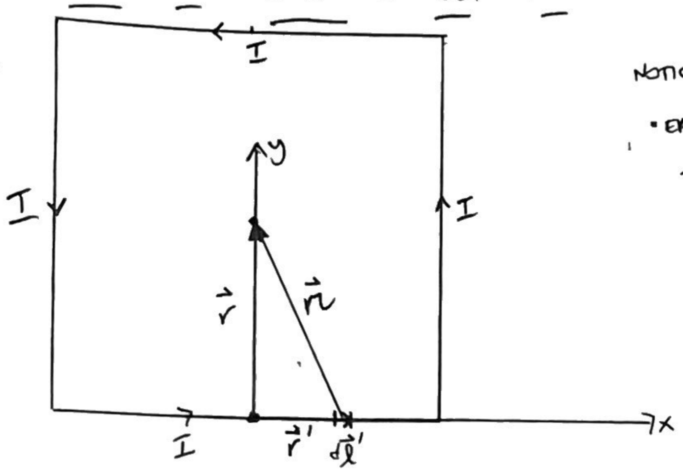
PHYS 311 HOMEWORK SET 10

1.



NOTICE:

- EACH OF THE FOUR SIDES WILL CONTRIBUTE AN EQUAL AMOUNT TO THE $\vec{B}$-FIELD @ THE CENTER OF THE LOOP

$\therefore$ CALCULATE THE $\vec{B}$-FIELD FROM ONE SIDE & MULTIPLY BY 4

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

NOW $\vec{r} = R\hat{y}$, $\vec{r}' = x'\hat{x}$ $\quad \therefore \vec{\mathscr{r}} = \vec{r} - \vec{r}' = R\hat{y} - x'\hat{x}$

$$\mathscr{r}^2 = R^2 + x'^2$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{R\hat{y} - x'\hat{x}}{\sqrt{R^2 + x'^2}}$$

ALSO

$$d\vec{l} = dx'\hat{x}$$

SO
$$d\vec{l} \times \hat{\mathscr{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{R^2+x'^2}} & \frac{R}{\sqrt{R^2+x'^2}} & 0 \end{vmatrix} = \frac{R\,dx'}{\sqrt{R^2+x'^2}}\hat{z}$$

SO

SINCE $1 + \tan^2\theta = \sec^2\theta$

$$\underset{\text{ONE SIDE}}{\vec{B}(\vec{r})} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{-R/2}^{R/2} \frac{R\,dx'}{(R^2+x'^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\hat{z}\,\frac{R^2}{R^3}\int_{\theta_1}^{\theta_2} \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{3/2}} = \frac{\mu_0 I}{4\pi R}\hat{z}\int_{\theta_1}^{\theta_2} \cos\theta\,d\theta$$

let $x' = R\tan\theta$

$dx' = R\sec^2\theta\,d\theta$

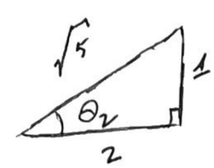
WHERE $\frac{R}{2} = R\tan\theta_2 \quad \therefore \quad \theta_2 = \tan^{-1}\left(\frac{1}{2}\right)$

$-\frac{R}{2} = R\tan\theta_2 \quad \therefore \quad \theta_1 = \tan^{-1}\left(\frac{-1}{2}\right)$

EVEN FUNCTION

$$= 2\frac{\mu_0 I}{4\pi R}\hat{z}\sin\theta\Big|_0^{\theta_2} = \frac{\mu_0 I}{2\pi R}\sin\theta_2\,\hat{z}$$

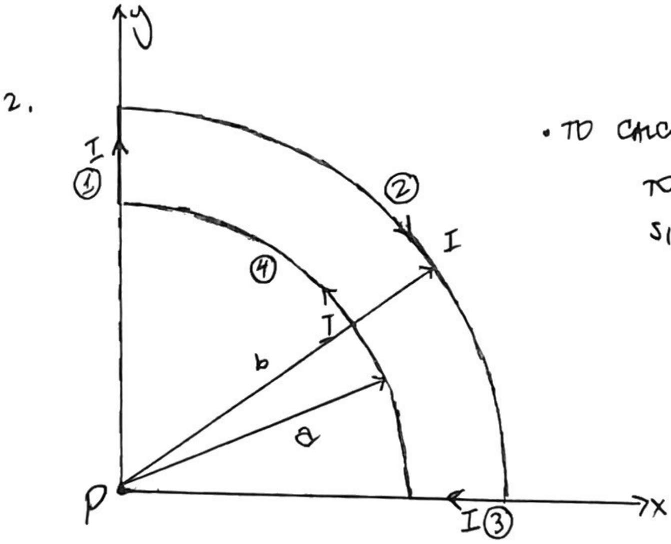
NOW



SO $\sin\theta_2 = \frac{1}{\sqrt{5}}$

$$= \frac{\mu_0 I}{2\sqrt{5}\pi R}\hat{z} \qquad \therefore \boxed{\vec{B}(\vec{r}) = \frac{2}{\sqrt{5}}\frac{\mu_0 I}{\pi R}\hat{z}}$$

2.



- TO CALCULATE THE TOTAL $\vec{B}$ FIELD @ POINT P, WE'LL NEED TO CALCULATE THE CONTRIBUTION OF EACH OF THE FOUR SIDES

$$B(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{\imath}}{\imath^2}$$

- SINCE WE'VE CHOSEN THE ORIGIN OF OUR COORDINATE SYSTEM TO BE AT THE FIELD POINT, WE HAVE $\vec{r} = 0$ FOR ALL SEGMENTS.

CONSIDER SIDES ① : ③ FIRST...

- NOTICE THAT NEITHER SEGMENT WILL CONTRIBUTE ANYTHING TO THE $\vec{B}$-FIELD

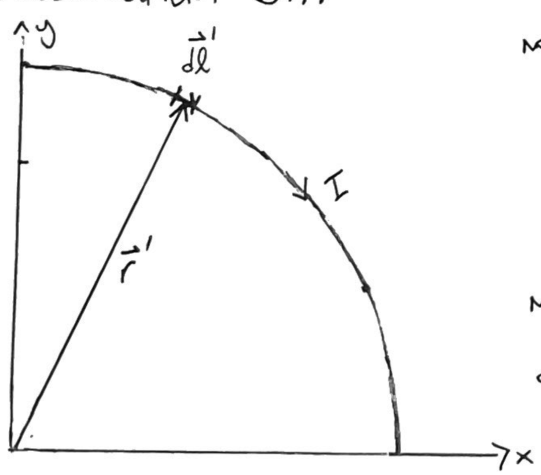
SPECIFICALLY LOOKING AT SEGMENT 1...

$d\vec{l}' = dy\,\hat{y}$ $\quad \vec{r} = 0$ $\quad \therefore \vec{\imath} = \vec{r} - \vec{r}' = y\,\hat{y}$ $\quad$ so $\hat{\imath} = \frac{\vec{\imath}}{\imath} = \hat{y}$

$\vec{r}' = -y\,\hat{y}$

$\therefore \quad d\vec{l} \times \hat{\imath} = dy\,\hat{y} \times \hat{y} = 0 \quad \therefore \vec{B}_1(\vec{r}) = \vec{B}_3(\vec{r}) = 0$

NOW CONSIDER SEGMENT ②...



NOW $d\vec{l}' = b\,d\varphi\,\hat{\varphi}$

$\vec{r} = 0$ $\quad \therefore \vec{\imath} = \vec{r} - \vec{r}' = -b\hat{s}$ $\quad \therefore \hat{\imath} = \frac{\vec{\imath}}{\imath} = -\hat{s}$

$\vec{r}' = +b\hat{s}$ $\quad \imath = b$

NOW

$d\vec{l}' \times \hat{\imath} = -b\,d\varphi(\hat{\varphi} \times \hat{s})$

NOW $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{s} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$

SO

$$\hat{\varphi} \times \hat{s} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\varphi & \cos\varphi & 0 \\ \cos\varphi & \sin\varphi & 0 \end{vmatrix} = \hat{z}(-\sin^2\varphi - \cos^2\varphi) = -\hat{z}$$

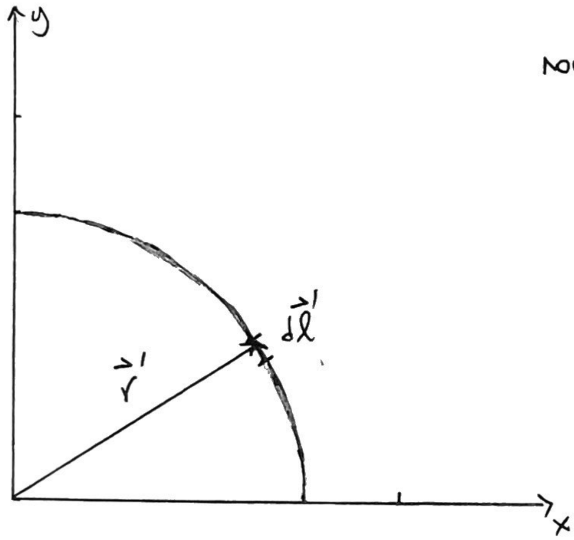
so $d\vec{l} \times \hat{\imath} = -b\,d\varphi(-\hat{z}) = b\,d\varphi\,\hat{z}$

NOW

$$\vec{B}_2(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi} \cdot +b\hat{z}\int_{\pi/2}^{0} \frac{d\varphi}{b^2} = +\frac{\mu_0 I}{4\pi b}\,\hat{z}\,\varphi\Big|_{\pi/2}^{0} = \frac{\mu_0 I}{4\pi b}\left(0 - \frac{\pi}{2}\right)\hat{z}$$

$$= -\frac{\mu_0 I}{8b}\hat{z}$$

NOW CONSIDER SEGMENT (4)...



NOW

$d\vec{l}' = a\,d\varphi\,\hat{\varphi}$

$\vec{r} = 0$ $\quad\therefore\ \vec{\mathscr{r}} = \vec{r} - \vec{r}' = -a\hat{s}$ $\quad\therefore\ \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = -\hat{s}$

$\vec{r}' = a\hat{s}$

$\mathscr{r} = a$

SO

$$d\vec{l}' \times \hat{\mathscr{r}} = a\,d\varphi\,\hat{\varphi} \times -\hat{s} = -a\,d\varphi(\hat{\varphi}\times\hat{s})$$

$$= a\,d\varphi\,\hat{z}$$

$\hat{\mathscr{r}}$ AS BEFORE

NOW

$$\vec{B}_4(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\,\frac{a}{a^2}\,\hat{z}\int_0^{\pi/2} d\varphi = \frac{\mu_0 I}{4\pi a}\left(\frac{\pi}{2} - 0\right)\hat{z} = \frac{\mu_0 I}{8a}\hat{z}$$

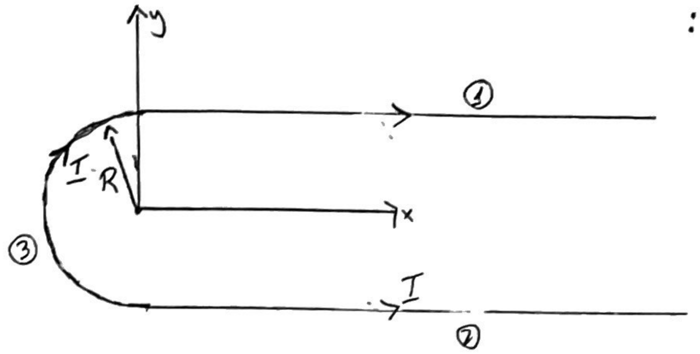
SO

$$\vec{B}(\vec{r}) = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 = 0 + 0 - \frac{\mu_0 I}{8b}\hat{z} + \frac{\mu_0 I}{8a}\hat{z}$$

$$\therefore \boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{z}}$$

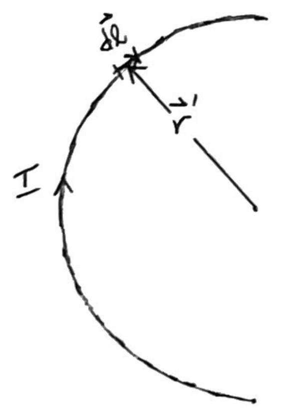
2b) CHOOSE THE ORIGIN OF THE COORDINATE SYSTEM TO ALIGN W/ POINT P

$\therefore \vec{r} = 0$



BREAK THE WIRE UP INTO THREE PIECES LABELED ①, ②, & ③

CONSIDER SECTION ③ FIRST, ...



NOW

$d\vec{l}' = R\,d\varphi\,\hat{\varphi}$

$\vec{r} = 0$

$\vec{r}' = R\hat{s}$ $\quad\therefore\ \vec{\mathscr{r}} = \vec{r} - \vec{r}' = -R\hat{s}$ $\quad\therefore\ \hat{\mathscr{r}} = -\hat{s}$

$\mathscr{r} = R$

NOW

$$d\vec{l}' \times \hat{\mathscr{r}} = -R\,d\varphi\underbrace{(\hat{\varphi} \times \hat{s})}_{-\hat{z}} = +R\,d\varphi\,\hat{z}$$

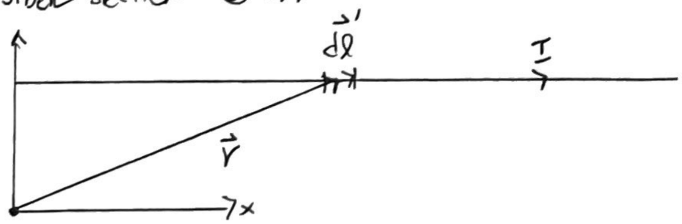
(AS WAS DONE IN 2a)

SO

$$\vec{B}_3(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi}\cdot\frac{+R\hat{z}}{R^2}\int_{3\pi/2}^{\pi/2} d\varphi = \frac{\mu_0 I}{4\pi R}\hat{z}\,\varphi\Big|_{3\pi/2}^{\pi/2} = \frac{\mu_0 I}{4\pi R}\hat{z}\left(\frac{\pi}{2} - \frac{3\pi}{2}\right)$$

$$= -\frac{\mu_0 I}{4R}\hat{z}$$

NOW CONSIDER SECTION ① ...



NOW

$d\vec{l}' = dx'\,\hat{x}$

$\vec{r} = 0$

$\vec{r}' = R\hat{y} + x'\hat{x}$ $\quad\therefore\ \vec{\mathscr{r}} = -R\hat{y} - x'\hat{x}$

$\mathscr{r} = \sqrt{R^2 + x'^2}$ $\quad\therefore\ \mathscr{r}^2 = R^2 + x'^2$

SO

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{-R\hat{y} - x'\hat{x}}{\sqrt{R^2 + x'^2}}$$

NOW

$$d\vec{l}' \times \hat{\imath} = dx'\hat{x} \times \frac{-(R\hat{y}+x'\hat{x})}{\sqrt{R^2+x'^2}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{R^2+x'^2}} & \frac{-R}{\sqrt{R^2+x'^2}} & 0 \end{vmatrix} = \frac{-R\,dx'}{\sqrt{R^2+x'^2}}\hat{z}$$

SO

$$\vec{B}_1(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\imath}}{\imath^2} = \frac{\mu_0 I}{4\pi}\cdot -R\hat{z}\int_0^\infty \frac{dx'}{(R^2+x'^2)^{3/2}} = -\frac{\mu_0 I R}{4\pi}\hat{z}\frac{R}{R^3}\int_0^{\pi/2}\frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$$

let $x' = R\tan\theta$

$dx' = R\sec^2\theta\, d\theta$

$$= -\frac{\mu_0 I}{4\pi R}\hat{z}\int_0^{\pi/2}\cos\theta\, d\theta = -\frac{\mu_0 I}{4\pi R}\hat{z}\sin\theta\Big|_0^{\pi/2} = -\frac{\mu_0 I}{4\pi R}\hat{z}$$

NOW CONSIDER SECTION ② ...

WHAT CHANGES COMPARED TO ①?

$d\vec{l}' = dx'\hat{x}$

$\vec{r} = 0$

$\vec{r}' = -R\hat{y} + x'\hat{x}$ $\quad\therefore\quad \vec{\imath} = R\hat{y} - x'\hat{x}$ $\quad\therefore\quad \hat{\imath} = \frac{R\hat{y}-x'\hat{x}}{\sqrt{R^2+x'^2}}$

$\imath = \sqrt{R^2+x'^2}$

NOW

$$d\vec{l}' \times \hat{\imath} = dx'\hat{x} \times \frac{R\hat{y}-x'\hat{x}}{\sqrt{R^2+x'^2}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{R^2+x'^2}} & \frac{R}{\sqrt{R^2+x'^2}} & 0 \end{vmatrix} = \frac{R\,dx'}{\sqrt{R^2+x'^2}}\hat{z}$$

SO

$$B_2(\vec{r}) = \frac{\mu_0 I}{4\pi}\cdot R\hat{z}\int_\infty^0 \frac{dx'}{(R^2+x'^2)^{3/2}}$$

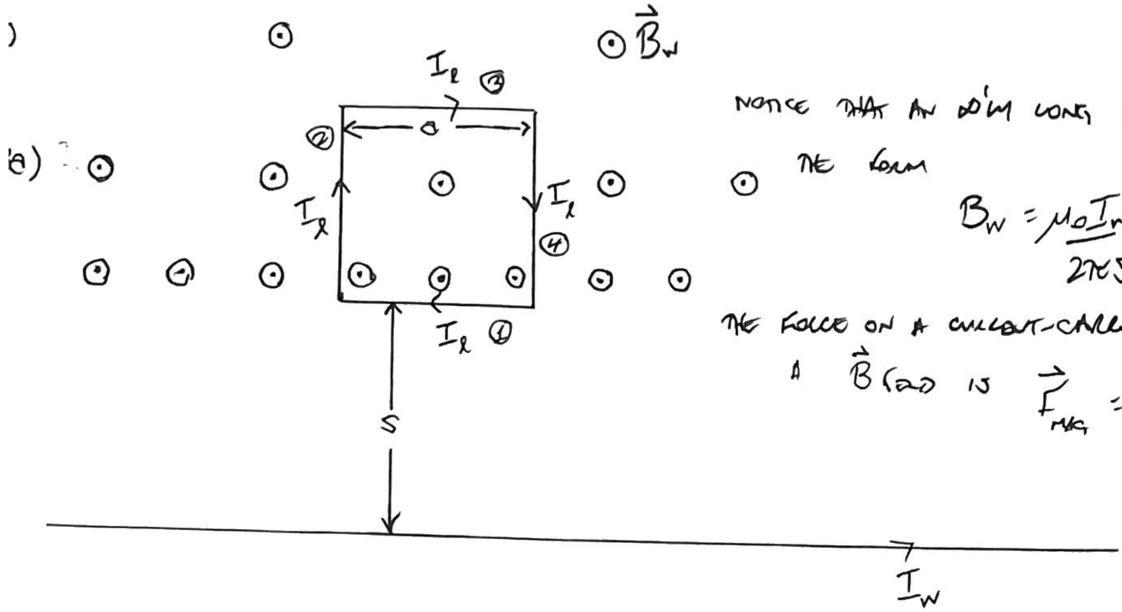
NOTE: UPON COMPARISON W/ THE INTEGRAL FOR $\vec{B}_1$ WE DIFFER BY

- A NEGATIVE SIGN
- LIMITS OF INTEGRATION ARE FLIPPED

$\therefore\ \vec{B}_1 = \vec{B}_2$

$$\boxed{\vec{B}(\vec{r}) = -\frac{\mu_0 I}{4R}\hat{z} - 2\frac{\mu_0 I}{4\pi R}\hat{z} = -\frac{\mu_0 I}{4R}\left(1+\frac{2}{\pi}\right)\hat{z}}$$

a)



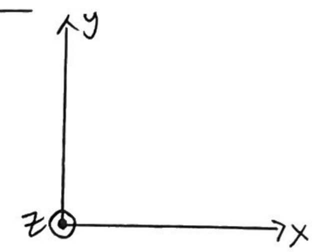
NOTICE THAT AN ∞'LY LONG WIRE GENERATES A $\vec{B}$-FIELD OF THE FORM

$$B_W = \frac{\mu_0 I_W}{2\pi s'}$$

WHERE s' IS THE PERPENDICULAR DISTANCE FROM THE WIRE

THE FORCE ON A CURRENT-CARRYING WIRE IN THE PRESENCE OF A $\vec{B}$ FIELD IS

$$\vec{F}_{MAG} = \int I\,(d\vec{\ell}' \times \vec{B})$$



CONSIDER SIDES ① : ③ FIRST...

NOTICE THAT

$$\vec{B}_W = \frac{\mu_0 I_W}{2\pi s}\hat{z} \quad \text{ALONG SEGMENT ①}$$

$$\vec{B}_W = \frac{\mu_0 I_W}{2\pi(s+a)}\hat{z} \quad \text{ALONG SEGMENT ③}$$

NOW, THE FORCE ON SEGMENT ① IS..

$$\vec{F}_{MAG,1} = I_\ell \vec{\ell} \times \vec{B}_W$$

(NO INTEGRATION NECESSARY SINCE $|\vec{B}_W|$ IS A CONSTANT ALONG EITHER SEGMENT)

NOW

$$\vec{\ell} = a(-\hat{x})$$

SO

$$\vec{F}_{MAG,1} = I_\ell a(-\hat{x}) \times \frac{\mu_0 I_W}{2\pi s}\hat{z} = \frac{\mu_0 I_\ell I_W a}{2\pi s}\hat{y}$$

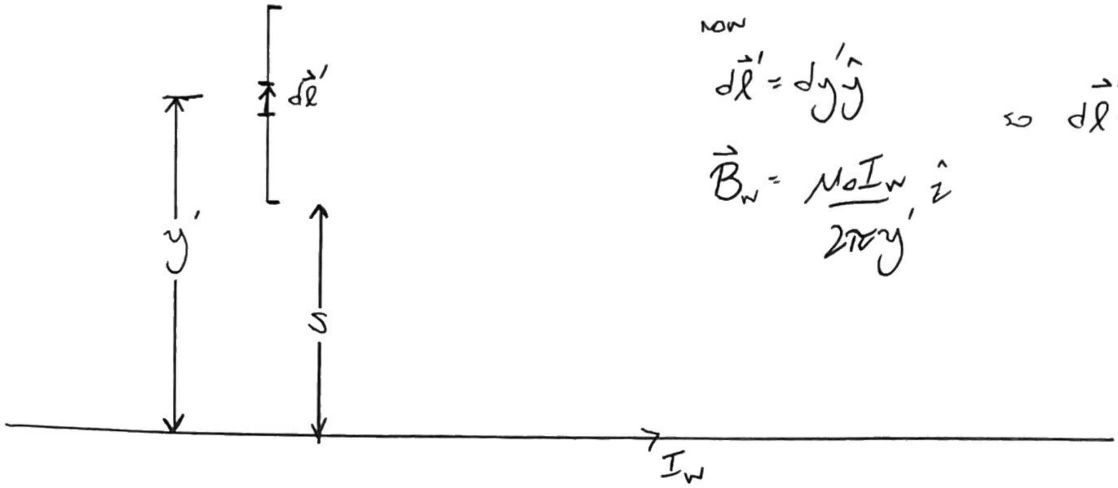
WHEREAS THE FORCE ON SEGMENT ③ IS..

$$\vec{F}_{MAG,3} = I_\ell \vec{\ell} \times \vec{B}_W \quad \text{WHERE } \vec{\ell} = a\hat{x}$$

SO

$$\vec{F}_{MAG,1} = I_\ell a\hat{x} \times \frac{\mu_0 I_W}{2\pi(s+a)}\hat{z} = -\frac{\mu_0 I_\ell I_W a}{2\pi(s+a)}\hat{y}$$

∴ NOW CONSIDER SEGMENT ② ....



NOW

$$d\vec{l}' = dy'\hat{y}$$

$$\vec{B}_w = \frac{\mu_0 I_w}{2\pi y'}\hat{z}$$

SO $$d\vec{l}' \times \vec{B}_w = \frac{\mu_0 I_w}{2\pi y'} dy' (\hat{y} \times \hat{z})$$

$$= \frac{\mu_0 I_w}{2\pi y'} dy' \hat{x}$$

SO

$$\vec{F}_{MAG,2} = \int I_l (d\vec{l}' \times \vec{B}_w) = \frac{\mu_0 I_l I_w}{2\pi}\hat{x} \int_s^{s+a} \frac{dy'}{y'} = \frac{\mu_0 I_l I_w}{2\pi}\hat{x} \ln y' \Big|_s^{s+a} = \frac{\mu_0 I_l I_w}{2\pi}\hat{x}\left(\ln(s+a) - \ln(s)\right)$$

$$= \frac{\mu_0 I_l I_w}{2\pi} \ln\left(1 + \frac{a}{s}\right)\hat{x}$$

NOW CONSIDER SEGMENT ④...

IN COMPARISON TO THE TREATMENT OF SEGMENT ②, THE ONLY DIFFERENCE LIES W/ THE LIMITS OF INTEGRATION

$$\vec{F}_{MAG,4} = \frac{\mu_0 I_l I_w}{2\pi}\hat{x} \int_{s+a}^{s} \frac{dy'}{y'} = -\frac{\mu_0 I_l I_w}{2\pi} \ln\left(1 + \frac{a}{s}\right)\hat{x}$$

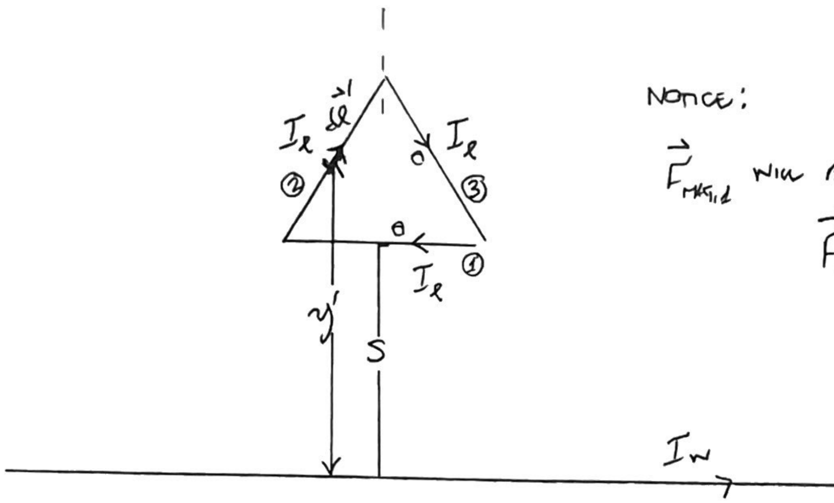
SO, THE NET FORCE ON THE SQUARE LOOP IS...

$$\vec{F}_{MAG} = \sum_{i=1}^{4} \vec{F}_{MAG,i} = \frac{\mu_0 I_l I_w a}{2\pi s}\hat{y} - \frac{\mu_0 I_l I_w a}{2\pi(s+a)}\hat{y} + \cancel{\frac{\mu_0 I_l I_w}{2\pi}\ln\left(1+\frac{a}{s}\right)\hat{x}} - \cancel{\frac{\mu_0 I_l I_w}{2\pi}\ln\left(1+\frac{a}{s}\right)\hat{x}}$$

$$= \frac{\mu_0 I_l I_w a}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right)\hat{y}$$

$$\left[\ \vec{F}_{MAG} = \frac{\mu_0 I_l I_w}{2\pi}\frac{a^2}{s(s+a)}\hat{y}\ \right]$$

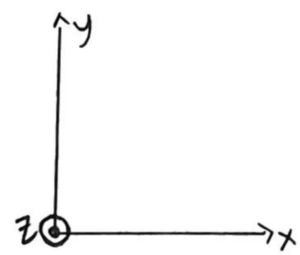
b)



NOTICE:

$\vec{F}_{MAG,1}$ WILL BE THE SAME AS THAT FOUND IN 2a)

$$\vec{F}_{MAG,1} = \frac{\mu_0 I_\ell I_w a}{2\pi S}\hat{y}$$



NOW CONSIDER SEGMENT ②...

$$d\vec{\ell}' = dx'\hat{x} + dy'\hat{y}$$

$$\vec{B}_w = \frac{\mu_0 I_w}{2\pi y'}\hat{z}$$

so
$$d\vec{\ell}' \times \vec{B}_w = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & dy' & 0 \\ 0 & 0 & \frac{\mu_0 I_w}{2\pi y'} \end{vmatrix} = \hat{x}\frac{\mu_0 I_w}{2\pi y'}dy' - \hat{y}\frac{\mu_0 I_w}{2\pi y'}dx'$$

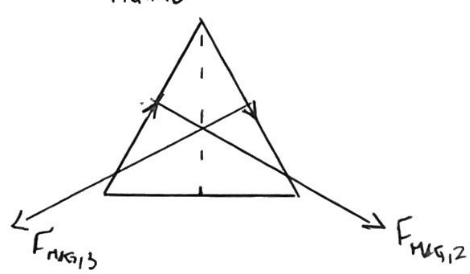
$$= \frac{\mu_0 I_w}{2\pi}\left(\hat{x}\frac{dy'}{y'} - \hat{y}\frac{dx'}{y'}\right)$$

so

$$\vec{F}_{MAG,2} = \int I_\ell (d\vec{\ell}' \times \vec{B}_w) = \frac{\mu_0 I_\ell I_w}{2\pi}\int\left(\hat{x}\frac{dy'}{y'} - \hat{y}\frac{dx'}{y'}\right)$$

$$= \frac{\mu_0 I_\ell I_w}{2\pi}\left[\hat{x}\int\frac{dy'}{y'} - \hat{y}\int\frac{dx'}{x'}\right]$$

FIGURE



BEFORE CALCULATING THE INTEGRAL, CONSIDER SEGMENT ③...

$$d\vec{\ell}' = dx'\hat{x} + dy'\hat{y}$$

$$\vec{B}_w = \frac{\mu_0 I_w}{2\pi y'}\hat{z} \quad \text{so} \quad d\vec{\ell}' \times \vec{B}_w = \frac{\mu_0 I_w}{2\pi}\left(\hat{x}\frac{dy'}{y'} - \hat{y}\frac{dx'}{y'}\right)$$

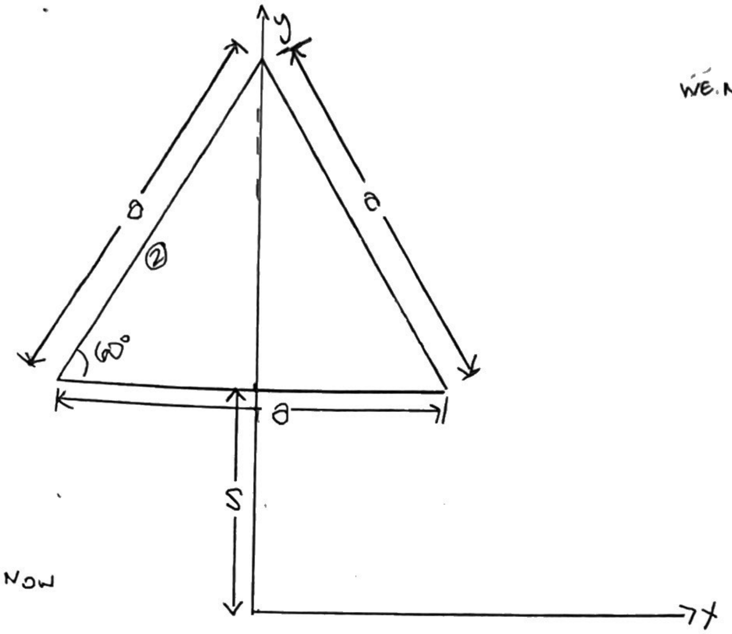
so

$$\vec{F}_{MAG,3} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\hat{x}\int\frac{dy'}{y'} - \hat{y}\int\frac{dx'}{y'}\right]$$

- THIS IS SAME FORM AS $\vec{F}_{MAG,2}$, ALTHOUGH THE LIMITS OF INTEGRATION ARE, IN GENERAL, DIFFERENT

• SEE *FIGURE* ABOVE FOR THESE FORCES...

NOTICE THAT THE X-COMPONENTS WILL CANCEL. THE y'-COMPONENTS OF THE FORCE SHOULD BE THE SAME.



WE NEED TO INTEGRATE ...

WE ALSO NEED THE EQUATION OF THE LINE

$$y(x) = mx + b$$

WHERE $b = a\sin(60°) + s = \frac{\sqrt{3}}{2}a + s$

$$m = \frac{a\sin(60°)}{a/2} = \frac{\sqrt{3}a}{2}\cdot\frac{2}{a} = \sqrt{3}$$

so

$$y(x) = \sqrt{3}x + \left(\frac{\sqrt{3}}{2}a + s\right) = \sqrt{3}x + b$$

NOW

$$(\vec{F}_{mag,3})_y = -\frac{\mu_0 I_\ell I_w}{2\pi}\hat{y}\int\frac{dx'}{y'} = -\frac{\mu_0 I_\ell I_w}{2\pi}\hat{y}\int_{-a/2}^{0}\frac{dx'}{\sqrt{3}x'+b} = -\frac{\mu_0 I_\ell I_w}{2\pi}\hat{y}\cdot\frac{1}{\sqrt{3}}\int_{-\frac{\sqrt{3}}{2}a+b}^{b}\frac{du}{u}$$

let $u = \sqrt{3}x' + b$

$du = \sqrt{3}dx'$

$$= -\frac{\mu_0 I_\ell I_w}{2\pi}\hat{y}\cdot\frac{1}{\sqrt{3}}\ln u\Big|_{-\frac{\sqrt{3}}{2}a+b}^{b}$$

$$= -\frac{\mu_0 I_\ell I_w}{2\pi}\hat{y}\cdot\frac{1}{\sqrt{3}}\left(\ln b - \ln\left(b - \frac{\sqrt{3}a}{2}\right)\right) = -\frac{\mu_0 I_\ell I_w}{2\pi\sqrt{3}}\ln\left(\frac{b}{b-\frac{\sqrt{3}}{2}a}\right)\hat{y}$$

$$= -\frac{\mu_0 I_\ell I_w}{2\sqrt{3}\pi}\ln\left(\frac{\sqrt{3}a/2 + s}{s}\right)\hat{y}$$

NOW $b = \frac{\sqrt{3}}{2}a + s$

so $b - \frac{\sqrt{3}}{2}a = s$

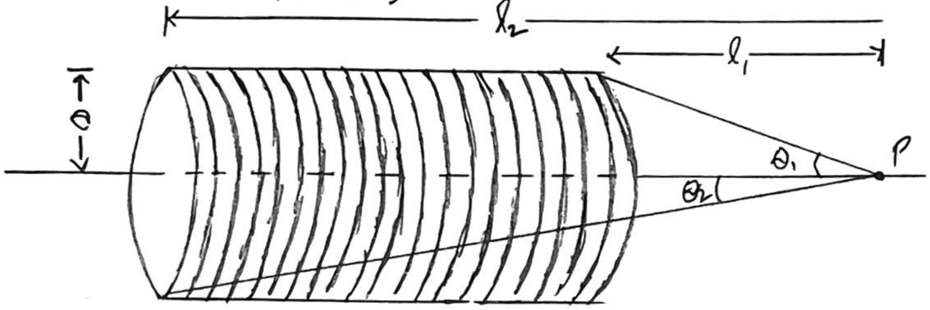
so

$$\vec{F}_{mag} = \frac{\mu_0 I_\ell I_w a}{2\pi s}\hat{y} - \frac{\mu_0 I_\ell I_w}{\sqrt{3}\pi}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)\hat{y}$$

$$= \frac{\mu_0 I_\ell I_w}{2\pi}\left[\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)\right]\hat{y}$$

4. IN EXAMPLE 5.6, THE $\vec{B}$-FIELD OF A SINGLE CIRCULAR LOOP OF WIRE A DISTANCE z FROM THE CENTER WAS FOUND TO BE..

a) $B(z)_{LOOP} = \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{3/2}}$ WHERE R IS THE RADIUS OF THE CIRCULAR LOOP.



$n = \frac{N}{L}$ W/ N TURNS IN A LENGTH L

THINK OF EACH LOOP AS AN INFINITESIMAL CONTRIBUTION TO THE TOTAL $\vec{B}$-FIELD..

$$d\vec{B}_{TOT} = B(z)\underbrace{\frac{N}{L}}\,dz \quad \text{SO} \quad B_{TOT} = \int_{\ell_1}^{\ell_2} B(z)\,n\,dz$$

PLAYS THE ROLE OF A 'MAGNETIC FIELD DENSITY'.

SO

SINCE $1+\tan^2\varphi = \sec^2\varphi$

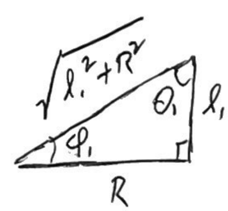
$$B_{TOT} = \frac{\mu_0 I}{2} a^2 n \int_{\ell_1}^{\ell_2} \frac{dz}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I}{2} a^2 n \frac{a}{a^3} \int_{\varphi_1}^{\varphi_2} \frac{\sec^2\varphi\, d\varphi}{(1+\tan^2\varphi)^{3/2}} = \frac{\mu_0 I}{2} n \int_{\varphi_1}^{\varphi_2} \cos\varphi\, d\varphi$$

LET $z = a\tan\varphi$ WHERE $\ell_2 = R\tan\varphi_2$ SO $\varphi_2 = \tan^{-1}\left(\frac{\ell_2}{R}\right)$

$dz = a\sec^2\varphi\, d\varphi$ $\quad \ell_1 = R\tan\varphi_1$ $\quad \varphi_1 = \tan^{-1}\left(\frac{\ell_1}{R}\right)$

$$= \frac{\mu_0 I n}{2} \sin\varphi \Big|_{\varphi_1}^{\varphi_2} = \frac{\mu_0 I n}{2}(\sin\varphi_2 - \sin\varphi_1)$$

$$= \frac{\mu_0 I n}{2}(\cos\theta_2 - \cos\theta_1)$$



SINCE $\tan\varphi_1 = \frac{\ell_1}{R}$

SO $\sin\varphi_1 = \frac{\ell_1}{\sqrt{\ell_1^2+R^2}}$

NOW NOTICE THAT $\sin\varphi_1 = \cos\theta_1$

b) FOR THE SOLENOID TO BECOME INFINITE IN BOTH DIRECTIONS, LET

$\theta_1 \to \pi$

$\theta_2 \to 0$

SO

$$B_{TOT} = \frac{\mu_0 I n}{2}(\cos 0 - \cos\pi) \quad \therefore \boxed{B = \mu_0 n I}$$