5.2 Properties of Linear Transformations

5.2 #4, 6, 26, 30

5.2.4

$T: \mathbb{R}^3 \to \mathbb{R}^3 \quad T(x,y,z) = (x-z, x-2y, y-z)$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$T[\vec{x}, \vec{y}, \vec{z}] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} = A$

$\text{Ker}(A) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\text{Ker}(T) = [0, 0, 0]$

5.2.6

$D^2: C^2 \to C \quad D^2(f) = f''$

$$\int \frac{D^2 f(t)}{dt^2} = \int 0$$

$$\int f'(t) = \int c_1$$

$$f(t) = c_1 t + c_2$$

$f(t) = c_1 t + c_2$

5.2.26

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Kernel: $\text{rref}(A) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

$K(T) = [0, 0]$

Image: $\text{Im}(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\vec{v}_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \vec{v}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Dimension: # of pivot columns in rref

$\text{Dim}(T) = 2$

$K(T) = [0, 0]$

$\text{Im}(T) = \vec{v}_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \vec{v}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\text{Dim}(T) = 2$

$K(T) = 0 \; \therefore$ Injective

$\text{Im}(T) = \mathbb{R}^2 \; \therefore$ Surjective

5.2.30

$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \end{bmatrix} = A$

Kernel: $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/4 \end{bmatrix}$

$x + 1/4 z = 0 \qquad x = -1/4 z$

$y + 1/4 z = 0 \qquad y = -1/4 z$

$K(T) = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

Image: Span $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\vec{v}_1 + \begin{bmatrix} 3 \\ 2 \end{bmatrix}\vec{v}_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\vec{v}_3 = \mathbb{R}^2$

Dimension: $\text{Im}(T)$: linearly independent vectors

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \# = 2 \; \therefore \text{Dim}(\text{Im}(T)) = 2$

$\text{Dim}(K(T)) = 1$

$K(T) = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$

$\text{Dim}(\text{Im}(T)) = 2$

$\text{Dim}(K(T)) = 1$

$K(T) \neq 0 \; \therefore$ not injective

$\mathbb{R}^2 \; \therefore$ surjective

5.2.12 $T: \mathbb{P}_2 \to \mathbb{P}_2$ $T(at^2+bt+c) = 2at+b$ Find the Kernel

$2at+b=0$

$a=0$

$b=0$

$P(t)=at^2+bt+c$

$P(t)=0+0+c$

$P(t)=c$

$P(t)=c$

5.2.13 $T: \mathbb{P}_2 \to \mathbb{P}_2$ $T(at^2+bt+c) = 2a$

$2a=0$

$a=0$

$P(t)= at^2+bt+c$

$P(0)=(0)t^2+bt+c$

$P(t)= bt+c$

$P(t)=bt+c$

5.2.14 $T: \mathbb{P}_2 \to \mathbb{P}_2$, $T(at^2+bt+c)=0$

$0=0$

a,b,c are arbitrary

$P(t)=at^2+bt+c$

5.2.15 $T: \mathbb{P}_3 \to \mathbb{P}_3$, $T(at^3+bt^2+ct+d)=6at+2b$

$6at+2b=0$

$a=0$

$b=0$

$P(t)=ct+d$