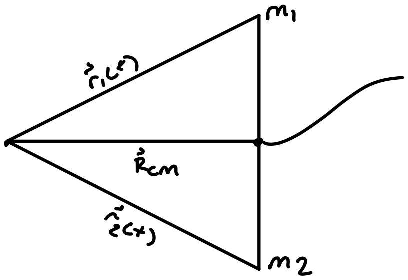
Ch. 8 Orbital Dynamics

10-25-18

Reduced mass 2 body → 1 body



Center of mass → $\vec{r}$

$$\vec{R}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

$\vec{r}_2 + \vec{r} = \vec{r}_1 \qquad \vec{r}_2 = \vec{r}_1 - \vec{r}$

$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad |\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

$$L = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 - u(r)$$

$R_{cm} = m_1(\vec{r}_2 + \vec{r}) + m_2(\vec{r}_2) = 0$

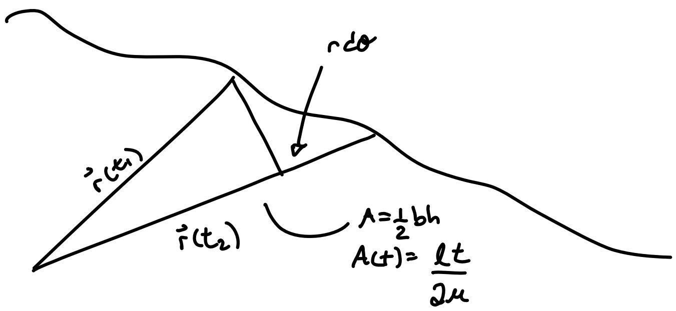
$\frac{(m_1+m_2)}{m_1}\vec{r}_2 = -\vec{r}$, $\vec{r}_2 = -\frac{m_1\vec{r}}{(m_1+m_2)}$, $m_1\vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0$, $\vec{r}_1 = \frac{m_2\vec{r}}{m_1+m_2}$, $\mu = \frac{m_1 m_2}{m_1+m_2}$

$L = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - u(r)$, Solving $\vec{r}(t)$ → gives $\vec{r}_1(t)$, $\vec{r}_2(t)$

$L = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - u(r) = \frac{1}{2}\mu\left[\dot{r}^2 - r^2\dot{\theta}^2\right] - u(r)$ if $\frac{\partial L}{\partial q_i} = 0$ then $P_i =$ Constant

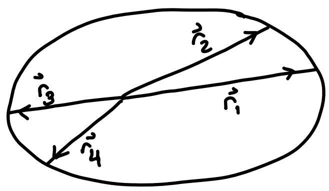
$\dot{P}_\theta = \frac{\partial L}{\partial \theta} = 0$, $\frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$, $\mu r^2\dot{\theta} = \ell$

$\dot{\theta} = \frac{L}{\mu r^2}$



$|\vec{r}_1(t_1)| \approx |\vec{r}_2(t_2)|$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\ell}{2\mu}$$



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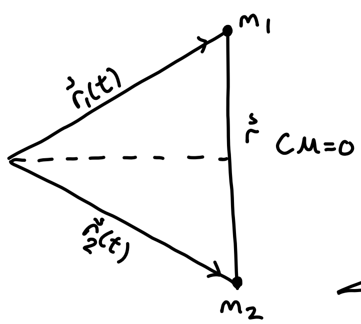
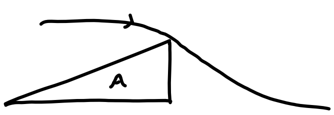


$\vec{r}_2 = -\frac{m_1}{m_1+m_2}\vec{r}$, $\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}$ $\mu = \frac{m_1 m_2}{m_1+m_2}$

$L = \frac{1}{2}\mu(\dot{r}^2 + \dot{r}^2\dot{\theta}^2) - u(r)$

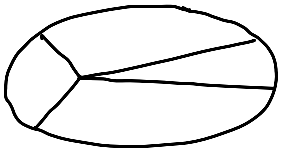
$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\mu r^2\dot{\theta} = 0$

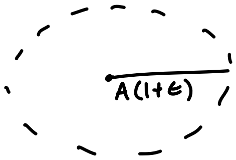


$\frac{dA}{dr} = \frac{1}{2}r^2\dot{\theta}$

→ $\mu r^2\dot{\theta} = \ell$ ← constant

$\dot{\theta} = \frac{\ell}{\mu r^2}$ → orbit





$l = m_e V_{ye}\, a(1+\epsilon)$

$$E = \frac{1}{2}\mu \dot r^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + u(r) \quad , \quad 2(E - u(r)) = \mu \dot r^2 + \frac{l^2}{\mu r^2} \qquad \pm\left(\frac{2(E-u(r))}{\mu} - \frac{l^2}{\mu^2 r^2}\right)^{1/2} = \dot r = \frac{dr}{dT}$$

$$d\theta = \frac{d\theta}{dT}\frac{dT}{dr}dr = \frac{\dot\theta}{\dot r}dr \quad \text{but} \quad \dot\theta = \frac{l}{\mu r^2} \quad \text{so} \quad \frac{\dot\theta}{\dot r}dr = \frac{l}{\mu r^2}\frac{dr}{\dot r} = d\theta \quad , \quad \dot r = \frac{l}{\mu r^2}\frac{dr}{d\theta}$$

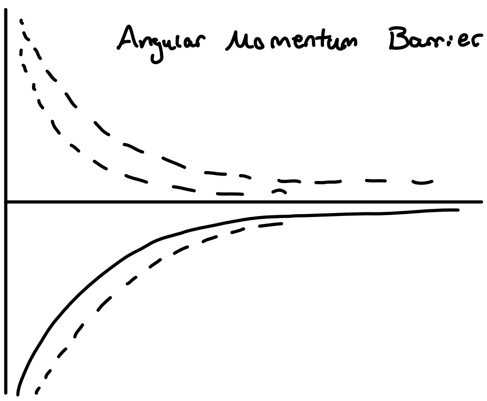
$$\pm\frac{l}{\mu r^2}\int\left[\left(\frac{2(E-u(r))}{\mu}\right) - \frac{l^2}{\mu^2 r^2}\right]^{-1/2} dr = \theta(r) \quad , \quad \pm\int\frac{\frac{l}{r^2}\,dr}{\sqrt{2\mu\left[E-u(r)-\frac{l^2}{2\mu r^2}\right]}} = \theta(r)$$

Only analytic for $\vec F(r) \propto r^n \hat r$ when $n = 1, -2, -3$ spring

$$F = \frac{GM_1M_2}{r^2}\hat r \; : \; u = -\frac{GM_1M_2}{r}$$

$\llcorner \frac{dF}{dr}$

$$E = \frac{1}{2}\mu\dot r^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + u(r) = \frac{1}{2}\mu\dot r^2 + \left[\frac{\alpha}{r^2} - \frac{\kappa}{r}\right]$$



Lagrangians

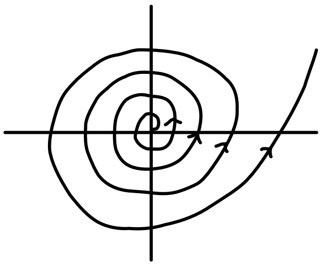
$$L = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - u(r)$$

$$\frac{\partial L}{\partial r} = \mu r\dot\theta^2 - \frac{du}{dr} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot r} = \mu\ddot r \; : \; \mu r\dot\theta^2 - \frac{du}{dr} = \mu \ddot r \; : \; \mu\ddot r - \mu r\dot\theta^2 = -\frac{du(r)}{dr} \; , \; \dot\theta = \frac{l}{\mu r^2} \; , \; V = \frac{1}{r}$$

$$\frac{dV}{d\theta} = \frac{dV}{dr}\frac{dr}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dT}\frac{dT}{d\theta} = -\frac{1}{r^2}\frac{\dot r}{\dot\theta} = -\frac{\mu}{l}\dot r$$

$$\frac{d^2V}{d\theta^2} = \frac{d}{d\theta}\left[-\frac{\mu}{l}\dot r\right] = -\frac{\mu}{l}\frac{d}{d\theta}\left[\frac{dr}{dt}\right] = \frac{dT}{d\theta}\frac{d}{dT}\left(-\frac{\mu}{l}\dot r\right) = \frac{-\mu}{l\dot\theta}\ddot r = \frac{-\mu^2 r^2}{l^2}\ddot r$$

$$\ddot r = \frac{-l^2}{\mu^2}V^2\frac{d^2V}{d\theta^2} \; : \; r\dot\theta^2 = \frac{l^2V^3}{\mu^2} \qquad \frac{d^2V}{d\theta^2} + V = -\frac{\mu}{l^2}\cdot\frac{1}{V^2}\cdot F\left(\frac{1}{V}\right) \quad \text{or} \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{l^2}r^2F(r)$$



$$r = ke^{\alpha\theta}, \quad r^{-1} = \frac{e^{-\alpha\theta}}{k} \quad \frac{d^2}{d\theta^2}(r^{-1}) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

$$F(r) = -\left(\frac{\alpha^2+1}{r^3}\right)\frac{\ell^2}{\mu} = F(r) = \frac{\beta}{r^3} = \beta r^{-3}$$

$$\dot\theta = \frac{\ell}{\mu r^2} = \frac{\ell}{\mu k^2 e^{2\alpha\theta}} = \frac{d\theta}{dt} \rightarrow \frac{\ell}{\mu k^2}\, dT = e^{2\alpha\theta} d\theta, \quad \frac{\ell T}{\mu k^2} = \frac{1}{2\alpha} e^{2\alpha\theta} + C$$

$$\frac{2\alpha\ell T}{\mu k^2} + \underbrace{c'}_{-2\alpha C} = e^{2\alpha\theta} = \theta(t) = \frac{1}{2\alpha}\ln\left[\frac{2\alpha\ell T}{\mu k^3} + c'\right], \quad r^2 = k^2 e^{2\alpha\theta}, \quad \frac{r^2}{k^2} = e^{2\alpha\theta} = \frac{2\alpha\ell T}{\mu k^2} + c^2$$

$$r(T) = \pm\sqrt{\frac{2\alpha l}{\mu}T + D}, \quad u(r) = -\int \vec F \cdot d\vec r = \frac{\ell^2}{\mu}(\alpha^2+1)\int r^{-3}\,dr = \frac{\ell^2}{2\mu r^2}(\alpha^2+1)$$

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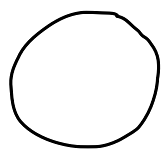
$$\dot r = \pm\sqrt{\frac{2}{\mu}(E - u(r)) - \frac{\ell^2}{\mu^2 r^2}} = 0$$

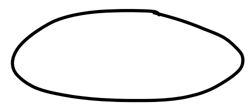
1 root



— Circular orbit

$$\Delta\theta(r) = 2\int_{min}^{max} \frac{\ell/r^2}{\sqrt{2\mu(E-u(r)) - \frac{\ell^2}{\mu^2 r^2}}}\,dr = 2\pi\left(\frac{a}{b}\right)^{\text{Integers}}$$

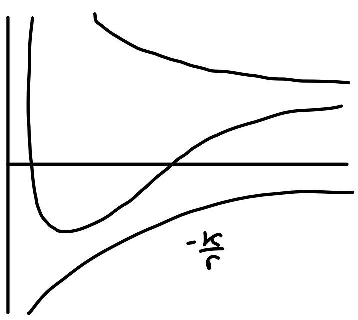
2 root



- Elliptic orbit

$$\frac{1}{2}\mu r^2\dot\theta^2 = \frac{\ell^2}{2\mu r^2} = U_{centripetal}, \quad \frac{-k}{r}$$

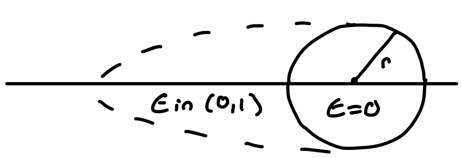
$$\vec F_c = -\vec\nabla u_c = \frac{\ell^2}{2\mu r^2} = \mu r\dot\theta^2, \quad V(r) = u(r) + \frac{\ell^2}{2\mu r^2} = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}$$

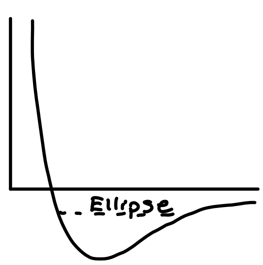


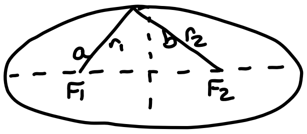
$$\theta(r) = \int \frac{\ell/r^2}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}}\,dr, \quad \cos(\theta) = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}, \quad \text{where } \alpha = \frac{\ell^2}{\mu k}, \quad \epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

Eccentricity

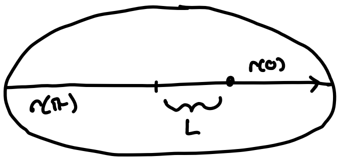
$$\frac{\alpha}{r} = 1 + \epsilon\cos(\theta), \quad r = \frac{\alpha}{1 + \epsilon\cos(\theta)}$$





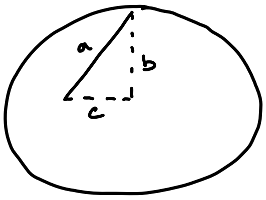


$r_1 + r_2 = 2a$, $\frac{\alpha}{r} = 1+\epsilon\cos(\theta)$, $\epsilon = \left(1+\frac{2E\ell^2}{\mu k^2}\right)^{\frac{1}{2}}$, $\alpha = \frac{\ell^2}{\mu k}$



$r(0) + r(\pi) = 2a$, $r = \frac{\alpha}{1+\epsilon\cos\theta}$, $r(0) = \frac{\alpha}{1+\epsilon}$, $r(\pi) = \frac{\alpha}{1-\epsilon}$

$\frac{\alpha}{1+\epsilon} + \frac{\alpha}{1-\epsilon} = \frac{2\alpha}{1-\epsilon^2} = 2a \quad \therefore \quad a = \frac{k}{2|E|}$



$b = \sqrt{a^2 - c^2} \quad \therefore \quad c = a - r(0) = a - \frac{\alpha}{1+\epsilon} = \alpha\left[\frac{1}{1-\epsilon^2} - \frac{1}{1+\epsilon}\right] \to \boxed{\frac{c}{\alpha} = \frac{\epsilon}{1-\epsilon^2}}$

$b = \frac{\alpha}{\sqrt{1-\epsilon^2}} \quad \therefore \quad \frac{b}{a} = \frac{\alpha}{(1-\epsilon^2)^{\frac{1}{2}}} \frac{(1-\epsilon^2)^1}{\alpha} = \sqrt{1-\epsilon^2}$

Eccentricities

Earth = 0.167 $\quad b = \sqrt{1-(0.167)^2} = 0.999861$

Pluto = 0.25

Halley's Comit = 0.96714

$b = \frac{\alpha}{(1-\epsilon^2)^{\frac{1}{2}}}$, $a = \frac{\alpha}{1-\epsilon^2}$

$r_{min} = \frac{\alpha}{1+\epsilon}$ - Perihelion $= a(1-\epsilon)$, $r_{max} = \frac{\alpha}{1-\epsilon}$ - Aphelion $= a(1+\epsilon)$

$\frac{dA}{dt} = \frac{\ell}{2\mu} \to dt = \frac{2\mu}{\ell} dA \to \int_0^\tau dt = \frac{2\mu}{\ell}\int dA = \frac{2\mu}{\ell}\pi ab$, $\tau = \pi k\sqrt{\frac{\mu}{2}}\,|E|^{-\frac{3}{2}} = \pi a^{\frac{3}{2}}\sqrt{\alpha}$

↑

$\frac{\ell^2}{\mu k}$

$\tau = \frac{2\mu}{\ell}\pi a^{\frac{3}{2}}\sqrt{\alpha}$, $\tau^2 = \frac{4\mu^2}{\ell^2}\pi^2 a^3 \frac{\ell^2}{\mu k} = \frac{4\mu\pi^2 a^3}{k} = \frac{\pi^2 a^3}{m_1+m_2 \cdot G}\cdot 4$

$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$, $\frac{dA}{dt} = \frac{\ell}{2\mu}$, $r = \frac{\alpha}{1+\epsilon\cos\theta}$, $\tau^2 = \frac{(2\pi a)^2}{\bar{v}^2}$, $\bar{v}^2 = \frac{4\pi^2 a^2}{\tau^2} = \frac{GM_1M_2}{a}$, $\bar{v} = \left(\frac{GM_1M_2}{a}\right)^{\frac{1}{2}}$

3rd Law ↑ 2nd Law ↑ 1st Law ↑

$\bar{V} \to r\dot{\theta}$, $\frac{V}{a} = \dot{\theta} = \left(\frac{G(m_1+m_2)}{a^3}\right)^{\frac{1}{2}}$

Keplerian rotation ↑

$\ell = \mu r^2\dot{\theta} = \mu\left[G(m_1+m_2)r\right]^{\frac{1}{2}}$, $\dot{\theta} = \sqrt{\frac{G(m_1+m_2)}{r^3}}$

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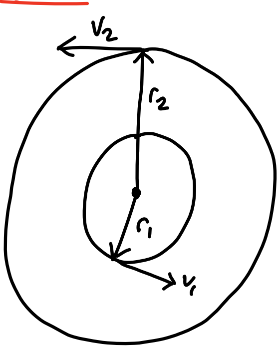


$E = \frac{-k}{2a} = \frac{1}{2}m\vec{v}_1^{\,2} - \frac{k}{r_1} = \frac{-k}{2r_1}$, $\vec{V} = \left(\frac{k}{mr_1}\right)^{\frac{1}{2}}$, $r_1 + r_2 = 2a_T$

at perihelion $E_T = \frac{-k}{r_1+r_2} = \frac{1}{2}mV_{T_1}^2 - \frac{k}{r_1}$

$V_{T_1} = \sqrt{\frac{2k}{mr_1}\left(\frac{r_2}{r_1+r_2}\right)}$

$\Delta V_1 = V_{T_1} - V_1$ Change in V to go from $a = r_1$, $r_1 + r_2 = 2a$

$V_{T2} = \sqrt{\frac{2k}{mr_2}\left(\frac{r_1}{r_1+r_2}\right)}$, Transfer $= \frac{\Upsilon}{2} \rightarrow \frac{m}{k} = \frac{1}{GM_{sun}} = 7.53\times10^{-21}\ \frac{s^2}{m^3}$

$\frac{K}{m} = 1.32\times10^{20}\ \frac{m^3}{s^2}$, $a_T = \frac{1}{2}(r_{E-S} + r_{m-s}) = 1.89\times10^{11}\ m$

$\Upsilon_t = \pi\left(\sqrt{\frac{m}{K}}\right)a_T^{3/2} = 259$ days $\rightarrow$ To get to Mars' orbit

$V = \frac{d}{t} = \frac{1.5\times10^{11}}{\pi\cdot10^7 s}$, $V_1 = 29.8\ \frac{km}{s}$, $V_{T_1} = 32.7\ \frac{km}{s}$ ~ need $2.9\ \frac{km}{s}$

$E_{add} = \frac{m}{2}(\Delta v)^2 = \frac{5000\ kg}{2}(3.1\times10^3\ m/s)^2 = \frac{2.25\times10^{10}}{0.5} \sim 10^{11}\ J$

Gravitational Slingshots

Analogy

ball $\longrightarrow$ $\longleftarrow$ Train

30 mph 50 mph

Train sees 80 mph

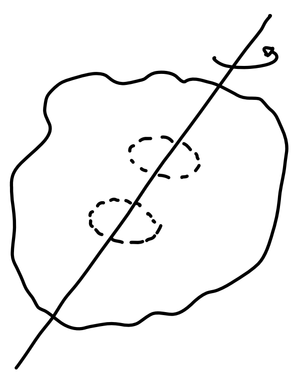
$V_{bf} = \frac{V_b(\cancel{m_b} - m_T) + 2m_TV_T}{\cancel{m_b}+m_T} = -V_b + 2V_T = (30 + 2(50)) = 180$ mph

Venus $= 3.5\times10^4\ \frac{m}{s} = 1.87\times10^{40}\ kg\cdot\frac{m}{s^2}$ distance to Sun $= 1.1\times10^{11}\ m$ $\Delta(\vec{r}\times\vec{p}) = 2mV_p d_{sun}$

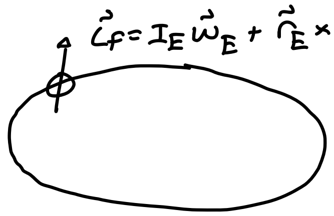
Cassini $= 2.5\times10^3\ kg = 1.9\times10^{14}\ kg\cdot\frac{m^2}{s}$

$\frac{\Delta(L_V)}{L_V} = 1.08\times10^{-21}\ kg\cdot\frac{m}{s^2}$, $E=0$ for Simplicity , $r_{V_{old}} = r_\alpha$, $r_{V_{new}} = \frac{(L-\Delta L)^2}{mk}$

$\frac{r_{new}}{r_{old}}$: $r_{new} = (1-2\cdot10^{-21})$



$\vec{L} = I\vec{\omega}$: $\vec{L} = \sum_{m=1}^{n} \vec{r}_i\times\vec{r}_j$



$\vec{L}_f = I_E\vec{\omega}_E + \vec{r}_E\times\vec{r}_E$

$|\vec{r}_E\times\vec{P}_E| = |M_EV_Ea_n|$

E.C Find $|\vec{L}|$ for typical protostellar System

Compare $\vec{L}_{ss}$ to $\vec{L}_s$

what happens to Sun if all $\vec{L}_{ss} \rightarrow$ into Sun

Binding Energy $= \frac{3}{5}\frac{GM^2}{R}$, $R = 6400\ km$

$$BE \approx \frac{6.7\times10^{-11}\cdot(2\cdot10^{30})^2}{6.4\times10^5} \approx \frac{42.4\times10^{49}}{6.4\times10^5} = 2.8\times10^{44}\ J$$

$$\vec{L}_m = I_m\vec{\omega}_m + \vec{r}_m\times\vec{P}_m$$

we slow down, moon moves outward

$$\phi\ \vec{L}_E = I_E\vec{\omega}_E \qquad \vec{L}_{E+M} = \vec{L}_E + \vec{L}_M = C$$

$$\vec{\omega}_E(T)\downarrow$$

$$\vec{E} = -\frac{m_m k^2}{2L_m^2} + \frac{(\overset{L+S}{J}-\overset{\text{orbit}}{L})^2}{2I} \qquad \frac{dE}{dL_m} = \frac{m_m k^2}{L^3} - \frac{(J-L)}{I} = 0 \rightarrow \frac{L}{mr^2} = \frac{S}{I}$$

$\dot{\omega}_{Earth} \sim 0.35\cdot10^{-21}$ rad/s , day ↑ 4.4×10^{-8} s/day

$\dot{r}_{moon} \sim 0.4$ cm/month , Tides stop at $r = 1.44\, r_0$

└ 1 meter = 250 month