Ch. 4 Quantum Mechanics in Three Dimensions

4-10-19

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi \quad : \quad \hat{H} = \frac{1}{2}m|\vec{v}|^2 + V(|\vec{r}|) = \frac{|\vec{P}|^2}{2m} + V(|\vec{r}|) = \left(\frac{\hat{P}_x^2}{2m} + \frac{\hat{P}_y^2}{2m} + \frac{P_z^2}{2m} + V(|\vec{r}|)\right)$$

$$\Psi_n(r,t) = \Psi(x,y,z,t) = \Psi(r,\theta,\varphi,t) \qquad \hat{P}_x = -i\hbar\frac{d}{dx} \quad \hat{P}_y = -i\hbar\frac{d}{dy} \quad \hat{P}_z = -i\hbar\frac{d}{dz}$$

$$[r_i, P_j] = i\hbar\delta_{ij} \quad : \quad [r_i, r_j] = 0 \quad : \quad [P_i, P_j] = 0$$

$$\Psi(x,y,z) = X(x)Y(y)Z(z) \quad : \quad \vec{P} = -i\hbar\vec{\nabla}$$

I.S.W Cube $V=0$ $\quad x,y,z \in [0,a]$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) = E\Psi \quad : \quad \Psi = XYZ = X(x)Y(y)Z(z)$$

$$-\frac{\hbar^2}{2m}\left(YZX'' + XZY'' + XYZ''\right) = E\cdot XYZ \quad : \quad -\frac{\hbar^2}{2m}\left[\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right] = E$$

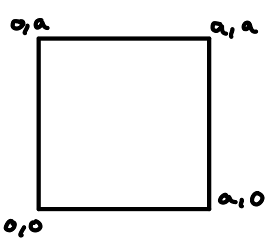
$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{-2mE}{\hbar^2} = -|\vec{k}|^2 \quad : \quad -\frac{2m}{\hbar^2}E = -\frac{2m}{\hbar^2}E_x - \frac{2m}{\hbar^2}E_y - \frac{2m}{\hbar^2}E_z$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k_x^2 - k_y^2 - k_z^2 \quad : \quad \frac{X''}{X} = -k_x^2 \rightarrow X'' + k_x^2 X = 0 \rightarrow A\cos(kx) + B\sin(kx)$$

$X(x) = A\cos(kx) + B\sin(kx)$ same for Y & Z

$Y(y) = C\cos(kx) + D\sin(kx)$

$Z(z) = E\cos(kx) + F\sin(kx)$



$X(0) = A + 0 = 0 \quad A = 0$

$X(a) = B\sin(ka) = 0$

$\therefore X(x) = B\sin(k_x x) \rightarrow k_x = \frac{n\pi}{a} \qquad B = (2/a)^{1/2}$

$Y(y) = D\sin(k_y y) \rightarrow k_y = \frac{n\pi}{a} \qquad D = (2/a)^{1/2}$

$Z(z) = F\sin(k_z z) \rightarrow k_z = \frac{n\pi}{a} \qquad F = (2/a)^{1/2}$

$$\Psi(x,y,z) = \left(\frac{2}{a}\right)^{3/2}\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

$$E = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} = \frac{1}{2m}\left(\frac{\pi\hbar^2}{a}\right)(n_x^2 + n_y^2 + n_z^2)$$

$n_x = n_y = n_z \quad E_1 = \frac{3}{2m}\left(\frac{\pi\hbar^2}{a}\right) \quad [(112), (121), (211)]$

$E_2 = \frac{6}{2m}\left(\frac{\pi\hbar^2}{a}\right) \quad [(221), (212), (122)]$

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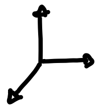
$$\Psi(x,y,z) = \left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{a}\right)\sin\left(\frac{n_z\pi z}{a}\right) \quad : \quad E = \frac{\hbar^2|\vec{k}|^2}{2m} = \frac{\hbar^2}{2m}(n_x^2 + n_y^2 + n_z^2)\frac{\pi^2}{a^2}$$

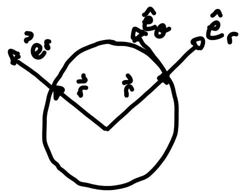
The Hydrogen atom

3D orthogonal coordinate system



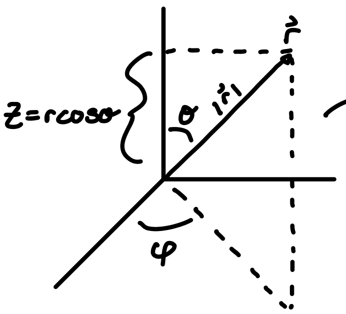
Curvilinea coordinate systems

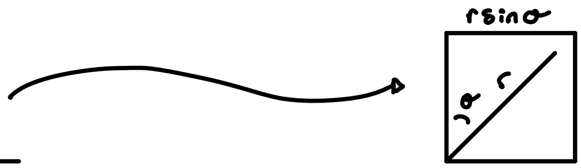
$$d\vec{s} = h_1 dx_1 \hat{e}_1 + h_2 dx_2 \hat{e}_2 + h_3 dx_3 \hat{e}_3$$



$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \qquad h_1 = h_2 = h_3 = 1$$
$$d\vec{s} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + dz\,\hat{e}_z$$
$$d\vec{s} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\varphi\,\hat{e}_\varphi$$

$$\vec{V}(|\vec{r}|) = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = \frac{-e^2}{4\pi\varepsilon_0 r}$$





$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\left[\frac{\partial}{\partial x} = \frac{\partial}{r}\frac{\partial r}{\partial x} + \frac{\partial}{\theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\varphi}\frac{\partial \varphi}{\partial x}\right]$$

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2 = ? \longrightarrow \nabla^2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}\frac{h_2 h_3}{h_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{h_1 h_3}{h_2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{h_1 h_2}{h_3}\frac{\partial}{\partial x_3}\right)$$

$$h_1 = 1,\quad h_2 = r,\quad h_3 = r\sin\theta$$

$$\nabla^2 = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(r^2\sin\theta)\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2mr^2\sin\theta}\left(\sin\theta\frac{\partial}{\partial r}(r^2)\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$= \frac{-\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2)\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

$$\hat{H}\psi = E\psi \;:\; \frac{-\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] + V(r)\psi = E\psi$$

$$\psi(r,\theta,\varphi) = R(r)\,Y(\theta,\varphi)$$

$$-\frac{\hbar^2}{2m}\left(\frac{Y}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta} + \frac{R}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right) + RYV = RYE \qquad \times\frac{1}{RY}$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{R}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \frac{1}{Y}\frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta} + \frac{1}{Y}\frac{1}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right) + V - E = 0 \qquad \times\frac{-2mr^2}{\hbar^2}$$

$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial Y}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]+\left[\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r}-\frac{2mr^2}{\hbar^2}(V-E)\right]=0$$

$$F(\theta,\varphi)=-\ell(\ell+1) \qquad F(r)=\ell(\ell+1)$$

(r,θ,φ,t) : $\varphi(t)=e^{-iEt/\hbar}$

$\psi=\sum C_{n,\ell,m}R_n(r)Y_{\ell,m}(\theta,\varphi)e^{\frac{-iE_{\ell,m,n}t}{\hbar}}$

$Y_{\ell,m}(\theta,\varphi)$

$Y_{\ell,m}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$

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$$\frac{1}{Y(\theta,\varphi)}\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial Y}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right]=-\ell(\ell+1)$$

$$Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$$

$$\underbrace{\frac{1}{\Theta}\left[\sin(\theta)\frac{d}{d\theta}\sin(\theta)\frac{d\Theta}{d\theta}+\ell(\ell+1)\sin^2(\theta)\right]}_{m^2}+\underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{-m^2}=0$$

$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m^2 \longrightarrow$ easy

$\Theta(\theta)=AP_\ell^m(\theta) \longrightarrow$ Associated Legendre polynomials

$P_\ell^m(x)=(1-x^2)^{|m|/2}\frac{d^m}{dx}P_\ell(x)$: $P_\ell(x)=\frac{1}{2^\ell\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell$: $x=\cos(\theta)$: $m\le|\ell|$, $\ell<n$

$\ell>n$

$Y_\ell^m(\theta,\varphi)=P_\ell^m(x)e^{-im\varphi}$

1 2 3 4 5
S P D F G

$n=1$ $\ell=0$ $m=0$

$P_0(x)=\frac{1}{2^0 0!}\left(\frac{d}{dx}\right)^0(x^2-1)^0=1$: $P_\ell^m(x)=(1-x^2)^0\left(\frac{d}{dx}\right)^0(1)=1$

$\psi_{n\ell m}=R_{n\ell}(r)Y_\ell^m(\varphi,\theta)e^{-iE_nt/\hbar}$

$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}(R_{n\ell}(r))^2(Y_\ell^m(\varphi,\theta))^2r^2\sin\theta\,dr\,d\theta\,d\varphi$: $\int_0^\infty|R|^2r^2dr\left[\int_0^{2\pi}\int_0^{\pi}|Y|^2\sin\theta\,d\theta\,d\varphi\right]$: Y_0^0

$\int_0^{2\pi}\int_0^{\pi}A^2\sin\theta\,d\theta\,d\varphi=1$: $4\pi A^2=1$ $\therefore$ $A=\frac{1}{\sqrt{4\pi}}$

$n\ell m\ell$

$n=2$: 200

2P : $21\text{-}1$, 210, 211

$$P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x \quad : \quad P_1(x) = x = \cos\theta$$

$$P_1^0(x) = (1-x^2)^0 \frac{d^0}{dx} P_1(x) = x \; : \; P_1^0(x) = x = \cos\theta \; : \; Y_1^0 = \cos\theta$$

$$P_1^1(x) = (1-x^2)^{\frac{1}{2}}\frac{d}{dx}(x) = (1-x^2)^{\frac{1}{2}} \; : \; P_1^1(x) = \sqrt{1-x^2} = \sin\theta \; : \; Y_1^1 = \sin\theta \; : \; Y_1^{-1} = \sin\theta e^{i\varphi}$$

$$P_2(x) = \frac{1}{2^2 2!}\left(\frac{d}{dx}\right)^2 (x^2-1)^2 = \frac{1}{8}\left(\frac{d^2}{dx^2}\right)(x^4-2x^2+1) = \frac{1}{8}(12x^2-4) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_2^2(x) = (1-x^2)^1 \frac{d^2}{dx}\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) = (1-x^2)(3) = 3\cdot\sin^2\theta \; : \; Y_2^2 = 3\sin^2\theta e^{-2i\varphi}$$

$$\Psi(r,t) = \sum C_{n\ell m} R_{n\ell}(r) Y_\ell^m(\theta,\varphi) e^{-iE_n t/\hbar}$$

$$R_{n\ell}(r) : \frac{1}{R}\left(\frac{d}{dr} r^2 \frac{dR}{dr} - \frac{2mr^2}{\hbar^2}(V-E)R\right) = \ell(\ell+1)$$

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell'}^{m'} Y_\ell^m \sin\theta\, d\theta\, d\varphi = \delta_{\ell'\ell}\delta_{m'm}$$

$V=0$: Infinite spherical well

$$\frac{1}{R^2}\left(\frac{d}{dr} r^2 \frac{dR}{dr} + \frac{2mr^2}{\hbar^2} E R\right) = \ell(\ell+1)$$

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$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \qquad Y_1^1 = -\left(\frac{3}{8\pi}\right)^{1/2}\cos\theta$$

$$\langle Y_1^0 | Y_1^1 \rangle = -\frac{3}{\pi}\frac{1}{\sqrt{32}}\int_0^{2\pi} e^{i\varphi} d\varphi \int_0^{\pi} \cos\theta(\sin^2\theta)\, d\theta = 0$$

$$Y_1^1 = -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{i\varphi}$$

$$\langle Y_1^1 | Y_1^1 \rangle = \frac{3}{8\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{3}{4}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{3}{4}\cdot\frac{4}{3} = 1$$

$\langle Y_{l'm'} | Y_{lm} \rangle = \delta_{l'l}\delta_{m'm}$: $e^{-iE_n t/\hbar}$, $Y_l^m(\theta,\varphi)$, $R_{nl}(r) = ?$, $u(r) = rR(r)$, $m \rightarrow \mu = \frac{m_1 m_2}{m_1+m_2}$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}(V(r)-E)R = l(l+1)\,R$$

$$ur^{-1} = R$$

$$\frac{d}{dr}(ur^{-1}) = \frac{dR}{dr} = \frac{u'}{r} - \frac{u}{r^2} \quad : \quad \frac{d}{dr} r^2\left[\frac{u'}{r} - \frac{u}{r^2}\right] = \frac{d}{dr}\left(u'r - u\right) = u''r + u' - u' = u''r$$

$$ru'' - \frac{2mr^2}{\hbar^2}(V(r)-E)\frac{u}{r} = l(l+1)\frac{u}{r} \quad : \quad u'' - \frac{2m}{\hbar^2}(V-E)u = l(l+1)\frac{u}{r^2}$$

$$-\frac{\hbar^2}{2m}u'' + \underbrace{(V(r) - E)}_{A}u = \underbrace{-\frac{\hbar^2}{2m}l(l+1)}_{B}\frac{u}{r^2}$$

$$-\frac{\hbar^2}{2m}u'' + \left[V + \frac{\hbar^2}{2mr^2}l(l+2)\right]u = Eu$$

Allows for larger orbits

Ultimately Coulomb Potential

$$-\frac{\hbar^2}{2m}u'' + V_{eff}\,u = Eu$$

$$V(r) = \begin{cases} 0 & r < a \\ \infty & r \geq \infty \end{cases}$$

$$u'' = \left(\frac{l(l+1)}{r^2} - k^2\right)u \qquad k = \frac{\sqrt{2mE}}{\hbar} \longrightarrow E > 0$$

$$l=0 : \quad u'' = -k^2 u \quad : \quad u = Ae^{ikr} + Be^{-ikr} \text{ or } C\cdot\cos(kr) + D\cdot\sin(kr)$$

$$\psi(r,\theta,\varphi) = \left(\frac{Ae^{ikr}}{r} + \frac{Be^{-ikr}}{r}\right)Y_{lm}(\theta,\varphi)e^{-iE_n t/\hbar}$$

$$R(r) = \cancel{\frac{C\cdot\cos(kr)}{r}}^{\infty} + \frac{D\sin(kr)}{r}$$

$$R(0)\,? \quad \frac{dR(r)}{dr} = \left.\frac{D\cdot k\cos(kr)}{1}\right|_{r=0} = D\cdot k$$

$$\frac{D\cdot\sin(ka)}{a} = 0 \qquad ka = n\pi \quad : \quad \hbar k = \frac{\hbar n\pi}{a} \quad : \quad E = \left(\frac{\hbar n\pi}{a}\right)^2 \cdot \frac{1}{2m}$$

$$R_{nl=0} = \frac{D\sin(kr)}{r} \longrightarrow \underset{\text{SISW}}{\psi_{100}} = \frac{\sin\left(\frac{n\pi}{a}r\right)}{r}\cdot\frac{1}{\sqrt{2\pi a}}$$

$$\psi_{100} = \frac{1}{\sqrt{2\pi a}}\,\frac{\sin\left(\frac{n\pi}{a}r\right)}{r}\,e^{-i\frac{n^2\pi^2\hbar}{2ma^2}t}$$

$l \neq 0 \;:\; u(r) = A r j_l(kr) + B r \eta_l(kr)$

Spherical bessel $\;\;$ neumann

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin(x)}{x} \;:\; \eta_l(x) = -(-x)^l \left(\frac{1}{x}\frac{1}{dx}\right)^l \frac{\cos(x)}{x}$$

$$\eta_0 = (-x)^0 \left(\frac{1}{x}\frac{d}{dx}\right)^0 \frac{\cos(x)}{x} \;:\; \eta_1 = (-x)^1 \left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{\cos(x)}{x}\right] = -1\left(-x^{-2}\cos(x) - x^{-1}\sin(x)\right)$$

$\therefore$ neumann functions are divergent @ 0 and no good

$$R_{nl}(r) = A \cdot j_l(kr) \;:\; \psi_{nlm} = A_{nl}\, j_l\left(\frac{B_{nl} r}{a}\right) Y_l^m(\theta, \varphi)$$

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$$-\frac{\hbar^2}{2m} u''(r) + \left[\frac{-e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r} + \frac{\hbar^2}{2m}\cdot\frac{l(l+1)}{r^2}\right] u(r) = E(r) \;:\; K = \frac{\sqrt{-2mE}}{\hbar} = \frac{\sqrt{2m|E|}}{\hbar}$$

$$E = \frac{-\hbar^2 K^2}{2m}$$

$$\frac{-\hbar^2}{2mE} u''(r) + \left[\frac{-e^2}{4\pi\varepsilon_0 E}\cdot\frac{1}{r} + \frac{\hbar^2}{2mE}\frac{l(l+1)}{r^2}\right] u(r) = u(r)$$

$$\frac{u''}{k^2} + \left[\frac{-e^2}{4\pi\varepsilon_0 E}\cdot\frac{1}{r} - \frac{l(l+1)}{(kr)^2}\right] u = u \;:\; \frac{u''}{k^2} = \left[1 + \frac{e^2}{4\pi\varepsilon_0 E r} + \frac{l(l+1)}{(kr)^2}\right] u$$

$$\frac{u''}{k^2} = \left[1 - \frac{m \cdot e^2}{2\pi\varepsilon_0 \hbar^2 k}\cdot\frac{1}{(kr)} + \frac{l(l+1)}{(kr)^2}\right] u \;:\; \rho_0 = \frac{me^2}{2\pi\varepsilon_0 \hbar^2 k} \;:\; \rho = kr$$

$$\frac{u''}{k^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u \;:\; u = \sum c_n \rho^{n+\lambda} \;:\; \frac{du}{dr} = \frac{du}{d\rho}\frac{d\rho}{dr} = k\frac{du}{d\rho}$$

$\rho = \infty$

$$\frac{d^2u}{dr^2} = k^2 \frac{d^2u}{d\rho^2}$$

Let $\;u(\rho) = F(\rho)\, G(\rho)\, V(\rho)$ $\quad$ Let $\rho = kr$

$\rho = 0$

$$F'' - F = 0 \;:\; F'' = F \;:\; F = Ae^{-\rho} + \cancel{Be^{\rho}} \to 0$$

$$G \to u'' - \frac{l(l+1)}{\rho^2} u = 0$$

$$u = \rho^a : u' = a\rho^{a-1} : u'' = a(a-1)\rho^{a-2}$$

$$a(a-1)\frac{\rho^a}{\rho^2} - \frac{l(l+1)}{\rho^2}\rho^2 = 0 \;:\; a(a-1) = l(l+1)$$

$$a = l+1 \,,\; a = -l$$

$$G = c\rho^{l+1} + \cancel{D\rho^{-l}} \to 0$$

$$u(\rho) = Ae^{-\rho} c \rho^{l+1} V(\rho)$$

$$RHS = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] e^{-\rho}\rho^{l+1} V(\rho)$$

$$\frac{du}{d\rho} = \frac{d}{d\rho}\left[e^{-\rho}\rho^{\ell+1}v(\rho)\right] = -e^{-\rho}\rho^{\ell+1}v(\rho) + (\ell+1)e^{-\rho}\rho^{\ell}v(\rho) - e^{-\rho}\rho^{\ell+1}v'(\rho)$$

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$$u(r) = rR(r), \quad \rho = \kappa r, \quad u'' = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]u, \quad \rho_0 = \frac{me^2}{2\pi\epsilon_0\hbar^2\kappa}, \quad \kappa = \frac{\sqrt{2m|E|}}{\hbar}$$

$$\frac{du}{d\rho} = \frac{d}{d\rho}\Big[\underset{A}{\rho^{\ell+1}}\,\underset{B}{e^{-\rho}}\,\underset{C}{v(\rho)}\Big] = A'BC + AB'C + ABC'$$

$$\frac{d^2u}{d\rho^2} = \frac{d}{d\rho}\left[A'BC + AB'C + ABC'\right] = A''BC + A'B'C + A'BC' + A'B'C + AB''C + AB'C' + A'BC' + AB'C' + ABC''$$

$$= A''BC + AB''C + ABC'' + 2A'B'C + 2A'BC' + 2AB'C'$$

$$A' = (\ell+1)\rho^{\ell} = (\ell+1)\frac{A}{\rho} \;:\; A'' = \ell(\ell+1)\rho^{\ell-1} = \ell(\ell+1)\frac{A}{\rho^2}$$

$$B' = -e^{-\rho} = -B \;:\; B'' = e^{-\rho} = B$$

$$C' = v'(\rho) \;:\; C'' = v''(\rho)$$

$$= \frac{\ell(\ell+1)}{\rho^2}(ABC) + ABC + ABC'' - \frac{2(\ell+1)}{\rho}ABC + \frac{2(\ell+1)}{\rho}ABC' - 2ABC' = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right]ABC$$

$$\cancel{\frac{\ell(\ell+1)}{\rho^2}} + \cancel{1} - \frac{2(\ell+1)}{\rho} - \cancel{1} + \frac{\rho_0}{\rho} - \cancel{\frac{\ell(\ell+1)}{\rho^2}} = \left[-\frac{2(\ell+1)}{\rho} + \frac{\rho_0}{\rho}\right]C$$

$$\frac{2(\ell+1)}{\rho} - 2 = 2\left[\frac{(\ell+1)}{\rho} - 1\right]C'$$

$$C'': \quad C'' + 2\left[\frac{(\ell+1)}{\rho} - 1\right]C' + \left[-\frac{2(\ell+1)}{\rho} + \frac{\rho_0}{\rho}\right]C = 0 \quad : \quad C \to V$$

$$V'' + 2\left[\frac{(\ell+1)}{\rho} - 1\right]V' + \left[\frac{\rho_0}{\rho} - \frac{2(\ell+1)}{\rho}\right]V = 0$$

$$\rho V'' + 2\left[\ell+1-\rho\right]V' + \left[\rho_0 - 2(\ell+1)\right]V = 0$$

$$V(\rho) = C_0\rho^0 + C_1\rho^1 + C_2\rho^2 + C_3\rho^3 + C_4\rho^4 + \ldots\ldots = \sum_{j=0} C_j\rho^j$$

$$V'(\rho) = C_1\rho^0 + 2C_2\rho^1 + 3C_3\rho^2 + 4C_4\rho^3 = \sum_{j=1} jC_j\rho^{j-1} = \sum_{j=0}(j+1)C_{j+1}\rho^j$$

$$V''(\rho) = 2C_2\rho^0 + 6C_3\rho^1 + 12C_4\rho^2 = \sum_{j=2} j(j-1)C_j\rho^{j-2} = \sum_{j=0}(j+2)(j+1)C_{j+2}\rho^j$$

$$\rho V''(\rho) = 6C_3\rho^0 + 24C_4\rho^1 = \sum_{j=0} j(j+1)C_{j+1}\rho^j$$

$$\rho V'(\rho) = C_1\rho^1 + 2C_2\rho^2 + 3C_3\rho^3 = \sum_{j=0} jC_j\rho^j$$

4-24-19

$u(r) = rR(r)$, $\rho = kr$, $\rho_0 = \frac{me^2}{2\pi\varepsilon_0\hbar^2 k}$, $k = \frac{\sqrt{2m|E|}}{\hbar}$, $u(r) = \rho^{l+1}e^{-\rho}v(\rho)$

$v(\rho) = \sum_{j=0} c_j \rho^j$, $v'(\rho) = \sum_{j=0}(j+1)c_{j+1}\rho^j$, $\rho v'(\rho) = \sum_{j=0} j c_j \rho^j$, $\rho v''(\rho) = \sum_{j=0} j(j+1)c_{j+1}\rho^j$

$$\sum_{j=0}\Big[\overset{\rho v''}{j(j+1)c_{j+1}} + \overset{v'}{2(l+1)(j+1)c_{j+1}} - \overset{\rho v'}{2jc_j} + \overset{v}{[\rho_0 - 2(l+1)]c_j}\Big]\rho^j = 0$$

$$[j(j+1)+2(l+1)]c_{j+1} = [2(j+l+1)-\rho_0]c_j$$

$$\therefore \boxed{c_{j+1} = \frac{[2(j+l+1)-\rho_0]}{[j(j+1)+2(l+1)]}c_j} \; : \; c_0 \longrightarrow \text{define this}$$

For huge c_j: $c_{j+1} = \frac{2}{(j+1)}c_j \approx \frac{2^j}{j!}c_j$

$c_1 = 2\cdot c_0$, $c_2 = 2\cdot c_0$, $c_3 = \frac{2}{3}\cdot c_2 = \frac{4}{3}c_0$, $c_4 = \frac{2}{4}c_3 = \frac{2}{3}c_0$

$c_5 = \frac{2}{5}c_4 = \frac{2}{5}\cdot\frac{2}{3}c_0 = \frac{4}{15}c_0$, $c_6 = \frac{2^5}{6\cdot5\cdot4\cdot3}c_0$, $c_{j+1} = \frac{2^j}{j!}c_0$

$v(\rho) \approx \sum \frac{2^j}{j!}\rho^j$: Very large j, $v(\rho) = c_0 e^{2\rho}$:

There exists a maximum : $2(j_{max}+l+1)-\rho_0 = 0$: $n = j_{max}+l+1$: $2n = \rho_0$

$\frac{\rho_0}{2} = n$: $\frac{me^2}{4\pi\varepsilon_0\hbar^2}\frac{\hbar}{\sqrt{-2mE}} = n$: $\frac{me^2}{4\pi\varepsilon_0\hbar}\cdot\frac{1}{n} = \sqrt{-2mE}$: $\frac{-m^2}{2m\cdot16}\left(\frac{e^2}{\pi\varepsilon_0\hbar}\right)^2\cdot\frac{1}{n^2} = E_n$

$E_n = \frac{-m}{32}\left(\frac{e^2}{\pi\varepsilon_0\hbar}\right)^2\cdot\frac{1}{n^2}$: $\rho_0 = 2n = \frac{me^2}{2\pi\varepsilon_0\hbar^2 k}$, $k = \frac{me^2}{2\pi\varepsilon_0\hbar^2}\cdot\frac{1}{n}$, $k = \frac{1}{a_0 n}$: $a_0 = 5.29\times10^{-11}m$

$\rho = kr = \frac{r}{a_0 n}$

$\psi_{nlm} = Y_l^m(\theta,\varphi)\rho^{l+1}\frac{e^{-\rho}v(\rho)}{r}$: $R_{nl}(r) = \frac{\left(\frac{r}{a_0 n}\right)^{l+1}e^{-r/a_0 n}v(r/a_0 n)}{r}$

ψ_{100} : $\frac{\rho^{0+1}e^{-\rho}v_1(\rho)}{r} = \frac{1}{r}\left(\frac{r}{a_0}e^{-r/a_0}\right)$: $c_1 = \frac{2(0+0+1)}{0(0+1)+2(0+1)}c_0$

4-26-19

$c_{j+1} = \frac{2(j+l+1)-\rho_0}{(j+1)(j+2l+2)}c_j$, $2(j_{max}+l+1)-\rho_0 = 0$

$R_{10} = \frac{1}{\sqrt{4\pi}}\cdot\frac{\rho e^{-\rho}v_1(\rho)}{r} = \frac{1}{\sqrt{4\pi}}\cdot\frac{r}{a}\cdot\frac{1}{r}\cdot e^{-r/a} = \frac{1}{\sqrt{4\pi}}\frac{e^{-r/a}}{a}c_0 = \frac{2}{\sqrt{4\pi}}\frac{e^{-r/a}}{a^{3/2}} = \frac{1}{\sqrt{\pi}}e^{-r/a}a^{-3/2}$

$V(\rho) = C_0 + C_1\rho^1$

$$C_{0+1} = \frac{2(0+0+1-1)}{2} = 0$$

$$\langle \psi_{100} | \psi_{100} \rangle = \frac{C_0^2}{4\pi a^2} \int_0^\infty e^{-2r/a} r^2 dr \quad : \quad \int_0^\infty x^n e^{-x/b} dx = n!\, b^{n+1}$$

$$\frac{2r}{a} = \frac{x}{b} \quad \therefore \quad b = \frac{a}{2} \qquad n=2$$

$$= \frac{C_0^2}{4\pi a^2} \cdot (2)\left(\frac{a}{2}\right)^3 = \frac{C_0^2}{4\pi a^2} \cdot \frac{a^3}{4} = \frac{C_0^2 a}{16\pi} \times 4\pi = \frac{C_0^2 a}{4}$$

From $d\varphi$ & $d\theta$

$$\langle \psi_{100} | \psi_{100} \rangle = \frac{C_0^2 a}{4} \quad \therefore \quad \frac{C_0^2 a}{4} = 1 \quad \therefore \quad C_0 = \frac{2}{\sqrt{a}}$$

$$R_{20} = \frac{\rho^1 e^{-\rho} V(\rho)}{r} = \frac{r}{2a} e^{-r/2a} V(\rho) \quad : \quad V(\rho) = C_0\rho_0 + C_1\rho_1 + C_2\rho_2$$

$$c_1 = \frac{2(0+0+1-2)}{2} C_0 = -C_0 \quad : \quad C_2 = 2(1+0+1-2)$$

$$R_{20} = \left(\frac{r}{2a}\right) e^{-r/2a}(1 - r/2a)\frac{C_0}{r}, \quad R_{21} = \frac{1}{r}\left(\frac{r}{2a}\right)^2 e^{-r/2a}, \quad c_1 = 2(0+1+1-2)$$

$$R_{21} = \frac{1}{r}\left(\frac{r}{2a}\right)^2 e^{-r/2a} C_0$$

$V(\rho)$ = associated Laguerre polynomials

$$= L_{n-l-1}^{2l+1}(2\rho) \quad : \quad L_{q-p}^{p}(x) = (-1)^n \left(\frac{d}{dx}\right)^p L_q(x)$$

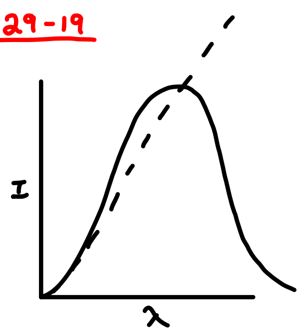
$$2l+1 = p$$
$$q - 2l + 1 = n - l - 1$$
$$q = n - l - 2$$
$$q = n + l - 2$$

$$= L_q(x) = e^x \left(\frac{d}{dx}\right)^q (e^{-x} \cdot x^q)$$

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l \left[L_{n-l-1}^{2l+1}(2r/na)\right] Y_l^m(\theta,\varphi)$$

$$C_{nlm} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^*_{nlm}\, \psi(r,\theta,\varphi,0)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi \quad : \quad \psi(r,\theta,\varphi) = \sum_n \sum_l \sum_m C_{nlm}\psi_{nlm}$$

4-29-19



$$I(\lambda) = \frac{\text{'' ''}}{e^{hc/\lambda kT} - 1} \quad : \quad E_n = \frac{-13.6\text{ eV}}{n^2} \quad : \quad E_u \to E_L \to \Delta E = \frac{-13.6\text{ eV}}{n_L^2} + \frac{(13.6\text{ eV})}{n_u^2}$$

$$|\Delta E| = 13.6\text{ eV}\left[\frac{1}{n_u^2} - \frac{1}{n_l^2}\right]$$

$$|\Delta E| = \frac{hc}{\lambda} = 13.6\text{ ev}\left[\frac{1}{n_\ell^2} - \frac{1}{n_u^2}\right] \quad : \quad \frac{1}{\lambda} = R\left[\frac{1}{n_\ell^2} - \frac{1}{n_u^2}\right] \longrightarrow R = 1.047\times10^7\,m^{-1}$$

5-1-19

$\vec{L} = \vec{r}\times\vec{p}$: $L_x = yP_z - zP_y$: $L_y = zP_x - xP_z$: $L_z = xP_y - yP_x$: $|\vec{L}|, L_z, E_n$

$[r_{ij} P_{ij}] = i\hbar\delta_{ij}$: $[r_i, r_j], [P_i, P_j] = 0$: $[AB, C] = A[B,C] + [A,C]B$

$[L_x, L_y] = [yP_z - zP_y, zP_x - yP_z]$

$[yP_z, \overset{1}{z}P_x] - [yP_z, \overset{2}{x}P_z] - [zP_y, \overset{3}{z}P_x] + [zP_y, \overset{4}{x}P_z]$

$\#1: [yP_z zP_x - zP_x yP_z]$: $\#2: [yP_z xP_z - xP_z yP_z]$: $\#3: [zP_y zP_x - zP_x zP_y]$

$\#4: [zP_y xP_z - xP_z zP_y]$ $\quad yP_x[P_z, z] = -i\hbar$: $xP_y[z, P_z] = i\hbar$

$[L_x, L_y] = yP_x(-i\hbar) + xP_y(i\hbar) = i\hbar[xP_y - yP_x] = i\hbar L_z$: $[L_x, L_z] = i\hbar L_y$: $[L_y, L_z] = i\hbar L_x$

$[|\vec{L}|^2, x] = [\cancel{L_x^2, L_x}]^{0} + [L_y^2, L_x] + [L_z^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y$

$= -i\hbar \cancel{L_y} L_z - i\hbar \cancel{L_z} L_y + i\hbar \cancel{L_z} L_y + i\hbar \cancel{L_y} L_z = 0 \quad \therefore \quad [|\vec{L}|^2, L_{x,y,z}] = 0$

$|\vec{L}|^2 F = \lambda F$: $L_z F = \mu F$: $\hat{L}_\pm = L_x \pm iL_y$: $[L_z, L_\pm] = \pm\hbar L_z$

$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = i\hbar L_y \pm \hbar L_x = \pm\hbar(L_x \pm iL_y)$

$L_z L_\pm = L_\pm L_z \pm \hbar L_\pm$: $[L^2, L_\pm] = 0$: $L^2 L_\pm = L_\pm L^2$

Claim : IF f is an eigenfunction of L^2, L_z So is $L_\pm F$

$\#1: L^2(L_\pm F) = L_\pm L^2 F = L_\pm \lambda F = \lambda L_\pm F$

$\#2: L_z L_\pm F = (L_\pm L_z \pm \hbar L_\pm)F = (L_\pm \mu \pm \hbar L_\pm)F = (\mu \pm \hbar)L_\pm F$

$L^2(L_\pm F) = \lambda(L_\pm F)$: $L_z(L_\pm F) = (\mu \pm \hbar)(L_\pm F)$: $L_z F_T = \rho\hbar F_T$: $L^2 F_T = \lambda F_T$

$m_\ell \le |\ell|$: $L_z F = m_\ell \hbar F$: $(2\ell+1) \longrightarrow \mu_{\ell s}$: Degeneracy from $E_n = 2n^2$ occurs From spin
$E \longrightarrow E_n \pm \mu_B m_\ell \hbar$

5-3-19

$|\vec{L}|F = \lambda F$: $L_z F = \mu F \longrightarrow \mu = m_\ell \hbar$: $L_\pm = L_x \pm iL_y$: $[|\vec{L}|^2, \vec{L}] = 0$: $[L_z, L_\pm] = \pm\hbar L_\pm$

$[L_i, L_j] = i\hbar L_K$: $F_{\ell m} = Y_\ell^m \longrightarrow 2(\ell+1)$: w/o spin degen. $= n^2$: W/ spin degen. $= 2n^2$

$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i\underbrace{(L_xL_y - L_yL_x)}_{i\hbar L_z}$

$\quad = L_x^2 + L_y^2 \mp (i)^2\hbar L_z$: $= L^2 - L_z^2 \mp (i)^2\hbar L_z$

$|\vec{L}|^2 F_T = (\cancel{L_-}L_+ + L_z^2 + \hbar L_z)F_T = \left[(\ell\hbar)(\ell\hbar) + \ell\hbar^2\right]F_T = \lambda F_T = \hbar^2\ell(\ell+1)F_T$

$|\vec{L}|^2 F_b = (\cancel{L_+}L_- + L_z^2 - \hbar L_z)F_b = (\hbar^2\bar{\ell}^2 - \hbar^2\bar{\ell})F_b = \hbar^2\bar{\ell}(\bar{\ell}-1)F_b = \lambda F_b$

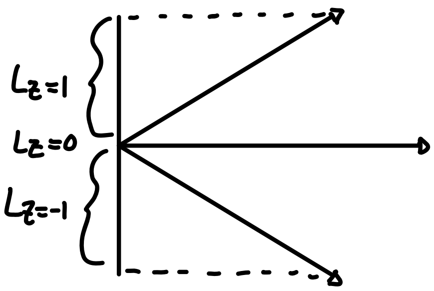
$\ell(\ell+1) = \bar{\ell}(\bar{\ell}-1)$: $\bar{\ell} = -\ell$ or $\bar{\ell} = \ell+1$

$|\vec{L}|^2\psi = \hbar^2\ell(\ell+1)$: $|\vec{L}|\psi = \hbar\sqrt{\ell(\ell+1)}$: $L_z\psi = m_\ell\hbar$: $m_\ell \le |\ell|$

$\ell = 0$: $|\vec{L}| = 0$

$\ell = 1$: $|\vec{L}| = \sqrt{2}\hbar$ $\quad m = -1, 0, 1$

$\ell = 2$: $|\vec{L}| = \sqrt{6}\hbar$ $\quad m = -2, -1, 0, 1, 2$



$L_z = r \times p = (xv_y - yv_x)\hat{k} \longrightarrow r, \theta, \varphi$ components : $\hat{p} = \frac{\hbar}{i}\vec{\nabla}$

$\vec{L} = \frac{\hbar}{i}(\vec{r} \times \vec{\nabla})$: $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$: $\xrightarrow[\vec{r}]{} = |\vec{r}|\hat{r}$

$\vec{L} = \frac{\hbar}{i}\left(r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + \frac{(r\hat{r}\times\hat{\theta})}{r}\frac{\partial}{\partial\theta} + \frac{(r\hat{r}\times\hat{\varphi})}{r\sin\theta}\frac{\partial}{\partial\varphi}\right) = \frac{\hbar}{i}\left(\hat{\varphi}\frac{\partial}{\partial\varphi} - \hat{\theta}\cdot\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$

$\hat{\theta} = \cos\theta\cos\varphi(\hat{i}) + \cos\theta\sin\varphi(\hat{j}) - \sin\theta(\hat{k})$: $\hat{\varphi} = -\sin\varphi(\hat{i}) + \cos\varphi(\hat{j})$

$\vec{L} = \frac{\hbar}{i}\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\cos\theta\frac{\partial}{\partial\varphi}\right)\hat{i} + \frac{\hbar}{i}\left(\cos\varphi\frac{\partial}{\partial\theta} - \sin\varphi\cos\theta\frac{\partial}{\partial\varphi}\right)\hat{j} + \frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi}\right)\hat{k}$

$L_z F_z = m_\ell\hbar F_z$: $\frac{\hbar}{i}\frac{\partial}{\partial\varphi}F_z = m_\ell\hbar F_z$: $\frac{dF}{d\varphi} = i m_\ell F \propto e^{im\varphi}$ $\quad m \le |\ell|$

$\vec{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] = -\frac{r^2}{\hbar^2}\nabla^2_{angular}$

$\hat{H} = \frac{1}{2mr^2}\left[-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + |\vec{L}|^2\right] + V_{eff}$

5-6-19

Spin classically is $I\vec{\omega}$: $|\vec{S}| = \hbar\sqrt{s(s+1)}$: $[S_i, S_j] = i\hbar$

Electron : $|\vec{S}|, s, m_s \longleftrightarrow (3/4)^{1/2}\hbar, \frac{1}{2}, \pm\frac{1}{2}$: $|\psi_s\rangle = |s, m_s\rangle$, $|\frac{1}{2}, \pm\frac{1}{2}\rangle$

$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$: $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$: $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\hat{S}_\pm|s\,m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s\pm1)}\,|s, m_{s\pm1}\rangle$: $\hat{S}_\pm = S_x \pm iS_y$: $S_x = \frac{1}{2}(S_+ + S_-)$: $S_y = \frac{1}{2i}(S_+ - S_-)$

$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ means $s\uparrow, m_s = +\frac{1}{2}$: $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$ means $s\downarrow, m_s = -\frac{1}{2}$

generic $a\chi_+ + b\chi_- = \begin{pmatrix} a \\ b \end{pmatrix}$ w/ $a^2+b^2=1$

$S^2\chi_+ = \frac{3}{4}\hbar^2 \chi_+$: $\longrightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S^2\chi_- = \frac{3}{4}\hbar^2 \chi$: $\longrightarrow \begin{pmatrix} c & d \\ e & f \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = \pm \hbar/2$

$\hat{S}_x$: $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$: $\lambda^2 - 1 = 0$: $(\lambda+1)(\lambda-1) = 0$ $\therefore$ $\lambda = \pm 1 \cdot \frac{\hbar}{2} = \pm\frac{\hbar}{2}$

$\hat{S}_y$: $\begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = 0$: $\lambda^2 - 1 = 0$: $(\lambda+1)(\lambda-1) = 0$ $\therefore$ $\lambda = \pm 1 \cdot \frac{\hbar}{2} = \pm\frac{\hbar}{2}$

$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$: $+\cancel{\frac{\hbar}{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = +\cancel{\frac{\hbar}{2}}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow$ $a_2 = (a_1)$

$-\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow$ $a_1 = -a_2$

$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \pm\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \longrightarrow$ $-ia_2 = \pm a_1$: $ia_1 = \pm a_2$

$a_2 = \pm i a_1$ $\quad$ $a_2 = \pm i a_1$

$\chi_+^{(y)} = \begin{pmatrix} a_1 \\ ia_1 \end{pmatrix}$: $\langle \chi_+^{(y)} | \chi_+^{(y)} \rangle = (a_1 \;\; -ia_1)\begin{pmatrix} a_1 \\ ia_1 \end{pmatrix} = 1$: $2a_1^2 = 1$ $\therefore$ $a_1 = \frac{1}{\sqrt{2}}$

$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

Generic Spinor : $|\chi\rangle = a\chi_+ + b\chi_-$

$|\chi\rangle$ in terms of $\chi_+^{(x)}, \chi_-^{(x)}$: $|\chi\rangle = \sum c_n \chi_n^{(y)}$: $c_n = \langle \psi_n | f(x) \rangle$

$c_n = \langle \chi_n^{(x)} | \chi_n \rangle = \frac{1}{\sqrt{2}}(1 \; 1)\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a+b)$: $c_n = \frac{1}{\sqrt{2}}(a \pm b)$

$|\chi\rangle = \frac{1}{\sqrt{2}}(a+b)\chi_+^{(x)} + \frac{1}{\sqrt{2}}(a-b)\chi_-^{(x)} = \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right)\left[(a+b)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (a-b)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$

$= \frac{1}{2}\left[\begin{pmatrix} a+b \\ a+b \end{pmatrix} + \begin{pmatrix} a-b \\ b-a \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 2a \\ 2b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Measure S_z : get $\uparrow \hbar/2$

$|s\, m_s\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\frac{1}{\sqrt{2}}\left(\chi_+^{(x)} + \chi_-^{(x)}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Measure S_x : get $+\hbar/2$

$$|s\, m_{s}-\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \longrightarrow \begin{pmatrix}-1\\0\end{pmatrix} : \frac{1}{\sqrt{2}}\left(\chi_+^{(x)} - \chi_-^{(x)}\right) = \begin{pmatrix}-1\\0\end{pmatrix}$$

5-8-19

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} : S_y = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} : S_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} : \quad S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\frac{\hbar}{2}\, S_z \rightarrow \begin{pmatrix}1\\0\end{pmatrix} \quad S_x \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad S_y \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\chi = \frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\2\end{pmatrix} = a\chi_+ + b\chi_- \; : \; P(\hbar/2) = a^2 = a^*a = \frac{(1-i)(1+i)}{6} = \frac{2}{6} = \frac{1}{3}$$

$$P(-\hbar/2) = \frac{2}{3}$$

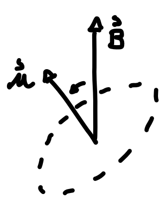
Measure S_z, get $\frac{\hbar}{2}$ $\longrightarrow \chi = \begin{pmatrix}1\\0\end{pmatrix}$

Measure S_x, $P(\hbar/2)$: $c = \langle \chi_+^{(x)} | \chi_+ \rangle = \frac{1}{\sqrt{2}}(1\;1)\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$

$$\langle \chi_-^{(x)} | \chi_+ \rangle = \frac{1}{\sqrt{2}}(1\;-1)\cdot\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\chi = c\chi_+^{(x)} + D\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Measure S_z, $P(-\hbar/2)$: $\chi = \begin{pmatrix}0\\1\end{pmatrix}$



$\vec{B} \longrightarrow$ in $\hat{z}$

$\vec{\mu} = \gamma \vec{S}$ gyromagnetic ratio : $\vec{\tau} = \vec{\mu} \times \vec{B}$ aligns $\vec{\mu}$ to $\vec{B}$

$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{B}\cdot\vec{S}$ Take $\vec{S} = \vec{S}_z$

$$\hat{H}\psi = E\psi \; : \; \vec{B} = B_0\hat{k} \; : \; \hat{H} = -\gamma B_0 \hat{S}_z = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$E_+ = -\frac{\gamma B_0 \hbar}{2}, \begin{pmatrix}1\\0\end{pmatrix} : E_- = \frac{\gamma B_0 \hbar}{2}, \begin{pmatrix}0\\1\end{pmatrix}$$

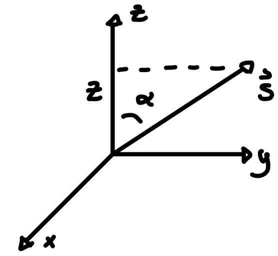
$$\chi = a\chi_+ + b\chi_- \; : \; \chi(t) = a\chi_+ e^{\frac{i\gamma B_0}{2}t} + b\chi_- e^{-\frac{i\gamma B_0}{2}t}$$

$$\chi(t) = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$\Psi(x,t) = \begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix} \sum_n \sum_l \sum_m C_n C_l C_m R_{nl}(r) Y_l^m(\theta,\varphi) e^{-iE_n t/\hbar}$$

$$\langle S_z \rangle = \begin{pmatrix} a e^{-i\gamma B_0 t/2} & b e^{i\gamma B_0 t/2} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a e^{i\gamma B_0 t/2} \\ b e^{-i\gamma B_0 t/2} \end{pmatrix} \quad : \quad a = \cos(\alpha/2), \; b = \sin(\alpha/2)$$

$$\langle S_z \rangle = \frac{\hbar}{2}\cos(\alpha) \longrightarrow$$



$$\langle S_y \rangle = -\frac{\hbar}{2}\sin(\alpha)\sin(\omega t)$$

$$\langle S_x \rangle = \frac{\hbar}{2}\sin(\alpha)\cos(\omega t)$$

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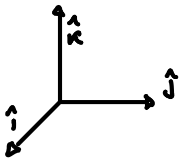
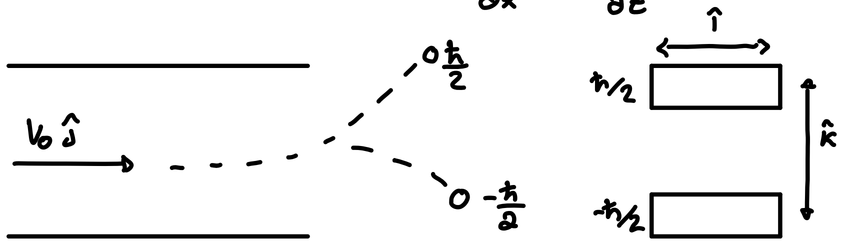


Stern Gerlach Inhomogeneous $\vec{B}$ Field : $\vec{\nabla}\cdot\vec{B} = 0$: $\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = 0$$



$$\vec{B}(x,y,z) = -\alpha x\hat{i} + (B_0 + \alpha z)\hat{k} \quad : \quad \vec{\mu} = \gamma(S_x\hat{i} + S_y\hat{j} + S_z\hat{k})$$

$$\vec{\mu}\cdot\vec{B} = -\alpha\gamma x S_x + \gamma(B_0 + \alpha z)S_z$$

$$\vec{\nabla}(\vec{\mu}\cdot\vec{B}) = -\alpha\gamma S_x\hat{i} + \gamma\alpha S_z\hat{k} \quad : \quad \vec{F}_z = \gamma\alpha S_z\hat{k}$$

$$H(t) = -\gamma(B_0 + \alpha z)S_z$$

$$\chi(t) = a\chi_+(t) + b\chi_-(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$H\chi = E\chi \longrightarrow \text{ or } \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(-\gamma)(B_0 + \alpha z)\begin{pmatrix} a \\ b \end{pmatrix} = E\begin{pmatrix} a \\ b \end{pmatrix}$$

$$E_\pm = \mp\frac{\gamma\hbar}{2}(B_0 + \alpha z)$$

$$\chi(t) = (a\chi_+ e^{i\gamma t B_0/2})e^{i(\alpha\gamma t)z/2} + (b\chi_- e^{-i\gamma t B_0/2})e^{-i(\alpha\gamma t)z/2}$$

$$F = \frac{P}{T} \longrightarrow \vec{P} = \int_0^T \vec{F}\,dt \quad : \quad P_z = F_z t = \pm\alpha\gamma t\frac{\hbar}{2}$$