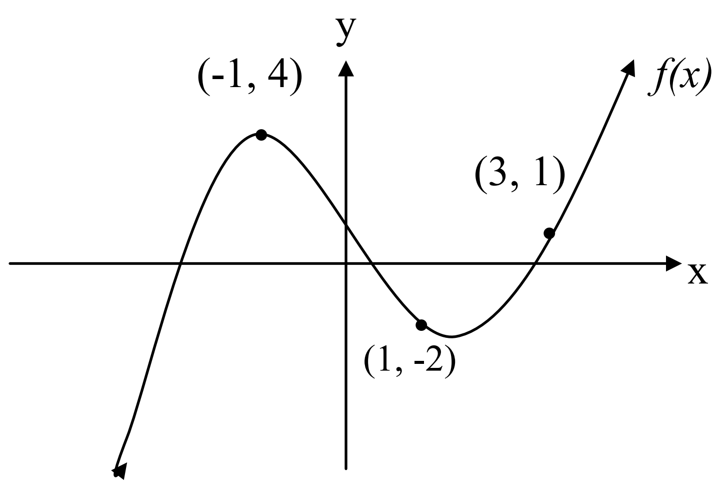
MAT 201
Larson/Edwards – Section 2.1
The Derivative and the Tangent Line Problem

What does it mean when one says that a line is tangent to a curve at a point?

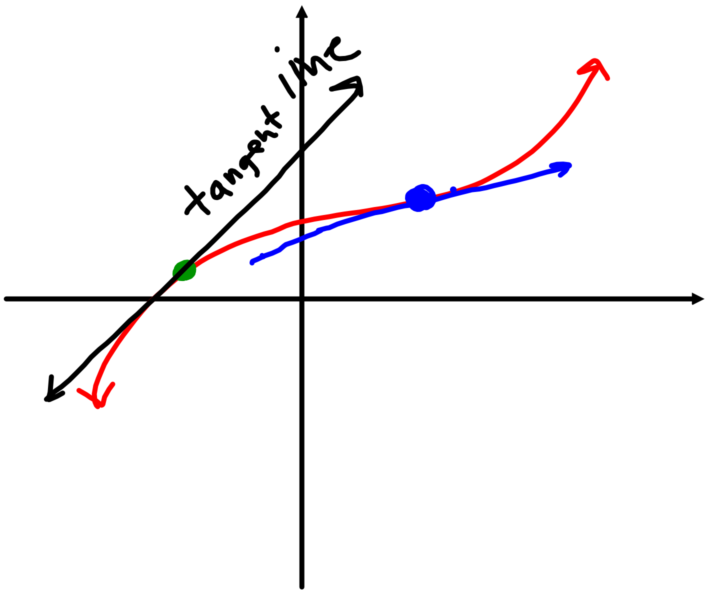
Suppose our function is given in the graph below. How would we sketch the tangent line at the point (-1, 4)? (1, -2)? (3, 1)?



Notice, that if the tangent lines through the points (-1, 4) and (1, -2) were extended, they would intersect the function *f* at another point. You will see that this may, or may not happen when sketching tangent lines.

Also, you may have noticed that the slopes of the tangent line change depending on where the point is on this particular function *f*!

What does it mean when one says that a line is tangent to a curve at a point?



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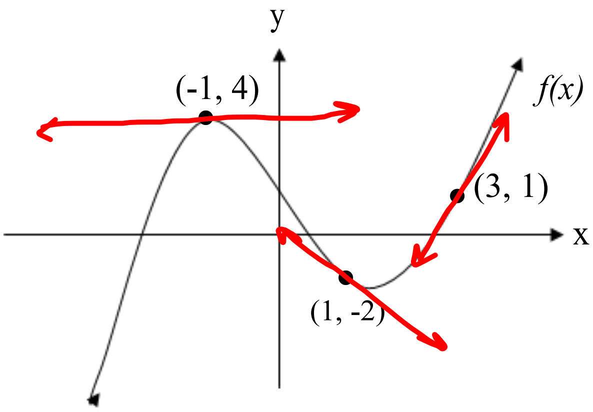


Does there exist a function in which the tangent line to it stays the same no matter what point on the function you use? If so, give an example.

If the tangent line changes for the given function, then how do we come up with a generalized equation for finding the slope of the tangent line at *ANY* given point? If we can do this, then we are able to find the equation of the tangent line at that given point

Remember the Difference Quotient? The difference quotient $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ gave us the slope of the secant line given two points $(x,\ f(x))$ and $(x+\Delta x,\ f(x+\Delta x))$. In order to approximate the slope of the tangent line at a given point, say $(x,\ f(x))$, we would find the slope of the secant line as Δx gets closer to 0. See figure 2.7 below.

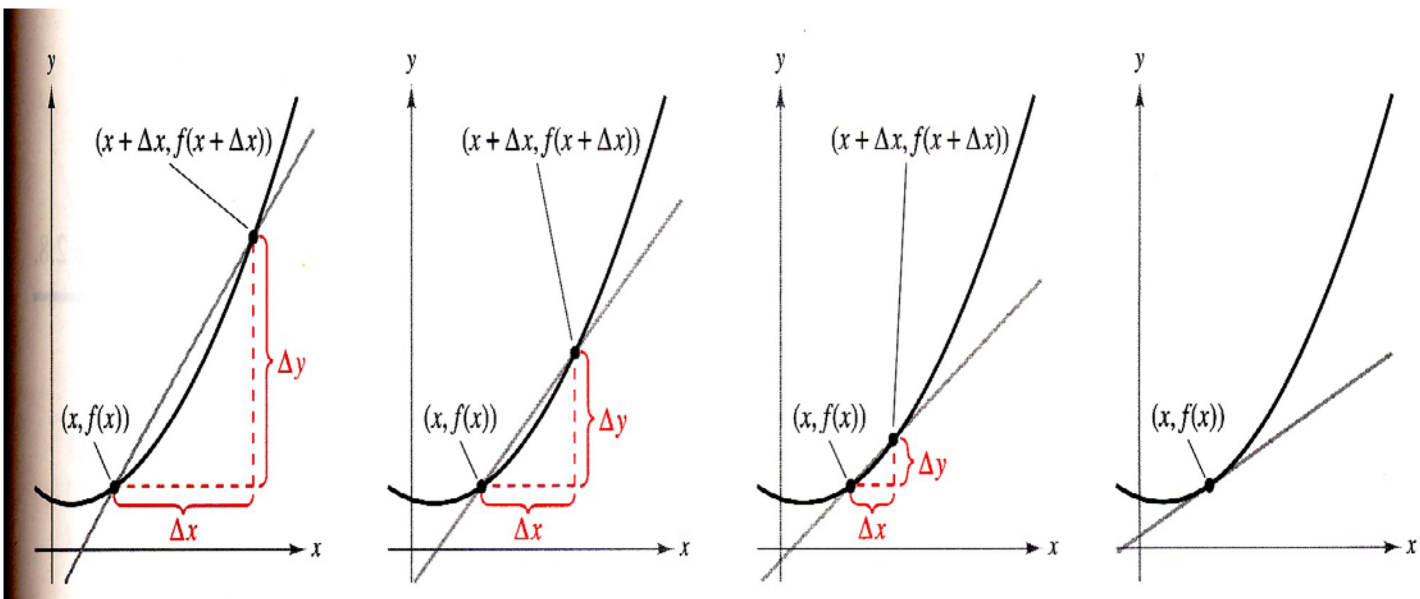
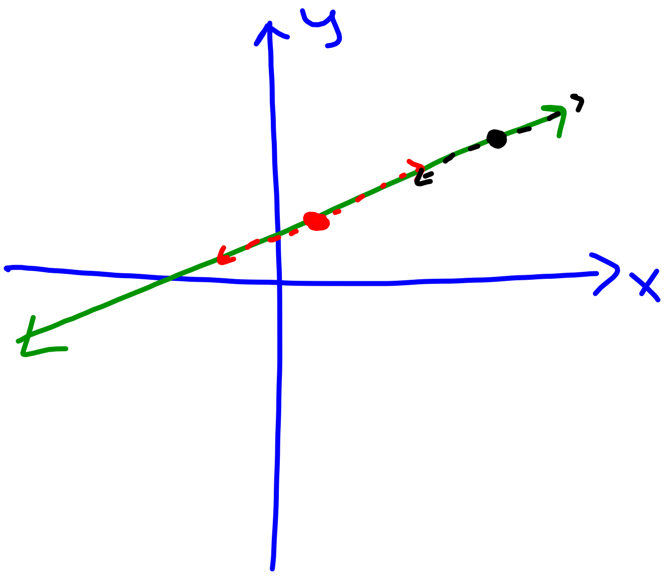


FIGURE 2.7 As Δx approaches 0, the secant lines approach the tangent line.

Does there exist a function in which the tangent line to it stays the same no matter what point on the function you use? If so, give an example.

A line.



In other words, in order for us to find the slope of the tangent line of the graph of f at the point $(x, f(x))$ we would need to find the limit as Δx approaches 0 of the function *f*.

Derivative of a Function: The ***derivative*** of *f* at *x* is given by $f'(x)$ (read: "*f* prime of *x*")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Provided this limit exists. A function is ***differentiable*** at *x* if its derivative exists at *x*. The process of finding derivatives is called ***differentiation***.

Note: in some books, the definition for the derivative is given to be $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, but to stay consistent with this book we will use $h = \Delta x$.

Multiple ways for writing the same thing! As if it isn't already confusing, we can write the derivative in multiple ways:

$$f'(x) = \frac{dy}{dx} = y' = \frac{d}{dx}[f(x)] = D_x[y]$$

NO MATTER HOW YOU WRITE IT…

Derivative = Slope of the tangent line to the graph of *f* at the point $(x, f(x))$.

Now we are finally ready to find the derivative!!!

Limit Process for finding $f'(x)$:

1. Compute $f(x + \Delta x)$.
2. Form the difference: $f(x + \Delta x) - f(x)$.
3. Form the quotient: $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$
4. Compute $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Ex: Use the limit process to find the derivative of the following functions:

a) $f(x) = 3x - 2$

b) $f(x) = 2x^2 - x$

Ex: Use the limit process to find the derivative of the following functions:

a) f(x) = 3x - 2

1) Find $f(x+\Delta x)$: $3(x+\Delta x)-2$

$3x+3\Delta x-2$

2) Find $f(x+\Delta x)-f(x)$: $3x+3\Delta x-2-(3x-2)$

$\cancel{3x}+3\Delta x\cancel{-2}\cancel{-3x}\cancel{+2}$

$3\Delta x$

3) Find $\frac{f(x+\Delta x)-f(x)}{\Delta x}$: $\frac{3\cancel{\Delta x}}{\cancel{\Delta x}}$

4) $\lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} 3 = \boxed{3}$

b) $f(x) = 2x^2 - x$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{2(x+\Delta x)^2 - (x+\Delta x) - (2x^2 - x)}{\Delta x}$$

$$= \frac{2(x^2 + 2x\Delta x + \Delta x^2) - x - \Delta x - 2x^2 + x}{\Delta x}$$

$$= \frac{\cancel{2x^2} + 4x\Delta x + 2\Delta x^2 \cancel{- x} - \Delta x \cancel{- 2x^2} \cancel{+ x}}{\Delta x}$$

$$= \frac{4x\overset{1}{\cancel{\Delta x}} + 2\overset{\Delta x}{\cancel{\Delta x^2}} - \overset{1}{\cancel{\Delta x}}}{\underset{1}{\cancel{\Delta x}}}$$

$$= 4x + 2\Delta x - 1$$

$$f'(x) = \lim_{\Delta x \to 0} (4x + 2\Delta x - 1) = \boxed{4x - 1}$$

$$\boxed{f'(x) = 4x - 1}$$

Ex: For the function $f(x) = 2x^2 - x$, find the equation of the tangent line at the point (1, 1).

Ex: For $f(x) = 2x^2 - x$, find x where the slope of the tangent line is equal to 0. Explain why your answer makes sense.

Not all functions are differentiable (the ability to differentiate, a.k.a. differentiability) at certain values in their domain. In fact, any function that makes an abrupt change of direction creating a "corner", that is, a "cusp" or a "node" at some point on its domain, then it is not differentiable at that point. Also, any function that has a vertical tangent or is discontinuous at some point in its domain is not differentiable at that point (see figure 2.11).

Ex: For the function $f(x) = 2x^2 - x$, find the equation of the tangent line at the point (1, 1).

Slope of the tangent line at any given value x is $f'(x) = 4x - 1$

$y = mx + b$

@ $x = 1$ $m = f'(1) = 4(1) - 1 = 3$

Find b:

$1 = 3(1) + b$

$-2 = b$

$$\boxed{y = 3x - 2}$$

Ex: For $f(x) = 2x^2 - x$, find x where the slope of the tangent line is equal to 0. Explain why your answer makes sense.

$f'(x) = 4x - 1$

$0 = 4x - 1$

$\boxed{\frac{1}{4} = x}$

Vertex:

$x = \frac{-b}{2a} = \frac{-(-1)}{2(2)}$

$= \frac{1}{4}$

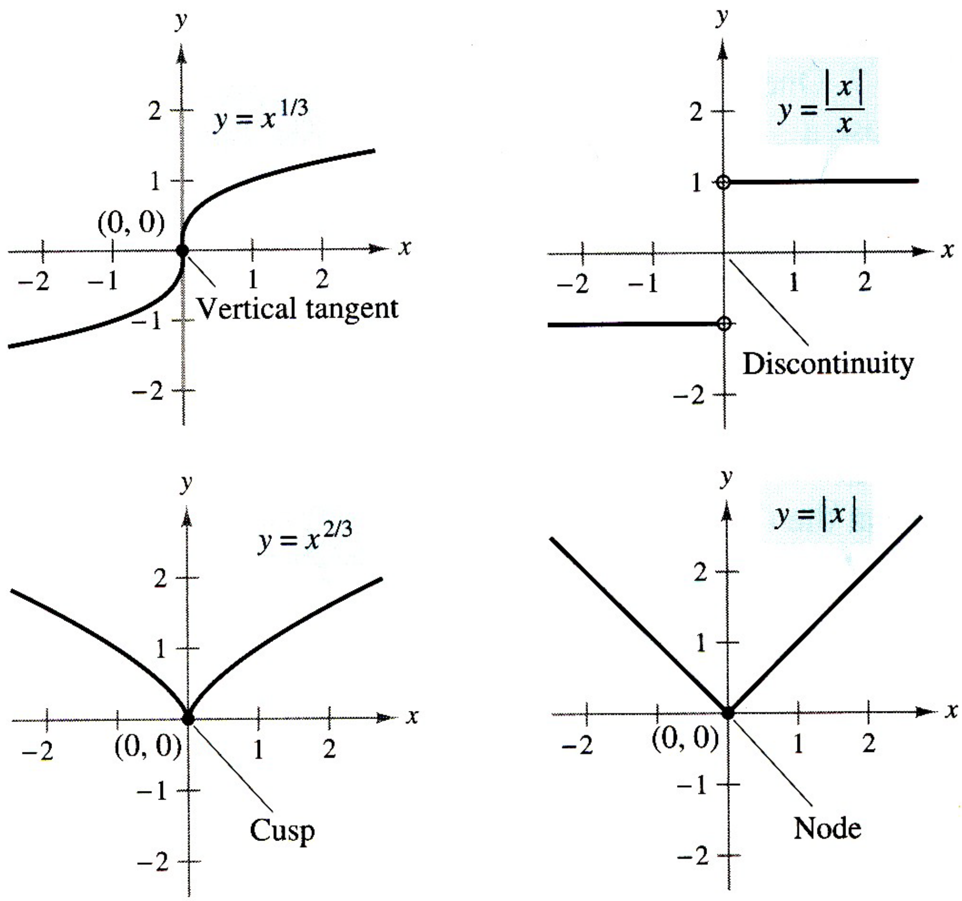


FIGURE 2.11 Functions That Are Not Differentiable at $x = 0$

<u>Differentiability Implies Continuity</u>: If f is differentiable at $x = c$, then f is continuous at $x = c$.
Note: the converse of this statement is not true!