MAT 202
Larson – Section 9.10
Taylor and Maclaurin Series

In section 9.8 we introduced power series. We will now define the Taylor Series and Maclaurin Series.

<u>The Form of a Convergent Power Series</u>: If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \text{and}$$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

This theorem says that if a power series converges to $f(x)$, then the series must be a Taylor series. The converse is not necessarily true.

<u>Definition of Taylor and Maclaurin Series</u>: If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the ***Taylor Series*** for $f(x)$ at c. Moreover, if $c = 0$, then the series is the ***Maclaurin Series*** for f.

The Form of a Convergent Power Series: If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

$$a_3 = \frac{f^{(3)}(c)}{3!} = \frac{6a_3}{6} = a_3$$

and

$$f(x) = \underbrace{f(c)}_{a_0} + \underbrace{f'(c)(x-c)}_{a_1(x-c)} + \underbrace{\frac{f''(c)}{2!}(x-c)^2}_{a_2(x-c)^2} + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

This theorem says that if a power series converges to $f(x)$, then the series must be a Taylor series. The converse is not necessarily true.

$$f(c) = \sum_{n=0}^{\infty} a_n (c-c)^n = a_0 \quad \Big\} \quad a_0 = \frac{f^{(0)}(c)}{0!}$$

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = f(c)$$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$f''(x) = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2$$

$$f(c) = 0!\,a_0$$

$$f'(c) = 1!\,a_1$$

$$f''(c) = 2!\,a_2$$

$$f'''(c) = \overset{3!}{6}\,a_3$$

Ex: Use the definition of Taylor series to find the Taylor series, centered at c, for the function. $f(x) = e^{2x}$, $c = 0$

Ex: Use the definition of Taylor series to find the Taylor series, centered at c, for the function. $f(x) = \dfrac{1}{1-x}$, $c = 2$

Ex: Use the definition of Taylor series to find the Taylor series, centered at c, for the function. $f(x) = e^{2x}$, $c = 0$

Maclaurin Series

$$e^{2x} = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!}(x)^n$$

Figure this out.

$f^{0}(x) = e^{2x}$ $\quad f(0) = 1$

$f'(x) = 2e^{2x}$ $\quad f'(0) = 2$

$f''(x) = 4e^{2x}$ $\quad f''(0) = 4$

$f'''(x) = 8e^{2x}$ $\quad f'''(0) = 8$

$\vdots$

$f^{(n)}(0) = 2^n$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}}$$

Ex: Use the definition of Taylor series to find the Taylor series, centered at c, for the function. $f(x) = \dfrac{1}{1-x}$, $c = 2$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(2)}}{n!}(x-2)^n$$

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{+2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

$$f^{0}(2) = -1 = -1 \cdot 0!$$

$$f'(2) = 1 = 1 \cdot 1!$$

$$f''(2) = -2 = -1 \cdot 2!$$

$$f'''(2) = 6 = 1 \cdot 3!$$

$$\vdots$$

$$f^{(n)}(2) = (-1)^{n+1} n!$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\cancel{n!}}{\cancel{n!}}(x-2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot (-1)\,(x-2)^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n}$$

<u>Deriving Taylor Series from a Basic List</u>: The list below provides the power series for several elementary functions with the corresponding intervals of convergence.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

Ex: Find the Maclaurin series for the function. Use the table of power series for elementary functions. $f(x) = \ln(1 + x^2)$

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From the Table:

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(1+x^2-1)^n}{n}$$

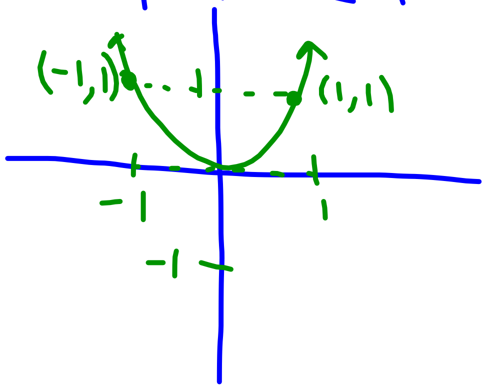
$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$$

Interval of Convergence

$0 < 1+x^2 \leq 2$

$-1 < x^2 \leq 1$

$\boxed{(-1, 1]}$



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$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$(1+x^2)$ $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$ $(1+x^2)$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
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* The convergence at $x = \pm 1$ depends on the value of k.