4.5 Variation of Parameters

4.5 # 6, 14

4.5.6) $y'' - 2y' + 2y = e^t \sin(t)$

y_h: $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(2)$
$= 4 - 8$
$= -4$

$y_h = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$ $\alpha = 1$ $\beta = 1$

$\alpha = \frac{-b}{2a} = \frac{2}{2(1)} = 1$

$\beta = \frac{\sqrt{4ac - b^2}}{2(a)} = \frac{\sqrt{4(1)(2) - (-2)^2}}{2(1)} = 1$

$8 - 4 = 4$ $\sqrt{4} = 2$ $\frac{2}{2} = 1$

$y_p = v_1 y_1 + v_2 y_2$

$e^t \cos(t)\left(\frac{\sin(t)\cos(t) - t}{2}\right) + e^t \sin(t)\left(\frac{\sin^2(t)}{2}\right)$

$y_h = e^t(c_1 \cos(t) + c_2 \sin(t))$

$y_1 = e^t c_1 \cos(t)$
$y_2 = e^t c_2 \sin(t)$

$$W = \left[\begin{array}{cc|c} e^t \cos(t) & e^t \sin(t) & 0 \\ e^t(\cos(t) - \sin(t)) & e^t(\sin(t) + \cos(t)) & e^t \sin(t) \end{array}\right]$$

$e^t \cos(t) - e^t \sin(t)$
$e^t \sin(t) + e^t \cos(t)$

rref: $-(\sin(t))^2$
$\sin(t)\cos(t)$

$v_1' = -(\sin(t))^2$
$v_2' = \sin(t)\cos(t)$

$v_1 = \int -(\sin(t))^2 \, dt$
$v_1 = \frac{\sin(t)\cos(t) - t}{2}$

$v_2 = \int \sin(t)\cos(t) \, dt$
$v_2 = \frac{\sin^2(t)}{2}$

$$\boxed{y = e^t(c_1 \cos(t) + c_2 \sin(t)) + e^t \cos(t)\left(\frac{\sin(t)\cos(t) - t}{2}\right) + e^t \sin(t)\left(\frac{\sin^2(t)}{2}\right)}$$

4.5.14) $t^2 y'' + t y' - 4y = t^2(1 + t^2)$

$y'' + \frac{1}{t} y' - \frac{4}{t^2} y = (1 + t^2)$

$y_1(t) = t^2$
$y_2(t) = t^{-2}$

$$W = \begin{bmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{bmatrix}$$

$y_h = c_1 y_1 + c_2 y_2$
$y_h = c_1 t^2 + c_2 t^{-2}$

$y_p =$ $y_1 v_1' + y_2 v_2' = 0$
$y_1' v_1' + y_2' v_2' = f$

$y_p = y_1 v_1 + y_2 v_2$

$W = -4/t$

$v_1' = \frac{-y_2 f}{W(y_1, y_2)}$ $v_2' = \frac{y_1 f}{W(y_1, y_2)}$

$W = -4/t$
$y_1 = t^2$ $y_2 = t^{-2}$ $f = (1 + t^2)$

$y_p = t^2\left(\frac{2\ln(t) + t^2}{8}\right) + t^{-2}\left(\frac{-t^4(2t^2 + 3)}{48}\right)$

$v_1' = \frac{-t^{-2}(1 + t^2)}{-4/t}$ $v_2' = \frac{t^2(1 + t^2)}{-4/t}$

$v_1' = \frac{-t^{-1}(1 + t^2)}{-4}$ $v_2' = \frac{t^3(1 + t^2)}{-4}$

$v_1' = \frac{(1 + t^2)}{4t}$ $v_2' = -\frac{t^3(1 + t^2)}{4}$

$$\boxed{y = c_1 t^2 + c_2 t^{-2} + t^2\left(\frac{2\ln(t) + t^2}{8}\right) + t^{-2}\left(\frac{-t^4(2t^2 + 3)}{48}\right)}$$

$v_1 = \frac{2\ln(t) + t^2}{8}$ $v_2 = \frac{-t^4(2t^2 + 3)}{48}$