MAT 202
Larson – Section 5.8
Hyperbolic Functions

In this section we will investigate a special class of ***exponential functions*** called ***hyperbolic functions***. Even though the algebraic representation of these functions looks and sounds trigonometric, they really are exponential. The name hyperbolic function arose from comparing the area under the semicircle $y = \sqrt{1 - x^2}$ with the area under the hyperbola $y = \sqrt{1 + x^2}$. In finding these values, the inverse sine function is needed, as well as the inverse hyperbolic sine function.

<u>Definitions of the Hyperbolic Functions</u>: (Read sinh x as "the hyperbolic sine of x", etc.)

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

So, what do these exponential functions look like?

<u>Definitions of the Hyperbolic Functions</u>: (Read sinh x as "the hyperbolic sine of x", etc.)

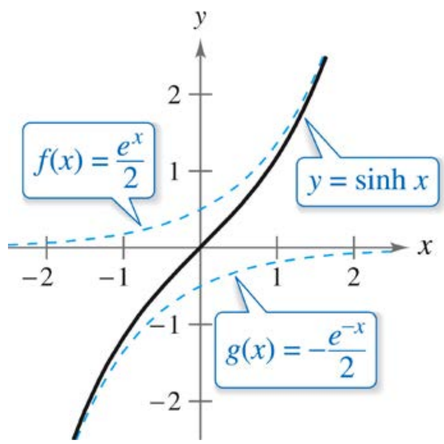
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0 \qquad \frac{2}{e^x - e^{-x}}$$

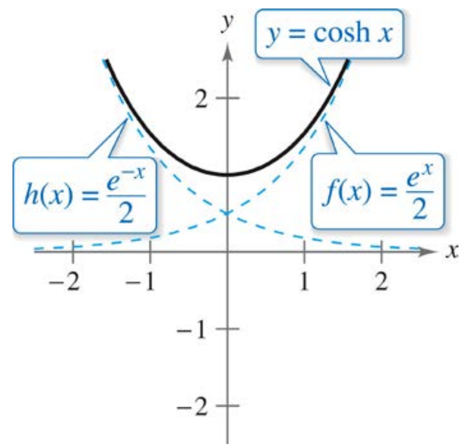
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \frac{2}{e^x + e^{-x}}$$

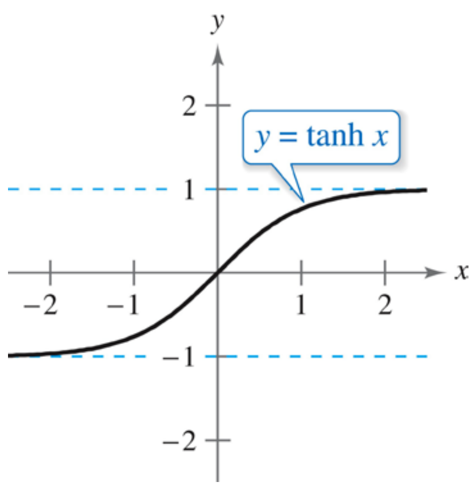
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0 \qquad \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

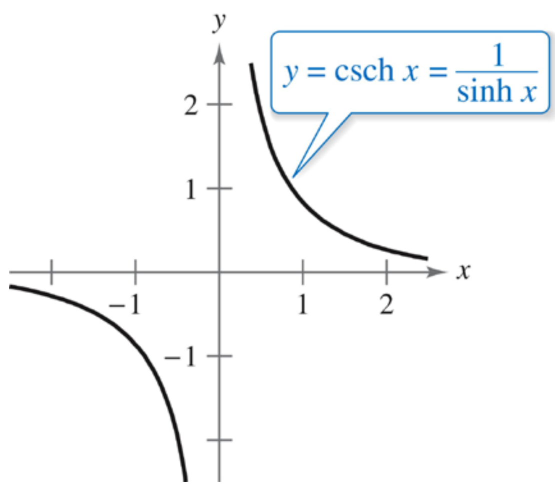
$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

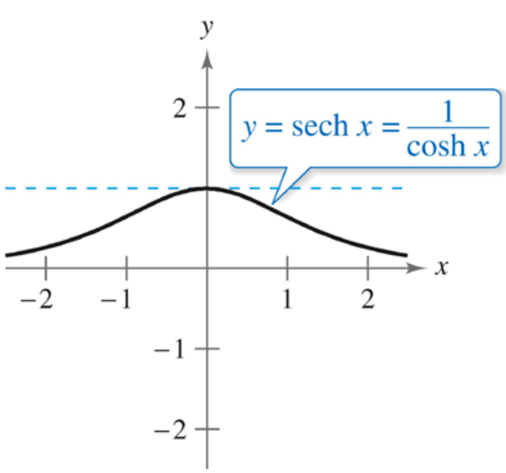
Hyperbolic Functions Defined Graphically:

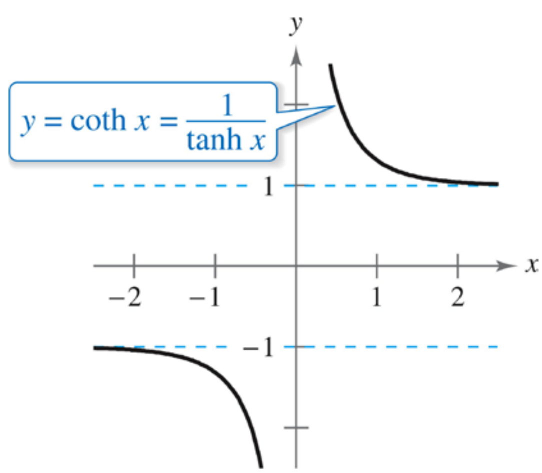












<u>Hyperbolic Identities</u>: (One of the many ways hyperbolic functions relate to trigonometric functions).

$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x + \operatorname{sech}^2 x = 1$$
$$\coth^2 x - \operatorname{csch}^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$
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$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2} \qquad \cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

Because the hyperbolic functions are written in terms of e^x and e^{-x}, we can easily find rules for their derivatives.

<u>Derivatives and Integrals of Hyperbolic Functions</u>: Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u \; du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u \; du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u \; du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u \; du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u \; du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u \; du = -\operatorname{csch} u + C$$

Even though we will not introduce, formally, the ***inverse hyperbolic function*** (which can be defined in a similar fashion as the inverse trigonometric function),we will use the following integration rules that can be derived by using inverse hyperbolic functions.

Additional Integral Formulas:

1. $\int \frac{1}{\sqrt{u^2 \pm a^2}}\,du = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$

2. $\int \frac{1}{a^2 - u^2}\,du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$

3. $\int \frac{1}{u\sqrt{a^2 \pm u^2}}\,du = -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 \pm u^2}}{|u|}\right) + C$

Ex: Use your calculator to compute the following. Round your answer to three decimal places:

a) cosh (-2)

b) csch (ln 2)

Ex: Verify the identity: $e^{2x} = \sinh 2x + \cosh 2x$

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$$\sinh(2x) + \cosh(2x) = \frac{e^{2x} - e^{-2x}}{2} + \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{e^{2x} \cancel{- e^{-2x}} + e^{2x} \cancel{+ e^{-2x}}}{2}$$

$$= \frac{\cancel{2}e^{2x}}{\cancel{2}} = e^{2x}$$

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Ex: Find the derivative of the following:

a) $h(x) = \frac{1}{4}\sinh 2x - \frac{x}{2}$

b) $g(x) = \operatorname{sech}^2 3x$

Ex: Find an equation of the tangent line to the graph of $y = (\cosh x - \sinh x)^2$ at the point (0, 1).

Ex: Find the derivative of the following:

a) $h(x) = \frac{1}{4}\sinh(2x) - \frac{x}{2}$ $=$ $h(x) = \frac{1}{4}\sinh(2x) - \frac{1}{2}x$

$h'(x) = \frac{1}{4}\cosh(2x) \cdot 2 - \frac{1}{2}$

$$\boxed{h'(x) = \frac{1}{2}\cosh(2x) - \frac{1}{2}}$$

b) $g(x) = \text{sech}^2 3x = (\text{sech}(3x))^2$

$g'(x) = 2(\text{sech}(3x))^1 \cdot (-\text{sech}(3x)\tanh(3x) \cdot 3)$

$\boxed{g'(x) = -6\,\text{sech}^2(3x)\tanh(3x)}$

Ex: Find an equation of the tangent line to the graph of

$y = (\cosh x - \sinh x)^2$ at the point (0, 1).

$$y' = 2(\cosh x - \sinh x)(\sinh x - \cosh x)$$

$$m = 2(\cosh(0) - \sinh(0))(\sinh(0) - \cosh(0))$$

$$m = 2(1 - 0)(0 - 1) = -2$$

$$\boxed{y = -2x + 1}$$

Ex: Find the following indefinite integrals:

a) $\int \cosh^2(x-1)\sinh(x-1)\,dx$

b) $\int \operatorname{sech}^2(2x-1)\,dx$

Ex: Evaluate the following: $\int_0^1 \cosh^2 x\,dx$

Ex: Find the following indefinite integrals:

a) $\int \cosh^2(x-1)\sinh(x-1)\,dx$

$$= \int \left(\cosh(x-1)\right)^2 \sinh(x-1)\,dx$$

$u = \cosh(x-1)$

$du = \sinh(x-1)\,dx$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \boxed{\frac{\cosh^3(x-1)}{3} + C}$$

b) $\int \operatorname{sech}^2(2x-1)\,dx = \frac{1}{2}\int \operatorname{sech}^2 u\,du$

$u = 2x-1$

$du = 2\,dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\tanh u + C$

$= \boxed{\frac{1}{2}\tanh(2x-1) + C}$

Ex: Evaluate the following: $\int_0^1 \cosh^2 x \, dx$

$$\int \cosh^2 x \, dx = \int \frac{1+\cosh 2x}{2} \, dx$$

$$= \int \frac{1}{2} \, dx + \frac{1}{2} \int \cosh 2x \, dx$$

$u = 2x$
$du = 2\,dx$
$\frac{1}{2} du = dx$

$$= \frac{1}{2}x + \frac{1}{2} \cdot \frac{1}{2} \int \cosh u \, du$$

$$= \frac{1}{2}x + \frac{1}{4}\sinh(2x) \Big]_0^1$$

$$= \left(\frac{1}{2}(1) + \frac{1}{4}\sinh(2)\right) - \left(0 + \frac{1}{4}\sinh(0)\right)$$

0

$$= \boxed{\frac{1}{2} + \frac{1}{4}\sinh(2)} \approx \boxed{1.407}$$