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PHYS 396 HW 11

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1,2,3,4

Problem 1

a.) $H_0 = 72 \frac{\cancel{km}}{Sec \cdot \cancel{MPC}} \times \frac{1000\, \cancel{m}}{1\, \cancel{km}} \times \frac{1\, \cancel{MPC}}{1 \times 10^6\, \cancel{pc}} \times \frac{1\, \cancel{pc}}{3.09 \times 10^{16}\, \cancel{m}} = 2.33 \times 10^{-18}\, s^{-1}$

$$t_h = \frac{1}{H_0} = \frac{1}{2.33 \times 10^{-18}\, s^{-1}} = 4.29185 \times 10^{17}\, s$$

$$t_h = 4.29185 \times 10^{17}\, s \times \frac{1\, \cancel{min}}{60\, s} \times \frac{1\, \cancel{hr}}{60\, \cancel{min}} \times \frac{1\, \cancel{day}}{24\, \cancel{hrs}} \times \frac{1\, year}{365\, \cancel{days}} = 13.6 \times 10^9\, yrs$$

$$\boxed{t_h = 13.6 \times 10^9\, yrs}$$

b.) $a(t) = a_0 t^{2/3(1+w)} \longrightarrow (1) \quad H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \longrightarrow (2) \quad H_0 = \frac{1}{t_h} \longrightarrow (3)$

$$\dot{a} = a_0 \cdot \frac{2}{3} \cdot \frac{1}{(1+w)} t^{\frac{2}{3}(1+w)-1} \Big|_{t=t_0} = a_0 \cdot \frac{2}{3} \cdot \frac{1}{(1+w)} t_0^{\frac{2}{3}(1+w)-1}$$

$$\frac{\dot{a}(t_0)}{a(t_0)} = \frac{a_0 \cdot 2/3\; t_0^{2/3(1+w)-1}}{a_0\, t_0^{2/3(1+w)}\, (1+w)} = \frac{\cancel{a_0} \cdot 2/3\; \cancel{t_0^{2/3(1+w)}}\, t_0^{-1}}{(1+w)\cancel{a_0}\, \cancel{t_0^{2/3(1+w)}}} = \frac{2}{3} \cdot \frac{1}{t_0(1+w)}$$

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{1}{t_h} : \quad \frac{2}{3} \cdot \frac{1}{t_0(1+w)} = \frac{1}{t_h} \quad \therefore \quad t_0 = \frac{2}{3} \frac{t_h}{(1+w)}$$

$$\boxed{t_0 = \frac{2}{3} \frac{t_h}{(1+w)}}$$

c.) $w = \frac{1}{3}$: $t_h = 13.6 \times 10^9\, yrs$

$$t_0 = \frac{2}{3} \frac{(13.6 \times 10^9\, yrs)}{(1 + 1/3)} = 6.8 \times 10^9\, yrs$$

$$\boxed{t_0 = 6.8 \times 10^9\, yrs}$$

d.) $w = 0$: $t_h = 13.6 \times 10^9\, yrs$

$$t_0 = \frac{2}{3} \frac{(13.6 \times 10^9\, yrs)}{(1+0)} = 9.07 \times 10^9\, yrs$$

$$\boxed{t_0 = 9.07 \times 10^9\, yrs}$$

Problem 2

a.) $x = \sinh(\chi)\sin(\sigma)\cos(\varphi)$: $-t^2 + x^2 + y^2 + z^2 = -1$

$y = \sinh(\chi)\sin(\sigma)\sin(\varphi)$

$z = \sinh(\chi)\cos(\sigma)$

$t = \cosh(\chi)$

$$-(\cosh^2(\chi)) + \sinh^2(\chi)\sin^2(\sigma)\cos^2(\varphi) + \sinh^2(\chi)\sin^2(\sigma)\sin^2(\varphi) + \sinh^2(\chi)\cos^2(\sigma) = -1$$

$$-\cosh^2(\chi) + \sinh^2(\chi)\sin^2(\sigma)\big(\underbrace{\cos^2(\varphi) + \sin^2(\varphi)}_{1}\big) + \sinh^2(\chi)\cos^2(\sigma) = -1$$

$$-\cosh^2(\chi) + \sinh^2(\chi)\big(\underbrace{\sin^2(\sigma) + \cos^2(\sigma)}_{1}\big) = -1$$

$$\cosh^2(\sigma) - \sinh^2(\sigma) = 1 \quad \therefore \quad -\cosh^2(\sigma) + \sinh^2(\sigma) = -1$$

$$-\cosh^2(\chi) + \sinh^2(\chi) = -1$$

$$\boxed{-1 = -1 \checkmark}$$

b.)

dx : $dx = \frac{dx}{d\chi} + \frac{dx}{d\sigma} + \frac{dx}{d\varphi}$ w/ R=1

$\frac{dx}{d\chi} = \cosh(\chi)\sin(\sigma)\cos(\varphi)$: $\frac{dx}{d\sigma} = \sinh(\chi)\cos(\sigma)\cos(\varphi)$: $\frac{dx}{d\varphi} = -\sinh(\chi)\sin(\sigma)\sin(\varphi)$

$\therefore$ $dx = \cosh(\chi)\sin(\sigma)\cos(\varphi)\,d\chi + \sinh(\chi)\cos(\sigma)\cos(\varphi)\,d\sigma - \sinh(\chi)\sin(\sigma)\sin(\varphi)\,d\varphi$

dy : $dy = \frac{dy}{d\chi} + \frac{dy}{d\sigma} + \frac{dy}{d\varphi}$

$\frac{dy}{d\chi} = \cosh(\chi)\sin(\sigma)\sin(\varphi)$: $\frac{dy}{d\sigma} = \sinh(\chi)\cos(\sigma)\sin(\varphi)$: $\frac{dy}{d\varphi} = \sinh(\chi)\sin(\sigma)\cos(\varphi)$

$\therefore$ $dy = \cosh(\chi)\sin(\sigma)\sin(\varphi)\,d\chi + \sinh(\chi)\cos(\sigma)\sin(\varphi)\,d\sigma + \sinh(\chi)\sin(\sigma)\cos(\varphi)\,d\varphi$

$\frac{dz}{d\chi} = \cosh(\chi)\cos(\sigma)$: $\frac{dz}{d\sigma} = -\sinh(\chi)\sin(\sigma)$

$\therefore$ $dz = \cosh(\chi)\cos(\sigma)\,d\chi - \sinh(\chi)\sin(\sigma)\,d\sigma$

$\frac{dt}{d\chi} = \sinh(\chi)$

$\therefore$ $dt = \sinh(\chi)\,d\chi$

$$\boxed{\begin{aligned} dx &= \cosh(\chi)\sin(\sigma)\cos(\varphi)\,d\chi + \sinh(\chi)\cos(\sigma)\cos(\varphi)\,d\sigma - \sinh(\chi)\sin(\sigma)\sin(\varphi)\,d\varphi \\ dy &= \cosh(\chi)\sin(\sigma)\sin(\varphi)\,d\chi + \sinh(\chi)\cos(\sigma)\sin(\varphi)\,d\sigma + \sinh(\chi)\sin(\sigma)\cos(\varphi)\,d\varphi \\ dz &= \cosh(\chi)\cos(\sigma)\,d\chi - \sinh(\chi)\sin(\sigma)\,d\sigma \\ dt &= \sinh(\chi)\,d\chi \end{aligned}}$$

Problem 2 | continued

c.) dx^2: $(\cosh(x)\sin(\sigma)\cos(\varphi)\,dx + \sinh(x)\cos(\sigma)\cos(\varphi)\,d\sigma - \sinh(x)\sin(\sigma)\sin(\varphi)\,d\varphi)$

$\times$

$(\cosh(x)\sin(\sigma)\cos(\varphi)\,dx + \sinh(x)\cos(\sigma)\cos(\varphi)\,d\sigma - \sinh(x)\sin(\sigma)\sin(\varphi)\,d\varphi)$

$\cosh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,dx^2 + \cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\cos^2(\varphi)\,dx\,d\sigma$

$-\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + \cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\cos^2(\varphi)\,dx\,d\sigma$

$+\sinh^2(x)\cos^2(\sigma)\cos^2(\varphi)\,d\sigma^2 - \sinh^2\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$

$-\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi - \sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$

$+\sinh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,d\varphi^2$

$$dx^2 = \cosh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,dx^2 + \sinh^2(x)\cos^2(\sigma)\cos^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,d\varphi^2$$

$$-2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\cos^2(\varphi)\,dx\,d\sigma$$

$$-2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$$

dy^2: $(\cosh(x)\sin(\sigma)\sin(\varphi)\,dx + \sinh(x)\cos(\sigma)\sin(\varphi)\,d\sigma + \sinh(x)\sin(\sigma)\cos(\varphi)\,d\varphi)$

$\times$

$(\cosh(x)\sin(\sigma)\sin(\varphi)\,dx + \sinh(x)\cos(\sigma)\sin(\varphi)\,d\sigma + \sinh(x)\sin(\sigma)\cos(\varphi)\,d\varphi)$

$\cosh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,dx^2 + \cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\sin^2(\varphi)\,dx\,d\sigma$

$\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + \cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\sin^2(\varphi)\,dx\,d\sigma$

$+\sinh^2(x)\cos^2(\sigma)\sin^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$

$+\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + \sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$

$+\sinh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,d\varphi^2$

$$dy^2 = \cosh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,dx^2 + \sinh^2(x)\cos^2(\sigma)\sin^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,d\varphi^2$$

$$+2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\sin^2(\varphi)\,dx\,d\sigma$$

$$+2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$$

$$dx^2 = \cosh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,dx^2 + \sinh^2(x)\cos^2(\sigma)\cos^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,d\varphi^2$$
$$-2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\cos^2(\varphi)\,dx\,d\sigma$$
$$-2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$$

$$dy^2 = \cosh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,dx^2 + \sinh^2(x)\cos^2(\sigma)\sin^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,d\varphi^2$$
$$+2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dx\,d\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\sin^2(\varphi)\,dx\,d\sigma$$
$$+2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi\,d\sigma$$

Problem 2 | Continued

d.) dz^2: $(\cosh(x)\cos(\sigma)dx - \sinh(x)\sin(\sigma)d\sigma) \times (\cosh(x)\cos(\sigma)dx - \sinh(x)\sin(\sigma)d\sigma)$

$$= \cosh^2(x)\cos^2(\sigma)dx^2 - 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)dxd\sigma - \sinh^2(x)\sin^2(\sigma)d\sigma^2$$

$$dz^2 = \cosh^2(x)\cos^2(\sigma)dx^2 - 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)dxd\sigma + \sinh^2(x)\sin^2(\sigma)d\sigma^2$$

$dx^2 + dy^2 + dz^2$:

$$\cosh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,dx^2 + \sinh^2(x)\cos^2(\sigma)\cos^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,d\varphi^2$$

$$- 2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dxd\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\cos^2(\varphi)\,dxd\sigma$$

$$- 2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi d\sigma + \cosh^2(x)\sin^2(\sigma)\sin^2(\varphi)\,dx^2$$

$$+ \sinh^2(x)\cos^2(\sigma)\sin^2(\varphi)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\cos^2(\varphi)\,d\varphi^2$$

$$+ 2\cosh(x)\sinh(x)\sin^2(\sigma)\sin(\varphi)\cos(\varphi)\,dxd\varphi + 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\sin^2(\varphi)\,dxd\sigma$$

$$+ 2\sinh^2(x)\sin(\sigma)\cos(\sigma)\sin(\varphi)\cos(\varphi)\,d\varphi d\sigma + \cosh^2(x)\cos^2(\sigma)\,dx^2$$

$$- 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\,dxd\sigma + \sinh^2(x)\sin^2(\sigma)\,d\sigma^2$$

$$= (\cosh^2(x)\sin^2(\sigma)(\sin^2(\varphi) + \cos^2(\varphi)) + \cosh^2(x)\cos^2(\sigma))\,dx^2 + \sinh^2\cos^2(\sigma)(\sin^2(\varphi) + \cos^2(\varphi))\,d\sigma^2$$

$$+ \sinh^2(x)\sin^2(\sigma)\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)(\sin^2(\varphi) + \cos^2(\varphi))\,d\varphi^2 +$$

$$2\cosh(x)\sin(x)\sin(\sigma)\cos(\sigma)(\sin^2(\varphi) + \cos^2(\varphi))\,dxd\sigma - 2\cosh(x)\sinh(x)\sin(\sigma)\cos(\sigma)\,dxd\sigma$$

$$= \cosh^2(x)(\sin^2(\sigma) + \cos^2(\sigma))\,dx^2 + \sinh^2(x)(\sin^2(\sigma) + \cos^2(\sigma))\,d\sigma^2 + \sinh^2(x)\sin^2(\sigma)\,d\varphi^2$$

$$\therefore \quad \boxed{dx^2 + dy^2 + dz^2 = \cosh^2(x)\,dx^2 + \sinh^2(x)(d\sigma^2 + \sin^2(\sigma))\,d\varphi^2}$$

e.) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$: $dt = \sinh(x)dx$

$$= -\sinh^2(x)dx^2 + \cosh^2(x)\,dx^2 + \sinh^2(x)(d\sigma^2 + \sin^2(\sigma))\,d\varphi^2$$

$$= (\cosh^2(x) - \sinh^2(x))\,dx^2 + \sinh^2(x)(d\sigma^2 + \sin^2(\sigma))\,d\varphi^2$$

$$\therefore \quad \boxed{ds^2 = dx^2 + \sinh^2(x)(d\sigma^2 + \sin^2(\sigma))\,d\varphi^2}$$

f.) $r = \sinh(x)$: $dr = \cosh(x)\,dx$ $\therefore$ $dx = \dfrac{dr}{\cosh(x)}$: $dx^2 = \dfrac{dr^2}{1 + \sinh^2(x)} = \dfrac{dr^2}{1 + r^2}$

$$\therefore \quad \boxed{ds^2 = \frac{dr^2}{1 + r^2} + r^2(d\sigma^2 + \sin^2\sigma\, d\varphi^2)}$$

Problem 3

$$d\sigma^2 = dx^2 + \sinh^2(x)(d\sigma^2 + \sin^2(\sigma)d\varphi^2) \quad : \quad ds^2 = \frac{dr^2}{1+r^2} + r^2(d\sigma^2 + \sin^2\sigma)d\varphi^2$$

a.) $\sigma = \pi/2 \quad : \quad d\sigma^2 = \dfrac{dr^2}{1+r^2} + r^2 d\varphi^2$

$$\therefore \quad \boxed{ds^2 = \frac{dr^2}{1+r^2} + r^2 d\varphi^2}$$

$$d\sigma^2 = dx^2 + \sinh^2(x)d\varphi^2$$

b.) $d\sigma^2 = dx^2 + \sinh^2(x)d\varphi^2 \qquad \cosh^2(\sigma) - \sinh^2(\sigma) = 1 \quad \therefore \quad \sinh^2(\sigma) = \cosh^2(\sigma) - 1$

$$z(r) : \quad dz = \frac{dz}{dr}dr \quad : \quad dz^2 = \left(\frac{dz}{dr}\right)^2 dr^2 \qquad ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

$$dr^2 + \cancel{r^2 d\varphi^2} + \left(\frac{dz}{dr}\right)^2 dr^2 = \frac{dr^2}{1+r^2} + \cancel{r^2 d\varphi^2}$$

$$\cancel{dr^2}\left(1 + \left(\frac{dz}{dr}\right)^2\right) = \frac{\cancel{dr^2}}{1+r^2} \quad : \quad 1 + \left(\frac{dz}{dr}\right)^2 = \frac{1}{1+r^2} \quad \therefore \quad \left(\frac{dz}{dr}\right)^2 = \frac{1}{1+r^2} - 1$$

$$\left(\frac{dz}{dr}\right)^2 = 1 - 1 - r^2 = -r^2 \Big|_{r = R\sinh(x)} \quad : \quad \left(\frac{dz}{dx}\right)^2 = -R^2\sinh^2(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \therefore \quad -\sinh^2(x) = 1 - \cosh^2(x)$$

$$\therefore \quad \left(\frac{dz}{dx}\right)^2 = R^2(1 - \cosh^2(x)) \quad \therefore \quad \frac{dz}{dx} = \pm R\sqrt{1 - \cosh^2(x)}$$

$$\boxed{\frac{dz}{dx} = \pm R\sqrt{1 - \cosh^2(x)}}$$

c.)

$$\frac{dz}{dx} = \pm R\sqrt{1 - \cosh^2(x)} = \pm R\sqrt{1 - \cosh^2(x)}\,dx = \pm R\sqrt{\sinh^2(x)}\,dx$$

$$z = \pm R\int_0^\infty \sinh(x)\,dx = \pm R\cosh(x)\Big|_0^\infty = \pm R\left(\cosh(\infty) - \cosh(0)\right) = \infty$$

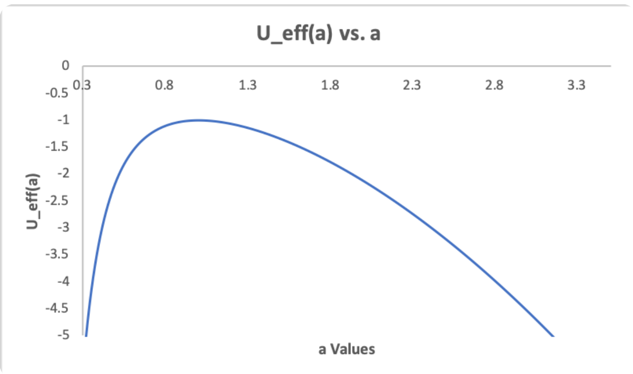
> Integrating the above differential equation yields an undetermined surface and thus it is imaginary and cannot be used to map a 2D surface to a 3D surface.

Problem 4

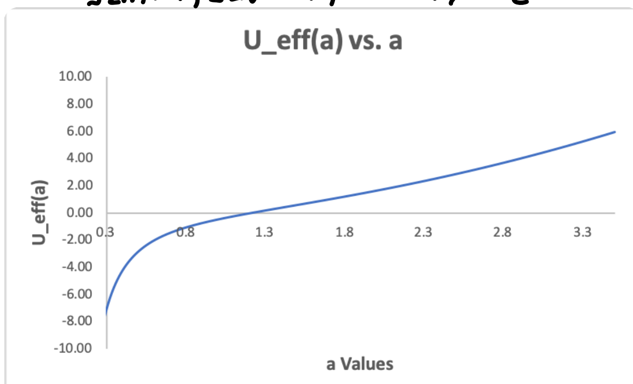
$$\frac{1}{2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 + U_{eff}(\tilde{a}) = \frac{\Omega_c}{2} \qquad U_{eff}(\tilde{a}) = -\frac{1}{2}\left(\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} + \frac{\Omega_r}{\tilde{a}^2}\right)$$

$$\Omega_v + \Omega_m + \Omega_r + \Omega_c = 1$$
$$0 \le \Omega_m, \Omega_r \le 1$$

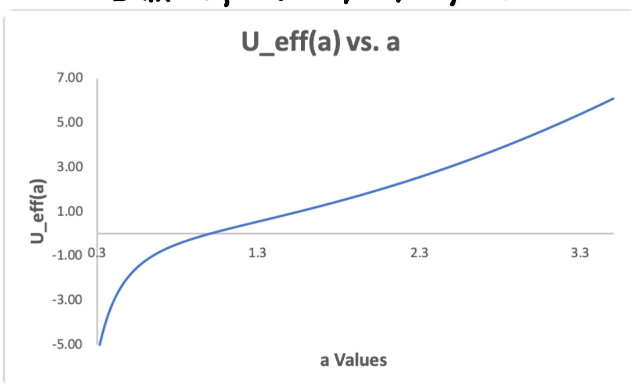
a.) $\Omega_m = 0, \Omega_v = 1, \Omega_r = 1, \Omega_c = -1$



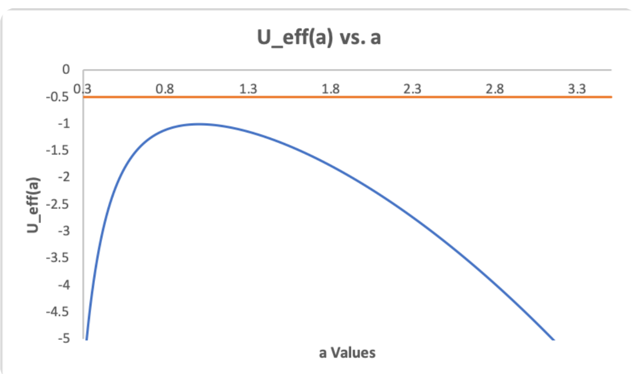
$\Omega_m = 1, \Omega_v = -1, \Omega_r = 1, \Omega_c = 0$



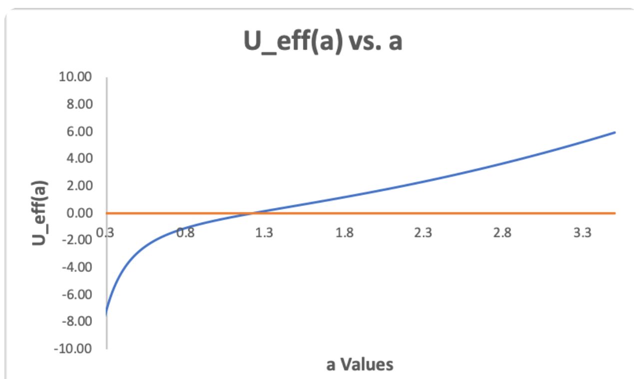
$\Omega_m = 0, \Omega_v = -1, \Omega_r = 1, \Omega_c = 1$



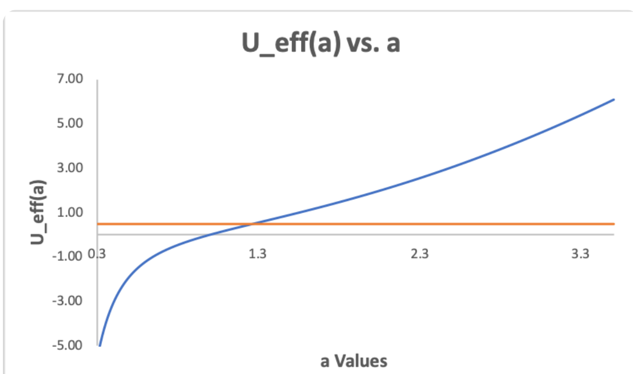
b.) $\Omega_m = 0, \Omega_v = 1, \Omega_r = 1, \Omega_c = -1$



$\Omega_m = 1, \Omega_v = -1, \Omega_r = 1, \Omega_c = 0$



$\Omega_m = 0, \Omega_v = -1, \Omega_r = 1, \Omega_c = 1$



Problem 4 Continued

c.) $U_{eff}(\tilde{a}) = \frac{-1}{2}\left(\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} + \frac{\Omega_r}{\tilde{a}^2}\right)$

$$\frac{du}{d\tilde{a}} = \frac{-1}{2}\left(2\Omega_v \tilde{a} - \frac{\Omega_m}{\tilde{a}^2} - \frac{2\Omega_r}{\tilde{a}^3}\right) = 0$$

$$2\Omega_v \tilde{a} - \frac{\Omega_m}{\tilde{a}^2} - \frac{2\Omega_r}{\tilde{a}^3} = 0 \quad : \quad 2\Omega_v \tilde{a}^4 - \Omega_m \tilde{a} - 2\Omega_r = 0$$

$$2\tilde{a}\left(\tilde{a}^3 \cdot \Omega_v - 0.5\Omega_m\right) = 2\Omega_r \qquad \tilde{a} = \Omega_r \longrightarrow \text{Maximum}$$

$$\tilde{a}^3 \cdot \Omega_v = 4\Omega_r + \Omega_m \qquad \tilde{a} = \left(\frac{4\Omega_r + \Omega_m}{\Omega_v}\right)^{1/3} \longrightarrow \text{Minimum}$$