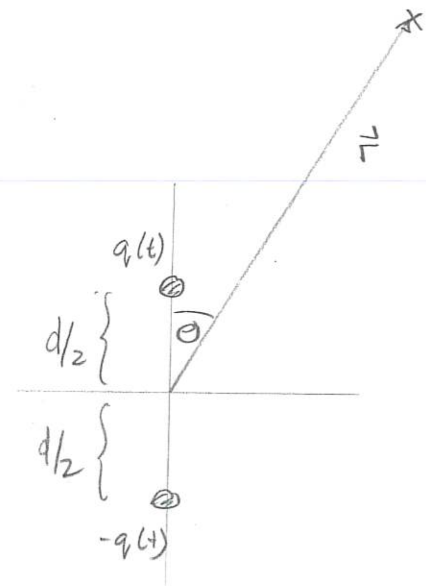
Tues: HW by 5pm

Thurs: 12.1.1, 12.1.3, 12.1.4.

Oscillating dipole

We have consider a dipole, arranged as illustrated and where the charge oscillates

$$q(t) = q_0 \cos(\omega t)$$



We use the dipole approximation

$$r \gg d$$

and the non-relativistic approximation

$$d \ll c/\omega$$

and this gives

$$V(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0} \frac{p_0 \cos\theta}{r} \left\{ -\frac{\omega}{c} \sin\left[\omega(t - r/c)\right] + \frac{1}{r}\cos\left[\omega(t-r/c)\right] \right\}$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega(t-r/c)\right] \hat{z}$$

where $p_0 = q_0 d$ is the amplitude of the dipole moment.

We now need to compute:

1) the electric field $\vec{E}$
2) the magnetic field $\vec{B}$
3) the Poynting vector $\vec{S}$

that this produces. We anticipate that it may yield waves that propagate spherically outwards. We can see that the potentials form sinusoidal waves with spherical wavefronts and presumably so will the fields.

We thus introduce the wavenumber

$$k = \omega / c$$

and the wavelength

$$\lambda = 2\pi / k.$$

Radiation zone

In the radiation zone approximation

$$r \gg \lambda$$

$$\Rightarrow r \gg \frac{2\pi}{k} = \frac{2\pi c}{\omega}$$

$$\Rightarrow \frac{1}{r} \ll \frac{\omega}{c} \frac{1}{2\pi}$$

This further simplifies the potentials.

Radiation zone approximation:

$$\boxed{\begin{aligned} V(\vec{r},t) &\approx \frac{-p_0\omega\cos\theta}{4\pi\epsilon_0 r\, c}\sin(\omega t - kr) \\ \vec{A}(\vec{r},t) &\approx \frac{-\mu_0 p_0\omega}{4\pi r}\sin(\omega t - kr)\,\hat{z} \end{aligned}}$$

We get the fields via:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

The results are:

If $r \gg d$ and $d \ll c/\omega$ and $r \gg \lambda$ then an oscillating dipole $\vec{p} = p_0\cos\omega t\,\hat{z}$ produces fields:

$$\vec{E} = \frac{-p_0 k^2}{4\pi\epsilon_0 r}\sin\theta\cos(\omega t - kr)\,\hat{\theta}$$

$$\vec{B} = \frac{-p_0 k^2}{4\pi\epsilon_0 r}\frac{1}{c}\sin\theta\cos(\omega t - kr)\,\hat{\phi}$$

Proof: First

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$= \left[\frac{p_0 \omega \cos\theta}{4\pi\epsilon_0 r^2 c}\sin(\omega t - kr) + \frac{p_0 \omega \overset{\omega/c}{k}}{4\pi\epsilon_0 r c}\cos\theta\cos(\omega t - kr)\right]\hat{r}$$

$$+ \frac{1}{r}\frac{p_0 \sin\theta\, \omega}{4\pi\epsilon_0 r c}\sin(\omega t - kr)\hat{\theta}$$

Then

$$\frac{\partial \vec{A}}{\partial t} = \frac{-\mu_0 p_0 \omega^2}{4\pi r}\cos(\omega t - kr)\hat{z}$$

$$= \frac{-\mu_0 p_0 \omega^2}{4\pi r}\cos(\omega t - kr)\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

$$= -\frac{p_0 \omega^2}{4\pi\epsilon_0 c^2 r}\cos(\omega t - kr)\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right].$$

Thus:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = -\frac{p_0 \cos\theta}{4\pi\epsilon_0 r^2}\frac{\omega}{c}\sin(\omega t - kr)\hat{r}$$

$$-\frac{p_0 \sin\theta}{4\pi\epsilon_0 r^2}\frac{\omega}{c}\sin(\omega t - kr)\hat{\theta}$$

$$-\frac{p_0 \sin\theta}{4\pi\epsilon_0 r}\frac{\omega^2}{c^2}\cos(\omega t - kr)\hat{\theta}$$

The first two terms have magnitudes:

$$\frac{1}{r^2}\frac{\omega}{c}$$

The last has

$$\frac{1}{r}\frac{\omega^2}{c^2}$$

The ratio of this to the first is $\frac{1}{r}\frac{\omega^2}{c^2} \Big/ \frac{1}{r^2}\frac{\omega}{c} = r\frac{\omega}{c} \gg 1.$

$rk \gg 1$

Thus

$$\vec{E} \approx \frac{-p_0}{4\pi\epsilon_0}\frac{k^2}{r}\sin\theta\cos(\omega t - kr)\,\hat{\theta}$$

Now for $\vec{B}$

$$\vec{A} = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin(\omega t - kr)\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

So

$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\cancel{\frac{\partial A_\theta}{\partial\phi}}\,\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(+\frac{\mu_0 p_0 \omega}{4\pi}\sin(\omega t - kr)\sin\theta\right) + \frac{\mu_0 p_0 \omega}{4\pi r}\sin(\omega t - kr)\frac{\partial}{\partial\theta}\cos\theta\right]\hat{\phi}$$

$1/\epsilon_0 c^2$

$$= \left[-\frac{\mu_0 p_0 \omega}{4\pi r}k\sin\theta\cos(\omega t - kr) - \frac{\mu_0 p_0 \omega}{4\pi r^2}\sin\theta\sin(\omega t - kr)\right]\hat{\phi}$$

$$= \left[\frac{-p_0}{4\pi\epsilon_0}\frac{\omega k}{r c^2}\cos(\omega t - kr) - \frac{p_0\omega}{4\pi\epsilon_0 r^2 c^2}\sin(\omega t - kr)\right]\sin\theta\,\hat{\phi}$$

now compare $\frac{k}{r}$ to $\frac{1}{r^2}$, i.e. $\frac{2\pi}{r\lambda}$ to $\frac{1}{r^2}$. If $r \gg \lambda$

then $r^2 \gg r\lambda \Rightarrow \frac{1}{r^2} \ll \frac{1}{r\lambda}$ and the k/r dominates. Thus

$$\vec{B} = \frac{p_0 \omega k}{4\pi\epsilon_0 r c^2} \sin\theta \cos(\omega t - kr)\, \hat{\phi}$$

Thus we see that:

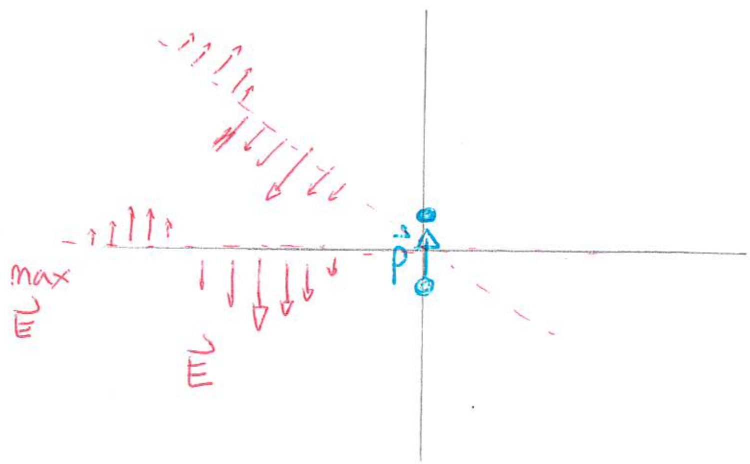
$\boxed{\text{An oscillating dipole produces electromagnetic waves}}$

These fields satisfy:

1) $\vec{B} = \frac{1}{c} \hat{r} \times \vec{E}$

2) the amplitude depends on the "angle of observation"

$\theta = 0 \Rightarrow \vec{E} = 0 \ , \vec{B} = 0$



$\theta = \pi \Rightarrow \vec{E} = 0 \ , \vec{B} = 0$

1 Poynting Vector for an Oscillating Electric Dipole

An electric dipole oscillates as $\mathbf{p} = p_0 \cos(\omega t)\hat{\mathbf{z}}$. The following refer to the radiation zone approximation.

a) Determine the Poynting vector

b) In which direction does the energy propagate?

c) Determine the time-averaged Poynting vector.

d) Determine the total rate at which energy radiates in all directions.

a) $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

$$= \frac{1}{\mu_0} \frac{p_0^2 k^4}{16\pi^2 \epsilon_0^2 r^2 c} \sin^2\theta \cos^2(\omega t - kr) \underbrace{\hat{\theta} \times \hat{\phi}}_{\hat{r}}$$

$$= \frac{1}{\mu_0} \frac{1}{\epsilon_0^2 c} \frac{p_0^2 k^4}{16\pi^2} \frac{1}{r^2} \sin^2\theta \cos^2(\omega t - kr)\, \hat{r}$$

Then $\epsilon_0 = \frac{1}{\mu_0 c^2}$ gives

$$\vec{S} = \frac{\mu_0 c^4}{c} \frac{p_0^2}{16\pi^2} \frac{\omega^4}{c^4} \frac{1}{r^2} \sin^2\theta \cos^2(\omega t - kr)\, \hat{r}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c} \frac{\sin^2\theta}{r^2} \cos^2(\omega t - kr)\, \hat{r}$$

b) outward along $\hat{r}$.

c) We need the time-average of $\cos^2(\omega t - kr)$. This is

$$\langle \cos^2(\omega t - kr) \rangle = \frac{1}{2}.$$

Thus

$$\langle \vec{S} \rangle = \frac{\mu_0 p^2 \omega^4}{32\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r}$$

d) $\int \langle \vec{S} \rangle \cdot d\vec{a}$ over a sphere of radius R $\Rightarrow$ $d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

So

$$\int \langle \vec{S} \rangle \cdot d\vec{a} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \frac{\mu_0 p^2 \omega^4}{32\pi^2 c} \sin^2\theta \sin\theta$$

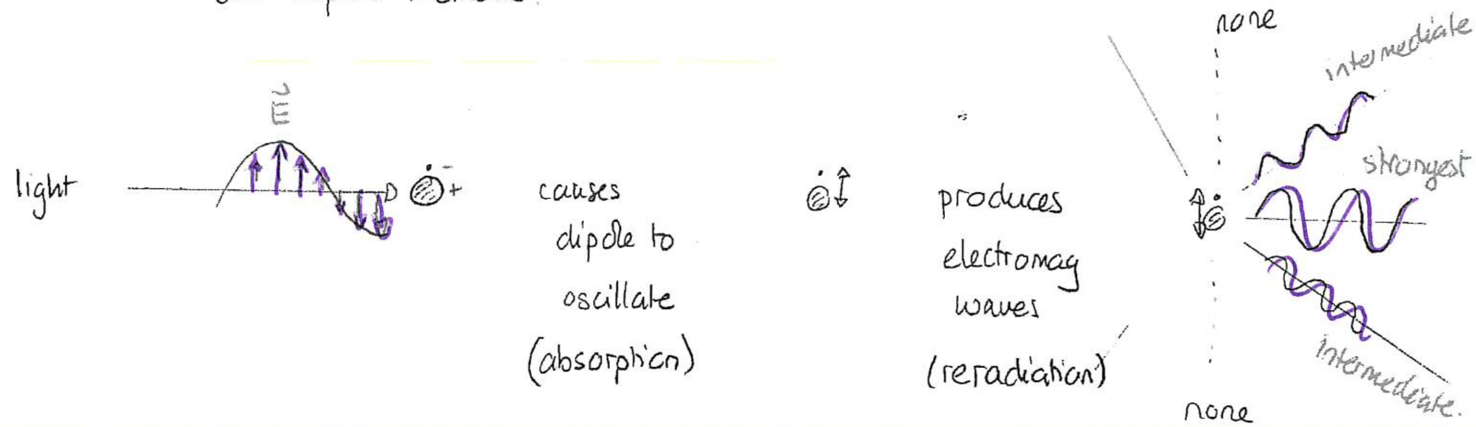
$$= \frac{2\pi \mu_0 p^2 \omega^4}{32\pi^2 c} \underbrace{\int_0^{\pi} \sin\theta (1 - \cos^2\theta) d\theta}_{\underbrace{-\cos\theta + \frac{1}{3}\cos^3\theta \Big|_0^{\pi}}_{= 1 - \frac{1}{3} - (-1 + \frac{1}{3}) = 4/3}}$$

$$\Rightarrow \int \langle \vec{S} \rangle \cdot d\vec{a} = \frac{\mu_0 p^2 \omega^4}{12\pi c} \quad \boxtimes$$

So we find that:

> For an oscillating dipole
>
> $$\langle \vec{S} \rangle = \frac{\mu_0 p^2 \omega^4}{32\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r}$$
>
> and the total power radiated is
>
> $$P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

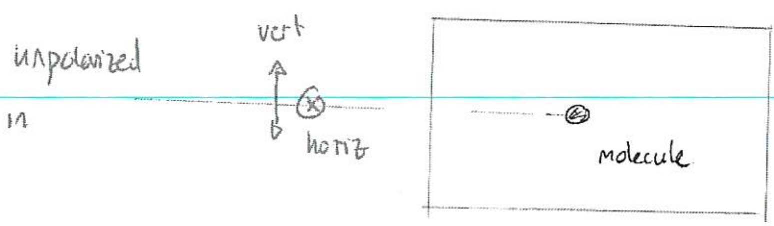
This can explain various phenomena in which electromagnetic waves interact with matter, particularly charged atoms. Such atoms have non-zero dipole moments.



Various phenomena can be explained using this model:

1) The reradiated (scattered) light is more intense as the frequency increases – blue light is scattered more strongly than red.

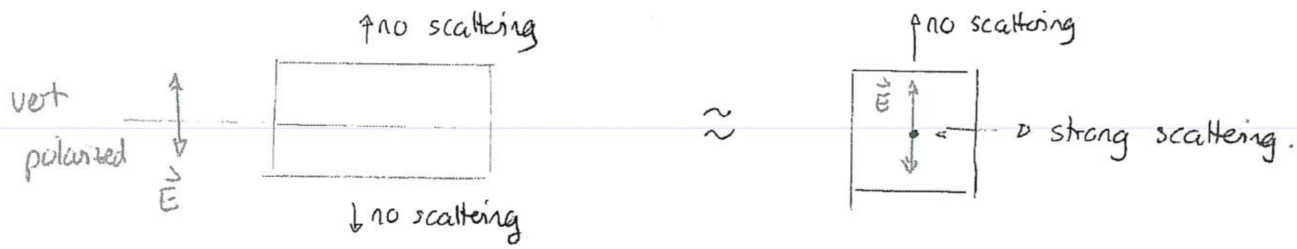
2) Scattered light can have polarization effects.



The horizontally polarized light causes horizontal oscillations – as the radiation along the direction of oscillation is zero, such radiation will be absent in the illustrated perspective. On the other hand vertical oscillations produced radiation strongly along this direction. This will be observed.

So, if the observer's line of sight is perpendicular to the direction of the incident beam, then this observer will predominantly detect scattered light that is polarized perpendicular to the plane containing the incident beam and the line of sight.

3) If the incident light is linearly polarized, then there will be no scattered light along the direction of polarization.



<u>Demo:</u> Demo last two with water, laser + polarizer.

Radiation from a moving point charge

We were able to determine the fields produced by a moving point charge.

For field at $\vec{r}$ at time t and trajectory $\vec{w}(t)$ get retarded time t_r via

$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

Retarded separation vector

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$$

Velocity

$$\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r}$$

$$\vec{u} = c\hat{\mathscr{r}}_r - \vec{v}$$

acceleration

$$\vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$$

Then the fields are:

$$\boxed{\begin{aligned} \vec{E} &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r\cdot\vec{u})^3}\left\{(c^2-v^2)\vec{u} + \vec{\mathscr{r}}_r \times (\vec{u}\times\vec{a})\right\} \\ \vec{B} &= \frac{1}{c}\,\hat{\mathscr{r}} \times \vec{E} \end{aligned}}$$

We want to determine whether this radiates energy and if so how. We address this via the Poynting vector

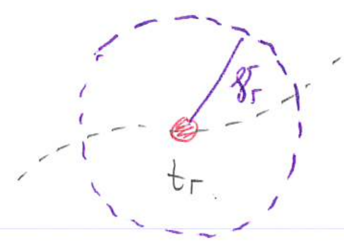
$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{1}{\mu_0}\vec{E}\times\frac{1}{c}(\hat{\mathscr{r}}_r\times\vec{E})$$

$$= \frac{1}{\mu_0 c}\left[\hat{\mathscr{r}}_r(\vec{E}\cdot\vec{E}) - \vec{E}(\hat{\mathscr{r}}_r\cdot\vec{E})\right]$$

$$\Rightarrow \qquad \vec{S} = \frac{1}{\mu_0 c}\left[E^2 \hat{\mathscr{r}}_r - (\hat{\mathscr{r}}_r \cdot \vec{E})\vec{E} \right].$$

We will need to integrate over a surface at one instant. We could do so over a spherical surface centered on the particle location at a retarded time t_r since all signals will arrive on this surface at time

$$t = t_r + \mathscr{r}_r/c$$



Then we let $\mathscr{r}_r \to \infty$. The power radiated is

$$P = \int \vec{S}\cdot d\vec{a} = \int \vec{S}\cdot\hat{\mathscr{r}}_r \, \mathscr{r}_r^2 \, d\theta \, d\phi$$

Then

$$\vec{S}\cdot\hat{\mathscr{r}}_r = \frac{1}{\mu_0 c}\left[E^2 - (\hat{\mathscr{r}}_r\cdot\vec{E})^2 \right]$$

and thus varies as E^2. In the expression for E there is an overall factor of $\frac{1}{\mathscr{r}_r^{\,2}}$ in front of both terms. Then

$$\underbrace{(c^2 - v^2)\vec{u}}_{\text{no factor of } \mathscr{r}_r} \qquad \text{vs} \qquad \underbrace{\underbrace{\vec{\mathscr{r}}_r \times (\vec{u}\times\vec{a})}_{\text{order } \mathscr{r}_r}}_{\text{dominates.}}$$

Thus the radiation part of the field

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}_r}\frac{1}{(\hat{\mathscr{r}}_r\cdot\vec{u})^3}\,\hat{\mathscr{r}}_r\times(\vec{u}\times\vec{a})$$

dominates.

Now consider

$$\vec{u} = c\,\hat{\mathscr{r}}_r - \vec{v}$$

If $c \gg v$ then $\vec{u} \approx c\,\hat{\mathscr{r}}_r$ and thus

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0}\,\frac{1}{\mathscr{r}_r}\,\frac{1}{c^3}\,\hat{\mathscr{r}}_r \times (\hat{\mathscr{r}}_r \times \vec{a})$$

$$= \frac{\mu_0 q}{4\pi\,\mathscr{r}_r}\left[\hat{\mathscr{r}}_r(\hat{\mathscr{r}}_r\cdot\vec{a}) - \vec{a}\right]$$

So

$$\hat{S}\cdot\hat{\mathscr{r}}_r = \frac{1}{\mu_0 c}\left[E^2_{rad} - \underbrace{(\hat{\mathscr{r}}_r\cdot\vec{E}_{rad})^2}_{=0}\right]$$

$$= \frac{1}{\mu_0 c}E^2_{rad} = \frac{\mu_0^2 q^2}{\mu_0 c\,16\pi^2}\,\frac{1}{\mathscr{r}_r^2}\left[(\hat{\mathscr{r}}_r\cdot\vec{a})^2 + a^2 - 2(\hat{\mathscr{r}}_r\cdot\vec{a})^2\right]$$

$$= \frac{\mu_0 q^2}{c\,16\pi^2}\,\frac{1}{\mathscr{r}_r^2}\left[a^2 - (\hat{\mathscr{r}}_r\cdot\vec{a})^2\right]$$

So in the radiation zone if $v \ll c$ then

$$\boxed{\vec{S} = \frac{\mu_0 q^2}{16\pi^2 c}\,\frac{1}{\mathscr{r}_r^2}\left[a^2 - (\hat{\mathscr{r}}_r\cdot\vec{a})^2\right]\hat{\mathscr{r}}_r}$$

We can choose the axes so that instantaneous the acceleration is along $\hat{z}$. Then:

$$\hat{\mathscr{r}}_r \cdot \vec{a} = a \cos\theta$$

$$\Rightarrow \quad \vec{S} = \frac{\mu_0 q^2}{16\pi^2 c} \frac{1}{\mathscr{r}_r^2} a^2 \sin^2\theta \, \hat{\mathscr{r}}_r$$

The total power radiated is:

$$P = \int\limits_{\text{sphere radius } \mathscr{r}_r} \vec{S} \cdot d\vec{a}$$

$$= \frac{\mu_0 q^2}{16\pi^2 c} \frac{1}{\mathscr{r}_r^2} a^2 \mathscr{r}_r^2 \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{4/3} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

This gives the <u>Larmor formula</u>:

> If $v \ll c$ then the power radiated by a charge is
>
> $$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$