Final: * Tues 18 May 10 am → 11:50am

* Final delivered electronically + submitted to D2L Drop Box

~~* Monitor via Zoom - use camera to show face.~~

* Covers Ch 10-12

  Lectures ~~16~~ → 17 → 28

  HW 14 → 23

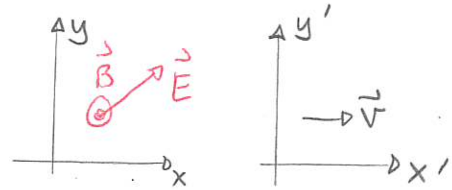
2012 Final All

2016 Final All

Fri: HW ~~24~~ 23 D2L

Field Transformations

Consider fields as viewed from two inertial frames. The fields transform as:



$$\boxed{\begin{aligned}
E_x' &= E_x \\
E_y' &= \gamma\left(E_y - vB_z\right) \\
E_z' &= \gamma\left(E_z + vB_y\right) \\[1em]
B_x' &= B_x \\
B_y' &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \\
B_z' &= \gamma\left(B_z - \frac{v}{c^2}E_y\right)
\end{aligned}}$$

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = v/c$$

We can immediately reach important general conclusions. Suppose for example that, in the unprimed frame, the sources only produce electric fields, i.e. $\vec{B}=0$. Then

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \frac{\partial\vec{E}}{\partial t} = -\frac{1}{\epsilon_0}\vec{J}$$

In this case

$$E_x' = E_x \qquad B_x' = 0$$

$$E_y' = \gamma E_y \qquad B_y' = \gamma\frac{v}{c^2}E_z = \frac{v}{c^2}E_z'$$

$$E_z' = \gamma E_z \qquad B_z' = -\gamma\frac{v}{c^2}E_y = -\frac{v}{c^2}E_y'$$

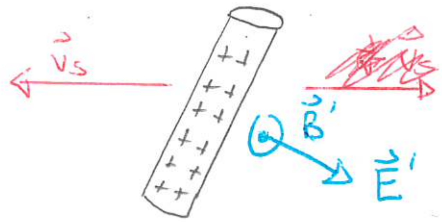
Now the velocity of the sources as observed from the primed frame is $\vec{V}_S = -v\hat{x}$. So

$$\vec{V}_S\times\vec{E}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -v & 0 & 0 \\ E_x' & E_y' & E_z' \end{vmatrix} = \hat{y}\,vE_z' - \hat{z}\,vE_y' = c^2\vec{B}'$$

Thus we get

$$\boxed{\begin{array}{l}\text{If the magnetic field as observed in } S \text{ is zero then in } S' \\ \text{the field is} \\ \qquad \vec{B}' = \frac{1}{c^2}\vec{V}_S\times\vec{E}' \\ \text{where } \vec{V}_S \text{ is the velocity of } S \text{ as observed from } S'\end{array}}$$

So if the sources are stationary in S then this lets us calculate $\vec{E}'$ and by relatively easy algebra $\vec{B}'$



1 Point charge moving with constant velocity

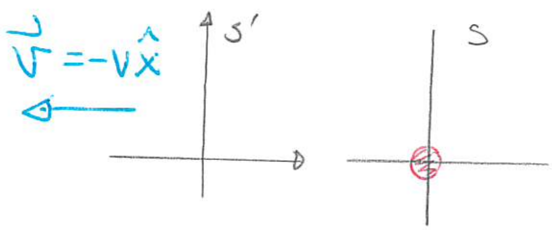
A point charge moves with constant velocity $\mathbf{v} = v\hat{\mathbf{x}}$ as observed from the S' frame. Let the S frame be the frame in which the particle is at rest.

a) Determine an expression for the electric and magnetic fields as observed from the S frame.

b) With what velocity must the S' frame move with respect to the S frame?

c) Use the field transformations to determine the fields in the S' frame. Express the results in terms of coordinates for the primed frame.

Answer Suppose particle is at rest in S frame at origin



a) $\vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\hat{r}}{r^2} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3}$ $\qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$r = (x^2+y^2+z^2)^{3/2}$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow E_x = \frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} \qquad B_x = B_y = B_z = 0$$

$$E_y = \frac{q}{4\pi\epsilon_0}\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$E_z = \frac{q}{4\pi\epsilon_0}\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

b) $\vec{v} = -v\hat{x}$

c) Here $E_x' = E_x$

$$E_y' = \gamma(E_y + vB_z) = \gamma E_y$$

$$E_z' = \gamma(E_z - vB_y) = \gamma E_z$$

Now $x' = x'^1 = \gamma(x' - \beta x^0)$

$$\Rightarrow \quad x^1 = \gamma(x'^1 + \beta x'^0)$$

$$\Rightarrow \quad x = \gamma\left(x' - \frac{v}{c} ct'\right) = \gamma(x' - vt')$$

Then $y' = y$

$z' = z$

gives:

$$x^2 + y^2 + z^2 = \gamma^2(x' - vt')^2 + y'^2 + z'^2$$

Thus

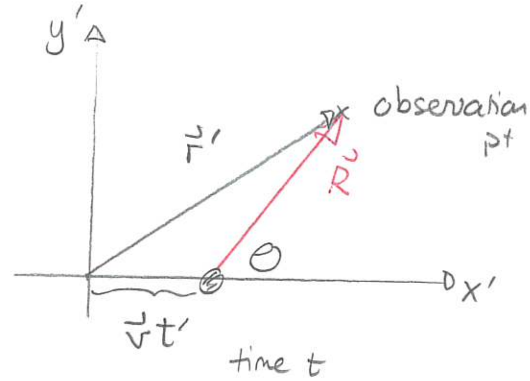
$$E_x' = \frac{q}{4\pi\epsilon_0} \frac{\gamma(x' - vt')}{\left[\gamma^2(x' - vt')^2 + y'^2 + z'^2\right]^{3/2}}$$

$$E_y' = \frac{q}{4\pi\epsilon_0} \frac{\gamma y'}{[\ \ldots\ ]^{3/2}}$$

$$E_z' = \frac{q}{4\pi\epsilon_0} \frac{\gamma z'}{[\ \ldots\ ]^{3/2}}$$

Note that $\vec{B}' = \frac{1}{c^2} \vec{v} \times \vec{E}'$

Note that we can use the vector $\vec{R}$ as defined below:



$$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$\vec{R} = \vec{r}' - \vec{v}t$$

Then

$$\vec{R} = \underbrace{(x' - vt)}_{R_x}\hat{x} + \underbrace{y'}_{R_y}\hat{y} + \underbrace{z'}_{R_z}\hat{z}.$$

So $\gamma(x'-vt')^2 + y'^2 + z'^2 = \gamma^2 R_x^2 + R_y^2 + R_z^2$

But $R_x = R\cos\theta$

$$\sqrt{R_y^2 + R_z^2} = R\sin\theta$$

give

$$\gamma(\dots)^2 + y'^2 + z'^2 = R^2\sin^2\theta + \gamma^2 R^2\cos^2\theta$$

$$= R^2\left[\sin^2\theta + \gamma^2\cos^2\theta\right]$$

$$= R^2\left[\sin^2\theta + \gamma^2(1-\sin^2\theta)\right]$$

$$= \gamma^2 R^2\left[1 + \sin^2\theta\left(\frac{1}{\gamma^2} - 1\right)\right]$$

$$= \gamma^2 R^2\left[1 + \sin^2\theta\left(1 - \frac{v^2}{c^2} - 1\right)\right]$$

$$= \gamma^2 R^2\left[1 - \frac{v^2}{c^2}\sin^2\theta\right]$$

Thus:

$$\left[\gamma^2 (x'-vt')^2 + y'^2 + z'^2\right]^{3/2} = \gamma^3 R^3 \left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}$$

So:

$$E_x' = \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0}\frac{R_x}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2} R^3}$$

$$E_y' = \frac{1}{\gamma^2}\frac{q}{4\pi\epsilon_0}\frac{R_y}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2} R^3}$$

$\vdots$

Thus

$$\vec{E}' = \frac{q}{4\pi\epsilon_0}\frac{(1-v^2/c^2)}{\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}\frac{\hat{R}}{R^2}$$

This is what we had obtained by direct calculation earlier.

Force Transformations

Do the field transformation rules generate the correct force transformation rules? We will consider this for the situation where

1) the source charges are stationary in frame S
2) the force is the relativistic force i.e.

$$F_x = \frac{dp^1}{dt} \qquad \text{where} \quad p^1 = \frac{1}{\sqrt{1-u^2/c^2}}\, p_x$$

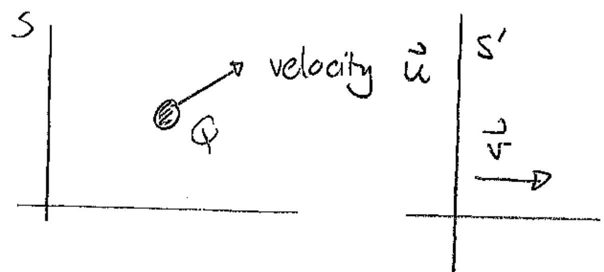
$$F_y = \frac{dp^2}{dt} \qquad \text{where} \quad p^2 = \frac{1}{\sqrt{1-u^2/c^2}}\, p_y$$

$$F_z = \frac{dp^3}{dt} \qquad \text{where} \quad p^3 = \frac{1}{\sqrt{1-u^2/c^2}}\, p_z$$

3) the Lorentz force law is correct.

Then in the S frame the sources only produce an electric field $\vec{E}$. The Lorentz force law gives

$$\vec{F} = Q\vec{E}$$



where the velocity of the particle in the S frame is $\vec{u}$.

Now suppose that this is viewed from a frame S′ that moves with velocity $\vec{v} = v\hat{x}$ w.r.t S. Then the force transformation law gives:

In the S′ frame the components of the force are:

$$F_x' = \left[F_x - \frac{\beta}{c}\,\vec{u}\cdot\vec{F}\right] / \left(1 - \frac{\beta}{c} u_x\right)$$

$$F_y' = \frac{1}{\gamma\left(1 - \frac{\beta}{c} u_x\right)}\, F_y$$

$$F_z' = \frac{1}{\gamma\left(1 - \frac{\beta}{c} u_x\right)}\, F_z$$

So we expect that

Based on the Lorentz force law in the S frame and the fields in the S frame, the force in the S' frame is:

$$F_x' = \frac{Q}{\left(1-\frac{\beta}{c}u_x\right)}\left[E_x - \frac{\beta}{c}\vec{u}\cdot\vec{E}\right]$$

$$F'_y = \frac{Q}{\gamma\left(1-\frac{\beta}{c}u_x\right)}E_y$$

$$F'_z = \frac{Q}{\gamma\left(1-\frac{\beta}{u}u_x\right)}E_z$$

On the other hand we could

- transform the fields
- transform the particle velocity
- use the Lorentz force law in S'.

The fields transform as:

$$E_x' = E_x \qquad B_x' = 0$$

$$E_y' = \gamma E_y \qquad B_y' = \gamma\frac{V}{c^2}E_z$$

$$E_z' = \gamma E_z \qquad B_z' = -\gamma\frac{V}{c^2}E_y$$

Then

$$\vec{F}' = Q\left(\vec{E}' + \vec{u}'\times\vec{B}'\right)$$

gives:

$$F'_x = Q\left(E'_x + u'_y B'_z - u'_z B'_y\right)$$

$$F'_y = Q\left(E'_y + \cancel{u'_z B'_x} - u'_x B'_z\right)$$

$$F'_z = Q\left(E'_z + u'_x B'_y - \cancel{u'_y B'_x}\right)$$

Consider the components:

1) The x component

$$F'_x = Q\left(E_x + u'_y\left(-\gamma \frac{v}{c^2} E_y\right) - u'_z \gamma \frac{v}{c^2} E_z\right)$$

$$= Q\left[E_x - \frac{\gamma v}{c^2}\left(u_y' E_y + u_z' E_z\right)\right]$$

We need the velocity transformations. These are

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{u_x - v}{1 - \frac{\beta}{c} u_x}$$

$$u_y' = \frac{u_y}{\gamma(1 - v u_x / c^2)} = \frac{u_y}{\gamma(1 - \frac{\beta}{c} u_x)}$$

$$u_z' = \frac{u_z}{\gamma(1 - v u_x / c^2)} = \frac{u_z}{\gamma(1 - \frac{\beta}{c} u_x)}$$

So $$F_x' = Q\left(E_x - \frac{\cancel{\gamma}\beta}{c} \frac{1}{\cancel{\gamma}(1 - \frac{\beta}{c} u_x)}\left[u_y E_y + u_z E_z\right]\right)$$

$$= \frac{Q}{(1 - \frac{\beta}{c} u_x)}\left[E_x - \frac{\beta}{c} E_x u_x - \frac{\beta}{c} u_y E_y - \frac{\beta}{c} u_z E_z\right]$$

Thus

$$F_x' = \frac{Q}{1-\frac{\beta}{c}u_x}\left[E_x - \frac{\beta}{c}\vec{u}\cdot\vec{E}\right]$$

Matches the force transformed from S !!

2) The y component

$$F_y' = Q\left[\gamma E_y + \gamma\frac{v}{c^2}u_x' E_y\right]$$

$$= \gamma Q E_y\left[1 + \frac{v}{c^2}u_x'\right]$$

$$= \gamma Q E_y\left[1 + \frac{\beta}{c}\frac{u_x - v}{1-\frac{\beta}{c}u_x}\right]$$

$$= \frac{\gamma Q E_y}{1-\frac{\beta}{c}u_x}\left[1 - \cancel{\frac{\beta}{c}u_x} + \cancel{\frac{\beta}{c}u_x} - \frac{\beta}{c}v\right]$$

$$= \frac{\gamma Q E_y}{1-\frac{\beta}{c}u_x}\left[1 - \frac{v^2}{c^2}\right] = \frac{\gamma Q E_y}{1-\frac{\beta}{c}u_x}\frac{1}{\gamma^2}$$

$$\Rightarrow F_y' = \frac{1}{\gamma}\frac{QE_y}{(1-\frac{\beta}{c}u_x)}$$

Matches the force transformed from S !!

Thus we have

$$\boxed{\text{The field transformations guarantee that for source charges stationary in one frame, the theory is relativistically invariant.}}$$