Evolution operators : action on bases

An evolution operator maps linearly

$$|\psi_0\rangle \to \cancel{\hat{U}|\psi_0\rangle} \quad |\Psi\rangle = U|\psi_0\rangle$$

so that

$$\alpha|\psi_1\rangle + \beta|\psi_2\rangle \xrightarrow{\hat{U}} \alpha\hat{U}|\psi_1\rangle + \beta\hat{U}|\psi_2\rangle$$

This means that one can describe evolution via action of the operator on basis states:

Given : want

$|\psi_0\rangle$: $U|\psi_0\rangle$

In general

$|\psi_0\rangle = a_0|0\rangle + a_1|1\rangle$

So $U|\psi_0\rangle$

$= U(a_0|0\rangle + a_1|1\rangle)$

$= a_0\hat{U}|0\rangle + a_1\hat{U}|1\rangle$

We only need $\hat{U}|0\rangle, \hat{U}|1\rangle$ to describe operation

Consider various examples where $\hat{U}$ is specified by action on standard bases.

Exercise: Suppose

$$\hat{U}|0\rangle = e^{-i\alpha/2}|0\rangle \qquad U\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-i\alpha/2}\\0\end{pmatrix}$$

$$\hat{U}|1\rangle = e^{+i\alpha/2}|1\rangle \qquad \begin{pmatrix}e^{-i\alpha/2} & -\\ 0 & -\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-i\alpha/2}\\0\end{pmatrix}$$

a) Determine how i) $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$

ii) $\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$

evolve under $\hat{U}$

b) Determine the matrix for $\hat{U}$. Show that $\hat{U}$ is unitary.

Exercise: Suppose

$$\hat{U}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$\hat{U}|1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle)$$

a) Determine how the following evolve under $\hat{U}$:

i) $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$

ii) $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

iii) $\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$

iv) $\frac{1}{\sqrt{2}}(|0\rangle-i|1\rangle)$

b) Determine the matrix for $\hat{U}$. Show that $\hat{U}$ is unitary.

Exercise: Suppose

$$\hat{u}\,|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right)$$

$$\hat{u}\,|1\rangle = \frac{1}{\sqrt{2}}\left(i\,|0\rangle + |1\rangle\right)$$

a) Determine how the following evolve under $\hat{u}$:

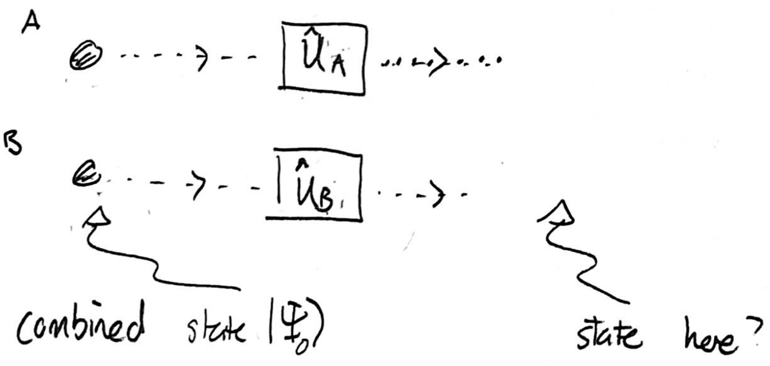
i) $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

ii) $\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

b) Determine the matrix for $\hat{u}$. Show that $\hat{u}$ is unitary.

Multiple qubit operators

What if we have



If the initial state is a product state

$$|\Psi_0\rangle = |\psi_{0A}\rangle\,|\psi_{0B}\rangle$$

$$\rightarrow \underbrace{\hat{U}_A|\psi_{0A}\rangle\;\hat{U}_B|\psi_{0B}\rangle}_{\text{another product state}}$$

We denote the joint operator by:

$$\hat{U}_A \otimes \hat{U}_B$$

tensor product

We ask – how to use this to calculate $|\Psi\rangle$

– what matrix represents $\hat{U}_A \otimes \hat{U}_B$

To calculate we just need

$$\left(\hat{U}_A \otimes \hat{U}_B\right) |0\rangle|0\rangle = U_A|0\rangle\, U_A|0\rangle$$

$$(\quad " \quad)\ |0\rangle|1\rangle =$$

$$(\quad " \quad)\ |1\rangle|0\rangle =$$

$$(\quad " \quad)\ |1\rangle|1\rangle =$$

Example: Consider

$$U_A = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \qquad U_B = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}$$

a) Determine $\hat{U}_A \otimes \hat{U}_B$ on

$|0\rangle|0\rangle$

$|0\rangle|1\rangle$

$|1\rangle|0\rangle$

$|1\rangle|1\rangle$

$U_A|0\rangle\ U_B|0\rangle$

$e^{-i\alpha/2}|0\rangle\ e^{-i\beta/2}|0\rangle = e^{-i(\alpha+\beta)/2}|0\rangle|0\rangle$

b) Determine $\hat{U}_A \otimes \hat{U}_B$ on $\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$

c) " " " $\frac{1}{2}\left(|00\rangle+|01\rangle+|10\rangle-|11\rangle\right)$

~~Example~~: Consider

Exercise

$$\hat{U}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} \qquad \hat{U}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$u_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

a) Determine $\hat{U}_A \otimes \hat{U}_B$ on
$|0\rangle|0\rangle$
$|0\rangle|1\rangle$
$|1\rangle|0\rangle$
$|1\rangle|1\rangle$

b ~~d~~) " " " $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

c) " " " $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

Tensor products

We can represent

$$|0\rangle|0\rangle \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |1\rangle|0\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|0\rangle|1\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad |1\rangle|1\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise a) Determine matrix for $\hat{U}_A \otimes \hat{U}_B$ for ~~previous~~ examples.

b) Show that we can construct $\hat{U}_A \otimes \hat{U}_B$ via.

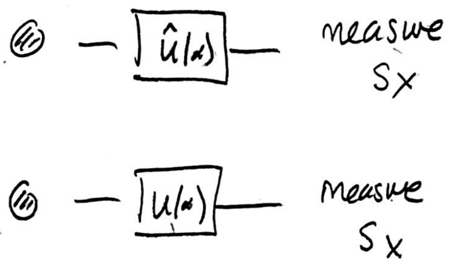
$$\hat{A} \otimes \hat{B} \equiv \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \left(\begin{array}{cc|cc} a_1 b_1 & a_1 b_2 & a_2 b_1 & a_2 b_2 \\ a_1 b_3 & a_1 b_4 & a_2 b_3 & a_2 b_4 \\ \hline a_3 b_1 & a_3 b_2 & a_4 b_1 & a_4 b_3 \\ a_3 b_3 & a_3 b_4 & a_4 b_2 & a_4 b.. \end{array}\right)$$

Estimation:

Consider estimating α in

$$\hat{U}(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

with two qubits.



For the following initial states determine FI.

a) $|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

b) $|\Psi_0\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$

outcomes

S_x^A	S_x^B	
+	+	$\lvert\langle +\hat{x}\rvert\langle +\hat{x}\rvert\,\lvert\psi\rangle\rvert^2$
+	−	
−	+	
−	−	