Thurs: Read 6.2.1, 6.2.2, 6.3

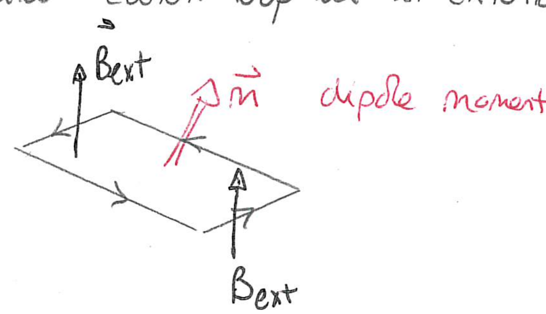
Fri:

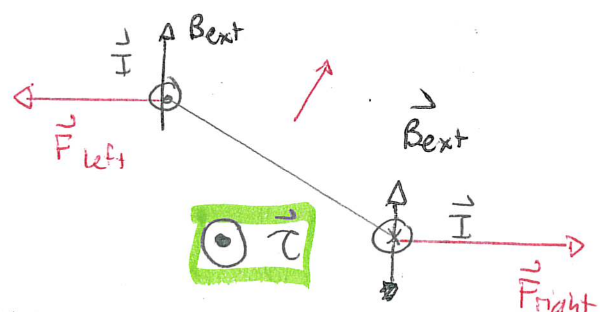
Magnetic dipole in external field 6.1

We will model magnetic materials as a collection of magnetic dipoles that respond to external magnetic fields. We therefore need to know how an individual dipole will respond to an external field.

As an example, consider a rectangular current loop in an external field.

The field will exert forces on the four arms of the loop. We can see that in general these forces will produce a torque on the loop. The diagram shows that in this case the direction of the torque is $\vec{m} \times \vec{B}$.





Separately in a uniform field there will be no net force on the loop. However in a field that is non-uniform there may be a net force and this will depend on the gradient of the field.

In general, we can show:

$\boxed{\begin{array}{l}\text{If a dipole } \vec{m} \text{ is placed in an external magnetic field } \vec{B} \text{ then}\\ \text{the force exerted by the field on the dipole is}\\ \qquad \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})\\ \text{and if the field is uniform, then the torque on the dipole is}\\ \qquad \vec{\tau} = \vec{m}\times\vec{B}\end{array}}$

Proof: Torque: In a uniform field - see Griffiths 6.2

Force: This requires several lemmas

Lemma 1: For any localized current density $\vec{J}(\vec{r}\,')$

$$\int_{\text{all space}} (\vec{r}\cdot\vec{J}(\vec{r}\,'))\,\vec{r}\,'\,d\tau' + \int_{\text{all space}} (\vec{r}\,'\cdot\vec{r})\,\vec{J}(\vec{r}\,')\,d\tau' = 0$$

Proof: Let $\vec{s}$, $\vec{r}$ be any vectors

$$\vec{\nabla}'\cdot\left[(\vec{r}\,'\cdot\vec{r})(\vec{r}\,'\cdot\vec{s})\,\vec{J}(\vec{r}\,')\right] = \vec{\nabla}'\left[(\vec{r}\,'\cdot\vec{r})(\vec{r}\,'\cdot\vec{s})\right]\cdot\vec{J}(\vec{r}\,')$$
$$+ (\vec{r}\,'\cdot\vec{r})(\vec{r}\,'\cdot\vec{s})\,\vec{\nabla}'\cdot\vec{J}(\vec{r}\,')$$

For any non-time varying current $\vec{\nabla}\cdot\vec{J} = 0$. Then also

$$\vec{\nabla}'(\vec{r}\,'\cdot\vec{r}) = \vec{r}$$
$$\vec{\nabla}'(\vec{r}\,'\cdot\vec{s}) = \vec{s}$$

gives:

$$\vec{\nabla}'\cdot[\ \ldots\ ] = \left[\vec{r}\cdot\vec{J}(\vec{r}\,')\right]\left[\vec{r}\,'\cdot\vec{s}\right] + \left[\vec{r}\,'\cdot\vec{r}\right]\left[\vec{s}\cdot\vec{J}(\vec{r}\,')\right]$$

Now integrating over all space gives

$$\int_{\text{all space}} \vec{\nabla}\cdot[\ldots]\,d\tau' = \oint_{\text{limiting surface}} [\ldots]\cdot d\vec{a}$$

On this surface $\vec{J}=0$ for a localized current. So

$$0 = \left[\int [\vec{r}\cdot\vec{J}(\vec{r}')]\,\vec{r}'\,d\tau' + \int (\vec{r}'\cdot\vec{r})\,\vec{J}(\vec{r}')\,d\tau'\right]\cdot\vec{s} = 0$$

This is true for all $\vec{s}$ and that proves the lemma. ■

Lemma 2 For any current $\vec{J}(\vec{r}')$

$$\vec{r}\times\int \vec{r}'\times\vec{J}(\vec{r}')\,d\tau' = \int [\vec{r}\cdot\vec{J}(\vec{r}')]\,\vec{r}'\,d\tau' - \int (\vec{r}'\cdot\vec{r})\,\vec{J}(\vec{r}')\,d\tau'$$

Proof: $\vec{r}\times(\vec{r}'\times\vec{J}(\vec{r}')) = \vec{r}'\,(\vec{r}\cdot\vec{J}(\vec{r}')) - \vec{J}(\vec{r}')(\vec{r}'\cdot\vec{r})$

Integrating gives the result. ■

Lemma 3 $\int (\vec{r}'\cdot\vec{r})\,\vec{J}(\vec{r}')\,d\tau' = -\,\vec{r}\times\vec{m}$

Proof: Subtract the result of lemma 2 from Lemma 1 and where the dipole moment is

$$\vec{m} = \frac{1}{2}\int \vec{r}'\times\vec{J}(\vec{r}')\,d\tau'$$ ■

Proof of main result:

In general the force on a current is

$$\vec{F} = \int_{\text{all space}} \vec{J}(\vec{r}') \times \vec{B}(\vec{r}')\, d\tau'$$

We now pick a reference point $\vec{r}$ and expand $\vec{B}(\vec{r}')$ about this.

Then:

$$\vec{B}(\vec{r}') = B(\vec{r}) + (\vec{r}' - \vec{r}) \cdot \vec{\nabla}\, \vec{B}(\vec{r}) + \dots$$

where the latter term means

$$(x'-x) \frac{\partial B(x,y,z)}{\partial x} + (y'-y) \frac{\partial B}{\partial y}(x,y,z) + \dots$$

Thus:

$$\vec{F} = \int \vec{J}(\vec{r}') \times B(\vec{r})\, d\tau' + \int \vec{J}(\vec{r}') \times \left[(\vec{r}' \cdot \vec{\nabla}) \vec{B} \right] d\tau' - \underbrace{\int \vec{J}(\vec{r}') \times \left[(\vec{r} \cdot \vec{\nabla}) \vec{B} \right]}_{\underbrace{\left[\int \vec{J}(\vec{r}')\, d\tau'\right]}_{0} \times \dots}$$

$$= \underbrace{\left[\int \vec{J}(\vec{r}')\, d\tau' \right]}_{=0} \times B(\vec{r}) + \dots$$

$$\Rightarrow \quad \vec{F} = \int_{\text{all space}} \vec{J}(\vec{r}') \times \left[(\vec{r}' \cdot \vec{\nabla}) \vec{B} \right] d\tau'$$

Now the integrand is

$$\vec{J}(\vec{r}') \times \left[x' \frac{\partial \vec{B}}{\partial x} + y' \frac{\partial \vec{B}}{\partial y} + z' \frac{\partial \vec{B}}{\partial z} \right]$$

$$= \left\{ \frac{\partial}{\partial x} \left[x' J(\vec{r}') \right] + \frac{\partial}{\partial y} \left[y' \vec{J}(\vec{r}') \right] + \frac{\partial}{\partial z} \left[z' \vec{J}(\vec{r}') \right] \right\} \times \vec{B}$$

Then with $x' = \hat{x} \cdot \vec{r}'$ we have:

$$\vec{F} = \frac{\partial}{\partial x} \int [\hat{x} \cdot \vec{r}'] \vec{J}(\vec{r}') d\tau' \times \vec{B}$$

$$+ \frac{\partial}{\partial y} \int [\hat{y} \cdot \vec{r}'] \vec{J}(\vec{r}') d\tau' \times \vec{B} + \cdots$$

By lemma 3

$$\vec{F} = \frac{\partial}{\partial x} \left[- (\hat{x} \times \vec{m}) \times \vec{B} \right] + \frac{\partial}{\partial y} \left[- (\hat{y} \times \vec{m}) \times \vec{B} \right] \cdots$$

$$= \frac{\partial}{\partial x} \left[\vec{B} \times (\hat{x} \times \vec{m}) \right] + \frac{\partial}{\partial y} \left[\vec{B} \times (\hat{y} \times \vec{m}) \right] + \frac{\partial}{\partial z} \left[\vec{B} \times (\hat{z} \times \vec{m}) \right]$$

$$= \frac{\partial}{\partial x} \left[\hat{x} (\vec{B} \cdot \vec{m}) - \vec{m} (\hat{x} \cdot \vec{B}) \right] + \cdots$$

$$= \vec{\nabla} (\vec{m} \cdot \vec{B}) - \vec{m} \underbrace{\cancel{(\vec{\nabla} \cdot \vec{B})}}_{0 \text{ in magnetism}}$$

This proves $\vec{F} = \vec{\nabla} (\vec{m} \cdot \vec{B})$ ∎

1 Magnetic dipole in a field with a gradient

Suppose that $\mathbf{B} = B(z)\hat{\mathbf{z}}$ where $B(z)$ is the field magnitude which depends on z only. A magnetic dipole is propelled along the $\hat{\mathbf{x}}$ direction and then enters this field. Determine the force exerted on this particle and describe how the process of observing the trajectory of the particle as it traverses the field can be used to determine a component of its dipole moment. This is the basic idea behind the famous Stern-Gerlach experiment.

Answer: $\vec{F} = (\vec{m} \cdot \vec{\nabla})\vec{B}$

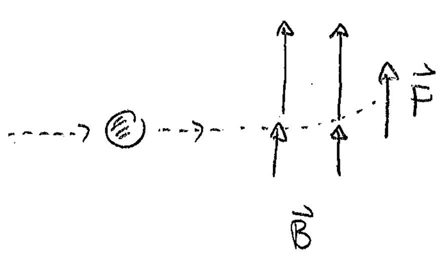
and $\vec{B} = B_z \hat{z}$

$$\vec{F} = \left[m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right] B_z(z)\, \hat{z}$$

$$= \left[m_x \overbrace{\cancel{\frac{\partial B_z(z)}{\partial x}}}^{0} + m_y \overbrace{\cancel{\frac{\partial B_z(z)}{\partial y}}}^{0} + m_z \frac{\partial B_z}{\partial z} \right] \hat{z}$$

$$= m_z \frac{\partial B_z}{\partial z} \hat{z}$$

If $\frac{\partial B_z}{\partial z} > 0$ then the force is along the $\hat{z}$ direction and is proportional to m_z. Thus the particle will be deflected along the z-direction. Then



$m_z > 0 \Rightarrow$ deflected up

$m_z < 0 \Rightarrow$ deflected down

The degree of deflection will be proportional to m_z. So if we measure the deflection we can obtain m_z.

2 Diamagnetic effects

An electron in an external uniform magnetic field will orbit in a circle. Suppose that the field points in the $\hat{\mathbf{z}}$ direction and the electron moves with speed v.

a) Determine an expression that relates the orbital speed to the radius of orbit and the magnitude of the field.

The orbiting electron does not exactly provide a steady current, but we will consider a simple approximate model which associates an effective current with the electron and the current magnitude is

$$I = \frac{e}{T}$$

where T is the orbital period of the electron.

b) Determine an expression for the effective magnetic dipole moment of the orbiting electron in terms of the external magnetic field. Which way does the dipole moment point? What happens to the dipole moment as the field increases?

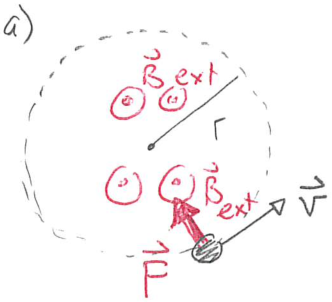
The magnetic field produced by a current loop (of radius r) at its center has magnitude

$$B = \frac{\mu_0 I}{2r}.$$

c) Determine an expression for the magnetic field produced by the orbiting electron. Which way does this field point?

d) Determine an expression for the ratio of the magnetic field produced by the orbiting electron to the external field.

Answer. a)



$$\vec{F} = q\vec{v} \times \vec{B}_{ext}$$

$$= -e\,\vec{v} \times \vec{B}_{ext}$$

$$\Rightarrow \quad F = e v B_{ext} = m_e a$$

$m_e \to$ mass electron

But $\quad a = v^2/r \quad \Rightarrow \quad e \cancel{v} B_{ext} = m_e \frac{v^{\cancel{2}}}{r}$

$$\Rightarrow \quad v = \frac{e B_{ext} r}{m_e}$$

b) $m = I a$

where a is the area of the loop.

$$\Rightarrow \quad m = \frac{e}{T} \pi r^2$$

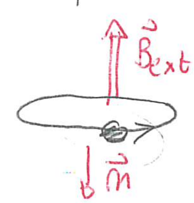
But $v = \frac{\Delta s}{\Delta t} = \frac{2\pi r}{T} \Rightarrow T = \frac{2\pi r}{v}$.

Thus

$$m = \frac{e \cancel{\pi} r^{\cancel{2}}}{2 \cancel{\pi} \cancel{r}} v = \frac{e r v}{2} = \frac{e r}{2} \frac{e B_{ext} r}{m_e}$$

$$\Rightarrow \quad \boxed{m = \frac{e^2 r^2}{2 m_e} B_{ext}}$$

The dipole moment is proportional to the field but points in the opposite direction



c) $B = \frac{\mu_0 I}{2r} = \frac{\mu_0 e}{2r\, T} = \frac{\mu_0 e}{2r\, 2\pi r} v = \frac{\mu_0 e}{4\pi r^2} v$

$$= \frac{\mu_0 e}{4\pi r^2} \frac{e B_{ext} r}{m_e} = \frac{\mu_0 e^2}{4\pi r} \frac{B_{ext}}{m_e} \Rightarrow B = \frac{\mu_0 e^2}{4\pi r m_e} B_{ext}$$

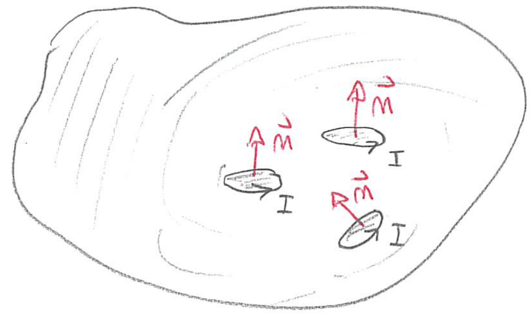
d) $\frac{B}{B_{ext}} = \frac{\mu_0 e^2}{4\pi r\, m_e} = \frac{4\pi \times 10^{-7}\ N/A^2 \times (1.6 \times 10^{-19} C)^2}{4\pi \times 9.11 \times 10^{-31} kg} \frac{1}{r}$ ($N = kg\, m/s^2$)

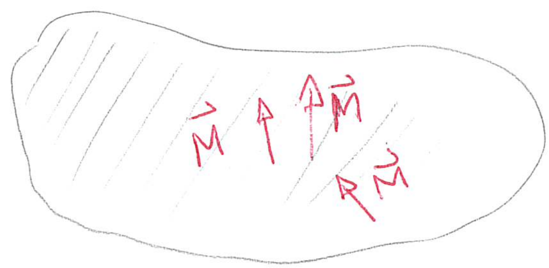
$$\frac{B}{B_{ext}} = 2.8 \times 10^{-15} m \ \frac{1}{r}$$

For molecules $r \approx 10^{-10} m \Rightarrow \frac{B}{B_{ext}} \approx 10^{-5}$, Small but observable!

Magnetization

Now consider a material which can be model (magnetically) as a collection of many perfect point magnetic dipoles. Rather than describe individual magnetic dipoles we will describe their distribution via a continuous variable called magnetization specifically.



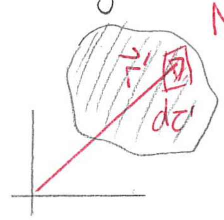


> The magnetization of a material $\vec{M}(\vec{r}\,')$ is defined such that the magnetic dipole moment of a small region with volume $d\tau'$ at the location $\vec{r}\,'$ is
>
> $$d\vec{m} = \vec{M}(\vec{r}\,')\,d\tau'$$

Units: A/m

We will be able to use the magnetization as follows:

given $\vec{M}(\vec{r}\,')$



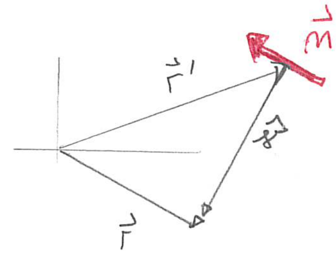
→ Compute magnetic vector potential $\vec{A}(\vec{r})$ at field location $\vec{r}$

→ Determine magnetic fied $\vec{B}(\vec{r})$ at field location $\vec{r}$

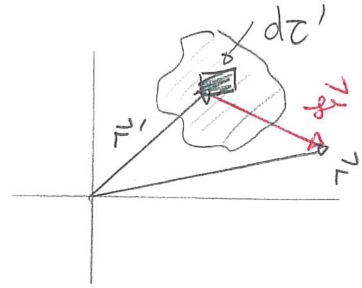
→ Compute bound surface current density and bound volume current density →

The basis for the first route is that, for a point dipole at $\vec{r}\,'$, the magnetic vector potential at $\vec{r}$ is

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$



Then over an extended distribution the exact vector potential will be assembled from contributions



$$d\vec{A} = \frac{\mu_0}{4\pi} \frac{d\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{\mu_0}{4\pi} \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

giving

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'}$$

This possibly gives one way to determine the potential and then the magnetic field. However, such direct calculations can be difficult and we consider the possibility of using current distributions derived from the magnetization. The basic idea here is that:

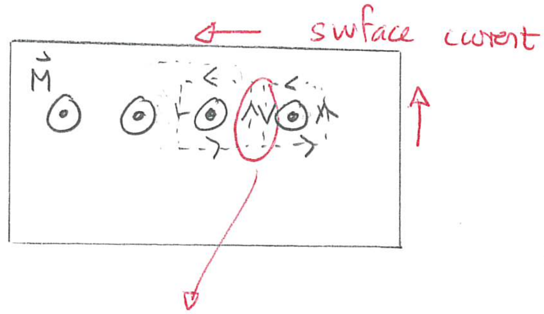
volume current distribution $\vec{J}(\vec{r}\,') \rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}} d\tau'$

surface " " $\vec{K}(\vec{r}\,') \rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,')}{\mathscr{r}} d\tau'$

We now use this to find appropriate bound current distributions.

To gain some insight consider the following simple examples.

Uniform magnetization

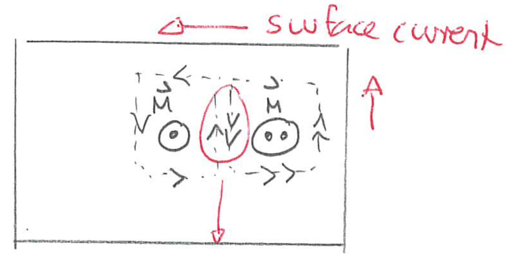


no bound volume current

bond surface current exists

perpendicular to normal (cross product)

Gradient magnetization

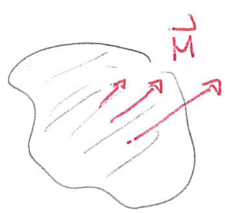


bound volume current exists

band surface current exists

In either case we aim for a scheme such that:

Known magnetization



→ Determine band current densities $\vec{J}_b$, $\vec{K}_b$.

↓

The fields produced by the magnetization satisfy

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_b \qquad \vec{\nabla}\cdot\vec{B} = 0$$

↙

Usual techniques for computing fields from currents - just use band currents.

We can prove that:

Given a magnetization $\vec{M}(\vec{r}\,')$ the magnetic vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_b(\vec{r}\,')}{\mathscr{r}}\, d\tau' + \frac{\mu_0}{4\pi}\int \frac{\vec{K}_b(\vec{r}\,')}{\mathscr{r}}\, da'$$

where $\vec{\mathscr{r}}, \vec{r}, \vec{r}\,'$ are defined as usual and the bound volume current density is:

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

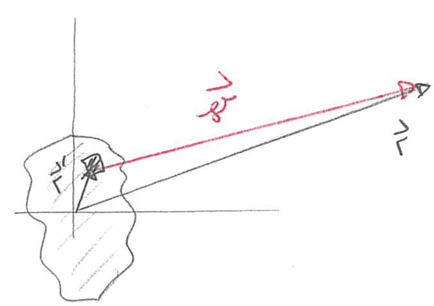
and the bound surface current density is

$$\vec{K}_b = \vec{M} \times \hat{n}$$

where $\hat{n}$ is the normal to the surface

Proof: Start with

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$



We first replace $\hat{\mathscr{r}}/\mathscr{r}^2$ using

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$$

where $\vec{\nabla}' = \hat{x}\frac{\partial}{\partial x'} + \hat{y}\frac{\partial}{\partial y'} + \hat{z}\frac{\partial}{\partial z'}$

$$\mathscr{r} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

We can then integrate by parts (in an effort to switch the $\vec{\nabla}'$ operator).

$$\vec{\nabla} \times (f\vec{C}) = f(\vec{\nabla}\times\vec{C}) - \vec{C}\times\vec{\nabla} f$$

$$\Rightarrow \quad \vec{C}\times\vec{\nabla} f = f(\vec{\nabla}\times\vec{C}) - \vec{\nabla}\times(f\vec{C}).$$

Thus:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \vec{\nabla}'\times\vec{M}(\vec{r}')\, d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}'\times\left(\frac{1}{\mathscr{r}}\vec{M}(\vec{r}')\right) d\tau'$$

We can restrict the integrals to regions where the magnetization is non-zero. Then:

$$\underset{\text{volume}}{\int} (\vec{\nabla}\times\vec{C})\, d\tau' = -\underset{\text{surface}}{\oint} \vec{C}\times d\vec{a}$$

gives:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}_b(\vec{r}')}{\mathscr{r}} d\tau' + \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \underbrace{\vec{M}(\vec{r}')\times d\vec{a}'}_{\vec{M}(\vec{r}')\times\hat{n}\, da'}$$

where $\vec{J}_b = \vec{\nabla}\times M(\vec{r})$. Then with

$\vec{K}_b = \vec{M}(\vec{r}')\times\hat{n}$

we get the result. $\boxtimes$