\tau) - (dr/d\tau)^2 + 2e(\cancel{dr}/d\tau) + 2(dr/d\tau)^2\right] = -1 - \frac{\ell^2}{r^2}$$

$$(1-2m/r)^{-1}\left[-e^2 + (dr/d\tau)^2\right] = -1 - \frac{\ell^2}{r^2}$$

$$-e^2 + \left(\frac{dr}{d\tau}\right)^2 = (1-2m/r)(-1 - \ell^2/r^2)$$

$$-e^2 + \left(\frac{dr}{d\tau}\right)^2 = -1 - \frac{\ell^2}{r^2} + \frac{2m}{r} + \frac{2m\ell^2}{r^3}$$

$$-e^2 + 1 = -\left(\frac{dr}{d\tau}\right)^2 - \frac{\ell^2}{r^2} + \frac{2m}{r} + \frac{2m\ell^2}{r^3}$$

$$e^2 - 1 = \left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r} + \frac{\ell^2}{r^2} - \frac{2m\ell^2}{r^3}$$

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{m}{r} + \frac{\ell^2}{2r^2} - \frac{m\ell^2}{r^3}$$

$$\mathcal{E} = \frac{e^2-1}{2} \quad : \quad V_{eff}(r) = -\frac{m}{r} + \frac{\ell^2}{2r^2} - \frac{m\ell^2}{r^3}$$

$$\boxed{\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)}$$

Problem 3

$$ds^2 = -(1-2m/r)dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \longrightarrow (17)$$

$$e = -\underset{\sim}{\xi}\cdot\underset{\sim}{u} = (1-2m/r)\,dt/d\tau \longrightarrow (18)$$

$$l = \underset{\sim}{\eta}\cdot\underset{\sim}{u} = r^2\sin^2\theta\, d\varphi/d\tau \longrightarrow (19)$$

The Killing vectors from (17) used in (18) and (19) are

$$\underset{\sim}{\xi} = (1,0,0,0)\ ,\ \underset{\sim}{\eta} = (0,0,0,1). \longrightarrow (20)$$

In Eddington-Finklestein coordinates, the $\underset{\sim}{u}^\alpha$ are

$$\underset{\sim}{u}^0 = \frac{dv}{d\tau},\ \underset{\sim}{u}^1 = \frac{dr}{d\tau},\ \underset{\sim}{u}^3 = \frac{d\varphi}{d\tau}. \longrightarrow (21)$$

Using the symmetries from (20) and the four-velocities from (21), (18) transforms to

$$e = -\underset{\sim}{\xi}\cdot\underset{\sim}{u} = -g_{\alpha\beta}\xi^\alpha u^\beta = -g_{0\beta}\xi^0 u^\beta = -g_{00}\xi^0 u^0 - g_{01}\xi^0 u^1$$

$$e = (1-2m/r)\frac{dv}{d\tau} - \frac{dr}{d\tau} \quad : \quad dt = \frac{dt}{dr}dr + \frac{dt}{dv}dv \quad : \quad \frac{dt}{d\tau} = \frac{dt}{dr}\frac{dr}{d\tau} + \frac{dt}{dv}\frac{dv}{d\tau}$$

$$t = v - r - 2m\ln|1 - r/2m| \quad : \quad \frac{dt}{dr} = -1 - \frac{2m}{r-2m} \quad : \quad \frac{dt}{dv} = 1$$

$$\frac{dt}{d\tau} = \left(-1 - \frac{2m}{r-2m}\right)\frac{dr}{d\tau} + \frac{dv}{d\tau} \quad : \quad \frac{dv}{d\tau} = \frac{dt}{d\tau} + \left(1 + \frac{2m}{r-2m}\right)\frac{dr}{d\tau} \quad : \quad \frac{dv}{d\tau} = \frac{dt}{d\tau} + \left(\frac{r}{r-2m}\right)\frac{dr}{d\tau}$$

$$e = (1-2m/r)\left[\frac{dt}{d\tau} + \left(\frac{r}{r-2m}\right)\frac{dr}{d\tau}\right] - \frac{dr}{d\tau} = \left(\frac{r-2m}{r}\right)\frac{dt}{d\tau} + \left(\frac{r-2m}{r}\right)\left(\frac{r}{r-2m}\right)\frac{dr}{d\tau} - \frac{dr}{d\tau}$$

$$e = (1-2m/r)\frac{dt}{d\tau} + \frac{dr}{d\tau} - \frac{dr}{d\tau} = (1-2m/r)\frac{dt}{d\tau}$$

$$\boxed{e = (1-2m/r)\frac{dt}{d\tau}}$$

And similarily (19) becomes,

$$l = \underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{33}u^3 = r^2\sin^2\theta\frac{d\varphi}{d\tau} \longrightarrow \text{unchanged}$$

$$\boxed{l = r^2\sin^2\theta\cdot\frac{d\varphi}{d\tau}}$$

Problem 4

$$ds^2 = -(1 - 2m/r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \longrightarrow (22)$$

a.) $r > 2m : v - 2(r + 2m\ln|r/2m - 1|) \longrightarrow (23)$

$r < 2m : v - 2(r + 2m\ln|r/2m - 1|) \longrightarrow (24)$

Radial light rays have no change in θ or φ $\therefore$ $d\theta = d\varphi = 0$. $ds^2 = 0$ as well. Therefore (22) becomes

$$0 = -(1 - 2m/r)dv^2 + 2dvdr \ : \quad 0 = dv(2dr - (1 - 2m/r)dv) \longrightarrow (25)$$

From (25) we can see there are two cases,

$$dv = 0 \ \vdots \ 2 \cdot dr - (1 - 2m/r)dv = 0 \longrightarrow (26)$$

(26) can be minipulated to

$$\frac{dv}{dr} = \frac{2}{(1 - 2m/r)} \longrightarrow (27)$$

(27) is a first order differential equation. (27) can be written as

$$\int dv = \int \frac{2}{(1 - 2m/r)} dr$$

$$V = 2\int \frac{1}{(1 - 2m/r)} dr \longrightarrow (28)$$

$$2\int (1 - 2m/r)^{-1} dr \ : \ \begin{array}{l} r = 2mx \\ dr = 2m\,dx \end{array} \ : \ 2\int \frac{2m}{(1 - \frac{2m}{2mx})} dx = 4m\int \frac{1}{(1 - 1/x)} dx \rightarrow (29)$$

(29) becomes : $4m \cdot (\ln|x - 1| + x)\Big|_{x = \frac{r}{2m}} + c \ : \ 4m \cdot (\ln|\frac{r}{2m} - 1| + \frac{r}{2m}) + c$

$= 4m\ln|r/2m - 1| + 2r + c \ :$ $\therefore$ (28) becomes

$$V = 2(r + 2m\ln|r/2m - 1|) + c \longrightarrow (30)$$

V is actually a function of r. That comes into play in part b.

$$\boxed{V(r) - 2(r + 2m\ln|r/2m - 1|) = c^*}$$

c^* - Constant of integration

b.) Setting $r = 2mx$, (30) becomes

$$V(2mx) = 2(2mx + 2m\ln|\tfrac{2mx}{2m} - 1|) + K$$

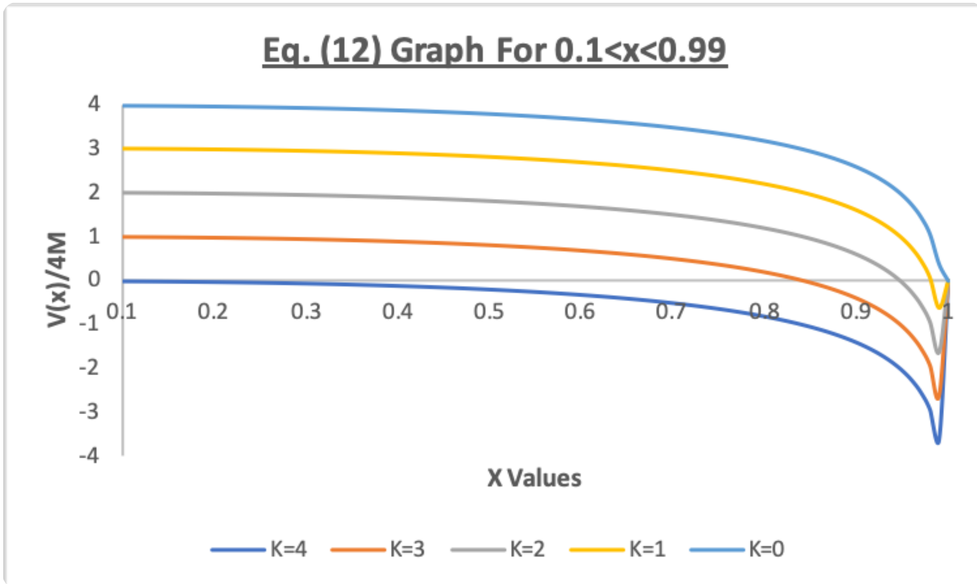
$$V(x) = 4mx + 4m \cdot \ln|x - 1| + K \longrightarrow (31)$$

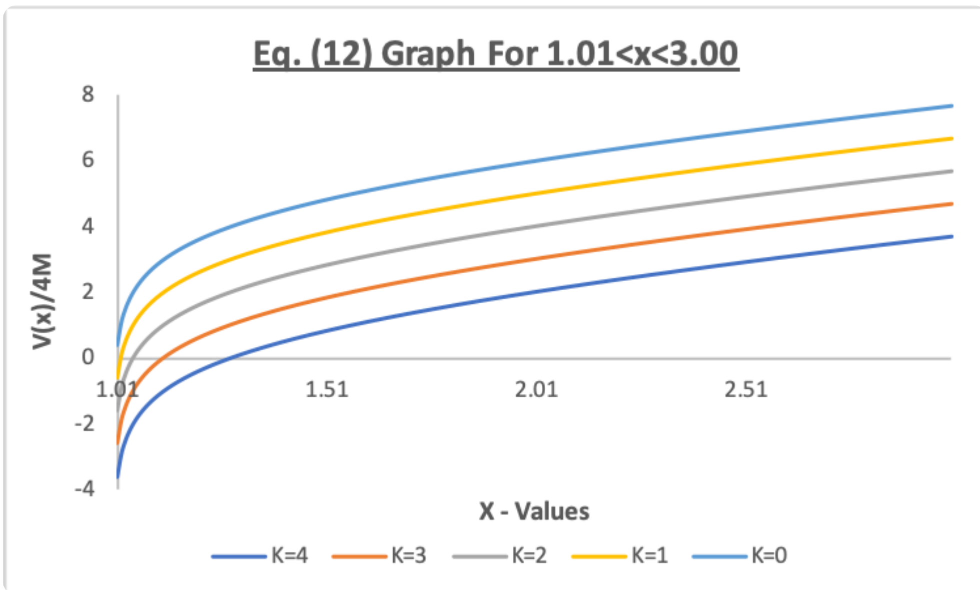
Problem 4) continued

dividing the entire term by 4M, (31) becomes:

$$\boxed{\frac{V(x)}{4M} = K + x + \ln|x-1|}$$

c.) Plotting equation (31) for $K=0,1,2,3,4$, the following plots are created





Problem 5

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \longrightarrow (32) \quad : \quad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

a.)

i.) $\mathcal{E} = \frac{1}{2}(0)^2 - \frac{M}{R} + \frac{\ell^2}{2R^2} - \frac{M\ell^2}{R^3} \quad \therefore \quad \boxed{\mathcal{E} = \text{negative}}$

ii.) $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right) - \cancel{\frac{M}{\infty}}^{0} + \cancel{\frac{\ell^2}{\infty}}^{0} - \cancel{\frac{M\ell^2}{\infty}}^{0} \quad \therefore \quad \boxed{\mathcal{E} = \text{positive}}$

iii.) $\mathcal{E} = \frac{1}{2}(0)^2 - \cancel{\frac{M}{\infty}}^{0} + \cancel{\frac{\ell^2}{\infty}}^{0} - \cancel{\frac{M\ell^2}{\infty}}^{0} \quad \therefore \quad \boxed{\mathcal{E} = 0}$

b.) Starting with (32)

$$2\mathcal{E} = \left(\frac{dr}{d\tau}\right)^2 + 2V_{eff}(r) \; : \; \ell = 0, \mathcal{E} = 0, \; : \; 0 = \left(\frac{dr}{d\tau}\right)^2 - \frac{2M}{r} \; : \; \frac{2M}{r} = \left(\frac{dr}{d\tau}\right)^2$$

$$\pm\sqrt{\frac{2M}{r}} = \frac{dr}{d\tau} \longrightarrow d\tau = \pm\sqrt{\frac{r}{2M}}\, dr \; : \; \tau = \int \frac{r^{1/2}}{\pm\sqrt{2M}}\, dr = \frac{1}{\pm\sqrt{2M}} \frac{2}{3} r^{3/2} + \tau_*$$

$$\boxed{\tau(r) = \tau_* \pm \frac{1}{\sqrt{2M}} \frac{2}{3} (r)^{3/2}} \longrightarrow (33)$$

c.) $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{dt}\frac{dt}{d\tau}\right)^2 + V_{eff}(r) \; : \; \mathcal{E} = \ell = 0 \; : \; 0 = \left(\frac{dr}{dt}\frac{dt}{d\tau}\right)^2 - \frac{2M}{r}$

$$\frac{2M}{r} = \left(\frac{dr}{dt}\frac{dt}{d\tau}\right)^2 \; : \; \frac{\sqrt{2M}}{\sqrt{r}} = \frac{dr}{dt}\frac{dt}{d\tau} \; : \; \frac{dr}{dt} = \frac{d\tau}{dt}\sqrt{\frac{2M}{r}} \longrightarrow (34)$$

rearranging $e = (1 - 2M/r)\frac{dt}{d\tau}$, $\frac{d\tau}{dt} = \frac{(1 - 2M/r)}{e}$

34 becomes : $\frac{dr}{dt} = \frac{(1-2M/r)}{e}\sqrt{\frac{2M}{r}} \; : \; \frac{dt}{dr} = \frac{e}{(1-2M/r)}\sqrt{\frac{r}{2M}}$

$$dt = \int \frac{e}{(1-2M/r)}\sqrt{\frac{r}{2M}}\, dr \longrightarrow (35)$$ using Wolframalpha, (35) becomes

$$\boxed{t(r) = \frac{\sqrt{2}\, e \sqrt{\frac{r}{M}} \left(\sqrt{r}(6M + r) - 6\sqrt{2} M^{3/2} \tanh^{-1}\left(\frac{\sqrt{r}}{\sqrt{2M}}\right)\right)}{3\sqrt{r}}} \longrightarrow (36)$$

Problem 6

a.) Time taken for observer @ rest from ∞: $r_f = 0$ $r_i = 2M$

$$\Delta\tau = \tau_f - \tau_i \; : \quad \tau_f = \tau(r_f) = \tau_*$$
$$\tau_i = \tau(r_i) = \tau_* - \frac{1}{\sqrt{2M}}\frac{2}{3}(2M)^{3/2}$$

$$\tau(r) = \tau_* \pm \frac{1}{\sqrt{2M}}\frac{2}{3}(r)^{3/2}$$

$$\Delta\tau = \tau_f - \tau_i = \tau_* - \tau_* + \frac{1}{(2M)^{1/2}}\frac{2}{3}(2M)^{3/2} = \frac{2}{3}(2M) = \frac{4}{3}M$$

$$\boxed{\Delta\tau = \frac{4}{3}M} \longrightarrow (37)$$

b.) Distance needed for $\Delta\tau$ to be the same? $r_i = R$, $r_f = 2M$

$$\Delta\tau = \tau_f - \tau_i \; : \quad \tau_f = \tau(r_f) = \tau_* - \frac{1}{\sqrt{2M}}\frac{2}{3}(2M)^{3/2}$$

$$\tau_i = \tau(r_i) = \tau_* - \frac{1}{\sqrt{2M}}\frac{2}{3}(R)^{3/2}$$

$$\Delta\tau = -\frac{1}{\sqrt{2M}}\frac{2}{3}(2M)^{3/2} + \frac{1}{\sqrt{2M}}\frac{2}{3}(R)^{3/2} \; : \quad \frac{4}{3}M = -\frac{4}{3}M + \frac{1}{\sqrt{2M}}\frac{2}{3}(R)^{3/2}$$

$$\frac{8}{3}M = \frac{1}{\sqrt{2M}}\frac{2}{3}(R)^{3/2} \; : \quad \frac{8}{3}M\cdot\sqrt{2M} = \frac{2}{3}(R)^{3/2} \; : \quad 4\cdot\sqrt{2}\,M^{3/2} = R^{(3/2)}$$

$$R = 4^{2/3}\,2^{1/3}M \qquad : \quad 2^{4/3}2^{1/3} = 2^{5/3}$$

$$\boxed{R = 2^{5/3}M} \longrightarrow (38)$$

c.) Take (38) and put into $\tau(r) = \frac{1}{\sqrt{2M}}\frac{2}{3}(r)^{3/2}$

$M = 2\cdot 10^{30}\,\text{kg}$

$\Delta\tau \to \Delta\tau\cdot c$

$M \to \frac{GM}{c^2}$

$$c\cdot\Delta\tau = \frac{4}{3}\frac{GM}{c^2} \; : \quad \Delta\tau = \frac{4}{3}\frac{(6.67\times10^{-11}\,\text{Nm}^2/\text{kg}^2)(2.0\times10^{30}\,\text{kg})}{(3.00\times10^{8}\,\text{m/s})^3} = 6.67\times10^{-6}\,\text{s}$$

$$\boxed{\Delta\tau = 6.67\,\mu\text{s}}$$