PHYS 311 HOMEWORK SET 2

1 THE DIVERGENCE OF A VECTOR FUNCTION IS...

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{IN CARTESIAN COORDINATES}$$

SO, FOR THE VECTOR FUNCTION...

$$\vec{v}_a = x^4\hat{x} + 4x^2z^2\hat{y} - 2y^3z\,\hat{z}$$

SO $$\vec{\nabla}\cdot\vec{v}_a = \frac{\partial}{\partial x}(x^4) + \frac{\partial}{\partial y}(4x^2z^2) + \frac{\partial}{\partial z}(-2y^3z)$$

$$\left[\vec{\nabla}\cdot\vec{v}_a = 4x^3 - 2y^3\right]$$

FOR THE VECTOR FUNCTION...

$$\vec{v}_b = x^2y\,\hat{x} + 3x^2z\,\hat{y} + 2yz^2\,\hat{z}$$

SO $$\vec{\nabla}\cdot\vec{v}_b = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(3x^2z) + \frac{\partial}{\partial z}(2yz^2)$$

$$\left[\vec{\nabla}\cdot\vec{v}_b = 2xy + 4yz\right]$$

FOR THE VECTOR FUNCTION...

$$\vec{v}_c = 4y^3\hat{x} + (2x^2z + z^3)\hat{y} + 2xy^2\hat{z}$$

SO $$\vec{\nabla}\cdot\vec{v}_c = \frac{\partial}{\partial x}(4y^3) + \frac{\partial}{\partial y}(2x^2z + z^3) + \frac{\partial}{\partial z}(2xy^2)$$

$$\left[\vec{\nabla}\cdot\vec{v}_c = 0\right]$$

2. THE CURL OF A VECTOR FUNCTION IS...

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

so

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^4 & 4x^2z^2 & -2y^3z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(-2y^3z) - \frac{\partial}{\partial z}(4x^2z^2)\right)$$

$$+ \hat{y}\left(\cancel{\frac{\partial}{\partial z}(x^4)}^{\to 0} - \cancel{\frac{\partial}{\partial x}(-2y^3z)}^{\to 0}\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}(4x^2z^2) - \cancel{\frac{\partial}{\partial y}(x^4)}^{\to 0}\right)$$

$$= \hat{x}\left(-6y^2z - 8x^2z\right) + \hat{z}\left(8xz^2\right)$$

so

$$\boxed{\vec{\nabla} \times \vec{v}_a = -2z(3y^2 + 4x^2)\hat{x} + 8xz^2\hat{z}}$$

WHEREAS

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & 3x^2z & 2yz^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(2yz^2) - \frac{\partial}{\partial z}(3x^2z)\right) + \hat{y}\left(\cancel{\frac{\partial}{\partial z}(x^2y)}^{\to 0} - \cancel{\frac{\partial}{\partial x}(2yz^2)}^{\to 0}\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}(3x^2z) - \frac{\partial}{\partial y}(x^2y)\right)$$

$$= \hat{x}\left(2z^2 - 3x^2\right) + \hat{z}\left(6xz - x^2\right)$$

so

$$\boxed{\vec{\nabla} \times \vec{v}_b = (2z^2 - 3x^2)\hat{x} + x(6z - x)\hat{z}}$$

LASTLY...

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y^3 & 2x^2z+z^3 & 2xy^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(2xy^2) - \frac{\partial}{\partial z}(2x^2z+z^3)\right)$$

$$+ \hat{y}\left(\frac{\partial}{\partial z}\cancelto{0}{(4y^3)} - \frac{\partial}{\partial x}(2xy^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(2x^2z+z^3) - \frac{\partial}{\partial y}(4y^3)\right)$$

$$= \hat{x}\left(4xy - (2x^2+3z^2)\right) - \hat{y}(2y^2) + \hat{z}\left(4xz - 12y^2\right)$$

$$\left[\vec{\nabla} \times \vec{v}_c = (4xy - 2x^2 - 3z^2)\hat{x} - (2y^2)\hat{y} + 4(xz - 3y^2)\hat{z}\right]$$

3. Let $\vec{v} = \vec{v}(x,y,z) = \hat{x} v_x + \hat{y} v_y + \hat{z} v_z$

NOW I WANT...

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

AND

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial}{\partial x} v_z\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$= 0\,\hat{x} + 0\,\hat{y} + 0\,\hat{z}$$

SO FOR A VECTOR FUNCTION THAT IS DIVERGENCELESS AND CURLLESS EQUATES TO...

1) $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$

2) $\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0$

3) $\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0$

) $\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$ ALL BEING SATISFIED!

4. WE NEED TO SHOW THAT..

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

FOR THE VECTOR FUNCTIONS...

$$\vec{A} = 3y\,\hat{x} + z\hat{y} + 2x\,\hat{z}$$
$$\vec{B} = 2x\,\hat{x} - 3z\hat{y}$$

FIRST, THE RIGHT-HAND SIDE...

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3y & z & 2x \\ 2x & -3z & 0 \end{vmatrix} = \hat{x}\left(0-(2x)(-3z)\right) + \hat{y}\left(4x^2-0\right) + \hat{z}\left(-9yz-2xz\right)$$
$$= (6xz)\hat{x} + (4x^2)\hat{y} - z(9y+2x)\hat{z}$$

SO

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xz & 4x^2 & -z(9y+2x) \end{vmatrix} = \hat{x}\left(-\frac{\partial}{\partial y}(9yz+\cancel{2xz}) - \cancel{\frac{\partial}{\partial z}(4x^2)}\right)$$
$$+\hat{y}\left(\frac{\partial}{\partial z}(6xz) + \frac{\partial}{\partial x}(9\cancel{z}y+2xz)\right)$$
$$+\hat{z}\left(\frac{\partial}{\partial x}(4x^2) - \cancel{\frac{\partial}{\partial y}(6xz)}\right)$$

$$= -(9z)\hat{x} + (6x+2z)\hat{y} + (8x)\hat{z}.$$

NOW THE LEFT HAND SIDE...

$$(\vec{B}\cdot\vec{\nabla})\vec{A} = \left[\hat{x}B_x + \hat{y}B_y + \hat{z}B_z\right]\cdot\left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right]\vec{A} = \left[B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right](\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)$$

$$= \left[2x\frac{\partial}{\partial x} - 3z\frac{\partial}{\partial y}\right](\hat{x}\,3y + \hat{y}z + \hat{z}\,2x) =$$

$$= 2x\frac{\partial}{\partial x}(\hat{x}\,3y + \hat{y}z + \hat{z}\,2x) - 3z\frac{\partial}{\partial y}(\hat{x}\,3y + \hat{y}z + \hat{z}\,2x)$$

$$= 2x\cdot 2\,\hat{z} - 3z\cdot 3\,\hat{x} = (4x)\hat{z} - (9z)\hat{x}$$

LIKEWISE..

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left[A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right](\hat{x}B_x + \hat{y}B_y + \hat{z}B_z) = \left[3y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}\right](2x\hat{x} - 3z\hat{y})$$

so

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = 3y\frac{\partial}{\partial x}(2x\hat{x} - 3z\hat{y}) + z\frac{\partial}{\partial y}(2x\hat{x} - 3z\hat{y}) + 2x\frac{\partial}{\partial z}(2x\hat{x} - 3z\hat{y})$$

$$= 3y\cdot 2\hat{x} + 2x\cdot -3\hat{y} = 6y\hat{x} - 6x\hat{y}$$

now

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3z) = 2$$

so

$$\vec{A}(\vec{\nabla}\cdot\vec{B}) = 6y\hat{x} + 2z\hat{y} + 4x\hat{z}$$

whereas

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(2x) = 0$$

so

$$\vec{B}(\vec{\nabla}\cdot\vec{A}) = 0$$

NOW PUTTING IT ALL TOGETHER...

$$(\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) = \left[(4x)\hat{z} - (9z)\hat{x}\right] - \left[6y\hat{x} - 6x\hat{y}\right] + \left[6y\hat{x} + 2z\hat{y} + 4x\hat{z}\right]$$

NOW GROUPING BY UNIT VECTORS...

$$= \left[-(9z) - \cancel{6y} + \cancel{6y}\right]\hat{x} + \left[6x + 2z\right]\hat{y} + \left[4x + 4x\right]\hat{z}$$

$$= -(9z)\hat{x} + (6x + 2z)\hat{y} + (8x)\hat{z} \quad ✓$$

5.

a) $\nabla^2 T_a = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(z^2 + 2zx + 3y + 5)$

$= \frac{\partial^2}{\partial x^2}(z^2+2zx+3y+5) + \frac{\partial^2}{\partial y^2}(z^2+2zx+3y+5) + \frac{\partial^2}{\partial z^2}(z^2+2zx+3y+5)$

$= 0 + 0 + 2$

so

$\boxed{\nabla^2 T_a = 2}$

b) $\nabla^2 T_b = \frac{\partial^2}{\partial x^2}(\sin x \cos y \sin z) + \frac{\partial^2}{\partial y^2}(\sin x \cos y \sin z) + \frac{\partial^2}{\partial z^2}(\sin x \cos y \sin z)$

$= -\sin x \cos y \sin z - \sin x \cos y \sin z - \sin x \cos y \sin z$

so

$\boxed{\nabla^2 T_b = -3 \sin x \cos y \sin z}$

c) $\nabla^2 T_c = \frac{\partial^2}{\partial x^2}(e^{-2z} \sin 3x \cos 4y) + \frac{\partial^2}{\partial y^2}(e^{-2z} \sin 3x \cos 4y) + \frac{\partial^2}{\partial z^2}(e^{-2z} \sin 3x \cos 4y)$

$= -9e^{-2z} \sin 3x \cos 4y - 16e^{-2z} \sin 3x \cos 4y + 4e^{-2z} \sin 3x \cos 4y$

$\boxed{\nabla^2 T_c = -21 e^{-2z} \sin 3x \cos 4y}$

d.) $\nabla^2 \vec{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(z^2\hat{x} + 2zy^2\hat{y} - zy\hat{z})$

$= \hat{x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)z^2 + \hat{y}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)2zy^2 - \hat{z}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)zy$

$= \hat{x} \cdot 2 + \hat{y} \cdot 4z$

so

$\boxed{\nabla^2 \vec{v} = 2\hat{x} + 4z\hat{y}}$

6. CONSIDER A GENERIC VECTOR FUNCTION

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

a) THE CURL OF $\vec{B}$ IS...

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

SO THE DIVERGENCE OF THE CURL OF B IS...

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right] \cdot \left[\hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 B_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 B_y}{\partial x \partial z}} + \cancel{\frac{\partial^2 B_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 B_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 B_y}{\partial z \partial x}} - \cancel{\frac{\partial^2 B_x}{\partial z \partial y}} = 0 \checkmark$$

b) CONSIDER A GENERIC SCALAR FUNCTION

$$g = g(x, y, z)$$

THE GRADIENT OF g IS...

$$\vec{\nabla} g = \hat{x}\frac{\partial g}{\partial x} + \hat{y}\frac{\partial g}{\partial y} + \hat{z}\frac{\partial g}{\partial z}$$

SO THE CURL OF THE GRADIENT IS...

$$\vec{\nabla} \times (\vec{\nabla} g) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial g/\partial x & \partial g/\partial y & \partial g/\partial z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial}{\partial z}\frac{\partial g}{\partial y}\right) + \hat{y}\left(\frac{\partial}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial}{\partial x}\frac{\partial g}{\partial z}\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial}{\partial y}\frac{\partial g}{\partial x}\right) = 0\hat{x} + 0\hat{y} + 0\hat{z}$$

SINCE

$$\frac{\partial}{\partial y}\frac{\partial}{\partial z} g = \frac{\partial}{\partial z}\frac{\partial}{\partial y} g = \frac{\partial^2}{\partial y \partial z} g \quad ..$$

7. CONSIDER THE VECTOR FUNCTION

$$\vec{v} = z^2\hat{x} + 2zy^2\hat{y} - zy\hat{z} \quad \text{so} \quad \vec{v}\cdot d\vec{l} = z^2 dx + 2zy^2 dy - zy\,dz$$

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

a) FROM $(0,0,0) \rightarrow (1,0,0)$ IMPLIES $dy = dz = 0$, $y = z = 0$

SO
$$\int_{(i)} \vec{v}\cdot d\vec{l} = \int_0^1 z^2 dx \Big|_{z=0} = 0$$

FROM $(1,0,0) \rightarrow (1,1,0)$ IMPLIES $dx = dz = 0$, $x = 1$, $z = 0$

SO
$$\int_{(ii)} \vec{v}\cdot d\vec{l} = \int_0^1 2zy^2 dy \Big|_{z=0} = 0$$

FROM $(1,1,0) \rightarrow (1,1,1)$ IMPLIES $dx = dy = 0$, $x = y = 1$

SO
$$\int_{(iii)} \vec{v}\cdot d\vec{l} = -\int_0^1 zy\,dz \Big|_{y=1} = -\int_0^1 z\,dz = -\frac{z^2}{2}\Big|_0^1 = \frac{-1}{2}$$

SO
$$\left[\int_{(0,0,0)}^{(1,1,1)} \vec{v}\cdot d\vec{l} = \frac{-1}{2}\right]$$ ALONG THE PATH IN a)

b) FROM $(0,0,0) \rightarrow (0,0,1)$ IMPLIES $dx = dy = 0$, $x = y = 0$

SO
$$\int_{(i)} \vec{v}\cdot d\vec{l} = -\int_0^1 zy\,dz \Big|_{y=0} = 0$$

FROM $(0,0,1) \rightarrow (0,1,1)$ IMPLIES $dx = dz = 0$, $x = 0$, $z = 1$

SO
$$\int_{(ii)} \vec{v}\cdot d\vec{l} = \int_0^1 2zy^2 dy \Big|_{z=1} = 2\int_0^1 y^2 dy = \frac{2}{3}y^3\Big|_0^1 = \frac{2}{3}$$

FROM $(0,1,1) \to (1,1,1)$ IMPLIES $dy = dz = 0$, $y = z = 1$

SO

$$\int_{(iii)} \vec{v} \cdot d\vec{l} = \int_0^1 z^2 dx \Big|_{z=1} = \int_0^1 dx = x \Big|_0^1 = 1$$

SO

$$\int_{(0,0,0)}^{(1,1,1)} \vec{v} \cdot d\vec{l} = 0 + \frac{2}{3} + 1 \qquad \therefore \boxed{\int_{(0,0,0)}^{(1,1,1)} \vec{v} \cdot d\vec{l} = \frac{5}{3}}$$ ALONG THE PATH IN b)

c) THE DIRECT STRAIGHT LINE IMPLIES

$$dx = dy = dz \quad , x = y = z$$

SO

$$\int \vec{v} \cdot d\vec{l} = \int (z^2 dx + 2zy^2 dy - zy\,dz)\Big|_{\substack{x=y=z \\ dx=dy=dz}} = \int_0^1 (\cancel{x^2 dx} + 2x^3 dx - \cancel{x^2 dx})$$

$$= 2\int_0^1 x^3 dx = \frac{1}{2} x^4 \Big|_0^1 = \frac{1}{2}$$

$$\therefore \boxed{\int_{(0,0,0)}^{(1,1,1)} \vec{v} \cdot d\vec{l} = \frac{1}{2}}$$ ALONG THE STRAIGHT-LINE PATH OF c)