MAT 202
Larson – Section 5.7 (Part II)
Inverse Trigonometric Functions: Integration

Suppose we would like to integrate the following: $\int \frac{1}{x^2 - 4x + 7}\,dx$. Notice that it doesn't fall into any one of the formulas given in this section. How would we proceed?

<u>Integrals Involving Inverse Trigonometric Functions</u>: Let u be a differentiable function of x and let $a > 0$.

1. $\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\frac{u}{a} + C$
2. $\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$
3. $\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}arc\sec\frac{|u|}{a} + C$

We can use ***completing the square*** to rewrite the integral:

$$\int \frac{1}{x^2 - 4x + 7}\,dx = \int \frac{1}{\left(x^2 - 4x + 4\right) - 4 + 7}\,dx = \int \frac{1}{(x-2)^2 + 3}\,dx$$

Now, we can use rule #2 above to integrate!

Suppose we would like to integrate the following:

$$\int \frac{1}{x^2 - 4x + 7}\,dx$$

. Notice that it doesn't fall into any one of the formulas given in this section. How would we proceed?

We would need to complete the square on the denominator to get the expression to look like $\frac{1}{(u)^2 + a^2}$

$$\int \frac{1}{(x^2-4x+4)+7-4}\,dx$$

$$= \int \frac{1}{(x-2)^2 + 3}\,dx$$

$u = x-2$
$du = dx$

$$= \int \frac{1}{u^2 + (\sqrt{3})^2}\,du$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C$$

$$= \boxed{\frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C}$$

Ex: Find the indefinite integral of the following:

a) $\int \frac{2x}{x^2 + 6x + 13} dx$

b) $\int \frac{x}{x^4 + 2x^2 + 2} dx$

Ex: Evaluate the following definite integrals: $\int_2^3 \frac{2x - 3}{\sqrt{4x - x^2}} dx$

Ex: Find the indefinite integral of the following:

a) $\int \frac{2x}{x^2+6x+13}dx = \int \frac{(2x+6)-6}{x^2+6x+13}dx$

$= \int \frac{2x+6}{x^2+6x+13}dx - 6\int \frac{1}{x^2+6x+13}dx$

$u = x^2+6x+13$
$du = 2x+6\ dx$

$- 6\int \frac{1}{x^2+6x+9+13-9}dx$

$= \int \frac{1}{u}du - 6\int \frac{1}{(x+3)^2+4}dx$

u $\quad 2^2 \quad a=2$

$= \ln|u|$

$= \ln|x^2+6x+13| - 6\cdot\frac{1}{2}\arctan\frac{x+3}{2} + C$

$= \boxed{\ln|x^2+6x+13| - 3\arctan\frac{x+3}{2} + C}$

b) $$\int \frac{x}{x^4 + 2x^2 + 2}\,dx = \frac{1}{2}\int \frac{1}{u^2+2u+2}\,du$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int \frac{1}{u^2+2u+1+2-1}\,du$$

$$= \frac{1}{2}\int \frac{1}{(u+1)^2+1}\,du$$

$$= \frac{1}{2}\cdot\frac{1}{1}\arctan\frac{u+1}{1} + C$$

$$= \frac{1}{2}\arctan(x^2+1) + C$$

$$\int \frac{x}{x^4+2x^2+1+2-1}\,dx$$

$$= \int \frac{x}{(x^2+1)^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$.

$$= \frac{1}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\arctan u + C$$

$$= \boxed{\frac{1}{2}\arctan(x^2+1) + C}$$

Ex: Evaluate the following definite integrals: $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

$$\int \frac{2x-3}{\sqrt{4x-x^2}}\,dx = \int \frac{2x-\cancel{3}4}{\sqrt{4x-x^2}}\,dx \,{}^{+4}_{-3}\int \frac{1}{\sqrt{4x-x^2}}\,dx$$

$u = 4x - x^2$

$du = 4 - 2x\; dx$

$du = -(2x-4)\; dx$

$-du = 2x - 4\; dx$

$$= \int \frac{2x+4}{\sqrt{4x-x^2}}\,dx + 1\int \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= -\int \frac{1}{\sqrt{u}}\,du + 1\int \frac{1}{\sqrt{-x^2+4x}}\,dx$$

$$= -\int u^{-1/2}\,du + 1\int \frac{1}{\sqrt{-(x^2-4x+4)+4}}\,dx$$

$$= -\frac{u^{1/2}}{1/2} + 1\int \frac{1}{\sqrt{4-(x-2)^2}}\,dx$$

$$= -2u^{1/2} + 1 \arcsin\left(\frac{x-2}{2}\right) + C$$

$$= -2\sqrt{4x-x^2} + \arcsin\left(\frac{x-2}{2}\right)\Bigg]_2^3$$

$$= -2\sqrt{3} + \arcsin\left(\tfrac{1}{2}\right) - \left(-2\sqrt{4} + \arcsin 0\right)$$

$$= -2\sqrt{3} + \frac{\pi}{6} + 4 - 1\cdot 0$$

$$= \boxed{-2\sqrt{3} + \frac{\pi}{6} + 4 \approx 1.059}$$