5.1 Linear Transformation

5.1 # 36,38

Linear Transformation

A linear transformation T on a vector space $\mathbb{V}$ to a vector space $\mathbb{W}$ is a function $T: \mathbb{V} \to \mathbb{W}$ that preserves scalar multiplication and vector addition. That is, for all $\vec{u}, \vec{v} \in \mathbb{V}$ and $c \in \mathbb{R}$

$$T(c\vec{u}) = cT(\vec{u})$$
$$T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v}$$

$\mathbb{V}$ is called the domain of T

Projection onto the x-y plane

$T(x,y,z) = (x,y,0)$

Show T is a linear transformation

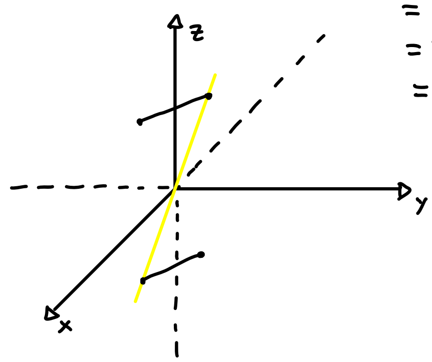
$$\begin{aligned} T(cx,cy,cz) &= (cx, cy, 0) \\ &= c(x,y,0) \\ &= cT(x,y,z) \end{aligned}$$

$$\begin{aligned} T(\vec{u}+\vec{v}) &= T(x_u+x_v,\ y_u+y_v,\ z_u+z_v) \\ &= (x_u+x_v,\ y_u+y_v,\ 0) \\ &= (x_u, y_u, 0) + (x_v, y_v, 0) \\ &= T(x_u,y_u,z_u) + T(x_v,y_v,z_v) \\ &= T(\vec{u}) + T(\vec{v}) \end{aligned}$$

Mapping R^3 to a line

$T(x,y,z) = (x,0,3z)$

$$\begin{aligned} T(c\vec{u}+d\vec{v}) &= T(c(x_u,y_u,z_u)+d(x_v,y_v,z_v)) \\ &= T((cx_u+dx_v,\ cy_u+dy_v,\ cz_u+dz_v)) \\ &= (cx_u+dx_v,\ 0,\ 3cx_u+3dx_v) \\ &= (cx_u,0,3cx_u)+(dx_v,0,3dy_v) \\ &= c(x_u,0,3x_u)+d(x_v,0,3y_v) \\ &= T(\vec{u})+d(\vec{v}) \end{aligned}$$



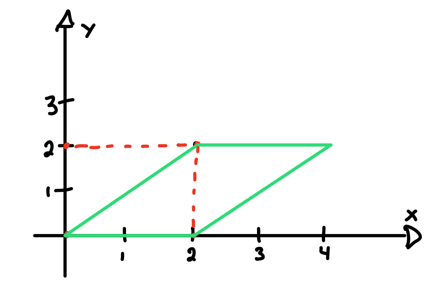
Derivative Operator $D: C'[a,b] \to C[a,b]$

$$D(cf) = cDf$$
$$D(f+g) = Df + Dg$$

Matrix Multiplication

Let A be an $n \times n$ matrix, x is a column vector, $n \times 1$

$$T(\vec{x}) = A\vec{x}$$



$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

The standard matrix for a linear transformation

$$A = [\,T(\vec{e}_1) \mid T(\vec{e}_2) \mid T(\vec{e}_3) \mid \ldots \mid T(\vec{e}_n)\,]$$

under basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n)$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$A = \left[\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]$$

Ex: 8 $\quad T(x,y) = (x-y,\ x+y,\ 2x)$

Basis vectors for $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A = \left[\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right] = \begin{bmatrix} 1-0 & 0-1 \\ 1+0 & 0+1 \\ 2 & 2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(\vec{v}) = A\vec{v}$, $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}$, $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1+v_2 \\ v_1-v_2 \\ 2v_1+v_2 \end{bmatrix}$$
$$= v_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$