Electromagnetic Theory II: Homework 2

Due: 2 February 2021

1 Energy of a dipole in a field

The force exerted by an external electric field $\mathbf{E}_{\text{ext}}$ on a dipole $\mathbf{p}$ can be shown to be

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\,\mathbf{E}.$$

Then the energy, U, associated with this dipole in the field must satisfy

$$\mathbf{F} = -\boldsymbol{\nabla} U.$$

a) Show that one form of the energy that agrees with this is

$$U = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}.$$

b) Suppose that an electric dipole is in a uniform external electric field. How must it be oriented so that the energy is a maximum? A minimum?

2 Energy of two dipoles

Consider two dipoles, with dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$, whose centers are separated by displacement $\mathbf{r}$.

a) Show that the potential energy associated with their interaction is

$$U = \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\,\left[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\mathbf{p}_1 \cdot \hat{\mathbf{r}})(\mathbf{p}_2 \cdot \hat{\mathbf{r}})\right].$$

b) Suppose that the two dipoles are oriented so that their dipole moments are parallel. One dipole is held fixed while the other is moved around, maintaining the same distance from the first. How should the two dipoles be situated (relative to their dipole moments) so that the potential energy is a maximum? A minimum? In each case, sketch the dipoles and their relative orientation of their centers.

c) Suppose that the two dipoles are oriented with their dipole moments parallel and that they do move relative to each other, maintaining the same distance, but with the direction between them changing randomly and rapidly. On average what will the energy of interaction between them be? To answer this convincingly, one has to average over all angles, which means integrating using the usual spherical coordinates and $\sin\theta\, \mathrm{d}\theta \mathrm{d}\phi$.

The magnetic version of this is important in nuclear magnetic resonance (NMR) in liquids. NMR has to do with (quantum) energies of magnetic dipoles within molecules. Traditionally one wants to information about the interactions between dipoles within a molecule but not between two molecules. If two molecules tumble relative to each other then the answer to the previous part simplifies the situation so that inter-molecular interactions are irrelevant.

d) Suppose that the two dipoles are oriented so that their dipole moments exactly opposite. One dipole is held fixed while the other is moved around, maintaining the same distance from the first. How should the two dipoles be situated (relative to their dipole moments) so that the potential energy is a maximum? A minimum? In each case, sketch the dipoles and their relative orientation of their centers.

e) Suppose that the dipole moments are either parallel or exactly opposite. Show that there is one particular angle at with the energy of interaction between the two is exactly zero in both cases. Find the angle.

The magnetic version of this is important in nuclear magnetic resonance (NMR) in solids and the angle is called the *the magic angle.*

3 Axially polarized sphere

A sphere with radius R has polarization, in spherical coordinates, of

$$\mathbf{P} = P(r)\ \hat{\boldsymbol{\phi}},$$

where $P(r)$ is an arbitrary function. Explain, without using calculations, whether the bound surface and volume charge densities are positive, negative or zero. *You could use a calculation as an aid but you must explain without using the calculation.*

4 Radially polarized sphere

A sphere with radius R has polarization, in spherical coordinates, of

$$\mathbf{P} = \alpha r \hat{\mathbf{r}},$$

where α is a constant with units of $\mathrm{C/m^3}$.

a) Determine the electric field at all points inside and outside the sphere.

b) Determine the total bound charge on the sphere.

5 Polarized cylinder

An infinitely long cylinder with radius R has polarization, given in cylindrical coordinates, of

$$\mathbf{P} = ks\ \hat{\mathbf{s}}$$

where k is constant.

a) Determine the bound volume charge and the bound surface charge.

b) Determine the electric field at all points inside and outside the cylinder.

6 Total bound charge

Consider any region that holds any polarization. Provide an (integral) expression for the net bound charge within and on the surface of the region and prove that the this total bound charge must always be zero. *(Hint: Using the correct vector calculus theorem should yield the proof in a few lines.)*