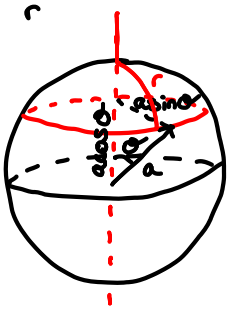
Ch. 2 Geometry as Physics

(1-25-19)

	Euclidean	2-Sphere
$\sum_{\text{vertices}} \left(\text{Interior Angles}\right) =$	π	$\pi + A/a^2 = \frac{3\pi}{2} = 270^\circ$
$\frac{C}{r} =$	2π	

a is radius of sphere

$$A = \tfrac{1}{8} A_{sph} = \frac{4\pi a^2}{8} = \frac{\pi}{2} a^2$$



$$\frac{C}{r} = \frac{2\pi a \sin\theta}{a\theta} = \frac{2\pi \sin\theta}{\theta} = \frac{2\pi \sin(r/a)}{r/a}$$

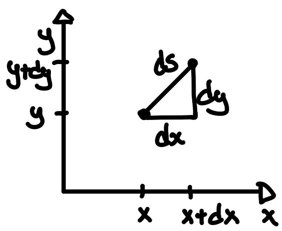
$: \; r = a\theta$

$$\lim_{r/a \ll 1} \frac{C}{r} \to 2\pi$$

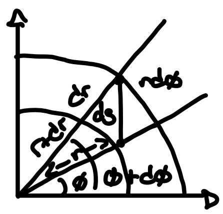
General Description of Gravity....

- Specify distance between nearby points (ds^2) ← Line Element which defines geometry
- Use differential and integral Calculus to get finite lengths

Euclidean Geometry of a Plane



$$ds^2 = dx^2 + dy^2$$

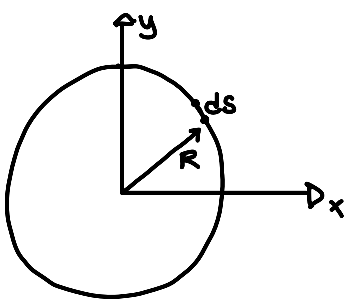


$$ds^2 = dr^2 + r^2 d\phi^2$$

$x = r\cos\phi$
$y = r\sin\phi$

(1-28-18)

Calculate the circumference of a circle.



$$C = \oint ds = \int \sqrt{dx^2 + dy^2}\,\Big|_{x^2+y^2=R^2} = 2\int_{-R}^{R} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = \sqrt{R^2 - x^2} \qquad \frac{dy}{dx} = \frac{1}{2}(R^2 - x^2)^{-1/2} \cdot -2x = \frac{-x}{\sqrt{R^2 - x^2}}$$

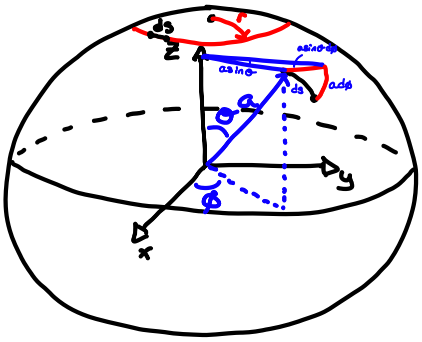
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}$$

$$\therefore\; C = 2\int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}}\,dx = 2\int_{-R}^{R} \frac{dx}{\sqrt{1 - x^2/R^2}} = 2\pi R$$

using polar coordinates.....

$$C = \oint ds = \int \sqrt{dr^2 + r^2 d\phi^2}\,\Big|_{\substack{r=R \\ dr=0}} = \int_0^{2\pi} R\,d\phi = R\int_0^{2\pi} d\phi = 2\pi R$$

Non-Euclidean Geometry of A Sphere



$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta\, d\phi^2 = a^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

Calculate $\frac{C}{r}$ of a circle on 2-sphere using ds^2

- Choose NP as center of circle $\therefore$ $\theta = \text{constant}$, $d\theta = 0$

  $\therefore ds = a\sin\theta\, d\phi$

$$C = \oint ds = \int_0^{2\pi} a\sin\theta\, d\phi = a\sin\theta \int_0^{2\pi} d\phi = 2\pi a \sin\theta$$

For Radius

- Choose Line of constant ϕ

  $ds = a\,d\theta$

$$r = \int_{NP}^{\text{circle}} ds = \int_0^{\theta'} a\,d\theta = a\int_0^{\theta'} d\theta = a\theta'$$

$$\frac{C}{r} = \frac{2\pi a \sin\theta}{a\theta} = \frac{2\pi \sin\theta}{\theta} = \frac{2\pi \sin(r/a)}{r/a}$$

In GR, usually confronted w/ Line Element : Have to figure out the properties of the geometry it represents.

Consider,

$$ds^2 = a^2(d\theta^2 + F^2(\theta)\, d\phi^2) \quad \text{if } F(\theta) = \sin\theta, \text{ then geometry of a 2-sphere}$$

if $\phi \to \phi + \text{constant}$, $d\phi \to d\phi$ $\therefore$ Line element is the same for all ϕ $\therefore$ Axisymmetric

- For circumference $C(\theta)$ of a circle .....

  $\theta = \text{const.}$ $\therefore$ $d\theta = 0$

  $ds = a\,F(\theta)\, d\phi$

  $C(\theta) = \oint ds = \int_0^{2\pi} a F(\theta)\, d\phi = 2\pi a F(\theta)$

- To calculate distance from pole-to-pole is ....

  $\phi = \text{const}$, $d\phi = 0$

  $ds = a\,d\theta$

$$d_{prop} \int_{NP}^{SP} ds = \int_0^{\pi} a\,d\phi = a\pi$$