4.5 Variation of Parameters

4.5 # 6, 14

Variation of Parameters

1) Find y_h

2) Solve $y_1v_1' + y_2v_2' = 0$

$y_1'v_1' + y_2'v_2' = f(t)$

$$v_1' = \frac{-y_2 f}{W(y_1,y_2)} \qquad v_2' = \frac{y_1 f}{W(y_1,y_2)}$$

3) Integrate step 2

4) $y_p = v_1y_1 + v_2y_2$

$y'' + p(t)y' + q(t)y = f(t)$

Ex: $y'' - 2y' + y = \frac{e^t}{1+t^2}$

① $y'' - 2y' + 1 = 0$

$(r-1)^2 = 0$

$y_h = c_1e^t + c_2te^t$

$y_1 = e^t \qquad y_2 = te^t$

② $e^tv_1' + te^tv_2' = 0$

$e^tv_1' + e^t(1+t)v_2' = \frac{e^t}{1+t^2}$

$W(e^t, te^t)$

$$\begin{vmatrix} e^t & te^t \\ e^t & e^t(1+t) \end{vmatrix} = e^{2t}$$

$e^t + te^t$

$$v_1' = \frac{-te^t\left(\frac{e^t}{1+t^2}\right)}{e^{2t}}$$

$$v_1' = \frac{t}{1+t^2}$$

$$v_2' = \frac{e^t\frac{e^t}{1+t^2}}{e^{2t}}$$

$$v_2' = \frac{1}{1+t^2}$$

$\int \frac{t}{1+t^2}\,dt$ $\quad u = 1+t^2$, $du = 2t\,dt$, $\frac{1}{2}du = t\,dt$

③ $v_1 = \frac{1}{2}\int \frac{1}{u}\,du$

$\frac{1}{2}\ln|1+t^2|$

$v_1 = -\frac{1}{2}\ln|1+t^2|$

$\int \frac{1}{1+t^2}\,dt$

$v_2 = \arctan(1+t^2)$

$v_2 = \tan^{-1}(t)$

④ $y = y_h + y_p$

$y = c_1e^t + c_2te^t - \frac{1}{2}\ln|1+t^2|e^t + \tan^{-1}(t)te^t$

$y'' - y' - 2y = 3t^2 - 1$

$y(t) = At^2 + Bt + C$

$y'(t) = 2At + B$

$y''(t) = 2A$

$2A - (2At+B) - 2(At^2+Bt+C) = 3t^2 - 1$

$2A - 2At - B - 2At^2 - 2Bt - 2C = 3t^2 - 1$

$-2At^2 - 2At - 2Bt + 2A - B - 2C = 3t^2 - 1$

$-2At^2 + t(-2A-2B) + (2A-B-2C) = 3t^2 - 1$

$-2A + 0 + 0 = 3 \quad A = -\frac{3}{2}$

$-2A - 2B + 0 = 0 \quad B = \frac{3}{2}$

$2A - B - 2C = -1 \quad C = -\frac{7}{4}$

$y = -\frac{3}{2}t^2 + \frac{3}{2}t - \frac{7}{4}$