APS/123-QED

Physics 231, Exam 2

Ch 5 and 6

(Dated: April 17, 2018)

Abstract

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DIRECTIONS: Show all work and make your reasoning clear. Make sure I can follow your steps and locate answers. Closed-book, only a calculator, notecard, and the attached reference sheet are permitted as resources.

Part I: shorter problems/conceptual: choose 5 out of 6 (w/ 2 points extra-credit possible.)

Part II: longer problems: choose 1 of 2 (w/ 10 points extra-credit possible.)

PACS numbers:

I. EXAM PROBLEMS TOTAL SCORE: 95

A. Shorter problems/conceptual: SCORE: 67

Choose 5 out of 6 problems, 15 pts each. **CIRCLE THE NUMBERS OF YOUR CHOICES!**

1.) Try to normalize the wavefunction $\Psi(x,t) = e^{i(kx-\omega t)}$ over all space. Explain what the result of your attempt means physically in the context of the uncertainty principle.

2.) Consider a solution to Schrödinger's equation, $\Psi(x,t)$ for a particle bound in some 1D potential well $V(x)$. What does the integral $\int_{x=a}^{x=b} \Psi^*(x,t)\Psi(x,t)dx$ mean physically?

EC 3.) Calculate the expectation value of the position, $\langle x \rangle$, of a quantum simple harmonic oscillator (SHO) in the states (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$.

4.) Sketch the eigenfunctions for the first four states of the quantum SHO on the figure below and indicate the values of the corresponding energies in terms of $\hbar\omega_o$, where $\omega_o = \sqrt{k/m}$ is the frequency of the equivalent classical oscillator.

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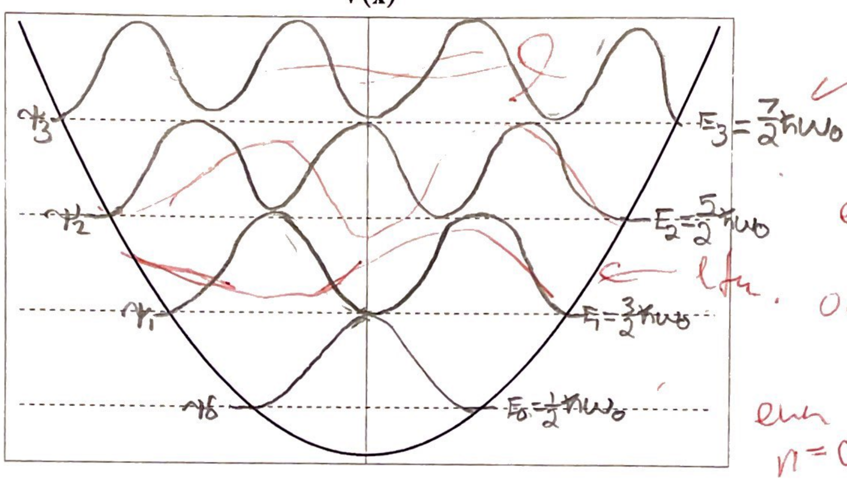


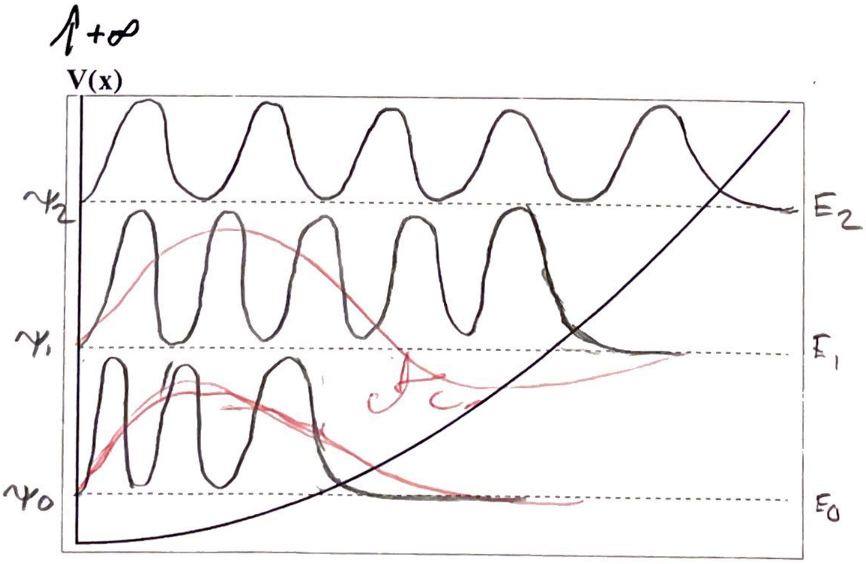
odd n=3

even n=2

odd n=1

even n=0

5.) The potential for a toy model of a restricted harmonic oscillator is shown below. It is just like a regular oscillator but restricted to displacements right of the equilibrium point only because the potential goes to infinity for $x \leq 0$. This has consequences for the symmetry! Sketch your best educated guesses for the first few wavefunctions.



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When $E > V_0$, ψ_n oscillates where the particle transmits toward the barrier

When $E < V_0$, ψ_n decays and the particle may penetrate the barrier

6.) Consider the barrier potential shown below where $E_1 = V_o - 4\hbar^2/mW^2$ and $E_2 = 10E_1$. Write down the number of the equation (see reference section at the end) that would be most appropriate to calculate the transmission probability next to the corresponding energy and justify your choice.

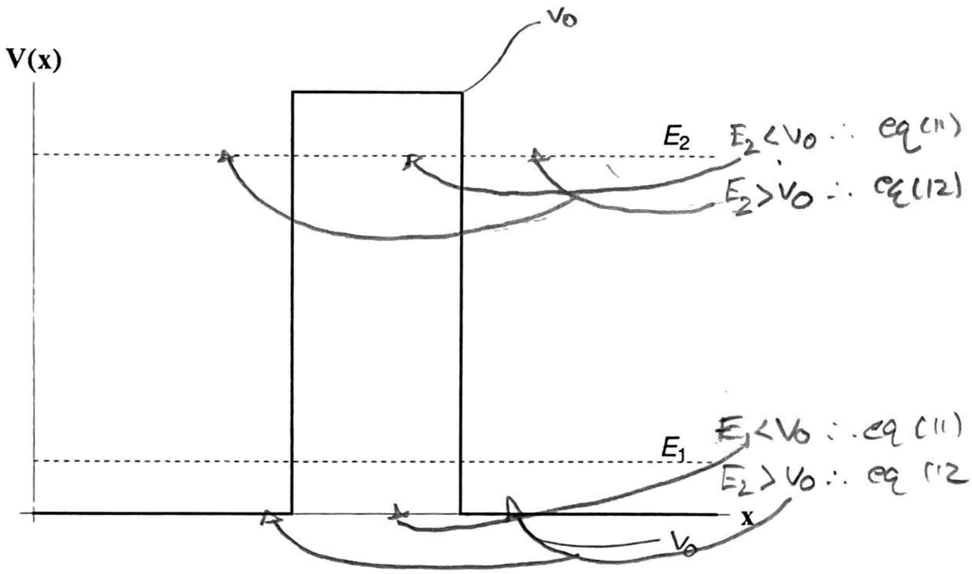

FIG. 1: Potential barrier of height V_0 and width W.

I have a scratch paper attached with my more thorough reasoning.

II. LONGER PROBLEMS: CHOOSE 1 (25 PTS) NUMBER:____
SCORE:28 (w/ EC)

Please choose one of the following for full credit and indicate your choice in the space above next to "NUMBER." If you attempt both problems, the second problem will be used as extra credit (worth a maximum of 10 points.) Attach your work on extra paper and make sure your name (and problem number) are indicated clearly on each extra sheet.

1. *Infinite square well problem*

Consider the solutions of the infinite square well potential of total width L centered about the origin. (a) Write down or find the general solutions to the time-independent Schrödinger equation, $\psi_n(x)$, and normalize them for the the $n = 1$, $n = 2$, and $n = 3$ cases. (15 pts) (b) Write down or find the energy eigenvalues and calculate the difference, $\Delta E = E_3 - E_1$. (5 pts) (c) Consider the integral $\langle n = 1|x|n = 3\rangle = \int_{-L/2}^{L/2} \Psi_1^*(x,t)x\Psi_3(x,t)dx$, where $\Psi_n(x,t)$ is a normalized solution to the time-dependent Schrödinger wave equation for the infinite square well. What is this integral associated with physically? (5 pts)

EC

2. *Estimating the ground state energy of the Morse potential using the uncertainty principle*

Assume that bound states correspond approximately to very small displacements from r_e, the equilibrium point in the Morse potential from textbook problems 63 and 64, $V(r) = D(1 - \exp\{-\alpha(r - r_e)\})^2$. Use the uncertainty principle to estimate the ground state energy for the quadratic approximation to the Morse potential and find an expression for the percent difference with the value from the perturbation theory energy estimate provided in problem 64, $E_n = \hbar\omega_m(n + 1/2) - \hbar^2\omega_m^2(n + 1/2)^2/4D$ where $\omega_m = \alpha\sqrt{2D/m}$.

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Exam 2

Part I

Problem 1

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$\Psi(x,t) = Ae^{i(kx-\omega t)}$

$\Psi^*(x,t) = Ae^{-i(kx-\omega t)}$

$\Psi(x,t) = A(e^{ikx-i\omega t})$

$\Psi^*(x,t) = Ae^{-ikx+i\omega t}$

Normalize: $\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\,dx = 1$

$$\int_{-\infty}^{\infty} Ae^{(-ikx+i\omega t)}\,Ae^{(ikx-i\omega t)}\,dx = 1$$

$$\int_{-\infty}^{\infty} A^2e^{(-\cancel{ikx}+\cancel{i\omega t}+\cancel{ikx}-\cancel{i\omega t})}\,dx = 1$$

$$\int_{-\infty}^{\infty} A^2e^{0}\,dx = 1$$

$$\boxed{\int_{-\infty}^{\infty} A^2\,dx = 1} \quad (1)$$

This integral is impossible, with the result of it being equal to 1 is impossible, Thus why we have to change to $(0, L)$. In context of the uncertainty principle, $\Delta p \Delta x \geq \hbar/2$, the position of this particle would be completely unknown

$$A^2x\Big|_{-\infty}^{\infty} = 1$$

Changing limits of integration $\{0, L\}$

$$A^2x\Big|_{0}^{L} = 1$$

$$A^2L = 1 \;\therefore\; A^2 = \frac{1}{L} \;\therefore\; A = \frac{1}{\sqrt{L}}$$

$$\boxed{\Psi = \frac{1}{\sqrt{L}}e^{i(kx-\omega t)}} \quad (2)$$

When we change limits

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Exam 2

Part I

Problem 2

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Infinite

$$V(x) = \begin{cases} \infty & x \le 0,\ x \ge L \\ 0 & 0 < x < L \end{cases}$$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = -k^2\psi$$

$$\psi_{(n)} = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$k_n = \frac{n\pi}{L}$$

Finite

$$V(x) = \begin{cases} V_0, & x \le 0 & R\,I \\ 0, & 0 < x < L & R\,II \\ V_0 & x \ge L & R\,III \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} = E - V_0 \quad R\,I,\ R\,III$$

$$-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} = E \quad R\,II$$

$$\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi_I = Ae^{\alpha x} \qquad \psi_{II} = Ce^{ikx} \qquad \psi_{III} = Be^{-\alpha x}$$

The integral: $\int_{x=a}^{x=b} \psi^*(x,t)\,\psi(x,t)\,dx$ is the probability of finding a particle of the wave function $\psi(x,t)$ in space between (a,b) at any point in time.

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Exam II

Extra credit Problem +2

Part I

Problem 3

$$\langle x \rangle = \int_{-\infty}^{\infty} x\Psi^*(x,t)\Psi(x,t)\,dx$$

a) $n=0$

$$\Psi_0(x) = \frac{a^{1/4}}{\pi}(1)e^{-\alpha x^2/2}$$

$$\Psi_0^*(x) = \frac{a^{1/4}}{\pi}(1)e^{-\alpha x^2/2}$$

$$\Psi^*\Psi = \frac{a^{1/2}}{\pi^2}e^{-\alpha x^2}$$

$$\langle x \rangle = \int_0^{\infty} x\frac{\sqrt{a}}{\pi^2}e^{-\alpha x^2}$$

4

0

$\boxed{\text{odd function, cannot } \langle x \rangle}$

b.)

$$\Psi_1(x) = \frac{a^{1/4}}{\pi}\sqrt{2a}\,x\,e^{-\alpha x^2/2}$$

$$\Psi_1^*(x) = \frac{a^{1/4}}{\pi}\sqrt{2a}\,x\,e^{-\alpha x^2/2}$$

3

$$\langle x \rangle = \int_0^{\infty} x\,x^2\frac{\sqrt{a}}{\pi^2}e^{-\alpha x^2} = \frac{1}{4a}\sqrt{\frac{\pi}{a}}\cdot\frac{\sqrt{a}}{\pi^2} = \frac{1}{4a}\frac{\sqrt{\pi}}{\pi^2}$$

3

$$\frac{1}{4a}\frac{\sqrt{\pi}}{\cancel{\sqrt{a}}}\frac{\cancel{\sqrt{a}}}{\pi^2} = \frac{1}{4a}\frac{\sqrt{\pi}}{\pi^2} = \frac{1}{4a}\pi^{-3/2}$$

$\boxed{\langle x \rangle,\ n=1,\ \frac{1}{4a}\pi^{-3/2}}$

c.)

$$\Psi_2(x) = \frac{a^{1/4}}{\pi}\frac{1}{\sqrt{2}}(2\alpha x^2-1)e^{-\alpha x^2/2}$$

$$(2\alpha x^2-1)(2\alpha x^2-1)$$

$$4\alpha^2x^4-4\alpha x^2+1$$

$$\Psi_2^*(x) = \frac{a^{1/4}}{\pi}\frac{1}{\sqrt{2}}(2\alpha x^2-1)e^{-\alpha x^2/2}$$

x

$$\langle x \rangle = \int_0^{\infty}\frac{\sqrt{a}}{\pi^2}(4\alpha^2x^4-4\alpha x^2+1)\frac{1}{4}e^{-\alpha x^2/2}$$

$\longrightarrow$ $\boxed{\text{no idea from here}}$

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Exam 2

Part I
Problem 4

See packet

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Part I
Problem 5 |

$E_n > V_0$, ψ_n oscillates

$E_n < V_0$, ψ_n decays

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Exam 2

Part I

Problem 6

$\frac{10}{15}$

$$E_1 = V_0 - \frac{2\hbar^2}{m\omega^2}$$

@ E_1, $E > V_0$ $\therefore$ eq (12)

$$\boxed{T = 16\frac{E}{V_0}\left(1 - \frac{E}{V_0}\right)e^{-2\kappa L}}$$

$$E_2 = V_0 - \frac{200\hbar^2}{m\omega^2}$$

@ E_2, $E < V_0$ $\therefore$ eq (11)

$$k_1 = \frac{2\hbar^2}{m\omega^2} \rightarrow kW = \sqrt{2m\,\text{[illegible]}}\,W$$

$$k_2 = \frac{200\hbar^2}{m\omega^2} \quad KW =$$

$$T = \left[1 + \frac{V_0^2 \sinh(\kappa L)^2}{4E(V_0 - E)}\right]^{-1}$$

$$\boxed{\begin{array}{l}\text{When } E < V_0\text{, we use equation (11)}\\ \text{When } E > V_0\text{, we use equation (12)}\end{array}}$$

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Exam 2
Part II

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Problem 1

a.) Normalized general solution : $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$

$\boxed{n=1 \quad \psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right)}$, $\boxed{n=2 \quad \psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L}x\right)}$, $\boxed{n=3 \quad \psi_3(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi}{L}x\right)}$

b.) Energy Eigenvalues

$\frac{\pi^2 \cdot h^2}{4\pi^2} = \frac{h^2}{4}$

$\hbar = \frac{h}{2\pi}$

$\hbar^2 = \frac{h^2}{4\pi^2}$

$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ $\qquad E_3 = \frac{(3)^2\pi^2\hbar^2}{2mL^2}$ $\qquad E_1 = \frac{(1)^2\pi^2\hbar^2}{2mL^2}$

$\Delta E = E_3 - E_1 = \frac{9h^2}{8mL^2} - \frac{h^2}{8mL^2} = \frac{9h^2 - h^2}{8mL^2} = \frac{8h^2}{8mL^2} = \frac{h^2}{8mL^2}$

$$\boxed{\Delta E = \frac{h^2}{8mL^2}}$$

c.)

$\int_0^L x\,\psi_1^*(x,t)\,\psi_3(\cancel{x,t})\,dx \iff \langle x \rangle = \int_{-\infty}^{\infty} x\,\psi^*(x,t)\,\psi(x,t)\,dx$

The expected result of the average of many measurements of a given quantity

The expected result of the average position between $\psi_1(x,t)$ and $\psi_3(x,t)$.

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Exam 2

Part II

Problem 2

Extra Credit problem

$$V(r) = D(1 - e^{-\alpha(r-r_e)})^2 \qquad r - r_e = x$$

a)

$$V(x) = e^{-x} \qquad V(0) = 1$$

$$V'(x) = -e^{-x} \qquad V''(x) = e^{-x}$$

$$V'(0) = -1 \qquad V''(0) = 1$$

$$\sum_n^\infty \frac{f^{(n)}(x-a)^n}{n!} = \underbrace{1 - x + \frac{x^2}{2}}_{(1-\alpha)}$$

+8 EC

$$V(r) = D(1 - (1-\alpha)(r - r_e))^2$$

$$V(r) = D(1 - 1 + \alpha(r - r_e))$$

$$V(r) = D(\alpha(r - r_e))^2$$

$$\boxed{V(r) = D\alpha^2(r - r_e)^2}$$

b.)

$$E = K + V \qquad K = \frac{p^2}{2m} \qquad V = \frac{kx^2}{2}$$

$$E = \frac{p^2}{2m} + \frac{kx^2}{2} \qquad \Delta p = \frac{\hbar}{2\Delta x} \qquad \Delta p^2 = \frac{\hbar^2}{4\Delta x^2}$$

$$= \frac{\left(\frac{\hbar^2}{4\Delta x^2}\right)}{2m} + \frac{k\Delta x^2}{2} \qquad \frac{\hbar^2}{8m}\Delta x^{-2} \quad -\frac{\hbar^2}{4m}\Delta x^{-3}$$

$$E = \frac{\hbar^2}{8m\Delta x^2} + \frac{k}{2}\Delta x^2$$

$$\frac{dE}{d\Delta x} = \frac{-\hbar^2}{4m\Delta x^3} + k\Delta x$$

$$0 = \frac{-\hbar^2}{4m\Delta x^3} + k\Delta x \rightarrow k\Delta x = \frac{\hbar^2}{4m\Delta x^3}$$

$$\Delta x^4 = \frac{\hbar^2}{4mk}$$

$$\Delta x^2 = \sqrt{\frac{\hbar^2}{4mk}}$$

$$E = \frac{\hbar^2}{8m\Delta x^2} + \frac{k}{2}\Delta x^2$$

$$= \frac{\hbar^2}{8m}\sqrt{\frac{4mk}{\hbar^2}} + \frac{k}{2}\sqrt{\frac{\hbar^2}{4mk}}$$

$$= \frac{\hbar\, 2\sqrt{mk}}{8m} + \frac{k}{2}\frac{\hbar}{2\sqrt{mk}}$$

$$= \frac{\hbar\sqrt{mk}}{4m} + \frac{k\hbar}{4\sqrt{mk}}$$

$$= \frac{4\hbar mk + 4\hbar mk}{16m\sqrt{mk}} = \frac{8\hbar mk}{16m\sqrt{mk}}$$

$$= \frac{\hbar k}{2\sqrt{mk}} = \frac{\hbar}{2}\sqrt{\frac{k}{m}} = \frac{\hbar}{2}\omega$$

$$\boxed{E_n = \frac{\hbar}{2}\omega_0} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

III. REFERENCE MATERIALS

A. Useful integrals and identities

1. For a symmetric interval centered about the origin of width a (where a may be finite or infinite),

For an even (symmetric about the origin) function, $f(x)$,

$$\int_{-a/2}^{a/2} f(x)dx = 2\int_{0}^{a/2} f(x)dx \tag{1}$$

For an odd (asymmetric about the origin) function, $g(x)$,

$$\int_{-a/2}^{a/2} g(x)dx = 0 \tag{2}$$

2. Other integrals

$$\int_{0}^{\infty} \exp\{-ax^2\}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{3}$$

$$\int_{0}^{\infty} x^2\exp\{-ax^2\}dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \tag{4}$$

$$\int \sin ax^2 dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{5}$$

ALSO: $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

B. Hermite polynomial expansions

For the Hermite polynomial factors in the normalized SHO eigenfunctions,

$$\psi_n(x) = \frac{\alpha}{\pi}^{1/4} H_n(x)\exp -\alpha x^2/2 \tag{6}$$

$H_n(x) = \sum_k^n a_k x^k$ (where $k = 0, 2, 4, ...n$, or $k = 1, 2, 3, ..., n$ if n is even or odd, respectively,

$$H_0 = 1 \tag{7}$$

$$H_1 = \sqrt{2\alpha}x \tag{8}$$

$$H_2 = \frac{1}{\sqrt{2}}(2\alpha x^2 - 1) \tag{9}$$

$$H_3 = \frac{1}{\sqrt{3}}(2\alpha^{3/2}x^3 - 3\alpha^{1/2}x) \tag{10}$$