Ch. 3 Space Time : Gravity in Newtonian Mechanics

1-30-19

What is an inertial reference frame..?

i.e. A free particle obeys

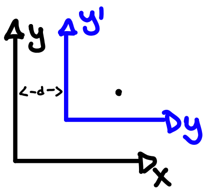
$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 0 \Rightarrow \vec{V} = \text{constant vector}$$

$$(*) \left[\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \right]$$

· Laws of Newtonian mechanics take their simplest form in inertial reference frame.

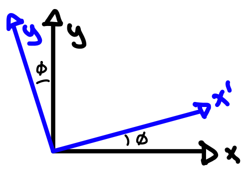
To define the rest

1.)



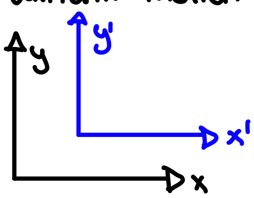
$$x' = x - d$$
$$y' = y$$
$$z' = z$$

2.) Rotation by ϕ



$$x' = (\cos\phi)x + (\sin\phi)y$$
$$y' = (-\sin\phi)x + (\cos\phi)y$$
$$z' = z$$

3.) Uniform motion by velocity V



$$x' = x - Vt$$
$$y' = y$$
$$z' = z$$

Central assumption of Newtonian mechanics is that there is absolute time, $t = t'$

Galilean Transformations $\rightarrow$ Not valid at speeds close to c.

$$x' = x - Vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

The Principle of Relativity

Identical experiments carried out indifferent intertial reference frames

If (*), then

$$\left[\frac{d^2x'}{dt^2} = \frac{d^2y'}{dt^2} = \frac{d^2z'}{dt^2} = 0 \right] \rightarrow \text{Invariant under Galilean Transformations}$$

- Principle of Relativity is possible only because the geometry of Euclidean Space shares these Symmetries.
- One can verify these Symmetries of Euclidean geometry by examining,

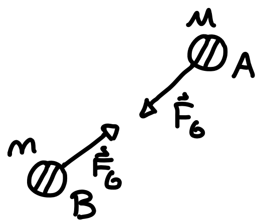
$$ds^2 = dx^2 + dy^2 + dz^2$$

i.e. Rotation

$$x = (\cos\phi)x' - (\sin\phi)y' \qquad dx = (\cos\phi)\,dx' - (\sin\phi)\,dy'$$
$$y = (\sin\phi)x' + (\cos\phi)y' \qquad dy = (\sin\phi)\,dx' - (\cos\phi)\,dy'$$
$$z = z' \qquad dz = dz'$$

$$ds^2 = (\cos\phi\, dx' - \sin\phi\, dy')^2 + (\sin\phi\, dx' + \cos\phi\, dy')^2 + dz^2$$
$$ds^2 = \cos^2\phi\, dx'^2 - 2\cos\phi\,\cancel{\sin\phi}\, dx'dy' + \sin^2\phi\, dy'^2 + \sin^2\phi\, dx'^2 + 2\cancel{\sin\phi}\cos\phi\, dx'dy' + \cos\phi^2 dy'^2 + dz^2$$
$$= dx'^2(\cos^2\phi + \sin^2\phi) + dy'^2(\cos^2\phi + \sin^2\phi) + dz'^2 = dx'^2 + dy'^2 + dz'^2$$

Newtonian Gravity



$$\boxed{\vec{F}_G = -\frac{GmM}{r^2}\hat{e}_r}$$ — Newton's universal law of gravitation

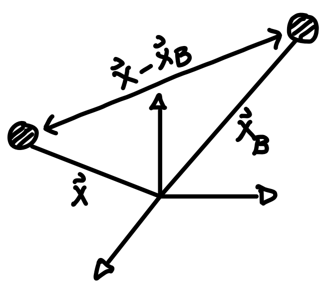
we can write this in terms of a Gravitational Potential

$\vec{F}_G = -m\vec{\nabla}\Phi(x_B)$ Similar for Electrostatics : $\vec{F}_E = -q\vec{\nabla}\phi$
$= qE$

$\Phi(\vec{x}_B)$ is the gravitational potential produced by A at B's position $\vec{x}_B$

$$\Phi(\vec{x}) = -\frac{GM}{r} = -\frac{GM}{|\vec{x} - x|}$$

2-1-19



- Gravitational potential @ B masses M_A , $A = 1, 2, 3, \ldots\ldots$

$$\Phi(\vec{x}) = -\sum_A \frac{GM_A}{|\vec{x} - x_A|}$$

For a Continuous Distribution of Mass Density $\mu(\vec{x}')$

$$\sum_A M_A \rightarrow \int \mu(\vec{x}')\,d^3x'$$

$$\left[\Phi(\vec{x}) = -\int d^3x' \cdot \frac{G\mu(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] \qquad \left(\text{Like } \Phi(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{\mathscr{r}} \text{ in E:M}\right)$$

Define The Newtonian Gravitational Field $\vec{g}(\vec{x})$ (Like $\vec{E}(\vec{x})$ in E:M)

$$\left[\vec{g}(\vec{x}) \equiv -\vec{\nabla}\Phi(\vec{x})\right] \quad \therefore \quad \vec{F}_{Grav} = m\vec{g}$$

The Differential Form of (*) . . . .

$$\left[\vec{\nabla}\cdot\vec{g}(x) = -4\pi G\mu(\vec{x})\right] \quad \left(\text{Like } \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}\right)$$

or

$$\left[\nabla^2\Phi(x) = 4\pi G\mu(\vec{x})\right] \text{ since } \vec{\nabla}\cdot\vec{\nabla}\Phi = \nabla^2\Phi \quad \left(\text{Like } \nabla^2\Phi = \frac{-\rho}{\varepsilon_0}\right)$$

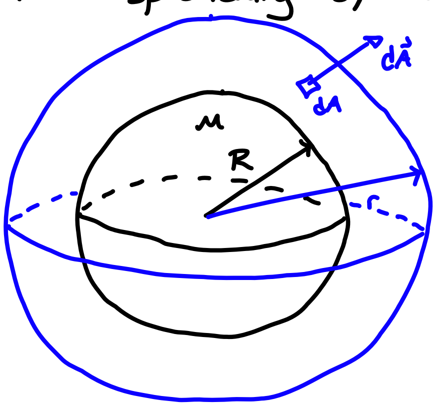
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \text{ is the Laplacian}$$

To arrive @ the integral form, integrate over the volume . . . .

$$\int_V \vec{\nabla}\cdot\vec{g}(\vec{x})\,d\vec{x}^3 = -4\pi G\underbrace{\int_V \mu(\vec{x})\,d^3x}_{M}$$

$$\int_V \vec{\nabla}\cdot\vec{g}(\vec{x})\,d^3x = \oint_S \vec{g}(x)\cdot d\vec{A} \longrightarrow \oint \vec{g}\cdot d\vec{A} = -4\pi GM \quad \left(\oint \vec{E}\cdot d\vec{a} = \frac{Q_{Enc}}{\varepsilon_0}\right)$$

For a spherically-symmetric mass . . . . . .



$$\oint_S \vec{g}\cdot d\vec{A} \qquad \vec{g}\cdot d\vec{A} = g\,dA\cos\theta\Big|_{\theta=180^\circ} = -g\,dA \ , \ \vec{g} = -g\,\hat{e}_r$$

$$\oint_S \vec{g}\cdot d\vec{A} = -\oint_S g\,dA = -g\oint_S dA = -g\cdot 4\pi r^2$$

$$-g\cdot 4\pi r^2 = -4\pi GM$$

$$g = \frac{GM}{r^2} \ , \ \vec{g} = -g\,\hat{e}_r = -\frac{GM}{r^2}\hat{e}_r$$

$$\vec{\nabla}\Phi = \frac{GM}{r^2}\hat{e}_r = \frac{\partial\Phi}{\partial r}\hat{e}_r \ , \ \frac{d\Phi}{dr} = \frac{GM}{r^2} \ , \ \Phi = -\frac{GM}{r}$$

Gravitational : Inertial Mass

$$\vec{F}_{Grav} = -m_G\vec{\nabla}\Phi(\vec{x}) = m_G\vec{g}$$

Gravitational mass – measures the strength of gravitational force between bodies

• Newton's 2nd Law of motion is .....

$$\vec{F} = m_I \vec{a}$$

Inertial mass - Governs the inertial properties of the body

Newton's 2nd Law for a body in a gravitational field ....

$$-m_G \vec{\nabla} \Phi = m_G \vec{g} = m_I \vec{a} \qquad \text{Since } [m_I = m_G]$$

$\vec{a} = -\vec{\nabla}\Phi = \vec{g}$ – All bodies fall with the same acceleration in a gravitational field