Tues: HW

Thurs: Read 10.3.1

Fri: HW

Potentials and electromagnetism

We saw that given known charge and current densities, we can find a scalar, λ, and a vector, $\vec{A}$, potential that satisfy

$$\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

and the resulting fields are

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

These fields will automatically satisfy Maxwell's equations and the continuity equation.

We were also able to show that, in electrostatics, if $V(\vec{r})$ satisfies

$$\nabla^2 V = -\rho/\epsilon_0$$

then so does $V(\vec{r}) + V_0$ where V_0 is constant. This will produce the same electric field as $V(\vec{r})$. Since potentials are related to fields by differentiation, there will always be multiple potentials giving the same fields.

Gauge transformations and gauge choices

In general a multiplicity of potentials give rise to the same fields. The scheme is:

One pair of potentials V_1 and $\vec{A}_1$ $\longrightarrow$ Produce fields $\vec{E} = -\vec{\nabla} V_1 - \frac{\partial \vec{A}_1}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}_1$

$\updownarrow$ If related by a legitimate gauge transformation. $\qquad\qquad$ $\updownarrow$ produce same fields

Another pair of potentials V_2 and $\vec{A}_2$ $\longrightarrow$ Produce fields $\vec{E} = -\vec{\nabla} V_2 - \frac{\partial \vec{A}_2}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}_2$

We aim to find out how these potentials are related in order to produce the same fields. If they do then

$$\vec{E} \text{ equal} \iff -\vec{\nabla} V_1 - \frac{\partial \vec{A}_1}{\partial t} = -\vec{\nabla} V_2 - \frac{\partial \vec{A}_2}{\partial t} \iff \vec{\nabla}(V_2 - V_1) + \frac{\partial}{\partial t}(\vec{A}_2 - \vec{A}_1) = 0$$

$$\vec{B} \text{ equal} \iff \vec{\nabla} \times \vec{A}_1 = \vec{\nabla} \times \vec{A}_2 \iff \vec{\nabla} \times (\vec{A}_2 - \vec{A}_1) = 0$$

The latter will be true if, for any scalar function, $\lambda(\vec{r}, t)$

$$\vec{A}_2 - \vec{A}_1 = \vec{\nabla} \lambda$$

$$\Rightarrow \quad \vec{A}_2 = \vec{A}_1 + \vec{\nabla} \lambda$$

Substitution into the former gives:

$$\vec{\nabla}(V_2 - V_1) + \frac{\partial}{\partial t}\vec{\nabla}\lambda = 0 \quad \Rightarrow \quad \vec{\nabla}\left[V_2 - V_1 + \frac{\partial \lambda}{\partial t}\right] = 0$$

A necessary and sufficient condition for this to be true is

$$V_2 - V_1 + \frac{\partial \lambda}{\partial t} = V_0 \equiv \text{constant}$$

$$\Rightarrow \quad V_2 = V_1 - \frac{\partial \lambda}{\partial t} + V_0$$

Without any real loss of generality we can choose $V_0 = 0$. Thus

Sufficient conditions for $V_2, \vec{A}_2$ to generate the same electric and magnetic fields as $V_1, \vec{A}_1$ are:

$$V_2 = V_1 - \frac{\partial \lambda}{\partial t}$$

$$\vec{A}_2 = \vec{A}_1 + \vec{\nabla}\lambda$$

for any sufficiently smooth scalar function $\lambda = \lambda(\vec{r}, t)$

The transformation from $V_1, \vec{A}_1$ to $V_2, \vec{A}_2$ is called a gauge transformation and the choice of pair of potentials that results is called a gauge choice

Coulomb gauge

Suppose that we find a potential such that $\vec{\nabla}\cdot\vec{A}=0$. Then the scalar and vector potentials satisfy:

$$\nabla^2 V = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\mu_0\epsilon_0\frac{\partial V}{\partial t} = -\mu_0\vec{J}$$

$$\Rightarrow \quad \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V = -\mu_0\vec{J}$$

We could use the first to solve Poisson's equation. This would give V and from that we could substitute into the second to obtain $\vec{A}$. This is a possible simplification. But is it possible?

Suppose we have $\vec{A}_1$ s.t. $\vec{\nabla}\cdot\vec{A}_1 \neq 0$. We can hope to find λ s.t. $\vec{A}_2 = \vec{A}_1 + \vec{\nabla}\lambda$ will satisfy $\vec{\nabla}\cdot\vec{A}_2 = 0$. This will be possible if

$$\vec{\nabla}\cdot\vec{A}_2 = \vec{\nabla}\cdot\vec{A}_1 + \nabla^2\lambda = 0 \quad \Rightarrow \quad \nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_1$$

The last equation is a differential equation for λ that does generally have a solution. Thus we can always find $\vec{A}$ so that $\vec{\nabla}\cdot\vec{A}=0$.

This is called the Coulomb gauge.

In the Coulomb gauge $\vec{A}$ satisfies

$$\vec{\nabla}\cdot\vec{A}=0$$

and then

$$\nabla^2 V = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V = -\mu_0\vec{J}$$

1 Coulomb gauge

Suppose that

$$V = \frac{x^2 B_0}{2}$$

and

$$\mathbf{A} = -xtB_0\hat{\mathbf{x}} + A_0 \sin(kx - \omega t)\hat{\mathbf{y}}$$

where A_0 and B_0 are constants.

a) Determine the electric and magnetic fields associated with these potentials.

b) Is **A** in the Coulomb gauge?

c) Determine a gauge transformation that transforms into the ~~Lorentz~~ Coulomb gauge. Determine expressions for the potentials in this gauge.

d) Determine the fields using the ~~Lorentz~~ Coulomb gauge potentials.

Answer: a)

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$= -xB_0\hat{x} - \frac{\partial}{\partial t}\left(-xtB_0\hat{x} + A_0\sin(kx-\omega t)\hat{y}\right)$$

$$= \cancel{-xB_0\hat{x}} + \cancel{xB_0\hat{x}} + \omega_0 A_0 \cos(kx-\omega t)\hat{y}$$

$$= \omega_0 A_0 \cos(kx-\omega t)\hat{y}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xtB_0 & A_0\sin(kx-\omega t) & 0 \end{vmatrix} = kA_0\cos(kx-\omega t)\hat{z}$$

This corresponds to an electromagnetic wave traveling along $+\hat{x}$

b) $$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \cancel{\frac{\partial A_y}{\partial y}} = -tB_0 \neq 0$$

c) Need to find $\lambda(\vec{r},t)$ so that

$$\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}$$

$$= -(-tB_0) \Rightarrow \nabla^2\lambda = tB_0$$

$$\Rightarrow \frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2} + \frac{\partial^2\lambda}{\partial z^2} = tB_0$$

There are many possiblities. In all cases

$$V \rightarrow V - \frac{\partial\lambda}{\partial t} = \frac{x^2 B_0}{2} - \frac{\partial\lambda}{\partial t}$$

We can in fact find λ s.t. $\frac{\partial\lambda}{\partial t} = \frac{x^2 B_0}{2} \Rightarrow \lambda = \frac{x^2 B_0 t}{2}$

For this $\nabla^2\lambda = tB_0$ as required. So

$$V = 0$$

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\lambda$$

$$= -xtB_0\hat{x} + A_0\sin(kx-\omega t)\hat{y} + \underbrace{\frac{\partial\lambda}{\partial x}\hat{x}}_{xB_0t\,\hat{x}} = A_0\sin(kx-\omega t)\hat{y}$$

$$\Rightarrow \quad V = 0$$

$$\vec{A} = A_0\sin(kx-\omega t)\hat{y}$$

d) $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{E} = +A_0\omega\cos(kx-\omega t)\hat{y}$

$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{B} = A_0 k\cos(kx-\omega t)\hat{z}$

Lorentz gauge

There is a choice of gauge that completely decouples the two potential equations. Specifically if we can find potentials such that

$$\vec{\nabla}\cdot\vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$$

then the equations for the potentials satisfy

$$\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0 \quad \Rightarrow \quad \nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(0) = -\mu_0 \vec{J} \quad \Rightarrow \quad \nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

These potentials are in the Lorentz gauge. So

> If the potentials satisfy
>
> $$\vec{\nabla}\cdot\vec{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$$
>
> then they are said to be in the Lorentz gauge. These satisfy:
>
> $$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$
>
> $$\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

What if we have a pair of potentials s.t. $\vec{\nabla}\cdot\vec{A} \neq -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$? We can transform

$$\vec{A} \to \vec{A} + \vec{\nabla}\lambda \equiv \vec{A}'$$

$$V \to V - \frac{\partial \lambda}{\partial t} \equiv V'$$

Then we require

$$\vec{\nabla}\cdot\vec{A}' + \mu_0\epsilon_0 \frac{\partial V'}{\partial t} = 0$$

$$\Leftrightarrow \quad \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda + \mu_0\epsilon_0\left(\frac{\partial V}{\partial t} - \frac{\partial^2\lambda}{\partial t^2}\right) = 0$$

$$\Leftrightarrow \quad \nabla^2\lambda - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \underbrace{-\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right)}_{\text{evaluate using original potentials}}$$

We can then solve the second order differential equation for λ. Such a solution fairly generally exists. So we can find a transformation that produces potentials in the Lorentz gauge. Given this fact

$\rho(\vec{r},t)$

charge density $\longrightarrow$ scalar potential via

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

Only charge density needed!

$\vec{J}(\vec{r},t)$

current density $\longrightarrow$ vector potential via

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Only current density needed!

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

Retarded potentials

The equations for the scalar potential and each component of the vector potential in the Lorentz gauge have the form

$$\nabla^2 f - \mu_0 \epsilon_0 \frac{\partial^2 f}{\partial t^2} = g(\vec{r},t)$$

where $f = V(\vec{r},t)$ or $A_x(\vec{r},t), \ldots$ Note that $\mu_0 \epsilon_0 = \frac{1}{c^2}$. These give equations of the form

$$\boxed{\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = g}$$

We know how to solve such equations when $g = 0$. In that case we get wavelike solutions. We aim for a more general form of solution

Conceptually we have

$$\underbrace{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]}_{\text{operator}} f = g$$

This is reminiscent of a vector like equation

$$\underbrace{M}_{\text{matrix}} \underbrace{\vec{u}}_{\cancel{\text{given}}\ \text{wanted}} = \underbrace{\vec{v}}_{\cancel{\text{wanted}}\ \text{given}} \quad \Rightarrow \quad \underbrace{\vec{u}}_{\text{wanted}} = \underbrace{M^{-1}}_{\text{inverse matrix}} \underbrace{\vec{v}}_{\text{given}}$$

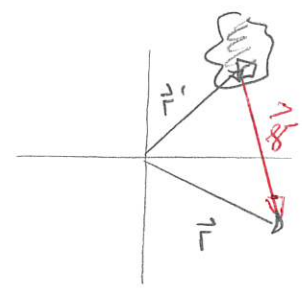
We need to find the "inverse of the operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$" and apply it to g. We expect some type of integral

Consider the analogous simpler version of this situation for electrostatics. Here

$$\underbrace{\nabla^2}_{\text{operator}} \underbrace{V}_{\text{wanted}} = \underbrace{-\rho/\epsilon_0}_{\text{given}}$$

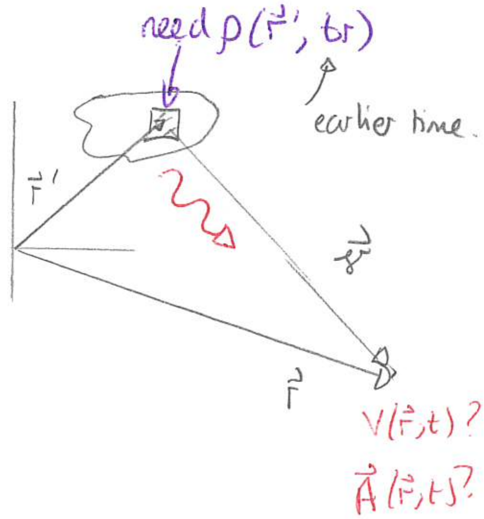
has solution

$$V(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\overbrace{\rho(\vec{r}\,')}^{\text{given.}}}{\mathscr{r}} \, d\tau'}_{\text{inverse operator}}$$



The situation here is similar with the only difference being that the operator involves derivatives w.r.t time as well as position. How do we incorporate time?

On physical grounds we might expect that at a fixed field point $\vec{r}$ the contribution from $\vec{r}\,'$ will take time to travel the distance $\mathscr{r}$. Therefore at time t the contribute to a potential from sources at $\vec{r}\,'$ would reflect the distribution at an earlier time $t_r < t$. This is called the retarded time. So



Need $V(\vec{r},t)$ requires $\rho(\vec{r}\,', t_r)$ at an earlier retarded time $t_r < t$. This would give a signal time to travel from the source to the field point.

A general result is:

A solution to

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r},t)$$

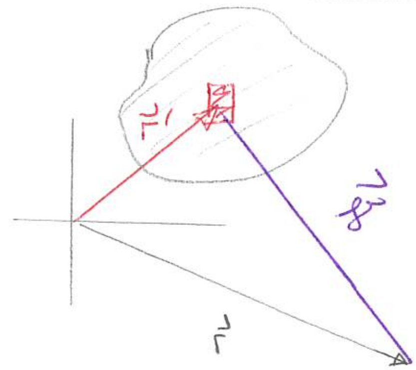
is

$$f = -\frac{1}{4\pi}\int \frac{g(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

where the retarded time is

$$t_r = t - \mathscr{r}/v$$

and $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$



Proof: Consider the possible solution and differentiate w.r.t. unprimed co-ordinates:

$$\vec{\nabla} f = -\frac{1}{4\pi}\int \vec{\nabla}\left(\frac{g(\vec{r}\,',t_r)}{\mathscr{r}}\right) d\tau'$$

$$= -\frac{1}{4\pi}\int\left[g(\vec{r}\,',t_r)\,\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\vec{\nabla} g(\vec{r}\,',t_r)\right] d\tau'$$

depends on $\vec{r}$ via $\vec{\mathscr{r}}$

Now $\vec{\nabla} g(\vec{r}\,',t_r) = \frac{\partial g}{\partial t}\frac{\partial t_r}{\partial x}\hat{x} + \frac{\partial g}{\partial t}\frac{\partial t_r}{\partial y}\hat{y} + \frac{\partial g}{\partial t}\frac{\partial t_r}{\partial z}\hat{z}$

unprimed

$$= \frac{\partial g}{\partial t}\vec{\nabla} t_r$$

Now

$$\nabla^2 f = -\frac{1}{4\pi}\int\left[\vec{\nabla}g\cdot\vec{\nabla}\frac{1}{\mathscr{r}} + g\,\nabla^2\left(\frac{1}{\mathscr{r}}\right) + \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\cdot\vec{\nabla}g + \frac{1}{\mathscr{r}}\nabla^2 g\right]d\tau$$

$$= -\frac{1}{4\pi}\int\left[g\,\nabla^2\left(\frac{1}{\mathscr{r}}\right) + 2\vec{\nabla}g\cdot\vec{\nabla}\frac{1}{\mathscr{r}} + \frac{1}{\mathscr{r}}\nabla^2 g\right]d\tau$$

Then $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}\cdot\left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$

$$= -4\pi\,\delta^3(\vec{\mathscr{r}})$$

$$\Rightarrow \nabla^2 f = \int g(\vec{r}',t_r)\,\delta^3(\vec{\mathscr{r}})\,d\tau' - \frac{1}{4\pi}\int\left[2\vec{\nabla}g\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 g\right]d\tau$$

$$= g(\vec{r},t) - \frac{1}{4\pi}\int\left[2\vec{\nabla}g\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 g\right]d\tau$$

$$= g(\vec{r},t) - \frac{1}{4\pi}\int\left[2\frac{\partial g}{\partial t}\vec{\nabla}t_r\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 g\right]d\tau.$$

But $\nabla^2 g = \vec{\nabla}\cdot\left(\frac{\partial g}{\partial t}\vec{\nabla}t_r\right)$

$$= \vec{\nabla}\left(\frac{\partial g}{\partial t}\right)\cdot\vec{\nabla}t_r + \frac{\partial g}{\partial t}\nabla^2 t_r = \frac{\partial^2 g}{\partial t^2}\vec{\nabla}t_r\cdot\vec{\nabla}t_r + \frac{\partial g}{\partial t}\nabla^2 t_r$$

$$\Rightarrow \nabla^2 f = g(\vec{r},t) - \frac{1}{4\pi}\int\left\{\frac{\partial g}{\partial t}\left[2\vec{\nabla}t_r\cdot\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 t_r\right]\right.$$
$$\left. + \frac{1}{\mathscr{r}}\frac{\partial^2 g}{\partial t^2}\left(\vec{\nabla}t_r\cdot\vec{\nabla}t_r\right)\right]d\tau'$$

Then $\frac{\partial f}{\partial t} = -\frac{1}{4\pi}\int\frac{1}{\mathscr{r}}\frac{\partial g}{\partial t}\frac{\partial t_r}{\partial t}\,d\tau'$

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4\pi}\int\frac{1}{\mathscr{r}}\left[\frac{\partial^2 g}{\partial t^2}\left(\frac{\partial t_r}{\partial t}\right)^2 + \frac{\partial g}{\partial t}\frac{\partial^2 t_r}{\partial t^2}\right]d\tau'$$

$$\Rightarrow \nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r},t) - \frac{1}{4\pi}\int \frac{\partial^2 g}{\partial t^2}\left[\frac{1}{\mathscr{r}}\vec{\nabla} t_r \cdot \vec{\nabla} t_r - \frac{1}{\mathscr{r}}\frac{1}{v^2}\left(\frac{\partial t_r}{\partial t}\right)^2\right] d\tau'$$

$$- \frac{1}{4\pi}\int \frac{\partial g}{\partial t}\left[2\vec{\nabla} t_r \cdot \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 t_r - \frac{1}{\mathscr{r}}\frac{1}{v^2}\frac{\partial^2 t_r}{\partial t^2}\right] d\tau$$

$$= g(\vec{r},t) - \frac{1}{4\pi}\int \frac{\partial^2 g}{\partial t^2}\frac{1}{\mathscr{r}}\left[\vec{\nabla} t_r \cdot \vec{\nabla} t_r - \frac{1}{v^2}\left(\frac{\partial t_r}{\partial t}\right)^2\right] d\tau'$$

$$- \frac{1}{4\pi}\int \frac{\partial g}{\partial t}\left[2\vec{\nabla} t_r \cdot \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) + \frac{1}{\mathscr{r}}\nabla^2 t_r - \frac{1}{\mathscr{r}}\frac{1}{v^2}\frac{\partial^2 t_r}{\partial t^2}\right] d\tau'$$

Then we can see that for

$$t_r = t - \frac{\mathscr{r}}{v} \Rightarrow \frac{\partial t_r}{\partial t} = 1$$

$$\Rightarrow \frac{\partial^2 t_r}{\partial t^2} = 0$$

$$\vec{\nabla} t_r = -\frac{1}{v}\vec{\nabla}(\mathscr{r}) = -\frac{1}{v}\hat{\mathscr{r}}$$

$$\nabla^2 t_r = -\frac{1}{v}(\vec{\nabla}\cdot\hat{\mathscr{r}}) = -\frac{1}{v}\frac{2}{\mathscr{r}}$$

Note if we have

$$t_r = t + h(\mathscr{r})$$

we get as required

$$\vec{\nabla} h \cdot \vec{\nabla} h = \frac{1}{v^2}$$

and:

$$2\vec{\nabla} h \cdot \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) + \frac{1}{\mathscr{r}}\nabla^2 h = 0$$

$$\Rightarrow \vec{\nabla} h \cdot \hat{\mathscr{r}} = \mathscr{r}\nabla^2 h$$

Substitution gives

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r},t) - \frac{1}{4\pi}\int \frac{\partial^2 g}{\partial t^2}\frac{1}{\mathscr{r}}\overbrace{\cancel{\left[\frac{1}{v^2} - \frac{1}{v^2}\right]}}^{0} d\tau'$$

$$- \frac{1}{4\pi}\int \frac{\partial g}{\partial t}\overbrace{\left[\cancel{-\frac{2}{v}\hat{\mathscr{r}}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \frac{2}{v\mathscr{r}^2}} - 0\right]}^{0} d\tau'$$

$$\Rightarrow \nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r},t)$$

as required. ☒

Thus we get

The potentials at location $\vec{r}$ at time t are:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ and the retarded time is

$$t_r = t - \frac{\mathscr{r}}{c}$$

So we see that the integral is over the charge density at times that are earlier by $\mathscr{r}/c$

Note that the integration variable $\mathscr{r}'$ appears in

$\vec{r}\,' \equiv$ spatial argument

$\mathscr{r} \equiv$ denominator

$t_r =$ time argument