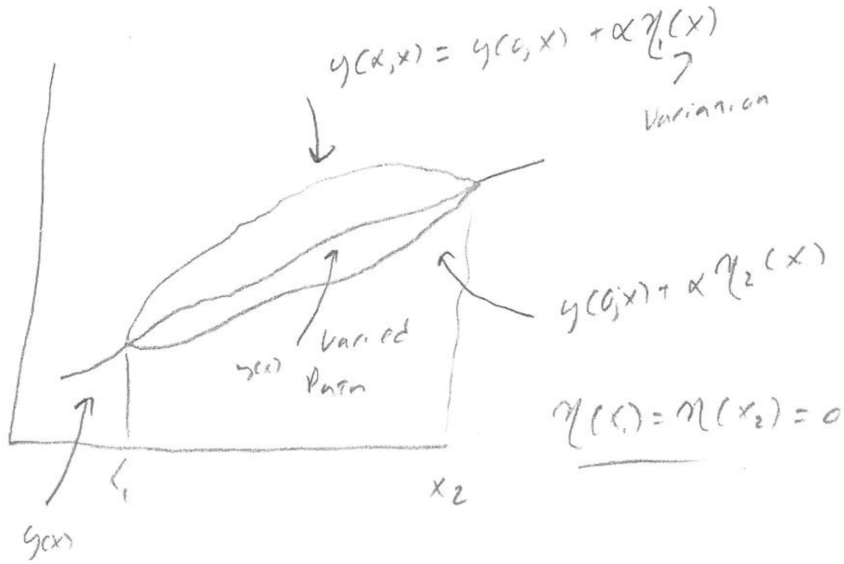
Chapter #6. Calculus of Variations

define $y(x)$ s.t. $J = \int_{x_1}^{x_2} F(y(x), y'(x); x)\,dx$ is an extremum (minimum)

$J \to$ Functional



Then $J(\alpha) = \int_{x_1}^{x_2} F(y(\alpha,x), y'(\alpha,x); x)\,dx \to$ goal, consider all variations to $y(x)$, pick path that is an extremum

$$\boxed{\text{require } \left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0}$$

$y(x) \to$ extremum path

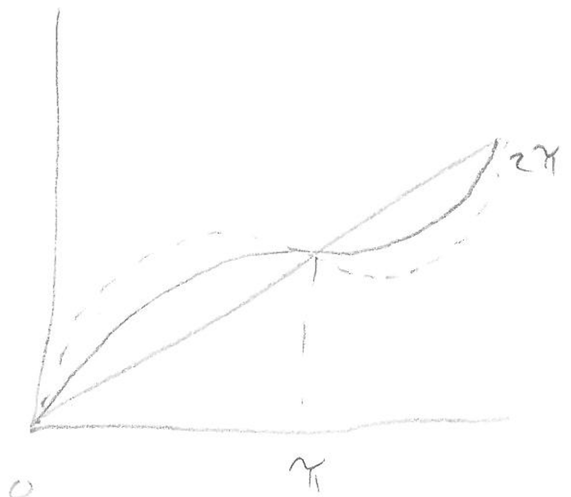
Ex. Consider $F = \left(\frac{dy}{dx}\right)^2$ where $y(x) = x \in [0, 2\pi]$

add $\eta(x) = \sin(x)$

So $y(\alpha, x) = x + \alpha \sin(x)$

Show minimum is $\alpha = 0$

For $J(\alpha)$



$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha \cos(x)$$

Then $F = \left(\frac{dy(\alpha, x)}{dx}\right)^2 = 1 + 2\alpha\cos(x) + \alpha^2\cos^2 x$

$$J(\alpha) = \int_0^{2\pi} \left(1 + 2\alpha\cos(x) + \alpha^2\cos^2(x)\right) dx = 2\pi + \alpha^2 \pi$$

So $\alpha \neq 0$ increases $J(\alpha)$

Euler's Equation

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} F(y,y';x)\,dx = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} F(y,y';x)\,dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)dx$$

$$\boxed{\begin{aligned} y(\alpha,x) &= y(0,x) + \alpha\eta(x) \\ y'(\alpha,x) &= y'(0,x) + \alpha\eta'(x)\end{aligned}}$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \qquad \frac{\partial y'}{\partial \alpha} = \eta'(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right]dx$$

Now $$\int_{x_1}^{x_2} \underbrace{\frac{\partial F}{\partial y'}}_{u}\underbrace{\frac{d\eta}{dx}\,dx}_{dv} = \underbrace{\frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2}}_{1} - \underbrace{\int \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx}_{2}$$

$1 = 0 \quad \eta(x_1) = \eta(x_2) \quad 2 \neq 0$

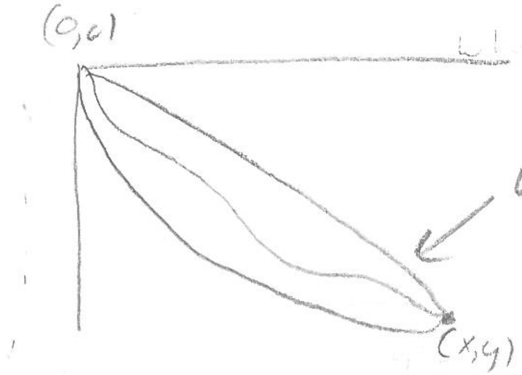
So $$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\right)dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx$$

$\eta(x) \rightarrow$ any need $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

Euler's Equation

brachistochrone



Move x_1, y_1 to $(0,0)$

Now $E = T + U = 0$ if $v_{initial} = 0$

or $0 = -mgy + \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \rightarrow 2gy = \left(\frac{ds}{dt}\right)^2$

$|\vec{V}| = \sqrt{2gy}$

$dt = \frac{ds}{\sqrt{2gy}}$ want to minimize $\int dt = \int \frac{ds}{\sqrt{2gy}}$

$ds = \sqrt{dx^2 + dy^2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = (1 + (y')^2)^{1/2}dx$

$$\boxed{\text{or minimize } \frac{1}{\sqrt{2g}} \int \frac{(1 + (y')^2)^{1/2}}{\sqrt{y}} dx}$$

$$T = J = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1+(y')^2}{y}}\, dx$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}$$

$$\frac{\partial F}{\partial y} = -\sqrt{\frac{1+(y')^2}{y}}\left(\frac{1}{2y}\right)$$

$$\frac{\partial F}{\partial y'} = \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{1+(y')^2}} \cdot 2y' = \frac{y'}{\sqrt{y(1+(y')^2)}}$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y(1+(y')^2)}}\left[\frac{y''}{1+(y')^2} - \frac{1}{2}\frac{(y')^2}{y}\right]$$ Let $A = 1+(y')^2$

$$\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'} \rightarrow -\frac{A^{1/2}}{\sqrt{y}} \cdot \frac{1}{2y} = \frac{1}{A^{1/2}\sqrt{y}}\left[\frac{y''}{A} - \frac{1}{2}\frac{(y')^2}{y}\right]$$

$$-\frac{A}{2y} = \frac{y''}{A} - \frac{1}{2}\frac{(y')^2}{y} \rightarrow \frac{(y')^2 - A}{2y} = \frac{y''}{A}$$

$$-\frac{1}{2y} = \frac{y''}{A} \rightarrow \boxed{y'' = -\frac{(1+(y')^2)}{2y}}$$

$$\boxed{1 + 2y(y'') + (y')^2 = 0}$$

Now $1+(y')^2\ 2yy''=0$

Let $y'=u$ $\quad y''=\frac{du}{dx}=\frac{du}{dy}y'=u\frac{du}{dy}$ $\quad$ So $2yy''=2yu\frac{du}{dy}$

$$1+u^2+2yu\frac{du}{dy}=0$$

$$(1+u^2)+\frac{y}{dy}2u\,du=0$$

$$\frac{dy}{y}+\frac{2u\,du}{(1+u^2)}=0$$

$$\ln(y)+\ln(1+u^2)=0$$

$$y(1+u^2)=D \rightarrow \frac{D}{y}-1=u^2$$

$$u=\left(\frac{D-y}{y}\right)^{1/2}=\frac{dy}{dx}$$

Now $\displaystyle\int\sqrt{\frac{y}{D-y}}\,dy=\int dx$

$$X+C=\int\sqrt{\frac{y}{D-y}}\,dy \qquad \text{Let } y=D\sin^2\theta$$

$$x + c = \int \sqrt{\frac{D\sin^2\theta}{D\cos^2\theta}}\; 2D\sin\theta\cos\theta\, d\theta$$

$$x + C = 2D\int \sin^2\theta\, d\theta = 2D\int \frac{1}{2} - \frac{\cos(2\theta)}{2}\, d\theta$$

$$x + C = \frac{1}{2}D\left[2\theta - \sin(2\theta)\right]$$

$$x = \frac{1}{2}D\left[2\theta - \sin(2\theta)\right] - C$$

$$y = D\sin^2\theta = \frac{D}{2}\left(1 - \cos(2\theta)\right)$$

Starts at 0,0 → C=0

$$x = \frac{D}{2}\left(2\theta - \sin(2\theta)\right) \qquad y = \frac{D}{2}\left(1 - \cos(2\theta)\right)$$

$$\frac{D}{2} = A \qquad 2\theta = \phi$$

$$x = A\left[\phi - \sin(\phi)\right] \qquad y = A\left[1 - \cos(\phi)\right]$$

o $\phi = \frac{\pi}{2}$