Electromagnetic Theory II: Homework 11

Due: 16 March 2021

1 General plane wave solution to the three dimensional wave equation

a) Show that
$$f(\mathbf{r},t) = g(\mathbf{k}\cdot\mathbf{r} \pm \omega t),$$
where $g(u)$ is any twice differentiable function of a single variable, is a solution to the three dimensional wave equation.

b) Suppose that
$$g(u) = \frac{A}{u^2+b^2}$$
where A and b are constants. If $\mathbf{k} = k\left(\hat{\mathbf{x}} + 2\hat{\mathbf{y}}\right)/\sqrt{5}$ and $b = 2$, sketch (in the xy plane) the location of the maximum displacement and the locations along which the displacement is half of the maximum value at $t = 0$. In which direction does the wave propagate?

2 Three dimensional waves

Consider the following candidates for three dimensional functions:

$$\begin{aligned}
f_1(\mathbf{r},t) &:= A(x+2y-vt)e^{-(x+2y-vt)^2/a^2}\\
f_2(\mathbf{r},t) &:= A(x-vt)e^{-(x+2y-vt)^2/a^2}\\
f_3(\mathbf{r},t) &:= A(x^2+y^2-vt)e^{-(x^2+y^2-vt)^2/a^2}\\
f_4(\mathbf{r},t) &:= A\cos\left(x+y-vt\right)\\
f_5(\mathbf{r},t) &:= A\cos\left(x^2+y^2-vt\right)
\end{aligned}$$

where $a > 0$. Explain which of these might represent plane waves and which do not represent plane waves.

3 Spherical waves

Consider the three dimensional wave equation
$$\nabla^2 f = \frac{1}{v^2}\,\frac{\partial^2 f}{\partial t^2}.$$

In spherical coordinates, this gives
$$\frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 f}{\partial\phi^2} = \frac{1}{v^2}\,\frac{\partial^2 f}{\partial t^2}.$$

Assume a spherically symmetrical solution of the form

$$f(\mathbf{r}, t) = \frac{h(r)}{r} e^{-i\omega t}$$

By substituting, determine a differential equation for $h(r)$ and solve this for $h(r)$ (use the complex representation when solving this).