Exercise 1

$$\hat{u}(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}$$

a.) $|\psi_0\rangle = |0\rangle$

$$|\psi(\lambda)\rangle = \hat{u}(\lambda)|\psi_0\rangle = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\lambda/2} \\ 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} |\psi(\lambda)\rangle = \begin{pmatrix} e^{-i\lambda/2} \\ 0 \end{pmatrix} \\ \hat{n}_0 : \theta = 0, \varphi = 0 \\ \hat{n} : \theta = 0, \varphi = 0 \end{array}}$$

b.) $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|\psi(\lambda)\rangle = \hat{u}(\lambda)|\psi_0\rangle = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\lambda/2} \\ e^{i\lambda/2} \end{pmatrix}$$

$$\boxed{\begin{array}{l} |\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\lambda/2} \\ e^{i\lambda/2} \end{pmatrix} \\ \hat{n}_0 : \theta = \pi/2, \varphi = 0 \\ \hat{n} : \theta = \pi/2, \varphi = \lambda \end{array}}$$

c.) $|\psi_0\rangle = \frac{1}{5}(4|0\rangle + 3|1\rangle)$

$$|\psi(\lambda)\rangle = \hat{u}(\lambda)|\psi_0\rangle = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} \cdot \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4e^{-i\lambda/2} \\ 3e^{i\lambda/2} \end{pmatrix}$$

$$\boxed{\begin{array}{l} |\psi(\lambda)\rangle = \frac{1}{5}\begin{pmatrix} 4e^{-i\lambda/2} \\ 3e^{i\lambda/2} \end{pmatrix} \\ \hat{n}_0 : \theta = 73.7^\circ, \varphi = 0 \\ \hat{n} : \theta = 73.7^\circ, \varphi = \lambda \end{array}}$$

d.) $|\psi(\lambda)\rangle = e^{-i\lambda/2}|0\rangle : \theta = 0, \varphi = 0, \quad |+\hat{z}\rangle = |0\rangle : \theta = \pi, \varphi = 0, \quad |-\hat{z}\rangle = |1\rangle$

$$\langle +\hat{z}|\psi(\lambda)\rangle = e^{-i\lambda/2}\cancelto{1}{\langle 0|0\rangle} + 0 \cdot \cancelto{1}{\langle 1|1\rangle} = e^{-i\lambda/2} \quad : \quad |\langle +\hat{z}|\psi(\lambda)\rangle|^2 = e^{-i\lambda/2} \cdot e^{i\lambda/2} = e^0 = 1$$

$$\langle -\hat{z}|\psi(\lambda)\rangle = 0 \cdot e^{-i\lambda/2}\cancelto{1}{\langle 0|0\rangle} + 0 \cdot \cancelto{1}{\langle 1|1\rangle} = 0 \quad : \quad |\langle -\hat{z}|\psi(\lambda)\rangle|^2 = 0^2 = 0$$

$$\boxed{F_z = 0}$$

Exercise 1 Continued

$\theta = \pi/2,\ \varphi = 0,\ |+\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$: $\theta = \pi/2,\ \varphi = \pi,\ |-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\langle +\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\cdot e^{-i\lambda/2}\cancel{\langle 0|0\rangle} + \frac{1}{\sqrt{2}}\cdot 0\,\cancel{\langle 1|1\rangle} = \frac{1}{\sqrt{2}}e^{-i\lambda/2}$; $|\langle +\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{\sqrt{2}}e^{-i\lambda/2}\cdot\frac{1}{\sqrt{2}}e^{i\lambda/2} = \frac{1}{2}$

$\langle -\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\cdot e^{-i\lambda/2}\cancel{\langle 0|0\rangle} - \frac{1}{\sqrt{2}}\cdot 0\,\cancel{\langle 1|1\rangle} = \frac{1}{\sqrt{2}}e^{-i\lambda/2}$: $|\langle -\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{\sqrt{2}}e^{-i\lambda/2}\cdot\frac{1}{\sqrt{2}}e^{i\lambda/2} = \frac{1}{2}$

$$\boxed{F_x = 0}$$

e.) $|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}(e^{-i\lambda/2}|0\rangle + e^{i\lambda/2}|1\rangle)$: $\theta = 0,\ \varphi = 0,\ |+\hat{z}\rangle = |0\rangle$: $\theta = \pi,\ \varphi = 0,\ |-\hat{z}\rangle = |1\rangle$

$\langle +\hat{z}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}e^{-i\lambda/2}\cancel{\langle 0|0\rangle} + 0\cdot\frac{1}{\sqrt{2}}e^{i\lambda/2}\cancel{\langle 1|1\rangle} = \frac{1}{\sqrt{2}}e^{-i\lambda/2}$

$|\langle +\hat{z}|\psi(\lambda)\rangle|^2 = \frac{1}{\sqrt{2}}e^{-i\lambda/2}\cdot\frac{1}{\sqrt{2}}e^{i\lambda/2} = \frac{1}{2}\cdot e^{0} = \frac{1}{2}$: $|\langle +\hat{z}|\psi(\lambda)\rangle|^2 = \frac{1}{2}$

$\langle -\hat{z}|\psi(\lambda)\rangle = 0\cdot\frac{1}{\sqrt{2}}e^{-i\lambda/2}\cancel{\langle 0|0\rangle} + \frac{1}{\sqrt{2}}e^{i\lambda/2}\cancel{\langle 1|1\rangle} = \frac{1}{\sqrt{2}}e^{i\lambda/2}$

$|\langle -\hat{z}|\psi(\lambda)\rangle|^2 = \frac{1}{\sqrt{2}}e^{i\lambda/2}\cdot\frac{1}{\sqrt{2}}e^{-i\lambda/2} = \frac{1}{2}e^{0} = \frac{1}{2}$: $|\langle -\hat{z}|\psi(\lambda)\rangle|^2 = \frac{1}{2}$

$$\boxed{F_z = 0}$$

$\theta = \pi/2,\ \varphi = 0,\ |+\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$: $\theta = \pi/2,\ \varphi = \pi,\ |-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\langle +\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\cdot\frac{1}{\sqrt{2}}(e^{-i\lambda/2}|0\rangle + e^{i\lambda/2}|1\rangle) = \frac{1}{2}(e^{-i\lambda/2}\cancel{\langle 0|0\rangle} + e^{i\lambda/2}\cancel{\langle 1|1\rangle})$

$\langle +\hat{x}|\psi(\lambda)\rangle = \frac{1}{2}(e^{-i\lambda/2} + e^{i\lambda/2})$

$|\langle +\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{4}(e^{-i\lambda/2} + e^{i\lambda/2})(e^{i\lambda/2} + e^{-i\lambda/2}) = \frac{1}{4}(e^{0} + e^{-i\lambda} + e^{i\lambda} + e^{0})$

$= \frac{1}{4}(2 + \cos(\lambda) - i\sin(\lambda) + \cos(\lambda) + i\sin(\lambda)) = \frac{1}{4}(2 + 2\cos(\lambda))$

$|\langle +\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{2}(1 + \cos(\lambda))$

$\langle -\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\cdot\frac{1}{\sqrt{2}}(e^{-i\lambda/2}|0\rangle + e^{i\lambda/2}|1\rangle) = \frac{1}{2}(e^{-i\lambda/2}\cancel{\langle 0|0\rangle} - e^{i\lambda/2}\cancel{\langle 1|1\rangle})$

$\langle -\hat{x}|\psi(\lambda)\rangle = \frac{1}{2}(e^{-i\lambda/2} - e^{i\lambda/2})$

$|\langle -\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{4}(e^{-i\lambda/2} - e^{i\lambda/2})(e^{i\lambda/2} - e^{-i\lambda/2}) = \frac{1}{4}(e^{0} - e^{-i\lambda} - e^{i\lambda} + e^{0})$

$= \frac{1}{4}(2 - \cos(\lambda) + i\sin(\lambda) - \cos(\lambda) - i\sin(\lambda)) = \frac{1}{4}(2 - 2\cos(\lambda))$

$|\langle -\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{2}(1 - \cos(\lambda))$

Exercise 1 Continued

$$\sum_{\text{All possible outcomes}}^{\infty} \frac{1}{\text{Prob(outcome)}}\left(\frac{\partial \text{Prob(outcome)}}{\partial P}\right)^2 = F$$

$F_+ : P_+ = \frac{1}{2}(1+\cos(\lambda)) \; : \; \frac{\partial P_+}{\partial \lambda} = \frac{-1}{2}\sin(\lambda) \; : \; \left(\frac{\partial P_+}{\partial \lambda}\right)^2 = \frac{1}{4}\sin^2(\lambda)$

$$F_+ = \frac{2}{1+\cos(\lambda)} \cdot \frac{1}{4}\sin^2(\lambda) = \frac{\sin^2(\lambda)}{2(1+\cos(\lambda))}$$

$F_- : P_- = \frac{1}{2}(1-\cos(\lambda)) \; : \; \frac{\partial P_-}{\partial \lambda} = \frac{1}{2}\sin(\lambda) \; : \; \left(\frac{\partial P_-}{\partial \lambda}\right)^2 = \frac{1}{4}\sin^2(\lambda)$

$$F_- = \frac{2}{1-\cos(\lambda)} \cdot \frac{1}{4}\sin^2(\lambda) = \frac{\sin^2(\lambda)}{2(1-\cos(\lambda))}$$

$$F = F_+ + F_- = \frac{1}{2}\left(\frac{\sin^2(\lambda)}{1+\cos(\lambda)} + \frac{\sin^2(\lambda)}{1-\cos(\lambda)}\right) = \frac{1}{2}\left(\frac{\sin^2(\lambda) - \cos(\lambda)\sin^2(\lambda) + \sin^2(\lambda) + \cos(\lambda)\sin^2(\lambda)}{1-\cos^2(\lambda)}\right)$$

$$= \frac{1}{2}\left(\frac{2\sin^2(\lambda)}{\sin^2(\lambda)}\right) = 1$$

$$\boxed{F_x = 1}$$

f.) $|\psi(\lambda)\rangle = \frac{1}{5}\left(4e^{-i\lambda/2}|0\rangle + 3e^{i\lambda/2}|1\rangle\right) \; : \; \theta = 0,\ \varphi = 0,\ |+\hat{z}\rangle = |0\rangle \; : \; \theta = \pi,\ \varphi = 0,\ |-\hat{z}\rangle = |1\rangle$

$$\langle +\hat{z}|\psi(\lambda)\rangle = \frac{4}{5}e^{-i\lambda/2}\cancelto{1}{\langle 0|0\rangle} + 0\cdot\frac{3}{5}e^{i\lambda/2}\cancelto{1}{\langle 1|1\rangle} = \frac{4}{5}e^{-i\lambda/2}$$

$$|\langle +\hat{z}|\psi(\lambda)\rangle|^2 = \frac{16}{25}e^{-i\lambda/2}\cdot e^{i\lambda/2} = \frac{16}{25}e^0 = \frac{16}{25}$$

$$\langle -\hat{z}|\psi(\lambda)\rangle = 0\cdot\frac{4}{5}e^{-i\lambda/2}\cancelto{1}{\langle 0|0\rangle} + \frac{3}{5}e^{i\lambda/2}\cancelto{1}{\langle 1|1\rangle} = \frac{3}{5}e^{i\lambda/2}$$

$$|\langle -\hat{z}|\psi(\lambda)\rangle|^2 = \frac{9}{25}e^{-i\lambda/2}\cdot e^{i\lambda/2} = \frac{9}{25}e^0 = \frac{9}{25}$$

$$\boxed{F_z = 0}$$

$\theta = \pi/2,\ \varphi = 0,\ |+\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) : \theta = \pi/2,\ \varphi = \pi,\ |-\hat{x}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\langle +\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\cdot\frac{1}{5}\left(4e^{-i\lambda/2}|0\rangle + 3e^{i\lambda/2}|1\rangle\right) = \frac{1}{5\sqrt{2}}\left(4e^{-i\lambda/2}\cancelto{1}{\langle 0|0\rangle} + 3e^{i\lambda/2}\cancelto{1}{\langle 1|1\rangle}\right)$$

$$\langle +\hat{x}|\psi(\lambda)\rangle = \frac{1}{5\sqrt{2}}\left(4e^{-i\lambda/2} + 3e^{i\lambda/2}\right)$$

$$|\langle +\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{50}\left(4e^{-i\lambda/2} + 3e^{i\lambda/2}\right)\left(4e^{i\lambda/2} + 3e^{-i\lambda/2}\right) = \frac{1}{50}\left(16e^0 + 12e^{-i\lambda} + 12e^{i\lambda} + 9e^0\right)$$

$$= \frac{1}{50}\left(25 + 12\cos(\lambda) - 12i\sin(\lambda) + 12\cos(\lambda) + 12i\sin(\lambda)\right)$$

Exercise 1 Continued

$$|\langle +\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{50}\left(25 + 24\cos(\lambda)\right)$$

$$\langle -\hat{x}|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right)\cdot\frac{1}{5}\left(4e^{-i\lambda/2}|0\rangle + 3e^{i\lambda/2}|1\rangle\right) = \frac{1}{5\sqrt{2}}\left(4e^{-i\lambda/2}\overset{1}{\cancel{\langle 0|0\rangle}} - 3e^{i\lambda/2}\overset{1}{\cancel{\langle 1|1\rangle}}\right)$$

$$\langle -\hat{x}|\psi(x)\rangle = \frac{1}{5\sqrt{2}}\left(4e^{-i\lambda/2} - 3e^{i\lambda/2}\right)$$

$$|\langle -\hat{x}|\psi(x)\rangle|^2 = \frac{1}{50}\left(4e^{-i\lambda/2} - 3e^{i\lambda/2}\right)\left(4e^{i\lambda/2} - 3e^{-i\lambda/2}\right) = \frac{1}{50}\left(16e^{0} - 12e^{-i\lambda} - 12e^{i\lambda} + 9e^{0}\right)$$

$$= \frac{1}{50}\left(25 - 12\cos(\lambda) + 12i\sin(\lambda) - 12\cos(\lambda) - 12i\sin(\lambda)\right)$$

$$|\langle -\hat{x}|\psi(\lambda)\rangle|^2 = \frac{1}{50}\left(25 - 24\cos(\lambda)\right)$$

$$\sum_{\substack{\text{All possible}\\ \text{outcomes}}}^{\infty} \frac{1}{\text{Prob(outcome)}}\left(\frac{\partial\,\text{Prob(outcome)}}{\partial P}\right)^2 = F$$

$$F_+ : P_+ = \frac{1}{50}\left(25 + 24\cos(\lambda)\right) \; : \; \frac{\partial P_+}{\partial \lambda} = -\frac{24}{50}\sin(\lambda) \; : \; \left(\frac{\partial P_+}{\partial \lambda}\right)^2 = \frac{144}{625}\sin^2(\lambda)$$

$$F_+ = \frac{50}{25 + 24\cos(\lambda)}\left(\frac{144}{625}\sin^2(\lambda)\right) = \frac{2}{(25 + 24\cos(\lambda))}\left(\frac{144}{25}\sin^2(\lambda)\right)$$

$$F_- : P_- = \frac{1}{50}\left(25 - 24\cos(\lambda)\right) \; : \; \frac{\partial P_-}{\partial \lambda} = \frac{24}{50}\sin(\lambda) \; : \; \left(\frac{\partial P_-}{\partial \lambda}\right)^2 = \frac{144}{625}\sin^2(\lambda)$$

$$F_- = \frac{50}{25 - 24\cos(\lambda)}\left(\frac{144}{625}\sin^2(\lambda)\right) = \frac{2}{25 - 24\cos(\lambda)}\left(\frac{144}{25}\sin^2(\lambda)\right)$$

$$F = F_+ + F_- = \frac{2}{(25 + 24\cos(\lambda))}\left(\frac{144}{25}\sin^2(\lambda)\right) + \frac{2}{25 - 24\cos(\lambda)}\left(\frac{144}{25}\sin^2(\lambda)\right)$$

$$= \frac{288}{25}\,\frac{\sin^2(\lambda)}{(25 + 24\cos(\lambda))} + \frac{288}{25}\,\frac{\sin^2(\lambda)}{(25 - 24\cos(\lambda))}$$

$$= \frac{288}{25}\left(\frac{25\sin^2(\lambda) - 24\cos(\lambda)\sin^2(\lambda) + 25\sin^2(\lambda) + 24\cos(\lambda)\sin^2(\lambda)}{625 - 576\cos^2(\lambda)}\right)$$

$$= \frac{288}{25}\left(\frac{50\sin^2(\lambda)}{625 - 576\cos^2(\lambda)}\right) = \frac{576\sin^2(\lambda)}{625 - 576\cos^2(\lambda)}$$

$$\boxed{F_x = \frac{576\sin^2(\lambda)}{625 - 576\cos^2(\lambda)}}$$

Exercise 2

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a.)

i.) $\hat{\sigma}_x^2 = \hat{\sigma}_x \cdot \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1(1) & 0(1) + 1\cdot 0 \\ 1\cdot 0 + 0(1) & 1(1) + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\hat{\sigma}_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

ii.) $\hat{\sigma}_y^2 = \hat{\sigma}_y \cdot \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 - i(i) & 0(-i) - i(0) \\ i\cdot 0 + 0(i) & -i(i) + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\hat{\sigma}_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

iii.) $\hat{\sigma}_z^2 = \hat{\sigma}_z \cdot \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1(1) + 0\cdot 0 & 1(0) - (0)1 \\ 0(1) - 1(0) & 0\cdot 0 - 1(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\hat{\sigma}_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \hat{I}}$$

b.)

i.) $\hat{\sigma}_x \hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1(i) & 0(-i) + 1(0) \\ 1\cdot 0 + 0(i) & -1(i) + 0(0) \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hat{\sigma}_z$

$$\boxed{\hat{\sigma}_x \hat{\sigma}_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\hat{\sigma}_z}$$

ii.) $\hat{\sigma}_y \hat{\sigma}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0(1) - i(0) & 0(0) + i(1) \\ i(1) + i(0) & i(0) - 0(1) \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hat{\sigma}_x$

$$\boxed{\hat{\sigma}_y \hat{\sigma}_z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\hat{\sigma}_x}$$

iii.) $\hat{\sigma}_z \hat{\sigma}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1(0) + 0(1) & 1(1) + 0\cdot 0 \\ 0\cdot 0 - 1(1) & 0(1) - 1\cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y$$

$$\boxed{\hat{\sigma}_z \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_y}$$

Exercise 2 Continued

c.)

i.) $\hat{\sigma}_x \hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + 1(i) & 0(-i)+1(0) \\ 1(0)+0(i) & -i(1)+0\cdot 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$-\hat{\sigma}_y \hat{\sigma}_x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\cdot 0 + i(1) & 0(1)+i(0) \\ -i\cdot 0+0(1) & -i(1)+0\cdot 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$$\boxed{\hat{\sigma}_x \hat{\sigma}_y = -\hat{\sigma}_y \hat{\sigma}_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}$$

ii.) $\hat{\sigma}_y \hat{\sigma}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0(1) - i(0) & 0\cdot 0 + i(1) \\ i(1)+0(0) & i(0)-1(0) \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$-\hat{\sigma}_z \hat{\sigma}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1(0)+i(0) & (1)i+0\cdot 0 \\ 0\cdot 0+1(i) & -i(0)+(1)0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$\boxed{\hat{\sigma}_y \hat{\sigma}_z = -\hat{\sigma}_z \hat{\sigma}_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}$$

d.)

$\hat{\sigma} = n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z$

$\sigma^2 = (n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z)(n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z) = (n_x^2 + n_y^2 + n_z^2)\hat{I}$

$= n_x^2 \hat{\sigma}_x^2 + n_x n_y \hat{\sigma}_x \hat{\sigma}_y + n_x n_z \hat{\sigma}_x \hat{\sigma}_z + n_y n_x \hat{\sigma}_y \hat{\sigma}_x + n_y^2 \hat{\sigma}_y^2 + n_y n_z \hat{\sigma}_y \hat{\sigma}_z + n_z n_x \hat{\sigma}_z \hat{\sigma}_x + n_z n_y \hat{\sigma}_z \hat{\sigma}_y + n_z^2 \hat{\sigma}_z^2$

$n_x n_y = n_y n_x$, $n_x n_z = n_z n_x$, $n_y n_z = n_z n_y$: $\hat{\sigma}_x \hat{\sigma}_y = -\hat{\sigma}_y \hat{\sigma}_x$, $\hat{\sigma}_x \hat{\sigma}_z = -\hat{\sigma}_z \hat{\sigma}_x$, $\hat{\sigma}_y \hat{\sigma}_z = -\hat{\sigma}_z \hat{\sigma}_y$

$= n_x^2 \hat{I} + n_y^2 \hat{I} + n_z^2 \hat{I} + \cancel{n_x n_y \hat{\sigma}_x \hat{\sigma}_y} - \cancel{n_y n_x \hat{\sigma}_x \hat{\sigma}_y} + \cancel{n_x n_z \hat{\sigma}_x \hat{\sigma}_z} - \cancel{n_z n_x \hat{\sigma}_x \hat{\sigma}_z} + \cancel{n_y n_z \hat{\sigma}_y \hat{\sigma}_z} - \cancel{n_z n_y \hat{\sigma}_z \hat{\sigma}_y}$

$\therefore$

$$\boxed{\sigma^2 = (n_x^2 + n_y^2 + n_z^2)\hat{I}}$$

Exercise 3

a.) $e^{-\frac{i\lambda}{2}\hat{\sigma}_z}$: $e^{-i\lambda\sigma_z} = \cos(\lambda)\hat{I} - i\sin(\lambda)\hat{\sigma}_z$

$$e^{-\frac{i\lambda}{2}\sigma_z} = \begin{pmatrix} \cos(\lambda/2) & 0 \\ 0 & \cos(\lambda/2) \end{pmatrix} - \begin{pmatrix} i\sin(\lambda/2) & 0 \\ 0 & -i\sin(\lambda/2) \end{pmatrix}$$

$$\boxed{e^{-\frac{i\lambda}{2}\sigma_z} = \begin{pmatrix} \cos(\lambda/2) - i\sin(\lambda/2) & 0 \\ 0 & \cos(\lambda/2) + i\sin(\lambda/2) \end{pmatrix}}$$

b.) $e^{-\frac{i\lambda}{2}\hat{\sigma}_x}$: $e^{-i\lambda\sigma_x} = \cos(\lambda)\hat{I} - i\sin(\lambda)\hat{\sigma}_x$

$$e^{-\frac{i\lambda}{2}\sigma_x} = \begin{pmatrix} \cos(\lambda/2) & 0 \\ 0 & \cos(\lambda/2) \end{pmatrix} - \begin{pmatrix} 0 & i\sin(\lambda/2) \\ i\sin(\lambda/2) & 0 \end{pmatrix}$$

$$\boxed{e^{-\frac{i\lambda}{2}\sigma_x} = \begin{pmatrix} \cos(\lambda/2) & -i\sin(\lambda/2) \\ -i\sin(\lambda/2) & \cos(\lambda/2) \end{pmatrix}}$$

c.) $e^{-\frac{i\lambda}{2}\hat{\sigma}_y}$: $e^{-i\lambda\sigma_y} = \cos(\lambda)\hat{I} - i\sin(\lambda)\hat{\sigma}_y$

$$e^{-\frac{i\lambda}{2}\sigma_y} = \begin{pmatrix} \cos(\lambda/2) & 0 \\ 0 & \cos(\lambda/2) \end{pmatrix} - \begin{pmatrix} 0 & -i^2\sin(\lambda/2) \\ i^2\sin(\lambda/2) & 0 \end{pmatrix}$$

$$\boxed{e^{-\frac{i\lambda}{2}\sigma_y} = \begin{pmatrix} \cos(\lambda/2) & -\sin(\lambda/2) \\ \sin(\lambda/2) & \cos(\lambda/2) \end{pmatrix}}$$