Ch 1.3: Algorithms and Convergence

- In this course we will be examining approximation procedures, called algorithms, involving a sequence of calculations.
- An **algorithm** is a procedure that describes a finite sequence of steps to be performed in a specified order.
- The object of the algorithm is to implement a procedure to solve a problem or approximate a solution to the problem.
- We will use **pseudocode** to describe the algorithms. This psuedocode specifies the form of the input to be supplied and the form of the desired output.

Outputs and Stopping Techniques

- Not all numerical procedures give satisfactory output for arbitrarily chosen input.
- As a consequence, a stopping technique independent of the numerical technique is incorporated into each algorithm to avoid infinite loops.
- The steps of the algorithms are arranged so that they can be translated from the pseudocode into most programming languages.

Chapter 1.3 Algorithms and Convergence

Example 1: An algorithm to compute

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + \cdots + x_N$$

INPUT $N, x_1, \ldots, x_N$.

OUTPUT $SUM = \sum_{i=1}^{N} x_i$.

Step 1 Set SUM = 0. (*Initialize accumulator*)

Step 2 For i = 1, 2, …, N do
Set SUM = SUM + x_i. (*Add next term*)

Step 3 OUTPUT(SUM);
STOP.

Stable Algorithms

- We are interested in choosing methods that will produce dependably accurate results for a wide range of problems.
- An algorithm is **stable** if small changes in initial data produce correspondingly small changes in the final results.
- Otherwise the algorithm is said to be **unstable**.
- Some algorithms are stable only for certain choices of initial data. These are called **conditionally stable**.
- We will characterize the stability properties of algorithms whenever possible.

Definition 1.17: Linear and Exponential Error Growth

- Suppose that $E_0 > 0$ denotes the initial error and E_n represents the magnitude of an error after n subsequent operations.
- If $E_n \approx CnE_0$, where C is a constant independent of n, then the growth if error is said to be **linear**.
- If $E_n \approx C^n E_0$ for some $C > 1$, then the growth is said to be **exponential**.

Table 1.5

n	Computed $\hat{p}_n$	Correct p_n	Relative Error
0	0.10000×10^1	0.10000×10^1	
1	0.33333×10^0	0.33333×10^0	
2	0.11110×10^0	0.11111×10^0	9×10^{-5}
3	0.37000×10^{-1}	0.37037×10^{-1}	1×10^{-3}
4	0.12230×10^{-1}	0.12346×10^{-1}	9×10^{-3}
5	0.37660×10^{-2}	0.41152×10^{-2}	8×10^{-2}
6	0.32300×10^{-3}	0.13717×10^{-2}	8×10^{-1}
7	-0.26893×10^{-2}	0.45725×10^{-3}	7×10^0
8	-0.92872×10^{-2}	0.15242×10^{-3}	6×10^1

Table 1.6

n	Computed $\hat{p}_n$	Correct p_n	Relative Error
0	0.10000×10^1	0.10000×10^1	
1	0.33333×10^0	0.33333×10^0	
2	-0.33330×10^0	-0.33333×10^0	9×10^{-5}
3	-0.10000×10^1	-0.10000×10^1	0
4	-0.16667×10^1	-0.16667×10^1	0
5	-0.23334×10^1	-0.23333×10^1	4×10^{-5}
6	-0.30000×10^1	-0.30000×10^1	0
7	-0.36667×10^1	-0.36667×10^1	0
8	-0.43334×10^1	-0.43333×10^1	2×10^{-5}

Definition 1.18 Suppose that $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive number K exists with

$$|\alpha_n - \alpha| \leq K|\beta_n|, \text{ for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with rate of convergence $O(\beta_n)$. This expression is read "big oh of β_n." It is written

$$\alpha_n = \alpha + O(\beta_n)$$

Typically $\{\frac{1}{n^p}\}$ is used for $\{\beta_n\}$, for some $p > 0$. We are usually interested in the largest value of p with $\alpha_n = \alpha + O(n^{-p})$

<u>Example</u> Suppose that

$$\alpha_n = \frac{n+1}{n^2} \quad \text{and} \quad \hat{\alpha}_n = \frac{n+3}{n^3}$$

Then $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \hat{\alpha}_n = 0$.

However, $\{\hat{\alpha}_n\}$ converges much faster. See Table 1.7.

Let $\beta_n = \frac{1}{n}$ and $\hat{\beta}_n = \frac{1}{n^2}$. Then

$$|\alpha_n - 0| = \frac{n+1}{n^2} \le \frac{n+n}{n^2} = \frac{2n}{n^2} = 2\left(\frac{1}{n}\right),$$

$$|\hat{\alpha}_n - 0| = \frac{n+3}{n^3} \le \frac{n+3n}{n^3} = \frac{4n}{n^3} = 4\left(\frac{1}{n^2}\right).$$

Thus

$$\alpha_n = 0 + O\left(\frac{1}{n}\right),$$

$$\hat{\alpha}_n = 0 + O\left(\frac{1}{n^2}\right)$$

Therefore $\{\alpha_n\}$ converges to 0 at about the rate $\{\frac{1}{n}\}$ converges to zero, and $\{\hat{\alpha}_n\}$ converges to 0 about as fast as $\{\frac{1}{n^2}\}$.

Ex Find the rate of convergence of the following sequence as $n \to \infty$.

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = 0.$$

Solution Let $\alpha_n = \ln\left(1+\frac{1}{n}\right)$, $\alpha = 0$.
We want to find k & β_n such that $k>0$ and $\lim_{n\to\infty} \beta_n = 0$ with

$$|\alpha_n - \alpha| \leq k\,|\beta_n|$$

To do so, examine $\alpha_n - \alpha$:

$$|\alpha_n - \alpha| = \left|\ln\left(1+\frac{1}{n}\right)\right| \overset{*}{\leq} \frac{1}{n}$$

Choose $k=1$ and $\beta_n = \frac{1}{n}$. Then

$$\left|\ln\left(1+\frac{1}{n}\right) - 0\right| \leq 1\cdot\left|\frac{1}{n}\right|.$$

Therefore $\ln\left(1+\frac{1}{n}\right) = 0 + O\left(\frac{1}{n}\right)$ and the rate of convergence is $O\left(\frac{1}{n}\right)$.

* We now show $\left|\ln\left(1+\frac{1}{n}\right)\right| \le \frac{1}{n}$.

Let $\underline{f(x) = x - \ln(1+x),\ x \ge 0}$.

If we can show $f(x) > 0$ on $(0,\infty)$, then we are done.

Taking derivative of f, we have

$$f'(x) = 1 - \frac{1}{1+x} > 0 \quad \text{on } (0,\infty).$$

Thus f is increasing on $(0,\infty)$.

Now
$$\begin{aligned} f(0) &= 0 - \ln(1+0) \\ &= -\ln(1) \\ &= 0 \end{aligned}$$

Therefore $f(x) > 0$ on $(0,\infty)$.

It follows that $\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) > 0$ for all n, and hence

$$\left|\ln\left(1+\frac{1}{n}\right)\right| < \left|\frac{1}{n}\right|$$

for all n.

Definition 1.19. Suppose that

$$\lim_{h\to 0} G(h) = 0 \quad \text{and} \quad \lim_{h\to 0} F(h) = L.$$

If a positive constant K exists with

$$|F(h) - L| \le K|G(h)|, \quad h \text{ sufficiently small},$$

then we write $F(h) = L + O(G(h))$ (function "big oh" notation).

The function $G(h)$ typically used for comparison has the form $G(h) = h^p$, where $p > 0$, p as large as possible.

Example Recall $\cos(h) = 1 - \frac{h^2}{2} + \frac{1}{24}h^4 \cos\xi(h)$ where $\xi(h)$ is between 0 and h. Thus

$$\cos(h) + \tfrac{1}{2}h^2 = 1 + \tfrac{1}{24}h^4 \cos\xi(h).$$

Since

$$\left|\cos(h) + \tfrac{1}{2}h^2 - 1\right| = \tfrac{1}{24}\left|h^4 \cos\xi(h)\right| \le \tfrac{1}{24}h^4,$$

we have $\cos(h) + \frac{1}{2}h^2 = 1 + O(h^4)$.

Hence $\cos(h) + \frac{1}{2}h^2$ converges to 1 as fast as $h^4 \to 0$.

<u>Ex</u> Find the rate of convergence for the following function as $h \to 0$.

$$\lim_{h \to 0} \frac{1-e^h}{h} = -1$$

<u>Solution</u> Let $F(h) = \frac{1-e^h}{h}$ and $L=-1$.
We want to find $k>0$ and $G(h)$ s.t.

$$|F(h) - L| \le k\,|G(h)|,$$

where $\lim_{h\to 0} |G(h)| = 0$.

To do so, examine $F(h) - L$:

$$|F(h)-L| = \left|\frac{1-e^h}{h} + 1\right| = \left|\frac{1-e^h}{h} + \frac{h}{h}\right|$$

$$= \left|\frac{1+h-e^h}{h}\right|$$

To obtain a usable simplification of the above result, consider Taylor expansions of e^h, expanded about $h=0$.

$$f(h) = e^h \Rightarrow f(0) = 1 \Rightarrow f(h) = 1 + R_0(h)$$

$$f'(h) = e^h \Rightarrow f'(0) = 1 \Rightarrow f(h) = 1 + h + R_1(h)$$

$$f''(h) = e^h \Rightarrow f''(0) = 1 \Rightarrow f(h) = 1 + h + \frac{h^2}{2!} + R_2(h)$$

$$f'''(h) = e^h \Rightarrow f'''(0) = 1 \Rightarrow f(h) = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + R_3(h)$$

Which expression should we use?

Recall

$$|F(h) - L| = \left| \frac{1 + h - e^h}{h} \right|$$

Because of the obvious cancellation, choose Taylor representation using $R_1(h)$:

$$|F(h) - L| = \left| \frac{-R_1(h)}{h} \right| = \left| \frac{f''(\xi(h))}{2! \cdot h} h^2 \right|$$

$$= \left| \frac{e^{\xi(h)}}{2} h \right|, \quad \xi(h) \text{ between } 0 \text{ and } h.$$

$$\leq \left| \frac{e^1}{2} h \right|, \quad \text{assuming } |h| \ll 1$$

$$\leq \frac{3}{2} \cdot |h|$$

From the previous slide, we have

$$\left|F(h) - L\right| = \left|\frac{1-e^h}{h} + 1\right| \le \frac{3}{2} \cdot |h|$$

Choose $k = 3/2$ and $G(h) = h$.
Then

$$\left|F(h) - L\right| \le k\,|G(h)|$$

or

$$\left|\frac{1-e^h}{h} + 1\right| \le \frac{3}{2}|h|$$

and hence

$$\frac{1-e^h}{h} = -1 + O(h)$$

It follows that the rate of convergence for $F(h) = \dfrac{1-e^h}{h}$ as $h \to 0$ is $O(h)$.