MAT 202
Larson – Section 10.2
Plane Curves & Parametric Equations

Until now, we have been representing a graph by a single equation involving two variables. For motion problems involving time, it is useful to introduce the equations in this form with a third variable t, called a ***parameter***. These types of equations are called ***parametric equations***.

Definition of a Plane Curve*:** If f and g are continuous functions of t on an interval I, then the equations given by $x = f(t)$ and $y = g(t)$ are ***parametric equations and t is the ***parameter***. The set of ordered pairs (x, y) obtained as t varies over the interval I is the ***graph*** of the parametric equations. Taken together, the parametric equations and the graph are a ***plane curve***, denoted by C.

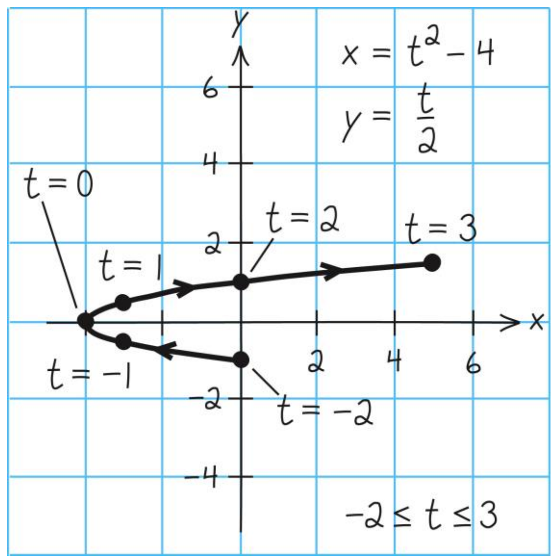
***Tools for Graphing Parametric Equations*:**

1. **Point plotting:** Substitute t-values (in increasing order) into the equations to find x, and y. By plotting the resulting points in order of increasing values of t, you trace the curve in a specific direction. This is called the ***orientation*** of the curve. For example, when sketching the curve given by the parametric equations

$x = t^2 - 4 \quad and \quad y = \frac{t}{2},\ where\ -2 \le t \le 3$, our table of values may look like:

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-½	0	½	1	3/2

The graph would then look like:



Notice the arrows are giving the graphs ***orientation***. Note: You may change your calculator settings to parametric equations under "MODE" so that you can check your graphs.

2. **Convert Parametric Equation to Rectangular Equation:** Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular equations (in x and y). The process of finding the rectangular equation is called ***eliminating the parameter***.

a) Start with your given set of parametric equations and solve for t in one equation.
b) Substitute the equation solved for t in part (a) into the second equation.
c) Simplify the remaining rectangular equation.

Ex: For the following parametric equations, (i) graph the curves using the specified interval and identify the graph (if possible) and (ii) eliminate the parameter and write the corresponding rectangular form.

a) $x = t + 2$, $y = t^2$; t is in $[0, 4]$
b) $x = 2 \cos t$, $y = 3 \sin t$, t is in $[0, 2\pi)$

Ex: For the following parametric equations, (i) graph the curves using the specified interval ar identify the graph and orientation (if possible) and (ii) eliminate the parameter and write the corresponding rectangular form.

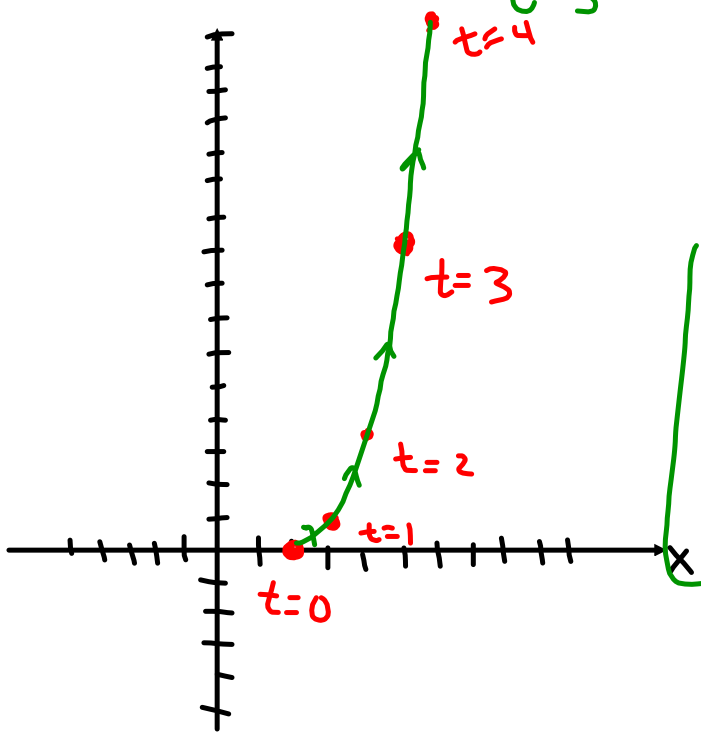
a) $x = t + 2, y = t^2$; t is in [0, 4]

$t = x - 2$

t	0	1	2	3	4
x	2	3	4	5	6
y	0	1	4	9	16

$(2,0)$, $(3,1)$, $(4,4)$, $(5,9)$, $(6,16)$

$t=0$ $t=1$ $t=2$ $t=3$ $t=4$

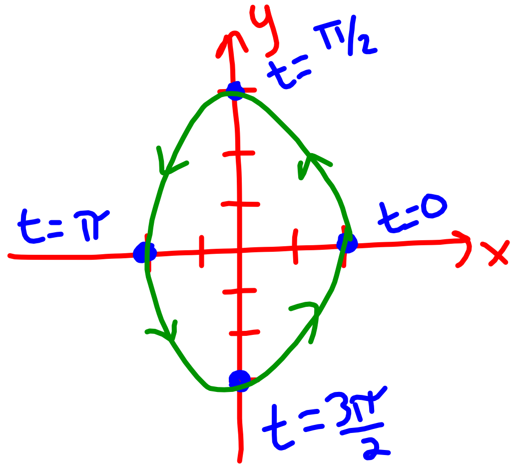


$t = x - 2$

$$\boxed{\begin{gathered} y = (x-2)^2 \\ 2 \le x \le 6 \\ 0 \le y \le 16 \end{gathered}}$$

b) $x = 2\cos t, y = 3\sin t$, t is in $[0, 2\pi)$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	-2	0
y	0	3	0	-3



$(2,0)$, $(0,3)$, $(-2,0)$, $(0,-3)$
$t=0$ $t=\pi/2$ $t=\pi$ $t=\frac{3\pi}{2}$

$(x)^2 = (2\cos t)^2 \quad (y)^2 = (3\sin t)^2$

$\frac{x^2}{4} = \frac{4\cos^2 t}{4} \quad \frac{y^2}{9} = \frac{9\sin^2 t}{9}$

$\frac{x^2}{4} = \cos^2 t \quad \frac{y^2}{9} = \sin^2 t$

$\sin^2 t + \cos^2 t = \frac{x^2}{4} + \frac{y^2}{9}$

$1 = \frac{x^2}{4} + \frac{y^2}{9}$

***Finding Parametric Equations for a Rectangular Equation*:** It is important to note that there are an infinite set of parametric equations for a given rectangular equation. For example

$$x = t^2 - 4 \quad and \quad y = \frac{t}{2}, \; where \; -2 \le t \le 3$$

will give the same graph as the equations

$$x = 4t^2 - 4 \quad and \quad y = t, \; where \; -1 \le t \le \frac{3}{2}.$$

It really all depends on what how we decide to rename x or y in terms of t.

Ex: Find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) t = x, (b) t = 3 – x, and (c) Using at least one trigonometric form, restricting the domain as needed.

$y = 6x^2 - 5$

Definition of a Smooth Curve: A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth when f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called piecewise smooth when it is smooth on each subinterval of some partition of I.

Ex: Find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) t = x, (b) t = 3 – x, and (c) Using at least one trigonometric form, restricting the domain as needed.

$y = 6x^2 - 5$

a) $\boxed{\begin{array}{l} x = t \\ y = 6t^2 - 5 \end{array}}$

b) $t = 3 - x$

$\boxed{\begin{array}{l} x = 3 - t \\ y = 6(3-t)^2 - 5 \end{array}}$

c) $\boxed{\begin{array}{l} x = \sin t \\ y = 6\sin^2 t - 5 \end{array}}$