Physics 396

Homework Set 1

1. For an object of mass M at a radial distance R, the dimensionless quantity

$$\frac{GM}{Rc^2} \tag{1}$$

indicates when relativistic gravity becomes important as this quantity takes on values near unity.

a) Working in SI units, show that the aforementioned relativistic quantity is in fact dimensionless.

b) Knowing that $c = 2.998 \times 10^8$ m/s and $G = 6.67 \times 10^{-11}$ Nm^2/kg^2, calculate the value of the above dimensionless relativistic quantity, to three significant figures,

i. on the surface of the Earth.

ii. on the surface of the Sun.

The *Schwarzschild radius* is the characteristic length scale for curvature in the Schwarzschild geometry (the geometry exterior to a static, spherically symmetric object of mass M) and is calculated via the expression

$$R_S = \frac{2GM}{c^2}. \tag{2}$$

c) Calculate the Schwarzschild radius (in meters) of

i. the Earth.

ii. the Sun.

d) Calculate the dimensionless ratios $R_{S,\odot}/R_\odot$ and $R_{S,\oplus}/R_\oplus$ where R_S is the Schwarzschild radius for each respective body and $R_\odot$, $R_\oplus$ are the radii of the Sun, Earth. Is the Schwarzschild radius, for each respective object, smaller or larger than the radius of each respective object?

2. In Cartesian coordinates (x, y), the square of the infinitesimal distance between two neighboring points is of the form

$$ds^2 = dx^2 + dy^2. \tag{3}$$

Similarly, in polar coordinates (r, ϕ) the square of the distance between two points separated by an infinitesimal amount is of the form

$$ds^2 = dr^2 + r^2 d\phi^2. \tag{4}$$

Since the two coordinate systems are simply different ways of labeling points in a plane, these two line elements must be equivalent. The *coordinate transformation* linking these two coordinate systems are

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi. \end{aligned} \tag{5}$$

a) Using Eq. (5), calculate the differentials dx and dy in terms of r and ϕ.

b) Substitute these differentials into Eq. (3) and simplify to arrive at the line element given in Eq. (4). Notice that since Eq. (4) can be arrived at from Eq. (3) through a simple coordinate transformation, this implies that the geometries are equivalent.

3. The line element in 3D spherical-polar coordinates (r, θ, ϕ) is of the form

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

If we restrict this line element to reside on the surface of a two-dimensional sphere of radius a, where $r = a$ and $dr = 0$, the aforementioned line element reduces to

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

which equates to the expression presented in Eq. (2.15). In Cartesian coordinates (x, y, z), the line element is of the form

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{8}$$

with the constraint that

$$x^2 + y^2 + z^2 = a^2, \tag{9}$$

if the line element resides on the surface of the sphere of radius a.

The *coordinate transformation* linking these two coordinate systems are

$$\begin{aligned} x &= a \sin\theta \cos\phi \\ y &= a \sin\theta \sin\phi \\ z &= a \cos\theta. \end{aligned} \tag{10}$$

a) Using Eq. (10), calculate the differentials dx, dy, and dz in terms of θ and ϕ.

b) Substitute these differentials into Eq. (8) and simplify to arrive at the line element given in Eq. (7).

4. Consider the *axisymmetric* line element

$$ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2), \tag{11}$$

with the surface specified by

$$f(\theta) = \sin\theta \cos^2\theta. \tag{12}$$

a) Explicitly write the line element on this surface for *constant* θ.

b) Calculate the circumference $C(\theta)$ of a circle of constant θ on this surface.

c) For what values of θ do we get a minimum or maximum circumference? Calculate the circumference for these angles.

5. In the "Map Projections" handout, we focused on understanding the *Mercator Projection*, which equated to a mapping of the surface of the Earth onto a 2D map where *angles* are preserved. In this problem, we're interested in understanding the *Equal-Area Projection*, which equates to a mapping of the surface of the Earth onto a 2D map where *areas* are preserved.

Warning: Do not attempt this problem until you have studied the "Map Projections" handout!

a) Construct the infinitesimal patch of area dA_{map} on the 2D map in terms of the Cartesian coordinates x and y.

b) Construct the infinitesimal patch of area dA_{earth} on the Earth, projected onto a 2D map, in terms of the Cartesian coordinates x and y. *Hint*: you can find this quantity in the handout!

c) By demanding that these infinitesimal patches are *equal*, construct the differential equation that yields $y(\lambda)$.

d) Integrate the aforementioned expression to find $y(\lambda)$. Invert to find $\lambda(y)$.

e) Lastly, plug these into (**) in the handout and show that the line element takes the form

$$ds^2 = \frac{1}{f^2(y)} dx^2 + f^2(y) dy^2, \tag{13}$$

where

$$f(y) \equiv \frac{a}{\sqrt{4\pi^2 a^4/L^2 - y^2}}. \tag{14}$$