PHYS 396 HOMEWORK SET 9 SOLUTIONS

1. NOW

$$t = v - r - 2M\ln\left(1 - \frac{r}{2M}\right) \quad \text{for } r<2M$$

$$dt = dv - dr - 2M\cdot\frac{1}{\left(1-\frac{r}{2M}\right)}\cdot -\frac{1}{2M}dr = dv - \left(1 - \frac{1}{1-r/2M}\right)dr$$

$$= dv + \frac{r}{2M}\cdot\frac{1}{\left(1-\frac{r}{2M}\right)}dr\,\frac{2M/r}{2M/r} = dv + \frac{dr}{\left(\frac{2M}{r}-1\right)} = dv - \frac{dr}{\left(1-\frac{2M}{r}\right)}$$

So

$$dt^2 = dv^2 - \frac{2dvdr}{(1-2M/r)} + \frac{dr^2}{(1-2M/r)^2}$$

PLUGGING THIS INTO THE SCHWARZSCHILD LINE ELEMENT YIELDS..

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left[dv^2 - 2\frac{dvdr}{(1-2M/r)} + \frac{\cancel{dr^2}}{\cancel{(1-2M/r)^2}}\right] + \frac{\cancel{1}}{\cancel{(1-2M/r)}}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)$$

$$= -(1-2M/r)dv^2 + 2dvdr + r^2(d\theta^2+\sin^2\theta d\phi^2)$$

2. $ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

a) NOW $v \to v + \text{const}$, LEAVES ds^2 INVARIANT $\therefore$ AN ASSOCIATED KILLING VECTOR $\underset{\sim}{\xi}$

WHERE $\xi^\alpha = (1, 0, 0, 0)$

SO THE FIRST INTEGRAL HAS THE FORM..

$$e \equiv -\underset{\sim}{\xi} \cdot \underset{\sim}{u} = -g_{\alpha\beta}\xi^\alpha u^\beta = -g_{v\beta}\xi^v u^\beta = -g_{vv}u^v - g_{vr}u^r$$

WHERE

$g_{vv} = -\left(1 - \frac{2M}{r}\right)$

$g_{vr} = 1$ (SINCE $g_{vr} + g_{rv} = 2 \;\therefore g_{vr} = g_{rv}$)

$g_{\theta\theta} = r^2$

$g_{\varphi\varphi} = r^2\sin^2\theta$

So

$$\boxed{e = +\left(1 - \frac{2M}{r}\right)\frac{dv}{d\tau} - \frac{dr}{d\tau}}$$

ALSO $\varphi \to \varphi + \text{const}$, LEAVES ds^2 INVARIANT $\therefore$ AN ASSOCIATED KILLING VECTOR $\underset{\sim}{\eta}$

WHERE $\eta^\alpha = (0, 0, 0, 1)$

SO THE FIRST INTEGRAL HAS THE FORM...

$$l \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{\varphi\beta}\eta^\varphi u^\beta = g_{\varphi\varphi}\eta^\varphi u^\varphi = r^2\sin^2\theta\frac{d\varphi}{d\tau}$$

So

$$\boxed{l = r^2\sin^2\theta\frac{d\varphi}{d\tau}}$$

b) SINCE $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$ ...

(*)
$$-1 = g_{\alpha\beta}u^\alpha u^\beta = g_{vv}(u^v)^2 + g_{vr}u^v u^r + g_{rv}u^r u^v + g_{\theta\theta}(u^\theta)^2 + g_{\varphi\varphi}(u^\varphi)^2$$

$$= -\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{d\tau}\right)^2 + 2\frac{dv}{d\tau}\frac{dr}{d\tau} + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

but $\theta = \pi/2 \;\therefore \frac{d\theta}{d\tau} = 0$ AND (*) BECOMES

(**)
$$-1 = -\left(1-\frac{2M}{r}\right)\left(\frac{dv}{d\tau}\right)^2 + 2\frac{dv}{d\tau}\frac{dr}{d\tau} + r^2\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\frac{dv}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1}\left(e+\frac{dr}{d\tau}\right) \quad \text{so} \quad \left(\frac{dv}{d\tau}\right)^2 = \left(1-\frac{2M}{r}\right)^{-2}\left(e^2+2e\frac{dr}{d\tau}+\left(\frac{dr}{d\tau}\right)^2\right)$$

$$\frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

PLUGGING THESE INTO (**) YIELDS...

$$-1 = -\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-2}\left(e^2+2e\frac{dr}{d\tau}+\left(\frac{dr}{d\tau}\right)^2\right) + 2\left(1-\frac{2M}{r}\right)^{-1}\left(e+\frac{dr}{d\tau}\right)\frac{dr}{d\tau} + r^2\cdot\frac{\ell^2}{r^4}$$

$$= -\left(1-\frac{2M}{r}\right)^{-1}\left(e^2+\cancel{2e\frac{dr}{d\tau}}+\left(\frac{dr}{d\tau}\right)^2\right) + \cancel{2\left(1-\frac{2M}{r}\right)^{-1}e\frac{dr}{d\tau}} + 2\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2}$$

$$= -\left(1-\frac{2M}{r}\right)^{-1}\left(e^2+\left(\frac{dr}{d\tau}\right)^2-2\left(\frac{dr}{d\tau}\right)^2\right)+\frac{\ell^2}{r^2}$$

or

$$-\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right) = -\left(e^2-\left(\frac{dr}{d\tau}\right)\right)^2$$

or

$$e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)$$

NOW DEFINE $\mathcal{E} \equiv \frac{1}{2}(e^2-1)$

SO

$$\mathcal{E} = \frac{1}{2}(e^2-1) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)-1\right]$$

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\cancel{1}-\frac{2M}{r}+\frac{\ell^2}{r^2}-\frac{2M\ell^2}{r^3}\cancel{-1}\right]$$

or

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad \text{WHERE} \quad V_{eff}(r) = -\frac{M}{r}+\frac{\ell^2}{2r^2}-\frac{M\ell^2}{r^3}$$

3. IN PROBLEM 1 WE SHOWED THAT...

$$dt = dv - \frac{dr}{\left(1 - \frac{2M}{r}\right)} \qquad \therefore \quad \frac{dt}{d\tau} = \frac{dv}{d\tau} - \frac{1}{\left(1 - \frac{2M}{r}\right)} \frac{dr}{d\tau}$$

NOW EQ (9) IS OF THE FORM..

$$e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)\left[\frac{dv}{d\tau} - \frac{1}{\left(1 - \frac{2M}{r}\right)} \frac{dr}{d\tau}\right]$$

$$= \left(1 - \frac{2M}{r}\right) \frac{dv}{d\tau} - \frac{dr}{d\tau} \checkmark$$ WHICH IS THAT THAT WAS FOUND IN PROBLEM 2a

$l = r^2 \sin^2\theta \frac{d\phi}{d\tau} \checkmark$ OF EQN (10) IS THAT THAT WAS FOUND IN PROBLEM 2a

4. $ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

FOR A RADIAL NULL RAY, $ds^2 = 0$: $d\theta = d\phi = 0$

THE LINE ELEMENT TAKES THE FORM

a) $0 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr$ WHICH HAS SOLUTIONS 1) $\frac{dv}{dr} = 0$

2) $\frac{dv}{dr} = \frac{2}{\left(1 - \frac{2M}{r}\right)}$ AND 3) $r = 2M$

LOOKING AT THE SOLUTION FOR OUTGOING RADIAL LIGHT RAYS..

$$\frac{dv}{dr} = \frac{2}{\left(1 - \frac{2M}{r}\right)} \quad \therefore \int dv = 2\int \frac{dr}{\left(1 - \frac{2M}{r}\right)}$$

WHICH YIELDS..

$$v(r) = C + 2\int \frac{dr}{\left(1 - \frac{2M}{r}\right)} = C + 2\int \frac{r\,dr}{(r - 2M)} = C + 2\int \frac{(u + 2M)\,du}{u}$$

now let $u = r - 2M$: $r = u + 2M$

$\therefore du = dr$

$$= C + 2\int \left(1 + \frac{2M}{u}\right)du = C + 2\left[u + 2M\ln u\right] = C + 2\left[(r - 2M) + 2M\ln(r - 2M)\right]$$

NOW SUBTRACTING THE BRACKETED TERM TO THE OTHER SIDE...

$$v(r) - 2\left(r + 2M\ln\left[2M\left(\frac{r}{2M} - 1\right)\right]\right) = C - 4M \quad \text{now} \quad \ln\left[2M\left(\frac{r}{2M} - 1\right)\right] = \ln(2M) + \ln\left(\frac{r}{2M} - 1\right)$$

so

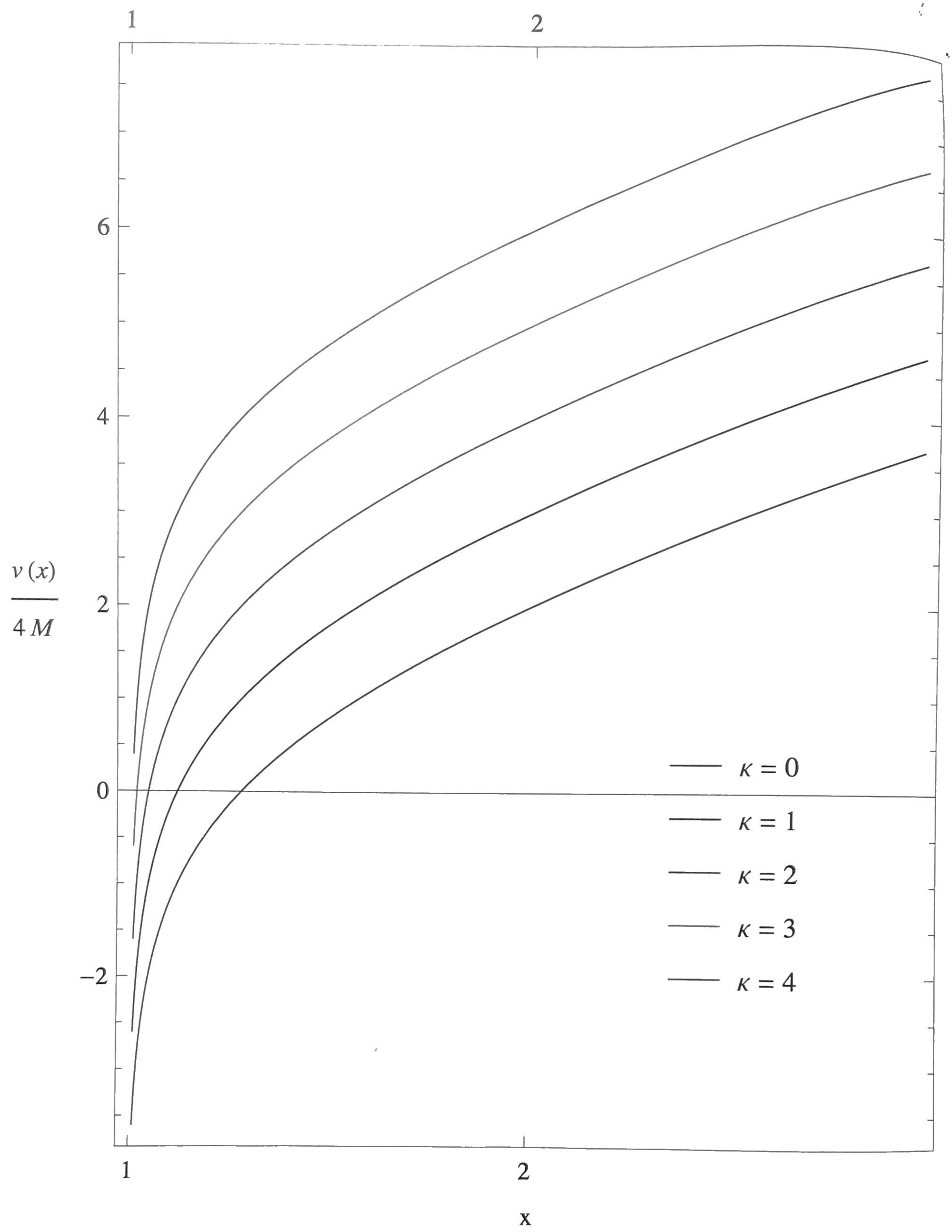
$$v(r) - 2\left(r + 2M\ln\left(\frac{r}{2M} - 1\right)\right) = C - 4M + 4M\ln(2M) = \text{CONST}$$

$$\therefore v(r) - 2\left(r + 2M\ln\left(\frac{r}{2M} - 1\right)\right) = \text{CONST} \checkmark$$

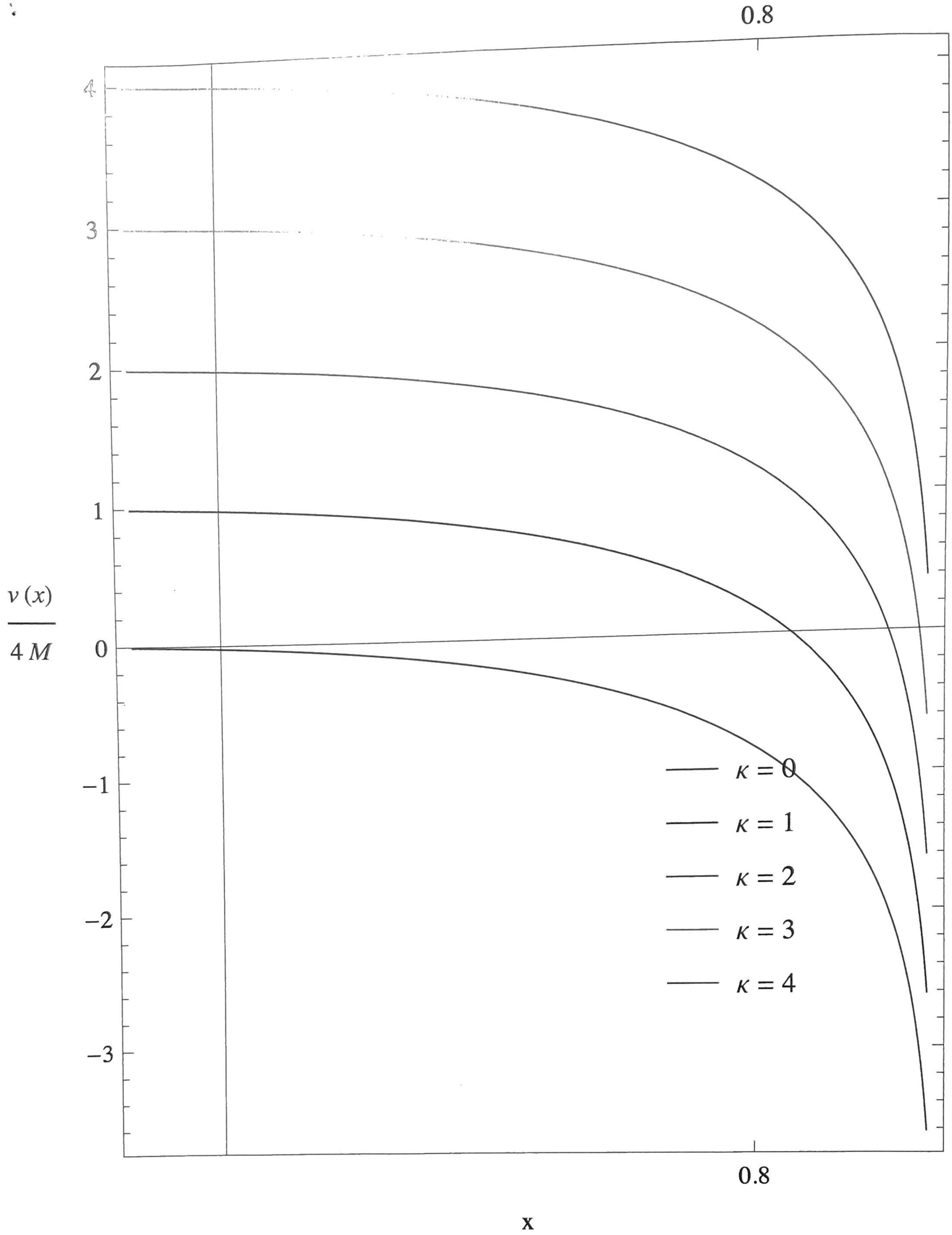
b) let $r = 2Mx$, then

$$\left[v(x) - 2\left(2Mx + 2M\ln(x - 1)\right) = \text{CONST.}\right]\frac{1}{4M}$$

$$\frac{v(x)}{4M} - \left(x + \ln(x - 1)\right) = K \quad \text{or} \quad \left[\frac{v(x)}{4M} = K + x + \ln(x - 1)\right]$$



1
2
6
4
2
0
−2
$\frac{v(x)}{4M}$
$\kappa = 0$
$\kappa = 1$
$\kappa = 2$
$\kappa = 3$
$\kappa = 4$
1
2
x



0.8
4
3
2
1
0
−1
−2
−3
$\frac{v(x)}{4M}$
$\kappa = 0$
$\kappa = 1$
$\kappa = 2$
$\kappa = 3$
$\kappa = 4$
0.8
x

5a) FOR A RADIAL TRAJECTORY, NOTICE THAT $\ell = 0$. EQNS (13) : (14) TAKE THE FORM...

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r}$$

i) FOR INITIAL CONDITIONS: $r(\tau = \tau_i) = R$; $\left.\frac{dr}{d\tau}\right|_{\tau=\tau_i} = 0$

$$\therefore \quad \mathcal{E} = -\frac{M}{R} \qquad \therefore \quad \mathcal{E} < 0$$

ii) FOR INITIAL CONDITIONS : $r(\tau = \tau_i) \to \infty$, $\left.\frac{dr}{d\tau}\right|_{\tau=\tau_i} \neq 0$

$$\therefore \quad \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \qquad \therefore \quad \mathcal{E} > 0$$

iii) FOR INITIAL CONDITIONS : $r(\tau = \tau_i) \to \infty$, $\left.\frac{dr}{d\tau}\right|_{\tau=\tau_i} = 0$

$$\therefore \quad \mathcal{E} = 0$$

b) $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r}$ WHERE $\mathcal{E} = 0$

SO

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \quad \text{OR} \quad \frac{dr}{d\tau} = \overset{\text{FOR RADIALLY INFALLING OBSERVER}}{-}\sqrt{\frac{2M}{r}} \quad \text{SO} \quad \int r^{1/2}\,dr = -\sqrt{2M}\int d\tau$$

$$\frac{2}{3}r^{3/2} = -\sqrt{2M}\,(\tau - \tau_*)$$

NOW SOLVING FOR $\tau = \tau(r)$..

$$\left[\tau = \tau_* - \frac{2}{3\sqrt{2M}}\,r^{3/2}\right] \qquad \text{NOW } \tau(r \to \infty) \to -\infty$$

c) NOW EQN. (13) BECOMES

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} = \frac{dr}{dt}\frac{dt}{d\tau} \qquad \text{NOW FROM EQN (9)}, \quad \frac{dt}{d\tau} = \frac{e}{\left(1 - \frac{2M}{r}\right)}$$

FOR THIS CASE OF INITIAL CONDITIONS, $\mathcal{E} = \frac{1}{2}(e^2 - 1) = 0$. $\therefore$ $e = 1$

SO

$$\frac{dr}{dt}\left(1 - \frac{2M}{r}\right)^{-1} = -\sqrt{\frac{2M}{r}} \quad \text{OR} \quad \frac{dr}{dt} = -\sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right) \checkmark$$

so

$$\int \frac{\sqrt{r}}{\left(1-\frac{2M}{r}\right)}\,dr = -\sqrt{2M}\int dt$$

or

WOLFRAM ALPHA!

$$-\sqrt{2M}\,(t-t_0) = \int \frac{r^{3/2}}{(r-2M)}\,dr = \frac{2}{3}r^{3/2} + 2\cdot 2M\,r^{1/2} + (2M)^{3/2}\ln\left(1-\sqrt{\frac{r}{2M}}\right) - (2M)^{3/2}\ln\left(1+\sqrt{\frac{r}{2M}}\right)$$

$$= \frac{2}{3}r^{3/2} + 4M\,r^{1/2} + (2M)^{3/2}\ln\left[\frac{1-\sqrt{r/2M}}{1+\sqrt{r/2M}}\right]$$

so

$$t(r) = t_0 - \frac{1}{\sqrt{2M}}\left[\frac{2}{3}r^{3/2} + 4M\,r^{1/2} + (2M)^{3/2}\ln\left[\frac{1-\sqrt{r/2M}}{1+\sqrt{r/2M}}\right]\right]$$

$$= t_0 + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} - \ln\left[\frac{1-\sqrt{r/2M}}{1+\sqrt{r/2M}}\right]\right]$$

6. FOR THE RADIAL IN-FALLING OBSERVER WHO STARTS FROM REST AT INFINITY...

$\ell = 0$

$\mathcal{E} = 0$ AND

a) $\frac{2}{3} r^{3/2} = -\sqrt{2M}(\tau - \tau_*)$ AS FOUND IN 5b. $\therefore \tau = \tau_* - \frac{1}{\sqrt{2M}} \cdot \frac{2}{3} r^{3/2}$

NOW, NOTICE THAT

$r_f = 0$ WHEN $\tau_f = \tau_*$

SO

$r_i = 2M$ WHEN $\tau_i = \tau_* - \frac{2}{3\sqrt{2M}}(2M)^{3/2} = \tau_* - \frac{2}{3}(2M)$

SO

$\Delta\tau = \tau_f - \tau_i = \tau_* - \left(\tau_* - \frac{2}{3}(2M)\right) = \frac{4}{3}M$

SO

$$\boxed{\Delta\tau = \frac{4}{3}M}$$

b) NOW, I NEED...

$r_f = 2M$ WHEN $\tau_f = \tau_* - \frac{2}{3}(2M)$

$r_i = R$ WHEN $\tau_i = \tau_* - \frac{1}{\sqrt{2M}} \cdot \frac{2}{3} r^{3/2}$

SO

$\Delta\tau = \tau_f - \tau_i = \tau_* - \frac{2}{3}(2M) - \left(\tau_* - \frac{1}{\sqrt{2M}} \cdot \frac{2}{3} R^{3/2}\right) = \frac{2}{3}\left(\frac{R^{3/2}}{\sqrt{2M}} - 2M\right) = \frac{4}{3}M$

NOW SOLVE FOR R..

$\frac{2}{3}\left(\frac{R^{3/2}}{\sqrt{2M}} - 2M\right) = \frac{4}{3}M \therefore \frac{R^{3/2}}{\sqrt{2M}} = 2M + 2M = 2 \cdot 2M \quad \therefore R^{3/2} = 2 \cdot (2M)^{3/2}$

$$\therefore \boxed{R = (2)^{2/3} \cdot 2M}$$

c) TO GO FROM GEOMETRIZED UNITS TO SI UNITS, LET $M \to \frac{GM}{c^2}$ & $\Delta\tau \to c\Delta\tau$

SO FROM PART a)

$c\Delta\tau = \frac{4}{3}\frac{GM}{c^2} \therefore \quad \Delta\tau = \frac{4}{3} \frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)\left(1.99 \times 10^{30} kg\right)}{\left(2.99 \times 10^8 m/s\right)^3} = 6.6 \times 10^{-6} s$!

$6.6 \mu s$!

$[\Delta\tau] = \frac{N\cancel{m^2}}{\cancel{kg^2}} \cdot \frac{\cancel{kg}}{m^{\cancel{3}}/s^3} = \frac{Ns^3}{m\,kg} = \cancel{kg}\frac{\cancel{m}}{\cancel{s^2}} \cdot \frac{s^{\cancel{3}}}{\cancel{m}\,\cancel{kg}} = s$ ✓