Math 361 **Numerical Analysis** Taylor Larrechea
11/7/19 Exam 2 Name

Answer the following questions in the space provided. **Show all work.** (100 pts. total.)

1. (a) Use algebraic manipulation to show that $f(x)=x^3-x-2$ has a root precisely where $g(x)=\sqrt[3]{x+2}$ has fixed point. (6 pts)

$f(x)=0$: $x^3-x-2=0$ $\therefore$ $x=\sqrt[3]{x+2}$

$$\boxed{x=\sqrt[3]{x+2}}$$

(b) Use fixed point iteration on $g(x)=\sqrt[3]{x+2}$ to find p_1, p_2, p_3 given that $p_0=1$. Be sure to specify how each iterate is obtained using the notation $p_{n+1}=g(p_n)$. (10 pts)

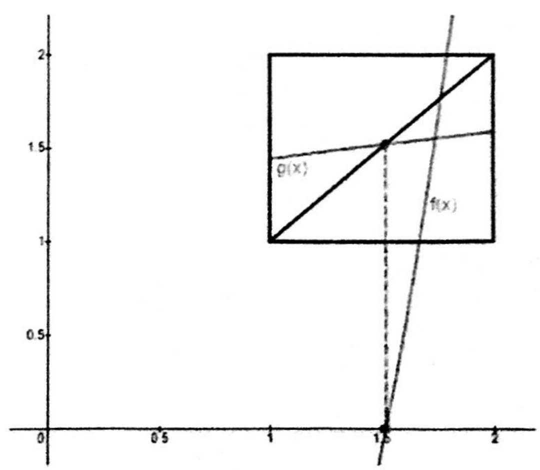
$P_1 = g(P_0) = 1.44224957$

$P_2 = g(P_1) = 1.509897449$

$P_3 = g(P_2) = 1.519724305$

(c) Determine whether the following statements are true or false and then circle your choice for each. The questions refer to the graphs of $f(x), g(x)$ and $y=x$ given below, the sequence of iterates $p_{n+1}=g(p_n)$, and results related to the Fixed Point Theorem. (1 pt each)

1) True or False $g(x)\in[1,2]$ for $x\in[1,2]$
2) True or False $g\notin C[1,2]$
3) True or False $|g'(x)|\le k<1$, for $x\in[1,2]$
4) True or False There is not a fixed point $p\in[1,2]$.
5) True or False The fixed point $p\in[1,2]$ is unique.
6) True or False $p_{n+1}\to p$
7) True or False $p_{n+1}\to p$ quadratically.
8) True or False $p_{n+1}\to p$ linearly.



2. Let $g(x) = \sqrt[3]{x+2}$. Define $\{p_n\}_{n=0}^{\infty}$ by $p_n = g(p_{n-1})$ and $p_0 = 1$. Use Steffensen's Method to find the following iterates. Be sure to specify how each iterate is obtained using an appropriately notated iterating function. (10 pts.)

$$A(x,y,z) = x - \frac{(x-y)^2}{z-2y+x}$$

$$p_0^{(0)} = 1$$

$$p_1^{(0)} = g(p_0^{(0)}) = 1.44224957$$

$$p_2^{(0)} = g(p_1^{(0)}) = 1.509897449$$

$$p_0^{(1)} = A(p_0^{(0)}, p_1^{(0)}, p_2^{(0)}) = 1.52211372$$

$$p_1^{(1)} = g(p_0^{(1)}) = 1.521485407$$

$$p_2^{(1)} = g(p_1^{(1)}) = 1.521394929$$

$$p_0^{(2)} = A(p_0^{(1)}, p_1^{(1)}, p_2^{(1)}) = 1.521379708$$

3. Let $P(x) = 2x^4 - 3x^3 + x^2 + 4x - 3$ on $[0, 2]$, and let $x_0 = 1$. Use Newton's Method and synthetic division (Horner's Method) to compute x_1. (10 pts.)

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\begin{array}{c|ccccc}
1 & 2 & -3 & 1 & 4 & -3 \\
 & \downarrow & 2 & -1 & 0 & 4 \\
\hline
 & 2 & -1 & 0 & 4 & \boxed{1} \rightarrow f(x) \\
 & \downarrow & 2 & 1 & 1 & \\
\hline
 & 2 & 1 & 1 & \boxed{5} \rightarrow f'(x) &
\end{array}$$

$$x_1 = g(x_0) = 1 - \frac{1}{5} = \frac{4}{5} = 0.8$$

$$\boxed{x_1 = 0.8}$$

4. Consider the equation $x^3 - x - 2 = 0$ for $x \in [1, 2]$.

(a) Given $p_0 = 1$, use Newton's Method to find p_1, p_2, p_3. Be sure to specify your iterating function $g(x)$. (10 pts.)

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - x - 2}{3x^2 - 1}$$

$P_1 = g(P_0) = 2$

$P_2 = g(P_1) = 1.636363636$

$P_3 = g(P_2) = 1.530392052$

(b) Given $p_0 = 1$, $p_1 = 2$, use the Secant Method to find p_2, p_3, p_4. Be sure to specify your iterating function $h(x, y)$. (10 pts.)

$$h(x,y) = x - \frac{(x^3 - x - 2)(x - y)}{(x^3 - x - 2) - (y^3 - y - 2)}$$

$P_2 = h(P_1, P_0) = 1.333333333$

$P_3 = h(P_2, P_1) = 1.462686567$

$P_4 = h(P_3, P_2) = 1.531169432$

2. Show that the sequence $p_n = (0.32)^n$ converges linearly to $p = 0$; that is, show that the order of convergence is $\alpha = 1$. (10 pts)

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lim_{n\to\infty} \frac{|(0.32)^{n+1} - 0|}{|(0.32)^n - 0|^1} = \lim_{n\to\infty} |(0.32)| = 0.32 = \lambda$$

$$\boxed{\text{Since } \lambda > 0 \text{ and finite, } p_n \to p \text{ on the order of } \alpha = 1 \quad \lambda = 0.32}$$

5. Use the following data to set up, but do not simplify, the associated second degree interpolating Lagrange polynomial. (10 pts.)

x	$f(x)$
0.1	3.1452
0.3	2.7384
0.5	1.3119

$$\boxed{L_2(x) = \frac{(x-0.3)(x-0.5)}{(0.1-0.3)(0.1-0.5)}(3.1452) + \frac{(x-0.1)(x-0.5)}{(0.3-0.1)(0.3-0.5)}(2.7384) + \frac{(x-0.1)(x-0.3)}{(0.5-0.1)(0.5-0.3)}(1.3119)}$$

6. Given that $f(x) = x\ln(x)$, fill in the second row of the table (the one that is not shaded in). Eqn 4.4 refers to the forward three-point formula and Eqn 4.5 refers to the centered three-point formula. You may use your calculator to obtain the values for the derivative, but partial credit can only be given for incorrect answers if appropriate supporting work is shown. For the error bounds, be sure to show all supporting work in the space provided below the table. (4 pts. each)

		f'(x)			
x	f(x)	Eqn 4.4	Eqn 4.5	Eqn 4.4 Error Bound	Eqn 4.5 Error Bound
6.4	11.8803				
6.5	12.1667	2.873	2.872	$7.889546351 \times 10^{-5}$	$4.069010417 \times 10^{-5}$
6.6	12.4547				
6.7	12.7441				

$f(x) = x\ln(x)$: $f'(x) = \ln(x) + 1$: $f''(x) = \frac{1}{x}$: $f'''(x) = \frac{-1}{x^2}$: $f^4(x) = \frac{2}{x^3}$

EQN 4.4 Bound $\quad E = \frac{h^2}{3} \cdot f'''(\xi(x))$ $[6.5, 6.7]$

Since $f^4(x)$ is positive and decreasing we will use to left endpoint to maximize $f'''(x)$ for our error bound.

$$E = \frac{(0.1)^2}{3} \cdot \left| \frac{-1}{(\xi(x))^2} \right| = \frac{(0.1)^2}{3} \cdot \frac{1}{(6.5)^2} = 7.889546381 \times 10^{-5}$$

$$\boxed{E = 7.889546351 \times 10^{-5}}$$

EQN 4.5 Bound $\quad E = \frac{h^2}{6} \cdot f'''(\xi(x))$ $[6.4, 6.6]$

Since $f^4(x)$ is positive and decreasing we will use to left endpoint to maximize $f'''(x)$ for our error bound.

$$E = \frac{(0.1)^2}{6} \cdot \left| \frac{-1}{(\xi(x))^2} \right| = \frac{(0.1)^2}{6} \cdot \frac{1}{(6.4)^2} = 4.069010417 \times 10^{-5}$$

$$\boxed{E = 4.069010417 \times 10^{-5}}$$