Statistical and Thermal Physics: Class Exam I

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Instructions

- There are 6 questions on 8 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$R = 8.31\,\text{J/mol K}$ $\quad N_A = 6.02 \times 10^{23}\,\text{mol}^{-1}$ $\quad k = 1.38 \times 10^{-23}\,\text{J/K}$ $\quad 1\,\text{atm} = 1.01 \times 10^5\,\text{Pa}$

Question 1

The atmosphere of Titan consists nearly entirely of molecular nitrogen (mass of 1 mol = 0.028 kg). The surface pressure is about 147×10^3 Pa and the surface temperature is about 94 K. Determine the density of Titan's atmosphere at its surface.

$$PV = NkT \; : \; \rho = \frac{m}{V} \; : \; V = \frac{m}{\rho}$$

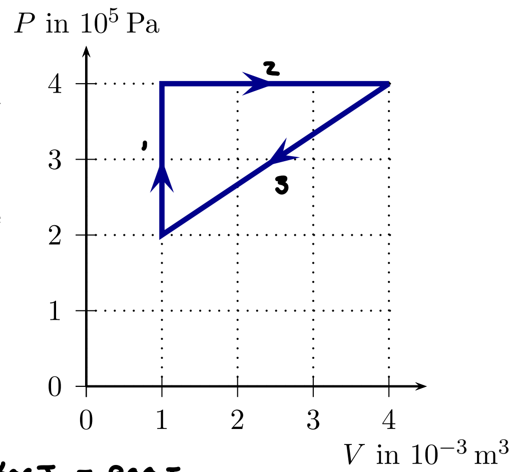
$$\frac{Pm}{\rho} = NkT \rightsquigarrow \rho = \frac{Pm}{NkT} = \frac{147\times10^3\,\text{Pa}\,(0.028\,\text{kg})}{6.02\times10^{23}\,(1.38\times10^{-23}\,\text{J/K})(94\,\text{K})} = 5.27\,\frac{\text{kg}}{\text{m}^3}$$

$$\boxed{\rho = 5.3\,\frac{\text{kg}}{\text{m}^3}}$$

/8

Question 2

A heat engine operates by having a monoatomic ideal gas undergo the process indicated on the PV diagram.



a) Determine the work done by the engine in one cycle.

$W_1 = 0$

$W_2 = -(4.0\times10^5 Pa)(3.0\times10^{-3} m^3) = -1200 J$

$W_3 = +\frac{1}{2}(2.0\times10^5 Pa)(3.0\times10^{-3} m^3) + (2.0\times10^5 Pa)(3.0\times10^{-3} m^3) = 300 J + 600 J = 900 J$

$W_1 = 0 J, \quad W_2 = -1200 J, \quad W_3 = 900 J$

$\boxed{W_{net} = -300 J}$

$-\frac{1}{2}(3.0\times10^{-3} m^3)(2.0\times10^5 Pa)$

$= -300 J$ ✓

b) Determine the efficiency of the engine.

$\Delta E = Q + W \quad \therefore \quad Q = \Delta E - W$

$Q_1 = \Delta E_1 - \cancel{W_1}^{0} = \frac{3}{2}(400J - 200J) = \frac{3}{2}(200 J) = 300 J$

$Q_2 = \Delta E_2 - W_2 = \frac{3}{2}(1600 J - 400 J) + 1200 J = \frac{3}{2}(1200 J) + 1200 J = 3000 J$

$Q_3 = \Delta E_3 - W_3 = \frac{3}{2}(200 J - 1600 J) - 900 J = -\frac{3}{2}(1400 J) - 900 J = -3000 J$

$Q_+ = 3300 J$

$\eta = \left|\frac{W_n}{Q_+}\right| = \left|\frac{300 J}{3300 J}\right| = \frac{1}{11}$

$\boxed{\eta = \frac{1}{11}}$

Question 2 continued . . .

/15

Question 3

Answer either part a) or part b) for full credit for this problem.

a) The isothermal compressibility of a gas is given by

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

and the thermal expansion coefficient is

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

Starting with an expression for dP in terms of dT and dV, show that

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha}{\kappa}.$$

$$\frac{\alpha}{\kappa} = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \cdot -V\left(\frac{\partial P}{\partial V}\right)_T = -\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P$$

$$\left(\frac{\partial z}{\partial y}\right)_x = -\left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_z, \quad z = P,\ y = T,\ x = V$$

$$\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P = \frac{\alpha}{\kappa}$$

$$\boxed{\left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha}{\kappa}}$$

Question 3 continued ...

b) Starting with $\mathrm{d}E = \delta Q - P\mathrm{d}V$ and the definition of enthalpy $H = E + PV$, show that

$$c_V = \left(\frac{\partial H}{\partial T}\right)_V - V\left(\frac{\partial P}{\partial T}\right)_V .$$

$$dE = Tds - Pdv \quad , \quad dH = dE + Pdv + vdP$$

$$dH = Tds - \cancel{Pdv} + \cancel{Pdv} + vdP = Tds + vdP \quad , \quad ds = \left(\frac{\partial s}{\partial T}\right)_P dT + \left(\frac{\partial s}{\partial P}\right)_T dP$$

$$dH = T\left(\frac{\partial s}{\partial T}\right)_P dT + \left(T\cancel{\left(\frac{\partial s}{\partial P}\right)_T}^{0} + v\right) dP \quad , \quad dH = \left(\frac{\partial H}{\partial T}\right)_v dT + \left(\frac{\partial H}{\partial P}\right)_T dP$$

$$\frac{ds}{dT} = \frac{C_v}{T}$$

$$dH = T\left(\frac{\partial s}{\partial T}\right)_P dT + vdP \quad : \quad \left(\frac{\partial H}{\partial T}\right)_v = T\cdot\frac{C_v}{T} + v\left(\frac{\partial P}{\partial T}\right)_v$$

$$\therefore \quad \boxed{C_v = \left(\frac{\partial H}{\partial T}\right)_v - v\left(\frac{\partial P}{\partial T}\right)_v}$$

/8

Question 4

The entropy for a particular gas is

$$S(E, V, N) = Nk \ln (E^{\beta}) + Nk \ln (V)$$

where β is a constant.

a) Determine an expression for the temperature of the gas and use it to determine a relationship between the energy and temperature of the gas.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \quad : \quad \frac{\partial S}{\partial E} = Nk \cdot \frac{\beta E^{\beta-1}}{\cancel{E}^{\beta}} = Nk \frac{\beta}{E}$$

$$\frac{1}{T} = Nk \frac{\beta}{E} \quad \therefore \quad \boxed{T = \frac{E}{\beta} \cdot \frac{1}{Nk}}$$

b) Determine an expression for the pressure of the gas and use this to obtain the pressure equation of state, i.e. $P = P(V, T)$.

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N} \quad : \quad \left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{Nk}{V} \quad \therefore \quad \boxed{P(V,T) = \frac{NkT}{V}}$$

/6

Question 5

An ideal gas undergoes a isothermal expansion. Which of the following (choose one) is true regarding the change in entropy of the gas, ΔS, in this process?

i) $\Delta S = 0$ for both monoatomic and diatomic gases.
ii) $\Delta S > 0$ for both monoatomic and diatomic gases.
iii) $\Delta S < 0$ for both monoatomic and diatomic gases.
iv) Whether $\Delta S = 0$, or $\Delta S > 0$ or $\Delta S < 0$ depends on whether the gas is monoatomic or diatomic.

Briefly explain your answer

$\Delta T = 0$, $V_f > V_i$

$\Delta S = Nk\ln(\cancel{T_f/T_i})^{0} + \underbrace{Nk\ln(V_f/V_i)}_{>0}$

$V_f > V_i \therefore \Delta S > 0$

/7

Question 6

Answer either part a) or part b) for full credit for this problem.

a) Use the second law of thermodynamics to show any process whose only result is the transfer of heat from a lower to a higher temperature reservoir is impossible.

$$\Delta S_{Tot} = \Delta S_{High} + \Delta S_{Low}$$

$$\Delta S_H = \frac{Q}{T_H} \;,\; \Delta S_L = -\frac{Q}{T_L} \quad \therefore \quad \Delta S_T = Q\left(\frac{1}{T_H} - \frac{1}{T_L}\right)$$

$$T_H > T_L \;\leadsto\; \Delta S_T = Q\left(\frac{1}{T_H} - \frac{1}{T_L}\right) < 0 \quad \therefore \quad \Delta S_T < 0$$

ΔS_T cannot < 0

Impossible!

b) The Helmholtz free energy is $F = E - TS$. Use this to show that

$$P = -\left(\frac{\partial F}{\partial V}\right)_S$$

and

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T .$$

$$dF = dE - TdS - SdT = \cancel{TdS} - PdV - \cancel{TdS} - SdT = -PdV - SdT$$

$$dF = \left(\frac{\partial F}{\partial V}\right)_S dV + \left(\frac{\partial F}{\partial T}\right)_P dT \;,\; \left(\frac{\partial F}{\partial V}\right)_S dV + \left(\frac{\partial F}{\partial T}\right)_P dT = -PdV - SdT$$

$$\therefore \boxed{P = -\left(\frac{\partial F}{\partial V}\right)_S dT} \;,\; S = -\left(\frac{\partial F}{\partial T}\right)_P$$

$$\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\partial^2 F}{\partial T \partial V} \;,\; \left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial^2 F}{\partial V \partial T} \;:\; \frac{\partial^2 F}{\partial T \partial V} = \frac{\partial^2 F}{\partial V \partial T} \;\therefore$$

$$\boxed{\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T}$$

/6