MAT 201
Larson/Edwards – Section 2.4
The Chain Rule

Ex: Find the derivative for the following functions:

a) $g(x) = (2x + 3)^3$

b) $g(x) = (2x + 3)^{300}$ **OUCH!!!**

There has to be an easier way to find the derivative of the function in part (b) above without expanding the expression!

Since *h* is a composite function (that is, a function that is built up from simpler functions: *h(x) = g(f(x))* where $g(x) = x^{300}$ and $f(x) = 2x + 3$ we can use the ***chain rule***!

Ex: Find the derivative for the following functions:

a) $g(x) = (2x + 3)^3$

$g(x) = (2x+3)(2x+3)(2x+3)$

$g(x) = (2x+3)(4x^2 + 12x + 9)$

$g(x) = 8x^3 + 24x^2 + 18x + 12x^2 + 36x + 27$

$g(x) = 8x^3 + 36x^2 + 54x + 27$

$\boxed{g'(x) = 24x^2 + 72x + 54}$

Chain Rule: If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Or equivalently:

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

A consequent of the chain rule is the ***general power rule***.

The General Power Rule: If the function f is differentiable and $h(x) = [f(x)]^n$ (n is a real number), then
$$h'(x) = \frac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1} \cdot f'(x)$$

Now we can find the derivative of $g(x) = (2x + 3)^{300}$.

b) $g(x) = (2x+3)^{300}$ **OUCH!!!**

$f(x) = (2x+3)$
$h(x) = x^{300}$

Using the chain rule:

$g(x) = h(f(x))$

"Derivative of outside times the derivative of the inside."

$g'(x) = 300(2x+3)^{299} \cdot 2$

$\boxed{g'(x) = 600(2x+3)^{299}}$

Ex: Find the derivatives of the following functions:

a) $h(x) = 5(x^2 + 2x)^3$

b) $f(x) = \sqrt{6x - 5}$

c) $f(x) = (1 - x^2)^3(2x + 1)^2$

Ex: Find the derivatives of the following functions:

a) $h(x) = 5\left(x^2 + 2x\right)^3$

$$\boxed{h'(x) = 15\left(x^2+2x\right)^2 \cdot (2x+2)}$$

b) $f(x) = \sqrt{6x - 5}$

$f(x) = (6x-5)^{1/2}$

$f'(x) = \frac{1}{2}(6x-5)^{-1/2} \cdot 6$

$\boxed{f'(x) = \frac{3}{\sqrt{6x-5}}}$

c) $f(x) = (1 - x^2)^3 (2x + 1)^2$

$$f'(x) = 3(1-x^2)^2 \cdot (-2x)(2x+1)^2 + 2(2x+1)' \cdot 2(1-x^2)^3$$

$$\boxed{\begin{gathered} f'(x) = -6x(1-x^2)^2(2x+1)^2 + 4(2x+1)(1-x^2)^3 \\ \text{or} \\ f'(x) = -2(1-x^2)^2(2x+1)\left[3x(2x+1) - 2(1-x^2)\right] \end{gathered}}$$

d) $g(x) = \sqrt{\dfrac{x^2}{3x - 1}}$

<u>Trigonometric Functions and the Chain Rule</u>: The "chain rule versions" of the derivatives of the six trigonometric functions are as follows:

$\dfrac{d}{dx}[\sin u] = (\cos u)u' \qquad \dfrac{d}{dx}[\cos u] = -(\sin u)u'$

$\dfrac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \dfrac{d}{dx}[\cot u] = -(\csc^2 u)u'$

$\dfrac{d}{dx}[\sec u] = (\sec u \tan u)u' \qquad \dfrac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

Ex: Find the derivative of the following function:

$f(x) = \sqrt{x} + \dfrac{1}{4}\sin\left((2x)^2\right)$

d) $g(x) = \sqrt{\dfrac{x^2}{3x-1}} = \left(\dfrac{x^2}{3x-1}\right)^{1/2}$

$$\boxed{\begin{aligned} g'(x) &= \frac{1}{2}\left(\frac{x^2}{3x-1}\right)^{-1/2} \cdot \left[\frac{2x(3x-1) - 3x^2}{[3x-1]^2}\right] \\ &\underline{\text{or}} \\ g'(x) &= \frac{\sqrt{3x-1}}{2\sqrt{x^2}} \cdot \frac{3x^2 - 2x}{[3x-1]^2} \end{aligned}}$$

Ex: Find the derivative of the following function:

$4x^2$

$f(x) = \sqrt{x} + \frac{1}{4}\sin\left((2x)^2\right)$

$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{4} \cdot \cos\left(4x^2\right) \cdot 8x$

or

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}} + 2x\cos\left(4x^2\right)}$$

Chain Rule: If $y = f(u)$ is a differentiable function of u, and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Or equivalently:

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$$