MAT 201
Larson/Edwards – Section 7.4 (part 2)
Arc Length and Surfaces of Revolution

We will now calculate surfaces of revolution (lateral surface area).

To generate a lateral surface area we rotate a specific arc length around an axis.

<u>Definition of Surface of Revolution</u>:
If the graph of a continuous function is revolved about a line, the resulting surface is a ***surface of revolution***.

To find the lateral surface area we can use the formula $S = 2\pi rL$ where r is the average radius of the solid and L is the length of the arc being rotated.

Our lateral surface area equation for a axis of revolution (both horizontal or vertical axis of revolution) will then take on the following transformation:

$$S = 2\pi rL$$

$$S = 2\pi(\text{average radius})(\text{length of arc})$$

$$\Delta S = 2\pi r(x)\sqrt{1 + [f'(x)]^2}\,\Delta x$$

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\ dx$$

Definition of the Area of a Surface of Revolution:
Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is:

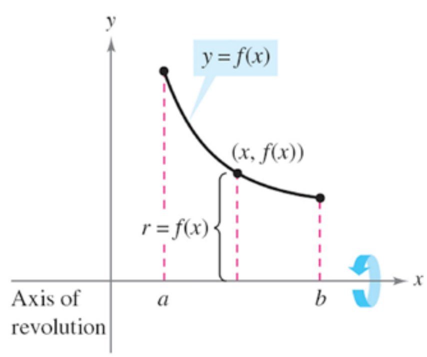
$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\ dx$$

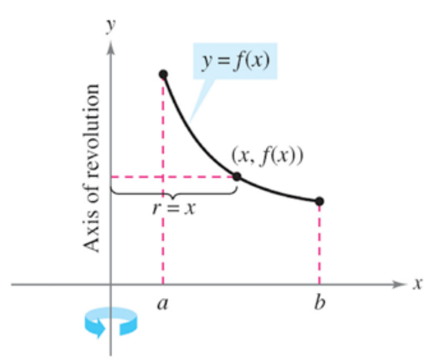
where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is:

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\ dy$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Note the differences in $r(x)$ depends on the axis of revolution below:





<u>Definition of the Area of a Surface of Revolution</u>:
Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is:

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where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface ~~area~~ is: of revolution

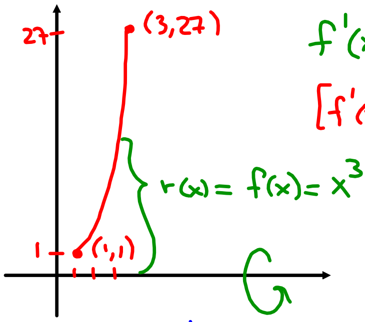
$$S = 2\pi \int_c^d r(y)\sqrt{1 + \left[g'(y)\right]^2}\, dy$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [1, 3] about the x-axis.

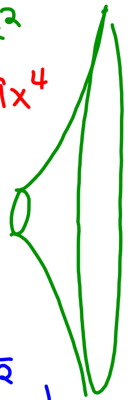
Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval [0, 1] about the y-axis.

Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [1, 3] about the x-axis.



$f'(x) = 3x^2$

$[f'(x)]^2 = 9x^4$



$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$= 2\pi \int_1^3 x^3 \sqrt{1 + 9x^4}\, dx$$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$\frac{1}{36} du = x^3\, dx$

$$\int \frac{1}{36} u^{1/2}\, du = \frac{1}{36} \cdot \frac{2}{3} u^{3/2} = \frac{1}{54}(1+9x^4)^{3/2}$$

$$= 2\pi \left[\frac{1}{54}(1+9x^4)^{3/2} \Big|_1^3 \right]$$

$$= 2\pi \left[\frac{1}{54}(1+729)^{3/2} - \frac{1}{54}(1+9)^{3/2} \right]$$

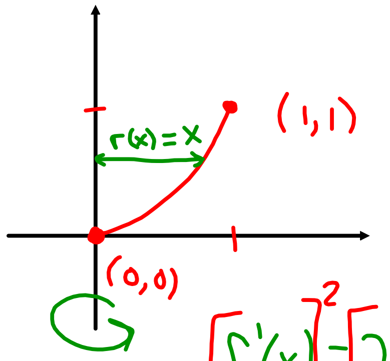
$$= 2\pi \left[\frac{1}{54} \cdot 730\sqrt{730} - \frac{1}{54} \cdot 10\sqrt{10} \right]$$

$$= \frac{1460\pi\sqrt{730}}{54} - \frac{20\pi\sqrt{10}}{54}$$

$$= \frac{1460\pi\sqrt{730} - 20\pi\sqrt{10}}{54}$$

$$= \boxed{\frac{730\pi\sqrt{730} - 10\pi\sqrt{10}}{27} \approx 2291.26 \text{ units}^2}$$

Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval [0, 1] about the y-axis.



$$S = 2\pi\int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= 2\pi\int_0^1 x\sqrt{1+4x^2}\,dx$$

$$[f'(x)]^2 = [2x]^2 = 4x^2$$

$$u = 1+4x^2$$
$$du = 8x\,dx$$
$$\frac{1}{8}du = x\,dx$$

$$= \int \frac{1}{8}u^{1/2}\,du$$

$$= \frac{1}{12}u^{3/2} \qquad \frac{1}{8}\cdot\frac{2}{3}u^{3/2}$$

$$= 2\pi\left[\frac{1}{12}(1+4x^2)^{3/2}\Big|_0^1\right]$$

$$= 2\pi\left[\frac{1}{12}(5)^{3/2} - \frac{1}{12}\right]$$

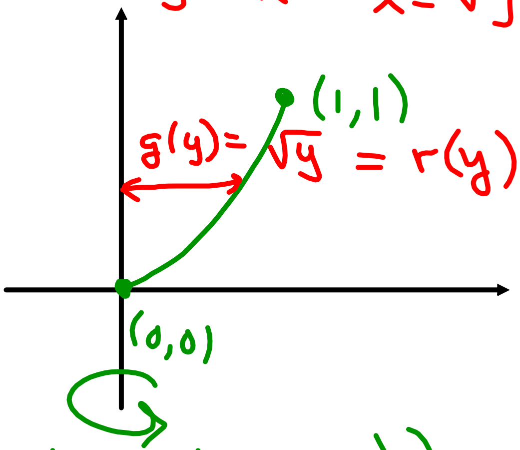
$$= 2\pi\left[\frac{5\sqrt{5}}{12} - \frac{1}{12}\right]$$

$$= \frac{10\pi\sqrt{5} - 2\pi}{12}$$

$$= \boxed{\frac{5\pi\sqrt{5} - \pi}{6} \approx 5.33 \text{ units}^2}$$

Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval [0, 1] about the y-axis.

$y = x^2 \quad x = \sqrt{y}$



$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2}\, dy$$

$$S = 2\pi \int_0^1 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy$$

$$(x')^2 = \left(\frac{1}{2} y^{-1/2}\right)^2 = \frac{1}{4} y^{-1}$$

≈ 5.33 units^2