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PHYS 321
HW7

✓ 4.3, ✓ 4.4, ✓ 4.7, ✓ 4.11, ✓ 4.13, ✓ 4.14, ✓ 4.15

## Problem 4.3

$$Y_\ell^m(\theta,\varphi) = \epsilon\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\, e^{im\varphi} P_\ell^m \cos(\theta) \longrightarrow (1) \quad P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_\ell(x) \longrightarrow (2)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \longrightarrow (3) \quad : \epsilon = (-1)^m \text{ for } m \geq 0, \quad \epsilon = 1 \text{ for } m \leq 0$$

a.) $Y_0^0$ : $m = \ell = 0$

$$(3): \quad P_1(x) = \frac{1}{2^0 0!}\left(\frac{d}{dx}\right)^0 (x^2-1)^0 = 1 \qquad (2): \quad P_1^1(x) = (1-x^2)^0\left(\frac{d}{dx}\right)^0 (1) = 1$$

$$(1): \quad Y_0^0(\theta,\varphi) = \epsilon\sqrt{\frac{(1)}{4\pi}\frac{(0)!}{0!}}\, e^0 (1) = (-1)^0\sqrt{\frac{1}{4\pi}} = \sqrt{\frac{1}{4\pi}}$$

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$\boxed{Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}}$$

b.) $Y_2^1$ : $\ell = 2 \quad m = 1$

$$(3): \quad P_2(x) = \frac{1}{2^2 2!}\left(\frac{d}{dx}\right)^2 (x^2-1)^2 = \frac{1}{8}\left(\frac{d^2}{dx^2}\right)(x^4-2x^2+1) = \frac{1}{8}(12x^2-4) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$(2): \quad P_2^1(x) = (1-x^2)^{1/2}\left(\frac{d}{dx}\right)^1 \frac{1}{2}(3x^2-1) = (1-x^2)^{1/2}\, 3x \Big|_{x=\cos\theta} = (\sin^2\theta)^{1/2} 3\cdot\cos\theta$$

$$1-\cos^2\theta = \sin^2\theta$$

$$P_2^1(x) = 3\cos\theta\sin\theta$$

$$(3): \quad Y_2^1(\theta,\varphi) = -\sqrt{\frac{(2(2)+1)}{4\pi}\frac{(1)!}{(3)!}}\, e^{i\varphi}\cdot 3\cos\theta\sin\theta = -3\sqrt{\frac{5}{24\pi}}\, e^{i\varphi}\cos\theta\sin\theta$$

$$\boxed{Y_2^1(\theta,\varphi) = -3\sqrt{\frac{5}{24\pi}}\, e^{i\varphi}\cos\theta\sin\theta}$$

c.) $Y_0^0 = \sqrt{\frac{1}{4\pi}}$ : $\int_0^{2\pi}\int_0^{\pi} |Y_0^0|^2 \sin\theta\, d\theta\, d\varphi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\varphi = \frac{1}{4\pi}\cdot 4\pi = 1$

$\boxed{Y_0^0 \text{ is normalized}}$

Problem 4.3 Continued

$$Y_2^1 = -3\sqrt{\frac{5}{24\pi}}\, e^{i\varphi}\cos\theta\sin\theta \quad : \int_0^{2\pi}\int_0^{\pi} |Y_2^1|^* |Y_2^1| \sin\theta \, d\theta \, d\varphi$$

$$= 9\cdot\frac{5}{24\pi}\int_0^{2\pi}\int_0^{\pi} \cos^2\theta\sin^2\theta\sin\theta \, d\theta \, d\varphi$$

$$= 9\cdot\frac{5}{24\pi}\cdot 2\pi\int_0^{\pi} \cos^2\theta\,(1-\cos^2\theta)\sin\theta \, d\theta$$

$$= \frac{45}{12}\int_0^{\pi}(\cos^2\theta - \cos^4\theta)\sin\theta \, d\theta \qquad u=\cos\theta \quad du=-\sin\theta\, d\theta$$

$$\theta=0: u=1 \qquad \theta=\pi: u=-1$$

$$= \frac{45}{12}\int_{-1}^{1} u^2 - u^4 \; du$$

$$= \frac{45}{12}\left(\frac{u^3}{3}-\frac{u^5}{5}\right)\Big|_{-1}^{1} = \frac{45}{12}\left[\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{-1}{3}+\frac{1}{5}\right)\right] = \frac{45}{12}\left(\frac{2}{3}-\frac{2}{5}\right)$$

$$= \frac{45}{12}\left(\frac{10-6}{15}\right) = \frac{45}{12}\left(\frac{4}{15}\right) = \frac{3}{3} = 1 \checkmark$$

$$\boxed{Y_2^1 \text{ is normalized}}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_0^0\, Y_2^1 \sin\theta\, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\pi}\frac{1}{\sqrt{4\pi}}\cdot -3\sqrt{\frac{5}{24\pi}}\, e^{i\varphi}\cos\theta\sin^2\theta \, d\theta\, d\varphi$$

$$= -\frac{3\sqrt{5}}{\sqrt{96}\,\pi}\int_0^{2\pi}\int_0^{\pi}\cos\theta\sin^2\theta\, e^{i\varphi}\, d\theta\, d\varphi \longrightarrow (4)$$

$$\int_0^{\pi}\cos\theta\sin^2\theta \, d\theta = \int_0^{\pi}\cos\theta\,(1-\cos^2\theta)\, d\theta$$

$$= \int_0^{\pi}\cancelto{0}{\cos\theta}\, d\theta - \int_0^{\pi}\cos^3\theta\, d\theta$$

$$= \int_{\pi}^{0}\cos^3\theta\, d\theta = \int_{\pi}^{0}\cos\theta\,(1-\sin^2\theta)\, d\theta \qquad u=\sin\theta$$

$$du=\cos\theta\, d\theta$$

$$= \int_0^0 (1-u^2)\, du \longrightarrow = 0 \qquad \theta=\pi : u=0$$

$$\theta=0 : u=0$$

$\therefore$ (4) = 0 and $Y_0^0 : Y_2^1$ are orthogonal

$$\boxed{Y_0^0 : Y_2^1 \text{ are orthogonal}}$$

## Problem 4.4

$$\Theta(\theta) = A\ln(\tan(\theta/2)) \longrightarrow (5) \qquad l=m=0$$

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + (l(l+1)\sin^2\theta - m^2)\Theta = 0 \longrightarrow (6)$$

Differentiating (5)

$$\frac{d\Theta}{d\theta} = \frac{A}{\tan(\theta/2)} \cdot \frac{1}{2\cos(\theta/2)^2} = \frac{A\cdot\cos(\theta/2)}{\sin(\theta/2)} \cdot \frac{1}{2\cos(\theta/2)^2} = \frac{A}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{A}{\sin(\theta)} \quad \therefore \quad \frac{d\Theta}{d\theta} = \frac{A}{\sin\theta} \longrightarrow (7)$$

Putting $l=m=0$ and (7) into (6), we have

$$\sin\theta \cdot \frac{d}{d\theta}\left(\sin\theta \cdot \frac{A}{\sin\theta}\right) + (0(0+1)\sin^2\theta - 0^2)\Theta = 0$$

$$\sin\theta \cdot \frac{d}{d\theta}(A) = 0 \longrightarrow (8)$$

It is evident from (8) that the only possible scenario to this is when

$$\theta = 0, \pi, \ldots\ldots n\pi \longrightarrow (9)$$

Putting the results of (9) into (5), we get

$$\Theta(0) = A\cdot\ln(\tan(0/2)) = A\cdot\ln(0) = \infty \longrightarrow (10)$$

$$\Theta(\pi) = A\cdot\ln(\tan(\pi/2)) = A\cdot\ln(\infty) = \infty \longrightarrow (11)$$

Looking at the results of (10) and (11), we see that there is divergent behavior for $\Theta$ at 0 and $\pi$. From this we can conclude that (5) is not a suitable solution due to its divergent behavior at the poles.

$$\boxed{\text{Divergent behavior @ } \theta = 0, \pi}$$

## Problem 4.7

$$n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos(x)}{x} \longrightarrow (12)$$

a.)

i.) $n_1$ :

$$n_1(x) = -(-x)^1 \left(\frac{1}{x}\frac{d}{dx}\right)^1 \frac{\cos(x)}{x} = x \cdot \left(\frac{1}{x}\frac{d}{dx}\right) \cos(x) \cdot x^{-1}$$

$$= -x^{-2}\cos(x) - x^{-1}\sin(x) : \frac{-\cos(x)}{x^2} - \frac{\sin(x)}{x}$$

$$\boxed{n_1(x) = \frac{-\cos(x)}{x^2} - \frac{\sin(x)}{x}} \longrightarrow (13)$$

ii.) $n_2$:

$$n_2(x) = -(-x)^2 \left(\frac{1}{x}\frac{d}{dx}\right)^2 \frac{\cos(x)}{x} = -x^2 \left(\frac{1}{x}\frac{d}{dx}\right)^2 \frac{\cos(x)}{x}$$

$$= -x^2 \left(\frac{1}{x}\frac{d}{dx}\right)\left(\frac{1}{x}\frac{d}{dx}\right) \frac{\cos(x)}{x} = -x^2 \left(\frac{1}{x}\frac{d}{dx}\right) \frac{1}{x} \cdot \left(\frac{-\cos(x)}{x^2} - \frac{\sin(x)}{x}\right)$$

$$= -x \left(\frac{d}{dx}\right)\left(\frac{-\cos(x)}{x^3} - \frac{\sin(x)}{x^2}\right) = x \left(\frac{d}{dx}\right)\left(\frac{\cos(x)}{x^3} + \frac{\sin(x)}{x^2}\right)$$

$$= x \cdot \left(-3x^{-4}\cos(x) - x^{-3}\sin(x) - 2x^{-3}\sin(x) + x^{-2}\cos(x)\right)$$

$$= -\frac{3\cos(x)}{x^3} - \frac{\sin(x)}{x^2} - \frac{2\sin(x)}{x^2} + \frac{\cos(x)}{x} = \frac{1}{x}\left(\cos(x) - \frac{3\cos(x)}{x^2} - \frac{3\sin(x)}{x}\right)$$

$$\boxed{n_2(x) = \frac{1}{x}\left(\cos(x) - \frac{3\cos(x)}{x^2} - \frac{3\sin(x)}{x}\right)} \longrightarrow (14)$$

b.)

i.) we need to Taylor series expand (13) for $\cos(x)$ and $\sin(x)$

lame

$$\text{Taylor} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^{(n)}}{n!}$$

Taylor $\cos(x)$ $a=0$ : $f^{(0)}(0) = 1$ : $f'(0) = 0$ ; $f''(x) = -\cos(x)$ $f''(0) = -1$

$$\hookrightarrow T(\cos a) = 1 - \frac{x^2}{2} \longrightarrow (15)$$

Taylor $\sin(x)$ $a=0$ : $f^{(0)}(0) = 0$ : $f'(x) = \cos(x)$ : $f'(0) = 1$ : $f''(0) = 0$ : $f'''(x) = -\cos(x)$ : $f'''(0) = -1$

$$\hookrightarrow T(\sin a) = x - \frac{x^3}{6} \longrightarrow (16)$$

we only need the first term of both (15) and (16)

(15) and (16) become : $T(\cos\theta)_1 = 1$ & $T(\sin\theta)_1 = x$ ⟶ (17)

Putting (17) into (13), we get

$$n_1(x) = -\frac{1}{x^2} - 1 \ : \ \lim_{x \to 0}(n_1(x)) = -\frac{1}{0^2} - 1 = -\infty \ : \ n_1 \text{ blows up at } 0 \longrightarrow (18)$$

and (17) into (14), we get

$$n_2(x) = \frac{1}{x}\left(1 - \frac{3}{x^2} - 3\right) = \frac{1}{x}\left(-2 - \frac{3}{x^2}\right) = -\frac{2}{x} - \frac{3}{x^3}$$

$$\lim_{x \to 0}(n_2(x)) = -\frac{2}{0} - \frac{3}{0^3} = -\infty \ : \ n_2 \text{ blows up at } 0 \longrightarrow (19)$$

From (18) and (19) we can see that $n_1(x)$ and $n_2(x)$ blow up at the origin.

## Problem 4.11

a.) $R_{20} = \frac{C_0}{2a}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{(-r/2a)}$

$$N = \int_0^{+\infty} |R_{20}|^2 r^2 dr = 1 \longrightarrow (20)$$

(20) becomes $\int_0^{+\infty} \frac{C_0^2}{4a^2}\left(1 - \frac{1}{2}\frac{r}{a}\right)^2 e^{-r/a} r^2 dr = 1$

$$\frac{C_0^2}{4a^2}\int_0^{+\infty}\left[1 - \frac{r}{a} + \frac{1}{4}\frac{r^2}{a^2}\right] r^2 e^{-r/a} dr = 1 \; : \; \frac{C_0^2}{4a^2}\int_0^{+\infty}\left[r^2 - \frac{r^3}{a} + \frac{1}{4}\frac{r^4}{a^2}\right] e^{-r/a} dr = 1$$

$$\frac{C_0^2}{4a^2}\left[\underbrace{\int_0^{\infty} r^2 e^{-r/a} dr}_{n=2} - \frac{1}{a}\underbrace{\int_0^{\infty} r^3 e^{-r/a} dr}_{n=3} + \frac{1}{4}\frac{1}{a^2}\underbrace{\int_0^{\infty} r^4 e^{-r/a} dr}_{n=4}\right] = 1 \longrightarrow (21)$$

using the common integral: $\int_0^{\infty} x^n e^{-x/a} dx = n!\, a^{n+1} \longrightarrow (22)$

we put (22) into (21):

$$\frac{C_0^2}{4a^2}\left[2!a^3 - \frac{1}{a}\cdot 3!a^4 + \frac{1}{4}\cdot\frac{1}{a^2}\cdot 4!a^5\right] = 1 \; : \; \frac{C_0^2}{4a^2}\left[2a^3 - 6a^3 + 6a^3\right] = 1 \; : \; \frac{C_0^2}{2}\cdot a = 1$$

$$\therefore \quad C_0 = \sqrt{\frac{2}{a}} \; : \; R_{20} = \frac{1}{2a}\cdot\sqrt{\frac{2}{a}}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{(-r/2a)} \qquad \sqrt{\frac{2}{4a^3}} = \frac{1}{\sqrt{2}a^{3/2}}$$

$$\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\varphi) \; : \; Y_l^m = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\, e^{im\varphi} P_l^m(\cos\theta)$$

$l = m = 0 \; : \; n = 2$

$$Y_0^0 = (-1)^0\sqrt{\frac{(1)}{4\pi}\frac{(0)!}{(0)!}}\, e^0 P_0^0\cos(\theta) = \sqrt{\frac{1}{4\pi}}\, P_0^0\cos(\theta) \longrightarrow (23)$$

Lagrangian table on page 138 allows us to write $P_0^0 = 1$

$\therefore$ (23) becomes : $Y_0^0 = \sqrt{\frac{1}{4\pi}}$

Thus

$$\boxed{\psi_{200} = \sqrt{\frac{1}{8\pi a^3}}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{(-r/2a)}}$$

b.) $R_{21} = \frac{C_0}{4a^2}\, r e^{-r/2a} \longrightarrow (24)$

i.)

Normalizing (24): $\frac{C_0^2}{16a^4}\int_0^{\infty} r^2 e^{-r/a} r^2 dr = \frac{C_0^2}{16a^4}\int_0^{\infty} r^4 e^{-r/a} dr \longrightarrow (25)$

Using (22) on (25): $\frac{C_0^2}{16a^4}(4!\,a^5)=1 \;:\; C_0^2\cdot\frac{3}{2}a=1 \;:\; C_0^2=\frac{2}{3a} \quad \therefore\; C_0=\sqrt{\frac{2}{3a}}$

$$R_{21}=\frac{1}{\sqrt{16a^4}}\frac{\sqrt{2}}{\sqrt{3a}}re^{-r/2a}=\sqrt{\frac{2}{48a^5}}\,re^{-r/2a}=\sqrt{\frac{1}{24a^5}}\,re^{-r/2a}$$

$$R_{21}=\sqrt{\frac{1}{24}}\,ra^{-5/2}e^{-r/2a} \;:\; n=2 \;:\; l=1 \;:\; m=\pm1,0$$

ii.)

$\psi_{211}=R_{21}(r)Y_1^1(\theta,\varphi)$ : $Y_1^1(\theta,\varphi)=-1\sqrt{\frac{(2+1)}{4\pi}\frac{(0)!}{(2)!}}\,e^{i\varphi}P_1^1(\cos\theta)=-\sqrt{\frac{3}{8\pi}}\,e^{i\varphi}P_1^1(\cos\theta)$

$n=2 : l=m=1$

$P_1^1=\sin\theta$ according to Pg. 138 $\therefore\; Y_1^1(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\,e^{i\varphi}\sin\theta$

$$\therefore\; \psi_{211}=\sqrt{\frac{1}{24}}\,ra^{-5/2}e^{-r/2a}\cdot-\sqrt{\frac{3}{8\pi}}\,e^{i\varphi}\sin\theta=-\sqrt{\frac{1}{64\pi}}\,ra^{-5/2}e^{-r/2a}e^{i\varphi}\sin\theta$$

$$\boxed{\psi_{211}=-\sqrt{\frac{1}{64\pi}}\,a^{-3/2}re^{-r/2a}e^{i\varphi}\sin\theta}$$

iii.)

$\psi_{210}=R_{21}(r)Y_1^0(\theta,\varphi)$ : $Y_1^0=(1)\sqrt{\frac{(3)}{4\pi}\frac{(1)!}{(1)!}}\,e^0P_1^0(\cos\theta)=\sqrt{\frac{3}{4\pi}}\,P_1^0(\cos\theta)$

$P_1^0=\cos\theta$ according to Pg. 138 $\therefore\; Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$

$$\therefore\; \psi_{210}=\sqrt{\frac{3}{4\pi}}\cdot\sqrt{\frac{1}{24}}\,a^{-5/2}re^{-r/2a}\cos\theta=\sqrt{\frac{1}{32\pi}}\,a^{-5/2}re^{-r/2a}\cos\theta$$

$$\boxed{\psi_{210}=\sqrt{\frac{1}{32\pi}}\,a^{-5/2}re^{-r/2a}\cos\theta}$$

iv.)

$\psi_{21-1}=R_{21}(r)Y_1^{-1}(\theta,\varphi)$ : $Y_1^{-1}=1\sqrt{\frac{(3)}{4\pi}\frac{0!}{2!}}\,e^{-i\varphi}P_1^{-1}(\cos\theta)=\sqrt{\frac{3}{8\pi}}\,e^{-i\varphi}P_1^{-1}(\cos\theta)$

$P_1^{-1}=\sin\theta$ according to Pg. 138 $\therefore\; Y_1^{-1}=\sqrt{\frac{3}{8\pi}}\,e^{-i\varphi}\sin\theta$

$$\therefore\; \psi_{21-1}=\sqrt{\frac{1}{24}}\,a^{-5/2}re^{-r/2a}\cdot\sqrt{\frac{3}{8\pi}}\,e^{-i\varphi}\sin\theta=\sqrt{\frac{1}{64\pi}}\,a^{-5/2}re^{-r/2a}e^{-i\varphi}\sin\theta$$

$$\boxed{\psi_{21-1}=\sqrt{\frac{1}{64\pi}}\,a^{-3/2}re^{-r/2a}e^{-i\varphi}\sin\theta}$$

**Problem 4.13**

$$\langle \hat{Q} \rangle = \int_{-\infty}^{\infty} \Psi^*(\hat{Q})\Psi\, dy \longrightarrow (26) : \Psi_{100}(r,\theta,\varphi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

a.)

i.) $\langle r \rangle$ :
$$\langle r \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-r/a} (r) \frac{1}{\sqrt{\pi a^3}} e^{-r/a}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 4\pi \cdot \frac{1}{\pi a^3} \int_0^{\infty} r^3 e^{-2r/a}\, dr \quad : \quad \int_0^{\infty} x^n e^{-x/b}\, dx = n!\,(b)^{n+1}$$

$$\frac{-2r}{a} = \frac{-r}{b} \quad : \quad \frac{a}{2r} = \frac{b}{r} \quad \therefore \quad b = \frac{a}{2}$$

$$= \frac{4}{a^3}\left[3!\left(\frac{a}{2}\right)^4\right] = \frac{4}{a^3}\left(\frac{6}{16}a^4\right) = \frac{3}{2}a$$

$$\boxed{\langle r \rangle = \frac{3}{2}a}$$

ii.) $\langle r^2 \rangle$ :
$$\langle r^2 \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-r/a} (r^2) \frac{1}{\sqrt{\pi a^3}} e^{-r/a}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{4\pi}{\pi a^3}\int_0^{\infty} r^4 e^{-2r/a}\, dr = \frac{4}{a^3}\left[4!\left(\frac{a}{2}\right)^5\right] = \frac{4}{a^3}\left[\frac{24}{32}a^5\right] = 3a^2$$

$$\boxed{\langle r^2 \rangle = 3a^2}$$

b.) $r^2 = \underbrace{x^2 + y^2 + z^2}_{z=y=0}$ : $x = r\sin\theta\cos\varphi$ : $y = r\sin\theta\sin\varphi$ : $z = r\cos\varphi$

i.) $\langle x \rangle$ :
$$\langle x \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \cdot r\sin\theta\cos\varphi \cdot \frac{1}{\sqrt{\pi a^3}} e^{-r/2a}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{\pi a^3}\underbrace{\int_0^{2\pi}\cos\varphi\, d\varphi}_{=0}\int_0^{\pi}\sin^2\theta\, d\theta\int_0^{\infty} r^3 e^{-r/a}\, dr \longrightarrow (27)$$

$$\therefore \quad (27) = 0 = \langle x \rangle$$

$$\boxed{\langle x \rangle = 0}$$

ii.) $\langle x^2 \rangle$ :
$$\langle x^2 \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\sqrt{\pi a^3}} e^{-r/a}\, r^2\sin^2\theta\cos^2\varphi\, \frac{1}{\sqrt{\pi a^3}} e^{-r/a}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{\pi a^3}\int_0^{2\pi}\cos^2\varphi\, d\varphi\int_0^{\pi}\sin^3\theta\, d\theta\int_0^{\infty} r^4 e^{-2r/a}\, dr \longrightarrow \text{using (22) again}$$

$$= \frac{1}{\pi a^3}\cdot\frac{1}{2}\int_0^{2\pi}(1+\cos(2\varphi))\,d\varphi\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta\left[4!\left(\frac{a}{2}\right)^5\right]$$

$$= \frac{1}{2\pi a^3}\cdot\frac{24a^5}{32}\left[\varphi+\frac{\sin(2\varphi)}{2}\right]_0^{2\pi}\cdot\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta \longrightarrow \begin{array}{l} u=\cos\theta \\ du=-\sin\theta\,d\theta \end{array}$$

$$= \frac{3}{8\pi}a^2(8\pi)\cdot\int_{-1}^{1}1-u^2\,du = \frac{3}{4}a^2\cdot\left(u-\frac{u^3}{3}\right)\Big|_{-1}^{1} \qquad \begin{array}{l} \theta=0 : u=1 \\ \theta=\pi : u=-1 \end{array}$$

$$= \frac{3}{4}a^2\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right] = \frac{3}{4}a^2\left(2-\frac{2}{3}\right) = \frac{3}{4}a^2\cdot\frac{4}{3} = a^2$$

$$\boxed{\langle x^2\rangle = a^2}$$

c.) $n=2 : l=m=1 : \psi_{211} = R_{nl}(r)\,Y_l^m(\theta,\varphi) : Y_l^m(\theta,\varphi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\,e^{im\varphi}P_l^m(\cos\theta)$

$Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi} \longrightarrow$ Table 4.3 Pg. 139

$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}re^{-r/2a}$

$$\therefore\ \psi_{211} = -\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta\, e^{i\varphi} \ :\ \psi_{211}^* = -\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta\, e^{-i\varphi}$$

$$\langle x^2\rangle :\quad \langle x^2\rangle = \frac{1}{64\pi}\cdot\frac{1}{a^5}\int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{\infty} r^2e^{-r/a}\cdot(\overbrace{r^2\sin^2\theta\cos^2\varphi}^{x^2})\cdot\sin^2\theta\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \frac{1}{64\pi}\frac{1}{a^5}\int_0^{2\pi}\cos^2\varphi\,d\varphi\int_0^{\pi}\sin^5\theta\,d\theta\int_0^{\infty}r^6e^{-r/a}\,dr \longrightarrow \text{using (22)}$$

$$= \frac{1}{64\pi}\frac{1}{a^5}\int_0^{2\pi}\frac{1}{2}(1+\cos(2\varphi))\,d\varphi\int_0^{\pi}\sin\theta(1-\cos^2\theta)^2\,d\theta\cdot\left[6!\,(a)^7\right]$$

$$= \frac{1}{128\pi}\cdot\frac{1}{a^5}\cdot\frac{720}{1}\cdot\frac{a^7}{1}\left[\varphi+\frac{\sin(2\varphi)}{2}\right]_0^{2\pi}\int_0^{\pi}\sin\theta(1-\cos^2\theta)^2\,d\theta$$

$$= \frac{45}{8\pi}a^2\cdot 2\pi\int_0^{\pi}\sin\theta(1-\cos^2\theta)^2\,d\theta = \frac{45}{4}a^2\int_0^{\pi}\sin\theta(1-\cos^2\theta)^2\,d\theta \qquad \begin{array}{l} u=\cos\theta : du=-\sin\theta\,d\theta \\ \theta=0 \to u=1 \\ \theta=\pi \to u=-1 \end{array}$$

$$= \frac{45}{4}a^2\int_{-1}^{1}(1-u^2)^2\,du = \frac{45}{4}a^2\int_{-1}^{1}u^4-2u^2+1\,du = \frac{45}{2}a^2\int_0^1 u^4-2u^2+1\,du$$

$$= \frac{45}{2}a^2\left(\frac{u^5}{5}-\frac{2u^3}{3}+u\right)\Big|_0^1 = \frac{45}{2}a^2\left(\frac{1}{5}+\frac{1}{3}\right) = \frac{45}{2}a^2\left(\frac{8}{15}\right) = 12a^2$$

$$\boxed{\langle x^2\rangle = 12a^2}$$

## Problem 4.14

The most probable result of a probability function is the derivative of the function and the location where the derivative is equal to zero. Namely

$$P = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} |f(r)|^2 \, r^2 \sin\theta \, dr\, d\theta\, d\phi \longrightarrow (28)$$

Ground state of electron : $\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \equiv f(r)$

(28) becomes : $P = 4\pi \cdot \frac{1}{\pi a^3} e^{-2r/a} \cdot r^2 = \frac{4}{a^3} r^2 e^{-\frac{2r}{a}} \longrightarrow (29)$

Differentiating (29) w.r.t $r$,

$$\frac{dP}{dr} = \frac{4}{a^3}\left[2re^{-\frac{2r}{a}} + r^2 \cdot \frac{-2}{a} e^{-\frac{2r}{a}}\right] = 0$$

$$= \frac{8r}{a^3}\left[e^{-\frac{2r}{a}} - \frac{r}{a} e^{-\frac{2r}{a}}\right] = 0 \quad ; \quad e^{-\frac{2r}{a}}\left[1 - \frac{r}{a}\right] = 0 \longrightarrow (30)$$

The exponential term in (30) is never zero, therefor

$$1 - \frac{r}{a} = 0 \quad : \quad a - r = 0 \quad \therefore \quad r = a$$

The most probable location is thus:

$$\boxed{r = a, \text{ Bohr radius}}$$

Problem 4.15

$$\Psi(r,0)=\frac{1}{\sqrt{2}}(\Psi_{211}+\Psi_{21-1}) \quad : \quad \Psi_{211}=-\frac{1}{8\sqrt{\pi}}a^{-3/2}re^{-r/2a}\sin\theta e^{i\varphi}$$

$$\Psi_{21-1}=\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta e^{-i\varphi}$$

a.) $\Psi(r,t)=\Psi(r,0)\cdot\varphi(t) \quad : \quad \varphi(t)=e^{\frac{-iE_nt}{\hbar}}$

$$\Psi(r,t)=\frac{1}{\sqrt{2}}\left[-\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta e^{i\varphi}+\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta e^{-i\varphi}\right]e^{\frac{-iE_nt}{\hbar}} \qquad n=2$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{1}{8\sqrt{\pi}}a^{-5/2}re^{-r/2a}\sin\theta e^{\frac{-iE_2t}{\hbar}}\left[-e^{i\varphi}+e^{-i\varphi}\right] \qquad e^{ix}=\cos(x)+i\sin(x)$$

$$=\frac{1}{8\sqrt{2\pi}}a^{-5/2}re^{-r/2a}\sin\theta e^{\frac{-iE_2t}{\hbar}}\left[-\cancel{\cos(\varphi)}-i\sin(\varphi)+\cancel{\cos(\varphi)}-i\sin(\varphi)\right]$$

$$=\frac{-i}{4\sqrt{2\pi}}a^{-5/2}\sin\theta\sin\varphi\, re^{-r/2a}e^{\frac{-iE_2t}{\hbar}} \quad : \quad E_2=\frac{-\hbar^2}{8ma^2}$$

$$=\frac{-i}{4\sqrt{2\pi}}a^{-5/2}\sin\theta\sin\varphi\, re^{\frac{-r}{2a}\cdot\frac{-i\hbar t}{8ma^2}}=\frac{1}{4\sqrt{2\pi}}a^{-5/2}\sin\theta\sin\varphi\cdot re^{\frac{ir\hbar t}{16ma^3}}$$

$$\boxed{\Psi(r,t)=\frac{-i}{4\sqrt{2\pi}}a^{-5/2}\sin\theta\sin\varphi\cdot re^{\frac{ir\hbar t}{16ma^3}}}$$

b.) $V=\frac{-e^2}{4\pi\varepsilon_0 r}$ The expectation value does not depend on time

$$\Psi(r,0)=\frac{-i}{4\sqrt{2\pi}}a^{-5/2}\sin\theta\sin\varphi\, re^{-r/2a}$$

i.)

$$\langle V\rangle : \langle V\rangle=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\Psi^*\cdot V\cdot\Psi\; r^2\sin\theta\, dr\,d\theta\,d\varphi$$

$$=\frac{1}{32\pi}\cdot\frac{1}{a^5}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{-e^2}{4\pi\varepsilon_0 r}\cdot r^2e^{-r/a}\cdot\sin^2\theta\sin^2\varphi\cdot r^2\sin\theta\, dr\,d\theta\,d\varphi$$

$$=\frac{-e^2}{128\pi^2\varepsilon_0}\cdot\frac{1}{a^5}\int_0^{2\pi}\sin^2\varphi\, d\varphi\int_0^{\pi}\sin^3\theta\, d\theta\int_0^{\infty}r^3e^{-r/a}dr$$

$$=\frac{-e^2}{128\pi^2\varepsilon_0}\cdot\frac{1}{a^5}\int_0^{2\pi}\frac{1}{2}(1-\cos(2\varphi))d\varphi\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta\cdot\left[3!(a)^4\right]$$

$$=\frac{-e^2}{256\pi^2\varepsilon_0}\cdot\frac{1}{a^5}\cdot\frac{6a^4}{1}\int_0^{2\pi}(1-\cos(2\varphi))d\varphi\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta$$

$$=\frac{-3e^2}{128\pi^2\varepsilon_0}\cdot\frac{1}{a}\cdot(2\pi)\cdot\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta \qquad \begin{array}{ll} u=\cos\theta & u=1\;\; \theta=0 \\ du=-\sin\theta\, d\theta & u=1\;\; \theta=\pi \end{array}$$

Problem 4.15 Continued

$$= \frac{-3e^2}{64\pi\varepsilon_0} \cdot \frac{1}{a} \cdot -\int_1^{-1} 1-u^2\, du = \frac{-3e^2}{64\pi\varepsilon_0 a}\int_{-1}^{1} 1-u^2\, du = \frac{-3e^2}{32\pi\varepsilon_0 a}\int_0^1 1-u^2\, du$$

$$= \frac{-3e^2}{32\pi\varepsilon_0 a}\left[u-\frac{u^3}{3}\right]_0^1 = -\frac{3e^2}{32\pi\varepsilon_0 a}\cdot\left[\frac{2}{3}\right] = -\frac{e^2}{16\pi\varepsilon_0 a} = \frac{1}{4}\cdot\frac{-e^2}{4\pi\varepsilon_0 a}$$

$$V = \frac{e^2}{4\pi\varepsilon_0} \rightarrow \frac{\hbar^2}{ma} \quad \therefore \quad \langle v\rangle = -\frac{1}{4}\cdot\frac{\hbar^2}{ma^2} = -\frac{\hbar^2}{4ma^2}$$

$$\boxed{\langle v\rangle = -\frac{\hbar^2}{4ma^2}}$$

ii.) $E_1 = -13.6\text{ eV} : E_2 = -3.4\text{ eV} = \frac{1}{4}E_1$

$$E = \underbrace{-\frac{\hbar^2}{2ma^2}}_{E_1} \quad \therefore \quad \langle v\rangle = \frac{1}{2}\cdot\frac{-\hbar^2}{2ma^2} = \frac{1}{2}\cdot(-13.6\text{ eV}) = -6.8\text{ eV}$$

[4.69]

$$\boxed{\langle v\rangle = -6.8\text{ eV}}$$