Taylor Research 17 Sept

Estimation question

Suppose that collection of N particles are each in state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|1\rangle$$

We want to estimate ϕ. We do so by subjecting each particle to the same measurement and recording measurement outcomes.

a) Describe what direction $SG\hat{n}$ must be oriented to give one particular outcome with certainty.

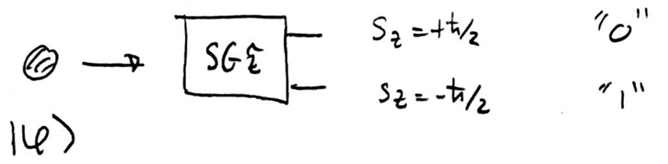
b) Suppose one measures $SG\hat{z}$ of each particle. Can one infer ϕ from outcomes? If so how?

c) Suppose one measures $SG\hat{x}$. Can one infer ϕ from outcomes? If so how.

d) One of these is worse than other for learning ϕ. For that one how is measurement direction related to $\hat{n}$ (from part a).

Measurement Operators

Suppose we measure SG$\hat{z}$ or $\{|0\rangle, |1\rangle\}$. Then



We list

Outcome	Associated State	Prob
0 $\sim +\hbar/2$	$|0\rangle$	$|\langle 0|\psi\rangle|^2$
1 $\sim -\hbar/2$	$|1\rangle$	$|\langle 1|\psi\rangle|^2$

Then

$\langle 0|\psi\rangle^\dagger = (\langle 0|\psi\rangle)^*$
$\langle \psi|0\rangle$

$$\text{Prob}(0) = |\langle 0|\psi\rangle|^2 = \langle 0|\psi\rangle^* \langle 0|\psi\rangle$$
$$= \langle \psi|\underbrace{0\rangle\langle 0}|\psi\rangle$$

We can extract the middle quantity $|0\rangle\langle 0|$ that refers to a measurement outcome. Then

$$\text{Prob}(0) = \underset{\text{bra}}{\langle \psi|} \underbrace{(|0\rangle\langle 0|)}_{\text{measurement operator}} \underset{\text{ket}}{|\psi\rangle}$$

bra

measurement operator.

ket

What is the measurement operator?

$|0\rangle\langle 0|$

$|0\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$

$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$

$\langle 0| \to \begin{pmatrix}1 & 0\end{pmatrix}$

$|0\rangle\langle 0| \to \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a_0\\a_1\end{pmatrix} = \begin{pmatrix}a_0\\0\end{pmatrix}$

$\begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\left[\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} + \begin{pmatrix}\cdot\\ \cdot\end{pmatrix}\right]$

$= \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} + \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}\cdot\\ \cdot\end{pmatrix}$

It can be regarded as a matrix. So define

$\hat{P}_0 := |0\rangle\langle 0| \equiv \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

$\hat{P}_1 := |1\rangle\langle 1| \equiv \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

Meas:

outcome	op	prob.
0	$\hat{P}_0$	$\langle\psi|\hat{P}_0|\psi\rangle$
1	$\hat{P}_1$	$\langle\psi|\hat{P}_1|\psi\rangle$

Then

$$\text{Prob}(0) = \langle\psi|\hat{P}_0|\psi\rangle$$

$$\text{Prob}(1) = \langle\psi|\hat{P}_1|\psi\rangle$$

Note that

$$\hat{P}_0|\psi\rangle = |0\rangle\langle 0|\psi\rangle = |0\rangle\underbrace{(\langle 0|\psi\rangle)}_{\text{number}}$$

state.

$\begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}\cdot\\ \cdot\end{pmatrix} = \begin{pmatrix}\cdot\\ \cdot\end{pmatrix}$

So $\hat{P}_0$ maps a ket $\to$ ket.

Exercise For each of the following states use the measurement operators to determine

a) the states $\hat{P}_0|\psi\rangle$

$P_1|\psi\rangle$.

b) the probabilities of outcomes $0, 1$.

States:

1) $|\psi\rangle = |0\rangle$

2) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

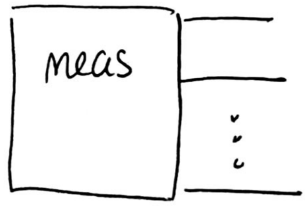
3) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

4) $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

5) $|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4i}{5}|1\rangle$

We can construct measurement operators for any measurement.

If



outcome	state
a_1	$\vert\phi_1\rangle$
a_2	$\vert\phi_2\rangle$
$\vdots$	$\vdots$
a_n	$\vert\phi_n\rangle$

the measurement operators are:

$$\hat{P}_1 = |\phi_1\rangle\langle\phi_1|$$

$$\hat{P}_2 = |\phi_2\rangle\langle\phi_2|$$

$$\vdots$$

state . state in.

and Prob (outcome a_i) = $\langle\psi|\,\hat{P}_i\,|\psi\rangle$

Exercise: a) Construct measurement operators for the measurement $SG\hat{y}$. Call these

$\hat{P}_+ \rightsquigarrow S_y = +\hbar/2$

$\hat{P}_- \rightsquigarrow S_y = -\hbar/2$

b) For the following pre-measurement states; determine probabilities:

1) $|0\rangle$

2) $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

3) $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

4) $\frac{4}{5}|0\rangle + \frac{i3}{5}|1\rangle$

These Measurement operators satisfy:

1) $\hat{P}_i \hat{P}_i = \hat{P}_i$

2) $\hat{P}_i \hat{P}_j = 0 \qquad i \neq j$

3) $\sum_{\text{all } i} P_i = \hat{I} \rightarrow$ identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4) Measure operators must be positive.

Exercise Check 1→3 for

a) operators for $SG\hat{z}$ $\{|0\rangle, |1\rangle\}$

b) " " $SG\hat{y}$

Estimation (classical)

Coin toss.

prob(heads) $= p_H = p$

prob(tails) $= p_T = 1-p$

- Want to estimate p.
- Flip coin N times. Count $n_H =$ number heads

  $n_T =$ " tails.

Estimate for p be? $\quad p_H = \frac{n_H}{N} = p_{estimate} \stackrel{?}{=} p$

- ~~Re~~ If repeat expt with N runs. – multiple times
  - get different n_H
  - get different $p_{estimate}$.

- Can check if ~~one~~ on average $p_{est} = p$
- Want minimal fluctuations in p_{est}.
- Can calculate mean of $p_{estimate}$. Call this $\overline{p_{estimate}}$

$$\overline{p_{estimate}} = \frac{1}{N}\overline{n_H}$$

- Can also calculate variance $= \langle (p_{est} - \overline{p_{est}})^2 \rangle$ average
  std dev

$$= \overline{p_{est}^2} - (\overline{p_{est}})^2$$

Need

$$\overline{p_{est}^2} = \overline{\left(\frac{n_H^2}{N^2}\right)} = \frac{1}{N^2}\overline{n_H^2}$$

$$\overline{p_{est}^2} = \frac{1}{N^2}\overline{n_H^2}$$

Need $\bar{n}_H$

$\overline{n_H^2}$

$$\bar{n}_H = \sum_{n_H=0}^{N} n_H \, \mathrm{prob}(n_H)$$

$$\overline{n_H^2} = \sum_{n_H=0} n_H^2 \, \mathrm{prob}(n_H)$$

Then

$$\mathrm{prob}(n_H) = \binom{N}{n_H} p^{n_H} (1-p)^{N-n_H}$$

$$\frac{N!}{n_H!\,(N-n_H)!}$$

$$\bar{n}_H = \sum_{n_H=0}^{N} n_H \binom{N}{n_H} p^{n_H} (1-p)^{N-n_H}$$

Binomial theorem:

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

Check

$$\sum_{n_H=0}^{N} \mathrm{prob}(n_H) = \sum_{n_H=0}^{N} \binom{N}{n_H} p^{n_H} (1-p)^{N-n_H} = [p+(1-p)]^N = 1$$

$$\sum_{k=0}^{N} \binom{N}{k} a^k \cancel{b}^{N-k}_{(1-a)} \, k = Na$$

$$\sum_{k=0}^{N} \binom{N}{k} a^k \cancel{b}^{N-k}_{(1-a)} \, k^2 = Na\{1+a(N-1)\}$$

Task:

* Determine $\overline{p_{est}} \overset{??}{=} p$

* Determine variance (p_{est}) small as possible

$?? \sqrt{\frac{p(1-p)}{N}}$

Was this the best one could do? Could do other things with n_H. Best requires

$$\overline{p_{est}} = p$$

$$\text{variance}(p_{est}) = \text{smallest}$$

Theorem

$$\text{variance}(p_{est}) \geq \frac{1}{F_{N\,toss}}$$ Fisher info.

Fisher info calculated via:

- probabilities for outcome of single coin toss and number of coin tosses (if these are independent)

- with N coin tosses

$$F_{N\,toss} = N\, F_{\text{single coin toss}}.$$

calculated via probabilities for outcomes of single meas.

$$\sum_{\substack{\text{all possible}\\ \text{outcomes}}} \frac{1}{\text{prob(outcome)}} \left(\frac{\partial\, \text{prob(outcome)}}{\partial p} \right)^2 = F$$

regardless of how one estimates

* Calculate $F_{N\,toss}$ + compare