Taylor Larrechea
Dr. Gustafson
MATH 366
CP Ch. 21.2

1) In contrast, a multistep method uses, in each step, values from two or more previous steps. These methods are motivated by the expectation that the additional information will increase accuracy and stability. But to get started, one needs values, say, y_0, y_1, y_2, y_3 in a 4-step method, obtained by Runge-Kutta or another accurate method. Thus, multistep methods are not self-starting. Such methods are obtained as follows.

2) Adams-Bashforth Methods

We consider an initial value problem

(1) $$y' = f(x,y), \quad y(x_0) = y_0$$

as before, with f such that the problem has a unique solution on some open interval containing x_0. We integrate $y' = f(x,y)$ from x_n to $x_{n+1} = x_n + h$. This gives

$$\int_{x_n}^{x_{n+1}} y'(x)\,dx = y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx.$$

Now comes the main idea. We replace $f(x, y(x))$ by an interpolation polynomial $p(x)$ (see sec. 19.3), so that we can later integrate. This gives approximations y_{n+1} of $y(x_{n+1})$ and y_n of $y(x_n)$,

(2) $$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p(x)\,dx.$$

Different choices of $p(x)$ will now produce different methods. We explain the principle by taking a cubic polynomial, namely, the polynomial $p_3(x)$ that at (equidistant)

$$x_n, \quad x_{n-1}, \quad x_{n-2}, \quad x_{n-3}$$

has the respective values

(3) $$\begin{aligned} f_n &= f(x_n, y_n) \\ f_{n-1} &= f(x_{n-1}, y_{n-1}) \\ f_{n-2} &= f(x_{n-2}, y_{n-2}) \\ f_{n-3} &= f(x_{n-3}, y_{n-3}). \end{aligned}$$

This will lead to a practically useful formula. We can obtain $p_3(x)$ from Newton's backward difference formula (18), sec. 19.3:

$$p_3(x) = f_n + r\nabla f_n + \frac{1}{2}r(r+1)\nabla^2 f_n + \frac{1}{6}r(r+1)(r+2)\nabla^3 f_n$$

where

$$r = \frac{x - x_n}{h}.$$

3)

$$\frac{1}{2}r(r+1)\nabla^2 f_n \quad : \quad r = \frac{x-x_n}{h} \qquad x_{n+1} = x_n + h \qquad x_n = x_{n+1} - h$$

$$\frac{1}{2}(r^2+r)\nabla^2 f_n = \frac{1}{2}\left(\left(\frac{x-x_n}{h}\right)^2 + \frac{x-x_n}{h}\right)\nabla^2 f_n = \frac{1}{2}\left(\frac{x^2-2x x_n + x_n^2}{h^2} + \frac{x-x_n}{h}\right)\nabla^2 f_n$$

$$= \frac{1}{2}\left(\frac{x^2-2x x_n + x_n^2 + h(x-x_n)}{h^2}\right)\nabla^2 f_n = \frac{1}{2}\left(\frac{(x-x_n)^2 + h(x-x_n)}{h^2}\right)\nabla^2 f_n$$

$$= (x-x_n)\left[(x-x_n)-h\right]\frac{\nabla^2 f_n}{2h^2} = (x-x_n)\left[(x-(x_{n+1}-h))-h\right]\frac{\nabla^2 f_n}{2h^2} = (x-x_n)(x-x_{n+1})\frac{\nabla^2 f_n}{2h^2}$$

4)

$$r = \frac{x-x_n}{h}, \quad x = x_n, \quad x = x_n + h$$

$$x = x_n, \quad r = \frac{x_n - x_n}{h} = \frac{0}{h} = 0 \quad : \quad r = 0$$

$$x = x_n + h, \quad r = \frac{x_n + h - x_n}{h} = \frac{h}{h} = 1 \quad : \quad r = 1$$

5)

$$\frac{1}{2}\int_0^1 r(r+1)\,dr : \quad \frac{1}{2}\int_0^1 r^2 + r\,dr = \frac{1}{2}\left[\frac{r^3}{3} + \frac{r^2}{2}\right]_0^1 = \frac{1}{2}\left[\frac{1}{3} + \frac{1}{2}\right] = \frac{1}{2}\left[\frac{5}{6}\right] = \frac{5}{12}$$

6)

$$\nabla f_n = f_n - f_{n-1}$$
$$\nabla^2 f_n = f_n - 2f_{n-1} + f_{n-2}$$
$$\nabla^3 f_n = f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}$$

7)

(5)
$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

8) Adams - Moulton Methods

Adams - Moulton methods are obtained if for $p(x)$ in (2) we chose a polynomial that interpolates $f(x, y(x))$ at $x_{n+1}, x_n, x_{n-1}, \dots$ (as opposed to $x_n, x_{n-1}, \dots$ used before; this is the main point).

9)

(7a)
$$y^*_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

10)

(7b)
$$y_{n+1} = y_n + \frac{h}{24}(9f^*_{n+1} + 19f_n - 5f_{n-1} + f_{n-2})$$