Phys 362
Spring 2020

Statistical and Thermal Physics: Homework 3

Due: 31 January 2020

1 Thermal expansion

The linear thermal expansion coefficient describes the increase in length of a homogenous material as its temperature changes and is defined by

$$\alpha_1 := \frac{1}{L}\frac{\partial L}{\partial T}$$

where L is the length of the material. *The subscript in α_1 is not standard notation; it is included to distinguish the linear thermal expansion coefficient from the volume thermal expansion coefficient.*

a) For copper, $\alpha_1 = 1.65\times 10^{-5}\,\mathrm{K}^{-1}$. Determine the amount by which a copper rod of length $10\,\mathrm{m}$ will expand if it is heated from -10° C to 30° C, assuming that α is independent of temperature.

b) A rectangular sheet of any material will expand as its temperature increases. Here the coefficient of area expansion is defined by

$$\gamma := \frac{1}{A}\frac{\partial A}{\partial T}$$

where A is the area of the material. Show that

$$\gamma = 2\alpha_1.$$

Determine the amount by which the area of a rectangular copper roof, whose sides are $10\,\mathrm{m}$ and $4\,\mathrm{m}$ increases if it is heated from -10° C to 30° C.

2 Thermal expansion coefficient for a van der Waals gas

Determine an expression for the thermal expansion coefficient for a van der Waals gas. *Hint: note that differentiation of V w.r.t. T will be difficult. There is an identity that we encountered in class that will make this easier.*

3 Isothermal compressibility of an ideal gas

Determine an expression for the isothermal compressibility of an ideal gas, in terms of N, P and T. Show that it is positive.

4 Equation of state for a solid

The state of a solid material can be described by the same variables as for a gas. Suppose that the equation of state of the solid is

$$V = V_0\,(1 + aT - bP)$$

where V_0 is a constant equal to the volume when pressure and temperature are zero and a and b are constants that are very small.

a) Determine expressions for the isothermal compressibility and the isobaric expansion coefficient.

b) Suppose that a and b are so small that at typical temperatures $aT \ll 1$ and $bP \ll 1$. Determine approximate expressions for the isothermal compressibility and the isobaric thermal expansion coefficient in this case. Are they approximately constant or not?

a) $\alpha_1 = \frac{1}{L}\frac{\partial L}{\partial T}$

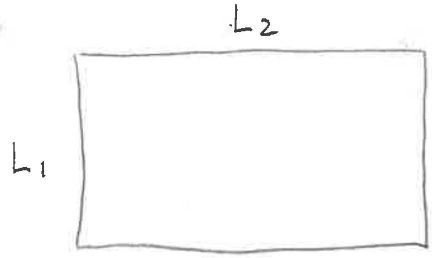
$\approx \frac{1}{L}\frac{\Delta L}{\Delta T} \Rightarrow$ $\Delta L \approx L\alpha_1 \Delta T$

$\Delta T = 30^\circ C - (-10^\circ C) = 40^\circ C$

$= 10m \times 1.65\times10^{-5}\,K^{-1} \times 40K$

$= 6.60 \times 10^{-3}\,m$

b)



area $= A = L_1 L_2$

$$\frac{\partial A}{\partial T} = \frac{\partial L_1}{\partial T} L_2 + L_1 \frac{\partial L_2}{\partial T}$$

$$\frac{1}{L_1}\frac{\partial L_1}{\partial T} = \alpha_1 \Rightarrow \frac{\partial L_1}{\partial T} = \alpha_1 L_1$$

Similarly $\frac{\partial L_2}{\partial T} = \alpha_1 L_2$

So $\frac{\partial A}{\partial T} = \alpha_1 L_1 L_2 + \alpha_1 L_1 L_2 = 2\alpha_1 L_1 L_2 = 2\alpha_1 A$

$\Rightarrow \frac{1}{A}\frac{\partial A}{\partial T} = 2\alpha_1 = \gamma$

Then $\gamma = \frac{1}{A}\frac{\partial A}{\partial T} \approx \frac{1}{A}\frac{\Delta A}{\Delta T}$ $\Rightarrow \Delta A = \gamma A \Delta T$

$= 2\alpha_1 A \Delta T$

$= 2 \times 1.65 \times 10^{-5}\,K^{-1} \times (40m^2) \times 40K$

$= 52.80 \times 10^{-3}\,m^2$

$= 5.28 \times 10^{-2}\,m^2$

Exact calculations:

a) $$\alpha_1 = \frac{1}{L}\frac{dL}{dT} \Rightarrow \alpha_1 dT = \frac{dL}{L}$$

$$\Rightarrow \alpha_1(T-T_0) = \ln L - \ln L_0 = \ln(L/L_0)$$

$$\Rightarrow e^{\alpha_1(T-T_0)} = L/L_0$$

$$\Rightarrow L_0 e^{\alpha_1(T-T_0)} = L$$

In part a) $T = 30°C$ $T_0 = -10°C$ $\Rightarrow$ $T - T_0 = 40°C = 40K$

So $$L = 10m \times e^{1.65\times10^{-5} K^{-1} \times 40 K} = 10m \times 1.00066$$

$$= 10.0066\, m.$$

So $\Delta L = L - L_0 = 0.0066 m$

b) A similar derivation yields $A = A_0 e^{\gamma(T-T_0)} = A_0 e^{2\alpha_1(T-T_0)}$.

Then $$\Delta A = A - A_0 = A_0\left(1 - e^{2\alpha_1(T-T_0)}\right)$$

$$= 40m^2\left(1 - e^{2\times1.65\times10^{-5} K^{-1} 40K}\right)$$

$$= 0.0528 m^2$$

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T}$$

Now $\dfrac{\partial V}{\partial T} = \dfrac{1}{\partial T/\partial V}$ $\Rightarrow$ $\alpha = 1\Big/\left(V\dfrac{\partial T}{\partial V}\right)$

and for the v.d.W gas

$$\left(P + \frac{N^2}{V^2}a\right)(V - Nb) = NkT.$$

$$\Rightarrow \quad T = \frac{1}{Nk}\left(P + \frac{N^2}{V^2}a\right)(V - Nb)$$

$$\Rightarrow \quad \frac{\partial T}{\partial V} = \frac{1}{Nk}\left\{\left(-\frac{N^2}{V^3}2a\right)(V - Nb) + \left(P + \frac{N^2}{V^2}a\right)\right\}$$

$$= \frac{1}{Nk}\left\{-\frac{N^2}{V^2}2a + \frac{N^3 2ab}{V^3} + P + \frac{N^2}{V^2}a\right\}.$$

$$= \frac{1}{Nk}\left\{P - \frac{N^2}{V^2}a + \frac{N^3 2ab}{V^3}\right\}$$

$$V\frac{\partial T}{\partial V} = \frac{1}{Nk}\left\{PV - \frac{N^2}{V}a + \frac{N^3}{V^2}2ab\right\}.$$

$$\Rightarrow \quad \alpha = Nk\Big/\left[PV - \frac{N^2}{V}a + \frac{N^3}{V^2}2ab\right]$$

Solving for P in terms of T for the eqn of state

$$\Rightarrow \quad P = \frac{NkT}{V - Nb} - \frac{N^2}{V^2}a \quad \Rightarrow \quad PV = -\frac{N^2}{V}a + \frac{NkTV}{V - Nb}$$

$$\Rightarrow \quad \alpha = Nk\Big/\left[-\frac{2N^2}{V}a + \frac{N^3}{V^2}2ab + \frac{NkT}{V - Nb}V\right]$$

$$= k\Big/\left[-\frac{2N}{V}a + \frac{N^2}{V^2}2ab + \frac{kTV}{V - Nb}\right]$$

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Def: $K = -\frac{1}{V}\frac{\partial V}{\partial P}$

Ideal gas: $PV = NkT$

$V = \frac{NkT}{P}$ $\qquad \frac{\partial V}{\partial P} = -\frac{NkT}{P^2}$

$K = -\frac{1}{V}\left(\frac{-NkT}{P^2}\right) = \frac{NkT}{P^2 V}$

Then $NkT = PV \Rightarrow K = \frac{PV}{P^2 V} \Rightarrow K = \frac{1}{P}$

a) Isothermal compressibility

$$K = -\frac{1}{V}\frac{\partial V}{\partial P}$$

$$= -\frac{1}{V_0(1+aT-bP)}\left(-bV_0\right)$$

$$\Rightarrow \boxed{K = \frac{b}{1+aT-bP}}$$

Thermal expansion

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T} = \frac{1}{\cancel{V_0}(1+aT-bP)}\, a\cancel{V_0}$$

$$\Rightarrow \boxed{\alpha = \frac{a}{1+aT-bP}}$$

b) $aT \ll 1$, $bP \ll 1$ $\Rightarrow$ $\boxed{\begin{array}{l} K \approx b \\ \alpha \approx a \end{array}}$ Yes constant