6.6 Matrix Exponential

Constant Matrix Exponential

Given a $n \times n$ matrix A.

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots + \frac{A^K}{K!}$$

Nilpotent Matrix

A square matrix A is called nilpotent if $A^n = 0$ for some positive integer n.

Ex 1) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $\quad A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$

$$e^{A} = I + A + \frac{A^2}{2!}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix}$$

$$\begin{bmatrix} 1 + a + a^2/2 \ldots & 0 \\ 0 & 1 + b + b^2/2 \ldots \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

Ex 2) $A = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{A} = I + A + \frac{1}{2!}A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Exponential Function

$$e^{At} = I + tA + t^2/2!\,A^2 + \cdots\cdots + t^K/K!\,A^K + \cdots$$

$$\frac{d}{dt} e^{At} = Ae^{At}$$

Ex 6) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \bar{F}(t) = \begin{bmatrix} t \\ 6 \end{bmatrix}$

$$\bar{x}' = A\bar{x} + \vec{F}(t)$$

$$\left[\begin{array}{cc|cc} \cos(t) & \sin(t) & 1 & 0 \\ -\sin(t) & \cos(t) & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cccc} 1 & \frac{\sin(t)}{\cos(t)} & \frac{1}{\cos(t)} & 0 \\ -1 & \frac{\cos(t)}{\sin(t)} & 0 & +\frac{1}{\sin(t)} \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & \frac{\sin t}{\cos t} & \frac{1}{\cos(t)} & 0 \\ 0 & 1 & \sin(t) & \cos(t) \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{\sin(t)}{\cos(t)} & \frac{1}{\cos(t)} & 0 \\ -\sin(t) & \cos(t) & 0 & 1 \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & \frac{\sin(t)}{\cos(t)} & \frac{1}{\cos(t)} & 0 \\ 0 & \frac{\sin(t)}{\cos(t)} + \frac{\cos(t)}{\sin(t)} & \frac{1}{\cos(t)} & \frac{1}{\sin(t)} \end{array}\right] \quad \left[\begin{array}{cc|cc} 1 & 0 & \cos(t) & -\sin(t) \\ 0 & 1 & \sin(t) & \cos(t) \end{array}\right]$$

$$\bar{x}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \int_0^t \begin{bmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix}$$

$uv - \int v\,du$

$u = s \quad dv = \cos(s)$

$du = 1 \quad v = \sin(s)$

$s\sin(s) - \int \sin(s)$

$s\sin(s) + \cos(s) + C$

$$\int_0^t \begin{bmatrix} s\cos(s) \\ s\sin(s) \end{bmatrix}$$

$$\begin{bmatrix} s\sin(s) + \cos(s) \\ -s\cos(s) + \sin(s) \end{bmatrix} \Bigg|_0^t$$

$$e^{At}\bar{c} + e^{At}\int_0^t e^{-As}\bar{f}(s)\,ds$$

$$\begin{bmatrix} t\sin(t) & \cos(t) \\ -t\cos(t) & \sin(t) \end{bmatrix}$$

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \left(\begin{bmatrix} t\sin(t) + \cos(t) - 1 \\ -t\cos(t) + \sin(t) \end{bmatrix} \right)$$

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 - \cos(t) \\ \sin(t) - t \end{bmatrix}$$