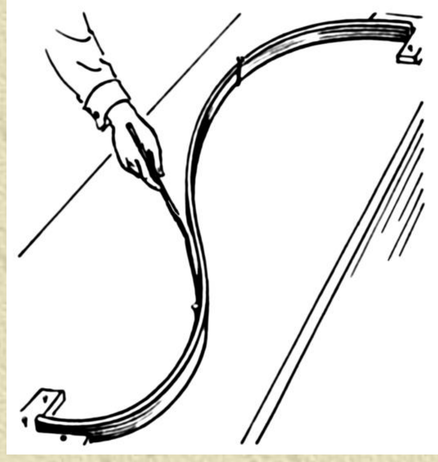
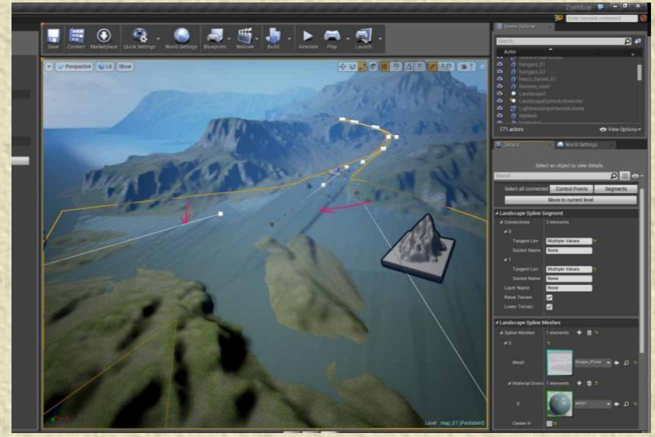
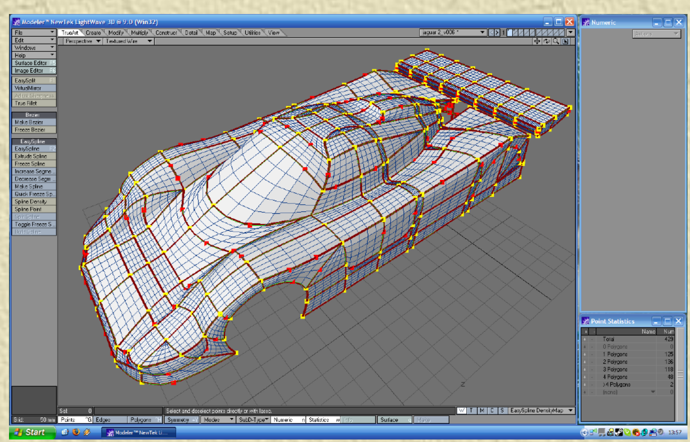
Ch 3.4: Cubic Spline Interpolation

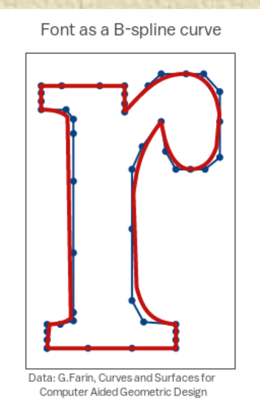
- Interpolation problems involve fitting a curve through a given set of data points.
- When draftsmen draw smooth curves through points, they often mark the points with a tack and fit a flexible elastic bar, called a *spline*, through the points as an sketching aid.
- In Ch 3.4 we consider the problem of fitting third-degree polynomials through given data points. These polynomial act as mathematical splines and are called *cubic splines*.

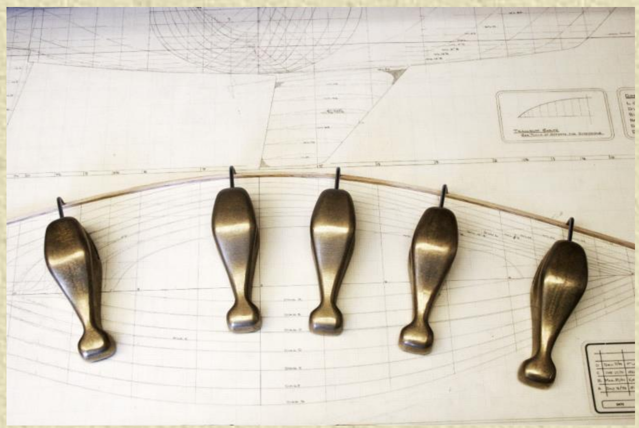


Ch 3.4: Cubic Spline Interpolation



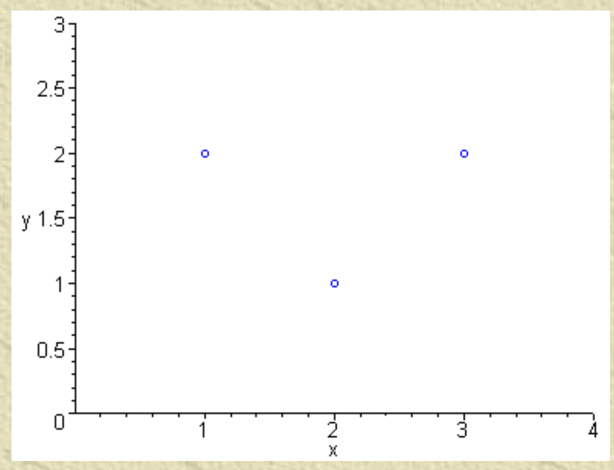






Linear Splines $y - y_0 = m(x - x_0)$

- Find the equations of the lines that pass through the given points, and label segments with the appropriate equation.

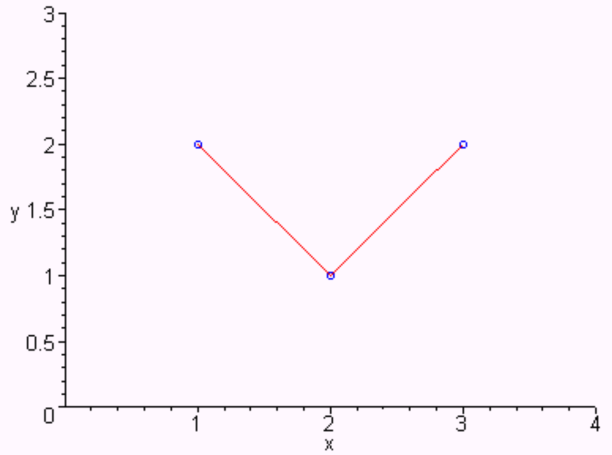


<u>Equations</u>:

$1 \le x \le 2$: ____________________

$2 \le x \le 3$: ____________________

Linear Splines $y - y_0 = m(x - x_0)$



- Is this graph smooth or are there places in which it is pointed or jagged? Is the graph differentiable at the pointy places? Explain.

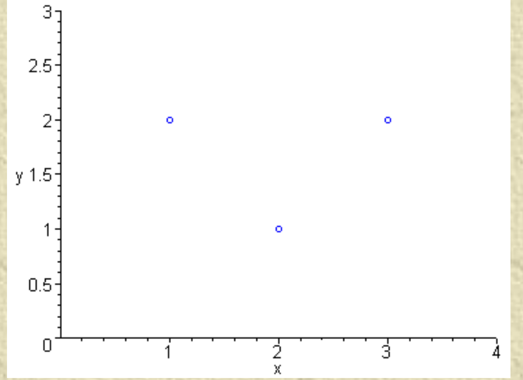
S(x) =

$$\begin{cases} 3 - x & x < 2 \\ -1 + x & otherwise \end{cases}$$

- Use equations to predict value of the phenomena generating this data when $x = 2.5$.

Cubic Splines $p(x) = a + bx + cx^2 + dx^3$

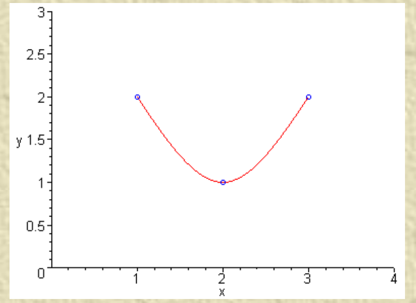
- Instead of trying to fit a piecewise linear model through a set of data points, we can try to fit a piecewise cubic polynomial model through the points.
- If we want a smooth curve (no jaggedness), we make sure that the resulting graph is differentiable everywhere.

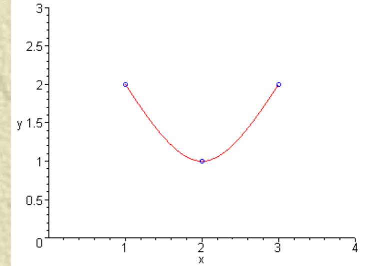


Cubic Splines $p(x) = a + bx + cx^2 + dx^3$

- Suppose that we want to fit cubic splines through the data points (1,2), (2,1), (3,2). Then we need to find two cubic polynomials with the given properties:
- $1 \le x \le 2$: $p_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$
- $2 \le x \le 3$: $p_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$
- $p_1(x)$ passes through (1,2) and (2,1)
- $p_2(x)$ passes through (2,1) and (3,2)

<u>Note</u>: Cubic polynomials have smooth 1st derivatives, while quadratics have linear 1st derivs.





Cubic Splines

- The polynomial equations we need are:

$$p_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3, \quad p_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

$$p_1'(x) = b_1 + 2c_1 x + 3d_1 x^2, \quad p_2'(x) = b_2 + 2c_2 x + 3d_2 x^2$$

$$p_1''(x) = 2c_1 + 6d_1 x, \quad p_2''(x) = 2c_2 + 6d_2 x$$

- Evaluating these at the appropriate points, we obtain

$$p_1(1) = a_1 + b_1 1 + c_1 1^2 + d_1 1^3 = 2$$

$$p_1(2) = a_1 + b_1 2 + c_1 2^2 + d_1 2^3 = 1$$

$$p_2(2) = a_2 + b_2 2 + c_2 2^2 + d_2 2^3 = 1$$

$$p_2(3) = a_2 + b_2 3 + c_2 3^2 + d_2 3^3 = 2$$

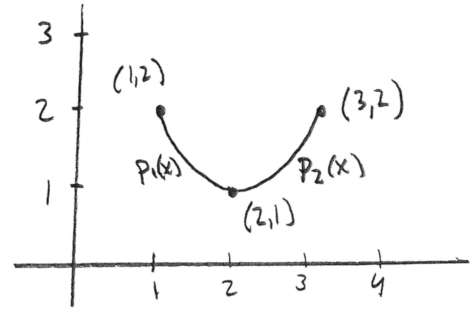
$$p_1'(2) = p_2'(2) \Rightarrow b_1 + 2c_1 2 + 3d_1 2^2 = b_2 + 2c_2 2 + 3d_2 2^2$$

$$p_1''(2) = p_2''(2) \Rightarrow 2c_1 + 6d_1 2 = 2c_2 + 6d_2 2$$

$$p_1''(1) = 2c_1 + 6d_1 1 = 0,$$

$$p_2''(3) = 2c_2 + 6d_2 3 = 0$$

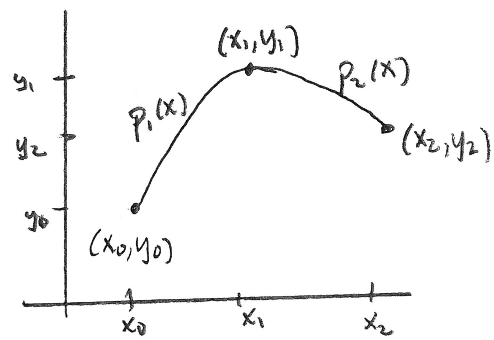
Cubic Splines



Polynomials

$$P_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$$

$$P_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$$



Equations

(1) $P_1(x_0) = a_1x_0^3 + b_1x_0^2 + c_1x_0 + d_1 = y_0$

(2) $P_1(x_1) = a_1x_1^3 + b_1x_1^2 + c_1x_1 + d_1 = y_1$

(3) $P_2(x_1) = a_2x_1^3 + b_2x_1^2 + c_2x_1 + d_1 = y_1$

(4) $P_2(x_2) = a_2x_2^3 + b_2x_2^2 + c_2x_2 + d_2 = y_2$

(5) $P_1'(x_1) = 3a_1x_1^2 + 2b_1x_1 + c_1 = 3a_2x_1^2 + 2b_2x_1^2 + c_2 = P_2'(x_1)$

(6) $P_1''(x_1) = 6a_1x_1 + 2b_1 = 6a_2x_1 + 2b_2 = P_2''(x_1)$

(7) $P_1''(x_0) = 6a_1x_0 + 2b_1 = 0$

(8) $P_2''(x_2) = 6a_2x_2 + 2b_2 = 0$

Matrix-Vector Formulation: $A\vec{x} = \vec{b}$

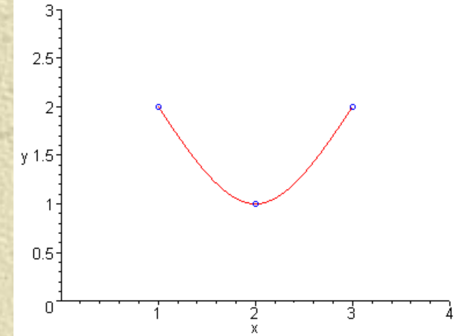
$$\begin{bmatrix} x_0^3 & x_0^2 & x_0 & 1 & 0 & 0 & 0 & 0 \\ x_1^3 & x_1^2 & x_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1^3 & x_1^2 & x_1 & 1 \\ 0 & 0 & 0 & 0 & x_2^3 & x_2^2 & x_2 & 1 \\ 0 & 0 & 0 & 0 & -3x_1^2 & -2x_1^2 & -1 & 0 \\ 3x_1^2 & 2x_1 & 1 & 0 & -6x_1 & -2 & 0 & 0 \\ 6x_1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6x_0 & 2 & 0 & 0 & 6x_2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_1 \\ y_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
>> x = [1,2,3];
>> y = [3,5,-1];
>> row1 = [x(1)^3, x(1)^2, x(1), 1, 0,0,0,0];
>> row2 = [x(2)^3, x(2)^2, x(2), 1, 0,0,0,0];
>> row3 = [0,0,0,0,x(2)^3, x(2)^2, x(2), 1];
>> row4 = [0,0,0,0,x(3)^3, x(3)^2, x(3), 1];
>> row5 = [3*x(2)^2,2*x(2),1,0,-3*x(2)^2, -2*x(2), -1, 0];
>> row6 = [6*x(2), 2,0,0,-6*x(2),-2,0,0];
>> row7 = [6*x(1), 2,0,0,0,0,0,0];
>> row8 = [0,0,0,0,6*x(3),2,0,0];
>> >> A = [row1;row2;row3;row4;row5;row6;row7;row8]

A =

     1     1     1     1     0     0     0     0
     8     4     2     1     0     0     0     0
     0     0     0     0     8     4     2     1
     0     0     0     0    27     9     3     1
    12     4     1     0   -12    -4    -1     0
    12     2     0     0   -12    -2     0     0
     6     2     0     0     0     0     0     0
     0     0     0     0    18     2     0     0

>> G = inv(A);
>> b = [y(1),y(2), y(2), y(3), 0,0,0,0]

b =

     3     5     5    -1     0     0     0     0

>> coeffs = G*b'

coeffs =

   -2.0000
    6.0000
   -2.0000
    1.0000
    2.0000
  -18.0000
   46.0000
  -31.0000
```

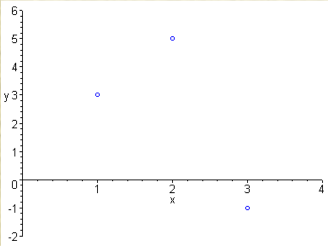
Cubic Splines

- We have 8 equations and 8 unknowns. Use Maple worksheet to solve equations and plot resulting splines.

$$p_1(1) = a_1 + b_1 1 + c_1 1^2 + d_1 1^3 = 2$$
$$p_1(2) = a_1 + b_1 2 + c_1 2^2 + d_1 2^3 = 1$$
$$p_2(2) = a_2 + b_2 2 + c_2 2^2 + d_2 2^3 = 1$$
$$p_2(3) = a_2 + b_2 3 + c_2 3^2 + d_2 3^3 = 2$$



$$p_1'(2) = p_2'(2) \Rightarrow b_1 + 2c_1 2 + 3d_1 2^2 = b_2 + 2c_2 2 + 3d_2 2^2$$
$$p_1''(2) = p_2''(2) \Rightarrow 2c_1 + 6d_1 2 = 2c_2 + 6d_2 2$$
$$p_1''(1) = 2c_1 + 6d_1 1 = 0,$$
$$p_2''(3) = 2c_2 + 6d_2 3 = 0$$



Cubic Splines Example

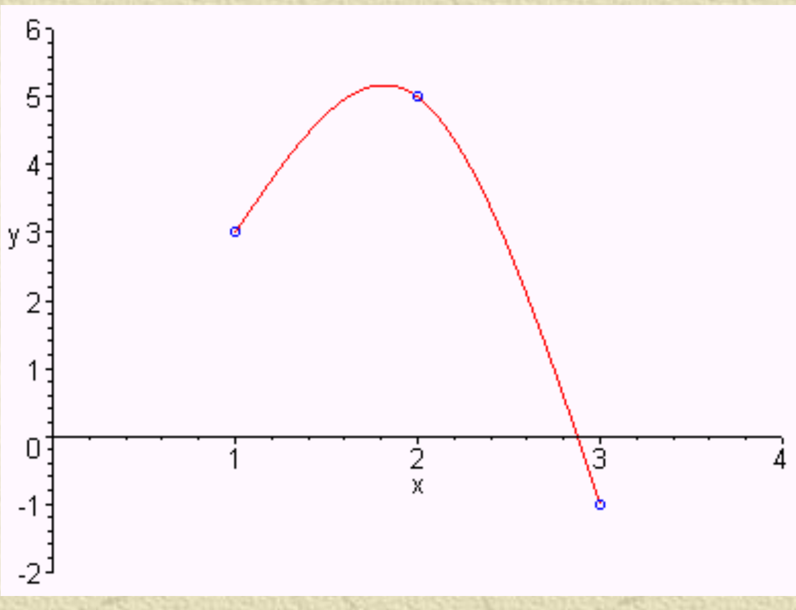
- Consider the data points (1,3), (2,5), (3,-1).
- In the space given, write the equations required to find the individual cubic spline polynomials, and then use Maple to solve the equations. Fill in the lines below for the overall spline $s(x)$ with the resulting equations of $p_1(x)$ and $p_2(x)$, making sure to associate each with the appropriate interval.

$$s(x) = \begin{cases} \underline{\hspace{6cm}}, & 1 \le x \le 2 \\ \underline{\hspace{6cm}}, & 2 \le x \le 3 \end{cases}$$

- Use Maple to graph the piecewise cubic function $s(x)$ found above, and sketch graph. Comment on how well $s(x)$ fits data.

Cubic Splines Example

- $s(x) = \begin{cases} 1 - 2x + 6x^2 - 2x^3 & x < 2 \\ -31 + 46x - 18x^2 + 2x^3 & otherwise \end{cases}$



Project

- Use Maple **spline** command, as given on Maple worksheet.