Linear Systems With Real Eigenvalues

#5-8, 9-22, 25-34
#52, 55, 56

(1) $ax'' + bx' + c = 0$

(2) $x' = Ax$

The roots of the characteristic are the eigenvalues of A.

$$\vec{x} = e^{\lambda t}\vec{v}$$
$$\lambda e^{\lambda t}\vec{v} = A e^{\lambda t}\vec{v}$$
$$e^{\lambda t}A\vec{v} - \lambda e^{\lambda t} I\vec{v} = \vec{0}$$
$$e^{\lambda t}(A - \lambda I)\vec{v} = \vec{0}$$
$$(A - \lambda I)\vec{v} = \vec{0}$$

Solving Homogeneous Linear 2x2 DE Systems

For a two-dimensional system of homogeneous linear differential equations $\vec{x}' = A\vec{x}$, where A is a matrix of constants that has eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$, we obtain two solutions:

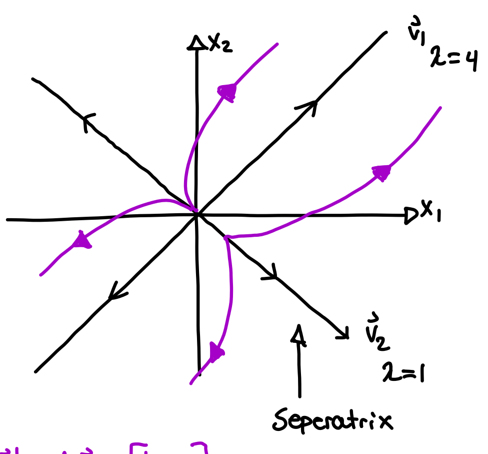
$$\vec{x}_1 = e^{\lambda_1 t}\vec{v}_1 \qquad \vec{x}_2 = e^{\lambda_2 t}\vec{v}_2$$

If $\lambda_1 \neq \lambda_2$, these two solutions are linearly independent and form a basis for the solution space. Thus the general solution

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

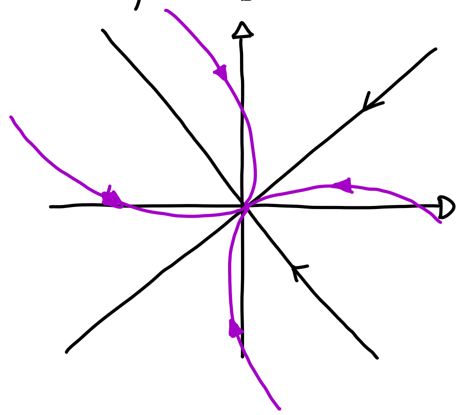
Ex 2) $\vec{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\vec{x}$

$\lambda = 4$, $\lambda = 1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



Ex 3) $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{x}$ $\quad \lambda_1 = -1$, $\lambda_2 = -3$

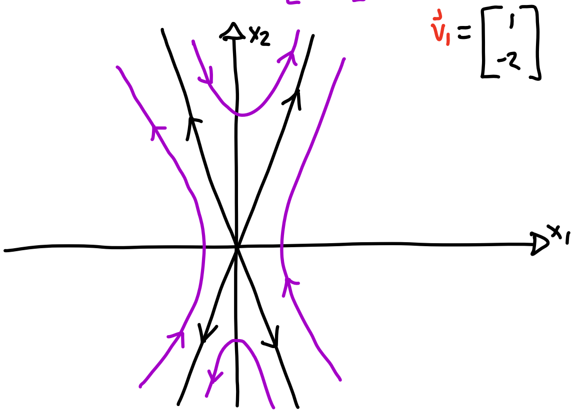
$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \vec{x} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



Ex 1) $\vec{x}' = A\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{x}$

$\lambda_1 = -1 \quad \lambda_2 = 3$

$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Ex 5) $\vec{x}' = A\vec{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\vec{x}$

$\lambda_1 = \lambda_2 = 3$

$$\vec{x} = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{3t} \end{bmatrix}$$

Ex 6) $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}\vec{x}$

$\bar{x}_1$ $\quad \lambda_1 = \lambda_2 = 4 \quad \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \vec{x}_1 = e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\bar{x}_2 = c_2 t e^{\lambda_1 t}\vec{v} + e^{\lambda_2 t}\vec{u} \qquad \bar{x}' = A\bar{x}$

$$e^{\lambda_1 t}\vec{v} + 4te^{\lambda_1 t}\vec{v} + 4e^{4t}\vec{u} = A(te^{4t}\vec{v} + e^{4t}\vec{u})$$

$4\vec{v} = A\vec{v} \qquad 0 = (A - 4I)\vec{v}$

$\vec{v} + 4\vec{u} = A\vec{u} \qquad \vec{v} = (A - 4I)\vec{u} \qquad \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = (A - 4I)$

$\left[\begin{array}{cc|c} -2 & -1 & 1 \\ 4 & 2 & -2 \end{array}\right]$ $\quad u_1 + \frac{1}{2}u_2 = -\frac{1}{2}$ $\quad u_1 = K$

$u_2 = -2K + 1$

$u = K\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Generalized Eigenvector

1.) Find an eigenvector corresponding to λ

2.) Find a nonzero $\bar{u}$

$$(A - \lambda I)\bar{u} = \bar{v}$$

3.) $\bar{x}(t) = c_1 e^{\lambda t}\bar{v} + c_2 e^{\lambda t}(t\bar{v} + \bar{u})$