Taylor Larrechea
Dr. Gustafson
MATH 366
CP Ch. 20.1

1) The first topic is how to solve linear systems of equations numerically. The second topic is how to solve eigen problems numerically.

2) Small pivots and roundoff error magnification.

5) The pivot a_{kk} (in step k) must be different from zero and should be large in absolute value to avoid roundoff magnification by the multiplication in the elimination.

6) **Example 1: Gauss Elimination. Partial Pivoting**

Solve the system

$$\begin{aligned} E_1 &: \quad 8x_2 + 2x_3 = -7 \\ E_2 &: \quad 3x_1 + 5x_2 + 2x_3 = 8 \\ E_3 &: \quad 6x_1 + 2x_2 + 8x_3 = 26 \end{aligned}$$

Solution. We must pivot since E_1 has no x_1-term. In column 1, equation E_3 has the largest coefficient. Hence we interchange E_1 and E_3

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 3x_1 + 5x_2 + 2x_3 &= 8 \\ 8x_2 + 2x_3 &= -7 \end{aligned}$$

Step 1. Elimination of x_1

It would suffice to show the augmented matrix and operate on it. We show both the equations and the augmented matrix. In the first step, the first equation is the pivot equation. Thus

$$\begin{aligned} \text{Pivot } 6 \rightarrow \quad & 6x_1 + 2x_2 + 8x_3 = 26 \\ \text{Eliminate} \rightarrow \quad & \boxed{3x_1} + 5x_2 + 2x_3 = 8 \\ & 8x_2 + 2x_3 = -7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 3 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

To eliminate x_1 from the other equations (here, from the second equation), do:
Subtract $(3/6) = (1/2)$ times the pivot equation from the second equation.

The result is,

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 \\ 4x_2 - 2x_3 &= -5 \\ 8x_2 + 2x_3 &= -7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

Step 2. Elimination of x_2

The largest coefficient in column 2 is 8. Hence we take the new third equation as the pivot equation, interchanging equations 2 and 3,

$$6x_1 + 2x_2 + 8x_3 = 26$$

Pivot 8 $\rightarrow$ $8x_2 + 2x_3 = -7$

Eliminate $\rightarrow$ $\boxed{4x_2} - 2x_3 = -5$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right]$$

To eliminate x_2 from the third equation, do:
Subtract ½ times the pivot equation from the third equation.

The resulting triangular system is shown below. This is the end of the forward elimination. Now comes the back substitution.

Back Substitution. Determination of x_3, x_2, x_1

The triangular system obtained in step 2 is

$$6x_1 + 2x_2 + 8x_3 = 26$$
$$8x_2 + 2x_3 = -7$$
$$-3x_3 = -3/2$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -\frac{3}{2} \end{array}\right]$$

From this system, taking the last equation, then the second equation, and finally the first equation, we compute the solution

$$x_3 = 1/2$$
$$x_2 = 1/8(-7 - 2x_3) = -1$$
$$x_1 = 1/6(26 - 2x_2 - 8x_3) = 4.$$

This agrees with the values given above, before the beginning of the example.

7) If $a_{kk} = 0$ in step k, we must pivot. If $|a_{kk}|$ is small, we should pivot because of roundoff error magnification that may seriously affect accuracy or even produce nonsensical results.

8)

$$0.0004x_1 + 1.402x_2 = 1.406$$
$$0.4003x_1 - 1.502x_2 = 2.501$$

$x_1 = 10, \ x_2 = 1$

9) $|a_{11}| = 0.0004$, $|a_{12}| = 1.402$, $m = a_{21}/a_{11} = 0.4003/0.0004 = 1001$, $x_1 = 12.5$, $x_2 = 0.9993$
The reason for the inaccuracy in the computation of x_1 is because $|a_{11}|$ is small compared to $|a_{12}|$. A small roundoff error in x_2 will lead to a large error in x_1.

10) $m = a_{21}/a_{11} = 0.0004/0.4003 = 0.0009993$, $x_1 = 10$, $x_2 = 1$
This success occurs because $|a_{21}|$ is not very small compared to $|a_{22}|$, so that a small roundoff error in x_2 would not lead to a large error in x_1.

11) Amount of storage, Amount of time (≡ number of operations), Effect of roundoff error

12) For instance, if an operation takes 10^{-9} sec, then the times needed are:

Algorithm	$n = 1000$	$n = 10000$
Elimination	0.7 sec	11 min
Back substitution	0.001 sec	0.1 sec