5.3 Eigenvalues and Eigenvectors

5.3 #6, 8, 20, 72, 74

5.3.6) $\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$ $|A - \lambda I|$

Eigenvalues: $|A - \lambda I|$

$$\left| \begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 3-\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix} = \begin{matrix} \lambda^2 - 5 = 0 \\ \lambda^2 = 5 \end{matrix}$$

$$\lambda = \pm\sqrt{5}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-3}{2} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ \frac{(3+\sqrt{5})}{2} \end{bmatrix}$$

Eigenvectors: $(A - \lambda I)\vec{v}_i$

$\sqrt{5}$

$$\left(\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \right) \vec{v}_1$$

$$\begin{bmatrix} 3-\sqrt{5} & 2 \\ -2 & -3-\sqrt{5} \end{bmatrix} \begin{bmatrix} v_{1x} \\ v_{2y} \end{bmatrix}$$

$$(3-\sqrt{5})x + 2y = 0$$

$$2y = (\sqrt{5}-3)x$$

$$y = \frac{(\sqrt{5}-3)}{2}x$$

$-\sqrt{5}$

$$\left(\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{bmatrix} \right) \vec{v}_2$$

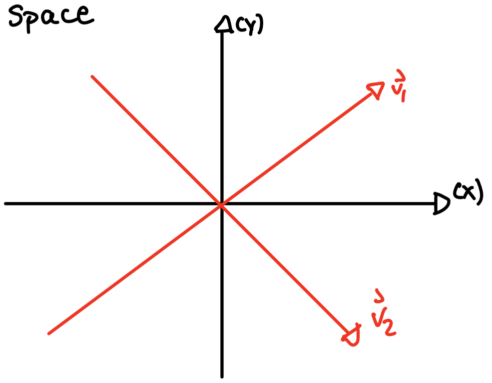
$$\begin{bmatrix} 3+\sqrt{5} & 2 \\ -2 & -3+\sqrt{5} \end{bmatrix} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix}$$

$$(3+\sqrt{5})x + (2)y = 0$$

$$2y = (3+\sqrt{5})x$$

$$y = \frac{(3+\sqrt{5})x}{2}$$

Space



Eigenvalues: $\lambda = \pm\sqrt{5}$

Eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-3}{2} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ \frac{3+\sqrt{5}}{2} \end{bmatrix}$$

5.3.8) $A = \begin{bmatrix} 12 & -6 \\ 15 & -7 \end{bmatrix}$

Eigenvalues: $|A - \lambda I| = 0$

$$\begin{bmatrix} 12 & -6 \\ 15 & -7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\left| \begin{bmatrix} 12-\lambda & -6 \\ 15 & -7-\lambda \end{bmatrix} \right| = \lambda^2 - 5\lambda + 6$$

$$(\lambda - 3)(\lambda - 2)$$

$$\lambda = 3, 2$$

Eigenvectors: $(A - \lambda I)\vec{v}_i$

$\vec{v}_1$: $\lambda = 3$

$$\begin{bmatrix} 12 & -6 \\ 15 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -6 \\ 15 & -10 \end{bmatrix} \begin{bmatrix} v_{1x} \\ v_{2y} \end{bmatrix}$$

$$\text{rref} = \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x = \tfrac{2}{3}y$$

$\vec{v}_2$: $\lambda = 2$

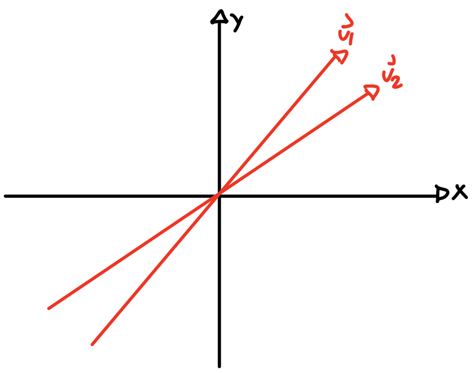
$$\begin{bmatrix} 12 & -6 \\ 15 & -7 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -6 \\ 15 & -9 \end{bmatrix} \begin{bmatrix} v_{2x} \\ v_{2y} \end{bmatrix}$$

$$\text{rref} = \begin{bmatrix} 1 & -3/5 \\ 0 & 0 \end{bmatrix} \quad x = \tfrac{3}{5}y$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3/5 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

Space



Values: $\lambda = 3, 2$

Vectors: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 3/5 \end{bmatrix}$

5.3.20

$A=\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}$

Eigenvalues : $|A-\lambda I|$ $\qquad \lambda=1,2,3$

$$\left|\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}1-\lambda&2&-1\\1&-\lambda&1\\4&-4&5-\lambda\end{bmatrix}\right|=-(\lambda-1)(\lambda^2-5\lambda+6)$$
$$(1-\lambda)(\lambda-3)(\lambda-2)$$

$\lambda=1,2,3$

Eigen vectors : $(A-\lambda I)$

$\lambda=1$ $\quad \left(\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\vec{v}_1$ $\qquad \vec{v}_1=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}$

$\begin{bmatrix}0&2&-1\\1&-1&1\\4&-4&4\end{bmatrix}\begin{bmatrix}V_x\\V_y\\V_z\end{bmatrix}$ $\quad$ rref $=\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-\frac{1}{2}\\0&0&0\end{bmatrix}$ $\quad x=-\frac{1}{2}z$, $y=\frac{1}{2}z$, $z=1$

$\lambda=3$ $\quad \left(\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}-\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}\right)\vec{v}_3$

$\begin{bmatrix}-2&2&-1\\1&-3&1\\4&-4&2\end{bmatrix}$ $\quad$ rref $=\begin{bmatrix}1&0&\frac{1}{4}\\0&1&-\frac{1}{4}\\0&0&0\end{bmatrix}$ $\quad x+\frac{1}{4}z=0$, $y-\frac{1}{4}z=0$

$x=-\frac{1}{4}z$, $z=1$
$y=\frac{1}{4}z$

$\vec{v}_3=\begin{bmatrix}-\frac{1}{4}\\\frac{1}{4}\\1\end{bmatrix}$

$\lambda=2$ $\quad \left(\begin{bmatrix}1&2&-1\\1&0&1\\4&-4&5\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\right)\vec{v}_2$ $\qquad \vec{v}_2=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{4}\\1\end{bmatrix}$

$\begin{bmatrix}-1&2&-1\\1&-2&1\\4&-4&3\end{bmatrix}$ $\quad$ rref $=\begin{bmatrix}1&0&\frac{1}{2}\\0&1&-\frac{1}{4}\\0&0&0\end{bmatrix}$ $\quad x=-\frac{1}{2}z$, $y=\frac{1}{4}z$, $z=1$

$\lambda=1,2,3$
$\vec{v}_3=\begin{bmatrix}-\frac{1}{4}\\\frac{1}{4}\\1\end{bmatrix}$ $\quad \vec{v}_2=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{4}\\1\end{bmatrix}$ $\quad \vec{v}_1=\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\end{bmatrix}$

5.3.72 $\quad y''-y'-2y=0$

$y''=y_2'$
$y_1'=y_2$
$y=y_1$

$y_1=y$
$y_2'-y_2-2y_1=0$
$y_2'=y_2+2y_1$
$y_1'=y_2$

$A=\begin{bmatrix}0&1\\2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$

$y''-y'-2y=0$
$(r-2)(r+1)=0$
$r=-1,2$

$r=-1,2$
$\lambda=-1,2$

Determinant check : $|A-\lambda I|$

$$\left|\begin{bmatrix}0&1\\2&1\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\right|=\lambda^2-\lambda-2$$
$$(\lambda-2)(\lambda+1)$$
$$\lambda=2,-1$$

$\lambda=-2,-1,1$
$r=-2,-1,1$

5.3.74 $\quad y'''+2y''-y'-2y=0$

$y'''=y_3'$
$y''=y_3$
$y'=y_2$
$y=y_1$

$y_3'+2y_3-y_2-2y_1=0$

$y_3'=2y_1+y_2-2y_3$
$y_2'=y_3$
$y_1'=y_2$

$A=\begin{bmatrix}0&1&0\\0&0&1\\2&1&-2\end{bmatrix}$

$r^3+2r^2-r-2=0$
$(r+2)(r+1)(r-1)$
$r=-2,-1,1$

$$\left|\begin{bmatrix}0&1&0\\0&0&1\\2&1&-2\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right|$$

$-\lambda=2$ $\quad \lambda^2=1$
$\lambda=-2$ $\quad \lambda=\pm1$

$$\left|\begin{bmatrix}-\lambda&1&0\\0&-\lambda&1\\2&1&-2-\lambda\end{bmatrix}\right|=(-\lambda-2)(\lambda^2-1)$$
$$\lambda=-2,-1,1$$