Taylor Larrechea

Physics 396

Exam 2

Show all correct work for credit!

1. A photon of energy 12.0 TeV (1TeV $= 10^{12}$ eV) strikes a particle of mass M_0 at rest. After the collision there is only a single final particle of mass M, moving at speed $(12/13)c$. Using TeV units, find

   a) the momentum of the final particle,

   b) the mass M,

   c) the mass M_0.

2. Consider the spacetime geometry with the line element

$$ds^2 = -(1 - Ar^2)^2 dt^2 + (1 - Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (1)$$

   a) Calculate the proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$.

   b) Calculate the area of a sphere of coordinate radius $r = R$.

   c) Calculate the three-volume of a sphere of coordinate radius $r = R$.

3. Consider the spacetime geometry with the line element

$$ds^2 = -(1 - Ar^2)^2 dt^2 + (1 - Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2)$$

   a) Calculate the light cone at a point (t, r).

   b) Calculate the slopes of the light cones positioned at locations $r = 1/\sqrt{2A}$, $r = 1/\sqrt{3A}$, and $r = 1/\sqrt{4A}$.

PHYS 396 EXAM 2 SOLUTIONS

INITIALLY.. FINALLY...

1.

WHERE

$E_\gamma = 12.0\,\text{TeV}$

$V = \frac{12}{13} c$

CONSERVATION OF ENERGY YIELDS...

$$E_\gamma + M_0 c^2 = \gamma M c^2$$

WHERE $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1-(12/13)^2}} = \frac{1}{5/13} = \frac{13}{5}$

SO

1) $E_\gamma + M_0 c^2 = \frac{13}{5} M c^2$

CONSERVATION OF MOMENTUM YIELDS...

2) $E_\gamma / c = \gamma M V = \frac{13}{5} M \cdot \frac{12}{13} c = \frac{12}{5} M c$ $\quad \therefore M = \frac{5}{12} E_\gamma / c^2 = \frac{5}{12}\left(\frac{12.0\,\text{TeV}}{c^2}\right) = 5.0 \frac{\text{TeV}}{c^2}$

$$\therefore \boxed{M = 5.0 \frac{\text{TeV}}{c^2}}$$

PLUGGING THIS INTO (1) YIELDS..

$$12.0\,\text{TeV} + M_0 c^2 = \frac{13}{5}\left(5.0\frac{\text{TeV}}{c^2}\right) c^2 = 13.0\,\text{TeV}$$

$$\therefore \boxed{M_0 = 1.0 \frac{\text{TeV}}{c^2}}$$

$$P_M = \gamma M V = \frac{13}{5} \cdot 5.0 \frac{\text{TeV}}{c^2} \cdot \frac{12}{13} c = 12.0 \frac{\text{TeV}}{c}$$

$$\boxed{P_M = 12.0\,\text{TeV}/c}$$

2. $ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

a) TO CALCULATE THE PROPER DISTANCE ALONG A RADIAL LINE FROM $r=0$ TO $r=R$

SET $\theta = $ CONST., $\varphi = $ CONST., $t = $ CONST.

SO

$ds^2 = (1-Ar^2)^2 dr^2 \therefore ds = (1-Ar^2)dr$

SO

$$S = \int_0^R ds = \int_0^R (1-Ar^2)dr = r - \frac{1}{3}Ar^3 \Big|_0^R = R - \frac{1}{3}AR^3 = R(1-\frac{1}{3}AR^2)$$

$$\therefore \boxed{S = R(1-\frac{1}{3}AR^2)}$$

b) TO CALCULATE THE AREA OF A SPHERE OF COORDINATE RADIUS $r=R$ ...

SET $t = $ CONST., $r=R$

$$dA = \sqrt{g_{22}g_{33}}\, dx^2 dx^3 = \sqrt{R^2 \cdot R^2 \sin^2\theta}\, d\theta\, d\varphi = R^2 \sin\theta\, d\theta\, d\varphi$$

SO

$$A = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta\, d\theta\, d\varphi = R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta = 4\pi R^2$$

$$\therefore \boxed{A = 4\pi R^2}$$

c) TO CALCULATE THE THREE-VOLUME OF A SPHERE OF COORDINATE RADIUS $r=R$ ...

$$dV = \sqrt{g_{11}g_{22}g_{33}}\, dx^1 dx^2 dx^3 = \sqrt{(1-Ar^2)^2 \cdot r^2 \cdot r^2 \sin^2\theta}\, dr\, d\theta\, d\varphi = r^2(1-Ar^2)\sin\theta\, dr\, d\theta\, d\varphi$$

SO

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^R r^2(1-Ar^2)\sin\theta\, dr\, d\theta\, d\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^R r^2(1-Ar^2)\, dr$$

$$= 4\pi \int_0^R (r^2 - Ar^4)dr = 4\pi\left(\frac{1}{3}r^3 - \frac{1}{5}Ar^5\right)\Big|_0^R = \frac{4}{3}\pi R^3\left(1-\frac{3}{5}AR^2\right)$$

$$\boxed{V = \frac{4}{3}\pi R^3\left(1-\frac{3}{5}AR^2\right)}$$

3. $ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$

a) CONSIDER A RADIAL NULL CURVE $(d\theta = d\phi = 0 \; ; \; ds^2 = 0)$

THE LINE ELEMENT BECOMES..

$$0 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2$$

so

$$\frac{dt^2}{dr^2} = 1 \qquad \therefore \quad \frac{dt}{dr} = \pm 1$$

b)