PHYS 311 HOMEWORK SET 8

10 a) LAPLACE'S EQN IN SPHERICAL-POLAR COORDINATES IS..

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

NOW, IF $V = V(r) \neq V(\theta,\phi)$, LAPLACE'S EQN REDUCES TO...

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

SO

$$r^2\frac{dV}{dr} = C_1 \quad \text{SO} \quad \frac{dV}{dr} = \frac{C_1}{r^2} \quad \text{SO} \quad V(r) = \int\frac{C_1}{r^2}$$

$$\therefore \left[V(r) = C_2 - \frac{C_1}{r}\right]$$

b) LAPLACE'S EQN IN CYLINDRICAL COORDINATES IS..

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

NOW, IF $V = V(s) \neq V(\phi, z)$, LAPLACE'S EQN REDUCES TO...

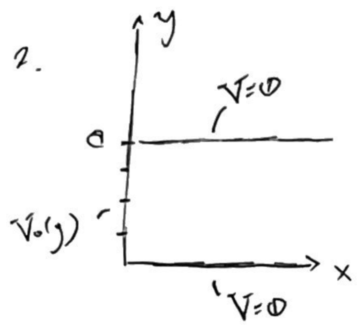
$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$

SO

$$s\frac{dV}{ds} = C_1 \quad \text{SO} \quad \frac{dV}{ds} = \frac{C_1}{s} \quad \text{SO} \quad V(s) = \int\frac{C_1}{s}\,ds$$

$$\therefore \left[V(s) = C_2 + C_1\ln s\right]$$

2.



(*)

WHERE $V_0(y) = V_0$ IF $0 < y < a/4$

$= -V_0/4$ IF $a/4 < y < a$

W/ BOUNDARY CONDITIONS

1) $V(x, y=0) = 0$

2) $V(x, y=a) = 0$

3) $V(x=0, y) = V_0(y)$

4) $\lim_{x \to \infty} V(x,y) \to 0$

IF WE APPLY BOUNDARY CONDITIONS 1), 2), ∴ 4) (AS DONE IN CLASS) WE ARRIVE AT..

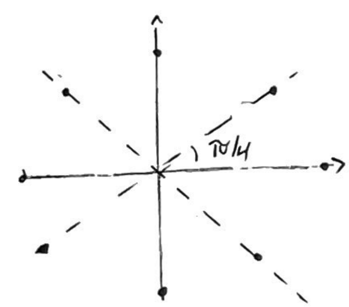
$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \quad \text{w/} \quad C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

NOW USING (*)..

$$C_n = \frac{2}{a}\left[\int_0^{a/4} V_0 \sin\left(\frac{n\pi y}{a}\right) dy + \int_{a/4}^{a} -\frac{V_0}{4} \sin\left(\frac{n\pi y}{a}\right) dy\right] =$$

$$= \frac{2}{a} V_0 \left[\frac{a}{n\pi} \cos\left(\frac{n\pi y}{a}\right)\Big|_{a/4}^{0} - \frac{1}{4}\frac{a}{n\pi}\cos\left(\frac{n\pi y}{a}\right)\Big|_{a}^{a/4}\right]$$

$$= \frac{2}{a} V_0 \frac{a}{n\pi}\left[1 - \cos\left(\frac{n\pi}{4}\right) - \frac{1}{4}\left(\cos\left(\frac{n\pi}{4}\right) - \cos(n\pi)\right)\right]$$



NOW

$\cos(n\pi) = +1$ IF n EVEN

-1 IF n ODD

WHEREAS

$\cos\left(\frac{n\pi}{4}\right) = \frac{1}{\sqrt{2}}$ IF $n = 1, 7, 9, \ldots$

$= 0$ IF $n = 2, 6, 10, \ldots$

$= -1$ IF $n = 4, 12, 20, \ldots$

$= +1$ IF $n = 0, 8, 16, \ldots$

$= -\frac{1}{\sqrt{2}}$ IF $n = 3, 5, 11, 13$

so...

$$C_n = \frac{2}{n\pi} V_0 \left[1 + \frac{1}{4}\cos(n\pi) - \frac{5}{4}\cos\left(\frac{n\pi}{4}\right) \right]$$

so for even n...

$$C_n = \frac{2}{n\pi} V_0 \frac{5}{4} \left[1 - \cos(n\pi/4) \right] \qquad n \text{ even}$$

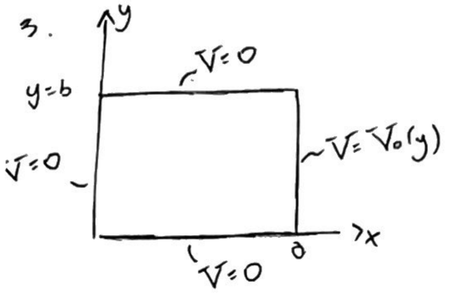
for odd n...

$$C_n = \frac{2}{n\pi} V_0 \cdot \frac{1}{4} \left[3 - 5\cos(n\pi/4) \right] \qquad n \text{ odd}$$

so

$$\boxed{V(x,y) = \frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 + \frac{1}{4}\cos(n\pi) - \frac{5}{4}\cos\left(\frac{n\pi}{4}\right) \right] e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)}$$

3.



THIS POTENTIAL MUST BE INDEPENDENT OF z SO $V = V(x,y)$

LAPLACE'S EQN IS OF THE FORM..

$$(*) \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

CHOOSING A PRODUCT SOLUTION OF THE FORM

$$V(x,y) = X(x)\,Y(y)$$

THE (*) EQN BECOMES..

$$\frac{1}{XY}\left[Y \frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0\right]$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{k^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k^2} = 0$$

so

$$\frac{d^2X}{dx^2} = k^2 X \quad \therefore \quad X(x) = Ae^{kx} + Be^{-kx}$$

$$\frac{d^2Y}{dy^2} = -k^2 Y \quad \therefore \quad Y(y) = C\sin(ky) + D\cos(ky)$$

so

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

NOW APPLY B.C.'S.

1) $V(x, y=0) = (Ae^{kx} + Be^{-kx})D = 0 \quad \therefore \quad D = 0$

2) $V(x, y=b) = (Ae^{kx} + Be^{-kx})C\sin(kb) \quad \therefore \quad kb = n\pi \quad \therefore \quad k_n = \frac{n\pi}{b}$

3) $V(x=0, y) = (A+B)C\sin(ky) = 0 \quad \therefore \quad B = -A$

SO, THE POTENTIAL TAKES THE FORM...

$$V_n(x,y) = A(e^{k_n x} - e^{-k_n x})\, C\sin(k_n y)$$

NOTICE: $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

so

$$V_n(x,y) = (2AC)_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

now let $2AC \to C$

so

$$V_n(x,y) = C_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

So, THE GENERAL SOLUTION IS..

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

NOW, THE LAST BOUNDARY CONDITION...

$$V(x=a,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = V_0(y)$$

MULTIPLY BOTH SIDES BY $\sin\left(\frac{n'\pi y}{b}\right) dy$: INTEGRATE FROM 0 TO b..

$$\int_0^b \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{n'\pi y}{b}\right) dy = \int_0^b V_0(y) \sin\left(\frac{n'\pi y}{b}\right) dy$$

$$= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right) \int_0^b \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{n'\pi y}{b}\right) dy = \frac{b}{2} C_{n'} \sinh\left(\frac{n'\pi a}{b}\right)$$

0 IF $n \neq n'$

$\frac{b}{2}$ IF $n = n'$

so

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \text{w/} \quad C_n = \frac{2}{b} \cdot \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b V_0(y) \sin\left(\frac{n\pi y}{b}\right) dy}$$

b) NOW w/ $V_0(y) = V_0$ ...

$$C_n = \frac{2}{b} \cdot \frac{V_0}{\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b \sin\left(\frac{n\pi y}{b}\right) dy = \frac{2}{b} \cdot \frac{V_0}{\sinh(n\pi a/b)} \cdot \frac{b}{n\pi} \cos\left(\frac{n\pi y}{b}\right)\Big|_b^0$$

$$= \frac{2V_0}{n\pi} \frac{1}{\sinh(n\pi a/b)} \left(1 - \cos(n\pi)\right)$$

so

$$C_n = \frac{4V_0}{n\pi} \cdot \frac{1}{\sinh(n\pi a/b)} \quad \text{for } n \text{ odd}$$

$$= 0 \quad \text{for } n \text{ even}$$

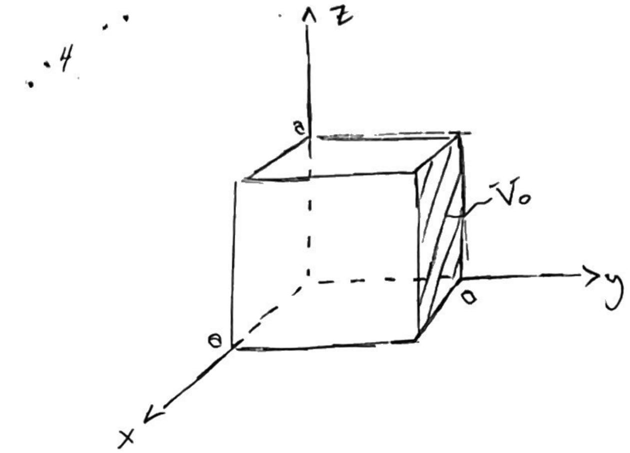
so

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n\ \text{odd}} \frac{1}{n} \cdot \frac{1}{\sinh(n\pi a/b)} \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

so

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n\ \text{odd}} \frac{1}{n} \frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} \sin\left(\frac{n\pi y}{b}\right)}$$

4



LAPLACE'S EQN TAKES THE FORM...

$$(*)\quad \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$

CHOOSE A PRODUCT SOLUTION OF THE FORM..

$$V(x,y,z) = X(x)Y(y)Z(z)$$

PLUGGING THIS PRODUCT SOLUTION INTO (*),..

$$\frac{1}{XYZ}\left[YZ\frac{d^2}{dx^2}X + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0\right]$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)=-k_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)=(k_x^2+k_y^2)} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{h(z)=-k_z^2} = 0$$

WHICH YIELD...

$$\frac{d^2X}{dx^2} = -k_x^2X \quad \therefore\ X(x) = A\sin(k_xx) + B\cos(k_xx)$$

$$\frac{d^2Y}{dy^2} = (k_x^2+k_z^2)Y \quad \therefore\ Y(y) = Ce^{\sqrt{k_x^2+k_z^2}\,y} + De^{-\sqrt{k_x^2+k_z^2}\,y}$$

$$\frac{d^2Z}{dz^2} = -k_z^2Z \quad \therefore\ Z(z) = E\sin(k_zz) + F\cos(k_zz)$$

NOW APPLY B.C.'S..

$$)\ V(x=0,y,z) = B\left(C\sin(k_yy) + D\cos(k_yy)\right)\left(Ee^{\sqrt{k_x^2+k_y^2}\,z} + Fe^{-\sqrt{k_x^2+k_y^2}\,z}\right) = 0 \quad \therefore\ B=0$$

LIKEWISE..

$$2)\ V(x,y,z=0) = 0 \quad \therefore\ F=0$$

$$)\ V(x,y=0,z) = \left(A\sin(k_xx) + B\cos(k_xx)\right)(C+D)\left(E\sin(k_zz) + F\cos(k_zz)\right) = 0$$

$$\therefore\ D = -C$$

THE SOLUTION TAKES THE FORM..

$$V(x,y,z) = A\sin(k_xx)C\left(e^{\sqrt{k_x^2+k_z^2}\,y} - e^{-\sqrt{k_x^2+k_z^2}\,y}\right)E\sin(k_zz)$$

so

$$V(x,y,z) = C \sin(k_x x) \sinh\left(\sqrt{k_x^2 + k_z^2}\, y\right) \sin(k_z z)$$

WHERE I LET $2ACE \to C$; USED THE FACT THAT

$$e^x - e^{-x} = 2\sinh(x)$$

NOW

4) $V(x=a, y, z) = C \sin(k_x a) \sinh\left(\sqrt{k_x^2 + k_z^2}\, y\right) \sin(k_z z) = 0$ $\quad \therefore k_x a = n_x \pi$

so $k_x = \dfrac{n_x \pi}{a}$

LIKEWISE...

5) $V(x,y,z=a) = 0 \quad \therefore k_z a = n_z \pi \quad$ so $\quad k_z = \dfrac{n_z \pi}{a}$

THE SOLUTION NOW TAKES THE FORM..

$$V(x,y,z) = C_{n_x n_z} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) \sinh\left(\sqrt{n_x^2 + n_z^2}\, \frac{\pi y}{a}\right)$$

SO THE GENERAL SOLUTION.

$$V(x,y,z) = \sum_{n_x=1}^{\infty} \sum_{n_z=1}^{\infty} C_{n_x n_z} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) \sinh\left(\sqrt{n_x^2 + n_z^2}\, \frac{\pi y}{a}\right)$$

NOW, THE LAST BOUNDARY CONDITION,

$$V(x,y=a,z) = \sum_{n_x=1}^{\infty} \sum_{n_z=1}^{\infty} C_{n_x n_z} \sinh\left(\sqrt{n_x^2 + n_z^2}\, \pi\right) \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right) = V_0$$

MULTIPLY BOTH SIDES OF THE EQUATION BY $\sin\left(\frac{n_x' \pi x}{a}\right) \sin\left(\frac{n_z' \pi z}{a}\right) dx\, dz$; INTEGRATE...

$$\sum_{n_x=1}^{\infty} \sum_{n_z=1}^{\infty} C_{n_x n_z} \sinh\left(\sqrt{n_x^2 + n_z^2}\, \pi\right) \underbrace{\int_0^a \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_x' \pi x}{a}\right) dx}_{\frac{a}{2}\delta_{n_x' n_x}} \underbrace{\int_0^a \sin\left(\frac{n_z \pi z}{a}\right) \sin\left(\frac{n_z' \pi z}{a}\right) dz}_{\frac{a}{2}\delta_{n_z' n_z}}$$

$$= V_0 \int_0^a \sin\left(\frac{n_x' \pi y}{a}\right) dx \int_0^a \sin\left(\frac{n_z' \pi z}{a}\right) dz$$

WHICH BECOMES..

$$C_{n_x' n_z'} \sinh\left(\sqrt{n_x^2 + n_z^2}\, \pi\right) \frac{a}{2} \cdot \frac{a}{2} = V_0 \frac{a}{n_x' \pi} \cos\left(\frac{n_x' \pi y}{a}\right)\Bigg|_0^a \cdot \frac{a}{n_z' \pi} \cos\left(\frac{n_z' \pi z}{a}\right)\Bigg|_0^a$$

$$= \frac{V_0}{n_x' n_z' \pi^2} \left(1 - \cos(n_x' \pi)\right)\left(1 - \cos(n_z' \pi)\right)$$

So...

$$C_{n_x n_z} = \frac{4V_0}{n_x n_z \pi^2} \cdot \frac{1}{\sinh\left(\sqrt{n_x^2+n_z^2}\,\pi\right)} \quad \text{for } n_x \text{ odd, } n_z \text{ odd}$$

$$= 0 \quad \text{for either } n_x \text{ or } n_z \text{ even (or both!)}$$

so

$$\boxed{V(x,y,z) = \frac{4V_0}{\pi^2} \sum_{n_x \text{ odd}} \sum_{n_z \text{ odd}} \frac{1}{n_x n_z} \cdot \frac{\sinh\left(\sqrt{n_x^2+n_z^2}\,\pi y/a\right)}{\sinh\left(\sqrt{n_x^2+n_z^2}\,\pi\right)} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)}$$