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MATH 360
CP Ch. 12.9

Vibrating Rectangular Membrane

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(2) $u = 0$ on the boundary

(3a) $u(x,y,0) = f(x,y)$

(3b) $u_t(x,y,0) = g(x,y)$

Solving The 2-D Vibrating Membrane

Step 1. By separating variables, first setting $u(x,y,t) = F(x,y)G(t)$ and later $F(x,y) = H(x)Q(y)$ we obtain from (1) an ODE (4) for G and later from a PDE (5) for F two ODE's (6) and (7) for H and Q.

Step 2. From the solutions of those ODE's we determine solutions (13) of (1) ("Eigenfunctions" u_{mn}) that satisfy the boundary condition (2).

Step 3. We compose the u_{mn} into a double series (14) solving the whole model (1), (2), (3).

Example 2 Vibration of a Rectangular Membrane

Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft (Fig. 305) if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft^2 (as for light rubber), the initial velocity is 0, and the initial displacement is,

(20) $$f(x,y) = 0.1(4x - x^2)(2y - y^2) \text{ ft}$$

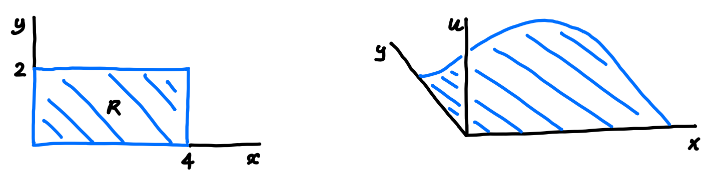

Fig. 305. Example 2

Solution: $c^2 = T/\rho = 12.5/2.5 = 5[\text{ft}^2/\text{s}^2]$. Also $B^*_{mn} = 0$ from (19). From (18) and (20),

$$B_{mn} = \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)\sin\left(\frac{m\pi}{4}x\right)\sin\left(\frac{n\pi}{2}y\right)\,dx\,dy$$

$$= \frac{1}{20}\int_0^4 (4x - x^2)\sin\left(\frac{m\pi}{4}x\right)dx\int_0^2 (2y - y^2)\sin\left(\frac{n\pi}{2}y\right)dy$$

Two integrations by parts gives for the first integral on the right

$$\frac{128}{m^3\pi^3}\left[1-(-1)^m\right] = \frac{256}{m^3\pi^3} \qquad (m \text{ odd})$$

and for the second integral

$$\frac{16}{n^3\pi^3}\left[1-(-1)^n\right] = \frac{32}{n^3\pi^3} \qquad (n \text{ odd})$$

For even m or n we get 0. Together with the factor 1/20 we thus have $B_{mn} = 0$ if m or n is even and

$$B_{mn} = \frac{256 \cdot 32}{20 m^3 n^3 \pi^6} = \frac{0.426050}{m^3 n^3} \qquad (m \text{ and } n \text{ both odd})$$

From this, (9), and (14) we obtain the answer

$$u(x,y,t) = 0.426050 \sum_{m,n \text{ odd}}\sum \frac{1}{m^3 n}\left(\cos\frac{\sqrt{5}\pi\sqrt{5}}{4}t \sin\frac{\pi x}{4}\sin\frac{\pi y}{2} + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{37}}{4}t \sin\frac{\pi x}{4}\sin\frac{3\pi y}{2}\right.$$

$$\left. + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{13}}{4}t \sin\frac{3\pi x}{4}\sin\frac{\pi y}{2} + \frac{1}{729}\cos\frac{\sqrt{5}\pi\sqrt{45}}{4}t \sin\frac{3\pi x}{4}\sin\frac{3\pi y}{2} + \cdots\right)$$

To discuss this solution, we note that the first term is very similar to the initial shape of the membrane, has no nodal lines, and is by far the dominating term because the coefficients of the next terms are much smaller. The second term has two horizontal nodal lines ($y = 2/3, 4/3$), the third term two vertical ones ($x = 4/3, 8/3$) the fourth term two horizontal and two vertical ones, and so on.