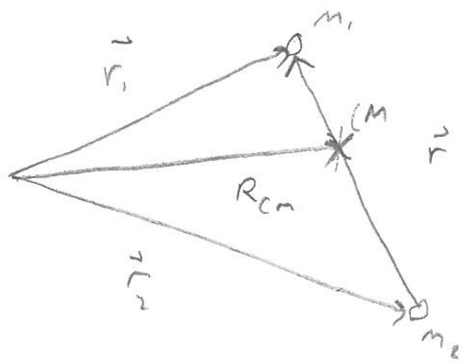
Chapter 8

Central Force Motion

Reduced Mass → 2 body to 1 body



$\vec{r}_2 + \vec{r} = \vec{r}_1 \qquad \vec{r}_2 = \vec{r}_1 - \vec{r}$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$|\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - U(r)$$

Let $\vec{R}_{cm} = 0$

Then $\dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0$ or $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$

Essentially we shift our origin

Then $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 = m_1 (\vec{r}_2 + \vec{r}) + m_2 (\vec{r}_2) = 0$

or $(m_1 + m_2)(\vec{r}_2) = -m_1 \vec{r}$ $\quad \boxed{\vec{r}_2 = -\dfrac{m_1}{(m_1 + m_2)} \vec{r}}$

and $m_1 \vec{r}_1 + m_2 (\vec{r}_1 - \vec{r}) = 0 \quad (m_1 + m_2) \vec{r}_1 = m_2 \vec{r}$

$$\boxed{\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}}$$

If $m_2 = m_1$

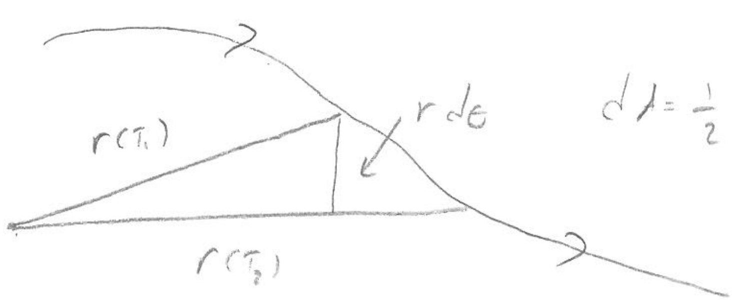
$$\vec{r}_1 = \frac{\frac{m_2}{m_1}}{\frac{m_1}{m_1} + \frac{m_2}{m_1}} = \frac{1}{2}\vec{r} \qquad \vec{r}_2 = -\frac{1}{2}\vec{r}$$

orbit

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\dot{P}_\theta = \frac{\partial L}{\partial \theta} = 0 \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$ angular momentum is conserved

$$P_\theta = \text{Constant} = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = |L|$$



$$dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r} \quad \text{and} \quad \vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}$$

gives $L = \frac{1}{2}\mu\left[|\dot{\vec{r}}|^2\right] - U(r)$

where $\mu = \dfrac{m_1 m_2}{m_1+m_2}$

Solve $r(t)$ get $\vec{r}_1(t)$ and $\vec{r}_2(t)$ for the price of one

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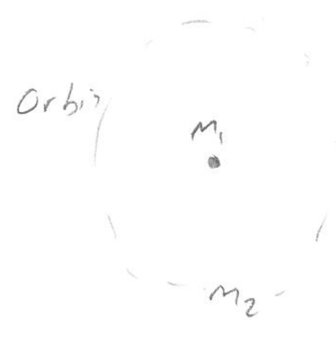
what if $m_1 >> m_2$

$$\mu = \frac{\frac{m_1 m_2}{m_1}}{\frac{m_1}{m_1}+\frac{m_2}{m_1}} = \frac{m_2}{1+\text{Tiny}} = m_2$$

$$\vec{r}_1 = \frac{\frac{m_2}{m_1}\vec{r}}{\frac{m_1}{m_1}+\frac{m_2}{m_1}} = 0 \qquad \vec{r}_2 = \frac{-\frac{m_1}{m_1}}{\frac{m_1}{m_1}+\frac{m_2}{m_1}}\vec{r} = -\vec{r}$$

So $|\vec{r}|$ essentially $|\vec{r}_2|$



$$\frac{1}{2} r^2 \dot{\theta} = \frac{|L|}{2\mu} = \text{constant}$$

keplers Second Law

$$\frac{dA}{dT} = \frac{|L|}{2\mu} = \text{Equal Areas in Equal Times}$$

Now $E = T + U$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{|L|^2}{\mu r^2} + U(r)$$

Motion?

Motion

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$2(E - U(r)) = \mu \dot{r}^2 + \frac{\ell^2}{\mu r^2}$$

$$\pm\left(\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}\right)^{1/2} = \frac{dr}{dt} = \dot{r}$$

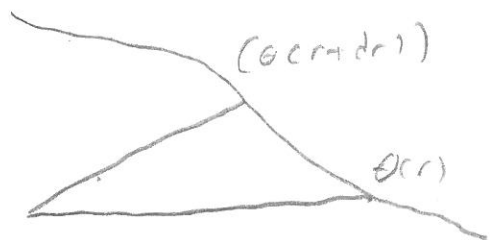
Trick

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \frac{\dot{\theta}}{\dot{r}}\,dr$$

but $\dot{\theta} = \dfrac{\ell}{\mu r^2}$

So $\dfrac{\dot{\theta}}{\dot{r}}\,dr = \dfrac{\ell}{\mu r^2}\cdot\dfrac{dr}{\dot{r}} = d\theta$

$$\pm\frac{\ell}{\mu r^2}\cdot\left(\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}\right)^{-1/2} dr = \theta(r)$$



$$\pm\frac{\ell}{\mu r^2}\,\frac{1}{\sqrt{\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}}}\,dr = \theta(r)$$

$$\pm\frac{\ell}{r^2}\cdot\frac{1}{\sqrt{2\mu(E-U) - \frac{\ell^2}{r^2}}}\,dr = \theta(r)$$

$$\pm\int\frac{\ell/r^2\;dr}{\sqrt{2\mu\left[E - U - \frac{\ell^2}{2\mu r^2}\right]}} = \theta(r)$$

$$\theta(r) = \int \frac{\pm(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

$\dot{\theta} \propto L \leftarrow$ constant same So $\theta \nearrow$ or $\downarrow$ only

Only analytic for $F(r) \propto r^n$ when $n = 1, -2, -3$ $\leftarrow$ HW?

Can also do via Lagrangian $\rightarrow$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \mu\ddot{r} \qquad \frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = -\frac{\partial U}{\partial r} = F(r)$$

Let $u = \frac{1}{r}$ $\dot{\theta} = \frac{\ell}{\mu r^2}$

a.) $$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{\ell}\dot{r}$$

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b.) $$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left[\frac{du}{d\theta}\right] = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\dot{r}\right) = -\frac{1}{\dot{\theta}}\frac{\mu}{\ell}\ddot{r}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\mu^2 r^2}{\ell^2}\ddot{r}$$

Then

$$\ddot{r} = -\frac{\ell^2}{\mu^2}\overset{\frac{1}{r^2}}{u^2}\frac{d^2u}{d\theta^2} \quad , \quad r\dot{\theta}^2 = \frac{\ell^2 u^3}{\mu^2}$$

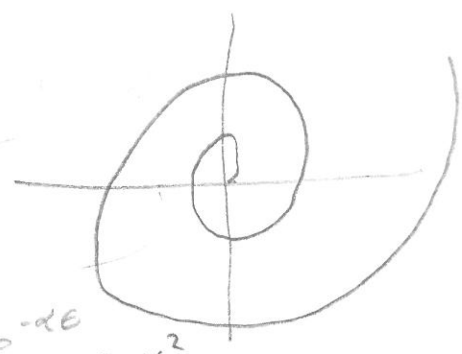
$$\text{Then } \mu\left[-\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\theta^2} - \frac{\ell^2 u^3}{\mu^2}\right] = F\left(\frac{1}{u}\right)$$

$$\text{or } \quad \frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(1/u)$$

$$\text{or } \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{\ell^2}\, r^2 F(r) \rightarrow \text{want } r(\theta)$$

Force which gives rise to orbit

Ex What Force allows $r = ke^{\alpha\theta}$



$$\frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{-\alpha}{k}e^{-\alpha\theta} \qquad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k}e^{-\alpha\theta} = \frac{\alpha^2}{r}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{r}(\alpha+1) = -\frac{\mu}{\ell^2}\, r^2 F(r)$$

$$F(r) = \frac{-1}{\underset{n=3}{r^3}}\frac{\ell^2}{\mu}(\alpha+1)$$

$r(\tau)$ and $\theta(\tau)$

$$\dot{\theta} = \frac{\ell}{\mu r^2} = \frac{\ell}{\mu k^2 e^{2\alpha\theta}} = \frac{d\theta}{d\tau} \rightarrow \frac{\ell}{\mu k^2} d\tau = e^{2\alpha\theta} d\theta$$

$u = 2\alpha\theta \quad du = 2\alpha\, d\theta$

$$\frac{\ell \tau}{\mu k^2} = \frac{1}{2\alpha} e^{2\alpha\theta} + c$$

$$\rightarrow \quad \frac{2\alpha \ell \tau}{\mu k^2} + \underset{-2\alpha c}{c'} = e^{2\alpha\theta} \rightarrow \frac{1}{2\alpha} \ln\left[\frac{2\alpha \ell \tau}{\mu k^2} + c\right] = \theta(\tau)$$

Then $r = k e^{\alpha\theta} \rightarrow r^2 = k^2 e^{2\alpha\theta} \rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$

So $\frac{r^2}{k^2} = \frac{2\alpha \ell \tau}{\mu k^2} + c' \rightarrow r^2 = \frac{2\alpha \ell \tau}{\mu} + D$

$$r(\tau) = \pm \sqrt{\frac{2\alpha\ell}{\mu}\tau + D} \qquad n = -3$$

Energy? $E = T + U$

U? $\quad U(r) = -\int \vec{F} \cdot d\vec{r} = +\frac{\ell^2}{\mu}(\alpha^2+1)\int r^{-3} dr = \frac{-\ell^2}{2\mu}\frac{(\alpha^2+1)}{r^2}$

$\dot{r}$? $\quad \dot{\theta} = \frac{d\theta}{dr}\frac{dr}{d\tau} = \frac{\ell}{\mu r^2} \rightarrow \quad \dot{r} = \frac{dr}{d\theta}\frac{\ell}{\mu r^2}$

$$\dot{r} = \alpha k e^{\alpha\theta}\frac{\ell}{\mu r^2} = \frac{\alpha \ell}{\mu r}$$

And $\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{\ell^2(\alpha^2+1)}{2\mu r^2} = E$

$$\frac{\alpha^2\ell^2}{\mu^2 r^2} \rightarrow \frac{1}{2}\frac{\alpha^2\ell^2}{\mu r^2} + \frac{\ell^2}{2\mu r^2} - \frac{1}{2}\frac{\alpha^2\ell^2}{\mu r^2} - \frac{\ell^2}{2\mu r^2} = E = 0$$

1 2 3 4

Central Field orbits

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}} = 0 \text{ when } = 0$$

Turning points between r_{min} and r_{max}

If only one root orbit is circular

If closed orbit repeats path if not, not.

See Figure 8-4

$\Delta\theta$ in $r_{min} \to r_{max} \to r_{min} = 2 \int_{r_{min}}^{r_{max}} \frac{\ell / r^2 \, dr}{\sqrt{2\mu\left[E - U - \frac{\ell^2}{2\mu r^2}\right]}}$

Closed for $\Delta\theta = 2\pi\left(\frac{a}{b}\right)$] - integers

Closed noncircular $n = -2, +1 \leftarrow$ harmonic oscilator

$U(r) \propto r^{n+1}$ $\quad\uparrow$ inverse square law gravity

r^{-1}, r^2

Centrifugal Energy, Potential with effective potential

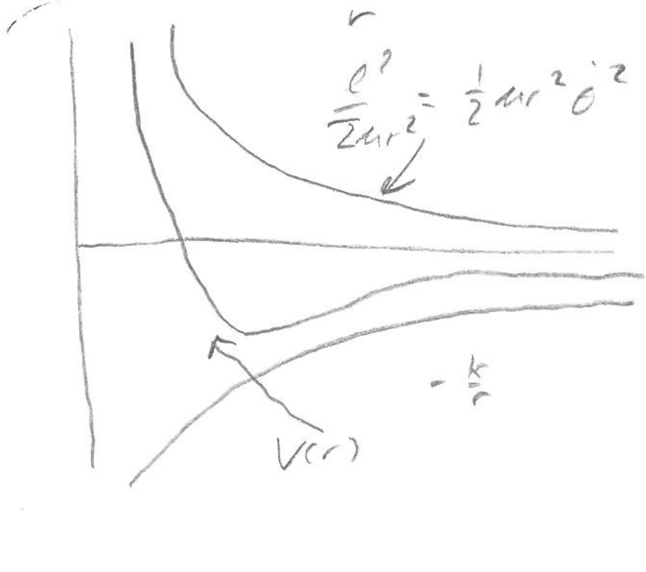
$\frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{\ell^2}{2\mu r^2} \to U_c = \frac{\ell^2}{2\mu r^2}$ $\quad F_c = -\frac{\partial U}{\partial r} = \frac{\ell^2}{\mu r^3} = \mu r\dot{\theta}^2$

$\uparrow$ Centripetal Force

$V(r) = U(r) + \frac{\ell^2}{2\mu r^2}$

$\nwarrow$ centrifugal barrier when $F(r) = -\frac{k}{r^2}$

and $U(r) = -\frac{k}{r}$ $\quad$ So $V(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}$



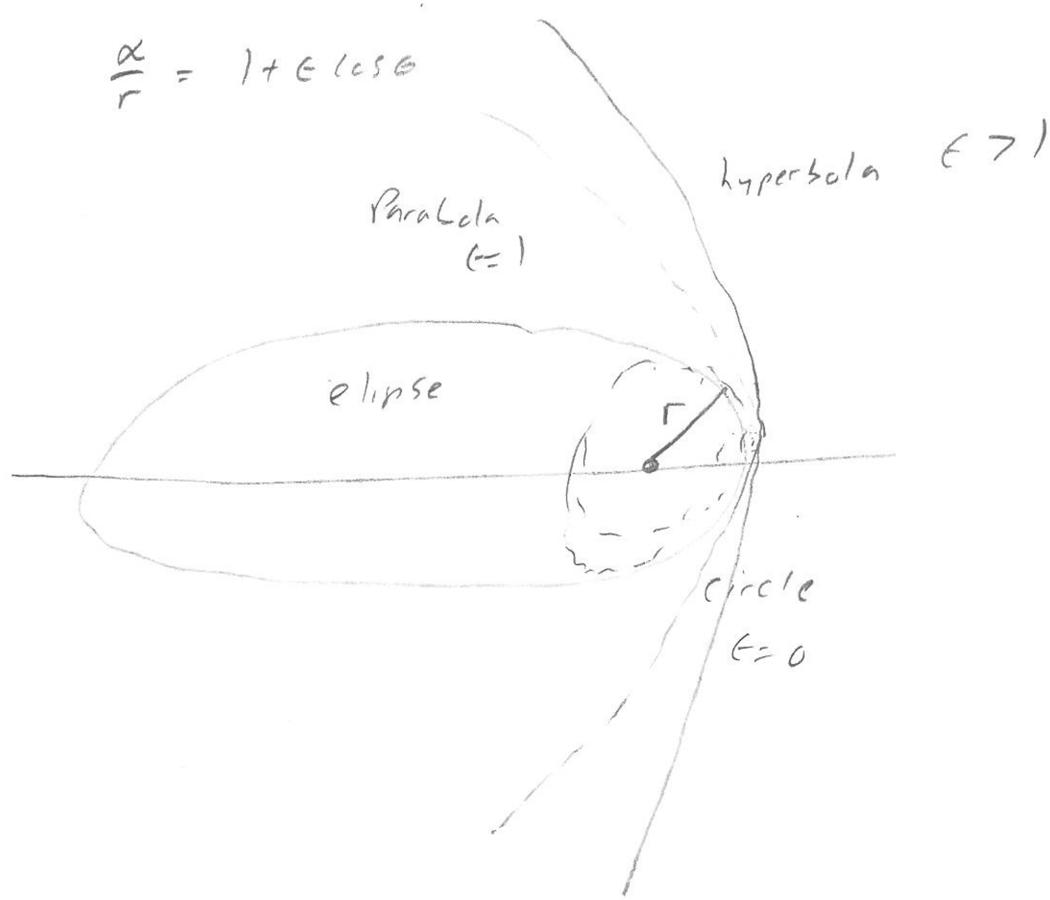
E1 → Free
Scattering
E↓ bound
E2 ← elipse
E3
r2
3
r4
E=0
orbit circular

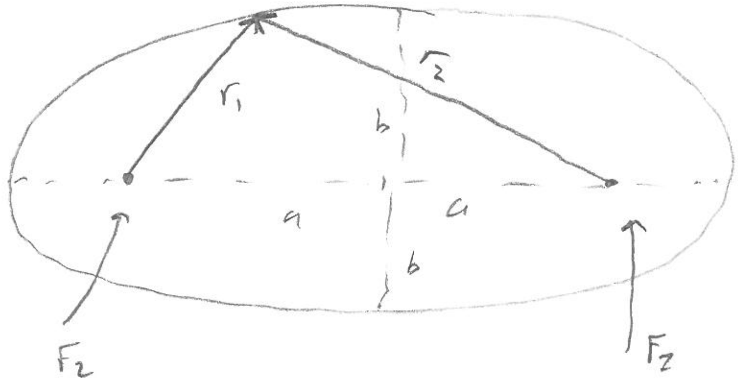
Orbits

$$\theta(r) = \int \frac{\ell/r^2 \; dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}} + c$$

$$\cos(\theta) = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} \quad \text{where} \quad \alpha = \frac{\ell^2}{\mu k}, \quad \epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

gives $\frac{\alpha}{r} = 1 + \epsilon\cos\theta$





$F_i \rightarrow$ Foci

$r_1 + r_2 = 2a$

$r_1 + r_2 = 2a$ a = semi major axis

b = semi minor axis

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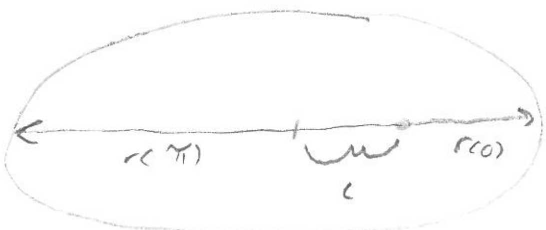
a: ?

also $\frac{\alpha}{r} = 1 + \epsilon \cos\theta$



$r(0) + r(\pi) = 2a$ $r = \frac{\alpha}{1+\epsilon\cos\theta} \rightarrow r_0 = \frac{\alpha}{1+\epsilon}$ $r(\pi) = \frac{\alpha}{1-\epsilon}$

$$r_0 + r_\pi = \frac{\alpha[1-\epsilon+1+\epsilon]}{(1+\epsilon)(1-\epsilon)} = \frac{2\alpha}{(1-\epsilon^2)}$$

$$\frac{r_0 + r_\pi}{2} = \frac{\alpha}{1-\epsilon^2} = \boxed{a = \frac{k}{2|E|}}$$

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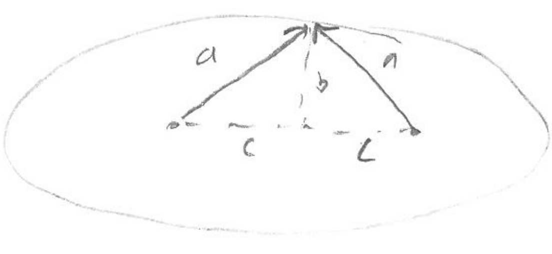
b: ?



$b = \sqrt{a^2 - c^2}$

$c = a - r(0)$

$c = a - \frac{\alpha}{1+\epsilon} = \alpha\left[\frac{1}{1-\epsilon^2} - \frac{1}{1+\epsilon}\right]$

$$\boxed{\frac{c}{\alpha} = \frac{1-(1-\epsilon)}{1-\epsilon^2} = \frac{\epsilon}{1-\epsilon^2}}$$

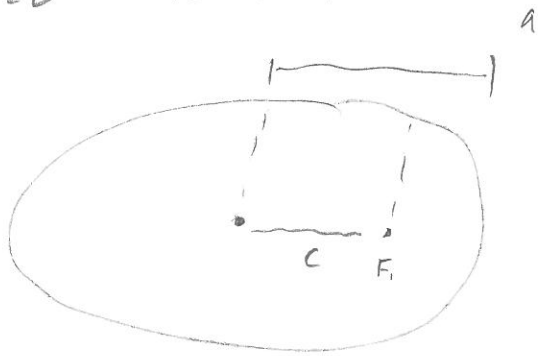
$$c = \frac{\alpha \epsilon}{1-\epsilon^2} \rightarrow \quad b = \left(\frac{\alpha^2}{(1-\epsilon^2)^2} - \frac{\alpha^2 \epsilon^2}{1-\epsilon^2} \right)^{1/2}$$

$$b = \left(\frac{1}{(1-\epsilon^2)^2} \cdot \alpha^2 (1-\epsilon^2) \right)^{1/2} = \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad \text{Perihelion} \qquad r_{max} = \frac{\alpha}{1-\epsilon} \quad \text{aphelion}$$

$$= a(1-\epsilon) \qquad\qquad = a(1+\epsilon)$$

if $\epsilon = 0$ $\quad r = a \rightarrow$ circle



$$c = \frac{\alpha \epsilon}{1-\epsilon^2} = \epsilon a$$

or $\frac{c}{a} = \epsilon$

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recall $\frac{dA}{dt} = \frac{\ell}{2\mu} \rightarrow \quad dt = \frac{2\mu}{\ell} dA$

$$\int_0^{\tau} dt = \frac{2\mu}{\ell} \int_0^{A} dA \rightarrow \tau = \frac{2\mu}{\ell} A = \frac{2\mu}{\ell} \pi a b$$

$$\tau = \frac{2\mu}{\ell} \pi a b = \pi k \sqrt{\frac{\mu}{2}} \, |E|^{-3/2}$$

or $\quad b = \sqrt{\alpha a} \qquad \pi a b = \pi a^{3/2} \sqrt{\alpha}$

$$\alpha = \frac{\ell^2}{\mu k}$$

$$\tau = \frac{2\mu}{\ell} \pi a b = \frac{2\mu}{\ell} \pi a^{3/2} \sqrt{\alpha}$$

$$\tau^2 = \frac{4\mu^2}{\ell^2} \pi^2 a^3 \frac{\ell^2}{\mu k} = \frac{4\mu \pi^2 a^3}{k}$$

For $F(r) = -\frac{G m_1 m_2}{r^2} = -\frac{k}{r}$ Keplers Third Law

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)} \rightarrow \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}, \quad \frac{\alpha}{r} = 1 + \epsilon \cos\theta, \quad \frac{dA}{dt} = \frac{\ell}{2\mu}$$

$$\tau^2 = \left(\frac{2\pi a}{v^2}\right)^2 \quad v^2 = \frac{4\pi^2 a^2}{\tau^2} = \frac{4\pi^2 a^2}{4\pi^2 a^3} G(m_1+m_2) \rightarrow v^2 = \frac{G(m_1+m_2)}{a}$$

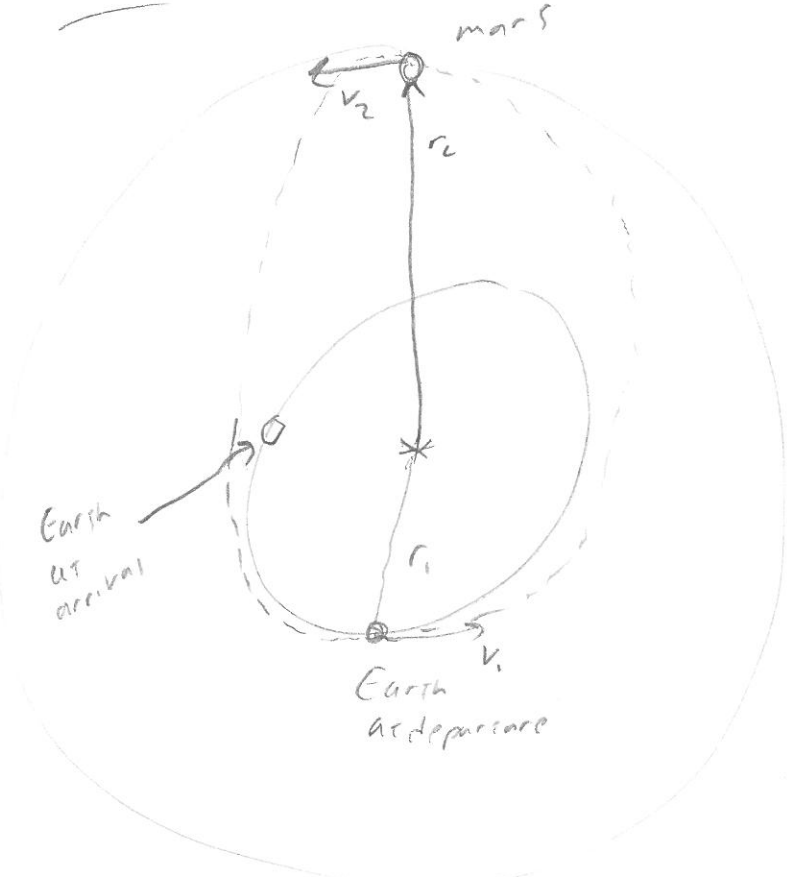
$$v = \left(\frac{G(m_1+m_2)}{a}\right)^{1/2} \rightarrow r\dot{\theta} = v \qquad r \rightarrow a$$

$$\frac{v}{a} = \dot{\theta} = \left(\frac{G(m_1+m_2)}{a^3}\right)^{1/2} \qquad r\uparrow \; v\downarrow$$

$$l = \mu r^2 \dot{\theta} = \mu\left[G(m_1+m_2)a\right]^{1/2} \qquad r\uparrow \; l\uparrow$$

Orbital dynamics

$E = -\frac{k}{2a}$ → For circle $E = \frac{1}{2}mv_1^2 - \frac{k}{r_1} = -\frac{k}{2r_1}$



$\frac{1}{2}mv_1^2 = \frac{k}{r_1}\left[1-\frac{1}{2}\right] = +\frac{k}{2r_1}$

$v_1 = \left(\frac{k}{mr_1}\right)^{1/2}$

make orbit into ellipse

$r_1 + r_2 = 2a_T$ ← semimajor axis transfer orbit

at perihelion $E_T = \frac{-k}{r_1+r_2} = \frac{1}{2}mv_{T2}^2 - \frac{k}{r_1}$

Or $v_{T2} = \sqrt{\frac{2k}{mr_1}\left(\frac{r_2}{r_1+r_2}\right)}$

$\Delta v_1 = v_{T2} - v_1$ Change in v to go from orbit

$a = r_1$ to $r_1+r_2 = 2a$

Similarly $\Delta V_2 = V_2 - V_{T2}$ → To go from orbit at r_2 to $2a = r_1 + r_2$

with $V_2 = \left(\frac{k}{mr_2}\right)^{1/2}$ $V_{T2} = \sqrt{\frac{2k}{mr_2}\left(\frac{r_1}{r_1+r_2}\right)}$

Total $\Delta V = \Delta V_1 + \Delta V_2$

Transfer Time $= \frac{\tau_T}{2}$ ← half period $= \pi\sqrt{\frac{m}{k}}\ a_T^{3/2}$

Ex. → Transfer from earth orbit to mars orbit

$\frac{m}{k} = \frac{m}{GmM_s} = \frac{1}{GM_s} = 7.53 \cdot 10^{-21} \frac{s^2}{m^3}$

$\frac{k}{m} = 1.33 \cdot 10^{20} \frac{m^3}{s^2}$

$a_T = \frac{1}{2}(r_{E-s} + r_{mars \to sun}) = 1.89 \cdot 10^{11} m$

$T_T = \pi\sqrt{\frac{m}{k}}\ a_T^{3/2} = 2.24 \cdot 10^7 s = 259$ days

and $V_{T1} = \sqrt{\frac{2k}{mr_1}\left(\frac{r_2}{r_1+r_2}\right)} = 32.7 \frac{km}{s}$

$V_1 = 29.8\ km/s$ $\Delta V = 2.9 \frac{km}{s}$

Use ORBITAL velocity make ↑ Small corrections

Hohmann Transfer to outer planets

launch in direction of Earth's orbit to gain it's orbital velocity.

To Inner Planets

launch → against direction of Earth velocity, it's ΔV that matters

See figure 8-11

Hohmann Transfers are Minimum energy Tours but not minimum time orbital Transfers

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To go Earth → Mars → Earth = 2.7 yrs

Enter
gravitational Sling Shots See figure 8-14, 8-13

Analogy

Boy throws a ball at a train. The ball goes 30mph Train 50mph

→ ←

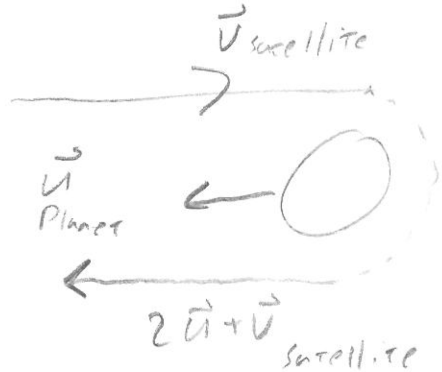
Imagine ball Thrown 30mph → Train 50mph ←

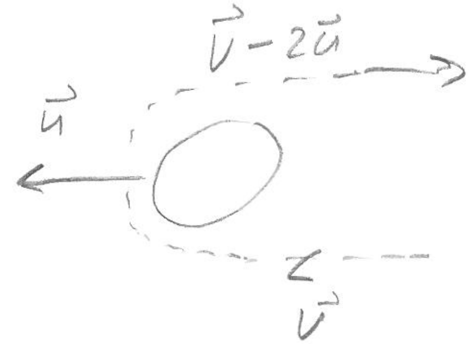
Train Sees ball coming at 80mph hits it

ball rebounds at 130mph

$$V_{bF} = \frac{V_b(m_b - m_T) + 2m_T V_T}{m_b + m_T} \rightarrow -V_b + 2V_T = V_{bF} = (30 + 2(50))\text{mph}$$

Similarly





Satellite gains $2m_s\vec{u}_p$ Loses $2m_s\vec{u}_p$

Planet loses angular momentum Planet gains angular momentum

Venus → $3.5\cdot10^4 \frac{m}{s}$ $\Delta(\vec{r} \times p) = m_s 2u_p d_{to\,sun}$

$d_{sun} \to 1.1\cdot10^{11}$ m

Mass Cassini → $2.5\cdot10^3$ kg $= 1.9\cdot10^{14} \frac{kg\,m^2}{s}$

Venus has $1.83\cdot 10^{40}\ \frac{kg\,m^2}{s^2}$

$$\frac{\Delta L_v}{L_v} = 1.05\cdot 10^{-21} \qquad \text{Now } \frac{L^2}{\mu k} = \alpha$$

assume $\frac{\alpha}{r} = 1 + \epsilon\cos\theta \Rightarrow$ let $\epsilon = 0$ for simplicity

Then $r = \alpha$ and $r_{new} = \frac{(L-\Delta L)^2}{\mu k}$ $\quad r_{old} = \frac{L^2}{\mu k}$

$$\frac{r_{new}}{r_{old}} = \frac{L^2}{(L^2-\Delta L)^2} = \frac{L^2}{L^2(1-\frac{\Delta L}{L})^2} \approx \frac{L^2}{L^2}\left(1-2\frac{\Delta L}{L}\right) = 1 - \frac{2\Delta L}{L}$$

$$r_{new} = r_{old}\left(1 - 2\cdot 10^{-21}\right)$$

Orbital Evolution of Moon - Earth

$$\vec{J} = \vec{L} + \vec{S} = \text{constant} = (L+S)\hat{L}$$

$m d_s^2 \omega_{orbit}$ $\quad \frac{2}{5} m r^2 \omega_{rotation}$

$I \omega_\omega$

show ecliptic node picture

Move to reference frame with earth at rest

$$E = \frac{1}{2} m v_m^2 - \frac{k}{r} + \frac{1}{2} I_m \omega_{rot}^2$$

assume circular orbit

$$\frac{mv^2}{r} = \frac{k}{r^2} \quad \text{and} \quad |\vec{S}_m| = I\omega_m$$

$$k = mvr = L$$

$$E = \frac{-mk^2}{2L^2} + \frac{S^2}{2I} \quad \text{but} \quad J = L + S = \text{constant}$$

so $$E = \frac{-mk^2}{2L^2} + \frac{(J-L)^2}{2I}$$

$$\frac{dE}{dL} = \frac{mk^2}{L^3} - \frac{(J-L)}{I} = 0 \rightarrow \frac{L}{m_c r^2} = \frac{S}{I}$$

$S = I\omega \qquad L = mr^2\Omega$

$\Omega = S$ orbit = day → Tidally locked

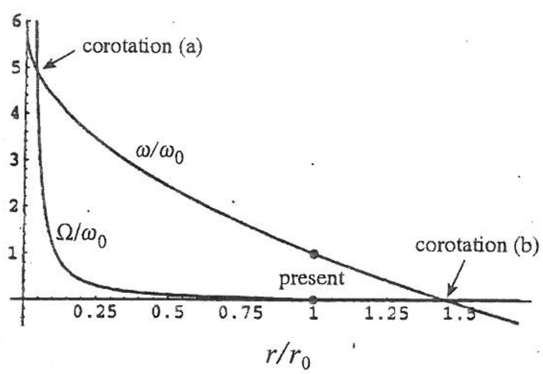


FIGURE 8-7. The spin angular velocity ω and the orbital angular velocity Ω for the earth-moon system as a function of orbital angular momentum. The subscript 0 denotes present value.

$\dot{\omega}_{earth} = -.85 \cdot 10^{-21}\ rad/s^2 \sim$ Earth day ↑ $4.4 \cdot 10^{-8}\ s/day$

$\dot{r}_{moon} \sim .4\ cm/month$ → Finger nail growth rate

Stops at $r = 1.44 r_0$ Tides End on Earth