Physics 396

Homework Set 7

1. In spherical-polar coordinates, the line element on the surface of the two-sphere takes the form

$$dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where a is the constant radius of the sphere. The *Christoffel symbols* can be computed from the metric and derivatives of the metric through the expression

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right), \tag{2}$$

where a repeated index implies a sum over that index. For diagonal line elements, Eq. (2) allows one to easily generate the Christoffel symbols.

a) Write down all possible combinations of $g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}$, which corresponds to the left hand side of Eq. (2), for the line element given in Eq. (1).

b) Using Eq. (2), *explicitly* calculate all Christoffel symbols.

2. The *geodesic equation*, which describes the motion of freely falling test particles in curved spacetime, is of the form

$$\frac{d^2x^{\alpha}}{d\tau^2} = -\Gamma^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau}, \tag{3}$$

where repeated indices imply sums over those indices.

a) Using your Christoffel symbols from the previous problem, construct the two equations of motion.[1] Show that these equations of motion take the form

$$\frac{d^2\theta}{dS^2} = \sin\theta\cos\theta\left(\frac{d\phi}{dS}\right)^2 \tag{4}$$

$$\frac{d^2\phi}{dS^2} = -2\,\frac{\cos\theta}{\sin\theta}\frac{d\theta}{dS}\frac{d\phi}{dS}. \tag{5}$$

b) The line element of Eq. (1) contains a symmetry. Identify this symmetry and *explicitly* show that by shifting the coordinate associated with this symmetry by a

[1] Notice in this problem we're studying a 2D space versus a 4D spacetime and therefore lack a time coordinate. Use the distance S to parameterize the two functions, so that $\theta = \theta(S)$, $\phi = \phi(S)$.

constant, the line element remains *unchanged*.

c) Construct the associated Killing vector, ξ, by identifying the coordinate basis components ξ^θ, ξ^ϕ.

d) Construct the *first integral* associated with this symmetry of the form

$$\ell \equiv \vec{\xi} \cdot \vec{u}, \tag{6}$$

where $\vec{u}$ is the two-velocity of the test particle and ℓ is a constant.

e) Show that this first integral is equivalent to the equation of motion given in Eq. (5).

3. Again consider the line element on the surface of the two sphere given by Eq. (1). Another *first integral* can be constructed from the fact that

$$\vec{u} \cdot \vec{u} = 1. \tag{7}$$

a) Using Eq. (1), construct this first integral in terms of θ, ϕ, and S.

b) By using the first integral generated by Eq. (6), show that the second first integral of Eq. (7) can be written in the form

$$\frac{d\theta}{dS} = \frac{1}{a}\left[1 - \frac{\ell^2}{a^2}\csc^2\theta\right]^{1/2}, \tag{8}$$

where we've effectively decoupled these differential equations.

c) Now using the fact that

$$\frac{d\theta}{d\phi} = \frac{d\theta/dS}{d\phi/dS}, \tag{9}$$

separate variables and integrate the above differential equation to arrive at an expression of the form

$$\phi(\theta) = \frac{\ell}{a}\int \frac{\csc^2\theta d\theta}{\sqrt{1 - (\ell^2/a^2)\csc^2\theta}}. \tag{10}$$

d) Integrate this expression and arrive at a solution of the form

$$\cot\theta = \left(\frac{a^2}{\ell^2} - 1\right)^{1/2}\sin(\phi_0 - \phi), \tag{11}$$

where ϕ_0 is a constant of integration.

e) By multiplying both sides of Eq. (11) by $a \sin\theta$, show that this solution can be written in the form

$$z = Ax - By, \tag{12}$$

where $A \equiv (a^2/\ell^2 - 1)^{1/2} \sin\phi_0$ and $B \equiv (a^2/\ell^2 - 1)^{1/2} \cos\phi_0$.

What does Eq. (12) describe?

4. Consider the line element of 3D flat spacetime in polar coordinates of the form

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\phi^2. \tag{13}$$

a) Write down all possible combinations of $g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}$, which corresponds to the left hand side of Eq. (2), for the line element given in Eq. (13).

b) Out of all of the possible combinations of Christoffel symbols, there are only *two* unique, non-vanishing Christoffel symbols for the line element of Eq. (13). Using Eq. (2) and some careful thinking, identify and calculate the two non-vanishing Christoffel symbols.

c) Using these non-vanishing Christoffel symbols, construct the three equations of motion for *light rays*, a.k.a. *null* geodesics. Show that these equations of motion take the form

$$\frac{d^2t}{d\lambda^2} = 0 \tag{14}$$

$$\frac{d^2r}{d\lambda^2} - r\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{15}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = 0, \tag{16}$$

where λ is a parameter.

d) A *first integral* can be constructed from the fact that light rays have *null* four-velocities, namely,

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = 0. \tag{17}$$

Using Eq. (13), construct this first integral.

e) The line element of Eq. (13) contains two symmetries associated with the t and ϕ coordinates. Construct the two *first integrals* associated with these symmetries of the form

$$-e \equiv \underset{\sim}{\xi} \cdot \underset{\sim}{u} \tag{18}$$

$$\ell \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u}, \tag{19}$$

where e and ℓ are constants, $\underset{\sim}{u}$ is the four-velocity of the light ray, and ξ, η are the Killing vectors associated with the t, ϕ coordinates, respectively.

5. Reconsider the *null geodesics* in the 3D flat spacetime of the previous problem.
   a) Show that the first integrals of Eqs. (18) and (19) are *equivalent* to the first and third equations of motion found in Eqs. (14) and (16).
   b) By using the first integrals of Eqs. (18) and (19), show that Eq. (17) can be written in the form

$$\frac{dr}{d\lambda} = \left[\frac{e^2}{c^2} - \frac{\ell^2}{r^2}\right]^{1/2}. \tag{20}$$

   c) Separate variables and integrate the above differential equation to arrive at $r(\lambda)$.
   d) By using the first integral of Eq. (19) and the solution to Eq. (20), find $\phi(\lambda)$.