6.5 Decoupling a Linear DE System

4, 6, 22

Diagonalization of a Matrix

$n \times n$ matrix with n eigenvalues an n linearly independent eigenvectors

- D, a diagonal matrix with diagonal elements that are the eigenvalues
- P, is the corresponding eigenvectors as columns

$$A = PDP^{-1}$$
$$D = P^{-1}AP$$

Decoupling a Homo Linear DE System

$$\bar{x}' = Ax$$

with diagonizable matrix A

$$\bar{x}' = P\bar{w}$$

to the decoupled

$$\bar{w}' = D\bar{w}$$

$\bar{x} = P\bar{w}$ $\quad w' = P^{-1}\bar{x}' = P^{-1}A\bar{x} = P^{-1}PDP^{-1}\bar{x} = D\bar{w}$

$\bar{w} = P^{-1}\bar{x}$ $\quad w' = D\bar{w}$

Ex) $\bar{x}' = A\bar{x} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\bar{x}$ $\quad \lambda_1 = 4$, $\lambda_2 = 1$ $\quad \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \bar{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

$$\bar{w}' = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\bar{w}$$

$w_1' = 4w_1$ $\quad w_1 = c_1e^{4t}$

$w_2' = 1w_2$ $\quad w_2 = c_2e^{t}$

$$\bar{w} = \begin{bmatrix} c_1e^{4t} \\ c_2e^{t} \end{bmatrix}$$

$$\bar{x} = P\bar{w}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{4t} \\ c_2e^{t} \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} c_1e^{4t} + 2c_2e^{t} \\ c_1e^{4t} - c_2e^{t} \end{bmatrix}$$

Ex:) $\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\bar{x}$ $\quad \lambda_1 = \sqrt{-1}$, $\lambda_2 = -\sqrt{-1}$

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right| = \lambda^2 + 1 = 0$$
$$\lambda = \pm\sqrt{-1}$$

$(A - \lambda I)\bar{v} = 0$

$\lambda = \sqrt{-1}$ $\quad \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{bmatrix}\right)\bar{v} = 0$

$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} -\sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{bmatrix}$ $\sqrt{-1}$

$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & +\sqrt{-1} \\ -1 & -\sqrt{-1} \end{bmatrix}$ $\quad R_2 + R_1$
$(-1 - \sqrt{-1}) + (1 + \sqrt{-1})$
$0 + 0$

$\left[\begin{array}{cc|c} 1 & +\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\quad x = -\sqrt{-1}\,y$ $\quad x = -\sqrt{-1}$, $y = 1$

$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} = 0$ $\quad R_2^* = R_1 + R_2$

$\begin{bmatrix} 1 & -i \\ -1 & i \end{bmatrix} = 0 = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $\quad x = iy$ $\quad \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\mathrm{Re}\; X(t) = e^{\alpha t}(\cos\beta t\,\bar{p} - \sin\beta t\,\bar{q})$

$\mathrm{Im}\; X(t) = e^{\alpha t}(\sin\beta t\,\bar{p} + \cos\beta t\,\bar{q})$

$V_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\alpha = \dfrac{-b}{2a}$ $\quad \alpha = 0$

$\beta = \dfrac{\sqrt{4ac - b^2}}{2a} = \dfrac{\sqrt{4}}{2} = 1$

$\bar{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \bar{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \bar{q}_1 = \begin{bmatrix} -i \\ 0 \end{bmatrix}$ $\quad \bar{q}_2 = \begin{bmatrix} i \\ 0 \end{bmatrix}$

$$x_{Real} = \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} -i \\ 0 \end{bmatrix}$$

$$x_{Im} = \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \cos(t)\begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \qquad P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

$$w' = Dw$$

$$\bar{x} = P\bar{w}$$

$$w' = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$w_1' = -iw_1$$

$$w_2' = iw_2$$

$$\frac{dw_1'}{dt} = -iw_1 = w_1 = ke^{-it}$$

$$\frac{dw_2'}{dt} = iw_2 = w_2 = ke^{it}$$

$$w_1 = ke^{-it}$$

$$w_2 = ke^{it}$$

$$D = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$P\bar{w}: \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-it} \\ c_2e^{it} \end{bmatrix} = \begin{matrix} -ic_1e^{-it} + ic_2e^{it} \\ c_1e^{it} + c_2e^{it} \end{matrix}$$

$$P = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$