Exercise 1

$\hat{u}|0\rangle = e^{-i\alpha/2}|0\rangle$, $\hat{u}|1\rangle = e^{i\alpha/2}|1\rangle$

a.) Determine how the following evolve under $\hat{u}$

i.) $\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle+|1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle+\hat{u}|1\rangle) = \frac{1}{\sqrt{2}}(e^{-i\alpha/2}|0\rangle + e^{i\alpha/2}|1\rangle)$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle+|1\rangle) = \frac{1}{\sqrt{2}}(e^{-i\alpha/2}|0\rangle + e^{i\alpha/2}|1\rangle)}$$

ii.) $\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle+i|1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle+\hat{u}i|1\rangle) = \frac{1}{\sqrt{2}}(e^{-i\alpha/2}|0\rangle + ie^{i\alpha/2}|1\rangle)$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle+i|1\rangle) = \frac{1}{\sqrt{2}}(e^{-i\alpha/2}|0\rangle + ie^{i\alpha/2}|1\rangle)}$$

b.) Determine the matrix for $\hat{u}$: Show that $\hat{u}$ is unitary

$\hat{u} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$: $\hat{u}^{\dagger}\hat{u} = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}\begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = \begin{pmatrix} e^{0} & 0 \\ 0 & e^{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\hat{u} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}, \ \hat{u} \text{ is unitary}}$$

Exercise 2

$$\hat{u}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad \hat{u}|1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

a.) Determine how the following evolve under $\hat{u}$:

i.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle + \hat{u}|1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}|1\rangle\right) = |1\rangle$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + |1\rangle) = |1\rangle}$$

ii.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - |1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle - \hat{u}|1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) - \frac{1}{\sqrt{2}}(-|0\rangle+|1\rangle)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}|0\rangle\right) = |0\rangle$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - |1\rangle) = |0\rangle}$$

iii.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + i|1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle + i\cdot\hat{u}|1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{i}{\sqrt{2}}(-|0\rangle+|1\rangle)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + i|1\rangle) = \frac{1}{2}\left((1-i)|0\rangle + (1+i)|1\rangle\right)$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle + i|1\rangle) = \frac{1}{2}\left((1-i)|0\rangle + (1+i)|1\rangle\right)}$$

iv.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - i|1\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\hat{u}|0\rangle - i\cdot\hat{u}|1\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) - \frac{i}{\sqrt{2}}(-|0\rangle+|1\rangle)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - i|1\rangle) = \frac{1}{2}\left((1+i)|0\rangle + (1-i)|1\rangle\right)$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}(|0\rangle - i|1\rangle) = \frac{1}{2}\left((1+i)|0\rangle + (1-i)|1\rangle\right)}$$

Exercise 2 Continued

b.) Determine the matrix for $\hat{u}$: Show that $\hat{u}$ is unitary

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad : \quad \hat{u}^{+}\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \ \hat{u} \text{ is unitary}}$$

Exercise 3

$$\hat{u}|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad \hat{u}|1\rangle = \frac{1}{\sqrt{2}}\left(i|0\rangle + |1\rangle\right)$$

a.) Determine how the following evolve under $\hat{u}$:

i.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle + |1\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(\hat{u}|0\rangle + \hat{u}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) + \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle + |1\rangle\right) = \frac{1}{2}\left((1+i)|0\rangle + (1+i)|1\rangle\right)$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle + |1\rangle\right) = \frac{1}{2}\left((1+i)|0\rangle + (1+i)|1\rangle\right)}$$

ii.) $$\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle - |1\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(\hat{u}|0\rangle + \hat{u}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) - \frac{1}{\sqrt{2}}(i|0\rangle + 1)\right)$$

$$\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle - |1\rangle\right) = \frac{1}{2}\left((1-i)|0\rangle + (i-1)|1\rangle\right)$$

$$\boxed{\frac{1}{\sqrt{2}}\cdot\hat{u}\left(|0\rangle - |1\rangle\right) = \frac{1}{2}\left((1-i)|0\rangle + (i-1)|1\rangle\right)}$$

b.) Determine the matrix for $\hat{u}$: Show that $\hat{u}$ is unitary

$$\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \; : \; \hat{u}^{\dagger}\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \; \hat{u} \text{ is unitary}}$$

Exercise 4

$$u_A = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}, \quad u_B = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}$$

a.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$

i.) $\hat{u}_A \otimes \hat{u}_B \, |0\rangle|0\rangle = \hat{u}_A|0\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A|0\rangle = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\alpha/2}|0\rangle \quad : \quad \hat{u}_B|0\rangle = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\beta/2}|0\rangle$$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = e^{-\frac{i}{2}(\alpha+\beta)}|0\rangle|0\rangle}$$

ii.) $\hat{u}_A \otimes \hat{u}_B \, |0\rangle|1\rangle = \hat{u}_A|0\rangle \, \hat{u}_B|1\rangle$

$$\hat{u}_A|0\rangle = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\alpha/2}|0\rangle \quad : \quad \hat{u}_B|1\rangle = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\beta/2}|1\rangle$$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = e^{\frac{i}{2}(\beta-\alpha)}|0\rangle|1\rangle}$$

iii.) $\hat{u}_A \otimes \hat{u}_B \, |1\rangle|0\rangle = \hat{u}_A|1\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A|1\rangle = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\alpha/2}|1\rangle \quad : \quad \hat{u}_B|0\rangle = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\beta/2}|0\rangle$$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = e^{\frac{i}{2}(\alpha-\beta)}|1\rangle|0\rangle}$$

iv.) $\hat{u}_A \otimes \hat{u}_B \, |1\rangle|1\rangle = \hat{u}_A|1\rangle \, \hat{u}_B|1\rangle$

$$\hat{u}_A|1\rangle = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\alpha/2}|1\rangle \quad : \quad \hat{u}_B|1\rangle = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\beta/2}|1\rangle$$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = e^{\frac{i}{2}(\alpha+\beta)}|1\rangle|1\rangle}$$

b.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$$

Exercise 4 Continued

$\hat{u}_A \otimes \hat{u}_B$ on $\frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$

$$= \frac{1}{2}\left(\hat{u}_A|0\rangle\hat{u}_B|0\rangle + \hat{u}_A|0\rangle\hat{u}_B|1\rangle + \hat{u}_A|1\rangle\hat{u}_B|0\rangle + \hat{u}_A|1\rangle\hat{u}_B|1\rangle\right)$$

$$= \frac{1}{2}\left(e^{-\frac{i}{2}(\alpha+\beta)}|0\rangle|0\rangle + e^{\frac{i}{2}(\beta-\alpha)}|0\rangle|1\rangle + e^{\frac{i}{2}(\alpha-\beta)}|1\rangle|0\rangle + e^{\frac{i}{2}(\alpha+\beta)}|1\rangle|1\rangle\right)$$

$$\boxed{\frac{1}{2}\left(e^{-\frac{i}{2}(\alpha+\beta)}|0\rangle|0\rangle + e^{\frac{i}{2}(\beta-\alpha)}|0\rangle|1\rangle + e^{\frac{i}{2}(\alpha-\beta)}|1\rangle|0\rangle + e^{\frac{i}{2}(\alpha+\beta)}|1\rangle|1\rangle\right)}$$

c.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $\frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)$

$$= \frac{1}{2}\left(\hat{u}_A|0\rangle\hat{u}_B|0\rangle + \hat{u}_A|0\rangle\hat{u}_B|1\rangle + \hat{u}_A|1\rangle\hat{u}_B|0\rangle - \hat{u}_A|1\rangle\hat{u}_B|1\rangle\right)$$

$$= \frac{1}{2}\left(e^{-\frac{i}{2}(\alpha+\beta)}|0\rangle|0\rangle + e^{\frac{i}{2}(\beta-\alpha)}|0\rangle|1\rangle + e^{\frac{i}{2}(\alpha-\beta)}|1\rangle|0\rangle - e^{\frac{i}{2}(\alpha+\beta)}|1\rangle|1\rangle\right)$$

$$\boxed{\frac{1}{2}\left(e^{-\frac{i}{2}(\alpha+\beta)}|0\rangle|0\rangle + e^{\frac{i}{2}(\beta-\alpha)}|0\rangle|1\rangle + e^{\frac{i}{2}(\alpha-\beta)}|1\rangle|0\rangle - e^{\frac{i}{2}(\alpha+\beta)}|1\rangle|1\rangle\right)}$$

Exercise 5

$$\hat{u}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \hat{u}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

a.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$

i.) $\hat{u}_A \otimes \hat{u}_B \, |0\rangle|0\rangle = \hat{u}_A|0\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \; : \; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|0\rangle \, \hat{u}_B|0\rangle = \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)}$$

ii.) $\hat{u}_A \otimes \hat{u}_B \, |0\rangle|1\rangle = \hat{u}_A|0\rangle \, \hat{u}_B|1\rangle$

$$\hat{u}_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \; : \; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|0\rangle \hat{u}_B|1\rangle = \frac{1}{2}\left(-|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle\right)}$$

iii.) $\hat{u}_A \otimes \hat{u}_B \, |1\rangle|0\rangle = \hat{u}_A|1\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle) \; : \; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|1\rangle \hat{u}_B|0\rangle = \frac{1}{2}\left(-|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)}$$

iv.) $\hat{u}_A \otimes \hat{u}_B \, |1\rangle|1\rangle = \hat{u}_A|1\rangle \, \hat{u}_B|1\rangle$

$$\hat{u}_A|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle) \; : \; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|1\rangle \hat{u}_B|1\rangle = \frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle\right)}$$

b.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{2}\left(|0\rangle|0\rangle + i|0\rangle|1\rangle + i|1\rangle|0\rangle - |1\rangle|1\rangle\right)$$

Exercise 5 Continued

i.) $\hat{u}_A \otimes \hat{u}_B \, |0\rangle|0\rangle = \hat{u}_A|0\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \; : \; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|0\rangle \, \hat{u}_B|0\rangle = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)}$$

ii.) $\hat{u}_A \otimes \hat{u}_B \, i|0\rangle|1\rangle = \hat{u}_A \cdot i|0\rangle \, \hat{u}_B|1\rangle$

$$\hat{u}_A \cdot i|0\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{i}{\sqrt{2}}(|0\rangle + |1\rangle) \; : \; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A \cdot i|0\rangle \, \hat{u}_B|1\rangle = \frac{1}{2}(-i|0\rangle|0\rangle + i|0\rangle|1\rangle - i|1\rangle|0\rangle + i|1\rangle|1\rangle)}$$

iii.) $\hat{u}_A \otimes \hat{u}_B \, i|1\rangle|0\rangle = \hat{u}_A \cdot i|1\rangle \, \hat{u}_B|0\rangle$

$$\hat{u}_A \cdot i|1\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}}(-|0\rangle + |1\rangle) \; : \; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A \cdot i|1\rangle \, u_B|0\rangle = \frac{1}{2}(-i|0\rangle|0\rangle - i|0\rangle|1\rangle + i|1\rangle|0\rangle + i|1\rangle|1\rangle)}$$

iv.) $\hat{u}_A \otimes \hat{u}_B \, -|1\rangle|1\rangle = -\hat{u}_A|1\rangle \, \hat{u}_B|1\rangle$

$$-\hat{u}_A|1\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \; : \; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{-\hat{u}_A|1\rangle \, \hat{u}_B|1\rangle = \frac{1}{2}(-|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)}$$

v.) Add them together

$$\boxed{-i|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + i|1\rangle|1\rangle}$$

Exercise 5 Continued

c.) Determine $\hat{u}_A \otimes \hat{u}_B$ on $\frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)$

i.) $\hat{u}_A \otimes \hat{u}_B |0\rangle|0\rangle = \hat{u}_A|0\rangle \hat{u}_B|0\rangle$

$$\hat{u}_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \;:\; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|0\rangle \hat{u}_B|0\rangle = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)}$$

ii.) $\hat{u}_A \otimes \hat{u}_B |0\rangle|1\rangle = \hat{u}_A|0\rangle \hat{u}_B|1\rangle$

$$\hat{u}_A|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \;:\; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|0\rangle \hat{u}_B|1\rangle = \frac{1}{2}(-|0\rangle|0\rangle + |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle)}$$

iii.) $\hat{u}_A \otimes \hat{u}_B |1\rangle|0\rangle = \hat{u}_A|1\rangle \hat{u}_B|0\rangle$

$$\hat{u}_A|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle) \;:\; \hat{u}_B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\boxed{\hat{u}_A|1\rangle \hat{u}_B|0\rangle = \frac{1}{2}(-|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)}$$

iv.) $\hat{u}_A \otimes \hat{u}_B \, -|1\rangle|1\rangle = -\hat{u}_A|1\rangle \hat{u}_B|1\rangle$

$$-\hat{u}_A|1\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \;:\; \hat{u}_B|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle)$$

$$\boxed{-\hat{u}_A|1\rangle \hat{u}_B|1\rangle = \frac{1}{2}(-|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)}$$

v.) Add them together

$$\boxed{-|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}$$

Exercise 6

a.) Determine matrix for $\hat{u}_A \otimes \hat{u}_B$ for previous examples

i.) $u_A = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$, $u_B = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}$

$$\begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i(\alpha+\beta)/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\beta-\alpha)/2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i(\beta-\alpha)/2} \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\beta-\alpha)/2} & 0 & 0 \\ 0 & 0 & e^{i(\alpha-\beta)/2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{i(\alpha-\beta)/2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\beta-\alpha)/2} & 0 & 0 \\ 0 & 0 & e^{i(\alpha-\beta)/2} & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)/2} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{i(\alpha+\beta)/2} \end{pmatrix}$$

$$\therefore \quad \boxed{\hat{u}_A \otimes \hat{u}_B = \begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\beta-\alpha)/2} & 0 & 0 \\ 0 & 0 & e^{i(\alpha-\beta)/2} & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)/2} \end{pmatrix}}$$

ii.) $\hat{u}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\hat{u}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad \therefore \quad \boxed{\hat{u}_A \otimes \hat{u}_B = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

Exercise 6 Continued

b.) Show that we can construct $\hat{u}_A \otimes \hat{u}_B$ via

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2 \\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2 \\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4 \end{pmatrix}$$

i.) $u_A = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$, $u_B = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix}$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = \begin{pmatrix} e^{-i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\beta-\alpha)/2} & 0 & 0 \\ 0 & 0 & e^{i(\alpha-\beta)/2} & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)/2} \end{pmatrix}}$$

ii.) $\hat{u}_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\hat{u}_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$\boxed{\hat{u}_A \otimes \hat{u}_B = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}$$

Exercise 7

$$\hat{u}(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

For the Following initial states determine FI

a.) $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|\psi_0\rangle = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)$$

$$\hat{u}(\alpha)|\psi_0\rangle = \frac{1}{2}(\hat{u}(\alpha)|0\rangle\hat{u}(\alpha)|0\rangle + \hat{u}(\alpha)|0\rangle\hat{u}(\alpha)|1\rangle + \hat{u}(\alpha)|1\rangle\hat{u}(\alpha)|0\rangle + \hat{u}(\alpha)|1\rangle\hat{u}(\alpha)|1\rangle)$$

$$\hat{u}(\alpha)|\psi_0\rangle = \frac{1}{2}(e^{-i\alpha}|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + e^{i\alpha}|1\rangle|1\rangle) = |\psi\rangle$$

SG $\hat{x}$ (A)	SG $\hat{x}$ (B)	State	$\langle\hat{x}\|\langle\hat{x}\|$
+	+	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}, \frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\|\langle 0\| + \langle 0\|\langle 1\| + \langle 1\|\langle 0\| + \langle 1\|\langle 1\|)$
+	-	$\frac{\|0\rangle+\|1\rangle}{\sqrt{2}}, \frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\|\langle 0\| - \langle 0\|\langle 1\| + \langle 1\|\langle 0\| - \langle 1\|\langle 1\|)$
-	+	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}, \frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\|\langle 0\| + \langle 0\|\langle 1\| - \langle 1\|\langle 0\| - \langle 1\|\langle 1\|)$
-	-	$\frac{\|0\rangle-\|1\rangle}{\sqrt{2}}, \frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\|\langle 0\| - \langle 0\|\langle 1\| - \langle 1\|\langle 0\| + \langle 1\|\langle 1\|)$

$$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{4}\Big(e^{-i\alpha}\overset{1}{\cancel{\langle 0|\langle 0|0\rangle|0\rangle}} + e^{-i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|0\rangle|0\rangle}} + e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|0\rangle|0\rangle}} + e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 1|0\rangle|0\rangle}} +$$

$$\overset{0}{\cancel{\langle 0|\langle 0|0\rangle|1\rangle}} + \overset{1}{\cancel{\langle 0|\langle 1|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 1|\langle 0|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 1|\langle 1|0\rangle|1\rangle}} + \overset{1}{\cancel{\langle 1|\langle 0|1\rangle|0\rangle}} + \overset{0}{\cancel{\langle 0|\langle 0|1\rangle|0\rangle}} + \overset{0}{\cancel{\langle 0|\langle 1|1\rangle|0\rangle}}$$

$$+ \overset{1}{\cancel{\langle 1|\langle 0|1\rangle|0\rangle}} + \overset{0}{\cancel{\langle 1|\langle 1|1\rangle|0\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 0|1\rangle|1\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|1\rangle|1\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|1\rangle|1\rangle}} + e^{i\alpha}\overset{1}{\cancel{\langle 1|\langle 1|1\rangle|1\rangle}}\Big)$$

$$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{4}(e^{-i\alpha} + 2 + e^{i\alpha}) = \frac{1}{4}(\cos(\alpha) - \cancel{i\sin(\alpha)} + 2 + \cos(\alpha) + \cancel{i\sin(\alpha)}) = \frac{1}{4}(2 + 2\cos(\alpha))$$

$$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{2}(1+\cos(\alpha)) \;:\; |\langle +\hat{x}|\langle +\hat{x}|\psi\rangle|^2 = \frac{1}{4}(1 + 2\cos(\alpha) + \cos^2(\alpha)) \longrightarrow P_a$$

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{4}\Big(e^{-i\alpha}\overset{1}{\cancel{\langle 0|\langle 0|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|0\rangle|0\rangle}} + e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 1|0\rangle|0\rangle}} +$$

$$\overset{0}{\cancel{\langle 0|\langle 0|0\rangle|1\rangle}} - \overset{1}{\cancel{\langle 0|\langle 1|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 1|\langle 0|0\rangle|1\rangle}} - \overset{0}{\cancel{\langle 1|\langle 1|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 0|\langle 0|1\rangle|0\rangle}} - \overset{0}{\cancel{\langle 0|\langle 1|1\rangle|0\rangle}} + \overset{1}{\cancel{\langle 1|\langle 0|1\rangle|0\rangle}}$$

$$- \overset{0}{\cancel{\langle 1|\langle 1|1\rangle|0\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 0|1\rangle|1\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|1\rangle|1\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|1\rangle|1\rangle}} - e^{i\alpha}\overset{1}{\cancel{\langle 1|\langle 1|1\rangle|1\rangle}}\Big)$$

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{4}(e^{-i\alpha} - e^{i\alpha}) = \frac{1}{4}(\cancel{\cos(\alpha)} - i\sin(\alpha) - \cancel{\cos(\alpha)} - i\sin(\alpha)) = \frac{1}{4}(-2i\sin(\alpha)) = \frac{-1}{2}(i\sin(\alpha))$$

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{-1}{2}(i\sin(\alpha)) \;:\; |\langle +\hat{x}|\langle -\hat{x}|\psi\rangle|^2 = \frac{1}{4}\sin^2(\alpha) \longrightarrow P_b$$

Exercise 7 Continued

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{4}\Big(e^{-i\alpha}\overset{1}{\cancel{\langle 0|\langle 0|0\rangle|0\rangle}} + e^{-i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 1|0\rangle|0\rangle}} +$$

$$\overset{0}{\cancel{\langle 0|\langle 0|0\rangle|1\rangle}} + \overset{1}{\cancel{\langle 0|\langle 1|0\rangle|1\rangle}} - \overset{0}{\cancel{\langle 1|\langle 0|0\rangle|1\rangle}} - \overset{0}{\cancel{\langle 1|\langle 1|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 0|\langle 0|1\rangle|0\rangle}} + \overset{0}{\cancel{\langle 0|\langle 1|1\rangle|0\rangle}} -$$

$$\overset{0}{\cancel{\langle 1|\langle 0|1\rangle|0\rangle}} - \overset{1}{\cancel{\langle 1|\langle 1|1\rangle|0\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 0|1\rangle|1\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|1\rangle|1\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|1\rangle|1\rangle}} - e^{i\alpha}\overset{1}{\cancel{\langle 1|\langle 1|1\rangle|1\rangle}}\Big)$$

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{4}\left(e^{-i\alpha} - e^{i\alpha}\right) = \frac{1}{4}\left(\cos(\alpha) - i\sin(\alpha) - \cos(\alpha) - i\sin(\alpha)\right) = \frac{1}{4}\left(-2i\sin(\alpha)\right) = \frac{-1}{2}\left(i\sin(\alpha)\right)$$

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = -\frac{1}{2}\left(i\sin(\alpha)\right) \;:\; |\langle -\hat{x}|\langle +\hat{x}|\psi\rangle|^2 = \frac{1}{4}\sin^2(\alpha) \longrightarrow P_c$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{4}\Big(e^{-i\alpha}\overset{1}{\cancel{\langle 0|\langle 0|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|0\rangle|0\rangle}} - e^{-i\alpha}\overset{0}{\cancel{\langle 1|\langle 1|0\rangle|0\rangle}} +$$

$$\overset{0}{\cancel{\langle 0|\langle 0|0\rangle|1\rangle}} - \overset{1}{\cancel{\langle 0|\langle 1|0\rangle|1\rangle}} - \overset{0}{\cancel{\langle 1|\langle 0|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 1|\langle 1|0\rangle|1\rangle}} + \overset{0}{\cancel{\langle 0|\langle 0|1\rangle|0\rangle}} - \overset{0}{\cancel{\langle 0|\langle 1|1\rangle|0\rangle}}$$

$$- \overset{1}{\cancel{\langle 1|\langle 0|1\rangle|0\rangle}} + \overset{0}{\cancel{\langle 1|\langle 1|1\rangle|0\rangle}} + e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 0|1\rangle|1\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 0|\langle 1|1\rangle|1\rangle}} - e^{i\alpha}\overset{0}{\cancel{\langle 1|\langle 0|1\rangle|1\rangle}} + e^{i\alpha}\overset{1}{\cancel{\langle 1|\langle 1|1\rangle|1\rangle}}\Big)$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{4}\left(e^{-i\alpha} - 1 - 1 + e^{i\alpha}\right) = \frac{1}{4}\left(-2 + \cos(\alpha) - \cancel{i\sin(\alpha)} + \cos(\alpha) + \cancel{i\sin(\alpha)}\right) = \frac{1}{4}\left(-2 + 2\cos(\alpha)\right)$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{2}\left(1 - \cos(\alpha)\right) \;:\; |\langle -\hat{x}|\langle -\hat{x}|\psi\rangle|^2 = \frac{1}{4}\left(1 - 2\cos(\alpha) + \cos^2(\alpha)\right) \longrightarrow P_d$$

Fisher information : $F = \sum_{\substack{\text{All possible}\\ \text{outcomes}}}^{\infty} \frac{1}{\text{Prob(outcome)}}\left(\frac{\partial\,\text{Prob(outcome)}}{\partial P}\right)^2 \;:\; F = F_a + F_b + F_c + F_d$

F_a: $P_a = \frac{1}{4}\left(1 + 2\cos(\alpha) + \cos^2(\alpha)\right) \;:\; \frac{\partial P_a}{\partial \alpha} = -\frac{1}{2}\sin(\alpha) - \frac{1}{2}\cos(\alpha)\sin(\alpha) = \frac{-1}{2}\sin(\alpha)\left(1 + \cos(\alpha)\right)$

$$\left(\frac{\partial P_a}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)\left(1 + 2\cos(\alpha) + \cos^2(\alpha)\right) \;:\; \frac{1}{P_a} = \frac{4}{\left(1 + 2\cos(\alpha) + \cos^2(\alpha)\right)}$$

$$F_a = \frac{\cancel{4}}{\cancel{\left(1 + 2\cos(\alpha) + \cos^2(\alpha)\right)}} \cdot \frac{1}{\cancel{4}}\sin^2(\alpha)\cancel{\left(1 + 2\cos(\alpha) + \cos^2(\alpha)\right)} = \sin^2(\alpha) \;:\; F_a = \sin^2(\alpha)$$

F_b: $P_b = \frac{1}{4}\left(\sin^2(\alpha)\right) \;:\; \frac{\partial P_b}{\partial \alpha} = \frac{1}{2}\sin(\alpha)\cos(\alpha) \;:\; \left(\frac{\partial P_b}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)\cos^2(\alpha)$

$$\frac{1}{P_b} = \frac{4}{\sin^2(\alpha)}$$

$$F_b = \frac{\cancel{4}}{\cancel{\sin^2(\alpha)}} \cdot \frac{\cancel{\sin^2(\alpha)}\cos^2(\alpha)}{\cancel{4}} = \cos^2(\alpha) \;:\; F_b = \cos^2(\alpha)$$

F_c: $P_c = \frac{1}{4}\left(\sin^2(\alpha)\right) \;:\; \frac{\partial P_c}{\partial \alpha} = \frac{1}{2}\sin(\alpha)\cos(\alpha) \;:\; \left(\frac{\partial P_c}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)\cos^2(\alpha)$

$$\frac{1}{P_c} = \frac{4}{\sin^2(\alpha)}$$

$$F_c = \frac{\cancel{4}}{\cancel{\sin^2(\alpha)}} \cdot \frac{\cancel{\sin^2(\alpha)}\cos^2(\alpha)}{\cancel{4}} = \cos^2(\alpha) \;:\; F_c = \cos^2(\alpha)$$

Exercise 7 Continued

F_d: $P_d = \frac{1}{4}(1-2\cos(\alpha)+\cos^2(\alpha))$: $\frac{\partial P_d}{\partial \alpha} = \frac{1}{2}\sin(\alpha) - \frac{1}{2}\cos(\alpha)\sin(\alpha) = \frac{1}{2}\sin(\alpha)(1-\cos(\alpha))$

$\left(\frac{\partial P_d}{\partial \alpha}\right)^2 = \frac{1}{4}\sin^2(\alpha)(1-2\cos(\alpha)+\cos^2(\alpha))$: $\frac{1}{P_d} = \frac{4}{(1-2\cos(\alpha)+\cos^2(\alpha))}$

$$F_d = \frac{\cancel{4}}{(1-\cancel{2\cos(\alpha)+\cos^2(\alpha)})} \cdot \frac{1}{\cancel{4}}\sin^2(\alpha)(1-\cancel{2\cos(\alpha)+\cos^2(\alpha)}) = \sin^2(\alpha) \quad : \quad F_d = \sin^2(\alpha)$$

$$F = F_a + F_b + F_c + F_d = \sin^2(\alpha)+\cos^2(\alpha)+\cos^2(\alpha)+\sin^2(\alpha) = 2\sin^2(\alpha)+2\cos^2(\alpha) = 2$$

$$\boxed{F = 2}$$

b.) $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

$$\hat{u}(\alpha)|\psi_0\rangle = \frac{1}{\sqrt{2}}(\hat{u}(\alpha)|0\rangle\hat{u}(\alpha)|0\rangle + \hat{u}(\alpha)|1\rangle\hat{u}(\alpha)|1\rangle)$$

$$\hat{u}(\alpha)|\psi_0\rangle = \frac{1}{\sqrt{2}}(e^{-i\alpha}|0\rangle|0\rangle + e^{i\alpha}|1\rangle|1\rangle) = |\psi\rangle$$

SG $\hat{x}$ (A)	SG $\hat{x}$ (B)	State	$\langle\hat{x}\vert\langle\hat{x}\vert$
+	+	$\frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}, \frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\vert\langle 0\vert + \langle 0\vert\langle 1\vert + \langle 1\vert\langle 0\vert + \langle 1\vert\langle 1\vert)$
+	-	$\frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}, \frac{\vert 0\rangle-\vert 1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\vert\langle 0\vert - \langle 0\vert\langle 1\vert + \langle 1\vert\langle 0\vert - \langle 1\vert\langle 1\vert)$
-	+	$\frac{\vert 0\rangle-\vert 1\rangle}{\sqrt{2}}, \frac{\vert 0\rangle+\vert 1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\vert\langle 0\vert + \langle 0\vert\langle 1\vert - \langle 1\vert\langle 0\vert - \langle 1\vert\langle 1\vert)$
-	-	$\frac{\vert 0\rangle-\vert 1\rangle}{\sqrt{2}}, \frac{\vert 0\rangle-\vert 1\rangle}{\sqrt{2}}$	$\frac{1}{2}(\langle 0\vert\langle 0\vert - \langle 0\vert\langle 1\vert - \langle 1\vert\langle 0\vert + \langle 1\vert\langle 1\vert)$

$$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(e^{-i\alpha}\langle 0|\overset{1}{\cancel{\langle 0|0\rangle}}|0\rangle + e^{-i\alpha}\langle 0|\overset{0}{\cancel{\langle 1|0\rangle}}|0\rangle + e^{-i\alpha}\langle 1|\overset{0}{\cancel{\langle 0|0\rangle}}|0\rangle + e^{-i\alpha}\langle 1|\overset{0}{\cancel{\langle 1|0\rangle}}|0\rangle$$
$$+ e^{i\alpha}\langle 0|\overset{0}{\cancel{\langle 0|1\rangle}}|1\rangle + e^{i\alpha}\langle 0|\overset{0}{\cancel{\langle 1|1\rangle}}|1\rangle + e^{i\alpha}\langle 1|\overset{0}{\cancel{\langle 0|1\rangle}}|1\rangle + e^{i\alpha}\langle 1|\overset{1}{\cancel{\langle 1|1\rangle}}|1\rangle\Big)$$

$$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}(e^{-i\alpha}+e^{i\alpha}) = \frac{1}{2\sqrt{2}}(\cos(\alpha) - i\cancel{\sin(\alpha)} + \cos(\alpha) + i\cancel{\sin(\alpha)}) = \frac{1}{2\sqrt{2}}(2\cos(\alpha))$$

$\langle +\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{\sqrt{2}}\cos(\alpha)$: $|\langle +\hat{x}|\langle +\hat{x}|\psi\rangle|^2 = \frac{1}{2}\cos^2(\alpha) \longrightarrow P_a$

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(e^{-i\alpha}\langle 0|\overset{1}{\cancel{\langle 0|0\rangle}}|0\rangle - e^{-i\alpha}\langle 0|\overset{0}{\cancel{\langle 1|0\rangle}}|0\rangle + e^{-i\alpha}\langle 1|\overset{0}{\cancel{\langle 0|0\rangle}}|0\rangle - e^{-i\alpha}\langle 1|\overset{0}{\cancel{\langle 1|0\rangle}}|0\rangle$$
$$+ e^{i\alpha}\langle 0|\overset{0}{\cancel{\langle 0|1\rangle}}|1\rangle - e^{i\alpha}\langle 0|\overset{0}{\cancel{\langle 1|1\rangle}}|1\rangle + e^{i\alpha}\langle 1|\overset{0}{\cancel{\langle 0|1\rangle}}|1\rangle - e^{i\alpha}\langle 1|\overset{1}{\cancel{\langle 1|1\rangle}}|1\rangle\Big)$$

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}(e^{-i\alpha}-e^{i\alpha}) = \frac{1}{2\sqrt{2}}(\cancel{\cos(\alpha)} - i\sin(\alpha) - \cancel{\cos(\alpha)} - i\sin(\alpha)) = \frac{1}{2\sqrt{2}}(-2i\sin(\alpha)) = \frac{-i}{\sqrt{2}}\sin(\alpha)$$

Exercise 7 Continued

$$\langle +\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{-i}{\sqrt{2}}\sin(\alpha) \;:\; |\langle +\hat{x}|\langle -\hat{x}|\psi\rangle|^2 = \frac{1}{2}\sin^2(\alpha) \longrightarrow P_b$$

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(e^{-i\alpha}\langle 0|\cancelto{1}{\langle 0|0\rangle}|0\rangle + e^{-i\alpha}\langle 0|\cancelto{0}{\langle 1|0\rangle}|0\rangle - e^{-i\alpha}\langle 1|\cancelto{0}{\langle 0|0\rangle}|0\rangle - e^{-i\alpha}\langle 1|\cancelto{0}{\langle 1|0\rangle}|0\rangle$$
$$+ e^{i\alpha}\langle 0|\cancelto{0}{\langle 0|1\rangle}|1\rangle + e^{i\alpha}\langle 0|\cancelto{0}{\langle 1|1\rangle}|1\rangle - e^{i\alpha}\langle 1|\cancelto{0}{\langle 0|1\rangle}|1\rangle - e^{i\alpha}\langle 1|\cancelto{1}{\langle 1|1\rangle}|1\rangle\Big)$$

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\left(e^{-i\alpha} - e^{i\alpha}\right) = \frac{1}{2\sqrt{2}}\left(\cancel{\cos(\alpha)} - i\sin(\alpha) - \cancel{\cos(\alpha)} - i\sin(\alpha)\right) = \frac{1}{2\sqrt{2}}\left(-2i\sin(\alpha)\right) = \frac{-i}{\sqrt{2}}\sin(\alpha)$$

$$\langle -\hat{x}|\langle +\hat{x}|\psi\rangle = \frac{-i}{\sqrt{2}}\sin(\alpha) \;:\; |\langle -\hat{x}|\langle +\hat{x}|\psi\rangle|^2 = \frac{1}{2}\sin^2(\alpha) \longrightarrow P_c$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\Big(e^{-i\alpha}\langle 0|\cancelto{1}{\langle 0|0\rangle}|0\rangle - e^{-i\alpha}\langle 0|\cancelto{0}{\langle 1|0\rangle}|0\rangle - e^{-i\alpha}\langle 1|\cancelto{0}{\langle 0|0\rangle}|0\rangle + e^{-i\alpha}\langle 1|\cancelto{0}{\langle 1|0\rangle}|0\rangle$$
$$+ e^{i\alpha}\langle 0|\cancelto{0}{\langle 0|1\rangle}|1\rangle - e^{i\alpha}\langle 0|\cancelto{0}{\langle 1|1\rangle}|1\rangle - e^{i\alpha}\langle 1|\cancelto{0}{\langle 0|1\rangle}|1\rangle + e^{i\alpha}\langle 1|\cancelto{0}{\langle 1|1\rangle}|1\rangle\Big)$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{2\sqrt{2}}\left(e^{-i\alpha} + e^{i\alpha}\right) = \frac{1}{2\sqrt{2}}\left(\cos(\alpha) - i\cancel{\sin(\alpha)} + \cos(\alpha) + i\cancel{\sin(\alpha)}\right) = \frac{1}{2\sqrt{2}}\left(2\cos(\alpha)\right) = \frac{1}{\sqrt{2}}\cos(\alpha)$$

$$\langle -\hat{x}|\langle -\hat{x}|\psi\rangle = \frac{1}{\sqrt{2}}\cos(\alpha) \;:\; |\langle -\hat{x}|\langle -\hat{x}|\psi\rangle|^2 = \frac{1}{2}\cos^2(\alpha) \longrightarrow P_d$$

Fisher information : $F = \sum_{\substack{\text{All possible}\\ \text{outcomes}}}^{\infty} \frac{1}{\text{Prob(outcome)}}\left(\frac{\partial\,\text{Prob(outcome)}}{\partial P}\right)^2$: $F = F_a + F_b + F_c + F_d$

F_a : $P_a = \frac{1}{2}\cos^2(\alpha)$: $\frac{\partial P_a}{\partial \alpha} = -\cos(\alpha)\sin(\alpha)$: $\left(\frac{\partial P_a}{\partial \alpha}\right)^2 = \cos^2(\alpha)\sin^2(\alpha)$

$$\frac{1}{P_a} = \frac{2}{\cos^2(\alpha)} \;:\; \frac{2}{\cancel{\cos^2(\alpha)}}\cdot\cancel{\cos^2(\alpha)}\sin^2(\alpha) = 2\sin^2(\alpha) \;:\; F_a = 2\sin^2(\alpha)$$

F_b : $P_b = \frac{1}{2}\sin^2(\alpha)$: $\frac{\partial P_b}{\partial \alpha} = \sin(\alpha)\cos(\alpha)$: $\left(\frac{\partial P_b}{\partial \alpha}\right)^2 = \sin^2(\alpha)\cos^2(\alpha)$

$$\frac{1}{P_b} = \frac{2}{\sin^2(\alpha)} \;:\; \frac{2}{\cancel{\sin^2(\alpha)}}\cdot\cancel{\sin^2(\alpha)}\cos^2(\alpha) = 2\cos^2(\alpha) \;:\; F_b = 2\cos^2(\alpha)$$

F_c : $P = \frac{1}{2}\sin^2(\alpha)$: $\frac{\partial P_c}{\partial \alpha} = \sin(\alpha)\cos(\alpha)$: $\left(\frac{\partial P_c}{\partial \alpha}\right)^2 = \sin^2(\alpha)\cos^2(\alpha)$

$$\frac{1}{P_c} = \frac{2}{\sin^2(\alpha)} \;:\; \frac{2}{\cancel{\sin^2(\alpha)}} : \cancel{\sin^2(\alpha)}\cos^2(\alpha) = 2\cos^2(\alpha) \;:\; F_c = 2\cos^2(\alpha)$$

F_d : $P = \frac{1}{2}\cos^2(\alpha)$: $\frac{\partial P_d}{\partial \alpha} = -\cos(\alpha)\sin(\alpha)$: $\left(\frac{\partial P_d}{\partial \alpha}\right)^2 = \cos^2(\alpha)\sin^2(\alpha)$

$$\frac{1}{P_d} = \frac{2}{\cos^2(\alpha)} \;:\; \frac{2}{\cancel{\cos^2(\alpha)}}\cdot\cancel{\cos^2(\alpha)}\sin^2(\alpha) = 2\sin^2(\alpha) \;:\; F_d = 2\sin^2(\alpha)$$

$$F = F_a + F_b + F_c + F_d = 2\sin^2(\alpha) + 2\cos^2(\alpha) + 2\cos^2(\alpha) + 2\sin^2(\alpha) = 4\sin^2(\alpha) + 4\cos^2(\alpha) = 4$$

$$\boxed{F = 4}$$