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PHYS 362 HW 14

Problem 1

a.) N_+ = Spin up , N_- = Spin down

$N_+ = 0, N_- = 16$	$N_+ = 9, N_- = 7$
$N_+ = 1, N_- = 15$	$N_+ = 10, N_- = 6$
$N_+ = 2, N_- = 14$	$N_+ = 11, N_- = 5$
$N_+ = 3, N_- = 13$	$N_+ = 12, N_- = 4$
$N_+ = 4, N_- = 12$	$N_+ = 13, N_- = 3$
$N_+ = 5, N_- = 11$	$N_+ = 14, N_- = 2$
$N_+ = 6, N_- = 10$	$N_+ = 15, N_- = 1$
$N_+ = 7, N_- = 9$	$N_+ = 16, N_- = 0$
$N_+ = 8, N_- = 8$	

b.) 2 possibilities, 16 particles : 2^N where $N = 16$ $\therefore$ $2^{16} = 65{,}536$

$$\boxed{65{,}536 \text{ microstates}}$$

c.)

$$\binom{N}{N_+} = \frac{N!}{N_+!\,(N - N_+)!}$$

Macrostates	# of associated Microstates
$N_+ = 0, N_- = 16$	1
$N_+ = 1, N_- = 15$	16
$N_+ = 2, N_- = 14$	120
$N_+ = 3, N_- = 13$	560
$N_+ = 4, N_- = 12$	1,820
$N_+ = 5, N_- = 11$	4,368
$N_+ = 6, N_- = 10$	8,008
$N_+ = 7, N_- = 9$	11,440
$N_+ = 8, N_- = 8$	12,870

Macrostates	# of associated Microstates
$N_+ = 9, N_- = 7$	11,440
$N_+ = 10, N_- = 6$	8,008
$N_+ = 11, N_- = 5$	4,368
$N_+ = 12, N_- = 4$	1,820
$N_+ = 13, N_- = 3$	560
$N_+ = 14, N_- = 2$	120
$N_+ = 15, N_- = 1$	16
$N_+ = 16, N_- = 0$	1

d.) 16 up : Prob (16 up) = $1/65{,}536$ ⋮ 12 up : Prob (12 up) = $1{,}820/65{,}536$

For the microstates that have 12 up and 4 down there are 1,820 possible combinations. The probability of having 12 up and 4 down is more likely than 16 up but this specific microstate is unique. This means that ↑↓↑↑↑↑↓↑↑↓↓↑↑↑↑ is just as probable as ↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑.

$$\boxed{\text{The probabilities are the Same.}}$$

Problem 1 Continued

e.) $\text{Prob}(\text{Macrostate}) = \dfrac{\Omega(N_+)}{\text{Microstates}}$: $\Omega(N_+) = \dfrac{N!}{N_+!\,(N-N_+)!}$: $\text{Prob}(\text{Macrostate}) = \dfrac{\frac{N!}{N_+!\,(N-N_+)!}}{2^{16}}$

Macrostates	Prob(Macrostates)
$N_+ = 0,\ N_- = 16$	1.526×10^{-5}
$N_+ = 1,\ N_- = 15$	2.441×10^{-4}
$N_+ = 2,\ N_- = 14$	1.831×10^{-3}
$N_+ = 3,\ N_- = 13$	8.545×10^{-3}
$N_+ = 4,\ N_- = 12$	2.777×10^{-2}
$N_+ = 5,\ N_- = 11$	6.665×10^{-2}
$N_+ = 6,\ N_- = 10$	0.122
$N_+ = 7,\ N_- = 9$	0.175
$N_+ = 8,\ N_- = 8$	0.196

Macrostates	Prob(Macrostates)
$N_+ = 9,\ N_- = 7$	0.175
$N_+ = 10,\ N_- = 6$	0.122
$N_+ = 11,\ N_- = 5$	6.665×10^{-2}
$N_+ = 12,\ N_- = 4$	2.777×10^{-2}
$N_+ = 13,\ N_- = 3$	8.545×10^{-3}
$N_+ = 14,\ N_- = 2$	1.831×10^{-3}
$N_+ = 15,\ N_- = 1$	2.441×10^{-4}
$N_+ = 16,\ N_- = 0$	1.526×10^{-5}

f.) The macrostate with $N_+ = 8$, $N_- = 8$ is the most probable

g.) Probability of less than $N_+ = 4$ $= 1.526\times10^{-5} + 2.441\times10^{-4} + 1.831\times10^{-3} + 8.545\times10^{-3} = 0.010635$

$\therefore$ Probability of at least $N_+ = 4$ $= 1 - 0.010635 = 0.989365$

$$\boxed{\text{Prob(At least } N_+ = 4) = 0.9894}$$

h.) Probability of at most $N_+ = 5$ $= 1.526\times10^{-5} + 2.441\times10^{-4} + 1.831\times10^{-3} + 8.545\times10^{-3} + 2.777\times10^{-2} + 6.665\times10^{-2}$

$= 0.105055$

$$\boxed{\text{Prob(At most } N_+ = 5) = 0.1051}$$

Problem 2

a.) $E = -2\mu B \rightsquigarrow$ 4 Spin up, 2 Spin down : N_+ = # of Spin up, N_- = # of spin down

$$\boxed{\text{Possible Macrostates : } N_+ = 4,\ N_- = 2}$$

b.) $2^6 = 64$ possible microstates

$\Omega(N_+) = \dfrac{6!}{4!\,(6-4)!}$ = 15 possible microstates for $N_+ = 4$: $1 \equiv$ Spin up, $0 \equiv$ spin down

111100	111010	011011
111001	110110	010111
110011	101110	101101
100111	011110	110101
001111	011101	101011

c.) There are 10 states with a particle in the left most position that is spin up. There are 15 total states for this macrostate.

$\therefore$ Prob(1 @ leftmost) $= \dfrac{10}{15} = \dfrac{2}{3} \approx 0.667$

$$\boxed{\text{Prob(1 @ leftmost)} = 0.667}$$

d.) There are 6 states with two particles that are in the left most position that are spin up. There are 15 total states for this macrostate.

$\therefore$ Prob(2 @ leftmost) $= \dfrac{6}{15} = \dfrac{2}{5} = 0.40$

$$\boxed{\text{Prob(2 @ leftmost)} = 0.40}$$

Problem 3

a.) $P = \binom{N}{N_+} p^{N_+} q^{N-N_+}$; $\binom{N}{N_+} = \dfrac{N!}{N_+!(N-N_+)!}$; $p = 3/4$, $q = 1/4$

Macrostates (N_+)	Probability of Macrostate	Energy (μB)
$N_+ = 0$	$\binom{8}{0}(3/4)^0(1/4)^8 = 1/65,536$	8
$N_+ = 1$	$\binom{8}{1}(3/4)^1(1/4)^7 = 3/8192$	6
$N_+ = 2$	$\binom{8}{2}(3/4)^2(1/4)^6 = 63/16384$	4
$N_+ = 3$	$\binom{8}{3}(3/4)^3(1/4)^5 = 189/8192$	2
$N_+ = 4$	$\binom{8}{4}(3/4)^4(1/4)^4 = 2835/32768$	0
$N_+ = 5$	$\binom{8}{5}(3/4)^5(1/4)^3 = 1701/8192$	-2
$N_+ = 6$	$\binom{8}{6}(3/4)^6(1/4)^2 = 5103/16884$	-4
$N_+ = 7$	$\binom{8}{7}(3/4)^7(1/4)^1 = 2187/8192$	-6
$N_+ = 8$	$\binom{8}{8}(3/4)^8(1/4)^0 = 6561/65536$	-8

b.) The most probable is when $N_+ = 6$, The probability is $\binom{8}{6}\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)^2 = \dfrac{5103}{16,834} \approx 0.311$

$$\boxed{N_+ = 6,\ \text{Prob}(N_+ = 6) = \frac{5103}{16,834} \approx 0.311}$$

c.) $$\bar{E} = \sum_{i=0}^{n} E_i P(i) = 8 \cdot \frac{1}{65,536} + 6 \cdot \frac{3}{8,192} + 4 \cdot \frac{63}{16,384} + 2 \cdot \frac{189}{8,192} - 2 \cdot \frac{1,701}{8,192}$$
$$- 4 \cdot \frac{5,103}{16,384} - 6 \cdot \frac{2,187}{8,192} - 8 \cdot \frac{6,561}{65,536} = -4$$

$$\boxed{\bar{E} = -4\mu B,\ N_+ = 6}$$

d.) $\sigma_E = \sqrt{(\bar{E^2}) - (\bar{E})^2}$

$$\bar{E^2} = \sum_{i=0}^{n} E_i^2 P(i) = 8^2 \cdot \frac{1}{65,536} + 6^2 \cdot \frac{3}{8,192} + 4^2 \cdot \frac{63}{16,384} + 2^2 \cdot \frac{189}{8,192} - 2^2 \cdot \frac{1,701}{8,192}$$
$$+ 4^2 \cdot \frac{5,103}{16,384} + 6^2 \cdot \frac{2,187}{8,192} + 8^2 \cdot \frac{6,561}{65,536} = 22$$

$\sigma_E = \sqrt{22 - 16} = \sqrt{6}$

$$\boxed{\sigma_E = \pm\sqrt{6}\ \mu B}$$

e.) The macrostate with energy that is the same as the mean energy is $N_+ = 6$. The microstates of this ensemble will all have the same energy as the mean energy. Thus the probability that the ensembles microstate has the same energy as the mean energy is

$$\text{Prob}(N_+ = 6) = \binom{8}{6}\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)^2 = \frac{8!}{6!(8-6)!}\left(\frac{3}{4}\right)^6\left(\frac{1}{4}\right)^2 = \frac{5,103}{16,834}$$

$$\boxed{\text{Prob} = \frac{5,103}{16,834}}$$

Problem 3 Continued

f.) $N_+ = 5$, $N_+ = 7$

$N_+ = 5 \to \frac{-2}{-4} = \frac{1}{2}$, $N_+ = 7 \to \frac{-6}{-4} = \frac{3}{2}$: $\frac{\sigma_E}{\bar{E}} = \frac{\pm\sqrt{6}}{-4} = \mp 0.612$

$$\boxed{N_+ = 5 \to \frac{1}{2}, \; N_+ = 7 \to \frac{3}{2}, \; \frac{\sigma_E}{\bar{E}} = \mp 0.612}$$

g.) $\text{Prob}(N_+=5) = \frac{1701}{8192}$, $\text{Prob}(N_+=6) = \frac{5103}{16384}$, $\text{Prob}(N_+=7) = \frac{2187}{8192}$

$$P = \text{Prob}(N_+=5) + \text{Prob}(N_+=6) + \text{Prob}(N_+=7) = \frac{1701}{8192} + \frac{5103}{16384} + \frac{2187}{8192} = \frac{12879}{16384} \approx 0.786$$

$$\boxed{P(N_+=5 + N_+=6 + N_+=7) = \frac{12879}{16384} \approx 0.786}$$

h.) $\bar{E} - \sigma_E = -6.45\,\mu B$, $\bar{E} + \sigma_E = -1.55\,\mu B$: The only macrostates within this range are $N_+=5, N_+=6, N_+=7$

$\text{Prob}(N_+=5) = \frac{1701}{8192}$, $\text{Prob}(N_+=6) = \frac{5103}{16384}$, $\text{Prob}(N_+=7) = \frac{2187}{8192}$

$$P = \text{Prob}(N_+=5) + \text{Prob}(N_+=6) + \text{Prob}(N_+=7) = \frac{1701}{8192} + \frac{5103}{16384} + \frac{2187}{8192} = \frac{12879}{16384} \approx 0.786$$

Thus

$$\boxed{P(\bar{E} - \sigma_E \text{ or } \bar{E} + \sigma_E) = \frac{12879}{16384} \approx 0.786}$$

i.) The energy of the ensemble is within 100% of the mean with probability $\frac{12879}{16384}$.

Problem 4

a.) $E=-BM$: $M=\mu(N_+-N_-)=\mu(N_+-(N-N_+))=\mu(2M_+-N)$: $P=\frac{1+\varepsilon}{2}$, $q=\frac{1-\varepsilon}{2}$

$E=-\mu B(2N_+-N)=\mu B(N-2N_+)$: $E=\mu B(N-2N_+)$

$\bar{E}=\mu B(\bar{N}-2\bar{N}_+)$: $\bar{N}=N$, $\bar{N}_+=PN=N\frac{(1+\varepsilon)}{2}$, $\bar{N}^2=N^2$, $\overline{N_+^2}=\bar{N}_+^2+pqN$

$\bar{E}=\mu B\left(N-2\cdot\frac{N(1+\varepsilon)}{2}\right)=\mu B(N-N(1+\varepsilon))=\mu B(\cancel{N}-\cancel{N}-N\varepsilon)=-\mu BN\varepsilon$

$$\boxed{\bar{E}=-\mu BN\varepsilon}$$

$\sigma_E=\sqrt{(\overline{E^2})-(\bar{E})^2}$: $(\bar{E})^2=\mu^2B^2N^2\varepsilon^2$

$\overline{E^2}=\mu^2B^2(4\overline{N_+^2}-4\bar{N}_+\bar{N}+\bar{N}^2)$

$\sigma_E=\sqrt{\mu^2B^2(4\overline{N_+^2}-4\bar{N}_+\bar{N}+\bar{N}^2)-\mu^2B^2(\bar{N}^2-4\bar{N}\bar{N}_++4\bar{N}_+^2)}$

$=\mu B\sqrt{(\cancel{N^2}-\cancel{4pN^2}+\cancel{4N^2p^2}+4pqN-\cancel{N^2}+\cancel{4pN^2}-\cancel{4p^2N^2})}=\mu B\sqrt{4pqN}$

$=\mu B\sqrt{\cancel{4}\frac{(1+\varepsilon)}{\cancel{2}}\frac{(1-\varepsilon)}{\cancel{2}}N}=\mu B\sqrt{(1-\varepsilon^2)N}$

$$\boxed{\sigma_E=\mu B\sqrt{(1-\varepsilon^2)N}}$$

b.) $\frac{\sigma_E}{\bar{E}}=\frac{\cancel{\mu B}\sqrt{(1-\varepsilon^2)N}}{-\cancel{\mu B}N\varepsilon}=-\frac{\sqrt{(1-\varepsilon^2)N}}{N\varepsilon}=-\frac{\sqrt{(1-\varepsilon^2)}}{\sqrt{N}\,\varepsilon}$

$$\boxed{\lim_{N\to\infty}\frac{\sigma_E}{\bar{E}}=-\frac{\sqrt{(1-\varepsilon^2)}}{\sqrt{N}\,\varepsilon}=0}$$

c.) $\bar{E}=-\mu BN\varepsilon=-\frac{1}{2}\cdot N\cdot\frac{1}{3}=-\frac{N}{6}$: $\bar{E}=-\frac{N}{6}$ ✓ : $\sigma_E=\mu B\sqrt{(1-\varepsilon^2)N}=\frac{1}{2}\sqrt{8/9\,N}=\sqrt{\frac{1}{4}\cdot\frac{8}{9}N}=\sqrt{2/9\,N}$ ✓

$\bar{E}=-\frac{N}{6}$, $\sigma_E=\sqrt{2N/9}$ ✓

$N_+=\frac{N}{2}-\frac{\bar{E}}{2\mu B}=\frac{N}{2}+\frac{N}{6}=\frac{4N}{6}=\frac{2N}{3}$: $N_+=\frac{2N}{3}$

$\bar{E}=-\mu B(2N_+-N)=-\frac{1}{2}\left(2\cdot\frac{2N}{3}-\frac{3N}{3}\right)=-\frac{1}{2}\left(\frac{4N}{3}-\frac{3N}{3}\right)=-\frac{1}{2}\left(\frac{N}{3}\right)=-\frac{N}{6}$ ✓

Because N_+ gives $\bar{E}=-\frac{N}{6}$, $N_+=\bar{E}\pm\sigma_E=\frac{2N}{3}\mp\sqrt{2N/9}$: $N_+=\frac{2N}{3}\mp\sqrt{2N/9}$ ✓

Parts d.)- f.) on next page

Problem 4 continued

N	E	σ	E-σ	E+σ	σ/E
21	-3.5	2.160247	-5.66025	-1.339753101	0.6172134
90	-15	4.472136	-19.4721	-10.52786405	0.298142397
150	-25	5.773503	-30.7735	-19.22649731	0.230940108

N+	N-	N	Energies	Probability	P(E+σ and E-σ)
0	21	21	10.5	9.55991E-11	0.754063801
1	20	21	9.5	4.01516E-09	**Fractional Diff**
2	19	21	8.5	8.03032E-08	-0.6172134
3	18	21	7.5	1.01717E-06	
4	17	21	6.5	9.15457E-06	
5	16	21	5.5	6.22511E-05	
6	15	21	4.5	0.000332006	
7	14	21	3.5	0.001422881	
8	13	21	2.5	0.004980084	
9	12	21	1.5	0.01438691	
10	11	21	0.5	0.034528584	

N+	N-	N	Energies	Probability	P(E+σ and E-σ)
0	90	90	45	1.14574E-43	0.685806183
1	89	90	44	2.06234E-41	**Fractional Diff**
2	88	90	43	1.83548E-39	-0.298142397
3	87	90	42	1.07681E-37	
4	86	90	41	4.68414E-36	
5	85	90	40	1.61135E-34	
6	84	90	39	4.56548E-33	
7	83	90	38	1.09572E-31	
8	82	90	37	2.27361E-30	
9	81	90	36	4.14302E-29	
10	80	90	35	6.71169E-28	

N+	N-	N	Energies	Probability	P(E+σ and E-σ)
0	150	150	75	2.70279E-72	0.659249611
1	149	150	74	8.10836E-70	**Fractional Diff**
2	148	150	73	1.20815E-67	-0.230940108
3	147	150	72	1.19204E-65	
4	146	150	71	8.76147E-64	
5	145	150	70	5.1167E-62	
6	144	150	69	2.47307E-60	
7	143	150	68	1.01749E-58	
8	142	150	67	3.63754E-57	
9	141	150	66	1.14784E-55	
10	140	150	65	3.23692E-54	

Problem 4 continued

g.) As N increases, the σ_E Value increases. Because σ_E is increasing, $\bar{E}-\sigma_E$ to $\bar{E}+\sigma_E$ becomes less precise.

h.) The probability of falling between $\bar{E}-\sigma_E$ and $\bar{E}+\sigma_E$ decreases as N increases. It looks like this probability is approaching 60.

Problem 5

a.) $q = 0,1,2,3,4,5 \; \ldots\ldots$ $\qquad E = \hbar\omega(q + N/2) = \hbar\omega(q + 5/2) = \hbar\omega(q + 2.5)$

q	E
0	$2.5\hbar\omega$
1	$3.5\hbar\omega$
2	$4.5\hbar\omega$
3	$5.5\hbar\omega$
4	$6.5\hbar\omega$
5	$7.5\hbar\omega$

$$\Omega(N,q) = \binom{N+q-1}{q} \; : \; \Omega(5,q) = \binom{4+q}{q} = \frac{(4+q)!}{q!\cdot 4!}$$

$$\Omega(5,0) = \binom{4}{0} = \frac{4!}{0!\cdot 4!} = 1 \; : \; \Omega(5,0) = 1$$

$$\Omega(5,1) = \binom{5}{1} = \frac{5!}{1!\cdot 4!} = 5 \; : \; \Omega(5,1) = 5$$

$$\Omega(5,2) = \binom{6}{2} = \frac{6!}{2!\cdot 4!} = 15 \; : \; \Omega(5,2) = 15$$

$$\Omega(5,3) = \binom{7}{3} = \frac{7!}{3!\,4!} = 35 \; : \; \Omega(5,3) = 35$$

q	E ($\hbar\omega$)	Ω
0	2.5	1
1	3.5	5
2	4.5	15
3	5.5	35

b.)

A	B	C	D	E
3	0	0	0	0
0	3	0	0	0
0	0	3	0	0
0	0	0	3	0
0	0	0	0	3

5 cases out of 35 possible microstates ∴

$$\text{Prob} = \frac{5}{35} = \frac{1}{7}$$

$$\boxed{\frac{1}{7}}$$

c.)

A	B	C	D	E
2	1	0	0	0
2	0	1	0	0
2	0	0	1	0
2	0	0	0	1

4 cases out of 35 possible microstates ∴

$$\text{Prob} = \frac{4}{35}$$

$$\boxed{\frac{4}{35}}$$

Problem 6

a.) 10 particles in A, 10 particles in B : $q_A = 9$, $q_B = 3$

$$\Omega = (N,q) = \binom{N+q-1}{q} \quad : \quad \Omega = (10,q) = \binom{9+q}{q}$$

$$\Omega_A = (10,9) = \binom{18}{9} = \frac{(18)!}{9!(18-9)!} = 48{,}620 \quad : \quad \Omega_B = (10,3) = \binom{12}{3} = \frac{(12)!}{3!(12-3)!} = 220$$

Total number of possible microstates is $\Omega = \Omega_A \Omega_B = 48{,}620 \cdot 220 = 10{,}696{,}400$

$$\boxed{\Omega = 10{,}696{,}400}$$

$$\boxed{\frac{q_A}{N_A} = \frac{9}{10}, \quad \frac{q_B}{N_B} = \frac{3}{10}}$$

b.)

The most probable state for A : $q_A = 6$
The most probable state for B : $q_B = 6$

The Energy per particle for A : $\frac{q_A}{N_A} = \frac{3}{5}$

The Energy per particle for B : $\frac{q_B}{N_B} = \frac{3}{5}$

c.) The energy flowed from high to low, in this case from A to B.

d.)

Prob (A to B) = 0.8742
Prob (B to A) = 0.1258
A to B : ≈ 7 times more likely

e.) Here $q = q_A + q_B = 20 + 10 = 30$: $q = 30$

Most probable state for A : $q_A = 25$ ∴ Most probable state for B : $q_B = 5$

$$\boxed{q_A = 25, \quad \frac{q_A}{N_A} = \frac{5}{4} \quad : \quad q_B = 5, \quad \frac{q_B}{N_B} = 1}$$

F.) In this instance, there was more energy in System A than that of System B. This means that energy was transfered from B to A. Thus energy was not transferred from high to low, it was transferred from low to high.