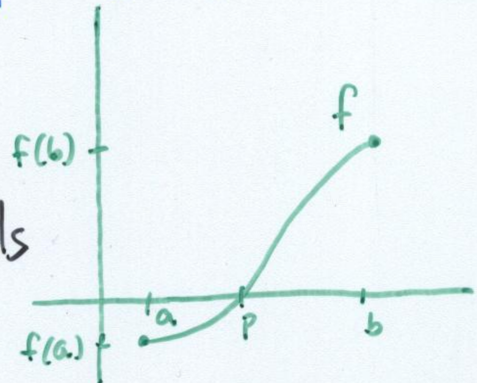
Section 2.1: The Bisection Method

Let $f \in C[a,b]$ with $f(a) \cdot f(b) < 0$

By IVT, $\exists\, p \in (a,b)$ s.t. $f(p) = 0$

The bisection method repeatedly halves subintervals of $[a,b]$.

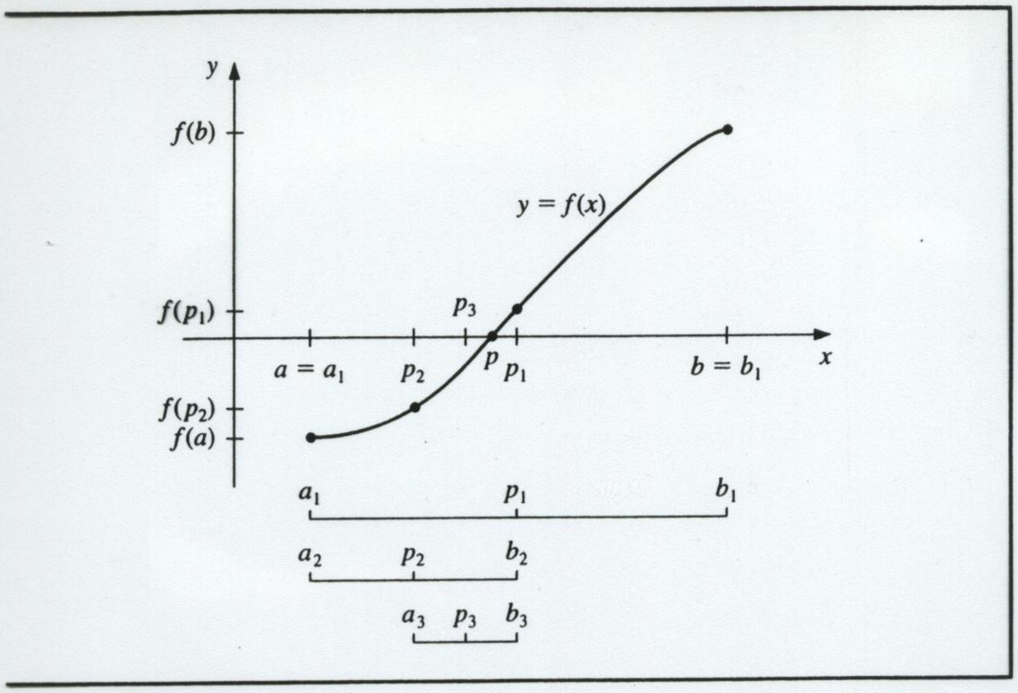


At each step, the subintervals contain p.

See graph on board.

To begin, set $a_1 = a$, $b_1 = b$, and $p_1 =$ midpoint of $[a,b]$:

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}$$



y
f(b)
y = f(x)
f(p1)
p3
a = a1
p2
p
p1
b = b1
x
f(p2)
f(a)
a1
p1
b1
a2
p2
b2
a3
p3
b3

Bisection Algorithm

INPUT: a, b, TOL, N_0.

OUTPUT p (or failure message).

Step 1 Set $i = 1$;
$FA = f(a)$.

Step 2 While $i \le N_0$ do steps 3-6.

Step 3 Set $p = a + (b-a)/2$; (Compute p_i)
$FP = f(p)$;

Step 4 If $FP = 0$ or $(b-a)/2 < TOL$ then
OUTPUT (p); (Procedure completed successfully)
stop.

Step 5 Set $i = i + 1$

Step 6 If $FA \cdot FP > 0$ then set $a = p$;
$FA = FP$
(compute a_i, b_i)
else set $b = p$.

Step 7 OUTPUT ('Method failed after ', N_0, ' iterations.');
STOP.

In step 4, the stopping procedure is

$$FP = 0 \quad \text{or} \quad \frac{(b-a)}{2} < TOL$$

Other stopping procedures include

$$|p_N - p_{N-1}| < \varepsilon$$

$$\left|\frac{p_N - p_{N-1}}{p_N}\right| < \varepsilon \qquad *$$

$$|f(p_N)| < \varepsilon$$

Cautions:

(1) $\{p_n\}$ can diverge even though

$$|p_n - p_{n-1}| \to 0$$

(2) $f(p_n) \approx 0$ even though

$$|p - p_n| >> 0 \quad \text{(possible)}$$

Inequality * the best criteria as it most closely resembles relative error.

Ch 2.1 Bisection Method: Extra Examples

① **Claim:** It is possible for a sequence $\{p_n\}$ to do the following:
$|p_n - p_{n-1}| \to 0$ but $\{p_n\}$ diverges

Example: Suppose $p_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n = 1, 2, 3, \ldots$

Then $p_1 = 1$

$p_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$p_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$

$\vdots$

n	p_n	$\lvert p_n - p_{n-1}\rvert$
1	1.0000000	
2	1.5000000	0.5000000
3	1.8333333	0.3333333
4	2.0833333	0.2500000
5	2.2833333	0.2000000
6	2.4500000	0.1666667
7	2.5928571	0.1428571
8	2.7178571	0.1250000
9	2.8289683	0.1111111
10	2.9289683	0.1000000

Now $|p_n - p_{n-1}| = \left|\left(\cancel{1} + \cancel{\tfrac{1}{2}} + \cdots + \cancel{\tfrac{1}{n-1}} + \tfrac{1}{n}\right) - \left(\cancel{1} + \cancel{\tfrac{1}{2}} + \cdots + \cancel{\tfrac{1}{n-1}}\right)\right|$

$= \frac{1}{n}$

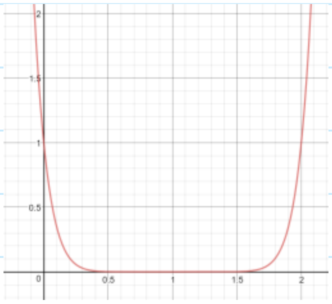
$\therefore |p_n - p_{n-1}| \to 0$ as $n \to \infty$.

However, $p_n = \sum_{k=1}^{n} \frac{1}{k}$, which is a divergent p-series $(p=1)$ (Harmonic Series)

$\therefore$ It is possible for $|p_n - p_{n-1}| \to 0$ but $p_n \to \infty$.

② **Claim:** It is possible for $f(p_n) \cong 0$ but $|p - p_n| >> 0$, where $f(p) = 0$.

Example Let $f(x) = (x-1)^{10}$; see graph.



$y = (x-1)^{10}$

Let $p_n = 1 + \frac{1}{n}$

Then $p_1 = 1 + \frac{1}{1} = 2$

$p_2 = 1 + \frac{1}{2} = 1.5$

$p_3 = 1 + \frac{1}{3} = 1.\overline{3}$

$\vdots$

n	p_n	$f(p_n)$	$\lvert p - p_n\rvert$
1	2.0000000	1.000E+00	1.0000000
2	1.5000000	9.766E-04	0.5000000
3	1.3333333	1.694E-05	0.3333333
4	1.2500000	9.537E-07	0.2500000
5	1.2000000	1.024E-07	0.2000000
998	1.0010020	1.020E-30	0.0010020
999	1.0010010	1.010E-30	0.0010010
1000	1.0010000	1.000E-30	0.0010000
1001	1.0009990	9.901E-31	0.0009990
1002	1.0009980	9.802E-31	0.0009980
1003	1.0009970	9.705E-31	0.0009970

Note that $f(p_2) < 10^{-3}$ but $|p - p_n| < 10^{-3}$ only when $n > 1000$

Example By IVT,

$$f(x) = x^3 + 4x^2 - 10$$

has a root in $[1,2]$, since $f \in C[1,2]$ &

$$f(1) \cdot f(2) = (-5)(14) < 0$$

Using a stopping criteria of

$$\frac{|p - p_n|}{|p|} < 10^{-4},$$

the Bisection Algorithm gives the values in Table 2.1

After 13 iterations,

$$|p - p_{13}| < |b_{14} - a_{14}| = 0.000122070$$

Since $|a_{14}| < |p|$,

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5}$$

Thus p_{13} is accurate to at least four significant digits.

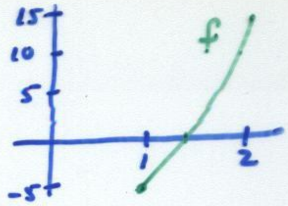
Note: $p \cong 1.365230013 \rightarrow$ closer to p_9 than p_{13}!

Table 2.1

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

Example (continued)

$f(x) = x^3 + 4x^2 - 10 \;,\; [1,2]$



$f(1) = -5 \;,\; f(2) = 14 \Rightarrow$ root in $[1,2]$ by IVT

$f \in C[1,2]$

$p_1 = 1.5 \;,\; f(p_1) \cong 2.375$

$p_2 = \dfrac{1+1.5}{2} = 1.25 \;,\; f(p_2) \cong -1.797$

$p_3 = \dfrac{1.25+1.5}{2} = 1.375, \; f(p_3) \cong 0.1621$

$p_4 = \dfrac{1.25+1.375}{2} = 1.3125, \; f(p_4) \cong -0.8484$

$p_5 = \dfrac{1.3125+1.375}{2} = 1.34375, \; f(p_5) \cong -0.3510$

$\vdots$

The Bisection method can be slow to converge, and a good intermediate approximation can be overlooked.

However, the Bisection method always converges to a solution, as we will see in the next Theorem.

It is for this reason that the Bisection method is often used as a starter for the more efficient methods given later in this chapter.

Theorem Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when } n \geq 1.$$

Proof We have, at each step,

$$b_n - a_n = \frac{1}{2^{n-1}}(b-a) \quad \& \quad p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$,

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}$$

Thus

$$|p_n - p| \leq \frac{1}{2^n}(b-a) \to 0$$

and hence $p_n = p + O(\frac{1}{2^n})$. ■

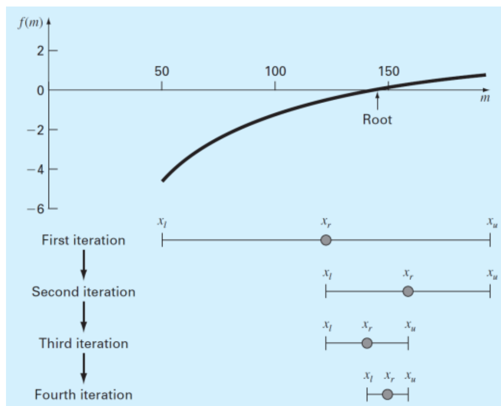
Note: Actual error may be much smaller. In previous example,

$$|p - p_9| \approx 4.4 \times 10^{-6} \leq \frac{2-1}{2^9} \approx 2 \times 10^{-3}$$

The bisection method requires that the initial interval $[a, b]$ bracket the root r, and similarly for all subsequent intervals. Thus we start with $r \in [a, b]$. The midpoint of this interval is $r_1 = \frac{a+b}{2}$. The distance from the midpoint r_1 to either endpoint is $\frac{b-a}{2}$. Since $r \in [a, b]$, we have $|r - r_1| < \frac{b-a}{2}$. The next bracketing subinterval will have length $\frac{b-a}{2}$, and the distance from the midpoint r_2 of this new subinterval to either endpoint will be $\frac{b-a}{4}$. Since r is in this subinterval, it follows that $|r - r_2| < \frac{b-a}{2^2}$. Repeating this argument, we conclude in general that $|r - r_n| < \frac{b-a}{2^n}$. See figure below, where $x_l = a$, $x_u = b$, and x_r denotes the midpoint.

Suppose we want to find the number of iterations n required to guarantee accuracy to within $10^{-3} = 0.001$. By our discussion above, we solve the following inequality for n:

$$\frac{b-a}{2^n} < 10^{-3}$$



First, we rewrite the fraction on left side:

$$(b-a)2^{-n} < 10^{-3}$$

Then divide both sides by $b - a$ to obtain

$$2^{-n} < \frac{10^{-3}}{b-a}$$

Next, take the logarithm of both sides to bring exponent down on left side:

$$-n\log(2) < \log\left(\frac{10^{-3}}{b-a}\right)$$

We can now solve for n:

$$n > \left(\frac{1}{\log(2)}\right) \times \log\left(\frac{10^{-3}}{b-a}\right)$$

Suppose that $[a, b] = [1, 2]$. Then $b - a = 1$ and $\log 10^{-3} = -3$. Thus

$$n > -\left(\frac{1}{\log(2)}\right) \times \log\left(\frac{10^{-3}}{b-a}\right) = \frac{3}{\log(2)} \cong 9.97$$

Thus after $n = 10$ iterations of the bisection method we are guaranteed that the absolute error will be within $10^{-3} = 0.001$. It is possible, and sometimes likely, to achieve this level of accuracy for one or more estimates before 10 iterations, but the algebra shown here provides a guarantee of when we will achieve a specified level of accuracy.

Example Recall $f(x) = x^3 + 4x^2 - 10$ has a zero in $[1,2]$. To determine the number of iterations necessary for accuracy 10^{-3} using $a_1 = 1$, $b_1 = 2$ requires solving for N:

$$|p_N - p| \le 2^{-N}(b-a) = 2^{-N} < 10^{-3}$$

Thus, using logs,

$$\log(2^{-N}) < \log(10^{-3})$$

$$-N \log 2 < -3 \log 10$$

$$-N < \frac{-3}{\log 2}$$

So we need to choose $N > \frac{+3}{\log 2} \cong 9.96$

Hence 10 iterations will ensure accuracy to within 10^{-3}.

Note from Table 2.1 that p_9 is accurate to within 10^{-4}.

The bound for the number of iterations for Bisection method assumes infinite digit arithmetic.

When implementing on computer, effects of roundoff error must be considered.

For example, when computing midpoint, we should use

$$p_n = a_n + \frac{b_n - a_n}{2}$$

instead of

$$p_n = \frac{a_n + b_n}{2}$$

The first equation adds a small correction to the known value of a_n. When $b_n - a_n$ is near maximum machine precision, this correction may be in error, but is small.

However, in this case, it may happen that

$$\frac{a_n + b_n}{2} \notin [a_n, b_n]$$

Our Bisection algorithm makes the calculation $f(a_n) \cdot f(b_n)$ at each iteration, checking for $f(a_n) \cdot f(b_n) < 0$.

To avoid risk of overflow or underflow, it is better to use

$$\operatorname{sgn}(f(a_n)) \cdot \operatorname{sgn}(f(b_n))$$

where

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

$\operatorname{sgn}(x)$ is known as the signum function