MAT 201
Larson/Edwards – Section 7.2 (Part 2)
Volume: The Disk Method

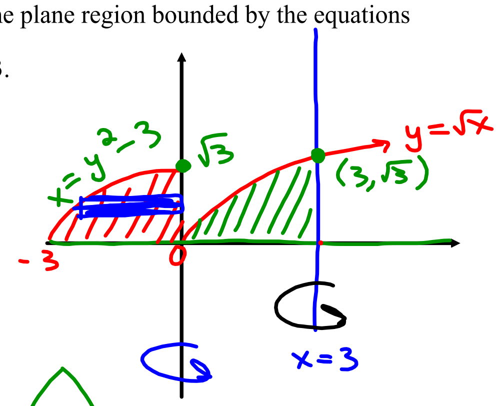
We will continue our investigation in solving volume problems. We will begin with an example in which we are revolving the area about a line that is not a coordinate axis (x- or y-axis):

Ex: Find the solid generated by revolving the plane region bounded by the equations $y = \sqrt{x}$, $y = 0$, $x = 3$, about the line $x = 3$.

Up to this point we have used the disk and washer method for finding volumes of solids of revolution. We can extend this method to solids of any shape as long as we know how to find the area of the base of the cross section that we are using (since volume of a solid with identical bases is equal to the area of the base times the height).

Ex: Find the solid generated by revolving the plane region bounded by the equations

$y = \sqrt{x}$, $y = 0$, $x = 3$, about the line $x = 3$.

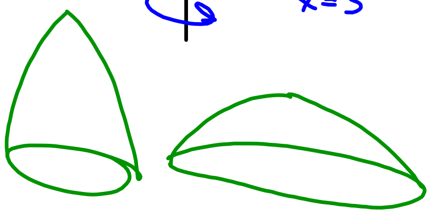


Shift $y=\sqrt{x}$ left 3

$$(y)^2 = \left(\sqrt{x+3}\right)^2$$

$$x = y^2 - 3$$

$$V = \pi \int_c^d [R(y)]^2 \, dy$$



$$V = \pi \int_0^{\sqrt{3}} (y^2-3)^2 \, dy$$

$$= \pi \int_0^{\sqrt{3}} (y^4 - 6y^2 + 9) \, dy$$

$$= \pi \left[\frac{y^5}{5} - 2y^3 + 9y \Big|_0^{\sqrt{3}} \right]$$

$$= \pi \left[\left(\frac{9\sqrt{3}}{5} - \frac{5 \cdot 6\sqrt{3}}{5} + \frac{5 \cdot 9\sqrt{3}}{5} \right) - (0) \right]$$

$$= \pi \left(\frac{24\sqrt{3}}{5} \right)$$

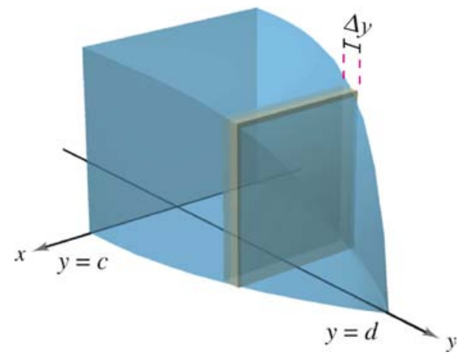
$$= \frac{24\pi\sqrt{3}}{5} \text{ units}^3$$

$$y = \sqrt{-x+3}$$

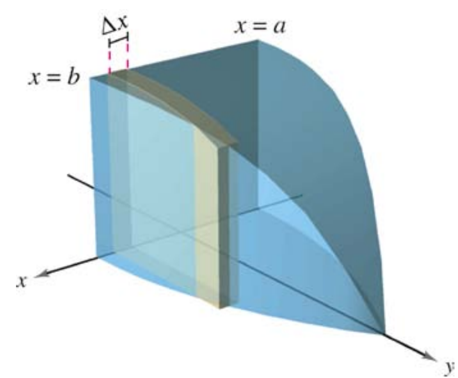
$$y^2 = -x + 3$$

$$x = 3 - y^2$$

For instance, consider the volume of a solid in which the base is the region bounded by $f(x) = x^2$ and y = 0. The cross sections that are perpendicular to the *y*-axis are squares. We will then have the solid given below:



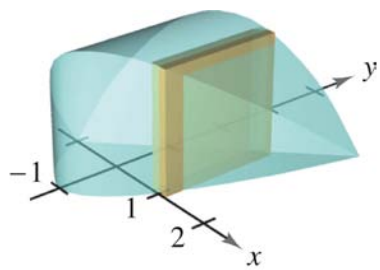
The volume of this shape would be more difficult to find if we used a cross section that is perpendicular to the *x*-axis since we have the downward curvature at the tip of the solid:



<u>Volumes of Solids With Known Cross Sections</u>:

1. For cross sections of area *A(x)* taken perpendicular to the *x*-axis, Volume = $\int_a^b A(x)\,dx$.
2. For cross sections of area A(y) taken perpendicular to the y-axis, Volume = $\int_c^d A(y)\,dy$.

Ex: Find the volume of the solid whose base is bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with cross sections that are squares taken perpendicular to the x-axis.



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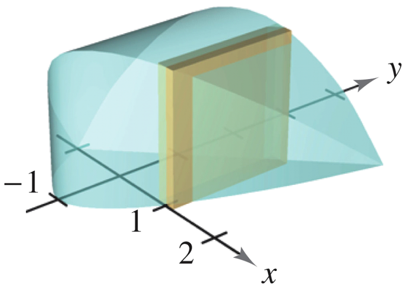
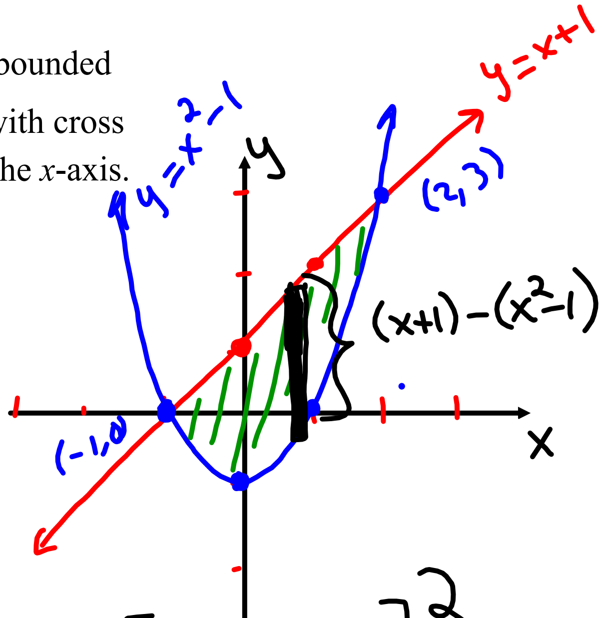




$$V = \int_a^b A(x)\,dx$$

$a=-1$, $b=2$

$$A(x) = \left[(x+1)-(x^2-1)\right]^2$$

$$= \left(-x^2+x+2\right)^2$$

$$(-x^2+x+2)(-x^2+x+2)$$

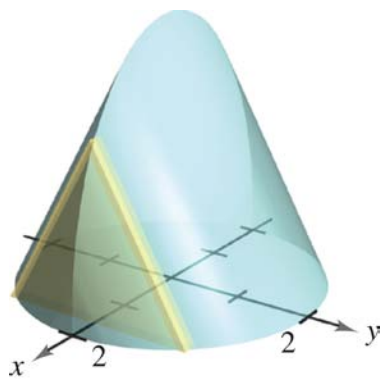
$$= x^4 - x^3 - 2x^2$$
$$- x^3 + x^2 + 2x$$
$$- 2x^2 + 2x + 4$$
$$= x^4 - 2x^3 - 3x^2 + 4x + 4$$

$$V = \int_{-1}^{2} \left(x^4 - 2x^3 - 3x^2 + 4x + 4\right) dx$$

$$= \frac{x^5}{5} - \frac{x^4}{2} - x^3 + 2x^2 + 4x \Big|_{-1}^{2}$$

$$\approx \boxed{8.1 \text{ units}^3}$$

Ex: Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$, with cross sections that are equilateral triangles taken perpendicular to the x-axis.



Ex: Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$, with cross sections that are equilateral triangles taken perpendicular to the *x*-axis.

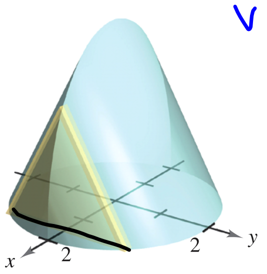
$\sqrt{y^2} = \pm\sqrt{4-x^2}$

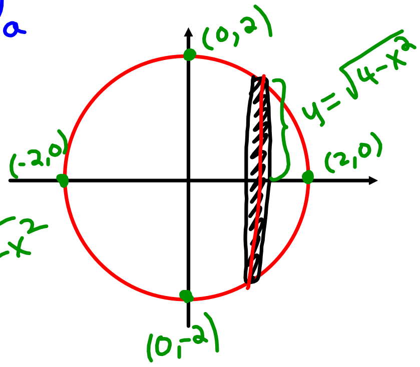
$y = \pm\sqrt{4-x^2}$

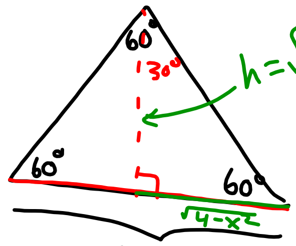
$a=-2$

$b=2$

$V = \int_a^b A(x)\,dx$







$h=\sqrt{3}\cdot\sqrt{4-x^2}$

$A = \frac{1}{2}b\cdot h$

Base $= 2\sqrt{4-x^2}$

$A(x) = \frac{1}{\cancel{2}}\cancel{2}\sqrt{4-x^2}\cdot\sqrt{3}\cdot\sqrt{4-x^2}$

$A(x) = \sqrt{3}\left(4-x^2\right)$

$$\sqrt{3}\int_{-2}^{2} 4-x^2\,dx$$

$$\sqrt{3}\left[4x-\frac{x^3}{3}\Big|_{-2}^{2}\right]$$

$$\sqrt{3}\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]$$

$$\sqrt{3}\left[8-\frac{8}{3}+8-\frac{8}{3}\right]$$

$$\sqrt{3}\left[\frac{32}{3}\right] = \frac{32\sqrt{3}}{3}\text{ units}^3$$