3.4 Fourier and Inverse Fourier Transforms

Exponential Fourier Expansion

$\omega = 2\pi/T$ $\quad h_n \Rightarrow$ order n of f on $[0,T]$

$$h_n(t) = \sum_{k=0}^{n} a_k \cos(k\omega t) + \sum_{k=1}^{n} b_k \sin(k\omega t)$$

$$h_n(t) = \sum_{k=-n}^{n} c_k e^{ik\omega t}$$

Expansion Coefficients c_k are given by,

$$c_k = \frac{1}{T}\int_0^T f(t) e^{-ik\omega t} \ , \quad k = 0, \pm 1, \pm 2, \dots, \pm n$$

Exponential and Trigonometric Versions

$$c_0 = a_0$$

$$c_k = \frac{a_k - ib_k}{2} \quad :: \quad c_k = \tfrac{1}{2}(a_k - ib_k)$$

$$c_{-k} = \bar{c}_k = \frac{a_k + ib_k}{2} \quad :: \quad \bar{c}_k = \tfrac{1}{2}(a_k + ib_k)$$

$$|c_{-k}| = |c_k| = \tfrac{1}{2}\sqrt{a_k^2 + b_k^2} \quad :: \quad |c_k| = \tfrac{1}{2}\sqrt{a_k^2 + b_k^2}$$

- Vector coefficients is known as the transform of f.

Orthogonality of Exponential Expansion Functions

$$g_n(t) = e^{2\pi i n t/T}$$

$$\langle g_m, g_n \rangle = \begin{cases} T, & m = n \\ 0, & m = n \end{cases}$$

$$\langle f, g \rangle = \int_0^T f(t)\overline{g(t)}\, dt$$

$$c_k = 1/T \int_0^T f(t) e^{-i2\pi k t/T} dt$$
$$= 1/T \int_0^T f(t)\cos(2\pi k t/T)\, dt - i\frac{1}{T}\int_0^T f(t)\sin(2\pi k t/T)\, dt$$

Fourier and Inverse Fourier Transform

f be a function defined on $(-\infty, \infty)$ that satisfies the dirchlet conditions

$$c(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi \omega t}\, dt, \quad \omega \in (-\infty, \infty)$$
$$f(t) = \int_{-\infty}^{\infty} c(\omega) e^{i2\pi \omega t}\, dt, \quad t \in (-\infty, \infty)$$