Phys 312 2021

Lecture 1

* Intro
* Course website
* Syllabus - note exams dates
* HW Tues / Fri - first one due Friday.

Thurs: Read. 4.1
4.2.1
4.2.2.

Scope of Phys 312

Phys 311 ended with arrival at Maxwell's equations. These are the fundamental equations that describe how source charges and currents produce electric and magnetic fields. Phys 311 included various methods for computing such fields from sources in various situations where the sources were stationary.

This course, Phys 312, will describe situations where the distributions are not necessarily stationary. We will start with Maxwell's equations and use these to determine fields produced in situations such as:

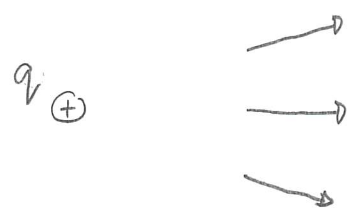
1) a vacuum ⤳ electromagnetic waves
2) moving point charges ⤳ electromagnetic radiation

The course will produce a substantial picture of how electromagnetic waves are generated, propagate and are detected. This was a great triumph of 19th century physics.

We will also cover a fundamental question that arose from the theory. How are different observers, moving relative to each other, to describe the same fields? This question was eventually resolved by the special theory of relativity. There is a deep relationship between relativity and electromagnetism.

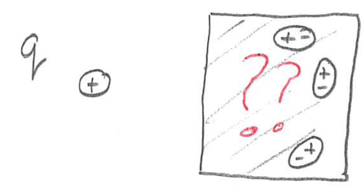
Initially the course will cover how electric and magnetic fields are produced in non-conducting materials that can respond to these fields.

Without material



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

With material

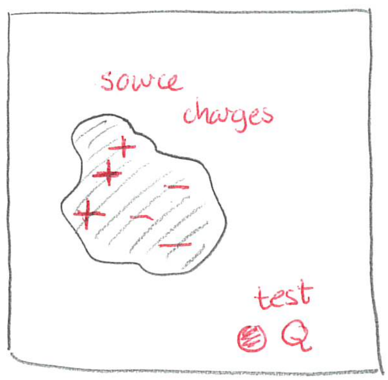


$\vec{E} = ??$

Broad classes of materials to have convenient response and which do allow some ease of description.

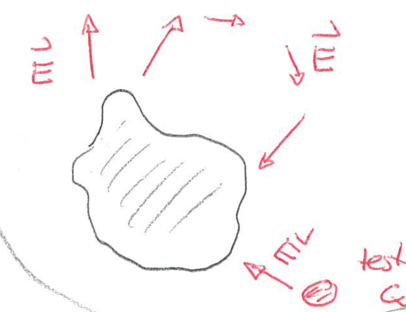
Review of Electrostatics

Electrostatics concerns situations where the source charges are stationary. The crucial task is to determine the electric fields that these produce and then the effect of the fields on a test charge



sources produce an electric field, $\vec{E}$

$\equiv$ one vector at each location in space



Electric field satisfies (in electrostatics)

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E} = 0$$

where $\rho(\vec{r})$ is the density of the charge distribution

Force on test charge

$$\vec{F} = Q\vec{E}$$

charge of test

electric field at test location

In practical terms the source charges are described by a charge density. This can often depend on the location in space. We will use:

$\vec{r} \equiv$ position vector that describes location at which field is computed

$\vec{r}' \equiv$ " " " " " of contributing source charge

Then:

> The charge density $\rho(\vec{r}\,')$ has units C/m^3 and depends on location $\vec{r}\,'$. This has the meaning that the charge in the region of volume $d\tau'$ at location $\vec{r}\,'$ is:
>
> $$dq = \rho(\vec{r}\,')\,d\tau'$$

Then the two equations that the field satisfies can be solved via:

1) solving the associated differential equations:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho(x,y,z)}{\epsilon_0}$$

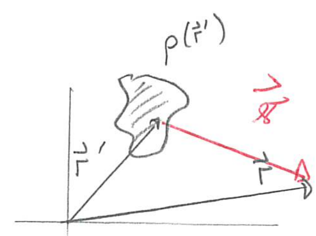
$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0 \qquad \dots$$

2) using Coulomb's law which can be derived from the Maxwell equations:

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho(\vec{r}\,')}{\mathfrak{r}^2}\,\hat{\mathfrak{r}}\,d\tau'}$$

where $\vec{\mathfrak{r}} = \vec{r} - \vec{r}\,'$ and

$$\hat{\mathfrak{r}} = \frac{\vec{\mathfrak{r}}}{\mathfrak{r}}$$

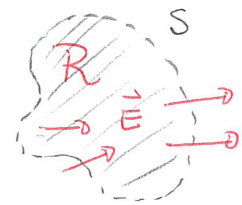


This requires decomposing the distribution into small pieces, multiplying the density by the volume element and eventually integrating...

3) using Gauss' Law, which can be derived from the Maxwell equations via the divergence theorem. This eventually states:

Let S be any close surface. Then

$$\oint_S \vec{E}(\vec{r}) \cdot d\vec{a} = q_{enc}/\epsilon_0$$



where q_{enc} is the charge enclosed by the surface and

$$q_{enc} = \int_R \rho(\vec{r}\,')\,d\tau'$$

where R is the region bounded by the surface.

1 Electric field produced by a charged sphere

A solid sphere with radius R has charge density given in spherical coordinates by

$$\rho(\mathbf{r}') = \alpha r'$$

where α is a constant with units $\mathrm{C/m^4}$. There is no charge beyond the sphere.

a) Determine the total charge, Q, in the sphere. Use this to rewrite α in terms of Q.

b) Determine the electric field for $r \leqslant R$. Express the answer in terms of Q.

c) Determine the electric field for $R \leqslant r$. Express the answer in terms of Q.

d) Check that the resulting fields satsify

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{E} = 0.$$

Answers: a) $$Q = \int_{\text{all space}} \rho(\vec{r}')\, d\tau'$$

$0 \leq r' \leq R$

$0 \leq \phi' \leq 2\pi$

$0 \leq \theta' \leq \pi$

$d\tau' = r'^2 \sin\theta' dr' d\phi' d\theta'$

$$\Rightarrow Q = \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' r'^2 \sin\theta' \, \alpha r'$$

$$= \alpha \underbrace{\int_0^R r'^3 dr'}_{R^4/4} \underbrace{\int_0^\pi \sin\theta' d\theta'}_{2} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} = \alpha \pi R^4$$

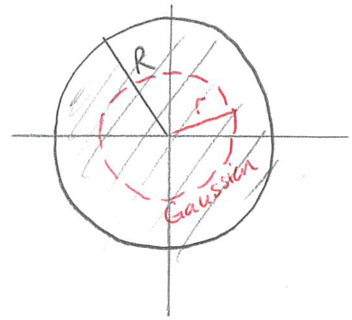
Thus $$\alpha = \frac{Q}{\pi R^4}$$

b) By symmetry the field only depends on r. It also points radially outward. Thus

$$\vec{E}(\vec{r}) = E_r(r)\hat{r}$$

We then use a Gaussian surface which is a sphere with radius r centered at the origin. On this surface

$$r = R$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$



$$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

So

$$\vec{E} \cdot d\vec{a} = E_r(r) \, r^2 \sin\theta \, d\theta \, d\phi$$

and

$$\oint \vec{E} \cdot d\vec{a} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \; E_r(r) \, r^2 \sin\theta = r^2 E_r(r) \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E_r(r) = \frac{q_{enc}}{\epsilon_0}$$

Thus in both cases $\boxed{E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}}$

When $r \leq R$ the enclosed charge is (similar to part a with $R \to r$)

$$q_{enc} = \alpha \underbrace{\int_0^{r} r'^3 dr' \int_0^{\pi} \sin\theta' \, d\theta' \int_0^{2\pi} d\phi'}_{4\pi} = \pi \alpha r^4 = \frac{\cancel{\pi} Q}{\cancel{\pi} R^4} r^4$$

Thus

$$E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{Q r^2}{R^4} \qquad r < R$$

c) In this case

$$q_{enc} = \alpha \int_0^R r'^3 dr' \int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' = \pi \alpha R^4 = Q$$

Then $$E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}$$

Thus we have

$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{Q r^2}{R^4} \hat{r} & r \leq R \\ \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} & r \geq R. \end{cases}$$

d) In spherical co-ordinates:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta}(\sin\theta \cancelto{0}{E_\theta}) + \frac{1}{r\sin\theta} \cancelto{0}{\frac{\partial E_\phi}{\partial \phi}}$$

Inside $(r \leq R)$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_0} \frac{Q}{R^4} r^4\right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^4} \frac{4r^3}{r^2} = \frac{1}{\epsilon_0} \frac{Q}{\pi R^4} r$$

$$= \frac{1}{\epsilon_0} \alpha r = \frac{\rho(r)}{\epsilon_0}$$

Outside $r^2 E_r$ does not depend on r $\Rightarrow$ $\vec{\nabla} \cdot \vec{E} = 0 = \frac{\rho(r)}{\epsilon_0}$

Then

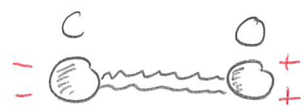
$$\vec{\nabla} \times \vec{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta \cancel{E_\phi}) - \frac{\partial \cancel{E_\theta}}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta} \cancelto{0}{\frac{\partial E_r}{\partial \phi}} - \frac{\partial}{\partial r}(r \cancelto{0}{E_\phi})\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r \cancelto{0}{E_\theta}) - \cancelto{0}{\frac{\partial E_r}{\partial \theta}}\right]\hat{\phi} = 0$$

The equations hold.

Electric Dipoles

A dipole is a charge distribute where one end is predominantly positively charged and the other predominantly negatively charged. Dipoles will form a basic part of the model for insulating materials which respond to electric fields since molecules often have this property.



Such dipoles:

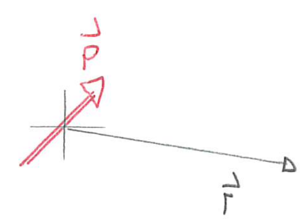
1) produce electric fields
2) respond to electric fields produced by other sources

These can both be described by adapting the basic rules of electromagnetism as follows:

The dipole can be described via a dipole moment $\vec{p}$

The electrostatic potential produced by the dipole at location $\vec{r}$ is

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}\cdot\vec{p}}{r^2}$$

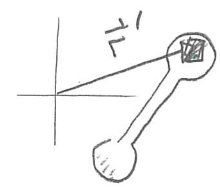


The electric field produced by the dipole is:

$$\vec{E}_{dip} = -\vec{\nabla} V_{dip}$$

For a general charge distribution the dipole moment is calculated by:

Given a charge density $\rho(\vec{r}')$, the electric dipole moment of this distribution is



$$\vec{p} = \int_{\text{all space.}} \rho(\vec{r}')\,\vec{r}'\,d\tau'$$

Units: Cm

This can be adapted to two- and one-dimensional charge distributions.
For point charge distributions:

$$\vec{p} = \sum_{\text{charges } i} q_i \vec{r}_i'$$

where $\vec{r}_i'$ is the position vector of charge i.

2 Electric dipole field

A point electric dipole has dipole moment

$$\mathbf{p} = p\hat{\mathbf{z}}.$$

Using spherical coordinates, determine the electric field produced by this dipole.

Answer: Need the potential

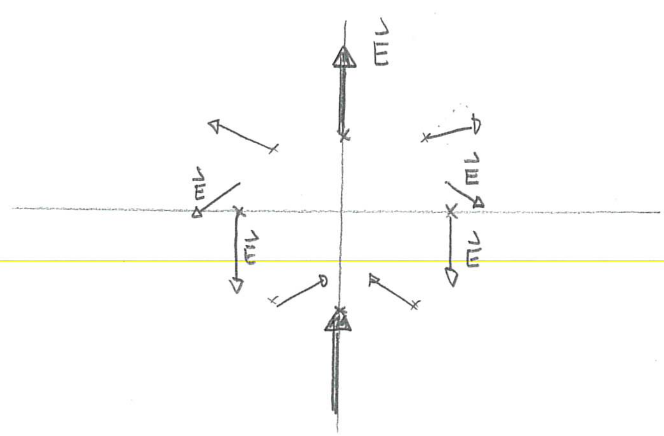
$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\underbrace{\hat{r}\cdot\hat{z}}_{\cos\theta} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$

Then:

$$\vec{E}_{dip} = -\vec{\nabla} V_{dip} = -\frac{\partial V_{dip}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_{dip}}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial V_{dip}}{\partial \phi}}\hat{\phi}$$

$$= \frac{2}{4\pi\epsilon_0}\frac{p}{r^3}\cos\theta\,\hat{r} - \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\underbrace{\frac{\partial}{\partial\theta}\cos\theta}_{-\sin\theta}\,\hat{\theta}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



Quite generally one can show that the field produced by a point dipole is:

$$\boxed{\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$