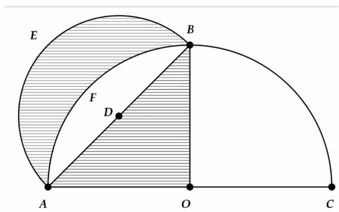
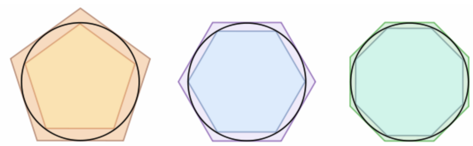
Ch4.3 - 4.7 Numerical Integration

- Historically known as quadrature, Pythagoreans in ancient Greece Computed the area of a region by finding a square that had the same area.
- Archimedes used the method of exhaustion as well, in contrast to the method of quadrature. (Wikipedia → Numerical Integration)

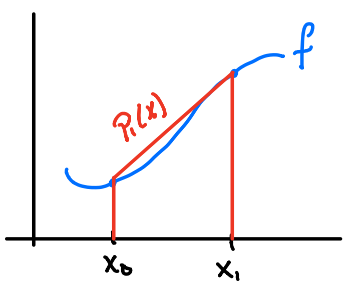


The lune of Hippocrates is the upper left shaded area. It has the same area as the lower right shaded triangle.



Archimedes used the method of exhaustion to compute the area inside a circle

As with numerical differentiation, we start with a Lagrange interpolating polynomial through our data points.



Degree 1 Case (Trapezoid rule)

$f(x) = P_1(x) + E_1(x)$

$$= \frac{(x-x_1)}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + \frac{f''(\xi(x))}{2!}(x-x_0)(x-x_1)$$

$$\therefore \int_{x_0}^{x_1} f(x)\,dx = f(x_0)\underbrace{\int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1}\,dx}_{C_0} + f(x_1)\underbrace{\int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}\,dx}_{C_1} + \underbrace{\int_{x_0}^{x_1} \frac{f''(\xi(x))}{2!}(x-x_0)(x-x_1)\,dx}_{E_1(x)}$$

$$\cong f(x_0)C_0 + f(x_1)C_1 \quad \leftarrow \text{quadrature rule (formula)}$$

$$= \sum_{k=0}^{1} f(x_k)C_k$$

If $f(x)$ is a polynomial of degree 1 or less, then $f''(x) = 0$ and $E_1(x) = 0$.

Ch4.3 Overview: Numerical Integration

The basic method involved in approximating $\int_a^b f(x)dx$ is called quadrature.

Typically we first select distinct nodes $\{x_0, x_1, \dots, x_n\}$ from $[a,b]$. Using the Lagrange interpolating polynomial,

$$f(x) = \sum_{k=0}^{n} f(x_k)L_k(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\prod_{k=0}^{n}(x-x_k)$$

Thus

$$\int_a^b f(x)dx = \sum_{k=0}^{n} a_k f(x_k) + \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi(x))\prod_{k=0}^{n}(x-x_k)dx$$

where $a_k = \int_a^b L_k(x)dx$, $k=0,1,\dots,n$.

Quadrature Formula: $\int_a^b f(x)dx \cong \sum_{k=0}^{n} a_k f(x_k)$

Error Formula: $E(f) = \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi(x))\prod_{k=0}^{n}(x-x_k)dx$

We have

$$\int_a^b f(x)\,dx = \underbrace{\sum_{k=0}^{n} a_k f(x_k)}_{\text{Quadrature Formula}} + \underbrace{\frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\xi(x))\prod_{k=0}^{n}(x-x_k)\,dx}_{\text{Error Formula } E(f)}$$

<u>Definition</u> The <u>degree of accuracy</u>, or <u>precision</u>, of a quadrature formula is the largest positive integer n for which the formula is exact for x^k, $k=0,1,2,\dots,n$.

Thus if a quadrature formula has precision n, then it is exact for all polynomials $p(x)$ of degree at most n.

Newton-Cotes formulas are a class of methods used for quadrature, and include the Trapezoid Rule and Simpson's Rule from Calculus.

Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi).$$

This is called the Trapezoidal rule because when f is a function with positive values, $\int_a^b f(x)\,dx$ is approximated by the area in a trapezoid, as shown in Figure 4.3.

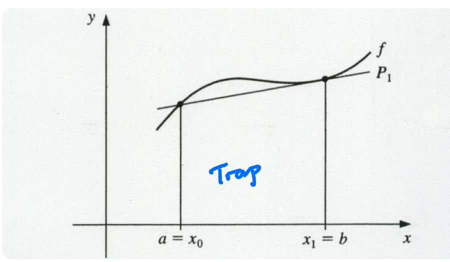
Degree of accuracy = 1

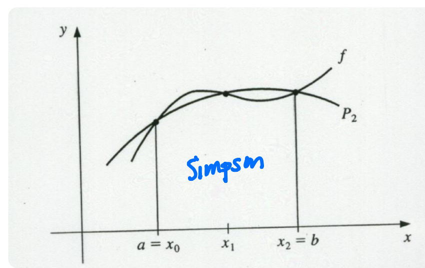
Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

Since the error term involves the fourth derivative of f, Simpson's rule gives exact results when applied to any polynomial of degree three or less.

Degree of accuracy = 3





Ch.4.4 Overview: Composite Numerical Integ.

- Newton-Cotes methods are generally unsuitable for use over large integration intervals.
- High-degree formulas ~~[illegible]~~ are required, and the values of the coefficients are difficult to obtain.
- Furthermore, interpolation polynomials based on equally spaced nodes over large intervals are typically highly oscillatory.
- Instead, a piecewise approach to numerical integration is taken, and uses low order Newton-Cotes formulas. see text examples.

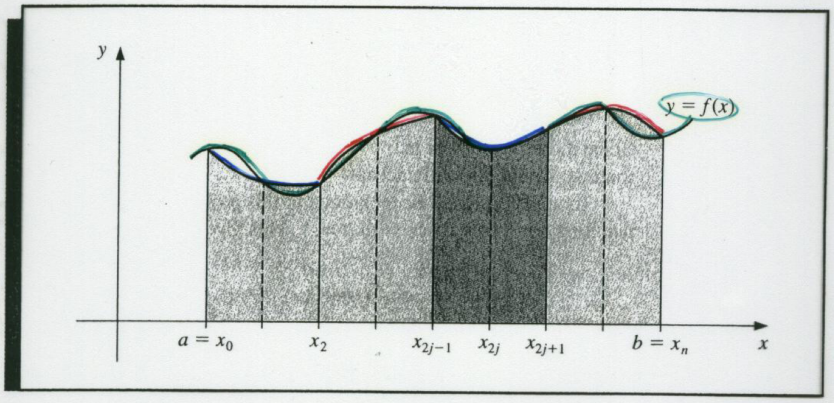
- Composite integration methods are stable with respect to roundoff error. Accumulated errors are bounded independent of h and n. (see p. ~~[illegible]~~ 203)
- Thus we can increase n and decrease h in order to improve accuracy, without increasing roundoff error. (It does increase # of computations) $\blacksquare$

Theorem 4.4

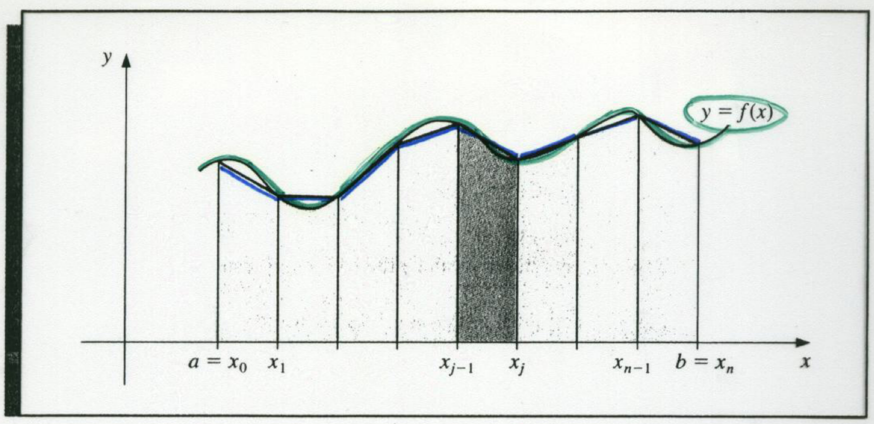
Let $f \in C^4[a, b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a, b)$ for which the **Composite Simpson's rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

■



Theorem 4.5



Let $f \in C^2[a, b]$, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a, b)$ for which the **Composite Trapezoidal rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

■

Ch4.5 Romberg Integration (Overview)

This method uses the Composite Trapezoid rule to give preliminary approximations, and then applies Richardson extrapolation to improve them.

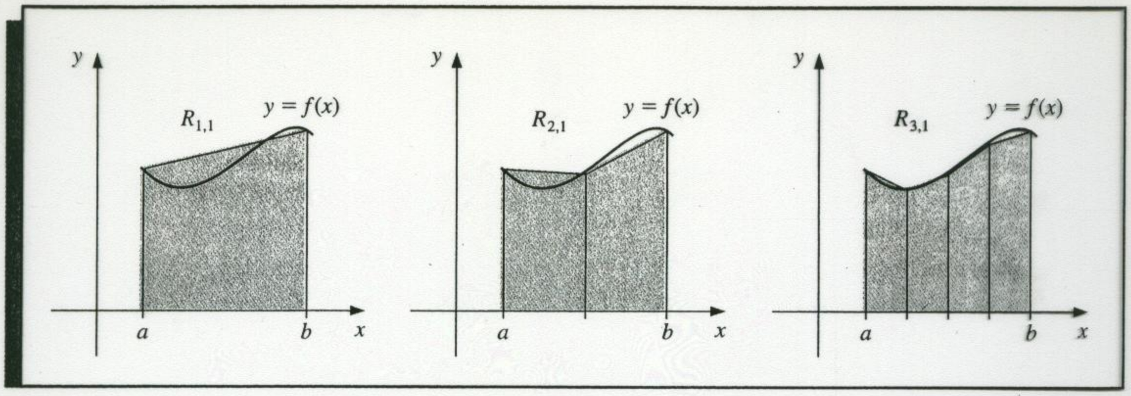
Ch4.6 Adaptive Quadrature Methods. (Overview)

Our composite formulas required equally spaced nodes. This is inappropriate when integrating a function with both large and small functional variation on [a,b].

If the approximation error is to be evenly distributed over [a,b], a smaller step size h is needed for large variation regions than for small variation regions.

Adaptive methods efficiently predict amount of variation, and adapt step size h accordingly.

Figure 4.10



$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$

The results that are generated from these formulas are shown in Table 4.9.

Table 4.9

$R_{1,1}$					
$R_{2,1}$	$R_{2,2}$				
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$