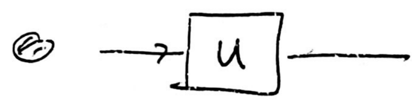
Density operator evolution

Recall that a unitary produces the following evolution



$\hat{\rho}_0$ initial state $\hat{\rho}$ final state

$$\hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^\dagger$$

One can evaluate these in terms of Pauli operators.

Exercise: Suppose

$$\hat{\rho}_0 = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)$$

$\hat{\sigma}_x\hat{\sigma}_x = I$
$\underbrace{\hat{\sigma}_x\hat{\sigma}_y}_{i\hat{\sigma}_z} = -\hat{\sigma}_y\hat{\sigma}_x$

with $0 \leq r \leq 1$. Use Pauli operators and algebra of these to evaluate $\hat{\rho}$ if

1) $\hat{U} = e^{-i\alpha\hat{\sigma}_x/2}$ $= (\cos\alpha/2\,\hat{I} - i\hat{\sigma}_x \sin\alpha/2)$

2) $\hat{U} = e^{-i\alpha\hat{\sigma}_y/2}$

Pure states

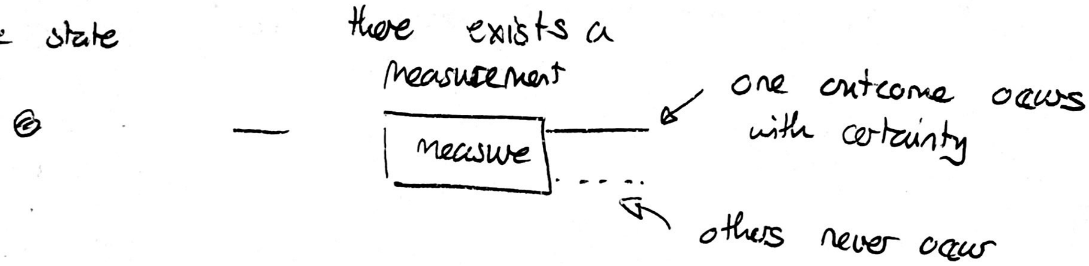
A system is in a pure state if the state of the system can be described by a ket $|\psi\rangle$. The corresponding density operator is

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

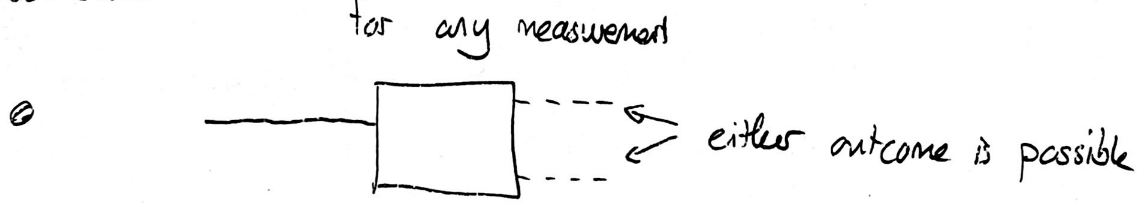
In this case one can show $\hat{\rho}^2 = \hat{\rho}$. If a system is not in a pure state, then the density operator cannot be written like this and $\hat{\rho}^2 \neq \hat{\rho}$.

In terms of measurements:

Pure state



Mixed state



Exercises:

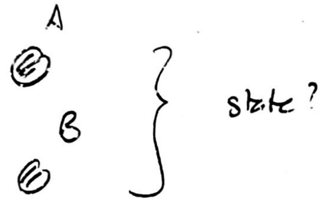
a) For $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ construct density op $\hat{\rho}$ and show $\hat{\rho}^2 = \hat{\rho}$

b) For $\hat{\rho} = \frac{1}{2}(\hat{I} + r\hat{\sigma}_y)$ find condition on r s.t. $\hat{\rho}$ is pure.

c) For $\hat{\rho} = \frac{1}{2}(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y)$ find condition on r_x, r_y s.t. $\hat{\rho}$ is pure.

Density operators for multiple particles

For two particles

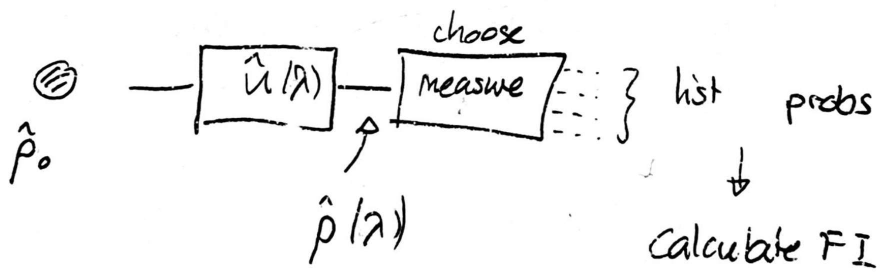


Density operator is 4×4 matrix $\hat{\rho}$ that satisfies usual properties. If the states of the particles are prepared separately then A will be in state $\hat{\rho}_A$ and B in state $\hat{\rho}_B$. Then

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$$

Quantum Fisher Information

We know that we can estimate via



Calculate FI, F $\quad \text{var} \geq \frac{1}{\sqrt{F}}$

The probabilities will depend on the measurement choice and thus so will F. Over all possible measurement choices is there an optimal choice that gives biggest F.

General result – classical Fisher info bounded by quantum Fisher info

$$F \leq H$$

independent of meas. choice
only depends on $\hat{\rho}(\lambda)$

where H is constructed via

$$\frac{\partial \hat{\rho}}{\partial \lambda} = \frac{1}{2}\left[\hat{L}\hat{\rho} + \hat{\rho}\hat{L}\right]$$

where $\hat{L}$ is symmetric logarithmic derivative. Then if one can find $\hat{L}$

SLD

$$H = \mathrm{Tr}\left[\hat{\rho}\,\hat{L}^2\right] = \mathrm{Tr}\left[\frac{\partial \hat{\rho}}{\partial \lambda}\hat{L}\right]$$

How to find $\hat{L}$ and H. If one can do eigenval decomposition

$$\hat{\rho}(\lambda) \to \text{eigenvals } p_j(\lambda) \to \text{eigenstate } |\phi_j(\lambda)\rangle$$

then

$$\hat{L} = 2\sum_{jk} \frac{\langle\phi_j|\frac{\partial \rho}{\partial \lambda}|\phi_k\rangle}{p_j + p_k}\,|\phi_j\rangle\langle\phi_j|$$

and then substitute.

If the state of the system is pure then one can show

$$\hat{L} = 2\frac{\partial \hat{\rho}}{\partial \lambda} \qquad \to \hat{\rho} = |\psi\rangle\langle\psi|$$

$$\Rightarrow H = 2\,\mathrm{Tr}\left[\frac{\partial \hat{\rho}}{\partial \lambda}\frac{\partial \hat{\rho}}{\partial \lambda}\right]$$

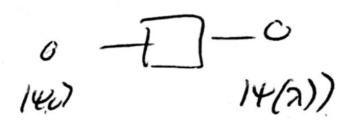
Now

$$\frac{\partial\hat{\rho}}{\partial\lambda} = \frac{\partial|\psi\rangle}{\partial\lambda}\langle\psi| + |\psi\rangle\frac{\partial\langle\psi|}{\partial\lambda}$$

$\Rightarrow$ ...

$$\boxed{H = 4\left[\frac{\partial\langle\psi|}{\partial\lambda}\frac{\partial|\psi\rangle}{\partial\lambda} + \left(\langle\psi|\frac{\partial|\psi\rangle}{\partial\lambda}\right)^2\right]}$$

pure state out only



Exercise: Single qubit phase shift

$$\hat{U}(\lambda) = e^{-i\lambda\hat{\sigma}_z/2}$$

estimate λ

Get QFI for

a) Initial state $|0\rangle$

b) " " $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

c) " " $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

d) General initial state $a_0|0\rangle + a_1|1\rangle$

$|a_0|^2 + |a_1|^2 = 1$

What is optimal?

<u>Exercise:</u> Single qubit phase shift two particles

~~a) Init~~ Get QFI for

a) Initial state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

b) " " $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\frac{1}{\sqrt{2}}(|0000\ldots0\rangle + |11\ldots1\rangle)$