<u>Trigonometric Substitution</u>: The goal for trigonometric substitution is to eliminate the radical in the integrand and rewrite it in the form of a trigonometric expression. Let a > 0,

1. For integrals involving $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$. Then

$\sqrt{a^2 - u^2} = a\cos\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ **Why?**

$u = a\sin\theta$

$$\sqrt{a^2 - (a\sin\theta)^2} = \sqrt{a^2 - a^2\sin^2\theta}$$
$$= \sqrt{a^2(1 - \sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta}$$
$$= a\cos\theta$$

2. For integrals involving $\sqrt{a^2+u^2}$, let $u = a\tan\theta$. Then $\sqrt{a^2+u^2} = a\sec\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. **Why?**

$u = a\tan\theta$

$$\sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$$= a\sec\theta$$

3. For integrals involving $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$. Then

$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta, \text{ for } u > a, \text{ where } 0 \le \theta \le \dfrac{\pi}{2} \\ -a\tan\theta, \text{ for } u < -a, \text{ where } \dfrac{\pi}{2} \le \theta \le \pi \end{cases}$$

. **Why?**

$u = a\sec\theta$

$$\sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{a^2\tan^2\theta}$$

$$= a\tan\theta$$

Notice that these are the same intervals over which arcsine, arctangent, and arcsecant are defined. This ensures one-to-one.

Ex: Integrate the following:

a) $\int \frac{-12}{x^2\sqrt{4 - x^2}}\,dx$

b) $\int \frac{\sqrt{x^2 - 9}}{x}\,dx$

c) $\int \frac{x^3}{\sqrt{4 + x^2}}\,dx$

Ex: Integrate the following:

a) $\displaystyle\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx$

$(2)^2$

$a=2$

Let $x = a\sin\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$$\int \frac{-12 \cdot 2\cos\theta}{(2\sin\theta)^2\sqrt{4-(2\sin\theta)^2}}\,d\theta$$

$$\sqrt{4-4\sin^2\theta}$$

$$\sqrt{4(1-\sin^2\theta)}$$

$$\sqrt{4\cdot\cos^2\theta}$$

$$\int \frac{\overset{-3}{\cancel{-24}}\cos\cancel{\theta}}{\cancel{4}\sin^2\theta\,(\cancel{2}\cancel{\cos\theta})}\,d\theta$$

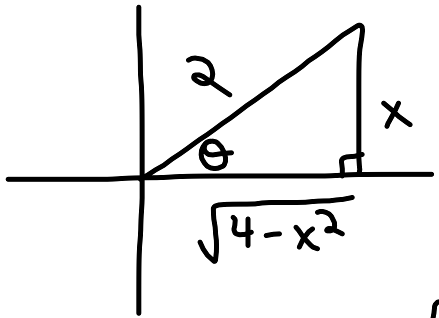
$$\int -3\csc^2\theta\, d\theta$$

$$-3\int \csc^2\theta\, d\theta$$

$$+3(\cot\theta) + C$$

$$= \boxed{\frac{3\sqrt{4-x^2}}{x} + C}$$

$\sin\theta = \dfrac{x}{2}$



$\cot\theta = \dfrac{\sqrt{4-x^2}}{x}$

b) $\displaystyle\int \frac{\sqrt{x^2-9}}{x}\,dx$

$x = a\sec\theta$

$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sqrt{(3\sec\theta)^2-9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{9\sec^2\theta-9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{9(\sec^2\theta-1)}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\tan\theta\,\cancel{3\sec\theta}\tan\theta}{\cancel{3\sec\theta}}\,d\theta$$

$$= \int 3\tan^2\theta\,d\theta$$

$$= \int 3(1+\sec^2\theta)\,d\theta$$

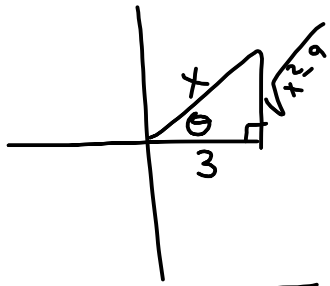
$$= \int 3\,d\theta + 3\int \sec^2\theta\,d\theta$$

$$= 3(\theta) + 3(\tan\theta) + C$$

$$= \boxed{3\sec^{-1}\left(\frac{x}{3}\right) + \sqrt{x^2-9} + C}$$

$\sec\theta = \frac{x}{3}$

$\theta = \sec^{-1}\left(\frac{x}{3}\right)$



$\tan\theta = \frac{\sqrt{x^2-9}}{3}$

c) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

$x = a\tan\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$\int \frac{(2\tan\theta)^3\cdot 2\sec^2\theta}{\sqrt{4+(2\tan\theta)^2}}\,d\theta$$

$$\int \frac{16\tan^3\theta\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,d\theta$$

$$\int \frac{\overset{8}{\cancel{16}}\tan^3\theta\,\sec^{\cancel{2}\,1}\theta}{\underset{1}{\cancel{2}}\,\underset{1}{\cancel{\sec\theta}}}\,d\theta$$

$$\int 8\tan^3\theta\sec\theta\,d\theta$$

$u = \sec\theta$

$$\int 8\tan^2\theta\,\sec\theta\tan\theta\,d\theta$$

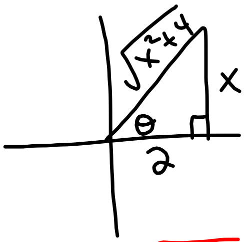
$du = \sec\theta\tan\theta\,d\theta$

$$\int 8(\sec^2\theta-1)\sec\theta\tan\theta\,d\theta$$

$$8\int u^2-1\,du$$

$$8\left[\frac{u^3}{3}-u\right]+C$$

$\tan\theta = \frac{x}{2}$



$$=\frac{8}{3}\sec^3\theta - 8\sec\theta + C$$

$$=\frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\frac{\sqrt{x^2+4}}{2}+C$$

$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

$$\boxed{\frac{(x^2+4)\sqrt{x^2+4}}{3} - 4\sqrt{x^2+4}+C}$$

Ex: Evaluate the following definite integral: $\displaystyle\int_0^1 \frac{6x^3}{\sqrt{16 + x^2}}\,dx$

Ex: Evaluate the following definite integral: $\int_0^1 \frac{6x^3}{\sqrt{16+x^2}}\,dx$

$$-62\sqrt{17}+256 \approx 0.367$$

$$6\int_0^1 \frac{x^3}{\sqrt{16+x^2}}\,dx$$