MAT 202
Larson – Section 8.2
Integration by Parts

Continuing on with our investigation of techniques of integration we will now learn a very powerful tool called ***Integration by Parts***.

Consider the following integrals: $\int x \ln x \, dx,\ \int xe^x \, dx,\ and\ \int e^x \sin x \, dx.$

Notice that the u-substitution approach for finding the integral is not an option. We will now develop a tool that will allow us integrate these types of integrals.

Deriving an Integration by Parts Formula:

1. We begin with the derivative of uv: $\frac{d}{dx}[uv] = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$ (Product rule)
2. Integrating both sides with respect to x, we get:

$$\int \frac{d}{dx}[uv]\,dx = \int u\,\frac{dv}{dx}\,dx + \int v\,\frac{du}{dx}\,dx$$

$$uv = \int u\ dv + \int v\ du$$

3. Solving for $\int u\ dv$, we get the formula: $\int u\ dv = uv - \int v\ du$

Choosing *u* & *dv*:
In integration by parts, you have the freedom of choosing "u", and "dv". Here are your options:
1) Let dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2) Let u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.
Note: *dv* will always include *dx* of the original integrand.

For example, with $\int xe^x\,dx$, we could choose: $u = e^x$ and $dv = x\,dx$ or we could choose it the other way around: $u = x$ and $dv = e^x\,dx$. In this case, the latter is better since the derivative of x is an easier function to deal with:

By letting $u = x$ and $dv = e^x\,dx$, we get $du = dx$ and $v = e^x$.

$$\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Ex: Integrate the following:

a) $\int xe^{-4x}\,dx$

b) $\int \frac{5x}{e^{2x}}\,dx$

c) $\int \frac{(\ln x)^2}{x}\,dx$

d) $\int \frac{\ln x}{x^3}\,dx$

Ex: Integrate the following:

$$\int u\,dv = uv - \int v\,du$$

a) $\displaystyle\int xe^{-4x}\,dx$

$u = x \qquad \int dv = \int e^{-4x}\,dx$

$du = dx \qquad v = -\frac{1}{4}e^{-4x}$

$$= x \cdot -\frac{1}{4}e^{-4x} - \int -\frac{1}{4}e^{-4x}\,dx$$

$$= -\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}\,dx$$

$$= \boxed{-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C}$$

b) $\int \frac{5x}{e^{2x}}\,dx = \int 5xe^{-2x}\,dx$

$u = 5x \quad dv = e^{-2x}\,dx$

$du = 5\,dx \quad v = -\frac{1}{2}e^{-2x}$

$$= 5x \cdot -\frac{1}{2}e^{-2x} - \int -\frac{1}{2}e^{-2x} \cdot 5\,dx$$

$$= -\frac{5}{2}xe^{-2x} + \frac{5}{2}\int e^{-2x}\,dx$$

$$= \boxed{-\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x} + C}$$

c) $\int \frac{(\ln x)^2}{x}\,dx$

We don't need "parts" here!

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \boxed{\frac{(\ln x)^3}{3} + C}$$

d) $\int \frac{\ln x}{x^3}dx = \int x^{-3}\ln x\, dx$

$u = \ln x \quad dv = x^{-3}dx$

$du = \frac{1}{x}dx \quad v = -\frac{1}{2}x^{-2}$

$= \ln x \cdot \frac{-1}{2}x^{-2} - \int -\frac{1}{2}x^{-2} \cdot \frac{1}{x}\, dx$

$= \frac{-\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\, dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2} \cdot \frac{x^{-2}}{-2} + C$

$= \boxed{\frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C}$

In some cases, integration by parts must be used multiple times.

Ex: Integrate the following:

a) $\int x^2 \cos x \, dx$

b) $\int x^3 \sin x \, dx$

Ex: Solve the differential equation: $y' = \ln x$

Ex: Integrate the following:

a) $\int x^2 \cos x \, dx$

$u = x^2 \quad dv = \cos x \, dx$

$du = 2x \, dx \quad v = \sin x$

$x^2 \cdot \sin x - \int 2x \sin x \, dx \leftarrow$

$u = 2x \quad dv = \sin x \, dx$

$du = 2 \, dx \quad v = -\cos x$

$x^2 \cdot \sin x - \left[-2x \cos x - \int -\cos x \cdot 2 \, dx \right]$

$x^2 \sin x - \left[-2x \cos x + 2 \int \cos x \, dx \right]$

$x^2 \sin x - \left[-2x \cos x + 2 \sin x \right] + C$

$\boxed{x^2 \sin x + 2x \cos x - 2 \sin x + C}$

b) $\int x^3 \sin x \, dx$

$u = x^3 \qquad dv = \sin x \, dx$

$du = 3x^2 \, dx \qquad v = -\cos x$

$-x^3 \cos x + \int 3x^2 \cos x \, dx$

$-x^3 \cos x + \left[3x^2 \sin x - \int 6x \sin x \, dx \right]$

$-x^3 \cos x + 3x^2 \sin x - \left[-6x \cos x - \int -6 \cos x \, dx \right]$

$-x^3 \cos x + 3x^2 \sin x - \left[-6x \cos x + 6 \sin x \right] + C$

$-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

Ex: Solve the differential equation: $y' = \ln x$

$$y = \int 1 \cdot \ln x \, dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$x \ln x - \int x \cdot \frac{1}{x} dx$$

$$x \ln x - \int 1 \, dx$$

$$\boxed{y = x \ln x - x + C}$$

$$x \ln x - x + C$$

<u>Knowing What to Look For</u>: The following are common integrals that need to be integrated by parts.

1. For integrals of the form: $\int x^n e^{ax}\,dx,\ \int x^n \sin ax\,dx,\ or\ \int x^n \cos ax\,dx.$

Let $u = x^n$ and $dv = e^{ax}dx,\ \sin ax\,dx,\ or\ \cos ax\,dx$.

2. For integrals of the form: $\int x^n \ln x\,dx,\ \int x^n \arcsin ax\,dx,\ or\ \int x^n \arctan ax\,dx.$

Let $u = \ln x,\ \arcsin ax,\ or\ \arctan ax$ and $dv = x^n\,dx.$

3. For integrals of the form: $\int e^{ax} \sin bx\,dx,\ or\ \int e^{ax} \cos bx\,dx.$

Let $u = \sin bx,\ or\ \cos bx$ and $dv = e^{ax}dx$.