PHYS 396 HOMEWORK SET 1

1a) $[G] = \frac{Nm^2}{kg^2}$, $[M] = kg$, $[R] = m$, $\therefore [c] = \frac{m}{s}$

so

$$\frac{[G][M]}{[R][c]^2} = \frac{\frac{N\cancel{m^2}}{kg^{\cancel{2}}} \cdot \cancel{kg}}{m \cdot \frac{\cancel{m^2}}{s^2}} = \frac{N}{kg\frac{m}{s^2}} = \frac{kg\,m/s^2}{kg\,m/s^2} = 1$$

SO $\frac{GM}{Rc^2}$ IS A DIMENSIONLESS QUANTITY

b) NOW $M_\oplus = 5.98 \times 10^{24}\,kg$, $R_\oplus = 6.37 \times 10^6\,m$ ...

i) $$\frac{GM_\oplus}{R_\oplus c^2} = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(5.98\times10^{24}\,kg)}{(6.37\times10^6\,m)(2.998\times10^8\,m/s)^2} = 0.697\times10^{-9} = 6.97\times10^{-10}$$

WHEREAS $M_\odot = 1.99\times10^{30}\,kg$, $R_\odot = 6.96\times10^8\,m$ ...

ii) $$\frac{GM_\odot}{R_\odot c^2} = \frac{(6.67\times10^{-11}\,Nm^2/kg^2)(1.99\times10^{30}\,kg)}{(6.96\times10^8\,m)(2.998\times10^8\,m/s)^2} = 2.12\times10^{-6}$$

c) i) $$R_{S,\oplus} = \frac{2GM_\oplus}{c^2} = \frac{2(6.67\times10^{-11}\,Nm^2/kg^2)(5.98\times10^{24}\,kg)}{(2.998\times10^8\,m/s)^2} = 8.88\times10^{-3}\,m$$

ii) $$R_{S,\odot} = \frac{2GM_\odot}{c^2} = \frac{2(6.67\times10^{-11}\,Nm^2/kg^2)(1.99\times10^{30}\,kg)}{(2.998\times10^8\,m/s)^2} = 2{,}950\,m$$

d) $\frac{R_{S,\odot}}{R_\odot} = \frac{2{,}950\,m}{6.96\times10^{8}\,m} = 4.24\times10^{-6}$ $\therefore R_{S,\odot} \ll R_\odot$

$\frac{R_{S,\oplus}}{R_\oplus} = \frac{8.88\times10^{-3}\,m}{6.37\times10^{6}\,m} = 1.39\times10^{-9}$ $\therefore R_{S,\oplus} \ll R_\oplus$

2. $x = r\cos\varphi$

$y = r\sin\varphi$

so

a) $dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi$

$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$

b) so $dx^2 = (\cos\varphi\, dr - r\sin\varphi\, d\varphi)^2 = \cos^2\varphi\, dr^2 + r^2\sin^2\varphi\, d\varphi^2 - 2r\sin\varphi\cos\varphi\, dr\, d\varphi$

$dy^2 = (\sin\varphi\, dr + r\cos\varphi\, d\varphi)^2 = \sin^2\varphi\, dr^2 + r^2\cos^2\varphi\, d\varphi^2 + 2r\sin\varphi\cos\varphi\, dr\, d\varphi$

so

$ds^2 = dx^2 + dy^2 = (\sin^2\varphi + \cos^2\varphi)\, dr^2 + r^2(\sin^2\varphi + \cos^2\varphi)\, d\varphi^2$

$= dr^2 + r^2 d\varphi^2$

3a) $x = a\sin\theta\cos\varphi$

$y = a\sin\theta\sin\varphi$

$z = a\cos\theta$

so

$$dx = a(\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi)$$

$$dy = a(\cos\theta\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\varphi)$$

$$dz = -a\sin\theta\, d\theta$$

b) so $dx^2 = a^2(\cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi)^2$

$$= a^2(\cos^2\theta\cos^2\varphi\, d\theta^2 + \sin^2\theta\sin^2\varphi\, d\varphi^2 - 2\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta d\varphi)$$

$dy^2 = a^2(\cos\theta\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\varphi)^2$

$$= a^2(\cos^2\theta\sin^2\varphi\, d\theta^2 + \sin^2\theta\cos^2\varphi\, d\varphi^2 + 2\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta d\varphi)$$

$dz^2 = a^2\sin^2\theta\, d\theta^2$

so

$$ds^2 = dx^2 + dy^2 + dz^2 = a^2\cos^2\theta\,(\cancel{\sin^2\varphi + \cos^2\varphi})^{\to 1}\, d\theta^2 + a^2\sin^2\theta\,(\cancel{\sin^2\varphi + \cos^2\varphi})^{\to 1}\, d\varphi^2 + a^2\sin^2\theta\, d\theta^2$$

$$= a^2(\cancel{\sin^2\theta + \cos^2\theta})_{\to 1}\, d\theta^2 + a^2\sin^2\theta\, d\varphi^2$$

$$= a^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad ✓$$

4. $ds^2 = a^2(d\theta^2 + f^2(\theta)\,d\varphi^2)$ WHERE $f(\theta) = \sin\theta\cos^2\theta$

a) FOR $\theta = \text{constant}$, $d\theta = 0$ AND THE LINE ELEMENT BECOMES...

$$ds^2 = a^2\sin^2\theta\cos^4\theta\,d\varphi^2$$

so

$$ds = a\sin\theta\cos^2\theta\,d\varphi$$

b) NOW $C(\theta) = \oint ds\Big|_{\theta=\text{CONSTANT}} = \int_0^{2\pi} a\sin\theta\cos^2\theta\,d\varphi = a\sin\theta\cos^2\theta\int_0^{2\pi} d\varphi$

so

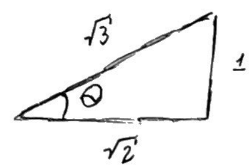
$$\left[C(\theta) = 2\pi a\sin\theta\cos^2\theta\right]$$

c) TO FIND MAXIMUM OR MINIMUM CIRCUMFERENCE, SET $\frac{dC(\theta)}{d\theta} = 0$: SOLVE FOR θ...

$$\frac{dC(\theta)}{d\theta} = 2\pi a\frac{d}{d\theta}(\sin\theta\cos^2\theta) = 2\pi a\left[\cos\theta\cdot\cos^2\theta + \sin\theta\cdot 2\cos\theta\cdot -\sin\theta\right]$$

$$= 2\pi a\cos\theta\left[\cos^2\theta - 2\sin^2\theta\right] \quad \text{NOW } \cos^2\theta = 1 - \sin^2\theta$$

$$= 2\pi a\cos\theta(1 - 3\sin^2\theta) = 0$$



so

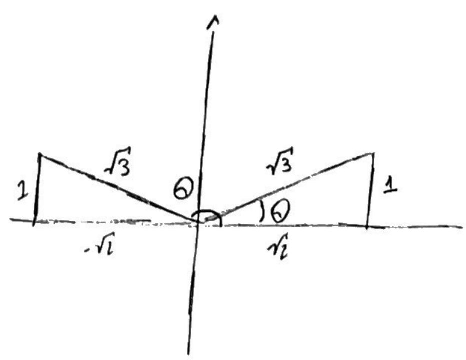
$\cos\theta = 0$ WHEN $\theta = \pi/2$ OR

$1 - 3\sin^2\theta = 0 \;\therefore\; \sin^2\theta = \frac{1}{3} \;\therefore\; \sin\theta = \pm\frac{1}{\sqrt{3}}$

$\therefore$ FOR $\sin\theta = +1/\sqrt{3}$, $\cos\theta = \sqrt{\frac{2}{3}}$

OR

$\cos\theta = -\sqrt{\frac{2}{3}}$



so $\theta \simeq 35.26^\circ$ OR 144.74°

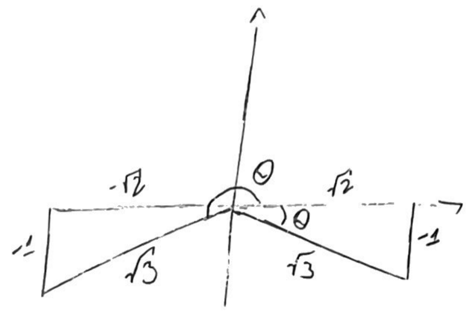
ALSO

$\therefore$ for $\sin\theta = -\frac{1}{\sqrt{3}}$, $\cos\theta = \sqrt{\frac{2}{3}}$

or

$\cos\theta = -\sqrt{\frac{2}{3}}$

so $\theta = -35.26°$ or $215.26°$



NOTICE THAT THESE ANGLES DO YIELD $\frac{dC(\theta)}{d\theta} = 0$, HOWEVER, THE DOMAIN OF θ IS $0 \leq \theta \leq \pi$

SO, THE POSSIBLE SOLUTIONS ARE...

1) $\theta = \pi/2$ $\therefore$ $C(\theta = \pi/2) = 2\pi a \sin(\pi/2)\cos^2(\pi/2) = 0$

2) $\theta = 35.26°$ $\therefore$ $C(\theta = 35.26°) = 2\pi a \sin(35.26°)\cos^2(35.26°)$

$$= 2\pi a \cdot \frac{1}{\sqrt{3}} \cdot \left(\sqrt{\frac{2}{3}}\right)^2 = 2\pi a \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3}$$

3) $\theta = 144.74°$ $\therefore$ $C(\theta = 144.74°) = 2\pi a \sin(144.74°)\cos^2(144.74°)$

$$= 2\pi a \cdot \frac{1}{\sqrt{3}} \cdot \left(-\sqrt{\frac{2}{3}}\right)^2 = 2\pi a \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3}$$

SO IN SUMMARY...

$C(\theta = \pi/2) = 0$ — A MINIMUM

$C(\theta = 35.26°) = C(\theta = 144.74°) = \frac{4\pi a}{3\sqrt{3}}$ — A MAXIMUM

5. THE LINE ELEMENT ON THE SURFACE OF A SPHERE OF RADIUS a IS OF THE FORM...

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad \text{WHERE} \qquad 0 \le \theta \le \pi \;;\; 0 \le \varphi \le 2\pi$$

LET $\lambda = \theta - \pi/2 \;\therefore\; d\lambda = d\theta$ SO $-\pi/2 \le \lambda \le \pi/2$

WHERE $\sin\theta = \sin(\lambda + \pi/2) = \sin\lambda \underbrace{\cos(\pi/2)}_{0} + \underbrace{\sin(\pi/2)}_{1}\cos\lambda = \cos\lambda$

SO

$$\sin^2\theta = \cos^2\lambda$$

SO

(*)

$$ds^2 = a^2(d\lambda^2 + \cos^2\lambda\, d\varphi^2)$$

AS WAS DONE IN THE HANDOUT, WE'LL DO A LINEAR MAPPING OF φ TO x VIA

$$\varphi = \frac{2\pi}{L}x$$

SO $\varphi = 0$ MAPS TO $x = 0$ AND $\varphi = 2\pi$ MAPS TO $x = L$

SO

$$d\varphi = \frac{2\pi}{L}dx$$

ALSO, WE'LL SET

$\lambda = \lambda(y)$ (THE LATITUDINAL COORDINATE IS ONLY A FUNCTION OF y)

SO

$$d\lambda = \frac{d\lambda}{dy}dy$$

PLUGGING THESE INTO (*)...

(**)

$$ds^2 = a^2\left(\left(\frac{d\lambda}{dy}\right)^2 dy^2 + \frac{4\pi^2}{L^2}\cos^2[\lambda(y)]\, dx^2\right)$$

NOW

a) $dA_{MAP} = dx\,dy$

b) $dA_{earth} = a\dfrac{d\lambda}{dy}dy \cdot \dfrac{2\pi}{L}a\cos[\lambda(y)]\,dx = \dfrac{2\pi}{L}a^2\dfrac{d\lambda}{dy}\cos[\lambda(y)]\,dx\,dy$

c) IF $dA_{earth} = dA_{MAP}$ THEN

$$\frac{2\pi}{L} a^2 \frac{d\lambda}{dy} \cos[\lambda(y)] = 1$$

a, BY SEPARATING VARIABLES...

d)
$$\frac{2\pi}{L} a^2 \int \cos(\lambda)\, d\lambda = \int dy$$

$$\frac{2\pi a^2}{L} \sin(\lambda) = y + \text{CONSTANT}$$

$\lambda = 0$ YIELDS $y = 0$ WHEN CONSTANT $= 0$

SO

$$\left[y(\lambda) = \frac{2\pi}{L} a^2 \sin(\lambda) \right] \quad \text{SO} \quad \left[\lambda(y) = \sin^{-1}\left(\frac{1}{2\pi} \cdot \frac{yL}{a^2} \right) \right]$$

e) NOW
$$\frac{d\lambda}{dy} = \frac{d}{dy} \sin^{-1}\left(\frac{1}{2\pi} \cdot \frac{yL}{a^2} \right) = \frac{1}{\sqrt{1 - \left(\frac{yL}{2\pi a^2} \right)^2}} \cdot \frac{1}{2\pi} \frac{L}{a^2}$$

PLUGGING THIS INTO (**) YIELDS...

$$ds^2 = a^2 \left[\frac{1}{\left(1 - \frac{y^2 L^2}{4\pi^2 a^4} \right)} \cdot \frac{L^2}{4\pi^2 a^4} dy^2 + \frac{4\pi^2}{L^2} \left(1 - \sin^2[\lambda(y)] \right) dx^2 \right]$$

BUT $\sin^2[\lambda(y)] = \frac{y^2 L^2}{4\pi^2 a^4}$

$$= a^2 \left[\frac{1}{\left(1 - \frac{y^2 L^2}{4\pi^2 a^4} \right)} \frac{L^2}{4\pi^2 a^4} dy^2 + \frac{4\pi^2}{L^2} \left(1 - \frac{y^2 L^2}{4\pi^2 a^4} \right) dx^2 \right]$$

$$= a^2 \left[\frac{1}{\left(\frac{4\pi^2 a^4}{L^2} - y^2 \right)} dy^2 + \frac{1}{a^4} \left(\frac{4\pi^2 a^4}{L^2} - y^2 \right) dx^2 \right]$$

WHERE
$$f(y) \equiv \frac{a}{\sqrt{4\pi^2 a^4 / L^2 - y^2}}$$

$$= \left[\frac{1}{a^2} \left(\frac{4\pi^2 a^4}{L^2} - y^2 \right) dx^2 + \frac{a^2}{\left(\frac{4\pi^2 a^4}{L^2} - y^2 \right)} dy^2 \right] = \frac{1}{f^2(y)} dx^2 + f^2(y) dy^2$$