Taylor Larrechea
Dr. Gustafson
MATH 366
CP Ch. 21.3

1) Example 1 Euler Method for a Second-order ODE. Mass-Spring System

Solve the initial value problem for a damped mass-spring system

$$y''+2y'+0.75y=0, \quad y(0)=3, \quad y'(0)=-2.5$$

by the Euler method for systems with step $h=0.2$ for x from 0 to 1 (where x is time).

Solution. The Euler Method (3), Sec. 21.1, generalizes to systems in the form

(5) $$y_{n+1} = y_n + hf(x_n, y_n)$$

in components

$$y_{1,n+1} = y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}),$$
$$y_{2,n+1} = y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n})$$

and similarly for systems of more than two equations. By (4) the given ODE converts to the system

$$y_1' = f_1(x, y_1, y_2) = y_2$$
$$y_2' = f_2(x, y_1, y_2) = -2y_2 - 0.75y_1.$$

Hence (5) becomes

$$y_{1,n+1} = y_{1,n} + 0.2y_{2,n}$$
$$y_{2,n+1} = y_{2,n} + 0.2(-2y_{2,n} - 0.75y_{1,n}).$$

The initial conditions are $y(0) = y_1(0) = 3$, $y'(0) = y_2(0) = -2.5$. The calculations are shown in Table 21.10. As for single ODEs, the results would not be accurate enough for practical purposes. The example merely serves to illustrate the method because the problem can be readily solved exactly,

$$y = y_1 = 2e^{-0.5x} + e^{-1.5x}, \quad \text{thus} \quad y' = y_2 = -e^{-0.5x} - 1.5e^{-1.5x}$$

2) Example 2 RK Method for Systems. Airy's Equation. Airy Function $Ai(x)$

Solve the initial value problem

$$y'' = xy, \quad y(0) = 1/(3^{2/3}\cdot\Gamma(2/3)) = 0.35502805, \quad y'(0) = -1/(3^{1/3}\cdot\Gamma(1/3)) = -0.25881940$$

by the Runge-Kutta method for systems with $h=0.2$; do 5 steps. This is Airy's equation[4], which arose in optics (see Ref. [A13], p. 188, listed in App. 1). Γ is the gamma function (see App. A3.1). The initial conditions are such that we obtain a standard solution, the Airy function $Ai(x)$, a special function has been thoroughly investigated: for numeric values, see Ref. [Gen Ref1], pp. 446, 475.

Solution. For $y'' = xy$, setting $y_1 = y$, $y_2 = y_1' = y'$ we obtain the system (4)

$$y_1' = y_2$$
$$y_2' = xy_1.$$

Hence $\mathbf{f} = [f_1, f_2]^T$ in (1) has the components $f_1(x, y) = y_2$, $f_2(x, y) = xy_1$. We now write (6) in components. The initial conditions (6a) are $y_{1,0} = 0.35502805$, $y_{2,0} = -0.25881940$. In (6b) we have fewer subscripts by simply writing $\mathbf{k}_1 = \mathbf{a}$, $\mathbf{k}_2 = \mathbf{b}$, $\mathbf{k}_3 = \mathbf{c}$, $\mathbf{k}_4 = \mathbf{d}$, so that $\mathbf{a} = [a_1, a_2]^T$, etc. Then (6b) takes the form

$$\mathbf{a} = h\begin{bmatrix} y_{2,n} \\ x_n y_{1,n} \end{bmatrix}$$

(6b*)
$$\mathbf{b} = h\begin{bmatrix} y_{2,n} + \tfrac{1}{2}a_2 \\ (x_n + \tfrac{1}{2}h)(y_{1,n} + \tfrac{1}{2}a_1) \end{bmatrix}$$

$$\mathbf{c} = h\begin{bmatrix} y_{2,n} + \tfrac{1}{2}b_2 \\ (x_n + \tfrac{1}{2}h)(y_{1,n} + \tfrac{1}{2}b_1) \end{bmatrix}$$

$$\mathbf{d} = h\begin{bmatrix} y_{2,n} + c_2 \\ (x_n + h)(y_{1,n} + c_1) \end{bmatrix}$$

For example, the second component of $\mathbf{b}$ is obtained as follows. $\mathbf{f}(x, \mathbf{y})$ has the second component $f_2(x, y) = xy_1$. Now in $\mathbf{b}$ $(= \mathbf{k}_2)$ the first argument is

$$x = x_n + \tfrac{1}{2}h.$$

The second argument in $\mathbf{b}$ is

$$\mathbf{y} = \mathbf{y}_n + \tfrac{1}{2}\mathbf{a},$$

and the first component of this is

$$y_1 = y_{1,n} + \tfrac{1}{2}a_1.$$

Together,

$$xy_1 = (x_n + \tfrac{1}{2}h)(y_{1,n} + \tfrac{1}{2}a_1).$$

Similarly for the other components in (6b*). Finally,

(6c*)
$$\mathbf{y}_{n+1} = \mathbf{y}_n + \tfrac{1}{6}(\mathbf{a} + 2\mathbf{b} + 2\mathbf{c} + \mathbf{d}).$$

Table 21.11 shows the values $y(x) = y_1(x)$ of the Airy function $Ai(x)$ and of its derivative $y'(x) = y_2(x)$ as well as of the (rather small!) error of $y(x)$.