MAT 201
Larson/Edwards – Section 2.3
The Product and Quotient Rules

In this section (and 2.4) we will investigate other rules for finding the derivative of a function. I will start with some examples.

Ex: Find $f'(x)$ for the following functions:

a) $f(x) = 6x^5 - 10x^3 + 18x - 4$

b) $f(x) = (x^2 - 3)(x^4 + 3x^2 + 9)$
How would we approach this one? Up to this point we haven't established any rule for taking the derivative of a product of functions $f(x) \cdot g(x)$.

ONE APPROACH: Multiply the expressions together, then find the derivative.

ANOTHER APPROACH: Use the product rule!
Product Rule: The derivative of two differentiable functions is equal to the first function times the derivative of the second plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Ex: Find $f'(x)$ for the following functions:

a) $f(x) = 6x^5 - 10x^3 + 18x - 4$

$$\boxed{f'(x) = 30x^4 - 30x^2 + 18}$$

$$f'(x) = 6\cdot 5x^{5-1} - 10\cdot 3x^{3-1} + 18\cdot 1x^{1-1} - 0$$

b) $f(x) = (x^2 - 3)(x^4 + 3x^2 + 9)$

$$f(x) = x^6 + \cancel{3x^4} + \cancel{9x^2} - \cancel{3x^4} - \cancel{9x^2} - 27$$

$$f(x) = x^6 - 27$$

$$\boxed{f'(x) = 6x^5}$$

Using the Product Rule:

"Derivative of first · second + Der. of Sec. · first."

$$f(x) = (x^2 - 3)(x^4 + 3x^2 + 9)$$

$$f'(x) = 2x(x^4 + 3x^2 + 9) + (4x^3 + 6x)(x^2 - 3)$$

$$f'(x) = 2x^5 + \cancel{6x^3} + \cancel{18x} + [4x^5 - \cancel{12x^3} + \cancel{6x^3} - \cancel{18x}]$$

$$f'(x) = 6x^5$$

c) $f(x) = \dfrac{x^6 + 1}{x^2 + 1}$

Again, we don't have a rule for this situation where we have one function divided by another $\left(\dfrac{f(x)}{g(x)}\right)$. One way to get around this obstacle is to simply reduce the fraction, if possible, or divide *g(x)* into the *f(x)*, then differentiate.

Sometimes the above approach isn't so easy, especially if we have complicated quotients.

ANOTHER APPROACH: Use the quotient rule!
Quotient Rule: The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

With the product and quotient rules we can find the derivatives of the remaining trigonometric functions:

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

c) $f(x) = \dfrac{\overset{h(x)}{x^6 + 1}}{\underset{g(x)}{x^2 + 1}}$

$$f'(x) = \frac{\overset{h'(x)}{6x^5}\,\overset{g(x)}{(x^2+1)} - \overset{g'(x)}{(2x)}\,\overset{h(x)}{(x^6+1)}}{\underset{[g(x)]^2}{[x^2+1]^2}}$$

$$f'(x) = \frac{6x^7 + 6x^5 - 2x^7 - 2x}{[x^2+1]^2}$$

$$\boxed{f'(x) = \frac{4x^7 + 6x^5 - 2x}{[x^2+1]^2}}$$

Ex: Find the derivative for the following functions by using the product or quotient rules. Write the derivative in simplest radical form, if radicals exist:

a) $f(x) = x^5 \cos x$

b) $h(t) = \dfrac{t^2 + 1}{\sqrt{t}}$

Ex: Prove, using the quotient rule, that $\dfrac{d}{dx}[\tan x] = \sec^2 x$.

Ex: Find the derivative for the following functions by using the product or quotient rules. Write the derivative in simplest radical form, if radicals exist:

a) $f(x) = \underbrace{\boxed{x^5}}_{g(x)} \underbrace{\boxed{\cos x}}_{h(x)}$

$f'(x) = \underbrace{5x^4}_{g'(x)} \underbrace{\cos x}_{h(x)} + \underbrace{-\sin x}_{h'(x)} \cdot \underbrace{x^5}_{g(x)}$

$\boxed{f'(x) = 5x^4 \cos x - x^5 \sin x}$

b) $h(t) = \dfrac{t^2+1}{\sqrt{t}}$ $\leftarrow f(t)$, $\leftarrow g(t)$

$\sqrt{t} = t^{1/2}$

$$h'(t) = \frac{2t \cdot t^{1/2} - \frac{1}{2}t^{-1/2} \cdot (t^2+1)}{(\sqrt{t})^2}$$

$$h'(t) = \frac{2t^{3/2} - \frac{1}{2}t^{3/2} - \frac{1}{2}t^{-1/2}}{t}$$

$$h'(t) = \frac{\frac{3}{2}t^{3/2} - \frac{1}{2}t^{-1/2}}{t^1}$$

$$h'(t) = \frac{3}{2}t^{3/2-1} - \frac{1}{2}t^{-1/2-1}$$

$$h'(t) = \frac{3}{2}t^{1/2} - \frac{1}{2}t^{-3/2}$$

$$h'(t) = \frac{3\sqrt{t}}{2} - \frac{1}{2\sqrt{t^3}}$$

$$h'(t) = \frac{3\sqrt{t}}{2} - \frac{1}{2t\sqrt{t}} \cdot \frac{\sqrt{t}}{\sqrt{t}}$$

$$\boxed{h'(t) = \frac{3\sqrt{t}}{2} - \frac{\sqrt{t}}{2t^2}}$$

or $\dfrac{3t^2\sqrt{t} - \sqrt{t}}{2t^2}$

or $\dfrac{\sqrt{t}\,(3t^2-1)}{2t^2}$

Ex: Prove, using the quotient rule, that

$$\frac{d}{dx}[\tan x] = \sec^2 x$$.

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}{[\cos x]^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \quad \blacksquare$$

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Higher-Order Derivatives

In section 2.2 we explored the idea of velocity in problem 97 of your homework set. Let's look at a problem similar to that problem.

Ex: The height s (in feet) at time t (in seconds) of a silver dollar dropped from the top of the Washington Monument is given by $s = -16t^2 + 555$. Find the derivative of s.

What does this result mean?

Suppose we wanted to take the derivative of the derivative of s with respect to t, that is $\frac{d}{dt}[s'(t)]$?

What does this result mean?

What we are doing is taking the second derivative of the function s. We denote the second derivative $\frac{d}{dt}[s'(t)] = s''(t)$. **How do you think we will represent the third derivative? How would we find it?**

Ex: The height *s* (in feet) at time *t* (in seconds) of a silver dollar dropped from the top of the Washington Monument is given by $s = -16t^2 + 555$. Find the derivative of s.

$s = -16t^2 + v_0 t + s_0$

$s' = \frac{ds}{dt} = \boxed{-32t}$

What does this result mean?

Velocity of the coin at any given time.

What we are doing is taking the second derivative of the function s. We denote the second derivative

$\frac{d}{dt}\left[s'(t)\right] = s''(t)$. **How do you think we will represent the third derivative? How would we find it?**

$s'''(t) \rightarrow$ take the derivative of $s''(t)$.

Notation for Higher-Order Derivatives: Given $y = f(x)$,

1. 1st Derivative: $f'(x) = \dfrac{dy}{dx} = y' = \dfrac{d}{dx}[f(x)] = D_x[y]$
2. 2nd Derivative: $f''(x) = \dfrac{d^2y}{dx^2} = y'' = \dfrac{d}{dx}[f'(x)] = \dfrac{d^2}{dx^2}[f(x)] = D_x^2[y]$
3. 3rd Derivative: $f'''(x) = \dfrac{d^3y}{dx^3} = y''' = \dfrac{d}{dx}[f''(x)] = \dfrac{d^3}{dx^3}[f(x)] = D_x^3[y]$
4. 4th Derivative: $f^{(4)}(x) = \dfrac{d^4y}{dx^4} = y^{(4)} = \dfrac{d}{dx}[f'''(x)] = \dfrac{d^4}{dx^4}[f(x)] = D_x^4[y]$
5. n^{th} Derivative: $f^{(n)}(x) = \dfrac{d^ny}{dx^n} = y^{(n)} = \dfrac{d}{dx}\left[f^{(n-1)}(x)\right] = \dfrac{d^n}{dx^n}[f(x)] = D_x^n[y]$

Acceleration: As in our previous example we saw that the acceleration of an object can be defined as the rate of change of the velocity with respect to time.

- $s = f(t)$ is the ***position function***, that is the height s of the object at any given time t.
- $\dfrac{ds}{dt} = f'(t)$ is the ***velocity function***, that is the velocity of the object at any given time t.
- $\dfrac{d^2s}{dt^2} = f''(t)$ is the ***acceleration function***, that is the acceleration of the object at any given time t.

Ex: Find the second derivative of the function: $f(x) = 4x^{3/2}$

Ex: Find the second derivative of the function:

$f(x) = 4x^{3/2}$

$f'(x) = 4 \cdot \frac{3}{2} x^{3/2 - 1} = 6x^{1/2}$

$f''(x) = 6 \cdot \frac{1}{2} x^{1/2 - 1} = 3x^{-1/2} = \boxed{\frac{3}{\sqrt{x}} \text{ or } \frac{3\sqrt{x}}{x}}$

Ex: Find the third derivative of the function:

$f(x) = x^5 - 3x^4$

$f'(x) = 5x^4 - 12x^3$

$f''(x) = 20x^3 - 36x^2$

$\boxed{f'''(x) = 60x^2 - 72x}$

Ex: Find the third derivative of the function: $f(x) = x^5 - 3x^4$

Ex: Find $f^{(6)}(x)$ given that $f^{(4)}(x) = 3x^2 - 5$.

Ex: The velocity (in feet per second) of an automobile starting from rest is modeled by $\frac{ds}{dt} = \frac{90t}{t + 10}$. Find the initial acceleration of the vehicle at $t = 0$ and the acceleration of the automobile after one minute.

Ex: Find $f^{(6)}(x)$ given that $f^{(4)}(x) = 3x^2 - 5$.

$$f^{(5)}(x) = 6x$$

$$\boxed{f^{(6)}(x) = 6}$$

Ex: The velocity (in feet per second) of an automobile starting from rest is modeled by $\frac{ds}{dt} = \frac{90t}{t+10}$. Find the initial acceleration of the vehicle at $t = 0$ and the acceleration of the automobile after one minute.

$t = 60$

$$\frac{d}{dt}\left[\frac{ds'}{dt}\right] = \frac{90(t+10) - 1 \cdot 90t}{(t+10)^2}$$

$$s''(t) = \frac{900}{(t+10)^2} \qquad ft/s/s$$

$$\boxed{\begin{aligned} s''(0) &= \frac{900}{(0+10)^2} = 9\ ft/s^2 \\ s''(60) &= \frac{900}{(60+10)^2} = \frac{900}{4900} \approx 0.184\ ft/s^2 \end{aligned}}$$