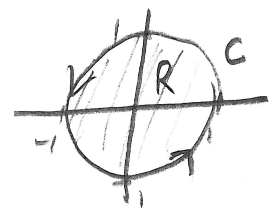
# Ch 10.4 Green's Theorem    In-Class Examples

(1) $\vec{F} = [y, 2x]$, $C: x^2+y^2=1$



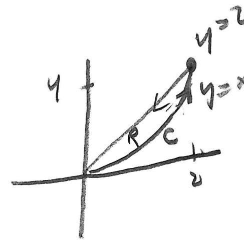

$$\oint_C \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \iint_R \left((F_2)_x - (F_1)_y\right)dx\,dy$$

$$= \iint_R (2-1)\,dx\,dy = \iint_R 1\cdot dA$$

$$= \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{2}r^2\Big|_0^1\,d\theta = \tfrac{1}{2}(\theta)\Big|_0^{2\pi} = \pi$$

**$2x\cosh(y) - x\cosh(y)$**

(2) $\vec{F} = [x\sinh y,\ x^2\cosh y]$, $x^2 \le y \le 2x$




**$x = \sqrt{y}$**
**$x = \frac{y}{2}$**

$$\oint \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \iint_R [(F_2)_x - (F_1)_y]\,dA$$

$$= \int_0^2\int_{x^2}^{2x} (2x\cosh y - x\cosh y)\,dy\,dx$$

$$= \int_0^2\int_{x^2}^{2x} x\cosh y\,dy\,dx = \int_0^2 x\sinh y\Big|_{x^2}^{2x}\,dx$$

$$= \int_0^2 [x\sinh(2x) - x\sinh(x^2)]\,dx$$

$\boxed{\begin{array}{ll} u = x & dv = \sinh(2x) \\ du = dx & v = \frac{\cosh(2x)}{2} \end{array}}$

$\boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{array} \Big\}\ \begin{array}{l} \text{when } x=0,\ u=0 \\ \text{when } x=2,\ u=4 \end{array}}$

$$= \frac{x}{2}\cosh(2x)\Big|_0^2 - \frac{1}{2}\int_0^2 \cosh(2x)\,dx - \frac{1}{2}\int_0^4 \sinh(u)\,du$$

$$= \cosh(4) - \frac{1}{4}\sinh(2x)\Big|_0^2 - \frac{1}{2}\cosh(u)\Big|_0^4$$

$$= \cosh(4) - \frac{1}{4}\sinh(4) - \frac{1}{2}\cosh(4) + \frac{1}{2}$$

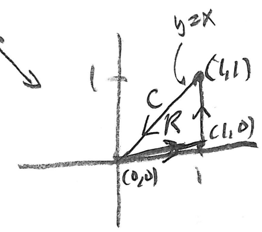
$$= \frac{1}{2}\cosh(4) - \frac{1}{4}\sinh(4) + \frac{1}{2}$$

$\left.\begin{array}{l} \cosh x = \frac{e^x + e^{-x}}{2} \\ \sinh x = \frac{e^x - e^{-x}}{2} \end{array}\right\}$

(3) Find $\int_C \frac{\partial w}{\partial n}\, ds$, using $\oint_C \frac{\partial w}{\partial n}\, ds = \iint_R \nabla^2 w \, dx\,dy$

$w = x^3 + y^3$

(a) $R =$ triangle with vertices $(0,0), (1,0), (1,1)$ $\leftrightarrow$



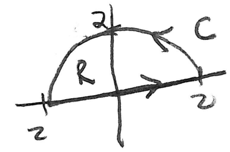

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 6x + 6y$$

$$\oint_C \frac{\partial w}{\partial n}\, ds = \iint_R (6x+6y)\, dA = \int_0^1\int_0^x (6x+6y)\, dy\, dx$$

$$= \int_0^1 6xy + 3y^2 \Big|_{y=0}^{y=x} dx = \int_0^1 (6x^2 + 3x^2)\, dx = \int_0^1 9x^2\, dx$$

$$= 3x^3\Big|_0^1 = 3$$

(b) $R: x^2 + y^2 \le 4,\ y \ge 0$ $\leftrightarrow$




$$\oint_C \frac{\partial w}{\partial n}\, ds = \iint_R \nabla^2 w\, dA$$

$$= \iint_R (6x+6y)\, dA = \int_0^\pi \int_0^2 (6r\cos\theta + 6r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi 4r^2(\cos\theta + \sin\theta)\Big|_{r=0}^{r=2} d\theta = \int_0^\pi 16(\cos\theta + \sin\theta)\, d\theta$$

$$= 16[\sin\theta - \cos\theta]\Big|_0^\pi = 16\left[(\overset{0}{\cancel{\sin\pi}} - \overset{-1}{\cancel{\cos\pi}}) - (\overset{0}{\cancel{\sin 0}} - \overset{1}{\cancel{\cos 0}})\right]$$

$$= 16[1 + 1] = 32$$

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Let $w(x,y)$ be continuous with continuous partials in $\mathbb{R}$;

see p.437.

Define $\vec{F}$ by $\vec{F} = \left[-\frac{\partial w}{\partial y}, \frac{\partial w}{\partial x}\right]$. Then

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w \quad (*)$$

Further, it is shown that $\vec{n} = \left[\frac{dy}{ds}, -\frac{dx}{ds}\right]$ is outward normal vector.

Then

$$\vec{\nabla} w \cdot \vec{n} = \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right] \cdot \left[\frac{dy}{ds}, \frac{dx}{ds}\right] = \frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds} \quad (**)$$

The quantity $\frac{\partial w}{\partial n}$ is defined as

$$\frac{\partial w}{\partial n} = \vec{\nabla} w \cdot \vec{n}$$

Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \overset{(*)}{=} \iint_R \nabla^2 w \, dx\,dy$$

$$\overset{\text{page 433}}{=} \oint_C F_1\,dx + F_2\,dy$$

$$= \oint_C \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds}\right) ds$$

$$= \oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right) ds$$

$$\overset{(**)}{=} \oint_C \frac{\partial w}{\partial n}\, ds$$