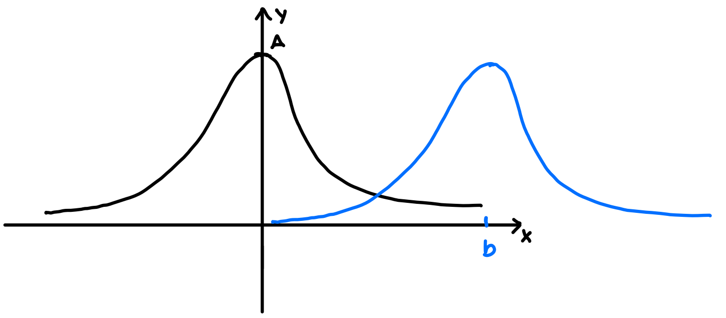
Chapter 5 Traveling waves

Consider a Gaussian Function to model wave pulse

$$y(x) = Ae^{-x^2/a^2} \qquad y(x=0) = A \qquad y(x=\pm a) = \frac{A}{e}$$

A is the height

a is a measure of its width



now shift $x \to x-b$, so

$$y(x) = Ae^{-(x-b)^2/a^2}$$ – shift gaussian to the right

Now let $b = vt$ where t's time : $[v] = m/s$

$$y(x,t) = Ae^{-(x-vt)^2/a^2}$$

- Decreases & gaussian that moves in the +x direction

Consider a generic function of x : t of the form

$$f(x,t) = f(x-vt)$$ – Function travel to the right

$$g(x,t) = g(x+vt)$$ – Function travel to the left

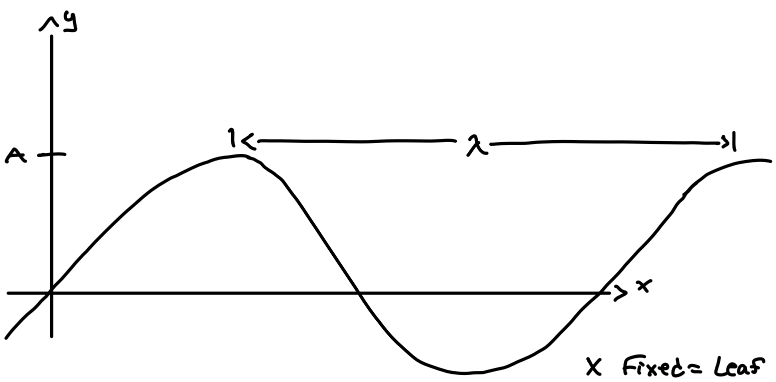
The generic form of any wave motion of a rope can be written as....

$$y(x,t) = f(x-vt) + g(x+vt)$$

Traveling Sinusoidal Waves

Consider the Function....

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$$



Notice:

- This is a transverse wave, Medium moves in ± y direction, Wave moves in the + x direction.

x Fixed = Leaf on a pond

t fixed = Photo of ripples on a pond

- Linearly Polarized, displacement lies in xy plane
- λ is the wavelength (or repeat distance)
- A is the amplitude
- The wave travels one wavelength in the time $t = T$

$$v = \frac{\lambda}{T} \qquad \bar{\nu} = \nu = \frac{1}{T} \qquad v = \lambda\nu$$

The Function again....

Define

$$y(x,t) = A\sin\left[\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t\right] \qquad \left[k \equiv \frac{2\pi}{\lambda}\right] \text{- wave number}$$

$$\left[2\pi\frac{v}{\lambda} = 2\pi\nu = \omega\right] \text{- Angular frequency}$$

Sinusoidal wave function becomes

$$y(x,t) = A\sin(kx - \omega t) = A\sin\left[k\left(x - \frac{\omega}{k}t\right)\right]$$

so

$$\left[v = \frac{\omega}{k}\right]$$

If the function is shifted by $\pi/2$....

$$y=(x,t) = A\cos(kx - \omega t)$$

The wave Equation

in 1D of the form....

$$* \left[\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\right] \text{- The solution to the 1D wave eqn is a 1D wave!}$$

The traveling Sinusoidal wave is a solution.....

$$y(x,t) = A\sin(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = kA\cos(kx - vt) \qquad \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - vt) \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$$

Plugging into (*) we find

$$-k^2 A\sin(kx - \omega t) = \frac{1}{v^2}\cdot -\omega^2 A\sin(kx - \omega t) \quad \therefore \quad k^2 = \frac{\omega^2}{v^2} \quad \therefore \quad v^2 = \frac{\omega^2}{k^2} \quad \therefore \quad v = \frac{\omega}{k}$$

$y=(x,t) = f(x-vt) + g(x+vt)$ is a solution

Proof:

For $f = f(x-vt) = f(u)$ where $u = x - vt$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} = \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial f}{\partial u}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(-v\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial t} = v^2\frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

$g = g(x+vt) = g(u)$ $\Rightarrow$ same process yields,

$u \equiv x + vt$

$$\frac{1}{V^2}\frac{\partial^2 g}{\partial t^2} = \frac{\partial^2 g}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2}{\partial t^2}y$$

Let $y = f + g$

$$\frac{\partial^2}{\partial x^2}(f+g) = \frac{1}{V^2}\frac{\partial^2}{\partial t^2}(f+g)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} + \frac{1}{V^2}\frac{\partial^2 g}{\partial t^2}$$

$$\frac{1}{V^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{V^2}\frac{\partial^2 g}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2}$$ - 1D wave eqn.

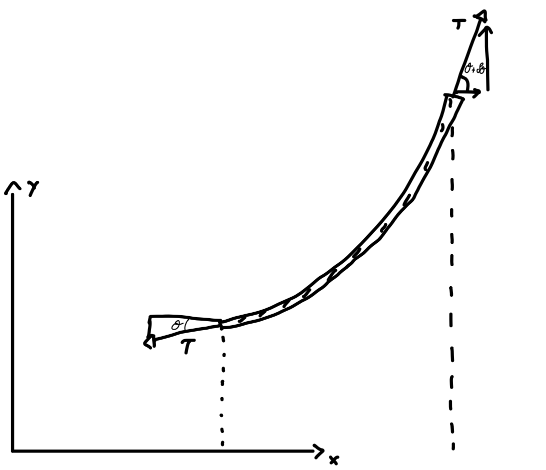
$$y(x,t) = f(x-vt) + g(x+vt)$$

$$V = \nu\lambda = \frac{\omega}{k}$$

$$k = \frac{2\pi}{\lambda}$$

The Equation of A Vibrating String

Consider a string of uniform mass per length, $\mu = m/\ell$ and uniform Tension, T



- The angle that the string makes x is θ : $x+\delta x$ is $\theta + \delta\theta$

The net force on the string in the +y-direction is ....

$$\vec{F}_{net,y} = m\vec{a}_y \qquad m = \mu\delta x$$

(*)

$$T\sin(\theta+\delta\theta) - T\sin\theta = \mu\delta x a_y$$

For small angles ....

$\sin\theta \simeq \theta \simeq \tan\theta = \frac{\partial y}{\partial x}$ - The slope of the string @ x

What about the slope of String @ $x+\delta x$?

(Slope @ $x+\delta x$) = (Slope @ x) + (Rate of change of slope) $\times\,\delta x$

$$\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} = \left.\frac{\partial y}{\partial x}\right|_{x} + \left.\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\right|_{x}\delta x \simeq \sin(\theta+\delta\theta)$$

The (*) Equation becomes...

$$T\left[\frac{\partial y}{\partial x}\Big|_x + \frac{\partial^2 y}{\partial x^2}\Big|_x \delta x - \frac{\partial y}{\partial x}\Big|_x^0\right] = \mu\delta x a_y$$

$$\frac{T\partial^2 y}{\partial x^2}\delta x = \mu\delta x\frac{\partial^2 y}{\partial t^2}$$

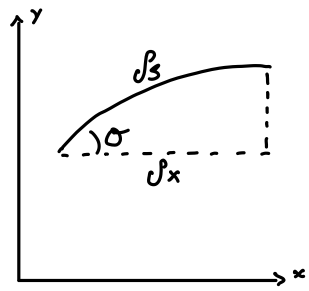
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

$$v^2 = \frac{T}{\mu} \text{ or } v = \sqrt{\frac{T}{\mu}}$$

$$\left[v = \sqrt{\frac{T}{\mu}}\right]$$ - Speed of a wave in a taught string

The Energy In A wave

- Here we are concerned with the rate at which Energy is conserved in a taught string
- Consider these waves on a taught string, String segment oscillates in the transverse direction.



$$K = \frac{1}{2}(\delta m)v^2 = \frac{1}{2}\mu\delta x\left(\frac{\partial y}{\partial x}\right)^2$$ - K of oscillating string of mass $\mu\delta x$

For the potential energy ......

- As the string stretches, there is U stored
- For a constant tension, T, the potential energy is

$$U = T(\delta s - \delta x)$$

$$\cos\theta = \frac{\delta x}{\delta s} \quad \therefore \delta s = \frac{\delta x}{\cos\theta} = \frac{\delta x}{(1-\sin^2\theta)^{1/2}} \simeq \frac{\delta x}{(1-\theta^2)^{1/2}}$$

$$= \delta x\left(1 + \frac{1}{2}\theta^2\right)$$

$$\theta = \tan\theta = \frac{\partial y}{\partial x} \qquad \delta s = \delta x\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right)$$

$$U = T\left(\delta x\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right) - \delta x\right) = \frac{1}{2}T\delta x\left(\frac{\partial y}{\partial x}\right)^2$$

The total Energy of a mass becomes $\delta m = \mu\delta x$ is ...

$$E = \frac{1}{2}\mu\delta x\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\delta x\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}\mu\delta x\left[\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T}{\mu}\left(\frac{\partial y}{\partial x}\right)^2\right] \qquad v^2 = \frac{T}{\mu}$$

$$= \frac{1}{2}\mu\delta x\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]$$

The total energy is ...

$$\left[E = \frac{1}{2}\mu\int_a^b dx\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right]\right]$$

Calculate the total energy in a 1λ segment of a string for a traveling sinusoidal wave

$$y(x,t) = A\sin(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = kA\cos(kx - wt)$$

$$\frac{\partial y}{\partial t} = -wA\cos(kx - wt)$$

The total energy in this segment is . . . . .

$$E_{TOT} = \frac{1}{2}\mu\int_0^{\lambda} dx\left[w^2A^2\cos^2(kx-wt) + v^2k^2A^2\cos^2(kx-wt)\right]$$

$$v^2k^2 = \frac{w^2}{k^2}\cdot k^2 = w^2$$

$$= \mu w^2A^2 \underbrace{\int_0^{\lambda} dx\left[\cos^2(kx-wt)\right]}_{1/2\,\lambda}$$

$$\left[E = \frac{1}{2}\lambda\mu w^2A^2\right]$$

- Depends on the Square of the Amplitude & Frequency