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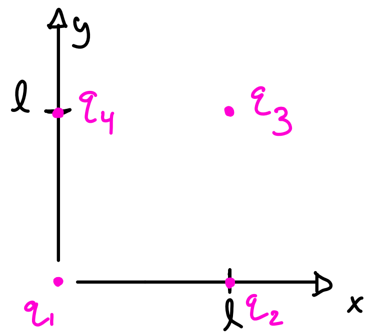
Dr. Middleton
PHYS 311 HW 4

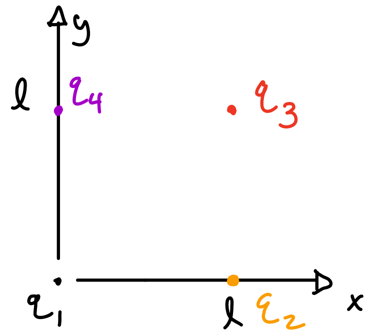
Problem 1

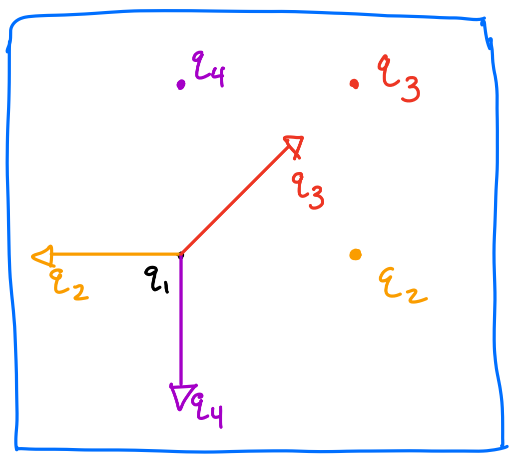
a.)



$q_1 = q$, $q_2 = 2q$
$q_3 = -q$, $q_4 = 3q$

b.)





c.) $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$ $\vec{\mathscr{r}} = r - r'$, $\hat{\mathscr{r}} = \frac{r - r'}{|\mathscr{r}|}$ r' — Source point, r — Field point

Q_1 is test charge

$\vec{F}_{q_2Q_1}$: $q_2 = (l,0)$, $Q_1 = (0,0)$ $\vec{\mathscr{r}} = [0,0] - [l,0]$: $\vec{\mathscr{r}} = -l\hat{x} + 0\hat{y}$ $q = 2q$, $Q = q$

$|\vec{\mathscr{r}}| = \sqrt{-l^2 + 0^2} = l$: $\hat{\mathscr{r}} = \frac{-l\hat{x}}{\hat{x}} = -\hat{x}$

$$F = -\frac{1}{4\pi\varepsilon_0}\frac{(q)(2q)}{l^2}\hat{x} = -\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{l^2}\hat{x}$$

$\vec{F}_{q_3Q_1}$: $q_3(l,l)$, $Q_1(0,0)$ $\vec{\mathscr{r}} = (l,l) - (0,0) = (l,l)$: $\vec{\mathscr{r}} = l\hat{x} + l\hat{y}$ $\hat{\mathscr{r}} = \frac{l\hat{x} + l\hat{y}}{\sqrt{2}\,l} = \frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}$

$|\vec{\mathscr{r}}| = \sqrt{l^2 + l^2} = \sqrt{2l^2} = \sqrt{2}\,l$ $q = -q$, $Q = q$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{(q)(-q)}{2l^2}\left(\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right) = \frac{-1}{4\pi\varepsilon_0}\frac{q^2}{2l^2}\left(\frac{\hat{x}}{\sqrt{2}} + \frac{\hat{y}}{\sqrt{2}}\right) = \frac{+q^2}{8\pi\sqrt{2}\varepsilon_0 l^2}\hat{x} + \frac{q^2}{8\pi\sqrt{2}\varepsilon_0 l^2}\hat{y}$$

$$+\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{l^2}\cdot\frac{1}{2\sqrt{2}} + \frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{l^2}\cdot\frac{1}{2\sqrt{2}}$$

Problem 1] Continued

$F_{q_4Q_1}$: $q_4(0,\ell)$ $\vec{r} = (0,0) - (0,\ell) = (0,-\ell)$: $\vec{r} = 0\hat{x} - \ell\hat{y}$ $\hat{r} = -\frac{\ell\hat{y}}{\ell} = -\hat{y}$ $q = 3e$
$Q_1(0,0)$ $|\vec{r}| = \sqrt{\ell^2+0} = \ell$ $Q = e$

$$F = -\frac{1}{4\pi\varepsilon_0}\frac{(3e)(e)}{\ell^2}\hat{y} = -\frac{3e^2}{4\pi\varepsilon_0\ell^2}\hat{y}$$

Law of Superposition :

$$\sum \vec{F}_x = -\frac{2e^2}{4\pi\varepsilon_0\ell^2} + \frac{e^2}{8\pi\varepsilon_0\sqrt{2}\ell^2}\hat{x} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{\ell^2}\left[-2 + \frac{1}{2\sqrt{2}}\right]\hat{x}$$

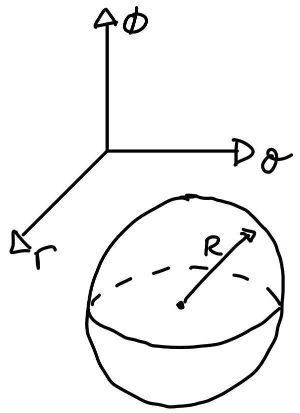
$$\sum F_y = \frac{-e^2}{8\pi\varepsilon_0\sqrt{2}\ell^2} - \frac{3e^2}{4\pi\varepsilon_0\ell^2} = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{\ell^2}\left[\frac{1}{2\sqrt{2}} - 3\right]\hat{y}$$

$$\therefore \boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{\ell^2}\left[\left(\frac{1}{2\sqrt{2}} - 2\right)\hat{x} + \left(\frac{1}{2\sqrt{2}} - 3\right)\hat{y}\right]}$$

Problem 2

$\rho(\theta) = \rho_0 \cos^2\theta$

a.)



$Q_{net} = \int_S \rho \, dv \qquad dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$Q = \rho_0 \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \cos^2(\theta) \sin(\theta) \, dr \, d\theta \, d\phi$$

$$\rho_0 \int_0^{2\pi} \int_0^{\pi} \left[\frac{r^3}{3} \cos^2(\theta) \sin\theta \right]_0^R d\theta \, d\phi$$

$$\frac{\rho_0 R^3}{3} \int_0^{2\pi} \int_0^{\pi} \cos^2(\theta) \sin\theta \, d\theta \, d\phi$$

$$-\frac{\rho_0 R^3}{9} \int_0^{2\pi} \cos^3(\theta) \Big|_0^{\pi} d\theta \, d\phi \qquad \cos^3(\pi) - \cos(0) = -2$$

$$\frac{2\rho_0 R^3}{9} \int_0^{2\pi} d\phi = \frac{4\pi \rho_0 R^3}{9}$$

$\int \cos^2(\theta) \sin(\theta)$

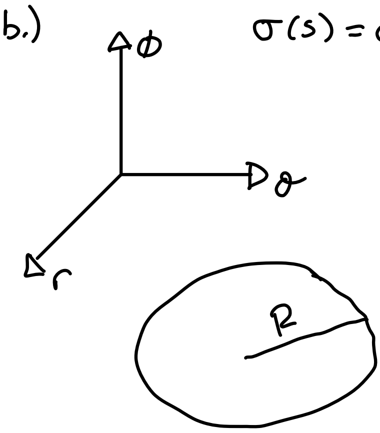
$u = \cos(\theta)$

$du = -\sin(\theta) \, d\theta$

$-\int u^2 \, du = -\frac{1}{3} \cos^3\theta$

$$\boxed{Q_{net} = \frac{4\pi \rho_0 R^3}{9} \; C}$$

b.)



$\sigma(s) = \sigma_0 \frac{R}{s}$

$Q_{net} = \int_S \sigma(s) \, dA \quad : \quad dA = s \, ds \, d\phi$

$$Q_{net} = Q_0 R \int_0^{2\pi} \int_0^{R} ds \, d\phi$$

$$Q_0 R \int_0^{2\pi} s \Big|_0^R d\phi$$

$$Q_0 R^2 \int_0^{2\pi} d\phi$$

$$Q_0 R^2 [\phi]_0^{2\pi}$$

$Q_{net} = 2\pi Q_0 R^2 \quad C$

$$\boxed{Q_{net} = 2\pi Q_0 R^2 \quad C}$$

Problem 2

a.)

$\rho(\theta) = \rho_0 \cos^2\theta$ Find Q_{net} : $\rho \equiv \frac{q}{V}$ $\therefore$ $q = \rho \cdot V$ $d\tau = r^2 \sin\theta \, dr d\theta d\phi$

$$\int_V \rho(\theta)\, d\tau = \rho_0 \int_0^{2\pi}\int_0^{\pi}\int_0^R r^2 \cdot \cos^2\theta \sin\theta \, dr d\theta d\phi$$

$$= \rho_0 \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}\Big|_0^R \cos^2\theta \sin\theta \, d\theta d\phi$$

$u = \cos(\theta)$

$du = -\sin(\theta)\, d\theta$

$\int u^2 = \frac{u^3}{3}$

$$= \frac{\rho_0 R^3}{3}\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta d\phi$$

$$= -\frac{\rho_0 R^3}{3}\int_0^{2\pi} \cos^3(\theta)\Big|_0^{\pi} d\theta d\phi$$

$\cos^3(\pi) - \cos^3(0)$

$-1 - 1 = -2$

$$= \frac{2}{3}\cdot \rho_0 R^3 \int_0^{2\pi} d\phi = \frac{2}{3}\rho_0 R^3 \, \phi\Big|_0^{2\pi} = \frac{4\pi}{3}\cdot \rho_0 R^3$$

$$\boxed{Q = \frac{4\pi}{3}\cdot \rho_0 R^3}$$

b.)

$\sigma(s) = \sigma_0 \cdot \frac{R}{s}$ $\sigma \equiv \frac{q}{A}$ $\therefore$ $q = \sigma \cdot A$: $da = s\, ds\, d\phi$

$$\int_S \sigma(s)\, dA = \sigma_0 \cdot R \int_0^{2\pi}\int_0^R ds\, d\phi$$

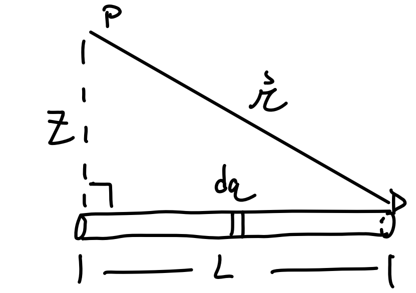
$$\sigma_0 \cdot R \int_0^{2\pi} s\Big|_0^R d\phi$$

$$\sigma_0 \cdot R^2 \int_0^{2\pi} d\phi$$

$$\sigma_0 \cdot R^2 \cdot \phi\Big|_0^{2\pi} = \sigma_0 \cdot 2\pi R^2$$

$$\boxed{Q = \sigma_0 \cdot 2\pi R^2}$$

Problem 31



$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{r} = [0, z]$, $\vec{r}\,' = [x, 0]$ $\quad dl' = dx$

$\vec{\mathscr{r}} = [0, z] - [x, 0] = -x\hat{x} + z\hat{z}$

$\mathscr{r} = \sqrt{x^2+z^2}$, $\mathscr{r}^2 = x^2+z^2$ $\quad \hat{\mathscr{r}} = \frac{-x\hat{x}+z\hat{z}}{\sqrt{x^2+z^2}}$

used calculator to integrate

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' \quad \vdots \quad \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda}{x^2+z^2}\cdot\frac{-x\hat{x}+z\hat{z}}{\sqrt{x^2+y^2}}\,dx$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[z\hat{z}\int_0^L \frac{1}{(x^2+z^2)^{3/2}}\,dx - \hat{x}\int_0^L \frac{x}{(x^2+z^2)^{3/2}}\,dx\right] \qquad u = \tan^{-1}\left(\frac{x}{z}\right)$$

$$z\hat{z}\int_0^L \frac{1}{(x^2+z^2)^{3/2}}\,dx \qquad x = z\tan u$$

$$\frac{dx}{du} = z\sec^2 u \quad \vdots \quad dx = z\cdot\sec^2 u\,du$$

$$z\hat{z}\int_0^L \frac{1}{(z^2+z^2\tan^2 u)^{3/2}}\,z\cdot\sec^2 u\,du$$

$$z\hat{z}\int_0^L \frac{z\cdot\sec^2 u}{(z^2(1+\tan^2 u))^{3/2}}\,du \qquad \sec^2 u - \tan^2 u = 1 \quad \therefore \quad \sec^2 u = 1+\tan^2 u$$

$$z\hat{z}\int_0^L \frac{z\cdot\sec^2 u}{z^3(\sec^2 u)^{3/2}}\,du = z\hat{z}\int_0^L \frac{z\cdot\sec^2 u}{z^3\cdot\sec^3 u} = z\hat{z}\int_0^L \frac{1}{z^2\sec(u)}\,du$$

$$\sec = \frac{1}{\cos} \quad \therefore \quad z\hat{z}\int_0^L \frac{\cos(u)}{z^2}\,du = \frac{1}{z}\Big[\sin(u)\Big]_0^L = \frac{1}{z}\Big[\sin\left(\tan^{-1}\left(\tfrac{x}{z}\right)\right)\Big]_0^L$$

$$\sin(\arctan(x)) = \frac{x\sqrt{1+x^2}}{1+x^2} \quad \vdots \quad \frac{1}{z}\left(\frac{\frac{x}{z}\sqrt{1+\left(\frac{x}{z}\right)^2}}{\left(1+\left(\frac{x}{z}\right)^2\right)}\right) = \frac{x}{z^2\sqrt{z^2+x^2}}\Bigg|_0^L$$

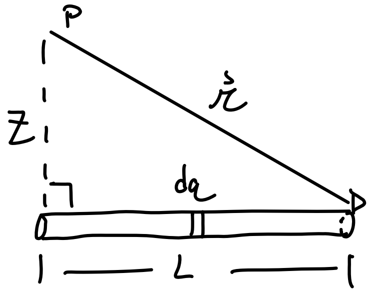
$$z\hat{z}\cdot\frac{x}{z^2\sqrt{z^2+x^2}}\Bigg|_0^L = \frac{L}{z\sqrt{z^2+L^2}}\hat{z} \quad \therefore \quad \boxed{\frac{1}{4\pi\varepsilon_0}\left[\frac{\lambda L}{z\sqrt{z^2+L^2}}\right]\hat{z}}$$

$$-\hat{x}\int_0^L \frac{x}{(x^2+z^2)^{3/2}}\,dx \qquad \begin{aligned} u &= x^2+z^2 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x}\end{aligned} \Rightarrow -\hat{x}\frac{1}{2}\int \frac{1}{u^{3/2}}\,du \Rightarrow -\frac{\hat{x}}{2}\int u^{-3/2}\,du$$

$$-\frac{\hat{x}}{2}\left[-2u^{-1/2}\right] \Rightarrow \hat{x}\Big[(x^2+z^2)^{-1/2}\Big]_0^L \Rightarrow \hat{x}\left[\frac{1}{\sqrt{x^2+z^2}}\right]_0^L \Rightarrow \boxed{\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right]\hat{x}}$$

$$\boxed{\vec{E}(r) = \frac{\lambda}{4\pi\varepsilon_0}\left[\left(\frac{L}{\sqrt{L^2+z^2}} - \frac{1}{z}\right)\hat{x} + \left(\frac{L}{z\sqrt{z^2+L^2}}\right)\hat{z}\right]}$$

Problem 3



Find $\vec{E}$ @ P $\qquad E(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl'$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, $\vec{r} = [0, z]$ $\quad \vec{r}\,' = [x, 0]$ $\quad dl' = dx$

$\vec{\mathscr{r}} = [0, z] - [x, 0] = [-x, z]$

$\mathscr{r}^2 = (x^2 + z^2)$, $\hat{\mathscr{r}} = \frac{-\hat{x}x + z\hat{z}}{\sqrt{x^2+z^2}}$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{-x\hat{x} + z\hat{z}}{(x^2+z^2)} \cdot \frac{\lambda}{\sqrt{x^2+z^2}}\,dx$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_0^L \frac{-x}{(x^2+z^2)^{3/2}}\,dx + z\int_0^L \frac{1}{(x^2+z^2)^{3/2}}\,dx\right]$$

$$z\int_0^L \frac{1}{(x^2+z^2)^{3/2}}\,dx\,\hat{z}$$

$x = z\tan(u)$
$dx = z\cdot\sec^2(u)\,du$

$\sin^2(u) + \cos^2(u) = 1$
$\sec^2(u) - \tan^2(u) = 1$
$\csc^2(u) - \cot^2(u) = 1$

$$z\int_0^L \frac{1}{(z^2\cdot\tan^2(u) + z^2)^{3/2}}\,dx = z\int_0^L \frac{1}{(z^2(\tan^2(u)+1))^{3/2}}\,dx$$

$$z\int_0^L \frac{1}{(z^2\cdot\sec^2(u))^{3/2}}\,dx = z\int_0^L \frac{z\cdot\sec^2(u)}{z^3\cdot\sec^3(u)}\,du = \int_0^L \frac{1}{z\cdot\sec(u)}\,du$$

$$\frac{1}{z}\int_0^L \cos(u)\,du = \frac{1}{z}\Big[\sin(u)\Big]_0^L \quad : \quad u = \tan^{-1}\left(\frac{x}{z}\right)$$

$$\frac{1}{z}\cdot\left[\sin\left(\tan^{-1}\left(\frac{x}{z}\right)\right)\right]_0^L = \frac{1}{z}\left[\frac{x/z}{\sqrt{(x/z)^2+1}}\right]_0^L = \frac{1}{z}\left[\frac{L/z}{\sqrt{L^2+z^2}}\right]\hat{z}$$

$$z\int_0^L \frac{1}{(x^2+z^2)^{3/2}}\,dx\,\hat{z} = \frac{L}{z^2\sqrt{L^2+z^2}}\,\hat{z}$$

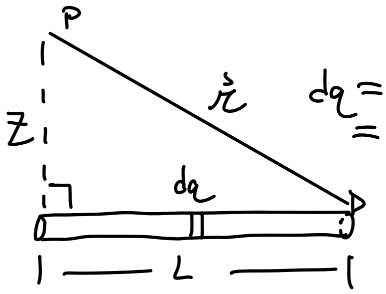
$V = x^2 + z^2$
$dv = 2x\,dx \; : \; dx = \frac{dv}{2x}$

$$\int_0^L \frac{-x}{(x^2+z^2)^{3/2}}\,dx\,\hat{x} = \frac{-1}{2}\int_0^L \frac{1}{(V)^{3/2}}\,dv = \frac{-1}{2}\int_0^L V^{-3/2}\,dv = \frac{-1}{2}\Big[-2V^{-1/2}\Big]_0^L = \frac{1}{\sqrt{V}} = \frac{1}{\sqrt{x^2+z^2}}\Bigg|_0^L$$

$$\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\,\hat{x}$$

$$\therefore \boxed{\vec{E} = \frac{\lambda}{4\pi\varepsilon_0}\left[\left[\frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}\right]\hat{x} + \left[\frac{L}{z^2\sqrt{L^2+z^2}}\right]\hat{z}\right]}$$

Problem 4



$$dq = \lambda\, dx = \lambda_0 \frac{x^2}{L^2}\, dx$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\, \hat{\mathscr{r}}\, dl' , \quad \lambda(x) = \lambda_0 \frac{x^2}{L^2}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' , \quad \vec{r} = [0, z] , \quad \vec{r}\,' = [x, 0]$$

$$\vec{\mathscr{r}} = [0, z] - [x, 0] = -x\hat{x} + z\hat{z}$$

$$\mathscr{r} = \sqrt{x^2+z^2} , \quad \mathscr{r}^2 = x^2 + z^2 , \quad \hat{\mathscr{r}} = \frac{-x\hat{x} + z\hat{z}}{\sqrt{x^2+z^2}}$$

(1) (2)

$$\vec{E}(r) = \frac{\lambda_0}{4\pi\varepsilon_0 L^2} \left[\hat{z}\, z \int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}\, dx - \hat{x} \int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}\, dx \right]$$

(1) $z\hat{z}\int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}\, dx$: $z\int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}\, dx$ used calculator to integrate

$$z\int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}\, dx = z\left(\frac{\sqrt{x^2+z^2} \cdot \ln(|\sqrt{x^2+z^2} + x|) - x}{\sqrt{x^2+z^2}} \right)\Bigg|_0^L \hat{z}$$

$$z\left(\left[\frac{\sqrt{L^2+z^2} \cdot \ln(|\sqrt{L^2+z^2} + L) - L}{\sqrt{L^2+z^2}} \right] - \left[\frac{z \cdot \ln(z)}{z} \right] \right) \hat{z}$$

$$\hookrightarrow = \boxed{\left[\frac{z\left(\sqrt{z^2+L^2} \cdot \ln(|\sqrt{z^2+L^2} + L|) - \sqrt{z^2+L^2} \cdot \ln(|z|) - L\right)}{\sqrt{z^2+L^2}} \right] \hat{z}}$$

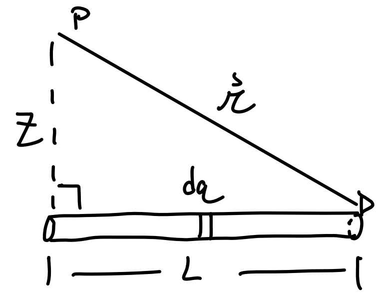
(2) $-\hat{x}\int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}\, dx$ used calculator to integrate

$$-\hat{x}\int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}\, dx = -\hat{x}\left(\frac{x^2+2z^2}{\sqrt{x^2+z^2}} \right)\Bigg|_0^L \Rightarrow -\hat{x}\left[\left(\frac{L^2+2z^2}{\sqrt{L^2+z^2}} \right) - \left(\frac{0^2+2z^2}{\sqrt{0^2+z^2}} \right) \right]$$

$$\hookrightarrow = \boxed{-\left[\left(\frac{L^2+2z^2}{\sqrt{L^2+z^2}} \right) - 2z \right] \hat{x}}$$

$$\boxed{\vec{E}(r) = \frac{\lambda_0}{4\pi\varepsilon_0 L^2} \left[-\left[\left(\frac{L^2+2z^2}{\sqrt{L^2+z^2}} \right) - 2z \right] \hat{x} + \left[\frac{z\left(\sqrt{z^2+L^2} \cdot \ln(|\sqrt{z^2+L^2} + L|) - \sqrt{z^2+L^2} \cdot \ln(|z|) - L\right)}{\sqrt{z^2+L^2}} \right] \hat{z} \right]}$$

Problem 4



$dq = \lambda\, dx$

$\lambda = \lambda_0 \dfrac{x^2}{L^2}$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = [0, z] - [x, 0] = [-x, z]$

$|\vec{\mathscr{r}}| = \sqrt{x^2+z^2}$

$\mathscr{r}^2 = x^2+z^2 \qquad \hat{\mathscr{r}} = \dfrac{-x\hat{x} + z\hat{z}}{\sqrt{x^2+z^2}}$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, dl' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\lambda_0}{L^2}\int_0^L \frac{-x\hat{x}+z\hat{z}}{(x^2+z^2)^{3/2}}\cdot x^2\, dx$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\lambda_0}{L^2}\left[-\hat{x}\int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}dx + z\int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}\hat{z}\, dx\right]$$

$-\hat{x}\displaystyle\int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}dx \qquad u = x^2+z^2 \quad x^2 = u - z^2$

$du = 2x\, dx$

$$\frac{-\hat{x}}{2}\int_0^L \frac{u-z^2}{(u)^{3/2}}du = \frac{-\hat{x}}{2}\left[\int_0^L u^{-1/2}du - z^2\int_0^L u^{-3/2}du\right]$$

$$\frac{-\hat{x}}{2}\left[2\cdot u^{1/2}\Big|_0^L + 2z^2\left(u^{-1/2}\right)\Big|_0^L\right]$$

$$-\frac{\hat{x}}{2}\left[2\sqrt{x^2+z^2}\Big|_0^L + \frac{2z^2}{\sqrt{x^2+z^2}}\Big|_0^L\right]$$

$$-\frac{\hat{x}}{2}\left[\left[2\sqrt{L^2+z^2} - 2z\right] + \left[\frac{2z^2}{\sqrt{L^2+z^2}} - 2z\right]\right]$$

$$\left[\left[\sqrt{L^2+z^2} - z\right] + \left[\frac{z^2}{\sqrt{L^2+z^2}} - z\right]\right] -\hat{x}$$

$$-\hat{x}\int_0^L \frac{x^3}{(x^2+z^2)^{3/2}}dx = \left[\frac{L^2+2z^2}{\sqrt{L^2+z^2}} - 2z\right]-\hat{x} = \left[2z - \frac{L^2+2z^2}{\sqrt{L^2+z^2}}\right]\hat{x}$$

$z\displaystyle\int_0^L \frac{x^2}{(x^2+z^2)^{3/2}}dx\,\hat{z} \qquad u = x^2 \qquad V' = (x^2+z^2)^{-3/2}$

$du = 2x\, dx \qquad V = \dfrac{x}{z\sqrt{x^2+z^2}}$

$\displaystyle\int \frac{1}{(x^2+z^2)^{3/2}}dx \qquad x = z\tan(u)$

$dx = z\sec^2(u)\, du$

$\displaystyle\int \frac{1}{(z^2\cdot\sec^2(u))^{3/2}}\cdot z\sec^2(u)\, du$

$\displaystyle\int \frac{z\cdot\sec^2(u)}{z^3\cdot\sec^3(u)}du$

$\dfrac{1}{z^2}\displaystyle\int \cos(u)\, du$

$\dfrac{\sin(u)}{z^2} \qquad u = \tan^{-1}\left(\frac{x}{z}\right)$

$\dfrac{x/z}{\sqrt{x^2/z^2+1}} = \dfrac{x}{z\sqrt{x^2+z^2}}$

$$\underline{\frac{x^3}{z\sqrt{x^2+z^2}}\Big|_0^L - \frac{2}{z}\int_0^L \frac{x^2}{\sqrt{x^2+z^2}}dx}$$

$\dfrac{2}{z}\displaystyle\int_0^L \frac{x^2}{\sqrt{x^2+z^2}}dx\,\hat{z} \qquad x = z\tan(u)$

$dx = z\cdot\sec^2(u)\, du$

$\dfrac{2}{z}\displaystyle\int_0^L \frac{z^2\cdot\tan^2(u)\cdot z\cdot\sec^2(u)\, du}{z\cdot\sec(u)}$

$\dfrac{2}{z}\displaystyle\int_0^L z\cdot\tan(u)\cdot\sec(u)\, du \qquad u = \tan^{-1}\left(\frac{x}{z}\right)$

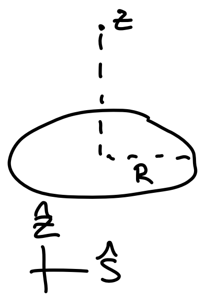
$$2\left[\sec(u)\right]_0^L = 2\left[\sec\left(\tan^{-1}\left(\frac{x}{z}\right)\right)\right]_0^L = \frac{2\sqrt{x^2+z^2}}{z}\Big|_0^L$$

$$\frac{x^3}{z\sqrt{x^2+z^2}}\Big|_0^L - \frac{2\sqrt{x^2+z^2}}{z}\Big|_0^L = \frac{L^3}{z\sqrt{L^2+z^2}} - \left(\frac{2\sqrt{L^2+z^2}}{z} - 2\right)\hat{z}$$

$$\frac{2}{z}\int_0^L \frac{x^2}{\sqrt{x^2+z^2}}\,dx\,\hat{z} = \frac{L^3}{z\sqrt{L^2+z^2}} - \frac{2\sqrt{L^2+z^2}}{z} + 2\,\hat{z}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{\lambda}{L^2}\left[\left[2z - \frac{L^2+2z^2}{\sqrt{L^2+z^2}}\right]\hat{x} + \left[\frac{L^3}{z\sqrt{L^2+z^2}} - \frac{2\sqrt{L^2+z^2}}{z} + 2\right]\hat{z}\right]}$$

Problem 5



$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' \qquad \sigma(r') = \sigma \qquad da' = s\,ds\,d\phi$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' \; : \; \vec{r} = [0, z] \quad \vec{r}' = [s, 0]$$

$$\vec{\mathscr{r}} = [0, z] - [s, 0] = -s\hat{s} + z\hat{z} \qquad |\vec{\mathscr{r}}| = \sqrt{s^2+z^2}, \; \mathscr{r}^2 = s^2+z^2$$

$$\hat{\mathscr{r}} = \frac{-s}{\sqrt{s^2+z^2}}\hat{s} + \frac{z}{\sqrt{s^2+z^2}}\hat{z}$$

<u>used calculator to integrate</u>

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{-\sigma\int_0^{2\pi}\int_0^R \frac{s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{s}}_{(1)} + \underbrace{\sigma z\int_0^{2\pi}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{z}}_{(2)}\Bigg]$$

$$(1)^* \quad -\sigma\int_0^{2\pi}\int_0^R \frac{s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{s} \rightarrow -\sigma\int_0^{2\pi}\left[\frac{\sqrt{s^2+z^2}\cdot\ln(|\sqrt{s^2+s}+s|) - s}{\sqrt{s^2+z^2}}\right]_0^R d\phi\,\hat{s}$$

$$-\sigma\int_0^{2\pi}\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\,d\phi\;\hat{s}$$

$$\rightarrow -\sigma\left[\left(\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\right)\phi\,\Big|_0^{2\pi}\right]\hat{s}$$

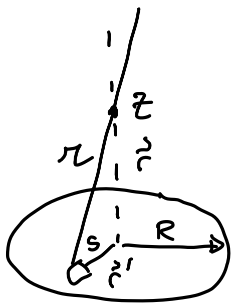
$$\rightarrow \boxed{-2\pi\sigma\left[\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\right]\hat{s}}$$

$$(2)^* \quad \sigma z\hat{z}\int_0^{2\pi}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{z} \rightarrow -\sigma z\hat{z}\int_0^{2\pi}\frac{1}{\sqrt{s^2+z^2}}\Big|_0^R d\phi\,\hat{z} \rightarrow -\sigma z\hat{z}\int_0^{2\pi}\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\,d\phi$$

$$-\sigma z\hat{z}\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]_0^{2\pi} \rightarrow \boxed{-2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{z}}$$

$$\boxed{\vec{E}(r) = \frac{\sigma}{2\varepsilon_0}\left[-\left[\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\right]\hat{s} - \left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}\right]}$$

Problem 5



$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' \qquad \sigma(r') = \sigma$$

$$da' = s\,ds\,d\phi$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' , \quad \vec{r} = z\hat{z} \quad \vec{r}\,' = s\hat{s}$$

$$\vec{\mathscr{r}} = [0, z] - [s, 0] = -s\hat{s} + z\hat{z}$$

$$|\vec{\mathscr{r}}| = \sqrt{s^2+z^2} , \quad \mathscr{r}^2 = s^2+z^2 , \quad \hat{\mathscr{r}} = \frac{-s\hat{s}+z\hat{z}}{\sqrt{s^2+z^2}}$$

$$\vec{E}(r) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{-s\hat{s}+z\hat{z}}{(s^2+z^2)^{3/2}}\, s\,ds\,d\phi$$

① ②

$$\frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{-s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{s} + \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{sz}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{z}$$

①

$$\frac{-\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{s} \qquad u = s^2 \quad du = 2s\,ds \qquad V' = (s^2+z^2)^{-3/2} \quad V = \frac{s}{z^2\sqrt{s^2+z^2}}$$

$$\int \frac{1}{(s^2+z^2)^{3/2}}\,ds \qquad s = z\tan\theta , \quad ds = z\cdot\sec^2(\theta)\,d\theta$$

$$s^2 = z^2\cdot\tan^2\theta \quad \therefore \quad s^2+z^2 = z^2\tan^2\theta + z^2 = z^2(1+\tan^2\theta) = z^2\sec^2(\theta)$$

$$\int \frac{z\cdot\sec^2(\theta)}{(z^2\cdot\sec^2(\theta))^{3/2}}\,d\theta = \int \frac{z\cdot\sec^2(\theta)}{z^3\cdot\sec^3(\theta)}\,d\theta = \frac{1}{z^2}\int\frac{1}{\sec\theta}\,d\theta = \frac{1}{z^2}\int\cos\theta\,d\theta$$

$$\frac{1}{z^2}\cdot\sin\theta\Big|_{\theta=\tan^{-1}(s/z)} = \frac{1}{z^2}\cdot\sin\left(\tan^{-1}\left(\frac{s}{z}\right)\right) = \frac{1}{z^2}\cdot\frac{s}{\sqrt{s^2+z^2}}$$

$$\frac{s^2\cdot s}{z^2\sqrt{s^2+z^2}}\Big|_0^R - 2\int_0^R \frac{s^2}{\sqrt{s^2+z^2}}\,ds \Longleftrightarrow \frac{R^3}{z^2\sqrt{R^2+z^2}} - 2\int_0^R \frac{s^2}{z^2\sqrt{s^2+z^2}}\,ds \qquad (1^*)$$

$$(1^*) \quad -2\int_0^R \frac{s^2}{z^2\sqrt{s^2+z^2}}\,ds \qquad s = z\tan(u) \quad s^2 = z^2\tan^2(u) \qquad ds = z\cdot\sec^2(u)\,du \qquad s^2+z^2 = z^2\sec^2(u)$$

Ignore til end

$$z^2\tan^2(u) + z^2$$

$$\frac{1}{z^2}\int_0^R \frac{z^2\tan^2(u)\cdot z\cdot\sec^2(u)}{z\cdot\sec(u)}\,du = \frac{1}{z^2}\int_0^R z^2\cdot\tan^2(u)\cdot\sec(u)\,du$$

$$\int_0^R \tan^2(u)\cdot\sec(u)\,du = -\int_0^R (\sec^2(u)-1)\sec u\,du$$

$$\int_0^R \sec^3(u)\,du - \int \sec(u)\,du \qquad \sec^2(u) - \tan^2(u) = 1$$

(A) (B)

$$(B) \quad -\int \sec(u)\,du = -\ln\left|\tan(u)+\sec(u)\right|_{u=\tan^{-1}(s/z)} \quad \therefore \quad -\ln\left|\frac{s}{z}+\frac{\sqrt{s^2+z^2}}{z}\right|$$

$$(B) \quad -\int \sec(u)\,du = -\ln\left|\frac{s+\sqrt{s^2+z^2}}{z}\right|\Big|_0^R = \underline{\underline{-\ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right|}}$$

$$(A)\ \int_0^R \sec^3(u)\,du = \frac{\sec^2(u)\sin(u)}{2} + \frac{1}{2}\int \sec(u)\,du$$

$$u = \tan^{-1}\left(\frac{s}{z}\right) \qquad \frac{\tan(u)\sec(u)}{2} + \frac{1}{2}\ln\left|\tan(u)+\sec(u)\right|$$

$$\frac{1}{2}\left[\tan\left(\tan^{-1}\left(\frac{s}{z}\right)\right)\sec\left(\tan^{-1}\left(\frac{s}{z}\right)\right) + \ln\left|\tan\left(\tan^{-1}\left(\frac{s}{z}\right)\right) + \sec\left(\tan^{-1}\left(\frac{s}{z}\right)\right)\right|\right]$$

$$\frac{1}{2}\left[\frac{s}{z}\cdot\frac{\sqrt{s^2+z^2}}{z} + \ln\left|\frac{s}{z} + \frac{\sqrt{s^2+z^2}}{z}\right|\right] = \frac{1}{2}\left[\frac{s\sqrt{s^2+z^2}}{z^2} + \ln\left|\frac{s+\sqrt{s^2+z^2}}{z}\right|\right]$$

$$\frac{1}{2}\left[\frac{s\sqrt{s^2+z^2}}{z^2} + \ln\left|\frac{s+\sqrt{s^2+z^2}}{z}\right|\right]_0^R = \frac{1}{2}\left[\frac{R\sqrt{R^2+z^2}}{z^2} + \ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right|\right]$$

$$(A)\ \int_0^R \sec^3(u)\,du = \frac{1}{2}\left[-\frac{R\sqrt{R^2+z^2}}{z^2} + \ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right|\right]$$

$$-2\int_0^R \sec^3(u)\,du = \left[\ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right| - \frac{R\sqrt{R^2+z^2}}{z^2}\right]$$

$$① = \frac{R^3}{z^2\sqrt{R^2+z^2}} + (1^*) = \frac{R^3}{z^2\sqrt{R^2+z^2}} + \ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right| - \frac{R\sqrt{R^2+z^2}}{z^2}$$

$$\boxed{① = \frac{-\sigma}{2\varepsilon_0}\left[\frac{R^3}{z^2\sqrt{R^2+z^2}} + \ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right| - \frac{R\sqrt{R^2+z^2}}{z^2}\right]\hat{s}}$$

②

$$\frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R \frac{sz}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{z}$$

$$u = s^2+z^2 \ :\ du = 2s\,ds$$

$$z\int_0^{2\pi}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds\,d\phi\,\hat{z}$$

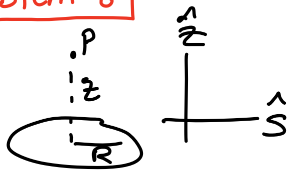
$$\frac{z}{2}\int_0^{2\pi}\int_0^R u^{-3/2}\,du\,d\phi\,\hat{z} = -z\int_0^{2\pi}\frac{1}{\sqrt{s^2+z^2}}\bigg|_0^R d\phi\,\hat{z} = -z\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right]\int_0^{2\pi} d\phi\,\hat{z}$$

$$\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}\,\phi\,\bigg|_0^{2\pi} = 2\pi\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}$$

$$\boxed{② = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}}$$

$$\boxed{\vec{E}(r) = \frac{-\sigma}{2\varepsilon_0}\left[\frac{R^3}{z^2\sqrt{R^2+z^2}} + \ln\left|\frac{R+\sqrt{R^2+z^2}}{z}\right| - \frac{R\sqrt{R^2+z^2}}{z^2}\right]\hat{s} + \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}}$$

Problem 6



$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da' \qquad \sigma(r') \equiv \sigma(s) = \sigma_0 \frac{s}{R} \qquad da' = s\,ds\,d\phi$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}', \quad \vec{r} = [0, z], \quad \vec{r}'[s, 0]$$

$$\vec{\mathscr{r}} = -s\hat{s} + z\hat{z} \qquad |\vec{\mathscr{r}}| = \sqrt{s^2+z^2} \qquad \mathscr{r} = s^2+z^2$$

$$\hat{\mathscr{r}} = \frac{-s}{\sqrt{s^2+z^2}}\hat{s} + \frac{z}{\sqrt{s^2+z^2}}\hat{z}$$

used calculator to integrate

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{-\sigma_0}{R}\hat{s}\int_0^{2\pi}\int_0^R \frac{s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi}_{(1^*)} + \underbrace{\frac{\sigma_0}{R} z\hat{z}\int_0^{2\pi}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds\,d\phi}_{(2^*)}\right]$$

$$(1^*)\ \frac{-\sigma_0}{R}\hat{s}\int_0^{2\pi}\int_0^R \frac{s^2}{(s^2+z^2)^{3/2}}\,ds\,d\phi \rightarrow \frac{-\sigma_0}{R}\hat{s}\int_0^{2\pi}\left[\frac{\sqrt{s^2+z^2}\cdot\ln(|\sqrt{s^2+z^2}+s|) - s}{\sqrt{s^2+z^2}}\right]_0^R$$

$$-\frac{\sigma_0}{R}\hat{s}\int_0^{2\pi}\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\,d\phi$$

$$\rightarrow \boxed{\frac{-2\pi\sigma_0}{R}\left[\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\right]\hat{s}}$$

$$(2^*)\ \frac{\sigma_0 z}{R}\hat{z}\int_0^{2\pi}\int_0^R \frac{s}{(s^2+z^2)^{3/2}}\,ds\,d\phi \rightarrow -\frac{\sigma_0 z}{R}\hat{z}\int_0^{2\pi}\left[\frac{1}{\sqrt{s^2+z^2}}\right]_0^R$$

$$-\frac{\sigma_0 z}{R}\hat{z}\int_0^{2\pi}\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\,d\phi \rightarrow \frac{-\sigma_0 z}{R}\hat{z}\left[\frac{\phi}{z} - \frac{\phi}{\sqrt{R^2+z^2}}\right]_0^{2\pi}$$

$$\rightarrow \boxed{-\frac{2\pi\sigma_0}{R}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}}$$

$$\boxed{\vec{E}(r) = \frac{\sigma_0}{2\varepsilon_0 R}\left[-\left(\frac{\sqrt{R^2+z^2}\cdot\ln(|\sqrt{R^2+z^2}+R|) - \sqrt{R^2+z^2}\cdot\ln(|z|) - R}{\sqrt{R^2+z^2}}\right)\hat{s} - \left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}\right]}$$

Problem 7 $\vec{E} = kr^5 \hat{r}$

a) $\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} \quad \therefore \quad \rho = \varepsilon_0 (\vec{\nabla}\cdot\vec{E})$

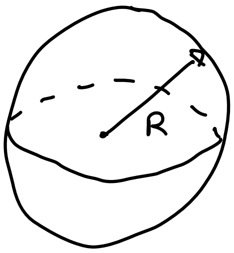
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \cancel{\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin\theta E_\theta)}^{\,0} + \cancel{\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}}^{\,0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(kr^7) = \frac{1}{r^2}(7kr^6) = 7kr^4 \equiv (\vec{\nabla}\cdot\vec{E})$$

$\rho = 7k\varepsilon_0 r^4$

$$\boxed{\rho = 7k\varepsilon_0 r^4}$$

b)
(i)



$$\frac{1}{\varepsilon_0}\int_V \rho\, d\tau = \frac{1}{\varepsilon_0} Q_{enc} \quad : \quad \int_V \rho\, d\tau = Q_{enc}$$

$$Q_{ENC} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 7k\varepsilon_0 r^4 \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$Q_{ENC} = 7k\varepsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^6 \sin\theta\, dr\, d\theta\, d\phi$$

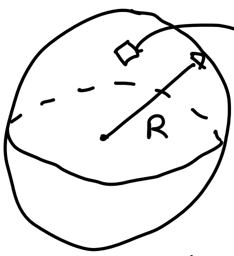
$$7k\varepsilon_0 \int_0^{2\pi}\int_0^{\pi} \frac{r^7}{7}\Big|_0^R \sin\theta\, d\theta\, d\phi$$

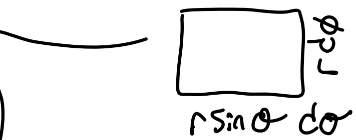
$$k\varepsilon_0 R^7 \int_0^{2\pi} -\cos\theta \Big|_0^{\pi} d\phi$$

$$2k\varepsilon_0 R^7 \int_0^{2\pi} \phi = 4\pi k\varepsilon_0 R^7$$

$$\boxed{Q_{ENC} = 4\pi k\varepsilon_0 R^7\ C}$$

(ii)





$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{ENC}$$

$$Q_{ENC} = \varepsilon_0 \oint_S \vec{E}\, d\vec{a}$$

$d\vec{a} = r^2 \sin\theta\, d\theta\, d\phi\ \hat{r}$

$\vec{E}\cdot d\vec{a} = kr^7 \sin\theta\, d\theta\, d\phi$

$$Q_{ENC} = R^7 k\varepsilon_0 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi \rightarrow R^7 k\varepsilon_0 \int_0^{2\pi} -\cos(\theta)\Big|_0^{\pi} d\phi$$

$$2k\varepsilon_0 R^7 \int_0^{2\pi} d\phi \rightarrow 4\pi k\varepsilon_0 R^7\ C \qquad \boxed{Q_{ENC} = 4\pi k\varepsilon_0 R^7\ C}$$