1.2) Solutions and Direction Fields

Differential Equation (DE) is an equation containing derivatives

$\frac{dy}{dt} = f(t,y)$ or $y' = f(t,y)$

Slope Field

- Make table of values for y & t
- Plug y & t values into equation to get slope at point
- Plot slope on graph

Graphically:

A Solution to a first-order differential equation is a function whose slope at each point is specified by the derivative

Stability

For a differential equation $y' = f(t,y)$, an equilibrium solution $y(t) \equiv C$ is called

Stable - if solutions near it tend toward it as $t \to \infty$

unstable - if solutions near it tend away from it as $t \to \infty$

Initial-Value Problem (IVP)

The combination of a first-order differential equation and an initial condition is called an initial value problem

$$\frac{dy}{dt} = f(t,y) \quad y(t_0) = y_0$$

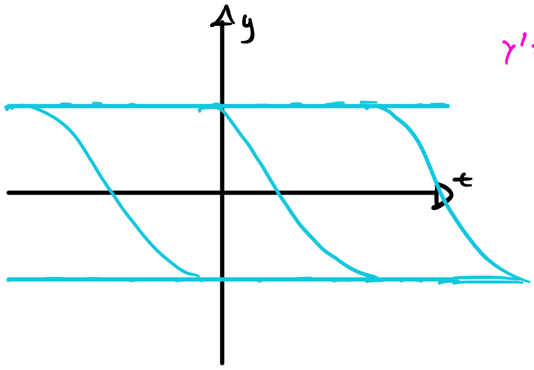
Equilibrium

set $y' = 0$

A differential equation, to have a solution be in equilibrium it cannot change over time.

For a first-order DE $y' = f(t,y)$, an equilibrium solution is always a horizontal line $y(t) \equiv C$, which can be attained by setting $y' = 0$.

Ex. 8



$y' = y^2 - 4$

$(y-2)(y+2)$

$y = \pm 2$

$y = -2$: stable, solutions near it tend toward it as $t \to \infty$, they funnel together.

$y = 2$: unstable, solutions near it tend away from it, these solutions spray apart

1.3) Seperation of Variables : Qualitative Analysis

Seperable DE's

A Seperable differential equation is one that can be written $y' = f(t)g(y)$. Constant Solutions $y \equiv C$ can be found by solving $g(y) = 0$.

Seperation of Variables Method

Step 1: Set $g(y) = 0$ and solve for equilibrium.

Step 2: Rewrite the equation in seperated or differential form.

$$\frac{dy}{g(y)} = f(t)\,dt$$

Step 3: Integrate each side if possible.

$$\int \frac{dy}{g(y)} = \int f(t)\,dt + C$$

Step 4: If possible, solve for y in terms of t, getting the explicit solution $y = y(t)$

Step 5: If you have an IVP, use the initial condition to evaluate C.

Ex. 1

a.) $dy/dt = -\frac{t}{y} \to y\,dy = -t\,dt$

b.) $dy/dt = t^2 y \to \frac{dy}{y} = t^2 dt$

c.) $dy/dt = y+1 \to \frac{dy}{y+1} = dt$

d.) $dy/dt = t+1$ (not seperable)

Ex. 2

$$\frac{dy}{dt} = \frac{3t^2+1}{1+2y} \quad y(0) = 1$$

$$1+2y\,dy = 3t^2+1\,dy$$

$$y + y^2 = t^3 + t + k$$

$y(0)=1$: $1 + (1)^2 = 0^3 + 0 + k$

$$2 = k$$

$$y + y^2 = t^3 + t + 2$$

$$y^2 + y - (t^3 + t + 2) = 0$$

Quadratic Formula

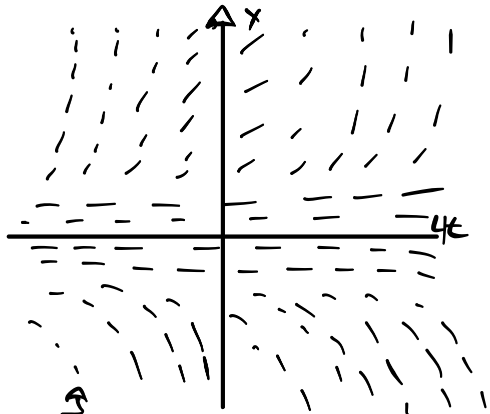
$$y = \frac{-1 \pm \sqrt{4t^3 + 4t + 9}}{2}$$

1.5) Picard's Theorem: Theoretical Analysis

Picard's Existence and Uniqueness Theorem

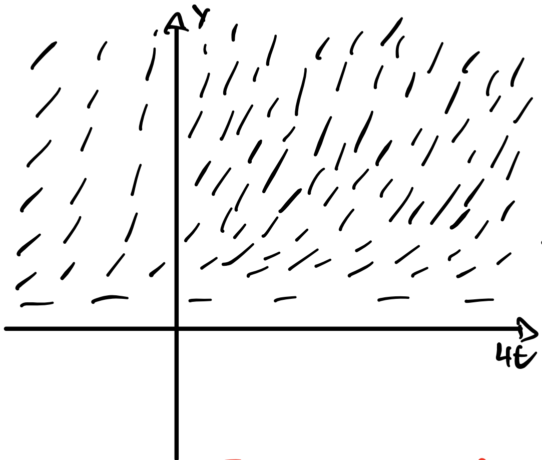
Suppose that the function $f(t,y)$ is continuous on the region $R=\{(t,y) \mid a<t<b,\ c<y<d\}$ and $(t_0,y_0)\in R$. Then there exists a positive number h such that the initial value problem $y'=f(t,y)$, $y(t_0)=y_0$ has a solution for t in the interval (t_0-h, t_0+h).

If $f_y(t,y)$ is also continuous in R, the solution is unique.



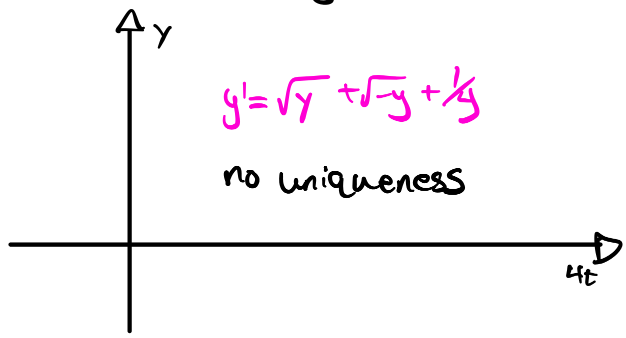
$y'=y$

unique through all



$y'=\sqrt{y}$

unique for $y>0$



$y'=\sqrt{y}+\sqrt{-y}+\frac{1}{y}$

no uniqueness

2.2) Solving the First-Order Linear Differential Equation

First-order linear differential equation $y'+p(t)y=f(t)$

Solution

$$y(t)=y_h+y_p=Ce^{-\int p(t)\,dt}+e^{-\int p(t)\,dt}\int f(t)e^{\int p(t)\,dt}\,dt$$

$$y_h=Ce^{-\int p(t)\,dt}$$

$$y_p=v(t)e^{-\int p(t)\,dt}\ :\ v(t)=\int f(t)e^{\int p(t)\,dt}\,dt$$

Ex 2.) $y'-y=t$

Integ. method

$$\mu=e^{\int(-1)\,dt}=e^{-t}$$

Multiply DE by $\mu(t)$

$$e^{-t}(y'-y)\ t(e^{-t})$$

$$\frac{d}{dt}e^{-t}y=te^{-t}$$

$$e^{-t}y=\int te^{-t}\,dt=e^{-t}(-t-1)+c$$

$$y=e^{t}(e^{-t})(-t-1)+ce^{t}=-t-1+ce^{t}$$

$$y=ke^{t}-t-1$$

Euler-Langrange Method for Linear ODE's

$y'+p(t)y=f(t)$

1. Solve $y'+p(t)y=0$ to obtain: $y_h=Ce^{-\int p(t)\,dt}$
2. Solve $v'(t)e^{-\int p(t)\,dt}=f(t)$ for $v(t)$ to obtain $y_p=v(t)e^{-\int p(t)\,dt}$
3. Combine steps 1 & 2 to form the general solution $y(t)=y_h+y_p$
4. Solve for IVP if applicable $y=y_h+y_p$

Integrating Factor Method $y'+p(t)y=f(t)$

1. Find the integrating factor $\mu(t)=e^{\int p(t)\,dt}$
2. Multiply each side of the differential equation by the integrating factor to get
$$e^{\int p(t)\,dt}[y'+p(t)y]=f(t)e^{\int p(t)\,dt}$$
which will always reduce to
$$\frac{d}{dt}\left[e^{\int p(t)\,dt}y(t)\right]=f(t)e^{\int p(t)\,dt}$$
3. Find the anti-derivative in 2 to get
$$e^{\int p(t)\,dt}y(t)=\int f(t)e^{\int p(t)\,dt}\,dt+C$$
4. Solve the final equation of 3 algebraically for to get the general solution
$$y=e^{-\int p(t)\,dt}\int f(t)e^{\int p(t)\,dt}\,dt+Ce^{-\int p(t)\,dt}$$

$y' - y = t$

$y_h: \ y' - y = 0$

$y' = y$

$\frac{dy}{dt} = y$

$\int \frac{dy}{y} = \int dt$

$\ln y = t + K$

$y_h = Ke^t$

$y = y_h + y_p$

$y_p: \ v'(t) = e^{\int p(t)dt} f(t)$

$e^{\int -1 dt} = e^{-t}$

$: \ e^{-t}(t)$

$v'(t) = te^{-t}$

$v(t) = e^{-t}(-t-1)$

$y_p = v(t) e^{-\int p(t)}$

$e^{-\int p(t)} = e^t$

$= e^{-t}(-t-1)e^t$

$y_p = -t-1$

$v'(t) = e^{\int p(t)dt} f(t)$

$y_p = v(t) e^{-\int p(t) dt}$

$y = y_h + y_p$

$u = t \quad dv = e^{-t}$

$du = 1 \quad v = -e^{-t}$

$-te^{-t} - \int -e^{-t} dt$

$-te^{-t} - e^{-t}$

$e^{-t}(-t-1)$

$y = Ke^t - t - 1$

3.3 The Inverse of A Matrix

The Inverse of A Matrix

If there exists, for an $n \times n$ matrix A, another matrix A^{-1} of the same order such that

$$A^{-1}A = AA^{-1} = I$$

then A^{-1} is called the inverse of matrix A, and A is said to be invertible.

The Inverse of A Matrix

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$$A^{-1}A = AA^{-1} = I$$

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Invertibility and Solutions

The matrix-vector equation $A\vec{x} = \vec{b}$, where A is an $n \times n$ matrix, has

- A unique solution $\vec{x} = A^{-1}\vec{b}$ if and only if A is invertible
- Either no solutions or infinitely many solutions if A is not invertible

Some Properties of Invertible Matrices

- If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$
- If A and B are invertible matrices of the same order, then their product AB is invertible. In fact, $(AB)^{-1} = B^{-1}A^{-1}$
- If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Inverse Matrix by RREF

For an $n \times n$ matrix A, the following procedure produces A^{-1}, or shows that A isn't invertible.

Step 1. Form the $n \times 2n$ matrix $M = [A|I]$.

Step 2. Transform M into it's RREF, R.

Step 3. If the first n columns of R form the identity matrix, then the last n columns form A^{-1}. If the first n columns of R do not form the identity matrix, then A is not invertible.

Some Properties of Invertible Matrices

- If A is invertible, then so is A^{-1}, and $(A^{-1})^{-1} = A$
- If A and B are invertible matrices of the same order, then their product AB is invertible. In fact, $(AB)^{-1} = B^{-1}A^{-1}$
- If A is invertible, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

A matrix is non invertible if the determinant of the matrix is zero!

Ex:

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$|A| = 1(2) - 2(1) = 2 - 2 = 0$

$|A| = 0 \therefore$ non invertible

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$|A| = 1(2) - 1(1) = 2 - 1 = 1$

$|A| = 1 \therefore$ Invertible

3.4 Determinants and Cramer's Rule

Determinant of A 2x2 Matrix

The determinant of a two-by-two matrix is the product of the diagonal elements minus the product of the off-diagonal elements:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Minors and Cofactors of a Matrix

Every element a_{ij} of an $n \times n$ matrix A has an associated minor and cofactor,

- The minor M_{ij} of a_{ij} is an $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A.
- The cofactor of a_{ij} is the scalar

$$C_{ij} = (-1)^{i+j}|M_{ij}|$$

Determinant of An nxn Matrix A

Choose any row or column and expand by the appropriate cofactor formula, using either expansion by the i^{th} row:

$$|A| = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}|$$

or, equivalently, expansion by the j^{th} column:

$$|A| = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}|$$

Repeat this process, obtaining smaller matrices each step.

Row Operations and Determinants

Let A be a square matrix.

- If two rows of A are interchanged to produce matrix B, then $|B| = -|A|$
- If one row of A is multiplied by a constant K and then added to another row to produce matrix B, then $|B| = |A|$
- If one row of A is multiplied by K to produce matrix B, then $|B| = K|A|$

Determinants of Products of Matrices

For $n \times n$ matrices A & B, the determinant AB is given by $|AB| = |A||B|$

Other Properties of Determinants

- $|A^T| = |A|$
- If $|A| \neq 0$, then $|A^{-1}| = \frac{1}{|A|}$
- If A is an upper triangular or lower triangle matrix the determinant is the product of the diagonal elements, $|A| = \prod_{i=1}^{M} a_{ij}$
- If one row or column of A consists entirely of zeros, then $|A| = 0$
- If two rows or columns of A are equal, then $|A| = 0$

Cramers Rule

For the $n \times n$ matrix A having $|A| \neq 0$, denote by A_i the matrix obtained from A by replacing its i^{th} column with the column vector $\vec{b}$. Then the i^{th} component of the solution of the system $A\vec{x} = \vec{b}$ is given by

$$x_i = \frac{|A_i|}{|A|}$$

Least Squares Method

The best-fit straight line for n data points (x_i, y_i), $i = 1, 2, \dots, n$, has intercept K and slope m as determined by the system.

Ex.)

$$A = \begin{bmatrix} 0 & 7 & 9 \\ 2 & 1 & -1 \\ 5 & 6 & 2 \end{bmatrix}$$

$$5 \cdot (-1)^4 (-16) + 6(-1)^5 (-18) + 2(-1)^6 (-14) = 0$$

$$\begin{bmatrix} 0 & 7 & 9 \\ 2 & 1 & -1 \\ 5 & 6 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 7 & 9 \\ 2 & 1 & -1 \\ 5 & 6 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 7 & 9 \\ 2 & 1 & -1 \\ 5 & 6 & 2 \end{bmatrix}$$

3.5 Vector Spaces and Subspaces

Vector Space

A vector space $\mathbb{V}$ is a nonempty collection of objects called vectors for which are defined operations

- Vector addition, denoted $\vec{x}+\vec{y}$, and
- Scalar multiplication (multiplication by a real constant) denoted $c\vec{x}$,

that satisfy the following properties for all $\vec{x},\vec{y},\vec{z} \in \mathbb{V}$ and $c,d \in \mathbb{R}$

Closure Properties

1. $\vec{x}+\vec{y} \in \mathbb{V}$
2. $c\vec{x} \in \mathbb{V}$

Addition Properties

3. There is a zero vector $\vec{0}$ in $\mathbb{V}$ such that $\vec{x}+\vec{0}=\vec{x}$ (Additive Identity)
4. For every vector $\vec{x} \in \mathbb{V}$, there is a vector $\vec{x}$ in $\mathbb{V}$ (its negative) such that $\vec{x}+(-\vec{x})=\vec{0}$ (Additive Inverse)
5. $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$ (Associativity)
6. $\vec{x}+\vec{y}=\vec{y}+\vec{x}$ (Commutativity)

Scalar Multiplication Properties.

7. $1\vec{x}=\vec{x}$ (Scalar Multiplicative Identity)
8. $c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y}$ (First Distributive Property)
9. $(c+d)\vec{x}=c\vec{x}+D\vec{y}$ (Second Distributive Property)
10. $c(d\vec{x})=(cd)\vec{x}$ (Associativity)

$$\begin{bmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} K \\ m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Closure under Linear Combination

0. $c\vec{x}+d\vec{y} \in \mathbb{V}$ whenever $\vec{x},\vec{y} \in \mathbb{V}$ and $c,d \in \mathbb{R}$

Prominent Vector Spaces

$\mathbb{R}^2$: The space of all ordered pairs (or $2\vec{v}$)

$\mathbb{R}^3$: The space of all ordered triples (or $3\vec{v}$)

$\mathbb{R}^n$: The space of all ordered n-tuples (or $n\vec{v}$)

$\mathbb{P}$: The space of all polynomials

$\mathbb{P}_n$: The space of all polynomials of degree less than or equal to n

$\mathbb{M}_{mn}$: The space of all $m\times n$ matrices

$C(I)$: The space of all continuous functions on the interval I (I may be an open or closed interval, finite or infinite)

$C^n(I)$: The space of all functions on interval I (as above) having n continuous derivatives; C^n with no I specified is understood to mean $C^n(-\infty,\infty)$

$\mathbb{C}^n$: The space of all ordered n-tuples of complex numbers $(a+bi, c+di, \ldots)$

Vector Subspace Theorem

A nonempty subset $\mathbb{W}$ of a vector space $\mathbb{V}$ is a subspace of $\mathbb{V}$ if it is closed under addition and scalar multiplication

(i) If $\vec{u},\vec{v} \in \mathbb{W}$, then $\vec{u}+\vec{v} \in \mathbb{W}$

(ii) If $\vec{u} \in \mathbb{W}$ and $c \in \mathbb{R}$, then $c\vec{u} \in \mathbb{W}$

Ex: The set of vectors in the first quadrant of the plane.

No because the closure property is violated.

$c\vec{x} \in V$

Also No Additive inverse

Ex. The set of all diagonal 2×2 matrices

No properties are violated

This is a vector space

Ex: The set of vectors in the first octant (x,y,z) space

No, closure property is violated

Closure proprty

Ex: The set of all pairs of real numbers (x,y) such that $x \geq y$

No, the negative does not exist i.e $[2,1] \Rightarrow [-2,-1]$ $x \ngeq y$

Negative of the set violates $x \geq y$

Ex: The set of all differentiable functions on $(-\infty,\infty)$

No properties violated

This is a vector space

3.6 Basis and Dimension

Span

The **span** of a set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of vectors in a vector space $\mathbb{V}$, denoted $\text{Span}\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$, is the set of all linear combinations of these vectors.

Span Theorem

For a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ in vector space $\mathbb{V}$, $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a subspace of $\mathbb{V}$.

Linear Independence

A set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of vectors in vector space $\mathbb{V}$ is **linearly independent** if no vector of the set can be written as a linear combination of the others. Otherwise it is **linearly dependent**.

Linear Independence of Vector Functions

A set of vector functions $\{\vec{v}_1(t), \vec{v}_2(t), \ldots \vec{v}_n(t)\}$ in a vector space $\mathbb{V}$ is **linearly independent** on an interval I if, for all t in I, the only solution of

$$c_1\vec{v}_1(t) + c_2\vec{v}_2(t) + \ldots + c_n\vec{v}_n(t) \equiv \vec{0}$$

for scalars $c_1, c_2, \ldots c_n$ is $c_i = 0$ for all i. If for any value t_0 of t there is any solution with $c_i \neq 0$, the vector functions $\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$ are linearly independent.

Wronskian of Functions f_1, f_2, f_3

$W[f_1, f_2, \ldots, f_n](t)$

$$W = \left| \begin{bmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{bmatrix} \right|$$

$|W| \neq 0 \;\therefore$ Linearly Independent

Basis of A Vector Space

The set $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a **basis** for a vector space $\mathbb{V}$ provided that

(i) $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent

(ii) $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\} = \mathbb{V}$

Properties of The Column Space of A Matrix

- The pivot columns of matrix A form a basis for Col A
- The dimension of the column space, called the **rank** of A, is the number of pivot columns in A.

$$\text{rank}\,A = \dim(\text{col}\,A)$$

Spanning Sets in $\mathbb{R}^n$

A vector $\vec{b}$ in $\mathbb{R}^n$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\}$, where $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are vectors in $\mathbb{R}^n$, provided that there is at least one solution of the matrix vector equation $A\vec{x} = \vec{b}$, where A is the matrix whose column vectors are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

Column Space

For any $m \times n$ matrix A, the **column space**, denoted Col A, is the span of the column vectors of A, and is a subspace of $\mathbb{R}^n$.

Testing For Linear Independence

The test for linear independence of a set of n vectors $\vec{v}_i$ in $\mathbb{R}^n$, we consider the system $(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0})$ in matrix-vector form

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{x}} = \vec{0}$$

The column vectors of A are linearly independent if and only if the solution $\vec{x} = \vec{0}$ is unique. Which means that $c_i = 0$ for all i. Recall from Sections 3.3 and 3.4 that any one of the following satisfies this condition for a unique solution:

- A is invertible
- A has n pivot columns
- $|A| \neq 0$

The Wronskian and Linear Independence Theorem

If $W[f_1, f_2, \ldots f_n](t) \neq 0$ for all t on the interval I, where $f_1, f_2, \ldots, f_n$ are defined, then $\{f_1, f_2, \ldots, f_n\}$ is a linearly independent set of functions on I^2.

Standard Basis For $\mathbb{R}^n$

The **Standard basis** for $\mathbb{R}^n$ is

$$\{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$$

where

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \vec{e}_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are the column vectors of the identity matrix I_n.

Invertible Matrix characterization (Basis for Col A)

Let A be an $n \times n$ matrix. The following statements are equivalent:

- A is invertible
- The column vectors of A are linearly independent
- Every column of A is a pivot column
- The column vectors of A form a basis for Col A.
- $\text{rank}\,A = n$

Ex) $W=\mathbb{R}^3$; $S=\{[2,-1,4],[4,-2,8]\}$

$$A=\left[\begin{array}{cc|c}2&4&0\\-1&-2&0\\4&8&0\end{array}\right]\quad rref(A)=\left[\begin{array}{cc|c}1&2&0\\0&0&0\\0&0&0\end{array}\right]$$

No identity matrix $\therefore$ not linearly independent

Ex) $W=\mathbb{R}^3$; $S=\{[1,1,8],[-3,4,2],[7,1,-3]\}$

$$A=\left[\begin{array}{ccc|c}1&-3&7&0\\1&4&1&0\\8&2&-3&0\end{array}\right]\quad rref(A)=\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

rref(A) yields an identity matrix and $\vec{x}=\vec{0}$ $\therefore$ linearly independent

Ex.)

$$3x_1+0x_2+6x_3+3x_4+9x_5=0$$
$$x_1+3x_2-4x_3-8x_4+3x_5=0$$
$$x_1-6x_2+14x_3+19x_4+3x_5=0$$

$$A=\left[\begin{array}{ccccc|c}3&0&6&3&9&0\\1&3&-4&-8&3&0\\1&-6&14&19&3&0\end{array}\right]$$

$$=\begin{bmatrix}1&0&2&1&3&0\\0&1&-2&-3&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Basis $=\begin{bmatrix}2&1&3\\-2&-3&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Dim $=3$

Ex)

$$\left\{\begin{bmatrix}e^t\\2e^t\\e^t\end{bmatrix},\begin{bmatrix}e^{-t}\\2e^{-t}\\e^t\end{bmatrix},\begin{bmatrix}e^{2t}\\3e^{2t}\\e^{2t}\end{bmatrix}\right\}$$

$$A=\begin{bmatrix}e^t&e^{-t}&e^{2t}\\2e^t&2e^{-t}&3e^{2t}\\e^t&e^t&e^{2t}\end{bmatrix}$$

$$rref(A)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$\det(A)=-e^{2t}(e^{2t}-1)\neq 0$$

$rref(A)=I$
$\det(A)\neq 0$
$\therefore$ linearly independent

4.1 Harmonic Oscillator

Linear second order DE : $m\ddot{x}+b\dot{x}+kx=0$

Simple Harmonic Oscillator

The simple harmonic oscillator equation is

$$m\ddot{x}+b\dot{x}+kx=f(t) \quad (1)$$

a second-order homogeneous linear differential equation with constant coefficients $m>0$, $k>0$, & $b\geq 0$

- When $b=0$, the motion is called undamped, otherwise it is damped.
- If $f(t)\equiv 0$, the equation is homogeneous,

$$m\ddot{x}+b\dot{x}+kx=0 \quad (2)$$

and the motion is called unforced, undriven, or free; otherwise the motion is forced or driven.

Solution to the Undamped Oscillator

For the undamped oscillator ($m\ddot{x}+kx=0$), solutions are

$$x(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t \quad (4)$$
$$\omega_0=\sqrt{\frac{k}{m}}$$

Where c_1 and c_2 are arbitrary constants determined by initial conditions.

Alternative Solution to the Undamped Unforced Oscillator

Solutions to the undamped unforced oscillator ($m\ddot{x}+kx=0$) may also be expressed as a family of sinusoidal oscillations given by $x(t)=A\cos(\omega_0 t-\delta)$ (5)

- Amplitude A and phase angle δ (measured in radians) are arbitrary (but meaningful) constants that can be determined by initial conditions.
- The motion has circular frequency $\omega_0=\sqrt{k/m}$ measured in radians per second or oscillations per 2π seconds, and natural frequency $f_0=\omega_0/2\pi$, measured in oscillations per second.
- The period T of the oscillation (measured in seconds) is given by

$$T=\frac{1}{f_0}=\frac{2\pi}{\omega_0}=2\pi\sqrt{m/k} \quad (6)$$

- The solution (5) is a horizontal translation of $A\cos(\omega_0 t)$ with phase shift δ/ω_0

Conversion of Solutions to the Undamped Unforced Oscillator

The translation between the two forms (4) and (5) of the solution to the undamped unforced oscillator is given by

$$A=\sqrt{c_1^2+c_2^2},\quad \tan\delta=c_2/c_1 \quad (7)$$

and

$$c_1=A\cos\delta,\quad c_2=A\sin\delta \quad (8)$$

Phase Portraits

For any autonomous second-order differential equation

$$\ddot{x} = F(x, \dot{x})$$

the **phase plane** is the two-dimensional graph wit x and $\dot{x}$ axes. The phase plane has a **vector field** specified by the DE, which at any point in the phase plane gives a direction vector with

horizontal component $dx/dt = \dot{x}$

vertical component $d\dot{x}/dt = \ddot{x}$

A **trajectory** is a path formed parametrically by the DE solutions $x(t)$ and $\dot{x}(t)$ as they follow the vector field.

A graph showing phase plane trajectories is called a **phase portrait**.

Conversion of a Second-Order Linear DE to a System

The Second-order DE

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

is equivalent to the system of first-order equations

$$\dot{x} = y$$
$$\dot{y} \equiv \ddot{x} = f(t) - (k/m)x - (b/m)y$$
$$\dot{y} = f(t) - (k/m)x - (b/m)y$$

Series Circuit Equation

$$L\ddot{Q} + R\dot{Q} + \tfrac{1}{C}Q = v(t) \qquad (17)$$

If we assume that there is no voltage source, so that $v(t) \equiv 0$

$$L\ddot{Q} + R\dot{Q} + \tfrac{1}{C}Q = 0 \qquad (18)$$

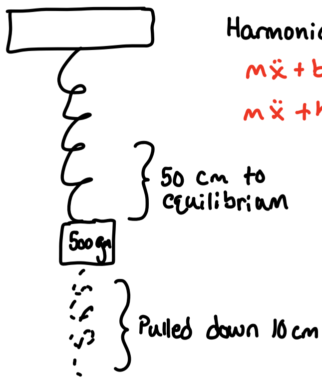
Series Circuit Equation (current)

$$L\ddot{I} + R\dot{I} + \tfrac{1}{C}I = \dot{V}(t) \qquad (19)$$

For $\dot{V}(t) \equiv 0$, we get the homogenous equation

$$L\ddot{I} + R\dot{I} + \tfrac{1}{C}I = 0 \qquad (20)$$

EX:)



Harmonic Oscillator

Frictionless = undamped

$m\ddot{x} + b\dot{x} + kx = f(t)$

$m\ddot{x} + kx = f(t)$ <= undamped

$K = \frac{m}{\Delta s} = \frac{500\text{ gm}}{50\text{ cm}}$

$K = 10$

a.) $m\ddot{x} + kx = 0$

$$\boxed{500\ddot{x} + 10x = 0}$$

$K = \frac{m}{\Delta s} = \frac{500}{50} = 10$

b.) $x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$, $\omega_0 = \sqrt{\frac{k}{m}}$

$500\ddot{x} + 10x = 0$

$K = 10$

$m = 500$

$\omega_0 = \sqrt{\frac{10}{500}} = \sqrt{\frac{1}{50}} = \frac{1}{\sqrt{50}} = \frac{1}{5\sqrt{2}}$

$c_1 = 10$

$c_2 = 0$

$x(t) = c_1\cos\left(\frac{t}{5\sqrt{2}}\right) + c_2\sin\left(\frac{t}{5\sqrt{2}}\right)$

$t = 0$

$10 = c_1\cos(0) + c_2\sin(0)$

$10 = c_1$

$\dot{x}(t) = -\frac{5}{\sqrt{2}}\sin\left(\frac{t}{5\sqrt{2}}\right) + c_2\left(\frac{1}{5\sqrt{2}}\right)\cos\left(\frac{t}{5\sqrt{2}}\right)$

$t = 0$

$0 = -\frac{5}{\sqrt{2}}\sin(0) + \frac{c_2}{\sqrt{50}}\cos(0)$

$0 = \frac{c_2}{\sqrt{50}}$ $\quad c_2 = 0$

$$\boxed{x(t) = 10\cos\left(\frac{t}{5\sqrt{2}}\right)}$$

c.) Amplitude: $A = \sqrt{c_1^2 + c_2^2}$

Phase Angle: $\tan\delta = c_2/c_1$

Frequency (natural): $f_0 = \omega_0/2\pi$

Period: $T = 2\pi\sqrt{\frac{m}{k}}$

$c_1 = A\cos\delta$

$c_2 = A\sin\delta$

$$\boxed{A = 10 \quad f_0 = \frac{1}{10\pi\sqrt{2}} \quad \delta = 0^\circ \quad T = 2\pi\sqrt{50}}$$

$A = \sqrt{c_1^2 + c_2^2}$ $\quad c_1 = 10, c_2 = 0$

$A = \sqrt{10^2}$

$A = 10$

$\tan\delta = c_2/c_1$

$\delta = \tan^{-1}(0/10)$

$\delta = 0$

$f_0 = \omega_0/2\pi$ $\quad \omega_0 = \frac{1}{5\sqrt{2}}$

$= \frac{1/5\sqrt{2}}{2\pi}$

$= 1/10\sqrt{2}\pi$

$f_0 = \frac{1}{10\sqrt{2}\pi}$

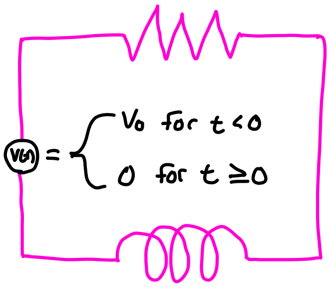
$T = 2\pi\sqrt{\frac{m}{k}}$ $\quad m = 500$, $K = 10$

$= 2\pi\sqrt{\frac{500}{10}}$

$2\pi\sqrt{50}$

$T = 2\pi\sqrt{50}$

Ex:



a.) The circuit will experience voltage drop,

$V_R(t) = RI(t)$, $V_L(t) = L\dot{I}(t)$

b.) Drop across Resistor: $V_R(t) = RI(t)$

c.) Drop across Inductor: $V_L(t) = L\dot{I}(t)$ $\qquad V = IR$, $I = \frac{V}{R}$

$V_{tot} = V_R(t) + V_L(t)$

$V = RI + L\dot{I}$, $V=0$

$0 = RI + L\dot{I}$

$-L\dot{I} = RI$

$\int \frac{\dot{I}}{I} = \int \frac{R}{-L}$

$\ln I = -(R/L)t + C$

$I = Ke^{-(R/L)t}$

$I = Ke^{-(R/L)t}$

$\frac{V}{R} = Ke^{-(R/L)t}$, $t=0$

$\frac{V}{R} = Ke^{0}$

$\frac{V}{R} = K$

$I = \frac{V}{R} e^{-(R/L)t}$

d.) $R = 40\Omega$

$L = 5h$

$V_0 = 10v$

$I = \frac{10}{40} e^{-(40/5)t}$

$I = \frac{1}{4} e^{-8t}$

$I = \frac{1}{4} e^{-8t}$

4.2 Real characteristic Roots

linear Second Order Homogeneous DE

$$ay'' + by' + cy = 0 \qquad (1)$$

(a,b,c) = constants $a \neq 0$

Characteristic equation

$$ar^2 + br + c = 0 \qquad (2)$$

Solution of $ay''+by'+cy=0$ with Distinct Real Characteristic Roots

For $\Delta = b^2 - 4ac > 0$, the characteristic roots of the DE are

$$r_1 = \frac{-b+\sqrt{\Delta}}{2a} \qquad r_2 = \frac{-b-\sqrt{\Delta}}{2a} \qquad (3)$$

The functions $e^{r_1 t}$ and $e^{r_2 t}$ are linealy independent solutions, and the generae solution is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (4)$$

where c_1 and c_2 are arbitrary constants determined by initial conditions. The set $\{e^{r_1 t}, e^{r_2 t}\}$ forms a basis for the Solution Space.

Solution of $ay''+by'+cy=0$ with Repeated Real Characteristic Roots

For $\Delta = b^2 - 4ac = 0$, the characteristic root of the DE is

$$r = \frac{-b}{2a} \qquad (6)$$

The functions e^{rt} and te^{rt} are linearly independent solutions, and the general solutions is given by

$$y(t) = c_1 e^{rt} + c_2 te^{rt} \qquad (7)$$

where c_1 and c_2 are arbitrary constants determined by initial conditions. The set $\{e^{rt}, te^{rt}\}$ is a basis for the solution space.

Critically Damped Mass-Spring System

The motion of a mass-spring system is called critically damped when $\Delta = b^2 - 4mk = 0$. The single characteristic root is negative,

$$r = \frac{-b}{2m}$$

Critically Damped Mass-Spring System

The motion of a mass-spring system is called critically damped when $\Delta = b^2 - 4mk = 0$. The single characteristic root is negative,

$$r = \frac{-b}{2m}$$

Solutions to the critically damped system, given by

$$x(t) = c_1 e^{rt} + c_2 te^{rt} \qquad (10)$$

tend to zero, crossing the t axis at most once.

Overdamped Mass-Spring System

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad (8)$$

The motion of a mass spring system (8) is called overdamped when we have $\Delta = b^2 - 4mk > 0$. Both characteristic roots are negative and solutions

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (9)$$

tend to zero without oscillation, crossing the t axis at most once.

Existance and Uniqueness Theorem (second-order V)

Let $p(t)$ and $q(t)$ be continuous on the open interval (a,b) containing t_0. For any A and B in $\mathbb{R}$, there exists a unique solution $y(t)$ defined on (a,b) to the initial-value problem

$$y'' + p(t)y' + q(t)y = 0 \qquad y(t_0) = A,\ y'(t_0) = B$$

Existance and Uniqueness Theorem (second-order V)

Let $p(t)$ and $q(t)$ be continuous on the open interval (a,b) containing t_0. For any A and B in $\mathbb{R}$, there exists a unique solution $y(t)$ defined on (a,b) to the initial-value problem

$$y'' + p(t)y' + q(t)y = 0 \qquad y(t_0) = A,\ y'(t_0) = B$$

Solutions of Homogeneous Linear DE'S (Second order version)

For any linear Second-order homogeneous DE on (a,b)

$$y''+p(t)y'+q(t)y=0$$

For which p and q are continuous on (a,b), any two linearly independent Solutions $\{y_1, y_2\}$ form a basis of the Solution Space S, and every Solution y on (a,b) can be written as

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

for some real numbers c_1 and c_2

Solutions of Homogeneous linear DE's (n^{th}-order Version)

For any linear n^{th}-order homogeneous DE on (a,b),

$$y^{(n)}(t)+p_1(t)y^{(n-1)}(t)+p_2(t)y^{(n-2)}(t)+\dots+p_n(t)y(t)=0$$

For which $p_1(t), p_2(t), \dots, p_n(t)$ are continuous on (a,b), any n linearly independent solution y on (a,b) can be written as

$$y(t)=c_1y_1(t)+c_2y_2(t)+\dots+c_ny_n(t)$$

For some real numbers $c_1, c_2, \dots c_n$

Solution Space Theorem (Second order version)

The Solution space S for a second-order homogeneous differential equation has Dim 2.

Existance and Uniqueness Theorem

Let $p_1(t), p_2(t), \dots, p_n(t)$ be continuous functions on the open interval (a,b) containg t_0. For any initial values $A_0, A_1, \dots A_{n-1} \in \mathbb{R}$, there exists a unique Solution $y(t)$ defined on (a,b) to the initial-value problem

$$y^{(n)}(t)+p_1(t)y^{(n-1)}(t)+p_2(t)y^{(n-2)}+\dots+p_n(t)y(t)=0 \quad (13)$$

$$y(t_0)=A_0,\ y'(t_0)=A_1, \dots y^{(n-1)}(t)=A_{n-1}$$

Solution Space Theorem

The solutions to homogeneous linear differential equation of order n for an n-dimensional vector space.

The Wronskian Test For linear Independence of DE Solutions

Suppose $\{y_1, y_2, \dots, y_n\}$ is a set of solutions on (a,b) of an n^{th}-order linear homogeneous DE.

$$L(y)=a_n(t)\frac{d^ny}{dt^n}+a_{(n-1)}\frac{d^{n-1}}{dt^{n-1}}+\dots+a_1(t)\frac{dy}{dt}+a_0(t)y\equiv 0$$

(i) IF $W[y_1,y_2,\dots y_n]\neq 0$ at any point t on (a,b), the set $\{y_1, y_2, \dots y_n\}$ is linearly independent.

(ii) IF $W[y_1,y_2,\dots y_n]\equiv 0$ on (a,b), the set $\{y_1, y_2, \dots y_n\}$ is linearly dependent

The Wronskian test works in "both directions" only for n Solutions to an n^{th}-order linear homogeneous DE.

Ex: $y''-9y=0 \quad y(0)=-1,\ y'(0)=0$

$\Delta=b^2-4ac:\ 0^2-4(1)(-9)$

$a=1 \quad \Delta=36$

$b=0 \quad \Delta>0 \ \therefore$

$c=-9$

$r_1=\dfrac{-b+\sqrt{\Delta}}{2a}, \quad r_2=\dfrac{-b-\sqrt{\Delta}}{2a}, \quad y(t)=c_1e^{r_1t}+c_2e^{r_2t}$

$r_1=3$

$r_2=-3$

$=\dfrac{0+\sqrt{36}}{2(1)} \quad = -\dfrac{\sqrt{36}}{2(1)}$

$=3 \quad =-3$

$y(t)=c_1e^{3t}+c_2e^{-3t}$

$-1=c_1e^0+c_2e^0 \ :\ y(0)=-1$

$-1=c_1+c_2$

$y'(t)=3c_1e^{3t}-3c_2e^{-3t}$

$0=3c_1e^0-3c_2e^0 \ :\ y'(0)=0$

$0=3c_1-3c_2$

$3c_2=3c_1$

$c_2=c_1$

$c_1+c_2=-1 \quad x+x=-1$

$c_1=c_2 \quad 2x=-1$

$x=-½$

$c_1=-½$

$c_2=-½$

$$y(t)=-\tfrac{1}{2}e^{3t}-\tfrac{1}{2}e^{-3t}$$

Ex: $y''-4y'+4y=0$

$(r-2)(r-2)$

$r=2$

$r=\dfrac{-b}{2a}=\dfrac{-(-4)}{2(1)}=\dfrac{4}{2}=2$

$y=c_1e^{rt}+c_2te^{rt}$

$$y=c_1e^{2t}-c_2te^{2t}$$

Ex: $\ddot{x}+3\dot{x}+2=0 \quad x(0)=1$
$\dot{x}(0)=0$

$(r+2)(r+1)$

$r=-1,-2$

$x=c_1e^{-t}+c_2e^{-2t}$

$1=c_1e^0+c_2e^0$

$1=c_1+c_2$

$\dot{x}=-c_1e^{-t}-2c_2e^{-2t}$

$0=-c_1e^0-2c_2e^0$

$0=-c_1-2c_2$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -2 & 0 \end{array}\right]$

$x+y=1 \quad y=-1 \quad c_1=2$

$x=1-y \quad c_2=-1$

$x=2$

$$x=2e^{-t}-e^{-2t}$$

4.3 Complex Characteristic Roots

Solution of "$ay''+by'+cy=0$" with Complex Characteristic Roots

For $\Delta = b^2-4ac<0$, the characteristic root of the DE are

$$r_1, r_2 = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a}, \quad \beta = \frac{\sqrt{-\Delta}}{2a} \qquad (5)$$

The functions $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$ are linearly independent solutions, and the general solution is given by

$$y(t) = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t) \qquad (6)$$

where c_1 and c_2 are arbitrary constants determined by initial conditions. The set $\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}$ forms a basis for the solution space.

Underdamped Mass-Spring System

$$m\ddot{x} + b\dot{x} + kx = 0 \quad m,b,k>0 \qquad (10)$$

The motion of a mass-spring system (10) is called underdamped when $\Delta = b^2-4mk<0$. Solutions are given by

$$x(t) = e^{\frac{-b}{2m}t}(c_1\cos w_d t + C_2\sin w_d t), \quad w_d = \frac{\sqrt{4mk-b^2}}{2m} \qquad (11)$$

Solutions to the Second-Order Linear DE with Constant Coefficients

The differential equation

$$ay''+by'+cy=0$$

has the characteristic equation

$$ar^2+br+c=0$$

The quadratic formula gives rise to three different general solutions y_h for the DE, depending on the value of the discriminant $\Delta = b^2-4ac$

Factoring Characteristic Equations

$$a_nr^n + a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \dots + a_1r + a_0 = 0 \qquad (17)$$

If the coefficients in (17) are integers, we can select a rational factor q of a_0/a_n and substitute $r=q$ into the characteristic equation

$$f(r) = a_nr^n + \dots + a_1r + a_0 = 0$$

to see if $f(q)=0$. If so, $r-q$ is a factor of $f(r)$

Alternate Solution to the Underdamped Unforced Oscillator

Solutions to the underdamped unforced oscillator may also be expressed as a family of sinusoidal oscillators with amplitude decreasing over time, given by

$$x(t) = A(t)\cos(w_d t - \delta) \qquad w_d = \frac{\sqrt{4mk-b^2}}{2m} \qquad (12)$$

where A and δ are arbitrary constants determined by initial conditions. The oscillator has

- Time-varying amplitude $A(t) = Ae^{-(b/2m)t}$
- Phase angle δ (measured in radians); and
- Phase shift $\varphi = \delta/w_d$

The motion is not strictly periodic; oscillating with

- Circular quasi-frequency $w_d = \frac{\sqrt{4mk-b^2}}{2m}$ (radians/sec)
- natural quasi-frequency $f_d = w_d/2\pi$ (in hertz); and
- quasi-period $T_d = \frac{1}{f_d} = \frac{2\pi}{w_d} = \frac{4\pi m}{\sqrt{4mk-b^2}}$ (measured in seconds)

The solution oscillates between two exponential curves $x(t) = \pm Ae^{-(b/2m)t}$, and the time required for the damped amplitude of the oscillation to decay from A to A/e is given by

- Time constant $\tau = 2m/b$

Case 1 $\Delta>0$	Real unequal roots: $r_1, r_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Overdamped motion: $y_h = C_1e^{r_1t} + C_2e^{r_2t}$
Case 2 $\Delta=0$	Real repeated root: $r = -\frac{b}{2a}$	Critically damped motion: $y_h = C_1e^{rt} + C_2te^{rt}$
Case 3 $\Delta<0$	Complex conjugate roots: $r_1, r_2 = \alpha \pm \beta i$, $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac-b^2}}{2a}$	underdamped motion: $y_h = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t)$

Ex: $y''-4y'+13y=0$

$\alpha = \frac{-b}{2a}$

$\beta = \frac{\sqrt{-\Delta}}{2a}$

$\Delta = -36$

$y = e^{\alpha t}($

$\alpha = 2$

$\beta = 3$

$$y = e^{2t}(c_1\cos(3t) + c_2\sin(3t))$$

Ex: $y''+2y'+4y=0 \quad y(0)=0, \quad y'(0)=1$

$\Delta = b^2-4ac$
$= 4-16$
$\Delta = -12$

$\alpha = -1$

$\beta = \frac{\sqrt{12}}{2} = \sqrt{3}$

$c_1 = 0$

$c_2 = 1/\sqrt{3}$

$$y = e^{-t}(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t))$$

$0 = e^0(c_1\cos(0) + c_2\sin(0))$

$0 = c_1$

$1 = \sqrt{3}c_2$

$-e^{-t}(c_1\cos\sqrt{3}t + c_2\sin\sqrt{3}t)$

$+e^{-t}(-\sqrt{3}c_1\sin\sqrt{3}t + \sqrt{3}c_2\cos\sqrt{3}t)$

$1 = -e^0(c_1\cos 0 + c_2\sin 0) + e^0(-\sqrt{3}c_1\sin 0 + \sqrt{3}c_2\cos 0)$

$1 = -c_1 + \sqrt{3}c_2 \qquad 1 = \sqrt{3}c_2$

$$y = \frac{e^{-t}}{\sqrt{3}}\sin(\sqrt{3}t)$$

4.4 Undetermined Coefficients

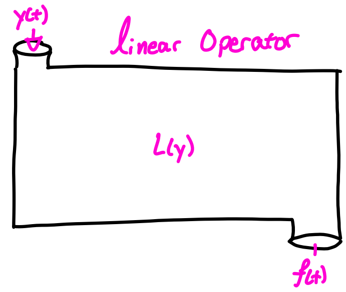
Superposition Principle for Nonhomogeneous Linear DE's

If $y_i(t)$ is a solution of $L(y)=f_i(t)$, for $i=1,2,\dots N$ & $c_1,c_2,\dots,c_n$ are constants, then

$$y(t)=c_1y_1(t)+c_2y_2(t)+\dots+c_ny_n(t)$$

is a solution of

$$L(y)=c_1f_1(t)+c_2f_2(t)+\dots+c_nf_n(t)$$



Nonhomogeneous Principle for Linear DE's

The general solution of the nonhomogeneous linear differential equation $L(y)=f$ is

$$y=y_h+y_p$$

where

- y_h is the general solution of $L(y)=0$
- y_p is a particular solution of $L(y)=f$

Forcing Terms for the Method of Undetermined Coefficients

- Polynomials in t
- Exponentials e^{at}
- Sinusoidal functions of the form $\cos kt$ & $\sin kt$

Predicted Forms of y_p for the Method of Undetermined coefficients

For a second-order linear DE

$$ay''+by'+cy=f(t)$$

the method of undetermined coefficients uses the form of $f(t)$ to predict the form of $y_p(t)$, as shown in Table 4.4.1

Table 4.4.1 Predicting Forms of Particular Solutions

	Forcing Function $f(t)$	$\Rightarrow$ Particular Solution $y_p(t)$
(i)	K	A_0
(ii)	$P_n(t)$	$A_n(t)$
(iii)	Ce^{Kt}	A_0e^{Kt}
(iv)	$C\cos wt+D\sin wt$	$A_0\cos wt+B_0\sin wt$
(v)	$P_n(t)e^{Kt}$	$A_n(t)e^{Kt}$
(vi)	$P_n(t)\cos wt+Q_n(t)\sin wt$	$A_n(t)\cos wt+B_n(t)\sin wt$
(vii)	$Ce^{Kt}\cos wt+De^{Kt}\sin wt$	$A_0e^{Kt}\cos wt+B_0e^{Kt}\sin wt$
(viii)	$P_n(t)e^{Kt}\cos wt+Q_n(t)e^{Kt}\sin wt$	$A_n(t)e^{Kt}\cos wt+B_n(t)e^{Kt}\sin wt$

- $P_n(t)$, $Q_n(t)$, $A_n(t)$, $B_n(t)\in P_n$ (hence $A_0,B_0\in P_0=\mathbb{R}$) and $K,w,C,$ and D are real constants
- In (iv)-(viii), both terms must be included in y_p, even if only one of the terms is present in $f(t)$

If any term or terms of y_p are found in y_h (i.e., if such terms are solutions of $ay''+by'+cy=0$), multiply the expression for y_p by t (or, if necessary, by t^2) to eliminate the duplication.

Example 3

a. $ay''+by'+cy=d$

When all constants, y_p can be solved by inspection.

$y_p=d/c$, $c\neq 0$

Example 5

$y''-y'-2y=3t^2-1$

$y_p=At^2+Bt+C$

$y'_p=2At+B \qquad y''_p=2A$

$2A-(2At+B)-2(At^2+Bt+C)=3t^2-1$

$(-2A)t^2+(-2A-2B)t+(2A-B-2C)=3t^2-1$

$-2A=3 \qquad -2A-2B=0 \qquad 2A-B-2C=-1$

$A=-3/2$, $B=3/2$, $C=-7/4$

$y_p=-\frac{3}{2}t^2+\frac{3}{2}t-\frac{7}{4}$

Example 6

$y''-y'-2y=2e^{-3t}$

$y_p=Ae^{-3t}$

$y'_p=-3Ae^{-3t}$

$y''_p=9Ae^{-3t}$

$9Ae^{-3t}+3Ae^{-3t}-2Ae^{-3t}=2e^{-3t}$

$10Ae^{-3t}=2e^{-3t}$

$A=1/5$

$y_p=\frac{1}{5}e^{-3t}$

Example 7

$y''-y'-2y=2\cos(3t) \qquad (8)$

$y_p=A\cos(3t)+B\sin(3t)$

$y'_p=-3A\sin(3t)+3B\cos(3t)$

$y''_p=-9A\cos(3t)+9B\sin(3t)$

Into (8) with simplification:

$(-11A-3B)\cos 3t+(3A-11B)\sin 3t=2\cos 3t$

$-11A-3B=2 \qquad A=-11/65$

$3A-11B=0 \qquad B=-3/65$

$y_p=\frac{-11}{65}\cos(3t)-\frac{3}{65}\sin(3t)$

Example 9

$y''-2y'+y=3e^t \qquad (1)$

$y_p=At^2e^t$

$y_p'=2Ate^t+At^2e^t$

$y_p''=2Ae^t+4Ate^t+At^2e^t$

Put into (1)

$2Ae^t=3e^t$

$A=3/2$

$y_p=\frac{3}{2}t^2e^t$

Example 8 $y''-y'-2y=t^2e^t \quad (10)$

$y_p=(At^2+Bt+C)e^t$

$\dot{y}_p=e^t[At^2+(2A+B)t+(B+C)]$

$\ddot{y}_p=e^t[At^2+(4A+B)t+(2A+2B+C)]$

Substituting into (10);

$e^t[t^2(-2A)+t(2A-2B)+(2A+B-2C)]=t^2e^t$

$t^2(-2A)+t(2A-2B)+(2A+B-2C)=t^2$

$-2A=1 \quad 2A-2B=0 \quad 2A+B-2C=0$

$A=-\frac{1}{2} \quad B=-\frac{1}{2} \quad C=-\frac{3}{4}$

$y_p=e^t\left(-\frac{1}{2}t^2-\frac{1}{2}t-\frac{3}{4}\right)$

4.5 Variation of Parameters

Method of Variation of Parameters for Determining a Particular Solution y_p for $L(y)=y''+p(t)y'+q(t)y=f(t)$

Step 1: Determine two linearly independent solutions y_1 and y_2 of the corresponding homogeneous equation $L(y)=0$

Step 2: Solve for v_1' and v_2' the system

$$y_1v_1'+y_2v_2'=0$$
$$y_1'v_1'+y_2'v_2'=f$$

or determine v_1' and v_2' from Cramers Rule

$$v_1'=\frac{\begin{vmatrix}0 & y_2\\ f & y_2'\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{-y_2f}{W(y_1,y_2)} \qquad v_2'=\frac{\begin{vmatrix}y_1 & 0\\ y_1' & f\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{y_1f}{W(y_1,y_2)}$$

where $W(y_1,y_2)=y_1y_2'-y_1'y_2$ is the wronskian

Step 3: Integrate the results of Step 2 to find v_1 & v_2

Step 4: Compute $y_p=v_1y_1+v_2y_2$

Note: The derivation is for an operator L having coefficient 1 for y'', so equations having a leading coefficient different from 1 must be divided by that coefficient in order to determine the standard form for $f(t)$.

Ex 1 $y''+y=\sec(t) \qquad |t|<\frac{\pi}{2}$

$y_h=c_1\cos\beta t+c_2\sin\beta t$

$y_1=\cos(t)$

$y_2=\sin(t)$

$v_1'=-y_2f=-\sin(t)\sec(t)=\frac{-\sin(t)}{\cos(t)}$

$v_2'=y_1f=\cos(t)\sec(t)=1$

$$W=\begin{vmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{vmatrix}=1$$

$y_p=y_1v_1+y_2v_2$

$y_p=\ln\cos(t)(\cos(t))+t(\sin(t))$

$v_1=\ln\cos(t)$

$v_2=t$

$y=c_1\cos(t)+c_2\sin(t)+\cos(t)\ln(\cos(t))+t\sin(t)$

Ex 2 $y''+y=4\sin(t)$

$y_h=c_1\cos(t)+c_2\sin(t) \qquad W=1 \qquad y_p: \quad y_p=y_1v_1+y_2v_2$

$y_1=\cos(t)$

$y_2=\sin(t)$

$v_1'=-y_2f$

$=-\sin(t)(4\sin(t))$

$v_1'=-4\sin^2(t)$

$v_1=-2t+\sin 2t$

$v_2'=y_1f$

$=\cos(t)(4\sin(t))$

$v_2'=4\sin(t)\cos(t)$

$v_2=-\cos 2t$

$y_p=\cos(t)(-2t+\sin 2t)+\sin(t)(-\cos 2t)$

$y=c_1\cos(t)+c_2\sin(t)+\cos(t)(-2t+\sin 2t)+\sin(t)(-\cos 2t)$

5.1 Linear Transformation

Linear Transformation

A linear transformation "T" on a vector space $\mathbb{V}$ to a vector space $\mathbb{W}$ is a function $T : \mathbb{V} \to \mathbb{W}$ that preserves scalar multiplication and vector addition. That is, for all $\vec{u}, \vec{v} \in \mathbb{V}$ and $c \in \mathbb{R}$

$$T(c\vec{u}) = cT(\vec{u}) \quad (1)$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \quad (2)$$

The vector space $\mathbb{V}$ is called the domain for T, and $\mathbb{W}$ is called the codomain (sometimes called the target)

Image of A Linear Transformation

The image, or range, of a linear transformation $T : \mathbb{V} \to \mathbb{W}$ is the set of vectors in $\mathbb{W}$ to which T maps the vectors in $\mathbb{V}$:

$$Im(T) = \{ \vec{w} \in \mathbb{W} \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in \mathbb{V} \}$$

The Standard Matrix for a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The standard matrix associated with T is defined by

$$A = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \dots \mid T(\vec{e}_n)]$$

where the columns $T(\vec{e}_j)$ are the images under T of the standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$

Ex 1) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x, y, z) = (x, y, 0)$

a.) $T(c(x,y,z)) = T(cx, cy, cz) = (cx, cy, 0) = c(x, y, 0)$
$= cT(x, y, 0)$

Passes Addition and multiplication

b.) $T[(x_1, y_1, z_1) + (x_2, y_2, z_2)] = T[x_1 + x_2, y_1 + y_2, z_1 + z_2]$
$= (x_1 + x_2, y_1 + y_2, 0)$
$= (x_1, y_1, 0) + (x_2, y_2, 0)$
$= T(x_1, y_1, 0) + T(x_2, y_2, 0)$

linear Transformation

Ex 8) $T(x, y) = (x - y, x + y, 2x)$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ x+y \\ 2x \end{bmatrix}$$

$$A \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| T\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Standard matrix

$\mathbb{P}_3 - \mathbb{P}_1$ $\quad ax^3 + bx^2 + cx + d$

$D^2(ax^3 + bx^2 + cx + d) = 6ax + 2b$

$$\left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 6a \\ 0 & 2 & 0 & 0 & 2b \end{array}\right]$$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

5.2 Properties of Linear Transformations

Injectivity

A function $f: \mathbb{X} \to \mathbb{Y}$ is one to one, or injective, provided it is true that $f(u) = f(v)$ implies that $u=v$. That is, different inputs give rise to different outputs

Surjectivity

The set of output values of a function $f: \mathbb{X} \to \mathbb{Y}$ is a subset of the codomain $\mathbb{Y}$ and is called the image of the function. If the image is all of $\mathbb{Y}$, the function f is said to map onto $\mathbb{Y}$ or to be surjective.

Image Theorem

Let $T: \mathbb{V} \to \mathbb{W}$ be a linear transformation from vector space $\mathbb{V}$ to vector space $\mathbb{W}$ with image $Im(T)$. Then

(i) $Im(T)$ is a subspace of $\mathbb{W}$

(ii) T is surjective if and only if $Im(T) = \mathbb{W}$

Rank of a Linear Transformation

The dimension of the image of a linear transformation T is called it's rank.

$$rank(T) \equiv dim(Im(T))$$

Rank of a Matrix Multiplication Operator

For any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$, where $A \in \mathbb{M}_{mn}$ and $\vec{x} \in \mathbb{V}$, the image of T is the column space of A; that is, $Im(T) = Col\, A$. The pivot columns of A form a basis for $Im(T)$. Consequently,

$$\begin{aligned} rank(T) &\equiv dim(Im(T)) \equiv dim(Col\, A) \\ &= \text{the number of pivot columns in } A \end{aligned}$$

Kernel of a Linear Transformation

The kernel (or nullspace) of a linear transformation $T: \mathbb{V} \to \mathbb{W}$, denoted $Ker(T)$, is the set of vectors in $\mathbb{V}$ mapped to the zero vector in $\mathbb{W}$:

$$Ker(T) = \{\vec{v} \text{ in } \mathbb{V} \mid T(\vec{v}) = \vec{0}\}$$

Kernel Theorem

Let $T: \mathbb{V} \to \mathbb{W}$ be a linear transformation from vector space $\mathbb{V}$ to vector space $\mathbb{W}$ with kernel $Ker(T)$. Then

(i) $Ker(T)$ is a subspace of $\mathbb{V}$:

(ii) T is injective if and only if $Ker(T) = \{\vec{0}\}$

Dimension Theorem

Let $T: \mathbb{V} \to \mathbb{W}$ be a linear transformation from a finite vector space $\mathbb{V}$. Then

$$dim(Ker(T)) + dim(Im(T)) = dim\, \mathbb{V}$$

Nonhomogeneous Principle for Differential Equations

The general solution for a nonhomogeneous differential equation can be expressed in terms of a particular solution and the general solution of the corresponding homogeneous equation.

Nonhomogeneous Principle for Linear Transformations

Let $T : \mathbb{V} \to \mathbb{W}$ be a linear transformation from vector space $\mathbb{V}$ to vector space $\mathbb{W}$. Suppose that $\vec{v}_p$ is any particular solution of the nonhomogeneous problem

$$T(\vec{v}) = \vec{b} \qquad (4)$$

Then the set S of all solutions of (4) is given by

$$S = \{ \vec{v}_p + \vec{v}_H \mid \vec{v}_H \in \ker(T) \}$$

Ex 2) Mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \qquad A\vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_1 - v_2 \\ 2v_1 + v_2 \end{bmatrix}$$

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u_1 = v_1 + v_2 \qquad 3u_1 + u_2 - 2u_3 = 0$$
$$u_2 = v_1 - v_2$$
$$u_3 = 2v_1 - v_2$$

Ex 3) Checking for Image

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \vec{w}$$

$$\vec{w} = \vec{v}_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \vec{v}_2 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \vec{v}_3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \vec{v}_4 \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\mathrm{Im}(T) = \mathrm{Span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \right\}$$

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$$

$$\dim(\mathrm{Im}(T)) = \dim(\mathrm{Col}\,A) = 2$$

dim = 2

Surjective

Ex 5) Kernel

$$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \vec{v}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ to } \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{matrix} v_1 + v_2 + 2v_3 \\ 2v_1 + 3v_2 + 5v_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 3 & 5 & 0 \end{bmatrix}$$

$$\mathrm{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad \begin{matrix} x = -z \\ y = -z \\ z = -1 \end{matrix}$$

$$K(T) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Ex 7) $A = \begin{bmatrix} 2 & -4 & 3 & 6 \\ -1 & 2 & -2 & -3 \end{bmatrix}$

$$\text{rref}(A) = \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} = \begin{bmatrix} 2r-3s \\ r \\ 0 \\ s \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Ker}(T) = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Ex 11) Nonhomogeneous

$$x_1 + x_2 + 3x_3 = 4$$
$$x_1 + 2x_2 + 5x_3 = 6$$

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$\text{rref}\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$x = 2 - z$
$y = 2 - 2z$
$z = z$

$$\vec{x} = z\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

5.4 coordinates and Diagonalization

Coordinates

Let $\vec{v}$ be a vector in the finite-dimensional vector space $\mathbb{V}$, with basis $B = \{\vec{b}_1, \vec{b}_2, \ldots\ldots, \vec{b}_n\}$ for $\mathbb{V}$. Then the coordinates of $\vec{v}$ relative to B are the unique real numbers $\beta_1, \beta_2, \ldots\ldots, \beta_n$ such that

$$\vec{v} = \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \ldots\ldots + \beta_n\vec{b}_n$$

Changing Bases in $\mathbb{R}^n$

Let $\vec{u}_B$ be the coordinate vector relative to basis $B = \{\vec{b}_1, \vec{b}_2, \ldots\ldots \vec{b}_n\}$, and $\vec{u}_S$ be the coordinate vector relative to the standard basis $S = [\vec{e}_1, \vec{e}_2, \ldots. \vec{e}_n]$

- To change coordinates from Basis B to the standard basis:

$$M_B\vec{u}_B = \vec{u}_S \quad \text{where} \quad M_B = [\vec{b}_1 | \vec{b}_2 | \ldots. | \vec{b}_n]$$

- To change coordinates from the standard basis to basis B:

$$M_S\vec{u}_S = \vec{u}_B \quad \text{where} \quad M_S = M_B^{-1} = [\vec{b}_1 | \vec{b}_2 | \ldots. | \vec{b}_n]^{-1}$$

M_B is called the change of coordinate matrix from basis B to the standard basis, while M_S is the change of coordinate matrix S to B

Diagonalization Theorem

An nxn matrix A is diagonalizable

(i) if and only if it has n linearly independent (real) eigenvectors

(ii) if and only if the sum of the dimensions of its eigenspace is n

Diagonalization of a Matrix

For an nxn matrix A with n linearly independent eigenvectors:

Step 1. Construct an nxn diagonal matrix D of the eigenvalues λ_i, for $1 \le i \le n$. (NOTE: An eigenvalue with multiplicity m appears m times.)

Step 2. Construct another nxn matrix P with eigenvectors $\vec{v}_i$ as columns, listed in order corresponding to the eigenvalues λ_i in D

$$AP = PD$$
$$A = PDP^{-1}$$
$$D = P^{-1}AP$$

$$P = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_n \\ | & | & | \end{bmatrix}$$

Similarity

A square matrix B is Similar to matrix A (in shorthand, $B \sim A$) if there exists an invertible matrix P such that $B = P^{-1}AP$

Ex: $S = [\vec{e}_1, \vec{e}_2, \vec{e}_3]$

$$B = \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix} \quad \begin{bmatrix}-1\\-1\\4\end{bmatrix}, \begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}3\\1\\4\end{bmatrix} \qquad \begin{bmatrix}-9\\-4\\3\end{bmatrix}, \begin{bmatrix}7\\3\\-4\end{bmatrix}, \begin{bmatrix}11\\4\\-3\end{bmatrix}$$

$\vec{u}_s = M_B \vec{u}_B$

$$M_B = \begin{bmatrix}1&0&2\\0&0&1\\0&1&-1\end{bmatrix} \vec{u}_B = \begin{bmatrix}-1\\-1\\-4\end{bmatrix} \quad M_B = \begin{bmatrix}1&0&2\\0&0&1\\0&1&-1\end{bmatrix} \vec{u}_B = \begin{bmatrix}1\\-1\\3\end{bmatrix} \quad M_B = \begin{bmatrix}1&0&2\\0&0&1\\0&1&-1\end{bmatrix} \quad \vec{u}_B = \begin{bmatrix}3\\1\\4\end{bmatrix}$$

$$M_B(\vec{u}_B) = \begin{bmatrix}-9\\-4\\3\end{bmatrix} \quad M_B(\vec{u}_B) = \begin{bmatrix}7\\3\\-4\end{bmatrix} \quad M_B(\vec{u}_B) = \begin{bmatrix}11\\4\\-3\end{bmatrix}$$

5.4.14) $S = [x^2, x, 1]$ $\quad P(x) = x^2 + 2x + 3$

$N = [2x^2 - x, x^2, x^2+1]$ $\quad Q(x) = x^2 - 2$

$r(x) = 4x - 5$

$$\begin{bmatrix}-2\\2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-2\end{bmatrix}, \begin{bmatrix}-4\\13\\-5\end{bmatrix}$$

$$M_B = \begin{bmatrix}2&1&1\\-1&0&0\\0&0&1\end{bmatrix}$$

$\vec{u}_B = (M_B^{-1})(\vec{u}_s)$

$$M_B^{-1} \begin{bmatrix}0&-1&0\\1&2&-1\\0&0&1\end{bmatrix} \quad P(x) = \begin{bmatrix}1\\2\\3\end{bmatrix} \quad Q(x) = \begin{bmatrix}1\\0\\-2\end{bmatrix} \quad r(x) = \begin{bmatrix}0\\4\\-5\end{bmatrix}$$

$$P(x): \quad M_B^{-1}(P(x)) = \begin{bmatrix}-2\\2\\3\end{bmatrix} \quad Q(x): \quad M_B^{-1}(Q(x)) = \begin{bmatrix}0\\3\\-2\end{bmatrix} \quad R(x): \quad M_B^{-1}(R(x)) = \begin{bmatrix}-4\\13\\-5\end{bmatrix}$$

5.4.40) $A = \begin{bmatrix}0&0&1\\0&1&2\\0&0&1\end{bmatrix}$ $\quad$ Diagonalization

$\lambda = 0,1,1$

One eigenvalue is repeated therefore this is not diagonizable.

$$|A - \lambda I| = \left|\begin{bmatrix}-\lambda&0&1\\0&1-\lambda&2\\0&0&1-\lambda\end{bmatrix}\right| = -\lambda(\lambda-1)^2$$

$\lambda = 0$

$\lambda = 1,1$

6.1 Linear Systems of Differential Equations

Linear First-order DE System

An n-dimensional linear first-order DE system on open interval I is one that can be written as a matrix-vector equation

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t) \qquad (1)$$

- $A(t)$ is an $n \times n$ matrix of continuous functions on I.
- $\vec{f}(t)$ is an $n \times 1$ vector of continuous functions on I.
- $\vec{x}(t)$ is an $n \times 1$ solution vector of differentiable functions on I that satisfies (1).

If $\vec{f}(t) \equiv \vec{0}$, the system is homogeneous.

$$\vec{x}'(t) = A(t)\vec{x}(t)$$

Initial Value Problem for a Linear DE System

For a linear DE system, an initial-value problem is the combination of system (1) and an initial value vector:

$$\vec{x}' = A(t)\vec{x} + \vec{f}(t), \qquad \vec{x}(t_0) = \vec{x}_0 = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

where $c_1, c_2, \ldots, c_n$ are real constants.

Existence and Uniqueness Theorem for Linear DE Systems

Given an $n \times n$ matrix function $A(t)$ and an $n \times 1$ vector function $\bar{f}(t)$, both continuous on an open interval I containing t_0, and a constant n-vector $\bar{x}_0$, there exists a unique vector function $\bar{x}(t)$ such that

$$\bar{x}' = A(t)\bar{x} + \bar{f}(t) \quad \text{and} \quad \bar{x}(t_0) = \bar{x}_0$$

The Superposition Principle for Homogeneous Linear DE Systems

Let $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ be solution vectors for the homogeneous equation

$$\bar{x}' = A(t)\bar{x} \quad \text{on } I \qquad (5)$$

Then, any linear combination of these solution vectors is also a solution vector for (5). That is,

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$$

Solution Space Theorem for Homogeneous Linear DE Systems

If,

$$\bar{x}' = A(t)\bar{x}$$

where A is an $n \times n$ matrix, then the set of solutions $\bar{x}(t)$ is a vector space of dimension n.

Solution Theorem for Homogeneous Linear DE Systems

For n linearly independent solutions $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ of

$$\bar{x}' = A(t)\bar{x}$$

the general solution is

$$\bar{x}_h = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$$

$$c_1, c_2, \ldots, c_n \in \mathbb{R}$$

Fundamental Matrix

For a basis of n linearly independent solutions of $\bar{x}' = A\bar{x}$, the matrix $X(t)$ whose columns are the vector solutions $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n$ is called a fundamental matrix for the system.

Graphs for Two-Dimensional DE Systems

- The tx and ty graphs showing the individual solution functions $x(t)$ and $y(t)$ are called component graphs, solution graphs, or time series.
- The xy graph is the phase plane. The trajectories in the phase plane are the parametric curves described by $x(t)$ and $y(t)$

Trajectories on a phase plane create a phase portrait

Ex:

$$x' = 3x - 2y$$
$$y' = x$$
$$z' = -x + y + 3z$$

$$\bar{x}' = A\bar{x}$$

$$\bar{x}' = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix}\bar{x} \qquad \bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad x_h = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

a.) Homogeneous

$$A\bar{x}_h = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 6e^{2t} - 2e^{2t} \\ 2e^{2t} \\ -2e^{2t} + e^{2t} + 3e^{2t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} = \bar{x}_h'$$

b.) Nonhomogeneous

$$x' = 3x - 2y + 2 + e^{-2t}$$
$$y' = x - e^{t}$$
$$z' = -x + y + 3z + e^{t} - 1$$

$$\bar{x}' = A\bar{x} + \bar{f}(t)$$

$$\bar{x}' = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix}\bar{x} + \begin{bmatrix} 2 - 2e^{t} \\ -e^{t} \\ e^{t} - 1 \end{bmatrix}$$

$$x_p = \begin{bmatrix} e^{t} \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{x}_p' - A\bar{x}_p = \begin{bmatrix} e^{t} \\ 1 \\ 0 \end{bmatrix}' - \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} e^{t} \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} - (3e^{t} - 2) \\ 0 - e^{t} \\ 0 - (-e^{t} + 1) \end{bmatrix} = \begin{bmatrix} 2 - 2e^{t} \\ -e^{t} \\ e^{t} - 1 \end{bmatrix} = \bar{f}(t)$$

6.2 Linear Systems with Real Eigenvalues

Solving Homogeneous Linear 2x2 DE Systems with Constant Coefficients

For a 2D system homogeneous linear DE's $\bar{x}' = A\bar{x}$, where A is a matrix of constants that has eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\bar{v}_1$ and $\bar{v}_2$, we obtain two Solutions:

$$e^{\lambda_1 t}\bar{v}_1 \quad \& \quad e^{\lambda_2 t}\bar{v}_2$$

If $\lambda_1 \neq \lambda_2$, these two solutions are linearly independent and form a basis for the solution space. Thus, the general solution, for arbitrary constants c_1 and c_2, is

$$x(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2$$

If $\lambda_1 = \lambda_2$, then there may be only one linearly independent eigenvector; additional tactics may be required to obtain a basis of two vectors for the solution space.

Phase Plane Role of Real Eigenvectors and Eigenvalues

For an autonomous and homogeneous 2D linear DE system

- Trajectories move toward or away from the equilibrium according to the sign of the eigenvalues (negative or positive, respectively) associated with the eigenvectors
- Along each eigenvector is a unique trajectory called a seperatrix that seperates trajectories curving one way from those curving another way
- The equilibrium occurs at the origin, and the phase portrait is symmetric about this point.

Creating a Generalized Eigenvector for a System with Insufficient Eigenvectors

If a homogeneous linear 2x2 system of first-order DE's has repeated eigenvalue λ with only a single eigenvector, a second linearly independent solution can be created as follows:

Step 1: Find an eigenvector $\bar{v}$ corresponding to λ.

Step 2: Find a nonzero vector $\bar{u}$ so that

$$(A - \lambda I)\bar{u} = \bar{v}$$

Step 3: Then $\bar{x}(t) = c_1 e^{\lambda t}\bar{v} + c_2 e^{\lambda t}(t\bar{v} + \bar{u})$

The vector $\bar{u}$ is called a generalized eigenvector of A corresponding to λ.

Solving n-Dimensional Homogeneous Linear DE Systems with Constant Coefficients

For an n-dimensional system of homogeneous linear differential equations $\bar{x} = A\bar{x}$, where A is a matrix of constants that has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$, we obtain Solutions:

$$e^{\lambda_1 t}\bar{v}_1, e^{\lambda_2 t}\bar{v}_2, \ldots, e^{\lambda_n t}\bar{v}_n$$

If $\lambda_i \neq \lambda_j$ for all $i \neq j$, these solutions are linearly independent and form a basis for the solution space. Thus, the general solution, for arbitrary constants $c_1, c_2, \ldots c_n \in \mathbb{R}^n$, is

$$\bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2 + \ldots + c_n e^{\lambda_n t}\bar{v}_n$$

The case of repeated eigenvalues ($\lambda_i = \lambda_j$ for some $i \neq j$) requires eithe independent eigenvectors or generalized eigenvectors.

Speed and Shape of Trajectories

- "Speed" along a trajectory in the direction of an eigenvector depends on the magnitude (absolute value) of the associated eigenvalue: "Fast" for the eigenvalue with the largest magnitude, or "slow" for the eigenvalue with the smallest magnitude.

- Trajectories become parallel to the fast eigenvectors farther away from the origin, and tangent to the slow eigenvectors - closer to the origin, in the cases of source or sink, farther from the origin for a saddle.

Ex: $\bar{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\bar{x} \qquad \bar{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\lambda = -1, -2$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x(t) = 2e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x(0) = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

Ex: $\bar{x}' = A\bar{x} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\bar{x}$

$\lambda_1 = \lambda_2 = 3$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\bar{x} = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{3t} \end{bmatrix}$

6.3 Linear Systems of Nonreal Eigenvalues

Complex Eigenvalues and Eigenvectors

For a real matrix A, nonreal eigenvalues come in complex conjugate pairs,

$$\lambda_1, \lambda_2 = \alpha \pm i\beta,$$

with α, β real numbers and $\beta \neq 0$.

The corresponding eigenvectors are also complex pairs and can be written

$$\bar{v}_1, \bar{v}_2 = \bar{p} \pm i\bar{q}$$

where $\bar{p}$ and $\bar{q}$ are real vectors.

Nullclines for a DE System

For a two-dimensional DE System

$$x' = f(x,y)$$
$$y' = g(x,y)$$

- The v-nullcline is the set of all points with vertical slope, which occur on the curve obtained by solving $x' = f(x,y) = 0$;
- The h-nullcline is the set of all points with horizontal slope, which occur on the curve obtained by solving $y' = g(x,y) = 0$

Solving a Two-Dimensional DE System $\bar{x}' = A\bar{x}$ with Nonreal Eigenvalues $\lambda_1, \lambda_2 = \alpha \pm i\beta$

Step 1. For one eigenvalue λ_1, find its corresponding eigenvector $\bar{v}_1$. The second eigenvalue λ_2 and $\bar{v}_2$ are complex conjugates of the first. The

Step 2. Construct the linearly independent real ($\bar{x}_{Re}$) and imaginary ($\bar{x}_{Im}$) parts of the solutions as follows:

$$\bar{x}_{Re} = e^{\alpha t}(\cos\beta t\,\bar{p} - \sin\beta t\,\bar{q})$$
$$\bar{x}_{Im} = e^{\alpha t}(\sin\beta t\,\bar{p} + \cos\beta t\,\bar{q})$$

Step 3. The general solution is

$$\bar{x}(t) = C_1\bar{x}_{Re}(t) + C_2\bar{x}_{Im}(t)$$

Second order Diffy Q

$$\bar{x}' = A\bar{x}$$

$$A = \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}$$

Ex: $\bar{x}' = A\bar{x} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\bar{x}$

$\lambda = \alpha \pm i\beta$

$\lambda_1 = -1 + 2i \qquad \alpha = -1$

$\lambda_2 = -1 - 2 \qquad \beta = 2$

$\lambda_1 = -1 + 2i \qquad (A - \lambda I)\bar{v}_1 = 0$

$$\begin{bmatrix} 1-2i & 1 \\ -5 & -1-2i \end{bmatrix}\bar{v} = 0$$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1+2i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \bar{p} + i\bar{q}$$

Real Solutions From Nonreal Eigenvalues

For $\bar{x}' = A\bar{x}$ with nonreal eigenvalues $\lambda_1, \lambda_2 = \alpha \pm i\beta$ and complex eigenvectors $\bar{v}_1, \bar{v}_2 = \bar{p} \pm \bar{q}i$, arrange the components of the solution as

$$\begin{bmatrix} x_{RE} \\ x_{Im} \end{bmatrix} = \underset{\text{Expansion}}{e^{\alpha t}} \underset{\text{rotation}}{\begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix}} \underset{\text{tilt and Shape}}{\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}} \qquad (13)$$

Each factor of equation (13) has a particular meaning.

(a) The first factor, $e^{\alpha t}$, determines expansion or contraction

- If $\alpha > 0$, trajectories spiral outward from the origin, representing solutions that grow without bound.
- If $\alpha < 0$, trajectories spiral inward toward the origin, representing solutions that decay to zero.
- If $\alpha = 0$, trajectories are closed loops, representing periodic solutions.

(b) The second factor is the familiar rotation matrix. The angle of rotation, βt, is ever-increasing as t increases, so trajectories spiral around the origin, counterclockwise for $\beta > 0$

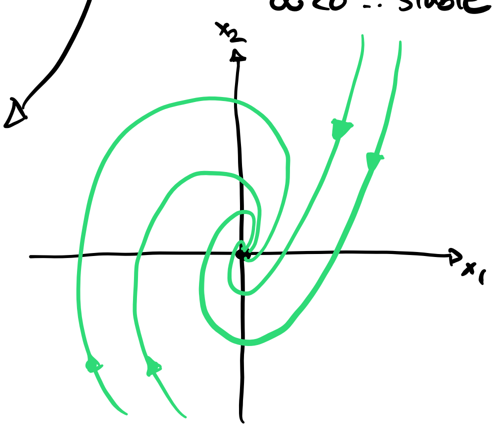
(c) The third factor, containing $\bar{p}$ and $\bar{q}$, determines tilt and shape of the elliptical trajectories that would result if $\alpha = 0$

$$\bar{x}_{Re}(t) = e^{-t}\cos(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-t}\sin(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-t}\cos(2t) \\ e^{-t}(-\cos(2t) - 2\sin(2t)) \end{bmatrix}$$

$$\bar{x}_{I}(t) = e^{-t}\sin(2t)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-t}\cos(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin(2t) \\ e^{-t}(-\sin(2t) + 2\cos(2t)) \end{bmatrix}$$

$$\bar{x}(t) = e^{-t}\left(c_1\begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix}\right)$$

$\alpha < 0 \therefore$ stable



6.4 Stability and Linear Classification

Equilibrium Solution

A constant solution $\bar{x} \equiv \bar{c}$ of the autonomous system $\bar{x}' = \bar{f}(\bar{x})$ (such that $\bar{f}(\bar{c}) = \bar{0}$) is called an equilibrium solution. An equilibrium solution in the phase plane is simply a point, called a fixed point.

Stability of Equilibrium Solutions

An equilibrium solution $\bar{x} \equiv \bar{c}$ of an autonomous system $\bar{x}' = \bar{f}(\bar{x})$ is Stable if solutions that start sufficiently near to $\bar{c}$ remain bounded.

- If nearby solutions not only remain close but actually tend to $\bar{c}$ as a limit as $t \to \infty$, the equilibrium solution is called asymptotically stable.
- If nearby solutions are neither attracted nor repelled, the equilibrium solution is called neutrally stable.

An equilibrium solution that is not stable is unstable.

Node Behaviors

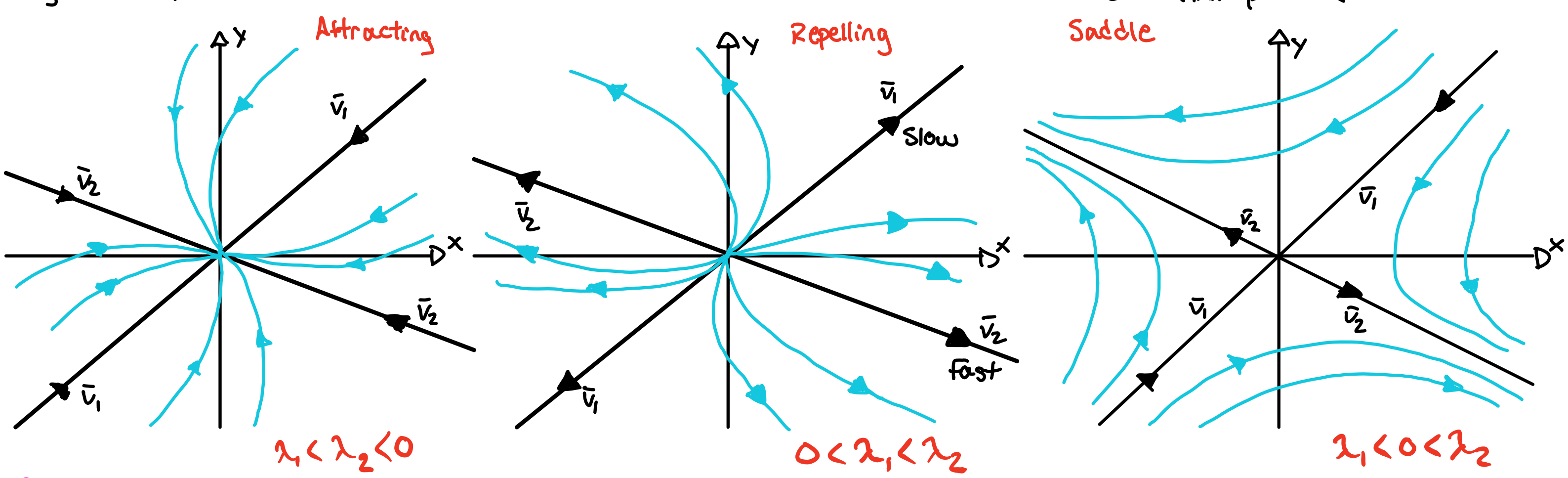
When $\Delta = (\mathrm{Tr}A)^2 - 4|A| > 0$ (in the shaded area of Fig. 6.4.3), we have real eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding linearly independent eigenvectors $\bar{v}_1$ and $\bar{v}_2$, and general solution

$$\bar{x} = c_1 e^{\lambda_1 t} \bar{v}_1 + c_2 e^{\lambda_2 t} \bar{v}_2 \qquad (3)$$

The signs of the eigenvalues direct the trajectory behavior in the phase portrait.

Attracting Node $(\lambda_1 < \lambda_2 < 0)$
Repelling Node $(0 < \lambda_1 < \lambda_2)$
Saddle Point $(\lambda_1 < 0 < \lambda_2)$

Figure 6.4.4



Spiraling Behaviors

When $\Delta = (\mathrm{Tr}A)^2 - 4|A| < 0$ (shaded area of Fig. 6.4.7), we get nonreal eigenvalues,

$$\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - \beta$$

where $\lambda = \mathrm{Tr}A/2$ and $\beta = \sqrt{-\Delta}$ notice that α and β are real, and $\beta \neq 0$. Nonreal solutions are given by

$$\bar{x}_{RE} = e^{\alpha t}(\cos\beta t\,\bar{p} - \sin\beta t\,\bar{q})$$
$$\bar{x}_{IM} = e^{\alpha t}(\sin\beta t\,\bar{p} + \cos\beta t\,\bar{q})$$

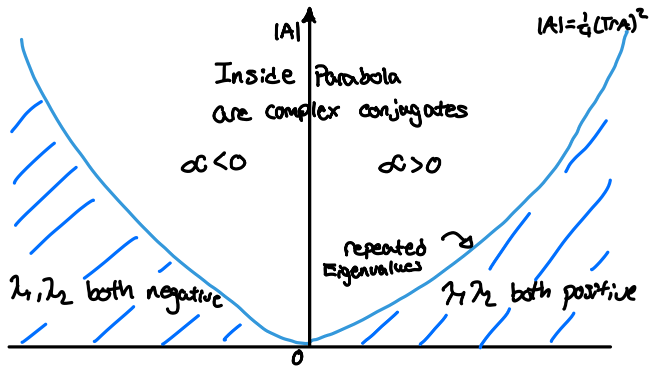
Stability is dependent on the sign of α.

Trace

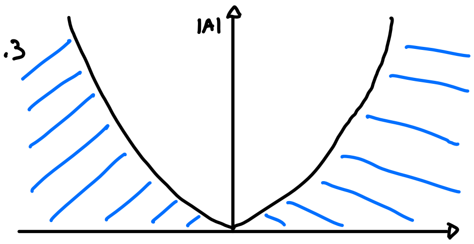
$$x' = ax + by \qquad \bar{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bar{x} = A\bar{x}$$
$$y' = cx + dy$$

$$|A - \lambda I| = \lambda^2 - \underbrace{(a+d)}_{\mathrm{Tr}A}\lambda + \underbrace{(ad-bc)}_{|A|} = 0$$

$$\lambda = \frac{\mathrm{Tr}A \pm \sqrt{(\mathrm{Tr}A)^2 - 4|A|}}{2} \qquad (2)$$

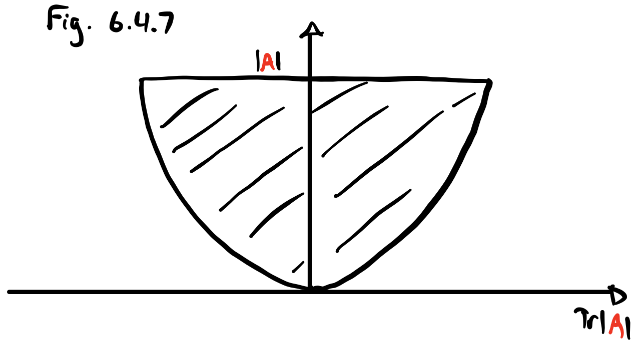


6.4.3



Real eigenvalues occur outside the Tr(|A|) parabola

Fig. 6.4.7



Attracting spiral ($\alpha < 0$)
Repelling Spiral ($\alpha > 0$)
Center ($\alpha = 0$)

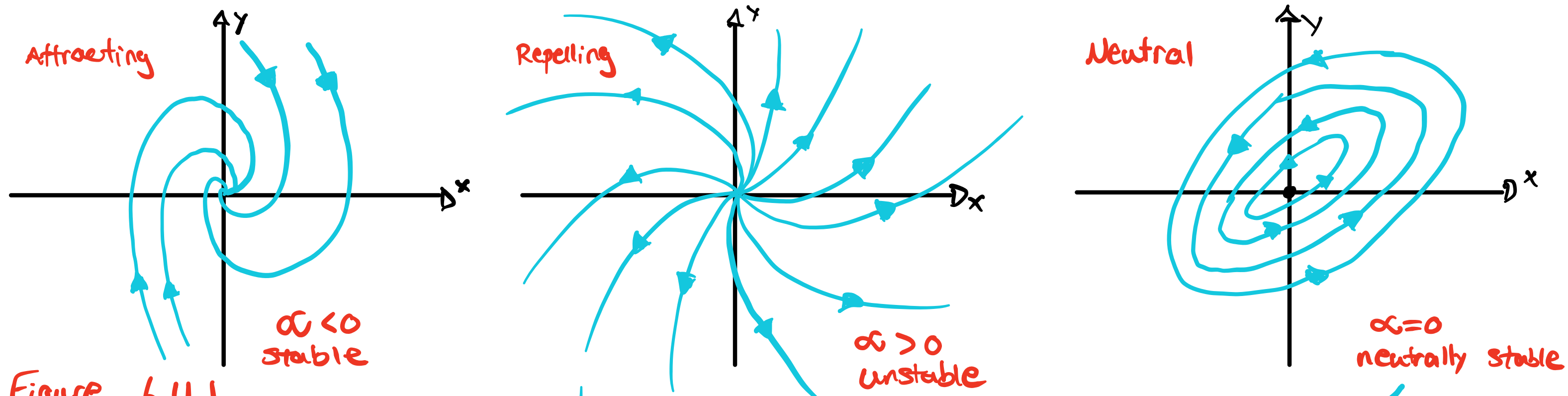


Figure 6.4.1

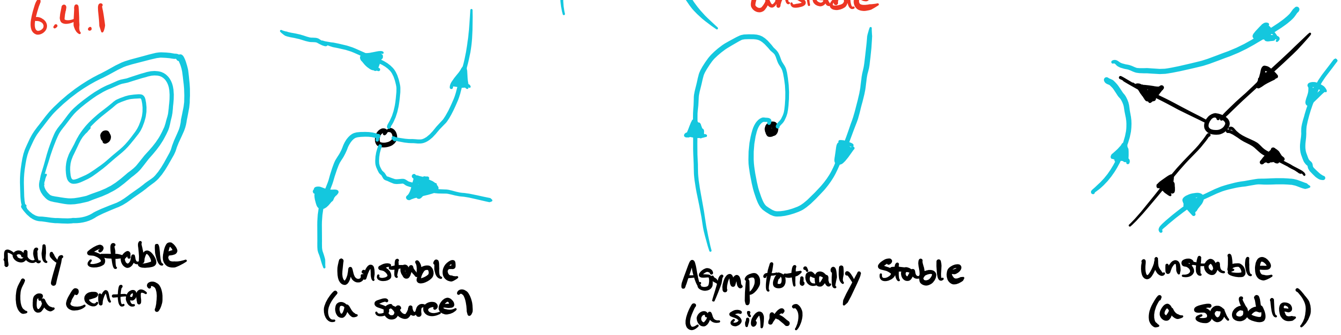
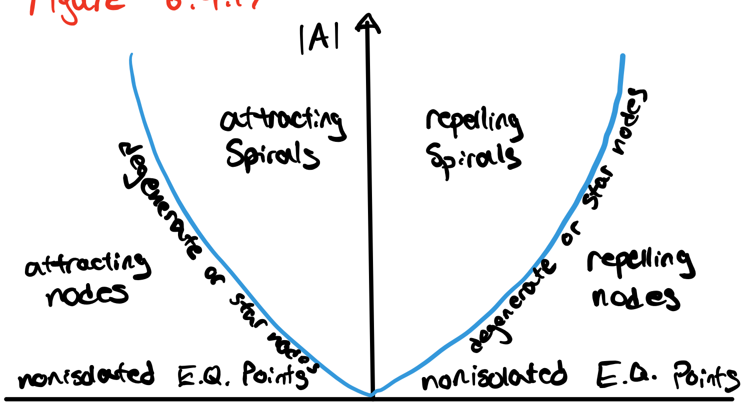
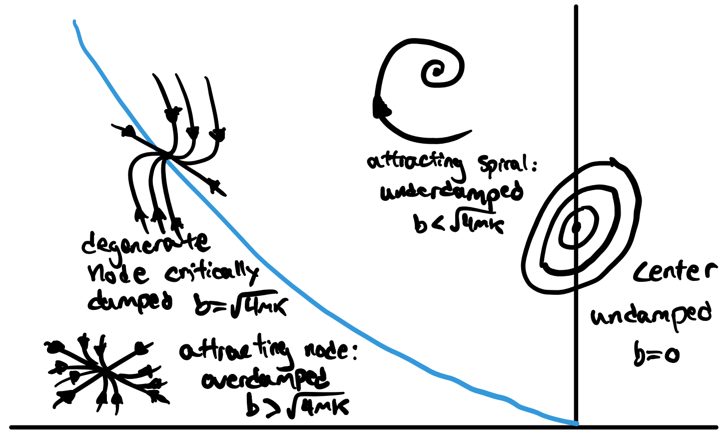


Figure 6.4.17





6.5 Decoupling a Linear DE System

Diagonalization of a Matrix

A diagonizable $n \times n$ matrix A has n eigenvalues and n linearly independent eigenvectors. We can construct:

- D, a diagonal matrix whose diagonal components are eigenvalues of A;
- P, a matrix whose columns are the eigenvectors, listed in the order corresponding to the order of the eigenvalues in D.

P is a change-of-basis matrix such that

$$A = PDP^{-1} \quad \text{and} \quad D = P^{-1}AP$$

We say that P diagonalizes A.

Decoupling a Homogeneous Linear DE System

For a linear DE system

$$\vec{x}' = A\vec{x} \qquad (2)$$

with diagonizable matrix A, the change of variables

$$\vec{x} = P\vec{w} \qquad (3)$$

transforms system (2) into a decoupled system

$$\vec{w}' = D\vec{w}$$

where each component equation involves a single variable and can be easily solved to find $\vec{w}$. The general solution $\vec{x}$ to (2) follows from (3).

Decoupling a Nonhomogeneous Linear DE System

To decouple a linear system

$$\bar{x}' = A\bar{x} + \bar{F}(t) \qquad (7)$$

where $n \times n$ matrix A has n linearly independent eigenvectors, proceed as follows.

Step 1. Calculate the eigenvalues and find the corresponding n independent eigenvectors of A.

Step 2. Form the diagonal matrix D whose diagonal elements are the eigenvalues and the matrix P whose columns are the n eigenvectors, listed in the same order as their corresponding eigenvalues. Then find P^{-1}.

Step 3. Let

$$\bar{x} = P\bar{w} \qquad (8)$$

and solve the decoupled system

$$\bar{w}' = D\bar{w} + P^{-1}\bar{F}(t) \qquad (9)$$

Step 4. Solve (7) using (8) and the solution to (9)

Ex:

$$x_1' = -3x_1 + x_2$$
$$x_2' = x_1 - 3x_2 + e^{-t}$$

$$\bar{w}' = D\bar{w} + P^{-1}f$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad \bar{F}(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\lambda_1 = -2 \quad \lambda = -4$$

$$P^{-1}f = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$w_1' = -2w_1 + \tfrac{1}{2}e^{-t} \qquad w_1 = c_1e^{-2t} + \tfrac{1}{2}e^{-t}$$
$$w_2' = -4w_2 - \tfrac{1}{2}e^{-t} \qquad w_2 = c_2e^{-4t} - \tfrac{1}{6}e^{-t}$$

$$\bar{x} = P\bar{w}$$

$$\bar{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{-2t} + \frac{1}{2}e^{-t} \\ c_2e^{-4t} - \frac{1}{6}e^{-t} \end{bmatrix} = \begin{bmatrix} w_1 + w_2 \\ w_1 - w_2 \end{bmatrix} = \begin{bmatrix} c_1e^{-2t} + c_2e^{-4t} + \frac{1}{3}e^{-t} \\ c_1e^{-2t} - c_2e^{-4t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

Ex:

$$\bar{x}' = A\bar{x} \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\bar{x}$$

$$\lambda_1 = 4 \quad \& \quad \lambda_2 = 1$$

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\bar{w}' = D\bar{w}$$

$$w_1' = 4w_1 \qquad w_1 = ke^{4t}$$
$$w_2' = w_2 \qquad w_2 = ke^{t}$$

$$\bar{x}(t) = P\bar{w} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{4t} \\ c_2e^{t} \end{bmatrix} = \begin{bmatrix} c_1e^{4t} + 2c_2e^{t} \\ c_1e^{4t} - c_2e^{t} \end{bmatrix}$$

6.6 Matrix Exponential

Constant Matrix Eponential

Given a constant $n \times n$ matrix A,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^K}{K!} + \cdots$$

Nilpotent Matrix

A square matrix A is called nilpotent if $A^n = 0$ for some positive integer n.

Properties of the Matrix Exponential e^A

(i) $e^0 = I_n$, where 0 is the $n \times n$ zero matrix

(ii) $(e^A)^{-1} = e^{-A}$

(iii) If $AB = BA$, then $e^{A+B} = e^Ae^B$

Differentiation of the Matrix Exponential Function

$$\frac{d}{dt}e^{At} = Ae^{At}$$

Matrix Exponential Solution of $\bar{x}' = A\bar{x}$

The general solution of

$$\bar{x}' = A\bar{x} \qquad (6)$$

where A is a constant $n \times n$ matrix, is given by

$$\bar{x} = e^{At}\bar{c} \qquad (7)$$

where $\bar{c}$ is an $n \times 1$ vector of arbitrary constants.

If an initial condition, $\bar{x}(0) = \bar{x}_0$, is added to (6), then the solution to the resulting IVP is

$$\bar{x} = e^{At}\bar{x}_0 \qquad (8)$$

Matrix Eponential

Give an $n \times n$ constant matrix A.

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^K}{K!}A^K$$

Matrix Exponential Solution to Nonhomogeneous DE Systems

If A is a constant $n\times n$ matrix and $\bar{f}(t)$ is an $n\times 1$ vector of functions, then the solution of the linear system

$$\bar{x}' = A\bar{x} + \bar{f}(t)$$

given in terms of the matrix exponential, is

$$\bar{x}(t) = e^{At}\bar{c} + e^{At}\int_0^t e^{-As}\bar{f}(s)\,ds \quad (12)$$

If initial conditions $\bar{x}(0) = \bar{x}_0$ are supplied, the unique solution is given by

Matrix Exponential from Eigenfunctions

For a diagonalizable matrix A,

$$e^{At} = Pe^{Dt}P^{-1} \quad (15)$$

where D is a diagonal matrix of all eigenvalues of A, and P is the matrix having the corresponding eigenvectors as columns.

Ex: 1 $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$

$$e^A = I + A + \frac{A^2}{2!} + \dots$$

$$e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a^2/2 & 0 \\ 0 & b^2/2 \end{bmatrix} + \dots$$

$$\begin{bmatrix} 1+a+a^2/2 \dots & 0 \\ 0 & 1+b+b^2/2 \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Diagonal matrices

$$e^A = \begin{bmatrix} e^{a_1} & 0 & \dots \\ 0 & e^{a_2} & \\ \vdots & & \ddots \ e^{a_n} \end{bmatrix}$$

Ex 4. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $x' = y$, $y' = -x$

$$A^2 = -I \quad A^3 = -A \quad A^4 = I \quad A^5 = A, \dots$$

$$e^{At} = I + tA - \frac{t^2}{2!}I - \frac{t^3}{3!}A + \frac{t^4}{4!}I + \dots$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \frac{t^2}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{t^3}{3!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} \dots \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

6.7 Nonhomogeneous Linear Systems

General Solution of $\bar{x}' = A(t)\bar{x} + \bar{f}(t)$

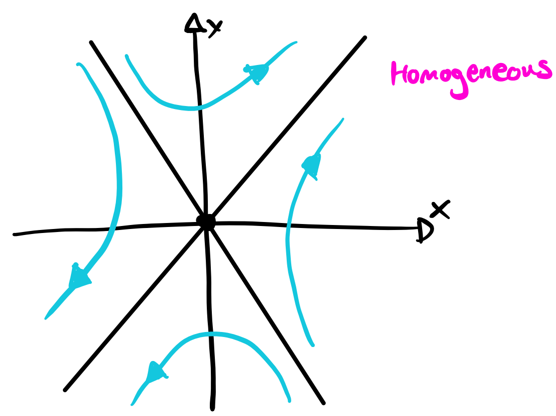
Let $A(t)$ be an $n\times n$ matrix whose elements are continuous functions on the interval under consideration, and let $\bar{f}(t)$ be an $n\times 1$ vector with continuous elements. If $X(t)$ is a fundamental matrix for $\bar{x}' = A(t)\bar{x}$, then the general solution of the nonhomogeneous linear system

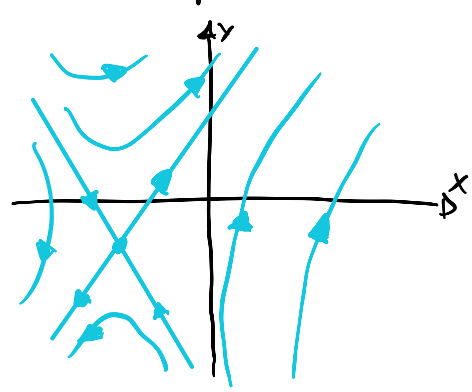
$$\bar{x}' = A(t)\bar{x} + \bar{f}(t) \qquad \bar{x}(0) = \bar{x}_0$$

is

$$\bar{x}(t) = x_h + x_p$$

$$x_p = X(t)\int_0^t X^{-1}(s)\bar{f}(s)\,ds \quad (15)$$





Ex2.) $\begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}\bar{x} + \begin{bmatrix}t-2\\ 4t-1\end{bmatrix}$

$$\bar{x}' = A\bar{x} + \bar{f}(t)$$

$$\bar{x}_p = \begin{bmatrix}at+b\\ ct+d\end{bmatrix} \quad \bar{x}'_p = \begin{bmatrix}a\\ c\end{bmatrix}$$

$$\begin{bmatrix}a\\ c\end{bmatrix} = \begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}\begin{bmatrix}at+b\\ ct+d\end{bmatrix} + \begin{bmatrix}t-2\\ 4t-1\end{bmatrix}$$

$$= \begin{bmatrix}at+b+ct+d\\ 4at+4b+ct+d\end{bmatrix} + \begin{bmatrix}t-2\\ 4t-1\end{bmatrix}$$

$$= \begin{bmatrix}at+b+ct+d+t-2\\ 4at+4b+ct+d+4t-1\end{bmatrix}$$

$$\begin{bmatrix}a\\ c\end{bmatrix} = t\begin{bmatrix}a+c+1\\ 4a+c+4\end{bmatrix} + \begin{bmatrix}b+d-2\\ 4b+d-1\end{bmatrix}$$

$a+c+1=0 \qquad b+d-2=a$

$4a+c+4=0 \qquad 4b+d-1=c$

$$\left[\begin{array}{cccc|c}1 & -1 & 0 & -1 & -2\\ 1 & 0 & 1 & 0 & -1\\ 0 & -4 & 1 & -1 & -1\\ 4 & 0 & 1 & 0 & -4\end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 1\end{array}\right]$$

$$\bar{x}_p = \begin{bmatrix}-t\\ 1\end{bmatrix}$$

$$x = x_h + x_p$$

$$x_h = c_1 e^{3t}\begin{bmatrix}1\\ 2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\ -2\end{bmatrix}$$

Given ↑

$$x_p = \begin{bmatrix}-t\\ 1\end{bmatrix}$$

$$\boxed{x(t) = c_1 e^{3t}\begin{bmatrix}1\\ 2\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\ -2\end{bmatrix} + \begin{bmatrix}-t\\ 1\end{bmatrix}}$$