PHYS 396 HOMEWORK SET 11

1a) $H_0 = 72.0 \frac{km}{s \cdot Mpc} \times \frac{1\,Mpc}{1\times10^6\,pc} \times \frac{1\,pc}{3.09\times10^{16}\,m} \times \frac{1000\,m}{1\,km} = 2.33\times10^{-18}\,s^{-1}$

so

$t_H = \frac{1}{H_0} = \frac{1}{2.33\times10^{-18}\,s^{-1}} = 4.3\times10^{17}\,s \times \frac{1\,hr}{3600\,s} \times \frac{1\,day}{24\,hrs} \times \frac{1\,yr}{365\,days} = 13.6\times10^{9}\,yrs$

$\therefore \left[t_H = 13.6\times10^{9}\,yrs \right]$

b) now

$a(t) = a_0 t^{2/3(1+w)}$

so

$\dot{a}(t) = \frac{2}{3(1+w)} a_0 t^{2/3(1+w)-1} = \frac{2}{3(1+w)} \cdot \frac{1}{t} \cdot a_0 t^{2/3(1+w)}$

so

$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = \frac{2}{3(1+w)} \cdot \frac{1}{t}\Big|_0 = \frac{1}{t_H} \quad \therefore \overset{(*)}{\left[t_0 = \frac{2}{3(1+w)} t_H \right]}$

c) for a universe containing only radiation, (*) takes the form..

$t_0\Big|_{w=1/3} = \frac{2}{3(1+\frac{1}{3})} t_H = \frac{1}{2} t_H = \frac{1}{2}(13.6\times10^{9}\,yrs)$

$\therefore \left[t_0\Big|_{w=1/3} = 6.8\times10^{9}\,yrs \right]$

d) for a universe containing only pressure-less matter, (*) takes the form...

$t_0\Big|_{w=0} = \frac{2}{3} t_H = \frac{2}{3}(13.6\times10^{9}\,yrs)$

$\left[t_0\Big|_{w=0} = 9.1\times10^{9}\,yrs \right]$

1 $x = R\sinh\chi \sin\theta\cos\varphi$

$y = R\sinh\chi\sin\theta\sin\varphi$

$z = R\sinh\chi\cos\theta$

$t = R\cosh\chi$

a) $-1 = -t^2+x^2+y^2+z^2 = -R^2\cosh^2\chi + R^2\sinh^2\chi\sin^2\theta\cos^2\varphi + R^2\sinh^2\chi\sin^2\theta\sin^2\varphi + R^2\sinh^2\chi\cos^2\theta$

$$= -R^2\cosh^2\chi + R^2\sinh^2\chi\sin^2\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1} + R^2\sinh^2\chi\cos^2\theta$$

$$= -R^2\cosh^2\chi + R^2\sinh^2\chi\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}$$

$$= -R^2\underbrace{(\cosh^2\chi-\sinh^2\chi)}_{1}$$

$$= -R^2$$

so

$-1 = -t^2+x^2+y^2+z^2$ for $R=1$

b) $dx = R[\sin\theta\cos\varphi\cosh\chi\, d\chi + \sinh\chi\cos\varphi\cos\theta\, d\theta - \sinh\chi\sin\theta\sin\varphi\, d\varphi]$

$dy = R[\sin\theta\sin\varphi\cosh\chi\, d\chi + \sinh\chi\sin\varphi\cos\theta\, d\theta + \sinh\chi\sin\theta\cos\varphi\, d\varphi]$

$dz = R[\cos\theta\cosh\chi\, d\chi - \sinh\chi\sin\theta\, d\theta]$

$dt = R\sinh\chi\, d\chi$

c) Now

$$dx^2+dy^2 = R^2[\sin^2\theta\cos^2\varphi\cosh^2\chi\, d\chi^2 + \sinh^2\chi\cos^2\varphi\cos^2\theta\, d\theta^2 + \sinh^2\chi\sin^2\theta\sin^2\varphi\, d\varphi^2$$
$$+2\sinh\chi\cosh\chi\sin\theta\cos\theta\cos^2\varphi\, d\chi\, d\theta - 2\sinh\chi\cosh\chi\sin\varphi\cos\varphi\sin^2\theta\, d\chi\, d\varphi$$
$$-2\sinh^2\chi\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi] \checkmark$$
$$+R^2[\sin^2\theta\sin^2\varphi\cosh^2\chi\, d\chi^2 + \sinh^2\chi\sin^2\varphi\cos^2\theta\, d\theta^2 + \sinh^2\chi\sin^2\theta\cos^2\varphi\, d\varphi^2$$
$$+2\sinh\chi\cosh\chi\sin\theta\cos\theta\sin^2\varphi\, d\chi\, d\theta + 2\sinh\chi\cosh\chi\sin\varphi\cos\varphi\sin^2\theta\, d\chi\, d\varphi$$
$$+2\sinh^2\chi\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi]$$

NOW GROUPING BY DIFFERENTIALS...

$$\therefore\ dx^2+dy^2 = R^2\Big[\sin^2\theta(\overbrace{\cancel{\cos^2\varphi+\sin^2\varphi}}^{=1})\cosh^2\chi\, d\chi^2 + \sinh^2\chi(\overbrace{\cancel{\cos^2\varphi+\sin^2\varphi}}^{=1})\cos^2\theta\, d\theta^2$$

$$+\sinh^2\chi\sin^2\theta(\underbrace{\cancel{\sin^2\varphi+\cos^2\varphi}}_{1})d\varphi^2 + 2\sinh\chi\cosh\chi\sin\theta\cos\theta(\underbrace{\cancel{\cos^2\varphi+\sin^2\varphi}}_{1})d\chi\, d\theta\Big]$$

$$dx^2+dy^2 = R^2\Big[\sin^2\theta\cosh^2\chi\, d\chi^2 + \sinh^2\chi\cos^2\theta\, d\theta^2 + \sinh^2\chi\sin^2\theta\, d\varphi^2 + 2\sinh\chi\cosh\chi\sin\theta\cos\theta\, d\chi\, d\theta\Big]$$

NOW

1) $$dz^2 = R^2\Big[\cosh^2\chi\cos^2\theta\, d\chi^2 + \sinh^2\chi\sin^2\theta\, d\theta^2 - 2\sinh\chi\cosh\chi\sin\theta\cos\theta\, d\chi\, d\theta\Big]$$

so

$$dx^2+dy^2+dz^2 = R^2\Big[\cosh^2\chi(\underbrace{\cancel{\sin^2\theta+\cos^2\theta}}_{1})d\chi^2 + \sinh^2\chi(\underbrace{\cancel{\cos^2\theta+\sin^2\theta}}_{1})d\theta^2 + \sinh^2\chi\sin^2\theta\, d\varphi^2\Big]$$

so

$$dx^2+dy^2+dz^2 = R^2\Big[\cosh^2\chi\, d\chi^2 + \sinh^2\chi\, d\theta^2 + \sinh^2\chi\sin^2\theta\, d\varphi^2\Big]$$

2) now $dt^2 = R^2\sinh^2\chi\, d\chi^2$

so

$$ds^2 = -dt^2+dx^2+dy^2+dz^2 = R^2\Big[(\underbrace{\cancel{\cosh^2\chi-\sinh^2\chi}}_{1})d\chi^2 + \sinh^2\chi\, d\theta^2 + \sinh^2\chi\sin^2\theta\, d\varphi^2\Big]$$

$$= R^2\Big[d\chi^2 + \sinh^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)\Big] = d\chi^2 + \sinh^2\chi(d\theta^2+\sin^2\theta\, d\varphi^2)$$

1) letting $r \equiv \sinh\chi$

$dr = \cosh\chi\, d\chi$ BUT $\cosh^2\chi - \sinh^2\chi = 1$

so $\cosh\chi = \sqrt{1+\sinh^2\chi}$

so

$$d\chi = \frac{dr}{\cosh\chi} = \frac{dr}{(1+\sinh^2\chi)^{1/2}}$$

$$= \frac{dr}{(1+r^2)^{1/2}} \quad \therefore\ d\chi^2 = \frac{dr^2}{1+r^2}$$

so

$$dS^2 = \frac{dr^2}{(1+r^2)} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$ ✓

3a) For $t = \text{const.}$, $\Theta = \pi/2$, EQN (9) TAKES THE FORM...

(*) $$d\Sigma^2 = d\chi^2 + \sinh^2\chi \, d\varphi^2$$

b) NOW CONSIDER THE 3D LINE ELEMENT OF EUCLIDEAN GEOMETRY IN CYLINDRICAL COORDINATES...

(**) $$dS^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

Let $\rho = \rho(\chi)$ & $z = z(\chi)$

so $$d\rho = \frac{d\rho}{d\chi} d\chi \quad \& \quad dz = \frac{dz}{d\chi} d\chi$$

NOW (**) TAKES THE FORM...

$$dS^2 = \left(\left(\frac{d\rho}{d\chi}\right)^2 + \left(\frac{dz}{d\chi}\right)^2\right) d\chi^2 + \rho^2 d\varphi^2$$

NOW, WE WANT TO EMBED A SURFACE (FIND $z(\chi)$) WHOSE SPATIAL GEOMETRY IS THE SAME AS THE 2D SPATIAL SLICE OF OUR FRW HOMOGENEOUS OPEN UNIVERSE

NOW SET $\rho = \sinh\chi$

so $$\frac{d\rho}{d\chi} = \cosh\chi$$

(**) THEN TAKES THE FORM..

$$dS^2 = \left[\cosh^2\chi + \left(\frac{dz}{d\chi}\right)^2\right] d\chi^2 + \sinh^2\chi \, d\varphi^2$$

NOTICE THAT (*) & (**) MATCH IF WE SET...

$$\cosh^2\chi + \left(\frac{dz}{d\chi}\right)^2 = 1$$

NOW SOLVING FOR $\frac{dz}{d\chi}$...

$$\frac{dz}{d\chi} = \pm\sqrt{1 - \cosh^2\chi}$$

c) TO FIND THIS SURFACE IN 3D EUCLIDEAN SPACE THAT HAS THE SAME SPATIAL CURVATURE AS THE 2D SLICE OF AN FRW OPEN UNIVERSE, SEPARATE & INTEGRATE

$$\frac{dz}{d\chi} = \pm\sqrt{1 - \cosh^2\chi}$$ AS $\cosh^2 x > 1$ FOR ANY x, THIS EXPRESSION IS IMAGINARY FOR ALL χ!

3a) FOR t=CONST., $\theta=\pi/2$, EQN (10) TAKES THE FORM..

(*)

$$d\Sigma^2 = \frac{dr^2}{1+r^2} + r^2 d\varphi^2$$

b) NOW CONSIDER THE 3D LINE ELEMENT OF EUCLIDEAN GEOMETRY IN CYLINDRICAL COORDINATES...

(**)

$$dS^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

Let $\rho = \rho(r)$; $z = z(r)$

so $d\rho = \frac{d\rho}{dr} dr$; $dz = \frac{dz}{dr} dr$

PLUGGING THESE INTO (**) YIELDS THE EQUIVALENT EXPRESSION...

(**)

$$dS^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2$$

NOW, WE WANT TO EMBED A SURFACE, DESCRIBED BY $z=z(r)$, WHICH HAS THE <u>SAME</u> SPATIAL GEOMETRY AS A 2D SPATIAL SLICE OF OUR NEGATIVELY CURVED FLRW COSMOLOGY

SETTING $\rho = r$, (**) TAKES THE FORM..

(**)

$$dS^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\varphi$$

NOTICE THAT FOR (*) ; (**) TO BE THE SAME, WE NEED

$$1 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1+r^2}$$

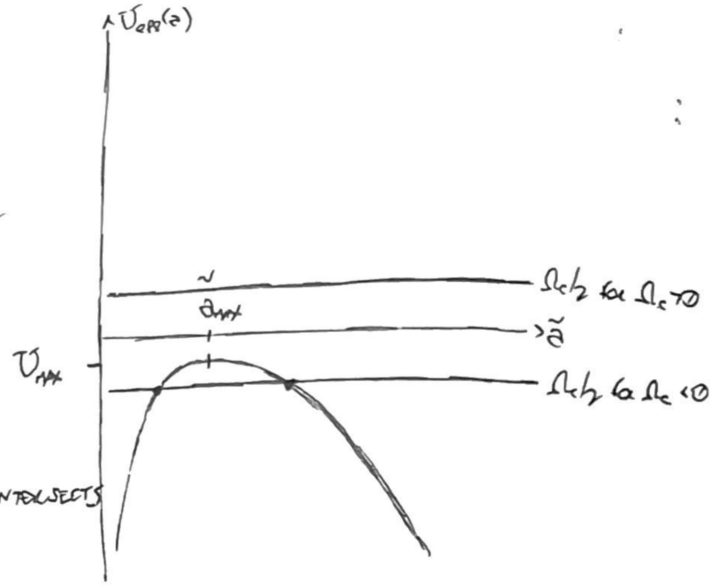
NOW SOLVE THIS DIFFERENTIAL EQN!

$$\left(\frac{dz}{dr}\right)^2 = \frac{1}{1+r^2} - 1 = \frac{-r^2}{1+r^2}$$

TAKING THE SQUARE ROOT..

$$\frac{dz}{dr} = \sqrt{\frac{-r^2}{1+r^2}}$$ – THIS EXPRESSION IS <u>IMAGINARY</u> FOR ALL r!

4. $\frac{1}{2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 + U_{eff}(\tilde{a}) = \frac{\Omega_c}{2}$



Notice that $\Omega_c/2$ plays the role of the conserved total energy, which is negative for $k>0$

$$\Omega_c = -\frac{k}{a_0^2 H_0^2} < 0 \text{ for } k>0 \therefore \text{ positive curvature}$$

We can get a solution exhibiting i. and ii. if the $\Omega_c/2$ line intersects the effective potential energy curve.

To find the maximum of the effective potential energy curve, set

(*) $$0 = \frac{dU_{eff}}{d\tilde a} = \frac{d}{d\tilde a}\left(\Omega_v \tilde a^2 + \frac{\Omega_m}{\tilde a} + \frac{\Omega_r}{\tilde a^2}\right) = 2\Omega_v\tilde a - \frac{\Omega_m}{\tilde a^2} - \frac{2\Omega_r}{\tilde a^3} = 0 \checkmark$$

Considering a universe void of radiation, set $\Omega_r = 0$ and (*) becomes..

$$0 = 2\Omega_v\tilde a - \frac{\Omega_m}{\tilde a^2} \quad \text{or} \quad \tilde a^3 = \frac{\Omega_m}{2\Omega_v} \quad \text{so} \quad \left[\tilde a_{max} = \left(\frac{\Omega_m}{2\Omega_v}\right)^{1/3}\right]$$

— value of $\tilde a$ that yields the maximum potential energy.

Now evaluating the effective potential energy at its maximum..

$$U_{eff}(\tilde a) = -\frac{1}{2}\left(\Omega_v\tilde a^2 + \frac{\Omega_m}{\tilde a}\right)\Big|_{\tilde a_{max}}$$

so

$$U_{eff}(\tilde a)\Big|_{\tilde a_{max}} = -\frac{1}{2\tilde a}\left(\Omega_v\tilde a^3+\Omega_m\right)\Big|_{\tilde a_{max}} = -\frac{1}{2}\left(\frac{2\Omega_v}{\Omega_m}\right)^{1/3}\left[\Omega_v\cdot\frac{\Omega_m}{2\Omega_v}+\Omega_m\right] = -\frac{1}{2}\left(\frac{2\Omega_v}{\Omega_m}\right)^{1/3}\cdot\frac{3}{2}\Omega_m$$

$$= -\frac{3}{4}\left(\frac{2\Omega_v}{\Omega_m}\cdot\Omega_m^3\right)^{1/3} = -\frac{3}{4}\left(2\Omega_v\Omega_m^2\right)^{1/3} \checkmark$$

Now I need...

$$U_{eff}(\tilde a)\Big|_{\tilde a_{max}} > \frac{\Omega_c}{2} = -\frac{|\Omega_c|}{2}$$ since $\Omega_c<0 \therefore \Omega_c = -|\Omega_c|$

or

$$-\frac{3}{4}\left(2\Omega_v\Omega_m^2\right)^{1/3} > -\frac{|\Omega_c|}{2}$$ cubing yields $$-\frac{27}{8}\cdot 2\Omega_v\Omega_m^2 > -|\Omega_c|^3$$

OR :

$$\begin{bmatrix} 27\Omega_v \Omega_m^2 < 4|\Omega_c|^3 \\ \text{AND} \\ 1 = \Omega_c + \Omega_m + \Omega_v \end{bmatrix} \text{ w/ } -1 \le \Omega_c \le 0$$

$\Rightarrow$ THESE TWO CONDITIONS MUST BE MET FOR SCENARIO (i) OR (ii) TO ARISE

SETTING $\Omega_c = -1$, THE CONDITIONS BECOME

$$\begin{matrix} 1) \\ 2) \end{matrix} \begin{bmatrix} 27\Omega_v\Omega_m^2 < 4 \\ 2 = \Omega_m + \Omega_v \end{bmatrix} \text{ OR } \begin{bmatrix} \Omega_v < \dfrac{4}{27\Omega_m^2} \\ \Omega_v = 2 - \Omega_m \end{bmatrix} \Bigg\}$$

SEE ATTACHED GRAPH FOR THE ALLOWED PARAMETER SPACE.

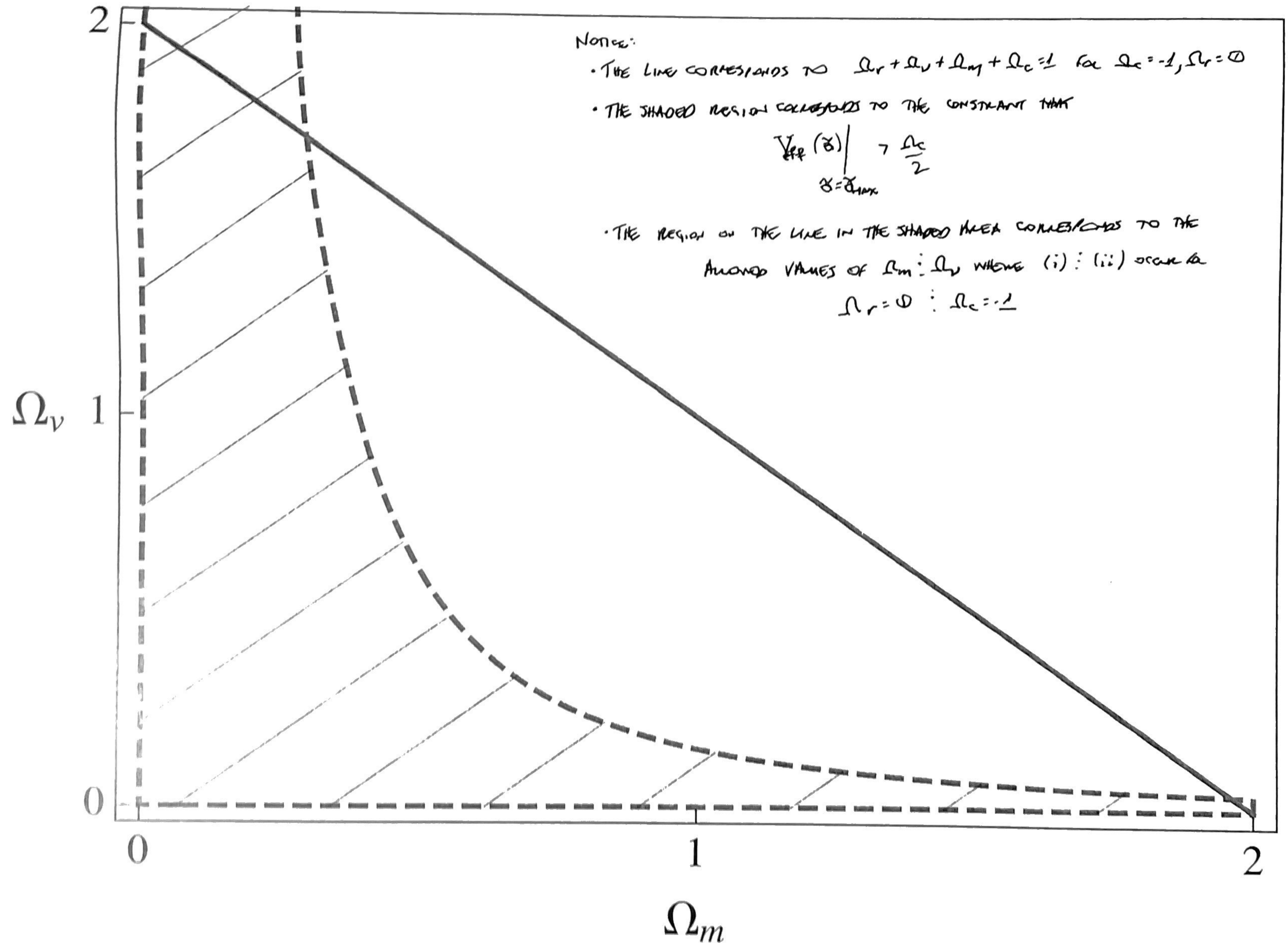
AS AN EXAMPLE, TRY $\Omega_m = 1.98$ & $\Omega_v = 0.02$

NOTICE THAT EQN 2) IS SATISFIED.

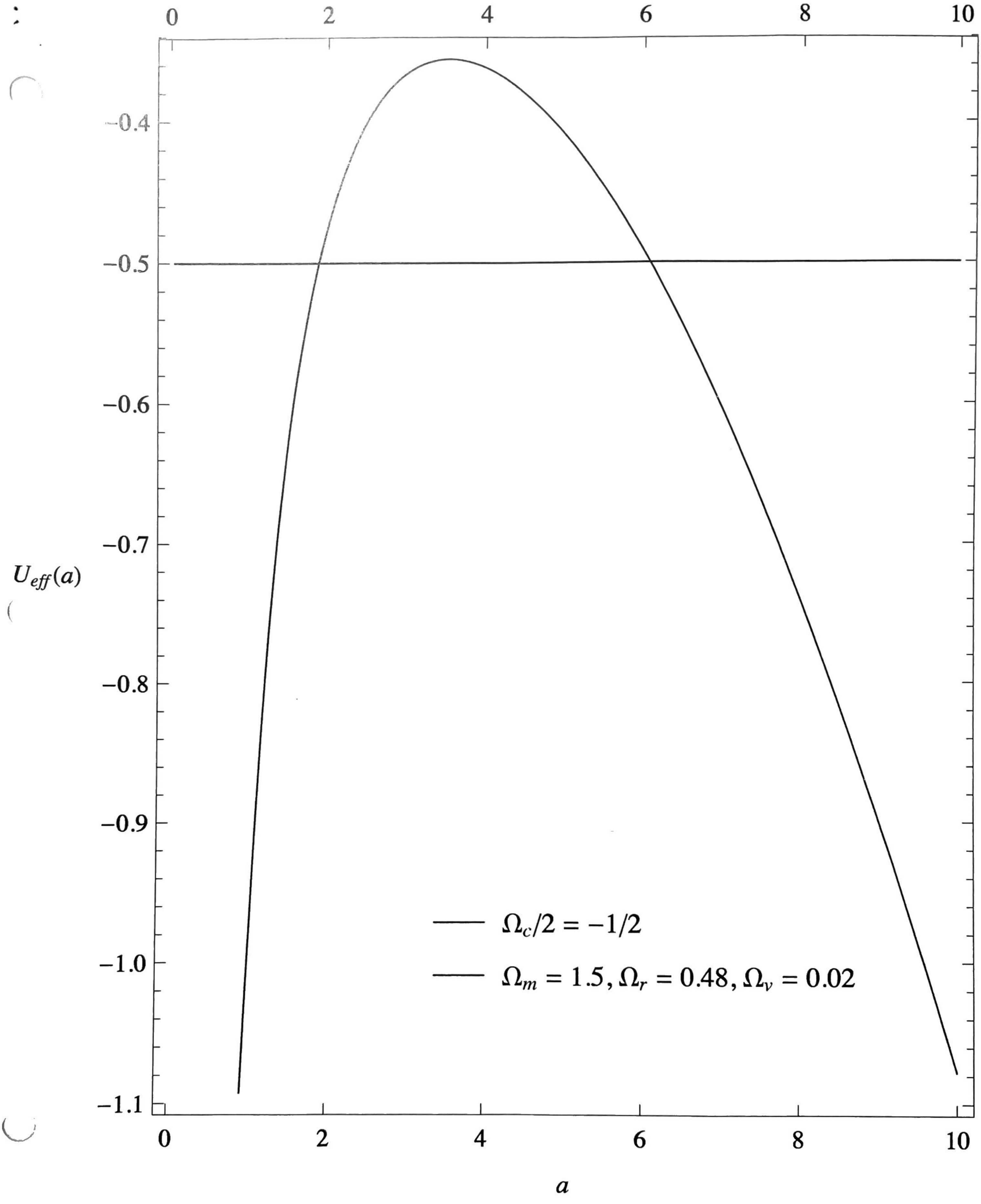
EQN 1) YIELDS

$$27 \cdot 0.02 \cdot (1.98)^2 = 2.12 < 4 \checkmark$$

*ALSO, SEE ATTACHED PLOT OF $U_{eff}(\tilde{a})$, $\Omega_c/2$ VS. $\tilde{a}$ FOR $\Omega_m = 1.5$, $\Omega_r = 0.48$, & $\Omega_v = 0.02$



NOTICE:
• THE LINE CORRESPONDS TO $\Omega_r + \Omega_v + \Omega_m + \Omega_c = 1$ FOR $\Omega_c = -1, \Omega_r = 0$
• THE SHADED REGION CORRESPONDS TO THE CONSTRAINT THAT
$V_{eff}(x)\big|_{x=x_{max}} > \frac{\Omega_c}{2}$
• THE REGION ON THE LINE IN THE SHADED AREA CORRESPONDS TO THE ALLOWED VALUES OF Ω_m & Ω_v WHERE (i) & (ii) OCCUR FOR
$\Omega_r = 0$ & $\Omega_c = -1$
Ω_v
Ω_m
0
1
2



0
2
4
6
8
10
−0.4
−0.5
−0.6
−0.7
−0.8
−0.9
−1.0
−1.1
$U_{eff}(a)$
$\Omega_c/2 = -1/2$
$\Omega_m = 1.5, \Omega_r = 0.48, \Omega_v = 0.02$
a