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PHYS 321
HW 5

✓ ✓ ✓ ✓
3.1, 3.2, 3.6, Extra Problems

Problem 3.1

a.)

Square Integratable: $F(x) = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \longrightarrow$ (1) : (1) is defined to be finite i.e $< \infty$

$H(x) = F(x) + G(x) \longleftarrow$ F(x) & G(x) are both S.I functions

H(x) is sum of S.I functions

$$H(x) = \int_{-\infty}^{\infty} H^*(x) \cdot H(x)\, dx = \int_{-\infty}^{\infty} (F^*(x) + G^*(x))(F(x) + G(x))\, dx \longrightarrow (2)$$

(2) becomes :

$$H(x) = \underbrace{\int_{-\infty}^{\infty} |F(x)|^2 dx}_{(3)} + \underbrace{\int_{-\infty}^{\infty} F^*(x) G(x)\, dx}_{(4)} + \underbrace{\int_{-\infty}^{\infty} F(x) G^*(x)\, dx}_{(5)} + \underbrace{\int_{-\infty}^{\infty} |G(x)|^2 dx}_{(6)}$$

(3) and (6) are both defined to be finite from (1). (4) and (5) now need to be proved to be finite.

$$\left| \int_a^b f(x)^* g(x)\, dx \right| \le \sqrt{\int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx} \longrightarrow (7)$$

using (7), (4) and (5) turn into

$$\sqrt{\int_{-\infty}^{\infty} |F(x)|^2 dx \int_{-\infty}^{\infty} |G(x)|^2 dx} + \sqrt{\int_{-\infty}^{\infty} |G(x)|^2 dx \int_{-\infty}^{\infty} |F(x)|^2 dx} \quad \therefore \quad H(x) \text{ is now}$$

$$\underbrace{\int_{-\infty}^{\infty} |F(x)|^2 dx}_{(8)} + \sqrt{\underbrace{\int_{-\infty}^{\infty} |F(x)|^2 dx}_{(9)} \underbrace{\int_{-\infty}^{\infty} |G(x)|^2 dx}_{(10)}} + \sqrt{\underbrace{\int_{-\infty}^{\infty} |G(x)|^2 dx}_{(11)} \underbrace{\int_{-\infty}^{\infty} |F(x)|^2 dx}_{(12)}} + \underbrace{\int_{-\infty}^{\infty} |G(x)|^2 dx}_{(13)}$$

Looking at H(x), (8) and (13) are both finite from (1), where as (9), (10), (11), (12) are also finite from (1). Therefore H(x) is finite which proves that the sum of a S.I. functions is itself square integratable.

$$\boxed{H(x) \text{ is S.I}}$$

b.) Conditions to be met : (i) $\langle \beta | \alpha \rangle = \langle \alpha | \beta \rangle^*$
(ii) $\langle \alpha | \alpha \rangle \ge 0$ & $\langle \alpha | \alpha \rangle = 0 \iff |\alpha\rangle = |0\rangle$
(iii) $\langle \alpha | (b|\beta\rangle + c|\gamma\rangle) = b\langle \alpha | \beta \rangle + c\langle \alpha | \gamma \rangle$

$$(i): \langle \alpha | \beta \rangle = \int_{-\infty}^{\infty} \alpha^*(x) \beta(x)\, dx = \int_{-\infty}^{\infty} \left((\alpha^*(x) \beta(x))^* dx \right)^* = \int_{-\infty}^{\infty} \left((\alpha(x) \beta^*(x))\, dx \right)^* = \langle \beta | \alpha \rangle^*$$

Condition 1 met. ✓

(ii) $\langle \alpha | \alpha \rangle = \int_{-\infty}^{\infty} \alpha^*(x)\alpha(x)\,dx = \int_{-\infty}^{\infty} |\alpha(x)|^2\,dx \longrightarrow$ (14)

(14) is only 0 if $|\alpha\rangle$ is 0.

Condition 2 met. ✓

(iii) $\langle \alpha | (b|\beta\rangle + c|\gamma\rangle) = \int_{-\infty}^{\infty} \alpha^*(x)\,b\,\beta(x) + \alpha^*(x)\,c\,\gamma(x)\,dx \longrightarrow$ (15)

(15) becomes : $\underbrace{\int_{-\infty}^{\infty} \alpha^*(x)\,b\,\beta(x)\,dx}_{(16)} + \underbrace{\int_{-\infty}^{\infty} \alpha^*(x)\,c\,\gamma(x)\,dx}_{(17)}$

(16) can be written as $b\int_{-\infty}^{\infty} \alpha^*(x)\beta(x)\,dx = b\langle \alpha | \beta \rangle \longrightarrow$ (18)

(17) is thus $c\int_{-\infty}^{\infty} \alpha^*(x)\gamma(x)\,dx = c\langle \alpha | \gamma \rangle \longrightarrow$ (19)

(18) and (19) combine to $b\langle \alpha|\beta\rangle + c\langle \alpha|\gamma\rangle$ $\therefore \langle \alpha | (b|\beta\rangle + c|\gamma\rangle) = b\langle \alpha|\beta\rangle + c\langle \alpha|\gamma\rangle$

Condition 3 met. ✓

$\boxed{\text{All conditions met. EQ. 3.6 is an inner product.}}$

Problem 3.2

a.) $f(x) = x^{v} \in (0,1) \longrightarrow$ Must be square integratable

$$\int_0^1 x^{2v}\,dx = \frac{x^{2v+1}}{2v+1}\Bigg|_0^1 = \frac{1^{2v+1}}{2v+1} - \frac{0^{2v+1}}{2v+1} \quad : \quad 2v+1 \neq 0 \quad \therefore \quad v \neq -\tfrac{1}{2}$$

Looking at when $v = -\frac{1}{2}$

$$\int_0^1 x^{-1}\,dx = \ln(x)\Big|_0^1 = \cancel{\ln(1)}^{\,0} - \cancel{\ln(0)}^{\,\infty} = -\infty \longrightarrow (20)$$

(20) shows that when $v = -\frac{1}{2}$, $f(x)$ is not square integratable since the result is not finite. From this we know that

$$\boxed{v > -\tfrac{1}{2}}$$

b.)

(i) $v = \frac{1}{2}$

$f(x) = x^{1/2} \in (0,1) \longrightarrow$ Must be square integratable

$$\int_0^1 x\,dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2} \longrightarrow (21)$$

The result of (21) shows that when $v = \frac{1}{2}$, the inner product is finite and thus in Hilbert Space.

$$\boxed{\text{In Hilbert Space}}$$

(ii) $xf(x)$ when $v = \frac{1}{2}$

$$xf(x) = x^{3/2} \in (0,1) \quad : \quad \int_0^1 x^{\frac{3}{2}}\,dx = \frac{2}{5}x^{\frac{5}{2}}\Bigg|_0^1 = \frac{2}{5} \quad : \quad \frac{2}{5} \neq \infty \therefore \text{ in Hilbert Space}$$

$$\boxed{\text{In Hilbert Space}}$$

(iii) $v = \frac{1}{2}$ $\quad f(x) = x^{1/2} \quad \frac{df}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad : \quad \frac{1}{4}\int_0^1 x^{-1}\,dx = \frac{1}{4}\ln(x)\Big|_0^1 = \infty \quad \therefore \notin$ in Hilbert space

$$\boxed{\text{Not in Hilbert Space}}$$

Problem 3.6

$$\hat{Q} = \frac{d^2}{d\varphi^2}$$

(i) is $\hat{Q}$ Hermetian

$$\langle f|\hat{Q}g\rangle = \int_0^{2\pi} i f^*\left(\frac{d^2}{d\varphi^2}\right)g\, d\varphi = i f^* g\Big|_0^{2\pi} - \int_0^{2\pi}\left(\frac{d^2}{d\varphi^2}\right)f^* g\, d\varphi = \langle \hat{Q}f|g\rangle$$

$\therefore$

$\boxed{Q \text{ is Hermetian.}}$

(ii) $Q|f\rangle = q|f\rangle$

$f(\varphi+2\pi n) = f(\varphi) \quad \therefore \quad |f\rangle = Ae^{\pm i\lambda\varphi} \longrightarrow$ (22)

$$\frac{\partial^2}{\partial\varphi^2}(f) = q(f) : \quad \frac{\partial^2}{\partial\varphi^2}\left(Ae^{\pm i\lambda\varphi}\right) = q\left(Ae^{\pm i\lambda\varphi}\right)$$

$$i^2\lambda^2 Ae^{\pm i\lambda\varphi} = qAe^{\pm i\lambda\varphi}$$

$$-\lambda^2 = q$$

$$\lambda = i\sqrt{q}$$

Putting into (22)

$|f\rangle = Ae^{\mp\sqrt{q}\varphi} \longrightarrow$ (23) : (23) are the eigen functions

$f(\varphi+2n\pi) = f(\varphi) : \quad f(\varphi) = Ae^{\mp\sqrt{q}\varphi} \quad : n \in \mathbb{N}$

$\sqrt{q}(2\pi) = 2\pi n i \quad : \quad \sqrt{q} = ni \quad : \quad q = -n^2 \longrightarrow$ (24) : (24) .... values

$$\boxed{\begin{aligned} |f\rangle &= Ae^{\mp\sqrt{q}\varphi} \\ q &= -n^2 \end{aligned}}$$

(iii)

Spectrum : All natural numbers This spectrum is degenerate

Extra Problems

#1) $\hat{A} = \begin{pmatrix} r & s \\ T & u \end{pmatrix}$, $\hat{B} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

$$\boxed{\hat{A}\hat{B} = \begin{pmatrix} (ra_1+sb_1) & (ra_2+sb_2) & (ra_3+sb_3) \\ (Ta_1+ub_1) & (Ta_2+ub_2) & (Ta_3+ub_3) \end{pmatrix}}$$

$\hat{B}\hat{A} \rightarrow$ Not possible.

#2) $\hat{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$, $\hat{B} = \begin{pmatrix} -9 & 0 & 6 \\ 21 & -3 & -24 \end{pmatrix} \longrightarrow$ $\boxed{\text{These matrices cannot be multiplied due to their size of } 2\times 3}$

#3) $\hat{A} = \begin{pmatrix} 1 & 3 & 5 \\ 6 & -7 & -2 \end{pmatrix}$

$$\boxed{A^T = \begin{pmatrix} 1 & 6 \\ 3 & -7 \\ 5 & -2 \end{pmatrix}}$$

#4) $\hat{A} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$, $\hat{A}^T = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}$

$$\boxed{\hat{A}\hat{A}^T = \begin{pmatrix} 5 & 1 \\ 1 & 26 \end{pmatrix}} \;—\; \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{vmatrix} \times \begin{vmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{vmatrix}$$

$$\boxed{\hat{A}^T\hat{A} = \begin{pmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{pmatrix}} \;——\; \begin{vmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{vmatrix} \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{vmatrix}$$

#5) $\hat{A} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \\ 1 & 5 & -2 \end{pmatrix}$ $\quad \det(A): \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \\ 1 & 5 & -2 \end{pmatrix} = (1)(-3) - (2)(3) + (3)(6)$

$$-3 - 6 + 18 = 9$$

$$\boxed{\det(A) = 9}$$

#6) $\vec{\nabla} = \frac{\partial}{\partial x}(\hat{i}) + \frac{\partial}{\partial y}(\hat{j}) + \frac{\partial}{\partial z}(\hat{k})$

$\vec{V} = V_x(\hat{i}) + V_y(\hat{j}) + V_z(\hat{k})$

$$\vec{\nabla}\times\vec{V} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{pmatrix} = \left(\frac{\partial}{\partial y}(V_z) - \frac{\partial}{\partial z}(V_y)\right)(\hat{i}) + \left(\frac{\partial}{\partial z}(V_x) - \frac{\partial}{\partial x}(V_z)\right)(\hat{j}) + \left(\frac{\partial}{\partial x}(V_y) - \frac{\partial}{\partial y}(V_x)\right)(\hat{k})$$

$$\boxed{\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial y}(V_z) - \frac{\partial}{\partial z}(V_y)\right)(\hat{i}) + \left(\frac{\partial}{\partial z}(V_x) - \frac{\partial}{\partial x}(V_z)\right)(\hat{j}) + \left(\frac{\partial}{\partial x}(V_y) - \frac{\partial}{\partial y}(V_x)\right)(\hat{k})}$$

#7) $\hat{A} = \begin{pmatrix} 2 & i & 0 \\ 0 & 1 & -5i \\ 1 & 1-i & 3 \end{pmatrix}$

$$\boxed{Adj(A) = \begin{pmatrix} (8+5i) & (-3i) & (5) \\ (-5i) & (6) & (10i) \\ (-1) & (3i-2) & (2) \end{pmatrix}}$$

#8) $\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$: $A^{+} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \longrightarrow$ Hermetian since $A = A^{+}$

$\hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow$ Not Hermetian since $A \neq A^{+}$

#9)

$\hat{A} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Eigen Values : $\left| \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \frac{3}{4}\hbar^2\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \right| = \frac{3}{4}\hbar^2(1-\lambda)^2 = 0$

$\lambda = 1 \quad \therefore$ Eigenvalue : $\lambda = 1$

$\lambda = 1$

$\frac{3}{4}\hbar^2\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix}$: $x = 0 \cdot y$

$$\boxed{\begin{array}{l} \text{Values} = 1 \\ \text{Vectors} = \frac{3}{4}\hbar^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ \& } \frac{3}{4}\hbar^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}}$$

#10 $\hat{a} = \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix}$ Eigenvalues: $\begin{vmatrix} A-\lambda & 0 & B \\ 0 & C-\lambda & 0 \\ B & 0 & A-\lambda \end{vmatrix} = (A-\lambda)((C-\lambda)(A-\lambda)) - B((C-\lambda)B)$

$= (C-\lambda)(A-\lambda)^2 - B^2(C-\lambda)$

$= ((A-\lambda)^2 - B^2)(C-\lambda) = 0$

$\rightarrow \lambda = C$, $(A-\lambda)^2 - B^2 = 0$ $\quad \lambda = A+B,\ \lambda = A-B$

$(A-\lambda)^2 = B^2$

$A-\lambda = \pm B$

$A \pm B = \lambda$

$\lambda = C$

$$\left[\begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix} - \begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix}\right]\begin{bmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} A-C & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A-C \end{bmatrix}\begin{bmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(A-C)V_{1x} + B(V_{1z}) = 0$: $(A-C)V_{1x} = -B(V_{1z})$: $V_{1x} = \dfrac{B}{(C-A)}(V_{1z})$

$BV_{1x} + (A-C)(V_{1z}) = 0$: $V_{1z} = \dfrac{B}{(C-A)}(V_{1x})$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{B}{C-A} \end{pmatrix}$$

$\lambda = A+B$

$$\left[\begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix} - \begin{pmatrix} A+B & 0 & 0 \\ 0 & A+B & 0 \\ 0 & 0 & A+B \end{pmatrix}\right]\begin{bmatrix} V_{2x} \\ V_{2y} \\ V_{2z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -B & 0 & B \\ 0 & C-A-B & 0 \\ B & 0 & -B \end{bmatrix}\begin{bmatrix} V_{2x} \\ V_{2y} \\ V_{2z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-BV_{2x} + BV_{2z} = 0$: $V_{2x} = -V_{2z}$

$((C-A-B)V_{2y} = 0$: $V_{2y} = 0$

$BV_{2x} - BV_{2z} = 0$

$$V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda = A-B$

$$\left[\begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix} - \begin{pmatrix} A-B & 0 & 0 \\ 0 & A-B & 0 \\ 0 & 0 & A-B \end{pmatrix}\right]\begin{bmatrix} V_{3x} \\ V_{3y} \\ V_{3z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} B & 0 & B \\ 0 & C-A+B & 0 \\ B & 0 & B \end{bmatrix}\begin{bmatrix} V_{3x} \\ V_{3y} \\ V_{3z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$BV_{3x} + BV_{3z} = 0$: $V_{3x} = -V_{3z}$

$(C-A+B)V_{3y} = 0$: $V_{3y} = 0$

$BV_{3x} + BV_{3z} = 0$

$$V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{array}{c} \lambda = C,\ \lambda = A \pm B \\ V_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{B}{C-A} \end{pmatrix},\ V_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \end{array}}$$