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MATH 360
CP - 9.9

The Curl

The curl of the vector function $\vec{v}$ or of the vector field given by $\vec{v}$ is defined by the "Symbolic" determinant

$$\text{Curl}(\vec{v}) = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

Example 1

Let $\vec{v} = [yz, 3zx, z] = yz\hat{\imath} + 3zx\hat{\jmath} + z\hat{k}$ with right-handed x,y,z then,

$$\text{curl}(\vec{v}) \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\hat{\imath} + y\hat{\jmath} + (3z - z)\hat{k} = -3x\hat{\imath} + y\hat{\jmath} + 2z\hat{k}$$

The curl has many applications. A typical example follows. More about the nature and significance of the curl will be considered in sec 10.9.

Theorem 1

Rotating Body and Curl

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Theorem 2

Grad, Div, Curl

Gradient fields are irrotational. That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,

$$\text{Curl}(\text{grad } f) = 0$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function $\vec{v}$ is zero.

$$\text{div}(\text{Curl}(\vec{v})) = 0$$

Example 3

The term "irrotational" for $\text{curl}(\vec{v}) = 0$ is suggested by the use of the curl for characterizing the rotation in a field. If a gradient field occurs elsewhere, not as a velocity field, it is usually called conservative. Relation (3) is plausible because of the interpretation of the curl as a rotation and of the divergence as a flux. Finally, since the curl is defined in terms of coordinates, we should do what we did for the gradient in sec. 9.7, namely, to find out whether the curl is a vector. This is true, as follows.