MAT 201
Larson/Edwards – Section 2.5
Implicit Differentiation

Up to this point we have differentiated functions that were written in their ***explicit form***, that is $y = f(x)$. Suppose that we are asked to find the derivative of y with respect to x, that is, $\frac{dy}{dx}$ for the following function: $x^2 y = 1$. **How would we proceed?**

One approach would be to re-write the equation in its explicit form, then find the derivative.

This method works great **IF** you are able to re-write the equation in its explicit form. What if we are asked to find $\frac{dy}{dx}$ for this relation: $y^2 + x^2 - 2y - 4x = 4$? This is not as easy to re-write in its explicit form!

Up to this point we have differentiated functions that were written in their ***explicit form***, that is $y = f(x)$. Suppose that we are asked to find the derivative of y with respect to x, that is, $\frac{dy}{dx}$ for the following function: $x^2 y = 1$. **How would we proceed?**

Rewrite in explicit form: $y = \frac{1}{x^2}$

$y = x^{-2}$

$$\boxed{\frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}}$$

Implicit Differentiation: Let's use implicit differentiation on our previous example: $x^2 y = 1$. At this point we know that our answer should be: $\frac{dy}{dx} = \frac{-2}{x^3}$.

Step 1: $\frac{d}{dx}\left[x^2 y\right] = \frac{d}{dx}[1]$ **(Think balancing technique in equations)**

Notice that on the left side of the equation we are finding the derivative of a product, so we must use the product rule. On the right side we are finding the derivative of a constant, therefore the derivative with respect to *x* is equal to 0.

Step 2: $\left(\frac{d}{dx}\left[x^2\right]\right) \cdot y + \frac{d}{dx}[y] \cdot x^2 = 0$

How do we evaluate $\frac{d}{dx}[y]$? We need to use the chain rule!!!

Remember: $\frac{d}{dx}\left[u^n\right] = n \cdot u^{n-1} \cdot u' = nu^{n-1} \cdot \frac{du}{dx}$. Using this rule we get $\frac{d}{dx}[y] = 1 \cdot \frac{dy}{dx}$

Step 3: $2xy + 1 \cdot \frac{dy}{dx} \cdot x^2 = 0$

Step 4: Rewrite the expression: $2xy + x^2 \frac{dy}{dx} = 0$

Step 5: Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{-2xy}{x^2} = \frac{-2y}{x}$.

WAIT A MINUTE!!! OUR ANSWERS ARE DIFFERENT!!! WHY???

Explicit: $\frac{dy}{dx} = \frac{-2}{x^3}$ **Implicit:** $\frac{dy}{dx} = \frac{-2y}{x}$

Ex: Use the derivatives that you found explicitly and implicitly for the equation $x^2y = 1$ to evaluate at the point $\left(-3, \frac{1}{9}\right)$. What do you notice? Is the result what you expected?

Ex: Find $\frac{dy}{dx}$ for $y^2 + x^2 - 2y - 4x = 4$ by implicit differentiation.

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Explicit: $\frac{dy}{dx} = \frac{-2}{x^3}$

$x = -3$

$\frac{dy}{dx} = \frac{-2}{(-3)^3} = \boxed{\frac{2}{27}}$

Implicit: $\frac{dy}{dx} = \frac{-2y}{x}$

$x = -3, \ y = \frac{1}{9}$

$\frac{dy}{dx} = \frac{-2\left(\frac{1}{9}\right)}{-3} = \frac{-2}{9} \div -3$

$= \boxed{\frac{2}{27}}$

Ex: Find $\frac{dy}{dx}$ for $y^2 + x^2 - 2y - 4x = 4$ by implicit differentiation.

$$\frac{d}{dx}[y^2] + \frac{d}{dx}[x^2] - \frac{d}{dx}[2y] - \frac{d}{dx}[4x] = \frac{d}{dx}[4]$$

$$2y \cdot \frac{dy}{dx} + \cancel{2x} - 2 \cdot \frac{dy}{dx} - \cancel{4} = 0$$
$$\cancel{-2x} \qquad \cancel{+4} \qquad -2x+4$$

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} = -2x+4$$

$$\frac{\frac{dy}{dx}\cancel{(2y-2)}}{\cancel{2y-2}} = \frac{-2x+4}{2y-2}$$

$$\frac{dy}{dx} = \frac{-2x+4}{2y-2} = \frac{\cancel{2}(-x+2)}{\cancel{2}(y-1)}$$

$$\boxed{\frac{dy}{dx} = \frac{-x+2}{y-1}}$$

What if we are asked to find $\frac{dy}{dx}$ for this relation: $y^2 + x^2 - 2y - 4x = 4$? This is not as easy to re-write in its explicit form!