Statistical and Thermal Physics: Homework 13

Due: 13 March 2020

1 Two dice

Two unbiased dice are rolled. Let V be the sum of their face values.

a) Determine the probability with which each value of V occurs.

b) Determine the mean value V. Is this equal to two times the mean value for a single die?

c) Determine the standard deviation of V. Is this equal to two times the standard deviation for a single die?

2 Multiple coin flips

This uses a computer simulation developed by the text authors; instructions for accessing this were given in Homework 1. Run the "launcher" and select Multiple Coin Toss.

The simulation flips a set number of coins (e.g. 100) and records the number of heads. This constitutes a single "trial" and the simulation can do repeated trials, record the results and display these in a histogram. Each "trial" of the simulation uses many (e.g. 100) coin tosses. The histogram plots the number trials that return various numbers of heads.

a) Set the probability to 0.5 and the number of coins to 100. In one run how many heads do you predict to appear?

b) Click Initialize and then Step. How many flips produced heads? What does the data yield for the average number of heads? Does this match your prediction?

c) Now repeat this many times by clicking Start and Stop until the simulation has done at least 500 trials. Using the Tools → Data Tools menu in the Histogram window, determine the most commonly occurring number of heads. What is the average number of heads returned over all the trials? Does this match your prediction better than the result with just one trial?

d) Now reset the simulation so that a single trial consists of 10000 flips and repeat steps a) and b). Do the results of the simulation match your predictions more or less accurately than when there were just 100 flips?

e) Repeat the previous part many times by clicking Start and Stop until the simulation has done at least 100 trials. What is the average number of heads returned over all the trials? Does this match your prediction better than the result with just one trial of 10000 flips? Does the histogram of the outcomes indicate a narrower or larger range of possibilities than when there were just 100 flips?

f) What does the animation indicate about the accuracy of statistical predictions: do more trials or fewer trials result in more accurate predictions?

3 Random walk

Consider a random walk with 6 steps.

a) If the probability of stepping right in a single step is 1/2, determine the probability with which the walker ends two steps left of where he started.

b) If the probability of stepping right in a single step is 1/3, determine the probability with which the walker ends two steps left of where he started.

4 Binomial distribution

Consider the binomial distribution whose outcomes are labeled $+1$ and -1. In a single trial the probability of attaining $+1$ is p and that of attaining -1 is q. Suppose that N trials are performed. The the probability with which one attains n outcomes of $+1$ is

$$P(n) = \binom{N}{n} p^n q^{N-n}.$$

a) Use the binomial theorem/identity to show that the probabilities satisfy

$$\sum_{n=0}^{N} P(n) = 1.$$

b) Use the fact that, for a binomial distribution

$$\overline{n^2} = N^2 p^2 + Npq$$

to determine the standard deviation σ for the distribution.

c) Consider the size of the standard deviation relative to the mean. If this is small, then it means that the mean captures most of the information about the distribution. A relevant measure of the size of the standard deviation to the mean is $\sigma/\overline{n}$. Determine an expression for this in terms of p and N. As $N \to \infty$, what does this approach?

5 Binomial distribution: large numbers

This uses a computer simulation developed by the text authors; instructions for accessing this were given in Homework 1. Run the "launcher" and select Binomial distribution. This produces two graphs of probability distributions. One is $P(n)$ versus n and the other is $P(n)$ versus $n/\langle n \rangle$. In this exercise you will use the graph of $P(n)$ versus n. Also left clicking the cursor on the graph gives the coordinates of any point.

Consider a collection of spin-1/2 particles, each of whose state can either be up or down. Let N be the total number of particles and n the number whose spins are up. Suppose that p is the probability with which any single particle has spin up.

a) Let $N = 6$ and suppose that $p = 1/2$. For each possible value of n, calculate the probability $P(n)$ with which n spins are up. Determine $\overline{n}$ and σ.

b) Let $N = 10$ and suppose that $p = 1/2$. Use the program to determine the probabilities $P(n)$ and to plot these. A measure of the width of the distribution is the Full Width Half Maximum. One attains this by determining the maximum value of $P(n)$ and then determining the two values n_1 and n_2 so that $P(n_1)$ and $P(n_2)$ are half of this maximum. The full width is then $|n_2 - n_1|$. Use the graph to estimate the FWHM for the distribution.

c) Repeat this for $N = 50, 200$ and 1000. Does the FWHM increase with N? If so does it scale linearly?

d) A crucial feature of the narrowness of such distributions is the value of the FHWM relative to $\overline{n}$. Does the ratio FWHM/$\overline{n}$ increase, decrease or stay constant as N increases?

a) For a single die the probability of each number is $1/6$

We consider all possibilities:

Die 1	Die 2	Sum	Prob
1	1	2	1/36
1	2	3	1/36
2	1	3	...
1	3	4	
2	2	4	all 1/36
3	1	4	
1	4	5	
2	3	5	
3	2	5	
4	1	5	
etc...	⋮		
2	6	8	
3	5	8	
4	4	8	
5	3	8	
6	2	8	

V = Sum	Prob
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

b) $\bar{V} = \sum V\, p(v) = \frac{1}{36}\left[2\times1 + 3\times2 + 4\times3 + 5\times4 + 6\times5 + 7\times6 + 8\times5 + 9\times4 + 10\times3 + 11\times2 + 12\times1\right]$

$= 7$ For a single die the mean is 3.5
So yes this is twice

c) Need $\sigma_V = \sqrt{\overline{V^2} - (\bar{V})^2}$

Then $\overline{V^2} = \sum V^2 p(v) = \frac{1}{36}\left[2^2\times1 + 3^2\times2 + \dots + 12^2\times1\right] = 54.8$

So $\sigma_V = \sqrt{54.8 - 7^2} = 2.41$ For a single die $\sigma = 1.71$ so not twice

a) We expect

$$\text{prob} = \frac{N_{heads}}{N_{total}} \Rightarrow N_{heads} = \text{prob} \times N_{total}$$

$$= 0.5 \times 100 = 50$$

b) The simulation gave 46 heads.

$\langle H \rangle = 46$

Does not match exactly

c) Simulation gives: (504 trials)

most commonly occuring : 50.5 (means 50)

average 50.246

It matches much better

d) Again predict 5000

we get 5010

Much more accurate than 100 flips

e) Did 104 trials.

$\langle H \rangle = 5005.683$

Slightly better than one trial

Narrower histogram

f) More trials $\Rightarrow$ more accurate...

Let N = total number steps

N_L = number left

N_R = " " right and $N = N_L + N_R$

Then position is $x = N_R - N_L$

$= N_R - (N - N_R)$

$= 2N_R - N$

a) We need $x = -2 \Rightarrow -2 = 2N_R - 6$

$\Rightarrow 4 = 2N_R \Rightarrow N_R = 2$

So we need probability that there are two steps right. The probability of k steps right is

$$p(k) = \binom{N}{k} p^k q^{N-k}$$

Here $p = q = 1/2$

$$p(2) = \binom{6}{2}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{N-k} = \binom{6}{2}\frac{1}{2^N}$$

$$= \frac{1}{2^6}\frac{6!}{4!\,2!} = \frac{1}{2^6}\frac{6\times5}{2} = \frac{15}{2^6}$$

$$p(2) = \frac{15}{64} = 0.23$$

b) $P(k) = \binom{N}{k} p^k q^{N-k}$

$p = 1/3$

$q = 1-p = 2/3$

$P(2) = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$

$= 15 \; \frac{1}{36} \; 2^4 = \frac{240}{9\times9\times9} = \frac{80}{243} = 0.33$

a) $$\sum P(n) = \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n}$$

But the binomial theorem gives:

$$(p+q)^N = \sum_{k=0}^{N} \binom{N}{k} p^k q^{N-k}.$$

and clearly

$$(p+q)^N = \sum_{n=0}^{N} P(n)$$

But $p+q=1$ here and thus

$$\sum_{n=0}^{N} P(n) = 1.$$

b) $$\sigma = \sqrt{\overline{n^2} - \bar{n}^2}$$

We have seen that

$$\bar{n} = Np$$

and

$$\overline{n^2} = \bar{n}^2 + pqN$$

Thus. $\overline{n^2} - \bar{n}^2 = pqN$

$$\Rightarrow \quad \sigma = \sqrt{pqN} = \sqrt{p(1-p)}\sqrt{N}$$

c)

$$\frac{\sigma}{\bar{n}} = \frac{\sqrt{p(1-p)}\,\sqrt{N}}{Np}$$

$$= \sqrt{\frac{1-p}{p}}\,\frac{1}{\sqrt{N}}$$

As $N \to \infty$ this approaches zero.

a) $P(n) = \binom{N}{n} p^n (1-p)^{N-n}$

$= \binom{6}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{6-n}$

$= \frac{1}{2^6}\binom{6}{n}$

$\binom{6}{2} = \frac{6!}{4!\,2!} = \frac{30}{2} = 15$

$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6.5.4}{6} = 20$

n	$\binom{6}{n}$	
0	1	$1/64$
1	6	$6/64$
2	15	$15/64$
3	20	$20/64$
4	15	$15/64$
5	6	$6/64$
6	1	$1/64$

$\bar{n} = \sum n P(n) = \frac{1}{64}\left[0\times1 + 1\times6 + 2\times15 + 3\times20 + 4\times15 + 5\times6 + 6\times1\right]$

$= \frac{1}{64}\left[6+30+60+60+30+6\right] = \frac{192}{64} = 3 \Rightarrow \bar{n} = 3$

$\sigma = \sqrt{\overline{n^2} - \bar{n}^2}$

From previous problem $\overline{n^2} = \bar{n}^2 + Np(1-p) \Rightarrow \overline{n^2} - \bar{n}^2 = Np(1-p)$

$= \frac{6}{4} = \frac{3}{2}$

$\sigma = \sqrt{\frac{3}{2}}$

b) max (at $n=5$) is 0.246

half max is 0.123

range for half max is $3.1 \to 6.9$

$FWHM = 3.8$

c)

N	range	FWHM	$\bar{n}$	FWHM/$\bar{n}$
50	$20.6 \to 29.4$	8.8	25	0.352
200	$91.5 \to 108.5$	19	100	0.19
1000	$481 \to 518$	37	500	0.074

It does increase, but not linearly

d) Using data above we see $\frac{FWHM}{\bar{n}}$ decreases as N increases