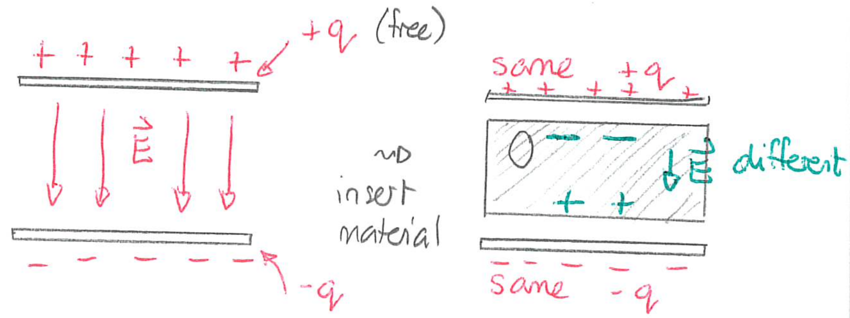
Tues: HW 2 due by 5pm

Thurs: Read 4.4.3, 6.1

Electrostatics in Matter

In many situations materials will respond to externally applied electric fields by altering their charge distribution and producing additional electric fields. We can model broad classes of such materials in terms of collections of electric dipoles.



The primary question is:

> Given a particular distribution of free charges in the vicinity of the material, determine the electric fields produced in and beyond the material

The process is:

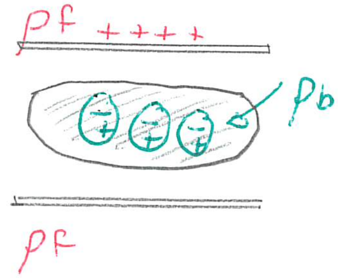
Free charge distribution (fixed) produces electric field and this induces a polarization in the material → Polarization produces bound charges in the material and these produce additional electric fields → Net electric field is

$$\vec{E} = \vec{E}_{\text{from free}} + \vec{E}_{\text{from bound}}$$

Electric Displacement.

The process of determining the combined effects of free and band charges is facilitated by an auxiliary quantity, the electric displacement, which can be calculated just from the free charges.

The total electric field $\vec{E}$ is produced from the entire charge density



$$\rho = \rho f + \rho b$$

where ρf is the free charge density (placed freely outside the material)
ρb " " band " "

This has to satisfy

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0 \qquad \text{and} \qquad \vec{\nabla} \times \vec{E} = 0$$

$$= \frac{\rho f}{\epsilon_0} + \frac{\rho b}{\epsilon_0}$$

But $\rho b = -\vec{\nabla} \cdot \vec{P}$ where $\vec{P}$ is the polarization. So

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho f - \vec{\nabla} \cdot \vec{P} \Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho f$$

Thus we define the electric displacement vector:

$$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$$

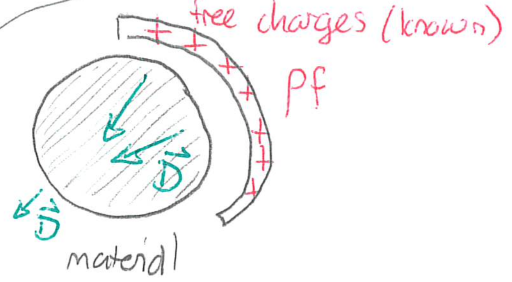
units C/m^2

and this always satisfies

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho f}$$

In principle we can solve this equation, using only the free charge density, to obtain the electric displacement

The conceptual scheme is then:



Knowing free charges determine electric displacement $\vec{D}$ via

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

Knowing electric displacement and polarization determine electric field via

$$\vec{E} = \frac{1}{\epsilon_0}\left(\vec{D} - \vec{P}\right)$$

We now need techniques to determine the electric displacement. In general we need both the divergence and curl of a field in order to obtain the field. Now

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \left(\epsilon_0 \vec{E} + \vec{P}\right) = \epsilon_0 \overbrace{\cancel{\vec{\nabla} \times \vec{E}}}^{0} + \vec{\nabla} \times \vec{P} \Rightarrow \boxed{\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}}$$

Thus

> The electric displacement satisfies
>
> $$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad\qquad \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$
>
> where ρ_f is the free charge density and $\vec{P}$ the polarization of the material

The strategies for solving for electric displacement then depend on the curl of the polarization. – whether this is zero or not.

If $\vec{\nabla} \times \vec{P} = 0$ then:

Any strategy used to determine electrostatic fields will work to determine electric displacement:

Coulomb's Law, Gauss' Law, Electrostatic Potential

$\longrightarrow$ replace

$\vec{E} \rightarrow \vec{D}/\epsilon_0$

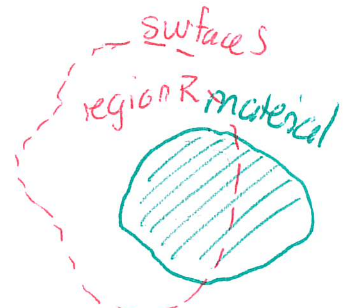
$\rho \rightarrow \rho_f$

In general if $\vec{\nabla} \times \vec{P} \neq 0$ then these will not necessarily work. Particularly trying to solve for a "bound electrostatic potential" using ρ_f alone will not work because it will result in $\vec{\nabla} \times \vec{D} = 0$ which is impossible if $\vec{\nabla} \times \vec{P} \neq 0$.

However, we can still use a version of Gauss' law.

For any closed region:

$$\int_{\text{region } R} \vec{\nabla} \cdot \vec{D} \, d\tau' = \oint_{\text{surface}} \vec{D} \cdot d\vec{a}$$



$$\Rightarrow \underbrace{\int_{\text{region } R} \rho_f \, d\tau'}_{\substack{\text{total enclosed} \\ \text{free charge. } q_{f\,enc}}} = \oint_{\text{surface}} \vec{D} \cdot d\vec{a}$$

Thus it is always true that:

For any closed surface

$$\oint_S \vec{D} \cdot d\vec{a} = q_{\text{free enc}}$$

where $q_{\text{free enc}}$ is the total free charge enclosed in the surface.

1 Electric field in a polarized sphere

A sphere of radius R contains no free charge but has polarization

$$\mathbf{P}(\mathbf{r}') = P(r')\,\hat{\mathbf{r}}$$

where $P(\vec{r})$ is an arbitrary function of $\vec{r}'$ only.

a) Determine an expression for the electric displacement.

b) Determine an expression for the electric field inside and outside the sphere.

Answer: a) Use

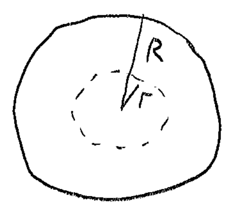
$$\oint \vec{D}\cdot d\vec{a} = q_{\text{free enc.}}$$

Here the symmetry will be such that both $\vec{E}(\vec{r})$ and $\vec{P}$ are spherically symmetric. So the usual symmetry argument gives:

$$\vec{D} = D_r(r)\,\hat{r}$$

The Gaussian surface will be a sphere of radius r, either outside or inside (illustrated).

On this sphere:



$$\left.\begin{array}{l} r' = r \\ 0 \leq \theta' \leq \pi \\ 0 \leq \phi' \leq 2\pi \end{array}\right\} \quad d\vec{a} = r'^2 \sin\theta'\, d\theta'\, d\phi'$$

$$= r^2 \sin\theta'\, d\theta'\, d\phi'$$

Then

$$\vec{D}\cdot d\vec{a} = D_r(r)\, r^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\Rightarrow \oint \vec{D}\cdot d\vec{a} = \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \sin\theta'\, D_r(r)\, r^2 = 4\pi r^2 D_r(r)$$

$$\Rightarrow 4\pi r^2 D_r(r) = q_{\text{free enc.}}$$

But

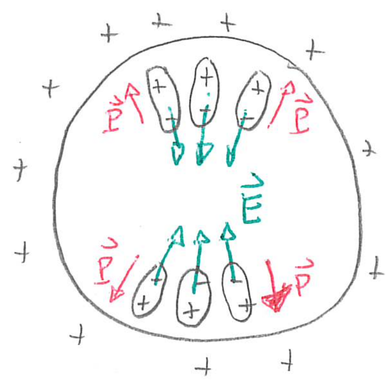
$$q_{free\ enc} = 0 \Rightarrow 4\pi r^2 D_r(r) = 0 \Rightarrow D_r(r) = 0$$

$$\Rightarrow \boxed{\vec{D}(\vec{r}) = 0} \text{ everywhere}$$

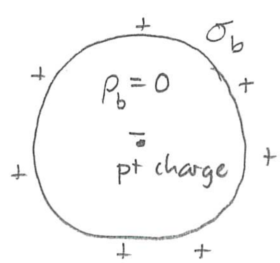
b) In general

$$\vec{D}(\vec{r}) = \varepsilon_0 \vec{E}(\vec{r}) + \vec{P} \Rightarrow 0 = \epsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{E} = -\frac{1}{\epsilon_0}\vec{P}$$

Note that if $\vec{P}$ points radially outward then:

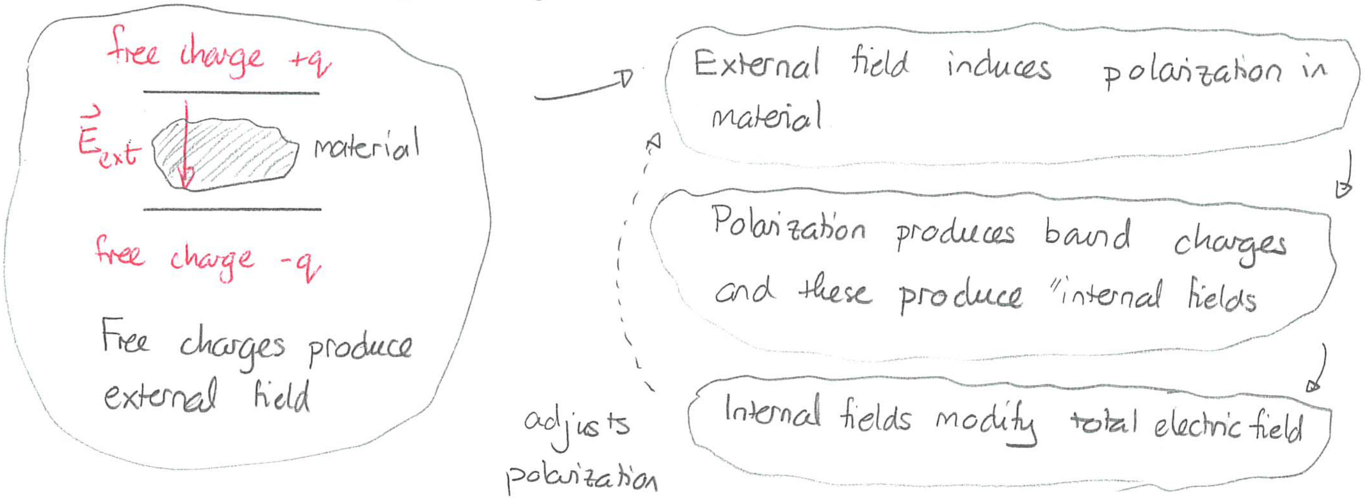


If $\vec{P}$ were constant $\vec{P} = P\hat{r}$ where $P > 0$ then this is a band charge distribution



Linear Dielectrics

We need to specify how the material will respond to fields produced by free charges. One might imagine a mechanism:



In general the eventual relationship between the electric field + polarization will settle. How this occurs depends on the material. We consider a broad class of materials where

$$\vec{P} \propto \vec{E} \leftarrow \text{total electric field}$$

This proportional relationship gives rise to the terminology <u>linear dielectric</u> and is usually expressed as:

$$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}}$$

where χ_e is called the electric susceptibility and has no units. The value of χ_e depends on the particular material. Thus, for a <u>linear dielectric</u>

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1+\chi_e) \vec{E}$$

We define the <u>permittivity</u> of the material as:

$$\epsilon = \epsilon_0 (1+\chi_e)$$

and then

$$\boxed{\vec{D} = \epsilon \vec{E}}$$

Thus

$$\boxed{\text{For a linear dielectric with permittivity } \epsilon \text{, the electric field can be determined from the electric displacement by } \vec{E} = \frac{1}{\epsilon}\vec{D}}$$

The dielectric constant of the material is:

$$\boxed{\epsilon_r = 1 + \chi_e}$$

TABLE 4.2

and so

$$\boxed{\epsilon = \epsilon_r \epsilon_0}$$

Thus we arrive at the process:

← free charges described by free charge density ρ_f

→ Electric displacement $\vec{D}$ satisfies:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

→ Solve for electric displacement, $\vec{D}$

For a linear dielectric

$$\vec{D} = \epsilon \vec{E}$$

where

$$\epsilon = (1+\chi_e)\epsilon_0$$

→ Get electric field

$$\vec{E} = \frac{1}{\epsilon}\vec{D}$$

A further special case is a <u>homogeneous linear dielectric</u>, where the permittivity does not depend on location. Then

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(\frac{\vec{D}}{\epsilon}\right) = \frac{1}{\epsilon}\vec{\nabla}\cdot\vec{D} = \frac{\rho_{free}}{\epsilon}.$$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\vec{D}/\epsilon\right) = \frac{1}{\epsilon}\vec{\nabla}\times\vec{E} = 0.$$

Thus,

For a linear homogeneous dielectric

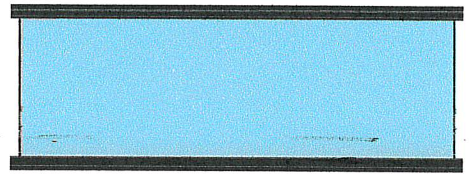
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{free}}{\epsilon}$$

$$\vec{\nabla}\times\vec{E} = 0$$

and the electric field can be determined using standard electrostatic techniques with $\rho \to \rho_{free}$ and $\epsilon_0 \to \epsilon$.

2 Parallel plate capacitor filled with a dielectric

A parallel plate capacitor has plates of area A separated by distance d. The gap between the plates is filled with a dielectric material with permittivity ϵ. Assume that the gap between the plates is very small compared to their horizontal extent.



a) Determine an expression for the electric field between the capacitor plates.

b) Determine an expression for the electrostatic potential difference between the plates.

c) Determine an expression for the capacitance of the capacitor.

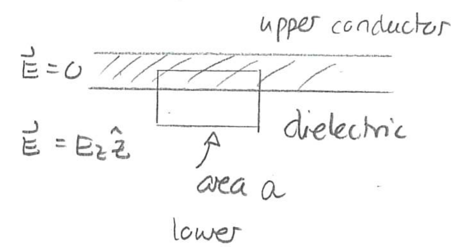
d) Suppose that initially the region between the plates is a vacuum but this is later filled by water (dielectric constant 80.4). By what factor will the capacitance change when the plates are filled by water?

e) If the free charge per unit area on the upper plate is $+\sigma_{\text{free}}$ and on the lower plate is $-\sigma_{\text{free}}$, determine expressions for the bound charge densities on the upper and lower surfaces of each dielectric.

Answer a) The field satisfies $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon}\rho_f \Rightarrow \oint \vec{E}\cdot d\vec{a} = \frac{q_{f\,enc}}{\epsilon_0}$

By symmetry $\vec{E} = E_z(z)\,\hat{z}$.

Use the illustrated pillbox



Then

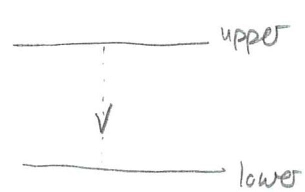
$$\oint \vec{E}\cdot d\vec{a} = \int_{\text{lower surface}} \vec{E}\cdot d\vec{a} = -E_z(z)a = \frac{q_{\text{free enc}}}{\epsilon}$$

$$\Rightarrow \quad E_z(z) = -\frac{\sigma_{\text{free enc}}\,a}{\epsilon\,a}$$

Thus between the plates $\quad \vec{E} = -\frac{\sigma_f}{\epsilon}\hat{z}$

b) Along any path

$$\Delta V = -\int \vec{E}\cdot d\vec{l}$$



Use the illustrated path: $d\vec{l} = -dz\,\hat{z}$

So $$\Delta V = -\int -\frac{\sigma_f}{\epsilon}\hat{z}\cdot d\vec{l}$$

$$= \frac{\sigma_f}{\epsilon}\int dz = \frac{\sigma_f}{\epsilon} d \quad \Rightarrow \quad \Delta V = \frac{\sigma_f}{\epsilon} d.$$

where d is the distance between the plates.

c) Suppose that the plates have area A. Then the total charge on one plate is $Q = \sigma_f A \Rightarrow \sigma_f = Q/A$

$$\Rightarrow \quad \Delta V = \frac{Q\,d}{\epsilon A} \Rightarrow Q = \frac{\epsilon A}{d}\Delta V$$

Thus the capacitance is $\boxed{C = \frac{\epsilon A}{d}}$

d) Without dielectric $\quad \epsilon = \epsilon_0 \Rightarrow C_{without} = \frac{\epsilon_0 A}{d}$

With dielectric $\quad \epsilon = \epsilon_r\epsilon_0 \Rightarrow C_{with} = \frac{\epsilon_r \epsilon_0 A}{d}$

$$\Rightarrow C_{with} = \epsilon_r C_{without}$$

Increases by a factor of $\epsilon_r = 80.4$.

e) In general $\quad \sigma_b = \vec{P}\cdot\hat{n}$

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

So we need $\vec{P}$. Now

$$\vec{D} = \epsilon\vec{E} = \epsilon_0\vec{E} + \vec{P} \Rightarrow \vec{P} = (\epsilon - \epsilon_0)\vec{E}$$

$$= (\epsilon_r\epsilon_0 - \epsilon_0)\vec{E}$$

$$= (\epsilon_r - 1)\epsilon_0\vec{E}.$$

Between the plates $\vec{E}$ is uniform. So $\quad \vec{\nabla}\cdot\vec{P} = (\epsilon_r - 1)\epsilon_0 \vec{\nabla}\cdot\vec{E} = 0$

$$\Rightarrow \rho_b = 0$$

on the upper surface

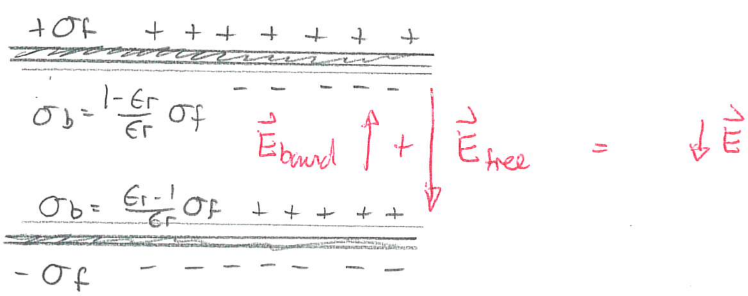
$$\sigma_b = \hat{z}\cdot\vec{P} = (\epsilon_r - 1)\epsilon_0\, \hat{z}\cdot\vec{E}$$

$$= (\epsilon_r - 1)\epsilon_0 \left(-\frac{\sigma_f}{\epsilon}\right)$$

$$= -\frac{(\epsilon_r - 1)\cancel{\epsilon_0}}{\epsilon_r \cancel{\epsilon_0}}\sigma_f$$

$$\sigma_b = \frac{1-\epsilon_r}{\epsilon_r}\sigma_f$$

Similarly on the lower surface $\quad \sigma_b = -\left(\frac{1-\epsilon_r}{\epsilon_r}\right)\sigma_f$



The dielectric has reduced the magnitude of the electric field, for the same free charge. So ΔV will have dropped and thus C increased.