Statistical and Thermal Physics: Class Exam II

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Instructions

- There are 6 questions on 9 pages.
- Show your reasoning and calculations and always justify your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\,K} \qquad N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01 \times 10^5\,\mathrm{Pa}$$

$$k = 1.38 \times 10^{-23}\,\mathrm{J/K} = 8.61 \times 10^{-5}\,\mathrm{eV/K} \qquad e = 1.60 \times 10^{-19}\,\mathrm{C}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N \ln N - N \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

Question 1

The multiplicity of a system in terms of its energy, E, is $\Omega(E)$. Briefly describe how this could be used to determine the temperature of the system in terms of its energy.

$S = k \ln(\Omega(E))$

$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N \longrightarrow \boxed{T = \frac{1}{\left(\frac{\partial S}{\partial E}\right)_N}}$

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Question 2

Consider a system that consists of five distinguishable non-interacting spin-1/2 particles, each with magnetic dipole μ and in a magnetic field with magnitude B. The energy of a single particle with spin up is $-\mu B$ and the energy of a particle with spin down is $+\mu B$.

a) The total energy of the ensemble is $+\mu B$. Determine the multiplicity of this state and list all possible microstates of the system of five particles.

$N_- = 3$, $N_+ = 2$, $N = 5$

$$\Omega(5,2) = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot 2!} = \frac{20}{2} = 10$$

$\Omega = 10$

1 → spin up , 0 → spin down

11000	00101
10100	00011
10010	00110
10001	01100
01001	01010

b) Suppose that you selected two of the spin-1/2 particles and measured their energies. Determine the probability that the sum of their energies is zero.

10 possible states, 6 w/ E = 0 ∴

$P_{E=0} = \frac{6}{10} = \frac{3}{5} = 60\%$

$\boxed{P_{E=0} = 60\%}$

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) A system of spin-1/2 particles consists of four particles. The energy of the system is zero. Two more particles are added to the system and its energy is still zero. Describe whether the entropy of the system increased, decreased or remained constant as a result of the addition of these particles. Explain your answer.

$$S = k\ln(\Omega)\ , \quad \Omega_i = \Omega(4,2) = \binom{4}{2} = \frac{4!}{2!\cdot 2!} = \frac{4\cdot 3\cdot \cancel{2!}}{\cancel{2!}\cdot 2!} = \frac{12}{2} = 6$$

$$S_i = k\ln(6)$$

$$\Omega_f = \Omega(6,3) = \binom{6}{3} = \frac{6!}{3!\cdot 3!} = \frac{6\cdot 5\cdot 4\cdot \cancel{3!}}{3!\cdot \cancel{3!}} = \frac{120}{6} = 20$$

$$S_f = k\ln(20)$$

$$\Delta S = S_f - S_i = k\ln(20) - k\ln(6) = k\ln(10/3)$$

$$\ln(10/3) > 1 \quad \therefore \quad \Delta S > 0$$

$$\boxed{\Delta S > 0}$$

Question 3 continued . . .

b) A single electron in a two dimensional square well with sides of length L has energy

$$E_{n_x,n_y} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2\right)$$

where $n_x, n_y \geqslant 1$ are integers representing its state. Let $\Gamma(E)$ be the number of states with energy less than or equal to E. Determine

$$\Gamma\left(50\frac{h^2}{8mL^2}\right).$$

$n_x^2 + n_y^2 \leq 50$, $n_x, n_y \leq 7$

$$\begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \\ 1 & 7 \end{bmatrix} \end{array} - 7 \qquad \begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 4 & 1 \\ 4 & 2 \\ \vdots & \\ 4 & 5 \end{bmatrix} \end{array} - 5$$

$$\begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ \vdots & \\ 2 & 6 \end{bmatrix} \end{array} - 6 \qquad \begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 5 & 1 \\ 5 & 2 \\ \vdots & \\ 5 & 4 \end{bmatrix} \end{array} - 4$$

$$\begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 3 & 1 \\ 3 & 2 \\ \vdots & \\ 3 & 6 \end{bmatrix} \end{array} - 6 \qquad \begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 6 & 1 \\ 6 & 2 \\ 6 & 3 \end{bmatrix} \end{array} - 3$$

$$\begin{array}{c} \begin{array}{cc} n_x & n_y \end{array} \\ \begin{bmatrix} 7 & 1 \end{bmatrix} \end{array} - 1$$

$$\boxed{\Gamma(E) = 32}$$

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Question 4

One Einstein solid, consisting of three oscillators is in thermal contact with another consisting of two oscillators. Each initially has five units of energy.

a) Show that the multiplicity of the system with three oscillators is $\Omega = (q+2)(q+1)/2$ where q is the number of energy units that it has.

$$\Omega(N,q) = \Omega(3,q) = \binom{2+q}{q} = \frac{(2+q)!}{q!(2!)} = \frac{(q+2)(q+1)\cancel{(q!)}}{\cancel{q!}(2)} = \frac{(q+2)(q+1)}{2}$$

$$\boxed{\Omega = \frac{(q+2)(q+1)}{2}}$$

b) Show that the multiplicity of the system with two oscillators is $\Omega = q + 1$ where q is the number of energy units that it has.

$$\Omega(N,q) = \Omega(2,q) = \binom{1+q}{q} = \frac{(1+q)!}{q!(1)!} = \frac{(1+q)\cancel{(q!)}}{\cancel{q!}} = 1+q$$

$$\boxed{\Omega = 1+q}$$

c) Determine the equilibrium macrostate of the two systems after they have been able to interact thermally.

$$\Omega_A = \frac{(q_A+2)(q_A+1)}{2}, \quad \Omega_B = 1+q_B$$

q_A	q_B	Ω_A	Ω_B	Ω
10	0	66	1	66
9	1	55	2	110
8	2	45	3	135
7	3	36	4	144 ← most probable
6	4	28	5	140
5	5	21	6	126
4	6	15	7	105
3	7	10	8	80
2	8	6	9	54
1	9	3	10	30
0	10	1	11	11

$$\boxed{\begin{array}{l}\text{Equilibrium state} \\ \Omega = 144 \\ q_A = 7,\ q_B = 3\end{array}}$$

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Question 5

Consider a system of N identical distinguishable one-dimensional quantum harmonic oscillators, each with frequency ω. Recall that the energy of the oscillator in the state labeled n is $\hbar\omega(n + 1/2)$.

a) Show that the partition function for the system is

$$Z = \frac{e^{-\hbar\omega\beta N/2}}{(1 - e^{-\hbar\omega\beta})^N}$$

where $\beta = 1/kT$.

$$Z = \sum e^{-\hbar\omega(n+\frac{1}{2})\beta} = \sum e^{-\hbar\omega\beta n} e^{-\hbar\omega\beta/2}$$

$$Z = e^{-\hbar\omega\beta/2} \sum e^{-\hbar\omega\beta n}$$

$$Z = e^{-\hbar\omega\beta/2} \underbrace{\left(1 + e^{-\hbar\omega\beta} + e^{-2\hbar\omega\beta} + \ldots\ldots e^{-\hbar\omega\beta n}\right)}_{\text{Geometric series}}$$

$$Z_{\text{Single}} = \frac{e^{-\hbar\omega\beta/2}}{(1 - e^{-\hbar\omega\beta})}, \quad Z_{\text{Ensemble}} = (Z_{\text{Single}})^N$$

$$Z_{\text{Ensemble}} = \frac{e^{-\hbar\omega\beta N/2}}{(1 - e^{-\hbar\omega\beta})^N}$$

$$\boxed{Z = \frac{e^{-\hbar\omega\beta N/2}}{(1 - e^{-\hbar\omega\beta})^N}}$$

Question 5 continued ...

b) Determine an expression for the mean energy of the system.

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln(Z) = -\frac{\partial}{\partial \beta} \ln\left(\left(\frac{e^{-\hbar\omega\beta/2}}{1-e^{-\hbar\omega\beta}}\right)^N\right) = -\frac{\partial}{\partial \beta} N \ln\left(\frac{e^{-\hbar\omega\beta/2}}{1-e^{-\hbar\omega\beta}}\right)$$

$$\bar{E} = -\frac{\partial}{\partial \beta} N \left[\ln(e^{-\hbar\omega\beta/2}) - \ln(1-e^{-\hbar\omega\beta})\right]$$

$$\bar{E} = -N \cdot \frac{\partial}{\partial \beta}\left(-\hbar\omega\beta/2 - \ln\left(1-e^{-\hbar\omega\beta}\right)\right) = -N\left(-\frac{\hbar\omega}{2} - \frac{\hbar\omega e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right)$$

$$\bar{E} = N\left(\frac{\hbar\omega - \hbar\omega e^{-\hbar\omega\beta} + 2\hbar\omega e^{-\hbar\omega\beta}}{2(1-e^{-\hbar\omega\beta})}\right) = \frac{N\hbar\omega}{2}\left(\frac{1+e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right)$$

$$\boxed{\bar{E} = \frac{N\hbar\omega}{2}\left(\frac{1+e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right)}$$

c) Suppose that $T \gg \hbar\omega/k$. Determine an approximation for the energy (to lowest non-zero order).

$$\frac{-\hbar\omega}{kT} \qquad 1 \gg \frac{\hbar\omega}{kT} \qquad \bar{E} = \frac{N\hbar\omega}{2}\left(\frac{e^{\hbar\omega\beta}+1}{e^{\hbar\omega\beta}-1}\right)$$

$$\bar{E} \approx \frac{N\cancel{\hbar\omega}}{2}\left(\frac{2+\hbar\omega\beta}{\cancel{\hbar\omega}\beta}\right) \approx \frac{N}{2}\left(\frac{2+\hbar\omega\beta}{\beta}\right) \quad : \ \hbar\omega\beta \approx 0 \ \therefore$$

$$\bar{E} \approx \frac{N\cdot 2}{\cancel{2}\beta} \approx NkT$$

$$\boxed{\bar{E} \approx NkT}$$

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Question 6

Answer either part a) or part b) for full credit for this problem.

a) An ensemble of spin-1/2 particles, each with magnetic dipole $\mu = 1.0 \times 10^{-4}\,\text{eV/T}$ is placed in a magnetic field with strength $8.0\,\text{T}$. The system is at temperature $300\,\text{K}$. Determine the fraction of particles in the spin down state.

$$P = \frac{e^{-E_s\beta}}{Z} \quad : \quad Z = e^{-\mu B\beta} + e^{\mu B\beta}$$

$$P = \frac{e^{-\mu B\beta}}{e^{-\mu B\beta} + e^{\mu B\beta}} = \frac{1}{1 + e^{2\mu B\beta}} \quad : \quad 2\mu B\beta = \frac{2(1.0\times10^{-4}\,\text{eV/T})(8.0\,\text{T})}{8.61\times10^{-5}\,\text{eV/K}\,(300\,\text{K})}$$

$$P_- = \frac{1}{1 + e^{0.061943}} \approx 0.484 \approx 0.48$$

$$\boxed{P_- = 0.48}$$

Question 6 continued ...

b) A particular atom can be in one of two states, whose energies differ by 0.50 eV. Determine how many more times probable it is that a single particle will be in the lower energy state than the higher energy state if the temperature is 1000 K.

$$\frac{P_L}{P_H} = \frac{e^{-E_L\beta}}{e^{-E_H\beta}} = e^{+(E_H - E_L)\beta} \quad : \; E_H - E_L = 0.50 \text{ eV}$$

$$\frac{P_L}{P_H} = e^{(0.50 \text{ eV})/(8.61\times10^{-5} \text{ eV/K} \cdot 1000 \text{ K})} = 332.687 \approx 333$$

$$\boxed{333}$$

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