Exercise 1

$$H = \frac{4r^2}{1-(1-2\lambda)^2 r^2}, \quad H = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}, \quad H = \frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}$$

Gain of second scenario over first

$$G = \frac{\dfrac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}{\dfrac{4r^2}{1-(1-2\lambda)^2r^2}} = \frac{(\overset{2}{\cancel{8r^2}})(1+r^2)(1-(1-2\lambda)^2r^2)}{(\cancel{4r^2})((1+r^2)^2-4r^2(1-2\lambda)^2)} = \frac{2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2}$$

$$\boxed{G = \frac{2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2}}$$

$$\frac{2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2} > 1 \quad : \quad 2(1+r^2)(1-(1-2\lambda)^2r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - 2r^2(1+r^2)(1-2\lambda)^2 > (1+r^2)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - (1+r^2)^2 > 2r^2(1+r^2)(1-2\lambda)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - (1+r^2)^2 > (1-2\lambda)^2(2r^2(1+r^2) - 4r^2)$$

$$(1-2\lambda) < \sqrt{\frac{2(1+r^2)-(1+r^2)^2}{2r^2(1+r^2)-4r^2}}$$

$$2\lambda - 1 > -\sqrt{\frac{2(1+r^2)-(1+r^2)^2}{2r^2(1+r^2)-4r^2}}$$

$$\lambda > -\frac{1}{2}\sqrt{\frac{2(1+r^2)-(1+r^2)^2}{2r^2(1+r^2)-4r^2}} + \frac{1}{2}$$

$$\boxed{\lambda > \frac{1}{2}\left(1 - \sqrt{\frac{2(1+r^2)-(1+r^2)^2}{2r^2(1+r^2)-4r^2}}\right)}$$

Gain of Third Scenario over First

$$G = \frac{\dfrac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2(1-2\alpha)^2}}{\dfrac{4r^2}{1-(1-2\lambda)^2r^2}} = \frac{\overset{2}{\cancel{8r^2}}(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2r^2)}{(\cancel{4r^2})((1+r^2)^2-4r^2(1-2\lambda)^2(1-2\alpha)^2)} = \frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2(1-2\alpha)^2}$$

$$\boxed{G = \frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2(1-2\alpha)^2}}$$

$$\frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2(1-2\alpha)^2} > 1 \quad : \quad 2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$2(1-2\alpha)^2(1+r^2) - 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$2(1-2\alpha)^2(1+r^2) - (1+r^2)^2 > 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$-(1+r^2)^2 > -2(1-2\alpha)^2(1+r^2) + 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$(1+r^2)^2 < 2(1-2\alpha)^2(1+r^2) - 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) + 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$(1+r^2)^2 < (1-2\alpha)^2(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2)$$

$$(1-2\alpha)^2 > \frac{(1+r^2)^2}{(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2)}$$

Exercise 1 continued

$$(1-2\alpha) > \sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}$$

$$2\alpha - 1 < -\sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}$$

$$\boxed{\alpha < \frac{1}{2}\left(1 - \sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}\right)}$$

Exercise 2

⊘ —— [Depolarizing λ] —— ⊘

a.) $\hat{\rho}_f = \frac{(1-\lambda)}{2} Tr[\hat{\rho}_i]\hat{I} + \lambda \hat{\rho}_i \quad : \quad \hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$

$$\hat{\rho}_f = \frac{(1-\lambda)}{2}(1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda & r\lambda \\ r\lambda & \lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & r\lambda \\ r\lambda & 1 \end{pmatrix}$$

$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & r\lambda \\ r\lambda & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & r\lambda \\ r\lambda & 1 \end{pmatrix}}$$

b.) Calculate QFI

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2 \cdot \frac{\partial \hat{A}}{\partial \lambda} - \frac{\partial \ln(\alpha)}{\partial \lambda}\hat{A}\right] + \frac{\partial}{\partial \lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I} \quad , \quad H_i = Tr\left[L_i \frac{\partial A_i}{\partial \lambda}\right]$$

i.) Find $\hat{L}_1$: $\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{4}\begin{pmatrix} 1 & r\lambda \\ r\lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & r\lambda \\ r\lambda & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r^2\lambda^2 & 2r\lambda \\ 2r\lambda & 1+r^2\lambda^2 \end{pmatrix} \quad : \quad Tr(\hat{A}_1^2) = \frac{1+r^2\lambda^2}{2}$$

$$Tr(\hat{A}_1) = 1 \quad : \quad Tr(\hat{A}_1)^2 = 1$$

$$\alpha = \frac{1+r^2\lambda^2}{2} - \frac{2}{2} = \frac{r^2\lambda^2 - 1}{2}$$

$$\frac{\partial \ln(\alpha)}{\partial \lambda} = \frac{\alpha'}{\alpha} \quad : \quad \alpha' = r^2\lambda \quad \therefore \quad \frac{\partial \ln(\alpha)}{\partial \lambda} = \frac{r^2\lambda}{\frac{r^2\lambda^2-1}{2}} = \frac{2r^2\lambda}{r^2\lambda^2-1} \longrightarrow \beta$$

$$\frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{2}\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}$$

$\hat{L}$ reduces to

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\frac{\partial \hat{A}}{\partial \lambda} - \frac{\partial \ln(\alpha)}{\partial \lambda}\hat{A}\right] + \frac{\partial \ln(\alpha)}{\partial \lambda}\hat{I}$$

$$\hat{L} = \left[\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \beta & \beta \cdot r\lambda \\ \beta \cdot r\lambda & \beta \end{pmatrix}\right] + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2r \\ 2r & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \beta & \beta r\lambda \\ \beta r\lambda & \beta \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2\beta & 0 \\ 0 & 2\beta \end{pmatrix}$$

$$\boxed{\hat{L}_1 = \frac{1}{2}\begin{pmatrix} \beta & 2r - \beta r\lambda \\ 2r - \beta r\lambda & \beta \end{pmatrix}}$$

Exercise 2 continued

ii.) Find H, : $H = \mathrm{Tr}\left[\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda}\right]$

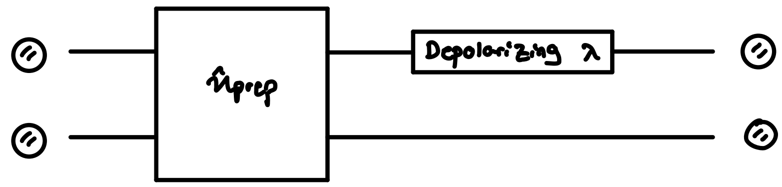
$$\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{2}\begin{pmatrix} \beta & 2r - \beta r \lambda \\ 2r - \beta r \lambda & \beta \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2r^2 - \beta r^2 \lambda & \beta r \\ \beta r & 2r^2 - \beta r^2 \lambda \end{pmatrix}$$

$$\mathrm{Tr}\left[\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda}\right] = \frac{1}{2}\left(2r^2 - \beta r^2 \lambda\right) = \frac{1}{2}\left(2r^2 - \frac{2r^2\lambda}{r^2\lambda^2 - 1}(r^2\lambda)\right) = \frac{1}{2}\left(\frac{\cancel{2r^4\lambda^2} - 2r^2 - \cancel{2r^4\lambda^2}}{r^2\lambda^2 - 1}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \frac{2r^2}{1 - r^2\lambda^2} = \frac{r^2}{1 - r^2\lambda^2}$$

$$\boxed{H = \frac{r^2}{1 - r^2\lambda^2}}$$

Exercise 3



a.) Write $\hat{\rho}_{prep} = \dots |0\rangle|0\rangle\langle 0|\langle 0| + |0\rangle|0\rangle\langle 0|\langle 1| \dots\dots$

$$\hat{\rho}_{prep} = \hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]}$$

b.) Calculate depolarizing channel : $\hat{\rho} = \frac{1-\lambda}{2}Tr[\rho]\hat{I} + \lambda\hat{\rho}_i$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (-8ir)|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| + (4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (8ir)|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

i.) $\hat{\rho}_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$

$$\hat{\rho}_f = \frac{1-\lambda}{2}Tr[\hat{\rho}_i]\hat{I} + \lambda\hat{\rho}_i = \frac{1-\lambda}{2}\hat{I} + \lambda\hat{\rho}_i = \begin{pmatrix} \frac{1-\lambda}{2} & 0 \\ 0 & \frac{1-\lambda}{2} \end{pmatrix} + \lambda\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$$

$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$: $\hat{\rho}_i \rightarrow \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}$

ii.) $\hat{\rho}_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$

$$\hat{\rho}_f = \frac{1-\lambda}{2}Tr[\hat{\rho}_i]\hat{I} + \lambda\hat{\rho}_i = \frac{1-\lambda}{2}\cdot 0\cdot\hat{I} + \lambda\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

$\hat{\rho}_f = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$: $\hat{\rho}_i \rightarrow \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

Exercise 3 Continued

iii.) $\hat{\rho}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$

$$\hat{\rho}_f = \frac{1-\lambda}{2} \mathrm{Tr}[\hat{\rho}_i]\hat{I} + \lambda\hat{\rho}_i = \frac{1-\lambda}{2} 0 \cdot \hat{I} + \lambda\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

$$\hat{\rho}_F = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} : \hat{\rho}_i \longrightarrow \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

iv.) $\hat{\rho}_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$

$$\hat{\rho}_F = \frac{1-\lambda}{2} \mathrm{Tr}[\hat{\rho}_i]\hat{I} + \lambda\hat{\rho}_i = \frac{1-\lambda}{2} \cdot \hat{I} + \lambda\hat{\rho}_i = \begin{pmatrix} \frac{1-\lambda}{2} & 0 \\ 0 & \frac{1-\lambda}{2} \end{pmatrix} + \lambda\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\hat{\rho}_F = \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix} : \hat{\rho}_i \longrightarrow \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix}$$

$$\boxed{\begin{array}{ll} \hat{\rho}_i = |0\rangle\langle 0| \longrightarrow \hat{\rho}_F = \frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} & : \hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_F = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \\ \hat{\rho}_i = |1\rangle\langle 0| \longrightarrow \hat{\rho}_F = \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} & : \hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_F = \frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix} \end{array}}$$

c.) Determine effect on first particle of $\hat{\rho}_{prep}$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\, \underline{\underline{|0\rangle\langle 0|}} \otimes |0\rangle\langle 0| + (-8ir)\, \underline{\underline{|0\rangle\langle 1|}} \otimes |0\rangle\langle 1| + (4-4r^2)\, \underline{\underline{|0\rangle\langle 0|}} \otimes |1\rangle\langle 1| +$$
$$(4-4r^2)\, \underline{\underline{|1\rangle\langle 1|}} \otimes |0\rangle\langle 0| + (8ir)\, \underline{\underline{|1\rangle\langle 0|}} \otimes |1\rangle\langle 0| + (4+4r^2)\, \underline{\underline{|1\rangle\langle 1|}} \otimes |1\rangle\langle 1|\Big]$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\cdot\frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \overset{1}{\otimes} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-8ir)\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \overset{2}{\otimes} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (4-4r^2)\cdot\frac{1}{2}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \overset{3}{\otimes} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$+ (4-4r^2)\cdot\frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix} \overset{4}{\otimes} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (8ir)\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \overset{5}{\otimes} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (4+4r^2)\cdot\frac{1}{2}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix} \overset{6}{\otimes} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Big]$$

$$1:\ (2+2r^2)\begin{pmatrix} 1+\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2:\ (-8ir)\begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 3:\ (2-2r^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$$4:\ (2-2r^2)\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1+\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 5:\ (8ir)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix} \quad 6:\ (2+2r^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1+\lambda \end{pmatrix}$$

Exercise 3 continued

$$\begin{pmatrix} (2+2r^2)(1+\lambda)+(2-2r^2)(1-\lambda) & 0 & 0 & (-8ir) \\ 0 & (2+2r^2)(1-\lambda)+(2-2r^2)(1+\lambda) & 0 & 0 \\ 0 & 0 & (2+2r^2)(1-\lambda)+(2-2r^2)(1+\lambda) & 0 \\ (8ir\lambda) & 0 & 0 & (2+2r^2)(1+\lambda)+(2-2r^2)(1-\lambda) \end{pmatrix}$$

$$(2+2r^2)(1+\lambda)+(2-2r^2)(1-\lambda) = 2(1+\cancel{\lambda}+\cancel{r^2}+r^2\lambda+1-\cancel{\lambda}-\cancel{r^2}+r^2\lambda) = 4(1+r^2\lambda)$$

$$(2+2r^2)(1-\lambda)+(2-2r^2)(1+\lambda) = 2(1-\cancel{\lambda}+\cancel{r^2}-r^2\lambda+1+\cancel{\lambda}-\cancel{r^2}-r^2\lambda) = 4(1-r^2\lambda)$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda) & 0 & 0 & (-2ir\lambda) \\ 0 & (1-r^2\lambda) & 0 & 0 \\ 0 & 0 & (1-r^2\lambda) & 0 \\ (2ir\lambda) & 0 & 0 & (1+r^2\lambda) \end{pmatrix}}$$

d.) Calculate QFI

$$H_i = \sum_m \frac{1}{\eta_m}\left(\frac{\partial \eta_m}{\partial \lambda}\right)^2 + 2\sum_{n=m} \frac{\eta_n-\eta_m}{\eta_n+\eta_m}\left|\langle \varphi_m | \frac{\partial |\varphi_n\rangle}{\partial \lambda}\right|^2 \quad : \quad \begin{matrix}\eta \equiv \text{eigenvalue} \\ \mu \equiv \text{eigenstate}\end{matrix}$$

$$\text{Eigenvalues} = \det(\hat{A}-\eta\hat{I}) = 0$$

$$\hat{A}_1 = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda) & (-2ir\lambda) \\ (2ir\lambda) & (1+r^2\lambda) \end{pmatrix}, \quad \hat{A}_2 = \frac{1}{4}\begin{pmatrix} (1-r^2\lambda) & 0 \\ 0 & (1-r^2\lambda) \end{pmatrix}$$

i.) Calculate H_1 :

$$\hat{A}_1 - \eta\hat{I} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda) & (-2ir\lambda) \\ (2ir\lambda) & (1+r^2\lambda) \end{pmatrix} - \eta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda) & (-2ir\lambda) \\ (2ir\lambda) & (1+r^2\lambda) \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4\eta & 0 \\ 0 & 4\eta \end{pmatrix}$$

$$\hat{A}_1 - \eta\hat{I} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda)-4\eta & (-2ir\lambda) \\ (2ir\lambda) & (1+r^2\lambda)-4\eta \end{pmatrix} \quad \det(A) = \frac{1}{4}\left(((1+r^2\lambda)-4\eta)^2 - 4r^2\lambda^2\right) = 0$$

$$((1+r^2\lambda)-4\eta)^2 - 4r^2\lambda^2 = 0 \quad : \quad ((1+r^2\lambda)-4\eta)^2 = 4r^2\lambda^2 \quad (1+r^2\lambda)-4\eta = \pm 2r\lambda$$

$$4\eta = (1+r^2\lambda)\pm 2r\lambda \quad : \quad \eta = \frac{1}{4}((1+r^2\lambda)\pm 2r\lambda) \quad : \quad \eta_1 = \frac{1}{4}((1+r^2\lambda)+2r\lambda), \quad \eta_2 = \frac{1}{4}((1+r^2\lambda)-2r\lambda)$$

$$\frac{\partial \eta_1}{\partial \lambda} = \frac{1}{4}(r^2+2r) \quad : \quad \left(\frac{\partial \eta_1}{\partial \lambda}\right)^2 = \frac{1}{16}(r^2+2r)^2 \quad : \quad \frac{4}{(1+r^2\lambda)+2r\lambda}\cdot\frac{1}{16}(r^2+2r)^2 = \frac{1}{4}\frac{(r^2+2r)^2}{(1+r^2\lambda)+2r\lambda}$$

$$\frac{\partial \eta_2}{\partial \lambda} = \frac{1}{4}(r^2-2r) \quad : \quad \left(\frac{\partial \eta_2}{\partial \lambda}\right)^2 = \frac{1}{16}(r^2-2r)^2 \quad : \quad \frac{4}{(1+r^2\lambda)-2r\lambda}\cdot\frac{1}{16}(r^2-2r)^2 = \frac{1}{4}\frac{(r^2-2r)^2}{(1+r^2\lambda)-2r\lambda}$$

$$H_1 = \frac{1}{4}\left(\frac{r^2(r+2)^2}{(1+r^2\lambda)+2r\lambda} + \frac{r^2(r-2)^2}{(1-r^2\lambda)-2r\lambda}\right) = \frac{r^2}{4}\left(\frac{(r+2)^2}{(1+r^2\lambda)+2r\lambda} + \frac{(r-2)^2}{(1+r^2\lambda)-2r\lambda}\right)$$

$$H_1 = \frac{r^2}{4}\left(\frac{(r+2)^2((1+r^2\lambda)-2r\lambda) + (r-2)^2((1+r^2\lambda)+2r\lambda)}{((1+r^2\lambda)+2r\lambda)((1+r^2\lambda)-2r\lambda)}\right)$$

Exercise 3 Continued

$$H_1 = \frac{r^2}{4}\left(\frac{(r+2)^2((1+r^2\lambda)-2r\lambda)+(r-2)^2((1+r^2\lambda)+2r\lambda)}{(1+r^2\lambda)^2-4r^2\lambda^2}\right)$$

$$\boxed{H_1 = \frac{r^2}{4}\left(\frac{(r+2)^2((1+r^2\lambda)-2r\lambda)+(r-2)^2((1+r^2\lambda)+2r\lambda)}{(1+r^2\lambda)^2-4r^2\lambda^2}\right)}$$

ii.) Calculate H_2

$$\hat{A}_2 - \eta\hat{I} = \frac{1}{4}\begin{pmatrix}(1-r^2\lambda) & 0\\ 0 & (1-r^2\lambda)\end{pmatrix} - \eta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}(1-r^2\lambda)-4\eta & 0\\ 0 & (1-r^2\lambda)-4\eta\end{pmatrix}$$

$$\det(A) = \frac{1}{4}\left((1-r^2\lambda)-4\eta\right)^2 = 0 \; : \; ((1-r^2\lambda)-4\eta)^2 = 0 \; : \; (1-r^2\lambda)-4\eta = 0$$

$$4\eta = (1-r^2\lambda) \; : \; \eta = \frac{1}{4}(1-r^2\lambda) \; : \; \eta_1 = \frac{1}{4}(1-r^2\lambda),\; \eta_2 = \frac{1}{4}(1-r^2\lambda)$$

$$\left(\frac{\partial\eta_1}{\partial\lambda}\right) = \frac{-r^2}{4} \; : \; \left(\frac{\partial\eta_1}{\partial\lambda}\right)^2 = \frac{r^4}{16} \; : \; \left(\frac{\partial\eta_2}{\partial\lambda}\right) = \frac{-r^2}{4} \; : \; \left(\frac{\partial\eta_2}{\partial\lambda}\right)^2 = \frac{r^4}{16}$$

$$\frac{4}{(1-r^2\lambda)}\cdot\frac{r^4}{16} = \frac{1}{4}\frac{r^4}{(1-r^2\lambda)} \quad \therefore \quad \boxed{H_2 = \frac{1}{2}\frac{r^4}{(1-r^2\lambda)}}$$

iii.) Calculate H : $H = H_1 + H_2$

$$H = \frac{r^2}{4}\left(\frac{(r+2)^2((1+r^2\lambda)-2r\lambda)+(r-2)^2((1+r^2\lambda)+2r\lambda)}{(1+r^2\lambda)^2-4r^2\lambda^2}\right) + \frac{2}{4}\frac{r^4}{(1-r^2\lambda)}$$

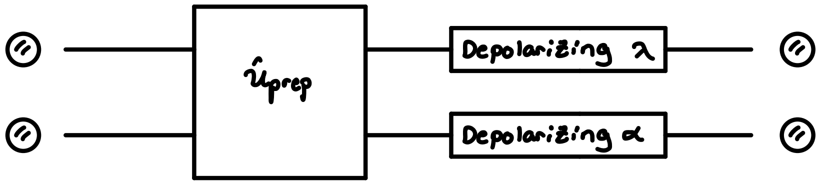
$$H = \frac{r^2}{4}\left(\frac{(1-r^2\lambda)\left((r+2)^2((1+r^2\lambda)-2r\lambda)+(r-2)^2((1+r^2\lambda)+2r\lambda)\right)+2r^2\left((1+r^2\lambda)^2-4r^2\lambda^2\right)}{(1-r^2\lambda)\left((1+r^2\lambda)^2-4r^2\lambda^2\right)}\right)$$

Maple

$$H = -\frac{r^2(r^2(1-4\lambda)+r^4\lambda+2)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(r^2\lambda-1)} = \frac{r^2(r^2(1-4\lambda)+r^4\lambda+2)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}$$

$$\boxed{H = \frac{r^2(r^2(1-4\lambda)+r^4\lambda+2)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}}$$

Exercise 4



a.) Write $\hat{\rho}_{prep} = \ldots |0\rangle|0\rangle\langle 0|\langle 0| + |0\rangle|0\rangle\langle 0|\langle 1| \ldots\ldots$

$$\hat{\rho}_{prep} = \hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| +$$
$$(4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]}$$

b.) Calculate depolarizing channel : $\hat{\rho} = \frac{1-\alpha}{2}\mathrm{Tr}[\hat{\rho}]\hat{I} + \alpha\hat{\rho}_i$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (-8ir)|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (8ir)|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

i.) $\hat{\rho}_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$

$$\hat{\rho}_f = \frac{1-\alpha}{2}\mathrm{Tr}[\hat{\rho}_i]\hat{I} + \alpha\hat{\rho}_i = \frac{1-\alpha}{2}\hat{I} + \alpha\hat{\rho}_i = \begin{pmatrix} \frac{1-\alpha}{2} & 0 \\ 0 & \frac{1-\alpha}{2} \end{pmatrix} + \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} \quad : \quad \hat{\rho}_i \rightarrow \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$$

ii.) $\hat{\rho}_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$

$$\hat{\rho}_f = \frac{1-\alpha}{2}\mathrm{Tr}[\hat{\rho}_i]\hat{I} + \alpha\hat{\rho}_i = \frac{1-\alpha}{2}\cdot 0\cdot\hat{I} + \alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \quad : \quad \hat{\rho}_i \rightarrow \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

Exercise 4 continued

iii.) $\hat{\rho}_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$

$$\hat{\rho}_f = \frac{1-\alpha}{2}\mathrm{Tr}[\hat{\rho}_i]\hat{I} + \alpha\hat{\rho}_i = \frac{1-\alpha}{2}\cdot 0 \cdot \hat{I} + \alpha\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} : \hat{\rho}_i \longrightarrow \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

iv.) $\hat{\rho}_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$

$$\hat{\rho}_f = \frac{1-\alpha}{2}\mathrm{Tr}[\hat{\rho}_i]\hat{I} + \alpha\hat{\rho}_i = \frac{1-\alpha}{2}\cdot\hat{I} + \alpha\hat{\rho}_i = \begin{pmatrix} \frac{1-\alpha}{2} & 0 \\ 0 & \frac{1-\alpha}{2} \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix} : \hat{\rho}_i \longrightarrow \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix}$$

$$\boxed{\begin{array}{l} \hat{\rho}_i = |0\rangle\langle 0| \longrightarrow \hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} : \hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \\ \hat{\rho}_i = |1\rangle\langle 0| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} : \hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix} \end{array}}$$

$$\hat{\rho}_i = |0\rangle\langle 0| \longrightarrow \hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} : \hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

$$\hat{\rho}_i = |1\rangle\langle 0| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} : \hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix}$$

can replace α w/ λ for first particle

c.) Determine effect on first and second particles of $\hat{\rho}_{prep}$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\,|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (-8ir)\,|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)\,|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)\,|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (8ir)\,|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)\,|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\cdot\frac{1}{4}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}^{1}\otimes\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} + (-8ir)\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}^{2}\otimes\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} + (4-4r^2)\frac{1}{4}\begin{pmatrix} 1+\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}^{3}\otimes\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix}$$
$$+ (4-4r^2)\cdot\frac{1}{4}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix}^{4}\otimes\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} + (8ir)\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}^{5}\otimes\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} + (4+4r^2)\frac{1}{4}\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1+\lambda \end{pmatrix}^{6}\otimes\begin{pmatrix} 1-\alpha & 0 \\ 0 & 1+\alpha \end{pmatrix}\Big]$$

1: $\begin{pmatrix} (1+r^2)(1+\lambda)(1+\alpha) & 0 & 0 & 0 \\ 0 & (1+r^2)(1+\lambda)(1-\alpha) & 0 & 0 \\ 0 & 0 & (1+r^2)(1-\lambda)(1+\alpha) & 0 \\ 0 & 0 & 0 & (1+r^2)(1-\lambda)(1-\alpha) \end{pmatrix}$

2: $\begin{pmatrix} 0 & 0 & 0 & (-8ir)(\lambda)(\alpha) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3: $\begin{pmatrix} (1-r^2)(1+\lambda)(1-\alpha) & 0 & 0 & 0 \\ 0 & (1-r^2)(1+\lambda)(1+\alpha) & 0 & 0 \\ 0 & 0 & (1-r^2)(1-\lambda)(1-\alpha) & 0 \\ 0 & 0 & 0 & (1-r^2)(1-\lambda)(1+\alpha) \end{pmatrix}$

Exercise 4 Continued

4: $\begin{pmatrix} (1-r^2)(1-\lambda)(1+\alpha) & 0 & 0 & 0 \\ 0 & (1-r^2)(1-\lambda)(1-\alpha) & 0 & 0 \\ 0 & 0 & (1-r^2)(1+\lambda)(1+\alpha) & 0 \\ 0 & 0 & 0 & (1-r^2)(1+\lambda)(1-\alpha) \end{pmatrix}$ 5: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (8ir)(\lambda)(\alpha) & 0 & 0 & 0 \end{pmatrix}$

6: $\begin{pmatrix} (1+r^2)(1-\lambda)(1-\alpha) & 0 & 0 & 0 \\ 0 & (1+r^2)(1-\lambda)(1+\alpha) & 0 & 0 \\ 0 & 0 & (1+r^2)(1+\lambda)(1-\alpha) & 0 \\ 0 & 0 & 0 & (1+r^2)(1+\lambda)(1+\alpha) \end{pmatrix}$

$[1,1] = (1+r^2)\big((1+\lambda)(1+\alpha)+(1-\lambda)(1-\alpha)\big) + (1-r^2)\big((1+\lambda)(1-\alpha)+(1-\lambda)(1+\alpha)\big)$
$= (1+r^2)(1+\cancel{\alpha}+\cancel{\lambda}+\lambda\alpha+1-\cancel{\alpha}-\cancel{\lambda}+\lambda\alpha)+(1-r^2)(1-\cancel{\alpha}+\cancel{\lambda}-\lambda\alpha+1+\cancel{\alpha}-\cancel{\lambda}-\lambda\alpha)$
$= 2(1+r^2)(1+\lambda\alpha)+2(1-r^2)(1-\lambda\alpha)$
$= 2(1+\cancel{\lambda\alpha}+\cancel{r^2}+r^2\lambda\alpha+1-\cancel{\lambda\alpha}-\cancel{r^2}+r^2\lambda\alpha) = 4(1+r^2\lambda\alpha)$
$[1,1] = 4(1+r^2\lambda\alpha)$

$[2,2] = (1+r^2)\big((1+\lambda)(1-\alpha)+(1-\lambda)(1+\alpha)\big) + (1-r^2)\big((1+\lambda)(1+\alpha)+(1-\lambda)(1-\alpha)\big)$
$= (1+r^2)(1-\cancel{\alpha}+\cancel{\lambda}-\lambda\alpha+1+\cancel{\alpha}-\cancel{\lambda}-\lambda\alpha)+(1-r^2)(1+\cancel{\alpha}+\cancel{\lambda}+\lambda\alpha+1-\cancel{\alpha}-\cancel{\lambda}+\lambda\alpha)$
$= 2(1+r^2)(1-\lambda\alpha)+2(1-r^2)(1+\lambda\alpha)$
$= 2(1-\cancel{\lambda\alpha}+\cancel{r^2}-r^2\lambda\alpha+1+\cancel{\lambda\alpha}-\cancel{r^2}-r^2\lambda\alpha) = 4(1-r^2\lambda\alpha)$
$[2,2] = 4(1-r^2\lambda\alpha)$

$[3,3] = (1+r^2)\big((1-\lambda)(1+\alpha)+(1+\lambda)(1-\alpha)\big) + (1-r^2)\big((1-\lambda)(1-\alpha)+(1+\lambda)(1+\alpha)\big)$
$= (1+r^2)(1+\cancel{\alpha}-\cancel{\lambda}-\lambda\alpha+1-\cancel{\alpha}+\cancel{\lambda}-\lambda\alpha)+(1-r^2)(1-\cancel{\alpha}-\cancel{\lambda}+\lambda\alpha+1+\cancel{\alpha}+\cancel{\lambda}+\lambda\alpha)$
$= 2(1+r^2)(1-\lambda\alpha)+2(1-r^2)(1+\lambda\alpha)$
$= 2(1-\cancel{\lambda\alpha}+\cancel{r^2}-r^2\lambda\alpha+1+\cancel{\lambda\alpha}-\cancel{r^2}-r^2\lambda\alpha) = 4(1-r^2\lambda\alpha)$
$[3,3] = 4(1-r^2\lambda\alpha)$

$[4,4] = (1+r^2)\big((1-\lambda)(1-\alpha)+(1+\lambda)(1+\alpha)\big) + (1-r^2)\big((1-\lambda)(1+\alpha)+(1+\lambda)(1-\alpha)\big)$
$= (1+r^2)(1-\cancel{\alpha}-\cancel{\lambda}+\lambda\alpha+1+\cancel{\alpha}+\cancel{\lambda}+\lambda\alpha)+(1-r^2)(1+\cancel{\alpha}-\cancel{\lambda}-\lambda\alpha+1-\cancel{\alpha}+\cancel{\lambda}-\lambda\alpha)$
$= 2(1+r^2)(1+\lambda\alpha)+2(1-r^2)(1-\lambda\alpha)$
$= 2(1+\cancel{\lambda\alpha}+\cancel{r^2}+r^2\lambda\alpha+1-\cancel{\lambda\alpha}-\cancel{r^2}+r^2\lambda\alpha) = 4(1+r^2\lambda\alpha)$
$[4,4] = 4(1+r^2\lambda\alpha)$

$[1,4] = (-8ir)(\lambda)(\alpha)$, $[4,1] = (8ir)(\lambda)(\alpha)$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda\alpha) & 0 & 0 & (-2ir)(\lambda)(\alpha) \\ 0 & (1-r^2\lambda\alpha) & 0 & 0 \\ 0 & 0 & (1-r^2\lambda\alpha) & 0 \\ (2ir)(\lambda)(\alpha) & 0 & 0 & (1+r^2\lambda\alpha) \end{pmatrix}}$$

d.) Calculate QFI

$$H_i = \sum_m \frac{1}{\eta_m}\left(\frac{\partial \eta_m}{\partial \lambda}\right)^2 + 2\sum_{n\neq m} \frac{\eta_n-\eta_m}{\eta_n+\eta_m}\left|\langle \varphi_m | \frac{\partial|\varphi_n\rangle}{\partial\lambda}\right|^2 \quad : \quad \eta \equiv \text{eigenvalue}, \ \mu \equiv \text{eigenstate}$$

$$\hat{A}_1 = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda\alpha) & (-2ir)(\lambda)(\alpha) \\ (2ir)(\lambda)(\alpha) & (1+r^2\lambda\alpha) \end{pmatrix}, \quad \hat{A}_2 = \frac{1}{4}\begin{pmatrix} (1-r^2\lambda\alpha) & 0 \\ 0 & (1-r^2\lambda\alpha) \end{pmatrix}$$

Exercise 4 continued

$\eta = \det(\hat{A} - \eta \hat{I}) = 0$

i.) Calculate H_1:

$$\hat{A}_1 - \eta \hat{I} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda\alpha) & (-2r)(\lambda)(\alpha) \\ (2r)(\lambda)(\alpha) & (1+r^2\lambda\alpha) \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4\eta & 0 \\ 0 & 4\eta \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (1+r^2\lambda\alpha) - 4\eta & (-2r)(\lambda)(\alpha) \\ (2r)(\lambda)(\alpha) & (1+r^2\lambda\alpha) - 4\eta \end{pmatrix}$$

$$\det(A) = \frac{1}{4}\left(((1+r^2\lambda\alpha) - 4\eta)^2 - 4r^2\alpha^2\lambda^2\right) = 0 \quad : \quad ((1+r^2\lambda\alpha) - 4\eta)^2 - 4r^2\alpha^2\lambda^2 = 0$$

$$((1+r^2\lambda\alpha) - 4\eta)^2 = 4r^2\alpha^2\lambda^2 \quad : \quad (1+r^2\lambda\alpha) - 4\eta = \pm 2r\alpha\lambda \quad : \quad 4\eta = (1+r^2\lambda\alpha) \pm 2r\alpha\lambda$$

$$\eta_1 = \frac{1}{4}\left((1+r^2\lambda\alpha) + 2r\alpha\lambda\right), \quad \eta_2 = \frac{1}{4}\left((1+r^2\lambda\alpha) - 2r\alpha\lambda\right)$$

$$\frac{\partial \eta_1}{\partial \lambda} = \frac{1}{4}(r^2 + 2r\alpha) = \frac{r}{4}(r+2\alpha) \quad : \quad \left(\frac{\partial \eta_1}{\partial \lambda}\right)^2 = \frac{r^2}{16}(r+2\alpha)^2 \quad : \quad \frac{4}{(1+r^2\lambda\alpha) + 2r\alpha\lambda} \cdot \frac{r^2(r+2\alpha)^2}{16}$$

$$h_{1A} = \frac{1}{4} \cdot \frac{r^2(r+2\alpha)^2}{(1+r^2\lambda\alpha) + 2r\alpha\lambda}$$

$$\frac{\partial \eta_2}{\partial \lambda} = \frac{1}{4}(r^2 - 2r\alpha) = \frac{r}{4}(r-2\alpha) \quad : \quad \left(\frac{\partial \eta_2}{\partial \lambda}\right)^2 = \frac{r^2}{16}(r-2\alpha)^2 \quad : \quad \frac{4}{(1+r^2\lambda\alpha) - 2r\alpha\lambda} \cdot \frac{r^2(r-2\alpha)^2}{16}$$

$$h_{1B} = \frac{1}{4} \cdot \frac{r^2(r-2\alpha)^2}{(1+r^2\lambda\alpha) - 2r\alpha\lambda} \quad : \quad H_1 = h_{1A} + h_{1B}$$

$$H_1 = \frac{1}{4} \cdot \frac{r^2(r+2\alpha)^2}{(1+r^2\lambda\alpha) + 2r\alpha\lambda} + \frac{1}{4} \cdot \frac{r^2(r-2\alpha)^2}{(1+r^2\lambda\alpha) - 2r\alpha\lambda} = \frac{r^2}{4}\left(\frac{(r+2\alpha)^2((1+r^2\lambda\alpha) - 2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha) + 2r\alpha\lambda)}{(1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2}\right)$$

$$\boxed{H_1 = \frac{r^2}{4}\left(\frac{(r+2\alpha)^2((1+r^2\lambda\alpha) - 2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha) + 2r\alpha\lambda)}{(1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2}\right)}$$

ii.) Calculate H_2:

$$\hat{A}_2 - \eta \hat{I} = \frac{1}{4}\begin{pmatrix} (1-r^2\lambda\alpha) & 0 \\ 0 & (1-r^2\lambda\alpha) \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4\eta & 0 \\ 0 & 4\eta \end{pmatrix} = \frac{1}{4}\begin{pmatrix} (1-r^2\lambda\alpha) - 4\eta & 0 \\ 0 & (1-r^2\lambda\alpha) - 4\eta \end{pmatrix}$$

$$\det(A) = \frac{1}{4}\left((1-r^2\lambda\alpha) - 4\eta\right)^2 = 0 \quad : \quad ((1-r^2\lambda\alpha) - 4\eta) = 0 \quad \therefore \quad \eta = \frac{1}{4}(1-r^2\lambda\alpha)$$

$$\eta_1 = \frac{1}{4}(1-r^2\lambda\alpha), \quad \eta_2 = \frac{1}{4}(1-r^2\lambda\alpha)$$

$$\frac{\partial \eta_1}{\partial \lambda} = \frac{-r^2\alpha}{4} \quad : \quad \left(\frac{\partial \eta_1}{\partial \lambda}\right)^2 = \frac{r^4\alpha^2}{16} \quad : \quad \frac{4}{(1-r^2\lambda\alpha)} \cdot \frac{r^4\alpha^2}{16} = \frac{1}{4} \cdot \frac{r^4\alpha^2}{(1-r^2\alpha)}$$

$$h_{2A} = \frac{1}{4} \cdot \frac{r^4\alpha^2}{(1-r^2\lambda\alpha)} \quad : \quad h_{2B} = \frac{1}{4} \cdot \frac{r^4\alpha^2}{(1-r^2\lambda\alpha)}$$

Exercise 4 Continued

$$H_2 = h_{2A} + h_{2B} = \frac{1}{2} \cdot \frac{r^4\alpha^2}{(1-r^2\lambda\alpha)}$$

$$\boxed{H_2 = \frac{r^2}{2} \cdot \frac{r^2\alpha^2}{(1-r^2\lambda\alpha)}}$$

iii.) Calculate H: $H = H_1 + H_2$

$$H = \frac{r^2}{4}\left(\frac{(r+2\alpha)^2((1+r^2\lambda\alpha)-2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha)+2r\alpha\lambda)}{(1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2}\right) + \frac{r^2}{4} \cdot \frac{2r^2\alpha^2}{(1-r^2\lambda\alpha)}$$

$$= \frac{r^2}{4}\left(\frac{(r+2\alpha)^2((1+r^2\lambda\alpha)-2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha)+2r\alpha\lambda)}{(1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2} + \frac{2r^2\alpha^2}{(1-r^2\lambda\alpha)}\right)$$

$$= \frac{r^2}{4}\left(\frac{(1-r^2\lambda\alpha)((r+2\alpha)^2((1+r^2\lambda\alpha)-2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha)+2r\alpha\lambda)) + 2r^2\alpha^2((1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2)}{(1-r^2\lambda\alpha)((1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2)}\right)$$

$$H = \frac{r^2}{4}\left(\frac{(1-r^2\lambda\alpha)((r+2\alpha)^2((1+r^2\lambda\alpha)-2r\lambda\alpha) + (r-2\alpha)^2((1+r^2\lambda\alpha)+2r\alpha\lambda)) + 2r^2\alpha^2((1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2)}{(1-r^2\lambda\alpha)((1+r^2\lambda\alpha)^2 - 4r^2\alpha^2\lambda^2)}\right)$$

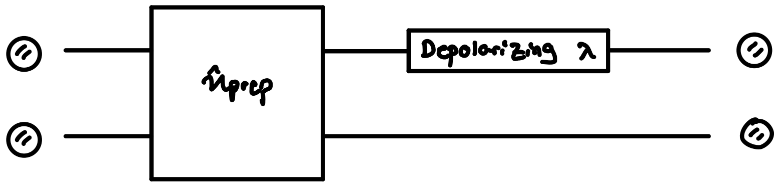
Maple

$$H = -\frac{r^2\alpha^2(\alpha\lambda r^4 - 4r^2\lambda\alpha + r^2 + 2)}{(r^2\lambda\alpha - 1)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)} = \frac{r^2\alpha^2(\alpha\lambda r^4 - 4r^2\lambda\alpha + r^2 + 2)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}$$

$$\boxed{H = \frac{r^2\alpha^2(\alpha\lambda r^4 - 4r^2\lambda\alpha + r^2 + 2)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}}$$

Exercise 5

a.) Solve gain for



,

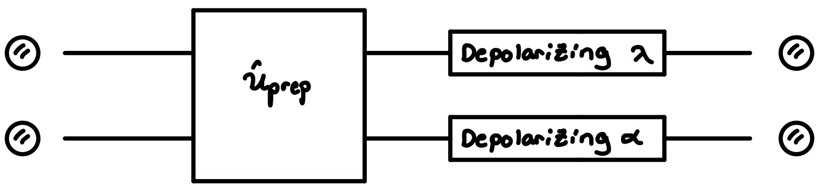
$$H = \frac{r^2\left(r^2(1-4\lambda)+r^4\lambda+2\right)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}$$

$$H = \frac{r^2}{1-r^2\lambda^2}$$

$$G = \frac{\dfrac{\cancel{r^2}\left(r^2(1-4\lambda)+r^4\lambda+2\right)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}}{\dfrac{\cancel{r^2}}{1-r^2\lambda^2}} = \frac{(1-r^2\lambda^2)\left(r^2(1-4\lambda)+r^4\lambda+2\right)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}$$

$$\boxed{G = \frac{(1-r^2\lambda^2)\left(r^2(1-4\lambda)+r^4\lambda+2\right)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}}$$

b.) Solve gain for



,

$$H = \frac{r^2\alpha^2\left(\alpha\lambda r^4-4r^2\lambda\alpha+r^2+2\right)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}$$

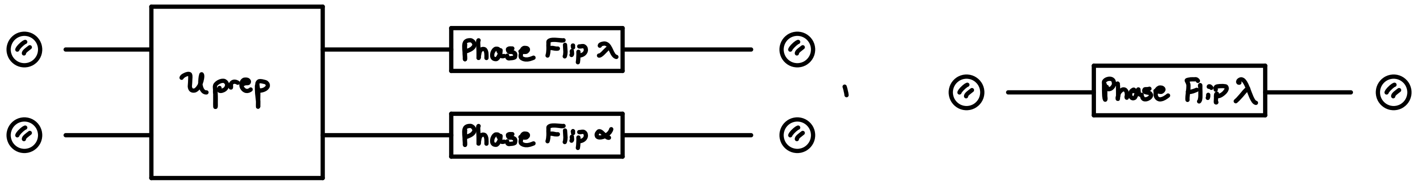
$$H = \frac{r^2}{1-r^2\lambda^2}$$

$$G = \frac{\dfrac{\cancel{r^2}\alpha^2\left(\alpha\lambda r^4-4r^2\lambda\alpha+r^2+2\right)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}}{\dfrac{\cancel{r^2}}{1-r^2\lambda^2}} = \frac{\alpha^2(1-r^2\lambda^2)\left(\alpha\lambda r^4-4r^2\lambda\alpha+r^2+2\right)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}$$

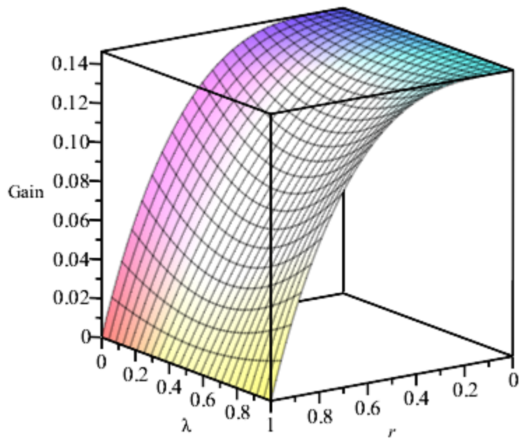
$$\boxed{G = \frac{\alpha^2(1-r^2\lambda^2)\left(\alpha\lambda r^4-4r^2\lambda\alpha+r^2+2\right)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}}$$

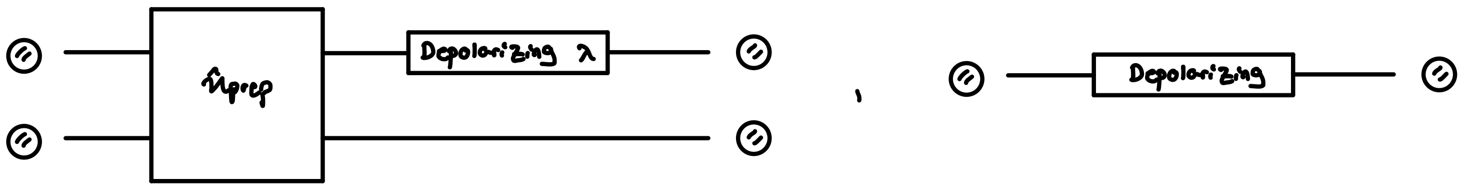
Exercise 6

Exercise 7

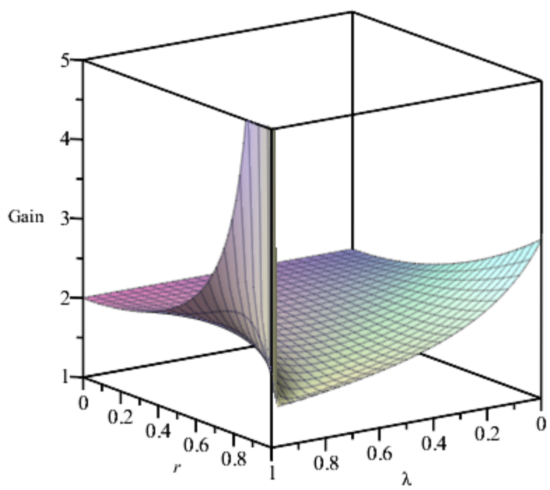


$$\alpha < \frac{1}{2}\left(1 - \sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}\right)$$

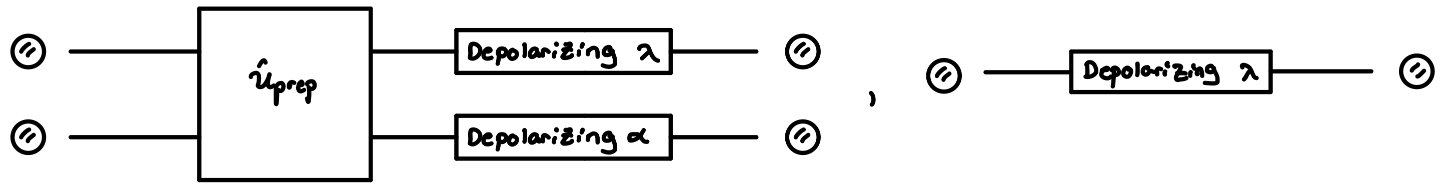




$$G = \frac{(1-r^2\lambda^2)\left(r^2(1-4\lambda)+r^4\lambda+2\right)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}$$

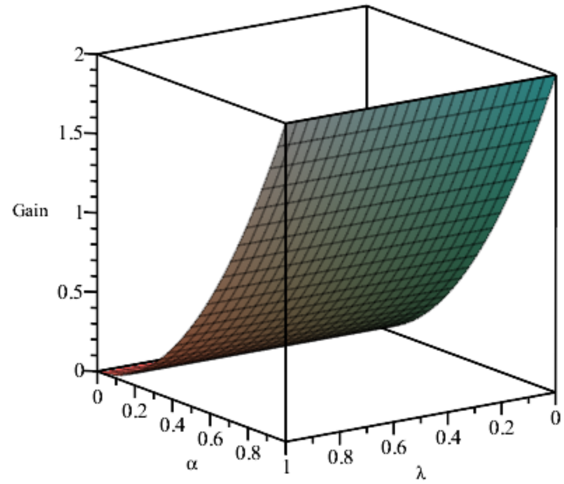


Exercise 7 continued



$$G = \frac{\alpha^2(1-r^2\lambda^2)(\alpha\lambda r^4 - 4r^2\lambda\alpha + r^2 + 2)}{(1-r^2\lambda\alpha)(1+r\alpha(r+2)\lambda)(1+r\alpha(r-2)\lambda)}$$

$r = 0.01$



$r = 1.0$