Electromagnetic Theory II: Homework 4

Due: 8 February 2021

1 Work and polarization

When a linear dielectric is placed in a field produced by free charges, the polarization of the dielectric changes and this will require work. Thus the work required to assemble the free charges, W_{free}, will be different to the "electrostatic" work,

$$W_{\text{electrostatic}} = \frac{\epsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}\tau,$$

required to assemble the entire charge distribution. This exercise analyzes this situation.

a) In general $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$. Using this, relate the work done to assemble the free charges to the electrostatic work and an integral involving the polarization. Assuming that the polarization is parallel to the electric field, is W_{free} larger than, smaller than or the same as $W_{\text{electrostatic}}$?

To explain this, consider the process of increasing the polarization. The simplest model for this is a dipole consisting of two point charges.

b) Consider a dipole consisting of two opposite point charges, whose charges are equal in magnitude. If an external force increases the dipole moment of this, does this force do positive or negative work? Explain your answer without doing any calculations.

c) Suppose that a dipole consists of two point charges with charges $+q$ and $-q$. If the dipole has dipole moment with magnitude p show that the work supplied to create the dipole by moving a point charge with charge $-q$ from infinity to the appropriate location near the point charge with charge $+q$ is

$$W = -\frac{q^3}{4\pi\epsilon_0 p}.$$

If the dipole is stretched without changing the charge, the dipole moment increases. What does the formula predict for the stored energy?

This should provide a qualitative explanation for the difference between W_{free} and $W_{\text{electrostatic}}$.

2 Energy required to charge a capacitor

A parallel plate capacitor has plates of area A separated by distance d.

a) The space between the capacitor is filled with a dielectric material of permittivity ϵ. Determine an expression for the work required to charge the capacitor if the potential difference across the capacitor is ΔV.

b) Now consider a similar capacitor but between whose plates the dielectric only fills half of the space.

Determine the work required to charge the capacitor if the potential difference across the capacitor is ΔV.

3 Force on a dielectric on a cylindrical capacitor

A cylindrical capacitor consists of an cylindrical rod of radius a surrounded by an outer cylindrical shell of radius b. Each have length $L \gg a, b$. The capacitor plates are connected to a battery which provides potential difference ΔV.

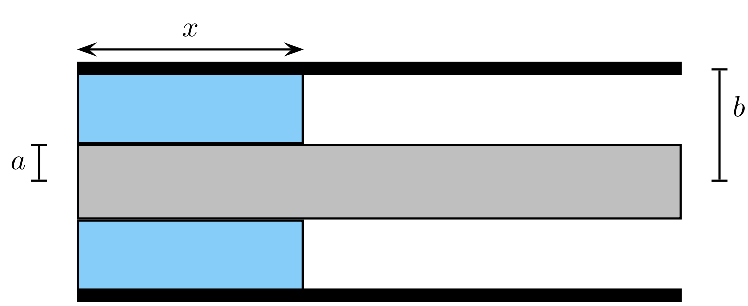
This problem uses the fact that the energy stored in a capacitor is always

$$W = \frac{1}{2}\, C \left(\Delta V\right)^2 .$$

a) Show that the capacitance of this arrangement is

$$C = 2\pi L \frac{\epsilon_0}{\ln\left(b/a\right)}.$$

Now suppose that the capacitor is partly filled by a dielectric with dielectric constant ϵ. The dielectric penetrates to depth x as illustrated.



b) Using qualitative arguments describe the direction of the force exerted by the capacitor on the dielectric.

c) Determine the capacitance of this arrangement. *Hint: This is effectively two capacitors connected in parallel.*

d) Determine the energy stored in this arrangement as a function of $x, L, a, b, \Delta V$ and the electric susceptibility.

e) In general the force associated with any energy, U, is

$$F = -\frac{dU}{dx}.$$

Use this to show that the force exerted by the capacitor on the dielectric has magnitude

$$F = \frac{\pi\epsilon_0}{\ln(b/a)} (\Delta V)^2 \chi_e.$$

f) Suppose that this capacitor is arranged vertically and that the bottom is immersed in water. The radius of the inner cylinder is 2.0 cm and the outer cylinder is 3.0 cm. Determine the voltage across the capacitor so that the water rises to a height of 5.0 mm.