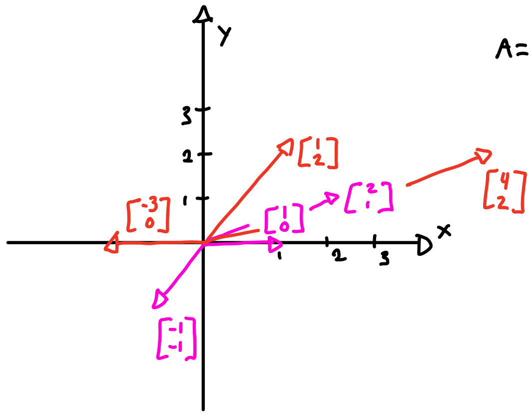
5.3 Eigenvalues and Eigenvectors

5.3 #6, 8, 20, 72, 74



$$A = \begin{bmatrix}1 & 2\\2 & -2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-3\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

Eigenvalue and Eigenvector

Let $T: V \to V$ be a linear transformation from vector space V into vector space V. A scalar λ is an eigenvalue of T if there is a nonzero vector $\vec{v} \in V$ such that

$$T(\vec{v}) = \lambda\vec{v}$$

Such a nonzero vector $\vec{v}$ is called an eigenvector of T corresponding to λ.

If the linear transformation T is represented by an $n \times n$ matrix A, where $V = \mathbb{R}^n$ and $T(\vec{v}) = A(\vec{v})$ then $\vec{v}$ and λ are represented by $A\vec{v} = \lambda\vec{v}$

Ex:

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} = \lambda I\vec{v}$

$A\vec{v} - \lambda I\vec{v} = 0$

$\vec{v}(A - \lambda I) = 0$

$A - \lambda I = 0$ (Solve to find eigenvalues)

$(A - \lambda I)v_i = 0$ (To find eigenvector)

$$A = \begin{bmatrix}1 & 1\\1 & 4\end{bmatrix}$$

$A - \lambda I = 0$

$$\begin{bmatrix}1 & 1\\1 & 4\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} = 0$$

$(1-\lambda)(4-\lambda)$

$4 - 5\lambda + \lambda^2$

$$\begin{bmatrix}1-\lambda & 1\\1 & 4-\lambda\end{bmatrix} = 0$$

$(\lambda^2 - 5\lambda + 4) - 1 = 0$

$\lambda^2 - 5\lambda + 3 = 0$

$$A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \quad \left|\begin{bmatrix}1-\lambda & 1 & 1\\0 & 1-\lambda & 1\\0 & 0 & 1-\lambda\end{bmatrix}\right| = (1-\lambda)(1-\lambda)^2 = 0$$

$\lambda = 1$

$$\begin{bmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\vec{v} = 0 \qquad \vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

$v_2 + v_3 = 0$

$v_3 = 0$

$v_1 =$ Free variable

$$\vec{e} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \text{Dim}(\vec{e}) = 1$$

Rotation (clockwise) by θ

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$$

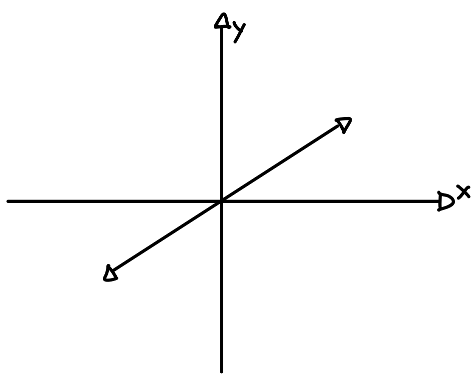
$$\left|\begin{bmatrix}\cos\theta - \lambda & -\sin\theta\\\sin\theta & \cos\theta - \lambda\end{bmatrix}\right| = 0$$

$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$(\cos\theta - \lambda)^2 = -\sin^2\theta$

$\cos\theta - \lambda = \pm i\sin\theta$

$\lambda = \pm i\sin\theta + \cos\theta$



$A\vec{x} = \lambda\vec{x}$

Ex 9 $y'' - y' - 2y = 0$

$(r^2 - r - 2) = 0$

$(r-2)(r+1) = 0$

$r = -1, 2$

$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$= c_1 e^{-t} + c_2 e^{2t}$

$y_h = c_1 e^{-t} + c_2 e^{2t}$

$x_1' = x_2$

$x_2' = 2x_1 + x_2$

$y'' - y' - 2y = 0$ $\quad x_2 = x_1'$ $\quad x_1' - x_2 - 2x_1 = 0$

$x_2' \quad x_2 \quad x_1$

$x_1' = x_2$

$x_2' = 2x_1 + x_2$

$\vec{x}' = A\vec{x}$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Eigenvalues of A

$$\left\| \begin{bmatrix} 0-\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix} \right\| = 0$$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1)$

$\lambda = 2, -1$

$\lambda = 2$

$x_1 = e^{2t}$

$x_2 = 2e^{2t}$

$$A\vec{x} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$$= 2 \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$