Tues: Exam 1

Covers Ch 4, 6, 7.3, 8.1

HW 1→8

2012, 2016 Exam 1.

Bring ½-Single Letter sheet Single Side

Given: Front / back covers of text.

Electric Stress Tensor

We have been able to prove that:

The net force on the material within a region R can be computed from the electric fields at the surface of the region using:

$$\vec{F}_{mech} = \left(\oint_S \vec{T}_x \cdot d\vec{a}\right)\hat{x} + \left(\oint_S \vec{T}_y \cdot d\vec{a}\right)\hat{y} + \left(\oint_S \vec{T}_z \cdot d\vec{a}\right)\hat{z}$$

where the electric stress vectors are

$$\vec{T}_x = \epsilon_0\left(E_xE_x - \tfrac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{x} + \epsilon_0 E_yE_x\,\hat{y} + \epsilon_0 E_zE_x\,\hat{z}$$

$$\hat{T}_y = \epsilon_0 E_xE_y\,\hat{x} + \epsilon_0\left(E_yE_y - \tfrac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{y} + \epsilon_0 E_zE_y\,\hat{z}$$

$$\hat{T}_z = \epsilon_0 E_xE_z\,\hat{x} + \epsilon_0 E_yE_z\,\hat{y} + \epsilon_0\left(E_zE_z - \tfrac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{z}$$

and $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$

We can arrange these in a matrix or tensor: (electric stress tensor)

$$\overleftrightarrow{T} = \epsilon_0 \begin{pmatrix} (E_x E_x - \frac{1}{2}\vec{E}\cdot\vec{E}) & E_x E_y & E_x E_z \\ E_y E_x & (E_y E_y - \frac{1}{2}\vec{E}\cdot\vec{E}) & E_y E_z \\ E_z E_x & E_z E_y & E_z E_z - \frac{1}{2}\vec{E}\cdot\vec{E}) \end{pmatrix}$$

Note for example that if $d\vec{a} = da_x \hat{x} \rightsquigarrow \begin{pmatrix} da_x \\ 0 \\ 0 \end{pmatrix}$

then

$$\overleftrightarrow{T} \cdot d\vec{a} = \epsilon_0 \begin{pmatrix} (E_x E_x - \frac{1}{2}\vec{E}\cdot\vec{E})\, da_x \\ (E_y E_x)\, da_x \\ E_z E_x)\, da_x \end{pmatrix}$$

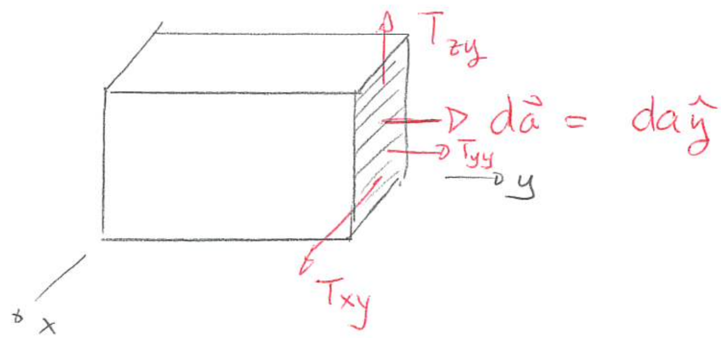
and

$$\oint_S \overleftrightarrow{T} \cdot d\vec{a} = \begin{pmatrix} \epsilon_0 \oint \vec{T}_x \cdot d\vec{a} \\ \epsilon_0 \oint \vec{T}_y \cdot d\vec{a} \\ \epsilon_0 \oint \overleftrightarrow{T}_z \cdot d\vec{a} \end{pmatrix}$$

In general the Maxwell electric stress tensor will be symmetric and has the form

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}.$$

We can consider a region with rectangular faces.



On the illustrated face

$\vec{T}_x \cdot d\vec{a} = T_{xy}$ ~ force component along x (shear)

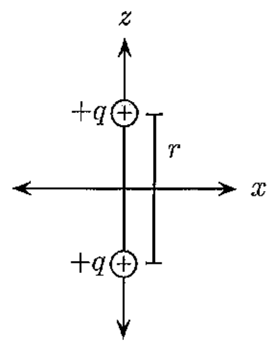
$\vec{T}_y \cdot d\vec{a} = T_{yy}$ ~ " " " y

$\vec{T}_z \cdot d\vec{a} = T_{zy}$ ~ " " " z

Thus some of these are shear forces, others are stresses.

1 Electric Stress Tensor

Consider two positive point charges along the z axis, Each has charge $+q$ and one is located at $z = +r/2$ while the other is at $z = -r/2$. The aim of this exercise is to use the electric stress tensor over a hemisphere with infinite radius and whose base lies in the xy plane.



a) Determine the electric field produced by both charges along the xy plane.

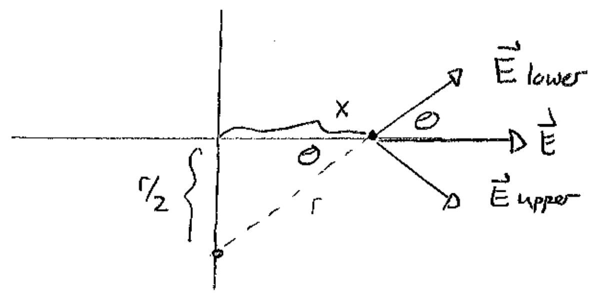
b) Determine the electric stress tensor along the xy plane.

c) Use the electric stress tensor to determine the force exerted on all the matter in the $z > 0$ region.

Answer:

a) Consider a location along the x-axis. Then from the lower charge

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{(x^2 + r^2/4)}$$



gives the magnitude. The horizontal component is $E\cos\theta$ and

$$\cos\theta = \frac{x}{\sqrt{x^2 + r^2/4}}$$

Thus the horizontal component from one charge is:

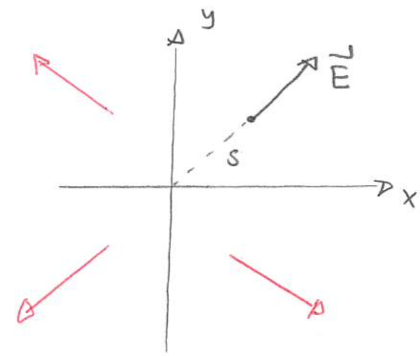
$$E = \frac{qx}{4\pi\epsilon_0 (x^2 + r^2/4)^{3/2}}$$

and at this location both charges contribute

$$\vec{E} = \frac{2qx}{4\pi\epsilon_0 (x^2 + r^2/4)^{3/2}}\hat{x}$$

The situation is symmetrical about the z-axis. Thus at distance s from the origin:

$$\vec{E} = \frac{2qs}{4\pi\epsilon_0 (s^2 + r^2/4)^{3/2}} \hat{s}$$



We then get $\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$. So

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{qs}{(s^2 + r^2/4)^{3/2}} \left[\cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right]$$

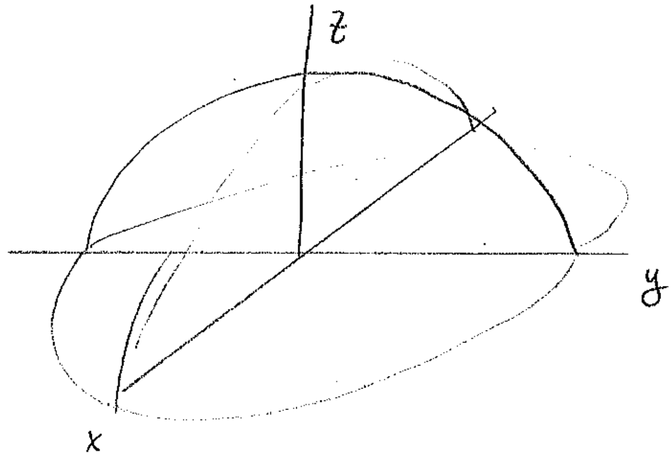
b) Clearly $E_z = 0$. Also

$$\vec{E}\cdot\vec{E} = \left(\frac{1}{2\pi\epsilon_0}\right)^2 \frac{(qs)^2}{(s^2+r^2/4)^3}$$

Thus:

$$T = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2+r^2/4)^3} \begin{pmatrix} \cos^2\phi - \frac{1}{2} & \cos\phi\sin\phi & 0 \\ \cos\phi\sin\phi & \sin^2\phi - 1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

c) We use a hemisphere



Then we need

$$\vec{F} = \left(\oint \vec{T}_x \cdot d\vec{a}\right)\hat{x} + \left(\oint \vec{T}_y \cdot d\vec{a}\right)\hat{y} + \left(\oint \vec{T}_z \cdot d\vec{a}\right)\hat{z}$$

over the whole surface. As the curved part $\to \infty$ this gives zero. So we need to integrate over the base.

$$0 < s \leq \infty \qquad d\vec{a} = -s\,ds\,d\phi\,\hat{z} \quad \text{outward!}$$
$$0 \leq \phi \leq 2\pi$$

We use:

$$\vec{T}_x = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left[\left(\cos^2\phi - \tfrac{1}{2}\right)\hat{x} + \cos\phi \sin\phi\, \hat{y} \right]$$

$$\vec{T}_y = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left[\cos\phi \sin\phi\, \hat{y} + \left(\sin^2\phi - \tfrac{1}{2}\right)\hat{y} \right]$$

$$\vec{T}_z = \frac{\epsilon_0}{(2\pi\epsilon_0)^2} \frac{q^2 s^2}{(s^2 + r^2/4)^3} \left(-\frac{1}{2}\right)\hat{z}$$

Clearly only $\vec{T}_z \cdot d\vec{a} \neq 0$. Thus

$$\vec{F} = +\frac{\epsilon_0 q^2}{2(2\pi\epsilon_0)^2} \int_0^\infty \frac{s^2}{(s^2 + r^2/4)^3}\, s\, d \int_0^{2\pi} d\phi \quad \hat{z}$$

$$= +\frac{q^2}{4\pi\epsilon_0} \int_0^\infty \frac{s^3}{(s^2 + r^2/4)^3}\, ds \quad \hat{z}$$

The relevant "u substitution" is

$$s = r/2 \, \tan\theta \qquad \Rightarrow \frac{ds}{d\theta} = \frac{r}{2} \frac{1}{\cos^2\theta}$$

$$\Rightarrow \vec{F} = + \frac{q^2}{4\pi\epsilon_0} \left(\frac{r}{2}\right)^4 \int_0^{\pi/2} \frac{\tan^3\theta}{\left(\frac{r}{2}\right)^6 (\tan^2\theta + 1)^3 \cos^2\theta} \, d\theta \quad \hat{z}$$

$$= + \frac{q^2}{4\pi\epsilon_0} \left(\frac{2}{r}\right)^2 \int_0^{\pi/2} \frac{\sin^3\theta / \cos^3\theta}{\cos^2\theta \left(\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}\right)^3} \, d\theta \quad \hat{z}$$

$$= \frac{+q^2}{4\pi\epsilon_0 r^2} \, 4 \int_0^{\pi/2} \cos\theta \sin^3\theta \, d\theta \quad \hat{z}$$

$$= + \frac{q^2}{4\pi\epsilon_0 r^2} \, 4 \left(\frac{1}{4}\right) \sin^4\theta \Big|_0^{\pi/2} \quad \hat{z}$$

$$\vec{F} = \frac{q^2}{4\pi\epsilon_0 r^2} \, \hat{z}$$

This is exactly what Coulomb's Law predicts.

Momentum conservation

In the most general non-static situation we found that the net force on all the material in a region is

$$\vec{F}_{mech} = -\epsilon_0 \frac{d}{dt}\int \vec{E}\times\vec{B}\, d\tau$$

$$+ \int' \Big\{\epsilon_0 (\vec{\nabla}\cdot\vec{E})\vec{E} - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$+ \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B})\vec{B} - \vec{B}\times(\vec{\nabla}\times\vec{B})\Big\} d\tau'$$

We now need to incorporate the terms such as $\vec{E}\times(\vec{\nabla}\times\vec{E})$. We know that

$$\vec{\nabla}(\vec{E}\cdot\vec{E}) = \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{E}\times(\vec{\nabla}\times\vec{E}) + (\vec{E}\cdot\vec{\nabla})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E}$$

Thus

$$\vec{E}\times(\vec{\nabla}\times\vec{E}) = \frac{1}{2}\vec{\nabla}(\vec{E}\cdot\vec{E}) - (\vec{E}\cdot\vec{\nabla})\vec{E}$$

Thus:

$$\epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} - \epsilon_0\vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$= \epsilon_0(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} + \frac{1}{2}\vec{\nabla}(\vec{E}\cdot\vec{E})$$

Thus

$$\vec{F}_{mech} = -\epsilon_0\frac{d}{dt}\int(\vec{E}\times\vec{B})\,d\tau + \epsilon_0\int\Big[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}(\vec{E}\cdot\vec{E})\Big]d\tau.$$

$$+ \frac{1}{\mu_0}\int\Big[(\vec{\nabla}\cdot\vec{B})\vec{E} + (\vec{B}\cdot\vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}(\vec{B}\cdot\vec{B})\Big]d\tau$$

To simplify this we consider components. For example the x component of the electric field term:

$$\boxed{\begin{aligned} &(\vec{\nabla}\cdot\vec{E})E_x + (\vec{E}\cdot\vec{\nabla})E_x - \frac{1}{2}\frac{\partial}{\partial x}(\vec{E}\cdot\vec{E}) \\ &= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z) \end{aligned}}$$

Proof:

$$(\vec{\nabla}\cdot\vec{E})E_x = \frac{\partial E_x}{\partial x}E_x + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z}E_x$$

$$(\vec{E}\cdot\vec{\nabla})E_x = E_x\frac{\partial E_x}{\partial x} + E_y\frac{\partial E_x}{\partial y} + E_z\frac{\partial E_x}{\partial z}.$$

Adding these and subtracting $\frac{1}{2}\frac{\partial}{\partial x}(\vec{E}\cdot\vec{E})$ gives

$$2E_x\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}E_x + \frac{\partial E_z}{\partial z}E_x$$

$$+ E_y\frac{\partial E_x}{\partial y} + E_z\frac{\partial E_x}{\partial z} - \frac{1}{2}\frac{\partial E_x^2}{\partial x} - \frac{1}{2}\frac{\partial E_y^2}{\partial x} - \frac{1}{2}\frac{\partial E_z^2}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{\partial}{\partial y}(E_y E_x) + \frac{\partial}{\partial z}(E_z E_x). \quad \square$$

Similarly

$$(\vec{\nabla}\cdot\vec{B})B_x + (\vec{B}\cdot\vec{\nabla})B_x - \frac{1}{2}\frac{\partial}{\partial x}(\vec{B}\cdot\vec{B})$$

$$= \frac{\partial}{\partial x}\left(B_x^2 - \frac{1}{2}\vec{B}\cdot\vec{B}\right) + \frac{\partial}{\partial y}(B_y B_x) + \frac{\partial}{\partial z}(B_z B_x)$$

Thus we define the Maxwell stress vectors:

$$\boxed{\begin{aligned}\vec{T}_x &:= \epsilon_0\left(E_x^2 - \tfrac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{x} + \epsilon_0 E_y E_x \hat{y} + \epsilon_0 E_z E_x \hat{z} \\ &\quad + \frac{1}{\mu_0}\left(B_x^2 - \tfrac{1}{2}\vec{B}\cdot\vec{B}\right)\hat{x} + \frac{1}{\mu_0} B_y E_x \hat{y} + \frac{1}{\mu_0} B_z B_x \hat{z} \\ \vec{T}_y &= \epsilon_0 E_x E_y \hat{x} + \epsilon_0\left(E_y^2 - \tfrac{1}{2}\vec{E}\cdot\vec{E}\right)\hat{y} + \epsilon_0 E_z E_y \hat{z} \\ &\quad + \frac{1}{\mu_0} B_x B_y \hat{x} + \frac{1}{\mu_0}\left(B_y^2 - \tfrac{1}{2}\vec{B}\cdot\vec{B}\right)\hat{y} + \frac{1}{\mu_0} B_z E_y \hat{z} \\ \vec{T}_z &= \ldots\end{aligned}}$$

Then this gives:

$$\vec{F}_{mech} = -\epsilon_0 \frac{d}{dt}\int (\vec{E}\times\vec{B})\, d\tau$$

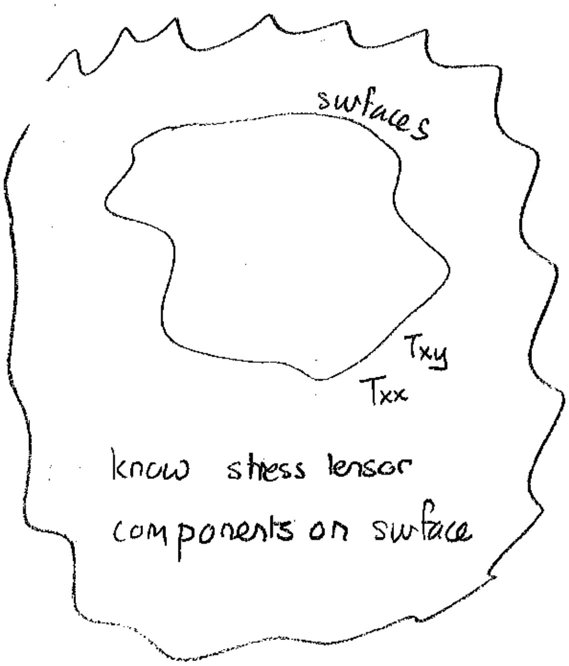
$$+ \int_R (\vec{\nabla}\cdot\vec{T}_x)\, d\tau\, \hat{x} + \int_R (\vec{\nabla}\cdot\vec{T}_y)\, d\tau\, \hat{y} + \int (\vec{\nabla}\cdot\vec{T}_z)\, d\tau\, \hat{z}$$

$$\Rightarrow \boxed{\begin{aligned}\vec{F}_{mech} &= -\epsilon_0 \frac{d}{dt}\int_R (\vec{E}\times\vec{B})\, d\tau \\ &\quad + \left(\oint_S \vec{T}_x\cdot d\vec{a}\right)\hat{x} + \left(\oint_S \vec{T}_y\cdot d\vec{a}\right)\hat{y} + \left(\oint \vec{T}_z\cdot d\vec{a}\right)\hat{z}\end{aligned}}$$

Again we can create a matrix that represents the Maxwell stress tensor:

$$\overleftrightarrow{T} = \begin{pmatrix} \epsilon_0\left(E_x^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{1}{\mu_0}\left(B_x^2 - \frac{1}{2}\vec{B}\cdot\vec{B}\right) & \epsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y & \epsilon_0 E_x E_z + \frac{1}{\mu_0} B_x B_z \\ \epsilon_0 E_y E_x + \frac{1}{\mu_0} B_y B_x & \epsilon_0\left(E_y^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{1}{\mu_0}\left(B_y^2 - \frac{1}{2}\vec{B}\cdot\vec{B}\right) & \epsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z \\ \epsilon_0 E_z E_x + \frac{1}{\mu_0} B_z B_x & \epsilon_0 E_z E_y + \frac{1}{\mu_0} B_z B_y & \epsilon_0\left(E_z^2 - \frac{1}{2}\vec{E}\cdot\vec{E}\right) + \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}\vec{B}\cdot\vec{B}\right) \end{pmatrix}$$

We get the conceptual scheme



$\rightarrow$ Net force on matter inside the surface is

$$\vec{F}_{mech} = -\epsilon_0 \frac{d}{dt}\int (\vec{E}\times\vec{B})\, d\tau + \left(\oint_S \vec{T}_x \cdot d\vec{a}\right)\hat{x} + \ldots$$

Finally we have that $\vec{F}_{mech} = \frac{d\vec{P}_{mech}}{dt}$. Thus, if we define the electromagnetic momentum as:

$$\boxed{\vec{P}_{elec} = \epsilon_0 \int_{region} \vec{E}\times\vec{B}\, d\tau = \epsilon_0\mu_0 \int_{region} \vec{S}\, d\tau}$$

we get a rule for the conservation of momentum:

$$\boxed{\frac{d}{dt}\left(\vec{P}_{mech} + \vec{P}_{elec}\right) = \left(\oint_S \vec{T}_x \cdot d\vec{a}\right)\hat{x} + \left(\oint \vec{T}_y \cdot d\vec{a}\right)\hat{y} + \left(\oint \vec{T}_z \cdot d\vec{a}\right)\hat{z}}$$

This gives conservation over all space in the case where $\vec{E}, \vec{B} \to 0$ as $r \to \infty$ because the surface integrals approach zero. But the conservation law now necessarily includes an electromagnetic component. The electromagnetic momentum density is then:

$$\boxed{\vec{g} = \epsilon_0 (\vec{E}\times\vec{B}) = \mu_0\epsilon_0 \vec{S}}$$

Here $\vec{S}$ is the Poynting vector.