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MATH 360
CP Ch. 5.3

Example 1 Euler – Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 Without a Logarithm
For the Euler-Cauchy equation (sec. 2.5)

$$x^2y'' + b_0xy' + c_0y = 0 \qquad (b_0, c_0 \text{ constant})$$

Substitution of $y = x^r$ gives the auxiliary equation

$$r(r-1) + b_0r + c_0 = 0$$

which is the indicial equation [and $y = x^r$ is a very special form of (2)!]. For different roots r_1, r_2 we get a basis $y_1 = x^{r_1}$, $y_2 = x^{r_2}$, and for a double root r we get a basis x^r, $x^r\ln(x)$. Accordingly, for this simple ODE, Case 3 plays no extra role.

Example 2 Illustration of Case 2 (Double Root)
Solve the ODE

$$x(x-1)y'' + (3x-1)y' + y = 0 \tag{11}$$

(This is a special hypergeometric equation, as we shall see in the problem set.)

Solution. Writing (11) in the standard form (1), we see that it satisfies the assumptions in Theorem 1. [What are b(x) and c(x) in (11)?] By inserting (2) and its derivatives (2*) into (11) we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} \\ + 3\sum_{m=0}^{\infty}(m+r)a_m x^{m+r} - \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty}a_m x^{m+r} = 0.\end{aligned} \tag{12}$$

The smallest power is x^{r-1}, occurring in the second and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]a_0 = 0 \quad \text{thus} \quad r^2 = 0.$$

Hence this indicial equation has the double root $r = 0$.

Second Solution. We get a second independent solution y_2 by the method of reduction of order (sec 2.1), substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9), sec. 2.1, which we shall use in this example, instead of starting reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x-1)/(x^2-x)$, the coefficient of y' in (11) in standard form. By partial fractions,

$$-\int p\,dx = -\int \frac{3x-1}{x(x-1)}\,dx = -\int\left(\frac{2}{x-1} + \frac{1}{x}\right)dx = -2\ln(x-1) - \ln(x).$$

Hence (9), sec. 2.1, becomes

$$u' = U = y_1^{-2} e^{-\int p\,dx} = \frac{(x-1)^2}{(x-1)^2 x} = \frac{1}{x}, \quad u = \ln(x), \quad y_2 = u y_1 = \frac{\ln(x)}{1-x}.$$

Y_1 and Y_2 are shown in Fig. 109. These Functions are linearly independent and thus a basis on the interval $0 < x < 1$ (as well as on $1 < x < \infty$).