3.6 Basis and Dimension

3.6 # 10, 11, 14, 21, 22, 28, 34, 40, 64

3.6.10) $W=\mathbb{R}^3$; $S=\{[2,-1,4],[4,-2,8]\}$

$$A=\left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \\ 4 & 8 & 0 \end{array}\right] \qquad \text{rref}(A)=\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

No identity matrix ∴
not linearly independent

3.6.11) $W=\mathbb{R}^3$; $S=\{[1,1,8],[-3,4,2],[7,1,-3]\}$

$$A=\left[\begin{array}{ccc|c} 1 & -3 & 7 & 0 \\ 1 & 4 & 1 & 0 \\ 8 & 2 & -3 & 0 \end{array}\right] \qquad \text{rref}(A)=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

rref(A) yields an identity matrix and
$\vec{x}=\vec{0}$ ∴ linearly independent

3.6.14) $W=\mathbb{P}_2$; $S=\{t, 1-t\}$

$$A=\left[\begin{array}{c|c} t & 0 \\ 1-t & 0 \end{array}\right] \qquad \text{rref}(A)=\left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \end{array}\right]$$

$K_1t + K_2(1-t)=0$
$K_1t + K_2 - K_2t = 0$
$K_2 + t(K_1-K_2)=0$
$K_2=0$
$K_1=0$

$K_1t - K_2t = 0$
$K_1t=0$
$K_1=0$

$K_2=0$
$K_1=0$

The rref of A yields an identity
matrix set equal to 0 ∴
linealy independent

3.6.21) $S=\{\sin t, \sin 2t, \sin 3t\}$

$$W=\begin{bmatrix} \sin t & \sin 2t & \sin 3t \\ \cos t & 2\cos 2t & 3\cos 3t \\ -\sin t & -4\sin 2t & -9\sin 3t \end{bmatrix}$$

$\sin 2t(3\cos 3t) - \sin 3t(2\cos 2t)$

$-\sin t\,(-1)^{3+1}\left|\sin 2t(3\cos 3t) - \sin 3t(2\cos 2t)\right|$

$-\sin t\,(\sin 2t(3\cos 3t) - \sin 3t(2\cos 2t))$

$|W| \neq 0$ ∴ linealy independent

linearly independent

3.6.22) $S=\{1, \sin^2 t, \cos^2 t\}$

$$W=\begin{bmatrix} 1 & \sin^2 t & \cos^2 t \\ 0 & 2\sin t\cos t & -2\sin t\cos t \\ 0 & 4\cos^2 t-2 & 2-4\cos^2 t \end{bmatrix}$$

$|W| = -2\sin^3 t\cos t - 2\sin t\cos^3 t$
$\sin^2 t(-2\sin t\cos t) + \cos^2 t(-2\sin t\cos t)$
$0^{(3+1)}((-2\sin t\cos t)(\sin^2 t + \cos^2 t))$
$0\,(-2\sin t\cos t)$
0

$|W|=0$ ∴
not linearly independent

3.6.28

$$\left\{ \begin{bmatrix} e^t \\ 2e^t \\ e^t \end{bmatrix}, \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^t \end{bmatrix}, \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix} \right\}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\det(A) = -e^{2t}(e^{2t}-1) \neq 0$$

$$A = \begin{bmatrix} e^t & e^{-t} & e^{2t} \\ 2e^t & 2e^{-t} & 3e^{2t} \\ e^t & e^t & e^{2t} \end{bmatrix}$$

rref(A) = I
det(A) ≠ 0
∴ linearly independent

3.6.34 $\{e^{at}, e^{bt}\}$

$$W = \begin{bmatrix} e^{at} & e^{bt} \\ ae^{at} & be^{bt} \end{bmatrix}$$

$$|W| = e^{at}(be^{bt}) - (ae^{at})e^{bt}$$
$$e^{at}e^{bt}(b-a) \neq 0$$

as long as b is not equal to A the determinant of the Wronskian will not equal zero and ∴ be linearly independent.

linearly independent
$a \neq b$

3.6.40 $\{3t^2-4, 2t, t^2-1\}$

$$W = \begin{bmatrix} 3t^2-4 & 2t & t^2-1 \\ 6t & 2 & 2t \\ 6 & 0 & 2 \end{bmatrix}$$

$$4t^2 - 2(t^2-1) \qquad -2(3t^2+4)$$
$$4t^2 - 2t^2 + 2 \qquad -6t^2 + 8$$
$$2t^2 + 2$$

$$6(-1)^{3+1}(2t^2+2) + 2(-1)^{3+3}(-6t^2+8)$$
$$12t^2+12 - 12t^2\ 16$$
$$-4 \neq 0 \quad \therefore \text{ linearly independent}$$

det(W) ≠ 0
∴ linearly independent

3.6.64

$$3x_1 + 0x_2 + 6x_3 + 3x_4 + 9x_5 = 0$$
$$x_1 + 3x_2 - 4x_3 - 8x_4 + 3x_5 = 0$$
$$x_1 - 6x_2 + 14x_3 + 19x_4 + 3x_5 = 0$$

$$A = \left[\begin{array}{ccccc|c} 3 & 0 & 6 & 3 & 9 & 0 \\ 1 & 3 & -4 & -8 & 3 & 0 \\ 1 & -6 & 14 & 19 & 3 & 0 \end{array}\right]$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis = $\begin{bmatrix} 2 & 1 & 3 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dim = 3