## Exercise 1

a.) Single Qubit $|\psi_0\rangle = a_0|0\rangle + a_1|1\rangle$ what is optimal $a_0, a_1$ and what is optimal QFI ?

$$H = 4\left[\frac{\partial\langle\psi|}{\partial\lambda}\frac{\partial|\psi\rangle}{\partial\lambda} + \left(\langle\psi|\cdot\frac{\partial|\psi\rangle}{\partial\lambda}\right)^2\right]$$

$$|\psi(\lambda)\rangle = \hat{u}(\lambda)|\psi\rangle = \begin{pmatrix} \cos(\lambda/2) - i\sin(\lambda/2) & 0 \\ 0 & \cos(\lambda/2) + i\sin(\lambda/2) \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$|\psi(\lambda)\rangle = \begin{pmatrix} a_0\left(\cos(\lambda/2) - i\sin(\lambda/2)\right) \\ a_1\left(\cos(\lambda/2) + i\sin(\lambda/2)\right) \end{pmatrix} : \langle\psi(\lambda)| = \left(a_0(\cos(\lambda/2) + i\sin(\lambda/2)) + a_1(\cos(\lambda/2) - i\sin(\lambda/2))\right)$$

$$\frac{\partial|\psi\rangle}{\partial\lambda} = \begin{pmatrix} a_0/2\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right) \\ a_1/2\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right) \end{pmatrix} : \frac{\partial\langle\psi|}{\partial\lambda} = \left(\frac{a_0}{2}(-\sin(\lambda/2) + i\cos(\lambda/2)) + \frac{a_1}{2}(-\sin(\lambda/2) - i\cos(\lambda/2))\right)$$

$$\frac{\partial\langle\psi|}{\partial\lambda}\cdot\frac{\partial|\psi\rangle}{\partial\lambda} = \left(\frac{a_0}{2}(-\sin(\lambda/2) + i\cos(\lambda/2)) + \frac{a_1}{2}(-\sin(\lambda/2) - i\cos(\lambda/2))\right)\begin{pmatrix} a_0/2\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right) \\ a_1/2\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right) \end{pmatrix}$$

$$= \frac{a_0^2}{4}\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right)\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right) + \frac{a_1^2}{4}\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right)\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right)$$

$$= \frac{a_0^2}{4}\left(\overbrace{\sin^2(\lambda/2) + \cos^2(\lambda/2)}^{1}\right) + \frac{a_1^2}{4}\left(\overbrace{\sin^2(\lambda/2) + \cos^2(\lambda/2)}^{1}\right) = \frac{a_0^2}{4} + \frac{a_1^2}{4} = \frac{1}{4}\left(a_0^2 + a_1^2\right)$$

$$\langle\psi|\cdot\frac{\partial|\psi\rangle}{\partial\lambda} = \left(a_0(\cos(\lambda/2) + i\sin(\lambda/2)) + a_1(\cos(\lambda/2) - i\sin(\lambda/2))\right)\begin{pmatrix} a_0/2\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right) \\ a_1/2\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right) \end{pmatrix}$$

$$= \frac{a_0^2}{2}\left(\cos(\lambda/2) + i\sin(\lambda/2)\right)\left(-\sin(\lambda/2) - i\cos(\lambda/2)\right) + \frac{a_1^2}{2}\left(\cos(\lambda/2) - i\sin(\lambda/2)\right)\left(-\sin(\lambda/2) + i\cos(\lambda/2)\right)$$

$$= \frac{a_0^2}{2}\left(-i\left(\overbrace{\cos^2(\lambda/2) + \sin^2(\lambda/2)}^{1}\right)\right) + \frac{a_1^2}{2}\left(i\left(\overbrace{\cos^2(\lambda/2) + \sin^2(\lambda/2)}^{1}\right)\right) = -\frac{i}{2}a_0^2 + \frac{i}{2}(a_1^2) = \frac{i}{2}(a_1^2 - a_0^2)$$

$$\left(\langle\psi|\cdot\frac{\partial|\psi\rangle}{\partial\lambda}\right)^2 = \frac{-1}{4}\left(a_1^2 - a_0^2\right)^2$$

$$\therefore\ H = 4\left(\frac{1}{4}(a_0^2 + a_1^2) - \frac{1}{4}(a_1^2 - a_0^2)^2\right) = \left(a_0^2 + a_1^2 - (a_1^2 - a_0^2)^2\right) = a_0^2 + a_1^2 - (a_1^4 + a_0^4 - 2a_0^2a_1^2)$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ H = a_0^2 + a_1^2 - (a_1^2 - a_0^2)^2}$$

Exercise 1 continued

b.) Multiple Qubits with independent channel use, get output state and QFI

$$\hat{U}(\lambda) = e^{-i\lambda\hat{\sigma}_z/2} = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} \quad : \quad |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|\psi(\lambda)\rangle = \hat{U}(\lambda)|+\rangle = \frac{1}{\sqrt{2}}\left(\hat{U}(\lambda)|0\rangle + \hat{U}(\lambda)|1\rangle\right) = \frac{1}{\sqrt{2}}\left(e^{-i\lambda/2}|0\rangle + e^{i\lambda/2}|1\rangle\right)$$

$$|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\lambda/2}|0\rangle + e^{i\lambda/2}|1\rangle\right) \quad : \quad \langle\psi(\lambda)| = \frac{1}{\sqrt{2}}\left(e^{i\lambda/2}\langle 0| + e^{-i\lambda/2}\langle 1|\right)$$

$$\frac{\partial\langle\psi(\lambda)|}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(\frac{i}{2}e^{i\lambda/2}\langle 0| - \frac{i}{2}e^{-i\lambda/2}\langle 1|\right) \quad : \quad \frac{\partial|\psi(\lambda)\rangle}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(\frac{-i}{2}e^{-i\lambda/2}|0\rangle + \frac{i}{2}e^{i\lambda/2}|1\rangle\right)$$

$$\frac{\partial\langle\psi(\lambda)|}{\partial\lambda}\cdot\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda} = \frac{1}{2}\left(\frac{1}{4}\overset{1}{\cancel{\langle 0|0\rangle}} + \frac{1}{4}\overset{1}{\cancel{\langle 1|1\rangle}}\right) = \frac{1}{2}\left(\frac{2}{4}\right) = \frac{1}{4} \quad : \text{ repeat } n \text{ times} : \quad \frac{n}{4}$$

$$\langle\psi(\lambda)|\cdot\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(e^{i\lambda/2}\langle 0| + e^{-i\lambda/2}\langle 1|\right)\cdot\frac{1}{\sqrt{2}}\left(-\frac{i}{2}e^{-i\lambda/2}|0\rangle + \frac{i}{2}e^{i\lambda/2}|1\rangle\right)$$

$$= \frac{1}{2}\left(-\frac{i}{2}\overset{1}{\cancel{\langle 0|0\rangle}} + \frac{i}{2}\overset{1}{\cancel{\langle 1|1\rangle}}\right) = \frac{1}{2}\left(-\frac{i}{2} + \frac{i}{2}\right) = 0 \quad : \text{ repeat } n \text{ times: } 0$$

$$\therefore \quad H = 4\left[\frac{n}{4} + 0\right] = n$$

$$\boxed{H = n}$$

# Exercise 1 Continued

c.) Multiple entangled Qubits.

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|00\ldots 0_n\rangle + |11\ldots 1_n\rangle\right)$$

$$|\psi(\lambda)\rangle = \hat{U}(\lambda)|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(\hat{U}|0\rangle\hat{U}|0\rangle\ldots\hat{U}|0_n\rangle + \hat{U}|1\rangle\hat{U}|1\rangle\ldots\hat{U}|1_n\rangle\right)$$

$$|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}}\left(e^{-ni\lambda/2}\cdot|0_n\rangle + e^{ni\lambda/2}|1_n\rangle\right)$$

$$\langle\psi(\lambda)| = \hat{U}(\lambda)\langle\psi_0| = \frac{1}{\sqrt{2}}\left(\hat{U}\langle 0|\hat{U}\langle 0|\ldots\hat{U}\langle 0_n| + \hat{U}\langle 1|\hat{U}\langle 1|\ldots\hat{U}\langle 1_n|\right)$$

$$\langle\psi(\lambda)| = \frac{1}{\sqrt{2}}\left(e^{ni\lambda/2}\langle 0_n| + e^{-ni\lambda/2}\langle 1_n|\right)$$

$$\frac{\partial\langle\psi|}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(\frac{ni}{2}e^{ni\lambda/2}\langle 0_n| - \frac{ni}{2}e^{-ni\lambda}\langle 1_n|\right) : \frac{\partial|\psi\rangle}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(-\frac{ni}{2}e^{-ni\lambda/2}|0_n\rangle + \frac{ni}{2}e^{ni\lambda/2}|1_n\rangle\right)$$

$$\frac{\partial\langle\psi|}{\partial\lambda}\cdot\frac{\partial|\psi\rangle}{\partial\lambda} = \frac{1}{2}\left(\frac{n^2}{4}\langle 0_n|0_n\rangle + \frac{n^2}{4}\langle 1_n|1_n\rangle\right) = \frac{1}{2}\left(\frac{2n^2}{4}\right) = \frac{n^2}{4} \quad \therefore \frac{\partial\langle\psi|}{\partial\lambda}\frac{\partial|\psi\rangle}{\partial\lambda} = \frac{n^2}{4}$$

$$\langle\psi|\cdot\frac{\partial|\psi\rangle}{\partial\lambda} = \frac{1}{\sqrt{2}}\left(e^{ni\lambda/2}\langle 0_n| + e^{-ni\lambda/2}\langle 1_n|\right)\cdot\frac{1}{\sqrt{2}}\left(-\frac{ni}{2}e^{-ni\lambda/2}|0_n\rangle + \frac{ni}{2}e^{ni\lambda/2}|1_n\rangle\right)$$

$$= \frac{1}{2}\left(-\frac{ni}{2}\langle 0_n|0_n\rangle + \frac{ni}{2}\langle 1_n|1_n\rangle\right) = \frac{n}{4}(-i+i) = 0 \quad \therefore \left(\langle\psi|\cdot\frac{\partial|\psi\rangle}{\partial\lambda}\right)^2 = 0$$

$$\therefore H = 4\left[\frac{n^2}{4} + 0\right] = n^2$$

$$\boxed{H = n^2}$$

## Exercise 2

a.) Suppose $\hat{\rho}_0 = \frac{1}{2}(\hat{I} + r\hat{\sigma}_x)$

$$\hat{\rho}_0 = \frac{1}{2}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + r\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix}$$

$$\boxed{\hat{\rho}_0 = \frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix}}$$

b.) Determine $\hat{\rho}(\lambda)$ : $\hat{u}(\lambda) = \begin{pmatrix}e^{-i\lambda/2} & 0\\ 0 & e^{i\lambda/2}\end{pmatrix}$

$$\hat{\rho}(\lambda) = \hat{u}(\lambda)\rho_0\hat{u}^\dagger(\lambda) = \begin{pmatrix}e^{-i\lambda/2} & 0\\ 0 & e^{i\lambda/2}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&r\\r&1\end{pmatrix}\begin{pmatrix}e^{i\lambda/2} & 0\\ 0 & e^{-i\lambda/2}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}e^{-i\lambda/2} & 0\\ 0 & e^{i\lambda/2}\end{pmatrix}\begin{pmatrix}e^{i\lambda/2} & re^{-i\lambda/2}\\ re^{i\lambda/2} & e^{-i\lambda/2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & re^{-i\lambda}\\ re^{i\lambda} & 1\end{pmatrix} : \hat{\rho}(\lambda) = \frac{1}{2}\begin{pmatrix}1 & re^{-i\lambda}\\ re^{i\lambda} & 1\end{pmatrix}$$

$$\boxed{\hat{\rho}(\lambda) = \frac{1}{2}\begin{pmatrix}1 & re^{-i\lambda}\\ re^{i\lambda} & 1\end{pmatrix}}$$

c.) Determine QFI

i.) $\alpha = Tr(\hat{A}^2) - (Tr\hat{A})^2$ : $\hat{A} \equiv \hat{\rho}(\lambda)$

$$\hat{\rho}^2(\lambda) = \frac{1}{4}\begin{pmatrix}1 & re^{-i\lambda}\\ re^{i\lambda} & 1\end{pmatrix}\begin{pmatrix}1 & re^{-i\lambda}\\ re^{i\lambda} & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1+r^2 & 2re^{-i\lambda}\\ 2re^{i\lambda} & 1+r^2\end{pmatrix}$$

$$Tr(\hat{\rho}^2) = \frac{1}{4}(1+r^2+1+r^2) = \frac{1}{4}(2+2r^2) = \frac{1}{2}(1+r^2) \; : \; (Tr(\hat{\rho}))^2 = (1)^2 = 1$$

$$Tr(\hat{\rho}^2) - (Tr(\hat{\rho}))^2 = \frac{1}{2} + \frac{r^2}{2} - 1 = \frac{r^2}{2} - \frac{1}{2} = \frac{1}{2}(r^2-1) \quad \therefore \quad \alpha = \frac{r^2-1}{2}$$

ii.) $$\hat{L} = \frac{1}{Tr(\hat{\rho})}\left[2\cdot\frac{\partial\hat{\rho}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{\rho}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \ln(Tr(\hat{\rho}))\right]\hat{I}$$

Since $\alpha$ does not depend upon $\lambda$, the $\hat{L}$ equation becomes

$$\hat{L} = \frac{1}{Tr(\hat{\rho})}\left[2\cdot\frac{\partial\hat{\rho}}{\partial\lambda} - \cancelto{0}{\frac{\partial \ln(\alpha)}{\partial\lambda}}\hat{\rho}\right] + \frac{\partial}{\partial\lambda}\left[\cancelto{0}{\ln(\alpha)} - \cancelto{0}{\ln(Tr(\hat{\rho}))}\right]\hat{I} = \frac{1}{Tr(\hat{\rho})}\cdot 2\cdot\frac{\partial\hat{\rho}}{\partial\lambda}$$

## Exercise 2 Continued

$$\hat{L} = \frac{1}{Tr(\hat{\rho})} \cdot 2\frac{\partial \hat{\rho}}{\partial \lambda} \quad : \quad Tr(\hat{\rho}) = 1 \quad : \quad \frac{\partial \hat{\rho}}{\partial \lambda} = \frac{1}{2}\begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix}$$

$$L = 1 \cdot 2 \cdot \frac{1}{2}\begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix} \quad \therefore \quad \hat{L} = \begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix}$$

iii.) Get QFI

$$H = Tr[\hat{\rho}\hat{L}^2]$$

$$\hat{L}^2 = \begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix}\begin{pmatrix} 0 & -ire^{-i\lambda} \\ ire^{i\lambda} & 0 \end{pmatrix} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \therefore \quad \hat{L}^2 = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\hat{\rho}\hat{L}^2 = \frac{1}{2}\begin{pmatrix} 1 & re^{-i\lambda} \\ re^{i\lambda} & 1 \end{pmatrix}\begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} r^2 & r^3e^{-i\lambda} \\ r^3e^{i\lambda} & r^2 \end{pmatrix} \quad \therefore \quad \hat{\rho}\hat{L}^2 = \frac{1}{2}\begin{pmatrix} r^2 & r^3e^{-i\lambda} \\ r^3e^{i\lambda} & r^2 \end{pmatrix}$$

$$Tr[\hat{\rho}\hat{L}^2] = Tr\left[\frac{1}{2}\begin{pmatrix} r^2 & r^3e^{-i\lambda} \\ r^3e^{i\lambda} & r^2 \end{pmatrix}\right] = \frac{1}{2} \cdot 2r^2 = r^2 \quad \therefore \quad H = r^2$$

$$\boxed{H = r^2}$$

## Exercise 3

$$\hat{u}_{prep} = \frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_c + \hat{\sigma}_c\otimes\hat{I} - \hat{\sigma}_c\otimes\hat{\sigma}_c\right) \;:\; \hat{\rho}_0 = \frac{1}{2}\left(\hat{I} + r\hat{\sigma}_x\right)$$

a.) Determine state after $\hat{U}_{prep}$ with $\hat{c} = \hat{y}$

i.) construct $\hat{u}_{prep}$

$$\hat{u}_{prep} = \frac{1}{2}\left(\hat{I}\otimes\hat{I} + \hat{I}\otimes\hat{\sigma}_y + \hat{\sigma}_y\otimes\hat{I} - \hat{\sigma}_y\otimes\hat{\sigma}_y\right) \;:\; \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\; \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{I}\otimes\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;:\; \hat{I}\otimes\hat{\sigma}_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\hat{\sigma}_y\otimes\hat{I} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \;:\; \hat{\sigma}_y\otimes\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{u}_{prep} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}\right]$$

$$\hat{u}_{prep} = \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 \\ i & 1 & -1 & -i \\ i & -1 & 1 & -i \\ 1 & i & i & 1 \end{pmatrix}$$

ii.) Construct $\hat{\rho} = \hat{\rho}_0\otimes\hat{\rho}_0$

$$\hat{\rho}_0 = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} r\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \;:\; \hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

$$\hat{\rho} = \hat{\rho}_0\otimes\hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & r & r & r^2 \\ r & 1 & r^2 & r \\ r & r^2 & 1 & r \\ r^2 & r & r & 1 \end{pmatrix}$$

$$\hat{\rho} = \frac{1}{4}\begin{pmatrix} 1 & r & r & r^2 \\ r & 1 & r^2 & r \\ r & r^2 & 1 & r \\ r^2 & r & r & 1 \end{pmatrix}$$

## Exercise 3 Continued

iii.) Calculate $\hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep}$

$$\hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep} = \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 \\ i & 1 & -1 & -i \\ i & -1 & 1 & -i \\ 1 & i & i & 1 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 1 & r & r & r^2 \\ r & 1 & r^2 & r \\ r & r^2 & 1 & r \\ r^2 & r & r & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 \\ i & 1 & -1 & -i \\ i & -1 & 1 & -i \\ 1 & i & i & 1 \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 1 & -i & -i & 1 \\ i & 1 & -1 & -i \\ i & -1 & 1 & -i \\ 1 & i & i & 1 \end{pmatrix}\begin{pmatrix} [r^2+2ir+1] & [-i+ir^2] & [-i+ir^2] & [r^2-2ir+1] \\ [2r+i+ir^2] & [1-r^2] & [r^2-1] & [2r-i-ir^2] \\ [2r+i+ir^2] & [r^2-1] & [1-r^2] & [8r-i-ir^2] \\ [r^2+2ir+1] & [i-ir^2] & [i-ir^2] & [r^2-2ir+1] \end{pmatrix}$$

$$= \begin{pmatrix} (1+\cancel{2i}r+r^2-\cancel{2i}r+1+r^2-\cancel{2i}r+1+r^2+r^2+\cancel{2i}r+1) \\ (\cancel{ir^2}-\cancel{2r}+\cancel{i}+\cancel{2r}+\cancel{i}+\cancel{ir^2}-\cancel{2r}-\cancel{i}-\cancel{ir^2}-\cancel{ir^2}+\cancel{2r}-\cancel{i}) \\ (\cancel{ir^2}-\cancel{2r}+\cancel{i}-\cancel{2r}-\cancel{i}-\cancel{ir^2}+\cancel{2r}+\cancel{i}+\cancel{ir^2}-\cancel{ir^2}+\cancel{2r}-\cancel{i}) \\ (\cancel{r^2}+2ir+\cancel{1}+2ir-\cancel{1}-\cancel{r^2}+2ir-\cancel{1}-\cancel{r^2}+\cancel{r^2}+2ir+\cancel{1}) \end{pmatrix} = \begin{pmatrix} (4+4r^2) \\ 0 \\ 0 \\ (8ir) \end{pmatrix} \longrightarrow \text{column } 1$$

$$= \begin{pmatrix} (\cancel{i}+\cancel{ir^2}-\cancel{i}+\cancel{ir^2}-\cancel{ir^2}+\cancel{i}+\cancel{i}-\cancel{ir^2}) \\ (1-r^2+1-r^2-r^2+1+1-r^2) \\ (\cancel{1}-\cancel{r^2}-\cancel{1}+\cancel{r^2}+\cancel{r^2}-\cancel{1}+\cancel{1}-\cancel{r^2}) \\ (-\cancel{i}+\cancel{ir^2}+\cancel{i}-\cancel{ir^2}+\cancel{ir^2}-\cancel{i}+\cancel{i}-\cancel{ir^2}) \end{pmatrix} = \begin{pmatrix} 0 \\ (4-4r^2) \\ 0 \\ 0 \end{pmatrix} \longrightarrow \text{column } 2$$

$$= \begin{pmatrix} (-\cancel{i}+\cancel{ir^2}-\cancel{ir^2}+\cancel{i}-\cancel{i}+\cancel{ir^2}+\cancel{i}-\cancel{ir^2}) \\ (\cancel{1}-\cancel{r^2}+\cancel{r^2}-\cancel{1}-\cancel{1}+\cancel{r^2}+\cancel{1}-\cancel{r^2}) \\ (1-r^2-r^2+1+1-r^2+1-r^2) \\ (-\cancel{i}+\cancel{ir^2}+\cancel{ir^2}-\cancel{i}+\cancel{i}-\cancel{ir^2}+\cancel{i}-\cancel{ir^2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (4-4r^2) \\ 0 \end{pmatrix} \longrightarrow \text{column } 3$$

$$= \begin{pmatrix} (\cancel{r^2}-2ir+\cancel{1}-2ir-\cancel{1}-\cancel{r^2}-2ir-\cancel{1}-\cancel{r^2}+\cancel{r^2}-2ir+\cancel{1}) \\ (\cancel{ir^2}+\cancel{2r}+\cancel{i}+\cancel{2r}-\cancel{i}-\cancel{ir^2}-\cancel{2r}+\cancel{i}+\cancel{ir^2}-\cancel{ir^2}-\cancel{2r}-\cancel{i}) \\ (\cancel{ir^2}+\cancel{2r}+\cancel{i}-\cancel{2r}+\cancel{i}+\cancel{ir^2}+\cancel{2r}-\cancel{i}-\cancel{ir^2}-\cancel{ir^2}-\cancel{2r}-\cancel{i}) \\ (r^2-\cancel{2i}r+1+\cancel{2i}r+1+r^2+\cancel{2i}r+1+r^2+r^2-\cancel{2i}r+1) \end{pmatrix} = \begin{pmatrix} (-8ir) \\ 0 \\ 0 \\ (4+4r^2) \end{pmatrix} \longrightarrow \text{column } 4$$

$$\boxed{\hat{\rho}_{prep} = \hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

## Exercise 3 Continued

b.) Determine state after unitary $\hat{u}(\lambda)$ acts on one

i.) Construct $\hat{u}$

$$\hat{u} = \hat{u}(\lambda) \otimes \hat{I} = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-i\lambda/2} & 0 & 0 & 0 \\ 0 & e^{-i\lambda/2} & 0 & 0 \\ 0 & 0 & e^{i\lambda/2} & 0 \\ 0 & 0 & 0 & e^{i\lambda/2} \end{pmatrix}$$

$$\hat{u} = \begin{pmatrix} e^{-i\lambda/2} & 0 & 0 & 0 \\ 0 & e^{-i\lambda/2} & 0 & 0 \\ 0 & 0 & e^{i\lambda/2} & 0 \\ 0 & 0 & 0 & e^{i\lambda/2} \end{pmatrix}$$

ii.) Calculate $\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger}$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \begin{pmatrix} e^{-i\lambda/2} & 0 & 0 & 0 \\ 0 & e^{-i\lambda/2} & 0 & 0 \\ 0 & 0 & e^{i\lambda/2} & 0 \\ 0 & 0 & 0 & e^{i\lambda/2} \end{pmatrix} \frac{1}{16} \begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix} \begin{pmatrix} e^{+i\lambda/2} & 0 & 0 & 0 \\ 0 & e^{i\lambda/2} & 0 & 0 \\ 0 & 0 & e^{-i\lambda/2} & 0 \\ 0 & 0 & 0 & e^{-i\lambda/2} \end{pmatrix}$$

$$= \frac{1}{16} \begin{pmatrix} e^{-i\lambda/2} & 0 & 0 & 0 \\ 0 & e^{-i\lambda/2} & 0 & 0 \\ 0 & 0 & e^{i\lambda/2} & 0 \\ 0 & 0 & 0 & e^{i\lambda/2} \end{pmatrix} \begin{pmatrix} (4+4r^2)e^{i\lambda/2} & 0 & 0 & (-8ir)e^{-i\lambda/2} \\ 0 & (4-4r^2)e^{i\lambda/2} & 0 & 0 \\ 0 & 0 & (4-4r^2)e^{-i\lambda/2} & 0 \\ (8ir)e^{i\lambda/2} & 0 & 0 & (4+4r^2)e^{-i\lambda/2} \end{pmatrix}$$

$$\boxed{\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16} \begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

c.) Calculate QFI after $\hat{u}(\lambda)$ has acted.

i.) $$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16} \begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}$$

This consists of two 2x2 matrices.

## Exercise 3 Continued

These matrices are

$$\hat{A}_1 = \frac{1}{16}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix}, \quad \hat{A}_2 = \frac{1}{16}\begin{pmatrix} 4+4r^2 & -8ire^{-i\lambda} \\ 8ire^{i\lambda} & 4+4r^2 \end{pmatrix}, \quad \hat{\rho}_1 = \frac{1}{16}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix}, \quad \hat{\rho}_2 = \frac{1}{16}\begin{pmatrix} 4+4r^2 & -8ir \\ 8ir & 4+4r^2 \end{pmatrix}$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial \ln(\alpha)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I}, \quad H_i = \hat{Tr}\left[\hat{\rho}_i \hat{L}_i^2\right]$$

We can find the QFI for both $\hat{A}_1, \hat{\rho}_1$

ii.) Calculate $\hat{L}_1$ : $\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix} = \frac{1}{16^2}\begin{pmatrix} (4-4r^2)^2 & 0 \\ 0 & (4-4r^2)^2 \end{pmatrix}$$

$$Tr(\hat{A}_1^2) = \frac{1}{16^2}\left[(4-4r^2)^2 + (4-4r^2)^2\right] = \frac{2}{16^2}\left[16 - 32r^2 + 16r^4\right]$$

$$Tr(\hat{A}_1) = \frac{1}{16}\left[4-4r^2+4-4r^2\right] = \frac{1}{16}\left[8-8r^2\right] = \frac{1}{2}\left[1-r^2\right] \;:\; Tr(\hat{A}_1)^2 = \frac{1}{4}\left[1-2r^2+r^4\right]$$

$$\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1) = \frac{1}{128}\left[16-32r^2+16r^4\right] - \frac{32}{128}\left[1-2r^2+r^4\right]$$

$$= \frac{16}{128} - \frac{32r^2}{128} + \frac{16r^4}{128} - \frac{32}{128} + \frac{64r^2}{128} - \frac{32r^4}{128} = -\frac{16}{128} + \frac{32r^2}{128} - \frac{16r^4}{128} = -\frac{1}{8} + \frac{r^2}{4} - \frac{r^4}{8}$$

$\alpha = -\frac{1}{8}(1-2r^2+r^4)$, Since $\alpha$ does not depend upon $\lambda$, $\hat{L}$ changes to

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \cancel{\frac{\partial \ln(\alpha)}{\partial\lambda}}^{\,0}\hat{A}\right] + \cancel{\frac{\partial}{\partial\lambda}}^{\,0}\left[\cancel{\ln(\alpha)} - \ln(\cancel{Tr(\hat{A})})\right]\hat{I} = \frac{2}{Tr(\hat{A}_1)}\cdot\frac{\partial\hat{A}}{\partial\lambda}$$

$\hat{A}_1$ does not depend on $\lambda$ $\therefore$ $\hat{L}_1 = 0$

$$\hat{L}_1 = 0$$

iii.) Calculate $\hat{L}_2$ : $\alpha = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix} 4+4r^2 & -8ire^{-i\lambda} \\ 8ire^{i\lambda} & 4+4r^2 \end{pmatrix}\begin{pmatrix} 4+4r^2 & -8ire^{-i\lambda} \\ 8ire^{i\lambda} & 4+4r^2 \end{pmatrix} = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)^2+64r^2 & -16ire^{-i\lambda}(4+4r^2) \\ 16ire^{i\lambda}(4+4r^2) & (4+4r^2)^2+64r^2 \end{pmatrix}$$

$$Tr(\hat{A}_2^2) = \frac{1}{16^2}\left(2(4+4r^2)^2+128r^2\right) = \frac{1}{128}\left((4+4r^2)^2+64r^2\right) = \frac{1}{128}\left(16+32r^2+16r^4+64r^2\right)$$

$$= \frac{1}{128}\left(16r^4+96r^2+16\right) = \frac{r^4}{8} + \frac{3r^2}{4} + \frac{1}{8} = \frac{1}{8}\left(r^4+6r^2+1\right)$$

$$Tr(\hat{A}_2^2) = \frac{1}{8}\left(r^4+6r^2+1\right)$$

Exercise 3 Continued

$$Tr(\hat{A}_2) = \frac{1}{16}(4+4r^2+4+4r^2) = \frac{1}{16}(8+8r^2) = \frac{1}{2}(1+r^2) \; : \; Tr(\hat{A}_2)^2 = \frac{1}{4}(1+r^2)^2 = \frac{1}{4}(1+2r^2+r^4)$$

$$\alpha = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2 = \frac{1}{8}(r^4+6r^2+1) - \frac{2}{8}(r^4+2r^2+1) = -\frac{1}{8}r^4 + \frac{2}{8}r^2 - \frac{1}{8} = -\frac{1}{8}(r^4-2r^2+1)$$

$\alpha = -\frac{1}{8}(r^4-2r^2+1)$, Since $\alpha$ does not depend upon $\lambda$ again, $\hat{L}$ reduces to

$$\hat{L}_2 = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial\lambda} - \overset{0}{\cancel{\frac{\partial \ln(\alpha)}{\partial\lambda}}}\hat{A}\right] + \overset{0}{\cancel{\frac{\partial}{\partial\lambda}}}\left[\cancel{\ln(\alpha)} - \cancel{\ln(Tr(\hat{A}))}\right]\hat{I} = \frac{2}{Tr(\hat{A}_2)}\cdot\frac{\partial \hat{A}_2}{\partial\lambda}$$

$$\hat{L}_2 = \frac{2}{Tr(\hat{A}_2)}\cdot\frac{\partial \hat{A}_2}{\partial\lambda} \; : \; \frac{\partial \hat{A}_2}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & 8re^{-i\lambda} \\ -8re^{i\lambda} & 0 \end{pmatrix} \; \therefore \; \hat{L}_2 = \frac{1}{(1+r^2)}\cdot\begin{pmatrix} 0 & 2re^{-i\lambda} \\ -2re^{i\lambda} & 0 \end{pmatrix}$$

$$\hat{L}_2 = \frac{1}{(1+r^2)}\begin{pmatrix} 0 & 2re^{-i\lambda} \\ -2re^{i\lambda} & 0 \end{pmatrix}$$

iv.) Calculate QFI :

$$H = H_1 + H_2 \; : \; H_1 = 0$$

$$H_2 = Tr(\hat{P}_2\cdot\hat{L}_2^2)$$

$$\hat{L}_2^2 = \frac{1}{(1+r^2)^2}\begin{pmatrix} 0 & 2re^{-i\lambda} \\ -2re^{i\lambda} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2re^{-i\lambda} \\ -2re^{i\lambda} & 0 \end{pmatrix} = \frac{1}{(1+r^2)^2}\begin{pmatrix} -4r^2 & 0 \\ 0 & -4r^2 \end{pmatrix}$$

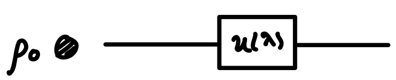
$$\hat{P}_2\cdot\hat{L}_2^2 = \frac{1}{16}\begin{pmatrix} 4+4r^2 & -8ir \\ 8ir & 4+4r^2 \end{pmatrix}\cdot\frac{1}{(1+r^2)^2}\begin{pmatrix} -4r^2 & 0 \\ 0 & -4r^2 \end{pmatrix} = \frac{1}{16(1+r^2)^2}\begin{pmatrix} (4+4r^2)(-4r^2) & 32ir^3 \\ -32ir^3 & (4+4r^2)(-4r^2) \end{pmatrix}$$

$$(\hat{P}_2\cdot\hat{L}_2^2) = \frac{1}{16(1+r^2)^2}\begin{pmatrix} -16r^2-16r^4 & 32ir^3 \\ -32ir^3 & -16r^2-16r^4 \end{pmatrix}$$

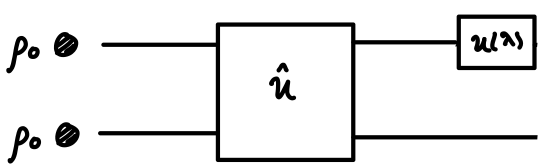
$$Tr(\hat{P}_2\cdot\hat{L}_2^2) = \frac{1}{16(1+r^2)^2}(-32r^2-32r^4) = \frac{-32r^2}{16(1+r^2)^{\cancel{2}}}\cdot\cancel{(1+r^2)} = \frac{-2r^2}{(1+r^2)} \; : \; |Tr(\hat{P}_2\cdot\hat{L}_2^2)| = \frac{2r^2}{(1+r^2)}$$

$$\boxed{H = \frac{2r^2}{(1+r^2)}}$$

d.)




$H = r^2$




$$H = \frac{2r^2}{(1+r^2)}$$