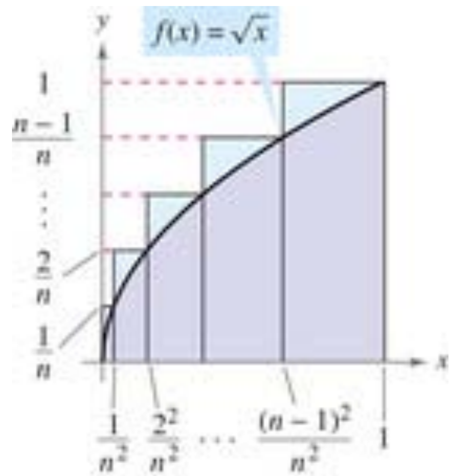
MAT 201
Larson/Edwards – Section 4.3
Riemann Sums and Definite Integrals

In section 4.2 we considered finding the area under a curve by adding rectangles of a equal width as those rectangles width gets smaller and smaller. In example 1 in this section you will see that it doesn't matter whether the partitions have subintervals that are of equal or unequal width. The area can still be found since the width of the largest subinterval is approaching 0.



The sums with partitions having subintervals of equal or unequal width (generalized) are known as ***Riemann Sums***, named after Georg Friedrich Bernhard Riemann (1826 – 1866).

$$\text{Equal width}: \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

$$\text{Generalized width}: \sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

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What's the difference???

Δx means that the width is constant.

Δx_i means the width of the i^{th} rectangle, which could be different than the $i+1^{th}$ rect.

Definition of Definite Integral:

If f is defined on the closed interval $[a, b]$ and the limit of the Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

exists, then f is said to be integrable on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx.$$

This limit is called the ***definite integral*** of f from a to b. The number a is the ***lower limit of integration***, and the number b is the ***upper limit of integration***. The width of the largest subinterval of a partition Δ is the ***norm*** of the partition and is denoted by $\|\Delta\|$. By saying $\|\Delta\| \to 0$ we are implying that $n \to \infty$.

Important points:

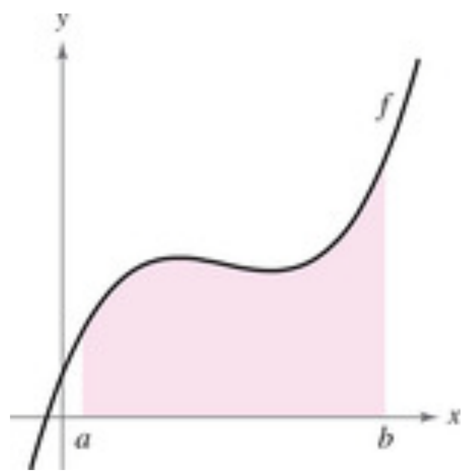
1. We have only defined the definite integral. This definition says nothing about area under a curve due to the fact that the limit could be negative!
2. A definite integral describes a number, whereas an indefinite integral describes a family of functions!

Continuity Implies Integrability (Theorem 4.4):

If a function f is continuous on the closed interval [a, b], then f is integrable. That is, $\int_a^b f(x)\,dx$ exists.

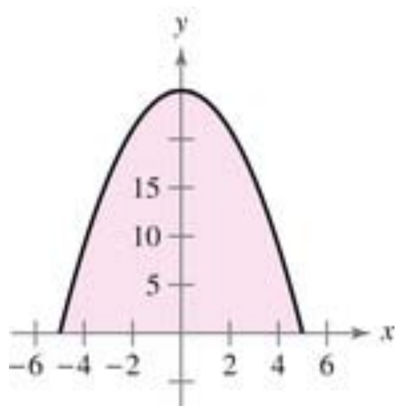
Definite Integral as the Area of a Region (Theorem 4.5): Let f be nonnegative and continuous on the closed interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is denoted by:

$$\text{Area} = \int_a^b f(x)\,dx$$



Notice that for a definite integral to be considered an area it must be continuous and nonnegative on $[a, b]$. This condition was not given in the definition of a definite integral!

Ex: Set up a definite integral that yields the area of the region. Do not evaluate the integral. $f(x) = 25 - x^2$



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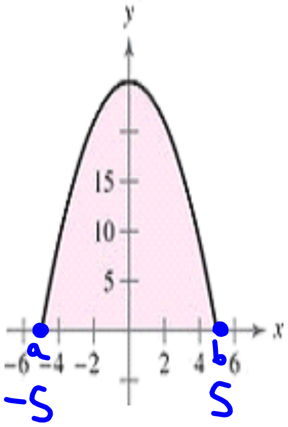


$$A = \int_a^b f(x)\,dx$$

$$= \boxed{\int_{-5}^{5} 25 - x^2 \; dx}$$

Ex: Sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

$$\int_0^4 \frac{x}{2}\,dx$$

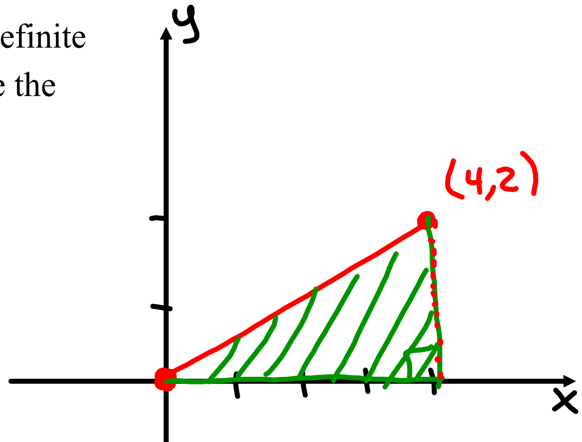
<u>Properties of Definite Integrals</u>: Let f and g be integrable on the closed interval $[a, b]$, $[a, c]$, and $[c, b]$.

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant.
2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $a < c < b$
4. $\int_a^a f(x)\,dx = 0$ f is defined at $x = a$.
5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex: Sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

$$\int_0^4 \frac{x}{2}\,dx$$

$f(x) = \frac{x}{2}$ $\quad [0, 4]$



$A = \frac{1}{2}(4)(2)$

$A = 4$ sq. units