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PHYS 396
HW 7

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Problem 1

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$$

a.) $g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma}$: $\quad g_{\theta\theta} = a^2 \quad g_{\varphi\varphi} = a^2\sin^2\theta \quad$ 3 Free Indices : α, β, γ

$$\boxed{\begin{array}{l} g_{\theta\delta}\Gamma^{\delta}{}_{\theta\theta},\ g_{\theta\delta}\Gamma^{\delta}{}_{\theta\varphi},\ g_{\theta\delta}\Gamma^{\delta}{}_{\varphi\varphi} \\ g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi\varphi},\ g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi\theta},\ g_{\varphi\delta}\Gamma^{\delta}{}_{\theta\theta} \end{array}}$$

b.)

$$g_{\theta\delta}\Gamma^{\delta}{}_{\theta\theta} \to g_{\theta\theta}\Gamma^{\theta}{}_{\theta\theta} + g_{\theta\varphi}\Gamma^{\varphi}{}_{\theta\theta}$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\theta\theta} = \frac{1}{2}\left(\cancel{\frac{\partial g_{\theta\theta}}{\partial\theta}}^{0} + \cancel{\frac{\partial g_{\theta\theta}}{\partial\theta}} - \cancel{\frac{\partial g_{\theta\theta}}{\partial\theta}}\right) = \frac{1}{2}(0) = 0$$

$$\cancel{g_{\theta\varphi}}^{0}\Gamma^{\varphi}{}_{\theta\theta} = \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial\theta} + \frac{\partial g_{\theta\theta}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial\theta}\right) = 0$$

$$g_{\theta\delta}\Gamma^{\delta}{}_{\theta\varphi} \to g_{\theta\theta}\Gamma^{\theta}{}_{\theta\varphi} + g_{\theta\varphi}\Gamma^{\varphi}{}_{\theta\varphi}$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\theta\varphi} = \frac{1}{2}\left(\cancel{\frac{\partial g_{\theta\theta}}{\partial\varphi}}^{0} + \cancel{\frac{\partial g_{\theta\varphi}}{\partial\theta}} - \cancel{\frac{\partial g_{\theta\varphi}}{\partial\theta}}\right) = \frac{1}{2}(0) = 0$$

$$\cancel{g_{\theta\varphi}}^{0}\Gamma^{\varphi}{}_{\theta\varphi} = \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial\varphi} + \frac{\partial g_{\theta\varphi}}{\partial\theta} - \frac{\partial g_{\theta\varphi}}{\partial\theta}\right) = \frac{1}{2}(0) = 0$$

$$g_{\theta\delta}\Gamma^{\delta}{}_{\varphi\varphi} \to g_{\theta\theta}\Gamma^{\theta}{}_{\varphi\varphi} + g_{\theta\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\varphi\varphi} = \frac{1}{2}\left(\cancel{\frac{\partial g_{\theta\varphi}}{\partial\varphi}} + \cancel{\frac{\partial g_{\theta\varphi}}{\partial\varphi}} - \frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{-1}{2}\left(2a^2\sin\theta\cos\theta\right) = -a^2\sin\theta\cos\theta$$

$$\cancel{g_{\theta\varphi}}^{0}\Gamma^{\varphi}{}_{\varphi\varphi} = \frac{1}{2}\left(\frac{\partial g_{\theta\varphi}}{\partial\varphi} + \frac{\partial g_{\theta\varphi}}{\partial\varphi} - \frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = 0 \qquad a^2\Gamma^{\theta}{}_{\varphi\varphi} = -a^2\sin\theta\cos\theta$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta$$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi\varphi} \to g_{\varphi\theta}\Gamma^{\theta}{}_{\varphi\varphi} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}$$

Problem 1] continued

$$\cancel{g_{\varphi\theta}}^{0}\Gamma^{\theta}{}_{\varphi\varphi} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial\varphi} + \frac{\partial g_{\varphi\varphi}}{\partial\varphi} - \frac{\partial g_{\varphi\varphi}}{\partial\varphi}\right) = 0$$

$$g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi\varphi} = \frac{1}{2}\left(\cancel{\frac{\partial g_{\varphi\varphi}}{\partial\varphi}}^{0} + \cancel{\frac{\partial g_{\varphi\varphi}}{\partial\varphi}} - \cancel{\frac{\partial g_{\varphi\varphi}}{\partial\varphi}}\right) = \frac{1}{2}(0) = 0$$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi\theta} \longrightarrow g_{\varphi\theta}\Gamma^{\theta}{}_{\varphi\theta} + g_{\varphi\varphi}\Gamma^{\theta}{}_{\varphi\theta}$$

$$\cancel{g_{\varphi\theta}}^{0}\Gamma^{\theta}{}_{\varphi\theta} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial\theta} + \frac{\partial g_{\varphi\theta}}{\partial\varphi} - \frac{\partial g_{\varphi\theta}}{\partial\varphi}\right) = 0$$

$$g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi\theta} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial\theta} + \cancel{\frac{\partial g_{\varphi\theta}}{\partial\varphi}} - \cancel{\frac{\partial g_{\varphi\theta}}{\partial\varphi}}\right) = \frac{1}{2}(2a^2\sin\theta\cos\theta) = a^2\sin\theta\cos\theta$$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{\theta\theta} \longrightarrow g_{\varphi\theta}\Gamma^{\theta}{}_{\theta\theta} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\theta\theta}$$

$$a^2\sin^2\theta\,\Gamma^{\varphi}{}_{\varphi\theta} = 2a^2\sin\theta\cos\theta$$

$$\Gamma^{\varphi}{}_{\varphi\theta} = 2\frac{\cos\theta}{\sin\theta}$$

$$\cancel{g_{\varphi\theta}}^{0}\Gamma^{\theta}{}_{\theta\theta} = \frac{1}{2}\left(\frac{\partial g_{\varphi\theta}}{\partial\theta} + \frac{\partial g_{\varphi\theta}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial\varphi}\right) = 0$$

$$g_{\varphi\varphi}\Gamma^{\varphi}{}_{\theta\theta} = \frac{1}{2}\left(\cancel{\frac{\partial g_{\varphi\theta}}{\partial\theta}} + \cancel{\frac{\partial g_{\varphi\theta}}{\partial\theta}} - \cancel{\frac{\partial g_{\theta\theta}}{\partial\varphi}}^{0}\right) = \frac{1}{2}(0) = 0$$

$$\boxed{\Gamma^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^{\varphi}{}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}}$$

Problem 2

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \ , \ \Gamma^\varphi_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$

a.) $$\frac{d^2\theta}{ds^2} = \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 , \quad \frac{d^2\varphi}{ds^2} = -2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{ds}\frac{d\varphi}{ds} , \quad \frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

$\alpha = \theta$: $$\frac{d^2\theta}{d\tau^2} = -\Gamma^\theta_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

$$= -\Gamma^\theta_{\theta\theta}\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} - \Gamma^\theta_{\varphi\theta}\cancel{\frac{d\varphi}{d\tau}}^{0}\frac{d\theta}{d\tau} - \Gamma^\theta_{\theta\varphi}\cancel{\frac{d\theta}{d\tau}}^{0}\frac{d\varphi}{d\tau} - \Gamma^\theta_{\varphi\varphi}\cancel{\frac{d\varphi}{d\tau}}^{0}\frac{d\varphi}{d\tau}$$

$$= -(-\sin\theta\cos\theta)\frac{d\theta^2}{d\tau^2} \quad : d\tau \to ds :$$

$$\boxed{\frac{d^2\theta}{ds^2} = \sin\theta\cos\theta\left(\frac{d\theta^2}{ds^2}\right)}$$

$$= -\Gamma^\varphi_{\theta\theta}\cancel{\frac{d\theta}{d\tau}}^{0}\frac{d\theta}{d\tau} - \Gamma^\varphi_{\varphi\theta}\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} - \Gamma^\varphi_{\theta\varphi}\cancel{\frac{d\theta}{d\tau}}^{0}\frac{d\varphi}{d\tau} - \Gamma^\varphi_{\varphi\varphi}\frac{d\varphi}{d\tau}\cancel{\frac{d\varphi}{d\tau}}^{0}$$

$$= -2\frac{\cos\theta}{\sin\theta}\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} \quad : d\tau \to ds :$$

$$\boxed{\frac{d^2\varphi}{ds^2} = -2\frac{\cos\theta}{\sin\theta}\frac{d\varphi}{ds}\frac{d\theta}{ds}}$$

b.) Equation (1) has a symmetry in φ

$\underline{x' = \varphi + \varphi_0}$ $\quad$ $\underline{dx' = d\varphi}$: $\underline{\text{line element remains unchanged}}$

c.) φ symmetry : $\boxed{\vec{\xi} = (0,1)}$

d.) $l = \vec{\xi}\cdot\vec{u}$: $\vec{\xi}\cdot\vec{u} = g_{\alpha\beta}\xi^\alpha u^\beta$

$$\cancel{g_{\theta\theta}\xi^\theta u^\theta}$$
$$\cancel{g_{\theta\varphi}\xi^\theta u^\varphi}$$
$$\cancel{g_{\varphi\theta}\xi^\varphi u^\theta}$$
$$g_{\varphi\varphi}\xi^\varphi u^\varphi = a^2\sin^2\theta\frac{d\varphi}{ds}$$

$$\boxed{l = a^2\sin^2\theta\frac{d\varphi}{ds}}$$

Problem 2 Continued

e.) $\frac{d\varphi}{ds} = \frac{l}{a^2} \cdot \csc^2\theta$: $\frac{d^2\varphi}{ds^2} = \frac{d}{ds} \frac{l}{a^2} \cdot \csc^2\theta$

$$= \frac{l}{a^2} \frac{d}{d\theta} \cdot \csc^2\theta \frac{d\theta}{ds}$$

$$= \frac{l}{a^2} (-2\cot\theta \csc^2\theta) \frac{d\theta}{ds}$$

$$= a^2 \sin^2\theta \frac{d\varphi}{ds} \cdot \frac{1}{a^2} (-2\cot\theta \csc^2\theta) \frac{d\theta}{ds}$$

$$\boxed{\frac{d^2\varphi}{ds^2} = -2\cot\theta \frac{d\theta}{ds} \frac{d\varphi}{ds}}$$

Problem 3

$$\vec{u}\cdot\vec{u} = 1 \quad : \quad ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

a.) $\vec{u} = \left(\frac{d\theta}{ds}, \frac{d\varphi}{ds}\right)$: $1 = \vec{u}\cdot\vec{u} = g_{\alpha\beta}u^\alpha u^\beta$

$$= g_{\theta\theta}u^\theta u^\theta + \cancel{g_{\theta\varphi}}^{\,0} u^\theta u^\varphi + \cancel{g_{\varphi\theta}}^{\,0} u^\varphi u^\theta + g_{\varphi\varphi}u^\varphi u^\varphi$$

$$= a^2\left(\frac{d\theta}{ds}\right)^2 + a^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2$$

$$\boxed{1 = a^2\left(\frac{d\theta}{ds}\right)^2 + a^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2}$$

b.) $a^2\sin^2\theta \frac{d\varphi}{ds} = \ell \longrightarrow$ (b)

$$\frac{d\varphi}{ds} = \frac{\ell}{a^2\sin^2\theta}$$

$$\frac{1}{a^2} = \left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2$$

$$\frac{1}{a^2} = \left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{\ell^2}{a^4\sin^4\theta}\right)$$

$$\frac{1}{a^2} = \left(\frac{d\theta}{ds}\right)^2 + \frac{\ell^2}{a^4\sin^2\theta}$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2} - \frac{\ell^2}{a^4\sin^2\theta}$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2}\left[1 - \frac{\ell^2\csc^2\theta}{a^2}\right]$$

$$\boxed{\frac{d\theta}{ds} = \frac{1}{a}\sqrt{1 - \frac{\ell^2\csc^2\theta}{a^2}}}$$

c.) $\frac{d\theta}{d\varphi} = \frac{d\theta/ds}{d\varphi/ds} \longrightarrow \dfrac{\frac{1}{a}\sqrt{1-\frac{\ell^2}{a^2}\csc^2\theta}}{\frac{\ell}{a^2\sin^2\theta}} \rightarrow \dfrac{\frac{\sqrt{1-(\ell^2/a^2)\csc^2\theta}}{a}}{\frac{\ell}{a^2\sin^2\theta}}$

$$\frac{d\theta}{d\varphi} = \frac{a^2\sin^2\theta\sqrt{1-(\ell^2/a^2)\csc^2\theta}}{a\ell} = \frac{a}{\ell}\sin^2\theta\sqrt{1-(\ell^2/a^2)\csc^2\theta}$$

$$\frac{d\varphi}{d\theta} = \frac{\ell}{a}\cdot\frac{\csc^2\theta}{\sqrt{1-(\ell^2/a^2)\csc^2\theta}} \rightarrow \int d\varphi = \frac{\ell}{a}\int\frac{\csc^2\theta}{\sqrt{1-(\ell^2/a^2)\csc^2\theta}}\,d\theta$$

$$\boxed{\varphi(\theta) = \frac{\ell}{a}\int\frac{\csc^2\theta}{\sqrt{1-(\ell^2/a^2)\csc^2\theta}}\,d\theta}$$

Problem 3 continued

d.) $\varphi(\theta) = \frac{\ell}{a}\int \frac{\csc^2\theta}{\sqrt{1-(\ell^2/a^2)\csc^2\theta}}\,d\theta$

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

$$= \frac{\ell}{a}\int \frac{\csc^2\theta}{\sqrt{1-(\ell^2/a^2)(1+\cot^2\theta)}}\,d\theta = \frac{\ell}{a}\int \frac{\csc^2\theta}{\sqrt{1-\ell^2/a^2+\ell^2/a^2\cot^2\theta}}\,d\theta$$

$$= \frac{\ell}{a}\int \frac{\csc^2\theta}{\frac{\ell}{a}\sqrt{a^2/\ell^2-1-\cot^2\theta}}\,d\theta = \int \frac{\csc^2\theta}{\sqrt{a^2/\ell^2-1-\cot^2\theta}}\,d\theta$$

$\gamma = \sqrt{a^2/\ell^2-1}$
$u = \cot\theta$
$du = -\csc^2\theta\, d\theta$:
$d\theta = \frac{-du}{\csc^2\theta}$

$$= \int \frac{-1}{\sqrt{\gamma^2-u^2}}\,du = -\sin^{-1}\left(\frac{u}{\gamma}\right) = -\sin^{-1}\left(\frac{\cot\theta}{\sqrt{a^2/\ell^2-1}}\right)$$

$$\varphi(\theta) = -\sin^{-1}\left(\frac{\cot\theta}{\sqrt{a^2/\ell^2-1}}\right)$$

$$\sin(\varphi(\theta)) = \frac{-\cot\theta}{\sqrt{a^2/\ell^2-1}}$$

$$-\cot\theta = \sqrt{a^2/\ell^2-1}\,(\sin(\varphi(\theta)))$$

$\varphi(\theta) \to \varphi - \varphi_0$

$$\boxed{\cot\theta = \sqrt{a^2/\ell^2-1}\,\sin(\varphi_0-\varphi)}$$

e.) $a\sin\theta$: $a\sin\theta\cot\theta = \sqrt{a^2/\ell^2-1}\,a\sin\theta\sin(\varphi_0-\varphi)$

$\sin(\varphi_0-\varphi) = \sin(\varphi_0)\cos(\varphi) - \sin(\varphi)\cos(\varphi_0)$

$$a\cos\theta = \sqrt{a^2/\ell^2-1}\,a\sin\theta\left(\sin(\varphi_0)\cos(\varphi) - \sin(\varphi)\cos(\varphi_0)\right)$$
$$= \sqrt{a^2/\ell^2-1}\left(a\sin(\theta)\sin(\varphi_0)\cos(\varphi) - a\sin(\theta)\sin(\varphi)\cos(\varphi_0)\right)$$

$x = a\sin(\theta)\cos(\varphi)$
$y = a\sin(\theta)\sin(\varphi)$
$z = a\cos\theta$

$$z = \sqrt{a^2/\ell^2-1}\left(x\cdot\sin(\varphi_0) - y\cos(\varphi_0)\right)$$
$$= \sqrt{a^2/\ell^2-1}\,\sin(\varphi_0)\,x - \sqrt{a^2/\ell^2-1}\,\cos(\varphi_0)\,y$$

$A = \sqrt{a^2/\ell^2-1}\,\sin(\varphi_0)$, $B = \sqrt{a^2/\ell^2-1}\,\cos(\varphi_0)$

$$\boxed{z = Ax - By}$$

Problem 4)

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\varphi^2$$

$$g_{tt} = -c^2 \quad g_{rr} = 1 \quad g_{\varphi\varphi} = r^2$$

a.) $g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}$

$\alpha = t,\ \beta = r,\ \gamma = \varphi$

$$g_{t\delta}\Gamma^{\delta}{}_{r\varphi} = g_{tt}\Gamma^{t}{}_{r\varphi} + g_{tr}\Gamma^{r}{}_{r\varphi} + g_{t\varphi}\Gamma^{\varphi}{}_{r\varphi}$$

$\alpha = t,\ \beta = \varphi,\ \gamma = r$

$$g_{t\delta}\Gamma^{\delta}_{\varphi r} = g_{tt}\Gamma^{t}{}_{\varphi r} + g_{tr}\Gamma^{r}{}_{\varphi r} + g_{t\varphi}\Gamma_{\varphi r}$$

$\alpha = t,\ \beta = r,\ \gamma = r$

$$g_{t\delta}\Gamma^{\delta}_{rr} = g_{tt}\Gamma^{t}{}_{rr} + g_{tr}\Gamma^{r}{}_{rr} + g_{t\varphi}\Gamma^{\varphi}{}_{rr}$$

$\alpha = t,\ \beta = \varphi,\ \gamma = \varphi$

$$g_{t\delta}\Gamma^{\delta}{}_{\varphi\varphi} = g_{tt}\Gamma^{t}{}_{\varphi\varphi} + g_{tr}\Gamma^{r}{}_{\varphi\varphi} + g_{t\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}$$

$\alpha = t,\ \beta = t,\ \gamma = r$

$$g_{t\delta}\Gamma^{\delta}{}_{tr} = g_{tt}\Gamma^{t}{}_{tr} + g_{tr}\Gamma^{r}{}_{tr} + g_{t\varphi}\Gamma^{\varphi}{}_{tr}$$

$\alpha = t,\ \beta = t,\ \gamma = \varphi$

$$g_{t\delta}\Gamma^{\delta}{}_{t\varphi} = g_{tt}\Gamma^{t}{}_{t\varphi} + g_{tr}\Gamma^{r}{}_{t\varphi} + g_{t\varphi}\Gamma^{\varphi}{}_{t\varphi}$$

$\alpha = t,\ \beta = r,\ \gamma = t$

$$g_{t\delta}\Gamma^{\delta}{}_{rt} = g_{tt}\Gamma^{t}{}_{rt} + g_{tr}\Gamma^{r}{}_{rt} + g_{t\varphi}\Gamma^{\varphi}{}_{rt}$$

$\alpha = t,\ \beta = \varphi,\ \gamma = t$

$$g_{t\delta}\Gamma^{\delta}{}_{\varphi t} = g_{tt}\Gamma^{t}{}_{\varphi t} + g_{tr}\Gamma^{r}{}_{\varphi t} + g_{t\varphi}\Gamma^{\varphi}{}_{\varphi t}$$

$\alpha = t,\ \beta = t,\ \gamma = t$

$$g_{t\delta}\Gamma^{\delta}{}_{tt} = g_{tt}\Gamma^{t}{}_{tt} + g_{tr}\Gamma^{r}{}_{tt} + g_{t\varphi}\Gamma^{\varphi}{}_{tt}$$

$\alpha = r,\ \beta = t,\ \gamma = \varphi$

$$g_{r\delta}\Gamma^{\delta}{}_{t\varphi} = g_{rt}\Gamma^{t}{}_{t\varphi} + g_{rr}\Gamma^{r}{}_{t\varphi} + g_{r\varphi}\Gamma^{\varphi}{}_{t\varphi}$$

$\alpha = r,\ \beta = \varphi,\ \gamma = t$

$$g_{r\delta}\Gamma^{\delta}{}_{\varphi t} = g_{rt}\Gamma^{t}{}_{\varphi t} + g_{rr}\Gamma^{r}{}_{\varphi t} + g_{r\varphi}\Gamma^{\varphi}{}_{\varphi t}$$

Problem 4 Continued

$\alpha = r$, $\beta = t$, $\gamma = t$

$$g_{r\delta}\Gamma^{\delta}{}_{tt} = g_{rt}\Gamma^{t}{}_{tt} + g_{rr}\Gamma^{r}{}_{tt} + g_{r\varphi}\Gamma^{\varphi}{}_{tt}$$

$\alpha = r$, $\beta = \varphi$, $\gamma = \varphi$

$$g_{r\delta}\Gamma^{\delta}{}_{\varphi\varphi} = g_{rt}\Gamma^{t}{}_{\varphi\varphi} + g_{rr}\Gamma^{r}{}_{\varphi\varphi} + g_{r\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}$$

$\alpha = r$, $\beta = r$, $\gamma = t$

$$g_{r\delta}\Gamma^{\delta}{}_{rt} = g_{rt}\Gamma^{t}{}_{rt} + g_{rr}\Gamma^{r}{}_{rt} + g_{r\varphi}\Gamma^{\varphi}{}_{rt}$$

$\alpha = r$, $\beta = r$, $\gamma = \varphi$

$$g_{r\delta}\Gamma^{\delta}{}_{r\varphi} = g_{rt}\Gamma^{t}{}_{r\varphi} + g_{rr}\Gamma^{r}{}_{r\varphi} + g_{r\varphi}\Gamma^{\varphi}{}_{r\varphi}$$

$\alpha = r$, $\beta = t$, $\gamma = r$

$$g_{r\delta}\Gamma^{\delta}{}_{tr} = g_{rt}\Gamma^{t}{}_{tr} + g_{rr}\Gamma^{r}{}_{tr} + g_{r\varphi}\Gamma^{\varphi}{}_{r\varphi}$$

$\alpha = r$, $\beta = \varphi$, $\gamma = r$

$$g_{r\delta}\Gamma^{\delta}{}_{\varphi r} = g_{rt}\Gamma^{t}{}_{\varphi r} + g_{rr}\Gamma^{r}{}_{\varphi r} + g_{r\varphi}\Gamma^{\varphi}{}_{\varphi r}$$

$\alpha = r$, $\beta = r$, $\gamma = r$

$$g_{r\delta}\Gamma^{\delta}{}_{rr} = g_{rt}\Gamma^{t}{}_{rr} + g_{rr}\Gamma^{r}{}_{rr} + g_{r\varphi}\Gamma^{\varphi}{}_{rr}$$

$\alpha = \varphi$, $\beta = r$, $\gamma = t$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{rt} = g_{\varphi t}\Gamma^{t}{}_{rt} + g_{\varphi r}\Gamma^{r}{}_{rt} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{rt}$$

$\alpha = \varphi$, $\beta = t$, $\gamma = r$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{tr} = g_{\varphi t}\Gamma^{t}{}_{tr} + g_{\varphi r}\Gamma^{r}{}_{tr} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{tr}$$

$\alpha = \varphi$, $\beta = r$, $\gamma = r$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{rr} = g_{\varphi t}\Gamma^{t}{}_{rr} + g_{\varphi r}\Gamma^{r}{}_{rr} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{rr}$$

$\alpha = \varphi$, $\beta = t$, $\gamma = t$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{tt} = g_{\varphi t}\Gamma^{t}{}_{tt} + g_{\varphi r}\Gamma^{r}{}_{tt} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{tt}$$

$\alpha = \varphi$, $\beta = \varphi$, $\gamma = r$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi r} = g_{\varphi t}\Gamma^{t}{}_{\varphi r} + g_{\varphi r}\Gamma^{r}{}_{\varphi r} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi r}$$

Problem 4] Continued

$\alpha = \varphi$, $\beta = \varphi$, $\gamma = t$

$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi t} = g_{\varphi t}\Gamma^{t}{}_{\varphi t} + g_{\varphi r}\Gamma^{r}{}_{\varphi t} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi t}$

$\alpha = \varphi$, $\beta = r$, $\gamma = \varphi$

$g_{\varphi\delta}\Gamma^{\delta}{}_{r\varphi} = g_{\varphi t}\Gamma^{t}{}_{r\varphi} + g_{\varphi r}\Gamma^{r}{}_{r\varphi} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{r\varphi}$

$\alpha = \varphi$, $\beta = t$, $\gamma = \varphi$

$g_{\varphi\delta}\Gamma^{\delta}{}_{t\varphi} = g_{\varphi t}\Gamma^{t}{}_{t\varphi} + g_{\varphi r}\Gamma^{r}{}_{t\varphi} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{t\varphi}$

$\alpha = \varphi$, $\beta = \varphi$, $\gamma = \varphi$

$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi\varphi} = g_{\varphi t}\Gamma^{t}{}_{\varphi\varphi} + g_{\varphi r}\Gamma^{r}{}_{\varphi\varphi} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}$

b.) $$g_{\alpha\delta}\Gamma^{\delta}{}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

$\alpha = \varphi$, $\beta = \varphi$, $\gamma = r$

$$g_{\varphi\delta}\Gamma^{\delta}{}_{\varphi r} = \cancel{g_{\varphi t}\Gamma^{t}{}_{\varphi r}}^{\,0} + \cancel{g_{\varphi r}\Gamma^{r}{}_{\varphi r}}^{\,0} + g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi r}$$

$$g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi r} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial r} + \cancel{\frac{\partial g_{\varphi r}}{\partial \varphi}} - \cancel{\frac{\partial g_{\varphi r}}{\partial \varphi}}\right) = \frac{1}{2}(2r) = r$$

$$g_{\varphi\varphi}\Gamma^{\varphi}{}_{\varphi r} = r \;:\; r^2\Gamma^{\varphi}{}_{\varphi r} = r \;:\; \Gamma^{\varphi}{}_{\varphi r} = \frac{1}{r}$$

$\alpha = r$, $\beta = \varphi$, $\gamma = \varphi$

$$g_{r\delta}\Gamma^{\delta}{}_{\varphi\varphi} = \cancel{g_{rt}\Gamma^{t}{}_{\varphi\varphi}}^{\,0} + g_{rr}\Gamma^{r}{}_{\varphi\varphi} + \cancel{g_{r\varphi}\Gamma^{\varphi}{}_{\varphi\varphi}}^{\,0}$$

$$g_{rr}\Gamma^{r}{}_{\varphi\varphi} = \frac{1}{2}\left(\frac{\partial g_{r\varphi}}{\partial \varphi} + \frac{\partial g_{r\varphi}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial r}\right) = \frac{1}{2}(-2r) = -r$$

$$g_{rr}\Gamma^{r}{}_{\varphi\varphi} = -r \;:\; (1)\Gamma^{r}{}_{\varphi\varphi} = -r \;:\; \Gamma^{r}{}_{\varphi\varphi} = -r$$

$$\boxed{\Gamma^{r}{}_{\varphi\varphi} = -r,\; \Gamma^{\varphi}{}_{\varphi r} = \frac{1}{r}}$$

c.) $$\frac{d^2x^{\alpha}}{d\tau^2} = -\Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau}$$

No $\Gamma^{\alpha}{}_{\beta\gamma}$ w/ $\alpha = t$ $\therefore$ $\boxed{\dfrac{d^2t}{d\tau^2} = 0}$

Problem 4 Continued

$\alpha = r \, , \; \beta = \varphi \, , \; \gamma = \varphi$

$$\frac{d^2 r}{d\lambda^2} = -(-r)\frac{d\varphi}{d\lambda}\frac{d\varphi}{d\lambda} \quad : \quad \frac{d^2 r}{d\lambda^2} = r\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\frac{d^2 r}{d\lambda^2} - r\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \quad : \quad r\left(\frac{d\varphi}{d\lambda}\right)^2 - r\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \quad \checkmark$$

$$\boxed{\frac{d^2 r}{d\lambda^2} - r\left(\frac{d\varphi}{d\lambda}\right)^2 = 0}$$

$\alpha = \varphi \, , \; \beta = \varphi \, , \; \gamma = r$

$$\frac{d^2 \varphi}{d\lambda^2} = -\Gamma^{\alpha}{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = -\Gamma^{\varphi}{}_{\varphi r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} - \Gamma^{\varphi}{}_{r\varphi}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$$

$$= -\frac{1}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} - \frac{1}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = \frac{-2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = \frac{d^2\varphi}{d\lambda^2}$$

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = -\frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\boxed{\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0}$$

d.) $\underset{\sim}{u}\cdot\underset{\sim}{u} = 0 \quad : \quad \underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta$

$$= g_{tt}u^t u^t + g_{rr}u^r u^r + g_{\varphi\varphi}u^\varphi u^\varphi = 0$$

$u^t = \frac{dx^t}{d\tau} \, , \; g_{tt} = -c^2$

$u^r = \frac{dx^r}{d\tau} \, , \; g_{rr} = 1$

$u^\varphi = \frac{dx^\varphi}{d\tau} \, , \; g_{\varphi\varphi} = r^2$

$$= (-c^2)\left(\frac{dx^t}{d\tau}\right)^2 + (1)\left(\frac{dx^r}{d\tau}\right)^2 + (r^2)\left(\frac{dx^\varphi}{d\tau}\right)^2$$

$$\boxed{\underset{\sim}{u}\cdot\underset{\sim}{u} = -c^2\left(\frac{dx^t}{d\tau}\right)^2 + \left(\frac{dx^r}{d\tau}\right)^2 + r^2\left(\frac{dx^\varphi}{d\tau}\right)^2}$$

e.) $\underset{\sim}{v} = (t, r, \varphi)$ $\qquad \underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta \, , \; \underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta$

$\underset{\sim}{\xi} = (1, 0, 0) \; \underset{\sim}{\eta} = (0, 0, 1)$

$$\underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{tt}\xi^t u^t + g_{rr}\cancel{\xi^r}^{\,0}u^r + g_{\varphi\varphi}\cancel{\xi^\varphi}^{\,0}u^\varphi = g_{tt}\xi^t u^t = -c^2\frac{dt}{ds} = -e$$

$$\underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{tt}\cancel{\eta^t}^{\,0}u^t + g_{rr}\cancel{\eta^r}^{\,0}u^r + g_{\varphi\varphi}\eta^\varphi u^\varphi = g_{\varphi\varphi}\eta^\varphi u^\varphi = r^2\frac{d\varphi}{ds} = \ell$$

$$\boxed{\ell = r^2\frac{d\varphi}{d\lambda} \, , \; -e = -c^2\frac{dt}{d\lambda}}$$

Problem 5

a.) $\frac{d^2t}{d\lambda^2}=0$: $\frac{dt}{d\lambda}=\frac{e}{c^2}$ $\quad \frac{d^2t}{d\lambda^2}=0$ ✓

$\frac{d\varphi}{d\lambda}=\frac{\ell}{r^2}$: $\frac{d^2\varphi}{d\lambda^2}=0$: $\frac{d}{d\lambda}(\ell)=\frac{d}{d\lambda}(r^2)\frac{d\varphi}{d\lambda}$: $0=\frac{d}{dr}(r^2)\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$

$\therefore\ 0+0=0$ ✓

$\frac{0}{r^2}=\frac{(2r)}{r^2}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$

$0=\frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$

$$\boxed{\frac{d^2t}{d\lambda^2}=0 : \frac{d^2\varphi}{d\lambda^2}+\frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}=0}$$

b.) $\frac{dr}{d\lambda}=\left[\frac{e^2}{c^2}-\frac{\ell^2}{r^2}\right]^{1/2}$: $0=-c^2\frac{dt^2}{d\lambda^2}+\frac{dr^2}{d\lambda^2}+r^2\frac{d\varphi^2}{d\lambda^2}$

$=-c^2\left(\frac{e}{c^2}\right)^2+r^2\left(\frac{\ell}{r^2}\right)^2+\frac{dr^2}{d\lambda^2}$

$\frac{e^2}{c^2}-\frac{\ell^2}{r^2}=\frac{dr^2}{d\lambda^2}$

$$\boxed{\frac{dr}{d\lambda}=\left[\frac{e^2}{c^2}-\frac{\ell^2}{r^2}\right]^{1/2}}$$ ✓

c.)

$\int d\lambda=\int\frac{dr}{\left(\frac{e^2}{c^2}-\frac{\ell^2}{r^2}\right)^{1/2}}$: $\lambda(r)=\lambda_0+\int\frac{r\,dr}{\left(\frac{e^2}{c^2}r^2-\ell^2\right)^{1/2}}=\lambda_0+\frac{c^2}{e^2}\left[\frac{e^2}{c^2}r^2-\ell^2\right]^{1/2}$

Solving for $r(\lambda)$

$\frac{e^4}{c^4}(\lambda-\lambda_0)^2=\frac{e^2}{c^2}r^2-\ell^2$: $r^2=\frac{c^2}{e^2}\left(\frac{e^4}{c^4}(\lambda-\lambda_0)^2+\ell^2\right)$

$r^2=\frac{e^2}{c^2}\left((\lambda-\lambda_0)^2+\frac{\ell^2c^4}{e^4}\right)$ $\therefore$ $$\boxed{r(\lambda)=\frac{e}{c}\left[(\lambda-\lambda_0)^2+\frac{\ell^2c^4}{e^4}\right]^{1/2}}$$

d.) $\frac{d\varphi}{d\lambda}=\frac{\ell}{r^2}$

$r^2=\frac{e^2}{c^2}\left[(\lambda-\lambda_0)+\frac{c^4\ell^2}{e^4}\right]$: $\frac{d\varphi}{d\lambda}=\frac{\ell}{\frac{e^2}{c^2}\left[(\lambda-\lambda_0)^2+\frac{c^4\ell^2}{e^4}\right]}$

$\int d\varphi=\frac{\ell c^2}{e^4}\int\frac{d\lambda}{\left[(\lambda-\lambda_0)^2+\frac{c^4\ell^2}{e^4}\right]}=\varphi_0+\mathrm{Tan}^{-1}\left(\frac{\lambda-\lambda_0}{k}\right)$

$$\boxed{\varphi(\lambda)=\varphi_0+\mathrm{Tan}^{-1}\left(\frac{\lambda-\lambda_0}{k}\right)}$$