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MATH 366
CP Ch. 21.4

1)
$$a u_{xx} + 2b u_{xy} + c u_{yy} = F(x, y, u, u_x, u_y)$$

Elliptic Type if $ac - b^2 > 0$
Parabolic Type if $ac - b^2 = 0$
Hyperbolic Type if $ac - b^2 < 0$

2)
- A first boundary value problem or Dirichlet Problem if u is prescribed on the boundary curve C of R.
- A second boundary value problem or Neumann Problem if $u_n = \partial u / \partial n$ (normal derivative of u) is prescribed on C.
- A third or mixed problem if u is prescribed on a part of C and u_n on the remaining part.

3)
(2) $\nabla^2 u = u_{xx} + u_{yy} = 0$: Laplace equation

(3) $\nabla^2 u = u_{xx} + u_{yy} = f(x,y)$: Poisson equation

4) The starting point for developing our numeric methods is the idea that we can replace the partial derivatives of these PDE's by corresponding difference quotients. Details are as follows: To develop this idea, we start with the Taylor Formula and obtain

(4) (a) $u(x+h, y) = u(x,y) + h u_x(x,y) + \frac{1}{2} h^2 u_{xx}(x,y) + \frac{1}{6} h^3 u_{xxx}(x,y) + \cdots$
(b) $u(x-h, y) = u(x,y) - h u_x(x,y) + \frac{1}{2} h^2 u_{xx}(x,y) - \frac{1}{6} h^3 u_{xxx}(x,y) + \cdots$

We subtract (4b) from (4a), neglect terms in $h^3, h^4, \ldots$, and solve for u_x. Then

(5a) $$u_x(x,y) \approx \frac{1}{2h}[u(x+h, y) - u(x-h, y)]$$

Similarily,
$$u(x, y+k) = u(x,y) + k u_y(x,y) + \tfrac{1}{2} k^2 u_{yy}(x,y) + \cdots$$

and
$$u(x, y-k) = u(x,y) - k u_y(x,y) + \tfrac{1}{2} k^2 u_{yy}(x,y) + \cdots$$

By subtracting, neglecting terms in $k^3, k^4, \ldots$, and solving for u_y we obtain

(5b) $$u_y(x,y) \approx \frac{1}{2k}[u(x, y+k) - u(x-h, y)].$$

We now turn to second derivatives. Adding (4a) and (4b) and neglecting terms in $h^4, h^5, \ldots$, we obtain $u(x+h, y) + u(x-h, y) \approx 2u(x,y) + h^2 u_{xx}(x,y)$. Solving for u_{xx} we have

(6a) $$u_{xx}(x,y) \approx \frac{1}{h^2}[u(x+h, y) - 2u(x,y) + u(x-h, y)].$$

Similarly,

(6b) $$u_{yy}(x,y) \approx \frac{1}{k^2}\left[u(x,y+k) - 2u(x,y) + u(x,y-k)\right].$$

(7) $$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2 f(x,y).$$

(8) $$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = 0.$$

5) h is called the **mesh size**.

6) Equation (8) relates u at (x,y) to u at the four neighboring points shown in Fig. 453b. It has a remarkable interpretation: u at (x,y) equals the mean of the values of u at the four neighboring points.

7)

(7) $$\left\{\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}\right\} u = h^2 f(x,y)$$

Example 1 Laplace Equation. Liebmann's Method:

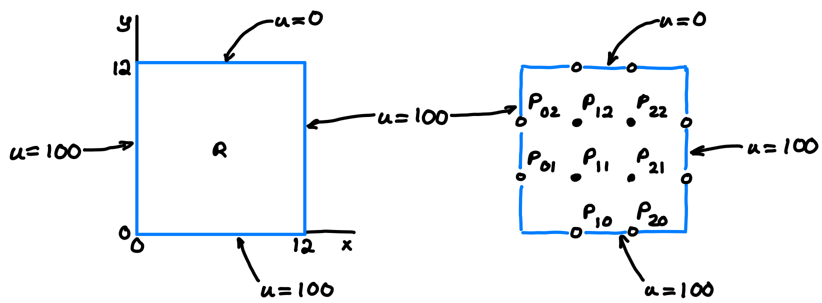
The four sides of a square plate of side 12 cm, made of homogeneous material, are kept at constant temperature 0°C and 100°C as shown in Fig. 455a. Using a (very wide) grid of mesh 4 cm and applying Liebmann's method (that is, Gauss-Seidel iteration), find the (steady-state) temperature at the mesh points.

Solution. In the case of independence of time, the heat equation (see Sec. 10.8)

$$u_t = c^2(u_{xx} + u_{yy})$$

reduces to the Laplace equation. Hence our problem is a Dirichlet problem for the latter. We choose the grid shown in Fig. 455b and consider the mesh points in the order $P_{11}, P_{21}, P_{12}, P_{22}$. We use (11) and, in each equation, take to the right all the terms resulting from the given boundary values. Then we obtain the system

$$\begin{aligned} -4u_{11} + u_{21} + u_{12} \qquad\qquad &= -200 \\ -u_{11} - 4u_{21} \qquad\qquad + u_{22} &= -200 \\ u_{11} \qquad\qquad - 4u_{12} + u_{22} &= -100 \\ u_{21} + u_{12} - 4u_{22} &= -100 \end{aligned}$$



(a) Given problem (b) Grid and mesh points

Fig. 455. Example 1

we write (12) in the Form (divide by -4 and take terms to the right)

$$\begin{aligned}
u_{11} &= \phantom{0.25u_{11}} \quad 0.25u_{21} + 0.25u_{12} \phantom{+ 0.25u_{22}} + 50 \\
u_{21} &= 0.25u_{11} \phantom{+ 0.25u_{21} + 0.25u_{12}} + 0.25u_{22} + 50 \\
u_{12} &= 0.25u_{11} \phantom{+ 0.25u_{21} + 0.25u_{12}} + 0.25u_{22} + 25 \\
u_{22} &= \phantom{0.25u_{11}} \quad 0.25u_{21} + 0.25u_{12} \phantom{+ 0.25u_{22}} + 25
\end{aligned}$$