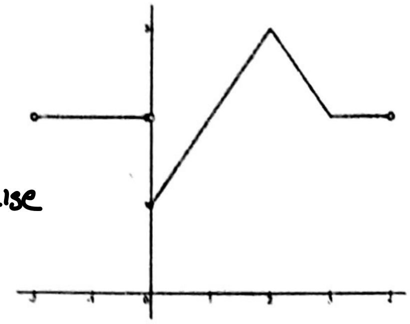
Math 361 **Numerical Analysis** Taylor Larrechea
9/26/19 Exam 1 Name

Answer all questions in the space provided. **Show all work.** **(100 pts. total)**

1. Use the graph of the function f below to answer the following questions. **(3 pts. each)**



(a) True or False (Circle one): $f \in C(-2,4)$ False

(b) True or False (Circle one): $\max_{-2\le x\le 4}|f(x)| = 3$ True

(c) True or False (Circle one): f is differentiable at $x = 2$. False

2. Suppose that $p = 4.112054$ is approximated by $p^* = 4.111610$.

(a) Find the relative error. (8 pts.)

$$RE = \frac{|P-P^*|}{|P|} = \frac{|4.112054-4.111610|}{|4.112054|} = \frac{0.000444}{4.112054} = 1.0797523357 \times 10^{-4}$$

$$\boxed{RE = 1.0797523357 \times 10^{-4}}$$

(b) To how many significant digits does p^* approximate p? Justify your answer. (5 pts.)

4 significant digits

3. The standard deviation s of a set of data values $\{x_1, x_2, \ldots, x_n\}$ is given by

$$s = \sqrt{\frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2}{n-1}}$$

Write an algorithm in pseudocode that computes s, where $\bar{x}$ is one of the inputs. (8 pts.)

Input : $\bar{x}, n, x_1, \ldots, x_n$
Output : s
Step 1 : Set $s=0$
Step 2 : For $i=1$; up to $i=n$; do
$s = s + (x_i - \bar{x})^2$
Step 3 : $s = \sqrt{\dfrac{s}{n-1}}$
Step 4 : Output s

STOP

4. Given that $\alpha_n = \dfrac{5n^2-1}{3n^2} \to \dfrac{5}{3}$ as $n \to \infty$, find the rate of convergence, expressing your answer using "big O" notation. Show work to justify your answer. (8 pts)

$$|\alpha_n - \alpha| = \left|\frac{5n^2-1}{3n^2} - \frac{5}{3}\right| = \left|-\frac{1}{3n^2}\right| = \frac{1}{3}\left|\frac{1}{n^2}\right| \qquad R.C \equiv \text{Rate of convergence}$$

$$\boxed{R.C = \frac{5}{3} + O\left(\frac{1}{n^2}\right)}$$

5. Let $f(x) = x^3 - 3x + 1$. Suppose we are interested in using the Bisection Method for finding the solution to $f(x) = 0$ on $[0,1]$.

(a) When evaluating $f(x)$ at a value of x, it is generally best to use which of the following methods? Identify the best choice. (2 pts.)

Direct computation   Nested computation

(b) Use the IVT to demonstrate that there is a solution to $f(x) = 0$ in $(0,1)$. (6 pts.)

$f(x)$ is a polynomial and thus it is continuous everywhere : $f \in C[0,1]$

$f(0) = 1$; $f(1) = -1$: $f(0) \cdot f(1) = 1 \cdot -1 = -1 < 0$ ✓

$\therefore$ $\boxed{\exists\, c \in [0,1] \text{ s.t. } f(c) = 0}$

(c) Implement the Bisection Method by completing the table below. (10 pts.)

n	a	b	p	sign f(a)	sign f(b)	sign f(p)
1	0	1	0.5	+1	-1	-1
2	0	0.5	0.25	+1	-1	+1
3	0.25	0.50	0.375	+1	-1	-1
4	0.25	0.375	0.3125	+1	-1	+1

(d) Use the theoretical results from Ch 2.1 (Theorem 2.1) to determine the minimum number of iterations required to guarantee an approximation with accuracy 10^{-5}. (4 pts.) $b=1, a=0$

$$\frac{b-a}{2^n} < 10^{-5} \rightarrow 2^n > \frac{b-a}{10^{-5}} \rightarrow \log_{10}(2^n) > \log_{10}\left(\frac{1-0}{10^{-5}}\right)$$

$$n > \frac{1}{\log_{10}(2)} \cdot \log_{10}\left(\frac{1}{10^{-5}}\right) \quad : \quad n > 16.6$$

$$\boxed{n > 17 \text{ iterations}}$$

6. Suppose we want to solve $1.01x^2 + 10.1x - 0.152 = 0$. $b \gg c$

(a) The two roots x_1, x_2 of the quadratic equation can be found using appropriate selections from the following formulas. For each root, identify the choice that provides the best computational approximation. (8 pts)

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad x_1 = \frac{-2c}{b+\sqrt{b^2-4ac}}, \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}, \quad x_2 = \frac{-2c}{b-\sqrt{b^2-4ac}}$$

(b) Find the roots of $1.01x^2 + 10.1x - 0.152 = 0$ using four digit rounding arithmetic on appropriate selections from the above formulas. Be sure to show intermediate steps. (10 pts)

x_1: $b^2 = (10.1)(10.1) = 102.01 \simeq 102.0$

$4a = 4(1.01) = 4.04$: $4ac = (4.04)(-0.152) = -0.6141$

$b^2 - 4ac = 102.0 - (-0.6141) = 102.6141 \simeq 102.6$

$\sqrt{b^2-4ac} = \sqrt{102.6} = 10.12916581 \simeq 10.13^*$

$b + \sqrt{b^2-4ac} = 10.1 + 10.13 = 20.23$

$-2c = -2(-0.152) = 0.304$

$$x_1 = \frac{0.304}{20.23} = 0.0150271873 \simeq 0.01503$$

$$\boxed{x_1 = 0.01503}$$

x_2: $-b - \sqrt{b^2-4ac} = -10.1 - 10.13 = -20.23$

$2a = 2(1.01) = 2.02$

$$x_2 = \frac{-20.23}{2.02} = -10.01485149 \simeq -10.01$$

$$\boxed{x_2 = -10.01}$$

7. Let $f(x) = 2 - e^{-x} - \sin(x)$. Answer the following questions.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

(a) Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$. (10 pts)

$f^{(0)}(0) = 2 - \cancel{e^0} - \cancel{\sin(0)} = 2-1 = 1 \quad : \quad f'(x) = e^{-x} - \cos(x)$

$f'(0) = \cancel{e^0} - \cancel{\cos(0)} = 1-1 = 0 \quad : \quad f''(x) = -e^{-x} + \sin(x)$

$f''(0) = -\cancel{e^0} + \cancel{\sin(0)} = -1+0 = -1 \quad : \quad f'''(x) = e^{-x} + \cos(x)$

$$\boxed{P_2(x) = 1 - \frac{x^2}{2}}$$

(b) Find $R_2(x)$. (4 pts.)

$$R_2(x) = \frac{f'''(x_0)(x-x_0)^3}{3!} = \frac{(e^{-x} + \cos(x))\,x^3}{6} \quad : \quad \boxed{R_2(x) = \frac{(e^{-\xi(x)} + \cos(\xi(x)))\,x^3}{6}}$$

$\xi(x)$ lies between x_0 and x

(c) Suppose we want to use $P_2(x)$ to approximate $f(x)$ at $x = 0.4$. Use $R_2(0.4)$ it to find a useful upper bound for the error $|f(0.4) - P_2(0.4)|$. Round your final answer to 4 decimal places. (8 pts.)

$$R_2(0.4) = \left| \frac{(e^{-\xi(0.4)} + \cos(\xi(0.4)))\,x^3}{6} \right|$$

0 ——— $\xi(0.4)$ ——— 0.4

want to maximize

$f'''(x)$ will be maximized at $\xi(0.4) = 0$ Since $f^4(x)$ is negative and decreasing, so therefore we choose the left endpoint

$$R_2(0.4) = \left| \frac{(e^0 + \cos(0))\,(0.4)^3}{6} \right| = \frac{1}{3}(0.4)^3 = 0.02133$$

$$\boxed{R_2(0.4) = 0.0213}$$