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PHYS 362 HW 18

Problem 1 $P_{speed}(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-v^2 m/2kT} = \alpha v^2 e^{-v^2\gamma}$

a.) $\frac{dP_{speed}}{dv} = 2\alpha v e^{-v^2\gamma} - 2\alpha\gamma v^3 e^{-v^2\gamma} = 0$

$\cancel{2\alpha} v \cancel{e^{-v^2\gamma}} = \cancel{2\alpha}\gamma v^3 \cancel{e^{-v^2\gamma}} \quad \therefore \quad 1 = \gamma v^2 \rightarrow v^2 = \frac{1}{\gamma}$

$v^2 = \frac{2kT}{m} \quad \therefore \quad v = \pm\sqrt{\frac{2kT}{m}}$

$$\boxed{v = \pm\sqrt{\frac{2kT}{m}}}$$

b.) $\bar{v} = \sum v_i P(v_i) = \int_0^\infty v P(v)\, dv = \alpha\int_0^\infty v^3 e^{-v^2\gamma} dv$

$\bar{v} = \alpha\int_0^\infty v^3 e^{-v^2\gamma} dv \quad : \quad \int_0^\infty x^n e^{-x^2 a} = \frac{k!}{2(a^{k+1})} \quad : \quad n = 2k+1 \quad \therefore \quad k = \frac{n-1}{2} \rightarrow k = \frac{3-1}{2} = 1$

$\bar{v} = \alpha\cdot\frac{1!}{2\gamma^2} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\cdot\frac{1}{2}\left(\frac{4k^2T^2}{m^2}\right) = 4\pi\left(\frac{m^3}{8\pi^3k^3T^3}\right)^{1/2}\cdot\frac{1}{2}\left(\frac{16k^4T^4}{m^4}\right)^{1/2}$

$= 2\pi\sqrt{\frac{m^3}{8\pi^3k^3T^3}\cdot\frac{16k^4T^4}{m^4}} = 2\pi\sqrt{\frac{2kT}{m\pi^3}} = \sqrt{4\pi^2}\sqrt{\frac{2kT}{m\pi^3}} = \sqrt{\frac{8\cancel{\pi^2}kT}{m\cancel{\pi^3}}} = \sqrt{\frac{8kT}{\pi m}}$

$\bar{v} = \sqrt{\frac{8kT}{\pi m}}, \quad V_{mp} = \pm\sqrt{\frac{2kT}{m}} \quad : \quad \frac{\bar{v}}{V_{mp}} = \sqrt{\frac{8kT}{\pi m}} \Big/ \sqrt{\frac{2kT}{m}} = \sqrt{\frac{\frac{8kT}{\pi m}}{\frac{2kT}{m}}} = \sqrt{\frac{8\cancel{kTm}}{2\cancel{kTm}\pi}}$

$\frac{\bar{v}}{V_{mp}} = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$

$$\boxed{\bar{v} = \sqrt{\frac{8kT}{\pi m}},\ \bar{v} \text{ differs } \frac{2}{\sqrt{\pi}} \text{ from the most probable speed}}$$

c.) As T increases, the most likely and mean speed both increase

d.) Average kinetic energy is $\bar{v^2}$

$\bar{v^2} = \int_0^\infty v^2 p(v)\, dv = \int_0^\infty v^2\cdot\alpha\cdot v^2 e^{-v^2\gamma} = \alpha\int_0^\infty v^4 e^{-v^2\gamma} dv \quad : \quad \int_0^\infty x^n e^{-x^2 a} : n = 2k+1 \quad \therefore \quad k = \frac{n-1}{2}$
$k = 3/2$

$\bar{v^2} = \alpha\frac{(3/2)!}{2\gamma^{5/2}} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{(3/2)!}{2}\left(\frac{2kT}{m}\right)^{5/2} = 2\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{3}{2}\right)!\left(\frac{2kT}{m}\right)^{5/2}$

$\bar{v^2} = 2\pi\left(\frac{3}{2}\right)!\sqrt{\left(\frac{m}{2\pi kT}\right)^3\cdot\left(\frac{2kT}{m}\right)^5} = 2\pi\left(\frac{3}{2}\right)!\sqrt{\frac{(2kT)^2}{\pi^3m^2}} \quad : \quad \left(\frac{3}{2}\right)! = \frac{3}{4}\sqrt{\pi}$

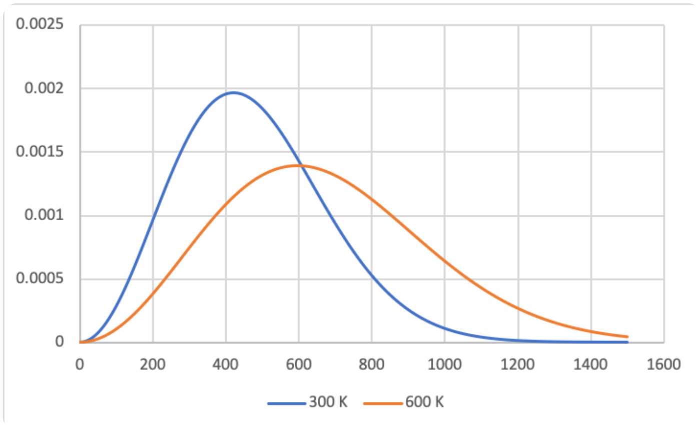
Problem 1 Continued

$$\bar{v}^2 = 2\pi \cdot \frac{3}{4}\sqrt{\pi} \cdot \sqrt{\frac{4k^2T^2}{\pi^3 m^2}} = \frac{3}{\cancel{2}} \cdot \cancel{\pi^{3/2}} \cdot \frac{\cancel{2}kT}{m} \cdot \frac{1}{\cancel{\pi^{3/2}}} = \frac{3kT}{m} \quad : \quad \bar{v}^2 = \frac{3kT}{m}$$

$$E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{3kT}{m}\right) = \frac{3}{2}kT$$

$$\boxed{\bar{v}^2 = \frac{3kT}{m}, \quad E = \frac{3}{2}kT}$$

e.)



There is a higher peak for the 300K line compared to that of the 600K line. As the speeds increase, the 300K line dives towards zero faster than the 600K line.

Problem 2

a.) $V_{mp} = \sqrt{\frac{2kT}{m}}$

at peak for gas A : $V_{mp} = 400$ m/s , at peak for gas B : $V_{mp} = 800$ m/s

$V_{mpA} = \sqrt{\frac{2kT_A}{m}}$ $\therefore$ $T_A = \frac{V_{mpA}^2 \cdot m}{2k}$: $V_{mpB} = \sqrt{\frac{2kT_B}{m}}$ $\therefore$ $T_B = \frac{V_{mpB}^2 \cdot m}{2k}$

$T_A = 160{,}000\ (m/s)^2 \cdot \frac{m}{2k}$ $\qquad T_B = 640{,}000\ (m/s)^2$ $\therefore$

$$\frac{T_B}{T_A} = \frac{640{,}000\ (m/s)^2}{160{,}000\ (m/s)^2} \frac{\cancel{m}}{\cancel{2k}} \cdot \frac{\cancel{2k}}{\cancel{m}} = \frac{4}{1}$$

$\boxed{\text{Gas B's temp is 4 times greater than that of Gas A's}}$

b.) $V_{mp} = \sqrt{\frac{2kT}{m}}$

at peak for gas A : $V_{mp} = 600$ m/s , at peak for gas B : $V_{mp} = 800$ m/s

$V_{mpA} = \sqrt{\frac{2kT}{m_A}}$ $\therefore$ $m_A = \frac{2kT}{V_{mpA}^2}$: $V_{mpB} = \sqrt{\frac{2kT}{m_B}}$ $\therefore$ $m_B = \frac{2kT}{V_{mpB}^2}$

$m_A = \frac{2kT}{360{,}000\ (m/s)^2}$, $m_B = \frac{2kT}{640{,}000\ (m/s)^2}$

$$\frac{m_B}{m_A} = \frac{\cancel{2kT}}{640{,}000\ \cancel{(m/s)^2}} \cdot \frac{360{,}000\ \cancel{(m/s)^2}}{\cancel{2kT}} = \frac{36}{64} = \frac{18}{32} = \frac{9}{16}$$

$\boxed{m_B \text{ is } 9/16 \text{ that of } m_A}$

Problem 3

a.) $P_{speed}(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-v^2 m/2kT}$: $P(v) = \int_0^\infty 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-v^2 m/2kT} dv$

$u = \frac{v}{\sqrt{2kT/m}}$: $v = u\sqrt{2kT/m}$: $\frac{du}{dv} = \frac{1}{\sqrt{2kT/m}} \longrightarrow dv = \sqrt{2kT/m}\, du$

$$P(v) = 4 \int_0^\infty \left(\frac{m^3 \cancel{\pi^2}}{8\pi^3 k^3 T^3}\right)^{1/2} u^2 \cdot \frac{2kT}{m}\, e^{-u^2(2k\cancel{T}/m)\cdot m/\cancel{2}kT} dv$$

$$P(v) = \frac{4}{\sqrt{\pi}} \int_0^\infty \left(\frac{m^3}{8k^3T^3}\right)^{1/2} \left(\frac{4k^2T^2}{m^2}\right)^{1/2} u^2 e^{-u^2} dv = \frac{4}{\sqrt{\pi}} \int_0^\infty \left(\frac{\cancel{m^3}}{2\,8k^{\cancel{3}}T^{\cancel{3}}} \cdot \frac{\cancel{4}k^{\cancel{2}}\cancel{T^2}}{\cancel{m^2}}\right)^{1/2} u^2 e^{-u^2} du$$

$$P(v) = \frac{4}{\sqrt{\pi}} \int_0^\infty \left(\frac{m}{2kT}\right)^{1/2} u^2 e^{-u^2} du = \frac{4}{\sqrt{\pi}} \int_{v_{min}/\sqrt{2kT/m}}^\infty \cancel{\left(\frac{m}{2kT}\right)^{1/2}} \cdot \cancel{\left(\frac{2kT}{m}\right)^{1/2}} u^2 e^{-u^2} du$$

$$\boxed{P(v) = \frac{4}{\sqrt{\pi}} \int_{v_{min}/\sqrt{2kT/m}}^\infty u^2 e^{-u^2} du}$$

b.) $m_{He} = 6.646\times10^{-27}$ kg , $V_{min} = 11{,}000$ m/s , $T = 300$ K , $k = 1.38\times10^{-23}$ J/K

$$\frac{V_{min}}{\sqrt{2kT/m}} = \frac{11{,}000 \text{ m/s}}{\sqrt{2\cdot 1.38\times10^{-23} \text{J/K}(300\text{ K})/6.646\times10^{-27}\text{kg}}} \approx 9.85 \longrightarrow \alpha$$

$$P(v) = \frac{4}{\sqrt{\pi}} \int_\alpha^\infty u^2 e^{-u^2} du = 7.396\times10^{-42} \longrightarrow \text{TI Nspire}$$

$$\boxed{P = 7.396\times10^{-42}}$$

c.) $m_{N_2} = 4.65\times10^{-26}$ kg , $V_{min} = 11{,}000$ m/s , $T = 300$ K , $k = 1.38\times10^{-23}$ J/K

$$\frac{V_{min}}{\sqrt{2kT/m}} = \frac{11{,}000 \text{ m/s}}{\sqrt{2\cdot 1.38\times10^{-23} \text{J/K}(300\text{ K})/4.65\times10^{-26}\text{kg}}} \approx 26.07 \longrightarrow \alpha$$

$$P(v) = \frac{4}{\sqrt{\pi}} \int_\alpha^\infty u^2 e^{-u^2} du = 2.256\times10^{-294} \longrightarrow \text{TI Nspire}$$

$$\frac{P_{He}}{P_{N_2}} = \frac{7.396\times10^{-42}}{2.256\times10^{-294}} = 3.278\times10^{252}$$

$$\boxed{P = 2.256\times10^{-294}}$$

A Helium molecule is 3.278×10^{252} more likely to escape Earth's atmosphere than Nitrogen. This tells us that there should be more Nitrogen than Helium in our atmosphere.

Problem 4

a.) $E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 r^2$, $Z_{single} = \sum_i \frac{e^{-E_i\beta}}{Z} = \alpha \int dx \int dy \int dz \int dp_x \int dp_y \int dp_z \; e^{-(p_x^2+p_y^2+p_z^2)\beta/2m}$

$$Z_{single} = \alpha \int_{-\infty}^{\infty} e^{-m\omega^2 x^2\beta/2} dx \int_{-\infty}^{\infty} e^{-P_x^2\beta/2m} dP_x \int_{-\infty}^{\infty} e^{-m\omega^2 y^2\beta/2} dy \int_{-\infty}^{\infty} e^{-P_y^2\beta/2m} dP_y \int_{-\infty}^{\infty} e^{-m\omega^2 z^2\beta/2} dz \int_{-\infty}^{\infty} e^{-P_z^2\beta/2m} dP_z$$

using $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$

$$Z_{single} = \alpha \sqrt{\frac{2\pi}{m\omega^2\beta}} \sqrt{\frac{2\pi}{m\omega^2\beta}} \sqrt{\frac{2\pi}{m\omega^2\beta}} \sqrt{\frac{2m\pi}{\beta}} \sqrt{\frac{2m\pi}{\beta}} \sqrt{\frac{2m\pi}{\beta}} = \left(\frac{2\pi}{m\omega^2\beta}\right)^{3/2} \left(\frac{2m\pi}{\beta}\right)^{3/2}$$

$$Z_{single} = \alpha \left(\frac{2\pi}{\cancel{m}\omega^2\beta} \cdot \frac{2\cancel{m}\pi}{\beta}\right)^{3/2} = \alpha \left(\frac{4\pi^2}{\omega^2\beta}\right)^{3/2} = \alpha \left(\frac{2\pi}{\omega\beta}\right)^3 \quad : \quad Z_{single} = \alpha \left(\frac{2\pi}{\omega\beta}\right)^3$$

$Z_{ensemble} = (Z_{single})^N \quad \therefore \quad \boxed{Z_{Ensemble} = \left(\alpha^N \left(\frac{2\pi}{\omega\beta}\right)^{3N}\right)}$

b.) $\bar{E} = -\frac{\partial}{\partial\beta} \ln(Z) = -\frac{\partial}{\partial\beta} \ln\left(\left(\alpha \left(\frac{2\pi}{\omega\beta}\right)^3\right)^N\right)$

$$\bar{E} = -\frac{\partial}{\partial\beta} \cdot N \ln\left(\alpha \left(\frac{2\pi}{\omega\beta}\right)^3\right) = N \cdot \frac{\partial}{\partial\beta} \ln\left(\frac{1}{\alpha}\left(\frac{\omega\beta}{2\pi}\right)^3\right) = N \cdot \frac{\partial}{\partial\beta}\left(\ln\left(\frac{1}{\alpha}\right) + \ln\left(\frac{\omega\beta}{2\pi}\right)^3\right)$$

$$\bar{E} = N \cdot \frac{\partial}{\partial\beta}\left(\overset{0}{\cancel{\ln(1)}} - \ln(\alpha) + 3\ln\left(\frac{\omega\beta}{2\pi}\right)\right) = N\frac{\partial}{\partial\beta}\left(3\ln\left(\frac{\omega\beta}{2\pi}\right) - \ln(\alpha)\right)$$

$$\bar{E} = N \cdot 3 \cdot \frac{\cancel{\omega}}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi}}{\cancel{\omega}\beta} = 3N \cdot \frac{1}{\beta} = 3N \cdot \frac{1}{\frac{1}{kT}} = 3NkT$$

$$\boxed{\bar{E} = 3NkT}$$

$C = \frac{\partial \bar{E}}{\partial T} = 3Nk$

$$\boxed{C = 3Nk}$$

c.) Dulong and Petit $C = 3 \cdot 6.02\times10^{23} \cdot 1.38\times10^{-23} \frac{J}{mol\,k} = 24.92 \frac{J}{mol\,k}$: $C_{DP} = 24.92 \frac{J}{mol\,k}$

$C_{Cu} = 385.1 \frac{J}{kg\,k} \cdot \frac{0.06355\,kg}{mol} = 24.47 \frac{J}{mol\,k}$: $PRE = \frac{|24.92 - 24.47|}{|24.92|} = 1.8\%$

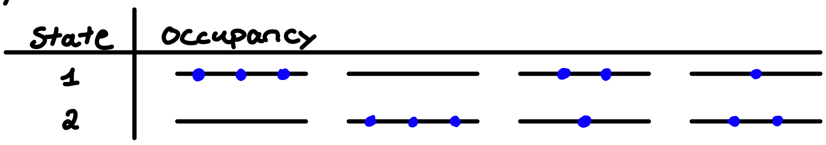
$C_{Al} = 897.1 \frac{J}{kg\,k} \cdot \frac{0.02698\,kg}{mol} = 24.20 \frac{J}{mol\,k}$: $PRE = \frac{|24.92 - 24.20|}{|24.92|} = 2.9\%$

$C_{Fe} = 449.3 \frac{J}{kg\,k} \cdot \frac{0.05585\,kg}{mol} = 25.09 \frac{J}{mol\,k}$: $PRE = \frac{|24.92 - 25.09|}{|24.92|} = 0.68\%$

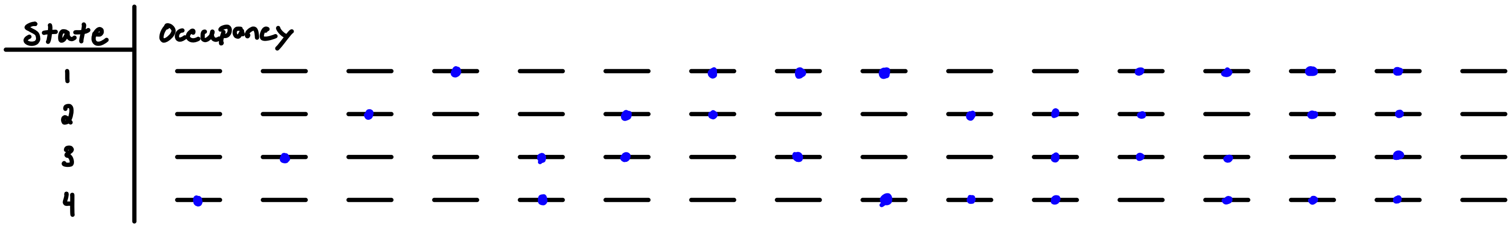
As we can see, the Dulong Petit law does reasonbly well for predicting heat capacity.

Problem 5

a.)



b.)



Problem 6

a.) $\bar{n}_k = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln[Z_{Gk}]$, $Z_{Gk} = \sum e^{-n_k(\epsilon_k-\mu)\beta}$, $\mu=0$

$$Z_{Gk} = \sum_i e^{-n_k(\epsilon_k-\mu)\beta} = \frac{1}{1-e^{-(\epsilon_k-\mu)\beta}}$$

$$\therefore \bar{n}_k = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\left[\frac{1}{1-e^{-(\epsilon_k-\mu)\beta}}\right] = -\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\left(1-e^{-(\epsilon_k-\mu)\beta}\right)$$

$$\bar{n}_k = \frac{-1}{\cancel{\beta}}\left(\frac{-\cancel{\beta} e^{-(\epsilon_k-\mu)\beta}}{1-e^{-(\epsilon_k-\mu)\beta}}\right) = \frac{e^{-(\epsilon_k-\mu)\beta}}{1-e^{-(\epsilon_k-\mu)\beta}} : \mu=0$$

$$\bar{n}_k = \frac{e^{-\epsilon_k\beta}}{1-e^{-\epsilon_k\beta}}\cdot\frac{e^{\epsilon_k\beta}}{e^{\epsilon_k\beta}} = \frac{1}{e^{\epsilon_k\beta}-1} : \bar{n}_k = \frac{1}{e^{\epsilon_k\beta}-1} = \frac{1}{e^{\epsilon_k/kT}-1}$$

$$\lim_{T\to 0}(\bar{n}_k) = \frac{1}{\infty-1} = 0$$

$$\boxed{T\to 0,\ \bar{n}_k=0}$$

$$\bar{E} = \sum \epsilon_k \bar{n}_k = \frac{\epsilon_k}{e^{\epsilon_k/kT}-1} : \lim_{T\to 0}(\bar{E}) = \frac{\epsilon_k}{\infty-1} = 0$$

$$\boxed{T\to 0,\ \bar{E}=0}$$

$$C = \frac{\partial\bar{E}}{\partial T} = \frac{\partial}{\partial T}\left(\epsilon_k\left(e^{\epsilon_k/kT}-1\right)^{-1}\right) = \frac{\epsilon^2}{4k\sinh(\epsilon/2kT)^2 t^2}$$

$$\lim_{T\to 0} C = \frac{\epsilon^2}{4k\sinh(\infty)\cdot 0} = \infty$$

$$\boxed{T\to 0,\ C=\infty}$$

b.) $\bar{n}_k = \frac{1}{e^{\epsilon_k\beta}-1} = \frac{1}{e^{\epsilon_k/kT}-1}$

$$\lim_{T\to\infty}(\bar{n}_k) = \frac{1}{1-1} = \frac{1}{0} = \infty$$

$$\boxed{T\to\infty,\ \bar{n}_k=\infty}$$

$$\bar{E} = \frac{\epsilon_k}{e^{\epsilon_k/kT}-1}$$

$$\lim_{T\to\infty}\bar{E} = \frac{\epsilon_k}{1-1} = \frac{\epsilon_k}{0} = \infty$$

$$\boxed{T\to\infty,\ \bar{E}=\infty}$$

$$C = \frac{\epsilon^2}{4k\sinh(\epsilon/2kT)^2 t^2}$$

$$\lim_{T\to\infty} C = \frac{\epsilon^2}{4k\sinh(0)\cdot\infty} = 0$$

$$\boxed{T\to\infty,\ C=0}$$

Problem 6 continued

c.) $\bar{E} = \epsilon$

$$\bar{E} = \frac{\epsilon_R}{e^{\epsilon_R/kT} - 1} \rightarrow \cancel{\bar{E}} = \frac{\cancel{\bar{E}}}{e^{\bar{E}/kT} - 1} \rightarrow 1 = \frac{1}{e^{\bar{E}/kT} - 1} \rightarrow e^{\bar{E}/kT} - 1 = 1$$

$$e^{\bar{E}/kT} = 2 \qquad \ln(2) = \frac{\bar{E}}{kT} \qquad \therefore \; T = \frac{\bar{E}}{k \ln(2)}$$

$$\boxed{T = \frac{\bar{E}}{k \ln(2)}}$$

Problem 7

a.) $\bar{n}_k = \frac{1}{\beta}\frac{\partial}{\partial \mu}\ln[Z_{Gk}]$, $Z_{Gk} = \sum e^{-n_k(\epsilon_k-\mu)\beta}$, $Z_{Gk} = \sum_0^1 e^{-n_k(\epsilon_k-\mu)\beta} = 1+e^{-(\epsilon-\mu)\beta}$

$Z_G = 1+e^{-(\epsilon-\mu)\beta}$: $\ln(Z_G) = \ln(1+e^{-(\epsilon-\mu)\beta})$

$$\frac{\partial}{\partial\mu}\ln(Z_G) = \frac{\beta e^{-(\epsilon-\mu)\beta}}{1+e^{-(\epsilon-\mu)\beta}} \quad \therefore \bar{n} = \frac{1}{\cancel{\beta}} \cdot \frac{\cancel{\beta} e^{-(\epsilon-\mu)\beta}}{1+e^{-(\epsilon-\mu)\beta}} \quad : \bar{n} = \frac{e^{-(\epsilon-\mu)\beta}}{1+e^{-(\epsilon-\mu)\beta}}$$

$$\bar{n} = \frac{e^{-(\epsilon-\mu)\beta}}{1+e^{-(\epsilon-\mu)\beta}} \cdot \frac{e^{(\epsilon-\mu)\beta}}{e^{(\epsilon-\mu)\beta}} = \frac{1}{e^{(\epsilon-\mu)\beta}+1} \qquad \lim_{T\to\infty} \bar{n} = \frac{1}{e^{(\epsilon-\mu)/kT}+1} = \frac{1}{e^0+1} = \frac{1}{2}$$

$$\boxed{\text{As } T\to\infty,\ \bar{n} = 1/2}$$

As $T\to\infty$, $\bar{n}$ does not depend upon μ. This tells us that the likelyhood of a Fermion occupying a state is about 50%.

b.)

if $\epsilon > \mu$, $T\to 0$ $\bar{n} = \frac{1}{e^{\infty}+1} = \frac{1}{\infty} = 0$ $\boxed{\bar{n}=0 \text{ as } T\to 0 \text{ for } \epsilon>\mu}$

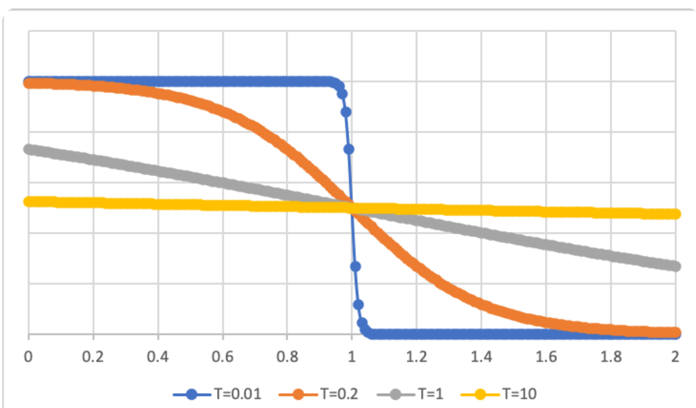
if $\epsilon < \mu$, $T\to 0$ $\bar{n} = \frac{1}{e^{-\infty}+1} = 1$ $\boxed{\bar{n}=1 \text{ as } T\to 0 \text{ for } \epsilon<\mu}$

This implies the likelyhood of Fermion occupying a state to be 0 if $\epsilon > \mu$ and 1 if $\epsilon < \mu$ for small temperatures.

c.) If $\epsilon = \mu$

$\bar{n} = \frac{1}{e^0+1} = \frac{1}{1+1} = \frac{1}{2}$ $\boxed{\bar{n} = \frac{1}{2}}$

d.)



e.) At very low temperatures the states will be either very populated or not populated. If $\epsilon' > 1 \to \bar{n} = 0$, $\epsilon' < 1$ $\bar{n} = 1$. And at very high temperatures $\bar{n} = 1/2$ regardless of ϵ' value.