PHYS 230 HOMEWORK SET 12 SOLUTIONS

1. $m = 0.550\,kg$

a) AS THE MASS IS OSCILLATING, THIS CORRESPONDS TO A LIGHTLY DAMPED OSCILLATOR W/

$$x(t) = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

IF TIME t' CORRESPONDS TO A MAXIMUM AMPLITUDE, THEN

$$x(t') = A_0 e^{-\gamma t'/2} \qquad \text{AS } \cos(\omega t' + \phi) = 1$$

AT TIME $t = t' + 6.00\,s$, WE ALSO HAVE A MAXIMUM AMPLITUDE SO

$$x(t' + 6.00s) = A_0 e^{-\gamma(t' + 6.00s)/2}$$

NOW, TAKING THE RATIO..

$$\frac{x(t' + 6.00s)}{x(t')} = \frac{A_0 e^{-\gamma(t' + 6.00s)/2}}{A_0 e^{-\gamma t'/2}} = e^{-\gamma(6.00s)/2} = \frac{1}{4}$$

SO

$$4 = e^{\gamma(3.00s)}$$

TAKING THE NATURAL LOG YIELDS..

$$\ln(4) = \gamma(3.00s) \quad \text{SO} \quad \gamma = \frac{\ln(4)}{3.00s} = 0.462\,s^{-1}$$

NOW

$$\gamma = \frac{b}{m} \quad \text{SO} \quad b = m\gamma = (0.550\,kg)(0.462\,s^{-1}) = 0.254\,kg/s$$

SO

$$\boxed{\begin{aligned} \gamma &= 0.462\,s^{-1} \\ b &= 0.254\,kg/s \end{aligned}}$$

b) THE ANGULAR FREQUENCY OF OSCILLATION IS...

$$\omega = \sqrt{\omega_0^2 - (\gamma/2)^2} = \sqrt{(9.15\,RAD/s)^2 - \left(\frac{0.462\,s^{-1}}{2}\right)^2} = 9.15\,RAD/s$$

THE SAME TO 3 SIGNIFICANT FIGURES!

c) $Q = \frac{\omega}{\gamma} = \frac{9.15 \text{ RAD/s}}{0.462 s^{-1}} = 19.8$

2. FOR THE LIGHTLY DAMPED OSCILLATOR ...

$$x(t) = A_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

NOW, THE INITIAL CONDITIONS ARE..

1) $x(t=0) = +5.50 \times 10^{-2}\,m$

2) $\dot{x}(t=0) = +0.450\,m/s$

a) NOW APPLYING THE INITIAL CONDITIONS YIELD..

1) $x(t=0) = +5.50 \times 10^{-2}\,m = A_0 \cos\phi$

TAKING A DERIVATIVE..

$$\frac{dx}{dt} = -\frac{\gamma}{2} A_0 e^{-\gamma t/2} \cos(\omega t + \phi) - \omega A_0 e^{-\gamma t/2} \sin(\omega t + \phi)$$

$$= -A_0 e^{-\gamma t/2} \left[\frac{\gamma}{2} \cos(\omega t + \phi) + \omega \sin(\omega t + \phi)\right] \simeq -\omega A_0 e^{-\gamma t/2} \sin(\omega t + \phi)$$

NOW $\frac{\gamma}{2} \ll \omega$ SO

SO

2) $\dot{x}(t=0) = -\omega A_0 \sin\phi = +0.450\,m/s$

IN SUMMARY, WE HAVE...

2) $-\omega A_0 \sin\phi = +0.450\,m/s$

1) $A_0 \cos\phi = 5.50 \times 10^{-2}\,m$

DIVIDING 2) BY 1) YIELDS..

$$-\omega \tan\phi = \frac{+0.450\,m/s}{5.5 \times 10^{-2}\,m} = 8.18\,s^{-1}$$

SO

$$\tan\phi = \frac{8.18\,s^{-1}}{-9.15\,s^{-1}} = -0.894 \quad \text{SO} \quad \phi = -41.8^\circ \times \frac{\pi\,\text{RAD}}{180^\circ} = -0.729\,\text{RAD}$$

PLUGGING THIS INTO 1) YIELDS..

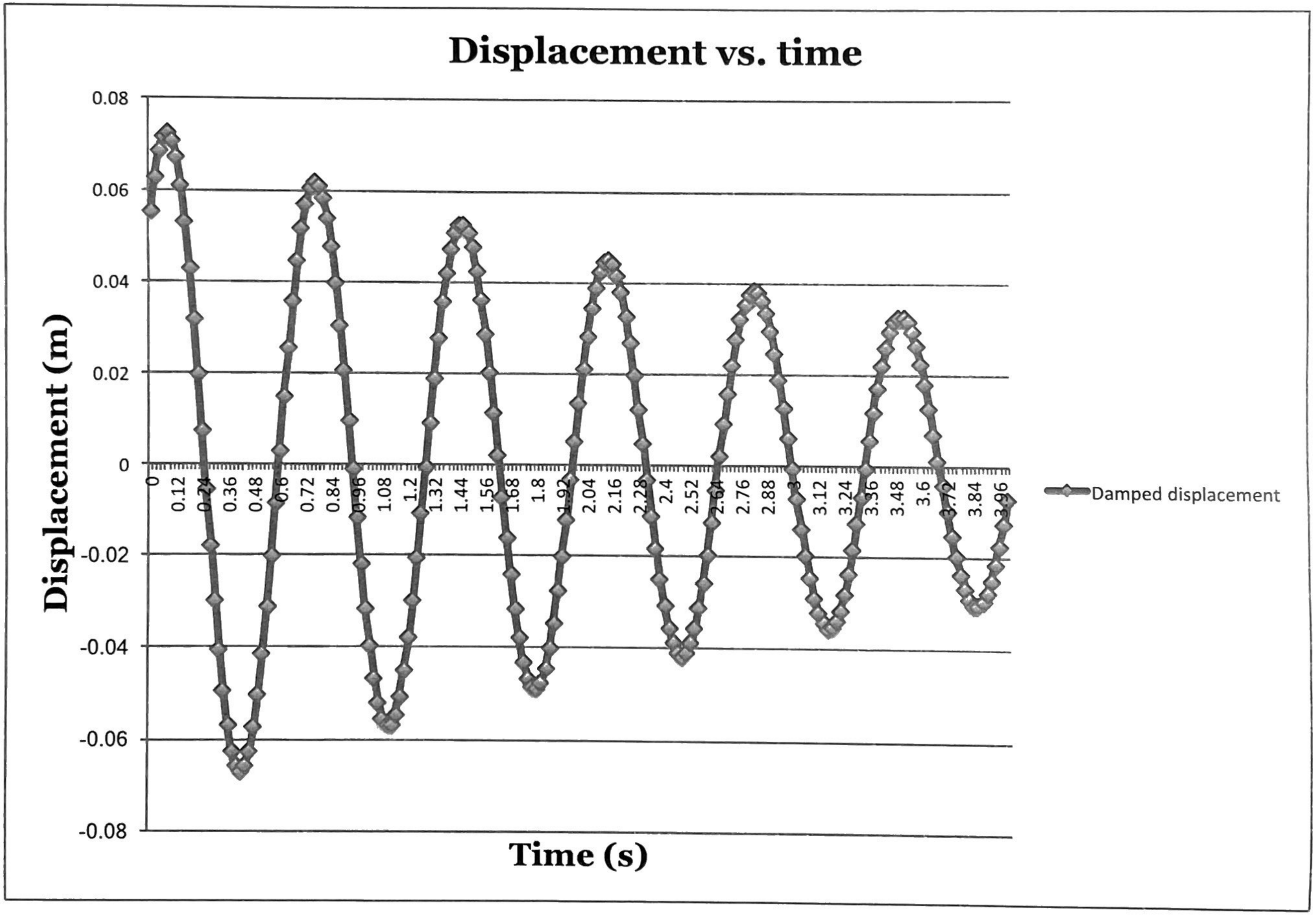
$$A_0 = \frac{5.5\times10^{-2}\,m}{\cos(-41.8^\circ)} = 7.38\times10^{-2}\,m$$

SO IN SUMMARY..

$$\left[\begin{array}{l} A_0 = 7.38\times10^{-2}\,m \\ \omega = 9.15\ \text{rad/s} \\ \gamma = 0.462\ s^{-1} \\ \phi = -41.8^\circ = -0.729\ \text{RAD} \end{array}\right]$$

A_0 =	0.0738	m
ω_0 =	9.15	1/s
γ =	0.462	1/s
ω =	9.15	rad/s
φ =	-0.729	

t(s)	$\gamma t/2$	$\exp(-\gamma t/2)$	$A_0*\cos(\omega t+\varphi)$	$A_0*\exp(-\gamma t/2)*\cos(\omega t+\varphi)$
0	0	1	0.055043058	0.055043058
0.02	0.00462	0.995391	0.063070103	0.062779391
0.04	0.00924	0.990803	0.068990881	0.068356342
0.06	0.01386	0.986236	0.072607665	0.071608264
0.08	0.01848	0.98169	0.073799668	0.072448374
0.1	0.0231	0.977165	0.072527084	0.070870911
0.12	0.02772	0.972661	0.068832411	0.066950579
0.14	0.03234	0.968177	0.062839035	0.06083933
0.16	0.03696	0.963715	0.05474711	0.052760593
0.18	0.04158	0.959273	0.044826869	0.043001187
0.2	0.0462	0.954851	0.033409606	0.031901195
0.22	0.05082	0.95045	0.020876608	0.019842166
0.24	0.05544	0.946069	0.007646422	0.007234041
0.26	0.06006	0.941708	-0.005839122	-0.005498748
0.28	0.06468	0.937367	-0.019129664	-0.017931522
0.3	0.0693	0.933047	-0.031781358	-0.029653492
0.32	0.07392	0.928746	-0.043371694	-0.040281287
0.34	0.07854	0.924465	-0.053513604	-0.049471458
0.36	0.08316	0.920204	-0.061868392	-0.056931536
0.38	0.08778	0.915962	-0.068157046	-0.062429289
0.4	0.0924	0.91174	-0.072169551	-0.065799894


Displacement vs. time
Displacement (m)
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0
0.12
0.24
0.36
0.48
0.6
0.72
0.84
0.96
1.08
1.2
1.32
1.44
1.56
1.68
1.8
1.92
2.04
2.16
2.28
2.4
2.52
2.64
2.76
2.88
3
3.12
3.24
3.36
3.48
3.6
3.72
3.84
3.96
Time (s)
Damped displacement

3. FOR A CRITICALLY DAMPED OSCILLATOR, WE HAVE...

a) $$\omega_0 = \frac{\gamma}{2} \quad \text{SO} \quad \gamma = 2\omega_0 = 2(9.15\,\text{RAD/s}) = 18.3\,s^{-1}$$

ALSO $$\gamma = \frac{b}{m} \quad \text{SO} \quad b = m\gamma = (0.550\,kg)(18.3\,s^{-1}) = 10.1\,\frac{kg}{s}$$

SO $$\boxed{\begin{aligned}\gamma &= 18.3\,s^{-1}\\ b &= 10.1\,kg/s\end{aligned}}$$

b) FOR A CRITICALLY DAMPED OSCILLATOR...

$$x(t) = (A + Bt)e^{-\gamma t/2}$$

NOW, EVALUATING INITIAL CONDITIONS..

1) $$x(t=0) = +5.50\times10^{-2}\,m = A$$

TAKING A TIME DERIVATIVE..

$$\dot{x} = \frac{dx}{dt} = Be^{-\gamma t/2} - \frac{\gamma}{2}(A + Bt)e^{-\gamma t/2} = \left[B - \frac{\gamma}{2}(A+Bt)\right]e^{-\gamma t/2}$$

SO

2) $$\dot{x}(t=0) = +0.450\,m/s = \left[B - \frac{\gamma}{2}A\right] \quad \text{SO} \quad B = \frac{\gamma}{2}A + 0.450\,m/s$$

$$= \left(\frac{18.3\,s^{-1}}{2}\right)(5.50\times10^{-2}\,m) + 0.450\,m/s$$

$$= 0.953\,m/s$$

SO

$$x(t) = (A+Bt)e^{-\gamma t/2}$$ WHERE

$$\boxed{\begin{aligned}A &= 5.50\times10^{-2}\,m\\ B &= 0.953\,m/s\\ \gamma &= 18.3\,s^{-1}\end{aligned}}$$

c) TO GET THE MAXIMUM DISPLACEMENT, SET

$\dot{x}(t') = 0$ AND SOLVE FOR t', THEN EVALUATE $x(t')$

SO

$$\dot{x}(t') = \left[B - \frac{\gamma}{2}(A + Bt')\right]e^{-\gamma t'/2} = 0 \quad \text{WHEN}$$

$$B - \frac{\gamma}{2}(A + Bt') = 0$$

SO

$$\frac{2B}{\gamma} = A + Bt' \quad \text{SO} \quad t' = \frac{2}{\gamma} - \frac{A}{B} = \frac{2}{18.3s^{-1}} - \frac{0.0550\,m}{0.953\,m/s}$$

$$\left[t' = 0.0516s\right]$$

SO

$$x(t') = (A + Bt')e^{-\gamma t'/2}$$

$$= \left[0.0550m + (0.953\tfrac{m}{s})(0.0516s)\right]\exp\left[-\frac{(18.3s^{-1})(0.0516s)}{2}\right]$$

$$\left[x(t') = 0.0650m\right]$$

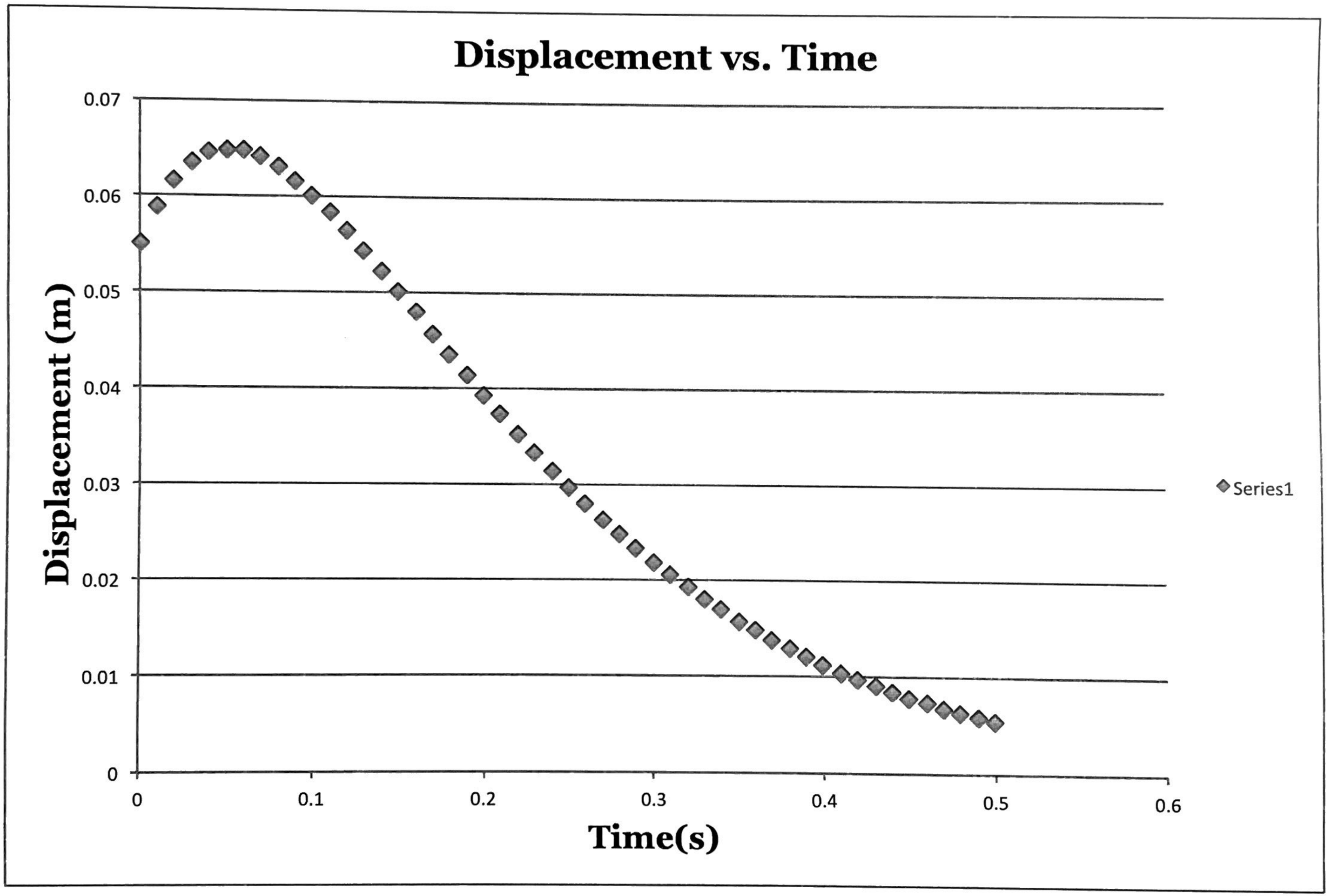
4. FROM MY EXCEL PLOT, I FIND A MAXIMUM DISPLACEMENT OF ...

$$x(t') = 0.0650\,m \quad @ \text{ time}$$

$$t' = 0.05\,s$$

A =	0.055	m
B =	0.953	m/s
γ =	18.3	1/s

t(s)	γt/2	exp(-γt/2)	(A+Bt)	(A+Bt)exp(-γt/2)
0	0	1	0.055	0.055
0.01	0.0915	0.91256132	0.065	0.058887582
0.02	0.183	0.83276816	0.074	0.06167481
0.03	0.2745	0.759952	0.084	0.063524388
0.04	0.366	0.6935028	0.093	0.064578981
0.05	0.4575	0.63286383	0.103	0.064963472
0.06	0.549	0.57752705	0.112	0.064786984
0.07	0.6405	0.52702884	0.122	0.064144681
0.08	0.732	0.48094614	0.131	0.063119371
0.09	0.8235	0.43889284	0.141	0.061782945
0.1	0.915	0.40051663	0.15	0.060197649
0.11	1.0065	0.36549598	0.16	0.058417222
0.12	1.098	0.33353749	0.169	0.05648791
0.13	1.1895	0.30437341	0.179	0.05444936
0.14	1.281	0.2777594	0.188	0.052335427
0.15	1.3725	0.25347249	0.198	0.050174879
0.16	1.464	0.23130919	0.207	0.04799203
0.17	1.5555	0.21108381	0.217	0.045807299
0.18	1.647	0.19262692	0.227	0.043637703
0.19	1.7385	0.17578388	0.236	0.0414973
0.2	1.83	0.16041357	0.246	0.039397572
0.21	1.9215	0.14638722	0.255	0.037347771
0.22	2.013	0.13358731	0.265	0.035355218
0.23	2.1045	0.12190661	0.274	0.033425574


Displacement vs. Time
Displacement (m)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.1
0.2
0.3
0.4
0.5
0.6
Time(s)
Series1

5. IN PROBLEM 3, WE FOUND A DAMPING PARAMETER OF

$$\gamma_3 = 18.3\,s^{-1} \quad \text{SO HERE}$$

$$\gamma = 3(18.3\,s^{-1}) = 54.9\,s^{-1}$$

W/ INITIAL CONDITIONS

$$x(t=0) = +0.0550\,m$$

$$\dot{x}(t=0) = +0.450\,m/s$$

a) NOW, FOR A HEAVILY DAMPED OSCILLATOR...

$$x(t) = e^{-\gamma t/2}\left[Ae^{\alpha t} + Be^{-\alpha t}\right] \quad \text{WHERE} \quad \alpha \equiv \sqrt{(\gamma/2)^2 - \omega_0^2} \qquad \text{WHERE} \quad \omega_0 = 9.15\,\text{RAD}/s$$

SO

$$\alpha = \sqrt{\left(54.9\,s^{-1}/2\right)^2 - (9.15\,\text{RAD}/s)^2} = 25.9\,s^{-1}$$

NOW

$$x(t=0) = A + B = +0.0550\,m$$

TO OBTAIN THE 2ND INITIAL CONDITIONS, FIND $v(t) = \dot{x}(t)$..

$$\frac{dx}{dt} = -\frac{\gamma}{2}e^{-\gamma t/2}\left[Ae^{\alpha t} + Be^{-\alpha t}\right] + e^{-\gamma t/2}\left[\alpha Ae^{\alpha t} - \alpha Be^{-\alpha t}\right]$$

$$= e^{-\gamma t/2}\left[\left(\alpha - \frac{\gamma}{2}\right)Ae^{\alpha t} - \left(\alpha + \frac{\gamma}{2}\right)Be^{-\alpha t}\right]$$

SO

$$\dot{x}(t=0) = \left(\alpha - \frac{\gamma}{2}\right)A - \left(\alpha + \frac{\gamma}{2}\right)B = +0.450\,m/s$$

SO, WE HAVE

1) $A + B = +0.0550\,m$

2) $(\alpha - \gamma/2)A - (\alpha + \gamma/2)B = +0.450\,m/s$ $\quad\therefore$ 2 EQNS & 2 UNKNOWNS

NOW

$$B = +0.0550\,m - A$$

PLUGGING THIS INTO 2)

$$(\alpha - \gamma/2)A - (\alpha + \gamma/2)(+0.0550\,m - A) = +0.450\,m/s$$

$$\therefore \left[\left(\alpha - \cancel{\gamma/2}\right) + \left(\alpha + \cancel{\gamma/2}\right)\right]A - \left(\alpha + \gamma/2\right)0.0550\,m = +0.450\,m/s$$

$$2\alpha A = 0.450\,m/s + \left(\alpha + \gamma/2\right)(0.0550\,m)$$

$$A = \frac{(0.450\,m/s) + \left(\alpha + \gamma/2\right)(0.0550\,m)}{2\alpha}$$

$$A = \frac{(0.450\,m/s) + \left(25.9\,s^{-1} + \left(54.9\,s^{-1}/2\right)\right)(0.0550\,m)}{2\left(25.9\,s^{-1}\right)}$$

$$\left[A = 0.0653\,m\right]$$

PLUGGING THIS BACK INTO 1) YIELDS..

$$B = +0.0550\,m - 0.0653\,m = -0.0103\,m$$

SO IN SUMMARY..

$$x(t) = e^{-\gamma t/2}\left[Ae^{\alpha t} + Be^{-\alpha t}\right]$$

WHERE

$$A = 0.0653\,m$$

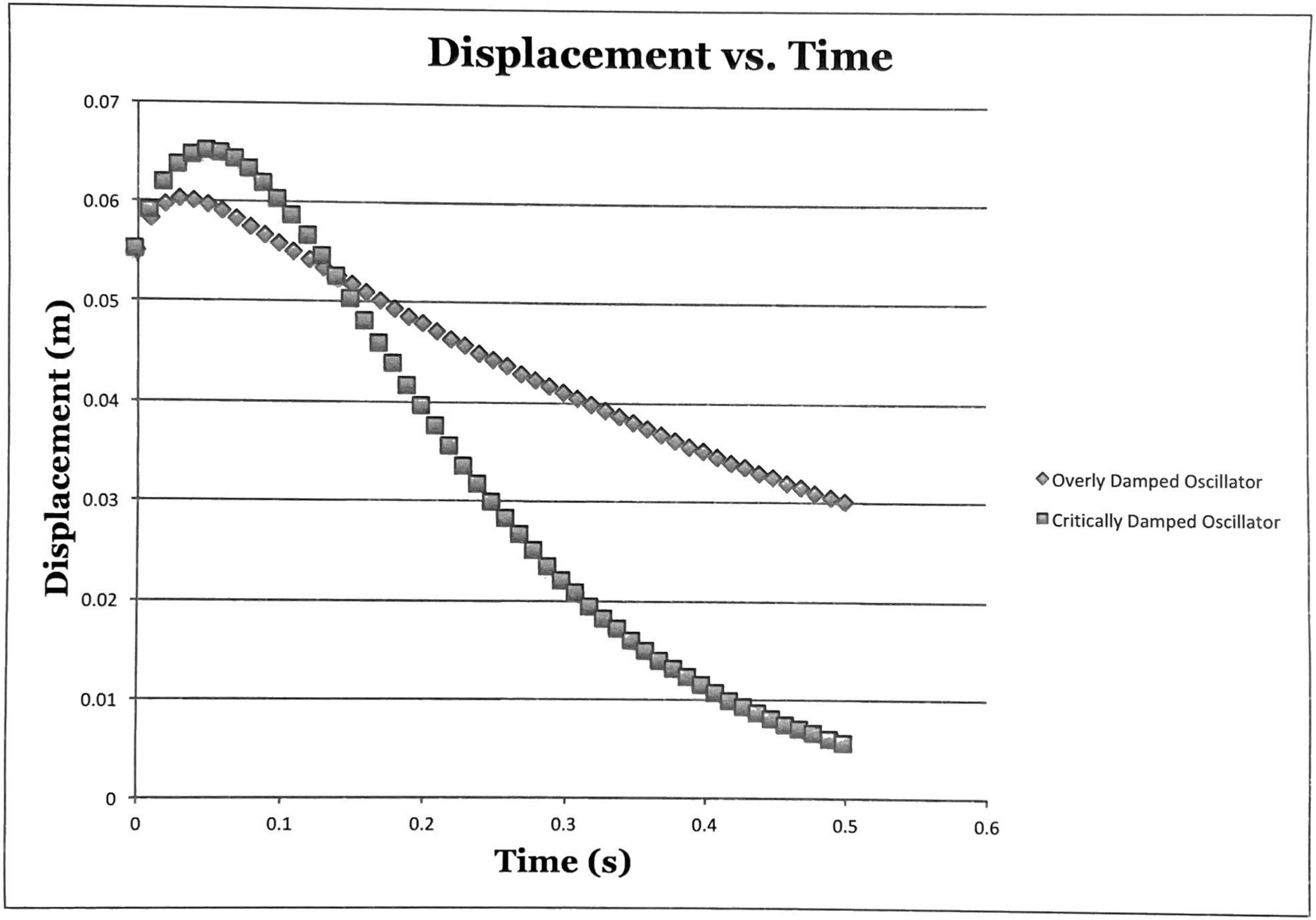
$$B = -0.0103\,m$$

$$\gamma = 54.9\,s^{-1}$$

$$\alpha = 25.9\,s^{-1}$$

A =	0.0653	m
B =	-0.0103	m
γ =	54.9	1/s
α =	25.9	1/s

t(s)	γt/2	exp(-γt/2)	exp(+αt)	exp(-αt)	(Aexp(+αt)+Bexp(-αt))	(Aexp(+αt)+Bexp(-αt))*exp(-γt/2)
0	0	1	1	1	0.055	0.055
0.01	0.2745	0.759952	1.295634	0.771823	0.07665511	0.058254205
0.02	0.549	0.577527	1.678667	0.595711	0.103481131	0.059763152
0.03	0.8235	0.438893	2.174938	0.459783	0.137287661	0.060254571
0.04	1.098	0.333537	2.817923	0.354871	0.180355181	0.060155215
0.05	1.3725	0.253472	3.650996	0.273898	0.235588889	0.059715301
0.06	1.647	0.192627	4.730354	0.211401	0.306714677	0.059081505
0.07	1.9215	0.146387	6.128806	0.163164	0.398530463	0.058339765
0.08	2.196	0.111247	7.940689	0.125934	0.51722985	0.057540403
0.09	2.4705	0.084543	10.28822	0.097198	0.670819923	0.056712845
0.1	2.745	0.064248	13.32977	0.07502	0.869661379	0.055874266
0.11	3.0195	0.048826	17.2705	0.057902	1.127167434	0.055034655
0.12	3.294	0.037105	22.37625	0.04469	1.460708628	0.054199786
0.13	3.5685	0.028198	28.99142	0.034493	1.892784592	0.053372965
0.14	3.843	0.021429	37.56227	0.026622	2.452541801	0.052556051
0.15	4.1175	0.016285	48.66694	0.020548	3.1777397	0.051750052
0.16	4.392	0.012376	63.05454	0.015859	4.117297839	0.050955483
0.17	4.6665	0.009405	81.69559	0.012241	5.334595829	0.050172567
0.18	4.941	0.007147	105.8476	0.009448	6.911748733	0.04940136
0.19	5.2155	0.005432	137.1397	0.007292	8.95514628	0.04864182
0.2	5.49	0.004128	177.6828	0.005628	11.60262959	0.047893847
0.21	5.7645	0.003137	230.2119	0.004344	15.03278949	0.047157311
0.22	6.039	0.002384	298.2703	0.003353	19.47701367	0.046432064


Displacement vs. Time
Displacement (m)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Time (s)
0
0.1
0.2
0.3
0.4
0.5
0.6
Overly Damped Oscillator
Critically Damped Oscillator