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MATH 366
CP Ch. 20.4

1) A computational problem is called ill-conditioned (or ill-posed) if "small" changes in the data (the input) cause "large" changes in the solution (the output). On the other hand, a problem is called well-conditioned (or well-posed) if "small" changes in the data cause only "small" changes in the solution.

These concepts are qualitative. We would certainly regard a magnification of inaccuracies by a factor 100 as "large," but could debate where to draw the line between "large" and "small," depending on the kind of problem and on our viewpoint. Double precision may sometimes help, but if data are measured inaccurately, one should attempt changing the mathematical setting of the problem to a well conditioned one.

Let us now turn to linear systems. Figure 445 explains that ill-conditioning occurs if and only if the two equations give two nearly parallel lines, so that their intersection point (the solution of the system) moves substantially if we raise or lower a line just a little. For larger systems the situation is similar in principle, although geometry no longer helps. We shall see that we may regard ill-conditioning as an approach to singularity of the matrix.

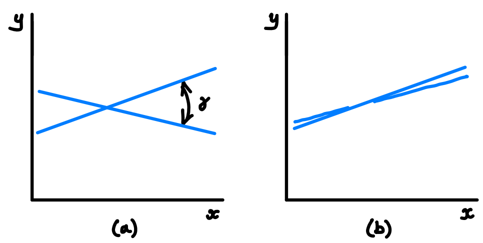

Fig. 445. (a) Well-conditioned and (b) ill-conditioned linear system of two equations in two unknowns.

Example 1 An Ill conditioned System

You may verify that the system

$$0.9999x - 1.0001y = 1$$
$$x - y = 1$$

has the solution $x = 0.5$, $y = -0.5$, whereas the system

$$0.9999x - 1.0001y = 1$$
$$x - y = 1 + \epsilon$$

has the solution $x = 0.5 + 5000.5\epsilon$, $y = -0.5 + 4999.5\epsilon$. This shows that the system is ill-conditioned because a change on the right of magnitude ϵ produces a change in the solution of magnitude 5000ϵ, approximately. We see that the lines given by the equations have nearly the same slope.

Well-conditioning can be asserted if the main diagonal entries of A have large absolute values compared to those of the other entries. Similarly if A^{-1} and A have maximum entries of about the same value.

Ill-Conditioning is indicated if A^{-1} has entries of large absolute value compared to those of the solution (about 5000 in Example 1) and if poor approximate solutions may still produce small residuals.

2)
$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

3)
$$K(A) = \|A\|\|A^{-1}\|$$

4) Theorem 1 Condition Number

A linear system of equations $Ax = b$ and its matrix A whose condition number (13) is small are well-conditioned. A large condition number indicates ill-conditioning.