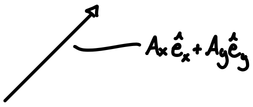
8-21-18

Taylor Series

$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots\ldots a_nx^n \iff F(x)$ about $x=a$ $= \sum \frac{f^n(a)(x-a)^n}{n!}$

Set of x^n's $\qquad a_n = \frac{f^n(a)}{n!}$

iff $a=0$: $\sum \frac{f^n(a)(x)^n}{n!}$



Ex: $f(x) = e^x$

$f^{I}(x) = e^x \quad f^{I}(0) = 1 \quad 1! = 1$

$f^{II}(x) = e^x \quad f^{II}(0) = 1 \quad 2! = 2$

$F(x) = 1 + x + \frac{x^2}{2} \quad \therefore \quad e^x = \sum \frac{x^n}{n!}$

Ex: $f(x) = \sin(x)$ @ $a=0$

$f^0(x) = \sin(x) \quad f^0(0) = 0$

$f^{I}(x) = \cos(x) \quad f^{I}(0) = 1$

$f^{II}(x) = -\sin(x) \quad f^{II}(0) = 0$

$f^{III}(x) = -\cos(x) \quad f^{III}(0) = -1$

$F(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \sim \sum \frac{x^{(2n+1)}}{(2n+1)!}$

Ex: $Ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}$

Ex : $f(x) = (1+x)^n$ @ $a=0$

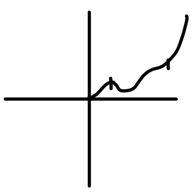
$f^0(x=0) = (1+0)^n = 1$

$f^{I}(x=0) = n(1+x)^{n-1}\Big|_{x=0} = n$

$f^{II}(x=0) = (n)(n-1)(1+x)^{n-2}\Big|_{x=0} = (n)(n-1)$

$f^{III}(x=0) = (n)(n-1)(n-2)$

$F(x) = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!}$



$h \to \Delta h$

$F(x+h) = F(x) + hf'(x) + \frac{1}{2}h^2F''(x) + \ldots\ldots \frac{1}{n!}h^nf^n(x)$

8-23-18

Sode's

2^{nd} order linear differential equation

With Constant Coefficients

$ay'' + by' + cy = 0$

$a\ddot{y} + b\dot{y} + cy = 0$

$y'' - 4 = 0$

$\frac{d}{dx} \to \hat{D}$

$y'' - 4y = 0$

$(D^2 - 4)y = 0$

$(\hat{D}-2)(\hat{D}+2)y = 0$

$(\hat{D}-2)y = 0$

$\frac{dy}{dx} - 2y = 0 \; : \; y = De^{2x}$ $\qquad\qquad$ $\frac{dy}{dx} + 2y = 0 \; : \; y = De^{-2x}$

$\int \frac{1}{y} dy = \int 2x \, dx$ $\qquad$ c and D are

$\ln(y) = 2x + C$ $\qquad$ Same constant

$y = De^{2x}$

<u>Ex</u>: $y'' + 4y = 0$

$(\hat{D}^2 + 4)y = 0$

$(\hat{D}+2i)(\hat{D}-2i)y = 0$

$y = Ae^{2ix} + Be^{-2ix} \to e^{i\theta} = \cos(\theta) \pm i\sin(\theta) \to A\cos(2x) + B\cos(2x)$

<u>Ex</u>: $Ay'' + By' + Cy = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = y$ Homogeneous

When, $a\ddot{y} + b\dot{y} + cy = F(y)$, we have $y = y_h + y_p$

<u>Ex</u>: $y'' - 4 = x$

$y_h = Ae^{-2x} + Be^{2x}$

$y_p = \frac{1}{6}x^3 + 2x^2$

y_p: $y = C_1x^3 + C_2x^2 + C_3x + C_4$

$y' = 3C_1x^2 + 2C_2x + C_3$

$y'' = 6C_1x + 2C_2$

$6C_1 = 1 \qquad 2C_2 - 4 = 0$

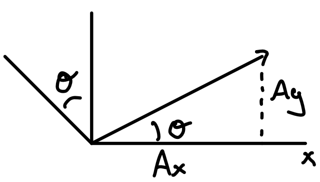
$C_1 = \frac{1}{6} \qquad C_2 = 2$

$C_3 = 0 \qquad C_4 = 0$

<u>Vector Calculus</u>

1. Scalar - Magnitude only
   : invariant under coordinate transforms
2. Vector $\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i = A_i e_i$
   <u>Ex</u>: (A_x, A_y, A_z), (A_r, A_θ, A_ϕ), $(A_{q_1}, A_{q_2}, A_{q_3})$
   $\vec{a}, \vec{v}, \vec{r}, \vec{p}, \vec{F}$

Depends on choice of C.S



<u>Rules For Vectors</u>

#1 if $\vec{A} = \vec{B}$, $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

#2 $\vec{A} + \vec{B} = \vec{C}$

#3 $c\vec{A} = cA_x + cA_y + cA_z$

#4 $\vec{0} = (0,0,0)$

#5 $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

$\hat{e}_1 = \langle 1,0,0 \rangle$

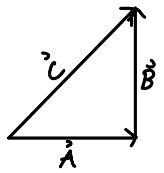
$\hat{e}_2 = \langle 0,1,0 \rangle$

$\hat{e}_3 = \langle 0,0,1 \rangle$

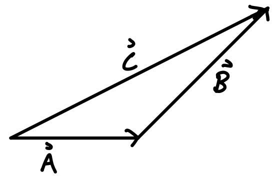
Scalar Products

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$$
$$= A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{\infty} A_iB_i$$

$$|\vec{A}| = (\vec{A}\cdot\vec{A})^{1/2} = (A_1^2 + A_2^2 + A_3^2)^{1/2}$$

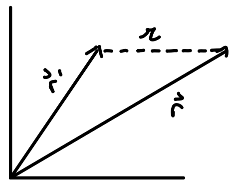


$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2$$



$$\vec{A}+\vec{B} = \vec{C}$$
$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = |\vec{C}|^2$$
$$\vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$
$$|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos(\theta) = |\vec{C}|^2$$

work: $W = \oint \vec{F}\cdot d\vec{r}$



$$r' + \vec{\mathscr{r}}' = \vec{r}$$
$$\vec{\mathscr{r}}' = \vec{r} - \vec{r}'$$
$$|\vec{\mathscr{r}}'| = \sqrt{(\vec{r}-r')\cdot(\vec{r}-r')}$$

Cross Product

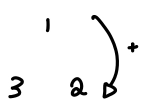
$$\vec{A}\times\vec{R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \langle A_yB_z - A_zB_y,\ A_zB_x - A_xB_z,\ A_xB_y - A_yB_x \rangle$$

$$\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$$
$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

$\vec{C} = \vec{A}\times\vec{B}$ $\quad C_i = \sum_j\sum_k \epsilon_{ijk} A_jB_k$: $\epsilon_{ijk} = 0$ iff $i=j,\ i=k,\ j=k$

$\epsilon_{ijk} = 1$ for $\epsilon_{123}, \epsilon_{231}, \epsilon_{312}$

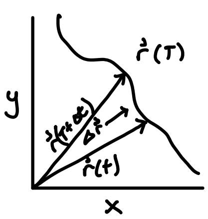
$\epsilon_{ijk} = -1$ for $\epsilon_{123}, \epsilon_{213}, \epsilon_{321}$



$$C_1 = \epsilon_{123}A_2B_3 + \epsilon_{132}A_3B_2$$
$$C_2 = \epsilon_{213}A_1B_3 + \epsilon_{231}A_3B_1$$
$$C_3 = \epsilon_{312}A_1B_2 + \epsilon_{321}A_2B_1$$

$$|\vec{A}\times\vec{B}|^2 = (\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B})$$
$$= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2$$
$$= |\vec{A}|^2|\vec{B}|^2 - |\vec{A}|^2|\vec{B}|^2\cos^2\theta$$
$$= |\vec{A}|^2|\vec{B}|^2\sin^2\theta$$
$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

8-28-18



$$\vec{r}(T) + \Delta\vec{T} = \vec{r}(T+\Delta T)$$
$$\frac{d\vec{r}}{dT} \lim_{\Delta T = 0} = \frac{(T+\Delta T) - r(T)}{\Delta T}$$
$$\frac{d\,r_x(r)_n}{dT} e_x + \frac{d\,r_y(r)_n}{dT} e_y$$

$$\frac{d}{dT}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dT} + \frac{d\vec{B}}{dT}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{B}\cdot\frac{d\vec{A}}{dT} + \vec{A}\cdot\frac{d\vec{B}}{dT}$$

$$\frac{d}{dT}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dT}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dT}$$

$$\frac{d\phi\vec{A}}{dT} = \frac{d\phi}{dT}\vec{A} + \phi\frac{d\vec{A}}{dT}$$

<u>Mechanics:</u>

$\vec{r}(T)$, $\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dT}$, $\vec{a} = \dot{\vec{V}} = \ddot{\vec{r}} = \frac{d^2\vec{r}}{dT^2}$

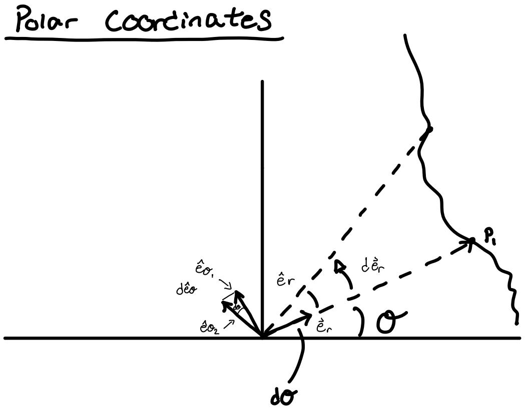
$\vec{F} = m\ddot{\vec{r}} \rightarrow$ get $\vec{r}(T)$

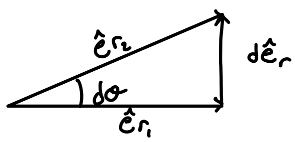
$\vec{r}(T) = x(T)\hat{e}_x + y(T)\hat{e}_y + z(T)\hat{e}_z \longrightarrow$

$\vec{v} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z$

$\vec{a} = \ddot{x}\vec{e}_x + \ddot{y}\vec{e}_y + \ddot{z}\vec{e}_z$

<u>Polar Coordinates</u>



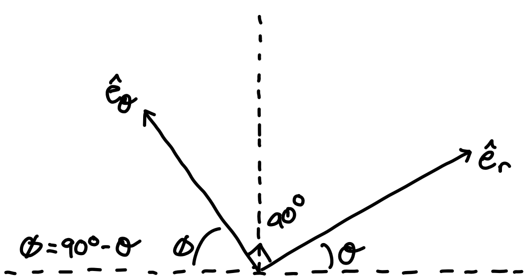


$d\hat{e}_r = \hat{e}_{r_2} - \hat{e}_{r_1} \parallel \hat{e}_\theta$

$d\hat{e}_r = d\theta\hat{e}_\theta$

$d\hat{e}_\theta = \hat{e}_{\theta_2} - \hat{e}_{\theta_1} \parallel -\hat{e}_r$

$d\hat{e}_\theta = -d\theta\hat{e}_r$



$\hat{e}_r = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$

$\hat{e}_\theta = -\cos(90-\theta)\hat{\imath} + \sin(90-\theta)\hat{\jmath}$

$\hat{e}_\theta = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}$

$d\hat{e}_r = -\sin(\theta)d\theta\,\hat{\imath} + \cos(\theta)d\theta\,\hat{\jmath} = d\theta\hat{e}_\theta$

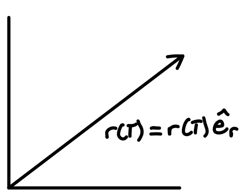
$d\hat{e}_\theta = -\cos(\theta)d\theta\,\hat{\imath} - \sin(\theta)d\theta\,\hat{\jmath} = -d\theta\hat{e}_r$

$\sin(A \pm B)$
$= \sin(A)\cos(B) \pm \cos(A)\sin(B)$

$\cos(A \pm B)$
$= \cos(A)\cos(B) \pm \sin(A)\sin(B)$

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dT} = \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dT} = -\dot{\theta}\hat{e}_r$$



$$\vec{v}(T) = \frac{d(r(T)\hat{e}_r)}{dT} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\vec{a}(T) = \frac{d}{dT}\left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right] = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

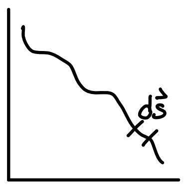
$$\vec{a}(T) = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{e}_r + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{e}_\theta$$

Linear Acceleration

Centripetal

Spin up and Spin down

$d\vec{s}$



$$d\vec{s} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$$

$$d\vec{s}\cdot d\vec{s} = dx^2 + dy^2 + dz^2$$

$$\frac{1}{2}mv^2 = |\vec{v}|^2 = \frac{d\vec{s}\cdot d\vec{s}}{dT^2} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\vec{V} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

Cylindrical

$$d\vec{s} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z$$

$$|\vec{v}|^2 = \frac{d\vec{s}\cdot d\vec{s}}{dT^2} = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{e}_z$$

Spherical

$$d\vec{s} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$$

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

ϕ = Scalar $F(x,y,z)$

$\vec{\nabla} = \sum \hat{e}_i \frac{\partial}{\partial x_i}$ ← cartesian

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \sum_{ijk} \epsilon_{ijk}\frac{\partial A_x}{\partial x_i}\hat{e} \Rightarrow \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}$$

i.e. $\nabla^2 \ln(r) =$

$$\nabla^2\ln(r) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left[\ln(r)\right] \qquad r = (x^2+y^2+z^2)^{1/2}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\ln(r))\right] \longrightarrow \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial x}\cdot\frac{1}{r}\right] \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \;:\; \frac{\partial r}{\partial y} = \frac{y}{r} \;:\; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) \to \frac{\partial}{\partial x}\left[(x)(x^2+y^2+z^2)^{-1}\right] = (x^2+y^2+z^2) - 2x(x^2+y^2+z^2)^{-2}(x)$$

$$= \frac{1}{r^2} - \frac{2x^2}{r^4} \;:\; \frac{1}{r^2} - \frac{2y^2}{r^4} \;:\; \frac{1}{r^2} - \frac{2z^2}{r^4}$$

$$\frac{3}{r^2} - \frac{2(x^2+y^2+z^2)}{r^4} = \frac{3}{r^2} - \frac{2r^2}{r^4} = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$