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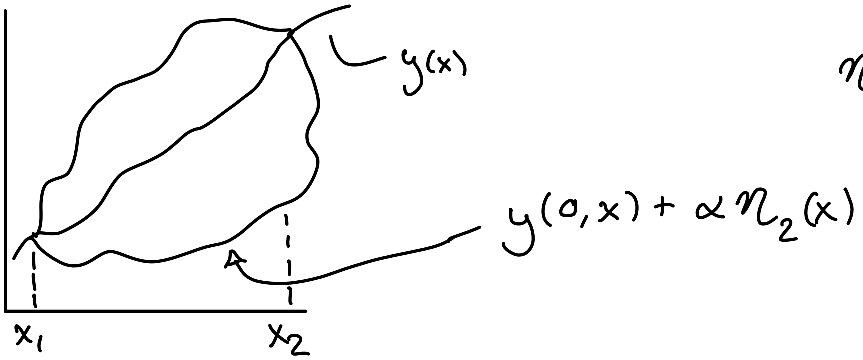
Calculus of Variations

Define a Function $y(x)$ Such that

$$J = \int_{x_1}^{x_2} F(y(x), y'(x), x)\, dx \quad \text{is an extremum}$$

Functional

$$y(\alpha, x) = y(0, x) + \alpha \eta(x_1)$$



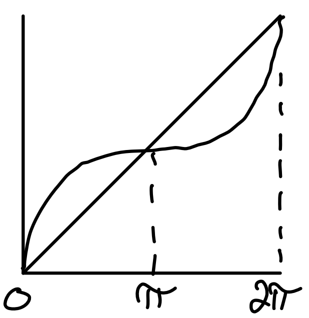
$$\eta(x_1) = \eta(x_2) = 0$$

Then $J(\alpha) = \int_{x_1}^{x_2} F[y(\alpha,x), y'(\alpha,x), x]\, dx$

$$\left.\frac{\partial J(\alpha)}{\partial(\alpha)}\right| = 0$$

i.e. $F = \left(\frac{dy}{dx}\right)^2 \quad y(x) = x \leftarrow [0, 2\pi]$

$\eta(x) = \sin(x) \qquad y(\alpha, x) = x + \alpha \sin(x)$



$$\frac{dy(\alpha,x)}{dx} = 1 + \alpha\cos(x) \Rightarrow (1 + \alpha\cos(x))^2$$

$$F = 1 + 2\alpha\cos(x) + \alpha^2\cos^2(x)$$

$$J = \int_0^{2\pi} F\, dx = 2\pi + \alpha^2\pi \qquad \alpha \neq 0 \uparrow$$

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial x}\int_{x_1}^{x_2} F(y, y', x)\, dx = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} F(y, y', \alpha)\, dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx$$

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$
$$y'(\alpha, x) = y'(0, x) + \alpha\eta'(x)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x), \quad \frac{\partial y'}{\partial \alpha} = \eta'(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right] dx$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \cdot \frac{\partial \eta}{\partial x}\, dx$$

$$uv\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} v\, du = \int_{x_1}^{x_2} u\, dv$$

$$u = \frac{\partial F}{\partial y'}, \quad du = \frac{d}{dx}\frac{\partial F}{\partial y'}, \quad dv = d\eta, \quad v = \eta$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \cdot \frac{\partial \eta}{\partial x}\, dx = \cancel{\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\frac{\partial F}{\partial y'}\, dx \quad *$$

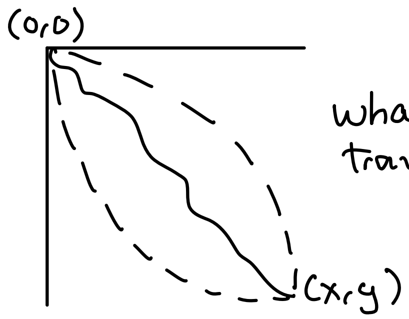
$\eta(x_2) - \eta(x_1) = 0 \quad 0$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\right) dx = 0$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\eta(x)\, dx = 0$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0, \qquad J = \int F(y, y', x)\, dx$$

Euler's equation



what path minimizes travel time

$E = T + u = 0$

$$0 = -mgy + \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

$$2gy = \left(\frac{ds}{dt}\right)^2 \qquad dt = \frac{ds}{\sqrt{2gy}}$$

$$\int dt = \int \frac{ds}{\sqrt{2gy}} \Rightarrow ds = \sqrt{dx^2 + dy^2} \Rightarrow \int dt = \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}} = \int\left(\sqrt{\frac{1+(y')^2}{2gy}}\right) dx$$

$$F = \frac{1}{\sqrt{2g}}\left(1 + (y')^2\right)^{\frac{1}{2}} y^{-\frac{1}{2}}$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \Rightarrow \frac{1}{\sqrt{2g}}\left(1 + (y')^2\right)^{\frac{1}{2}}\left(-\tfrac{1}{2}\right) y^{-\frac{3}{2}}$$

$$\frac{\partial F}{\partial y} = -\sqrt{\frac{1+(y')^2}{y}}\left(\frac{1}{2y}\right)\left(\frac{1}{\sqrt{2g}}\right) \qquad \frac{\partial F}{\partial y'} = \cancel{\frac{1}{\sqrt{2g}}}\, y^{-\frac{1}{2}}\left(\frac{1}{2}\right)\left(1 + (y')^2\right)^{-\frac{1}{2}} \cdot 2y'$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{y(1+(y'))^2}} \; : \; \frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{d}{dx}(y')(y)^{-\frac{1}{2}}\left(1+(y')^2\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{y(1+(y'))^2}}\left[\frac{y''}{1+(y')^2} - \frac{1}{2}\frac{(y')^2}{y}\right]$$

$$\frac{1}{\sqrt{y(1+(y')^2)}}\left[\frac{y''}{1+(y')^2}-\frac{1}{2}\frac{(y')^2}{y}\right] \Rightarrow \frac{-1}{2y}\sqrt{\frac{1+(y')^2}{y}}$$

Let $A = 1+(y')^2$

$$\frac{1}{A^{\frac{1}{2}}\sqrt{y}}\left[\frac{y''}{A}-\frac{1}{2}\frac{(y')^2}{y}\right] = -\frac{A^{\frac{1}{2}}}{\sqrt{2y}}\cdot\frac{1}{2y} \Rightarrow \frac{-A}{2y} = \frac{y''}{A} - \frac{1}{2}(y')^2 \Rightarrow \frac{y''}{A} = \frac{(y')^2 - A}{2y}$$

$$\frac{y''}{A} = \frac{-(1+(y')^2)}{2y} \Rightarrow \underline{1+2y(y'')+(y')^2=0}$$

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$$g = \int F(x,y,z)\,dx$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$$

$$1+2y(y'')+(y')^2=0$$

$$y'=u, \quad y'' = u' = \frac{du}{dx} = \frac{du}{dy}y' = u\frac{du}{dy}$$

$$2yy'' = 2y\frac{du}{dy}$$

$$(1+u^2) + 2yu\frac{du}{dy} = 0$$

$$(1+u^2) + 2yu\frac{du}{dy} = 0$$

$$\frac{dy}{y} + \frac{2u\,du}{1+u^2} = 0$$

$$\ln(y) + \ln(1+u^2) = C$$

$$y(1+u^2) = D$$

$$\frac{D}{y} - 1 = u^2 \quad : \quad u = \left(\frac{D-y}{y}\right)^{\frac{1}{2}} = \frac{dy}{dx}$$

$$\int\left(\frac{y}{D-y}\right)^{\frac{1}{2}} dy = \int dx$$

$$y = D\sin^2\theta$$

$$x + C = \int \sqrt{\frac{D\sin^2\theta}{D\cos^2\theta}}\, 2D\sin\theta\cos\theta \, d\theta$$

$$x + C = 2D \int \sin^2\theta \, d\theta$$

$$x + C = 2D \int \frac{1}{2} - \frac{1}{2}\cos(2\theta)\, d\theta$$

$$x + C = 2D\left[\frac{1}{2}\theta - \frac{\sin(2\theta)}{4}\right]$$

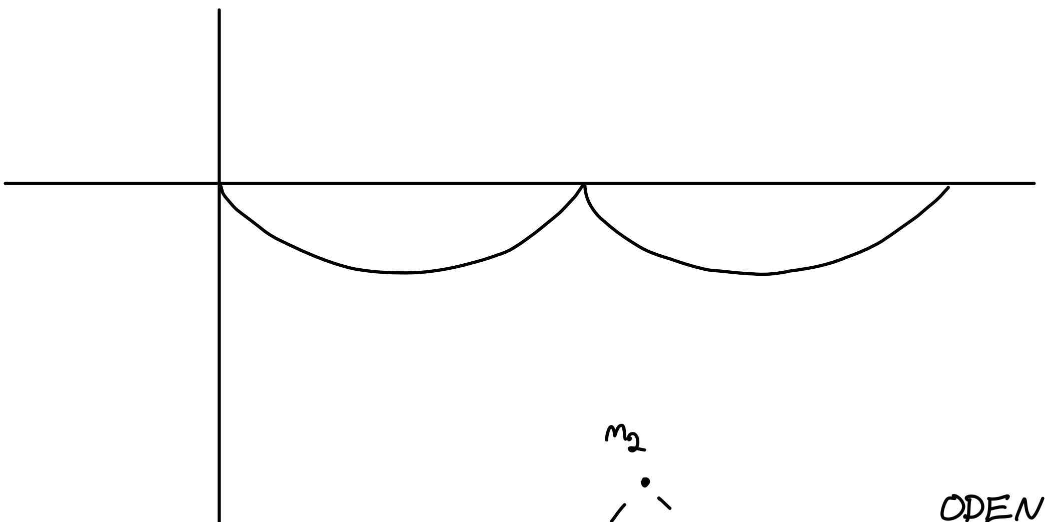
$$x = \frac{1}{2}D\left[2\theta - \sin(2\theta)\right] \qquad \text{Start @ } (0,0) \text{ so } C=0$$

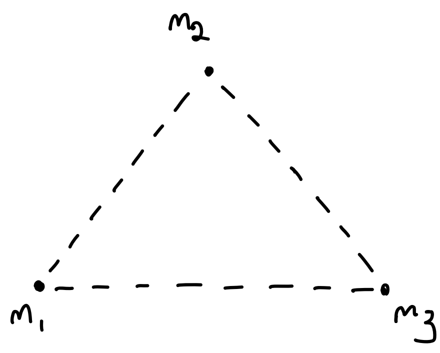
$$y = D\cdot\sin^2(\theta)$$

$$x = \frac{D}{2}\left(2\theta - \sin(2\theta)\right), \quad y = \frac{D}{2}\left(1 - \cos(2\theta)\right)$$

$$\frac{D}{2} = A \qquad 2\theta = \phi$$

$$x = A\left[\phi - \sin(\phi)\right], \quad y = A\left[1 - \cos(\phi)\right] \quad \phi \in [0, 2\pi]$$





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