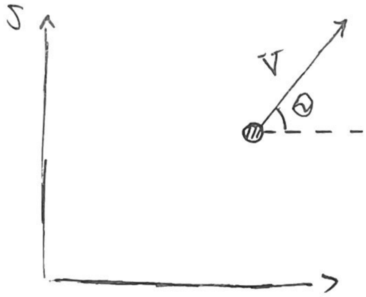
PHYS 396 HOMEWORK SET 4



$$V = \tfrac{3}{5}c \ , \quad \theta = \tfrac{\pi}{3}\text{ RAD} = 60^\circ$$

SO $V_x = V\cos(60^\circ) = \tfrac{3}{5}c \cdot \tfrac{1}{2} = \tfrac{3}{10}c$

$V_y = V\sin(60^\circ) = \tfrac{3}{5}c \cdot \tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{3}}{10}c$

NOW

a) $u^\alpha = (\gamma, \gamma\vec{V})$ WHERE $\vec{V}$ IS THE VELOCITY 3-VECTOR

NOW

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{1}{4/5} = \frac{5}{4}$$

SO $\gamma V_x = \tfrac{5}{4}\cdot\tfrac{3}{10}c = \tfrac{3}{8}c$

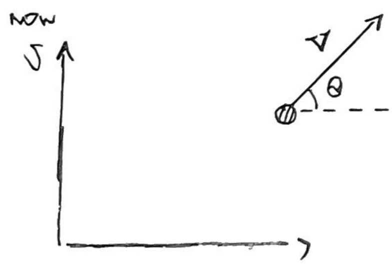
$\gamma V_y = \tfrac{5}{4}\cdot\tfrac{3\sqrt{3}}{10} = \tfrac{3\sqrt{3}}{8}c$

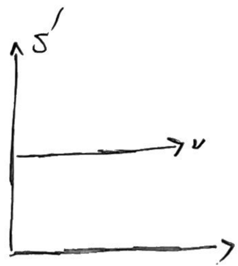
SO

$$\boxed{u^\alpha = \left(\frac{5}{4}, \frac{3}{8}, \frac{3\sqrt{3}}{8}, 0\right)}$$

b) $$\underset{\sim}{u}\cdot\underset{\sim}{u} = \eta_{\alpha\beta}u^\alpha u^\beta = -\left(\frac{5}{4}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3\sqrt{3}}{8}\right)^2 = -\frac{25}{16} + \frac{9}{64} + \frac{27}{64} = \frac{-100+9+27}{64} = -\frac{64}{64} = -1$$

NOW





WHERE $v = \tfrac{4}{5}$

WE NEED TO USE THE LORENTZ TRANSFORMATIONS FOR VELOCITIES..

$$V^{x'} = \frac{V^x - v}{1 - vV_x/c^2} = \frac{\frac{3}{10} - \frac{4}{5}}{1 - \frac{3}{10}\cdot\frac{4}{5}} = \frac{\frac{3}{10} - \frac{8}{10}}{1 - \frac{12}{50}} = \frac{-\frac{5}{10}}{\frac{38}{50}} = -\frac{1}{2}\cdot\frac{50}{38} = -\frac{25}{38}$$

$$V^{y'} = \frac{V^y}{1 - vV^x/c^2}\sqrt{1 - v^2/c^2} = \frac{\frac{3\sqrt{3}}{10}}{1 - \frac{4}{5}\cdot\frac{3}{10}}\sqrt{1-(4/5)^2} = \frac{\frac{3\sqrt{3}}{10}}{\frac{38}{50}}\cdot\frac{3}{5} = \frac{3\sqrt{3}}{10}\cdot\frac{50}{38}\cdot\frac{3}{5} = \frac{9\sqrt{3}}{38}$$

$$V^{z'} = \frac{V^z}{1 - vV^x/c^2}\sqrt{1 - v^2/c^2} = 0 \quad \text{SINCE } V^z = 0$$

NOW

$$V'^2 = V_x'^2 + V_y'^2 + V_z'^2 = \left(\frac{-25}{38}\right)^2 + \left(\frac{9\sqrt{3}}{38}\right)^2 + 0 = \frac{+625+243}{38^2} = \frac{868}{38^2} = \frac{217}{19^2}$$

SO ← GAMMA FACTOR FOR PARTICLE RELATIVE TO THE PRIMED FRAME,

$$\gamma' = \frac{1}{\sqrt{1-V'^2}} = \frac{1}{\sqrt{1-\frac{217}{19^2}}} = \frac{1}{\sqrt{\frac{19^2-217}{19^2}}} = \frac{1}{\sqrt{\frac{144}{19^2}}} = \frac{19}{12}$$

SO

$$\gamma' V_x' = \frac{19}{12}\cdot\frac{-25}{38} = \frac{-25}{24}$$

$$\gamma' V_y' = \frac{19}{12}\cdot\frac{9\sqrt{3}}{38} = \frac{9\sqrt{3}}{24}$$

$$\gamma' V_z' = 0$$

SO

$$\boxed{u'^{\alpha} = (\gamma', \gamma'\vec{V}') = \left(\frac{19}{12}, \frac{-25}{24}, \frac{9\sqrt{3}}{24}, 0\right)}$$

ALTERNATIVELY ...

← GAMMA FACTOR FOR PRIMED FRAME RELATIVE TO UNPRIMED FRAME.

$$u^{t'} = \gamma_v(u^t - vu^x) \qquad \text{WHERE} \quad \gamma_v = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-(4/5)^2}} = \frac{1}{3/5} = \frac{5}{3}$$

$$u^{x'} = \gamma(u^x - vu^t)$$

So

$$u^{t'} = \frac{5}{3}\left(\frac{5}{4} - \frac{4}{5}\cdot\frac{3}{8}\right) = \frac{5}{3}\left(\frac{5}{4} - \frac{3}{10}\right) = \frac{5}{3}\cdot\left(\frac{50-12}{40}\right) = \frac{5}{3}\cdot\frac{38}{40} = \frac{38}{24} = \frac{19}{12}$$

$$u^{x'} = \frac{5}{3}\left(\frac{3}{8} - \frac{4}{5}\cdot\frac{5}{4}\right) = \frac{5}{3}\left(\frac{3}{8} - 1\right) = \frac{5}{3}\cdot\frac{-5}{8} = -\frac{25}{24}$$

$$u^{y'} = u^y = \frac{3\sqrt{3}}{8}, \quad u^{z'} = u^z = 0$$

SO

$$\boxed{u'^{\alpha} = \left(\frac{19}{12}, \frac{-25}{24}, \frac{3\sqrt{3}}{8}, 0\right)}$$

d) $\underset{\sim}{u}' \cdot \underset{\sim}{u}' = \eta_{\alpha\beta} u^{\alpha'} u^{\beta'} = -\left(\frac{19}{12}\right)^2 + \left(\frac{-25}{24}\right)^2 + \left(\frac{9\sqrt{3}}{24}\right)^2 = \frac{-4(19)^2 + 625 + 243}{24^2} = -\frac{576}{24^2} = -1$ ✓

2. SINCE $u^\alpha(t) = \left(\sqrt{1+f^2(t)},\ f(t),\ 0,\ 0\right)$

$$\therefore \frac{dt}{d\tau} = \sqrt{1+f^2(t)}$$

$$\frac{dx}{d\tau} = f(t)$$

$$\frac{dy}{d\tau} = 0 = \frac{dz}{d\tau}$$

a) $\underset{\sim}{u}\cdot\underset{\sim}{u} = \eta_{\alpha\beta}u^\alpha u^\beta = -\left(\sqrt{1+f^2(t)}\right)^2 + f^2(t) = -\left(1+f^2(t)\right) + f^2(t) = -1$

b) NOW $a^\alpha(t) = \dfrac{du^\alpha}{d\tau}$

so

$$a^0(t) = \frac{d}{d\tau}u^0 = \frac{d}{d\tau}\sqrt{1+f^2(t)} = \frac{dt}{d\tau}\frac{d}{dt}\sqrt{1+f^2(t)} = \sqrt{1+f^2(t)}\cdot\frac{1}{2}\left(1+f^2(t)\right)^{-1/2}\cdot 2f\dot{f}$$

$$= f\dot{f}$$

$$a^1(t) = \frac{d}{d\tau}u^1 = \frac{d}{d\tau}f(t) = \frac{dt}{d\tau}\frac{df}{dt} = \sqrt{1+f^2(t)}\cdot\dot{f}$$

$$a^2(t) = \frac{d}{d\tau}u^2 = 0$$

$$a^3(t) = \frac{d}{d\tau}u^3 = 0$$

so

$$\boxed{a^\alpha(t) = \left(f\dot{f},\ \dot{f}\sqrt{1+f^2},\ 0,\ 0\right)}$$

c) $\underset{\sim}{u}\cdot\underset{\sim}{a} = \eta_{\alpha\beta}u^\alpha a^\beta = -u^0a^0 + u^1a^1 + u^2a^2 + u^3a^3 = -\sqrt{1+f^2}\,f\dot{f} + f\dot{f}\sqrt{1+f^2} = 0$ ✓

d) $V(t) = \dfrac{dx}{dt} = \dfrac{dx}{d\tau}\dfrac{d\tau}{dt} = f\left(1+f^2(t)\right)^{-1/2} = \dfrac{f}{\sqrt{1+f^2(t)}}$ so $\boxed{V(t) = \dfrac{f(t)}{\sqrt{1+f^2(t)}}}$

so

$$V^2 = \frac{f^2}{1+f^2} \quad \therefore\ V^2(1+f^2) = f^2$$

$$\therefore\ V^2 = f^2(1-V^2) \quad \therefore\ f^2 = \frac{V^2}{1-V^2}$$

$$\therefore\ \boxed{f(t) = \frac{V(t)}{\sqrt{1-V^2(t)}}}$$

e) now

$$\frac{dt}{d\tau} = \sqrt{1 + f^2(t)} \quad \text{so} \quad d\tau = \frac{dt}{\sqrt{1+f^2(t)}}$$

so

$$\boxed{\tau(t) = \int \frac{dt}{\sqrt{1+f^2(t)}}}$$

Also

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = f(t)$$

so

$$\frac{dx}{dt} \cdot \sqrt{1+f^2(t)} = f(t) \quad \text{so} \quad \frac{dx}{dt} = \frac{f(t)}{\sqrt{1+f^2(t)}}$$

so

$$x(t) = \int \frac{f(t)\,dt}{\sqrt{1+f^2(t)}}$$

f) now $\underset{\sim}{f} = m\underset{\sim}{a}$ so $f^\alpha = m a^\alpha$

so

$$\boxed{f^\alpha = (m f \dot{f},\ m\sqrt{1+f^2}\,\dot{f},\ 0,\ 0)}$$

whereas

$$\vec{F} = \frac{d}{dt}\vec{p} \quad \text{w/} \quad \vec{p} = \gamma m \vec{v} = m\vec{u} \quad \text{where} \quad \vec{u} = \gamma\vec{v}$$

so

$$F^x = m\frac{du^x}{dt} = m\frac{d}{dt}f(t) = m\dot{f}$$

$$F^y = m\frac{du^y}{dt} = 0 \quad \text{likewise} \quad F^z = m\frac{du^z}{dt} = 0$$

so $\boxed{\vec{F} = (m\dot{f}, 0, 0)}$

3. $t(\tau) = \frac{1}{\alpha} \ln[\tan(\alpha\tau) + \sec(\alpha\tau)]$

$x(\tau) = -\frac{1}{\alpha} \ln(\cos(\alpha\tau))$

a) NOW
$$ds^2 = -dt^2 + dx^2 = -d\tau^2$$

SO
$$dt = \frac{1}{\alpha} \cdot \frac{1}{\tan(\alpha\tau) + \sec(\alpha\tau)} d[\tan(\alpha\tau) + \sec(\alpha\tau)] = \frac{1}{\alpha} \cdot \frac{1}{\tan(\alpha\tau) + \sec(\alpha\tau)} [\alpha\sec^2(\alpha\tau) + \alpha\sec(\alpha\tau)\tan(\alpha\tau)] d\tau$$

$$= \frac{1}{\alpha} \cdot \frac{1}{[\tan(\alpha\tau) + \sec(\alpha\tau)]} \alpha\sec(\alpha\tau)[\sec(\alpha\tau) + \tan(\alpha\tau)] d\tau$$

$$\therefore \boxed{dt = \sec(\alpha\tau)\, d\tau}$$

WHEREAS
$$dx = -\frac{1}{\alpha} \cdot \frac{1}{\cos(\alpha\tau)} d(\cos(\alpha\tau)) = -\frac{1}{\alpha} \cdot \frac{1}{\cos(\alpha\tau)} \alpha \cdot -\sin(\alpha\tau) d\tau = +\tan(\alpha\tau)\, d\tau$$

$$\boxed{dx = \tan(\alpha\tau)\, d\tau}$$

NOW
$$ds^2 = -dt^2 + dx^2 = -\sec^2(\alpha\tau) d\tau^2 + \tan^2(\alpha\tau) d\tau^2 = +[\tan^2(\alpha\tau) - \sec^2(\alpha\tau)] d\tau^2$$

NOW
$$\sin^2\theta + \cos^2\theta = 1 \quad \text{SO} \quad \tan^2\theta + 1 = \frac{1}{\cos^2\theta} = \sec^2\theta$$

SO
$$\tan^2\theta - \sec^2\theta = -1$$

$\therefore ds^2 = -d\tau^2$ $\therefore$ τ IS THE PROPER TIME

NOW
$$u^\alpha = \frac{dx^\alpha}{d\tau} \quad \text{SO} \quad u^t = \frac{dt}{d\tau} = \sec(\alpha\tau)$$

$$u^x = \frac{dx}{d\tau} = \tan(\alpha\tau)$$

$$u^y = u^z = 0$$

SO
$$\boxed{u^\alpha = (\sec(\alpha\tau), \tan(\alpha\tau), 0, 0)}$$

whereas

$$a^\alpha = \frac{du^\alpha}{d\tau}$$ so $a^t = \frac{du^t}{d\tau} = \frac{d}{d\tau}\left(\sec(\alpha\tau)\right) = \alpha \cdot -\frac{-\sin(\alpha\tau)}{\cos^2(\alpha\tau)} = \alpha \tan(\alpha\tau)\sec(\alpha\tau)$

$$a^x = \frac{du^x}{d\tau} = \frac{d}{d\tau}\left(\tan(\alpha\tau)\right) = \alpha \sec^2(\alpha\tau)$$

$$\frac{d}{d\tau}\left(\frac{\sin(\alpha\tau)}{\cos(\alpha\tau)}\right) = \alpha\frac{\cos(\alpha\tau)}{\cos(\alpha\tau)} + \alpha\frac{\sin^2(\alpha\tau)}{\cos^2(\alpha\tau)} = \alpha\left(1+\tan^2(\alpha\tau)\right)$$
$$= \alpha\sec^2(\alpha\tau)$$

$$a^y = a^z = 0$$

so

$$\boxed{a^\alpha = \left(\alpha\tan(\alpha\tau)\sec(\alpha\tau),\ \alpha\sec^2(\alpha\tau), 0, 0\right)}$$

:) $u^{t'} = \gamma_v(u^t - v u^x)$ where $\gamma_v = \frac{1}{\sqrt{1-v^2}}$

$u^{x'} = \gamma_v(u^x - v u^t)$

so

$$u^{t'} = \gamma_v\left(\sec(\alpha\tau) - v\tan(\alpha\tau)\right)$$
$$u^{x'} = \gamma_v\left(\tan(\alpha\tau) - v\sec(\alpha\tau)\right)$$

so

$$\boxed{u'^\alpha = \left(\gamma_v\left(\sec(\alpha\tau)-v\tan(\alpha\tau)\right), \gamma_v\left(\tan(\alpha\tau)-v\sec(\alpha\tau)\right), 0, 0\right)}$$

whereas

$$a^{t'} = \gamma_v(a^t - v a^x) = \gamma_v\left(\alpha\tan(\alpha\tau)\sec(\alpha\tau) - v\alpha\sec^2(\alpha\tau)\right) = \gamma_v\alpha\sec(\alpha\tau)\left(\tan(\alpha\tau) - v\sec(\alpha\tau)\right)$$
$$a^{x'} = \gamma_v(a^x - v a^t) = \gamma_v\left(\alpha\sec^2(\alpha\tau) - v\alpha\tan(\alpha\tau)\sec(\alpha\tau)\right) = \gamma_v\alpha\sec(\alpha\tau)\left(\sec(\alpha\tau) - v\tan(\alpha\tau)\right)$$

so

$$\boxed{a'^\alpha = \gamma_v\alpha\sec(\alpha\tau)\Big(\left(\tan(\alpha\tau)-v\sec(\alpha\tau)\right), \left(\sec(\alpha\tau)-v\tan(\alpha\tau)\right), 0, 0\Big)}$$

NOW

$$1)\ \underset{\sim}{u}\cdot\underset{\sim}{u} = \eta_{\alpha\beta}u^{\alpha}u^{\beta} = -\gamma_v^2\left(\sec(\alpha\tau)-v\tan(\alpha\tau)\right)^2+\gamma_v^2\left(\tan(\alpha\tau)-v\sec(\alpha\tau)\right)^2$$

$$= -\gamma_v^2\Big[\sec^2(\alpha\tau)+v^2\tan^2(\alpha\tau)-2v\cancelto{0}{\sec(\alpha\tau)\tan(\alpha\tau)}$$

$$-\left[\tan^2(\alpha\tau)+v^2\sec^2(\alpha\tau)-2v\cancelto{0}{\sec(\alpha\tau)\tan(\alpha\tau)}\right]\Big]$$

$$= \gamma_v^2\left[\left(\tan^2(\alpha\tau)-\sec^2(\alpha\tau)\right)-v^2\left(\tan^2(\alpha\tau)-\sec^2(\alpha\tau)\right)\right]$$

$\sin^2\theta+\cos^2\theta=1$

so

$\tan^2\theta+1=\sec^2\theta \quad$ so $\tan^2\theta-\sec^2\theta=-1$

$$= -\gamma_v^2(1-v^2) = -1 \quad ✓$$

WHEREAS

$$\underset{\sim}{u}\cdot\underset{\sim}{a} = \eta_{\alpha\beta}u^{\alpha}a^{\beta} = -\gamma_v\left(\sec(\alpha\tau)-v\tan(\alpha\tau)\right)\cdot\gamma_v\alpha\sec(\alpha\tau)\left(\tan(\alpha\tau)-v\sec(\alpha\tau)\right)$$

$$+\gamma_v\left(\tan(\alpha\tau)-v\sec(\alpha\tau)\right)\cdot\gamma_v\alpha\sec(\alpha\tau)\left(\sec(\alpha\tau)-v\tan(\alpha\tau)\right) = 0 ✓$$

4. INITIALLY...

FINALLY...

WHERE $v = \frac{3}{5}c$

$E_\gamma = 12{,}000\,\text{MeV}$

CONSERVATION OF ENERGY YIELDS...

$$1) \quad \gamma M c^2 = M_0 c^2 + E_\gamma$$

CONSERVATION OF MOMENTUM YIELDS...

$$2) \quad \gamma M v = \frac{E_\gamma}{c}$$

NOW $\gamma = \dfrac{1}{\sqrt{1 - (3/5)^2}} = \dfrac{1}{4/5} = \dfrac{5}{4}$

(SO 2) YIELDS..

$$\frac{5}{4} \cdot M \cdot \frac{3}{5} c = \frac{E_\gamma}{c} = \frac{3}{4} M c \quad \therefore \quad M = \frac{4}{3} \frac{E_\gamma}{c^2} = \frac{4}{3} \cdot 12{,}000 \frac{\text{MeV}}{c^2}$$

$$\left[M = 16{,}000 \frac{\text{MeV}}{c^2} \right]$$

WHEREAS 1) YIELDS..

$$) \quad M_0 = \gamma M - \frac{E_\gamma}{c^2} = \frac{5}{4} \cdot 16{,}000 \frac{\text{MeV}}{c^2} - 12{,}000 \frac{\text{MeV}}{c^2} = 8{,}000 \frac{\text{MeV}}{c^2}$$

$$\therefore \left[M_0 = 8{,}000 \frac{\text{MeV}}{c^2} \right]$$

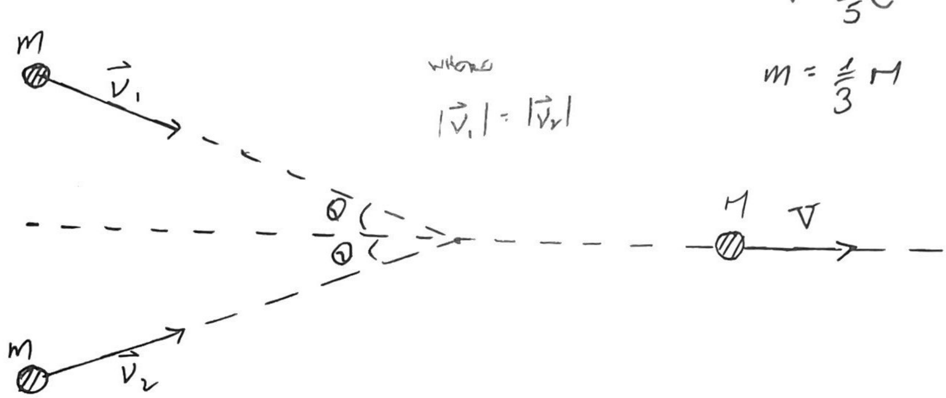
$$) \quad p_1 = \gamma M v = \frac{5}{4} \cdot 16{,}000 \frac{\text{MeV}}{c^2} \cdot \frac{3}{5} c = 12{,}000 \frac{\text{MeV}}{c}$$

$$\left[p_1 = 12{,}000\,\text{MeV}/c \right]$$

Conservation of Momentum yields...

$V = \frac{4}{5}c$

$m = \frac{1}{3}M$



$$\gamma_v m v \cos\theta + \gamma_v m v \cos\theta = \gamma_V M V$$

where $\gamma_V = \frac{1}{\sqrt{1-(4/5)^2}} = \frac{1}{3/5} = \frac{5}{3}$

$\gamma_v = \frac{1}{\sqrt{1-v^2}}$

so

$$2\gamma_v \cdot \frac{1}{3}M \cdot v\cos\theta = \frac{5}{3}M \cdot \frac{4}{5}c = \frac{4}{3}Mc$$

$$\frac{2}{3}\frac{v}{\sqrt{1-v^2/c^2}}\cos\theta = \frac{4}{3}c$$

so

$$(*)\left[\frac{v/c}{\sqrt{1-v^2/c^2}} = \frac{2}{\cos\theta}\right]$$

Conservation of Energy yields

$$\gamma_v mc^2 + \gamma_v mc^2 = \gamma_V Mc^2$$

$$2 \cdot \frac{1}{3}M \cdot \frac{1}{\sqrt{1-v^2/c^2}} = \frac{5}{3} \cdot M$$

so

$$\frac{1}{\sqrt{1-v^2/c^2}} = \frac{5}{3}\cdot\frac{3}{2} = \frac{5}{2}$$

so $\sqrt{1-v^2/c^2} = \frac{2}{5}$

$1 - v^2/c^2 = \frac{2^2}{5^2}$ $\therefore \frac{v^2}{c^2} = 1 - \frac{2^2}{5^2} = \frac{21}{25}$

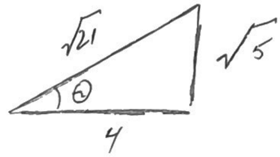
$\therefore \frac{v}{c} = \frac{\sqrt{21}}{5}$

So $\boxed{v = \frac{\sqrt{21}}{5} c}$

GGING THIS INTO (*)...

$$\sqrt{21}/5 \cdot \frac{5}{2} = \frac{2}{\cos\theta} \quad \therefore \quad \cos\theta = 2 \cdot \frac{2}{5} \cdot \frac{5}{\sqrt{21}} = \frac{4}{\sqrt{21}}$$

$$\boxed{\theta = 29.2^\circ}$$



So

$$v_x = v\cos\theta = \frac{\sqrt{21}}{5} c \cdot \frac{4}{\sqrt{21}} = \frac{4}{5} c$$

$$\cos\theta = \frac{4}{\sqrt{21}}$$

$$v_y = v\sin\theta = \frac{\sqrt{21}}{5} c \sqrt{\frac{5}{21}} = \frac{\sqrt{5}}{5} c = \frac{1}{\sqrt{5}} c$$

$$\sin\theta = \sqrt{\frac{5}{21}}$$

So

$$\left[\begin{aligned} \vec{v}_1 &= \left(\frac{4}{5}c, -\frac{1}{\sqrt{5}}, 0\right) \\ \vec{v}_2 &= \left(\frac{4}{5}c, \frac{1}{\sqrt{5}}, 0\right) \end{aligned}\right]$$