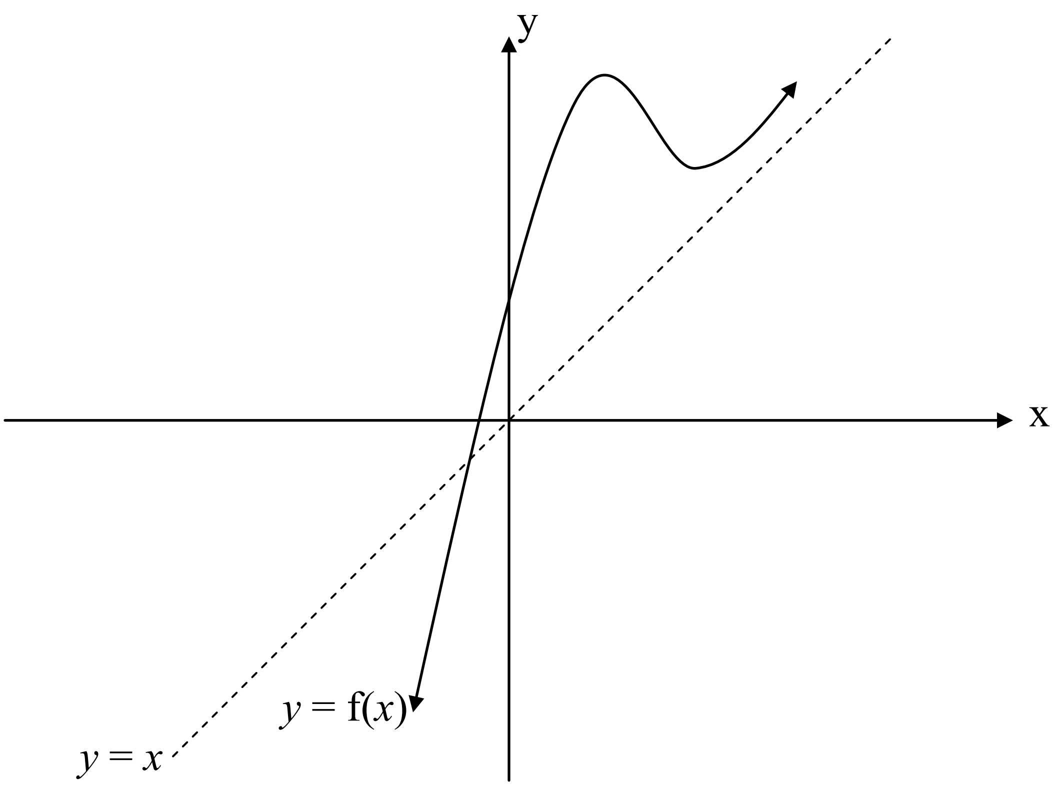
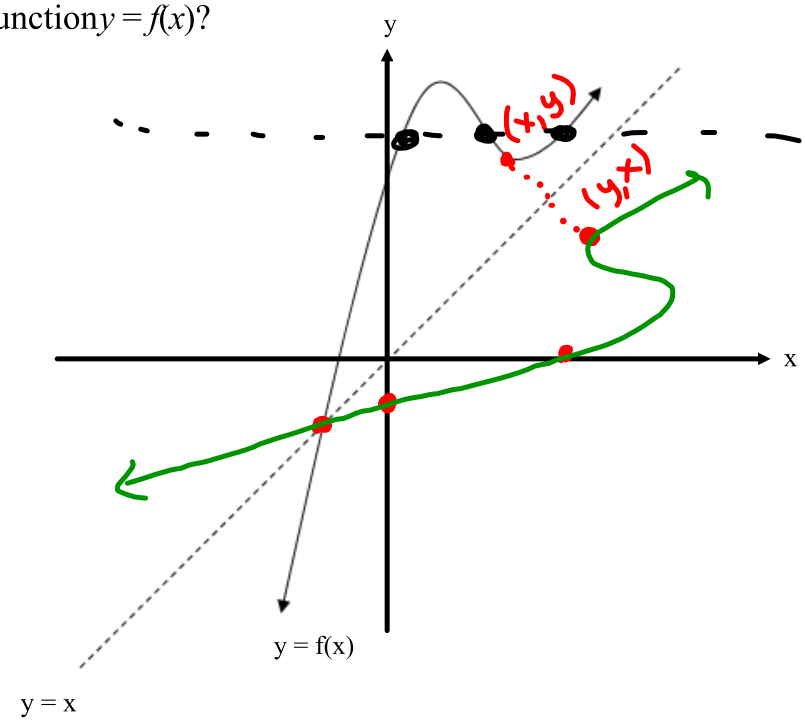
MAT 202
Larson – Section 5.3
Inverse Functions

Recall from College Algebra:
Suppose we have a function $y = f(x)$ given on the graph below. How would we proceed to find the inverse of the function $y = f(x)$?



What's the relationship between the ordered pairs of f and its inverse function?

Suppose we have a function $y = f(x)$ given on the graph below. How would we proceed to find the inverse of the function $y = f(x)$?



What's the relationship between the ordered pairs of f and its inverse function?

If (x,y) satisfies the function, then (y,x) satisfies the inverse function.

<u>Inverse Functions</u>:

Let f and g be two functions such that:

- $f(g(x)) = x$ for every x in the domain of g, **AND**
- $g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the inverse of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f\left(f^{-1}(x)\right) = x \quad AND \quad f^{-1}\left(f(x)\right) = x$$

Note: The domain of f must be the range of f^{-1} and the range of f must be the domain of f^{-1}.

Ex: Do all functions have inverses that are themselves functions? If not, find a function that does not have an inverse function.

How can we determine, algebraically, if a function has an inverse function?

- We must first check to see if the function is **one-to-one**. That is, if for a and b in the domain of f, $f(a) = f(b)$ implies that $a = b$.
- If the function f is one-to-one, then the function has an inverse function f^{-1}.
- If the function f is **strictly monotonic** (increasing or decreasing on its entire domain), then the function has an inverse function. We can use the derivative to determine whether or not a function is strictly monotonic.

How about graphically?

- You can show that a function f is or is not one-to-one by using the **horizontal line test**, that is, if a horizontal line intersects the graph of f in two or more places, then f is not one-to-one, therefore does not have an inverse function.

Ex: Show, algebraically, that the following functions are one-to-one, and if so, find their inverse functions algebraically:

a) $f(x) = 3x + 1$

Ex: Do all functions have inverses that are themselves functions? If not, find a function that does not have an inverse function.

No; $f(x) = x^2$

$f(x) = x^2$; $x \geq 0$

Ex: Show, algebraically, that the following functions are one-to-one, and if so, find their inverse functions algebraically:

a) $f(x) = 3x + 1$

One-to-one:

$f(a) = f(b)$

$3a + \cancel{1} = 3b + \cancel{1}$

$-1 \qquad -1$

$\frac{\cancel{3}a}{\cancel{3}} = \frac{\cancel{3}b}{3}$

$a = b$ $\blacksquare$

Inverse function:

$y = 3x + 1$

$x = 3y + 1$

$-1 \qquad -1$

$\frac{x-1}{3} = \frac{3y}{3}$

$y = \frac{x-1}{3}$

$f^{-1}(x) = \frac{x-1}{3}$

b) $f(x) = \sqrt[3]{x}$

Ex: For each, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse:

a) $f(x) = \frac{x^4}{4} - 2x^2$

b) $f(x) = x^5 + 2x^3$

<u>Continuity and Differentiability of Inverse Functions</u>:

Let f be a function whose domain is an interval *I*. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

b) $f(x)=\sqrt[3]{x}$

One-to-one:

$f(a)=f(b)$

$(\sqrt[3]{a})^3=(\sqrt[3]{b})^3$

$a=b$

Inverse function:

$y=\sqrt[3]{x}$

$(x)^3=(\sqrt[3]{y})^3$

$y=x^3$

$$\boxed{f^{-1}(x)=x^3}$$

Ex: For each, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse:

a) $f(x) = \dfrac{x^4}{4} - 2x^2$

Determine intervals of increase/decrease.

$f'(x) = x^3 - 4x$

$0 = x(x^2 - 4)$

$0 = x(x+2)(x-2)$

C.N.'s: $x = 0, -2, 2$

$(-\infty, -2)$ $(-2, 0)$ $(0, 2)$ $(2, \infty)$

<0 dec. >0 inc.

$\boxed{\text{Not strictly monotonic} \therefore \text{Does not have an inverse function}}$

b) $f(x) = x^5 + 2x^3$

$f'(x) = 5x^4 + 6x^2$

$0 = x^2(5x^2 + 6)$

C.N.'s: $x = 0$

$(-\infty, 0)$ $(0, \infty)$

$+$ $+$

Strictly monotonic & has an inverse function

<u>The Derivative of an Inverse Function</u>:
Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Ex: For the following function, verify that f has an inverse. Then use the function f and the given real number a to find $\left(f^{-1}\right)'(a)$.

$f(x) = \frac{1}{27}\left(x^5 + 2x^3\right); \quad a = -11$

Ex: Show that the slopes of the graphs of f and f^{1} are reciprocals at the given points.

$f(x) = x^3 \qquad \left(\frac{1}{2}, \frac{1}{8}\right)$

$f^{-1}(x) = \sqrt[3]{x} \qquad \left(\frac{1}{8}, \frac{1}{2}\right)$

The Derivative of an Inverse Function:
Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

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We know: If f & g are inverses of each other, then $f(g(x)) = x$

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{d}{dx}[x]$$

$$\frac{\cancel{f'(g(x))} \cdot g'(x)}{\cancel{f'(g(x))}} = \frac{1}{f'(g(x))}$$

$$g'(x) = \frac{1}{f'(g(x))}$$

Ex: For the following function, verify that f has an inverse. Then use the function f and the given real number a to find $\left(f^{-1}\right)'(a)$. $\Rightarrow$ $g'(a)$

$$f(x)=\frac{1}{27}\left(x^5+2x^3\right);\quad a=-11$$

$$f'(x)=\frac{1}{27}\left(5x^4+6x^2\right)$$

Strictly monotonic
& has an inverse function.

$$f^{-1}(x)=g(x)$$

$$g'(x)=\frac{1}{f'(g(x))}$$

$$g'(-11)=\frac{1}{f'(g(-11))}=\frac{1}{f'(-3)}=\boxed{\frac{1}{17}}$$

What is $g(-11)$?

$$-11=\frac{1}{27}\left(x^5+2x^3\right)$$

(using our calculators we get $x=-3$)

$$g(-11)=-3$$

Ex: Show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

$f(x) = x^3 \quad \left(\frac{1}{2}, \frac{1}{8}\right)$ $\quad f'(x) = 3x^2 \quad f'\left(\frac{1}{2}\right) = \frac{3}{4}$

$f^{-1}(x) = \sqrt[3]{x} \quad \left(\frac{1}{8}, \frac{1}{2}\right)$ $\quad \left(f^{-1}(x)\right)' = \frac{1}{3}x^{-2/3}$ $\quad \updownarrow$

$\left(f^{-1}\left(\frac{1}{8}\right)\right)' = \frac{1}{3}\left(\frac{1}{8}\right)^{-2/3} = \frac{4}{3}$