MAT 202
Larson – Section 6.3
Separation of Variables and the Logistic Equation

In section 6.2 we introduced the strategy of ***separation of variables*** to solve differential equations. In this section we will learn how to recognize when we can use this method.

Ex: Which of the following differential equations can be solved by separation of variables? Explain why or why not.

a) $\dfrac{dy}{dx} = xy + x$ b) $\dfrac{dy}{dx} = \dfrac{x}{x+y}$ c) $\dfrac{dy}{dx} = xy^2e^{x^2}$ d) $\dfrac{dy}{dx} = \dfrac{-2x}{3y}$

<u>Recognizing Differential Equations that can be solved by Separation of Variables</u>:

Consider a differential equation of the form: $M(x) + N(y)\dfrac{dy}{dx} = 0$ where M is a continuous function of x alone and N is a continuous function of y alone. Such equations are said to be ***separable***, and the solution procedure is called ***separation of variables***

1) ***Rewrite*** the differential equation as: $N(y)dy = M(x)dx$. You do this by manipulating the equation algebraically.

2) ***Integrate*** both sides: $\int N(y)dy = \int M(x)dx$ to get $n(y) = m(x) + C_1$.3) The ***solution*** may be written in multiple ways, due to algebraic manipulations; however, $g(y) = h(x) + C_1$ would suffice.

Note: This procedure is the same as how it was introduced in section 6.2.

Ex: Which of the following differential equations can be solved by separation of variables? Explain why or why not.

a) $\frac{dy}{dx} = xy + x$ Yes

$x(y+1)$

$\frac{1}{y+1}\,dy = x\,dx$

b) $\frac{dy}{dx} = \frac{x}{x+y}$ NO

$(x+y)\,dy = x\,dx$

c) $\frac{dy}{dx} = xy^2e^{x^2}$ Yes

$\frac{1}{y^2}\,dy = xe^{x^2}\,dx$

d) $\frac{dy}{dx} = \frac{-2x}{3y}$ Yes

$3y\,dy = -2x\,dx$

NO e) $\frac{dy}{dx} = 2x - 3y$

$+3y$

$dx\left(3y + \frac{dy}{dx} = 2x\right)$

$3y\,dx + dy = 2x\,dx$

Ex: Find the general solution of the differential equation: $\sqrt{1-4x^2}\,y' = x$

Ex: Find the particular solution of the differential equation $yy' - 2e^x = 0$ with the initial condition $y(0) = 3$.

Ex: Find an equation of the graph that passes through the point (9, 1) and has a slope of $y' = \dfrac{y}{2x}$.

Ex: Find the general solution of the differential equation:

$$\sqrt{1-4x^2}\,y' = x$$

$$\sqrt{1-4x^2}\,\frac{dy}{dx} = x$$

$$\int 1\,dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$u = 1-4x^2$

$du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$$y = -\frac{1}{8}\int u^{-1/2}\,du$$

$$y = -\frac{1}{8}\frac{u^{1/2}}{1/2} + C$$

$$y = -\frac{1}{4}\sqrt{1-4x^2} + C$$

Ex: Find the particular solution of the differential equation

$yy' - 2e^x = 0$ with the initial condition $y(0) = 3$.

$(0, 3)$

$$y\frac{dy}{dx} = 2e^x$$

$$\int y\,dy = \int 2e^x\,dx$$

$$\frac{y^2}{2} = 2e^x + C$$

$$\frac{(3)^2}{2} = 2e^0 + C$$

$$\frac{9}{2} = 2 + C$$

$$\frac{5}{2} = C$$

Particular Solution

$$\boxed{\frac{y^2}{2} = 2e^x + \frac{5}{2} \quad \text{or} \quad y^2 = 4e^x + 5}$$

Ex: Find an equation of the graph that passes through the point (9, 1) and has a slope of $y' = \dfrac{y}{2x}$.

$$\int \frac{1}{y}\,dy = \int \frac{1}{2x}\,dx$$

$$\ln|y| = \frac{1}{2}\ln|x| + C$$

$$\ln|1| = \frac{1}{2}\ln|9| + C$$

$$0 = \frac{1}{2}\ln 9 + C$$

$$C = -\frac{1}{2}\ln 9$$

$$\underbrace{-\frac{1}{2}\ln 3^2} = -\ln 3$$

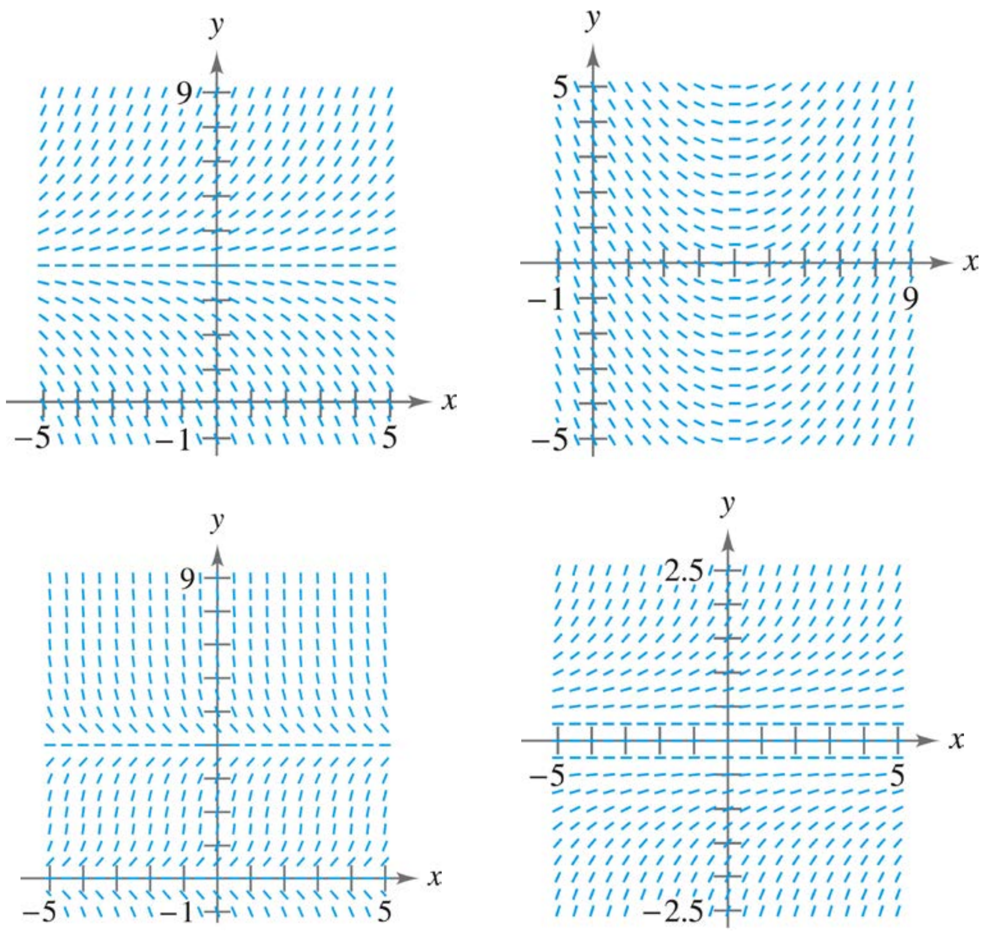
$$\ln|y| = \frac{1}{2}\ln|x| - \frac{1}{2}\ln 9$$

$$\ln|y| = \overbrace{\frac{1}{2}\ln|x|}^{\ln\sqrt{|x|}} - \ln 3$$

$$e^{\ln|y|} = e^{\ln\left(\frac{\sqrt{|x|}}{3}\right)}$$

$$\boxed{y = \frac{\sqrt{x}}{3}}$$

Ex: The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4. Write a differential equation for the statement and match the differential equation with one of the following slope fields.



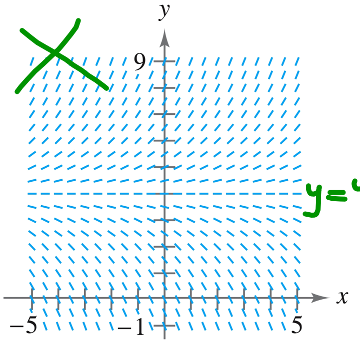
Ex: The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4. Write a differential equation for the statement and match the differential equation with one of the following slope fields.

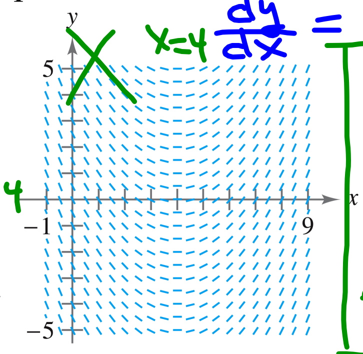
dy/dx

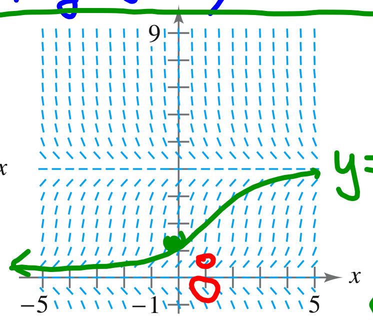
$y(y-4)$

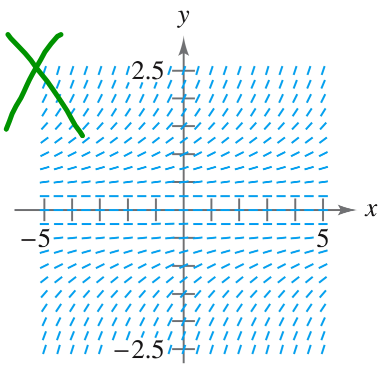
A is proportional to B → A=KB

$$\frac{dy}{dx} = Ky(y-4)$$









x	y	dy/dx
Any #	4	0
Any #	0	0
1	½	−7/4 K
1	−½	9/4 K

<u>Logistic Differential Equation</u>: In section 6.2 we discussed exponential growth, but often, when describing population, there exists some upper limit (call it L) for which growth cannot bypass. This upper limit L is called the ***carrying capacity***, which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the ***logistic differential equation***:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

<u>The General Solution to the Logistic Differential Equation</u>: The general solution to the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$ is:

$$y = \frac{L}{1 + be^{-kt}}$$

Note, by separation of variables, $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$ turns into $\frac{1}{y\left(1 - \frac{y}{L}\right)}\,dy = k\ dt$, which requires a technique called "partial fractions" to integrate. We will revisit this in Chapter 8.5. For now, we will assume that the general solution is valid. You may follow the derivation in the textbook on page 419.

The General Solution to the Logistic Differential Equation: The general solution to the differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$ is:

$$y = \frac{L}{1 + be^{-kt}}$$

$$\frac{dy}{dt} = \frac{0\cdot(1+be^{-kt}) - (0-kbe^{-kt})\cdot L}{(1+be^{-kt})^2}$$

$$\frac{dy}{dt} = \frac{kLbe^{-kt}}{(1+be^{-kt})^2}$$

$$\frac{kLbe^{-kt}}{(1+be^{-kt})^2} = k\left(\frac{L}{1+be^{-kt}}\right)\left(1 - \frac{\left(\frac{L}{1+be^{-kt}}\right)}{L}\right)$$

$$= \left(\frac{kL}{1+be^{-kt}}\right)\left(1 - \frac{1}{1+be^{-kt}}\right)$$

$$= \left(\frac{kL}{1+be^{-kt}}\right)\left(\frac{\cancel{1}+be^{-kt}-\cancel{1}}{1+be^{-kt}}\right)$$

$$= \frac{kL}{1+be^{-kt}} \cdot \frac{be^{-kt}}{1+be^{-kt}}$$

$$= \frac{kLbe^{-kt}}{(1+be^{-kt})^2}$$

Ex: Given the logistic differential equation $\frac{dP}{dt} = 3P\left(1 - \frac{P}{100}\right)$.

a) Write the particular solution of the differential equation that passes through (1, 25).
b) Find P when $t = 3.2$.

Homogeneous Functions & Homogeneous Differential Equations:
1) A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$.
2) A homogeneous differential equation is an equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. To solve an equation of this form by the method of separation of variables, use substitutions $\boldsymbol{y = vx}$ and $\boldsymbol{dy = x\,dv + v\,dx}$.

Ex: Determine whether the function is homogeneous, and if it is, determine its degree: $f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 - y^2}}$.

Ex: Solve the following homogeneous differential equation:
$xy\,dx + \left(y^2 - x^2\right)dy = 0$

Ex: Given the logistic differential equation

$$\frac{dP}{dt} = \overset{k}{3}P\left(1 - \frac{P}{\underset{L}{100}}\right).$$

$k=3$

$L=100$

$y = \dfrac{L}{1+be^{-kt}}$

a) Write the particular solution of the differential equation that passes through $(\overset{t}{1}, \overset{P}{25})$.

b) Find P when $t = 3.2$.

a) $y = \dfrac{L}{1+be^{-kt}} \rightarrow P = \dfrac{100}{1+be^{-3t}}$

$$25 = \frac{100}{1+be^{(-3\cdot 1)}}$$

$$\left(25 = \frac{100}{1+be^{-3}}\right)(1+be^{-3})$$

$$\frac{25(1+be^{-3})}{25} = \frac{100}{25}$$

$$1+be^{-3} = 4$$

$$b = \frac{3}{e^{-3}} = 3e^{3}$$

Particular Solution:

$$P = \frac{100}{1+3e^{3}\cdot e^{-3t}}$$

or

a) $$P = \frac{100}{1+3e^{(3-3t)}}$$

b) $P = \dfrac{100}{1+3e^{(3-3\cdot 3.2)}} \approx \boxed{99.59}$

Ex: Determine whether the function is homogeneous, and

$$f(x, y) = \frac{x^2 y^2}{\sqrt{x^2 - y^2}}.$$

if it is, determine its degree:

$$f(tx, ty) = \frac{(tx)^2 (ty)^2}{\sqrt{(tx)^2 - (ty)^2}} = \frac{t^2 x^2 t^2 y^2}{\sqrt{t^2 x^2 - t^2 y^2}}$$

$$= \frac{t^4 x^2 y^2}{\sqrt{t^2 (x^2 - y^2)}} = \frac{t^{\cancel{4}\,3} x^2 y^2}{\cancel{t} \sqrt{x^2 - y^2}}$$

$$= t^3 \cdot \frac{x^2 y^2}{\sqrt{x^2 - y^2}} = f(x,y)$$

$$= t^3 \cdot f(x,y)$$

homogeneous
degree 3

Ex: Solve the following homogeneous differential equation: $xy\,dx + \left(y^2 - x^2\right)dy = 0$

$M(x,y)\,dx + N(x,y)\,dy = 0$

Is $M(x,y)$ homogeneous? $tx \cdot ty = t^2 \cdot xy$
$= t^2 \cdot M(x,y)$
$\boxed{\text{Degree 2 homog.}}$

Is $N(x,y)$ homogeneous? $(ty)^2 - (tx)^2 = t^2y^2 - t^2x^2$
$= t^2(y^2 - x^2)$
$= t^2\, N(x,y)$
$\boxed{\text{Degree 2 homog.}}$

Substitute $y = vx$ & $dy = x\,dv + v\,dx$
into $xy\,dx + (y^2 - x^2)\,dy = 0$
& attempt separation of variables.

$(x \cdot vx)\,dx + ((vx)^2 - x^2)(x\,dv + v\,dx) = 0$

$x^2v\,dx + (v^2x^2 - x^2)(x\,dv + v\,dx) = 0$

$\cancel{x^2v\,dx} + v^2x^3\,dv + v^3x^2\,dx - x^3\,dv - \cancel{x^2v\,dx} = 0$

$v^2x^3\,dv + v^3x^2\,dx - x^3\,dv = 0$

$v^3x^2\,dx = x^3\,dv - v^2x^3\,dv$

$v^3x^2\,dx = x^3(1 - v^2)\,dv$

$\int \frac{1}{x}\,dx = \int \frac{1-v^2}{v^3}\,dv$

$\ln|x| = \int v^{-3} - v^{-1}\,dv$

$\ln|x| = \frac{-1v^{-2}}{2} - \ln|v| + C$

$\ln|x| + C_1 = -\frac{1}{2v^2} - \ln|v|$

$\ln|x| + \ln|C_2| = -\frac{1}{2v^2} - \ln|v|$ $\quad y = xv$

$\ln|xC_2| = -\frac{1}{2v^2} - \ln|v|$ $\quad v = \frac{y}{x}$

$\ln|xC_2| = \frac{-1}{2\left(\frac{y}{x}\right)^2} - \ln\left|\frac{y}{x}\right|$

$\ln|xC_2| = -\frac{x^2}{2y^2} - \ln\left|\frac{y}{x}\right|$

$\ln|xC_2| + \ln\left|\frac{y}{x}\right| = \frac{-x^2}{2y^2}$

$\ln|C_2y| = \frac{-x^2}{2y^2}$

$e \qquad e$

$C_2y = e^{\frac{-x^2}{2y^2}}$

$\boxed{y = Ce^{\frac{-x^2}{2y^2}}}$