Exercise 1

ρ_i —— $\boxed{\text{Phase Flip}}$ —— ρ_f QFI ? $\hat{\rho}_F = (1-\lambda)\hat{\rho}_i + \lambda\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z$

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a.) Calculate phase flip

$$\hat{\rho}_F = \frac{(1-\lambda)}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (1-\lambda) & r(1-\lambda) \\ r(1-\lambda) & (1-\lambda) \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -r \\ r & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1-\lambda) & r(1-\lambda) \\ r(1-\lambda) & (1-\lambda) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \lambda & -r\lambda \\ -r\lambda & \lambda \end{pmatrix}$$

$$\hat{\rho}_F = \frac{1}{2}\begin{pmatrix} 1 & r(1-2\lambda) \\ r(1-2\lambda) & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_F = \frac{1}{2}\begin{pmatrix} 1 & r(1-2\lambda) \\ r(1-2\lambda) & 1 \end{pmatrix}}$$

b.) Calculate QFI

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I}, \quad H_i = Tr\left[L_i\frac{\partial A_i}{\partial\lambda}\right]$$

i.) Calculate $\hat{L}$: $\alpha = Tr(\hat{A}^2) - Tr(\hat{A})^2$

$$\hat{A}^2 = \frac{1}{4}\begin{pmatrix} 1 & r(1-2\lambda) \\ r(1-2\lambda) & 1 \end{pmatrix}\begin{pmatrix} 1 & r(1-2\lambda) \\ r(1-2\lambda) & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+r^2(1-2\lambda)^2 & 2r(1-2\lambda) \\ 2r(1-2\lambda) & 1+r^2(1-2\lambda)^2 \end{pmatrix}$$

$$Tr(\hat{A}^2) = \frac{1}{2}\left(1+r^2(1-2\lambda)^2\right)$$

$Tr(\hat{A}) = 1 \quad \therefore \quad Tr(\hat{A})^2 = 1$

$$\frac{d}{d\lambda}\ln(\alpha) = \frac{4(2\lambda-1)r^2}{(2\lambda-1)^2r^2-1} \longrightarrow \beta$$

$$\alpha = \frac{1}{2}\left((2\lambda-1)^2r^2-1\right)$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial\ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \cancel{\ln(Tr(\hat{A}))}^{\,0}\right]\hat{I}$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial\ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\ln(\alpha)\,\hat{I} \quad : \quad \frac{\partial\ln(\alpha)}{\partial\lambda} = \frac{4(2\lambda-1)r^2}{(2\lambda-1)^2r^2-1} \longrightarrow \beta$$

$$\frac{\partial\hat{A}}{\partial\lambda} = \frac{1}{2}\begin{pmatrix} 0 & -2r \\ -2r & 0 \end{pmatrix} = \begin{pmatrix} 0 & -r \\ -r & 0 \end{pmatrix} : \frac{\partial\hat{A}}{\partial\lambda} = \begin{pmatrix} 0 & -r \\ -r & 0 \end{pmatrix} :$$

Exercise 1 continued

$$\hat{L} = \left[2\begin{pmatrix} 0 & -r \\ -r & 0 \end{pmatrix} - \beta \cdot \frac{1}{2}\begin{pmatrix} 1 & r(1-2\lambda) \\ r(1-2\lambda) & 1 \end{pmatrix} \right] + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left[\frac{1}{2}\begin{pmatrix} 0 & -4r \\ -4r & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \beta & \beta r(1-2\lambda) \\ \beta r(1-2\lambda) & \beta \end{pmatrix} \right] + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -\beta & -\beta r(1-2\lambda) - 4r \\ -\beta r(1-2\lambda) - 4r & -\beta \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \beta & -\beta r(1-2\lambda) - 4r \\ -\beta r(1-2\lambda) - 4r & \beta \end{pmatrix}$$

$$\boxed{\hat{L} = \frac{1}{2}\begin{pmatrix} \beta & -\beta r(1-2\lambda) - 4r \\ -\beta r(1-2\lambda) - 4r & \beta \end{pmatrix}}$$

ii.) Calculate $H = \mathrm{Tr}\left[\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda}\right]$

$$\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{2}\begin{pmatrix} \beta & -\beta r(1-2\lambda) - 4r \\ -\beta r(1-2\lambda) - 4r & \beta \end{pmatrix} \cdot \begin{pmatrix} 0 & -r \\ -r & 0 \end{pmatrix}$$

$$\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{2}\begin{pmatrix} \beta r^2(1-2\lambda) + 4r^2 & -\beta r \\ -\beta r & \beta r^2(1-2\lambda) + 4r^2 \end{pmatrix}$$

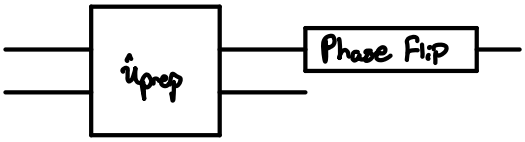
$$\mathrm{Tr}\left[\hat{L} \cdot \frac{\partial \hat{A}}{\partial \lambda}\right] = \beta r^2(1-2\lambda) + 4r^2 \quad \therefore \quad H = \beta r^2(1-2\lambda) + 4r^2 = -\beta r^2(2\lambda - 1) + 4r^2$$

$$H = -\frac{4(2\lambda-1)r^2}{(2\lambda-1)^2 r^2 - 1} \cdot r^2(2\lambda-1) + 4r^2 = \frac{-4r^4\cancel{(2\lambda-1)^2} + 4r^4\cancel{(2\lambda-1)^2} - 4r^2}{(2\lambda-1)^2 r^2 - 1} = \frac{-4r^2}{(2\lambda-1)^2 r^2 - 1}$$

$$H = \frac{4r^2}{1-(2\lambda-1)^2 r^2} = \frac{4r^2}{1-(1-2\lambda)^2 r^2}$$

$$\boxed{H = \frac{4r^2}{1-(1-2\lambda)^2 r^2}}$$

Exercise 2



a.) Write out $\hat{\rho}_{prep}$ in terms of $|0\rangle|0\rangle\langle 0|\langle 0|$ .....

$$\hat{\rho}_{prep} = \hat{U}_{prep}\hat{\rho}\hat{U}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]}$$

b.) Determine phase flip on $\hat{\rho}_{prep}$

$$\hat{\rho}_{prep} = \hat{U}_{prep}\hat{\rho}\hat{U}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

Phase flip acts on the first particle only:

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\underline{|0\rangle\langle 0|}\otimes|0\rangle\langle 0| + (-8ir)\underline{|0\rangle\langle 1|}\otimes|0\rangle\langle 1| + (4-4r^2)\underline{|0\rangle\langle 0|}\otimes|1\rangle\langle 1| + (4-4r^2)\underline{|1\rangle\langle 1|}\otimes|0\rangle\langle 0| + (8ir)\underline{|1\rangle\langle 0|}\otimes|1\rangle\langle 0| + (4+4r^2)\underline{|1\rangle\langle 1|}\otimes|1\rangle\langle 1|\Big]$$

$\hat{\rho}_i = |0\rangle\langle 0| : \hat{\rho}_f = |0\rangle\langle 0|$, $\hat{\rho}_i = |1\rangle\langle 1| : \hat{\rho}_f = |1\rangle\langle 1|$, $\hat{\rho}_i = |0\rangle\langle 1| : \hat{\rho}_f = (1-2\lambda)|0\rangle\langle 1|$, $\hat{\rho}_i = |1\rangle\langle 0| : \hat{\rho}_f = (1-2\lambda)|1\rangle\langle 0|$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (1-2\lambda)(-8ir)|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| + (4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (1-2\lambda)(8ir)|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (1-2\lambda)(-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (1-2\lambda)(8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

Exercise 2 Continued

c.) Calculate QFI

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I} \quad , \quad H_i = Tr\left[L_i \frac{\partial A_i}{\partial\lambda}\right]$$

$$\alpha = Tr(\hat{A}^2) - Tr(\hat{A})^2 : \quad \hat{A}_1 = \frac{1}{16}\begin{pmatrix} (4+4r^2) & (1-2\lambda)(-8ir) \\ (1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}, \quad \hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4-4r^2) & 0 \\ 0 & (4-4r^2) \end{pmatrix}$$

i.) Find $\hat{L}_1$: $\alpha_1 = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2) & (1-2\lambda)(-8ir) \\ (1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}\begin{pmatrix} (4+4r^2) & (1-2\lambda)(-8ir) \\ (1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}$$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)^2 + 64r^2(1-2\lambda)^2 & 2(4+4r^2)(1-2\lambda)(-8ir) \\ 2(4+4r^2)(1-2\lambda)(8ir) & (4+4r^2)^2 + 64r^2(1-2\lambda)^2 \end{pmatrix} : \quad Tr(\hat{A}_1^2) = \frac{1}{16^2}\left(2(4+4r^2)^2 + 128r^2(1-2\lambda)^2\right)$$

$$Tr(\hat{A}_1) = \frac{1}{8}(4+4r^2) \quad : \quad Tr(\hat{A}_1)^2 = \frac{1}{64}(4+4r^2)^2$$

$$\alpha = \frac{1}{16^2}\left(2(4+4r^2)^2 + 128r^2(1-2\lambda)^2\right) - \frac{1}{64}(4+4r^2)^2$$

not dependent upon λ!

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \cancel{\ln(Tr(\hat{A}))}\right]\hat{I}$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha)\right]\hat{I}$$

$$\frac{\partial}{\partial\lambda}\ln(\alpha) = \frac{\alpha'}{\alpha} : \quad \alpha' = \frac{1}{16^2}\cdot -256r^2(1-2\lambda)\cdot 2 = -2r^2(1-2\lambda)$$

$$\alpha = \frac{1}{16^2}\left(2(4+4r^2)^2 + 128r^2(1-2\lambda)^2\right) - \frac{1}{64}(4+4r^2)^2 = \frac{1}{256}\left(2\cdot 16(1+r^2)^2 + 128r^2(1-2\lambda)^2\right) - \frac{1}{64}16(1+r^2)^2$$

$$\alpha = \frac{1}{8}\left((1+r^2)^2 + 4r^2(1-2\lambda)^2\right) - \frac{1}{4}(1+r^2)^2 = \frac{1}{2}r^2(1-2\lambda^2) - \frac{1}{8}(1+r^2)^2 = \frac{4r^2(1-2\lambda)^2 - (1+r^2)^2}{8}$$

$$\frac{\partial \ln(\alpha)}{\partial\lambda} = \frac{-16r^2(1-2\lambda)}{4r^2(1-2\lambda)^2 - (1+r^2)^2} = \frac{16r^2(1-2\lambda)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2} \longrightarrow \beta$$

$$\hat{L}_1 = \frac{1}{Tr(\hat{A}_1)}\left[2\cdot\frac{\partial \hat{A}_1}{\partial\lambda} - \beta\hat{A}\right] + \beta\hat{I}$$

$$\hat{A}_1 = \frac{1}{16}\begin{pmatrix} (4+4r^2) & (1-2\lambda)(-8ir) \\ (1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}$$

$$\frac{\partial \hat{A}_1}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & 16ir \\ -16ir & 0 \end{pmatrix} = \begin{pmatrix} 0 & ir \\ -ir & 0 \end{pmatrix} \quad : \quad Tr(\hat{A}_1) = \frac{(1+r^2)}{2}$$

$$\hat{L}_1 = \frac{2}{(1+r^2)}\left[\begin{pmatrix} 0 & 2ir \\ -2ir & 0 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} \beta(4+4r^2) & \beta(1-2\lambda)(-8ir) \\ \beta(1-2\lambda)(8ir) & \beta(4+4r^2) \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}\right]$$

Exercise 2 Continued

$$\hat{L}_1 = \frac{2}{(1+r^2)}\left[\frac{1}{16}\begin{pmatrix} 0 & 32ir \\ -32ir & 0 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} \beta(4+4r^2) & \beta(1-2\lambda)(-8ir) \\ \beta(1-2\lambda)(8ir) & \beta(4+4r^2) \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 16\beta & 0 \\ 0 & 16\beta \end{pmatrix}\right]$$

$$= \frac{2}{(1+r^2)}\left[\frac{1}{16}\begin{pmatrix} 16\beta - \beta(4+4r^2) & 32ir - \beta(1-2\lambda)(-8ir) \\ -32ir - \beta(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}\right]$$

$$\hat{L}_1 = \frac{1}{8(1+r^2)}\begin{pmatrix} 16\beta - \beta(4+4r^2) & 32ir - \beta(1-2\lambda)(-8ir) \\ -32ir - \beta(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}$$

$$\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda} = \frac{1}{8(1+r^2)}\begin{pmatrix} 16\beta - \beta(4+4r^2) & 32ir - \beta(1-2\lambda)(-8ir) \\ -32ir - \beta(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}\begin{pmatrix} 0 & ir \\ -ir & 0 \end{pmatrix}$$

$$[1,1] = (16\beta - \beta(\cancelto{0}{4+4r^2}))(0) + (32ir + \beta(1-2\lambda)(8ir))(-ir)$$
$$32r^2 + \beta(1-2\lambda)(8r^2) \checkmark$$

$$[2,1] = (-32ir - \beta(\cancelto{0}{1-2\lambda})(8ir))(0) + (16\beta - \beta(4+4r^2))(-ir) \checkmark$$

$$[1,2] = (16\beta - \beta(4+4r^2))(ir) + (32ir - \beta(1-2\lambda)(\cancelto{0}{-8ir}))(0)$$

$$[2,2] = (-32ir - \beta(1-2\lambda)(8ir))(ir) + (16\beta - \beta(\cancelto{0}{4+4r^2}))(0)$$
$$32r^2 + \beta(1-2\lambda)(8r^2)$$

$$\boxed{\hat{L}_1 \cdot \frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{8(1+r^2)}\begin{pmatrix} 32r^2 + \beta(1-2\lambda)(8r^2) & (16\beta - \beta(4+4r^2))(ir) \\ (16\beta - \beta(4+4r^2))(-ir) & 32r^2 + \beta(1-2\lambda)(8r^2) \end{pmatrix}}$$

ii.) Calculate $H_1 = \mathrm{Tr}\left[L_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda}\right]$

$$\hat{L}_1 \cdot \frac{\partial \hat{A}}{\partial \lambda} = \frac{1}{8(1+r^2)}\begin{pmatrix} 32r^2 + \beta(1-2\lambda)(8r^2) & (16\beta - \beta(4+4r^2))(ir) \\ (16\beta - \beta(4+4r^2))(-ir) & 32r^2 + \beta(1-2\lambda)(8r^2) \end{pmatrix}$$

$$\mathrm{Trace}\left[\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda}\right] = \frac{1}{8(1+r^2)}\left[64r^2 + \beta(1-2\lambda)(16r^2)\right] = \frac{1}{(1+r^2)}\left(8r^2 + \beta(1-2\lambda)(2r^2)\right)$$

$$= \frac{1}{(1+r^2)}\left[8r^2 + \frac{16r^2(1-2\lambda)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}(1-2\lambda)2r^2\right]$$

$$= \frac{1}{(1+r^2)}\left[8r^2 + \frac{32r^4(1-2\lambda)^2}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}\right]$$

$$= \frac{1}{(1+r^2)}\left[\frac{8r^2(1+r^2)^2 - \cancel{32r^4(1-2\lambda)^2} + \cancel{32r^4(1-2\lambda)^2}}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}\right]$$

$$= \frac{1}{\cancel{(1+r^2)}} \cdot \frac{8r^2(1+r^2)^{\cancel{2}}}{(1+r^2)^2 - 4r^2(1-2\lambda)^2} = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}$$

$$\boxed{H_1 = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}$$

Exercise 2 Continued

iii.) Find $\hat{L}_2$: $\alpha = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix}(4-4r^2) & 0\\ 0 & (4-4r^2)\end{pmatrix}\begin{pmatrix}(4-4r^2) & 0\\ 0 & (4-4r^2)\end{pmatrix} = \frac{1}{16^2}\begin{pmatrix}(4-4r^2)^2 & 0\\ 0 & (4-4r^2)^2\end{pmatrix}$$

$$Tr(\hat{A}_2^2) = \frac{1}{128}(4-4r^2)^2 \quad : \quad Tr(\hat{A}_2) = \frac{1}{8}(4-4r^2) \quad : \quad Tr(\hat{A}_2)^2 = \frac{1}{64}(4-4r^2)^2$$

$\alpha = -\frac{1}{128}(4-4r^2)^2$, Since α is not dependent upon λ, $\hat{L}_2$ becomes

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \cancelto{0}{\frac{\partial \ln(\alpha)}{\partial\lambda}}\hat{A}\right] + \cancelto{0}{\frac{\partial}{\partial\lambda}}\left[\cancelto{0}{\ln(\alpha)} - \cancelto{0}{\ln(Tr(\hat{A}))}\right]\hat{I} = \frac{1}{Tr(\hat{A}_2)}\cdot 2\cdot\frac{\partial\hat{A}_2}{\partial\lambda}$$

$\frac{\partial\hat{A}_2}{\partial\lambda} = 0 \quad \therefore \quad \hat{L}_1$ and thus $\boxed{H_2 = 0}$

iv.) Calculate $H = H_1 + H_2$: $H_2 = 0 \quad \therefore \quad H = H_1$

$$\boxed{H = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}$$

Proof :

$$H(\lambda) = \frac{m^2(1-2\lambda)^{2m-2}}{2^{n-1}}\sum_{j=0}^{n}\binom{n}{j}\frac{c_j^2 d_j}{d_j^2-(1-2\lambda)^{2m}c_j^2}$$

$$c_j = (1+r)^j(1-r)^{n-j} - (1+r)^{n-j}(1-r)^j$$
$$d_j = (1+r)^j(1-r)^{n-j} + (1+r)^{n-j}(1-r)^j$$

$m=1$, $n=2$, $j=0$ $\qquad \frac{1}{2}\sum_{j=0}^{2}\binom{2}{j}\frac{c_j^2 d_j}{d_j^2-(1-2\lambda)^2 c_j^2}$

$j=0$: $c_0 = (1-r)^2 - (1+r)^2 = \cancel{1} - 2r + \cancel{r^2} - \cancel{1} - 2r - \cancel{r^2} = -4r$
: $d_0 = (1-r)^2 + (1+r)^2 = 1 - \cancel{2r} + r^2 + 1 + \cancel{2r} + r^2 = 2(1+r^2)$

$j=1$: $c_1 = (1+r)(1-r) - (1+r)(1-r) = 0$
$d_1 = (1+r)(1-r) + (1+r)(1-r) = 2(1-r^2)$

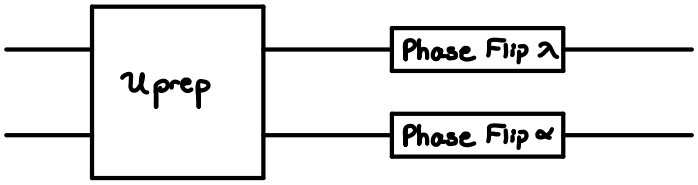
$j=2$: $c_2 = (1+r)^2 - (1-r)^2 = \cancel{1} + 2r + \cancel{r^2} - \cancel{1} + 2r - \cancel{r^2} = 4r$
$d_2 = (1+r)^2 + (1-r)^2 = 1 + \cancel{2r} + r^2 + 1 - \cancel{2r} + r^2 = 2(1+r^2)$

$$\frac{1}{2}\binom{2}{0}\frac{c_0^2 d_0}{d_0^2-(1-2\lambda)^2 c_0^2} + \frac{1}{2}\binom{2}{1}\cancelto{0}{\frac{c_1^2 d_1}{d_1^2-(1-2\lambda)^2 c_1^2}} + \frac{1}{2}\binom{2}{2}\frac{c_2^2 d_2}{d_2^2-(1-2\lambda)^2 c_2^2}$$

$$= \frac{1}{2}\frac{32r^2(1+r^2)}{4(1+r^2)^2-(1-2\lambda)^2(16r^2)} + \frac{1}{2}\frac{32r^2(1+r^2)}{4(1+r^2)^2-(1-2\lambda)^2(16r^2)} = \frac{8r^2(1+r^2)}{(1+r^2)^2-4r^2(1-2\lambda)^2}$$

$$\boxed{H = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}$$

Exercise 3



a.) Write out $\hat{\rho}_{prep}$ in terms of $|0\rangle|0\rangle\langle 0|\langle 0|$ .....

$$\hat{\rho}_{prep} = \hat{U}_{prep}\,\hat{\rho}\,\hat{U}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]}$$

b.) Determine phase flip on first particle

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)\underline{\underline{|0\rangle\langle 0|}}\otimes|0\rangle\langle 0| + (-8ir)\underline{\underline{|0\rangle\langle 1|}}\otimes|0\rangle\langle 1| + (4-4r^2)\underline{\underline{|0\rangle\langle 0|}}\otimes|1\rangle\langle 1| + (4-4r^2)\underline{\underline{|1\rangle\langle 1|}}\otimes|0\rangle\langle 0| + (8ir)\underline{\underline{|1\rangle\langle 0|}}\otimes|1\rangle\langle 0| + (4+4r^2)\underline{\underline{|1\rangle\langle 1|}}\otimes|1\rangle\langle 1|\Big]$$

$\hat{\rho}_i = |0\rangle\langle 0| : \hat{\rho}_F = |0\rangle\langle 0|$, $\hat{\rho}_i = |1\rangle\langle 1| : \hat{\rho}_F = |1\rangle\langle 1|$, $\hat{\rho}_i = |0\rangle\langle 1| : \hat{\rho}_F = (1-2\lambda)|0\rangle\langle 1|$, $\hat{\rho}_i = |1\rangle\langle 0| : \hat{\rho}_F = (1-2\lambda)|1\rangle\langle 0|$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (1-2\lambda)(-8ir)|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| + (4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (1-2\lambda)(8ir)|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (1-2\lambda)(-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (1-2\lambda)(8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

c.) Determine phase flip on second particle

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes\underline{\underline{|0\rangle\langle 0|}} + (1-2\lambda)(-8ir)|0\rangle\langle 1|\otimes\underline{\underline{|0\rangle\langle 1|}} + (4-4r^2)|0\rangle\langle 0|\otimes\underline{\underline{|1\rangle\langle 1|}} + (4-4r^2)|1\rangle\langle 1|\otimes\underline{\underline{|0\rangle\langle 0|}} + (1-2\lambda)(8ir)|1\rangle\langle 0|\otimes\underline{\underline{|1\rangle\langle 0|}} + (4+4r^2)|1\rangle\langle 1|\otimes\underline{\underline{|1\rangle\langle 1|}}\Big]$$

Exercise 3 Continued

$\hat{\rho}_i = |0\rangle\langle 0| : \hat{\rho}_F = |0\rangle\langle 0|$, $\hat{\rho}_i = |1\rangle\langle 1| : \hat{\rho}_F = |1\rangle\langle 1|$, $\hat{\rho}_i = |0\rangle\langle 1| : \hat{\rho}_F = (1-2\alpha)|0\rangle\langle 1|$, $\hat{\rho}_i = |1\rangle\langle 0| : \hat{\rho}_F = (1-2\alpha)|1\rangle\langle 0|$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (1-2\alpha)(1-2\lambda)(-8ir)|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (1-2\alpha)(1-2\lambda)(8ir)|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (1-2\alpha)(1-2\lambda)(-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (1-2\alpha)(1-2\lambda)(8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

d.) Calculate QFI

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I} \quad , \quad H_i = Tr\left[L_i\frac{\partial A_i}{\partial\lambda}\right]$$

$$\alpha = Tr(\hat{A}^2) - Tr(\hat{A})^2 \; : \; \hat{A}_1 = \frac{1}{16}\begin{pmatrix} (4+4r^2) & (1-2\alpha)(1-2\lambda)(-8ir) \\ (1-2\alpha)(1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix} \; \hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4-4r^2) & 0 \\ 0 & (4-4r^2) \end{pmatrix}$$

i.) Find $\hat{L}_1$: $\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2) & (1-2\alpha)(1-2\lambda)(-8ir) \\ (1-2\alpha)(1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}\begin{pmatrix} (4+4r^2) & (1-2\alpha)(1-2\lambda)(-8ir) \\ (1-2\alpha)(1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix}$$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)^2 + (1-2\alpha)^2(1-2\lambda)^2(64r^2) & 2(4+4r^2)(1-2\alpha)(1-2\lambda)(-8ir) \\ 2(4+4r^2)(1-2\alpha)(1-2\lambda)(8ir) & (4+4r^2)^2 + (1-2\alpha)^2(1-2\lambda)^2(64r^2) \end{pmatrix}$$

$$Tr(\hat{A}_1^2) = \frac{1}{128}\left((4+4r^2)^2 + (1-2\alpha)^2(1-2\lambda)^2(64r^2)\right) \; : \; Tr(\hat{A}_1) = \frac{1}{8}(4+4r^2) \; : \; Tr(\hat{A}_1)^2 = \frac{1}{64}(4+4r^2)^2$$

$$\alpha = \frac{1}{128}\left((4+4r^2)^2 + (1-2\alpha)^2(1-2\lambda)^2(64r^2)\right) - \frac{1}{64}(4+4r^2)^2 = \frac{1}{128}\left(16(1+r^2)^2 + (64r^2)(1-2\lambda)^2(1-2\alpha)^2\right) - \frac{16}{64}(1+r^2)^2$$
$$= \frac{1}{8}\left((1+r^2)^2 + (4r^2)(1-2\lambda)^2(1-2\alpha)^2 - \frac{1}{4}(1+r^2)^2 = \frac{1}{8}\left(4r^2(1-2\lambda)^2(1-2\alpha)^2 - (1+r^2)^2\right)\right.$$

$$\alpha = \frac{1}{8}\left(4r^2(1-2\lambda)^2(1-2\alpha)^2 - (1+r^2)^2\right)$$

$$\hat{L}_1 = \frac{1}{Tr(\hat{A}_1)}\left[2\cdot\frac{\partial\hat{A}_1}{\partial\lambda} - \frac{\partial\ln(\alpha)}{\partial\lambda}\hat{A}_1\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \cancelto{0}{\ln(Tr(\hat{A}_1))}\right]\hat{I}$$

$$\hat{L}_1 = \frac{1}{Tr(\hat{A}_1)}\left[2\cdot\frac{\partial\hat{A}_1}{\partial\lambda} - \frac{\partial\ln(\alpha)}{\partial\lambda}\hat{A}_1\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha)\right]\hat{I}$$

$$\frac{\partial\ln(\alpha)}{\partial\lambda} = \frac{\alpha'}{\alpha} \; : \; \alpha' = -8r^2(1-2\lambda)(1-2\alpha)^2 \; : \; \alpha = \frac{1}{8}\left(4r^2(1-2\lambda)^2(1-2\alpha)^2 - (1+r^2)^2\right)$$

Exercise 3 Continued

$$\frac{\partial \ln(\alpha)}{\partial \lambda} = \frac{-2r^2(1-2\lambda)(1-2\alpha)^2}{\frac{(4r^2(1-2\lambda)^2(1-2\alpha)^2-(1+r^2)^2)}{8}} = \frac{16r^2(1-2\lambda)(1-2\alpha)^2}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2} \longrightarrow \beta$$

$$\frac{\partial \hat{A}_1}{\partial \lambda} = \frac{1}{16}\begin{pmatrix} 0 & (16ir)(1-2\alpha) \\ (-16ir)(1-2\alpha) & 0 \end{pmatrix} = \begin{pmatrix} 0 & (ir)(1-2\alpha) \\ (-ir)(1-2\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_1 = \frac{2}{(r^2+1)}\left[2\begin{pmatrix} 0 & (ir)(1-2\alpha) \\ (-ir)(1-2\alpha) & 0 \end{pmatrix} - \beta\frac{1}{16}\begin{pmatrix} (4+4r^2) & (1-2\alpha)(1-2\lambda)(-8ir) \\ (1-2\alpha)(1-2\lambda)(8ir) & (4+4r^2) \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}\right]$$

$$= \frac{2}{(r^2+1)}\left[\frac{1}{16}\begin{pmatrix} 0 & (32ir)(1-2\alpha) \\ (-32ir)(1-2\alpha) & 0 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} \beta(4+4r^2) & -\beta(1-2\alpha)(1-2\lambda)(8ir) \\ \beta(1-2\alpha)(1-2\lambda)(8ir) & \beta(4+4r^2) \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 16\beta & 0 \\ 0 & 16\beta \end{pmatrix}\right]$$

$$= \frac{2}{(r^2+1)}\left[\frac{1}{16}\begin{pmatrix} 16\beta - \beta(4+4r^2) & (32ir)(1-2\alpha) + \beta(1-2\alpha)(1-2\lambda)(8ir) \\ (-32ir)(1-2\alpha) - \beta(1-2\alpha)(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}\right]$$

$$\boxed{\hat{L}_1 = \frac{1}{8(1+r^2)}\begin{pmatrix} 16\beta - \beta(4+4r^2) & (32ir)(1-2\alpha) + \beta(1-2\alpha)(1-2\lambda)(8ir) \\ (-32ir)(1-2\alpha) - \beta(1-2\alpha)(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}}$$

ii.) Calculate $H_1 = \mathrm{Tr}[\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda}]$

$$\hat{L}_1 = \frac{1}{8(1+r^2)}\begin{pmatrix} 16\beta - \beta(4+4r^2) & (32ir)(1-2\alpha) + \beta(1-2\alpha)(1-2\lambda)(8ir) \\ (-32ir)(1-2\alpha) - \beta(1-2\alpha)(1-2\lambda)(8ir) & 16\beta - \beta(4+4r^2) \end{pmatrix}$$

$$\frac{\partial \hat{A}_1}{\partial \lambda} = \begin{pmatrix} 0 & (ir)(1-2\alpha) \\ (-ir)(1-2\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda} : [1,1] = (16\beta - \cancelto{0}{\beta(4+4r^2)})(0) + ((32ir)(1-2\alpha) + \beta(1-2\alpha)(1-2\lambda)(8ir))((-ir)(1-2\alpha))$$

$$32r^2(1-2\alpha)^2 + 8r^2\beta(1-2\alpha)^2(1-2\lambda) \checkmark$$

$$[2,1] = ((-32ir)(1-2\alpha) - \cancelto{0}{\beta(1-2\alpha)}(1-2\lambda)(8ir))(0) + (16\beta - \beta(4+4r^2))(-ir)(1-2\alpha)$$

$$(16\beta - \beta(4+4r^2))(-ir)(1-2\alpha) \checkmark$$

$$[1,2] = (16\beta - \beta(4+4r^2))((ir)(1-2\alpha)) + ((32ir)(1-2\alpha) + \beta(1-2\alpha)\cancelto{0}{(1-2\lambda)}(8ir))(0)$$

$$(16\beta - \beta(4+4r^2))((ir)(1-2\alpha)) \checkmark$$

$$[2,2] = ((-32ir)(1-2\alpha) - \beta(1-2\alpha)(1-2\lambda)(8ir))((ir)(1-2\alpha)) + (16\beta - \cancelto{0}{\beta(4+4r^2)})(0)$$

$$32r^2(1-2\alpha)^2 + 8r^2\beta(1-2\alpha)^2(1-2\lambda) \checkmark$$

$$\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda} = \frac{1}{8(1+r^2)}\begin{pmatrix} 32r^2(1-2\alpha)^2 + 8r^2\beta(1-2\alpha)^2(1-2\lambda) & (16\beta - \beta(4+4r^2))((ir)(1-2\alpha)) \\ (16\beta - \beta(4+4r^2))(-ir)(1-2\alpha) & 32r^2(1-2\alpha)^2 + 8r^2\beta(1-2\alpha)^2(1-2\lambda) \end{pmatrix}$$

$$\mathrm{Tr}[\hat{L}_1 \cdot \frac{\partial \hat{A}_1}{\partial \lambda}] = \frac{1}{8(1+r^2)}\left[64r^2(1-2\alpha)^2 + 16r^2\beta(1-2\alpha)^2(1-2\lambda)\right]$$

$$= \frac{1}{(1+r^2)}\left[8r^2(1-2\alpha)^2 + 2r^2\beta(1-2\alpha)^2(1-2\lambda)\right]$$

$$= \frac{1}{(1+r^2)}\left(8r^2(1-2\alpha)^2 + \frac{32r^4(1-2\lambda)(1-2\alpha)^2}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2} \cdot (1-2\alpha)^2(1-2\lambda)\right)$$

$$= \frac{1}{(1+r^2)}\left(\frac{8r^2(1-2\alpha)^2(1+r^2)^2 - 32r^4\cancel{(1-2\lambda)^2}(1-2\alpha)^4 + 32r^4\cancel{(1-2\lambda)^2}(1-2\alpha)^4}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}\right)$$

Exercise 3 Continued

$$H_1 = \frac{1}{(1+\cancel{r^2})}\left(\frac{8r^2(1-2\alpha)^2(1+r^2)^{\cancel{2}}}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}\right) = \left(\frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}\right)$$

$$\boxed{H_1 = \frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}}$$

iii.) Find $\hat{L}_2$: $\alpha = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix}(4-4r^2) & 0\\ 0 & (4-4r^2)\end{pmatrix}\begin{pmatrix}(4-4r^2) & 0\\ 0 & (4-4r^2)\end{pmatrix} = \frac{1}{16^2}\begin{pmatrix}(4-4r^2)^2 & 0\\ 0 & (4-4r^2)^2\end{pmatrix}$$

$$Tr(\hat{A}_2^2) = \frac{1}{128}(4-4r^2)^2 \quad : \quad Tr(\hat{A}_2) = \frac{1}{8}(4-4r^2) \quad : \quad Tr(\hat{A}_2)^2 = \frac{1}{64}(4-4r^2)^2$$

$\alpha = -\frac{1}{128}(4-4r^2)^2$, Since α is not dependent upon λ, $\hat{L}_2$ becomes

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \cancelto{0}{\frac{\partial \ln(\alpha)}{\partial\lambda}}\hat{A}\right] + \cancelto{0}{\frac{\partial}{\partial\lambda}}\left[\cancelto{0}{\ln(\alpha)} - \cancelto{0}{\ln(Tr(\hat{A}))}\right]\hat{I} = \frac{1}{Tr(\hat{A}_2)}\cdot 2\cdot\frac{\partial\hat{A}_2}{\partial\lambda}$$

$\frac{\partial\hat{A}_2}{\partial\lambda} = 0 \quad \therefore \quad \hat{L}_1$ and thus $\boxed{H_2 = 0}$

iv.) Calculate $H = H_1 + H_2$, $H = H_1 + \cancelto{0}{H_2}$ $\therefore$ $H = H_1$

$$\boxed{H = \frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}}$$

Exercise 4

$$H = \frac{4r^2}{1-(1-2\lambda)^2 r^2}, \quad H = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}, \quad H = \frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}$$

Gain of second scenario over first

$$G = \frac{\dfrac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}{\dfrac{4r^2}{1-(1-2\lambda)^2 r^2}} = \frac{(\cancel{8r^2}^{\,2})(1+r^2)(1-(1-2\lambda)^2 r^2)}{(\cancel{4r^2})((1+r^2)^2 - 4r^2(1-2\lambda)^2)} = \frac{2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}$$

$$\boxed{G = \frac{2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}}$$

$$\frac{2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2} > 1 \quad : \quad 2(1+r^2)(1-(1-2\lambda)^2 r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - 2r^2(1+r^2)(1-2\lambda)^2 > (1+r^2)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - (1+r^2)^2 > 2r^2(1+r^2)(1-2\lambda)^2 - 4r^2(1-2\lambda)^2$$

$$2(1+r^2) - (1+r^2)^2 > (1-2\lambda)^2(2r^2(1+r^2) - 4r^2)$$

$$(1-2\lambda) < \sqrt{\frac{2(1+r^2) - (1+r^2)^2}{2r^2(1+r^2) - 4r^2}}$$

$$2\lambda - 1 > -\sqrt{\frac{2(1+r^2) - (1+r^2)^2}{2r^2(1+r^2) - 4r^2}}$$

$$\lambda > -\frac{1}{2}\sqrt{\frac{2(1+r^2) - (1+r^2)^2}{2r^2(1+r^2) - 4r^2}} + \frac{1}{2}$$

$$\boxed{\lambda > \frac{1}{2}\left(1 - \sqrt{\frac{2(1+r^2) - (1+r^2)^2}{2r^2(1+r^2) - 4r^2}}\right)}$$

Gain of Third Scenario over First

$$G = \frac{\dfrac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}}{\dfrac{4r^2}{1-(1-2\lambda)^2 r^2}} = \frac{(\cancel{8r^2}^{\,2})(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(\cancel{4r^2})((1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2)} = \frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}$$

$$\boxed{G = \frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}}$$

$$\frac{2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2 r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2} > 1 \quad : \quad 2(1-2\alpha)^2(1+r^2)(1-(1-2\lambda)^2 r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$2(1-2\alpha)^2(1+r^2) - 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) > (1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$2(1-2\alpha)^2(1+r^2) - (1+r^2)^2 > 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$-(1+r^2)^2 > -2(1-2\alpha)^2(1+r^2) + 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) - 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$(1+r^2)^2 < 2(1-2\alpha)^2(1+r^2) - 2r^2(1-2\lambda)^2(1-2\alpha)^2(1+r^2) + 4r^2(1-2\lambda)^2(1-2\alpha)^2$$

$$(1+r^2)^2 < (1-2\alpha)^2(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2)$$

$$(1-2\alpha)^2 > \frac{(1+r^2)^2}{(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2)}$$

Exercise 4 continued

$$(1-2\alpha) > \sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}$$

$$2\alpha - 1 < -\sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}$$

$$\boxed{\alpha < \frac{1}{2}\left(1 - \sqrt{\frac{(1+r^2)^2}{\left(2(1+r^2) - 2r^2(1-2\lambda)^2(1+r^2) + 4r^2(1-2\lambda)^2\right)}}\right)}$$