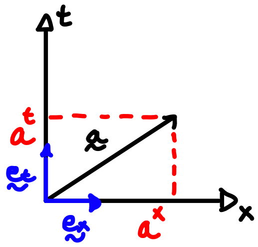
Ch. 5 Special Relativistic Mechanics

2-13-19



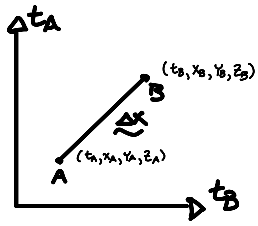
$$\underset{\sim}{a} = a^t \underset{\sim}{e}_t + a^x \underset{\sim}{e}_x + a^y \underset{\sim}{e}_y + a^z \underset{\sim}{e}_z$$

$$= a^0 \underset{\sim}{e}_t + a^1 \underset{\sim}{e}_1 + a^2 \underset{\sim}{e}_2 + a^3 \underset{\sim}{e}_3 = \sum_{\alpha=0}^{3} a^\alpha \underset{\sim}{e}_\alpha$$

— Repeated indices mean dummy index

$$\underset{\sim}{a} = a^\alpha \underset{\sim}{e}_\alpha = a^\rho \underset{\sim}{e}_\rho = a^\gamma \underset{\sim}{e}_\gamma$$

i.e Displacement Four Vector



$$\Delta \underset{\sim}{x} = (t_B - t_A, x_B - x_A, y_B - y_A, z_B - z_A)$$

$$\left[\Delta x^\alpha = x_B^\alpha - x_A^\alpha\right]$$ i.e. $x_B^0 = t_B$: α is a free index

Components of a four-vector transform between inertial reference frames just like the components of a displacement four-vector

$$a^{t'} = \gamma(a^t - v a^x)$$
$$a^{x'} = \gamma(a^x - v a^t)$$
$$a^{y'} = a^y$$
$$a^{z'} = a^z$$

Scalar Product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^\alpha \underset{\sim}{e}_\alpha \cdot b^\beta \underset{\sim}{e}_\beta = (\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_b) a^\alpha b^\beta$$

$$\left[\text{Define}: \ \eta_{\alpha\beta} \equiv \underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta\right]$$ — Metric of Flat Spacetime

$$\therefore \ \underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\alpha\beta} a^\alpha b^\beta \ : \ \Delta s^2 = \Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x}$$

$\eta_{\alpha\beta}$ is determined by the scalar product of the displacement four vector with itself.

$$(\Delta s)^2 = \Delta \underset{\sim}{x} \cdot \Delta \underset{\sim}{x} = \eta_{\alpha\beta} \Delta x^\alpha \Delta x^\beta$$

However

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \quad \therefore \quad \eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we can now write the line element as...

$$\left[ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta\right]$$

3D Example....

$$\vec{a} \cdot \vec{b} = \delta_{ij} a^i b^j \quad \text{where} \quad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

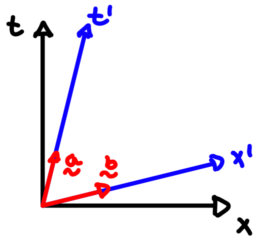
$$= \delta_{1j}\, a^1 b^j + \delta_{2j}\, a^2 b^j + \delta_{3j}\, a^3 b^j$$

$$= \delta_{11}\, a^1 b^1 + \underbrace{\delta_{12}}_{0} a^2 b^1 + \underbrace{\delta_{13}}_{0} a^3 b^1 + \underbrace{\delta_{21}}_{0} a^2 b^1 + \dots\dots + \delta_{33}\, a^3 b^3$$

$$= \delta_{11}\, a^1 b^1 + \delta_{22}\, a^2 b^2 + \delta_{33}\, a^3 b^3 = a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = -a^0 b^0 + \vec{a} \cdot \vec{b}$$

i.e 5.2



Are (t', x') axes orthogonal?

Perform calculation in the (t, x) Frame.

$$a^{\alpha'} = (1,0,0,0)\,, \quad b^{\beta'} = (0,1,0,0)$$

$$a^t = \gamma(\underbrace{a^{t'}}_{1} + v\underbrace{a^{x'}}_{0}) = \gamma\,, \quad a^x = \gamma(\underbrace{a^{x'}}_{0} + v\underbrace{a^{t'}}_{1}) = \gamma \quad \therefore\ a^\alpha = (\gamma, \gamma v, 0, 0)$$

$$b^\beta = (\gamma v, \gamma, 0, 0)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\alpha\beta}\, a^\alpha b^\beta = -a^0 b^0 + a^1 b^1$$
$$= -\gamma \cdot \gamma v + \gamma v \cdot \gamma = 0$$

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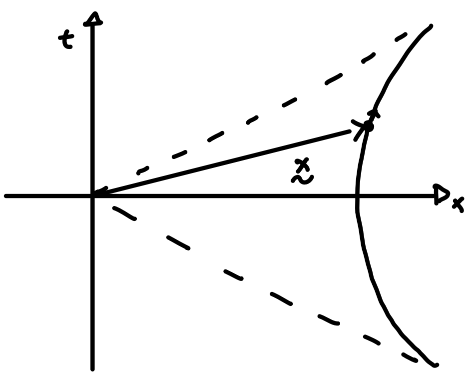
Special Relativistic Kinematics



$x^\alpha(t)$

$$\left[x^\alpha = x^\alpha(\tau)\right]$$

i.e 5.3

A particle moves on the x-axis described by

$$(*)\ \left[\begin{aligned} t(\tau) &= \frac{1}{a}\sinh(a\tau) \\ x(\tau) &= \frac{1}{a}\cosh(a\tau) \end{aligned}\right]$$

$$ds'^2 = ds^2$$

$$-d\tau^2 = -dt^2 + dx^2\,, \quad dt = \cosh(a\tau)\,d\tau\,, \quad dx = \sinh(a\tau)$$

$$-d\tau^2 = -\cosh^2(a\tau)d\tau^2 + \sinh^2(a\tau)d\tau^2$$
$$-d\tau^2 = -d\tau^2(\cosh^2(a\tau) - \sinh^2(a\tau)) = -d\tau^2$$

$x^2 - t^2 = \frac{1}{a^2}$ · Hyperbola

Four-Velocity $\underset{\sim}{u}$: $\left[u^\alpha = \frac{dx^\alpha}{d\tau} = (\gamma, \gamma\vec{v}) \right]$ Four velocity in terms of the three velocity

Calculate components in terms of the three-velocity $\vec{v}$....

$u^0 \equiv u^t$
$u^1 \equiv u^x$
$u^2 \equiv u^y$
$u^3 \equiv u^z$

$$u^t = u^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\vec{v}^2}} = \gamma$$

$$\therefore \ d\tau = \sqrt{1-\vec{v}^2}\,dt$$

$$u^x = u^1 = \frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \gamma v^x$$

Calculate

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = \eta_{\alpha\beta}u^\alpha u^\beta = -(u^0)^2 + (u^x)^2 + (u^y)^2 + (u^z)^2 = -\gamma^2 + \gamma^2\vec{v}^2 \ \therefore \ \left[\underset{\sim}{u}\cdot\underset{\sim}{u} = -1\right]$$

- The four velocity is always a timelike unit vector

i.e continued....

$$u^t = \frac{dt}{d\tau} = \cosh(a\tau)$$
$$u^x = \frac{dx}{d\tau} = \sinh(a\tau)$$
$$u^y = u^z = 0$$

now

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = -(u^t)^2 + (u^x)^2 = -\left[\cosh^2(a\tau) - \sinh^2(a\tau)\right] = -1$$

Special Relativistic Dynamics

Newton's 1st Law

$$\underset{\sim}{a} = \frac{d\underset{\sim}{u}}{d\tau} = 0$$

Newton's 2nd Law

$$\left[\underset{\sim}{f} = m\frac{d\underset{\sim}{u}}{d\tau}\right] \ \therefore \ \underset{\sim}{f} = m\underset{\sim}{a}$$

Rest mass

Four Force

Since $\underset{\sim}{u}\cdot\underset{\sim}{u}=-1$

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u})=\underset{\sim}{a}\cdot\underset{\sim}{u}=0 \quad \therefore \quad \left[\underset{\sim}{f}\cdot\underset{\sim}{u}=0\right]$$

i.e Continued....

$$a^t=\frac{du^t}{d\tau}=\frac{d}{d\tau}\Big(\cosh(a\tau)\Big)=a\sinh(a\tau)$$

$$a^x=\frac{du^x}{d\tau}=\frac{d}{d\tau}\Big(\sinh(a\tau)\Big)=a\cosh(a\tau)$$

What is $\underset{\sim}{a}\cdot\underset{\sim}{a}$?

$$\underset{\sim}{a}\cdot\underset{\sim}{a}=\eta_{\alpha\beta}a^\alpha a^\beta=-(a^t)^2+(a^x)^2=-a^2\sinh^2(a\tau)+a^2\cosh^2(a\tau)=a^2\left[\cosh^2(a\tau)-\sinh^2(a\tau)\right]$$

$$\underset{\sim}{a}\cdot\underset{\sim}{a}=a^2$$

<u>Energy-Momentum</u>

$$\left[\underset{\sim}{p}=m\underset{\sim}{u}\right]$$

$$\underset{\sim}{f}=m\frac{d\underset{\sim}{u}}{d\tau}=\frac{d}{d\tau}(m\underset{\sim}{u})=\frac{d\underset{\sim}{p}}{d\tau} \quad : \quad \left[\underset{\sim}{f}=\frac{d\underset{\sim}{p}}{d\tau}\right]$$

$$\underset{\sim}{p}\cdot\underset{\sim}{p}=\eta_{\alpha\beta}p^\alpha p^\beta=-(p^t)^2+(p^x)^2+(p^y)^2+(p^z)^2=-(mu^t)^2+(m\vec{u})^2=-(\gamma m)^2+(\gamma m\vec{v})^2=-\gamma^2m^2(1-\vec{v}^2)$$

$$\left[\underset{\sim}{p}\cdot\underset{\sim}{p}=-m^2\right]$$

An invariant quantity

$$p^t=\gamma m=\frac{m}{\sqrt{1-\vec{v}^2}}, \quad \vec{p}=m\vec{u}=\gamma m\vec{v}=\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}}$$

When $v \ll 1$......

$$\frac{1}{\sqrt{1-\vec{v}^2}}\cong 1+\frac{1}{2}\vec{v}^2$$

$$p^t\cong m\left(1+\frac{1}{2}\vec{v}^2\right)=m+\frac{1}{2}m\vec{v}^2$$

$$\vec{p}=m\vec{v}\left(1+\frac{1}{2}\vec{v}^2\right)$$

$\underset{\sim}{p}$ is Energy-momentum Four Vector w/ components

$$p^\alpha=(m\gamma,\gamma m\vec{v})=(E,\vec{p})$$

Three momentum

Energy is p^t

(**) Becomes .....

$$\underset{\sim}{P} \cdot \underset{\sim}{P} = -(\gamma m)^2 + (\gamma m \vec{v})^2$$

$$= -E^2 + \vec{p}^{\,2} = -m^2$$

$$E^2 = m^2 + \vec{p}^{\,2} \rightarrow \boxed{E^2 = (mc^2)^2 + (pc)^2}$$

The total four-momentum is a conserved quantity

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$$x^\alpha = x^\alpha(\tau)$$

$$u^\alpha = \frac{dx^\alpha}{d\tau} \quad w/ \quad \underset{\sim}{u} \cdot \underset{\sim}{u} = -1$$

$$\frac{d\underset{\sim}{u}}{d\tau} = 0 \quad : \text{Newton's } 1^{st} \text{ Law}$$

$$\underset{\sim}{f} = m\frac{d\underset{\sim}{u}}{d\tau} = \frac{d\underset{\sim}{p}}{d\tau} \quad w/ \quad \underset{\sim}{f} \cdot \underset{\sim}{u} = 0$$

$$\underset{\sim}{p} \cdot \underset{\sim}{p} = -m^2$$

$$E = p^t = \gamma m, \quad \vec{p} = \gamma m \vec{v}$$

What are the components of the four-force in terms of the three-force?

Three-Force

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{where} \quad \vec{p} = \gamma m \vec{v}$$

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \gamma \vec{F}$$

$$u^\alpha = (\gamma, \gamma \vec{v})$$

using,

$$\underset{\sim}{f} \cdot \underset{\sim}{u} = 0 = -f^t \gamma + \vec{f} \cdot \gamma \vec{v} \quad \therefore f^t = \vec{f} \cdot \vec{v} = \gamma \vec{F} \cdot \vec{v}$$

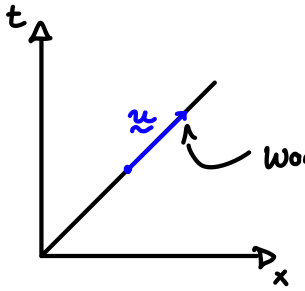
$$\boxed{\underset{\sim}{f} = (\gamma \vec{F} \cdot \vec{v}, \gamma \vec{F})}$$

<u>Light Rays</u>

Proper time for a massive particle, For a massless particle (Light Rays)

$$ds^2 = -d\tau^2$$

But $ds^2 = 0$ for Light Rays



World line of photon

$x = t$

How about parameterizing with λ where

$$x^\alpha = u^\alpha \lambda \quad , \quad u^\alpha = (1,1,0,0)$$

Affine parameter

$\underset{\sim}{u}$ is a null vector

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$$

where

$$\boxed{\frac{d\underset{\sim}{u}}{d\lambda} = 0}$$

Energy, Momentum, Frequency : wave vector

$$\boxed{E = \hbar \omega}$$

Notice that

$\frac{\vec{P}}{E} = \vec{V}$ For a photon, $|\vec{V}| = 1$ $\therefore$ $|\vec{P}| = E$

we can write ......

$$\boxed{\vec{P} = \hbar \vec{k}} \quad \therefore \quad |\vec{k}| = \omega$$

The wave three-vector

: In any I.R.F. ....

$$P^\alpha = (E, \vec{P}) = (\hbar\omega, \hbar\vec{k}) = \hbar(\omega, \vec{k}) = \hbar k^\alpha$$

wave Four-vector

What is $\underset{\sim}{p} \cdot \underset{\sim}{p}$?

$$\underset{\sim}{P} \cdot \underset{\sim}{P} = \eta_{\alpha\beta} P^\alpha P^\beta = -\hbar^2\omega^2 + \hbar^2 |\vec{k}|^2 = 0 \quad , \quad \boxed{\underset{\sim}{P} \cdot \underset{\sim}{P} = \underset{\sim}{K} \cdot \underset{\sim}{K} = 0}$$ - Photons have zero mass....

The equation of motion can be written as ....

$$\frac{d\underset{\sim}{P}}{d\lambda} = \frac{d\underset{\sim}{u}}{d\lambda} = 0$$