Ch. 5 Magnetostatics

10-19-18

Up to now, we only considered Electrostatics (Source Charges @ rest), now allow for the charges to be in motion.

Magnetic Forces

Consider a charge Q moving w/velocity $\vec{v}$ in a $\vec{B}$ field...

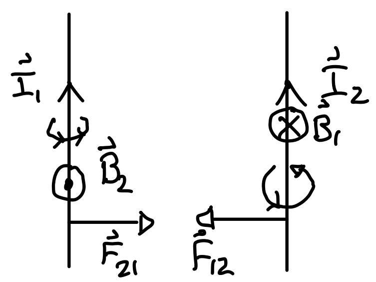
$$\vec{F}_{mag} = Q(\vec{v}\times\vec{B})$$

Also allow for $\vec{E}$-Field...

$$\vec{F} = Q\vec{E} + Q(\vec{v}\times\vec{B})$$

$$\boxed{\vec{F} = Q[\vec{E} + (\vec{v}\times\vec{B})]}$$ - The Lorentz Force law

Consider 2 current carrying wires ....



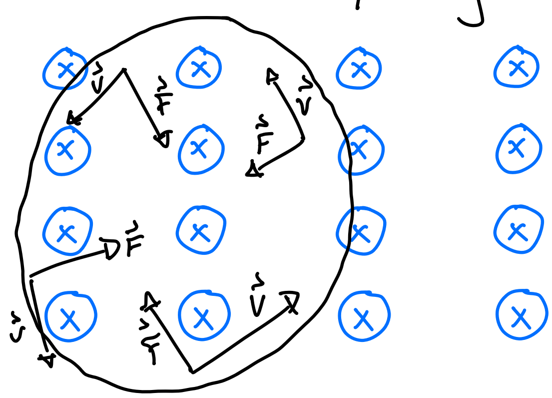
$\vec{B}$ Field is determined by the RHR

∴ 2 current carrying wires w/ $\vec{I}_1 \propto \vec{I}_2$ attract

∴ 2 current carrying wires w/ $\vec{I}_1 \propto \vec{I}_2$ Repel

i.e Cyclotron Motion

Consider a uniform $\vec{B}$-Field pointing into the page $\vec{B} = -B_0\hat{z}$



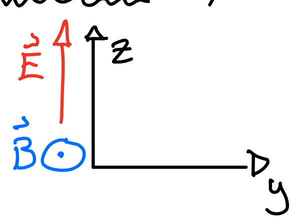
$$F_{net} = m\vec{a}$$

$$\vec{Q}\times\vec{B} = \frac{mv^2}{r}$$

$$\boxed{mv = QBr}$$

If we know Q and B, measure r, determine mv

i.e 5.2 Cycloid Motion Consider a uniform $\vec{E}:\vec{B}$ field $\vec{E} = E\hat{z}$: $\vec{B} = B\hat{x}$



A particle of Charge Q is initially at rest. $V(t=0) = 0$
What path will it follow?

Now

$F = m\vec{a}$ where $\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$

Note: The Force is never in the $\hat{x}$ direction, Since $v(t=0)=0$, the velocity will never be in the x-direction so,

$$\vec{v}(t) = \dot{y}\hat{y} + \dot{z}\hat{z}$$

So

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = \hat{x}(0) + \hat{y}(\dot{z}\cdot B) + \hat{z}(-\dot{y}B) \qquad \dot{f} = \frac{dF}{dt}$$

$$\vec{F} = Q(E\hat{z} + \dot{z}B\hat{y} - \dot{y}B\hat{z}) = QB\dot{z}\hat{y} + Q(E-\dot{y}B)\hat{z} = m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z})$$

$$m\ddot{x} = 0, \quad m\ddot{y} = QB\dot{z}, \quad m\ddot{z} = Q(E-B\dot{y})$$

$$x(t) = 0$$

$$\ddot{y} = \frac{QB\dot{z}}{m}, \quad \ddot{z} = \frac{Q(E-B\dot{y})}{m} \qquad \text{definitions: } \frac{QB}{m} = \omega \quad \frac{QE}{m} = \alpha$$

(*) $$\begin{bmatrix} \ddot{y} = \omega\dot{z} \\ \ddot{z} = \alpha - \omega\dot{y} \end{bmatrix}$$

differentiate both equations

$$\dddot{y} = \omega\ddot{z}, \quad \dddot{z} = -\omega\ddot{y}$$

$$\dddot{y} = \omega(\alpha - \omega\dot{y}) \qquad \dddot{z} = -\omega^2\dot{z}$$

$$\dddot{y} + \omega^2\dot{y} = \alpha\omega \qquad \dddot{z} + \omega^2\dot{z} = 0 \qquad \text{define } f = \dot{z},\ g = \dot{y}$$

$$\ddot{g} + \omega^2 g = \alpha\omega \ (1) \qquad \ddot{f} + \omega^2 f = 0 \ (2)$$

(2) $f = A_1\cos(\omega t) + A_2\sin(\omega t) \Rightarrow \dot{z} = A_1\cos(\omega t) + A_2\sin(\omega t)$

(1) $g = A_3\cos(\omega t) + A_4\sin(\omega t) + \frac{\alpha}{\omega} \quad \dot{y} = A_3\cos(\omega t) + A_4\sin(\omega t) + h$

$\dot{y}(t=0) = A_3 + \frac{\alpha}{\omega} = 0 \quad \therefore A_3 = -\frac{\alpha}{\omega} \qquad \therefore \dot{y} = \frac{\alpha}{\omega}(1-\cos(\omega t) + A_4\sin(\omega t))$

$\dot{z}(t=0) = A_1 = 0 \qquad \therefore \dot{z} = A_2\sin(\omega t)$

Taking derivatives and putting into (*)

$$\ddot{y} = \frac{\alpha}{\omega}(\omega\sin(\omega t) + \omega A_4\cos(\omega t)) = \omega A_2\sin(\omega t) \quad \therefore A_4 = 0 \quad A_2 = \frac{\alpha}{\omega}$$

$$\ddot{z} = A_2\omega\cos(\omega t) = \alpha - \omega\left[\frac{\alpha}{\omega}(1-\cos(\omega t) + A_4\sin(\omega t))\right] = \alpha\cos(\omega t) - \omega A_4\sin(\omega t)$$

$$\boxed{\dot{y} = \frac{\alpha}{\omega}(1-\cos(\omega t)), \quad \dot{z} = \frac{\alpha}{\omega}\sin(\omega t)}$$

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$$dz = \frac{\alpha}{w}\sin(wt)\,dt \quad : \quad \int dz = \frac{\alpha}{w}\int \sin(wt)\,dt$$

$$z = -\frac{\alpha}{w^2}\cos(wt) + A_5$$

$$dy = \frac{\alpha}{w}(1-\cos(wt))dt \; : \; \int dy = \frac{\alpha}{w}\int 1-\cos(wt)\,dt$$

$$y = \frac{\alpha}{w}\left[t - \frac{\sin(wt)}{w}\right] + A_6$$

I.C's

$$z(t=0) = -\frac{\alpha}{w^2} + A_5 = 0 \quad : \quad A_5 = \frac{\alpha}{w^2}$$

$$y(t=0) = A_6 = 0 \quad : \quad A_6 = 0$$

$$\left[\, y(t) = \frac{\alpha}{w^2}(wt - \sin(wt)) \;,\; z(t) = \frac{\alpha}{w^2}(1-\cos(wt)) \,\right]$$ - Curve generated is called a cycloid.

The particle moves as if it were a spot on the rim of a wheel, rolling along the y-axis w/ speed.

Notice: Magnetic forces do no work

Proof: Allow a charge Q to move an amount $d\vec{l} = \vec{v}\,dt$

The work done is .....

$$dW_{mag} = \vec{F}_{mag}\cdot d\vec{l} = Q(\vec{v}\times\vec{B})\cdot\vec{v}\,dt = Q[\vec{B}\cdot(\vec{v}\times\vec{v})]\,dt$$ - Scalar triple product

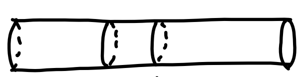
$$dW_{mag} = 0$$

Note: Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.

Currents

$I = \frac{dq}{dt}$ - Current in a wire is the charge per unit time passing a given point.

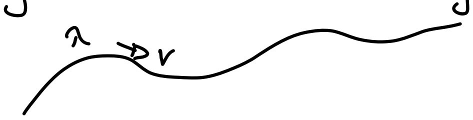
$[I] = \frac{[Q]}{[t]} = \frac{C}{S} = A$, current is a vector...., $I = \frac{dq}{dt} = \frac{dq}{dl}\frac{dl}{dt} = \lambda v \;\therefore\; \boxed{\vec{I} = \lambda\vec{v}}$



|- dl -|

Now, the magnetic force on a segment of a current-carrying wire is....

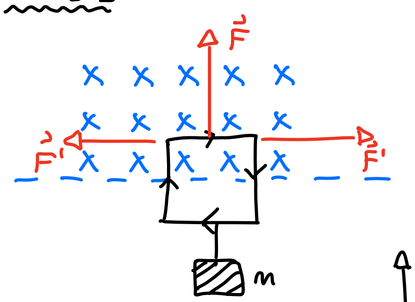
$d\vec{F}_{mag} = dq(\vec{v}\times\vec{B})$, so

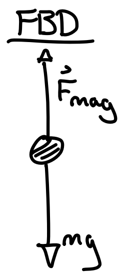


$$\vec{F}_{mag} = \int dq(\vec{v}\times\vec{B}) = \int \lambda\, d\ell(\vec{v}\times\vec{B}) = \underset{\vec{I}=\lambda\vec{v}}{\int d\ell(\vec{I}\times\vec{B})} = \int I(d\vec{l}\times\vec{B})$$

$$\boxed{\vec{F}_{mag} = \int I(d\vec{l}\times\vec{B})}$$

i.e 5.3



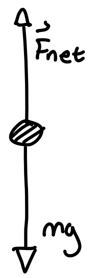


$F_{net_y} = m\cancel{a_y}^{0}$

$IaB - mg = 0$

$$\boxed{I = \frac{mg}{aB}}$$

If $I\uparrow$, $F_{mag}\uparrow$ $\therefore$



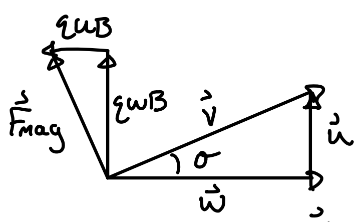
$\therefore \vec{a}\uparrow$

Q: Isn't the magnetic force not doing work?

$W_{mag} = F_{mag}h = IaBh$? $\therefore I = \frac{mg}{aB}$

No!

When the loop starts to rise, the charges no longer move horizontally



Let $\vec{w} = w\hat{x}$ Be the horizontal component of the velocity
$\vec{v} = v\hat{y}$ Be the vertical component of the velocity

Note: $\vec{F}_{mag} \perp$ Displacement

$$\vec{F}_{mag} = \vec{F}_{vert} + \vec{F}_{Hor} = qwB\hat{y} - quB\hat{x}$$

where $\vec{F}_{vert} = qwB = \lambda awB$
$q = \lambda a$ For upper segment of wire

$\vec{F}_{Hor} = quB$ - Opposes the flow of current

$\therefore$ The battery must push against the horizontal force $= \lambda uaB$

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Magnetostatics thus far....

$\vec{F} = Q[\vec{E} + \vec{v}\times\vec{B}]$, $\vec{I} = \lambda\vec{v}$, $\vec{F}_{mag} = \int I(d\vec{l}\times\vec{B})$ - Magnetic force on a segment of current carrying wire

$W_{mag} = F_{mag}h = IaBh$?

$F_{Hor} = \lambda uaB \longleftrightarrow$ opposes the flow of current

$\therefore$ The battery must push against the horizontal force $= \lambda uaB \longrightarrow$ Battery is doing work

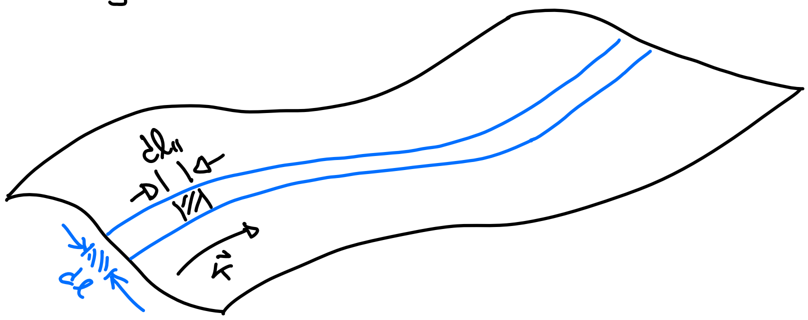
$F_{Hor} = \lambda a u B$

So the work done by the battery against this horizontal force is ....

$$W_{Bat} = \int F\,dx = \int \lambda a u B \cdot w\,dt \quad \text{as} \quad dx = w \cdot dt\,, \quad \lambda = Iw\,, \quad dh = u\,dt$$
$$= \int I a u B\,dt = \int I a B\,dh = I a B h$$

$W_{Bat} = IaBh$

Consider charge flowing over a surface .....



Define the surface current density to be .....

$$\left[\vec{K} \equiv \frac{d\vec{I}}{d\ell_{\perp}}\right] \text{ - Current per unit width } (\perp \text{ to flow})$$

$$\vec{K} = \frac{d\vec{I}}{d\ell_{\perp}} = \frac{d}{d\ell_{\perp}}\left(\frac{d\vec{q}}{dt}\right) = \underbrace{\frac{d}{d\ell_{\perp}}\left(\frac{dq}{d\ell_{\parallel}}}_{\frac{dq}{da}}\frac{d\ell_{\parallel}}{dt}\right) = \sigma\vec{v} \rightarrow \left[\vec{K} = \sigma\vec{v}\right]$$

Surface charge density (σ)

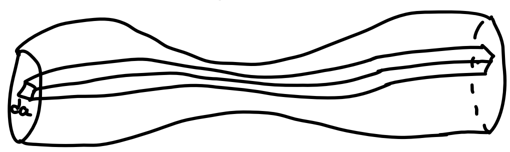
surface current density ($\vec{K} = \sigma\vec{v}$)

The magnetic force on the surface current ....

$$\vec{F}_{mag} = \int dq\,(\vec{v} \times \vec{B}) = \int \sigma(\vec{v} \times \vec{B})\,da = \int (\vec{K} \times \vec{B})\,da$$

$dq = \sigma\,da$

Consider charge flowing through a volume



Define the volume current density to be ...

$$\left[\vec{J} \equiv \frac{d\vec{I}}{da_{\perp}}\right] \text{ - Current per unit area } (\perp \text{ to flow})$$

Now,

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}} = \frac{d}{da_{\perp}}\left(\frac{d\vec{q}}{dt}\right) = \frac{d}{da_{\perp}}\left(\frac{dq}{d\ell_{\parallel}}\frac{d\ell_{\parallel}}{dt}\right) = \frac{dq}{d\tau}\vec{v} = \rho\vec{v}\,, \quad \left[\vec{J} = \rho\vec{v}\right] \text{ - Volume Current density}$$

Volume charge density (ρ)

The magnetic force on the volume current .....

$$\vec{F}_{mag} = \int dq\,(\vec{v} \times \vec{B}) = \int \rho(\vec{v} \times \vec{B})\,d\tau = \int (\vec{J} \times \vec{B})\,d\tau$$

$dq = \rho\,d\tau$

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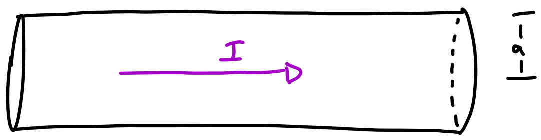
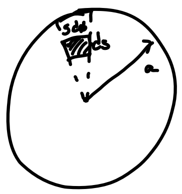
i.e 5.4

a)



$$J = \frac{dI}{da_\perp} = \frac{I}{\pi a^2}$$

b.) $J = KS$



$$J = \frac{dI}{da_\perp} \quad \therefore \quad \int dI = \int J\, da_\perp \; ; \; I = \int J\, da_\perp : \; I = K\int_0^{2\pi}\int_0^{a} s^2\, ds\, d\phi$$

$$I = K \cdot 2\pi \cdot \frac{s^3}{3} = K \cdot 2\pi \frac{s^3}{3} \qquad I = 2\pi k \cdot \frac{s^3}{3}$$

The current crossing a surface S

$$I = \int_S J\, da_\perp = \int_S \vec{J} \cdot d\vec{a}$$

So the total charge per unit time (I) leaving a volume $\mathcal{V}$ is .....

$$\oint_S \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{J}\, d\tau$$

As charge is conserved, this comes at the expense of the charge inside the volume...

$$\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{J})\, d\tau = -\frac{dq}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \frac{\partial}{\partial t}\rho\, d\tau$$

Since this applies to any volume $\mathcal{V}$, we have

$$\left[\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}\right] \text{— Continuity equation}$$

The Biot-Savart Law

Stationary charges ⟶ Gave rise to constant $\vec{E}$ fields (Electrostatics)
Steady currents ⟶ Gave rise to constant $\vec{B}$-Field (Magnetostatics)

When a steady current flows in a wire, the current "I" must be the same all along the wire.

$$\frac{\partial \rho}{\partial t} = 0 \quad \therefore \quad \vec{\nabla} \cdot \vec{J} = 0$$

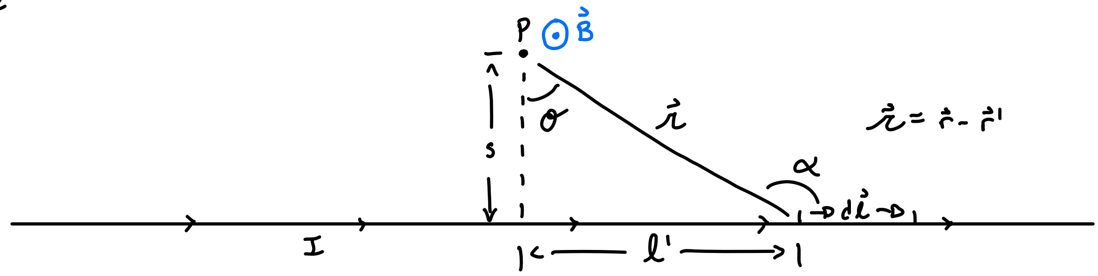
$\vec{B}$- Field of a Steady Current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$ — Biot Savart Law —(Analogous to Coulombs law)

Note:

- Integration along the current path in direction of flow
- $d\vec{l}'$ is an element of Length along wire
- $\hat{\mathscr{r}}$ is a vector From Source to field point
- $\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$ - Permeability of free space

i.e. 5.5



$$d\vec{l}' \times \vec{\mathscr{r}} = dl' \cdot \mathscr{r} \cdot \sin(\alpha) \odot = dl' \sin(180-\alpha) \odot = dl' \cos(\theta) \odot$$

$$\sin(180-\alpha) = \cos(\theta)$$

I want to integrate over θ vs. over l' ....

$$\tan\theta = \frac{l'}{s} \;\therefore\; l' = s \cdot \tan\theta, \quad dl' = s \cdot \sec^2\theta \, d\theta \qquad \cos\theta = \frac{s}{\mathscr{r}} \;\therefore\; \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$$

$$\frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{dl' \cos\theta}{\mathscr{r}^2} \odot = s \cdot \sec^2\theta \cdot \cos\theta \cdot \frac{\cos^2\theta}{s^2} = \frac{\cos\theta \, d\theta}{s} \odot$$

$$\vec{B}(s) = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos(\theta)}{s} d\theta \odot = \frac{\mu_0 I}{4\pi} \cdot \frac{1}{s} \odot \int_{\theta_1}^{\theta_2} \cos(\theta) \, d\theta$$

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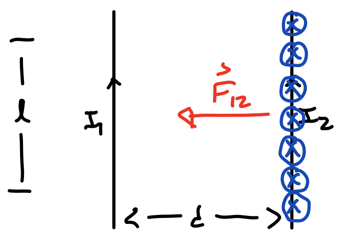
$$\vec{B}(s) = \frac{\mu_0 I}{4\pi s} \odot \sin\theta \Big|_{\theta_1}^{\theta_2} = \frac{\mu_0 I}{4\pi} \frac{1}{s} \odot \left(\sin\theta_2 - \sin\theta_1\right)$$

To obtain an infinitely long wire, let $\theta_1 \to -\pi/2$, $\theta_2 \to \pi/2$

$$\vec{B}_{wire}(\vec{r}) = \frac{\mu_0 I}{4\pi s} \odot \left(\sin(\pi/2) - \sin(-\pi/2)\right) = \frac{\mu_0 I}{2\pi s} \qquad \boxed{\vec{B}_{wire}(\vec{r}) = \frac{\mu_0 I}{2\pi s} \odot}$$

Now consider two long current carrying wires......



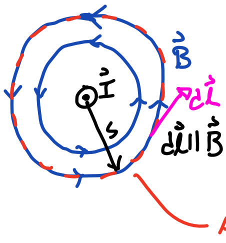
$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi d} \otimes \qquad \vec{F}_{12} = I_2 \vec{l} \times \vec{B}_1$$

$$\vec{F}_{12} = I_2 l B_1 (-\hat{x}) = \frac{\mu_0 I_1 I_2}{2\pi d} l (-\hat{x}) \; : \; F \to \infty \text{ as } l \to \infty$$

$$f \equiv \frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$$ Force per unit length

The Divergence and Curl of $\vec{B}$

Consider the $\vec{B}$-Field of an ∞ straight wire, w/ a steady current I....



Using cylindrical coordinates w/ z coming out of the board......

$d\vec{l} \parallel \vec{B}$ $\quad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

Calculate the line integral of $\vec{B}$ around a circular path of radius s

$$\oint \vec{B}\cdot d\vec{l}, \quad d\vec{l} = s\, d\phi\, \phi, \quad \vec{B} = \frac{\mu_0 I}{2\pi s}\phi \quad \therefore \quad \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\hat{\phi}\cdot s\, d\phi\, \phi = \frac{\mu_0 I}{2\pi}d\phi$$

$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi}d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$

Now suppose we have a bundle of straight wires.

Each wire contributes $\mu_0 I$ – (That passes through the loop)

So,

(*) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{ENC}$ Total current enclosed by the integration path

now

$I_{ENC} = \int \vec{J}\cdot d\vec{a}$

So,

$\oint \vec{B}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{B})\cdot d\vec{a}$ – By Stokes' theorem

So (*) Becomes.....

$$\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\int\vec{J}\cdot d\vec{a}$$

so

$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$ – Ampere's law in differential form

$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{ENC}}$ – Ampere's law in integral form

Note: This is the general formula for the Curl of $\vec{B}$

The Divergence of $\vec{B}$

The Biot-Savart law For the general case of a volume current is....

$$B(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{J(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$\vec{B}$-Field at a Field point $\vec{r}$ in terms of an integral over the current distribution $\vec{J}$ @ Source point.

Now the divergence of $\vec{B}$ (W.R.T unprimed coordinates)

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau$$

Now using product rule (6)

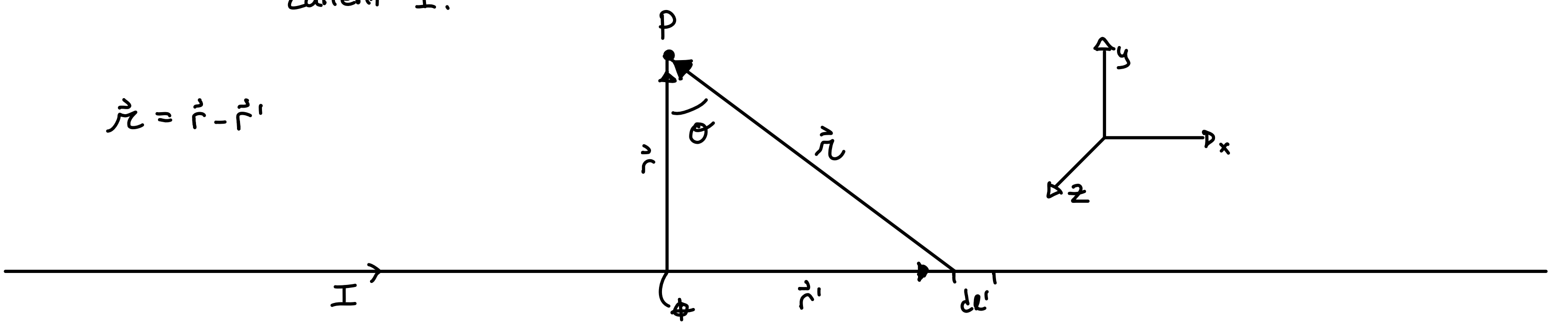
$$\vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\overset{0}{\cancel{(\vec{\nabla}\times\vec{J})}} - \vec{J}\cdot\overset{0}{\cancel{\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}} = 0$$

$\vec{J}$ isn't a function of $(\vec{r})$, its $(\vec{r}')$

$$\therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

i.e 5.5 In Depth: Find the $\vec{B}$-Field a distance s from a long straight wire carrying a steady current I.

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$



So: $\vec{r} = s\hat{y}$, $\vec{r}\,' = x'\hat{x}$ now $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (0, s) - (x', 0) = -x'\hat{x} + s\hat{y}$ $\therefore \mathscr{r} = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}} = \sqrt{s^2 + x'^2}$

$\hat{\mathscr{r}} = \dfrac{-x\hat{x} + s\hat{y}}{\sqrt{s^2+x^2}}$ $\quad d\vec{l}\,' = dx'\hat{x}$

$\vec{B}$-Field @ $\vec{r}$ is ....

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,'\times\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \text{where} \quad d\vec{l}\,'\times\hat{\mathscr{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{s^2+x'^2}} & \frac{s}{\sqrt{s^2+x^2}} & 0 \end{vmatrix} = \hat{z}\left(\frac{s}{\sqrt{s^2+x'^2}}\,dx'\right)$$

$\mathscr{r}^2 = (s^2 + x'^2)$

$$\text{So,} \quad \frac{d\vec{l}\,'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \hat{z}\left(\frac{s}{\sqrt{s^2+x'^2}}\,dx'\right)\cdot\frac{1}{(s^2+x'^2)} = \hat{z}\left(\frac{s}{(s^2+x'^2)^{3/2}}\,dx'\right)$$

$\sin^2\theta + \cos^2\theta = 1$
$\sec^2\theta - \tan^2\theta = 1$
$\csc^2\theta - \cot^2\theta = 1$

So,

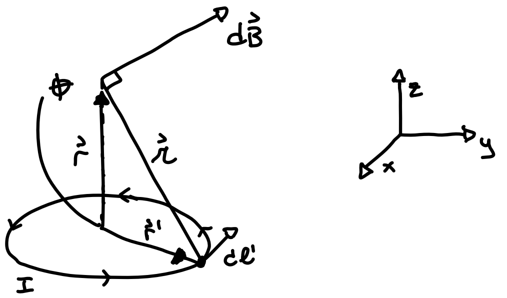
$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\cdot s\cdot\hat{z}\int_{-\infty}^{\infty}\frac{dx'}{(s^2+x'^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\cdot s\cdot\hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{s\cdot\sec^2\theta}{(s^2+s^2\tan^2\theta)^{3/2}}\,d\theta$$

$x' = s\tan\theta$ $\quad$ now $x' \to \infty$ when $\theta = \pi/2$
$dx' = s\cdot\sec^2\theta$ $\quad x' \to \infty$ when $\theta = -\pi/2$

$$= \frac{\mu_0 I s}{4\pi}\hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{s\cdot\sec^2\theta}{(s^2\sec^2\theta)^{3/2}}\,d\theta = \frac{\mu_0 I s}{4\pi}\hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{s}{s^3}\cdot\frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \frac{\mu_0 I}{4\pi s}\hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\sec\theta}\,d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\hat{z}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}\hat{z}\cdot\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\mu_0 I}{2\pi s}\hat{z} \qquad \boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}\hat{z}}$$

i.e 5.6 In Depth : Find $B(\vec{r})$ a distance z above the center of a circular loop of radius R, which carries a steady current I.



$$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

Now: $\vec{r} = z\hat{z}$ $\quad \vec{\mathscr{r}} = z\hat{z} - R\hat{s} = -R\hat{s} + z\hat{z}$ $\quad \therefore \mathscr{r} = \sqrt{R^2+z^2}$ $\quad \therefore \mathscr{r}^2 = z^2+R^2$

$\vec{r}' = R\hat{s}$ $\quad$ So, $\hat{\mathscr{r}} = \dfrac{-R\hat{s}+z\hat{z}}{\sqrt{z^2+R^2}}$ $\quad d\vec{l}' = R\,d\phi\,\hat{\phi}$

Now

(*)
$$d\vec{l}' \times \hat{\mathscr{r}} = R\,d\phi\,\hat{\phi} \times \left(\frac{-R\hat{s}+z\hat{z}}{\sqrt{z^2+R^2}}\right) = \frac{Rz\,d\phi}{\sqrt{z^2+R^2}}(\phi \times \hat{z}) - \frac{R^2 d\phi}{\sqrt{z^2+R^2}}(\hat{\phi}\times\hat{s})$$

Now: $\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ $\quad$: $\quad \hat{\phi}\times\hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\phi = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

$$\hat{\phi}\times\hat{s} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = (-\sin^2\phi - \cos^2\phi)\hat{z} = -\hat{z}$$

$$d\vec{l}' \times \hat{\mathscr{r}} = \frac{Rz\,d\phi}{\sqrt{z^2+R^2}}(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \frac{R^2 d\phi}{\sqrt{z^2+R^2}}\hat{z}$$

$$\text{so, } \vec{B}(s\hat{z}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\left[\frac{Rz\,d\phi}{(z^2+R^2)^{3/2}}(\cos\phi\,\hat{x}+\sin\phi\,\hat{y}) + \frac{R^2 d\phi}{(z^2+R^2)^{3/2}}\hat{z}\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{Rz}{(z^2+R^2)^{3/2}}\left(\hat{x}\overbrace{\cancel{\int_0^{2\pi}\cos\phi\,d\phi}}^{0} + \hat{y}\overbrace{\cancel{\int_0^{2\pi}\sin\phi\,d\phi}}^{0}\right) + \frac{R^2}{(z^2+R^2)^{3/2}}\hat{z}\int_0^{2\pi}d\phi\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{R^2}{(z^2+R^2)^{3/2}}\hat{z}\cdot 2\pi\right] = \frac{\mu_0 I}{2}\cdot\frac{R^2}{(z^2+R^2)^{3/2}} \qquad \boxed{\vec{B}(sz) = \frac{\mu_0 I}{2}\cdot\frac{R^2}{(z^2+R^2)^{3/2}}\hat{z}}$$

11-2-18

Last time....

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I(d\vec{l}'\times\hat{\mathscr{r}})}{\mathscr{r}^2}$ - Biot-Savart , $\quad \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{Enc}$ - Amperes law in $\int$ Form

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ - Amperes law in d/dx Form

Applications of Ampere's Law

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{Enc}$ where $I_{Enc} = \int \vec{J}\cdot d\vec{a}$ - Total current passing through the surface (I enclosed by Amperian Loop)

- Valid for arbitrary steady currents

Analogy to Electrostatics . . . . .

Electrostatics : Coulomb's law $\longrightarrow$ Gauss' Law

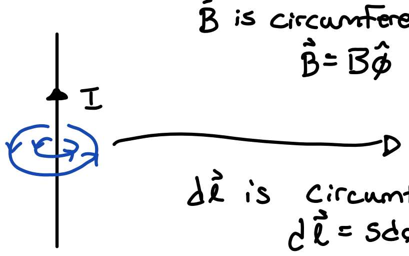
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}} \longrightarrow \oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V \rho(\vec{r}\,')\,d\tau'$$

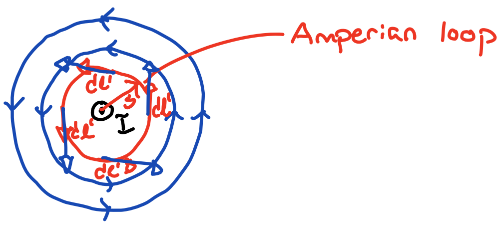
Magnetostatics : Biot - Savart law $\longrightarrow$ Ampere's law

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' \longrightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu_0\int_S \vec{J}\cdot d\vec{a}$$

i.e 5.7

$\vec{B}$ is circumferential,
$\vec{B} = B\hat{\phi}$





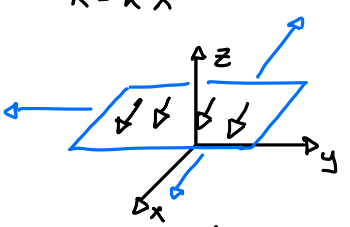
$d\vec{l}$ is circumferential
$d\vec{l} = s\,d\phi\,\hat{\phi}$

$$\oint \vec{B}\cdot d\vec{l} = \int_0^{2\pi} Bs\,d\phi = Bs\phi\Big|_0^{2\pi} = B\cdot 2\pi s$$

$$B\cdot 2\pi s = \mu_0 I \quad \therefore \left[\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}\right]$$

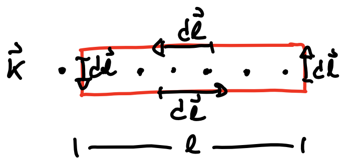
i.e 5.8

$\vec{K} = K\hat{x}$



Direction of $\vec{B}$?

- No x-component ($\perp$ to $\vec{J}$)
- No z-Component (Assume + B_z Component, switch $\vec{J}$ direction $\therefore$ $-B_z$ component, now look for other direction...)
- $\vec{B}$ must only have a component



$$\oint_C \vec{B}\cdot d\vec{l} = \oint_{Top}\vec{B}\cdot d\vec{l} + \oint_{Bottom}\vec{B}\cdot d\vec{l} = 2\int_{Top}\vec{B}\cdot d\vec{l} = 2\int_{Top} B\,dl$$

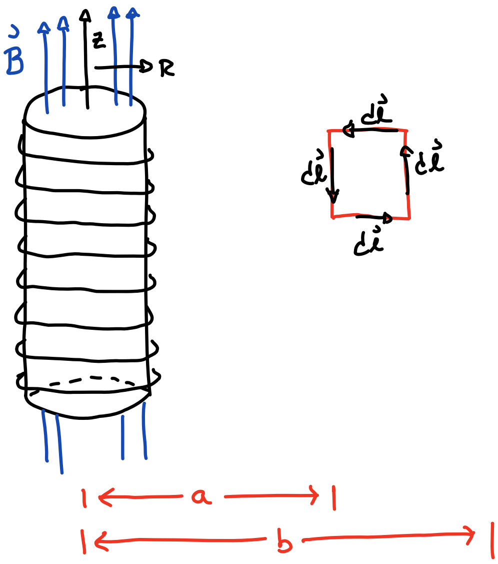
$$2B\int dl = 2Bl \qquad K = \frac{dI}{dl_\perp} \;\therefore\; dI = K\,dl \;\therefore\; I = Kl$$

Now by amperes law . . . .

$$2Bl = \mu_0 I_{enc}$$
$$2Bl = \mu_0 K l$$
$$B = \frac{\mu_0 K}{2}$$

so
$$\left[\begin{aligned} \vec{B} &= \frac{\mu_0 K}{2}\hat{y} \quad \text{for } z<0 \\ &= -\frac{\mu_0 K}{2}\hat{y} \quad \text{for } z>0 \end{aligned}\right]$$

i.e 5.9



Consider L_1 First

$$\oint_C \vec{B}\cdot d\vec{l} = \int_{s=a} \vec{B}\cdot d\vec{l} + \int_{s=b} \vec{B}\cdot d\vec{l} = -\int_{s=a} B\,dl + \int_{s=b} B\,dl$$

$$= -B\int_{s=a} dl + B\int_{s=b} dl = -B(s=a)\,l + B(s=b)\,l$$

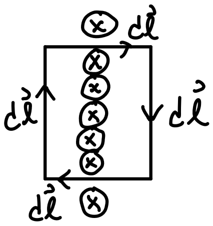
Amperes law,

$$-B(s=a)\,l + B(s=b)\,l = 0 \quad \therefore \quad B(s=a) = B(s=b) = 0$$

$$\lim_{s\to\infty} B(s) \to 0$$

11-5-18

Now consider L_2 .....

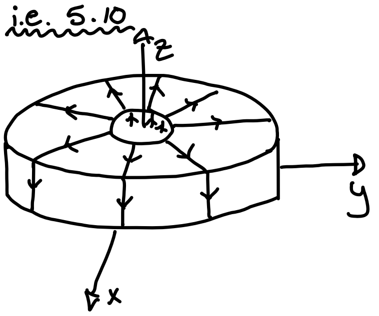


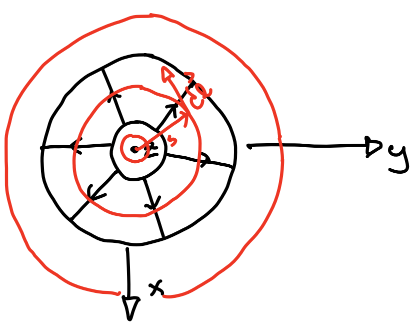
$$\oint \vec{B}\cdot d\vec{l} = \int_{\text{Inside}} \vec{B}\cdot d\vec{l} = \int B\,dl = B\int dl = Bl$$

By ampere's law .....

$$Bl = \frac{\mu_0 I_{Enc}}{l} = \mu_0 N I \quad \text{or} \quad \left[\begin{array}{ll} \vec{B} = \mu_0 N I\,\hat{z} & s<R \\ = 0 & s>R \end{array}\right]$$

i.e. 5.10





$\vec{B} = B\hat{\phi}$

Now, using Amperes law

$$\oint \vec{B}\cdot d\vec{l} = \oint Bs\,d\phi = Bs\int_0^{2\pi} d\phi = B\cdot 2\pi s$$

$\vec{B} = B\hat{\phi}$

$d\vec{l} = s\,d\phi\,\hat{\phi} \quad \therefore \quad \vec{B}\cdot d\vec{l} = Bs\,d\phi$

$$= \mu_0 N I = \mu_0 I_{Enc}$$

$$\left[\begin{array}{ll} \vec{B} = \dfrac{\mu_0 N I}{2\pi s}\hat{\phi} & \text{Inside coil} \\ = 0 & \text{Outside coil} \end{array}\right]$$

Comparison of Magnetostatics : Electrostatics

For Electrostatics .....

$$\left.\begin{array}{l} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \text{ - Gauss' Law} \\ \nabla\times\vec{E} = 0 \end{array}\right\}$$ Maxwell's equations for Electrostatics
↳ Determine $\vec{E}$ if ρ is Given $\lim_{r\to\infty} \vec{E}\to 0$

• Same info as Coulomb's law + principle of superposition

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

For Magnetostatics .....

$$\left.\begin{aligned}\vec{\nabla}\times\vec{B} &= \mu_0\vec{J} \text{ - Ampere's law}\\ \vec{\nabla}\cdot\vec{B} &= 0\end{aligned}\right\}$$

Maxwell's equations for magnetostatics $\longrightarrow$ Determine $\vec{B}$ if $\vec{J}$ is given w/ $\lim_{r\to\infty} \vec{B}\to 0$

These plus the Lorentz Force law .....

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

Completely describe Electro & Magnetostatics

<u>Magnetic Vector Potential</u>

Scaler potential

Ambiguity

Since $\vec{\nabla}\times\vec{E} = 0$, $\vec{E} \equiv -\vec{\nabla}V$ notice $V \to V+C$, Since $\vec{E} = -\vec{\nabla}V \to -\vec{\nabla}(V+C) = -\vec{\nabla}V = \vec{E}$

Since $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{B} \equiv \vec{\nabla}\times\vec{A}$

vector Potential since $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ For any $\vec{A}$

Notice : $\vec{A}$ also has a built in Ambiguity ....

$\vec{A} \to \vec{A} + \vec{\nabla}\lambda$ since $\vec{B} = \vec{\nabla}\times\vec{A} \longrightarrow \vec{\nabla}\times(\vec{A}+\vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} + \cancelto{0}{\vec{\nabla}\times\vec{\nabla}\lambda} = \vec{\nabla}\times\vec{A} = \vec{B}$

Now, plugging this definition of $\vec{A}$ into Amperes Law yields .....

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$
$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot A) - \nabla^2\vec{A} = \mu_0 J$$

Note: we can always use the ambiguity in $\vec{A}$ to set $[\vec{\nabla}\cdot\vec{A} = 0]$

$$\vec{\nabla}\cdot\vec{A} \to \vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\lambda) = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda$$

Now, choose λ to satisfy

$$\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}$$

Exploiting this freedom to choose the $\vec{\nabla}\cdot\vec{A} = 0$, yields

$$\left[\nabla^2\vec{A} = -\mu_0\vec{J}\right]$$ - Poisson's equation (one for each component)

$$\left[\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\,d\tau'}{\mathscr{r}}\right]$$

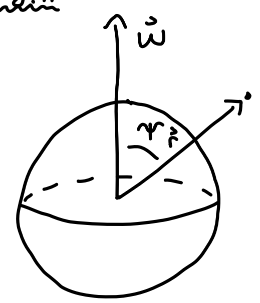
so long as $\vec{J}$ goes to zero at ∞

Compare to .....

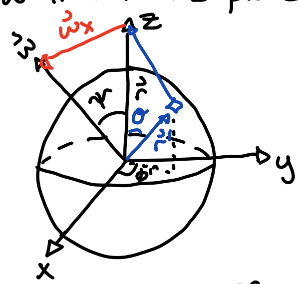
$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}, \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')\,d\tau'}{\mathscr{r}}$$

11-7-18

[illegible]



Rotate the axis such that
- z-axis is aligned w/ $\vec{r}$
- $\vec{\omega}$ lies in xz plane



where

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{K(\vec{r}')}{\mathscr{r}}\, da'$$

$\vec{r}' = R\hat{r},\ \vec{r} = r\hat{z}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = r\hat{z} - R\hat{r}$: $\mathscr{r}^2 = (r\hat{z} - R\hat{r})\cdot(r\hat{z} - R\hat{r}) = R^2 + r^2 - 2rR\,\hat{r}\cdot\hat{z}$

$|\vec{\mathscr{r}}| = \sqrt{R^2+r^2-2rR\cos\theta}$ $\qquad \hat{r}\cdot\hat{z} = \cos\theta$

$= R^2+r^2-2rR\cos\theta$

$\vec{K} = \sigma\vec{v}$, $\vec{v} = \vec{\omega}\times\vec{r}$

$\vec{\omega} = \vec{\omega}_x + \vec{\omega}_z = \omega\sin(\gamma)\hat{x} + \omega\cos(\gamma)\hat{z} = \omega(\sin(\gamma)\hat{x} + \cos(\gamma)\hat{\phi})$

$\vec{r}' = R\sin\theta\cos\phi\,\hat{x} + R\sin\theta\sin\phi\,\hat{y} + R\cos\theta\,\hat{z}$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin(\gamma) & 0 & \omega\cos(\gamma) \\ R\sin\theta\cos\phi & R\sin\theta\sin\phi & R\cos\theta \end{vmatrix}$$

$\vec{\omega}\times\vec{r} = \hat{x}(-R\omega\cos(\gamma)\sin\theta\sin\phi) + \hat{y}R\omega(\cos(\gamma)\sin\theta\cos\phi - \sin(\gamma)\cos\theta) + \hat{z}(R\omega\sin(\gamma)\sin\theta\sin\phi)$

$$\vec{A}(\vec{r}') = \frac{\mu_0}{4\pi}\sigma R\omega\cdot R^2 * \int_0^{2\pi}\int_0^{\pi}\Big[\hat{x}(\underbrace{-\cos(\gamma)\sin\theta\sin\phi}_{0}) + \hat{y}(\underbrace{\cos(\gamma)\sin\theta\cos\phi}_{0} - \sin(\gamma)\cos\theta) + \hat{z}(\underbrace{\sin(\gamma)\sin\theta\sin\phi}_{0})\Big]\sin\theta\, d\theta\, d\phi$$

$$\vec{A}(\vec{r}') = \frac{\mu_0}{4\pi}\sigma\cdot R^3\omega\sin(\gamma)\hat{y}\int_0^{2\pi}\int_0^{\pi}\frac{\cos\theta\sin\theta}{\sqrt{R^2+r^2-2rR\cos\theta}} = \frac{1}{2}\mu_0\sigma\omega R^3\sin(\gamma)\hat{y}\underbrace{\int_1^{-1}\frac{u\,du}{\sqrt{R^2+r^2-2rRu}}}_{I}$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\left[\begin{aligned} I &= -\frac{2}{3}\frac{r}{R^2} && \text{for } r<R \\ &= \frac{-2R}{3r^2} && \text{for } r>R \end{aligned}\right]$$

Also note that

$$-\omega r\sin(\gamma)\hat{y} = \vec{\omega}\times\vec{r}$$

So,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sigma\omega\frac{R^2}{r}\,\vec{\omega}\times\vec{r}\int I$$

$$= -\frac{1}{2}\mu_0\sigma\frac{R^3}{r}(\vec{\omega}\times\vec{r})\cdot -\frac{2}{3}\frac{r}{R^2} \quad \text{Inside}$$

$$= \frac{1}{3}\mu_0\sigma R(\vec{\omega}\times\vec{r}) \quad \text{for} \quad r<R$$

$$\vec{A}(\vec{r}) = -\frac{1}{2}\mu_0\sigma\frac{R^3}{r}(\vec{\omega}\times\vec{r})\cdot -\frac{2}{3}\frac{R}{r^2} \quad \text{For } r>R$$

$$= \frac{1}{3}\mu_0\sigma\frac{R^4}{r^3}(\vec{\omega}\times\vec{r})$$

Now shifting back to $\vec{\omega} = \omega\hat{z}$

then $\vec{\omega}\times\vec{r} = \omega r\sin\theta\,\hat{\phi}$

$$\boxed{\begin{aligned}\vec{A}(r,\theta,\phi) &= \frac{1}{3}\mu_0\sigma R\omega r\sin\theta\,\hat{\phi} & r\le R\\ &= \frac{1}{3}\mu_0\sigma R^4\omega\frac{\sin\theta}{r^2}\,\hat{\phi} & r>R\end{aligned}}$$

Now to get the magnetic Field inside....

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}$$

$$= \frac{1}{3}\mu_0\sigma R\omega\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\theta)\hat{\theta}\right]$$

$$= \frac{1}{3}\mu_0\sigma R\omega\left[2\cos\theta\,\hat{r} - 2\sin\theta\,\hat{\theta}\right]$$

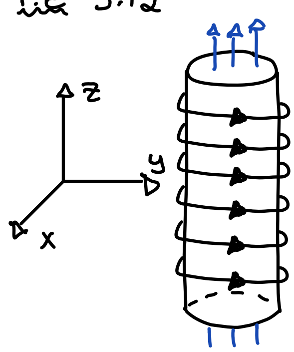
$$= \frac{2}{3}\mu_0\sigma R\omega\left[\cos\theta(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}) - \sin\theta(\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z})\right]$$

$$= \frac{2}{3}\mu_0\sigma R\omega\,\hat{z}$$

$$\boxed{\vec{B} = \frac{2}{3}\mu_0 R\omega\,\hat{z}}$$

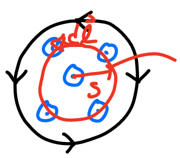
11-9-18

ie 5.12



$n = \frac{N}{l}$

$\vec{B} = \mu_0 n I\,\hat{z}$

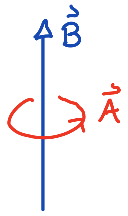


Amperian loop

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\,d\tau'}{r}$$

Magnetic flux through a loop....

$$\Phi_B = \int_S \vec{B}\cdot d\vec{a} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{l}$$



- use an amperian loop of radius s

Inside....

$$\oint \vec{A}\cdot d\vec{l} = \int_0^{2\pi} A s\, d\phi = As\int_0^{2\pi} d\phi = As\cdot 2\pi \quad \text{also} \quad \oint \vec{B}\cdot d\vec{a}$$

$\vec{B} = B\hat{z}$
$d\vec{a} = s\,ds\,d\phi\,\hat{z}$

$\vec{A} = A\hat{\phi}$
$d\vec{l} = s\,d\phi\,\hat{\phi}$
$\vec{A}\cdot d\vec{l} = As\,d\phi$

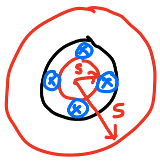
$$\oint \vec{B}\cdot d\vec{a} = B\int_0^{2\pi}\int_0^{s} s'\,ds'\,d\phi = B\pi s^2$$
$$= \mu_0 n I \pi s^2$$

So,

$$2\pi\cdot s\cdot A = \mu_0 n I \pi s^2$$

$$\boxed{\vec{A} = \frac{\mu_0 n I s}{2}\,\hat{\phi}}$$

outside.....



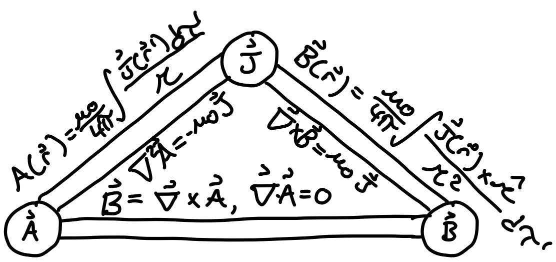
$$\oint \vec{A}\cdot d\vec{l} = A\cdot 2\pi\cdot s$$

$$\oint \vec{B}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{R} B s'\,ds'\,d\phi = B\cdot\pi R^2 = \mu_0 n I\cdot\pi\cdot R^2$$

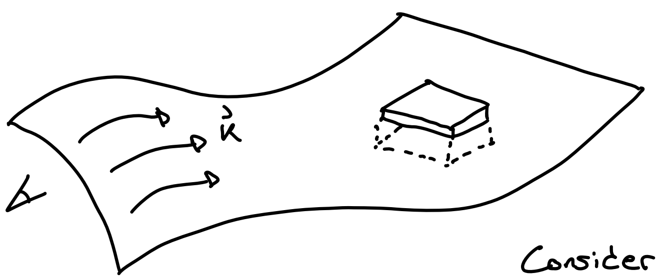
$$A\cdot 2\pi s = \mu_0 n I \pi R^2$$

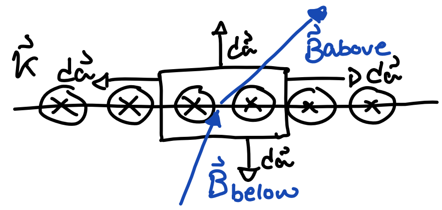
$$\boxed{\vec{A} = \frac{\mu_0 n I R^2}{s}\,\hat{\phi}}$$

<u>Summary</u>



The $\vec{B}$-Field is <u>discontinuous</u> at a surface current......





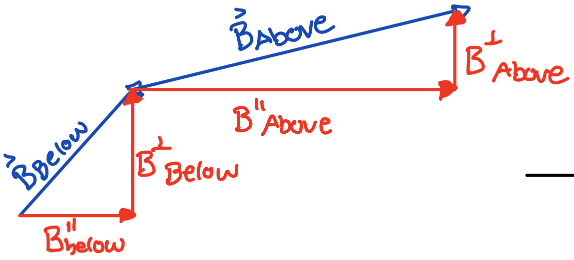
Consider the integral relation....

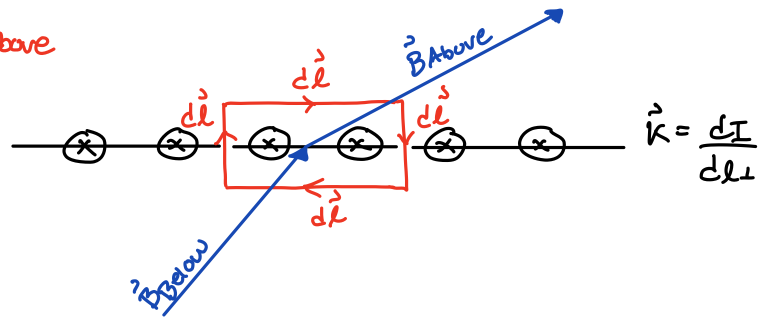
$$\oint \vec{B}\cdot d\vec{a} = 0$$

And a gaussian pillbox ......

So $\oint \vec{B}\cdot d\vec{a} = \int_{above} \vec{B}\cdot d\vec{a} + \int_{below} \vec{B}\cdot d\vec{a} + \cancelto{0}{\int_{sides} \vec{B}\cdot d\vec{a}}$ (In the limit that the pillbox is wafer thin)

$$= \int_{Above} \vec{B}^{\perp}_{above} \cdot d\vec{a} - \int_{below} \vec{B}^{\perp} \cdot d\vec{a} = B^{\perp}_{Above} \int da - B^{\perp}_{Below} \int da = (B^{\perp}_{Above} - B^{\perp}_{Below}) A = 0 \quad \therefore \quad \left[B^{\perp}_{Above} = B^{\perp}_{Below} \right]$$





now consider the integral relation ....

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{ENC}$ and an amperian loop $\perp$ to $\vec{K}$

$$= \int_{Above} \vec{B} \cdot d\vec{\ell} + \int_{Below} \vec{B} \cdot d\vec{\ell} + \underbrace{\int_{Sides} \vec{B} \cdot d\vec{\ell}}_{0} = \int_{Above} B^{\parallel}_{Above} \, d\ell - \int_{Below} B^{\parallel}_{Below} \, d\ell = B_{above}^{\parallel} \int d\ell = (B_{above}^{\parallel} - B_{below}^{\parallel}) \ell$$

$\mu_0 I_{ENC} = \mu_0 K \ell$

$$\left[B_{above}^{\parallel} - B_{below}^{\parallel} = \mu_0 K \right]$$