Phys 312 2021

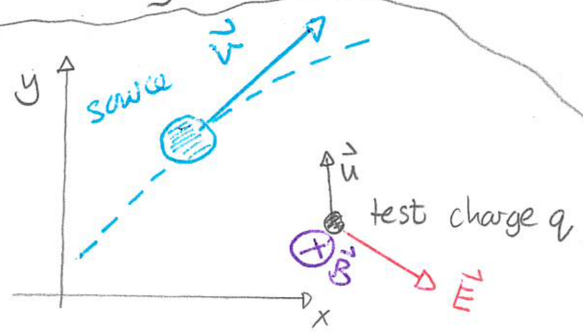
Lecture 24

Fri: HW 5pm

Tues: ~~Pr~~ 12.1.4, 12.2.1, 12.2.2.

Electromagnetism and relativity

The entire development of electromagnetic theory here has assumed a single frame of reference. We could illustrate the framework for a single rigid, non-rotating source



frame S

charge density $\rho(\vec{r}',t)$

current density $\vec{J}(\vec{r}',t) = \rho(\vec{r}',t)\,\vec{v}$

Determine fields produced by source: $\vec{E},\vec{B}$ satisfy

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Maxwell's equations

These fields exert a force on a test charge q:

$$\vec{F} = q(\vec{E}+\vec{u}\times\vec{B})$$

where $\vec{u}$ = test charge velocity

Lorentz Force Law

We have developed a series of techniques for solving Maxwell's equations for various sources. All of these were developed in a single frame.

We can view or analyze the situation from various other frames of reference. For example we could use a frame of reference in which the source is at rest. If admissible this would allow an easier analysis, as the resulting fields would consist of a time-independent electric field only. Given such a possibility we would need to transform the fields back to the original frame.

In classical physics we can use inertial frames to analyze the physical situation. Any inertial frame is equally admissible. This then results in questions such as:

1) If Maxwell's equations are correct in one frame will they be correct in another inertial frame?

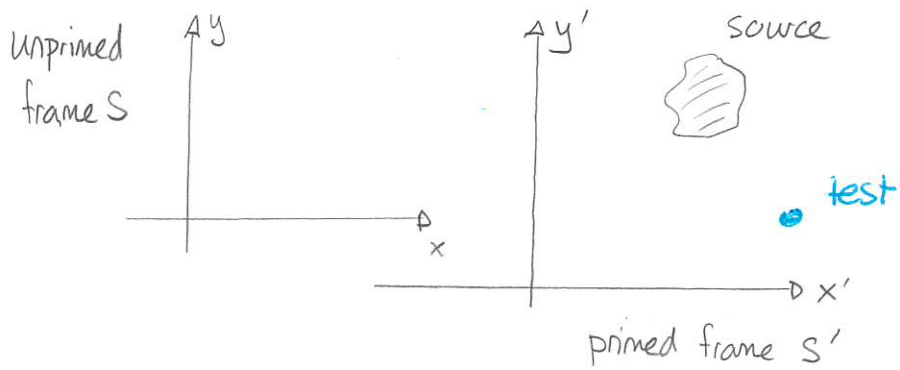
2) How are the electric and magnetic fields in one frame related to those in another frame?

3) Is the Lorentz force law correct in all frames?

A fundamental consideration will be that:

$$\boxed{\text{The laws of physics must be the same in all inertial frames.}}$$

The set up for analyzing this in electromagnetism is.



The various physical quantities are:

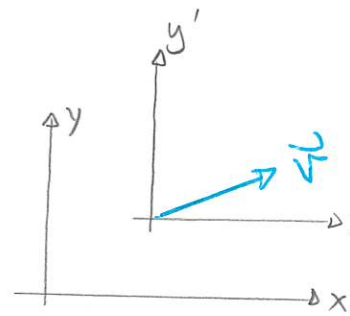
	Unprimed frame S	Primed frame S'
co-ordinates	$\vec{r} = (x, y, z)$	$\vec{r}' = (x', y', z')$
time	t	t'
test particle velocity	$\vec{u}$	$\vec{u}'$
momentum	$\vec{p}$	$\vec{p}'$
charge density	$\rho(\vec{r}, t)$	$\rho'(\vec{r}', t')$
current density	$\vec{J}(\vec{r}, t)$	$\vec{J}'(\vec{r}', t')$
fields	$\vec{E}, \vec{B}$	$\vec{E}', B'$
force	$\vec{F}$	$\vec{F}'$

If the structure of electromagnetism is correct then the laws of electromagnetism should be the same in both frames.

	unprimed	primed.
Maxwell's Laws	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \dots = \frac{\rho(x,y,z)}{\epsilon_0}$	$\vec{\nabla}' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0} \Rightarrow \frac{\partial E_x'}{\partial x'} + \frac{\partial E_y'}{\partial y'} + \frac{\partial E_z'}{\partial z'} = \rho'/\epsilon_0$
	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\vec{\nabla}' \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'}$
	...	...
Lorentz force law	$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$	$\vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}')$
	$\equiv$	$\equiv$
	$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \times \vec{B})$	$\frac{d\vec{p}'}{dt'} = q(\vec{E}' + \vec{u}' \times \vec{B}')$

Classical Galilean relativity

In classical physics the two inertial frames are related by a velocity (of the origin of S' as observed from S), $\vec{v}$, and /or a rotation. We ignore rotations as the vector calculus operations provide invariance under rotations. Then the co-ordinates in the two frames are related by:



$$x' = x - v_x t$$
$$y' = y - v_y t$$
$$z' = z - v_z t$$
$$t' = t$$

We separately check whether the following laws are the same in the two frames:

1) Newton's Second Law

If an object has momentum $\vec{p}$ is S and momentum $\vec{p}'$ in S' then how must the net forces in the two frames $\vec{F}, \vec{F}'$ be related so that

$$\underbrace{\frac{d\vec{p}}{dt} = \vec{F}} \quad \text{and} \quad \underbrace{\frac{d\vec{p}'}{dt} = \vec{F}'}$$

same form Newton's 2nd Law

2) Maxwell's Equations + Lorentz Force law

If Maxwell's equations are valid in each frame, is the Lorentz force law also true?

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}) \quad \overset{??}{\text{vs}} \quad \vec{F}' = q(\vec{E}' + \vec{u}' \times \vec{B}')$$

1 Newton's Second Law in two inertial frames

Suppose that a test particle has velocity $\mathbf{u}$ as observed from the unprimed frame. The velocity of the primed frame relative to the unprimed frame is $\mathbf{v}$.

a) Relate the velocity as observed from the primed frame, $\mathbf{u}'$, to $\mathbf{u}$.

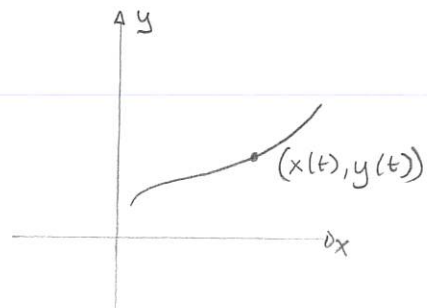
b) Relate the momentum as observed from the primed frame, $\mathbf{p}'$, to that as observed from the unprimed frame, $\mathbf{p}$.

c) Relate the force as observed from the primed frame, $\mathbf{F}'$, to that as observed from the unprimed frame, $\mathbf{F}$.

Answers: a) $u_x' = \dfrac{dx'}{dt'}$ $\qquad u_x = \dfrac{dx}{dt}$

$u_y' = \dfrac{dy'}{dt'}$ $\qquad u_y = \dfrac{dy}{dt}$

$u_z' = \dfrac{dz'}{dt'}$ $\qquad u_z = \dfrac{dz}{dt}$



Now $\left.\begin{array}{l} x' = x - v_x t \\ t' = t \end{array}\right\} \Rightarrow \begin{array}{l} x = x' + v_x t' \\ t = t' \end{array}$

Then given $x(t)$ we can get $u_x = \frac{dx}{dt}$. In the primed frame we need to describe the same trajectory via $x'(t')$ and form

$u_x' = \dfrac{dx'}{dt'}$

Now $\left.\begin{array}{l} x(t) \\ \text{and} \\ x' = x - v_x t \\ t' = t = t(t') \end{array}\right\} \Rightarrow \begin{array}{l} x' = x(t) - v_x t \\ x'(t') = x(t(t')) - v_x t(t') \end{array}$

So $\dfrac{dx'}{dt'} = \underbrace{\dfrac{dx}{dt}}_{u_x} \underbrace{\dfrac{dt}{dt'}}_{1} - v_x \underbrace{\dfrac{dt}{dt'}}_{1} \Rightarrow u_x' = u_x - v_x$

Similar derivations give

$$\left.\begin{aligned} u_x' &= u_x - v_x \\ u_y' &= u_y - v_y \\ u_z' &= u_z - v_z \end{aligned}\right\} \Rightarrow \vec{u}' = \vec{u} - \vec{v}$$

b) momentum is $\vec{p} = m\vec{u}$. Then we see

$$\vec{p}' = m\vec{u}' = m\vec{u} - m\vec{v} \quad \Rightarrow \quad \vec{p}' = \vec{p} - m\vec{v}$$

c) In the unprimed frame

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

and in the primed frame

$$\vec{F}' = \frac{d\vec{p}'}{dt'} = \frac{d\vec{p}'}{dt}\underbrace{\frac{dt}{dt'}}_{1} = \frac{d\vec{p}'}{dt} = \frac{d\vec{p}}{dt} - \underbrace{m\cancel{\frac{d\vec{v}}{dt}}}_{0} \text{ since inertial}$$

$$\Rightarrow \vec{F}' = \frac{d\vec{p}}{dt} = \vec{F}$$

Thus we have shown that:

If Galilean relativity applies and Newton's Laws are the same in all inertial reference frames then; the velocities of a particle are related by

$$\vec{u}' = \vec{u} - \vec{v}$$

and the forces are identical:

$$\vec{F}' = \vec{F} \iff \vec{F}' = \frac{d\vec{p}'}{dt'} \text{ and } \vec{F} = \frac{d\vec{p}}{dt}.$$

Now consider Maxwell's Laws and the Lorentz force law. We assume that

Maxwell's equations are valid in all inertial reference frames.

We will also assume

Inertial frames are related by Galilean transforms

+

Newton's second law is valid in all inertial frames

$\Downarrow$

Forces same in all frames

We try

Lorentz force law valid in all frames

Check if these are consistent.

2 Line of charges

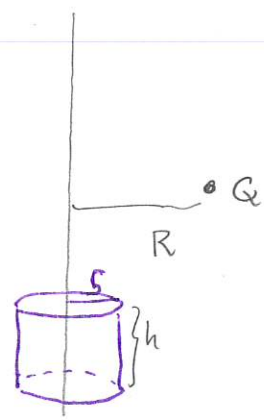
An infinite line carries charge with uniform linear charge density λ as observed from the rest frame (i.e. the line consists of infinitesimal point charges at rest as observed in this frame). A test particle with charge Q is at rest relative to the line of charge and is distance R from the line of charge.

a) Using the rest frame of the line of charge determine the fields produced by the line. Determine the force exerted on the test charge.

b) Using a frame moving with velocity $\mathbf{v}$ along the line of charge, determine the fields and the forces.

c) Are your results consistent with Galilean relativity and Newton's second law?

Answer a)



Maxwell's equations and Ampère's law clearly imply that

$$\vec{B} = 0$$

We calculate the electric field using Gauss' law (derived from $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$). Then

$$\oint \vec{E} \cdot d\vec{a} = \rho_{enc}/\epsilon_0 \Rightarrow E_s(s)\, 2\pi s h = \frac{\lambda h}{\epsilon_0}$$

$$\Rightarrow E_s(s) = \frac{\lambda}{2\pi\epsilon_0 s}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

At the test charge location

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 R}\hat{s}$$

$$\vec{B} = 0$$

$$\vec{F} = Q(\vec{E} + \overset{\shortparallel 0}{\vec{u}} \times \overset{\shortparallel 0}{\vec{B}})$$

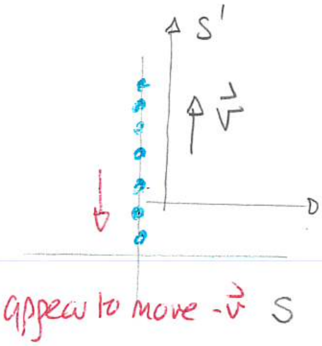
$$\Rightarrow \boxed{F = \frac{\lambda Q}{2\pi\epsilon_0 R}\hat{s}}$$

b) The charge density is the same an we get

$$\vec{E}' = \frac{\lambda}{2\pi\epsilon_0 s'}\hat{s}$$

However, there is now a current that results from the apparent motion of the charges with velocity $-\vec{v}$. Thus there will be a current

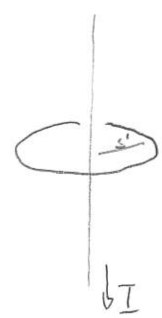
$$\vec{I} = -\lambda\vec{v}$$



Then Ampère's Law (derived from $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$)

gives

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$



$$\Rightarrow \quad B_\phi(s')\,2\pi s' = -\mu_0\lambda v$$

$$\Rightarrow \quad \vec{B} = -\frac{\mu_0\lambda v}{2\pi s}\hat{\phi} = -\frac{\lambda v}{2\pi\epsilon_0 c^2 s'}\hat{\phi}$$

Now at the test charge location

$$\vec{E}' = \frac{\lambda}{2\pi\epsilon_0 R}\hat{s} \qquad \vec{B}' = \frac{-\lambda v}{2\pi\epsilon_0 c^2 R}\hat{\phi}$$

the test charge moves with velocity $\vec{u}' = -\vec{v} = -v\hat{z}$

$$\vec{F}' = Q\left(\vec{E}' + \vec{u}'\times\vec{B}'\right) = Q\left[\frac{\lambda}{2\pi\epsilon_0 R}\hat{s} + \frac{\lambda v^2}{2\pi\epsilon_0 c^2 R}\underbrace{\hat{z}\times\hat{\phi}}_{-\hat{s}}\right]$$

$$\Rightarrow \vec{F}' = \frac{Q\lambda}{2\pi\epsilon_0 R}\left(1 - \frac{v^2}{c^2}\right)\hat{s}$$

c) This requires $\vec{F}' = \vec{F}$ But this is not true! Not consistent.

Thus we have the following possibilities:

Newton's second law

$$\frac{d\vec{p}}{dt} = \vec{F}$$

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Galilean relativity for two frames

$$\vec{r}\,' = \vec{r} - \vec{v}t$$

$$t' = t$$

Lorentz force law

$$\vec{F} = Q(\vec{E} + \vec{u} \times \vec{B})$$

Forces are same in both frames

inconsistent

Forces are not same in both frames

It is therefore impossible that all four assumptions are simultaneously correct. We need to abandon at least one.

What will occur is:

- replace the Galilean transformations with Lorentz transformations
- redefine momentum and forces and correct Newton's 2nd law

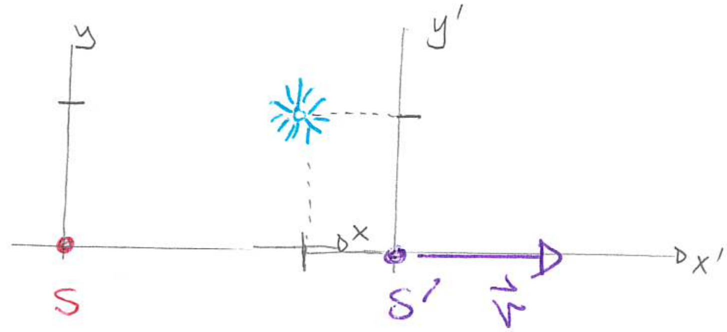
We will keep

- Maxwell's equations
- Lorentz force law.

Note that this does not present substantial issues when $v \ll c$.

Lorentz transformations

Consider again two inertial observers. Assume that S' moves along the x axis of S with velocity v as observed from S. They observe a single event and ascribe co-ordinates to this event



observer	position	time
S	x, y, z	t
S'	x', y', z'	t'

With a few simple assumptions we can relate these. One assumption is that the speed of light is the same in all frames. If we assumed that Maxwell's equations are correct in all frames with the same ϵ_0, μ_0 then this must be true since we would mathematically derive the existence of electromagnetic waves with $c = 1/\sqrt{\epsilon_0 \mu_0}$ in each frame.

A derivation gives: the Lorentz transformations:

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ t' &= \gamma\left(t - \tfrac{v}{c^2}x\right) & t &= \gamma\left(t' + \tfrac{v}{c^2}x'\right) \end{aligned} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

One consequence is that the time elapsed between two events is not the same in both frames. So we cannot use time to parameterize events in the same way as in Galilean relativity. This has implications for momenta, accelerations, etc.... There are also consequences for distances and thus charge and current densities.

3 Charge densities as observed by two inertial observers

Observer S' moves relative to observer S with velocity v along the x axis. A collection of point charges, each with charge q, are at rest in the S' frame and are evenly spaced along the x' axis. Let λ' be the charge per unit length as observed in the S' frame. The aim of this exercise is to determine an expression for the charge density λ as observed in the S frame.

a) Determine an expression for the distance, L', between adjacent charges as observed in the S' frame.

b) Using the Lorentz transformations determine an expression for the distance L between adjacent charges as observed in the S frame. Use the result to determine an expression for the charge density λ as observed in the S frame.

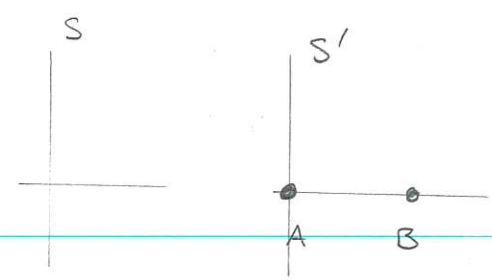
c) How are the charge densities related when $v \ll c$?

Answer: a) Let N' be the number of charges per unit length. Then

$$\lambda' = N'q$$

Then $N'L' = 1 \Rightarrow N' = 1/L' \Rightarrow \lambda' = \frac{q}{L'}$

b) Consider two adjacent particles A, B. In each frame we need to record the positions at the same instant (these can differ in between the frames)



Particle	Unprimed		Primed	
A	$x_A =$	$t_A = 0$	$x_A' = 0$	$t_A' = ?$
B	$x_B =$	$t_B = 0$	$x_B' = L'$	$t_B' = ?$

Note!

$$x_A' = \gamma(x_A - v\cancel{t_A}) \Rightarrow 0 = \gamma x_A \Rightarrow x_A = 0$$

$$x_B' = \gamma(x_B - v\cancel{t_B}) \Rightarrow L' = \gamma x_B \Rightarrow x_B = \frac{1}{\gamma} L'$$

Thus we get

moving frame — rest frame

$$x_B - x_A = L = \frac{1}{\gamma} L' \quad \Rightarrow \quad \boxed{L = \sqrt{1 - v^2/c^2}\, L'}$$

This is length contraction

Then $\lambda = \frac{q}{L} = \gamma \frac{q}{L'} \quad \Rightarrow \quad \boxed{\lambda = \gamma \lambda'}$

Thus the charge density transforms - it is larger in the frame in which the particles appear to move.

c) $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1 \quad \Rightarrow \quad \lambda \approx \lambda'$