MAT 202
Larson – Section 6.1
Differential Equations: Slope Fields

Up to this point we've dabbled a little with differential equations, but never formally introduced them. We will do that now. You must keep in mind, however, that there is an entire semester long math class dedicated to this topic, so what you see here is just the tip of the iceberg. A ***differential equation*** in x and y is an equation that involves x, y, and the derivatives of y.

$2xy' - 3y = 0, \quad y' + 2y = 0, \quad y' = x - y, \textit{ and } \quad y'' - y = 0$ are all examples of differential equations.

A **solution** of a differential equation is a function $y = f(x)$ such that when y and its derivatives are substituted into the differential equation, the result is a true equality.

Ex: Determine whether each function is a solution of the differential equation $y'' - y = 0$.

a) $y = \sin x$

b) $y = 4e^{-x}$

c) $y = Ce^{-x}$

If the solution involves an unknown constant, such as (c) above, then the solution is called the ***general solution***.

Ex: Determine whether each function is a solution of the differential equation $y'' - y = 0$.

a) $y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$-\sin x - \sin x \stackrel{?}{=} 0$

$-2\sin x \neq 0$

$y = \sin x$ is not a solution.

$$y'' - y = 0$$

b) $y = 4e^{-x}$

$y' = -4e^{-x}$

$y'' = 4e^{-x}$

$4e^{-x} - 4e^{-x} \stackrel{?}{=} 0$

$0 = 0$ ✓

$\boxed{y = 4e^{-x} \text{ is a solution.}}$

c) $y = Ce^{-x}$

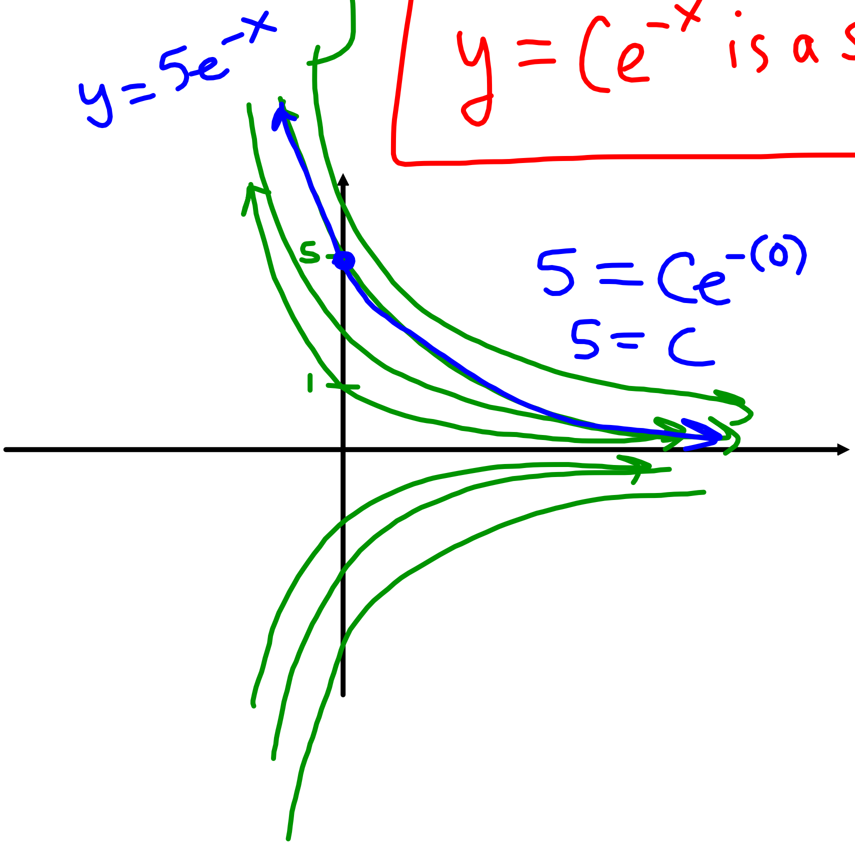
$y' = -Ce^{-x}$

$y'' = Ce^{-x}$

$Ce^{-x} - Ce^{-x} \stackrel{?}{=} 0$

$0 = 0$ ✓

$\boxed{y = Ce^{-x} \text{ is a solution}}$



$5 = Ce^{-(0)}$

$5 = C$

The ***order*** of a differential equation is determined by the highest-order derivative in the equation. For example, $2xy' - 3y = 0$ is a first-order differential equation, while $y'' - y = 0$ is a second-order differential equation.

If the general solution satisfies a first-order differential equation, then graphically it represents a family of curves known as ***solution curves***.

A ***particular solution*** of a differential equation is obtained from having ***initial conditions*** that give the values of the dependent variable or one of its derivatives for particular values of the independent variable. For example, the extra credit problem on your homework itinerary is an example of a second-order differential equation that is to be solved with initial conditions $f'(2) = 0,\ and\ \ f(2) = 3$. **Note: To determine a particular solution, the number of initial conditions, must match the number of constants in the general solution.**

Ex: Verify the particular solution $y = 6x - 4\sin x + 1$ of the differential equation $y' = 6 - 4\cos x$ with the initial condition $y(0) = 1$.

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$$y' = 6 - 4\cos x$$

$$y = \int 6 - 4\cos x \, dx$$

$$y = 6x - 4\sin x + C \qquad y(0) = 1$$

$$1 = 6(0) - 4\sin(0) + C$$

$$1 = 0 - 0 + C$$

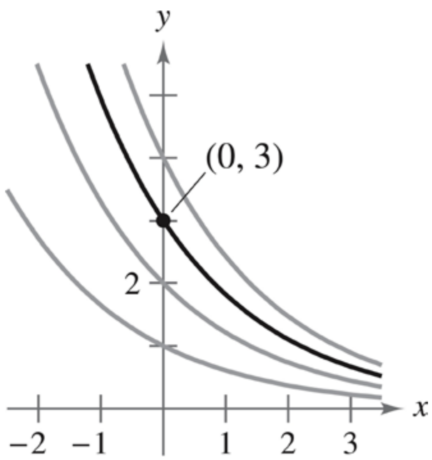
$$1 = C$$

$$\boxed{y = 6x - 4\sin x + 1}$$

Ex: Some of the curves corresponding to different values of C in the general solution of the differential equation are shown in the graph. Find the particular solution that passes through the point shown on the graph.

$$y = Ce^{-x/2}$$

$$2y' + y = 0$$



Ex: Use integration to find the general solution of the given differential equations:

a) $\dfrac{dy}{dx} = x\cos x^2$

b) $y' = 2x\sqrt{4x^2 + 1}$

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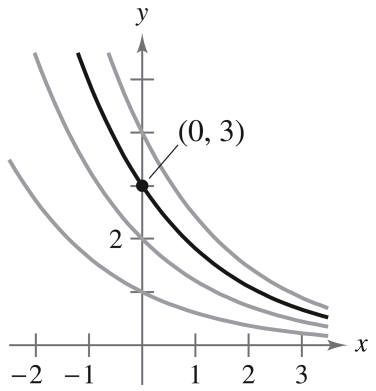


$y = Ce^{-x/2}$

$3 = Ce^{-0/2}$

$3 = C$

$\boxed{y = 3e^{-x/2}}$

Ex: Use integration to find the general solution of the given differential equations:

a) $$\frac{dy}{dx} = x\cos x^2$$

$$\int 1\,dy = \int x\cos x^2\,dx$$

$$u = x^2$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$y = \frac{1}{2}\int \cos u\,du$$

$$y = \frac{1}{2}\sin u + C$$

$$\boxed{y = \frac{1}{2}\sin x^2 + C}$$

b) $y' = 2x\sqrt{4x^2 + 1}$

$u = 4x^2+1$

$du = 8x\,dx$

$\frac{1}{4}du = 2x\,dx$

$\int dy = \int 2x\sqrt{4x^2+1}\,dx$

$y = \frac{1}{4}\int u^{1/2}\,du$

$y = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} + C$

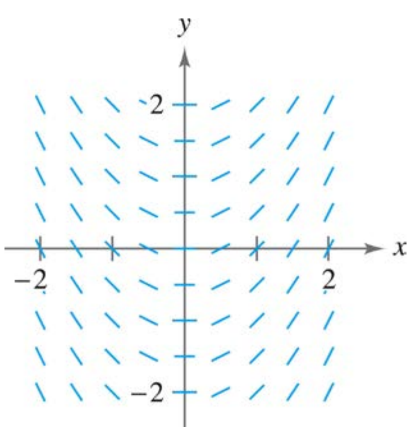
$\boxed{y = \frac{1}{6}(4x^2+1)^{3/2} + C}$

Determining Solutions To Differential Equations Graphically:
Solving differential equations analytically can sometimes be difficult or even impossible. There is a graphical approach that we can use to learn about the solution of the differential equation.

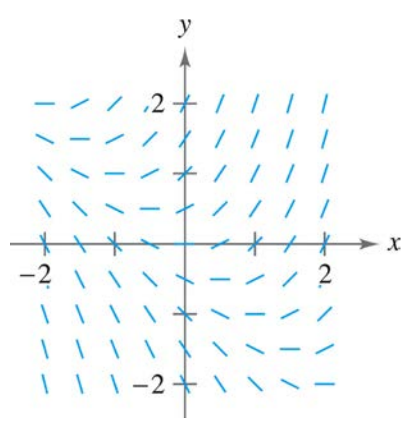
Consider a differential equation of the form $y' = F(x, y)$, where $F(x, y)$ is some expression in x and y. At each point (x, y) in the Cartesian plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If we draw short line segments with slope $F(x, y)$ at these selected point (x, y) in the domain of F, then these line segments form a ***slope field*** (AKA ***direction field***), for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

Ex: Match each slope field with its differential equation.

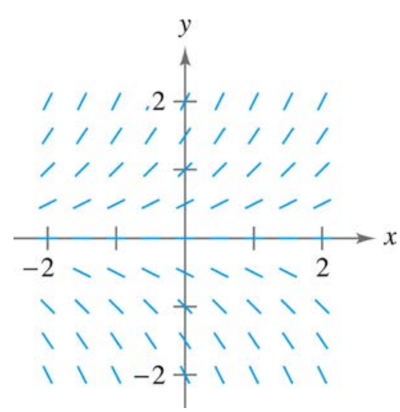
a)



b)



c)



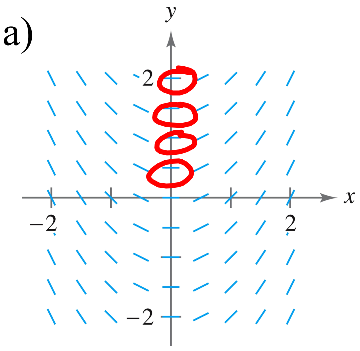
i. $y' = x + y$

ii. $y' = x$

iii. $y' = y$

Ex: Match each slope field with its differential equation.

a)



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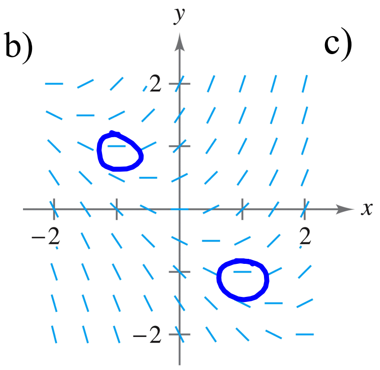
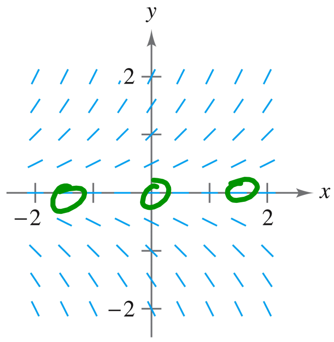


c)



i. $y' = x + y$ b

ii. $y' = x$ a

iii. $y' = y$ c

i)

x	1	1	-1	-1
y	1	-1	1	-1
y'	2	0	0	-2

ii)

x	0	0	0
y	1	2	3
y'	0	0	0