Taylor ...

+77/100

Physics 396

Exam 4

Show all correct work for credit!

1. Consider a homogeneous, isotropic cosmological model described by the line element

$$ds^2 = -dt^2 + \left(\frac{t}{t_0}\right)\left[dx^2 + dy^2 + dz^2\right], \tag{1}$$

where t_0 is a constant.

a) Is this model open, closed, or flat?

b) Is this a radiation-dominated universe? Explain.

c) Assuming the Friedmann equation holds for this universe, find $\rho(t)$.

2. a) Using the *First Law of Thermodynamics for Cosmology*

$$\frac{d}{dt}[\rho(t)a^3(t)] = -p(t)\frac{d}{dt}[a^3(t)], \tag{2}$$

explicitly find the density as a function of the scale factor, $\rho(a)$, for a universe containing only *pressureless* matter (void of any radiation or vacuum energy).

b) Using the Friedmann equation for a *flat* FRW model and the results from part a), show that the scale factor describing the expansion is of the form

$$a(t) = (t/t_0)^{2/3}, \tag{3}$$

where t_0 is a constant and t is the cosmological time from the singularity.

3. a) Show that the scale factor of the form

$$a(t) = e^{H(t-t_0)} \tag{4}$$

solves the Friedmann equation for a *flat* FRW model with *only* a vacuum energy density.

b) What is the relationship between H and ρ_v?

4. Consider a *flat* FRW model with only one fluid component. It can be shown that

$$a(t) = a_0 t^{2/(3(1+w))} \tag{5}$$

is a solution to the Friedmann equation, where w is the equation of state parameter where

$$p = w\rho, \tag{6}$$

where $w = 0, 1/3, -1$ for a matter-dominated, radiation-dominated, and vacuum energy-dominated universe, respectively.

a) Calculate the Hubble parameter, $H(t) \equiv \dot{a}/a$ for the solution given by Eq. (5).

b) Calculate the *acceleration* of this universe, $\ddot{a}/a$, for the solution given by Eq. (5).

c) For what values of the equation of state parameter, w, does one exhibit *expansion*? How about *contraction*?

d) For what values of the equation of state parameter, w, does one exhibit *accelerated expansion*?

PHYS 396 EXAM 4 SOLUTIONS

$$ds^2 = -dt^2 + \left(\frac{t}{t_0}\right)[dx^2 + dy^2 + dz^2]$$

THE LINE ELEMENT FOR A HOMOGENEOUS, ISOTROPIC FRW COSMOLOGY IS OF THE GENERIC FORM

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

a) NOW $dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\therefore$ $k = 0$ $\therefore$ FLAT!

b) $a^2(t) = \left(\frac{t}{t_0}\right)$ $\therefore$ $a(t) = \left(\frac{t}{t_0}\right)^{1/2}$

NOW IN GENERAL..

$$a(t) = a_0 t^{2/3(1+w)}$$ SO HERE $\frac{2}{3(1+w)} = \frac{1}{2}$ OR $\frac{4}{3} = 1 + w$ OR $w = 1/3$

$\therefore$ RADIATION-DOMINATED!

c) NOW FOR A FLAT FRW COSMOLOGY, THE FRIEDMANN EQN TAKES THE FORM...

$$\dot{a}^2 - \frac{8\pi}{3}\rho a^2 = 0 \quad \text{so} \quad \rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2}$$

NOW

$$\dot{a} = \frac{d}{dt}\left(\frac{t}{t_0}\right)^{1/2} = \frac{1}{2}\left(\frac{t}{t_0}\right)^{-1/2}\cdot\frac{1}{t_0}$$

SO

$$\frac{\dot{a}}{a} = \frac{1}{2t_0}\left(\frac{t}{t_0}\right)^{-1/2}\cdot\left(\frac{t}{t_0}\right)^{-1/2} = \frac{1}{2t_0}\left(\frac{t_0}{t}\right) = \frac{1}{2t}$$

SO

$$\left[\rho(t) = \frac{3}{8\pi}\cdot\frac{1}{4t^2} = \frac{3}{32\pi}\cdot\frac{1}{t^2}\right]$$

2. a) $\frac{d}{dt}\left[\rho(t)\, a^3(t)\right] = -p(t)\frac{d}{dt}\left[a^3(t)\right]$

NOW FOR PRESSURELESS-MATTER, $p(t) = 0$ $\therefore$

$$\frac{d}{dt}\left[\rho(t)\, a^3(t)\right] = 0 \quad \therefore \quad \rho(t)\, a^3(t) = c$$

$$\therefore \boxed{\rho(t) = \frac{c}{a^3(t)}}$$

b) THE FRIEDMANN EQN FOR A FLAT FRW MODEL IS OF THE FORM..

$$\dot{a}^2 - \frac{8\pi}{3}\rho\, a^2 = 0$$

NOW FOR $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$ SO $\rho(t) = \frac{c}{a^3(t)} = c\left(\frac{t}{t_0}\right)^{-2} = c\,\frac{t_0^2}{t^2}$

$$\dot{a} = \frac{2}{3}\left(\frac{t}{t_0}\right)^{-1/3} \cdot \frac{1}{t_0} = \frac{2}{3t_0}\left(\frac{t}{t_0}\right)^{-1/3} \quad \text{so} \quad \dot{a}^2 = \frac{4}{9t_0^2}\left(\frac{t}{t_0}\right)^{-2/3}$$

PLUGGING THESE IN TO THE FLAT FRIEDMANN EQN YIELDS...

$$\frac{4}{9t_0^2}\left(\frac{t}{t_0}\right)^{-2/3} - \frac{8\pi}{3}c\left(\frac{t}{t_0}\right)^{-2}\left(\frac{t}{t_0}\right)^{4/3} = \frac{4}{9t_0^2}\left(\frac{t}{t_0}\right)^{-2/3} - \frac{8\pi}{3}c\left(\frac{t}{t_0}\right)^{-2/3}$$

$$= \frac{4}{3}\left(\frac{t}{t_0}\right)^{-2/3}\left[\frac{1}{3t_0^2} - 2\pi c\right] = 0 \quad \therefore \quad \boxed{c = \frac{1}{6\pi t_0^2}}$$

3 a) $a(t) = e^{H(t-t_0)}$ so $\dot{a} = H e^{H(t-t_0)}$ $\therefore \frac{\dot{a}}{a} = H$

THE FRIEDMANN EQN FOR A FLAT, FRW COSMOLOGY W/ ONLY A VACUUM ENERGY DENSITY IS OF THE FORM..

$$\dot{a}^2 - \frac{8\pi}{3}\rho_v a^2 = 0$$

PLUGGING IN THE SOLUTION YIELDS...

$$H^2 e^{2H(t-t_0)} - \frac{8\pi}{3}\rho_v e^{2H(t-t_0)} = e^{2H(t-t_0)}\left[H^2 - \frac{8\pi}{3}\rho_v\right] = 0$$

b) $\therefore H = \sqrt{\frac{8\pi}{3}\rho_v}$

4. a) $a(t) = a_0 t^{2/3(1+w)}$

$$\dot{a} = \frac{2}{3(1+w)} a_0 t^{2/3(1+w)-1} = \frac{2}{3(1+w)} \cdot \frac{1}{t} a \qquad \therefore \left[\frac{\dot{a}}{a} = \frac{2}{3(1+w)} \cdot \frac{1}{t}\right]$$

b) now $\frac{2}{3(1+w)} - 1 = \frac{1}{3(1+w)}\left[2 - 3(1+w)\right] = \frac{-(1+3w)}{3(1+w)}$

so

$$\dot{a} = \frac{2}{3(1+w)} a_0 t^{-(1+3w)/3(1+w)}$$

so

$$\ddot{a} = \frac{2}{3(1+w)} \cdot -\frac{(1+3w)}{3(1+w)} a_0 t^{2/3(1+w)-2} = \frac{-2(1+3w)}{9(1+w)^2} \cdot \frac{1}{t^2} a(t)$$

$$\therefore \left[\frac{\ddot{a}}{a} = \frac{-2(1+3w)}{9(1+w)^2} \cdot \frac{1}{t^2}\right]$$

c) EXPANSION WHEN...

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(1+w)} \cdot \frac{1}{t} > 0 \quad \text{or when} \quad w+1 > 0 \quad \text{or} \quad [w > -1]$$

CONTRACTION WHEN...

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3(1+w)} \cdot \frac{1}{t} < 0 \quad \text{or when} \quad w+1 < 0 \quad \text{or} \quad [w < -1]$$

d) ACCELERATION WHEN...

$$\frac{\ddot{a}}{a} = \frac{-2(1+3w)}{9(1+w)^2} \cdot \frac{1}{t^2} > 0 \quad \text{or when} \quad 1+3w < 0 \quad \text{or when} \quad [w < -1/3]$$

SO ACCELERATED EXPANSION WHEN..

$$[-1 < w < -1/3]$$