4.6 Forced Oscillations

4.6 # 11, 14, 24

Ex: $\ddot{x} + x = 2\cos(3t)$ $\quad x(0)=0$, $\dot{x}(0)=0$

$r^2+1=0$ $\quad y_h = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$

$r = \pm 1i$ $\quad \alpha = \frac{-b}{2m}$ $\quad \beta = \frac{\sqrt{4mk-b^2}}{2m}$

$\alpha = 0$ $\quad \beta = 1$

$y_h = c_1\cos(t) + c_2\sin(t)$ $\quad x_p = A\sin(3t) + B\cos(3t)$ $\quad x_p = -\frac{1}{4}\cos(3t)$

$\dot{x}_p = 3A\cos 3t - 3B\sin 3t$

$\ddot{x}_p = -9A\sin 3t - 9B\cos 3t$ $\quad x_G = x_h + x_p$

$-9A\sin 3t - 9B\cos 3t + A\sin(3t) + B\cos(3t) = 2\cos(3t)$

$A=0$, no sin on right ∴ no sin on left $\quad x_h = c_1\cos(t) + c_2\sin(t)$

$-8B\cos 3t = 2\cos(3t)$ $\quad x_p = -\frac{1}{4}\cos(3t)$

$B = -\frac{1}{4}$

$x = c_1\cos(t) + c_2\sin(t) - \frac{1}{4}\cos(3t)$ $\quad x(0)=0$ $\quad c_1 = \frac{1}{4}$

$0 = c_1\cos(0) - \frac{1}{4}\cos(0)$ $\quad \dot{x}(0)=0$ $\quad c_2 = 0$

$\frac{1}{4} = c_1$

$\dot{x} = -c_1\sin(t) + c_2\cos(t) + \frac{3}{4}\sin(3t)$ $\quad x = \frac{1}{4}\cos(t) - \frac{1}{4}\cos(3t)$

$0 = c_2\cos(0)$

$0 = c_2$

Ex 4 $\quad \ddot{x} + 4\dot{x} + 5x = 10\cos(3t)$ $\quad x(0)=0$, $\dot{x}(0)=0$

$x_h: (r^2+4r+5)=0$ $\quad \Delta = 16-4(1)(5)$

$\Delta = -4$

$x_h = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$ $\quad x_p = A\cos(3t) + B\sin(3t)$

$\alpha = \frac{-4}{2(1)}$ $\quad \beta = \frac{\sqrt{4(1)(5)-(4)^2}}{2(1)}$ $\quad \dot{x}_p = -3A\sin(3t) + 3B\cos(3t)$

$\ddot{x}_p = -9A\cos(3t) - 9B\sin(3t)$

$\alpha = -2$ $\quad \beta = 1$

$x_h = e^{-2t}(c_1\cos(t) + c_2\sin(t))$

$-9A\cos(3t) - 9B\sin(3t) + 4(-3A\sin(3t) + 3B\cos(3t)) + 5(A\cos(3t) + B\sin(3t)) = 10\cos(3t)$

$\cos(3t)(-9A + 12B + 5A) + \sin(3t)(-9B - 12A + 5B) = 10\cos(3t)$

$-4A + 12B = 10$ $\quad \left[\begin{array}{cc|c} -4 & 12 & 0 \\ -12 & -4 & 0 \end{array}\right]$ $\quad A = -\frac{1}{4}$

$-12A - 4B = 0$ $\quad B = \frac{3}{4}$

$x(t) = e^{-2t}(c_1\cos t + c_2\sin t) - \frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t$

$\dot{x}(t) = -2e^{-2t}(c_1\cos t + c_2\sin t) + e^{-2t}(-c_1\sin t + c_2\cos t) + \frac{3}{4}\sin 3t + \frac{9}{4}\cos 3t$

$c_1 = -\frac{3}{4}$

$c_2 = \frac{1}{4}$ $\quad x(t) = e^{-2t}\left(-\frac{7}{4}\sin(t) + \frac{1}{4}\cos(t)\right) - \frac{1}{4}\cos(3t) + \frac{3}{4}\sin(3t)$

Undamped Forced Oscillator

$m\ddot{x} + Kx = F_0\cos(w_0 t)$

$x_h = c_1\cos(w_0 t) + c_2\sin(w_0 t)$

$x_p = A\cos(w_f t) + B\sin(w_f t)$

$w_0 = \sqrt{K/m}$

Case 1 $\quad w_0 \neq w_f$

$A = \frac{F_0}{m(w_0^2 - w_f^2)}$, $B = 0$

$w_0 \neq w_f$ $\quad x = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \frac{F_0}{m(w_0^2 - w_f^2)}\cos(w_f t)$

Case 2 $\quad w_0 = w_f$

$x_p = At\cos(w_0 t) + Bt\sin(w_0 t)$

$A = 0$

$B = \frac{F_0}{2m w_0}$

$x = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \frac{F_0}{2m w_0} t\sin(w_0 t)$

$x(0) = 0$, $\dot{x}(0) = 0$

$x(t) = \frac{2F_0}{m(w_0^2 - w_f^2)}\sin\frac{(w_0 - w_f)t}{2}\sin\frac{(w_0 + w_f)t}{2}$