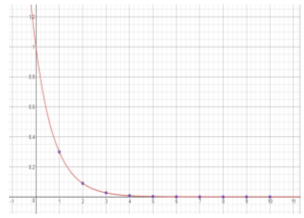
Math 361 Numerical Analysis **Ch 2.5 Extra Example**

Consider the sequence $p_n = (0.3)^n$, with $\lim_{n\to\infty} p_n = \lim_{n\to\infty}(0.3)^n = 0$. Thus convergence to zero is clear, but what is the rate of convergence?

$p(n) = (.3)^n$

x_1	$p(x_1)$
1	0.3
2	0.09
3	0.027
4	0.0081
5	0.00243
6	7.29×10^{-4}
7	2.187×10^{-4}
8	6.561×10^{-5}
9	1.9683×10^{-5}
10	5.9049×10^{-6}



Check for linear rate of convergence:

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty}\frac{(0.3)^{n+1}-0}{(0.3)^n-0} = \lim_{n\to\infty}\frac{(0.3)^n(0.3)}{(0.3)^n} = 0.3 = \lambda,\ \ 0<\lambda<1$$

$\therefore p_n \to p$ linearly, since $\lambda \in (0,1)$.

Check for quadratic rate of convergence:

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^2} = \lim_{n\to\infty}\frac{(0.3)^{n+1}}{(0.3)^{2n}} = \lim_{n\to\infty}\frac{(0.3)^n(0.3)}{(0.3)^n(0.3)^n} = \lim_{n\to\infty}\frac{1}{(0.3)^{n-1}} = 0 = \lambda$$

$\therefore p_n$ does not converge to p quadratically, since $\lambda \notin (0,1)$.

Check for other rates of convergence:

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lim_{n\to\infty}\frac{(0.3)^{n+1}}{(0.3)^{n\alpha}} = \lim_{n\to\infty}(0.3)^{n+1-n\alpha} = \lim_{n\to\infty}(0.3)^{n+1-n\alpha} = \lim_{n\to\infty}(0.3)^{n(1-\alpha)+1} = \begin{cases} 0.3, & \alpha = 1 \\ 0, & \alpha > 1 \\ \infty & \alpha < 1 \end{cases}$$

$\therefore p_n \to p$ linearly, and this is the only rate of convergence possible.

We can use Steffensen's Method to accelerate the convergence of $p_n = (0.3)^n$ to zero:

$$A(p_n) = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

n	p_n	$p_n^{(k)}$
0	1	1.00000
1	0.3	0.30000
2	0.09	0.09000
3	0.027	0.00000
4	0.0081	0.00000
5	0.00243	0.00000
6	0.000729	#DIV/0!
7	0.0002187	#DIV/0!
8	0.00006561	#DIV/0!
9	0.000019683	#DIV/0!
10	5.9049E-06	#DIV/0!
11	1.77147E-06	#DIV/0!
12	5.31441E-07	#DIV/0!

	A	B	C
1	n	p_n	$p_n^{(k)}$
2	0	=0.3^A2	=B2
3	1	=0.3^A3	=B3
4	2	=0.3^A4	=B4
5	3	=0.3^A5	=C2-(C3-C2)^2/(C4-2*C3+C2)
6	4	=0.3^A6	=C5^A6
7	5	=0.3^A7	=C6^A7
8	6	=0.3^A8	=C5-(C6-C5)^2/(C7-2*C6+C5)
9	7	=0.3^A9	=C8^A9
10	8	=0.3^A10	=C9^A10
11	9	=0.3^A11	=C8-(C9-C8)^2/(C10-2*C9+C8)
12	10	=0.3^A12	=C11^A12
13	11	=0.3^A13	=C12^A13
14	12	=0.3^A14	=C11-(C12-C11)^2/(C13-2*C12+C11)

Ch2.5 Accelerating Convergence

How might we obtain quadratic convergence in general?

Aitken's Δ^2 Method accelerates a linearly convergent sequence.

Suppose $\{p_n\}_{n=0}^{\infty}$ is linearly convergent, with limit p, and suppose

$$p_n - p, \quad p_{n+1} - p, \quad p_{n+2} - p$$

all have the same sign. Also, assume

$$\frac{p_{n+1} - p}{p_n - p} \cong \frac{p_{n+2} - p}{p_{n+1} - p}, \quad n \text{ large enough}$$

Then

$$(p_{n+1} - p)^2 \cong (p_{n+2} - p)(p_n - p)$$

Solving for p, and simplifying, we have

$$p \cong p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

From the previous page,

$$p \cong p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

Aitken's Δ^2 Method defines

$$\boxed{\hat{p}_n = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}}$$

As we will see, $\hat{p}_n \to p$ faster than $p_n \to p$.

<u>Example</u> Consider the sequence $\{p_n\}_{n=1}^{\infty}$ with $p_n = \cos(\frac{1}{n})$, which converges linearly to $p = 1$. (Show)

$p_1 = \cos(1) \cong 0.54030$

$p_2 = \cos(\frac{1}{2}) \cong 0.87758$

$p_3 = \cos(\frac{1}{3}) \cong 0.94496$

$$\hat{p}_1 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1} \cong 0.96178$$

Etc - See Table 2.11 for results.

<u>Definition</u> For a given sequence $\{p_n\}_{n=0}^{\infty}$ the forward difference Δp_n is

$$\boxed{\Delta p_n = p_{n+1} - p_n \,, \quad n \geq 0.}$$

Higher powers $\Delta^k p_n$ are defined recursively by $\Delta^k p_n = \Delta(\Delta^{k-1} p_n)$, $k \geq 2$

Thus

$$\begin{aligned} \Delta^2 p_n &= \Delta(\Delta p_n) \\ &= \Delta(p_{n+1} - p_n) \\ &= (p_{n+2} - p_{n+1}) - (p_{n+1} - p_n) \\ &= p_{n+2} - 2p_{n+1} + p_n \end{aligned}$$

Aitken's Δ^2 Method iterates are thus

$$\begin{aligned} \hat{p}_n &= p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_n + p_n} \\ &= p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \,, \quad n \geq 0 \end{aligned}$$

Theorem 2.13 Suppose that a sequence $\{p_n\}_{n=0}^{\infty}$ converges linearly to p and that for all sufficiently large values of n, we have

$$(p_n - p)(p_{n+1} - p) > 0.$$

Then the sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}_{n=0}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0.$$

What about quadratic convergence? To accomplish this, we consider Steffensen's method, which modifies Aitken's Δ^2 method.

Aitken's Δ^2 Method

$$\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

p_0

$p_1 = g(p_0)$

$p_2 = g(p_1)$

$\hat{p}_0 = p_0 - \frac{(\Delta p_0)^2}{\Delta^2 p_0}$

$p_3 = g(p_2)$

$\hat{p}_1 = p_1 - \frac{(\Delta p_1)^2}{\Delta^2 p_1}$

$\vdots$

Steffensen's Method

p_0

$p_1 = g(p_0)$

$p_2 = g(p_1)$

$\tilde{p}_0 = p_0 - \frac{(\Delta p_0)^2}{\Delta^2 p_0}$

$\tilde{p}_1 = g(\tilde{p}_0)$

$\tilde{p}_2 = g(\tilde{p}_1)$

$\tilde{p}_3 = \tilde{p}_0 - \frac{(\Delta \tilde{p}_0)^2}{\Delta^2 \tilde{p}_0}$

$\vdots$

Book's Notation for Steffensen method:

$$p_0^{(0)},\ p_1^{(0)},\ p_2^{(0)},\ \underset{\tilde{p}_0}{\underset{\uparrow}{p_0^{(1)}}},\ \underset{\tilde{p}_1}{\underset{\uparrow}{p_1^{(1)}}},\ \underset{\tilde{p}_2}{\underset{\uparrow}{p_2^{(1)}}},\ \underset{\tilde{p}_3}{\underset{\uparrow}{p_0^{(2)}}},\ p_1^{(2)},\ p_2^{(2)},\ \dots$$

$\dots$

Steffensen's

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p_1 = g(p_0)$; (*Compute* $p_1^{(i-1)}$.)
$p_2 = g(p_1)$; (*Compute* $p_2^{(i-1)}$.)
$p = p_0 - (p_1 - p_0)^2/(p_2 - 2p_1 + p_0)$. (*Compute* $p_0^{(i)}$.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*Procedure completed successfully.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*Procedure completed unsuccessfully.*)
STOP.

Example Solve $x^3 + 4x^2 - 10 = 0$ using Steffensen's Method.

Recall from Section 2.2, Example 3(d) that we can rewrite this equation as

$$x = g(x) = \left(\frac{10}{x+4}\right)^{1/2}$$

Start with $p_0 = 1.5$. Then

$p_0^{(0)} = 1.5$

$p_1^{(0)} = g(p_0^{(0)}) = 1.348399725$

$p_2^{(0)} = g(p_1^{(0)}) = 1.367376372$

Note: $p \cong 1.365230013$

$$p_0^{(1)} = p_0^{(0)} - \frac{(\Delta p_0^{(0)})^2}{\Delta^2 (p_0^{(0)})} = p_0^{(0)} - \frac{(p_1^{(0)} - p_0^{(0)})^2}{(p_2^{(0)} - 2p_1^{(0)} + p_0^{(0)})}$$

$= 1.365265224$

$p_1^{(1)} = g(p_0^{(1)}) = 1.365225534$

$p_2^{(1)} = g(p_1^{(1)}) = 1.365230583$

$$p_0^{(2)} = p_0^{(1)} - \frac{(\Delta p_0^{(1)})^2}{\Delta^2 (p_0^{(1)})} = 1.365230013$$

$\vdots$

Steffensen's Method results for this example,

$$f(x) = x^3 + 4x^2 - 10 = 0$$

$$\Leftrightarrow \quad x = g(x) = \left(\frac{10}{x+4}\right)^{1/2},$$

are comparable to Newton's Method for this problem in Section 2.4 (Ex 4).

The rapid convergence suggests quadratic convergence, without evaluating a derivative.

<u>Theorem 2.14</u> Suppose $x = g(x)$ has the solution p with $g'(p) \neq 1$. If $\exists\, \delta > 0$ s.t. $g \in C^3[p-\delta, p+\delta]$, then Steffensen's Method gives quadratic convergence for any $p_0 \in [p-\delta, p+\delta]$.

Example Let $g(x) = 1 + e^{-x}$

Use Steffensen's Method to find $p_0^{(1)}, p_0^{(2)}$.

Solution Recall

$$p_n = g(p_{n-1})$$

$$\hat{p}_n = A(p_n) = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

$$p_0^{(0)} = p_0 = 1$$

$$p_1^{(0)} = p_1 = g(p_0) = 1.367879441171$$

$$p_2^{(0)} = p_2 = g(p_1) = 1.25464638044$$

$$p_0^{(1)} = A(p_0^{(0)}) = A(p_0) = 1.281296542056$$

$$p_1^{(1)} = g(p_0^{(1)}) = 1.277677046993$$

$$p_2^{(1)} = g(p_1^{(1)}) = 1.278683918788$$

$$p_0^{(2)} = A(p_0^{(1)}) = 1.278464785530$$

Math 361 Numerical Analysis **Ch 2.5 Extra Example**

In our class notes for Ch 2.4, one of the examples sought to solve $f(x) = e^x - x - 1 = 0$. Since $f(0) = f'(0) = 0$ and $f''(0) \neq 0$, it follows that $p = 0$ is a zero of multiplicity two. See graph. In the work below, we use Steffensen's Method where $g_1(x)$ is the Newton's method iterating function. In the Excel table, the modified Newton results for $g_2(x)$ are also shown for comparison.

$$f(x) = e^x - x - 1$$

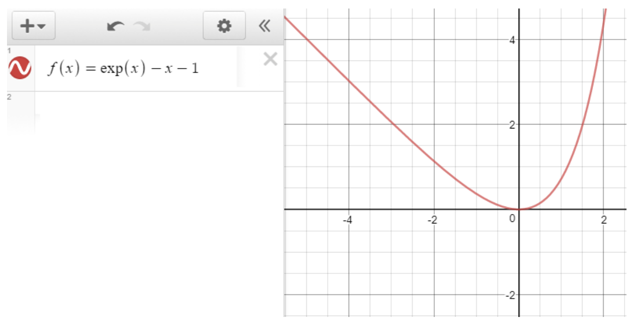
$$f'(x) = e^x - 1$$

$$f''(x) = e^x$$

$$g_1(x) = x - \frac{f(x)}{f'(x)}$$

$$g_2(x) = x - \frac{f(x)f'(x)}{\left[f'(x)\right]^2 - f(x)f''(x)}$$

$$A(p_n) = p_n - \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$



	A	B	C	D	E	F
1		Find Root	f(x) = e^x - x - 1			
2			g1(x) = x - (e^x - x - 1)/(e^x - 1)			
3			g2(x) = x - (e^x-x-1)(e^x-1)/((e^x-1)^2-(e^x-x-1)e^x)			
4			A(x,y,x) = x - (x-y)^2/(z-2y+x)			
5						
6	*n*	g1	g2	p(n,k)		
7	0	1	1	1		
8	1	0.581976707	-0.234210614	0.581976707		
9	2	0.319055041	-0.00845828	0.319055041		
10	3	0.167996173	-1.18902E-05	-0.126638558		
11	4	0.086348874	-4.21859E-11	-0.061983192		
12	5	0.043795704	-4.21859E-11	-0.030671457		
13	6	0.022057685	-4.21859E-11	-0.001267797		
14	7	0.011069387		-0.000633764		
15	8	0.005544905		-0.000316849		
16	9	0.002775014		-1.33872E-07		
17	10	0.001388149		-6.58683E-08		
18	11	0.000694235		-3.21579E-08		
19	12	0.000347158		9.79286E-10		
20	13	0.000173589		9.79286E-10		
21	14	8.6797E-05		9.79286E-10		
22	15	4.33991E-05		#DIV/0!		
23	16	2.16997E-05				