PHYS 396 HOMEWORK SET 2

1 a) $x(t) = x_0 + v_x t$

$y(t) = y_0$

$z(t) = z_0 + v_z t$

$\frac{dx}{dt} = v_x$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = v_z$

$\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = 0$, $\frac{d^2z}{dt^2} = 0$

$\therefore \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$

) $x'(t) = \cos(\omega t)x + \sin(\omega t)y$

$y'(t) = -\sin(\omega t)x + \cos(\omega t)y$

$z'(t) = z_0$

so

$$\frac{dx'}{dt} = -\omega\sin(\omega t)x + \cos(\omega t)\frac{dx}{dt} + \omega\cos(\omega t)y + \sin(\omega t)\frac{dy}{dt}$$

$$\frac{d^2x'}{dt^2} = -\omega^2\cos(\omega t)x - \omega\sin(\omega t)\frac{dx}{dt} - \omega\sin(\omega t)\frac{dx}{dt} + \cos(\omega t)\cancel{\frac{d^2x}{dt^2}}^{\to 0} \text{ since } \frac{d^2x}{dt^2} = 0$$

$$-\omega^2\sin(\omega t)y + \omega\cos(\omega t)\frac{dy}{dt} + \omega\cos(\omega t)\frac{dy}{dt} + \sin(\omega t)\cancel{\frac{d^2y}{dt^2}}^{\to 0} \text{ since } \frac{d^2y}{dt^2} = 0$$

$$= -\omega^2(\cos(\omega t)x + \sin(\omega t)y) - 2\omega\sin(\omega t)\frac{dx}{dt} + 2\omega\cos(\omega t)\cancel{\frac{dy}{dt}}^{\to 0}$$

$$= -\omega^2[\cos(\omega t)(x_0 + v_x t) + \sin(\omega t)y_0] - 2\omega\sin(\omega t)v_x$$

so

$$\frac{d^2x'}{dt^2} = -\omega^2[x_0\cos(\omega t) + y_0\sin(\omega t)] - \omega[v_x(\omega t)\cos(\omega t) + 2v_x\sin(\omega t)]$$

WHEREAS

$$\frac{dy'}{dt} = -\omega\cos(\omega t)x - \sin(\omega t)\frac{dx}{dt} - \omega\sin(\omega t)y + \cos(\omega t)\frac{dy}{dt} \checkmark$$

$$\frac{d^2y'}{dt^2} = \omega^2\sin(\omega t)x - \omega\cos(\omega t)\frac{dx}{dt} - \omega\cos(\omega t)\frac{dx}{dt} - \sin(\omega t)\cancel{\frac{d^2x}{dt^2}}^{\to 0} \quad \text{SINCE } \frac{d^2x}{dt^2} = 0$$

$$- \omega^2\cos(\omega t)y - \omega\sin(\omega t)\frac{dy}{dt} - \omega\sin(\omega t)\frac{dy}{dt} + \cos(\omega t)\cancel{\frac{d^2y}{dt^2}}^{\to 0} \checkmark \quad \text{SINCE } \frac{d^2y}{dt^2} = 0$$

$$= \omega^2\left(\sin(\omega t)x - \cos(\omega t)y\right) - 2\omega\cos(\omega t)\frac{dx}{dt} - 2\omega\sin(\omega t)\cancel{\frac{dy}{dt}}^{\to 0} \quad \text{SINCE } \frac{dy}{dt} = 0$$

$$= \omega^2\left[\sin(\omega t)(x_0 + v_x t) - \cos(\omega t)y_0\right] - 2\omega\cos(\omega t)v_x \checkmark$$

SO

$$\frac{d^2y'}{dt^2} = +\omega^2\left[x_0\sin(\omega t) - y_0\cos(\omega t)\right] + \omega\left[v_x(\omega t)\sin(\omega t) - 2v_x\cos(\omega t)\right] \checkmark$$

WHEREAS

$$\frac{dz'}{dt} = 0 \quad \therefore \quad \frac{d^2z'}{dt^2} = 0$$

IN SUMMARY, THE EQUATIONS OF MOTION FOR OUR FREE PARTICLE, ACCORDING TO THE NON-INERTIAL REFERENCE FRAME,

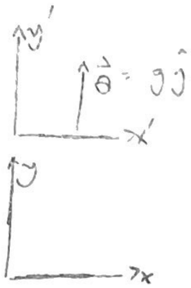
ARE

$$\frac{d^2x'}{dt^2} = -\omega^2\left[x_0\cos(\omega t) + y_0\sin(\omega t)\right] - \omega v_x\left[(\omega t)\cos(\omega t) + 2\sin(\omega t)\right]$$

$$\frac{d^2y'}{dt^2} = -\omega^2\left[-x_0\sin(\omega t) + y_0\cos(\omega t)\right] + \omega v_x\left[(\omega t)\sin(\omega t) - 2\cos(\omega t)\right]$$

$$\frac{d^2z'}{dt^2} = 0$$

2.



THE ORIGIN OF THE NON-INERTIAL REFERENCE FRAME, DESCRIBED BY THE INERTIAL REFERENCE FRAME, OBEYS

$$\frac{d^2y}{dt^2} = g = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

NOW, SEPARATING VARIABLES AND INTEGRATING ...

$$\int d\left(\frac{dy}{dt}\right) = \int g\,dt$$

$$\frac{dy}{dt} = \cancel{v_{0y}} + gt$$

0 SINCE THE NON-INERTIAL REFERENCE FRAME IS MOMENTARILY AT REST @ $t = t' = 0$

SO

$$\frac{dy}{dt} = gt$$

AGAIN, SEPARATING VARIABLES : INTEGRATING YIELDS..

$$\int dy = \int g\,t\,dt$$

SO

$$y(t) = \cancel{y_0} + \frac{1}{2}gt^2$$

0 THIS IDENTIFIES THE LOCATION OF THE ORIGIN OF THE NONINERTIAL REFERENCE FRAME. THESE TWO FRAMES WOULD THEREFORE IDENTIFY THE LOCATION OF SOME POINT AS

$$x'(t) = x(t)$$

$$y'(t) = y(t) - \frac{1}{2}gt^2$$

$$z'(t) = z(t)$$

b) SO

$$\frac{dx'}{dt} = \frac{dx}{dt}, \quad \frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} = 0$$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(y - \frac{1}{2}gt^2\right) = \frac{dy}{dt} - gt$$

$$\frac{d^2y'}{dt^2} = \cancel{\frac{d^2y}{dt^2}} - g$$

$$\frac{dz'}{dt} = \frac{dz}{dt}, \qquad \frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2} = 0$$

SO, IN SUMMARY...

(*)

$$\left[\begin{array}{l} \dfrac{d^2x'}{dt^2} = \dfrac{d^2z'}{dt^2} = 0 \\ \dfrac{d^2y'}{dt^2} = -g \end{array}\right]$$

NEWTON'S 2^{ND} TAKES THE FORM...

3. SINCE $\vec{F} = -m_G g\hat{j} = m_I\left(\frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}\right)$

a) SO, EQUATING LIKE COMPONENTS...

(**)
$$\frac{d^2x}{dt^2} = 0 = \frac{d^2z}{dt^2}$$
$$\frac{d^2y}{dt^2} = -\frac{m_G}{m_I}g$$

b) SETTING $m_G = m_I$, (**) TAKES THE FORM...

$$\frac{d^2x}{dt^2} = \frac{d^2z}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$ WHICH IS EXACTLY THE SAME OF THE SAME FORM AS (*) FROM PROBLEM 2!

AN OBSERVER IN A NONINERTIAL REFERENCE FRAME THAT EXPERIENCES A CONSTANT ACCELERATION WOULD DESCRIBE THE MOTION OF A FREE PARTICLE TO BE EXACTLY THE SAME AS THAT OF AN INERTIAL REFERENCE FRAME OBSERVER DESCRIBING THE MOTION OF A PARTICLE UNDER THE INFLUENCE OF A CONSTANT FORCE.

4. CONSIDER A SPHERICALLY-SYMMETRIC MASSIVE OBJECT OF RADIUS R, MASS M, $\therefore$ MASS DENSITY

$$\mu(r) = kr^{n} \quad \text{for } r<R$$

a) $M = \int \mu \, d^3x$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} kr^{n}\cdot r^{2}\sin\theta \, dr\, d\theta\, d\varphi = k\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{R} r^{n+2}\, dr$$

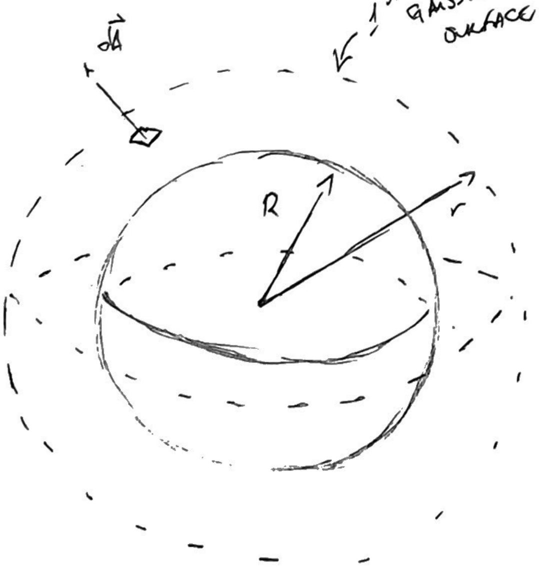
$$= k\cdot 2\pi \cdot 2 \; \frac{r^{n+3}}{(n+3)}\Bigg|_0^{R} = \frac{4\pi k}{(n+3)}\left[R^{n+3} - 0^{n+3}\right]$$

$$\therefore \boxed{M = \frac{4\pi k}{(n+3)} R^{n+3}}$$

b) NOTICE THAT FOR $n+3>0$, THE INTEGRAL OF PART a REMAINS FINITE

SO $\boxed{n>-3}$

c) FIRST CONSIDER $r>R$...



$$\boxed{\oint \vec{g}\cdot d\vec{A} = -4\pi G M_{enc}}$$ - GAUSS' LAW FOR GRAVITATION IN INTEGRAL FORM

NOW

$\vec{g} = -g\hat{e}_r$ SO $\vec{g}\cdot d\vec{A} = -g\, dA$

$d\vec{A} = dA\, \hat{e}_r$

SO THE LEFT HAND SIDE BECOMES...

$$\oint \vec{g}\cdot d\vec{A} = -\oint g\, dA = -g\int dA = -g\, 4\pi r^2$$

g IS A CONSTANT AT GIVEN r DUE TO THE SPHERICAL SYMMETRY OF THE PROBLEM.

SO GAUSS' LAW BECOMES..

$$-g\,4\pi r^2 = -4\pi G M_{enc}$$

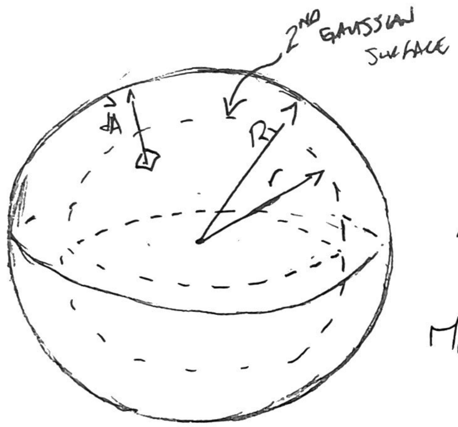
$$= -4\pi G M$$

SINCE ALL OF THE MASS IS ENCLOSED BY OUR GAUSSIAN SPHERE

SO

$$g = \frac{GM}{r^2} \quad \therefore \quad \boxed{\vec{g} = -\frac{GM}{r^2}\hat{e}_r}$$

NOW CONSIDER $r<R$...



NOTICE THAT THE LEFT-HAND-SIDE OF GAUSS' LAW YIELDS THE SAME RESULTS FROM THE PREVIOUS PART...

$$\oint \vec{g}\cdot d\vec{A} = -g\,4\pi r^2$$

THE TOTAL MASS ENCLOSED DIFFERS FROM THE PREVIOUS PART...

$$M_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} k r^n \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= k\int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta\, d\theta \int_0^{r} r^{n+2}\,dr = k\cdot 2\pi\cdot 2\cdot \left.\frac{r^{n+3}}{(n+3)}\right|_0^{r}$$

$$= \frac{4\pi k}{(n+3)} r^{n+3} = M\frac{r^{n+3}}{R^{n+3}}$$

SO GAUSS' LAW YIELDS..

$$-g\,4\pi r^2 = -4\pi G M\frac{r^{n+3}}{R^{n+3}} \quad \text{so} \quad g = GM\frac{r^{n+1}}{R^{n+3}} \quad \therefore \quad \vec{g} = -GM\frac{r^{n+1}}{R^{n+3}}\hat{e}_r$$

SO IN SUMMARY...

$$\left[\begin{aligned} \vec{g}(\vec{r}) &= -GM\frac{r^{n+1}}{R^{n+3}}\hat{e}_r \quad \text{for} \quad r<R \\ &= -\frac{GM}{r^2}\hat{e}_r \quad \text{for} \quad r>R \end{aligned}\right]$$

d) Now

$$\vec{g} = -\vec{\nabla}\Phi = -\left[\frac{\partial \Phi}{\partial r}\hat{e}_r + \frac{1}{r}\cancel{\frac{\partial \Phi}{\partial \theta}}\hat{\theta} + \frac{1}{r\sin\theta}\cancel{\frac{\partial \Phi}{\partial \phi}}\hat{\phi}\right]$$

$\to 0$ $\quad$ $\to 0$

since $\vec{g} = g\hat{e}_r$

so

$$g = -\frac{d\Phi}{dr} \quad \text{so} \quad \Phi(r) = -\int g\,dr$$

so for $r < R$..

for $n \neq -2$

$$\Phi(r) = -\int -\frac{GMr^{n+1}}{R^{n+3}}\,dr = \frac{GM}{R^{n+3}}\int r^{n+1}\,dr = \frac{GM}{R^{n+3}}\frac{r^{n+2}}{(n+2)}$$

for $r > R$...

$$\Phi(r) = -\int -\frac{GM}{r^2}\,dr = GM\int r^{-2}\,dr = -\frac{GM}{r}$$

so

$$\Phi(r) = \frac{1}{(n+2)}\frac{GM}{R^{n+3}}r^{n+2} \quad \text{for } r < R$$

$$= -\frac{GM}{r} \quad \text{for } r > R$$