Ch 3 Interpolation and Polynomial Approximation

Recall Taylor Series Approximation to a function f about a point x_0:

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

Here, we require $f \in C^n[a,b]$ if we use a degree n Taylor polynomial P_n to approximate f on $[a,b]$.

For example, consider $f(x) = e^x$.

See next slide, and Maple worksheet.

Note that the Taylor Approx. Polyn is accurate only for x close to x_0 (typically).

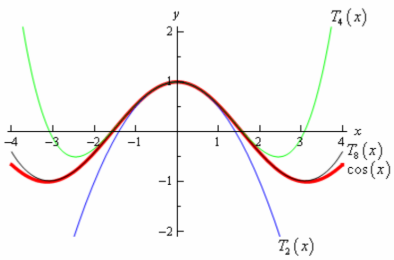
Sometimes $P_n(x)$ does not approx. $f(x)$ even for large n (See $f(x) = \frac{1}{x}$ example, Maple)

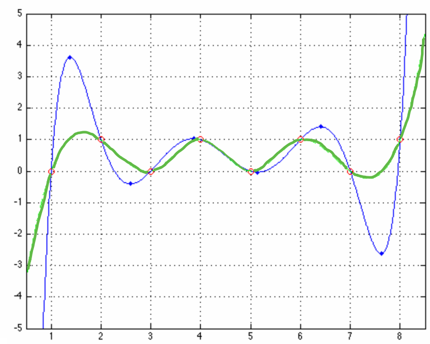
Taylor Polynomial: Shape-fitted interpolation at one point:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}\left(\xi(x)\right)}{(n+1)!}(x-x_0)^{n+1}$$

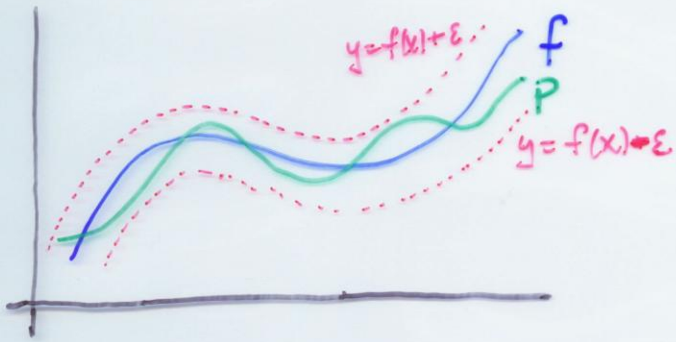
Lagrange Interpolating Polynomial: Interpolation across many points, not shape fitted:

$$f(x)=\sum_{k=0}^{n}f(x_k)L_{n,k}(x)+\frac{f^{(n+1)}\left(\xi(x)\right)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n),\quad \xi(x)\in(a,b)$$





It would be nice to have a polynomial approximation that uniformly approx's a function f over the interval $[a,b]$.



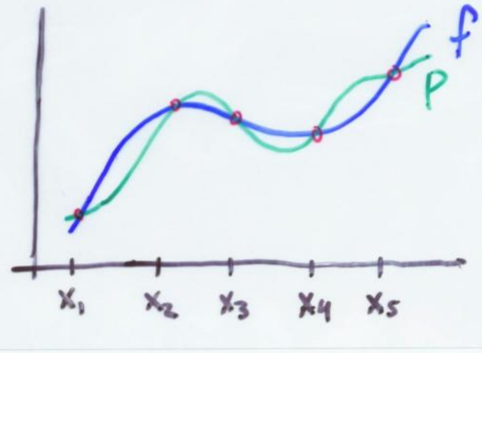
<u>Weierstrass</u> <u>Approximation</u> <u>Theorem</u>

Let $f \in C[a,b]$. For each $\varepsilon > 0$, $\exists P$ polynomial such that

$$|f(x) - P(x)| < \varepsilon \quad , \quad x \in [a,b]$$

Typically polynomials are chosen to <u>interpolate</u> a function f at a fixed set of points.

Taylor Polyns. are not suited for interpolation.



Suppose we are given $n+1$ points on graph of f:

$$(x_0, f(x_0)),\ (x_1, f(x_1)),\ \ldots,\ (x_n, f(x_n))$$

Define ~~the n-th Lagrange Polynomial by~~

$$L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

For example, suppose our data is

$$(1,5),\ (2,6),\ (3,7)$$

Then

$$L_{2,0}(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x-2)(x-3)$$

$$L_{2,1}(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -1(x-1)(x-3)$$

$$L_{2,2}(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x-1)(x-2)$$

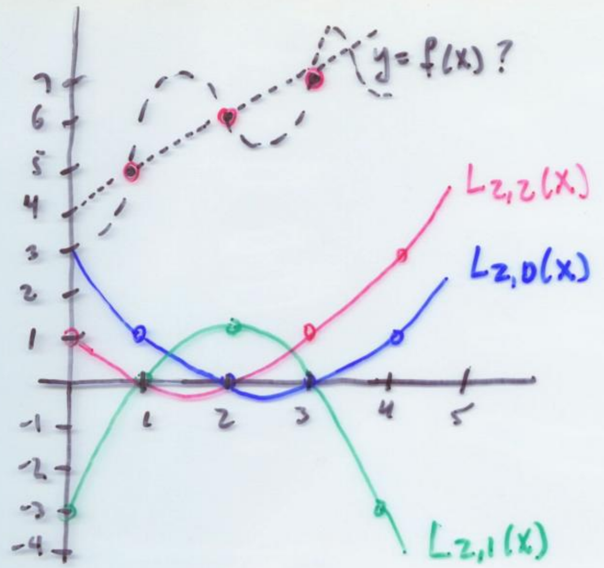
Note:

$$\left.\begin{array}{l} L_{2,0}(x_0) = L_{2,0}(1) = 1 \\ L_{2,1}(x_1) = L_{2,1}(2) = 1 \\ L_{2,2}(x_2) = L_{2,2}(3) = 1 \end{array}\right\}$$

In general,

$L_{n,k}(x_k) = 1$

$L_{n,k}(x_j) = 0,\ j \neq k$



$L_{2,k}(x_k) = 1$

$L_{2,k}(x_j) = 0,$ for $j \neq k$

To create our polynomial that interpolates the data, define

$$P(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$ ← nth Lagrange Interpolating polynomial

$$= f(x_0) L_{n,0}(x) + f(x_1) L_{n,1}(x) + \cdots + f(x_n) L_{n,n}(x)$$

In our example above,

$$P(x) = 5 L_{2,0}(x) + 6 L_{2,1}(x) + 7 L_{2,2}(x)$$

$$= 5\left[\tfrac{1}{2}(x-2)(x-3)\right] + 6\left[-(x-1)(x-3)\right] + 7\left[\tfrac{1}{2}(x-1)(x-2)\right]$$

$$= x + 4$$

See Maple Worksheet.

<u>Theorem</u> Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in $[a, b]$ and $f \in C^{n+1}[a,b]$. Then for each $x \in [a,b]$, $\exists\, \xi(x) \in (a,b)$ such that

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

where $P(x)$ is the nth Lagrange Interp Polyn.

(Compare with Taylor's Theorem)

As before, the error formula is used to develop error bounds.

We often do not know what f is, or its (n+1)-th derivative. Indeed, we may only have a table of values for f. In this case, we do not have an error expression. Nonetheless, the above error bound is useful both theoretically and practically. See text.

Example (a) Use appropriate Lagrange interpolating polynomials of degrees one, two and three to approximate $f(0.9)$, if we are given the following data:

$f(0.6) = -0.17694460$, $f(0.7) = 0.01375227$,
$f(0.8) = 0.22363362$, $f(1.0) = 0.65809197$.

Convert to table format:

k	x(k)	f(x)
0	0.6	-0.17694460
1	0.7	0.01375227
2	0.8	0.22363362
3	1.0	0.65809197

Degree One Interpolation (Use x_2, x_3)

$$P(x) = \frac{(x-1)}{(0.8-1)}(0.22363362) + \frac{(x-0.8)}{(1-0.8)}(0.65809197)$$

$$= 2.17229175x - 1.514199780 \quad \text{(Maple)}$$

$$P(0.9) = 0.440862795$$

k	x(k)	f(x)
0	0.6	-0.17694460
1	0.7	0.01375227
2	0.8	0.22363362
3	1.0	0.65809197

<u>Degree Two Interpolation</u> (One possibility: Use x_1, x_2, x_3)

$$P(x) = \frac{(x-0.8)(x-1.0)}{(0.7-0.8)(0.7-1.0)}(0.01375227)$$

$$+ \frac{(x-0.7)(x-1.0)}{(0.8-0.7)(0.8-1.0)}(0.22363362)$$

$$+ \frac{(x-0.7)(x-0.8)}{(1.0-0.7)(1.0-0.8)}(0.65809197)$$

$$= 0.2449275\,x^2 + 1.73142225\,x - 1.31825778$$

(Maple)

$$P(0.9) = 0.4384135 2$$

Example (a) Use appropriate Lagrange interpolating polynomials of degrees one, two and three to approximate $f(0.9)$, if we are given the following data:

$f(0.6)= -0.17694460, f(0.7) = 0.01375227, f(0.8) = 0.22363362, f(1.0) = 0.65809197.$

Convert to table format:

k	x(k)	f(x)
0	0.6	-0.17694460
1	0.7	0.01375227
2	0.8	0.22363362
3	1.0	0.65809197

<u>Degree One Interpolation</u> (Use x_2, x_3)

<u>Degree Two Interpolation</u> (One possibility: Use x_1, x_2, x_3)

<u>Degree Three Interpolation</u>

k	x(k)	f(x)
0	0.6	-0.17694460
1	0.7	0.01375227
2	0.8	0.22363362
3	1.0	0.65809197

Degree Three Interpolation

$$P(x) = \frac{(x-0.7)(x-0.8)(x-1.0)}{(0.6-0.7)(0.6-0.8)(0.6-1.0)}(-0.17694460)$$

$$+ \frac{(x-0.6)(x-0.8)(x-1.0)}{(0.7-0.6)(0.7-0.8)(0.7-1.0)}(0.01375227)$$

$$+ \frac{(x-0.6)(x-0.7)(x-1.0)}{(0.8-0.6)(0.8-0.7)(0.8-1.0)}(0.22363362)$$

$$+ \frac{(x-0.6)(x-0.7)(x-0.8)}{(1.0-0.6)(1.0-0.7)(1.0-0.8)}(0.65809197)$$

$$= -1.78574125x^3 + 4.709280625x^2 - 03318242680 - 1.947204725x$$

(Maple)

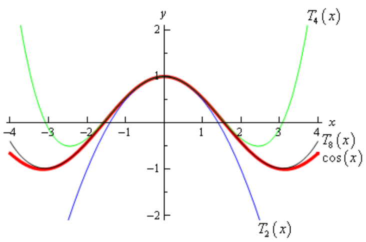
$$P(0.9) = 0.441985003$$

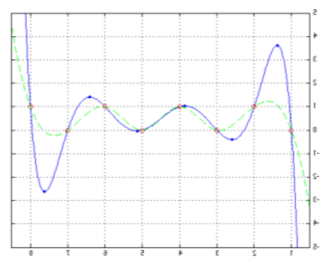
(b) Taylor Polynomial: Shape-fitted interpolation at one point:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}\left(\xi(x)\right)}{\left(n+1\right)!}(x-x_0)^{n+1}$$

Lagrange Interpolating Polynomial: Interpolation across many points, not shape fitted:

$$f(x)=\sum_{k=0}^{n}f(x_k)L_{n,k}(x)+\frac{f^{(n+1)}\left(\xi(x)\right)}{\left(n+1\right)!}(x-x_0)(x-x_1)\cdots(x-x_n),\quad \xi(x)\in(a,b)$$





Given that the data in part (a) was generated using $f(x)=\sin\left(e^x-2\right)$, use the error formula to find a bound for the error, and compute the bound to the actual error for the cases $n = 1$ and $n = 2$.

(b) Given that the above data was generated using $f(x) = \sin(e^x - 2)$, use the error formula to find a bound for the error, and compute the bound to the actual error for the cases $n = 1$ and $n = 2$.

Error $\dfrac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$

where $\xi(x) \in (a,b)$.

Here, $f(x) = \sin(e^x - 2)$

$f'(x) = \cos(e^x-2)\cdot e^x$

$f''(x) = -\sin(e^x-2)e^{2x} + \cos(e^x-2)e^x$

$f'''(x) = -\cos(e^x-2)e^{3x} - 3\sin(e^x-2)e^{2x} + \cos(e^x-2)e^x$

$n=1$ Error $= \left|\dfrac{\cos(e^{\xi(0.9)}-2)e^{\xi(0.9)} - \sin(e^{\xi(0.9)}-2)e^{2\xi(0.9)}}{2}\right|$

$\times \left|(0.9-0.8)(0.9-1.0)\right|$, where $\xi(0.9) \in (0.8, 1.0)$

$\leq \left(\dfrac{e^1+e^2}{2}\right)(0.1)(0.1) = 0.05054$

Actual Error : $|0.443592438 6 - 0.4408627 95| = 0.0027296436$

(c) Use Neville's Method to obtain the approximations found in part (a). (Use Excel.)

x	k	x(k)	f(x)	Q(k,1)	Q(k,2)	Q(k,3)
0.9	0	0.6	-0.17694460			
	1	0.7	0.01375227	$Q_{1,1}$		
	2	0.8	0.22363362	$Q_{2,1}$	$Q_{2,2}$	
	3	1.0	0.65809197	$Q_{3,1}$	$Q_{3,2}$	$Q_{3,3}$

$$Q_{1,1} = \frac{(0.9-0.6)(0.01375227)-(0.9-0.7)(-0.17694460)}{0.7-0.6}$$

$$= 0.39514601$$

$$Q_{2,1} = \frac{(0.9-0.7)(0.22363362)-(0.9-0.8)(0.01375227)}{0.8-0.7}$$

$$= 0.43351497$$

$$Q_{2,2} = \frac{(0.9-0.6)(0.43351497)-(0.9-0.8)(0.39514601)}{0.8-0.6}$$

$\vdots$

Definition Let f be a function defined at $x_0, x_1, \dots, x_n$, and suppose that $m_1, m_2, \dots, m_k$ are distinct integers such that $0 \le m_i \le n$, $i = 1, \dots, k$. The Lagrange polynomial that agrees with $f(x)$ at $x_{m_1}, x_{m_2}, \dots, x_{m_k}$ is denoted by $P_{m_1, m_2, \dots, m_k}(x)$.

Ex Let $f(x) = \cos(x)$, and

$x_0 = 1$, $x_1 = 3$, $x_2 = 5$, $x_3 = 7$

Then

$$P_{0,2,3}(x) = \frac{(x-5)(x-7)}{(1-5)(1-7)}\cos(1) + \frac{(x-1)(x-7)}{(5-1)(5-7)}\cos(5)$$

$$+ \frac{(x-1)(x-5)}{(7-1)(7-5)}\cos(7)$$

Theorem The nth Lagrange Polynomial that interpolates f at $x_0, x_1, \dots, x_k$ is given by

$$P(x) = \frac{(x-x_j)P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x-x_i)P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{x_i - x_j}$$

where $x_i \neq x_j$.

Example (Neville's Method)

$$P_0(x) = Q_{0,0}(x_0) = f(x_0)$$

$$P_1(x_1) = Q_{1,0}(x_1) = f(x_1)$$

$$P_{0,1}(x) = Q_{1,1}(x) = \frac{(x-x_0)\,Q_{1,0}(\cancel{x}) - (x-x_1)\,Q_{0,0}(\cancel{x})}{x_1 - x_0}$$

$$P_2(x_2) = Q_{2,0}(x_2) = f(x_2)$$

$$P_{1,2}(x) = Q_{2,1}(x) = \frac{(x-x_1)\,Q_{2,0}(\cancel{x}) - (x-x_2)\,Q_{1,0}(\cancel{x})}{x_2 - x_1}$$

$$P_{0,1,2}(x) = Q_{2,2}(x) = \frac{(x-x_0)\,Q_{2,1}(\cancel{x}) - (x-x_2)\,Q_{1,1}(\cancel{x})}{x_2 - x_0}$$

$\vdots$

where $\boxed{Q_{i,j} = P_{i-j,\, i-j+1, \ldots, i}}$

In table form:

x_0	$P_0 = Q_{0,0}$			
x_1	$P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$		
x_2	$P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$	
x_3	$P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$
$\vdots$	$\vdots$			

Example (Bessel Function, 1st Kind, Order 0)

$$f(x) = J_{1,0}(x)$$

(See Excel Worksheet), Maple

Given $J_{1,0}(x_i)$ values in Table:

x	f(x)
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.2818186
2.2	0.1103623

Find (approximate):

$$f(1.5) = J_{1,0}(1.5) \cong 0.5118277$$

Comments

1) Best linear approx will likely use $x_1 = 1.3$ and $x_2 = 1.6$.
2) Different choices of nodes are possible for quadratic, cubic and quartic approximations.

Example $f(x) = \ln(x)$ (See Excel Wksht)

Given: Table of Values. Approx: $f(2.1)$

K	x_K	$f(x_K)$	$Q_{k,1}$	$Q_{k,2}$
0	2.0	0.6931		
1	2.2	0.7885	0.7410	
2	2.3	0.8329	0.7441	0.7420

Thus our quadratic approximation is

$P(2.1) = 0.7420$,

using four-digit arithmetic.

The actual value is 0.7419, to four decimal places. Absolute Error:

Abs Error $= |.7419 - 0.7420| = 0.0001 = \underline{\underline{10^{-4}}}$

Compare with Lagrange Error Formula:

$f(x) = \ln x$, $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$.

Thus

$$|f(2.1) - P(2.1)| = \left|\frac{f'''(\xi(2.1))}{3!}\right| \cdot \left|(2.1-2.0)(2.1-2.2)(2.1-2.3)\right|$$

$$= \frac{1}{3\xi(2.1)^3} \cdot |(0.1)(-0.1)(-0.2)| \leq \underline{\underline{8.\bar{3} \times 10^{-5}}}$$

where $\xi(2.1) \in (2, 2.3)$.

(Assumes infinite precision)