Tues: HW by 5pm

Note IBM Q Experience ..

Two qubit gates

Recall that two qubit gates will be represented by operators acting on both qubits. These operators will be represented by 4×4 matrices. One class of such operators are tensor products of unitaries for individual qubits. The generic example would be

$$\hat{u} = \hat{A} \otimes \hat{B}$$

where $\hat{A}$ and $\hat{B}$ are unitary. Note that this has the following effect on a tensor product of states:

$$\hat{u}\, |\varphi\rangle |\psi\rangle = \hat{A} \otimes \hat{B}\, |\varphi\rangle |\psi\rangle = \hat{A} |\varphi\rangle \otimes \hat{B} |\psi\rangle$$

and this always produces a tensor product. Thus

$$\boxed{\text{A tensor product unitary maps a product state onto a product state.}}$$

1 Tensor product of operators

Consider $\hat{U} = \hat{\sigma}_z \otimes \hat{H}$ where

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the Hadamard operator.

a) Suppose that the state of the system prior to the evolution is $|\Psi_0\rangle = |10\rangle$. Determine the state after the evolution.

b) Suppose that the state of the system prior to the evolution is $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. This is entangled. Determine the state after the evolution and check whether it is entangled.

Answer:

a) $\hat{U}|\Psi_0\rangle = \sigma_z \otimes \hat{H} \; |1\rangle|0\rangle$

$= \sigma_z |1\rangle \otimes \hat{H}|0\rangle$

$= -|1\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = -\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$

b) $\hat{U}|\Psi_0\rangle = \hat{\sigma}_z \otimes \hat{H} \; \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \frac{1}{\sqrt{2}}\left[\sigma_z|0\rangle \otimes \hat{H}|0\rangle + \sigma_z|1\rangle \otimes \hat{H}|1\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[|0\rangle \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - |1\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$

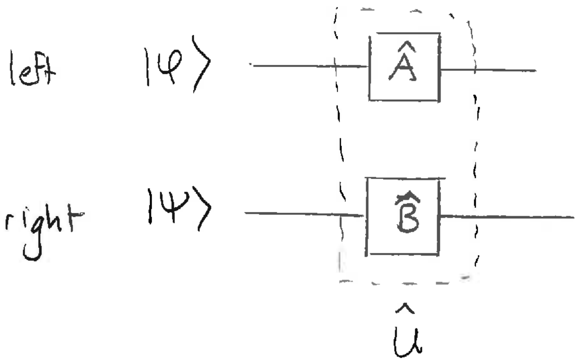
$= \frac{1}{2}\left[\underset{a_0}{|00\rangle} + \underset{a_1}{|01\rangle} - \underset{a_2}{|10\rangle} + \underset{a_3}{|11\rangle}\right]$

This is entangled. since $a_0 a_3 = 1$

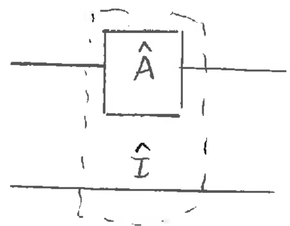
$a_1 a_2 = -1$

$a_0 a_3 \neq a_1 a_2$

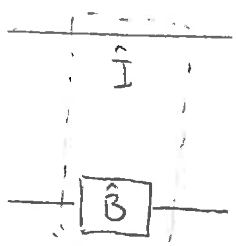
Such tensor product operators can be represented via



So then



$$\hat{U} = \hat{A} \otimes \hat{I}$$



$$\hat{U} = \hat{I} \otimes \hat{B}$$

The previous operations all involved tensor products of two unitaries. But crucial operations in quantum information will not be tensor products. The most important example is the controlled-NOT.

This can be described by its operation on standard basis states

$$\hat{U}_{CNOT}\,|\overset{\text{control}}{0}\overset{\text{target}}{0}\rangle := |00\rangle$$

$$\hat{U}_{CNOT}\,|01\rangle := |01\rangle$$

do nothing to target if control $= 0$

$$\hat{U}_{CNOT}\,|10\rangle := |11\rangle$$

$$\hat{U}_{CNOT}\,|11\rangle := |10\rangle$$

do NOT to target if control $= 1$.

We can represent this in various ways. First the matrix representation in $\{|00\rangle, \ldots |11\rangle\}$ is

$$\boxed{\hat{U}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

Secondly in terms of tensor products

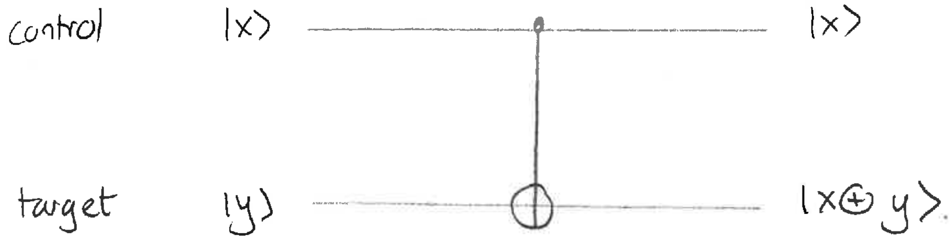
$$\boxed{\hat{U}_{CNOT} = \underbrace{|0\rangle\langle 0|}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \otimes \underbrace{\hat{I}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underbrace{|1\rangle\langle 1|}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \otimes \underbrace{\hat{\sigma}_x}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}$$

Finally we see that for $x, y \in \{0,1\}$

$$\hat{U}_{CNOT}\,|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$$

and we see that it can implement the classical XOR.

The circuit representation for this gate is:



On a general product state

$$|\Psi_0\rangle = |\varphi\rangle|\psi\rangle \xrightarrow{\hat{U}_{CNOT}} (|0\rangle\langle 0| \otimes \hat{I})|\varphi\rangle|\psi\rangle + (|1\rangle\langle 1| \otimes \hat{\sigma}_x)|\varphi\rangle|\psi\rangle$$

We will see that

$$\text{unentangled input } |\Psi_0\rangle \xrightarrow{\hat{U}_{CNOT}} \text{sometimes unentangled, sometimes entangled}$$

$$\text{entangled input } |\Psi_0\rangle \xrightarrow{\hat{U}_{CNOT}} \text{sometimes unentangled, sometimes entangled}$$

2 CNOT gate

a) Suppose that the initial state prior to a CNOT is $|\Psi_0\rangle = |0\rangle \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. Determine the state after the gate. Is this entangled?

b) Suppose that the initial state prior to a CNOT is $|\Psi_0\rangle = |1\rangle \frac{1}{\sqrt{5}} (|0\rangle + 2|1\rangle)$. Determine the state after the gate. Is this entangled?

c) Suppose that the initial state prior to a CNOT is $|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |0\rangle$. Determine the state after the gate. Is this entangled?

d) Suppose that the initial state prior to a CNOT is $|\Psi_0\rangle = \frac{1}{2} (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$. Determine the state after the gate. Is this entangled?

e) Suppose that the initial state prior to a CNOT is $|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$. Determine the state after the gate. Is this entangled?

f) Using algebraic manipulations show that the CNOT is unitary.

Answer: a) $|\Psi_0\rangle = \frac{1}{\sqrt{2}} |00\rangle + |01\rangle.$

$$\rightarrow \frac{1}{\sqrt{2}} |00\rangle + |01\rangle = |0\rangle \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \quad \text{not}$$

b) $$|\Psi_0\rangle = \frac{1}{\sqrt{5}} (|10\rangle + 2|11\rangle) \rightarrow \frac{1}{\sqrt{5}} (|11\rangle + 2|10\rangle)$$

$$= |1\rangle \frac{1}{\sqrt{5}} (|1\rangle + 2|0\rangle)$$

$$= |1\rangle \frac{1}{\sqrt{5}} (2|0\rangle + |1\rangle) \quad \text{not}$$

c) $$|\Psi_0\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \rightarrow \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad \text{is entangled}$$

d) $$|\Psi_0\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \rightarrow \frac{1}{2} (|00\rangle + |01\rangle + |11\rangle + |10\rangle)$$

$$= \text{same state} \quad \text{not}$$

e) $$|\Psi_0\rangle \rightarrow \frac{1}{\sqrt{2}} (|00\rangle - |10\rangle) = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \otimes |0\rangle \quad \text{is not entangled}$$

f) $$\hat{U}^{+} = (|0\rangle\langle 0|)^{+} \otimes \hat{I}^{+} + (|1\rangle\langle 1|)^{+} \otimes \sigma_x^{+}$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$$

So $$\hat{U}^{\dagger}\hat{U} = \left[|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right]\left[|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right]$$

$$= |0\rangle\overset{1}{\langle 0|0\rangle}\langle 0| \otimes \hat{I}^2 + |0\rangle\cancel{\overset{0}{\langle 0|1\rangle}}\langle 1| \otimes \underbrace{\hat{I}\hat{\sigma}_x}_{\hat{\sigma}_x}$$

$$+ |1\rangle\cancel{\underset{0}{\langle 1|0\rangle}}\langle 0| \otimes \underbrace{\hat{\sigma}_x\hat{I}}_{\hat{\sigma}_x} + |1\rangle\cancel{\overset{1}{\langle 1|1\rangle}}\langle 1| \otimes \underbrace{\hat{\sigma}_x\hat{\sigma}_x}_{\hat{I}}$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{I}$$

$$= \left[|0\rangle\langle 0| + |1\rangle\langle 1|\right] \otimes \hat{I} = \hat{I} \otimes \hat{I} = \hat{I} \qquad \square$$

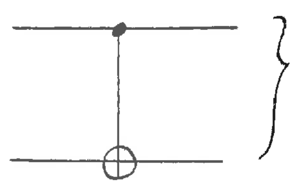
Specifically we can see that the CNOT generates entangled states and can converted entangled states back into product states.

So

$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|0\rangle$

product



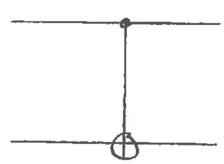
$\frac{1}{\sqrt{2}} |00\rangle + |11\rangle$

entangled

and

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

entangled



$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|0\rangle$

product

We can see that this maps the Bell states

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \longrightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}\, |0\rangle$$

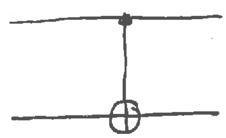
$$\frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}\, |0\rangle$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \longrightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}\, |1\rangle$$

$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}\, |1\rangle$$

So to measure in the Bell basis

Bell state



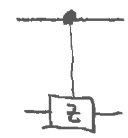
measure in $\left\{ \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \right\}$ basis

measure in $\{ |0\rangle, |1\rangle \}$ basis

Gate decomposition

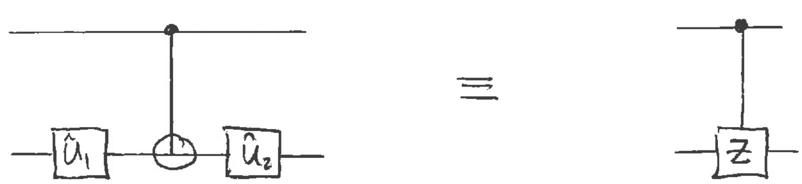
Suppose that we aim to construct a controlled-Z gate:

$$\hat{U} = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_z$$



Can we do this from a CNOT and single qubit gates?

We aim to try:



Consider the rotation gate

$$\hat{R}_y(\theta) = e^{-i\theta\hat{\sigma}_y/2} = \cos\frac{\theta}{2}\hat{I} - i\sin\frac{\theta}{2}\hat{\sigma}_y$$

for $\theta = -\pi/2$. This gives:

$$\hat{R}_y\left(\pm\frac{\pi}{2}\right) = \cos(-\pi/4)\,\hat{I} - i\sin(\pm\pi/4)\,\hat{\sigma}_y$$

$$= \frac{1}{\sqrt{2}}\left(\hat{I} \mp i\hat{\sigma}_y\right)$$

Then let $\hat{U}_1 = \hat{R}_y(+\pi/2)$

$\hat{U}_2 = \hat{R}_y(-\pi/2)$

and the sequence is

last — middle — first — input state

$$\left[\hat{I} \otimes \hat{R}_y(-\pi/2)\right]\left[|0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{\sigma}_x\right]\left[\hat{I} \otimes \hat{R}_y(\pi/2)\right]$$

$$= \left[\hat{I} \otimes \hat{R}_y(-\pi/2)\right]\left[|0\rangle\langle 0| \otimes \hat{R}_y(\pi/2) + |1\rangle\langle 1| \otimes \sigma_x \hat{R}_y(\pi/2)\right]$$

$$= |0\rangle\langle 0| \otimes \underbrace{\hat{R}_y(-\pi/2)\hat{R}_y(\pi/2)}_{=\hat{I}} + |1\rangle\langle 1| \otimes \hat{R}_y(-\pi/2)\hat{\sigma}_x \hat{R}_y(\pi/2)$$

$$= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{R}_y(-\pi/2)\hat{\sigma}_x \hat{R}_y(\pi/2)$$

Now

$$\hat{R}_y(-\pi/2)\hat{\sigma}_x \hat{R}_y(\pi/2) = \frac{1}{\sqrt{2}}\left(\hat{I} + i\hat{\sigma}_y\right)\hat{\sigma}_x \frac{1}{\sqrt{2}}\left(\hat{I} - i\hat{\sigma}_y\right)$$

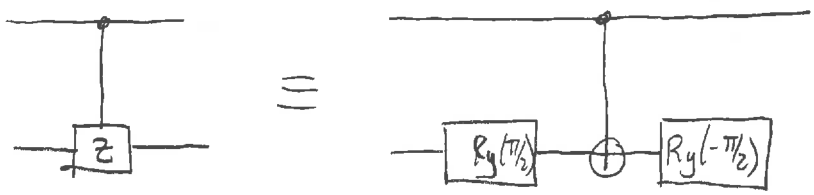
$$= \frac{1}{2}\left(\hat{I}\hat{\sigma}_x\hat{I} - i\,I\hat{\sigma}_x\hat{\sigma}_y + i\hat{\sigma}_y\hat{\sigma}_x\hat{I} + \hat{\sigma}_y\hat{\sigma}_x\hat{\sigma}_y\right)$$

$$= \frac{1}{2}\left(\hat{\sigma}_x - i(i\hat{\sigma}_z) + i(-i\hat{\sigma}_z) - \hat{\sigma}_x\underbrace{\hat{\sigma}_y\hat{\sigma}_y}_{\hat{I}}\right)$$

$$= -\hat{\sigma}_z$$

and this works

So we find that it is possible to decompose the gate



We can prove a general theorem:

> Any two qubit gate can be decomposed into a sequence of CNOT and single qubit gates

So we only need to be able to construct a CNOT plus arbitrary single qubit gates. Additionally we can show that

> Any single qubit gate can be constructed as a sequence of rotations about y and z

So we only need three gates

to construct any two qubit gate