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MATH 360
CP Ch. 10.4

Equations (1) and (1')

Let R be a closed bounded region in the xy-plane whose boundary C consists of infinitely many smooth curves. Let $F_1(x,y)$ and $F_2(x,y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ everywhere in some domain containing R. Then

$$(1) \quad \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint (F_1 dx + F_2 dy)$$

Setting $\vec{F} = [F_1, F_2] = F_1\hat{i} + F_2\hat{j}$ and using (1), we obtain (1) in vectorial form,

$$(1') \quad \iint_R (\text{curl} F) \cdot k \; dx\,dy = \oint_C \vec{F} \cdot d\vec{r}$$

Example 1 Verification of Green's Theorem in the Plane

Green's theorem in the plane will be quite important in our further work. Before proving it, let us get used to it by verifying it for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$, and the circle $x^2 + y^2 = 1$.

Solution. In (1) on the left we get

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R [(2y+2) - (2y-7)]\,dx\,dy = 9\iint_R dx\,dy = 9\pi$$

Since the circular disk R has area π.

We now show that the line integral in (1) on the right gives the same value, 9π. We must orient C counterclockwise, say, $\hat{r}(t) = [\cos(t), \sin(t)]$. Then $r'(t) = [-\sin(t), \cos(t)]$, and on C,

$$F_1 = y^2 - 7y = \sin^2(t) - 7\sin(t), \qquad F_2 = 2xy + 2x = 2\cos(t)\sin(t) + 2\cos(t)$$

Hence the line integral in (1) becomes, verifying Green's theorem.

$$\oint_C (F_1 x' + F_2 y')\,dt = \int_0^{2\pi} [(\sin^2(t) - 7\sin(t))(-\sin(t)) + 2(\cos(t)\sin(t) + \cos(t))(\cos(t))]\,dt$$

$$= \int_0^{2\pi} (-\sin^3(t) + 7\sin^2(t) + 2\cos^2(t)\sin(t) + 2\cos^2(t))\,dt$$

$$= 0 + 7\pi - 0 + 2\pi = 9\pi$$