Exercise 1

Problem 1 | α in $|\psi\rangle = \cos(\alpha/2)|0\rangle + \sin(\alpha/2)|1\rangle$

a.)

We wish to determine the Fisher information for this state $|\psi\rangle$. The equation for this is

$$\sum_{\substack{\text{All possible}\\ \text{outcomes}}}^{\infty} \frac{1}{\text{Prob(outcome)}}\left(\frac{\partial \text{Prob(outcome)}}{\partial P}\right)^2 = F$$

The possible outcomes are measuring $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ for the $S\sigma\hat{x}$ and $S\sigma\hat{z}$ directions. We begin with the $\hat{z}$ direction.

$$F = F_+ + F_-$$

F_+ : $P_+ = \cos^2(\alpha/2)$

$$\frac{\partial P_+}{\partial \alpha} = -2\cdot\cos(\alpha/2)\cdot\sin(\alpha/2)\cdot\frac{1}{2} = -\cos(\alpha/2)\cdot\sin(\alpha/2) \quad : \quad \left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \cos^2(\alpha/2)\cdot\sin^2(\alpha/2)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{1}{\cos^2(\alpha/2)}\left(\cos^2(\alpha/2)\cdot\sin^2(\alpha/2)\right) = \sin^2(\alpha/2)$$

F_- : $P_- = \sin^2(\alpha/2)$

$$\frac{\partial P_-}{\partial \alpha} = 2\cdot\sin(\alpha/2)\cdot\cos(\alpha/2)\cdot\frac{1}{2} = \sin(\alpha/2)\cdot\cos(\alpha/2) \quad : \quad \left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \sin^2(\alpha/2)\cdot\cos^2(\alpha/2)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{1}{\sin^2(\alpha/2)}\left(\sin^2(\alpha/2)\cdot\cos^2(\alpha/2)\right) = \cos^2(\alpha/2)$$

$$\therefore \; F = \sin^2(\alpha/2) + \cos^2(\alpha/2) = 1$$

$$\boxed{F_z = 1}$$

Now the x-direction

F_+ : $P_+ = \frac{1}{2}(1+\sin(\alpha))$

F_- : $P_- = \frac{1}{2}(1-\sin(\alpha))$

$$\frac{\partial P_+}{\partial \alpha} = \frac{1}{2}\cos(\alpha) \quad : \quad \left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{1}{4}\cos^2(\alpha)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{2}{1+\sin(\alpha)}\cdot\frac{1}{4}\cos^2(\alpha) \quad : \quad F_+ = \frac{\cos^2(\alpha)}{2(1+\sin(\alpha))}$$

F_- : $P_- = \frac{1}{2}(1-\sin(\alpha))$

$$\frac{\partial P_-}{\partial \alpha} = -\frac{1}{2}\cos(\alpha) \quad : \quad \left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{1}{4}\cos^2(\alpha)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{2}{1-\sin(\alpha)}\cdot\frac{1}{4}\cos^2(\alpha) \quad : \quad F_- = \frac{\cos^2(\alpha)}{2(1-\sin(\alpha))}$$

Exercise 1 Continued

$$F = \frac{\cos^2(\alpha)}{2(1+\sin(\alpha))} + \frac{\cos^2(\alpha)}{2(1-\sin(\alpha))} = \frac{1}{2}\left(\frac{\cos^2(\alpha)(1-\sin(\alpha)) + \cos^2(\alpha)(1+\sin(\alpha))}{1-\sin^2(\alpha)}\right)$$

$$= \frac{1}{2}\left(\frac{\cos^2(\alpha)(1-\sin(\alpha)) + \cos^2(\alpha)(1+\sin(\alpha))}{\cos^2(\alpha)}\right) = \frac{1}{2}\left(1-\sin(\alpha) + 1 + \sin(\alpha)\right) = \frac{1}{2}(2) = 1$$

$$\boxed{F_x = 1}$$

b.)

This means that at the very least our estimates will vary by 1.

Problem 2

φ in $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$

a.) For this state we will only be examining the x-direction.

$F_+ : P_+ = \frac{1}{2}(1+\cos(\alpha))$

$$\frac{\partial P_+}{\partial \alpha} = -\frac{1}{2}\sin(\alpha) \quad : \quad \left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_+}{\partial \alpha}\right)^2 = \frac{2}{1+\cos(\alpha)}\left(\frac{1}{4}\sin^2(\alpha)\right) = \frac{\sin^2(\alpha)}{2(1+\cos(\alpha))} \quad : \quad F_+ = \frac{\sin^2(\alpha)}{2(1+\cos(\alpha))}$$

$F_- : P_- = \frac{1}{2}(1-\cos(\alpha))$

$$\frac{\partial P_-}{\partial \alpha} = \frac{1}{2}\sin(\alpha) \quad : \quad \left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)$$

$$\frac{1}{P(\text{outcome})}\left(\frac{\partial P_-}{\partial \alpha}\right)^2 = \frac{2}{1-\cos(\alpha)}\left(\frac{1}{4}\sin^2(\alpha)\right) = \frac{\sin^2(\alpha)}{2(1-\cos(\alpha))} \quad : \quad F_- = \frac{\sin^2(\alpha)}{2(1-\cos(\alpha))}$$

$$F = \frac{\sin^2(\alpha)}{2(1+\cos(\alpha))} + \frac{\sin^2(\alpha)}{2(1-\cos(\alpha))} = \frac{1}{2}\left(\frac{\sin^2(\alpha)(1-\cos(\alpha)) + \sin^2(\alpha)(1+\cos(\alpha))}{1-\cos^2(\alpha)}\right)$$

$$= \frac{1}{2}\left(\frac{\sin^2(\alpha)(1-\cos(\alpha)) + \sin^2(\alpha)(1+\cos(\alpha))}{\sin^2(\alpha)}\right) = \frac{1}{2}\left(1-\cos(\alpha) + 1 + \cos(\alpha)\right) = \frac{1}{2}(2) = 1$$

$$\boxed{F_x = 1}$$

b.)

This means that at the very least our estimates will vary by 1.

Exercise 2

$$\hat{u} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a.) Show $\hat{u}$ is unitary

$$\hat{u}^\dagger \hat{u} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \therefore \quad \boxed{\hat{u}^\dagger \hat{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

b.)

i.)

$$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = |0\rangle}$$

ii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}$$

iii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)}$$

c.)

i.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = |0\rangle, \ \theta = 0, \ \varphi = 0 \ : \ |\hat{n}_0\rangle = |0\rangle \\ |\psi\rangle = |0\rangle, \ \theta = 0, \ \varphi = 0 \ : \ |\hat{n}\rangle = |0\rangle \end{array}}$$

ii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle), \ \theta = \pi/2, \ \varphi = 0 \ : \ |\hat{n}_0\rangle = \cos(\pi/4)|0\rangle + e^{0}\sin(\pi/4)|1\rangle \\ |\psi\rangle = 1/\sqrt{2}\,(|0\rangle - |1\rangle), \ \theta = \pi/2, \ \varphi = \pi \ : \ |\hat{n}\rangle = \cos(\pi/4)|0\rangle + e^{i\pi}\sin(\pi/4)|1\rangle \end{array}}$$

iii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}\,(|0\rangle + i|1\rangle), \ \theta = \pi/2, \ \varphi = \pi/2 \ : \ |\hat{n}_0\rangle = \cos(\pi/4)|0\rangle + e^{\frac{i\pi}{2}}\sin(\pi/4)|1\rangle \\ |\psi\rangle = 1/\sqrt{2}\,(|0\rangle - i|1\rangle), \ \theta = \pi/2, \ \varphi = 3\pi/2 \ : \ |\hat{n}\rangle = \cos(\pi/4)|0\rangle + e^{\frac{3\pi i}{2}}\sin(\pi/4)|1\rangle \end{array}}$$

Exercise 3

$$\hat{u} = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

a.) Show $\hat{u}$ is unitary

$$\hat{u}^\dagger \hat{u} = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} = \begin{pmatrix} e^{0} & 0 \\ 0 & e^{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{u}^\dagger \hat{u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

b.)

i.)

$$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{i\varphi/2} \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = e^{i\varphi/2}|0\rangle}$$

ii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi/2} \\ e^{-i\varphi/2} \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi/2}|0\rangle + e^{-i\varphi/2}|1\rangle\right)}$$

iii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi/2} \\ ie^{-i\varphi/2} \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi/2}|0\rangle + ie^{-i\varphi/2}|1\rangle\right)}$$

c.)

i.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = |0\rangle,\ \theta = 0,\ \varphi = 0 \ :\ |\hat{n}_0\rangle = |0\rangle \\ |\psi\rangle = e^{i\varphi/2}|0\rangle,\ \theta = 0,\ \varphi = \text{D.M}:\ |\hat{n}\rangle = e^{i\phi/2}|0\rangle \end{array}}$$

ii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle),\ \theta = \pi/2,\ \varphi = 0 \ :\ |\hat{n}_0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle) \\ |\psi\rangle = 1/\sqrt{2}(e^{i\varphi/2}|0\rangle + e^{-i\varphi/2}|1\rangle),\ \theta = \frac{\pi}{2},\ \varphi = -\phi:\ |\hat{n}\rangle = e^{-\frac{i\phi}{2}}(\cos(\pi/4)|0\rangle + e^{i\varphi}\sin(\pi/4)) \end{array}}$$

iii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle),\ \theta = \pi/2,\ \varphi = \pi/2 \ :\ |\hat{n}_0\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle) \\ |\psi\rangle = 1/\sqrt{2}(e^{i\varphi/2}|0\rangle + ie^{-i\varphi/2}|1\rangle),\ \theta = \pi/2,\ \phi = \pi/2 - \varphi:\ |\hat{n}\rangle = e^{-\frac{i\phi}{2}}(\cos(\pi/4)|0\rangle + e^{i(\frac{\pi}{2}-\phi)}\sin(\pi/4)|1\rangle) \end{array}}$$

Exercise 4

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

a.) Show $\hat{u}$ is unitary

$$u^{\dagger}u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \therefore \quad \boxed{u^{\dagger}u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

b.)

i.)

$$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)}$$

ii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = \frac{1}{2}\big((1+i)|0\rangle + (1+i)|1\rangle\big)}$$

iii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 2i \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} \quad \therefore \quad \boxed{|\psi\rangle = i|1\rangle}$$

c.)

$$(1+i) = \sqrt{2}\,e^{i\pi/4}$$

i.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = |0\rangle, \; \theta = 0, \; \varphi = 0 \; : \; |\hat{n}_0\rangle = |0\rangle \\ |\psi\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle), \; \theta = \pi/2, \; \varphi = \pi/2 : |\hat{n}\rangle = \cos(\pi/4)|0\rangle + e^{i\pi/2}\sin(\pi/4)|1\rangle \end{array}}$$

ii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle), \; \theta = \pi/2, \; \varphi = 0 \; : \; |\hat{n}_0\rangle = \cos(\pi/4)|0\rangle + e^{0}\sin(\pi/4)|1\rangle \\ |\psi\rangle = 1/2((1+i)|0\rangle + (1+i)|1\rangle), \; \theta = \frac{\pi}{2}, \; \varphi = 0 : |\hat{n}\rangle = \cos(\pi/4)|0\rangle + e^{0}\sin(\pi/4)|1\rangle \end{array}}$$

iii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle), \; \theta = \pi/2, \; \varphi = \pi/2 \; : \; |\hat{n}_0\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle) \\ |\psi\rangle = i|1\rangle, \quad \theta = \pi, \; \varphi = \text{DM} : |\hat{n}\rangle = \cos(\pi/2)|0\rangle + e^{i\varphi}\sin(\pi/2)|1\rangle \end{array}}$$

Exercise 5

a.) Show $\hat{u}$ is unitary

$$\hat{u} = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$$

$$u^\dagger u = \begin{pmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore$$

$$\boxed{u^\dagger u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

b.)

i.)

$$|\psi_0\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) \\ i\sin(\alpha/2) \end{pmatrix} \therefore \boxed{|\psi\rangle = \cos(\alpha/2)|0\rangle + i\sin(\alpha/2)|1\rangle}$$

ii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\alpha/2) + i\sin(\alpha/2) \\ i\sin(\alpha/2) + \cos(\alpha/2) \end{pmatrix}$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left((\cos(\alpha/2) + i\sin(\alpha/2))|0\rangle + (i\sin(\alpha/2) + \cos(\alpha/2))|1\rangle\right)}$$

iii.)

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|\psi\rangle = \hat{u}|\psi_0\rangle = \begin{pmatrix} \cos(\alpha/2) & i\sin(\alpha/2) \\ i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos(\alpha/2) - \sin(\alpha/2) \\ i\sin(\alpha/2) + i\cos(\alpha/2) \end{pmatrix}$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(((\cos(\alpha/2) - \sin(\alpha/2))|0\rangle + i(\cos(\alpha/2) + \sin(\alpha/2)))|1\rangle)}$$

c.)

i.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = |0\rangle,\ \theta = \alpha,\ \varphi = 0 : |\hat{n}_0\rangle = |0\rangle \\ |\psi\rangle = \cos(\alpha/2)|0\rangle + i\sin(\alpha/2)|1\rangle,\ \theta = \alpha,\ \varphi = \frac{\pi}{2} : |\hat{n}\rangle = \cos(\alpha/2)|0\rangle + e^{i\pi/2}\sin(\alpha/2)|1\rangle \end{array}}$$

ii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ \theta = \alpha = \frac{\pi}{2},\ \varphi = 0 : |\hat{n}_0\rangle = \cos(\pi/4)|0\rangle + e^{0}\sin(\pi/4)|1\rangle \\ |\psi\rangle = \frac{1}{\sqrt{2}}(\cos(\alpha/2) + i\sin(\alpha/2))(|0\rangle + |1\rangle),\ \theta = \frac{\pi}{2},\ \varphi = 0 : \\ |\hat{n}\rangle = \frac{1}{\sqrt{2}}(\cos(\pi/4) + i\sin(\pi/4))(|0\rangle + |1\rangle) \end{array}}$$

iii.)

$$\boxed{\begin{array}{l} |\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle),\ \theta = \frac{\pi}{2},\ \varphi = \frac{\pi}{2} : |\hat{n}_0\rangle = \cos(\pi/4)|0\rangle + e^{\frac{i\pi}{2}}\sin(\pi/4)|1\rangle \\ |\psi\rangle = \frac{1}{\sqrt{2}}(((\cos(\alpha/2) - \sin(\alpha/2))|0\rangle + i((\cos(\alpha/2) + \sin(\alpha/2))|1\rangle),\ \theta = \alpha + \frac{\pi}{2},\ \varphi = \frac{\pi}{2} \\ |\hat{n}\rangle = (((\cos(\pi/4) - \sin(\pi/4))|0\rangle + i((\cos(\pi/4) + \sin(\pi/4))|1\rangle) \end{array}}$$

Exercise 6

a.) $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\hat{\sigma}_x^\dagger \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \boxed{\sigma_x^\dagger \sigma_x = \hat{I}}$$

b.) $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\hat{\sigma}_y^\dagger \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \boxed{\sigma_y^\dagger \sigma_y = \hat{I}}$$

c.) $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{\sigma}_z^\dagger \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \boxed{\sigma_z^\dagger \sigma_z = \hat{I}}$$

d.) $\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\hat{u}^\dagger \hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \boxed{\hat{u}^\dagger \hat{u} = \hat{I}}$$