Chapter 9: The Geometry Outside of a Spherical Star

3-27-19

Schwarzschild Geometry

The line element is ....

$$\left[ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \right]$$

- These coordinates are called the Schwarzschild Coordinates
- $g_{\alpha\beta}(x)$ is called the Schwarzschild Metric

Notes:

1. Time independent : $t \to t + \text{const.}$ Leaves ds^2 invariant

   $\Rightarrow$ Associated killing vector $\underset{\sim}{\xi}$ where $\underset{\sim}{\xi}^{\alpha} = (1,0,0,\varphi)$

2. Spherically Symmetric for $t = \text{const.}$, $r = \text{const.}$

   $d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$
   - Geometry of a sphere in flat 3D Space

3. $\varphi \to \varphi + \text{Const.}$ Leaves ds^2 invariant

   $\Rightarrow$ Associated killing vector $\underset{\sim}{\eta}$ w/

   $\eta^{\alpha} = (0,0,0,1)$

Note: r is not the distance from any center

$A = 4\pi r^2$: $r = \left(\frac{A}{4\pi}\right)^{1/2}$ where A is the area of the 2-sphere

4. Mass M

   when $\frac{2GM}{c^2} \ll r$: $ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 + \frac{2GM}{c^2 r}\right) + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$

   $\Rightarrow$ Weak, Static metric w/ a Newtonian Gravitational Potential

   $$\left[\Phi = \frac{-GM}{r} \right] \text{ (See 6.6)}$$

   $\Rightarrow$ M is the total mass of the source of Curvature

5. Schwarzschild Radius
   - $r = 0 \Rightarrow$ Real physical Singularity
   - $r = 2GM/c^2 \Rightarrow$ Schwarzschild radius : Not real

4-1-19

The Schwarzchild Line element....

$$ds^2 = -\left(1-\frac{2GM}{r}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Killing vectors

1. $\underset{\sim}{\xi}$ w/ $\xi^\alpha = (1,0,0,0)$
2. $\underset{\sim}{\eta}$ w/ $\eta^\alpha = (0,0,0,1)$

Schwarzchild Radius

- $r=0 \Rightarrow$ Physical Singularity
- $r=\frac{2GM}{c^2} \Rightarrow$ Coordinate Singularity (Schwarzchild radius)

  $\Rightarrow$ Characteristic length scale for the curvature @ $r=\frac{2GM}{c^2}$

  $\hookrightarrow$ Surface joins different geometries

Define Geometerized Units .....

- A system of units convenient for GR, which also puts $G=1$ $(c=1)$

$$M(\text{in cm}) = M(\text{in g})\cdot\frac{G}{c^2} = 0.742\times10^{-28}\,\frac{\text{cm}}{\text{g}} \times M(\text{in g})$$

i.e $M_\odot = 1.47$ km
$M_\oplus = 0.44$ cm

In Geometerized units....

$$ds^2 = -(1-2M/r)\,dt^2 + (dr^2)/(1-2M/r) + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & & & \\ & (1-2M/r)^{-1} & & \\ & & (r^2) & \\ & & & (r^2\sin^2\theta) \end{pmatrix} \rightarrow \text{zero's elsewhere}$$

The Gravitational Redshift

observer @ $r=R$, emit signal ω^* (measured by the stationary observer)

Observer @ $r=\infty$, Absorb ω_∞ : $\omega_\infty < \omega_*$

Conserved Quantity is .....

$\underset{\sim}{\xi}\cdot\underset{\sim}{P}$ where $\underset{\sim}{P}$ is the photon's four-momentum

The energy of a photon measured by an observer w/ Four - Velocity $\underset{\sim}{u}_{OBS}$

$$E_\gamma = \hbar\omega = -\underset{\sim}{p}\cdot\underset{\sim}{u}_{OBS}$$

Now

$$u^i_{OBS.} = 0$$

So

$$\underset{\sim}{u}_{OBS}\cdot\underset{\sim}{u}_{OBS} = -1 = g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t_{OBS})^2 = -1(1-{}^{2M}/_r)(u^t_{OBS})^2$$

$$\therefore\ u^t_{OBS} = \frac{1}{\sqrt{1-2M/r}} \qquad \therefore\ u^\alpha_{OBS} = \left(\frac{1}{\sqrt{1-2M/r}}, 0, 0, 0\right) = (1-{}^{2M}/_r)^{-1/2}\xi^\alpha$$

$$\underset{\sim}{u}_{OBS}(r) = \frac{\underset{\sim}{\xi}}{\sqrt{1-2M/r}}$$

@ $r=R$

$$\hbar\omega_* = \frac{1}{\sqrt{1-2M/R}}(-\underset{\sim}{p}\cdot\underset{\sim}{\xi})_R$$

@ $r=\infty$

$$\hbar\omega_\infty = (-\underset{\sim}{p}\cdot\underset{\sim}{\xi})_\infty$$

However, $(\underset{\sim}{\xi}\cdot\underset{\sim}{p})_\infty = (\underset{\sim}{\xi}\cdot\underset{\sim}{p})_R$ Since $\underset{\sim}{\xi}\cdot\underset{\sim}{p}$ is conserved along the photon's Geodesic

$$\therefore\ \omega_* = \frac{\omega_\infty}{\sqrt{1-2M/R}} \Rightarrow \boxed{\omega_\infty = \sqrt{1-{}^{2M}/_r}\ \omega_*} \rightarrow \text{Gravitational redshift}$$

Particle Orbits - Precession of the Perihelion

Since ds^2 is independent of t : φ

$$-e \equiv \underset{\sim}{\xi}\cdot\underset{\sim}{u} \quad \text{for } \underset{\sim}{\xi} \text{ w/ } \xi^\alpha = (1,0,0,0)$$

$$l \equiv \underset{\sim}{\eta}\cdot\underset{\sim}{u} \quad \text{for } \underset{\sim}{\eta} \text{ w/ } \eta^\alpha = (0,0,0,1)$$

$\underset{\sim}{u}$ is the four-velocity of the particle

$$e = -g_{\alpha\beta}\xi^\alpha u^\beta = -g_{tt}\xi^t u^t = (1-{}^{2M}/_r)\,{}^{dt}/_{d\tau}$$

$$l = g_{\alpha\beta}\eta^\alpha u^\beta = g_{\varphi\varphi}\eta^\varphi u^\varphi = r^2\sin^2\theta\frac{d\varphi}{d\tau}$$ ⇒ Particle lies in a plane

· Orientate axes such that $\left[\theta = {}^{\pi}/_2 : u^\theta = {}^{d\theta}/_{d\tau} = 0\right]$

4-3-19

Another "first integral" comes from ....

$$-1 = \underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = -\left(1-\frac{2M}{r}\right)(u^t)^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}(u^r)^2 + r^2\underbrace{\cancel{(u^\theta)^2}}_{0} + r^2\underbrace{\cancel{\sin^2\theta}}_{1}(u^\varphi)^2$$

$$-1 = -(1-2M/r)\left(\frac{dt}{d\tau}\right)^2 + (1-2M/r)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\therefore\ \frac{dt}{d\tau} = e(1-2M/r)^{-1}, \quad \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

Plugging into (*) yields....

$$-1 = -(1-2M/r)^{-1}e^2 + (1-2M/r)^{-1}(dr/d\tau)^2 + (\ell/r)^2$$

$$-(1+\ell^2/r^2)(1-2M/r) = -e^2 + (dr/d\tau)^2 \ : \ e^2 = (dr/d\tau)^2 + (1+\ell^2/r^2)(1-2M/r)$$

Defining a new constant....

$$\mathcal{E} = (1/2)(e^2-1) = (1/2)(dr/d\tau)^2 + (1/2)\left[(1+\ell^2/r^2)(1-2M/r) - 1\right]$$

$$\mathcal{E} = (1/2)(dr/d\tau)^2 + (1/2)\left[1 - 2M/r + \ell^2/r^2 - 2M\ell^2/r^3 - 1\right]$$

$$\mathcal{E} = (1/2)(dr/d\tau)^2 + \underbrace{\left[\overbrace{-M/r + \ell^2/2r^2}^{\text{Newtonian}} \overbrace{- M\ell^2/r^3}^{\text{GR correction term}}\right]}_{V_{eff}(r)}$$

In Newtonian Gravitation....

- $\mathcal{E} > 0$, unbounded particle (Parabolas, Hyperbolas)
- $\mathcal{E} < 0$, bounded particles (Ellipses, circle)

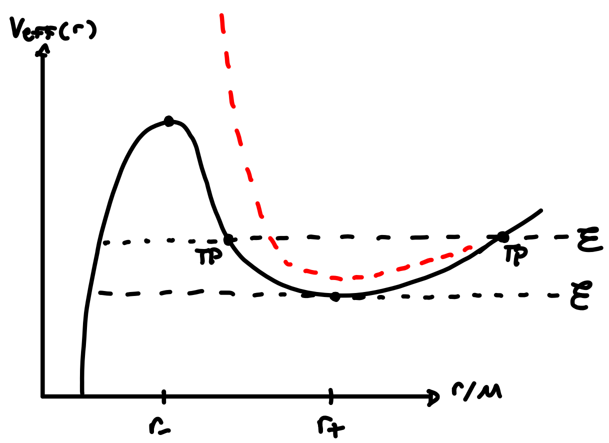
Put back c & G's

Let t & $\tau \longrightarrow ct$ & $c\tau$

$$M \to \frac{GM}{c^2}$$

$$\ell \to \frac{\ell}{c}$$

$$V_{eff}(r) = 1/c^2\left[-GM/r + \ell^2/2r^2 - GM\ell^2/c^2r^3\right]$$



Equilibrium points?

$$\frac{dV_{eff}}{dr} = 0 \longrightarrow V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$\frac{dV_{eff}}{dr} = \frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4} = 0 \qquad \frac{1}{r^4}\left[Mr^2 - \ell^2 r + 3M\ell^2\right] = 0$$

$$r_{\pm} = \frac{\ell^2 \pm \sqrt{\ell^4 - 4M(3M\ell^2)}}{2(M)} = \frac{\ell^2}{2(M)}\left[1 \pm \underbrace{\sqrt{1 - 12M^2/\ell^2}}_{0}\right]$$

Notice:

- If $1 - 12M^2/\ell^2 < 0 \Rightarrow \frac{\ell}{M} < \sqrt{12} = 3.46$

  $\Rightarrow$ There are no real extrema: $V_{eff} < 0$ for all r

- If $1 - 12M^2/\ell^2 > 0 \Rightarrow \frac{\ell}{M} > \sqrt{12}$

  $\Rightarrow$ 1 max : 1 min $\qquad r_{\substack{min \\ max}} = \frac{\ell^2}{2M}\left[1 \pm \sqrt{1 - 12M^2/\ell^2}\right] \qquad \left[\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)\right]^{(*)}$

4-5-19

Turning points? $\frac{dr}{d\tau} = 0$

- when $\ell/M < \sqrt{12}$, there are no r_{tp} for $\mathcal{E} > 0$

Radial Plunge Orbit

$\ell = 0$, start particle @ $r \to \infty$ w/ $\left.\frac{dr}{d\tau}\right|_{r\to\infty} = 0 \quad \therefore \quad \mathcal{E} = 0$, $e = 1$

$$e = (1 - 2M/r)(dt/d\tau)$$

Negative for radially in falling trajector

(*) Becomes....

$$0 = \tfrac{1}{2}(dr/d\tau)^2 - M/r \quad \therefore \quad \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \rightarrow \int r^{1/2}\,dr = -\sqrt{2M}\int d\tau$$

$$= \frac{2}{3} r^{3/2} = -\sqrt{2M}(\tau - \tau_*) \quad : \quad \tau_* \to \text{Arbitrary integration constant}$$

$$\left[r(\tau) = (3/2)^{2/3}(2M)^{1/3}(\tau_* - \tau)\right] \longrightarrow$$

τ_* Such that....
$\tau \to -\infty$, $r \to \infty$
$\tau = \tau_* \quad \therefore \; r = 0$

$$e = 1 = (1 - 2M/r)\,dt/d\tau \quad \therefore \quad \frac{dt}{d\tau} = (1 - 2M/r)^{-1}$$

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

$[r = 2Mx]$ Homework hint

So

$$-\sqrt{\frac{2m}{r}} = \frac{dr}{dt}\left(1-\frac{2m}{r}\right)^{-1} \quad \therefore \quad \frac{dr}{dt} = -\sqrt{\frac{2m}{r}}\left(1-\frac{2m}{r}\right) \quad : \quad \int dt = -\int\left(\frac{2m}{r}\right)^{-\frac{1}{2}}(1-2m/r)^{-1}dr$$

Integrating Yields ....

$$t = t_* + 2m\left[-2/3\,(r/2m)^{3/2} - 2\,(r/2m)^{1/2} + \ln\left(\frac{(r/2m)^{1/2}+1}{(r/2m)^{1/2}+1}\right)\right]$$

Notice : $T(r=2m)$: Finite

$$t(r=2m) = t_* + 2m\left[-2/3 - 2 + \ln(2) - \underbrace{\ln(0)}_{-\infty}\right]$$

__Stable Circular Orbits__

$$r_{min} = \frac{\ell^2}{2m}\left[1+\sqrt{1-12(m^2/\ell^2)}\right]$$

as (ℓ/m) gets smaller, r_{min} also gets smaller

$$1-12(m/\ell)^2 = 0 \quad \therefore \quad \ell/m = \sqrt{12} \longrightarrow r_{min}\Big|_{\frac{\ell}{m}=\sqrt{12}} = \frac{\ell^2}{2m^2}m = \frac{1}{2}\cdot 12m = 6m^{\Delta}$$

$\Delta \to$ Innermost Stable Circular Orbit (ISCO)

$$\left[V_{ISCO} = 6M\right]$$

The angular velocity of a particle in a circular orbit, Ω,

W.R.T. The Schwarzschild coordinate time, t, is ....

$$\Omega \equiv \frac{d\varphi}{d\tau} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\ell}{r^2}\cdot\frac{(1-2m/r)}{e} = \frac{1}{r^2}(1-2m/r)\frac{\ell}{e}$$

Circular orbits of radius r_{min} have ℓ : e determined by

1. $V_{eff}(r_{min}) = \mathcal{E}$
2. $r_{min} = \frac{\ell^2}{2m}\left[1+\sqrt{1-12m^2/\ell^2}\right]$

One Finds ....

$$\left[\Omega^2 = \frac{M}{r^3}\right]$$ — Angular velocity in stable circular orbit $\Rightarrow$ Nonrelativistic Keplers 3rd Law

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{(1-2m/r)} + r^2(d\theta^2 + \sin^2\varphi\, d\varphi^2)$$

4-8-19

Light Ray Orbits

$$u^\alpha \equiv \frac{dx^\alpha}{d\lambda}$$

Affine parameter

Symmetry in t : φ Directions $\Rightarrow$ Killing vector $\underset{\sim}{\xi}$: $\underset{\sim}{\eta}$ w/ Conserved quantities

- Two "First integrals"....

$$e \equiv -\underset{\sim}{\xi}\cdot\underset{\sim}{u} = (1-2m/r)\,dt/d\lambda$$

$$\ell = \underset{\sim}{\eta}\cdot\underset{\sim}{u} = r^2\sin^2\theta \frac{d\varphi}{d\lambda}$$

- Third First integral from requirement that four velocity vector is null

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = 0 = g_{\alpha\beta}u^\alpha u^\beta = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + g_{\varphi\varphi}\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$0 = -(1-2m/r)\left(\frac{dt}{d\lambda}\right)^2 + (1-2m/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\underbrace{\cancel{\left(\frac{d\theta}{d\lambda}\right)^2}}_{0} + r^2\underbrace{\cancel{\sin^2\theta}}_{1}\left(\frac{d\varphi}{d\lambda}\right) \qquad \theta = \pi/2$$

$$= -(1-2m/r)\left(\frac{dt}{d\lambda}\right)^2 + (1-2m/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 \quad : \quad \frac{dt}{d\lambda} = e(1-2m/r)^{-1}$$

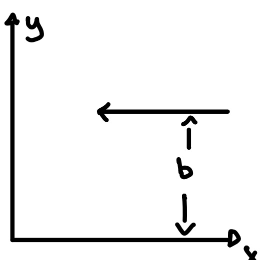
$$\frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$$

$$\left[0 = -e^2(1-2m/r)^{-1} + (1-2m/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}\right](1-2m/r)$$

$$\left[e^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}(1-2m/r)\right]\frac{1}{\ell^2} \qquad \frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r) \quad : \quad W_{eff}(r) = \frac{1}{r^2}(1-2m/r)$$

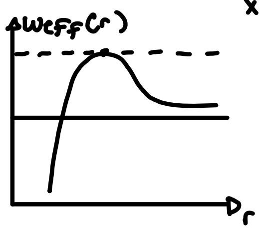
$$\therefore\ b = \left|\frac{e}{\ell}\right|$$

b is the impact parameter of a light ray that reaches infinity



$$W_{eff}(r) = \frac{1}{r^2}\left(1-\frac{2m}{r}\right)$$

where $x = r\cos\varphi$
$y = r\sin\varphi$



$$\frac{e^2}{\ell^2} = \frac{1}{b^2}$$

Notice: $\lim_{r\to\infty} W_{eff}(r) \longrightarrow 0$

$$\left.\frac{dW_{eff}}{dr}\right|_{r=r'} = \frac{d}{dr}\left(\frac{1}{r^2} - \frac{2m}{r^3}\right)$$

$$\frac{dW_{eff}}{dr} = \frac{-2}{r'^3} + \frac{6M}{r'^4} = \frac{2}{r'^4}\left(-r' + 3M\right) \quad : \quad \left[r' = 3M\right]$$

$$W_{eff}(r=3M) = \frac{1}{(3M)^2}\left(1 - \frac{2M}{3M}\right) = \frac{1}{27M^2}$$

Circular orbit? yes! $r = 3M$

$$\left[\frac{1}{b^2} = \frac{1}{27M^2}\right]$$

For our Sun ....

$$r' = 3M_\odot \simeq 4.54 \text{ km} \ll R_\odot$$