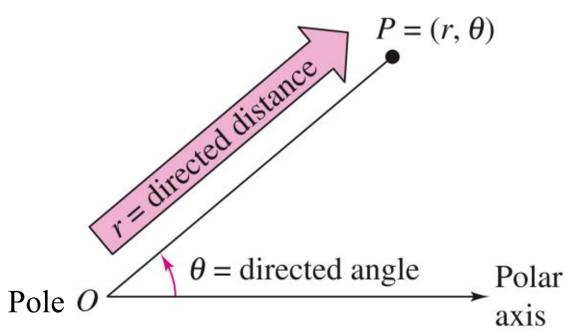
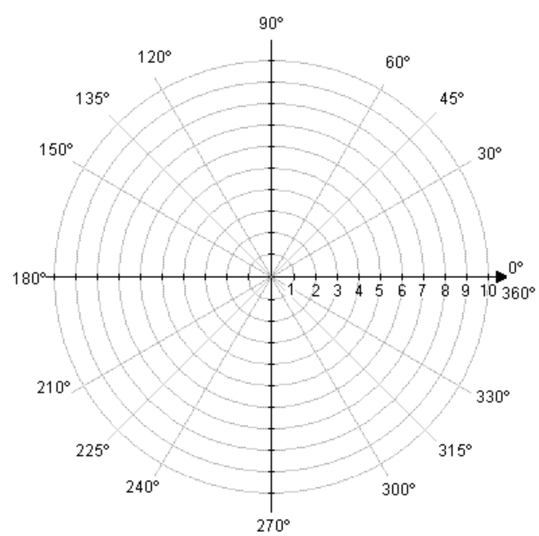
MAT 202
Larson – Section 10.4
Polar Coordinates & Polar Graphs

Up to this point we have been representing graphs of equations as collections of points (x, y) in the Cartesian plane. In this section we will introduce another coordinate system called the ***polar coordinate system***.

Plotting Points in the Polar Coordinate System: To form the polar coordinate system in the plane, fix a point O, called the ***pole*** (or ***origin***), and construct from O an initial ray called the ***polar axis***. Then each point P in the plane can be assigned polar coordinates (r, θ) as follows.

1. r = directed distance from O to P
2. θ = directed angle, counterclockwise from the polar axis to segment $\overline{OP}$





Since polar coordinates represent a directed distance and an angle of rotation, the polar coordinate plane is circular.

Ex: Plot the following polar coordinates in the polar coordinate plane: a) $\left(8, \frac{5\pi}{4}\right)$ b) $\left(-5, \frac{\pi}{6}\right)$

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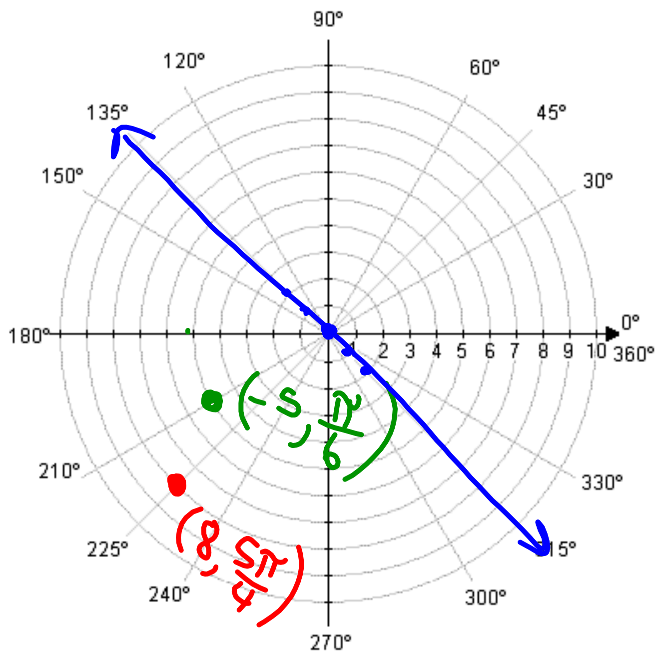


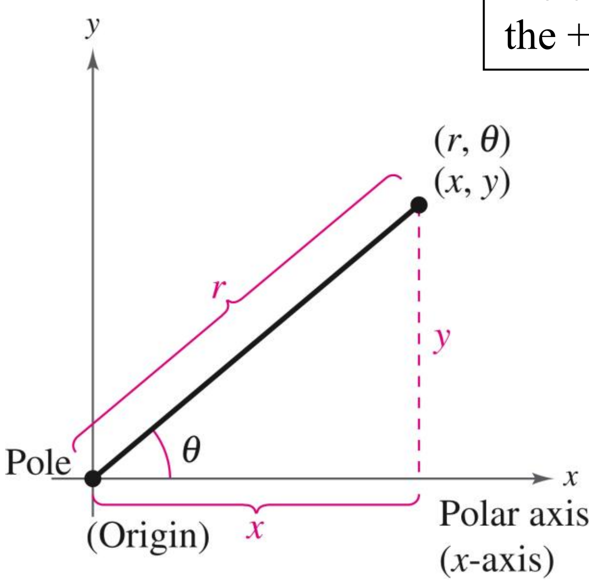
In rectangular coordinates, each point (x, y) has a unique representation. This is not true for polar coordinates. There are an infinite number of representations for one point in the polar coordinate system. For example, the point $(2, \pi)$ can be represented by $(2, 3\pi)$ or $(2, -\pi)$ or $(-2, 0)$, etc.

Different Representations for (r, θ):

$$(r, \theta) = (r, \theta \pm 2n\pi) \quad \text{or} \quad (r, \theta) = (-r, \theta \pm (2n + 1)\pi)$$

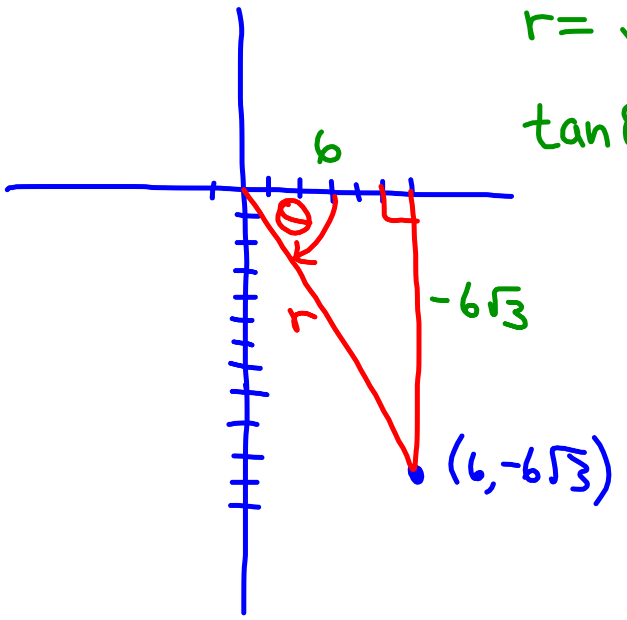
Coordinate Conversion: The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

Polar-to-Rectangular: $x = r\cos\theta$ $y = r\sin\theta$	***Rectangular-to-Polar***: $\tan\theta = \frac{y}{x}$ *or* $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $r^2 = x^2 + y^2$ *or* $r = \pm\sqrt{x^2 + y^2}$ Note: Pay attention to what quadrant the coordinate lies to help determine the + or – nature of *r* and the angle θ.



Ex: Convert the rectangular coordinate $\left(6, -6\sqrt{3}\right)$ to its polar coordinate.

Ex: Convert the rectangular coordinate $\left(6, -6\sqrt{3}\right)$ to its polar coordinate. $\rightarrow \boxed{\left(12, -\frac{\pi}{3}\right)}$



$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{36 + 108} = 12$

$\theta = -\frac{\pi}{3}$

Ex: Convert the polar coordinate $\left(-2, \frac{7\pi}{6}\right)$ to its rectangular coordinate.

__Equation Conversion__: We can convert polar equations to rectangular form and vice-versa by using the same tools we use for coordinate conversion.

Ex: For the following polar equations (i) describe the graph, and (ii) find the corresponding rectangular equation.

a) $r = 3$

b) $\theta = -\frac{\pi}{4}$

Ex: Convert the polar coordinate $\left(-2, \frac{7\pi}{6}\right)$ to its rectangular coordinate.

$x = r\cos\theta = -2\cos\frac{7\pi}{6} = -2 \cdot -\frac{\sqrt{3}}{2} = \sqrt{3}$

$y = r\sin\theta = -2\sin\frac{7\pi}{6} = -2 \cdot -\frac{1}{2} = 1$

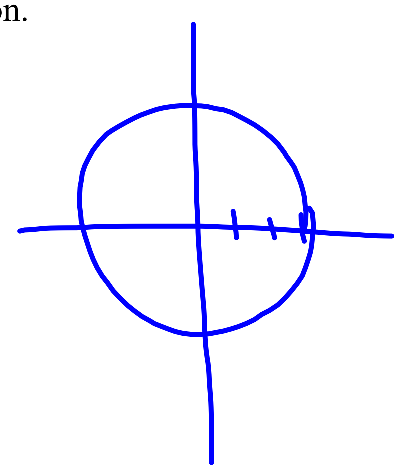
$\boxed{(\sqrt{3}, 1)}$

Ex: For the following polar equations (i) describe the graph, and (ii) find the corresponding rectangular equation.

a) $r = 3$ $\longrightarrow$

r	θ
3	0
3	π
3	$\pi/2$
3	$3\pi/2$
3	2π

$\boxed{\text{Circle}}$



$$r = \pm\sqrt{x^2+y^2}$$

$$(3)^2 = \left(\pm\sqrt{x^2+y^2}\right)^2$$

$$9 = x^2+y^2$$

b) $\theta = -\dfrac{\pi}{4}$

r	θ
0	$-\frac{\pi}{4}$
1	$-\frac{\pi}{4}$
2	$-\frac{\pi}{4}$
-1	$-\frac{\pi}{4}$
-2	$-\frac{\pi}{4}$

$\boxed{\text{Line}}$

$$\theta = -\frac{\pi}{4}$$

$$\tan\theta = \frac{y}{x}$$

$$\tan -\frac{\pi}{4} = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$$\boxed{y = -x}$$

***Tools for Graphing Polar Equations*:**

1. **Point plotting:** Substitute "familiar" angles in for θ, then find the corresponding r-value until you've recognized a pattern. Note: You may change your calculator settings to polar equations under "MODE" so that you can check your graphs.

2. **Symmetry:** The graph of a polar equation is symmetric with respect to the following when the given substitution yields an equivalent equation.

    a) The line $\theta = \frac{\pi}{2}$: Replace (r, θ) by (r, π – θ) or (-r, -θ)

    b) The polar axis: Replace (r, θ) by (r, -θ) or (-r, π – θ)

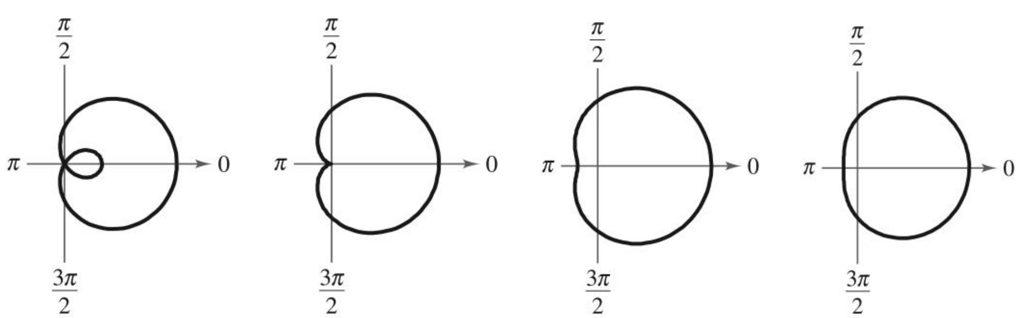
    c) The pole: Replace (r, θ) by (r, π + θ) or (-r, θ)

Note: A polar graph can exhibit symmetry even when the test fails to indicate symmetry. If all tests fail, then the results for symmetry are inconclusive. In general, the graph r = *f*(sin θ) is symmetric with respect to the line $\theta = \frac{\pi}{2}$ and the graph r = *f*(cos θ) is symmetric with respect to the polar axis.

3. **Zeros:** A helpful sketching aid is knowing the θ-values for which r = 0. Substitute r = 0 into the polar equation, then solve for θ.

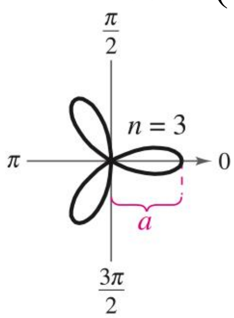
4. **Special Polar Graphs:**

    a) Limaçons: r = a ± b cos θ, r = a ± b sin θ (a > 0, b > 0)

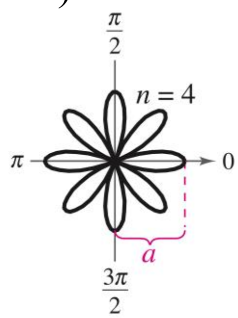


$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Limaçon with inner loop	Cardioid (heart shaped)	Dimpled Limaçon	Convex Limaçon

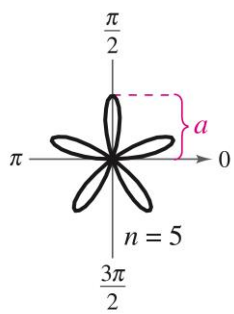
b) Rose Curves: n petals when n is odd, 2n petals when n is even ($n \geq 2$)



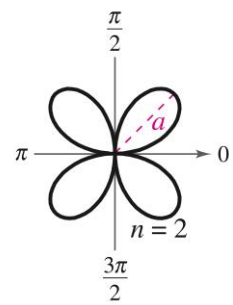
r = a cos nθ



r = a cos nθ

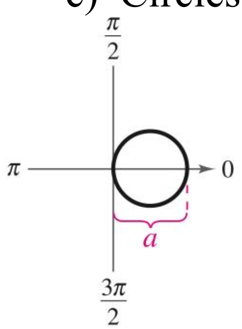


r = a sin nθ

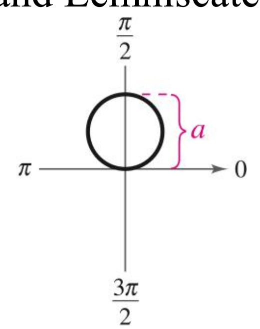


r = a sin nθ

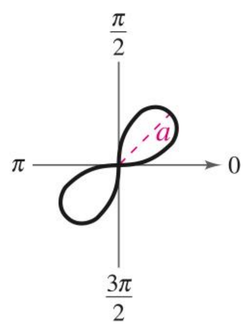
c) Circles and Lemniscates



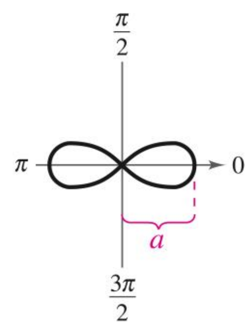
r = a cos θ



r = a sin θ



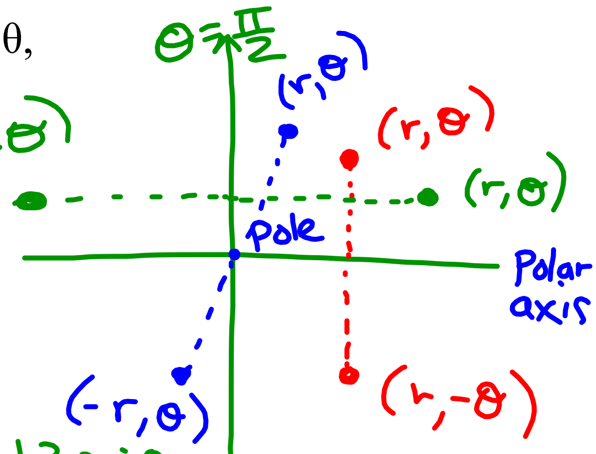
$r^2 = a^2 \sin 2\theta$



$r^2 = a^2 \cos 2\theta$

Ex: For the polar equation r = 3 + 3 sin θ,
Sketch a graph by hand, then check your answer with a graphing calculator.

Ex: For the polar equation $r = 3 + 3\sin\theta$,
a) Test for symmetry



Polar Axis: $\boxed{\text{Inconclusive}}$

$(r) = 3 + 3\sin(-\theta)$

$r = 3 - 3\sin\theta \neq r = 3 + 3\sin\theta$

Pole: $\boxed{\text{Inconclusive}}$

$(-r) = 3 + 3\sin(\theta)$

$-r = 3 + 3\sin\theta$

$r = -3 - 3\sin\theta \neq r = 3 + 3\sin\theta$

$\theta = \frac{\pi}{2}$: $\boxed{\text{Inconclusive}}$

$(-r) = 3 + 3\sin(-\theta)$

$-r = 3 - 3\sin\theta$

$r = -3 + 3\sin\theta \neq r = 3 + 3\sin\theta$

b) Find the zeros (if any)

$r = 0$

$0 = 3 + 3\sin\theta$

$-3 = 3\sin\theta$

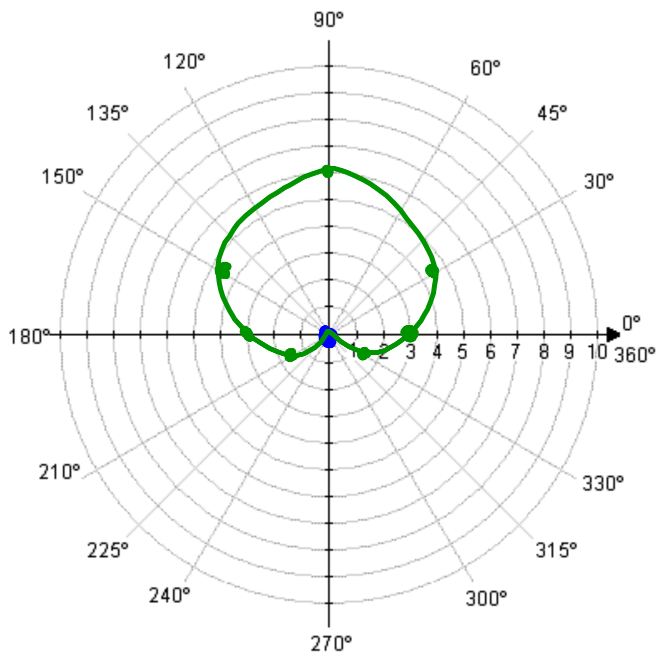
$-1 = \sin\theta$

$\frac{3\pi}{2} = \theta$

$\boxed{\left(0, \frac{3\pi}{2}\right)}$

c) Sketch the graph Point Plotting $r = 3 + 3\sin\theta$

r	0	4.5	1.5	3	6	4.5	3
θ	$\frac{3\pi}{2}$	$\pi/6$	$-\frac{\pi}{6}$	0	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π



***<u>Slope In Polar Form</u>*:** If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}, \quad \frac{dx}{d\theta} \neq 0 \; at \; (r, \theta)$$

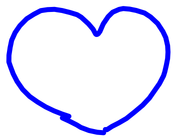
***<u>Horizontal & Vertical Tangents</u>*:**

1. Solutions of $\frac{dy}{d\theta} = 0$ yield **horizontal tangents**, provided that $\frac{dx}{d\theta} \neq 0$.
2. Solutions of $\frac{dx}{d\theta} = 0$ yield **vertical tangents**, provided that $\frac{dy}{d\theta} \neq 0$.

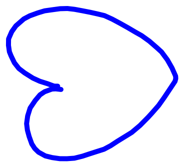
If $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are simultaneously 0, then no conclusion can be drawn about tangent lines.

How would the graph in the previous example change if...

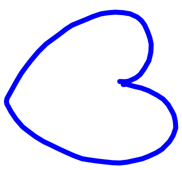
a) The equation were $r = 3 - 3\sin\theta$



b) The equation were $r = 3 + 3\cos\theta$



c) The equation were $r = 3 - 3\cos\theta$



$r = f(\theta)$

$x = f(\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$

$y = f(\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$