Physics 311

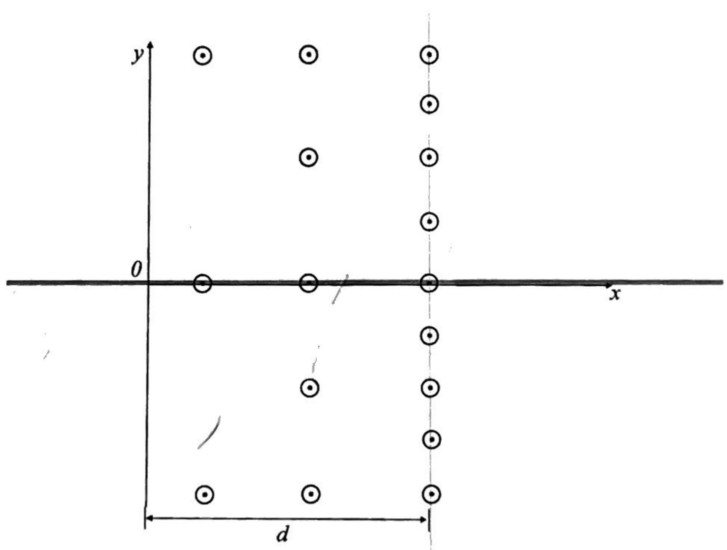
Exam 4

Show all work for full credit!

1. A long wire carries a current I in the $+x$-direction through an external *non-uniform* magnetic field of the form

$$\begin{aligned} \vec{B} &= B_0 \frac{x}{d}\,\hat{k} \text{ for } 0 \leq x \leq d \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \tag{1}$$

Calculate the net force, $\vec{F}$, on the wire indicating both magnitude and direction.

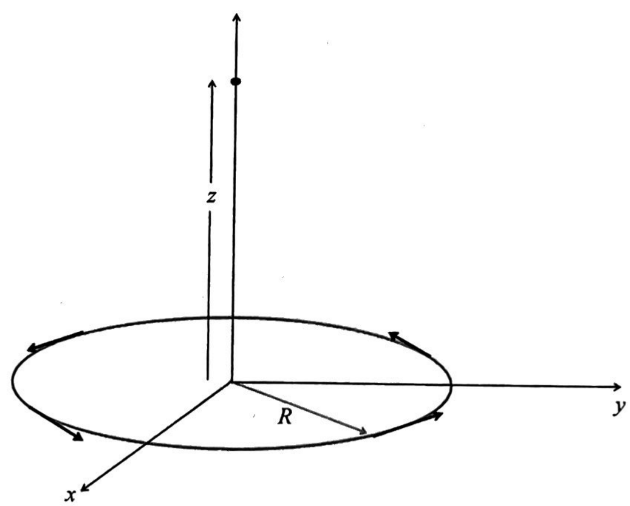


2. A circular loop of radius R lies in the xy-plane and carries a current $\vec{I} = I\hat{\phi}$.

Using the Law of Biot-Savart, find the $\vec{B}$-field

a) at the center of the loop, at $z = 0$

b) at a distance z above the center of loop. Indicate both magnitude and direction for each.



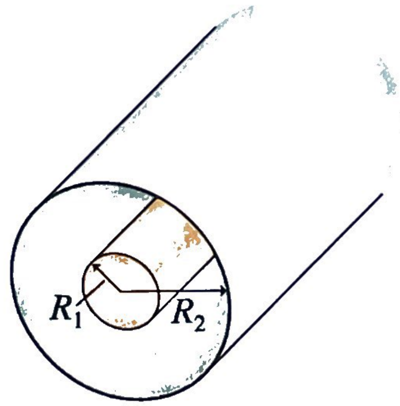
3. The coaxial cable, shown in the figure below, consists of a *solid* inner conductor of radius R_1, surrounded by a *hollow*, thin outer conductor of radius R_2. The two carry equal currents I in the *same* direction, where the current on the inner conductor flows *into* the page. The current density is *uniformly* distributed over and through each conductor.

Find the $\vec{B}$-field in each of the three regions:

a) inside the solid inner conductor, where $s < R_1$.

b) in the space between conductors, where $R_1 < s < R_2$.

c) outside the outer conductor, where $R_2 < s$.



4. A flat circular disk of radius R lies in the xy-plane and carries a *non-uniform* surface charge density given by the expression

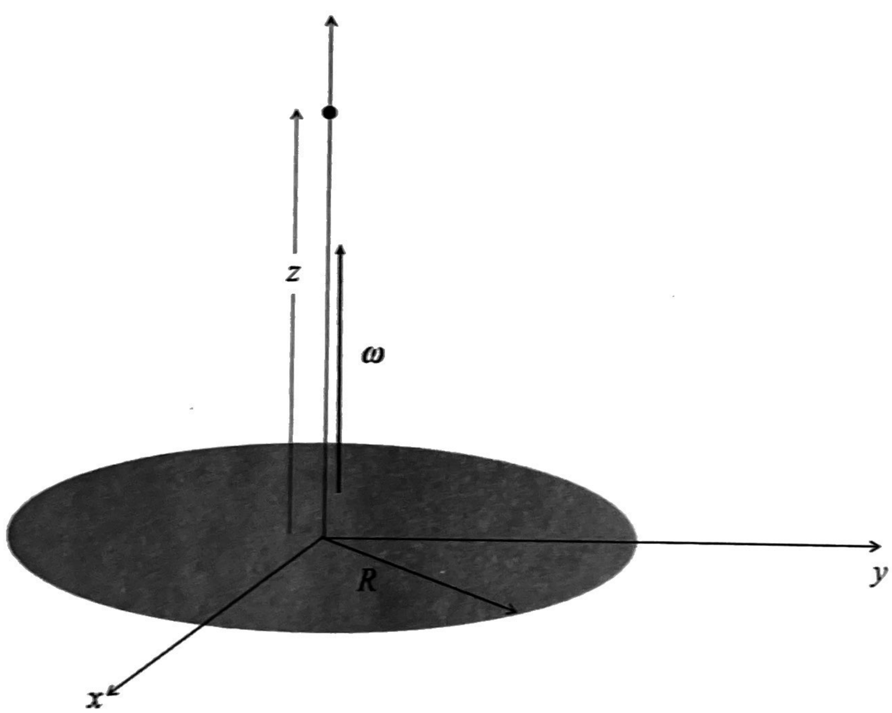
$$\sigma(s) = \sigma_0 \frac{R}{s}. \tag{2}$$

The disk rotates about it's center at a constant angular velocity ω.

a) What is the surface current density, $\vec{K}$, at a distance s from the center?

b) Find the vector potential, $\vec{A}$, it produces a distance z above the center on the axis.

Indicate both magnitude and direction for each.



PHYS 311 EXAM 4 SOLUTIONS

1. Here $\vec{B} = B_0 \frac{x}{d} \hat{z}$ for $0 \le x \le d$

$$d\vec{l} = dx\,\hat{x}$$

so

$$I(d\vec{l} \times \vec{B}) = I \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx & 0 & 0 \\ 0 & 0 & B_0 \frac{x}{d} \end{vmatrix} = -\frac{IB_0}{d} x\,dx\,\hat{y}$$

so

$$\vec{F}_{MAG} = \int_0^d I(d\vec{l} \times \vec{B}) = -\frac{IB_0}{d}\hat{y}\int_0^d x\,dx = -\frac{IB_0}{d}\hat{y}\left.\frac{x^2}{2}\right|_0^d$$

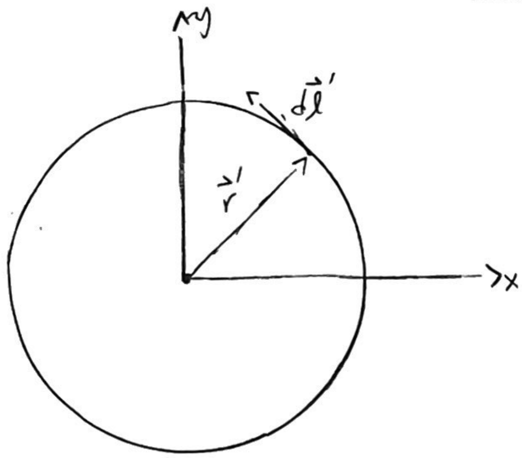
so

$$\boxed{\vec{F}_{MAG} = -\frac{1}{2} I B_0 d\,\hat{y}}$$

2.

LOOKING DOWN THE +Z-AXIS...

a)



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

NOW, IN CYLINDRICAL COORDINATES...

$\vec{r} = 0$ SO $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -R\hat{s}$

$\vec{r}' = R\hat{s}$

$\mathscr{r}^2 = R^2 \therefore \mathscr{r} = R$

so $\hat{\mathscr{r}} = -\hat{s}$

WHEREAS

$d\vec{l}' = R\, d\varphi\, \hat{\varphi}$

so

$$\frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{R\, d\varphi\, \hat{\varphi} \times -\hat{s}}{R^2} = \frac{d\varphi}{R}(\hat{s} \times \hat{\varphi})$$

NOW

$\hat{s} = \cos\varphi\, \hat{x} + \sin\varphi\, \hat{y}$

$\hat{\varphi} = -\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y}$

so

$$\hat{s} \times \hat{\varphi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \end{vmatrix} = (\cos^2\varphi + \sin^2\varphi)\hat{z} = \hat{z}$$

so

$$\frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{d\varphi}{R}\hat{z}$$

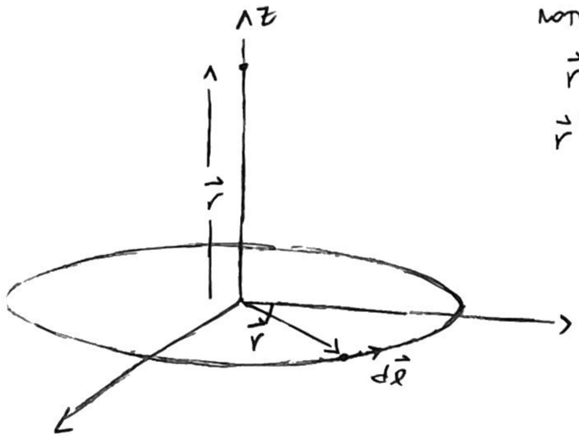
so

$$\vec{B}(\vec{r} = 0) = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{d\varphi}{R}\hat{z} = \frac{\mu_0 I}{4\pi R}\hat{z}\int_0^{2\pi} d\varphi = \frac{\mu_0 I}{2R}\hat{z}$$

so

$$\vec{B}(\vec{r} = 0) = \frac{\mu_0 I}{2R}\hat{z}$$

b)



NOTICE:

$\vec{r} = z\hat{z}$ so $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - R\hat{s}$

$\vec{r}' = R\hat{s}$

$\mathscr{r}^2 = z^2 + R^2$ so $\mathscr{r} = \sqrt{z^2+R^2}$

$$\therefore \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{z\hat{z} - R\hat{s}}{\sqrt{z^2+R^2}}$$

WHEREAS

$d\vec{\ell} = R\,d\varphi\,\hat{\varphi}$

so $$d\vec{\ell} \times \hat{\mathscr{r}} = R\,d\varphi\,\hat{\varphi} \times \frac{z\hat{z} - R\hat{s}}{\sqrt{z^2+R^2}}$$

$$= \frac{R\,d\varphi}{\sqrt{z^2+R^2}}\left[z(\hat{\varphi}\times\hat{z}) - R(\hat{\varphi}\times\hat{s})\right]$$

$$= \frac{R\,d\varphi}{\sqrt{z^2+R^2}}\left[z(\hat{\varphi}\times\hat{z}) + R(\hat{s}\times\hat{\varphi})\right]$$

NOW, FROM PART a)

$\hat{s} \times \hat{\varphi} = \hat{z}$

WHEREAS

$$\hat{\varphi} \times \hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\varphi)\hat{x} + (\sin\varphi)\hat{y}$$

so

$$d\vec{\ell} \times \hat{\mathscr{r}} = \frac{R\,d\varphi}{\sqrt{z^2+R^2}}\left[z\left(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}\right) + R\hat{z}\right]$$

so

$$\vec{B}(\vec{r} = z\hat{z}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} I \int_0^{2\pi} \frac{R\,d\varphi}{(z^2+R^2)^{3/2}}\left[z(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}) + R\hat{z}\right]$$

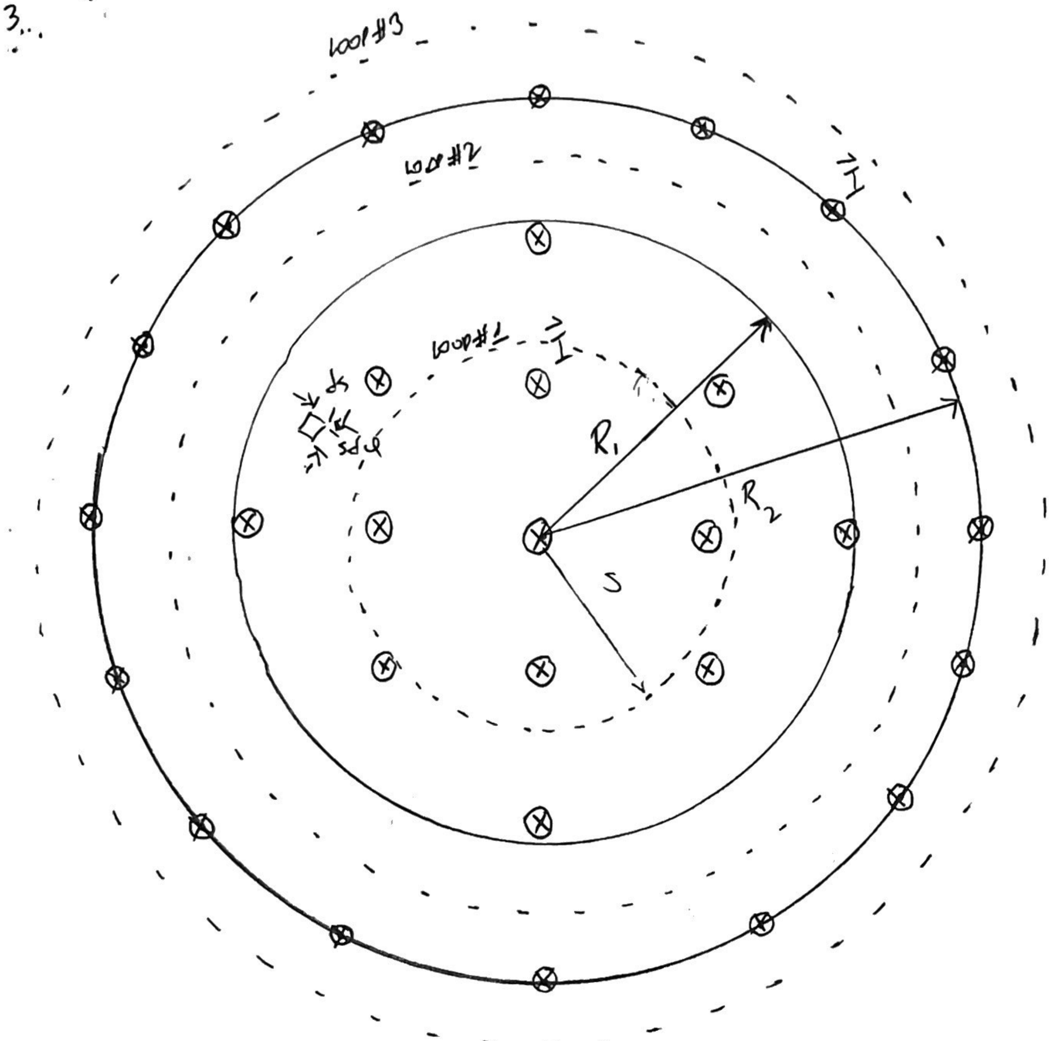
$$= \frac{\mu_0 I}{4\pi} \cdot \frac{R}{(z^2+R^2)^{3/2}}\left[z\hat{x}\underbrace{\int_0^{2\pi}\cos\varphi\,d\varphi}_{0} + z\hat{y}\underbrace{\int_0^{2\pi}\sin\varphi\,d\varphi}_{0} + R\hat{z}\int_0^{2\pi} d\varphi\right]$$

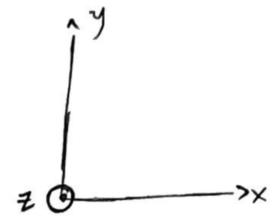
so

$$\vec{B}(\vec{r}=z\hat{z}) = \frac{\mu_0 I}{4\pi} \cdot \frac{R^2}{(z^2+R^2)^{3/2}} \hat{z} \int_0^{2\pi} d\varphi$$

$$\therefore \boxed{B(\vec{r}=z\hat{z}) = \frac{1}{2} \mu_0 I \frac{R^2}{(z^2+R^2)^{3/2}} \hat{z}}$$

3...





Let $\vec{J} = -J\hat{z}$ for the inner wire of radius R_1,

$$d\vec{a} = s\,ds\,d\varphi\,(-\hat{z})$$

so

$$\vec{J}\cdot d\vec{a} = J\,s\,ds\,d\varphi$$

The total current in the wire of radius R_1 is...

$$I = \int \vec{J}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{R_1} J\,s\,ds\,d\varphi$$

$$= J\frac{s^2}{2}\Big|_0^{R_1}\cdot 2\pi = J\pi R_1^2$$

so $\quad J = \dfrac{I}{\pi R_1^2}$

Consider the 3 Amperian loops, drawn as dashed lines, each of radius s...

$$\oint \vec{B}\cdot d\vec{l} = B2\pi s \quad \text{for any of the 3 loops.}$$

a) Consider Amperian loop #1...

$$I_{enc} = \int \vec{J}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{s} J\,s\,ds\,d\varphi = J\int_0^{2\pi} d\varphi \int_0^{s} s\,ds = \frac{I}{\pi R_1^2}\cdot 2\pi\cdot\frac{1}{2}s^2 = I\frac{s^2}{R_1^2}$$

$$= J\pi s^2$$

so Amperes law gives..

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B2\pi s = \mu_0 I\frac{s^2}{R_1^2} \quad \therefore\ B = \frac{\mu_0 I}{2\pi}\frac{s}{R_1^2}$$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I}{2\pi}\frac{s}{R_1^2}(-\hat{\varphi})}$$

$$= \frac{1}{2}\mu_0 J s\,(-\hat{\varphi})$$

b) HERE CONSIDER AMPERIAN LOOP #2...

$I_{enc} = I$ so $B 2\pi s = \mu_0 I$ $\therefore$ $B = \dfrac{\mu_0 I}{2\pi s}$

so

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi s}(-\hat{\varphi})}$$

c) NOW CONSIDER AMPERIAN LOOP #3...

$I_{enc} = 2I$ so $B 2\pi s = \mu_0 2I$ $\therefore$ $B = \dfrac{\mu_0 I}{\pi s}$

so

$$\boxed{\vec{B} = \frac{\mu_0 I}{\pi s}(-\hat{\varphi})}$$

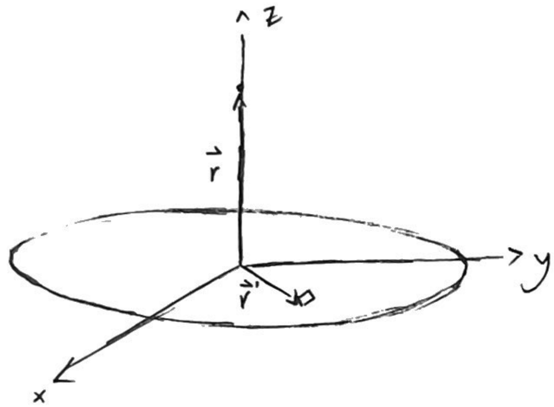
4 a) $\sigma(s) = \sigma_0 \frac{R}{s}$

now $\vec{K} = \sigma \vec{v} = \sigma_0 \frac{R}{s} s\omega \hat{\varphi}$

$\vec{v} = s\omega \hat{\varphi}$

$\therefore \boxed{\vec{K} = \sigma R \omega \hat{\varphi}}$

b)



$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}')}{\mathscr{r}} da'$

so

$\vec{A}(\vec{r} = z\hat{z}) = \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^R \frac{\sigma R \omega \hat{\varphi}}{\sqrt{z^2 + s^2}} s\, ds\, d\varphi$

$= \frac{\mu_0}{4\pi} \sigma R \omega \hat{\varphi} \int_0^{2\pi} d\varphi \int_0^R \frac{s\, ds}{\sqrt{s^2 + z^2}}$

let $u = s^2 + z^2$

$du = 2s\, ds$

$da = s\, ds\, d\varphi$

now

$\vec{r} = z\hat{z}$ so $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = z\hat{z} - s\hat{s}$

$\vec{r}' = s\hat{s}$ so $\mathscr{r} = \sqrt{z^2 + s^2}$

so

$\vec{A}(\vec{r} = z\hat{z}) = \frac{\mu_0}{4\pi} \sigma R \omega \hat{\varphi} \cdot 2\pi \cdot \frac{1}{2} \int_{z^2}^{R^2+z^2} \frac{du}{u^{1/2}} = \frac{\mu_0}{4} \sigma R \omega \hat{\varphi} \left. \frac{u^{1/2}}{1/2} \right|_{z^2}^{R^2+z^2}$

$$\boxed{\vec{A}(\vec{r} = z\hat{z}) = \frac{1}{2} \mu_0 \sigma R \omega \left[\sqrt{R^2 + z^2} - |z| \right] \hat{\varphi}}$$