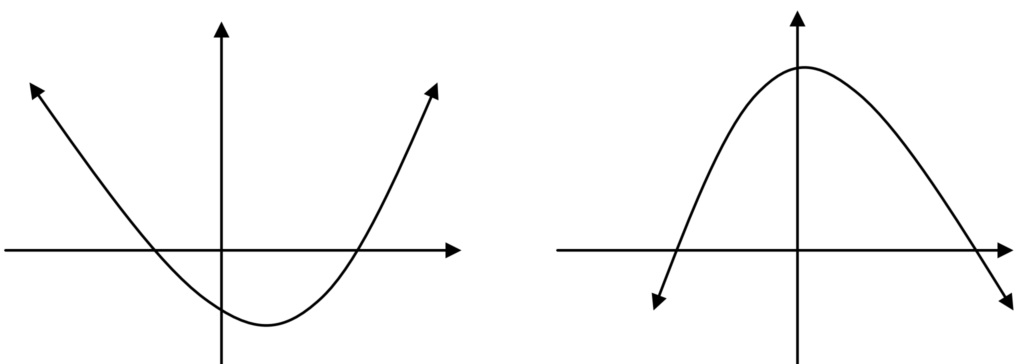
MAT 201
Larson/Edwards – Section 3.4
Concavity and the Second-Derivative Test

So far we have identified intervals in which functions are increasing or decreasing, as well as relative extrema. In this section we will learn how to identify the concavity of a function as well as where the function changes concavity.

Well…what is "concavity"??? No, it's not the result of a bad trip to the dentist!

Definition of Concavity:
Let the function *f* be differentiable on an interval (a, b). Then the graph of *f* is:

1. ***Concave upward*** on (a, b) if f' is increasing on (a, b).
2. ***Concave downward*** on (a, b) if f' is decreasing on (a, b).



Concave upward   Concave downward

How do we determine if f' is increasing or decreasing? The same as we would to determine if *f* is increasing or decreasing!

Test for Concavity:

Let *f* be a function whose second derivative exists on an open interval (a, b).

1. If $f''(x) > 0$ for all x in (a, b), then *f* is concave upward on (a, b).
2. If $f''(x) < 0$ for all x in (a, b), then *f* is concave downward on (a, b).

Ex: Find the open intervals on which the graph of the function is concave upward and concave downward:

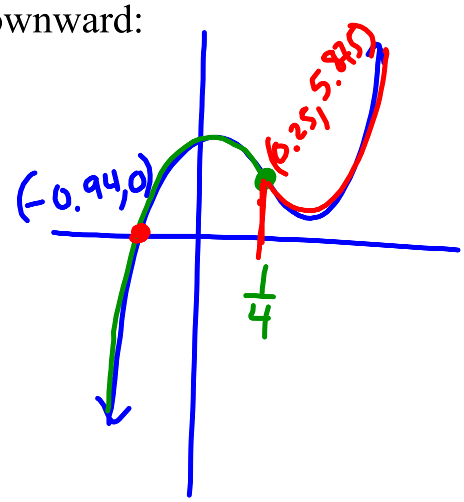
$f(x) = 4x^3 - 3x^2 + 6$

Inflection Points: A point on the graph of a function *f* where the tangent line exists and where the concavity changes (from upward to downward or vice versa) is called a ***point of inflection.***

How do we find points of inflection?

Ex: Find the open intervals on which the graph of the function is concave upward and concave downward:

$f(x) = 4x^3 - 3x^2 + 6$



$f'(x) = 12x^2 - 6x$

$f''(x) = 24x - 6$

$0 = 24x - 6$

$\frac{1}{4} = x$ ← critical #.

$\left(-\infty, \frac{1}{4}\right)$ $\left(\frac{1}{4}, \infty\right)$

$f''(0) = -6 < 0$ $f''(1) = 18 > 0$

Concave down Concave up

Concave down on $\left(-\infty, \frac{1}{4}\right)$

Concave up on $\left(\frac{1}{4}, \infty\right)$

Since there was a change in concavity @ $x = \frac{1}{4}$ we have a point of inflection.

$\left(\frac{1}{4}, \frac{47}{8}\right)$ or $(0.25, 5.875)$

Inflection Points: A point on the graph of a function *f* where the tangent line exists and where the concavity changes (from upward to downward or vice versa) is called a ***point of inflection.***

How do we find points of inflection?

Look @ concavity on intervals.
If it changes concavity, then at that point of transition you have a point of inflection.

Property of Points of Inflection:
If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Tools for Finding Inflection Points:

1. Find $f''(x)$.
2. Determine x values for which $f''(x) = 0$ or $f''(x)$ is undefined.
3. Determine the sign of $f''(x)$ to the left and right of each x value found in (2). If there is a change in sign, then $(c, f(c))$ is an inflection point.

Ex: Find the point(s) of inflection for the function
$f(x) = 3x^4 - 6x^3 + x - 8$

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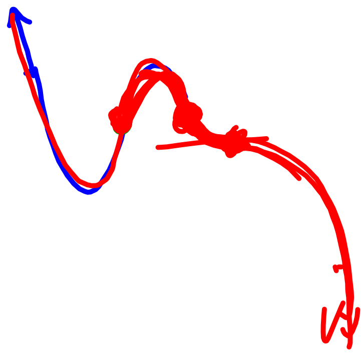

$f'(x) = 12x^3 - 18x^2 + 1$

$f''(x) = 36x^2 - 36x$

$0 = 36x^2 - 36x$

$0 = 36x(x-1)$

Critical #'s:
$x = 1, 0$

$(-\infty, 0)$ $(0, 1)$ $(1, \infty)$

$f''(-1) > 0$ $f''(0.5) < 0$ $f''(2) > 0$

Concave up ⟷ Concave down ⟷ Concave up

Changes changes

$\boxed{\text{P.O.I. @ } (0, -8), (1, -10)}$

There's another way to determine relative extrema!

Second-Derivative Test:

Let $f'(c) = 0$, and let f'' exist on an open interval containing c.

1. If $f''(c) > 0$, then *f(c)* is a relative minimum.
2. If $f''(c) < 0$, then *f(c)* is a relative maximum.
3. If $f''(c) = 0$, then the test fails. In such cases, you can use the First-Derivative Test to determine whether *f(c)* is a relative minimum, a relative maximum, or neither.

Ex: Use the second derivative test to find the relative extrema for the function $f(x) = 2x^3 - 6x^2 + x - 8$