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MATH 366
CP Ch. 21.5

1) Applications involving elliptic equations usually lead to boundary value problems in a region R, called a first boundary value problem or Dirchlet problem if u is prescribed on the boundary curve C of R.

2) A second boundary value problem or Neumann problem if $u_n = \partial u/\partial n$ (normal derivative of u) is prescribed on C, and a third or mixed problem if u is prescribed on a part of C and u_n on the remaining part.

3)
$$\nabla^2 u = u_{xx} + u_{yy} = f(x,y) = 12xy$$

Elliptic PDE's — $ac - b^2 > 0$.

$a = 1,\ b = 0,\ c = 1 \ :\ (1)(1) - 0^2 = 1 \ :\ 1 > 0 \ \therefore$ This is an Elliptic PDE

4) Example 1 Mixed Boundary Value Problem for a Poisson Equation

Solve the mixed boundary value problem for the Poisson equation

$$\nabla^2 u = u_{xx} + u_{yy} = f(x,y) = 12xy$$

Shown in Fig. 458a.

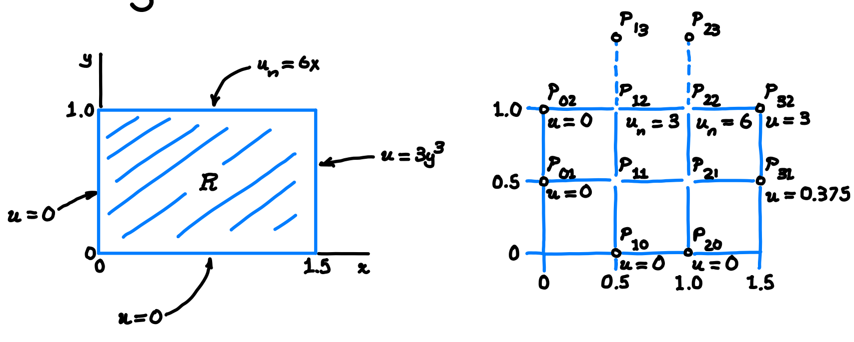

Fig. 458. Mixed boundary value problem in Example 1

Solution. We use the grid shown in Fig. 458b, where h=0.5. We recall (7) in Sec. 21.4 has the right side $h^2 f(x,y) = 0.5^2 \cdot 12xy = 3xy$. From the formulas $u = 3y^3$ and $u_n = 6x$ given on the boundary we compute the boundary data

(1) $u_{31} = 0.375,\quad u_{32} = 3,\quad \dfrac{\partial u_{12}}{\partial n} = \dfrac{\partial u_{12}}{\partial y} = 6 \cdot 0.5 = 3.\quad \dfrac{\partial u_{22}}{\partial n} = \dfrac{\partial u_{22}}{\partial y} = 6 \cdot 1 = 6.$

P_{11} and P_{21} are internal mesh points and can be handled as in the last section. Indeed, from (7), Sec. 21.4, with $h^2 = 0.25$ and $h^2 f(x,y) = 3xy$ and from the given boundary values we obtain two equations corresponding to P_{11} and P_{21}, as follows (with -0 resulting from the left boundary).

(2a)
$$\begin{aligned} -4u_{11} + u_{21} + u_{12} \quad\quad &= 12(0.5 \cdot 0.5)\cdot 1/4 - 0 = 0.75 \\ u_{11} - 4u_{21} \quad + u_{22} &= 12(1 \cdot 0.5)\cdot 1/4 - 0.375 = 1.125 \end{aligned}$$

The only difficulty with these equations seems to be that they involve the unknown values u_{12} and u_{22} of u at P_{12} and P_{22} on the boundary, where the normal derivative $u_n = \partial u/\partial n = \partial u/\partial y$ is given, instead of u; but we shall overcome this difficulty as follows.

We consider P_{12} and P_{22}. The idea that will help us here is. We imagine the region R to be extended above to the first row of external mesh points (corresponding to $y = 1.5$), and we assume that the Poisson equation also holds in the extended region. Then we can write down two more equations as before (Fig. 458b)

$$\text{(2b)} \qquad \begin{aligned} u_{11} \quad - 4u_{12} + u_{22} + u_{13} \qquad &= 1.5 - 0 = 1.5 \\ u_{21} + u_{12} - 4u_{22} \qquad + u_{23} &= 3 - 3 = 0. \end{aligned}$$

On the right, 1.5 is $12xyh^2$ at $(0.5, 1)$ and 3 is $12xyh^2$ at $(1,1)$ and 0 (at P_{02}) and 3 (at P_{32}) are given boundary values. We remember that we have not yet used the boundary condition on the upper part of the boundary of R, and we also notice that in (2b) we have introduced two more unknowns u_{13}, u_{23}. But we can now use that condition and get rid of u_{13}, u_{23} by applying the central difference formula for du/dy. From (1) we then obtain (see Fig. 458b)

$$3 = \frac{\partial u_{12}}{\partial y} \approx \frac{u_{13} - u_{11}}{2h} = u_{13} - u_{11}, \quad \text{hence} \quad u_{13} = u_{11} + 3$$

$$6 = \frac{\partial u_{22}}{\partial y} \approx \frac{u_{23} - u_{21}}{2h} = u_{23} - u_{21}, \quad \text{hence} \quad u_{23} = u_{21} + 6.$$

Substituting these results into (2b) and simplifying, we have

$$\begin{aligned} 2u_{11} \qquad - 4u_{12} + u_{22} &= 1.5 - 3 = -1.5 \\ 2u_{21} + u_{12} - 4u_{22} &= 3 - 3 - 6 = -6. \end{aligned}$$

Together with (2a) this yields, written in matrix form,

$$\text{(3)} \qquad \begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.125 \\ 1.5 - 3 \\ 0 - 6 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.125 \\ -1.5 \\ -6 \end{bmatrix}.$$

(The entries 2 come from u_{13} and u_{23}, and so do -3 and -6 on the right). The solution of (3) (obtained by Gauss elimination) is as follows; the exact values of the problem are given in parentheses.

$$\begin{aligned} u_{12} &= 0.866 \text{ (exact 1)} \qquad & u_{22} &= 1.812 \text{ (exact 2)} \\ u_{11} &= 0.077 \text{ (exact 0.125)} \qquad & u_{21} &= 0.191 \text{ (exact 0.25)}. \end{aligned}$$

5) Irregular Boundary

We continue our discussion of boundary value problems for elliptic PDE's in a region R in the xy-plane. If R has a simple geometric shape, we can usually arrange for certain mesh points to lie on the boundary C of R, and then we can approximate partial derivatives as explained in the last section. However, if C intersects the grid at points that are not mesh points, then at points close to the boundary we must proceed differently, as follows.

The mesh point O in Fig. 459 is of that kind. For O and its neighbors A and P we obtain from Taylor's theorem.

$$
(4) \qquad \begin{aligned}
&\text{(a)} \quad u_A = u_0 + ah\frac{\partial u_0}{\partial x} + \frac{1}{2}(ah)^2\frac{\partial^2 u_0}{\partial x^2} + \cdots\cdots \\
&\text{(b)} \quad u_P = u_0 - h\frac{\partial u_0}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 u_0}{\partial x^2} + \cdots\cdots .
\end{aligned}
$$

We disregard the terms marked by dots and eliminate $\partial u_0/\partial x$. Equation (4b) times a plus equation (4a) gives

$$u_A + au_P \approx (1+a)u_0 + \frac{1}{2}a(a+1)h^2\frac{\partial^2 u_0}{\partial x^2}.$$

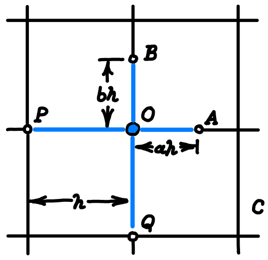

Fig. 459. Curved boundary C of a region R, a mesh point O near C, and neighbors A, B, P, Q.

We solve this last equation algebraically for the derivative, obtaining

$$\frac{\partial^2 u_0}{\partial x^2} \approx \frac{2}{h^2}\left[\frac{1}{a(1+a)}u_A + \frac{1}{1+a}u_P - \frac{1}{a}u_0\right].$$

Similarily, by considering the points O, B, and Q,

$$\frac{\partial^2 u_0}{\partial y^2} \approx \frac{2}{h^2}\left[\frac{1}{b(1+b)}u_B + \frac{1}{1+b}u_Q - \frac{1}{b}u_0\right].$$

By addition

$$(5) \qquad \nabla^2 u_0 \approx \frac{2}{h^2}\left[\frac{u_A}{a(1+a)} + \frac{u_B}{b(1+b)} + \frac{u_P}{1+a} + \frac{u_Q}{1+b} - \frac{(a+b)u_0}{ab}\right].$$

For example, if $a = \frac{1}{2}$, $b = \frac{1}{2}$, instead of the stencil (see Sec. 21.4)

$$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} \quad \text{we now have} \quad \left\{\begin{matrix} & \frac{4}{3} & \\ \frac{2}{3} & -4 & \frac{4}{3} \\ & \frac{2}{3} & \end{matrix}\right\}.$$

because $1/[a(1+a)] = 4/3$, etc. The sum of all five terms still being zero (which is useful for checking).

Using the same ideas, you may show that in the case of Fig. 460.

$$(6) \qquad \nabla^2 u_O \approx \frac{2}{h^2}\left[\frac{u_A}{a(a+p)} + \frac{u_B}{b(b+q)} + \frac{u_P}{p(p+a)} + \frac{u_Q}{q(q+b)} - \frac{ap+bq}{abpq}u_O\right],$$

a formula that takes care of all conceivable cases.

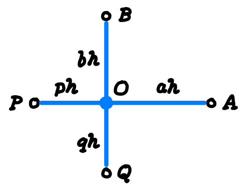

Fig. 460. Neighboring points A, B, P, Q of a mesh point O and notations in formula (6)