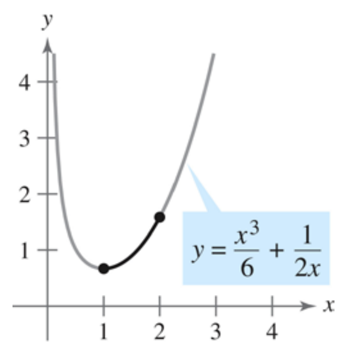
MAT 201
Larson/Edwards – Section 7.4 (part 1)
Arc Length and Surfaces of Revolution

In this section we will investigate finding the length of an arc of a function and the surface area of a solid of revolution. We will begin with arc length.

Suppose we would like to find the length of the arc on a ***rectifiable curve*** given by $f(x)=\frac{x^3}{6}+\frac{1}{2x}$ on the interval [1, 2] (i.e., the function f has a finite arc length on the interval [1, 2], and f' is continuous on [1, 2]). Such a function is ***continuously differentiable*** on [1, 2], and its graph on the interval [1, 2] is a ***smooth curve*** as in the figure below:



How would we go about finding the arc length of $f(x)=\frac{x^3}{6}+\frac{1}{2x}$ on the interval [1, 2]?

Now we can find the arc length of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on the interval [1, 2].

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f'(x) = \tfrac{1}{2}x^2 - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

$$S = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$$

$$S = \int_1^2 \sqrt{1 + \left[\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right]}\, dx$$

$$S = \int_1^2 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx$$

$$S = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$

$$S = \int_1^2 \tfrac{1}{2}x^2 + \tfrac{1}{2}x^{-2}\, dx$$

$$= \tfrac{1}{6}x^3 - \tfrac{1}{2}x^{-1} \Big|_1^2$$

$$= \left(\tfrac{1}{6}(2)^3 - \frac{1}{2(2)}\right) - \left(\tfrac{1}{6} - \tfrac{1}{2}\right)$$

$$= \left(\tfrac{4}{3} - \tfrac{1}{4}\right) - \left(\tfrac{1}{6} - \tfrac{1}{2}\right)$$

$$= \tfrac{4}{3} - \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{2} = \frac{17}{12} = 1.41\overline{6}$$

units

In area problems we were able to find the total area by adding rectangles as the width of those rectangles tended to 0. In volume problems we were able to find the total volume by adding the volumes of disks (or shells) as the width of the representative rectangle of the disks (or shells) tended to 0. Using this same principle we can add line segments from one point to the next on the curve as the line segments tend to 0.

What is the length of the line segment passing through the points (x_1, y_1) and (x_2, y_2)?

By using the distance formula we get the length of the line segment to be: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. We want to add these line segments as the points (x_1, y_1) and (x_2, y_2) get closer to each other. By letting $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, we can approximate the length of the arc, s, by:

$$\begin{aligned} s &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i) \end{aligned}$$

To make the approximation better for s, we take the limit as n goes to infinity. Also note, since $f'(x)$ exists for each x in (x_i, x_{i-1}), the Mean Value Theorem guarantees the existence of c_i in (x_i, x_{i-1}) such that $f'(c_i) = \dfrac{\Delta y_i}{\Delta x_i}$.

Finally we can develop a formula for finding the arc length of a function f on the interval $[a, b]$!

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\,(\Delta x_i) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

<u>Definition of Arc Length</u>:
Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of f between a and b is:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx\,.$$

Similarly, for a smooth curve given by $x = g(y)$, the arc length of g between c and d is:

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy\,.$$

Note: Choosing whether we use y as a function of x or x as a function of y is critical with these problems!!!

Now we can find the arc length of $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1, 2]$.

Ex: Find the arc length of the graph of $y = x^{2/3}$ over the interval [0, 8].

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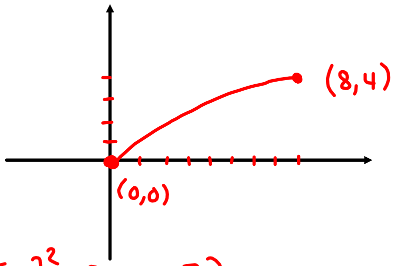

$$s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$s = \int_c^d \sqrt{1+[g'(y)]^2}\,dy$$

$$[y']^2 = \left[\frac{2}{3}x^{-1/3}\right]^2 = \frac{4}{9}x^{-2/3}$$

$$s = \int_0^8 \underbrace{\sqrt{1+\frac{4}{9}x^{-2/3}}}_{\text{sucks!}}\,dx$$

$$x = y^{3/2}$$

$$[x']^2 = \left[\frac{3}{2}y^{1/2}\right]^2 = \frac{9}{4}y$$

$$s = \int_0^4 \sqrt{1+\frac{9}{4}y}\,dy$$

$$u = 1+\frac{9}{4}y$$
$$du = \frac{9}{4}dy$$
$$\frac{4}{9}du = dy$$

$$\int \frac{4}{9}u^{1/2}\,du$$

$$\frac{2}{3}\cdot\frac{4}{9}u^{3/2}$$

$$\frac{8}{27}\left(1+\frac{9}{4}y\right)^{3/2}\Bigg|_0^4$$

$$= \frac{8}{27}(1+9)^{3/2} - \frac{8}{27}(1+0)^{3/2}$$

$$= \frac{8}{27}(10^{3/2}) - \frac{8}{27}(1)^{3/2}$$

$$= \frac{8\cdot 10\sqrt{10}}{27} - \frac{8}{27}$$

$$= \boxed{\frac{80\sqrt{10}-8}{27} \approx 9.07 \text{ units}}$$

TOL NOT MET