MAT 201
Larson/Edwards – Section 4.5
Integration by Substitution (Part 1)

In this section we will focus on some techniques for evaluating integrals of more complicated functions. For example, how would we evaluate the following indefinite integral $\int 2(1+2x)^2\,dx$?

What do we notice about the integrand $2(1+2x)^2$?

We are looking for a function who's derivative is $2(1+2x)^2$. One way to proceed to evaluate the integral $\int 2(1+2x)^2\,dx$ would be to use ***Pattern Recognition.***

Recall the Chain Rule: $\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$

The antiderivative is as follows: $\int F'(g(x))g'(x) = F(g(x)) + C.$

So the integral $\int 2(1+2x)^2\,dx$ can be evaluated by recognizing that $f(g(x)) = (1+2x)^2$, $g'(x) = 2$,where $(1 + 2x)$ is the inside function, and 2 is the derivative of the inside function. So by recognizing the pattern, $\int 2(1+2x)^2\,dx = \frac{1}{3}(1+2x)^3 + C$

Since pattern recognition requires a "mental substitution of variables" it is often easier to write out the steps by using a ***u*-*substitution***. This method is called ***change of variables*** or the ***substitution method***.

$$\int f(g(x))g'(x)dx = \int f(u)\, du. = F(u) + C$$

Ex: Use the substitution method to find: $\int 2(1 + 2x)^2 dx$

Let $u = 1 + 2x$, then $du = 2\, dx$. Substituting these values we get:

$\int 2(1 + 2x)^2 dx = \int u^2 \; du = \dfrac{1}{3}u^3 + C.$ By substituting $u = 1 + 2x$

back into our antiderivative we get: $\dfrac{1}{3}(1 + 2x)^3 + C$

One of the most common u-substitutions involves quantities in the integrand that are raised to a power. The ***General Power Rule for Integration*** is a very helpful tool.

<u>General Power Rule for Integration</u>: If g is a differential function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C,\; n \neq -1.$$

$$Equivalently,\ if\ \ u = g(x),\ then$$

$$\int u^n \; du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Note: When using the general power rule, you must confirm:

1. u^n (a factor u of the integrand raised to a power n) exists.
2. $\frac{du}{dx}$ is in fact the derivative of u and is a factor of the integrand.

Steps for Integrating by Substitution:

1. **Let *u* be a function of *x*.**
2. **Differentiate *u* with respect to *x*.**
3. **Solve for *x* and *dx* in terms of *u* and *du*.**
4. **Substitute the values found in step (2) into the original integral. Your entire integral should now be in terms of *u*.**
5. **After integrating, rewrite the antiderivative as a function of *x*. (Re-substitute your original values.)**
6. **Check your answer by differentiating. You should get back to where you started!**

Ex: Find each indefinite integral by pattern recognition:

a) $\int (3x^2 + 6)(x^3 + 6x)^2 \, dx$

b) $\int 2x\sqrt{x^2 - 2} \, dx$

Ex: Find each indefinite integral by pattern recognition:

a) $\int \underbrace{(3x^2+6)}_{g'(x)}\underbrace{(x^3+6x)^2}_{f(g(x))}\,dx = \dfrac{(x^3+6x)^3}{3} + C$

Check: $\dfrac{\cancel{3}(x^3+6x)^2}{\cancel{3}} \cdot (3x^2+6) + 0$

$(x^3+6x)^2(3x^2+6)$

b) $\int 2x\sqrt{x^2 - 2}\, dx$ $= \int \underbrace{2x}_{g'(x)} \underbrace{(x^2-2)^{1/2}}_{f(g(x))} dx$

$= \dfrac{(x^2-2)^{3/2}}{3/2} + C$

$= \boxed{\dfrac{2(x^2-2)^{3/2}}{3} + C}$

Check:

$\dfrac{\cancel{2}}{\cancel{3}} \cdot \dfrac{\cancel{3}}{\cancel{2}} (x^2-2)^{1/2} \cdot 2x$

c) $\int x^3\left(3x^4+1\right)^2\,dx$

d) $\int 2\left(3x^4+1\right)^2\,dx$

Ex: Integrate the following by substitution:

a) $\int \sqrt{1-2x}\,dx$

c) $\int \underbrace{x^3}_{\frac{1}{12}g'(x)} \underbrace{\left(3x^4+1\right)^2}_{f(g(x))} dx$

$$= \frac{1}{12} \cdot \frac{(3x^4+1)^3}{3} + C$$

$$= \boxed{\frac{(3x^4+1)^3}{36} + C}$$

Check: $\frac{\cancel{3}}{\cancel{36}}(3x^4+1)^2 \cdot \cancel{12}x^3$

d) $\int \underset{g'(x)}{2} \underbrace{\left(3x^4+1\right)^2}_{f(g(x))} dx$ No Pattern!

Expand.

$$\int 2\left(9x^8+6x^4+1\right)dx$$

$$\int 18x^8+12x^4+2 \; dx$$

$$= \frac{18x^9}{9}+\frac{12x^5}{5}+2x+C$$

$$= \boxed{2x^9+\frac{12}{5}x^5+2x+C}$$

Ex: Integrate the following by substitution:

a) $\int \sqrt{1-2x}\,dx$

$$= \int (1-2x)^{1/2}\,dx$$

$u = 1-2x$

$dx\left(\frac{du}{dx} = -2\right)$

$du = -2\,dx$

$-\frac{1}{2}du = dx$

$$= \int -\frac{1}{2}u^{1/2}\,du$$

$$= -\frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= -\frac{1}{3}u^{3/2} + C$$

$$= \boxed{-\frac{1}{3}(1-2x)^{3/2} + C}$$

Check: $\cancel{-}\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(1-2x)^{1/2}(\cancel{-2})$

$\sqrt{1-2x}$

b) $\int x\sqrt{x^2 + 4}\, dx$

c) $\int 5x\sqrt{x^2 - 1}\, dx$

b) $\int x\sqrt{x^2+4}\,dx$

$$\int x\,(x^2+4)^{1/2}\,dx$$

u

$\frac{1}{2}du$

$$u = x^2+4$$

$$\frac{du}{2} = \frac{2x}{2}\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} + C}$$

Check: $\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(x^2+4)^{1/2}\cdot\cancel{2}x$

c) $\int 5x\sqrt{x^2-1}\,dx$

$u = x^2 - 1$

$\frac{du}{2} = \frac{2x\,dx}{2}$

$\frac{1}{2}du = x\,dx$

$$\int 5x\,(\overbrace{x^2-1}^{u})^{1/2}\,dx$$

$\frac{1}{2}du$

$$\int \frac{5}{2}u^{1/2}\,du$$

$$= \frac{5}{2}\,\frac{u^{3/2}}{3/2} + C$$

$$= \frac{5}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(x^2-1)^{3/2} + C$$

$$= \boxed{\frac{5}{3}(x^2-1)^{3/2} + C}$$

Check: $\frac{5}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(x^2-1)^{1/2}\cdot\cancel{2}x$