Exercise 1

$$\hat{\rho}_i = \frac{1}{2}\left(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right) = \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + r_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + r_y\begin{pmatrix}0&-i\\i&0\end{pmatrix} + r_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&r_x\\r_x&0\end{pmatrix} + \begin{pmatrix}0&-ir_y\\ir_y&0\end{pmatrix} + \begin{pmatrix}r_z&0\\0&-r_z\end{pmatrix}\right) \quad : \quad \hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

$$\hat{U}\hat{\rho}_i\hat{U}^\dagger = \begin{pmatrix}e^{-i\lambda/2}&0\\0&e^{i\lambda/2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\begin{pmatrix}e^{i\lambda/2}&0\\0&e^{-i\lambda/2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\lambda/2}&0\\0&e^{i\lambda/2}\end{pmatrix}\begin{pmatrix}e^{i\lambda/2}(1+r_z) & e^{-i\lambda/2}(r_x-ir_y)\\ e^{i\lambda/2}(r_x+ir_y) & e^{-i\lambda/2}(1-r_z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+r_z & e^{-i\lambda}(r_x-ir_y)\\ e^{i\lambda}(r_x+ir_y) & 1-r_z\end{pmatrix}$$

$$e^{i\lambda} = \cos(\lambda) + i\sin(\lambda) \quad , \quad e^{-i\lambda} = \cos(\lambda) - i\sin(\lambda)$$

$$= \frac{1}{2}\begin{pmatrix}(1+r_z) & r_x\cos(\lambda) - ir_y\cos(\lambda) - ir_x\sin(\lambda) - r_y\sin(\lambda)\\ r_x\cos(\lambda) + ir_y\cos(\lambda) + ir_x\sin(\lambda) - r_y\sin(\lambda) & (1-r_z)\end{pmatrix}$$

$$\boxed{\begin{gathered}\hat{\rho}_F = \frac{1}{2}\begin{pmatrix}(1+r_z) & r_x(\cos(\lambda) - i\sin(\lambda)) - r_y(i\cos(\lambda) + \sin(\lambda))\\ r_x(\cos(\lambda) + i\sin(\lambda)) + r_y(i\cos(\lambda) - \sin(\lambda)) & (1-r_z)\end{pmatrix}\\[1em] \hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\end{gathered}}$$

Exercise 2

a.) Find $\langle\sigma_{x_i}\rangle, \langle\sigma_{y_i}\rangle, \langle\sigma_{z_i}\rangle$

i.) $\langle\sigma_{x_i}\rangle = Tr(\hat{\rho}_i\hat{\sigma}_x) = Tr\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix}r_x-ir_y & 1+r_z\\ 1-r_z & r_x+ir_y\end{pmatrix}\right) = \frac{1}{2}(r_x-\cancel{ir_y}+r_x+\cancel{ir_y}) = r_x$

$\boxed{\langle\sigma_{x_i}\rangle = r_x}$

ii.) $\langle\sigma_{y_i}\rangle = Tr(\hat{\rho}_i\hat{\sigma}_y) = Tr\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix}r_y+ir_x & -i(1+r_z)\\ i(1-r_z) & r_y-ir_x\end{pmatrix}\right) = \frac{1}{2}(r_y+\cancel{ir_x}+r_y-\cancel{ir_x}) = r_y$

$\boxed{\langle\sigma_{y_i}\rangle = r_y}$

iii.) $\langle\sigma_{z_i}\rangle = Tr(\hat{\rho}_i\cdot\hat{\sigma}_z) = Tr\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix}1+r_z & ir_y-r_x\\ r_x+ir_y & r_z-1\end{pmatrix}\right) = \frac{1}{2}(\cancel{1}+r_z+r_z-\cancel{1}) = r_z$

$\boxed{\langle\sigma_{z_i}\rangle = r_z}$

b.) Find $\langle\sigma_{xf},\sigma_{yf},\sigma_{zf}\rangle$

$$\hat{\rho}_F = \frac{1}{2}\begin{pmatrix}(1+r_z) & r_x(\cos(\lambda)-i\sin(\lambda))-r_y(i\cos(\lambda)+\sin(\lambda))\\ r_x(\cos(\lambda)+i\sin(\lambda))+r_y(i\cos(\lambda)-\sin(\lambda)) & (1-r_z)\end{pmatrix}$$

$\alpha = \cos(\lambda)-i\sin(\lambda)$, $\beta = i\cos(\lambda)+\sin(\lambda)$, $\delta = \cos(\lambda)+i\sin(\lambda)$, $\gamma = i\cos(\lambda)-\sin(\lambda)$

$$\hat{\rho}_F = \frac{1}{2}\begin{pmatrix}(1+r_z) & r_x\cdot\alpha-r_y\cdot\beta\\ r_x\cdot\delta+r_y\cdot\gamma & (1-r_z)\end{pmatrix}$$

i.) $\langle\sigma_{xf}\rangle = Tr(\hat{\rho}_F\cdot\sigma_x) = Tr\left(\frac{1}{2}\begin{pmatrix}(1+r_z) & r_x\cdot\alpha-r_y\cdot\beta\\ r_x\cdot\delta+r_y\cdot\gamma & (1-r_z)\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix}r_x\cdot\alpha-r_y\cdot\beta & (1+r_z)\\ (1-r_z) & r_x\cdot\delta+r_y\gamma\end{pmatrix}\right) = \frac{1}{2}(r_x\cdot\alpha+r_x\delta+r_y\cdot\gamma-r_y\cdot\beta)$

$= \frac{1}{2}\left(r_x(\cos(\lambda)-i\cancel{\sin(\lambda)}+\cos(\lambda)+i\cancel{\sin(\lambda)})+r_y(i\cancel{\cos(\lambda)}-\sin(\lambda)-i\cancel{\cos(\lambda)}-\sin(\lambda)\right) = r_x\cdot\cos(\lambda)-r_y\cdot\sin(\lambda)$

$\boxed{\langle\sigma_{xF}\rangle = r_x\cos(\lambda)-r_y\sin(\lambda)}$

Exercise 2 continued

ii.) $\langle \sigma_{yf} \rangle = \mathrm{Tr}(\hat{\rho}_f \cdot \sigma_y) = \mathrm{Tr}\left(\frac{1}{2} \begin{pmatrix} (1+r_z) & r_x \cdot \alpha - r_y \cdot \beta \\ r_x \cdot \delta + r_y \cdot \gamma & (1-r_z) \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right)$

$$= \mathrm{Tr}\left(\frac{1}{2} \begin{pmatrix} i(r_x\alpha - r_y\beta) & -i(1+r_z) \\ i(1-r_z) & -i(r_x\delta + r_y\gamma) \end{pmatrix} \right) = \frac{1}{2}\left(ir_x(\alpha - \delta) - ir_y(\beta + \gamma) \right)$$

$$= \frac{1}{2}\left(ir_x\left(\cancel{\cos(\lambda)} - i\sin(\lambda) - \cancel{\cos(\lambda)} - i\sin(\lambda)\right) - ir_y\left(i\cos(\lambda) + \cancel{\sin(\lambda)} + i\cos(\lambda) - \cancel{\sin(\lambda)}\right) \right)$$

$$= ir_x(-i\sin(\lambda)) - ir_y(i\cos(\lambda)) = r_x \sin(\lambda) + r_y \cos(\lambda)$$

$$\boxed{\langle \sigma_{yf} \rangle = r_x \sin(\lambda) + r_y \cos(\lambda)}$$

iii.) $\langle \sigma_{zf} \rangle = \mathrm{Tr}(\hat{\rho}_f \cdot \sigma_z) = \mathrm{Tr}\left(\frac{1}{2} \begin{pmatrix} (1+r_z) & r_x \cdot \alpha - r_y \cdot \beta \\ r_x \cdot \delta + r_y \cdot \gamma & (1-r_z) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

$$= \mathrm{Tr}\left(\frac{1}{2} \begin{pmatrix} 1+r_z & r_x \cdot \alpha - r_y \cdot \beta \\ r_x \cdot \delta + r_y \cdot \gamma & r_z - 1 \end{pmatrix} \right) = \frac{1}{2}\left(\cancel{1} + r_z + r_z - \cancel{1} \right) = r_z$$

$$\boxed{\langle \sigma_{zf} \rangle = r_z}$$

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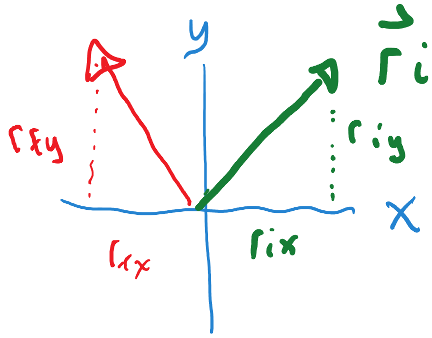
$$\rho_i = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix}$$

$$\hat{U} = e^{-i\lambda\hat{\sigma}_z/2} = \begin{pmatrix} e^{-i\lambda/2} & \\ 0 & e^{i\lambda/2} \end{pmatrix}$$

$$\hat{U}\hat{\rho}_i\hat{U}^{\dagger} \rightsquigarrow \hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 + \underset{r_{fz}}{a} & \underset{r_{fx}}{b} - i\underset{r_{fy}}{c} \\ b + ic & 1 - a \end{pmatrix}$$

$$\vec{r}_f = r_{fx}\hat{x} + r_{fy}\hat{y} + r_{fz}\hat{z}$$



$$\underline{r_{fx} = \cos(\lambda)\, r_{ix} + \sin(\lambda)\, r_{iy}}$$

$$\langle\sigma_z\rangle = \text{prob}(0) - \text{prob}(1)$$
$$= \text{Tr}\left(\hat{\rho}\,|0\rangle\langle 0|\right) - \text{Tr}\left[\hat{\rho}\,|1\rangle\langle 1|\right]$$

$$= \mathrm{Tr}\left[\hat{\rho}\,|0\rangle\langle 0| - \hat{\rho}\,|1\rangle\langle 1|\right]$$

$$= \mathrm{Tr}\Big[\hat{\rho}\,\underbrace{\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)}_{\sigma_z}\Big]$$

$$\langle \sigma_z \rangle = \mathrm{Tr}\left[\hat{\rho}\,\hat{\sigma}_z\right] \stackrel{?}{=} r_{z}$$

$$\hat{\sigma}_x \hat{\sigma}_y = i\,\hat{\sigma}_z \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\,\hat{\sigma}_z$$

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right) \leftarrow \qquad \langle \sigma_x \rangle = \underline{r_x}$$

$$\hat{\rho}\,\hat{\sigma}_z = \frac{1}{2}\left(\hat{I} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right)\sigma_z$$

$$= \frac{1}{2}\Big(\hat{\sigma}_z + r_x\underbrace{\sigma_x\sigma_z}_{-i\sigma_y} + r_y\underbrace{\sigma_y\sigma_z}_{i\sigma_x} + r_z\underbrace{\sigma_z\sigma_z}_{I}\Big)$$

$$\rho\,\hat{\sigma}_z = \frac{1}{2}\left[\hat{\sigma}_z - i r_x\sigma_y + i r_y\sigma_x + r_z\hat{I}\right]$$

$$\mathrm{Tr}(\ldots) = \frac{1}{2}\Big[\cancel{\mathrm{Tr}\,\sigma_z} - i r_x\cancel{\mathrm{Tr}\,\sigma_y} + i r_y\cancel{\mathrm{Tr}\,\sigma_x} + r_z\underbrace{\mathrm{Tr}\,\hat{I}}_{2}\Big]$$