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MATH 360
HW Ch. 5.1

5.1 # 2, 3, 11, 12

Problem 5.1.2

$$\sum_{m=0}^{\infty} (m+1) m x^m \qquad \lim_{m \to \infty} \left| \frac{a_m}{a_{m+1}} \right| \to \lim_{m \to \infty} \left| \frac{m\cancel{(m+1)}}{\cancel{(m+1)}(m+2)} \right| = \lim_{m \to \infty} \left| \frac{m}{(m+2)} \right| = \lim_{m \to \infty} \left| \frac{1}{1+\frac{2}{m}} \right|$$

$a_m = m(m+1)$
$a_{m+1} = (m+2)(m+1)$

$$\lim_{m \to \infty} \left| \frac{1}{1+\frac{2}{m}} \right| = \left| \frac{1}{1+0} \right| = 1$$

$$\boxed{R = 1}$$

Problem 5.1.2

$$\sum_{m=0}^{\infty}(m+1)mx^{m} : \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty}\left|\frac{(\cancel{m+1})(m)}{(m+1)(\cancel{m+2})}\right| = \lim_{m\to\infty}\left|\frac{m}{m+1}\right| = \lim_{m\to\infty}\left|\frac{1}{1+\frac{1}{m}}\right|$$

$$\lim_{m\to\infty}\left|\frac{1}{1+\frac{1}{m}}\right| = 1$$

$$\boxed{R=\sqrt{1}}$$

Problem 5.1.3

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} t^m \qquad t = x^2 \quad \vdots \quad \lim_{m \to \infty} \frac{1}{\sqrt[m]{a_m}} = \frac{1}{\sqrt[m]{\left(\frac{-1}{k}\right)^m}} = \frac{1}{\frac{1}{\sqrt{k}}} \quad \therefore \quad R = \sqrt{k}$$

$$a_m = \frac{(-1)^m}{k^m} = \left(\frac{-1}{k}\right)^m$$

$$\boxed{R = \sqrt{|k|}}$$

Problem 5.1.3

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m} \; : \; \lim_{m\to\infty} \frac{1}{\sqrt[m]{a_m}} = \frac{1}{\sqrt[m]{\left(\frac{-1}{k}\right)^m}} = \frac{1}{1/\sqrt{k}} = \sqrt{k}$$

$$\boxed{R = \sqrt{k}}$$

Problem 5.1.11

$y''-y'-x^2y=0$ Assume: $y=\sum_{m=0}^{\infty} a_m x^m$, $y'= m a_m x^{m-1}$, $y''= m(m-1) a_m x^{m-2}$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=1}^{\infty} m a_m x^{m-1} - x^2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m - \sum_{m=0}^{\infty} a_m x^{m+2} = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m - \sum_{m=2}^{\infty} a_{m-2} x^m = 0$$

$$(2)(1)a_2 + (3)(2)a_3 x - (a_1 + 2a_2 x) - \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2}x^m - (m+1)a_{m+1}x^m - a_{m-2}x^m\right] = 0$$

$2(a_2) - a_1 = 0 \quad \vdots \quad a_1 = 2a_2 \quad : \quad a_1 = 6a_3$

$6(a_3) - 2a_2 = 0 \quad \vdots \quad a_2 = 3a_3$

$$-\sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - a_{m-2}\right]x^m = 0$$

$$(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - a_{m-2} = 0$$

$$(m+2)(m+1)a_{m+2} = (m+1)a_{m+1} + a_{m-2}$$

$$\boxed{a_{m+2} = \frac{(m+1)a_{m+1} + a_{m-2}}{(m+2)(m+1)}}$$

Problem 5.1.11

$$y'' - y' + x^2 y = 0 \quad \text{Assume: } y = \sum_{m=0}^{\infty} a_m x^m$$

$$y'' = (m)(m-1)a_m x^{m-2}, \quad y' = m(a_m)x^{m-1}, \quad y = a_m x^m$$

$$\sum_{m=2}^{\infty} (m)(m-1)a_m x^{m-2} - \sum_{m=1}^{\infty} m(a_m)x^{m-1} + x^2 \sum_{m=0}^{\infty} a_m x^m$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m + \sum_{m=0}^{\infty} a_m x^{m+2}$$

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} - (m+1)a_{m+1}\right]x^m + \sum_{m=2}^{\infty} a_{m-2} x^m$$

$$(2)(1)a_2 - a_1 + (3)(2)a_3 x - (2)a_2 x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} + a_{m-2}\right]x^m = 0$$

$$2a_2 = a_1 \quad \therefore \quad a_2 = a_1/2$$
$$6a_3 = 2a_2 \quad \therefore \quad a_3 = a_2/3 = a_1/3\cdot 2$$

$$(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} + a_{m-2} = 0$$

$$(m+2)(m+1)a_{m+2} = (m+1)a_{m+1} - a_{m-2}$$

$$\boxed{a_{m+2} = \frac{(m+1)a_{m+1} - a_{m-2}}{(m+2)(m+1)}}$$ — Starts at $m=2$

$$a_4 = \frac{3a_3 - a_0}{(4)(3)} = \frac{\cancel{3}a_3}{4(\cancel{3})} - \frac{a_0}{4(3)} = \frac{a_1}{4\cdot 3\cdot 2} - \frac{a_0}{4\cdot 3}$$

$$a_5 = \frac{4a_4 - a_1}{(5)(4)} = \frac{4a_4}{(5)(4)} - \frac{a_1}{(5)(4)} = \frac{1}{(5)}\left[\frac{a_1}{4\cdot 3\cdot 2} - \frac{a_0}{4\cdot 3}\right] - \frac{a_1}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)} - \frac{a_0}{(5)(4)(3)} - \frac{a_1}{(5)(4)}$$

$$= \frac{a_1}{5\cdot 4\cdot 3\cdot 2} - \frac{a_0}{5\cdot 4\cdot 3} - \frac{6a_1}{5\cdot 4\cdot 3\cdot 2} = -\frac{5a_1}{5\cdot 4\cdot 3\cdot 2} - \frac{a_0}{5\cdot 4\cdot 3} = -\frac{a_1}{4\cdot 3\cdot 2} - \frac{a_0}{5\cdot 4\cdot 3}$$

$$a_5 = -\left[\frac{a_1}{4\cdot 3\cdot 2} + \frac{a_0}{5\cdot 4\cdot 3}\right]$$

$$\boxed{y(x) = a_0\left[1 - \frac{x^4}{12} - \frac{x^5}{60} \dots\right] + a_1\left[x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{24} \dots\right]}$$

Example

$y'' + y' - x^2 y = 0$ Assume $\sum_{m=0}^{\infty} a_m x^m$

$y'' = m(m-1)a_m x^{m-2}$, $y' = m a_m x^{m-1}$, $y = a_m x^m$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^{m+2}$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + (m+1)a_{m+1} x^m - \sum_{m=2}^{\infty} a_{m-2} x^m$$

$$(2)(1)a_2 + (3)(2)a_3 x + a_1 + 2a_2 x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} + (m+1)a_{m+1} - a_{m-2}\right] x^m = 0$$

$2a_2 + a_1 = 0$: $a_2 = -\frac{1}{2}a_1$

$6a_3 + 2a_2 = 0$: $a_3 = -\frac{1}{3}a_2$: $a_3 = a_1/3\cdot 2$

$$(m+2)(m+1)a_{m+2} + (m+1)a_{m+1} - a_{m-2} = 0$$

$$(m+2)(m+1)a_{m+2} = a_{m-2} - (m+1)a_{m+1}$$

$$\boxed{a_{m+2} = \frac{a_{m-2} - (m+1)a_{m+1}}{(m+2)(m+1)}}$$ Starts at $m=2$

$a_0 = \text{arb}$

$a_1 = \text{arb}$

$a_2 = -\frac{a_1}{2}$

$a_3 = \frac{a_1}{3\cdot 2}$

$$a_4 = \frac{a_0 - (3)a_3}{(4)(3)} = \frac{a_0}{4(3)} - \frac{\cancel{3}a_3}{(4)(\cancel{3})} = \frac{a_0}{4(3)} - \frac{a_1}{4(3)2}$$

$$a_5 = \frac{a_1 - (4)a_4}{(5)(4)} = \frac{a_1}{5(4)} - \frac{\cancel{4}(a_4)}{(5)(\cancel{4})} = \frac{a_1}{5(4)} - \frac{1}{5}\left(\frac{a_0}{4(3)} - \frac{a_1}{4(3)2}\right) = \frac{a_1}{(5)(4)} - \frac{a_0}{(5)(4)(3)} + \frac{a_1}{(5)(4)(3)(2)}$$

$$= \frac{6a_1}{(5)(4)(3)(2)} - \frac{a_0}{(5)(4)(3)} + \frac{a_1}{(5)(4)(3)(2)} = \frac{7a_1}{(5)(4)(3)(2)} - \frac{a_0}{(5)(4)(3)}$$

$$a_5 = \frac{7a_1}{(5)(4)(3)(2)} - \frac{a_0}{(5)(4)(3)}$$

$$\boxed{y(x) = a_0\left[1 + \frac{x^4}{12} - \frac{x^5}{60} + \cdots\right] + a_1\left[x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{7x^5}{120} - \cdots\right]}$$

Problem 5.1.12

$(1-x^2)y'' - 2xy' + 2y = 0$ Assume: $y = \sum_{m=0}^{\infty} a_m x^m$, $y' = m a_m x^{m-1}$, $y'' = m(m-1)a_m x^{m-2}$

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} 2a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=1}^{\infty} 2m a_m x^m + \sum_{m=0}^{\infty} 2a_m x^m = 0$$

$$(2)(1)a_2 + (3)(2)a_3 x + \sum_{m=2}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2a_1 x - \sum_{m=2}^{\infty} 2m a_m x^m$$
$$+ 2a_0 + 2a_1 x + \sum_{m=0}^{\infty} 2a_m x^m = 0$$

$a_2 + 2a_0 = 0 \quad \therefore a_2 = -2a_0$

$6(a_3) - 2a_1 + 2a_1 = 0 \quad \therefore a_3 = 0$

$$\sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - m(m-1)a_m - 2m a_m + 2a_m\right]x^m = 0$$

$$(m+2)(m+1)a_{m+2} - m(m-1)a_m - 2m a_m + 2a_m = 0$$

$$\boxed{a_{m+2} = \frac{m(m-1)a_m + 2m a_m - 2a_m}{(m+2)(m+1)}}$$

$a_0 = \text{arb}$

$a_1 = \text{arb}$

$a_2 = \dfrac{2a_2 + 4a_2 - 2a_2}{4(3)} = \dfrac{a_2}{(3)}$

$a_3 = \dfrac{6a_3 + 6a_3 - 2a_3}{5(4)} = \dfrac{a_3}{(4)}$

$a_4 = \dfrac{12a_4 + 8a_4 - 2a_4}{6(5)} = \dfrac{18a_4}{6(5)} = \dfrac{3a_4}{(5)}$

Example

$y'' + y = 0$ Assume $y = \sum_{m=0}^{\infty} a_m x^m$

$y'' = m(m-1)a_m x^{m-2}$, $y = a_m x^m$

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m$$

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} + a_m\right] x^m = 0$$

$$(m+2)(m+1)a_{m+2} + a_m = 0$$

$$\boxed{a_{m+2} = \frac{-a_m}{(m+2)(m+1)}}$$

starts at $m=0$

$a_0 = \text{arb}$

$a_1 = \text{arb}$

$a_2 = \frac{-a_0}{(2)(1)}$

$a_3 = \frac{-a_1}{3(2)}$

$a_4 = \frac{-a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}$ $\quad a_{2k} = \frac{(-1)^k a_0}{(2k)!} \longrightarrow \cos(\theta)$

$a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}$ $\quad a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!} \longrightarrow \sin(\theta)$

$\therefore$ $\boxed{y(x) = a_0 \cos(x) + a_1 \sin(x)}$

Example

$y'' - xy = 0$ Assume $y = \sum_{m=0}^{\infty} a_m x^m$

$y'' = (m)(m-1)a_m x^{m-2}$, $y = a_m x^m$

$$\sum_{m=2}^{\infty} (m)(m-1)a_m x^{m-2} - x \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - a_m x^{m+1} = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$(2)(1)a_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_{m-1}\right] x^m = 0$$

$2a_2 = 0 \therefore a_2 = 0$

$$(m+2)(m+1)a_{m+2} - a_{m-1} = 0$$

$$\boxed{a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}}$$ Starts at $m=1$

$a_0 = \text{arb}$
$a_1 = \text{arb}$
$a_2 = 0$

$a_3 = \dfrac{a_0}{(3)(2)}$

$a_4 = \dfrac{a_1}{(4)(3)}$

$a_5 = \dfrac{a_2}{(5)(4)} = 0$

$a_6 = \dfrac{a_3}{(6)(5)} = \dfrac{a_0}{(6)(5)(3)(2)}$ $a_{3k} = \dfrac{a_0}{(2)(3)(5)(6)(3k)(3k-1)}$ $a_{3k+2} = 0$

$a_7 = \dfrac{a_4}{(7)(6)} = \dfrac{a_1}{(7)(6)(4)(3)}$ $a_{3k+1} = \dfrac{a_1}{(3)(4)(6)(7)(3k)(3k+1)}$

$$\boxed{y(x) = a_0\left[1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2\cdot 3\cdot 5\cdot 6\cdot (3k-1)(3k)}\right] + a_1\left[\sum_{k=1}^{\infty} \frac{x^{3k+1}}{3\cdot 4\cdot 6\cdot 7\cdot (3k+1)(3k)}\right]}$$