6.5 Decoupling a Linear DE System

#4,6,22, 1-16

6.5.1

$\bar{x}' = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \bar{x}$

Eigen: $|A - \lambda I| = 0$

$$\left| \begin{bmatrix} -1-\lambda & -2 \\ -2 & 2-\lambda \end{bmatrix} \right| = 0$$

$(-1-\lambda)(2-\lambda) + 2(-2)$ $\quad \lambda_1 = -2$

$-2 - \lambda + \lambda^2 - 4$ $\quad \lambda_2 = 3$

$\lambda^2 - \lambda - 6$

$(\lambda - 3)(\lambda + 2)$

vectors: $(A - \lambda I) v_i$

$\lambda = -2$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad x = 2y \quad v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = 3$

$(A - \lambda I) v_i$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix} \quad x = -\tfrac{1}{2} y \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} [B]$$

$$x(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0.2 & -0.4 \\ 0.4 & 0.2 \end{bmatrix} \qquad D = P^{-1} A P = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$w' = Dw$

$$\begin{matrix} w_1' = \\ w_2' = \end{matrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \begin{matrix} w_1' = 3w_1 \\ w_2' = -2w_2 \end{matrix}$$

$\frac{dw}{dt} = 3w_1 \;:\; \frac{dw}{w_1} = 3dt \;:\; \ln w = 3t + c$

$w_1 = ke^{3t}$

$\frac{dw}{dt} = -2w_2 \;:\; \frac{dw}{w_2} = -2dt \;:\; \ln w_2 = -2t + c$

$w_2 = ke^{-2t}$

$$x(t) = c_1 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad \begin{matrix} w_1 = ke^{3t} \\ w_2 = ke^{-2t} \end{matrix}$

6.5.3

$\bar{x}' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \bar{x}$

Eigenvectors:

$|A - \lambda I|$

$$\left| \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} \right| = 0 \quad \begin{matrix} \lambda_1 = -1 \\ \lambda_1 = 1 \end{matrix}$$

$\lambda^2 - (-1)(-1)$

$\lambda^2 - 1 = 0$

$\lambda^2 = 1$

$(A - \lambda I) \bar{v}_i \quad \lambda = -1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x = y \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$(A - \lambda I) v_i \quad \lambda = 1$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x = -y \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$w' = Dw$

$$\begin{matrix} w_1' = \\ w_2' = \end{matrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \begin{matrix} w_1' = -w_1 \\ w_2' = w_2 \end{matrix} \qquad \begin{matrix} w_1 = ke^{-t} \\ w_2 = ke^{t} \end{matrix}$$

$$D = P^{-1} A P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{matrix} w_1 = ke^{-t} \\ w_2 = ke^{t} \end{matrix}$

6.5.4 $\bar{x}' = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \bar{x}$

Eigen: $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\left| \begin{bmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{bmatrix} \right|$ $(4-\lambda)(2-\lambda) - 3(1)$

$8 - 6\lambda + \lambda^2 - 3$

$\lambda^2 - 6\lambda + 5$

$(\lambda-5)(\lambda-1) = 0$

$\lambda = 5$

$\lambda = 1$

Vectors

$(A-\lambda I)$: $\lambda=1$ $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \bar{v}$ rref $= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $x + 3y = 0$ $y = -\cancel{x}_3$

$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda = 5$

$(A - \lambda I)$: $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$

rref $\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x - y = 0$ $x = y$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$D = P^{-1}AP$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ $P = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$

$\bar{x}' = A\bar{x}$

$x_1' = 2x_1 + 3x_2$

$x_2' = x_1 + 4x_2$

$w' = D\bar{w}$

$w' = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$w_1' = 1$ $w_1 = c_1 e^t$

$w_2' = 5$ $w_2 = c_2 e^{5t}$

$x(t) = P\bar{w} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{5t} \end{bmatrix}$

$x(t) = c_1 e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w_1 = c_1 e^t$ $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$w_2 = c_2 e^{5t}$

6.5.6 $\bar{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{x}$

Eigen: $|A - \lambda I| = 0$

$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$

$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(-\lambda) - 1(1)$

$\lambda^2 - 1$

$\lambda = \pm 1$

$D = P^{-1}AP$

[F] $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

[H] $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ [G] $P^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$

vectors: $(A - \lambda I)v_i = 0$

$\lambda = -1$ [b]

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

[C]

rref $= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$ $x = -y$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1$ [D] [E]

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

rref $= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x = y$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w' = Dw$

$w' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $w_1' = -w_1$: $w_1 = c_1 e^{-t}$

$w_2' = w_2$: $w_2 = c_2 e^t$

$x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$w_1 = c_1 e^{-t}$ $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$w_2 = c_2 e^t$

6.5.5 $\bar{x}' = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}\bar{x}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{bmatrix} \right| = 0$

$(2-\lambda)(-5-\lambda) + 3(2)$

$-10 + 3\lambda + \lambda^2 + 6$

$\lambda^2 + 3\lambda - 4$

$(\lambda - 1)(\lambda + 4)$

$\lambda = -4, 1$

$(A - \lambda I)\bar{v}_i$ $\quad \lambda = -4$

$\begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $\quad x = \frac{1}{2}y \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 1$

$\begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ $\quad x = 3y \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \quad D = P^{-1}AP = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

$w_1 = Dw \;:\; \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$w_1' = -4w_1$

$w_2' = w_2$

$w_1 = ke^{-4t}$

$w_2 = ke^{t}$

$x(t) = c_1 e^{-4t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad w_1 = ke^{-4t}$

$w_2 = ke^{t}$

6.5.11 $\bar{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\bar{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} ke^{-t} \\ ke^{t} - 1 \end{bmatrix}$

$-ke^{-t} + ke^{-t} - 1 = -1$

$ke^{-t} + ke^{t} - 1 = ke^{-t} + ke^{t} - 1 \quad = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right| = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$(A - \lambda I)v_i$

$\lambda = -1$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x = -y \quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 1$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x = y \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \quad D = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad x(t) = P\bar{w}(t)$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + P^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad w_2' - w_2 = 1 \quad \mu(t) = e^{\int(-1)dt} = e^{-t}$

$\int \frac{d}{dt} e^{-t} w_2 = \int e^{-t}$

$w_1' = -w_1 + 0 \quad w_1 = ke^{-t}$

$w_2' = w_2 + 1 \quad w_2 = ke^{t} - 1$

$e^{-t} w_2 = -e^{-t} + C$

$w_2 = -1 + ke^{t}$

$x(t) = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

6.5.15

$$\bar{x}' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\bar{x} + \begin{bmatrix} t \\ 2t \end{bmatrix}$$

$$w' = D\bar{w} + P^{-1}\bar{f}(t)$$
$$x = P\bar{w}$$

$$|A - \lambda I|$$
$$\left|\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}\right| = 0 \qquad (1-\lambda)(3-\lambda) - 4(2)$$
$$3 - 4\lambda + \lambda^2 - 8$$
$$\lambda^2 - 4\lambda - 5$$
$$(\lambda - 5)(\lambda + 1)$$
$$\lambda = -1, 5$$

$$(A - \lambda I)$$
$$\lambda = -1$$
$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad x + 2y = 0, \quad x = -2y \qquad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = 5$$
$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad x - y = 0, \quad x = y \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad D = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \qquad w' = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t \\ \frac{5}{3}t \end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} t \\ 2t \end{bmatrix} = \frac{1}{3}\begin{bmatrix} t \\ 5t \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ \frac{5}{3}t \end{bmatrix}$$

$$w_1' = -w_1 + \frac{1}{3}t \qquad w_2' = 5w_2 + \frac{5}{3}t$$

$$y' + y = \frac{1}{3}t \qquad y' - 5y = \frac{5}{3}t$$

$$e^{\int 1\,dt} = e^t$$
$$e^t(y' + y) = e^t\left(\frac{1}{3}t\right)$$
$$\frac{d}{dt}(ye^t) = e^t\left(\frac{1}{3}t\right) \qquad u = t \quad dv = e^t, \quad du = 1 \quad v = e^t$$
$$ye^t = \frac{1}{3}\int e^t t\,dt \qquad te^t - \int e^t\,dt$$
$$te^t - e^t$$
$$ye^t = \frac{1}{3}(te^t - e^t + k)$$
$$y = \frac{1}{3}e^{-t}(t - 1 + k)$$
$$y = Ke^{-t} + \frac{1}{3}e^{-t}(t-1)$$

$$x(t) = c_1e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -t + \frac{3}{5} \\ -\frac{2}{5} \end{bmatrix}$$

$$w_2' - 5w_2 = \frac{5}{3}t$$
$$\mu = e^{\int -5\,dt} = e^{-5t}$$
$$\frac{d}{dt}(-5ye^{-5t}) = e^{-5t}\left(\frac{5}{3}t\right)$$
$$-5ye^{-5t} = \int e^{-5t}\left(\frac{5}{3}t\right)dt$$
$$ye^{-5t} = \frac{5}{3}\int te^{-5t}\,dt$$
$$\frac{5}{3}\int te^{-5t} \qquad u = t \quad dv = e^{-5t}, \quad du = 1 \quad v = -\frac{1}{5}e^{-5t}$$
$$\frac{5}{3}\left(-\frac{t}{5}e^{-5t} + \frac{1}{5}\int e^{-5t}\,dt\right)$$
$$\frac{5}{3}\left(-\frac{t}{5}e^{-5t} - \frac{1}{25}e^{-5t} + c\right)$$
$$\left(-\frac{t}{3}e^{-5t} - \frac{1}{15}e^{-5t}\right) + C_2$$
$$ye^{-5t} = -\frac{t}{3}e^{-5t} - \frac{1}{15}e^{-5t} + C_2$$
$$y = Ke^{5t} - t/3 - 1/15$$

6.5.22) $\bar{x}' = A\bar{x} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}\bar{x}$

a.) Eigen: $\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}}_{}$ [A] $- \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$

$\left|\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}\right| = 0$ $\quad \lambda = 3$

$(-\lambda+4)(-\lambda+2) - (1)(-1)$

$\lambda^2 - 6\lambda + 8 + 1$

$\lambda^2 - 6\lambda + 9$

$(\lambda-3)(\lambda-3)$

$\lambda = 3$

$(A - \lambda I)\bar{v} = 0$

[b]

$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

rref $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x = y$

[c]

$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = [\bar{v} \mid \bar{w}]$

[B]

$\parallel$

$(A - \lambda I)\bar{w} = \bar{v}$

$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$\lambda = 3$

$P^{-1}AP = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = J$

$\boxed{\begin{array}{l} \lambda = 3 \quad \bar{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \; P^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ P^{-1}AP = J \checkmark \end{array}}$

b.) $\bar{w}' = J\bar{w}$

$J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

$\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$\bar{w}' = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$w_1' = 3w_1 + w_2$

$w_2' = 0w_1 + 3w_2$

$\frac{dw_2}{dt} = 3w_2$

$\int \frac{dw_2}{w_2} = \int 3\,dt$

$\ln(w_2) = 3t$

$w_2 = e^{3t}$

$\frac{dw_1}{dt} = 3w_1 + e^{3t}$

$w_1' - 3w_1 = e^{3t}$

$\boxed{\begin{array}{l} w_1 = c_1 e^{3t} \\ w_2 = c_2 e^{3t} \end{array}}$

$y = y_h + y_p$

$w' = 3w_1$

$w_1 = ke^{3t}$

$y_h = ke^{3t}$ $\qquad y = ke^{3t}$

$y_p:\ Ae^{3t} = e^{3t}$

$\dot{y}_p = 3Ae^{3t}$

$y_p:\ 3Ae^{3t} - 3Ae^{3t} = e^{3t}$

$0 = e^{3t}$

$y_p = 0$