Exercise 1

$$\hat{\rho}_i = \frac{1}{2}(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z) = \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + r_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + r_y\begin{pmatrix}0&-i\\i&0\end{pmatrix} + r_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&r_x\\r_x&0\end{pmatrix} + \begin{pmatrix}0&-ir_y\\ir_y&0\end{pmatrix} + \begin{pmatrix}r_z&0\\0&-r_z\end{pmatrix}\right) \quad \therefore \quad \hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

$$\hat{U}\hat{\rho}_i\hat{U}^\dagger = \begin{pmatrix}e^{-i\lambda/2} & 0\\ 0 & e^{i\lambda/2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\begin{pmatrix}e^{i\lambda/2} & 0\\ 0 & e^{-i\lambda/2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\lambda/2} & 0\\ 0 & e^{i\lambda/2}\end{pmatrix}\begin{pmatrix}e^{i\lambda/2}(1+r_z) & e^{-i\lambda/2}(r_x-ir_y)\\ e^{i\lambda/2}(r_x+ir_y) & e^{-i\lambda/2}(1-r_z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+r_z & e^{-i\lambda}(r_x-ir_y)\\ e^{i\lambda}(r_x+ir_y) & 1-r_z\end{pmatrix}$$

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

a.) $\hat{\rho}_f = (1-\lambda)\hat{\rho}_i + \lambda\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z$

$$\hat{\rho}_f = \frac{(1-\lambda)}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix} + \lambda\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \frac{(1-\lambda)}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1+r_z & ir_y-r_x\\ r_x+ir_y & r_z-1\end{pmatrix}$$

$$= \frac{(1-\lambda)}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix}1+r_z & ir_y-r_x\\ -r_x-ir_y & 1-r_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}(1-\lambda)(1+r_z) & (1-\lambda)(r_x-ir_y)\\ (1-\lambda)(r_x+ir_y) & (1-\lambda)(1-r_z)\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\lambda(1+r_z) & \lambda(ir_y-r_x)\\ -\lambda(r_x+ir_y) & \lambda(1-r_z)\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}(1-\lambda)(1+r_z)+\lambda(1+r_z) & (1-\lambda)(r_x-ir_y)-\lambda(r_x-ir_y)\\ (1-\lambda)(r_x+ir_y)-\lambda(r_x+ir_y) & (1-\lambda)(1-r_z)+\lambda(1-r_z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}(1+r_z) & (1-2\lambda)(r_x-ir_y)\\ (1-2\lambda)(r_x+ir_y) & (1-r_z)\end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\begin{pmatrix}(1+r_z) & (1-2\lambda)(r_x-ir_y)\\ (1-2\lambda)(r_x+ir_y) & (1-r_z)\end{pmatrix}}$$

b.) Find $\langle\sigma_{x_i}\rangle, \langle\sigma_{y_i}\rangle, \langle\sigma_{z_i}\rangle$

i.) $\langle\sigma_{x_i}\rangle = Tr(\hat{\rho}_i\hat{\sigma}_x) = Tr\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$

$$= Tr\left(\frac{1}{2}\begin{pmatrix}r_x-ir_y & 1+r_z\\ 1-r_z & r_x+ir_y\end{pmatrix}\right) = \frac{1}{2}(r_x - \cancel{ir_y} + r_x + \cancel{ir_y}) = r_x$$

$$\boxed{\langle\sigma_{x_i}\rangle = r_x}$$

Exercise 1 Continued

ii.) $\langle \sigma_{y_i} \rangle = Tr(\hat{\rho}_i \cdot \hat{\sigma}_y) = Tr\left(\frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1-r_z \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix} r_y + i r_x & -i(1+r_z) \\ i(1-r_z) & r_y - i r_x \end{pmatrix}\right) = \frac{1}{2}\left(r_y + \cancel{i r_x} + r_y - \cancel{i r_x}\right) = r_y$

$$\boxed{\langle \sigma_{y_i} \rangle = r_y}$$

iii.) $\langle \sigma_{z_i} \rangle = Tr(\hat{\rho}_i \cdot \hat{\sigma}_z) = Tr\left(\frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1-r_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix} 1+r_z & i r_y - r_x \\ r_x + i r_y & r_z - 1 \end{pmatrix}\right) = \frac{1}{2}\left(\cancel{1} + r_z + r_z - \cancel{1}\right) = r_z$

$$\boxed{\langle \sigma_{z_i} \rangle = r_z}$$

c.) Find $\langle \sigma_{xf} \rangle, \langle \sigma_{yf} \rangle, \langle \sigma_{zf} \rangle$

i.) $\langle \sigma_{xf} \rangle = Tr(\hat{\rho}_f \cdot \hat{\sigma}_x) = Tr\left(\frac{1}{2}\begin{pmatrix} (1+r_z) & (1-2\lambda)(r_x - i r_y) \\ (1-2\lambda)(r_x + i r_y) & (1-r_z) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix} (1-2\lambda)(r_x - i r_y) & (1+r_z) \\ (1-r_z) & (1-2\lambda)(r_x + i r_y) \end{pmatrix}\right) = \frac{1}{2}\left((1-2\lambda)(r_x - \cancel{i r_y} + r_x + \cancel{i r_y})\right)$

$$\boxed{\langle \sigma_{xF} \rangle = r_x(1-2\lambda)}$$

ii.) $\langle \sigma_{yf} \rangle = Tr(\hat{\rho}_f \cdot \hat{\sigma}_y) = Tr\left(\frac{1}{2}\begin{pmatrix} (1+r_z) & (1-2\lambda)(r_x - i r_y) \\ (1-2\lambda)(r_x + i r_y) & (1-r_z) \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$

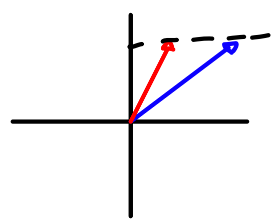
$= Tr\left(\frac{1}{2}\begin{pmatrix} i(1-2\lambda)(r_x - i r_y) & -i(1+r_z) \\ i(1-r_z) & -i(1-2\lambda)(r_x + i r_y) \end{pmatrix}\right) = \frac{1}{2}\left(i(1-2\lambda)(\cancel{r_x} - i r_y - i r_y - \cancel{r_x})\right) = r_y(1-2\lambda)$

$$\boxed{\langle \sigma_{yF} \rangle = r_y(1-2\lambda)}$$

iii.) $\langle \sigma_{zF} \rangle = Tr(\hat{\rho}_f \cdot \hat{\sigma}_z) = Tr\left(\frac{1}{2}\begin{pmatrix} (1+r_z) & (1-2\lambda)(r_x - i r_y) \\ (1-2\lambda)(r_x + i r_y) & (1-r_z) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

$= Tr\left(\frac{1}{2}\begin{pmatrix} (1+r_z) & (1-2\lambda)(i r_y - r_x) \\ (1-2\lambda)(r_x + i r_y) & (r_z - 1) \end{pmatrix}\right) = \frac{1}{2}\left(\cancel{1} + r_z + r_z - \cancel{1}\right) = r_z$

$$\boxed{\langle \sigma_{zF} \rangle = r_z}$$



Exercise 2

$$\hat{\rho}_i = \frac{1}{2}\left(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right) = \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + r_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + r_y\begin{pmatrix}0&-i\\i&0\end{pmatrix} + r_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&r_x\\r_x&0\end{pmatrix} + \begin{pmatrix}0&-i\cdot r_y\\i\cdot r_y&0\end{pmatrix} + \begin{pmatrix}r_z&0\\0&-r_z\end{pmatrix}\right) \therefore \hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

$$\hat{U}\hat{\rho}_i\hat{U}^\dagger = \begin{pmatrix}e^{-i\lambda/2}&0\\0&e^{i\lambda/2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\begin{pmatrix}e^{i\lambda/2}&0\\0&e^{-i\lambda/2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\lambda/2}&0\\0&e^{i\lambda/2}\end{pmatrix}\begin{pmatrix}e^{i\lambda/2}(1+r_z) & e^{-i\lambda/2}(r_x-ir_y)\\ e^{i\lambda/2}(r_x+ir_y) & e^{-i\lambda/2}(1-r_z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+r_z & e^{-i\lambda}(r_x-ir_y)\\ e^{i\lambda}(r_x+ir_y) & 1-r_z\end{pmatrix}$$

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

a.) $\hat{\rho}_F = \dfrac{(1-\lambda)}{2}\,\mathrm{Tr}[\hat{\rho}_i]\,\hat{I} + \lambda\hat{\rho}_i$

$$\hat{\rho}_F = \frac{(1-\lambda)}{2}\,\mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix} + \lambda\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}$$

$$= \frac{(1-\lambda)}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1-\lambda&0\\0&1-\lambda\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\lambda(1+r_z) & \lambda(r_x-ir_y)\\ \lambda(r_x+ir_y) & \lambda(1-r_z)\end{pmatrix}$$

$$\boxed{\hat{\rho}_F = \frac{1}{2}\begin{pmatrix}(1-\lambda)+\lambda(1+r_z) & \lambda(r_x-ir_y)\\ \lambda(r_x+ir_y) & (1-\lambda)+\lambda(1-r_z)\end{pmatrix}}$$

b.) Find $\langle\sigma_{x_i}\rangle, \langle\sigma_{y_i}\rangle, \langle\sigma_{z_i}\rangle$

i.) $$\langle\sigma_{x_i}\rangle = \mathrm{Tr}(\hat{\rho}_i\hat{\sigma}_x) = \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$

$$= \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix}r_x-ir_y & 1+r_z\\ 1-r_z & r_x+ir_y\end{pmatrix}\right) = \frac{1}{2}\left(r_x-\cancel{ir_y}+r_x+\cancel{ir_y}\right) = r_x$$

$$\boxed{\langle\sigma_{x_i}\rangle = r_x}$$

ii.) $$\langle\sigma_{y_i}\rangle = \mathrm{Tr}(\hat{\rho}_i\hat{\sigma}_y) = \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix}1+r_z & r_x-ir_y\\ r_x+ir_y & 1-r_z\end{pmatrix}\cdot\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)$$

$$= \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix}r_y+ir_x & -i(1+r_z)\\ i(1-r_z) & r_y-ir_x\end{pmatrix}\right) = \frac{1}{2}\left(r_y+\cancel{ir_x}+r_y-\cancel{ir_x}\right) = r_y$$

$$\boxed{\langle\sigma_{y_i}\rangle = r_y}$$

Exercise 2 Continued

iii.) $\langle \sigma_{z_i} \rangle = Tr(\hat{\rho}_i \cdot \hat{\sigma}_z) = Tr\left(\frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

$$= Tr\left(\frac{1}{2}\begin{pmatrix} 1+r_z & ir_y - r_x \\ r_x + ir_y & r_z - 1 \end{pmatrix}\right) = \frac{1}{2}(\cancel{1} + r_z + r_z - \cancel{1}) = r_z$$

$$\boxed{\langle \sigma_{z_i} \rangle = r_z}$$

c.) Find $\langle \sigma_{xf} \rangle, \langle \sigma_{yf} \rangle, \langle \sigma_{zf} \rangle$

i.) $\langle \sigma_{xf} \rangle = Tr(\hat{\rho}_f \cdot \hat{\sigma}_x) = Tr\left(\frac{1}{2}\begin{pmatrix} (1-\lambda) + \lambda(1+r_z) & \lambda(r_x - ir_y) \\ \lambda(r_x + ir_y) & (1-\lambda) + \lambda(1-r_z) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

$$= Tr\left(\frac{1}{2}\begin{pmatrix} \lambda(r_x - ir_y) & (1-\lambda) + \lambda(1+r_z) \\ (1-\lambda) + \lambda(1-r_z) & \lambda(r_x + ir_y) \end{pmatrix}\right) = \frac{1}{2}(\lambda(r_x - \cancel{ir_y}) + \lambda(r_x + \cancel{ir_y})) = \lambda r_x$$

$$\boxed{\langle \sigma_{xf} \rangle = \lambda r_x}$$

ii.) $\langle \sigma_{y_i} \rangle = Tr(\hat{\rho}_i \cdot \hat{\sigma}_y) = Tr\left(\frac{1}{2}\begin{pmatrix} (1-\lambda) + \lambda(1+r_z) & \lambda(r_x - ir_y) \\ \lambda(r_x + ir_y) & (1-\lambda) + \lambda(1-r_z) \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right)$

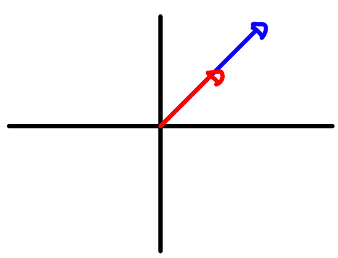
$$= Tr\left(\frac{1}{2}\begin{pmatrix} i\lambda(r_x - ir_y) & -i((1-\lambda) + \lambda(1+r_z)) \\ i((1-\lambda) + \lambda(1-r_z)) & -i\lambda(r_x + ir_y) \end{pmatrix}\right) = \frac{1}{2}(i\lambda(\cancel{r_x} - ir_y) - i\lambda(\cancel{r_x} + ir_y))\ \lambda r_y$$

$$\boxed{\langle \sigma_{yf} \rangle = \lambda r_y}$$

iii.) $\langle \sigma_{zf} \rangle = Tr(\hat{\rho}_f \cdot \hat{\sigma}_z) = Tr\left(\frac{1}{2}\begin{pmatrix} (1-\lambda) + \lambda(1+r_z) & \lambda(r_x - ir_y) \\ \lambda(r_x + ir_y) & (1-\lambda) + \lambda(1-r_z) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

$$= Tr\left(\frac{1}{2}\begin{pmatrix} (1-\lambda) + \lambda(1+r_z) & -\lambda(r_x - ir_y) \\ \lambda(r_x + ir_y) & (\lambda-1) + \lambda(r_z - 1) \end{pmatrix}\right) = \frac{1}{2}((1-\cancel{\lambda}) + \lambda(\cancel{1}+r_z) + (\cancel{\lambda}-1) + \lambda(r_z - \cancel{1})) = \lambda r_z$$

$$\boxed{\langle \sigma_{zf} \rangle = \lambda r_z}$$



$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 + r_z & c.c \\ (r_x \cos\lambda - r_y \sin\lambda) + i\,(r_x \sin\lambda + r_y \cos\lambda) & 1 - r_z \end{pmatrix}$$

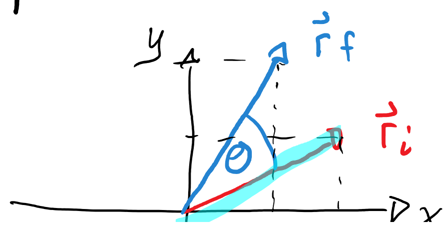
$$= \frac{1}{2}\begin{pmatrix} 1 + r_{fz} & r_{fx} - i\,r_{fy} \\ r_{fx} + i\,r_{fy} & 1 - r_{fz} \end{pmatrix}$$

$$\hat{\rho}_f = \frac{1}{2}\left(\hat{I} + \vec{r}_f \cdot \hat{\vec{\sigma}}\right)$$

$$\boxed{\begin{aligned} r_{fz} &= r_{iz} \\ r_{fx} &= r_{ix}\cos\lambda - r_{iy}\sin\lambda \\ r_{fy} &= r_{ix}\sin\lambda + r_{iy}\cos\lambda \end{aligned}}$$

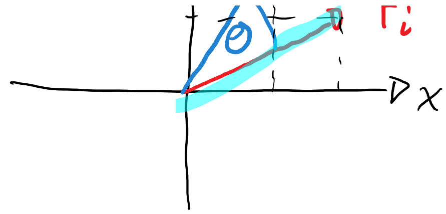
z component constant - rotate about $\hat{z}$

x, y components rotated.

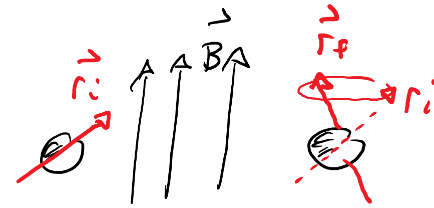


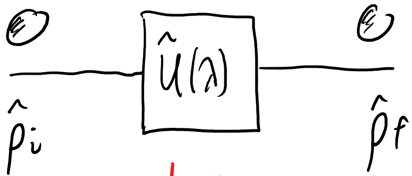
$$r_{fx} = r_{ix}\cos\theta - r_{iy}\sin\theta$$

$$r_{fy} = r_{ix}\sin\theta + r_{iy}\cos\theta$$

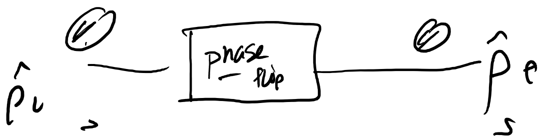


$r_{f_y} = r_{ix}\sin\theta + r_{iy}\cos\theta$





$\vec{r}_i$ phase shift. $\vec{r}_f$ = initial $\vec{r}_i$ rotated through λ about $\hat{z}$



$\vec{r}_i$ $\vec{r}_f = ??$

$$\hat{\rho}_f = (1-\lambda)\hat{\rho}_i + \lambda\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z$$

$$\frac{1}{2}\begin{pmatrix} 1+r_{iz} & \cdots \\ r_{ix}+ir_{iy} & \cdots \end{pmatrix}$$

NMR $\langle\sigma_x\rangle$ $\langle\sigma_y\rangle$

$\underbrace{\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2}_{\text{max strength.}}$ stays constant for phase shift