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MATH 360
CP 11.1

Equations 5 & 6

Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) \quad (5)$$

Fourier Coefficients:

(0) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

(a) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

(b) $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$$

Example 1: Periodic Rectangular Wave

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 260. The formula is

(7) $$f(x) = \begin{cases} -K & \text{if } -\pi < x < 0 \\ K & \text{if } 0 < x < \pi \end{cases} \quad \text{and} \quad f(x+2\pi) = f(x)$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x=0$ and $x=\pm\pi$.)

Solution. From (6.0) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π (taken with a minus sign where $f(x)$ is negative) is zero. From (6a) we obtain the coefficients $a_1, a_2 \ldots$ of the cosine terms. Since $f(x)$ is given by two expressions, the integrals from $-\pi$ to π split into two integrals:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-K)\cos(nx)\, dx + \int_{0}^{\pi} (K)\cos(nx)\, dx\right]$$

$$= \frac{1}{\pi}\left[(-K)\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + (K)\frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right] = 0$$

because $\sin(nx) = 0$ at $-\pi, 0$, and π for all $n = 1, 2, \ldots$. We see that all these cosine coefficients are zero. That is, the Fourier series of (7) has no cosine terms, just sine terms, it is a Fourier Sine Series with coefficients $b_1, b_2, \ldots$ obtained from (6b);

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-K)\cos(nx)\, dx + \int_{0}^{\pi} (K)\cos(nx)dx\right]$$

$$= \frac{1}{\pi}\left[-K\cdot\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + K\cdot\frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right] = 0$$

because $\sin(nx)=0$ at $-\pi, 0,$ and π for all $n=1,2,\dots,$ We see that all these cosine coefficients are zero. That is, the Fourier series of (7) has no cosine terms, just sine terms, it is a Fourier sine series with coefficients $b_1, b_2, \dots$ Obtained from (6b):

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-K)\sin(nx)\,dx + \int_{0}^{\pi}(K)\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[K\cdot\frac{\cos(nx)}{n}\Big|_{-\pi}^{0} - K\cdot\frac{\cos(nx)}{n}\Big|_{0}^{\pi}\right]$$

Since $\cos(-\alpha)=\cos(\alpha)$ and $\cos(0)=1$, this yields

$$b_n = \frac{K}{n\pi}\left[\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)\right] \frac{2K}{n\pi}(1-\cos(n\pi))$$

Now, $\cos(\pi)=-1$, $\cos(2\pi)=1$, $\cos(3\pi)=-1$, etc: in general,

$$\cos(n\pi) = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \quad \text{and thus} \quad 1-\cos(n\pi) = \begin{cases} 2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4K}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4K}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4K}{5\pi} \dots$$

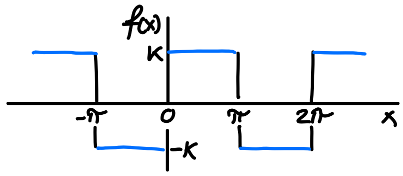

Fig. 260. Given function f(x) (Periodic rectangular wave)

Since the a_n are zero, the Fourier series of $f(x)$ is

(8) $$\frac{4K}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots +\right)$$