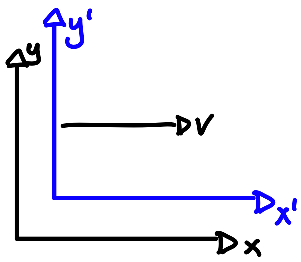
Ch. 4 Principles of Special Relativity

2-4-19



$\vec{V} = (V_x, V_y, V_z)$
$\vec{V}' = (V_x', V_y', V_z')$

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right] \text{ Galilean Transformation}$$

Not valid for speeds close to c! Thus special relativity!

$$V_x' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) = \frac{dx}{dt} - v = V_x - V$$
$$V_y' = \frac{dy'}{dt'} = \frac{d}{dt}(y) = \frac{dy}{dt} = V_y$$
$$V_z' = \frac{dz'}{dt'} = \frac{d}{dt}(z) = \frac{dz}{dt} = V_z$$

Einstein Builds Special Relativity on Two Postulates

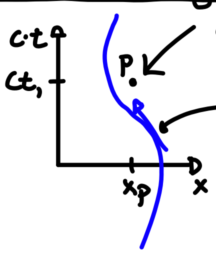
1. Principle of Relativity
2. The speed of light is a constant in all inertial reference frames

Spacetime

Inertial frames are spanned by four cartesian coordinates (t, x, y, z)

A different inertial reference frame has a different set of coordinates (t', x', y', z')

Spacetime Diagram



Labels an event

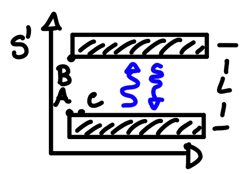
Worldline: The slope of the tangent of the worldline is ....

$$\frac{\Delta(ct)}{\Delta x} = c\left(\frac{\Delta t}{\Delta x}\right) = \frac{c}{V_x}$$

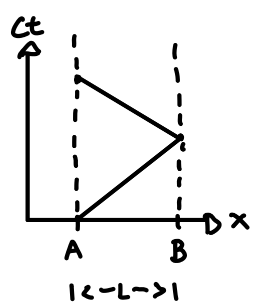
The worldlines of light rays have unity slope

The Geometry of Flat Spacetime

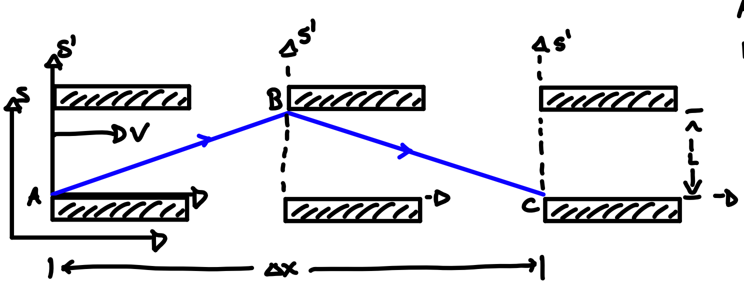
A light clock



$c = \frac{2L}{\Delta t'}$, $\Rightarrow \Delta t' = \frac{2L}{c}$



- A, B, C are three unique events



Total distance traveled $\Delta d = 2\sqrt{L^2 + \left(\frac{\Delta x}{2}\right)^2}$

$$c = \frac{\Delta d}{\Delta t} = 2\frac{\sqrt{L^2 + (\Delta x/2)^2}}{\Delta t}$$

$$\left[\Delta t = \frac{2\sqrt{L^2 + (\Delta x/2)^2}}{c}\right]$$

Consider the Following:

$$-(c\Delta t)^2 + (\Delta x)^2 = -4\left(L^2 + \cancel{(\Delta x/2)^2}\right) + \cancel{\Delta x^2} = -4L^2 = -c^2\Delta t'^2$$

Since $\Delta x' = 0$ Δy's : Δz's are Zero

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \Delta s'^2$$

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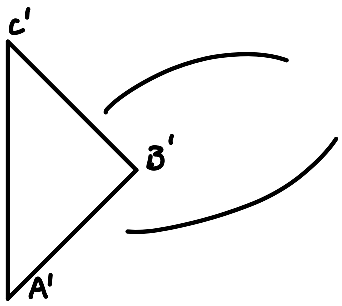
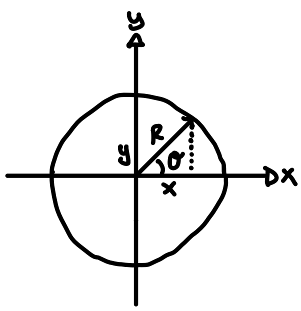
For motion in one Spatial dimension....

$$(\Delta s)^2 = -(c\Delta t)^2 + \Delta x^2$$



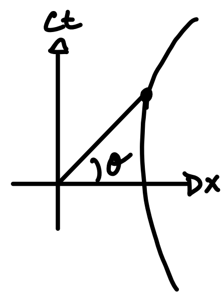
The path of light since $|A' \to B'|$ & $|B' \to C'|$ are both equal to zero

1.) A Circle



$x = R\cos\theta$
$y = R\sin\theta$
$x^2 + y^2 = R^2(\cos^2\theta + \sin^2\theta)$
$= R^2(1)$

A circle in 2D minkowski Spacetime



$x^2 - c^2t^2 = R^2$ in 2D minkowski Spacetime

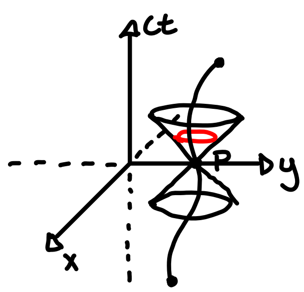
Equation of a hyperbola

$x = R\cosh\theta$, $y = R\sinh\theta$

$$x^2 - c^2t^2 = R^2(\cosh^2\theta - \sinh^2\theta) = R^2$$

Light Cones

$(\Delta s)^2$ can be positive, negative, or zero



$\Delta s^2 > 0$: Spacelike (i.e $\Delta t = 0$: $\Delta x \neq 0$)
$\Delta s^2 < 0$: Timelike (i.e $\Delta t \neq 0$: $\Delta x = 0$)
$\Delta s^2 = 0$: Lightlike {Null} (i.e $c = \Delta x / \Delta t$ <-> world lines of light rays)

To measure the spacetime distance along a Particle's worldline ($dx = 0$)

$ds^2 = -c^2 d\tau^2 \therefore \left[d\tau^2 = \frac{-ds^2}{c^2} \right]$ - Proper Time
- Time measured by a clock along the worldline

$ds'^2 = ds^2$ - Line Element is an invariant quantity

Let the prime frame be the frame which measures proper time
$(dx' = dy' = dz' = 0)$

$$-c^2 d\tau^2 = -c^2 dt^2 + (dx^2 + dy^2 + dz^2)$$

$$d\tau^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) = dt^2\left[1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right)\right]$$

$$d\tau^2 = dt^2\left[1 - \frac{1}{c^2}\vec{V}^2\right] \rightarrow \left[d\tau = dt\sqrt{1 - \vec{V}^2/c^2}\right] \longrightarrow \text{Time dilation}$$

Integrating between two points A : B on a timelike Worldline

$$\int_{\tau_A}^{\tau_B} d\tau = \int dt\sqrt{1 - V^2(t)/c^2} \quad : \quad \left[\tau_{AB} = \int dt\sqrt{1 - V^2/c^2}\right] \text{ - Time Dilation}$$

- Holds for accelerating clocks
- τ_{AB} is shorter than the interval $t_A - t_B$ because $\sqrt{1 - V^2/c^2} \leq 1$

For small intervals of time ....

$$\left[\Delta\tau_{AB} = \Delta t\sqrt{1 - \vec{V}^2/c^2}\right]$$

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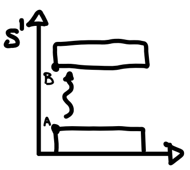
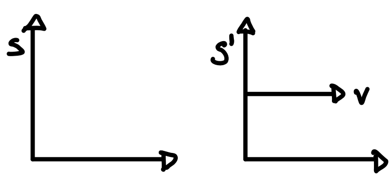
The Light Clock Revisited ....



$\Delta t' = \frac{2L}{c}$



$\Delta t = \frac{2}{c}\sqrt{L^2 + (\Delta x/2)^2}$ where $\Delta x = v\Delta t$

s' measures the proper time since $\Delta x' = 0$

$\Delta\tau = \Delta t' = \frac{2L}{c}$

$$\Delta t = \frac{2}{c}\sqrt{L^2 + \frac{1}{4}v^2\Delta t^2}$$

$$\frac{1}{4}c^2\Delta t^2 = L^2 + \frac{1}{4}v^2\Delta t^2$$

$$\frac{1}{4}(c^2 - v^2)\Delta t^2 = L^2$$

$$\frac{c^2}{4}\left(1-\frac{v^2}{c^2}\right)\Delta t^2 = L^2$$

$$\Delta t^2 = \frac{4L^2}{c^2}\cdot\frac{1}{(1-v^2/c^2)}$$

$$\Delta t = \frac{2L}{c}\cdot\frac{1}{\sqrt{1-v^2/c^2}} = \frac{\Delta T}{\sqrt{1-v^2/c^2}} \quad : \quad \Delta T = \Delta t\sqrt{1-v^2/c^2}$$

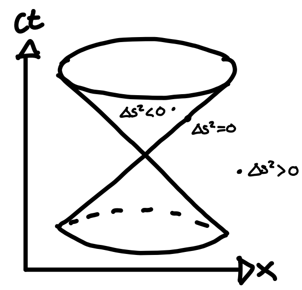
Last time,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

proper time : $\Delta x = \Delta y = \Delta z = 0$

$$d\tau^2 = \frac{-ds^2}{c^2}$$

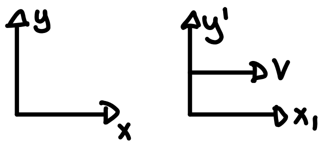
$$\tau_{AB} = \int_{t_A}^{t_B} dt\sqrt{1-\vec{v}^2(t)/c^2} \longrightarrow$$



Lorentz Boosts

What are these Lorentz transformations

$\Rightarrow$ Analog of rotation between space : time (Lorentz Boosts)



Claim:

$$\begin{bmatrix} ct' = (\cosh\theta)ct - (\sinh\theta)x \\ x' = -(\sinh\theta)ct + (\cosh\theta)x \\ y' = y \\ z' = z \end{bmatrix}$$ – Lorentz Boost in x-direction

Proof

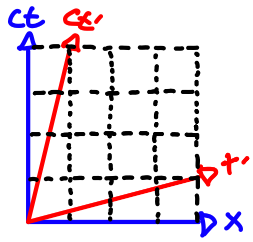
$$ds'^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 = -\left[\cosh\theta\, cdt - \sinh\theta\, dx\right]^2 + \left[-\sinh\theta\, cdt + \cosh\theta\, dx\right]^2 + dy^2 + dz^2$$

$$= -\left[\cosh^2\theta\, c^2dt^2 + \sinh^2\theta\, dx^2 - 2\sinh\theta\cosh\theta\, cdtdx\right]$$

$$+ \sinh^2\theta\, c^2dt^2 + \cosh^2\theta\, dx^2 - 2\sinh\theta\cosh\theta\, cdtdx + dy^2 + dz^2$$

$$= -(\cosh^2\theta - \sinh^2\theta)c^2dt^2 + (\cosh^2\theta - \sinh^2\theta)dx^2 + dy^2 + dz^2$$

$$= -c^2dt^2 + dx^2 + dy^2 + dz^2$$



$$\Delta x' = 0 = -(\sinh\theta)\, c\Delta t + \cosh\theta\, \Delta x$$

$$\frac{\Delta x}{\Delta t} = c\tanh\theta \rightarrow \frac{\Delta x}{\Delta t} \equiv v \rightarrow \frac{v}{c} = \tanh\theta$$

Plot ct' vs x' on (ct, x) Axes

$$\frac{v^2}{c^2} = \tanh^2\theta : \quad \cosh^2\theta - \sinh^2\theta = 1$$

$$1 - \tanh^2\theta = \frac{1}{\cosh^2\theta} = 1 - \frac{v^2}{c^2}$$

- Particle in (ct', x') I.R.F.

$$\left[\cosh\theta = \frac{1}{\sqrt{1 - v^2/c^2}}\right]$$

$$\sinh^2\theta = \cosh^2\theta - 1 = \frac{1}{1 - v^2/c^2} - 1$$

$$\sinh^2\theta = \frac{1 - (1 - v^2/c^2)}{1 - v^2/c^2} = \frac{v^2/c^2}{1 - v^2/c^2} \qquad \Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$$

$$\left[\sinh\theta = \frac{v/c}{\sqrt{1 - v^2/c^2}}\right]$$

$$ct' = \frac{1}{\sqrt{1-v^2/c^2}}\, ct - \frac{v/c}{\sqrt{1-v^2/c^2}}\, x = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct - \frac{v}{c}x\right) \; : \; t' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t - \frac{v}{c^2}x\right)$$

$$x' = -\frac{v/c}{\sqrt{1-v^2/c^2}}\, ct + \frac{1}{\sqrt{1-v^2/c^2}}\, x = \frac{1}{\sqrt{1-v^2/c^2}}(x - vt)$$

The Lorentz transformations are ....

$$\left[\begin{aligned} t' &= \gamma(t - v/c^2 \cdot x) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right] \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

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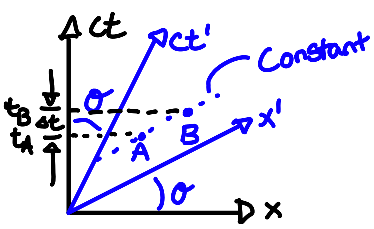
The Relativity of Simultaneity

- 2 Events (A : B) Simultaneous in (ct', x') $\therefore$ $\Delta t' = t_B' - t_A' = 0$



Inverse Transformations

$v \rightarrow -v$

Primes $\rightleftarrows$ unprimes

$$\left[\begin{aligned} t &= \gamma(t' + vx'/c^2) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}\right]$$

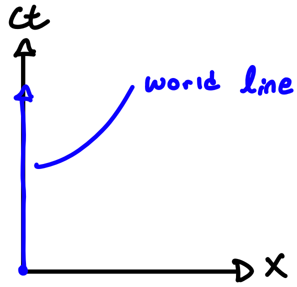
$$t_2' = \gamma(t_2 - Vx_2/c^2)$$

$$t_1' = \gamma(t_1 - Vx_1/c^2)$$

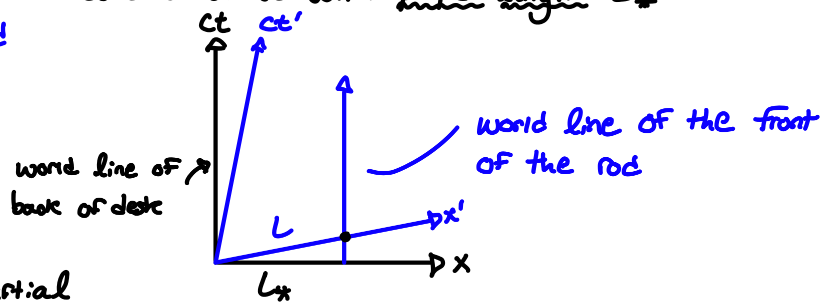
$$t_2' - t_1' = \gamma\left(t_2 - t_1 - \frac{V}{c^2}(x_2 - x_1)\right)$$

$$\Delta t' = \gamma(\Delta t - V\Delta x/c^2)$$

Length Contraction



Consider a rod with proper length L_*



What is the length measured in another inertial reference frame? Length of rod in (ct', x') is distance betwee two simultaneous events

$$\Delta t' = 0$$

$$\Delta s^2 = \Delta s'^2$$

$$-c^2\Delta t^2 + \underbrace{\Delta x^2}_{L_*^2} = \underbrace{c'^2\Delta t'^2}_{0} + \underbrace{\Delta x'^2}_{L^2}$$

$$-c^2\Delta t^2 + L_*^2 = L^2 \qquad \Delta t' = 0 = \gamma(\Delta t - V\Delta x/c^2) \rightarrow \Delta t = \frac{V\Delta x}{c^2} = \frac{VL_*}{c^2}$$

$$L^2 = L_*^2 - c^2 \cdot \frac{V^2 \cdot L_*^2}{c^4}$$

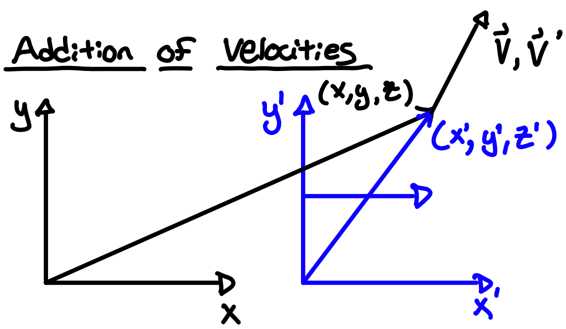
$$= L_*^2(1 - V^2/c^2)$$

$$L = L_*\sqrt{1 - V^2/c^2} \longrightarrow \left[\Delta x' = \Delta x\sqrt{1 - V^2/c^2}\right]$$

notice $L \leq L_*$ or $\Delta x' = \frac{\Delta x}{\gamma}$

Length contraction

Addition of Velocities



What is the relationship between $\vec{v} : \vec{v}'$?

$$v_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - Vdt)}{\gamma(dt - \frac{Vdx}{c^2})} = \frac{(v_x - V)}{(1 - Vv_x/c^2)}$$

$$v_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - Vdx/c^2)} = \frac{1}{\gamma}\frac{dy/dt}{(1 - \frac{V}{c^2}\frac{dx}{dt})} = \sqrt{1 - V^2/c^2} \cdot \frac{v_y}{1 - V/c^2 v_x}$$

Example 4.5

$$V_x = c \quad , \quad V_x' = \frac{c - v}{1 - v/c} = \frac{c(1 - v/c)}{(1 - v/c)} = c$$