Statistical and Thermal Physics: Class Exam II

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Instructions

- There are 6 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\ K} \qquad N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01 \times 10^5\,\mathrm{Pa}$$

$$k = 1.38 \times 10^{-23}\,\mathrm{J/K} = 8.61 \times 10^{-5}\,\mathrm{eV/K} \qquad e = 1.60 \times 10^{-19}\,\mathrm{C}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N \ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

Question 1

An isolated Einstein solid consists of four oscillators, labeled A, B, C and D. Any microstate of the solid can be described by listing the *energy level* n for each. These are denoted $n_A, n_B, n_C, \ldots$. Consider the following microstates:

State	n_A	n_B	n_C	n_D
microstate 1	1	1	1	1
microstate 2	4	0	0	0
microstate 3	0	2	0	2

a) Provide a value for q for each microstate.

$q_1 = q_2 = q_3 = 4$

b) Rank the microstates in order of increasing probability, indicating equality whenever this occurs.

$P_1 = P_2 = P_3$ since $E_1 = E_2 = E_3$

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Question 2

A system consists of four non-interacting identical spin-1/2 particles. The bulk observables for the system are the total energy of the ensemble and the particle number. For any *individual particle* the following are the possible states (where $\epsilon > 0$):

State	Energy	Probability
spin up	$-\epsilon$	$\frac{7}{10}$
spin down	ϵ	$\frac{3}{10}$

a) List all possible macrostates and the probabilities with which each occurs.

$p = 7/10$, $q = 3/10$

N_+	N_-	Probability
4	0	$\binom{4}{4} p^4 q^0 = 24.01\%$
3	1	$\binom{4}{3} p^3 q^1 = 41.16\%$
2	2	$\binom{4}{2} p^2 q^2 = 26.46\%$
1	3	$\binom{4}{1} p^1 q^3 = 7.56\%$
0	4	$\binom{4}{0} p^0 q^4 = 0.81\%$

b) Determine the energy of the equilibrium state of this system.

Equilibrium state is most probable. $N_+ = 3$, $N_- = 1$

$$\boxed{E = -2\epsilon}$$

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) Two Einstein solids, A and B, interact with each other but are otherwise isolated. In the high temperature limit the multiplicity of A is $\Omega_A = \kappa_A q_A^{N_A}$ where N_A is the number of particles in solid A, q_A is the number of energy units in solid A and κ_A is a term that is independent of q_A. Similarly for B, $\Omega_B = \kappa_B q_B^{N_B}$. Determine an expression for the entropy of the combined system (in terms of $N_A, N_B, q_A, q_B, \kappa_A$ and κ_B). Describe *how* you could use this to determine the ratio in which the energy is shared between the two solids in the equilibrium state.

$S = k \ln[\Omega] = k \ln[\kappa_A q_A^{N_A} \kappa_B q_B^{N_B}]$

$\boxed{S = k \ln[\kappa_A q_A^{N_A} \kappa_B q_B^{N_B}]}$

$S = k\left[\ln[\kappa_A \cdot \kappa_B] + N_A \ln[q_A] + N_B \ln[q_B]\right] \quad q_B = q - q_A$

$\frac{\partial S}{\partial q_A} = \cancel{k} \cdot \frac{N_A}{q_A} - \cancel{k} \frac{N_B}{q_B} = 0 \quad \therefore \quad \boxed{\frac{N_A}{N_B} = \frac{q_A}{q_B}}$

Question 3 continued ...

b) Consider a single two dimensional quantum oscillator for which the states are labeled by integers n_x and n_y; these can independently take on the values $0, 1, 2, \ldots$. The energy for this is

$$E = \hbar\omega\ (n_x + n_y + 1)\,.$$

List the three macrostates with lowest energy and all the microstates for each. Describe *how* you would find the entropy, S, for an ensemble of N such identical, distinguishable, non-interacting, oscillators as a function of energy, E.

n_x	n_y	E
0	0	$\hbar\omega$
0	1	$2\hbar\omega$
1	0	$2\hbar\omega$
1	1	$3\hbar\omega$
0	2	$3\hbar\omega$
2	0	$3\hbar\omega$

Count all states with energy E $\rightsquigarrow$ $\underbrace{\text{multiplicity}}$ $\Omega(E,N)$

$\rightsquigarrow$ $S = k \ln[\Omega(E,N)]$

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Question 4

Consider an Einstein solid, with N oscillators, for which $q \gg N \gg 1$ (this is the high temperature limit). Then the multiplicity of any macrostate is

$$\Omega \approx \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$$

and the energy of this state is $E = \hbar\omega\,(q + N/2)$.

a) Determine an expression for the entropy of the system.

$S = k \ln\left[(eq/N)^N \cancel{\frac{1}{\sqrt{2\pi N}}}\right]$ o

$S = k\left[N \ln(eq/N) - \cancel{\ln(\sqrt{2\pi N})}\right] = kN\left[\ln(e) + \ln(q/N)\right]$

$S = kN\left[1 + \ln(q) - \ln(N)\right]$

$$\boxed{S = kN\left[1 + \ln(q) - \ln(N)\right]}$$

Question 4 continued . . .

b) Determine an expression for the energy of the solid, E, in terms of temperature and particle number.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N \qquad S = kN\left[1 + \ln(q) - \ln(N)\right]$$

$$E = \hbar\omega(q + N/2) \quad \therefore \quad q = \frac{E}{\hbar\omega} - \frac{N}{2}$$

$$\left(\frac{\partial S}{\partial E}\right) = \left(\frac{\partial S}{\partial q}\right)\left(\frac{\partial q}{\partial E}\right)$$

$$\left(\frac{\partial S}{\partial q}\right) = \frac{kN}{q}, \quad \left(\frac{\partial q}{\partial E}\right) = \frac{1}{\hbar\omega}$$

$$\frac{1}{T} = \frac{Nk}{q} \cdot \frac{1}{\hbar\omega} \quad \therefore \quad T = \frac{q\hbar\omega}{Nk}$$

$$q\hbar\omega = E - \frac{N\hbar\omega}{2}$$

$$T = \frac{1}{Nk}\left[E - \frac{N\hbar\omega}{2}\right] \quad \therefore \quad E = NkT + \frac{N\hbar\omega}{2}$$

$$\boxed{E = NkT + \frac{N\hbar\omega}{2}}$$

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Question 5

An ensemble of N identical, distinguishable, non-interacting spin-1/2 particles are all in a magnetic field with magnitude B. The energy of a single particle in the spin-up state is $-\mu B$ and that of a single particle in the spin down state is $+\mu B$. These are in contact with a bath at temperature T.

a) Determine an expression for the partition function for the entire system.

$$Z = \sum_s e^{-E_s \beta} = e^{\mu B \beta} + e^{-\mu B \beta}$$

$$\boxed{Z = e^{\mu B \beta} + e^{-\mu B \beta}}$$

Question 5 continued ...

b) Determine an expression for the mean value of the energy of the system.

$$Z = e^{\mu B\beta} + e^{-\mu B\beta} \quad , \quad \bar{E} = -\frac{\partial}{\partial\beta} \ln(Z)$$

$$\bar{E} = -\frac{\partial}{\partial\beta} \ln\left(e^{\mu B\beta} + e^{-\mu B\beta}\right) = -\left(\frac{\mu B e^{\mu B\beta} - \mu B e^{-\mu B\beta}}{e^{\mu B\beta} + e^{-\mu B\beta}}\right)$$

$$\bar{E} = -\mu B\left(\frac{e^{\mu B\beta} - e^{-\mu B\beta}}{e^{\mu B\beta} + e^{-\mu B\beta}}\right)$$

$$\boxed{\bar{E} = -\mu B \operatorname{Tanh}(\mu B\beta)}$$

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Question 6

Answer either part a) or part b) for full credit for this problem.

a) Consider an ensemble of identical, non-interacting particles at temperature T. Each can be in one of four distinct states; the energies of these states are $0, \epsilon, \epsilon, \epsilon$ where $\epsilon = kT/4$. Determine the probability with which any single particle will be in the lowest energy state.

$$P = \frac{e^{-E_s\beta}}{Z} \quad : \quad Z = 1 + 3e^{-\epsilon\beta}$$

$$P = \frac{1}{1+3e^{-\epsilon\beta}} \quad : \quad \epsilon \cdot \frac{1}{kT} = \frac{kT}{4} \cdot \frac{1}{kT} = \frac{1}{4}$$

$$P = \frac{1}{1+3e^{-1/4}} = 0.2997 \approx 0.30$$

$$\boxed{P = 0.30}$$

Question 6 continued ...

b) Using the canonical ensemble, show that the heat capacity for a system is

$$c = \frac{1}{kT^2}\frac{\partial^2 \ln Z}{\partial \beta^2}$$

and use this to show that for a system consisting of N identical, distinguishable, non-interacting particles that the heat capacity is proportional to N.

$$C = \frac{\partial \bar{E}}{\partial T} = \frac{\partial \bar{E}}{\partial \beta} \cdot \frac{\partial \beta}{\partial T} \quad , \quad \beta = \frac{1}{kT}$$

$$\frac{\partial \beta}{\partial T} = \frac{-1}{kT^2} \quad , \quad \bar{E} = -\frac{\partial}{\partial \beta} \ln(Z)$$

$$C = \frac{-1}{k^2} \cdot \frac{\partial}{\partial \beta} \cdot -\frac{\partial}{\partial \beta} \ln(Z) = \frac{1}{kT^2} \frac{\partial^2}{\partial \beta^2} \ln(Z)$$

$$\boxed{C = \frac{1}{kT^2} \frac{\partial^2}{\partial \beta^2} \ln(Z)}$$

$$C = \frac{1}{kT^2} \frac{\partial^2}{\partial \beta^2} \ln(Z) \quad , \quad Z = (Z_{single})^N \quad , \quad C = \frac{1}{kT^2} \frac{\partial^2}{\partial \beta^2} \ln(Z_{single})^N$$

$$C = \frac{N}{kT^2} \cdot \underbrace{\frac{\partial^2}{\partial \beta^2} \ln(Z_{single})}_{}$$

not dependent upon N

$$C = N \cdot \underbrace{\frac{1}{kT^2} \cdot \frac{\partial^2}{\partial \beta^2} \ln(Z_{single})}_{}$$

" - . . . - . . "

$$\therefore \boxed{C \propto N}$$

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