Taylor Larrechea
Dr. Gustafson
MATH 366
CP Ch. 19.4

1) Given data (function values, points in the xy-plane) $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$ can be interpolated by a polynomial $P_n(x)$ of degree n or less so that the curve of $P_n(x)$ passes through these $(n+1)$ points (x_j, f_j): here $f_0 = f(x_0), \ldots, f_n = f(x_n)$, see sec. 19.3.
Now if n is large, there may be trouble: $P_n(x)$ may tend to oscillate for x between the nodes $x_0, \ldots, x_n$. Hence we must be prepared for numeric instability (sec. 19.1).

2)

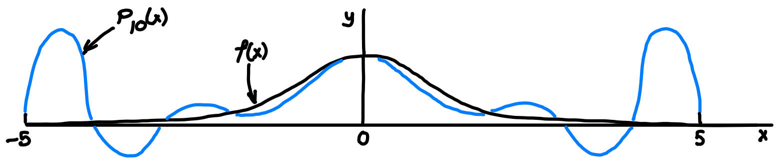

Fig. 434. Runge's example $f(x) = 1/(1+x^2)$ and interpolating polynomial $P_{10}(x)$.

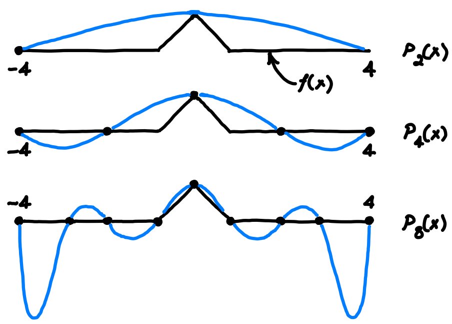

Fig. 435. Piecewise linear function $f(x)$ and interpolation polynomials of increasing degrees.

3) Instead of using a single high-degree polynomial P_n over the entire interval $a \leqq x \leqq b$ in which the nodes lie, that is,

(1) $$a = x_0 < x_1 < \ldots < x_n = b,$$

we use n low-degree, e.g., cubic, polynomials

$$q_0(x), \quad q_1(x), \quad \ldots, \quad q_{n-1}(x),$$

one over each subinterval between adjacent nodes, hence q_0 from x_0 to x_1, then q_1 from x_1 to x_2, and so on. From this we compose an interpolation function $g(x)$, called a spline, by fitting these polynomials together into a single continuous curve passing through the data points, that is,

(2) $$g(x_0) = f(x_0) = f_0, \quad g(x_1) = f(x_1) = f_1, \quad \ldots, \quad g(x_n) = f(x_n) = f_n.$$

Note that $g(x) = q_0(x)$ when $x_0 \leqq x \leqq x_1$, then $g(x) = q_1(x)$ when $x_1 \leqq x \leqq x_2$, and so on, according to our construction of g.
Thus spline interpolation is piecewise polynomial interpolation.
The simplest q_j's would be linear polynomials. However, the curve of a piecewise linear continuous function has corners and would be of little interest in general - think of

designing the body of a car or a ship.

We shall consider cubic splines because these are the most important ones in applications. By definition, a **cubic spline** $g(x)$ interpolating given data $(x_0, f_0), \dots, (x_n, f_n)$ is a continuous function on the interval $a = x_0 \leqq x \leqq x_n = b$ that has continuous first and second derivatives and satisfies the interpolation condition (2); furthermore, between adjacent nodes, $g(x)$ is given by a polynomial $q_j(x)$ of degree 3 or less.

4) Condition (3) includes the **clamped conditions**

(10) $$g'(x_0) = f'(x_0), \quad g'(x_n) = f'(x_n),$$

in which the tangent directions $f'(x_0)$ and $f'(x_n)$ at the ends are given. Other conditions of practical interest are the **free** or **natural conditions**

(11) $$g''(x_0) = 0, \quad g''(x_n) = 0$$

(geometrically: zero curvature at the ends, as for the draftman's spline), giving a **natural spline**. These names are motivated by Fig. 293 in Problem Set 12.3.

5)

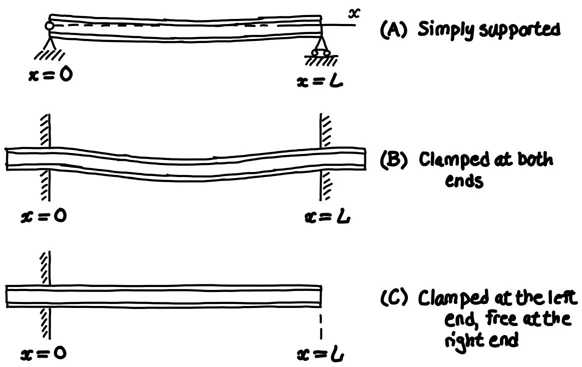

Fig. 293. Supports of a beam

6) The figure shows the corresponding interpolation polynomial of 12^{th} degree, which is useless because of its oscillation. (Because of roundoff your software will also give you small error terms involving odd powers of x.)