3.3 Fourier Expansions

Sinusoidal

$$f(t) \cong \sum_{k=0}^{n} a_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{n} b_k \sin\left(\frac{2\pi k}{T} t\right)$$

$$\int_0^T \cos^2\left(\frac{2\pi \cdot k \cdot t}{T}\right) dt = T, \quad k=0$$

$$\int_0^T \cos^2\left(\frac{2\pi \cdot k \cdot t}{T}\right) dt = T/2, \quad k \geq 1$$

$$\int_0^T \sin^2\left(\frac{2\pi \cdot k \cdot t}{T}\right) dt = T/2, \quad k \geq 1$$

Fourier Expansion & Fourier Coefficients

$$h_n(t) = \sum_{k=0}^{n} a_k \cos\left(\frac{2\pi k}{T} t\right) + \sum_{k=1}^{n} b_k \sin\left(\frac{2\pi k}{T} t\right)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi k}{T} t\right) dt$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi k}{T} t\right) dt$$

Basic Function Fourier Coefficients

1. Box Function

$$a_0 = \frac{1}{2}, \quad a_k = 0, \quad b_k = \begin{cases} \frac{2}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \quad k = 1,2,3,\ldots$$

2. Square Wave Function

$$a_0 = 0, \quad a_k = 0, \quad b_k = \begin{cases} \frac{4}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \quad k = 1,2,3,\ldots$$

3. Sawtooth function

$$a_0 = 0, \quad a_k = 0, \quad b_k = -\frac{2}{\pi k}, \quad k = 1,2,3$$

4. Tent Function

$$a_0 = \frac{1}{2}, \quad a_k = \begin{cases} -\frac{4}{(\pi k)^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \quad b_k = 0 \quad k = 1,2,3\ldots$$

5. Sharp Wave Function

$$a_0 = 0, \quad a_k = 0, \quad b_k = \begin{cases} \frac{(-1)^k \cdot 2}{(\pi k)^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \quad k = 1,2,3,\ldots$$

Fourier Series

$$h(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi k t/T) + \sum_{k=1}^{\infty} b_k \sin(2\pi k t/T), \quad t \in [0,T]$$

$$h(t) = \lim_{n \to \infty} h_n(t)$$

Gibbs Phenomenom

- Tendency to overshoot at jump discontinuity $\approx 9\%$

Pointwise Convergence

(a) all points of continuity in the time interval the expansion converges pointwise to f.

(b) anywhere there is a discontinuity in the graph, the expansion converges to the average of the left and right hand limits.

(c) at the endpoints of the graph, the expansion converges to the left and right hand limits of the periodic extension of f.

(d) outside the time interval, the expansion converges to the periodic extension in the same manner as in (a), (b), (c)

(e) If f is continuous and periodic with period T, then $f = f_T$ on $(-\infty,\infty)$ and the expansion converges pointwise to f on $(-\infty,\infty)$