1.) Area of the Surface?

part of the plane $4x+5y+z=20$, cylinder $x^2+y^2=16$

$z=-4x-5y+20$

$f_x=-4 \quad f_y=-5$

$x^2=16-y^2$

$x=\pm\sqrt{16-y^2}$

$16-y^2=0$
$y^2=16$
$y=\pm 4$

$x=(-\sqrt{16-y^2},\sqrt{16-y^2})$
$y=(-4,4)$

$$A(S)=\iint_D \sqrt{(f_x(x,y))^2+(f_y(x,y))^2+1}$$

$\sqrt{16+25+1}$

$\sqrt{42}$

$16\pi\sqrt{42}$

$$\int_{-4}^{4}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\sqrt{42}\,dx\,dy$$

$$\int_{-4}^{4}2\int_{0}^{\sqrt{16-y^2}}\sqrt{42}\,dx\,dy$$

$$\int_{-4}^{4}2(\sqrt{42}x)\Big|_0^{\sqrt{16-y^2}}dy$$

$$\int_{-4}^{4}2\sqrt{42}\sqrt{16-y^2}\,dy$$

$$2\int_0^4 2\sqrt{42}\sqrt{16-y^2}\,dy$$

$$4\sqrt{42}\int_0^4\sqrt{16-y^2}\,dy=16\pi\sqrt{42}$$

2.) Part of the plane $13x+3y+z=39$, first octant.

$z=-13x-3y+39$

$\frac{\partial z}{\partial x}=-13 \quad \frac{\partial z}{\partial y}=-3$

$\sqrt{-13^2-3^2+1}$

$\sqrt{169+9+1}$

$\sqrt{179}$

$z=0$

$0=-13x-3y+39$

$-39+3y=-13x$

$x=\frac{-39+3y}{-13}$

$-39+3y=0$

$3y=39$

$y=13$

$0\le x\le\frac{-39+3y}{-13}$

$0\le y\le 13$

$$\int_0^{13}\int_0^{\frac{-39+3y}{-13}}\sqrt{179}\,dx\,dy$$

$$\int_0^{13}\left[\sqrt{179}\,x\right]_0^{\frac{-39+3y}{-13}}dy$$

$$\int_0^{13}\sqrt{179}\left(\frac{-39+3y}{-13}\right)dy$$

$$\sqrt{179}\int_0^{13}\frac{-39}{-13}+\frac{3y}{-13}\,dy$$

$$\sqrt{179}\int_0^{13}3-\frac{3y}{13}\,dy$$

$$\sqrt{179}\int_0^{13}3\,dy-\sqrt{179}\int_0^{13}\frac{3y}{13}\,dy$$

$$\sqrt{179}\,(3y)\Big|_0^{13}-\sqrt{179}\left(\frac{3}{13}\right)\int_0^{13}y\,dy$$

$$\sqrt{179}\,(39)-\frac{3\sqrt{179}}{13}\left[\frac{1}{2}y^2\right]_0^{13}$$

$$\frac{3\sqrt{179}}{13}\left(\frac{169}{2}\right)$$

$$\sqrt{179}(39)-\frac{507\sqrt{179}}{26}$$

$$\frac{39\sqrt{179}}{2}$$

$\frac{39\sqrt{179}}{2}$

3.) Find the area of the Surface

part of the sphere $x^2+y^2+z^2=a^2$ Above xy plane

Lies within the cylinder $x^2+y^2=ax$

$z^2=-x^2-y^2+a^2$

$z=\sqrt{-x^2-y^2+a^2}$

$x^2+y^2=ax$

$r^2=a(r\cos\theta)$

$r=a(\cos\theta)$

$x=r\cos\theta \quad x^2+y^2=r^2$

$y=r\sin\theta$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$0\le r\le a\cos\theta$

$$\frac{\partial z}{\partial x}=\frac{1}{2}(-x^2-y^2+a^2)^{-\frac{1}{2}}(-2x) \qquad \frac{\partial z}{\partial y}=\frac{1}{2}(-x^2-y^2+a^2)^{-\frac{1}{2}}(-2y)$$

$$=-x(-x^2-y^2+a^2)^{-\frac{1}{2}} \qquad =-y(-x^2-y^2+a^2)^{-\frac{1}{2}}$$

$$A=\sqrt{(-x(-x^2-y^2+a^2)^{-\frac{1}{2}})^2+(-y(-x^2-y^2+a^2)^{-\frac{1}{2}})^2+1}$$

$$A=\sqrt{1+\left(\frac{x^2+y^2}{a^2-(x^2+y^2)}\right)}\,dA$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}r\sqrt{1+\left(\frac{r^2}{a^2-r^2}\right)}\,dr\,d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} r\sqrt{1+\left(\frac{r^2}{a^2-r^2}\right)}\,dr\,d\theta$$

$u=a^2-r^2$
$du=-2r\,dr$
$-\frac{1}{2}du=r\,dr$

$a^2-(a\cos\theta)^2=a^2$
$a^2-a^2\cos^2\theta$
$a^2(1-\cos^2\theta)$
$a^2\sin^2\theta$

$\cos^2\theta=1-\sin^2\theta$
$\sin^2\theta=1-\cos^2\theta$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{a^2}^{a^2\sin^2\theta}\sqrt{1+\left(\frac{a^2-u}{u}\right)}\,\frac{du}{-2}\,d\theta$$

$\Downarrow$

$$\int -\frac{1}{2}\int\sqrt{1+\left(\frac{a^2}{u}\right)-\frac{u}{u}}\,du\,d\theta$$

$$\int -\frac{1}{2}\int\sqrt{a^2u^{-1}}\,du\,d\theta$$

$$\int -\frac{1}{2}\int a\,u^{-\frac{1}{2}}\,du\,d\theta$$

$$\int -\frac{a}{2}\int u^{-\frac{1}{2}}\,du\,d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{a}{2}\left[2u^{\frac{1}{2}}\right]_{a^2}^{a^2\sin^2\theta}d\theta$$

$$a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(|a\sin\theta|-|a|\right)d\theta$$

$$-2a\int_0^{\frac{\pi}{2}} a\sin\theta-a\,d\theta$$

$$-2a\left[-a\cos\theta-a\theta\right]_0^{\frac{\pi}{2}}$$

$$-2a\left[\left(-a\cos\tfrac{\pi}{2}-a\tfrac{\pi}{2}\right)-\left(-a\cos(0)-a(0)\right)\right]$$

$$-2a\left[\left(0-\frac{a\pi}{2}\right)+a\right]$$

$$-2a\left(-\frac{a\pi}{2}+a\right)$$

$$2a^2\left(\frac{\pi}{2}-1\right)$$

$$a^2(\pi-2)$$