Ch. #7 Lagrangian and Hamiltonian Dynamics

Side Note → $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \eta(x)\, dx$ recall $\frac{\partial y}{\partial \alpha} = \eta(x)$

Then $\frac{\partial J}{\partial \alpha} d\alpha = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{\partial y}{\partial \alpha} d\alpha\, dx$

or $\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta y\, dx$

Where $\frac{\partial J}{\partial \alpha} d\alpha = \delta J$

δ notation

$\frac{\partial y}{\partial \alpha} d\alpha = \delta y$

Then $\delta J = \delta \int_{x_1}^{x_2} F(y, y'; x)\, dx = 0$ → extremum

$\delta J = \int_{x_1}^{x_2} \delta F\, dx = \int_{x_1} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' \right) dx$

$\delta y' = \delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$

So $\delta J = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \underbrace{\frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y)}_{2} \right] dx$

Integrate 2 by parts

$\delta J = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta y\, dx$ As before

Hamilton's Principle

Of all possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

or $\delta \int_{t_1}^{t_2} (T-U)\,dt = 0$ where $T = T(\dot{x})$, $U = U(x)$

Let $L = T - U = L(x_i, \dot{x}_i)$

Then $\delta \int_{x_1}^{x_2} F(y, y'; x)\,dx = \delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\,dt$

$x \to t$

$y_i(x) \to x_i(t)$

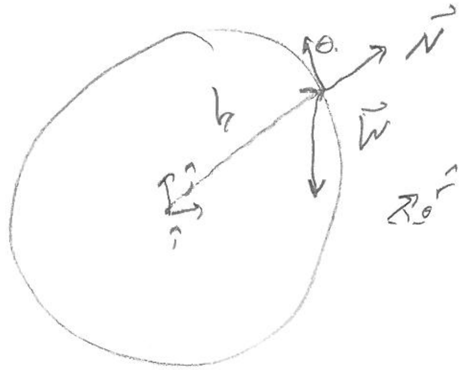
$y_i'(x) = \dot{x}_i(t)$

And Lagrange's Equations become

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \quad i = 1, 2, 3}$$

Why? We have $\vec{F} = m\vec{a}$

Ex: Constrained Motion



$\vec{W} = (\vec{W}\cdot\hat{r})\hat{r} + (\vec{W}\cdot\hat{\theta})\hat{\theta}$

$\vec{W} = (-mg\hat{j}\cdot\hat{r})\hat{r} + (-mg\hat{j}\cdot\hat{\theta})\hat{\theta}$

$\vec{W} = -[mg\sin\theta\,\hat{r} + mg\cos\theta\,\hat{\theta}]$

$\vec{N} = |N|\hat{r}$

$\vec{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -mg([\sin\theta + \vec{N}])\hat{r} +$
$(-mg\cos\theta)\hat{\theta}$

$r = b \quad \ddot{r} = \dot{r} = 0$

So $-mb\dot{\theta}^2 = N - mg\sin\theta \quad \underline{\underline{2}}$

$\dot{\theta}[b\ddot{\theta}] = -[g\cos\theta]\dot{\theta}\,\underline{\underline{2}} \rightarrow b\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -g\frac{d}{dt}\sin\theta$

or $\frac{b\dot{\theta}^2}{2} = -g\sin\theta + C \quad$ when $\theta = \frac{\pi}{2} \quad \dot{\theta} = 0 \quad C = g$

$b\dot{\theta}^2 = 2g(1 - \sin\theta)$

So $N = mg(3\sin\theta - 2) = 0 \quad$ For $\theta = \sin^{-1}\left(\frac{2}{3}\right)$

or $E_i = E_f$

$$mgb = mgb\sin\theta + \frac{1}{2}mv^2$$

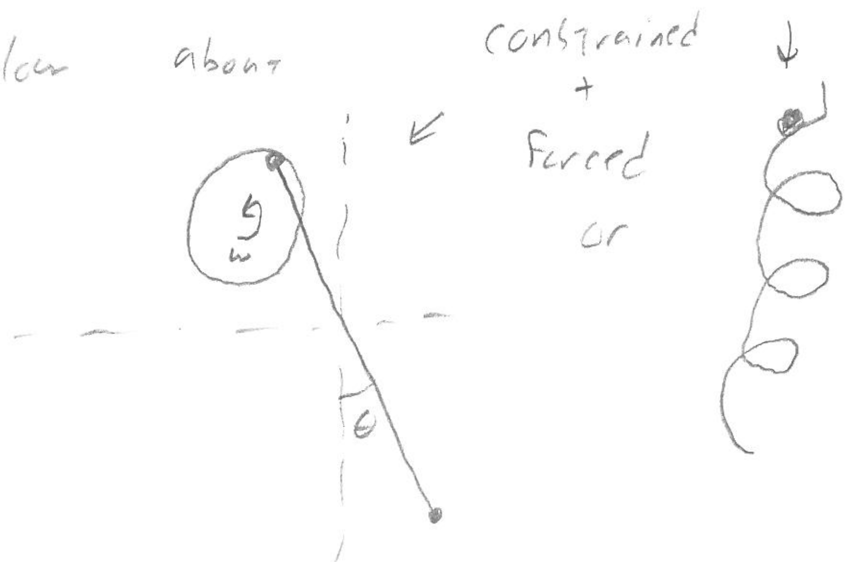
So $v^2 = 2gb(1-\sin\theta)$

and $\vec{F} = m\vec{a} = \frac{v^2}{b}\hat{r} - \frac{dv}{dt}\hat{\theta} = \vec{w} + \vec{N} = (N - mg\sin\theta)\hat{r} - mg\cos\theta\,\hat{\theta}$

$$\frac{v^2}{r}\hat{N} + \frac{dv}{dt}\hat{T}$$

$$\rightarrow \quad \frac{v^2}{b} = N - mg\sin\theta$$

How about



Energy is much easier

Find $d\vec{s} \Rightarrow \frac{d\vec{s}\cdot d\vec{s}}{dt^2}$, $U(q_i)$

Reduce To $L = T - U$ Stick in

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

gives $\vec{F} = m\vec{a}$

$\sim \nabla U$ $\sim m\vec{a}$

Ex. #1. $U = \frac{1}{2}kx^2$ $v = \dot{x}$ $d\vec{s} = x\hat{i}$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}m\dot{x} = m\ddot{x}$$

$$m\ddot{x} + kx = 0 \rightarrow m\ddot{x} = -kx$$

Ex #2.

$L(1 - \cos\theta) = h$

$mgL(1 - \cos\theta) = U$

$d\vec{s} = dr\hat{r} + r\,d\theta\,\hat{\theta}$ but $r = L$ $dr = 0$

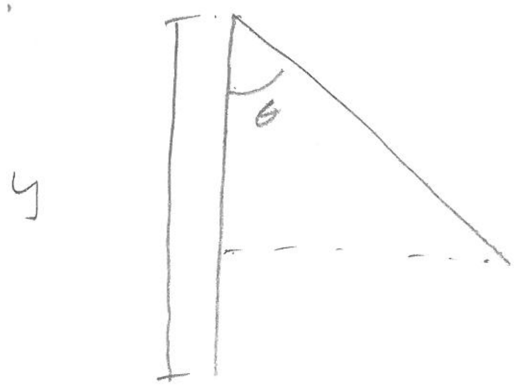
$$d\vec{s} \cdot d\vec{s} = r^2 d^2\theta \rightarrow \left(\frac{|ds|}{dt}\right)^2 = r^2\dot{\theta}^2$$

$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mg(1-\cos\theta) \qquad \frac{\partial L}{\partial \theta} - \frac{d}{dT}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$

$\frac{\partial L}{\partial \theta} = +(mg)(-\mathrm{SIN}\,\theta) = -mg\,\mathrm{SIN}\,\theta$

$\frac{d}{dT}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} \qquad m l^2 \ddot{\theta} + mg\,\mathrm{SIN}\,\theta = 0$

Ex:



$d\vec{s} = dr\hat{r} + r\,d\theta\,\hat{\theta}$

$r = \ell \quad d\vec{s} = \ell\, d\theta\,\hat{\theta}$

$d\vec{s}\cdot d\vec{s} = \ell^2 d\theta^2$

$T = \frac{1}{2} m \ell^2 \dot{\theta}^2$

$U = mg\ell(1-\cos\theta)$

$$L = T - U = \frac{1}{2} m \ell^2 \dot{\theta}^2 - mg\ell(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mg\ell \sin\theta \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \ddot{\theta}$$

or $m\ell^2 \ddot{\theta} + mg\ell \sin\theta = 0$

Note $I\alpha = \tau \qquad \tau = \vec{r}\times\vec{F} = -mg\sin\theta(\ell)$

$$I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d}{dt}(I\vec{\omega}) = \frac{d\vec{L}}{dt} \qquad L = m\ell^2\omega$$

Generalized Coordinates

For n particles there are $3n$ possible coordinates

However there are possibly m equations of constraint

leaving $s = 3n - m$ total coordinates.

Generalized Coordinates $\rightarrow$ you pick

really $x_{\alpha,i}(q_1, q_2, q_3 \ldots n)$

$\alpha \in [1, n]$

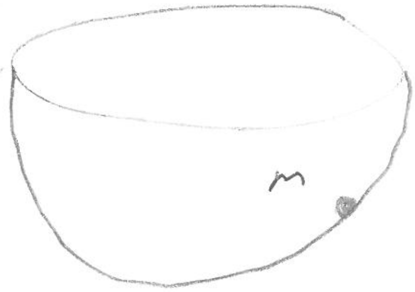
$i = 1,2,3$

$x_{\alpha,i}(q_j, t) \quad j \in [1, s]$

$x_i \rightarrow q \qquad x(q,t),\ \dot{x}(q,t)$

How to pick? <u>Experience</u>

<u>Example</u>



hemisphere radius R

$\downarrow \vec{g}$

$x^2+y^2+z^2=R^2$ let $x^2+y^2+z^2-R^2=0$

Then $q_1 = \frac{x}{R}$ $q_2 = \frac{y}{R}$ $q_3 = \frac{z}{R}$ and $q_1^2+q_2^2+q_3^2=1$

doesn't work if $q_i = q_i(q_j, q_k)$ Needs to be independent

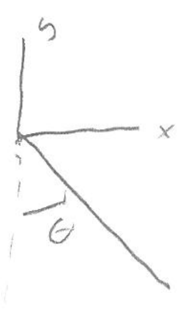
2 q's, 1 equation of constraint

$\underbrace{3 \cdot 1} - 1 = 2$

Better $\rightarrow$ $d\vec{s} = dr\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$T = \left(r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)\frac{1}{2}m$

<u>Example</u>



$d\vec{r} = dx\hat{i} + dy\hat{j}$

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$

$U = mgy$

$L = T - U$

$x = l \sin\theta \quad y = -l\cos\theta$

$\dot{x}^2 = l^2\dot{\theta}^2\cos^2\theta \quad \dot{y}^2 = l^2\dot{\theta}^2\sin^2\theta$

$T = \frac{1}{2}m(l^2\dot{\theta}^2[\cos^2\theta + \sin^2\theta]) = \frac{1}{2}ml^2\dot{\theta}^2$

$U = -mgl\cos\theta$

$L \rightarrow$ <u>Same</u> Equations $\ddot{\theta} + \frac{g}{l}\sin\theta$

MUST BE !! $\quad S = 3n - m \rightarrow$ need S, q's

Now $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad j = 1, 2, 3, \ldots S$

2 conditions

1 Forces must be derivable from potentials

2 Equations of constraint $f_k(x_{\alpha,i}, T)$ connect coordinates and may be functions of time

2 $\rightarrow$ holonomic
if constraint not $f(T)$ $\rightarrow$ Scleronomic

moving constraints $\rightarrow$ rheonomic

Projectile Motion

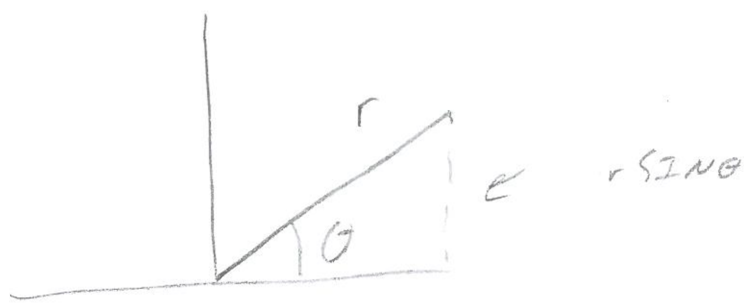
$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad U = mgy$

$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

$\frac{\partial L}{\partial x} = 0 \rightarrow m\ddot{x} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}m\dot{q} = m\ddot{q}$

$\frac{\partial L}{\partial y} = -mg \rightarrow m\ddot{y} + mg = 0$

How about



$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$

$U = +mgr\,\mathrm{SIN}\theta$

$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\,\mathrm{SIN}\theta$

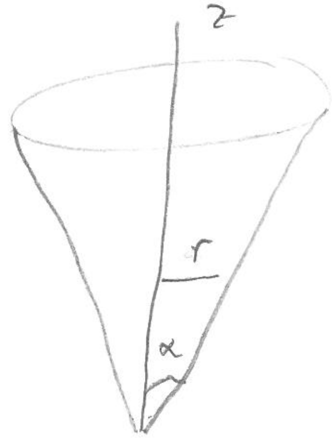
$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\,\mathrm{SIN}\theta \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$

or $mr\dot{\theta}^2 - mg\,\mathrm{SIN}\theta - m\ddot{r} = 0$

$\frac{\partial L}{\partial \theta} = -mgr\cos\theta \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(mr^2\dot{\theta}) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$

$\underbrace{-mgr\cos\theta}_{\tau} \underbrace{- 2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta}}_{-I\alpha} = 0$

Example



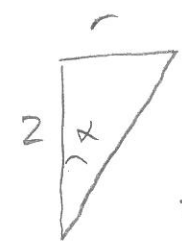
$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{z}$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

$U = mgz$

how many degrees of freedom?

$3 - 1 = 2$



$\frac{z}{r} = \cot\alpha$

$= F(q_i)$

eliminate either z or r

we'll kill z

$U = mgr\cot\alpha$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha) = \frac{1}{2}m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2)$

$\sin^2\alpha + \cos^2\alpha = 1 \rightarrow$ divide by $\sin^2\alpha$

Now $L = \frac{1}{2}m(\dot{r}^2\csc^2\alpha + r^2\dot{\theta}^2) - mgr\cot\alpha$

$\frac{\partial L}{\partial \theta} = 0$ So $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} = 0$ and $mr^2\dot{\theta} = \frac{\partial L}{\partial \dot{\theta}} = \text{constant}$

or $I\omega = L = \text{constant} \rightarrow$ angular momentum is conserved

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cot\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}\csc^2\alpha$$

$$m\ddot{r}\csc^2\alpha - mr\dot{\theta}^2 + mg\frac{\cos(\alpha)}{\sin(\alpha)} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2\sin^2\alpha + mg\cos(\alpha)\sin(\alpha) = 0$$ We'll come back To This

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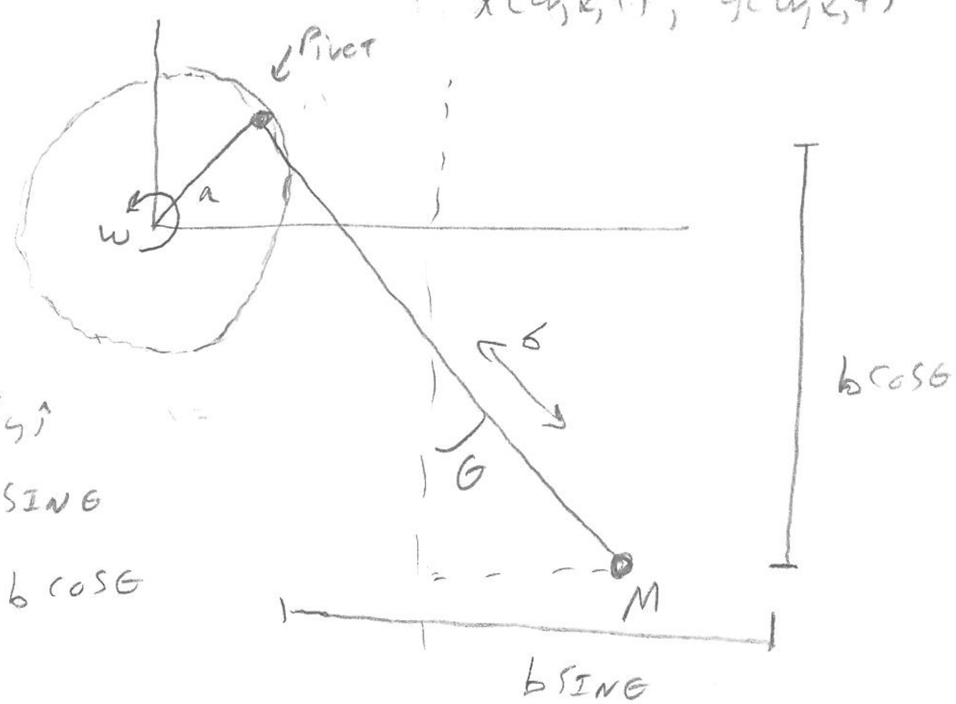
Example

$x(\omega, \alpha, t)$, $y(\omega, \alpha, t)$



$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$$

$$x = a\cos(\omega t) + b\sin\theta$$

$$y = a\sin(\omega t) - b\cos\theta$$

$$dx = -a\omega\sin(\omega t) + b\dot{\theta}\cos\theta$$

$$dy = a\omega\cos(\omega t) + b\dot{\theta}\sin\theta$$

$\rightarrow$ θ is the only Free coordinate

$$T = \frac{1}{2}m(-a\omega\sin(\omega t) + b\dot{\theta}\cos\theta)^2 + \frac{1}{2}m(a\omega\cos(\omega t) + b\dot{\theta}\sin\theta)^2$$

$$= \frac{1}{2}m\left[a^2\omega^2\sin^2(\omega t) + b^2\dot{\theta}^2\cos^2\theta - 2ab\omega\dot{\theta}\sin(\omega t)\cos\theta\right]$$
$$+ \frac{1}{2}m\left[a^2\omega^2\cos^2(\omega t) + b^2\dot{\theta}^2\sin^2\theta + 2ab\omega\dot{\theta}\cos(\omega t)\sin(\theta)\right]$$

$$T = \frac{1}{2}m\left[a^2w^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}aw\,\mathrm{SIN}(\theta - wT)\right]$$

$$U = mgy = mg(a\,\mathrm{SIN}(wT) - b\cos\theta)$$

$$L = T - U$$

$$\frac{\partial L}{\partial \theta} = -mgb\,\mathrm{SIN}\,\theta + mb\dot{\theta}aw\cos(\theta - wT)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} + baw\,\mathrm{SIN}(\theta - wT)$$

$$\frac{d}{dT}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mbaw\cos(\theta - wT)(\dot{\theta} - w)$$

$$mb^2\ddot{\theta} + mbaw(\cancel{\dot{\theta}} - w)\cos(\theta - wT) - \cancel{mb\dot{\theta}aw\cos(\theta - wT)} + mgb\,\mathrm{SIN}\,\theta$$

$$= 0$$

$$mb^2\ddot{\theta} - mbaw^2\cos(\theta - wT) + mgb\,\mathrm{SIN}\,\theta = 0$$

$$\ddot{\theta} - \frac{aw^2}{b}\cos(\theta - wT) + \frac{g}{b}\,\mathrm{SIN}\,\theta = 0$$

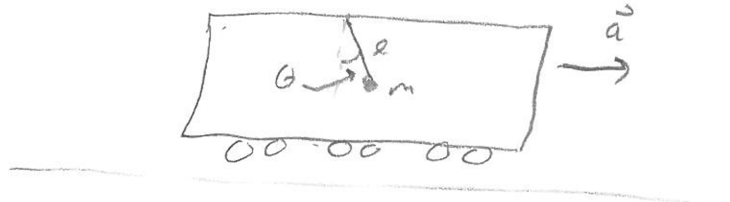
$$\ddot{\theta} = \frac{aw^2}{b}\cos(\theta - wT) - \frac{g}{b}\,\mathrm{SIN}\,\theta = 0$$

This is a DRIVEN Pendulum

Super duper non-linear

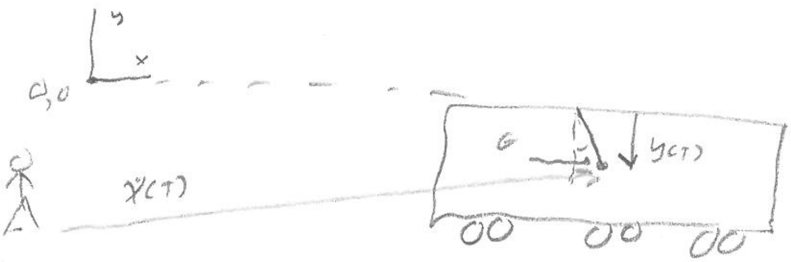
Example

begins with v_0



ω for θ small

Must go into inertial reference frame



$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$$

$$x(t) = v_0 t + \frac{1}{2}at^2 + l\sin\theta \qquad \frac{dx}{dt} = v_0 + at + l\dot{\theta}\cos\theta = \dot{x}(t)$$

$$y(t) = -l\cos\theta \qquad \frac{dy}{dt} = l\dot{\theta}\sin\theta = \dot{y}(t)$$

$$T = \frac{m}{2}(v_0 + at + l\dot{\theta}\cos\theta)^2 + \frac{m}{2}l^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{m}{2}(v_0^2 + a^2t^2 + l^2\dot{\theta}^2\cos^2\theta + 2v_0at + 2atl\dot{\theta}\cos\theta + 2v_0l\dot{\theta}\cos\theta)$$

$$+ \frac{m}{2}l^2\dot{\theta}^2\sin^2\theta$$

$$T = \frac{1}{2}mv_0^2 + \frac{1}{2}ml^2\dot{\theta}^2 + a^2t^2 + \frac{m}{2}(2v_0at + 2atl\dot{\theta}\cos\theta + 2v_0l\dot{\theta}\cos\theta)$$

$L = T - U \qquad U = -mgl\cos\theta \qquad$ dropped a^2T^2 doesn't matter

$$L = \frac{1}{2}m(V_0^2 + l^2\dot\theta^2) + m\left[V_0 aT + aTl\dot\theta\cos\theta + V_0 l\dot\theta\cos\theta\right] + ma^2T^2$$

$$+ mgl\cos\theta$$

$$\frac{\partial L}{\partial\theta} = maTl\dot\theta\sin\theta - mV_0 l\dot\theta\sin\theta - mgl\sin\theta$$

1 $$\frac{\partial L}{\partial\theta} = -\sin(\theta)\left[mlaT\dot\theta + mV_0 l\dot\theta + mgl\right] = -\sin(\theta)\left[m(V_0+aT)l\dot\theta + mgl\right]$$

$$\frac{d}{dT}\left[\frac{\partial L}{\partial\dot\theta}\right] = \frac{d}{dT}\left[ml^2\dot\theta + (aT+V_0)ml\cos\theta\right]$$

2 $$\frac{d}{dT}\left[\frac{\partial L}{\partial\dot\theta}\right] = ml^2\ddot\theta - (aT+V_0)ml\dot\theta\sin\theta + mal\cos\theta$$

$$1 = 2 \qquad \cancel{-\sin\theta\left[m(V_0+aT)l\dot\theta + mgl\right]} = ml^2\ddot\theta \cancel{\; - ml\dot\theta\sin\theta(aT+V_0)} + mal\cos\theta$$

$$-\sin\theta\, mgl = ml^2\ddot\theta + mal\cos\theta$$

$$ml^2\ddot\theta = -mal\cos\theta - \sin\theta(mgl)$$

$$\ddot\theta = -\frac{a}{l}\cos\theta - \frac{g}{l}\sin\theta$$

and $m\frac{d^2s}{dt^2} = -a\cos\theta - g\sin\theta$

$$\frac{d^2s}{dt^2} = \frac{d^2(\ell\theta)}{dt^2} = \ell\ddot{\theta}$$

$$m\ell\ddot{\theta} = -ma\cos\theta - mg\sin\theta \rightarrow \ddot{\theta} = -\frac{a}{\ell}\cos\theta - \frac{g}{\ell}\sin\theta$$

Now θ_e when $ma\cos\theta = -mg\sin\theta$

or $\tan^{-1}\left(\frac{-a}{g}\right) = \theta_{eq}$

Frequency of small oscillations?

Let $\theta = \theta_{eq} + \eta \qquad \ddot{\theta} = \ddot{\eta}$

$$\ddot{\eta} = -\frac{g}{\ell}\sin(\theta_{eq} + \eta) - \frac{a}{\ell}\cos(\theta_{eq} + \eta)$$

$$\ddot{\eta} = -\frac{g}{\ell}\left[\sin\theta_e\cos\eta + \cos\theta_e\sin\eta\right] - \frac{a}{\ell}\left(\cos\theta_e\cos\eta - \sin\theta_e\sin\eta\right)$$

$\cos(\eta) \sim 1 \qquad \sin(\eta) \approx \eta$

$$\ddot{\eta} = -\frac{g}{\ell}\left[\sin\theta_e + \eta\cos\theta_e\right] - \frac{a}{\ell}\left[\cos\theta_e - \eta\sin\theta_e\right]$$

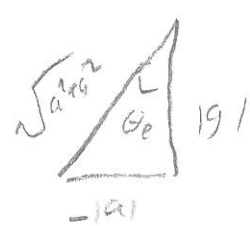
$$\ddot{\eta} = -\frac{1}{\ell}\Big[\underbrace{(g\sin\theta_e + a\cos\theta_e)}_{0} + \eta\left(g\cos\theta_e - a\sin\theta_e\right)\Big]$$

by definition

$g\sin\theta_e + a\cos\theta_e = 0$

So $\ddot{\eta} = -\frac{1}{\ell}\eta\left(g\cos\theta_e - a\sin\theta_e\right)$

but $\theta_e = \tan^{-1}\left(\frac{-a}{g}\right)$



$$\ddot{\eta} = -\frac{1}{\ell}\eta\left[\frac{g^2}{\sqrt{a^2+g^2}} + \frac{a^2}{\sqrt{a^2+g^2}}\right] = -\frac{1}{\ell}\eta\left[\frac{g^2+a^2}{\sqrt{a^2+g^2}}\right]$$

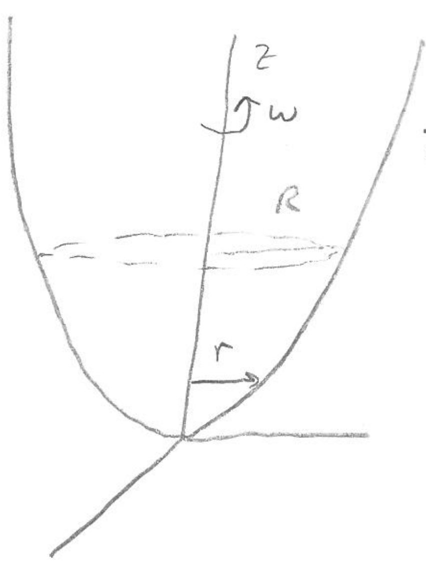
$$\ddot{\eta} = -\frac{\eta}{\ell}\left(\frac{g^2+a^2}{\sqrt{a^2+g^2}}\right) = -\frac{\eta}{\ell}\,\frac{\left((g^2+a^2)^{1/2}\right)^2}{(g^2+a^2)^{1/2}} = -\frac{\eta}{\ell}\sqrt{g^2+a^2}$$

$$\ddot{\eta} + \eta\underbrace{\left[\frac{\sqrt{g^2+a^2}}{\ell}\right]}_{\omega^2} \qquad \omega = \frac{g}{\ell} \text{ For } \vec{a} = 0$$

Ex. Bead on a rotating wire

$z = cr^2$, ω = rotation frequency, bead stays at $r = R$

For $c = ?$



$z = cr^2$

$$d\vec{s} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{z}$$

$$T = \left(\dot{r}^2 + r^2\underbrace{\dot{\theta}^2}_{\omega^2} + \dot{z}^2\right)\frac{1}{2}m$$

$$U = mgz$$

$$U = mgcr^2$$

$$\dot{z} = 2cr\dot{r}$$

$$L = T - U = \frac{m}{2}\left(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right) - mgcr^2$$

$$\frac{\partial L}{\partial r} = 4mc^2 r\dot{r}^2 + mr\omega^2 - 2mgcr$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} + 4mc^2 r^2 \dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\left[\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r\dot{r}^2\right]$$

$$\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r\dot{r}^2 = 4c^2 r\dot{r}^2 + r\omega^2 - 2gcr$$

$$\ddot{r}\left[1 + 4c^2 r^2\right] + (4c^2 r)\dot{r}^2 + (\omega^2 - 2gc)r = 0$$

If $r =$ constant $\ddot{r} = \dot{r}^2 = 0$

$(\omega^2 - 2gc)R = 0 \qquad c = \frac{\omega^2}{2g}$

Lagrange's Equations with Undetermined multipliers

Holonomic → constraints that are algebraic relationships amongst the coordinates, integrable (solvable)

If constraints are of the form $F(x_i, \dot{x}_i, T) = 0$ ← Velocity — Non-Holonomic, Not generally integrable

Consider a constraint of the form $\sum_i A_i \dot{x}_i + B = 0$

if $A_i = \frac{\partial F}{\partial x_i}$ and $B = \frac{\partial F}{\partial T}$

The $\frac{\partial F}{\partial x}\frac{\partial x}{\partial T} + \frac{\partial F}{\partial T} = \frac{dF}{dT} = 0$ → integrable

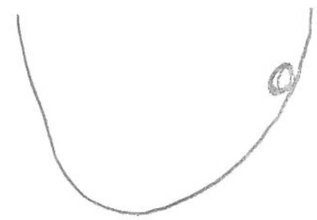
or $F(x,T) - C = 0$ ← makes $F(x,\dot{x},T) \to F(x,T) - C$

So Constraints of $\sum_j \frac{\partial F_i}{\partial q_j} dq_j + \frac{\partial F}{\partial T} dT = 0$

We can add This to $\frac{\partial L}{\partial q_j} - \frac{d}{dT}\frac{\partial L}{\partial \dot{q}_j} = 0$ without changing it

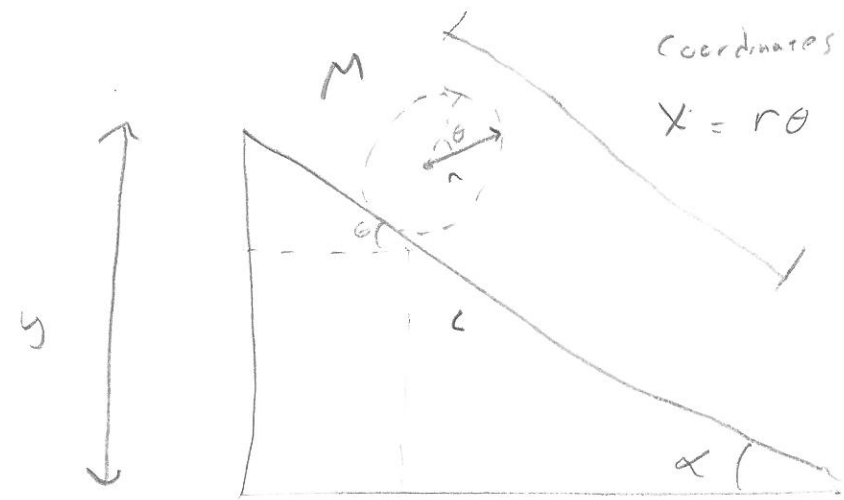
as $$\frac{\partial L}{\partial q_j} - \frac{d}{dT}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(T) \frac{\partial F_k}{\partial q_j} = 0$$

Allows for the determination of forces of constraint



$$Q_j = \sum_k \lambda_k \frac{\partial F_k}{\partial q_j}$$ $k \rightarrow$ # eq's constraint

constraint



Coordinates

$X = r\theta \rightarrow X - r\theta = 0$

$d\vec{s} = dx\,\hat{i} \rightarrow T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \quad U = mgy$

$U = mg(L-x)\sin\theta$

$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(L-x)\sin\alpha$

$F(x,\theta) = x - r\theta = 0$

Now $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial F}{\partial x} = 0$ 2

$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial F}{\partial \theta} = 0$ 2

$mg\sin\alpha - M\ddot{x} + \lambda = 0$

$-\frac{I\ddot{\theta}}{2} - \lambda r = 0$

Also $\ddot{x} = r\ddot{\theta}$ so $\frac{\ddot{x}}{r} = \ddot{\theta}$

or $\frac{Mr^2}{2}\frac{\ddot{x}}{r} = -\lambda r \rightarrow \lambda = -\frac{1}{2}M\ddot{x}$

For which $Mg\sin(\alpha) - \frac{3}{2}M\ddot{x} = 0$

$$\boxed{\frac{2}{3}g\sin(\alpha) = \ddot{x}}$$

and $\boxed{\lambda = -\frac{1}{3}Mg\sin(\alpha)}$

$$\boxed{\ddot{\theta} = \frac{2g}{3r}\sin(\alpha)}$$

$$\left.\begin{aligned} Q_x &= \lambda\frac{\partial F}{\partial x} = \lambda = -\frac{Mg}{3}\sin(\alpha) \\ Q_\theta &= \lambda\frac{\partial F}{\partial \theta} = -\lambda R = \frac{Mgr\sin(\alpha)}{3} \end{aligned}\right]$$ rolling w/out slipping

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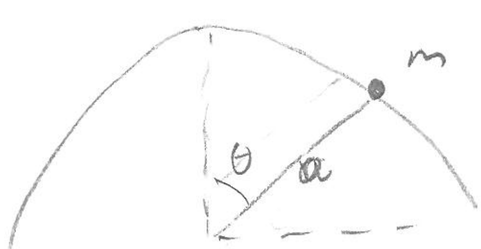

$F(r,\theta) = r - a = 0 \quad \frac{\partial F}{\partial r} = 1 \qquad r = a \quad \dot{r} = 0 \quad \ddot{r} = 0$

$T = \frac{1}{2} m (\cancel{\dot{r}^2}_{0} + r^2\dot{\theta}^2) = \frac{1}{2} m r^2 \dot{\theta}^2$

$U = mgr\cos\theta$

$L = \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\cos\theta$

$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \qquad mgr\sin\theta - mr^2\ddot{\theta} = 0$

$mga\sin\theta - ma^2\ddot{\theta} = 0$

$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda = 0 \qquad mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$

$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$

$\ddot{\theta} = \frac{g}{a}\sin\theta \qquad \ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$

$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{a}\sin\theta \rightarrow \int \dot{\theta}\, d\dot{\theta} = \int \frac{g}{a}\sin\theta\, d\theta$

$\frac{\dot{\theta}^2}{2} = -\frac{g}{a}\cos\theta + \frac{g}{a}$ and $\lambda = mg\cos\theta - 2ma\left(\frac{g}{a}\right)\left[1 - \cos\theta\right]$

$\lambda = 3mg\cos\theta - 2mg = mg(3\cos\theta - 2)$

$\frac{\partial F}{\partial r} = 1 \quad \lambda =$ Normal Force