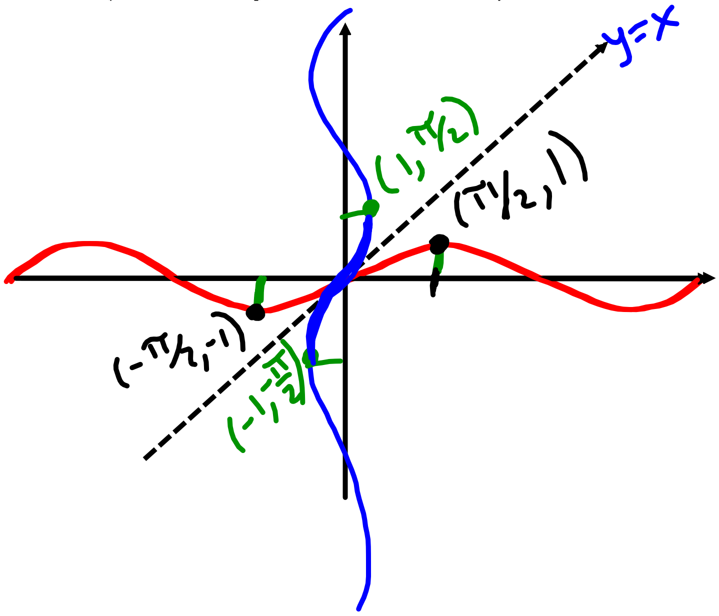
Recall: The inverse of f is the reflection of f in the line $y = x$.
Now consider the function $y = \sin x$. What would the inverse,

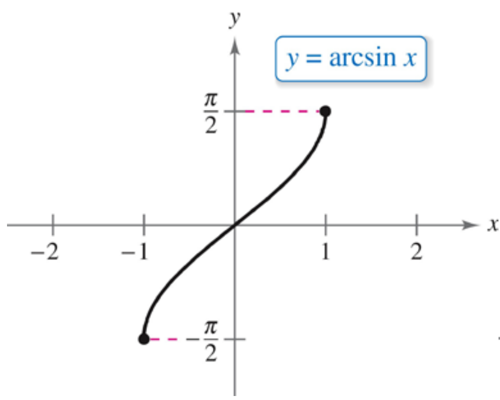
$y = \sin^{-1} x$ (a.k.a. $y = \arcsin x$) look like?

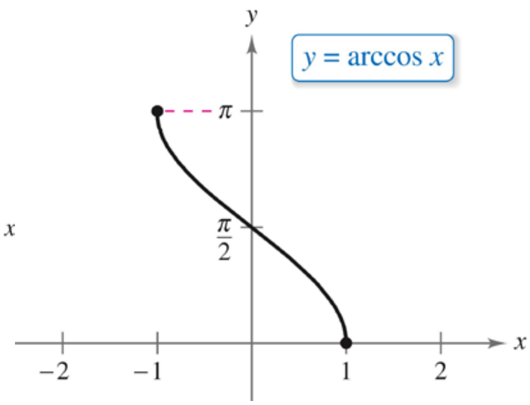


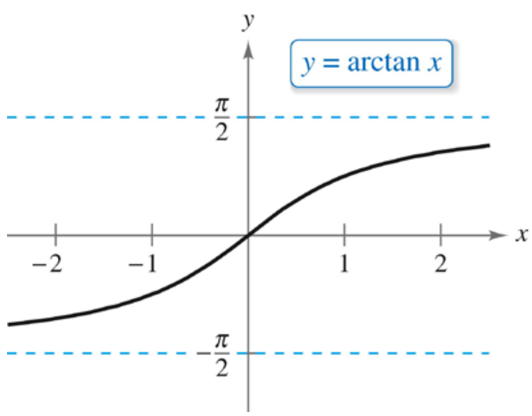
$y = \sin^{-1} x$

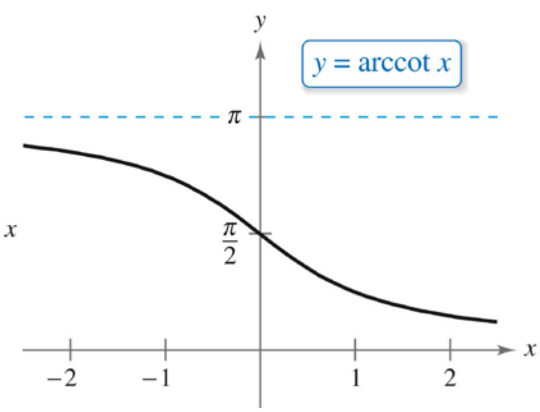
D: $[-1, 1]$

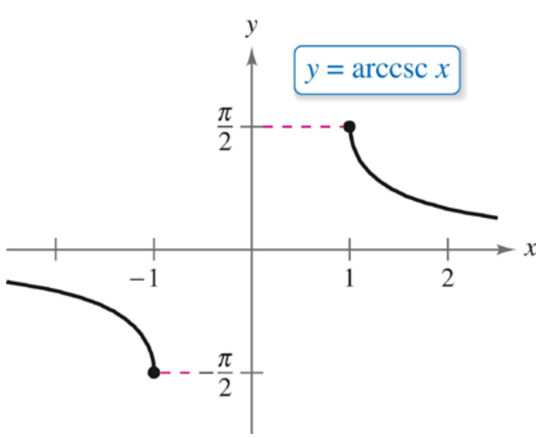
R: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

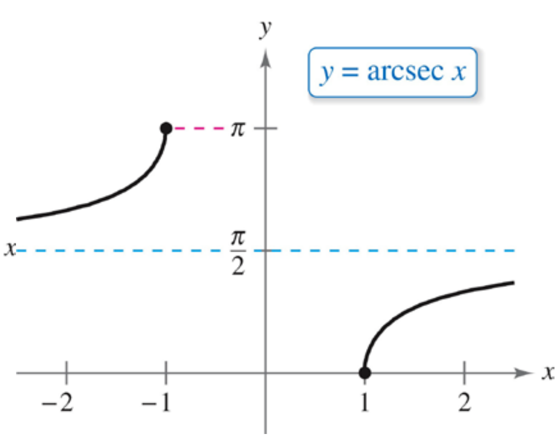

y
y = arcsin x
π/2
−π/2
x
−2
−1
1
2


y
y = arccos x
π
π/2
x
−2
−1
1
2


y
y = arctan x
π/2
−π/2
x
−2
−1
1
2


y
y = arccot x
π
π/2
x
−2
−1
1
2


y
y = arccsc x
π/2
−π/2
x
−1
1
2


y
y = arcsec x
π
π/2
x
−2
−1
1
2

<u>Definitions of Inverse Trigonometric Functions</u>:

1. $y = \sin^{-1} x \;\; or \;\; y = \arcsin x$ if and only if $\sin y = x$.
   a. Domain: $-1 \le x \le 1$
   b. Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

2. $y = \cos^{-1} x \;\; or \;\; y = \arccos x$ if and only if $\cos y = x$.
   a. Domain: $-1 \le x \le 1$
   b. Range: $0 \le y \le \pi$

3. $y = \tan^{-1} x \;\; or \;\; y = \arctan x$ if and only if $\tan y = x$.
   a. Domain: All real numbers or $-\infty < x < \infty$
   b. Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$

4. $y = \cot^{-1} x \;\; or \;\; y = arc\cot x$ if and only if $\cot y = x$.
   a. Domain: All real numbers or $-\infty < x < \infty$
   b. Range: $0 < y < \pi$

5. $y = \csc^{-1} x \;\; or \;\; y = arc\csc x$ if and only if $\csc y = x$.
   a. Domain: $x \le -1 \;\; or \;\; x \ge 1$
   b. Range: $-\frac{\pi}{2} \le y < 0 \;\; or \;\; 0 < y \le \frac{\pi}{2}$

6. $y = \sec^{-1} x \;\; or \;\; y = arc\sec x$ if and only if $\sec y = x$.
   a. Domain: $x \le -1 \;\; or \;\; x \ge 1$
   b. Range: $0 \le y < \frac{\pi}{2} \;\; or \;\; \frac{\pi}{2} < y \le \pi$

Inverse Properties:

1. If $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, then
   a. $\sin(\arcsin x) = x$
   b. $\arcsin(\sin y) = y$

2. If $-1 \le x \le 1$ and $0 \le y \le \pi$, then
   a. $\cos(\arccos x) = x$
   b. $\arccos(\cos y) = y$

3. If $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$, then
   a. $\tan(\arctan x) = x$
   b. $\arctan(\tan y) = y$

4. If $-\infty < x < \infty$ and $0 < y < \pi$, then
   a. $\cot(arc\cot x) = x$
   b. $arc\cot(\cot y) = y$

5. If $x \le -1 \quad or \quad x \ge 1$ and $-\frac{\pi}{2} \le y < 0 \quad or \quad 0 < y \le \frac{\pi}{2}$, then
   a. $\csc(arc\csc x) = x$
   b. $arc\csc(\csc y) = y$

6. If $x \le -1 \quad or \quad x \ge 1$ and $0 \le y < \frac{\pi}{2} \quad or \quad \frac{\pi}{2} < y \le \pi$, then
   a. $\sec(arc\sec x) = x$
   b. $arc\sec(\sec y) = y$

Ex: Find the exact value without a calculator: $arc\cot\left(-\sqrt{3}\right)$

Ex: Write the expression in algebraic form: $\sec(\arctan(4x))$

We have seen in an earlier section that the derivative of $f(x) = \ln x$ is $f'(x) = \dfrac{1}{x}$, that is, the derivative of a transcendental function turns out to be an algebraic function. You will now see that the derivatives of the inverse trigonometric functions also are algebraic functions.

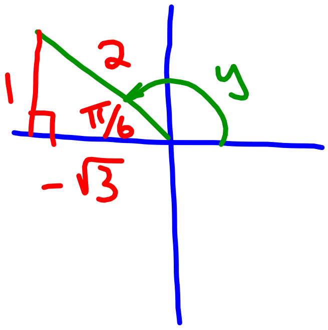
Ex: Find the exact value without a calculator:

$$arc\cot\left(-\sqrt{3}\right) = y = \frac{5\pi}{6}$$

D: $(-\infty, \infty)$

R: $(0, \pi)$

$\cot y = -\sqrt{3}$

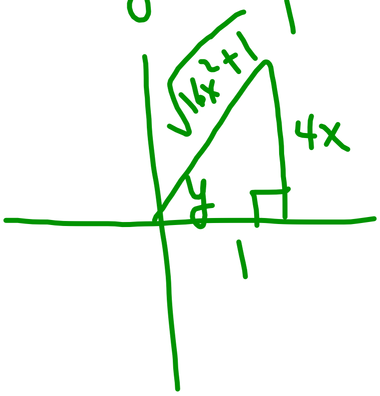


Ex: Write the expression in algebraic form:

$$\sec\left(\underbrace{\arctan(4x)}_{y}\right) = \sec(y) = \sqrt{16x^2+1}$$

$$\arctan(4x) = y$$

$$\tan y = \frac{4x}{1}$$



<u>Derivatives of Inverse Trigonometric Functions</u>: Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1-u^2}}$

2. $\dfrac{d}{dx}[\arccos u] = -\dfrac{u'}{\sqrt{1-u^2}}$

3. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1+u^2}$

4. $\dfrac{d}{dx}[arc\cot u] = -\dfrac{u'}{1+u^2}$

5. $\dfrac{d}{dx}[arc\sec u] = \dfrac{u'}{|u|\sqrt{u^2-1}}$

6. $\dfrac{d}{dx}[arc\csc u] = -\dfrac{u'}{|u|\sqrt{u^2-1}}$

Notice that the derivatives of arcos u, arccot u, and arccsc u are the negatives of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

For a list of all of the basic differentiation rules, go to page 371.

Ex: Find the derivative of the function: $y = 2x\arccos x - 2\sqrt{1-x^2}$

Ex: Find an equation of the tangent line to the graph of $y = \arctan\dfrac{x}{2}$ at the point $\left(2, \dfrac{\pi}{4}\right)$.

Derivatives of Inverse Trigonometric Functions: Let u be a differentiable function of x.

1. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}[\arccos u] = -\frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

3. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
4. $\frac{d}{dx}[\operatorname{arc}\cot u] = -\frac{u'}{1+u^2}$
5. $\frac{d}{dx}[\operatorname{arc}\sec u] = \frac{u'}{|u|\sqrt{u^2-1}}$
6. $\frac{d}{dx}[\operatorname{arc}\csc u] = -\frac{u'}{|u|\sqrt{u^2-1}}$

$$g'(x) = \frac{1}{f'(g(x))}$$

derivative of inverse function

derivative of the function evaluated at its inverse

$f(x) = \sin x \qquad f'(x) = \cos x$

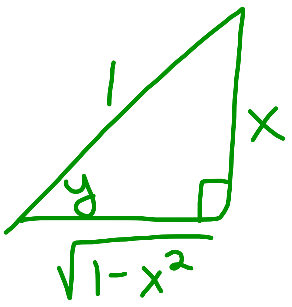
$g(x) = \sin^{-1} x$

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\cos\left[\sin^{-1} x\right]} = \frac{1}{\sqrt{1-x^2}}$$

$$\cos(\underbrace{\sin^{-1} x}_{y}) = \cos y = \sqrt{1-x^2}$$

$y = \sin^{-1} x$

$\sin y = \frac{x}{1}$



Ex: Find the derivative of the function:

$$y = 2x\arccos x - 2\sqrt{1-x^2}$$

$$y = 2x \cdot \arccos x - 2(1-x^2)^{1/2}$$

$$y' = 2\arccos x + 2x \cdot -\frac{1}{\sqrt{1-x^2}} - (1-x^2)^{-1/2}(-2x)$$

$$y' = 2\arccos x - \cancel{\frac{2x}{\sqrt{1-x^2}}} + \cancel{\frac{2x}{\sqrt{1-x^2}}}$$

$$\boxed{y' = 2\arccos x}$$

Ex: Find an equation of the tangent line to the graph of

$y = \arctan \frac{x}{2}$ at the point $\left(2, \frac{\pi}{4}\right)$.

$$m = y' = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)}$$

@ $x=2$ $\quad m = \frac{1}{4}$

$$\boxed{y - \frac{\pi}{4} = \frac{1}{4}(x-2)}$$

Ex: Find any relative extrema of the function $h(x) = \arcsin x - 2\arctan x$.

Ex: Use implicit differentiation to find an equation of the tangent line to the graph of the equation $\arcsin x + \arcsin y = \dfrac{\pi}{2}$ at the point $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

Ex: Find any relative extrema of the function

$$h(x) = \arcsin x - 2\arctan x$$

$$h'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2}{1+x^2}$$

$$0 = \frac{1}{\sqrt{1-x^2}} - \frac{2}{1+x^2}$$

$$\frac{2}{1+x^2} = \frac{1}{\sqrt{1-x^2}}$$

$$\left(2\sqrt{1-x^2}\right)^2 = \left(1+x^2\right)^2$$

$$4(1-x^2) = 1+2x^2+x^4$$

$$4-4x^2 = 1+2x^2+x^4$$

$$0 = x^4+6x^2-3$$

$$x^2 = \frac{-6 \pm \sqrt{6^2-4(1)(-3)}}{2(1)}$$

$$\sqrt{x^2} = \pm\sqrt{\frac{-6+\sqrt{48}}{2}} \quad \text{or} \quad \cancel{\sqrt{x^2} = \pm\sqrt{\frac{-6-\sqrt{48}}{2}}}$$

$$x \approx \pm 0.681$$

$(-1, -0.681)$ $(-0.681, 0.681)$ $(0.681, 1)$

$x=-.7 \to >0$ $x=0 \to <0$ $x=0.7 \to >0$

$\nearrow$ $\searrow$ $\nearrow$

$$\boxed{\begin{array}{l}\text{Relative Max: } (-0.681, 0.447) \\ \text{Relative Min: } (0.681, -0.447)\end{array}}$$

Ex: Use implicit differentiation to find an equation of the tangent line to the graph of the equation $\arcsin x + \arcsin y = \dfrac{\pi}{2}$ at the point $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

$$\frac{d}{dx}\left[\arcsin x + \arcsin y\right] = \frac{d}{dx}\left[\frac{\pi}{2}\right]$$

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\sqrt{1-y^2}\left(\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}\right)$$

$$m = \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$m = -\frac{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}}{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}} = -1$$

$$\boxed{y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)}$$