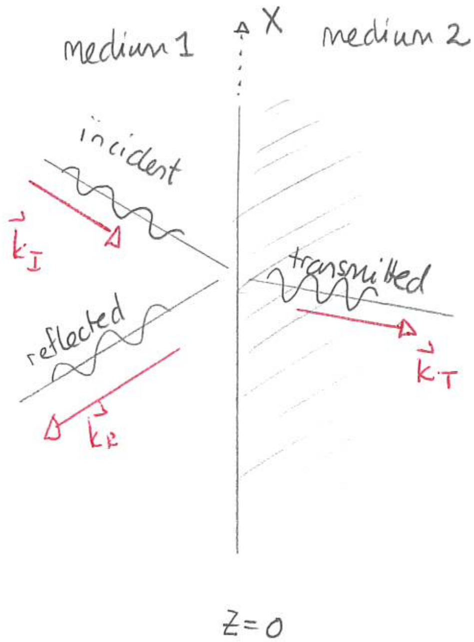
Thurs: 10.1

Fri HW

Reflection and Transmission at an Interface



Consider a situation where sinusoidal plane waves are incident on a boundary and the reflected and transmitted waves are also sinusoidal. The three waves are represented by:

1) incident

$$\vec{E}_I = \vec{E}_{oI}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}$$

2) reflected

$$\vec{E}_R = \vec{E}_{oR}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}$$

3) transmitted

$$\vec{E}_T = \vec{E}_{oT}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$$

Then in medium 1 the electric field is

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R$$

$$\Rightarrow \vec{E}_1 = \vec{E}_{oI}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} + \vec{E}_{oR}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}$$

In medium 2 the electric field is

$$\vec{E}_2 = \vec{E}_T$$

$$\Rightarrow \vec{E}_2 = \vec{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$$

There are comparable expressions for the magnetic field and these are attained via

$$\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$$

for each sinusoidal component.

At the boundary $(z=0)$ the boundary conditions give:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$B_1^{\perp} = B_2^{\perp}$$

$$\frac{1}{\mu_1} \vec{B}_1^{\parallel} = \frac{1}{\mu_2} \vec{B}_2^{\parallel}$$

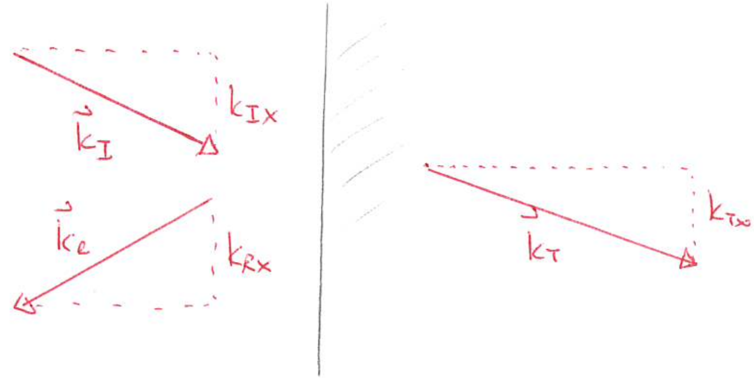
Applying the electric field boundary conditions gives:

For sinusoidal plane waves

1) the frequencies of all three waves are identical
2) the three wave vectors all lie in a single plane, called the plane of incidence. This is perpendicular to the boundary.
3) the components of each wavenumber vector parallel to the boundary are identical

Thus in terms of wavenumber vectors (using xz as plane of incidence

$$k_{Ix} = k_{Rx} = k_{Tx}$$



We can use these facts and the fact that the speed of each wave is

$$v = \omega / k$$

to determine relationships between the directions in which the waves propagate. Again recall that these results emerge entirely from Maxwell's equations.

1 Directions of reflected and transmitted waves

a) Use the dispersion relation $\omega = kv$ to relate the magnitudes of the reflected wavenumber to the incident wavenumber. Repeat this for the transmitted wavenumber.

b) Use these to relate the z components of the wavenumber vectors, k_{Rz} and k_{Iz}. Sketch the directions of $\mathbf{k}_R$ and $\mathbf{k}_I$.

c) Derive the law of reflection from the previous result.

d) Relate the z components of the wavenumber vectors, k_{Tz} and k_{Iz}. Sketch the directions of $\mathbf{k}_T$ and $\mathbf{k}_I$ for the case where $v_2 < v_1$.

e) Derive Snell's law from the previous result.

Answer: a) The reflected and incident wave both travel with speed v_1

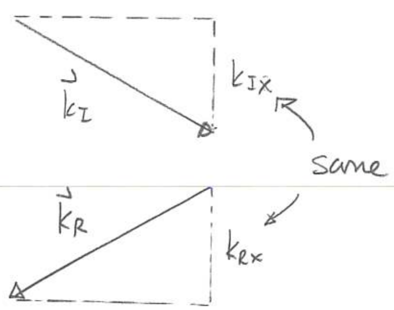
$$k = \frac{\omega}{v} \Rightarrow k_I = \frac{\omega}{v_1} \quad k_R = \frac{\omega}{v_1} \Rightarrow k_I = k_R$$

For the transmitted wave:

$$k_T = \frac{\omega}{v_2} \Rightarrow k_T / k_I = \frac{\omega}{v_2} \Big/ \frac{\omega}{v_1} \Rightarrow \frac{k_T}{k_I} = \frac{v_1}{v_2}$$

$$\Rightarrow k_T = k_I \frac{v_1}{v_2}$$

b)



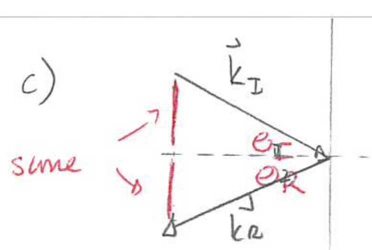
Same magnitudes and same x components means

$$|k_{Iz}| = |k_{Rz}|$$

We see

$$k_{Rz} = -k_{Iz}$$

c)



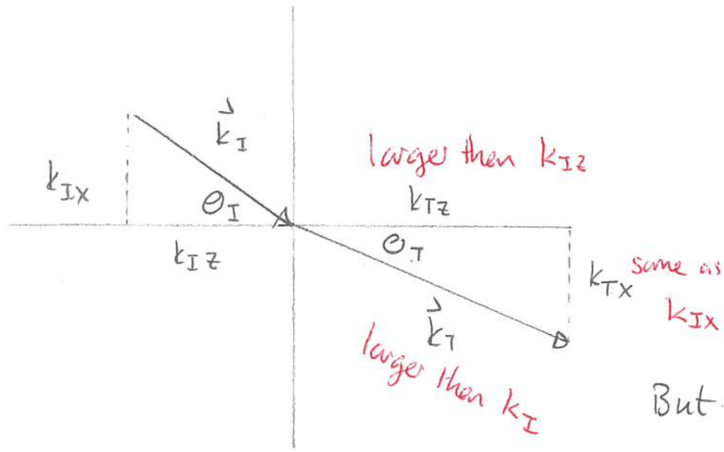
Geometry implies that

$$\theta_I = \theta_R$$

which is the law of reflection.

d) If $v_2 < v_1$ then $k_T > k_I$. Then

$$k_I^2 = k_{Ix}^2 + k_{Iz}^2$$



$$k_T^2 = k_{Tx}^2 + k_{Tz}^2$$

$$= k_{Ix}^2 + k_{Tz}^2$$

But $k_T = k_I \frac{v_1}{v_2}$ gives

$$k_I^2 \left(\frac{v_1}{v_2}\right)^2 = k_{Ix}^2 + k_{Tz}^2$$

$$\Rightarrow \left(k_{Ix}^2 + k_{Iz}^2\right)\left(\frac{v_1}{v_2}\right)^2 = k_{Ix}^2 + k_{Tz}^2$$

$$\Rightarrow k_{Tz}^2 = \left(k_{Ix}^2 + k_{Iz}^2\right)\left(\frac{v_1}{v_2}\right)^2 - k_{Ix}^2$$

e) Here $k_{Ix} = k_I \sin\theta_I$

$k_{Tx} = k_T \sin\theta_T$

and $k_{Ix} = k_{Tx} \Rightarrow k_I \sin\theta_I = k_T \sin\theta_T$

$$\Rightarrow k_I \sin\theta_I = k_I \frac{v_1}{v_2} \sin\theta_T$$

$$\Rightarrow v_2 \sin\theta_I = v_1 \sin\theta_T$$

$$\Rightarrow \frac{1}{v_1} \sin\theta_I = \frac{1}{v_2} \sin\theta_T$$

Let the index of refraction be $n = \frac{c}{v}$. We get

$$n_1 \sin\theta_I = n_2 \sin\theta_T$$

Thus:

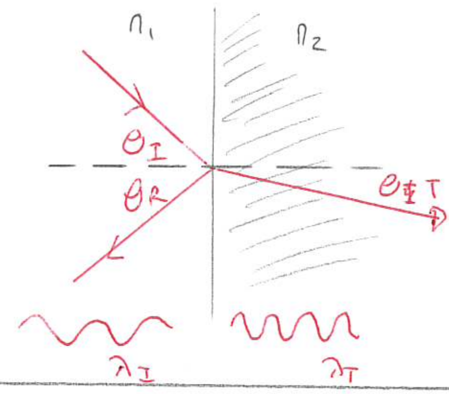
The electromagnetic boundary conditions imply that the directions of the reflected and transmitted waves obey

1) the law of reflection

$$\theta_I = \theta_R$$

2) Snell's law (law of refraction)

$$n_2 \sin\theta_T = n_1 \sin\theta_I$$



We can also see that

1) the frequencies of all waves are identical.

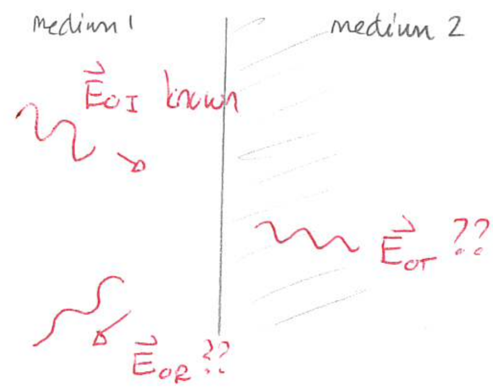
2) there is a wavelength shift across the boundary

$$k_T = k_I \frac{v_1}{v_2} \Rightarrow \frac{2\pi}{\lambda_T} = \frac{2\pi}{\lambda_I}\frac{v_1}{v_2}$$

$$\Rightarrow \boxed{\lambda_T = \lambda_I \frac{v_2}{v_1} \Rightarrow \lambda_T = \lambda_I \frac{n_1}{n_2}}$$

Normal incidence

The two laws regarding the directions of the transmitted and reflected wave apply generally. We now consider rules that describe the intensities of the waves and also their polarizations. These will require relating electric and magnetic fields. We will see that the boundary conditions can accomplish this.

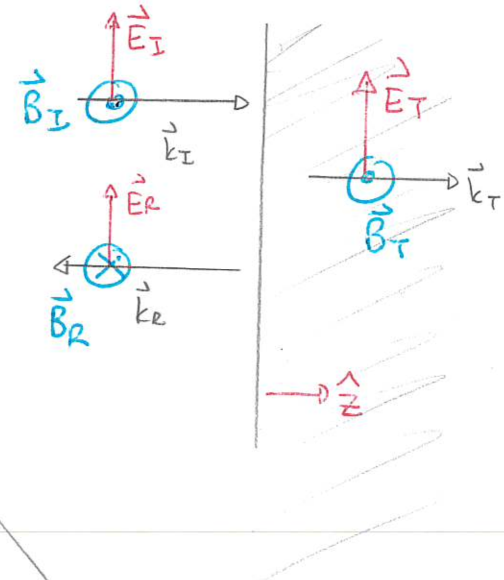


We first consider the simplest situation where the three waves are perpendicular to the surface. This is the normal incidence case. Then

$$\vec{k}_I = k\hat{z}$$

$$\vec{k}_R = -k\hat{z}$$

$$\vec{k}_T = \frac{v_1}{v_2}k\hat{z}$$



Then the electric fields are:

$$\vec{E}_I = \vec{E}_{0I}\, e^{i(kz-\omega t)}$$

$$\vec{E}_R = \vec{E}_{0R}\, e^{i(-kz-\omega t)}$$

$$\vec{E}_T = \vec{E}_{0T}\, e^{i(k_T z-\omega t)} \qquad k_T = k\frac{v_1}{v_2}$$

Then the generic rule

$$\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E}$$

can give the magnetic fields.

Thus

$$\left.\begin{aligned}
\vec{B}_I &= \frac{1}{\omega} k \hat{z} \times \vec{E}_I = \frac{1}{V_1} \hat{z} \times \vec{E}_I \\
\vec{B}_R &= -\frac{1}{\omega} k \hat{z} \times \vec{E}_R = -\frac{1}{V_1} \hat{z} \times \vec{E}_R \\
\vec{B}_T &= \frac{1}{\omega} k \frac{V_1}{V_2} \hat{z} \times \vec{E}_T = \frac{1}{V_2} \hat{z} \times \vec{E}_T
\end{aligned}\right\} \Rightarrow \boxed{\begin{aligned}
\vec{B}_I &= \frac{1}{V_1} \hat{z} \times \vec{E}_I \\
\vec{B}_R &= -\frac{1}{V_1} \hat{z} \times \vec{E}_R \\
\vec{B}_T &= \frac{1}{V_2} \hat{z} \times \vec{E}_T
\end{aligned}}$$

The boundary conditions involve perpendicular and parallel components. Since the waves are transverse the electric fields and magnetic fields are entirely parallel to the surface. So $E_1^{\perp} = E_2^{\perp} = 0$ and $B_1^{\perp} = B_2^{\perp} = 0$.

The only meaningful boundary conditions are then

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$\frac{1}{\mu_1} \vec{B}_1^{\parallel} = \frac{1}{\mu_2} \vec{B}_2^{\parallel}$$

2 Polarization of reflected and transmitted waves: normal incidence

Consider a monochromatic plane wave traveling in the $+z$ direction and incident on a surface lying in the xy plane. Suppose that the wave is linearly polarized in the xy plane. The complex function that represents the associated electric and magnetic fields is:

$$\tilde{\mathbf{E}}_I = \tilde{\mathbf{E}}_{0I}\, e^{i(kz-\omega t)}$$
$$\tilde{\mathbf{B}}_I = \tilde{\mathbf{B}}_{0I}\, e^{i(kz-\omega t)}$$

where E_{0I} is real. The reflected wave is represented by

$$\tilde{\mathbf{E}}_R = \tilde{\mathbf{E}}_{0R}\, e^{i(-kz-\omega t)}$$
$$\tilde{\mathbf{B}}_R = \tilde{\mathbf{B}}_{0R}\, e^{i(-kz-\omega t)}$$

and the transmitted wave by

$$\tilde{\mathbf{E}}_T = \tilde{\mathbf{E}}_{0T}\, e^{i(k_T z-\omega t)}$$
$$\tilde{\mathbf{B}}_T = \tilde{\mathbf{B}}_{0T}\, e^{i(k_T z-\omega t)}$$

where k is the wavenumber of in the incident medium and k_T is the wavenumber in the transmitted medium.

a) Apply the boundary conditions and the relationships between the electric and magnetic fields in monochromatic plane waves to show that

$$\tilde{\mathbf{E}}_{0I} + \tilde{\mathbf{E}}_{0R} = \tilde{\mathbf{E}}_{0T}$$
$$\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_{0I} - \hat{\mathbf{z}} \times \tilde{\mathbf{E}}_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\, \hat{\mathbf{z}} \times \tilde{\mathbf{E}}_{0T}$$

b) Take the cross product of the second equation with $\hat{\mathbf{z}}$, use a vector algebra identity to obtain a relationship of the form

$$\tilde{\mathbf{E}}_{0I} - \tilde{\mathbf{E}}_{0R} = \cdots$$

Combine the two relationships to show that

$$\tilde{\mathbf{E}}_{0R} = \text{scalar} \times \tilde{\mathbf{E}}_{0I}$$
$$\tilde{\mathbf{E}}_{0T} = \text{scalar} \times \tilde{\mathbf{E}}_{0I}$$

c) Suppose that the incident electric field is linearly polarized along the x direction. What does the previous result imply about the polarization of the reflected and transmitted fields?

d) Suppose that the incident electric field is linearly polarized along any direction in the xy plane. What does the previous result imply about the polarization of the reflected and transmitted fields?

Answer

a) $$\vec{E}_1^{\parallel} = \vec{E}_I^{\parallel} + \vec{E}_R^{\parallel} = \vec{E}_I + \vec{E}_R$$

$$\vec{E}_2^{\parallel} = \vec{E}_T^{\parallel} = \vec{E}_T$$

At $z=0$

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \Rightarrow \vec{E}_{0I} e^{i(k\cancel{z}^{\,0}-\omega t)} + \vec{E}_{0R} e^{i(-k\cancel{z}^{\,0}-\omega t)} = \vec{E}_{0T} e^{i(k_T\cancel{z}^{\,0}-\omega t)}$$

The $e^{-i\omega t}$ terms cancel and we get

$$\vec{E}_{0I} + \vec{E}_{0R} = \vec{E}_{0T}$$

Now consider the magnetic fields

$$\vec{B}_1 = \frac{1}{v_1}\hat{z}\times\vec{E}_I - \frac{1}{v_1}\hat{z}\times\vec{E}_R = \frac{1}{v_1}\left(\hat{z}\times\vec{E}_I - \hat{z}\times\vec{E}_R\right)$$

$$\vec{B}_2 = \frac{1}{v_2}\hat{z}\times\vec{E}_T$$

Then

$$\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

$$\Rightarrow \frac{1}{\mu_1}\frac{1}{v_1}\left(\hat{z}\times\vec{E}_I - \hat{z}\times\vec{E}_R\right) = \frac{1}{\mu_2}\frac{1}{v_2}\hat{z}\times\vec{E}_T$$

$$\Rightarrow \hat{z}\times\vec{E}_{0I} e^{i(k\cancel{z}^{\,0}-\omega t)} - \hat{z}\times\vec{E}_{0R} e^{i(-kz-\omega t)} = \frac{\mu_1}{\mu_2}\frac{v_1}{v_2}\hat{z}\times\vec{E}_{0T} e^{i(k\cancel{z}-\omega t)}$$

Again the $e^{-i\omega t}$ terms cancel. We get:

$$\hat{z}\times\vec{E}_{0I} - \hat{z}\times\vec{E}_{0R} = \frac{\mu_1}{\mu_2}\frac{v_1}{v_2}\hat{z}\times\vec{E}_{0T}$$

b) we use

$$\hat{z} \times (\hat{z} \times \vec{A}) = \hat{z}(\hat{z} \cdot \vec{A}) - \vec{A}(\cancel{\hat{z} \cdot \hat{z}})^{1}$$

Since $\vec{E}$ is perpendicular to $\hat{z}$ we get

$$\hat{z} \times (\hat{z} \times \vec{E}) = -\vec{E}$$

Thus:

$$-\vec{E}_{0I} + \vec{E}_{0R} = -\frac{\mu_1}{\mu_2}\frac{v_1}{v_2}\vec{E}_{0T}$$

$$\Rightarrow \quad \vec{E}_{0I} - \vec{E}_{0R} = \frac{\mu_1}{\mu_2}\frac{v_1}{v_2}\vec{E}_{0T}$$

Adding the two gives:

$$2\vec{E}_{0I} = \left(1 + \frac{\mu_1 v_1}{\mu_2 v_2}\right)\vec{E}_{0T} \quad \Rightarrow \quad \vec{E}_{0T} = \frac{2}{1+\beta}\vec{E}_{0I}$$

where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

Subtracting gives:

$$2\vec{E}_{0R} = (1-\beta)\vec{E}_{0T}$$

$$= \frac{1-\beta}{1+\beta} 2\vec{E}_{0I} \quad \Rightarrow \quad \vec{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\vec{E}_{0I}$$

c) These are also polarized along the $\hat{x}$ direction (as the diagram illustrates)

d) The direction of polarization is the same.

Thus we have found:

$$\boxed{\begin{array}{l} \text{For normal incidence} \\[1em] \vec{E}_{0R} = \left(\dfrac{1-\beta}{1+\beta}\right) \vec{E}_{0I} \\[1em] \vec{E}_{0T} = \dfrac{2}{1+\beta} \vec{E}_{0I} \\[1em] \text{where} \quad \beta = \dfrac{\mu_1}{\mu_2} \dfrac{v_1}{v_2} \end{array}}$$

We can eventually translate these into statements about intensity of the waves. In general

$$I = \frac{1}{2} \epsilon v E^2$$

where ϵ is the permittivity of the material and v is speed of the waves in that material. We can form the ratio of intensities and these give reflection

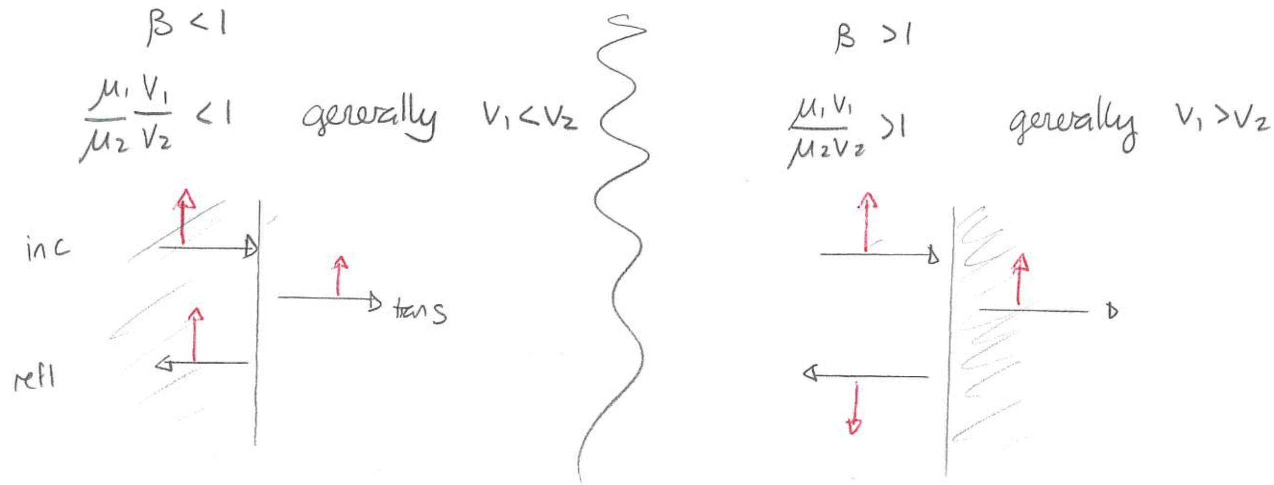
$$R = \frac{I_R}{I_I}$$

and transmission coefficients

$$T = \frac{I_T}{I_I}$$

Such coefficients describe how the energy is apportioned between the reflected and transmitted waves.

Separately we can see that phase shifts for the reflected wave are possible. If $\beta < 1$ then $\vec{E}_{0R}$ and $\vec{E}_{0I}$ have the same direction. If $\beta > 1$ then they have opposite directions.



Such reflection shifts manifest themselves in beam splitters.

The same situation can be analyzed when the incident wave is not normal to the surface. In these cases the electric fields of the reflected and transmitted wave depend on the angle of incidence and the polarization of the incident light.