Taylor Research

15 Oct 2019

Two qubit unitaries

If two qubits interact then the evolution will be described by an operator that acts on the pair of qubits.

For example consider CNOT defined via

$|0\rangle|0\rangle \to |0\rangle|0\rangle$

$|0\rangle|1\rangle \to |0\rangle|1\rangle$

$|1\rangle|0\rangle \to |1\rangle|1\rangle$

$|1\rangle|1\rangle \to |1\rangle|0\rangle$

Exercise: a) Using

$|0\rangle|0\rangle \rightsquigarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $|0\rangle|1\rangle \rightsquigarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$|1\rangle|0\rangle \to \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$\begin{pmatrix}- & - & - & -\\- & - & - & -\\- & - & - & -\\- & - & - & -\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Find the 4x4 matrix for CNOT. Show that it is unitary

b) Consider

$|\Psi_0\rangle = |0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

as input state into CNOT.

Find $|\Psi\rangle = U_{CNOT}|\Psi_0\rangle$. Is this entangled?

c) $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$

Find $|\Psi\rangle = U_{CNOT}|\Psi_0\rangle$ Is this entangled?

d) Let $|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$

Find

$|\Psi\rangle = \hat{U}_{CNOT}\,|\Psi_0\rangle$

is this entangled or not?

e) Let $|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)$

same questions ...

$\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}2&0\\0&0\end{pmatrix}$

Exercise: Show that $U_{CNOT} = \frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes\hat{I} + \frac{1}{2}(\hat{I}-\hat{\sigma}_z)\otimes\hat{\sigma}_x$

$|0\rangle|0\rangle \to \frac{1}{2}(\hat{I}+\hat{\sigma}_z)\otimes I\,|0\rangle|0\rangle + \frac{1}{2}(I-\hat{\sigma}_z)\otimes\sigma_x|0\rangle|0\rangle$

$\frac{1}{2}(I+\hat{\sigma}_z)|0\rangle \otimes \hat{I}|0\rangle$

$-\frac{1}{2}(I-\hat{\sigma}_z)|0\rangle\,\sigma_x|0\rangle$

$= \frac{1}{2}\left[\hat{I}\otimes\hat{I} + \hat{\sigma}_z\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x - \hat{\sigma}_z\otimes\hat{\sigma}_x\right]$

Exercise: One important class of unitaries has form:

$\frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_n + \hat{\sigma}_n\otimes\hat{I} - \hat{\sigma}_n\otimes\hat{\sigma}_n\right)$

where n is some direction.

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$\sigma_x|0\rangle = |1\rangle$

$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\sigma_z|0\rangle = |0\rangle$

a) Determine the matrix for

$\frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_z + \hat{\sigma}_z\otimes\hat{I} - \hat{\sigma}_z\otimes\hat{\sigma}_z\right)$

i) Determine action on

$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$

entangled or not?

b) Determine matrix for

$$\frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_x + \hat{\sigma}_x\otimes\hat{I} - \hat{\sigma}_x\otimes\hat{\sigma}_x\right)$$

and action on

$$|\Psi_0\rangle = |0\rangle|0\rangle$$

Is resulting state entangled or not?

Density operators

Suppose that one does not know state with certainty and one just knows

Either state is $|\psi_1\rangle$ with prob q_1

or " " $|\psi_2\rangle$ " " q_2

Suppose we do a measurement with

outcome	operator	state
m_0	$\hat{P}_0$	$|\varphi_0\rangle$
m_1	$\hat{P}_1$	$|\varphi_1\rangle$

Then

$$\text{Prob}(m_0) = \text{Prob state is } |\psi_1\rangle \times \text{prob } |\psi_1\rangle \text{ gives } m_0$$

$$+ \text{ Prob state is } |\psi_2\rangle \times \text{prob } |\psi_2\rangle \text{ gives } m_0$$

$$= q_1 |\langle\varphi_0|\psi_1\rangle|^2 + q_2 |\langle\varphi_0|\psi_2\rangle|^2$$

$$= q_1 \langle\varphi_0|\psi_1\rangle\langle\psi_1|\varphi_0\rangle + q_2 \langle\varphi_0|\psi_2\rangle\langle\psi_2|\varphi_0\rangle$$

$$= \langle\varphi_0| \underbrace{\left[q_1 |\psi_1\rangle\langle\psi_1| + q_2 |\psi_2\rangle\langle\psi_2| \right]}_{\text{represents system being measured.}} |\varphi_0\rangle$$

associated with meas outcome

We say that the <u>density operator</u> of the system prior to measurement is

$$\hat{\rho} = a_1 |\psi_1\rangle\langle\psi_1| + a_2 |\psi_2\rangle\langle\psi_2|$$

and then

$\hat{P}_0 \doteq |\varphi_0\rangle\langle\varphi_0|$

$$\text{Prob}(m_0) = \langle\varphi_0|\hat{\rho}|\varphi_0\rangle = \text{Tr}(\hat{P}_0\hat{\rho}) \qquad \text{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$$

$$\text{Prob}(m_1) = \langle\varphi_1|\hat{\rho}|\varphi_1\rangle$$

<u>Exercise</u> Suppose single qubit has density op:

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$$

where $0 \leq r \leq 1$. For each of the following meas. determine probs of outcomes

1) $SG\hat{z}$
2) $SG\hat{x}$

<u>Exercise</u> Density op is

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

For each of following determine probs of outcomes:

1) $SG\hat{z}$
2) $SG\hat{x}$

Constructing density operators

Exercise For the following states + probs construct density op matrix

1) $|0\rangle$ with prob q
   $|1\rangle$ " " $1-q$

2) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ prob q
   $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ " $1-q$

We can construct using Pauli ops:

1) Show that

a) $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_z)$

b) $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)$

Density operators

Satisfy:

1) $\hat{\rho}^{\dagger} = \hat{\rho}$

2) $\mathrm{Tr}[\hat{\rho}] = 1$

3) $\hat{\rho}$ is positive (eigenvals are positive)

If do measurement:

outcomes	meas operator	prob
m_0	$\hat{P}_0$	$\mathrm{Tr}[\hat{\rho}\hat{P}_0]$
m_1	$\hat{P}_1$	$\mathrm{Tr}[\hat{\rho} P_1]$
m_2	$\hat{P}_2$	
$\vdots$		

Exercise: $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r & 0 \\ 0 & 1-r \end{pmatrix}$

use

to get probabilities for

a) $SG\hat{x}$
b) SGy

__Exercise:__ same for $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$.

__General form for any qubit density operator__

$$\boxed{\hat{\rho} = \frac{1}{2}\left(\hat{I} + r_x \hat{\sigma}_x + r_y \hat{\sigma}_y + r_z \hat{\sigma}_z\right)}$$

where $0 \leq \sqrt{r_x^2 + r_y^2 + r_z^2} \leq 1$

can write $r_x \hat{\sigma}_x + r_y \hat{\sigma}_y + r_z \hat{\sigma}_z = \vec{r} \cdot \vec{\sigma}$

$$\hat{\rho} = \frac{1}{2}\left(I + \vec{r} \cdot \hat{\vec{\sigma}}\right)$$

magnitude of $\vec{r}$ is purity.

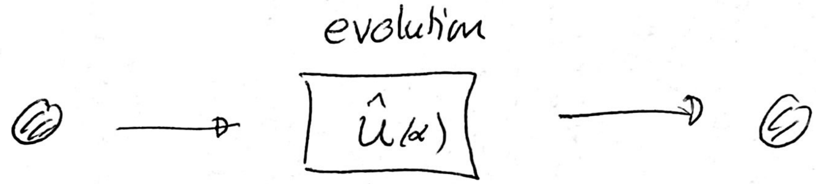
__Exercise__ a) $\hat{\rho} = \frac{1}{2}\left(\hat{I} + \hat{\sigma}_z\right)$

can find $|\psi\rangle$ s.t. $\hat{\rho} = |\psi\rangle\langle\psi|$

b) $\hat{\rho} = \frac{1}{2}\left(\hat{I} + \hat{\sigma}_x\right)$

can find $|\psi\rangle$ s.t. $\hat{\rho} = |\psi\rangle\langle\psi|$.

How does density op. evolve?



known state

$\hat{\rho}_0$

state after

$\hat{\rho} = ?$

Rule

$$\boxed{\hat{\rho}(\alpha) = \hat{U}(\alpha)\,\hat{\rho}_0\,\hat{U}^{+}(\alpha)}$$

Exercise:

$$\hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{2}\left(\hat{I} + \hat{\sigma}_x\right)$$

$$\hat{U} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = \cos\alpha/2\,\hat{I} - i\sin\alpha/2\,\hat{\sigma}_z$$

Find $\hat{\rho}(\alpha)$

Two particles

A ∈ $\hat\rho_A$

B ∈ $\hat\rho_B$ $\Big\}$ state both $\hat\rho_A \otimes \hat\rho_B$

product state.

$$\hat\rho = \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} \stackrel{?}{=} \hat\rho_A \otimes \hat\rho_B$$

$$= p_1\, \hat\rho_{A1} \otimes \hat\rho_{B1} + p_2\, \hat\rho_{A2} \otimes \hat\rho_{B2} + \dots$$

Estimation:

$\hat\rho = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$ A ∈ — $\boxed{\hat U(\alpha)}$ — ??

$\hat\rho = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$ B ∈ — $\boxed{\hat U(\alpha)}$ —

Exercise

For just one particle

$\hat\rho = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$ A ∈ — $\boxed{U(\alpha)}$ — meas SG $\hat x$ → Fisher info

→ $\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$