Phys 312 2021

Lecture 7

Thurs: Read 8.1

Maxwell's equations in matter

Whenever all source charges and currents are known, then the fields that they produce are described by Maxwell's equations.

Source currents and charges produce electric and magnetic fields that satisfy:

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

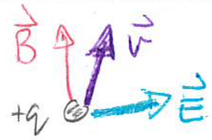
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

where $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ are the source charge density and source current densities respectively

We then just need to solve the associated differential equations to obtain the electric and magnetic fields. Given these fields the force exerted on any test charge is given by the Lorentz Force Law.



The force on a charge particle with charge q and velocity $\vec{v}$ is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

In the presence of matter which produces band charge and current distributions we can modify those so that they only refer to free source charges and currents. We have already seen that in electrostatic situations:

free charge density, ρ_f

Polarization, $\vec{P}$

Electric displacement $\vec{D}$ s.t.

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$$

Then in magnetostatic situations:

free current density $\vec{J}_f$

magnetization, $\vec{M}$

Auxiliary (magnetic) field

$$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f$$

These are connected to electric and magnetic fields by:

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

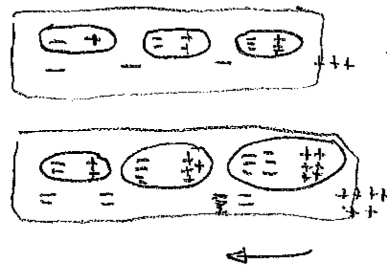
We now rework Maxwell's equations for time-varying fields. The original equations refered to the overall charge + current densities. The modifications refered to static bound charge + current densities. What would the equivalent time varying band current density be?

Consider a time-varying polarization, for example, $\vec{P} = xt\,\hat{x}$. Then the bound volume charge density is:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$= -\frac{\partial P_x}{\partial x} - \cancel{\frac{\partial P_y}{\partial y}} - \cancel{\frac{\partial P_z}{\partial z}}$$

$$= -xt$$



negative charge has flowed this way

← current.

This varies with time and an illustration suggests a bound current density as illustrated.

More precisely

$$\vec{\nabla}\times\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \vec{\nabla}\times\vec{M}$$

and
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{M} = \vec{J}_b$$

give:

$$\vec{\nabla}\times\vec{H} = \vec{J} - \vec{J}_b + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Then $\vec{J} = \vec{J}_b + \vec{J}_f$ gives:

$$\vec{\nabla}\times\vec{H} = \vec{J} - \vec{J}_b + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

But $\vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$

$$\Rightarrow \frac{\partial\vec{E}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial\vec{D}}{\partial t} - \frac{1}{\epsilon_0}\frac{\partial\vec{P}}{\partial t}$$

Thus

$$\vec{\nabla} \times \vec{H} = \vec{J} - \vec{J}_b + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\text{"Faraday's" Law in Matter}} - \frac{\partial \vec{P}}{\partial t}$$

↳ additional bond current source.

Thus we define:

$$\boxed{\begin{array}{l}\text{The polarization current is} \\ \vec{J}_p := \dfrac{\partial \vec{P}}{\partial t}.\end{array}}$$

Then:

$$\vec{\nabla} \times \vec{H} = \vec{J} - \vec{J}_b - \vec{J}_p + \frac{\partial \vec{D}}{\partial t}$$

Then $\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$ gives:

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$$

This is the version of Maxwell's equation that only refers to $\vec{D}, \vec{H}$ and the free current. Thus we get the <u>Maxwell equations in matter</u>

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_f \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

So now:

Known free charge density ρ_f

" " current " $\vec{J}_f$

Relations between

* $\vec{D}$ and $\vec{E}$
* $\vec{H}$ and $\vec{B}$

Solve

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

$\rightarrow$ Gives $\vec{D}, \vec{H}$ $\rightarrow$ Gives $\vec{E}, \vec{B}$

For linear homogeneous media

$$\vec{E} = \vec{D}/\epsilon$$

$$\vec{B} = \mu\vec{H}$$

give:

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

$$\vec{\nabla}\times\vec{D} = -\epsilon\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

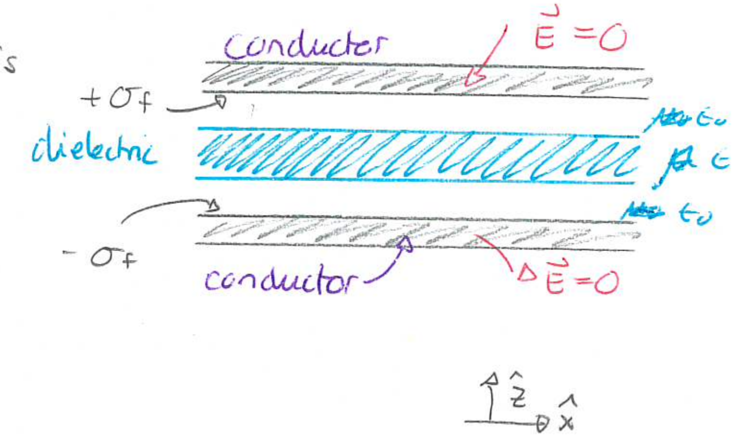
$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

These can be solved for $\vec{D}, \vec{H}$.

Boundary Conditions

We have often seen that there can be jumps in fields across boundaries. Consider electrostatic situations such as

a parallel plate capacitor whose gap is contains a linear dielectric.



We know that

1) inside the conductor plates

$$\vec{D}=0 \quad \text{and} \quad \vec{E}=0$$

2) the free charge only resides on the "inner" surfaces of the conductors.

3) between the plates

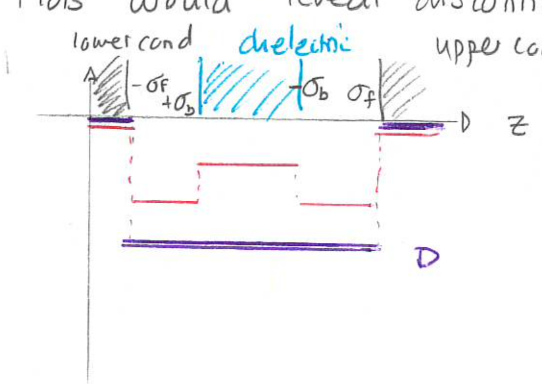
$$\vec{D} = -\sigma_f \hat{z}$$

4) between the plates:

$$\vec{E} = -\frac{\sigma_f}{\epsilon_0}\hat{z} \qquad \text{outside the dielectric}$$

$$\vec{E} = -\frac{\sigma_f}{\epsilon}\hat{z} \qquad \text{inside " "}$$

Plots would reveal discontinuities. There is a jump every time that a new sheet of surface charge is encountered:

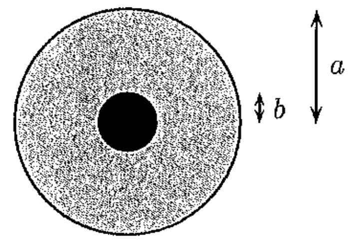


1) sheet of free charge
$\Rightarrow$ jump in $\vec{D}$, jump in $\vec{E}$

2) sheet of bound charge
$\Rightarrow$ jump in $\vec{E}$

1 Boundary conditions for electric displacement and fields

An infinite conducting cylindrical shell of radius a is concentric with an infinite conducting rod of radius $b < a$. The region between the conductors is filled with a linear dielectric with constant ϵ. Suppose that the surface charge density on the inner conductor is uniform and $\sigma_f > 0$.



a) Determine the electric displacement and electric field, in terms of σ_f, for all $S < a$.

b) Determine an expression for the change in the perpendicular component of the electric displacement across the inner boundary ($s = b$).

c) Determine an expression for the change in the tangential component of the electric displacement across the inner boundary.

d) Determine an expression for the change in the perpendicular component of the electric field across the inner boundary in terms of the free charge density on that surface.

e) Determine an expression for the change in the tangential component of the electric field across the inner boundary.

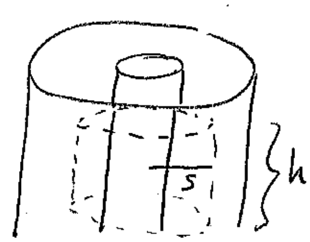
f) Determine an expression for the bound surface charge on the inner surface of the dielectric in terms of the free charge density on the neighboring conductor. Determine the total charge density at the inner interface.

g) Determine an expression for the change in the perpendicular component of the electric field across the inner boundary in terms of the total charge density at the inner interface.

Answer: a) By symmetry $\vec{D} = D_s(s')\hat{s}$. We use

$$\oint \vec{D}\cdot d\vec{a} = q_{\text{free enc}}$$

The surface is a cylinder. On sides.



$$\left.\begin{array}{l} s' = s \\ 0 \le z' \le h \\ 0 \le \phi' \le 2\pi \end{array}\right\} \begin{array}{l} d\vec{a} = s'\,d\phi'\,dz'\,\hat{s} \\ \quad = s\,d\phi'\,dz'\,\hat{s} \end{array}$$

Then $\vec{D}\cdot d\vec{a} = D_s(s)\,s\,d\phi'\,dz'$ on sides. On top $\vec{D}\cdot d\vec{a} = 0$

$$\oint \vec{D}\cdot d\vec{a} = q_{\text{free enc}} = \sigma_f\, h\, 2\pi\, b$$

$$D_s(s)\,s \int_0^{2\pi} d\phi' \int_0^h dz' = D_s(s)\,s\,\cancel{2\pi h} = \cancel{2\pi h}\, b\,\sigma_f \Rightarrow \boxed{\vec{D} = \sigma_f \frac{b}{s}\hat{s}}$$

Now, inside the conductor $\vec{D} = 0$. Thus

$$\vec{D} = \begin{cases} 0 & s < b \\ \sigma_f \frac{b}{s} \hat{s} & a < s < b \end{cases}$$

Then inside the conductor $\vec{E} = 0$. Outside

$$\vec{E} = \frac{1}{\epsilon} \vec{D} = \frac{\sigma_f}{\epsilon} \frac{b}{s} \hat{s}$$

Thus

$$\vec{E} = \begin{cases} 0 & s < b \\ \frac{\sigma_f}{\epsilon} \frac{b}{s} \hat{s} & b < s < a \end{cases}$$

b) The perpendicular component is $\vec{D} \cdot \hat{s}$. Then let

Region 1 = inside conductor
Region 2 = outside "

So $D_{1\perp} = \vec{D} \cdot \hat{s} = 0$

$$D_{2\perp} = \vec{D} \cdot \hat{s} = \sigma_f \frac{s}{s} = \sigma_f$$

$$\Rightarrow \boxed{D_{2\perp} - D_{1\perp} = \sigma_f}$$

c) The tangential components are both zero. So

$$\vec{D}_{2\parallel} = \vec{D}_{1\parallel} = 0 \quad \Rightarrow \quad \boxed{\vec{D}_{2\parallel} = \vec{D}_{1\parallel}}$$

d) $E_\perp = \vec{E}\cdot\hat{s}$

So inside $\quad E_{1\perp} = 0$

outside $\quad E_{2\perp} = \dfrac{\sigma_f}{\epsilon}\dfrac{s}{s}\,\hat{s}\cdot\hat{s} = \dfrac{\sigma_f}{\epsilon}.$

Thus $\quad \boxed{\epsilon E_{2\perp} - E_{1\perp} = \sigma_f}$

e) Since $\quad \vec{E}_{1\parallel} = \vec{E}_{2\parallel} = 0 \quad$ we get $\quad \boxed{\vec{E}_{2\parallel} = \vec{E}_{1\parallel}}$

f) We have $\quad \sigma_b = \hat{n}\cdot\vec{P}$. Then $\quad \vec{D} = \epsilon_0\vec{E} + \vec{P}$

$$\Rightarrow \quad \vec{P} = \vec{D} - \epsilon_0\vec{E}$$

On the inner surface the normal (outward) is $\hat{n} = -\hat{s}$. Thus

$$\sigma_b = -\hat{s}\cdot\vec{P} = -\hat{s}\cdot\vec{D} + \hat{s}\cdot\epsilon_0\vec{E}$$

$$= -\sigma_f + \frac{\epsilon_0}{\epsilon}\sigma_f = \sigma_f\left(\frac{\epsilon_0}{\epsilon} - 1\right).$$

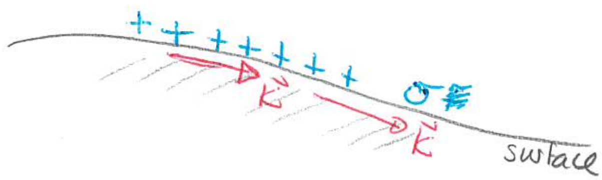
g) The total charge density is $\sigma = \sigma_b + \sigma_f = \sigma_f\left(\frac{\epsilon_0}{\epsilon}\right)$. Then:

$$\sigma_f = \frac{\epsilon}{\epsilon_0}\sigma \quad \Rightarrow \quad E_{2\perp} = \frac{\cancel{\epsilon}}{\epsilon_0}\frac{\sigma}{\cancel{\epsilon}} = \frac{\sigma}{\epsilon_0} \quad \Rightarrow \epsilon_0 E_{2\perp} = \sigma$$

Thus $\quad \boxed{\epsilon_0 E_{2\perp} - E_{1\perp} = \sigma}$

General boundary conditions

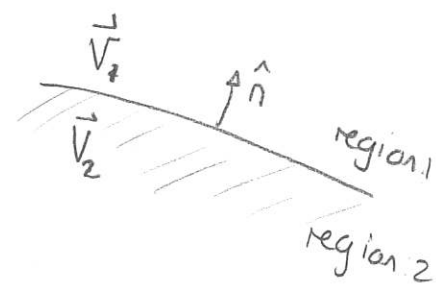
The previous example illustrates how surface charges can be related to discontinuous jumps in fields. There are general results that describe these.



Jump in $\vec{E}$, $\vec{D}$ due to σ

Jump in $\vec{B}, \vec{H}$ due to K.

Such discontinuities involve the perpendicular and tangential components of the field. Consider a general vector field $\vec{V}$. Then consider a surface with normal vector $\hat{n}$ point from region 1 to region 2. Let $\vec{V}_2$ be the field immediately above the surface in region 2. Similarly for $\vec{V}_1$. Then the perpendicular component of $\vec{V}_2$ is



$$V_{2\perp} = \vec{V}_2 \cdot \hat{n}$$

and the perpendicular component of $\vec{V}_1$ is

$$V_{1\perp} = \vec{V}_1 \cdot \hat{n}.$$

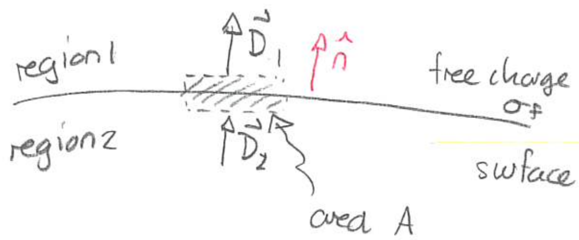
We will establish rules for $V_{1\perp} - V_{2\perp}$ in various situations.

Separately we will establish rules for the tangential components of any field $\vec{V}$. This is:

$$\vec{V}_{\parallel} = \vec{V} - \underbrace{(\vec{V}\cdot\hat{n})\,\hat{n}}_{\text{perpendicular component.}}$$

First consider discontinuities in $\vec{D}$.

Then for the pillbox



$$\oint \vec{D} \cdot d\vec{a} = Q_{free\ enc}$$

$$\underbrace{\cancel{\int \vec{D}\, d\vec{a}}}_{\substack{side \\ 0}} + \underbrace{\int \vec{D}\, d\vec{a}}_{top} + \underbrace{\int \vec{D}\, d\vec{a}}_{bottom} = \sigma_f A \qquad \text{(for infinitesimal area)}$$

sides disappear

$$\Rightarrow \int \vec{D}_1 \cdot \hat{n}\, da + \int \vec{D}_2 \cdot (-\hat{n})\, d\vec{a} = \sigma_f A$$

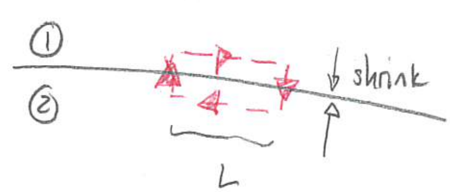
$$\Rightarrow D_{1\perp} A - D_{2\perp} A = \sigma_f A. \qquad \Rightarrow \boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f}$$

In terms of vectors,

> Let $\hat{n}$ be a normal from $2 \to 1$. Then
>
> $$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \sigma_f$$
>
> where σ_f is the free surface charge density

Now consider the tangential component. In this case the electric field discontinuity is more relevant. Here

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$



Using the Ampèrian loop

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{a} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$$

If we shrink the loop perpendicular to the surface then

$$\oint \vec{E}\cdot d\vec{l} = \int_{top} \vec{E}\cdot d\vec{l} + \int_{bottom} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \to 0$$

$$\Rightarrow \quad (E_1^{\parallel} - E_2^{\parallel})\,L \to 0 \qquad \Rightarrow \qquad E_1^{\parallel} = E_2^{\parallel}$$

We can do this for any component of $\vec{E}$ that is parallel to the surface. Then we get

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \qquad \Rightarrow \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$

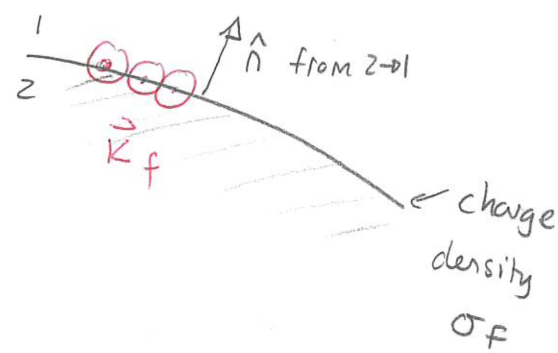
We can reason similarly for $\vec{H}$ and $\vec{B}$. We arrive at:

$$(\vec{D}_1 - \vec{D}_2)\cdot\hat{n} = \sigma_f \Leftrightarrow D_1^{\perp} - D_2^{\perp} = \sigma_f$$

$$(\vec{E}_1 - \vec{E}_2)\times\hat{n} = 0 \Leftrightarrow \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$(\vec{B}_1 - \vec{B}_2)\cdot\hat{n} = 0 \Leftrightarrow B_1^{\perp} = B_2^{\perp}$$

$$\hat{n}\times(\vec{H}_1 - \vec{H}_2) = \vec{K}_f \Leftrightarrow \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$$



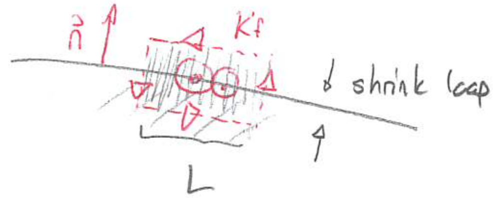
<u>Proof:</u> <u>For perpendicular components of $\vec{B}$</u>

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \oint_{\text{closed surface}} \vec{B}\cdot d\vec{a} = 0$$

The same reasoning for $\vec{D}$ gives $\quad B_1^{\perp} = B_2^{\perp}$

<u>For tangential components of $\vec{H}$</u>

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$



$$\int_{\text{loop surface}} \vec{\nabla}\times\vec{H}\cdot d\vec{a} = \int_{\text{surface}} \vec{J}_f\cdot d\vec{a} + \frac{\partial}{\partial t}\int_{\text{loop surface}} \vec{D}\cdot d\vec{a}$$

$$\Rightarrow \oint_{\text{loop}} \vec{H}\cdot d\vec{l} = \int_{\text{loop surface}} \vec{K}_f\cdot d\vec{a} + \frac{\partial}{\partial t}\underbrace{\int \vec{D}\cdot d\vec{a}}_{\to 0}$$

$$\int_{\text{top}} \vec{H}\cdot d\vec{l} + \int_{\text{bottom}} \vec{H}\cdot d\vec{l} = \int_{\text{loop surface}} \vec{K}_f\cdot d\vec{a}.$$

Assume that the loop is perpendicular to $\vec{K}_f$. Let $\hat{k}$ be along $\vec{K}_f$. On the top $\quad d\vec{l} = \hat{k}\times\hat{n}\, dl \quad \Rightarrow \quad \int_{\text{top}} \vec{H}\cdot d\vec{l} = \vec{H}_1\cdot(\hat{k}\times\hat{n})\, L$

On the bottom $\quad d\vec{l} = -\hat{k}\times\hat{n}\, dl \quad \Rightarrow \quad \int_{\text{bottom}} \vec{H}\, d\vec{l} = -\vec{H}_2\cdot(\hat{k}\times\hat{n})\, L.$

Thus $\quad (\vec{H}_1 - \vec{H}_2)(\hat{k}\times\hat{n})\cancel{L} = K_f \cancel{L}$

$$\Rightarrow \hat{n}\cdot\left[(\vec{H}_1-\vec{H}_2)\times\hat{k}\right] = K_f$$

$$\Rightarrow \hat{k}\cdot\left[(\vec{H}_1-\vec{H}_2)\times\hat{n}\right] = K_f = \vec{K}_f\cdot\hat{k}$$

When the loop is parallel to $\vec{K}$ the l.h.s $= 0$. Thus we get:

$$\hat{n} \times (\vec{H}_1 - \vec{H}_2) = \vec{K}_f \qquad \blacksquare$$

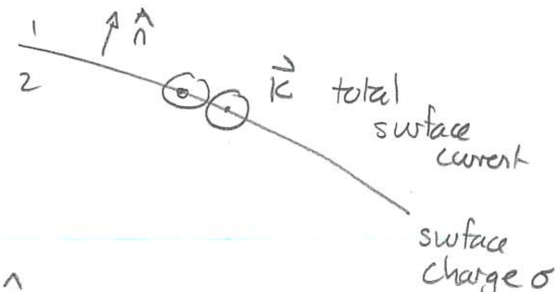
We can use similar arguments for Maxwell's eqns in general.

$$(\vec{E}_1 - \vec{E}_2) \cdot \hat{n} = \frac{\sigma}{\epsilon_0} \Rightarrow E_1^{\perp} - E_2^{\perp} = \sigma/\epsilon_0$$

$$(\vec{E}_1 - \vec{E}_2) \times \hat{n} = 0 \Rightarrow \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{n} = 0 \Rightarrow B_1^{\perp} = B_2^{\perp}$$

$$\hat{n} \times (\vec{B}_1 - \vec{B}_2) = \mu_0 \vec{K} \Rightarrow \vec{B}_1^{\parallel} - \vec{B}_2^{\parallel} = \mu_0 \vec{K} \times \hat{n}$$



Examples: Infinite sheet

$+\sigma$

$$\uparrow\uparrow \quad \vec{E}_1 = \frac{\sigma}{2\epsilon_0} \hat{z}$$

$$\downarrow\downarrow \quad \vec{E}_2 = -\frac{\sigma}{2\epsilon_0} \hat{z}$$

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} = 0$$

$$E_1^{\perp} - E_2^{\perp} = \sigma/\epsilon_0$$

$$\odot \quad \vec{B}_1 = \frac{\mu_0 K}{2} \hat{x}$$

$$\otimes \quad \vec{B}_2 = -\frac{\mu_0 K}{2} \hat{x}$$

$$B_1^{\perp} = B_2^{\perp} = 0$$

$$\vec{B}_1^{\parallel} - \vec{B}_2^{\parallel} = \mu_0 K \hat{x} = \mu_0 \vec{K}$$

For potentials similar reasoning gives:

$$V_1 = V_2$$

$$(\vec{\nabla} V_1 - \vec{\nabla} V_2) \cdot \hat{n} = -\sigma/\epsilon_0$$

$$(\vec{\nabla} V_1 - \vec{\nabla} V_2) \times \hat{n} = 0$$

$$\vec{A}_1 = \vec{A}_2$$

$$\frac{\partial \vec{A}_1}{\partial n} - \frac{\partial \vec{A}_2}{\partial n} = -\mu_0 \vec{K}$$