Ch 1.1 Taylor Polynomials

(1) Open Desmos, then graph $f(x) = x^2$ and $g(x) = (x-1)^2$. Save file as "361 Ch1.1 Taylor Polyns"

(2) Insert folder in Desmos and title it as "Warm Up" Then drag $f(x)$ & $g(x)$ into this folder.

(3) Compute $f'(x)$ and $g'(x)$ by hand in the space below, and using chain rule for $g'(x)$.

$$f'(x) = 2x$$
$$g'(x) = 2(x-1)' \cdot 1$$

(4) Insert new folder and title it "Taylor Polynomial."
In this folder, enter the following:

$$h(x) = \ln x$$
$$P_3(x) = h(x_0) + h'(x_0)(x-x_0) + \frac{h''(x_0)}{2}(x-x_0)^2 + \frac{h'''(x_0)}{3!}(x-x_0)^3$$

create slider for x_0

In the graph, you can see that moving the slider for x_0 will re-center Taylor Polynomial at the point $(x_0, h(x_0))$ on graph of h.

In general, the idea is to represent a function $f(x)$ with an "infinite polynomial" $p(x)$ at the point $(x_0, f(x_0))$ on graph of f.

$$f(x) = p(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3 + e(x-x_0)^4 + \cdots$$

The choices required for $a, b, c, d, e, \ldots$ can be found as follows:

(1) $f(x_0) = a$

$\therefore a = f(x_0)$

(2) $f'(x) = b + 2c(x-x_0) + 3d(x-x_0)^2 + 4e(x-x_0)^3 + \cdots$

$f'(x_0) = b$

$\therefore b = f'(x_0)$

(3) $f''(x) = 2c + 3\cdot 2\, d(x-x_0) + 4\cdot 3\, e(x-x_0)^2 + \cdots$

$f''(x_0) = 2c$

$\therefore c = \dfrac{f''(x_0)}{2}$

(4) $f'''(x) = 3\cdot 2\cdot d + 4\cdot 3\cdot 2\, e(x-x_0) + \cdots$

$f'''(x_0) = 3\cdot 2\cdot d$

$\therefore d = \dfrac{f'''(x_0)}{3!}$

(5) $f^{(4)}(x) = 4!\, e + \cdots$

$f^{(4)}(x_0) = 4!\, e$

$\therefore e = \dfrac{f^{(4)}(x_0)}{4!}$

In Class Examples

We will find Taylor Polynomials / Taylor Series for

(1) $f(x) = e^x$, $x_0 = 0$

(2) $f(x) = \ln x$, $x_0 = 1$

(3) $f(x) = \sin(x)$, $x_0 = 0$

(1)
$$\left.\begin{array}{ll} f(x) = e^x & \Rightarrow f(0) = 1 \\ f'(x) = e^x & \Rightarrow f'(0) = 1 \\ f''(x) = e^x & \Rightarrow f''(0) = 1 \\ f'''(x) = e^x & \Rightarrow f'''(0) = 1 \\ \vdots & \end{array}\right\} \quad P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

In general,

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

(2) $(x_0 = 1)$

$$f(x) = \ln x \Rightarrow f(x_0) = \ln 1 = 0$$

$$f'(x) = x^{-1} \Rightarrow f'(x_0) = 1$$

$$f''(x) = -x^{-2} \Rightarrow f''(x_0) = -1$$

$$f'''(x) = 2x^{-3} \Rightarrow f'''(x_0) = 2$$

$$f^{(4)}(x) = -3 \cdot 2x^{-4} \Rightarrow f^{(4)}(x_0) = -3!$$

$$f^{(5)}(x) = 4!\, x^{-5} \Rightarrow f^{(5)}(x_0) = 4!$$

$$\vdots$$

Taylor Series: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Khan Academy: Taylor Series ~~for~~ $\sin x$ & $\cos x$, etc