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Preface

This Instructor's Manual contains the solutions to all the end-of-chapter problems (but not the appendices) from *Classical Dynamics of Particles and Systems*, Fifth Edition, by Stephen T. Thornton and Jerry B. Marion. It is intended for use only by instructors using *Classical Dynamics* as a textbook, and it is not available to students in any form. A Student Solutions Manual containing solutions to about 25% of the end-of-chapter problems is available for sale to students. The problem numbers of those solutions in the Student Solutions Manual are listed on the next page.

As a result of surveys received from users, I continue to add more worked out examples in the text and add additional problems. There are now 509 problems, a significant number over the 4th edition.

The instructor will find a large array of problems ranging in difficulty from the simple "plug and chug" to the type worthy of the Ph.D. qualifying examinations in classical mechanics. A few of the problems are quite challenging. Many of them require numerical methods. Having this solutions manual should provide a greater appreciation of what the authors intended to accomplish by the statement of the problem in those cases where the problem statement is not completely clear. Please inform me when either the problem statement or solutions can be improved. Specific help is encouraged. The instructor will also be able to pick and choose different levels of difficulty when assigning homework problems. And since students may occasionally need hints to work some problems, this manual will allow the instructor to take a quick peek to see how the students can be helped.

It is absolutely forbidden for the students to have access to this manual. Please do not give students solutions from this manual. Posting these solutions on the Internet will result in widespread distribution of the solutions and will ultimately result in the decrease of the usefulness of the text.

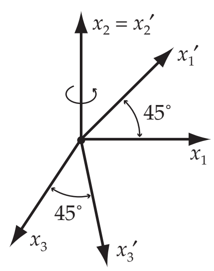
The author would like to acknowledge the assistance of Tran ngoc Khanh (5th edition), Warren Griffith (4th edition), and Brian Giambattista (3rd edition), who checked the solutions of previous versions, went over user comments, and worked out solutions for new problems. Without their help, this manual would not be possible. The author would appreciate receiving reports of suggested improvements and suspected errors. Comments can be sent by email to stt@virginia.edu, the more detailed the better.

Stephen T. Thornton
Charlottesville, Virginia

CHAPTER 1

Matrices, Vectors, and Vector Calculus

1-1.



Axes x_1' and x_3' lie in the x_1x_3 plane.

The transformation equations are:

$$x_1' = x_1 \cos 45° - x_3 \cos 45°$$

$$x_2' = x_2$$

$$x_3' = x_3 \cos 45° + x_1 \cos 45°$$

$$x_1' = \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_3$$

$$x_2' = x_2$$

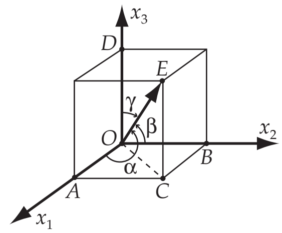
$$x_3' = \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_3$$

So the transformation matrix is:

$$\boxed{\begin{pmatrix} \dfrac{1}{\sqrt{2}} & 0 & -\dfrac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix}}$$

1-2.

a)



From this diagram, we have

$$\overline{OE}\cos\alpha = \overline{OA}$$

$$\overline{OE}\cos\beta = \overline{OB} \tag{1}$$

$$\overline{OE}\cos\gamma = \overline{OD}$$

Taking the square of each equation in (1) and adding, we find

$$\overline{OE}^2\left[\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right] = \overline{OA}^2 + \overline{OB}^2 + \overline{OD}^2 \tag{2}$$

But

$$\overline{OA}^2 + \overline{OB}^2 = \overline{OC}^2 \tag{3}$$

and

$$\overline{OC}^2 + \overline{OD}^2 = \overline{OE}^2 \tag{4}$$

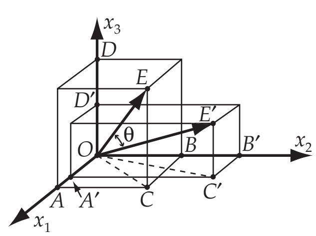
Therefore,

$$\overline{OA}^2 + \overline{OB}^2 + \overline{OD}^2 = \overline{OE}^2 \tag{5}$$

Thus,

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1} \tag{6}$$

b)



First, we have the following trigonometric relation:

$$\overline{OE}^2 + \overline{OE}'^2 - 2\overline{OE}\,\overline{OE}'\cos\theta = \overline{EE}'^2 \tag{7}$$

But,

$$\overline{EE}'^2 = \left[\overline{OB}' - \overline{OB}\right]^2 + \left[\overline{OA}' - \overline{OA}\right]^2 + \left[\overline{OD}' - \overline{OD}\right]^2$$

$$= \left[\overline{OE}' \cos\beta' - \overline{OE}\cos\beta\right]^2 + \left[\overline{OE}' \cos\alpha' - \overline{OE}\cos\alpha\right]^2$$

$$+ \left[\overline{OE}' \cos\gamma' - \overline{OE}\cos\gamma\right]^2 \tag{8}$$

or,

$$\overline{EE}'^2 = \overline{OE}'^2\left[\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma'\right] + \overline{OE}^2\left[\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right]$$

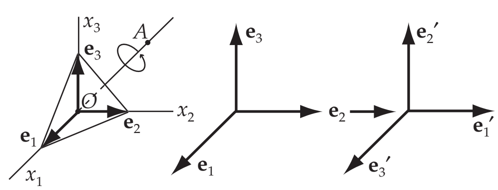
$$- 2\overline{OE}'\ \overline{OE}\left[\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'\right]$$

$$= OE'^2 + OE^2 - 2\overline{OE}\,\overline{OE}'\left[\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'\right] \tag{9}$$

Comparing (9) with (7), we find

$$\boxed{\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'} \tag{10}$$

1-3.



Denote the original axes by x_1, x_2, x_3, and the corresponding unit vectors by $\mathbf{e}_1, \mathbf{e}_2$, $\mathbf{e}_3$. Denote the new axes by x_1', x_2', x_3' and the corresponding unit vectors by $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$. The effect of the rotation is $\mathbf{e}_1 \to \mathbf{e}_3'$, $\mathbf{e}_2 \to \mathbf{e}_1'$, $\mathbf{e}_3 \to \mathbf{e}_2'$. Therefore, the transformation matrix is written as:

$$\lambda = \begin{bmatrix} \cos(\mathbf{e}_1', \mathbf{e}_1) & \cos(\mathbf{e}_1', \mathbf{e}_2) & \cos(\mathbf{e}_1', \mathbf{e}_3) \\ \cos(\mathbf{e}_2', \mathbf{e}_1) & \cos(\mathbf{e}_2', \mathbf{e}_2) & \cos(\mathbf{e}_2', \mathbf{e}_3) \\ \cos(\mathbf{e}_3', \mathbf{e}_1) & \cos(\mathbf{e}_3', \mathbf{e}_2) & \cos(\mathbf{e}_3', \mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

1-4.

a) Let **C** = **AB** where **A**, **B**, and **C** are matrices. Then,

$$C_{ij} = \sum_k A_{ik}\, B_{kj} \tag{1}$$

$$\left(C^t\right)_{ij} = C_{ji} = \sum_k A_{jk}\, B_{ki} = \sum_k B_{ki}\, A_{jk}$$

Identifying $B_{ki} = \left(B^t\right)_{ik}$ and $A_{jk} = \left(A^t\right)_{kj}$,

$$\left(C^t\right)_{ij} = \sum_k \left(B^t\right)_{ik} \left(A^t\right)_{kj} \tag{2}$$

or,

$$C^t = \left(AB\right)^t = B^t A^t \tag{3}$$

b) To show that $\left(AB\right)^{-1} = B^{-1}A^{-1}$,

$$\left(AB\right)B^{-1}A^{-1} = I = \left(B^{-1}A^{-1}\right)AB \tag{4}$$

That is,

$$\left(AB\right)B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I \tag{5}$$

$$\left(B^{-1}A^{-1}\right)\left(AB\right) = B^{-1}IB = B^{-1}B = I \tag{6}$$

1-5. Take λ to be a two-dimensional matrix:

$$|\lambda| = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \tag{1}$$

Then,

$$\begin{aligned} |\lambda|^2 &= \lambda_{11}^2\lambda_{22}^2 - 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21} + \lambda_{12}^2\lambda_{21}^2 + \left(\lambda_{11}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2\right) - \left(\lambda_{11}^2\lambda_{21}^2 + \lambda_{12}^2\lambda_{22}^2\right) \\ &= \lambda_{22}^2\left(\lambda_{11}^2 + \lambda_{12}^2\right) + \lambda_{21}^2\left(\lambda_{11}^2 + \lambda_{12}^2\right) - \left(\lambda_{11}^2\lambda_{21}^2 + 2\lambda_{11}\lambda_{22}\lambda_{12}\lambda_{21} + \lambda_{12}^2\lambda_{22}^2\right) \\ &= \left(\lambda_{11}^2 + \lambda_{12}^2\right)\left(\lambda_{22}^2 + \lambda_{21}^2\right) - \left(\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22}\right)^2 \end{aligned} \tag{2}$$

But since λ is an orthogonal transformation matrix, $\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}$.

Thus,

$$\begin{aligned} &\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2 + \lambda_{22}^2 = 1 \\ &\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0 \end{aligned} \tag{3}$$

Therefore, (2) becomes

$$|\lambda|^2 = 1 \tag{4}$$

1-6. The lengths of line segments in the x_j and x_j' systems are

$$L = \sqrt{\sum_j x_j^2}\ ;\ L' = \sqrt{\sum_i x_i'^2} \tag{1}$$

If $L = L'$, then

$$\sum_j x_j^2 = \sum_i x_i'^2 \tag{2}$$

The transformation is

$$x_i' = \sum_j \lambda_{ij} x_j \tag{3}$$

Then,

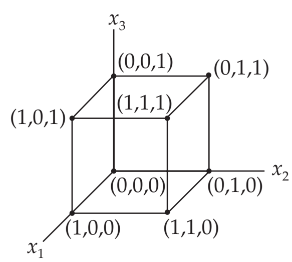
$$\begin{aligned}\sum_j x_j^2 &= \sum_i \left(\sum_k \lambda_{ik} x_k\right)\left(\sum_\ell \lambda_{i\ell} x_\ell\right) \\ &= \sum_{k,\ell} x_k x_\ell \left(\sum_i \lambda_{ik}\lambda_{i\ell}\right)\end{aligned} \tag{4}$$

But this can be true only if

$$\boxed{\sum_i \lambda_{ik}\lambda_{i\ell} = \delta_{k\ell}} \tag{5}$$

which is the desired result.

1-7.



There are 4 diagonals:

$\mathbf{D}_1$, from (0,0,0) to (1,1,1), so (1,1,1) – (0,0,0) = (1,1,1) = $\mathbf{D}_1$;

$\mathbf{D}_2$, from (1,0,0) to (0,1,1), so (0,1,1) – (1,0,0) = (–1,1,1) = $\mathbf{D}_2$;

$\mathbf{D}_3$, from (0,0,1) to (1,1,0), so (1,1,0) – (0,0,1) = (1,1,–1) = $\mathbf{D}_3$; and

$\mathbf{D}_4$, from (0,1,0) to (1,0,1), so (1,0,1) – (0,1,0) = (1,–1,1) = $\mathbf{D}_4$.

The magnitudes of the diagonal vectors are

$$|\mathbf{D}_1| = |\mathbf{D}_2| = |\mathbf{D}_3| = |\mathbf{D}_4| = \sqrt{3}$$

The angle between any two of these diagonal vectors is, for example,

$$\frac{\mathbf{D}_1 \cdot \mathbf{D}_2}{|\mathbf{D}_1||\mathbf{D}_2|} = \cos\theta = \frac{(1,1,1)\cdot(-1,1,1)}{3} = \frac{1}{3}$$

so that

$$\theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.5°$$

Similarly,

$$\frac{\mathbf{D}_1 \cdot \mathbf{D}_3}{|\mathbf{D}_1||\mathbf{D}_3|} = \frac{\mathbf{D}_1 \cdot \mathbf{D}_4}{|\mathbf{D}_1||\mathbf{D}_4|} = \frac{\mathbf{D}_2 \cdot \mathbf{D}_3}{|\mathbf{D}_2||\mathbf{D}_3|} = \frac{\mathbf{D}_2 \cdot \mathbf{D}_4}{|\mathbf{D}_2||\mathbf{D}_4|} = \frac{\mathbf{D}_3 \cdot \mathbf{D}_4}{|\mathbf{D}_3||\mathbf{D}_4|} = \pm\frac{1}{3}$$

1-8. Let θ be the angle between **A** and **r**. Then, $\mathbf{A} \cdot \mathbf{r} = A^2$ can be written as

$$Ar \cos \theta = A^2$$

or,

$$r \cos \theta = A \tag{1}$$

This implies

$$QPO = \frac{\pi}{2} \tag{2}$$

Therefore, the end point of **r** must be on a plane perpendicular to **A** and passing through P.

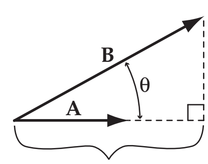
1-9. $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \qquad \mathbf{B} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

a) $\boxed{\mathbf{A} - \mathbf{B} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}}$

$$|\mathbf{A} - \mathbf{B}| = \left[(3)^2 + (-1)^2 + (-2)^2\right]^{1/2}$$

$$\boxed{|\mathbf{A} - \mathbf{B}| = \sqrt{14}}$$

b)



component of **B** along **A**

The length of the component of **B** along **A** is $B \cos \theta$.

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

$$B \cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{A} = \frac{-2 + 6 - 1}{\sqrt{6}} = \frac{3}{\sqrt{6}} \text{ or } \frac{\sqrt{6}}{2}$$

The direction is, of course, along **A**. A unit vector in the **A** direction is

$$\frac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

So the component of **B** along **A** is

$$\boxed{\frac{1}{2}(\mathbf{i}+2\mathbf{j}-\mathbf{k})}$$

c) $$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{AB}=\frac{3}{\sqrt{6}\sqrt{14}}=\frac{\sqrt{3}}{2\sqrt{7}}\,;\ \theta=\cos^{-1}\frac{\sqrt{3}}{2\sqrt{7}}$$

$$\boxed{\theta\simeq 71°}$$

d) $$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-2&3&1\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&-1\\3&1\end{vmatrix}-\mathbf{j}\begin{vmatrix}1&-1\\-2&1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1&2\\-2&3\end{vmatrix}$$

$$\boxed{\mathbf{A}\times\mathbf{B}=5\mathbf{i}+\mathbf{j}+7\mathbf{k}}$$

e) $$\mathbf{A}-\mathbf{B}=3\mathbf{i}-\mathbf{j}-2\mathbf{k}\qquad \mathbf{A}+\mathbf{B}=-\mathbf{i}+5\mathbf{j}$$

$$(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-1&-2\\-1&5&0\end{vmatrix}$$

$$\boxed{(\mathbf{A}-\mathbf{B})\times(\mathbf{A}+\mathbf{B})=10\mathbf{i}+2\mathbf{j}+14\mathbf{k}}$$

1-10. $\mathbf{r}=2b\sin\omega t\,\mathbf{i}+b\cos\omega t\,\mathbf{j}$

a) $$\boxed{\begin{aligned}&\mathbf{v}=\dot{\mathbf{r}}=2b\omega\cos\omega t\,\mathbf{i}-b\omega\sin\omega t\,\mathbf{j}\\&\mathbf{a}=\dot{\mathbf{v}}=-2b\omega^2\sin\omega t\,\mathbf{i}-b\omega^2\cos\omega t\,\mathbf{j}=-\omega^2\mathbf{r}\end{aligned}}$$

$$\text{speed}=|\mathbf{v}|=\left[4b^2\omega^2\cos^2\omega t+b^2\omega^2\sin^2\omega t\right]^{1/2}$$

$$=b\omega\left[4\cos^2\omega t+\sin^2\omega t\right]^{1/2}$$

$$\boxed{\text{speed}=b\omega\left[3\cos^2\omega t+1\right]^{1/2}}$$

b) At $t=\pi/2\omega$, $\sin\omega t=1$, $\cos\omega t=0$

So, at this time, $\mathbf{v}=-b\omega\ \mathbf{j}$, $\mathbf{a}=-2b\omega^2\ \mathbf{i}$

So, $\boxed{\theta\simeq 90°}$

1-11.

a) Since $(\mathbf{A}\times\mathbf{B})_i = \sum_{jk} \varepsilon_{ijk} A_j B_k$, we have

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \sum_i \sum_{j,k} \varepsilon_{ijk} A_j B_k C_i$$

$$= C_1\left(A_2B_3 - A_3B_2\right) - C_2\left(A_1B_3 - A_3B_1\right) + C_3\left(A_1B_2 - A_2B_1\right) \tag{1}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = -\begin{vmatrix} A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

We can also write

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) \tag{2}$$

We notice from this result that an even number of permutations leaves the determinant unchanged.

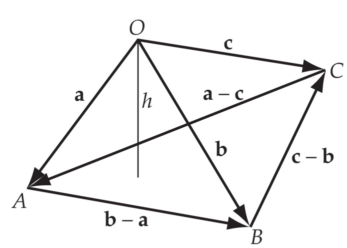
b) Consider vectors **A** and **B** in the plane defined by $\mathbf{e}_1$, $\mathbf{e}_2$. Since the figure defined by **A**, **B**, **C** is a parallelepiped, $\mathbf{A}\times\mathbf{B} = \mathbf{e}_3 \times$ area of the base, but $\mathbf{e}_3\cdot\mathbf{C} =$ altitude of the parallelepiped. Then,

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\cdot\mathbf{e}_3)\times \text{ area of the base}$$

$$= \text{altitude } \times \text{ area of the base}$$

$$= \text{volume of the parallelepiped}$$

1-12.



The distance h from the origin O to the plane defined by A, B, C is

$$h = \frac{\left|\mathbf{a}\cdot(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})\right|}{\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})\right|}$$

$$= \frac{\left|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b})\right|}{\left|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}\right|}$$

$$= \frac{\left|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}\right|}{\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\right|} \tag{1}$$

The area of the triangle ABC is:

$$A = \frac{1}{2}\left|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})\right| = \frac{1}{2}\left|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{a})\right| = \frac{1}{2}\left|(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{c})\right| \tag{2}$$

1-13. Using the Eq. (1.82) in the text, we have

$$\mathbf{A}\times\mathbf{B} = \mathbf{A}\times(\mathbf{A}\times\mathbf{X}) = (\mathbf{X}\cdot\mathbf{A})\mathbf{A} - (\mathbf{A}\cdot\mathbf{A})\mathbf{X} = \phi\mathbf{A} - A^2\mathbf{X}$$

from which

$$\mathbf{X} = \frac{(\mathbf{B}\times\mathbf{A}) + \phi\mathbf{A}}{A^2}$$

1-14.

a)

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{bmatrix}$$

Expand by the first row.

$$|\mathbf{AB}| = 1\begin{vmatrix} -2 & 9 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 9 \\ 5 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$\boxed{|\mathbf{AB}| = -104}$$

b)

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{bmatrix}$$

$$\boxed{\mathbf{AC} = \begin{bmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{bmatrix}}$$

c)

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 5 \\ -2 & -3 \\ 9 & 4 \end{bmatrix}$$

$$\boxed{\mathbf{ABC} = \begin{bmatrix} -5 & -5 \\ 3 & -5 \\ 25 & 14 \end{bmatrix}}$$

d)

$$\mathbf{AB} - \mathbf{B}^t\mathbf{A}^t = ?$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{bmatrix} \qquad \text{(from part } \mathbf{a}\text{)}$$

$$\mathbf{B}^t\mathbf{A}^t = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ -2 & -2 & 3 \\ 1 & 9 & 3 \end{bmatrix}$$

$$\boxed{\mathbf{AB} - \mathbf{B}^t\mathbf{A}^t = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}}$$

1-15. If **A** is an orthogonal matrix, then

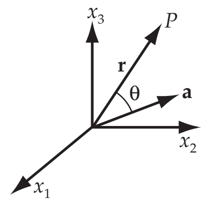
$$\mathbf{A}^t\mathbf{A} = 1$$

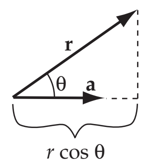
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & a \\ 0 & -a & a \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -a \\ 0 & a & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2a^2 & 0 \\ 0 & 0 & 2a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{a = \frac{1}{\sqrt{2}}}$$

1-16.





$$\mathbf{r} \cdot \mathbf{a} = \text{constant}$$

$$ra \cos \theta = \text{constant}$$

It is given that **a** is constant, so we know that

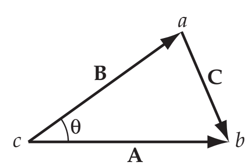
$$r \cos \theta = \text{constant}$$

But $r \cos \theta$ is the magnitude of the component of **r** along **a**.

The set of vectors that satisfy $\mathbf{r} \cdot \mathbf{a} = \text{constant}$ all have the same component along **a**; however, the component perpendicular to **a** is arbitrary.

Thus the surface represented by $\mathbf{r} \cdot \mathbf{a} = \text{constant}$ is a plane perpendicular to **a**.

1-17.



Consider the triangle a, b, c which is formed by the vectors **A**, **B**, **C**. Since

$$\mathbf{C} = \mathbf{A} - \mathbf{B}$$

$$|\mathbf{C}|^2 = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \tag{1}$$

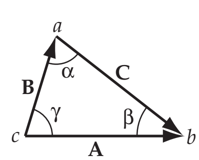
$$= A^2 - 2\mathbf{A} \cdot \mathbf{B} + B^2$$

or,

$$\boxed{|\mathbf{C}|^2 = A^2 + B^2 - 2AB \cos \theta} \tag{2}$$

which is the cosine law of plane trigonometry.

1-18. Consider the triangle a, b, c which is formed by the vectors **A**, **B**, **C**.



$$\mathbf{C} = \mathbf{A} - \mathbf{B} \tag{1}$$

so that

$$\mathbf{C} \times \mathbf{B} = (\mathbf{A} - \mathbf{B}) \times \mathbf{B} \tag{2}$$

but the left-hand side and the right-hand side of (2) are written as:

$$\mathbf{C} \times \mathbf{B} = BC \sin \alpha \, \mathbf{e}_3 \tag{3}$$

and

$$(\mathbf{A} - \mathbf{B}) \times \mathbf{B} = \mathbf{A} \times \mathbf{B} - \mathbf{B} \times \mathbf{B} = \mathbf{A} \times \mathbf{B} = AB \sin \gamma \, \mathbf{e}_3 \tag{4}$$

where $\mathbf{e}_3$ is the unit vector perpendicular to the triangle *abc*. Therefore,

$$BC \sin \alpha = AB \sin \gamma \tag{5}$$

or,

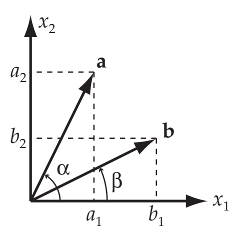
$$\frac{C}{\sin \gamma} = \frac{A}{\sin \alpha}$$

Similarly,

$$\boxed{\frac{C}{\sin \gamma} = \frac{A}{\sin \alpha} = \frac{B}{\sin \beta}} \tag{6}$$

which is the sine law of plane trigonometry.

1-19.



a) We begin by noting that

$$|\mathbf{a} - \mathbf{b}|^2 = a^2 + b^2 - 2ab \cos(\alpha - \beta) \tag{1}$$

We can also write that

$$\begin{aligned}
|\mathbf{a} - \mathbf{b}|^2 &= (a_1 - b_1)^2 + (a_2 - b_2)^2 \\
&= (a \cos \alpha - b \cos \beta)^2 + (a \sin \alpha - b \sin \beta)^2 \\
&= a^2 (\sin^2 \alpha + \cos^2 \alpha) + b^2 (\sin^2 \beta + \cos^2 \beta) - 2ab(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\
&= a^2 + b^2 - 2ab(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \qquad (2)
\end{aligned}$$

Thus, comparing (1) and (2), we conclude that

$$\boxed{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta} \tag{3}$$

b) Using (3), we can find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta)&=\sqrt{1-\cos^2(\alpha-\beta)}\\ &=\sqrt{1-\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta-2\cos\alpha\sin\alpha\cos\beta\sin\beta}\\ &=\sqrt{1-\cos^2\alpha\left(1-\sin^2\beta\right)-\sin^2\alpha\left(1-\cos^2\beta\right)-2\cos\alpha\sin\alpha\cos\beta\sin\beta}\\ &=\sqrt{\sin^2\alpha\cos^2\beta-2\sin\alpha\sin\beta\cos\alpha\cos\beta+\cos^2\alpha\sin^2\beta}\\ &=\sqrt{(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2}\end{aligned} \tag{4}$$

so that

$$\boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta} \tag{5}$$

1-20.

a) Consider the following two cases:

When $i\neq j$ $\quad \delta_{ij}=0 \quad$ but $\varepsilon_{ijk}\neq 0$.

When $i=j$ $\quad \delta_{ij}\neq 0 \quad$ but $\varepsilon_{ijk}=0$.

Therefore,

$$\boxed{\sum_{ij}\varepsilon_{ijk}\,\delta_{ij}=0} \tag{1}$$

b) We proceed in the following way:

When $j=k$, $\varepsilon_{ijk}=\varepsilon_{ijj}=0$.

Terms such as $\varepsilon_{j11}\,\varepsilon_{\ell 11}=0$. Then,

$$\sum_{jk}\varepsilon_{ijk}\,\varepsilon_{\ell jk}=\varepsilon_{i12}\,\varepsilon_{\ell 12}+\varepsilon_{i13}\,\varepsilon_{\ell 13}+\varepsilon_{i21}\,\varepsilon_{\ell 21}+\varepsilon_{i31}\,\varepsilon_{\ell 31}+\varepsilon_{i32}\,\varepsilon_{\ell 32}+\varepsilon_{i23}\,\varepsilon_{\ell 23}$$

Now, suppose $i=\ell=1$, then,

$$\sum_{jk}=\varepsilon_{123}\,\varepsilon_{123}+\varepsilon_{132}\,\varepsilon_{132}=1+1=2$$

for $i=\ell=2$, $\sum_{jk} = \varepsilon_{213}\ \varepsilon_{213} + \varepsilon_{231}\ \varepsilon_{231} = 1+1=2$. For $i=\ell=3$, $\sum_{jk} = \varepsilon_{312}\ \varepsilon_{312} + \varepsilon_{321}\ \varepsilon_{321} = 2$. But i = 1, $\ell=2$ gives $\sum_{jk} = 0$. Likewise for i = 2, $\ell=1$; i = 1, $\ell=3$; i = 3, $\ell=1$; i = 2, $\ell=3$; i = 3, $\ell=2$. Therefore,

$$\boxed{\sum_{j,k} \varepsilon_{ijk}\ \varepsilon_{\ell jk} = 2\delta_{i\ell}} \tag{2}$$

c) $\sum_{ijk} \varepsilon_{ijk}\ \varepsilon_{ijk} = \varepsilon_{123}\ \varepsilon_{123} + \varepsilon_{312}\ \varepsilon_{312} + \varepsilon_{321}\ \varepsilon_{321} + \varepsilon_{132}\ \varepsilon_{132} + \varepsilon_{213}\ \varepsilon_{213} + \varepsilon_{231}\ \varepsilon_{231}$

$$= 1\cdot 1 + 1\cdot 1 + (-1)\cdot(-1) + (-1)\cdot(-1) + (-1)\cdot(-1) + (1)\cdot(1)$$

or,

$$\boxed{\sum_{ijk} \varepsilon_{ijk}\ \varepsilon_{ijk} = 6} \tag{3}$$

1-21.
$$(\mathbf{A}\times\mathbf{B})_i = \sum_{jk} \varepsilon_{ijk}\ A_j B_k \tag{1}$$

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \sum_{i}\ \sum_{jk} \varepsilon_{ijk}\ A_j B_k C_i \tag{2}$$

By an even permutation, we find

$$\boxed{\mathbf{ABC} = \sum_{ijk} \varepsilon_{ijk}\ A_i B_j C_k} \tag{3}$$

1-22. To evaluate $\sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell mk}$ we consider the following cases:

a) $i=j$: $\sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_{k} \varepsilon_{iik}\ \varepsilon_{\ell mk} = 0$ for all i, ℓ, m

b) $i=\ell$: $\sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_{k} \varepsilon_{ijk}\ \varepsilon_{imk} = 1$ for $j=m$ and $k\neq i,j$

$$= 0 \text{ for } j\neq m$$

c) $i=m$: $\sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell ik} = 0$ for $j\neq\ell$

$$= -1 \text{ for } j=\ell \text{ and } k\neq i,j$$

d) $j=\ell$: $\sum_{k} \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_{k} \varepsilon_{ijk}\ \varepsilon_{jmk} = 0$ for $m\neq i$

$$= -1 \text{ for } m=i \text{ and } k\neq i,j$$

e) $j = m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell jk} = 0$ for $i \neq \ell$

$= 1$ for $i = \ell$ and $k \neq i, j$

f) $\ell = m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell\ell k} = 0$ for all i, j, k

g) $i \neq \ell$ or m: This implies that $i = k$ or $i = j$ or $m = k$.

Then, $\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = 0$ for all i, j, ℓ, m

h) $j \neq \ell$ or $m: \sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = 0$ for all i, j, ℓ, m

Now, consider $\delta_{i\ell}\ \delta_{jm} - \delta_{im}\ \delta_{j\ell}$ and examine it under the same conditions. If this quantity behaves in the same way as the sum above, we have verified the equation

$$\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \delta_{i\ell}\ \delta_{jm} - \delta_{im}\ \delta_{j\ell}$$

a) $i = j: \ \delta_{i\ell}\ \delta_{im} - \delta_{im}\ \delta_{i\ell} = 0$ for all i, ℓ, m

b) $i = \ell: \ \delta_{ii}\ \delta_{jm} - \delta_{im}\ \delta_{ji} = 1$ if $j = m,\ i \neq j, m$

$= 0$ if $j \neq m$

c) $i = m: \ \delta_{i\ell}\ \delta_{ji} - \delta_{ii}\ \delta_{j\ell} = -1$ if $j = \ell,\ i \neq j, \ell$

$= 0$ if $j \neq \ell$

d) $j = \ell: \ \delta_{i\ell}\ \delta_{\ell m} - \delta_{im}\ \delta_{\ell\ell} = -1$ if $i = m,\ i \neq \ell$

$= 0$ if $i \neq m$

e) $j = m: \ \delta_{i\ell}\ \delta_{mm} - \delta_{im}\ \delta_{m\ell} = 1$ if $i = \ell,\ m \neq \ell$

$= 0$ if $i \neq \ell$

f) $\ell = m: \ \delta_{i\ell}\ \delta_{j\ell} - \delta_{il}\ \delta_{j\ell} = 0$ for all i, j, ℓ

g) $i \neq \ell, m: \ \delta_{i\ell}\ \delta_{jm} - \delta_{im}\ \delta_{j\ell} = 0$ for all i, j, ℓ, m

h) $j \neq \ell, m: \ \delta_{i\ell}\ \delta_{jm} - \delta_{im}\ \delta_{i\ell} = 0$ for all i, j, ℓ, m

Therefore,

$$\boxed{\sum_k \varepsilon_{ijk}\ \varepsilon_{\ell mk} = \delta_{i\ell}\ \delta_{jm} - \delta_{im}\ \delta_{j\ell}} \tag{1}$$

Using this result we can prove that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

First $(\mathbf{B}\times\mathbf{C})_i = \sum_{jk} \varepsilon_{ijk}\, B_j C_k$. Then,

$$\begin{aligned}
\left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right]_\ell &= \sum_{mn} \varepsilon_{\ell mn}\, A_m (B\times C)_n = \sum_{mn} \varepsilon_{\ell mn}\, A_m \sum_{jk} \varepsilon_{njk}\, B_j C_k \\
&= \sum_{jkmn} \varepsilon_{\ell mn}\, \varepsilon_{njk}\, A_m B_j C_k = \sum_{jkmn} \varepsilon_{\ell mn}\, \varepsilon_{jkn}\, A_m B_j C_k \\
&= \sum_{jkm} \left(\sum_n \varepsilon_{lmn}\, \varepsilon_{jkn} \right) A_m B_j C_k \\
&= \sum_{jkm} \left(\delta_{jl}\delta_{km} - \delta_{k\ell}\delta_{jm} \right) A_m B_j C_k \\
&= \sum_m A_m B_\ell C_m - \sum_m A_m B_m C_\ell = B_\ell \left(\sum_m A_m C_m \right) - C_\ell \left(\sum_m A_m B_m \right) \\
&= (\mathbf{A}\cdot\mathbf{C}) B_\ell - (\mathbf{A}\cdot\mathbf{B}) C_\ell
\end{aligned}$$

Therefore,

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}} \tag{2}$$

1-23. Write

$$(\mathbf{A}\times\mathbf{B})_j = \sum_{\ell m} \varepsilon_{j\ell m}\, A_\ell B_m$$

$$(\mathbf{C}\times\mathbf{D})_k = \sum_{rs} \varepsilon_{krs}\, C_r D_s$$

Then,

$$\begin{aligned}
\left[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})\right]_i &= \sum_{jk}\varepsilon_{ijk}\left(\sum_{\ell m}\varepsilon_{j\ell m}\,A_\ell B_m\right)\left(\sum_{rs}\varepsilon_{krs}\,C_r D_s\right)\\
&= \sum_{jk\ell mrs}\varepsilon_{ijk}\,\varepsilon_{j\ell m}\,\varepsilon_{krs}\,A_\ell B_m C_r D_s\\
&= \sum_{j\ell mrs}\varepsilon_{j\ell m}\left(\sum_k\varepsilon_{ijk}\,\varepsilon_{rsk}\right)A_\ell B_m C_r D_s\\
&= \sum_{j\ell mrs}\varepsilon_{j\ell m}\left(\delta_{ir}\,\delta_{js}-\delta_{is}\,\delta_{jr}\right)A_\ell B_m C_r D_s\\
&= \sum_{j\ell m}\varepsilon_{j\ell m}\left(A_\ell B_m C_i D_j - A_\ell B_m D_i C_j\right)\\
&= \left(\sum_{j\ell m}\varepsilon_{j\ell m}\,D_j A_\ell B_m\right)C_i - \left(\sum_{j\ell m}\varepsilon_{j\ell m}\,C_j A_\ell B_m\right)D_i\\
&= (\mathbf{ABD})C_i - (\mathbf{ABC})D_i
\end{aligned}$$

Therefore,

$$\boxed{\left[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})\right] = (\mathbf{ABD})\mathbf{C}-(\mathbf{ABC})\mathbf{D}}$$

1-24. Expanding the triple vector product, we have

$$\mathbf{e}\times(\mathbf{A}\times\mathbf{e}) = \mathbf{A}(\mathbf{e}\cdot\mathbf{e}) - \mathbf{e}(\mathbf{A}\cdot\mathbf{e}) \tag{1}$$

But,

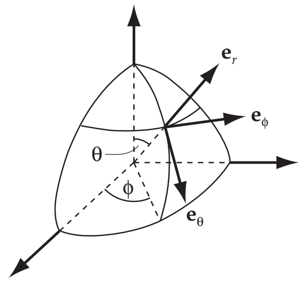
$$\mathbf{A}(\mathbf{e}\cdot\mathbf{e}) = \mathbf{A} \tag{2}$$

Thus,

$$\boxed{\mathbf{A} = \mathbf{e}(\mathbf{A}\cdot\mathbf{e}) + \mathbf{e}\times(\mathbf{A}\times\mathbf{e})} \tag{3}$$

$\mathbf{e}(\mathbf{A}\cdot\mathbf{e})$ is the component of $\mathbf{A}$ in the $\mathbf{e}$ direction, while $\mathbf{e}\times(\mathbf{A}\times\mathbf{e})$ is the component of $\mathbf{A}$ perpendicular to $\mathbf{e}$.

1-25.



The unit vectors in spherical coordinates are expressed in terms of rectangular coordinates by

$$\left.\begin{aligned}\mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\\ \mathbf{e}_\phi &= (-\sin\phi, \cos\phi, 0)\\ \mathbf{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\end{aligned}\right] \tag{1}$$

Thus,

$$\dot{\mathbf{e}}_\theta = (-\dot\phi\cos\theta\sin\phi - \dot\theta\sin\theta\cos\phi, \dot\phi\cos\theta\cos\phi - \dot\theta\sin\theta\sin\phi, -\dot\theta\cos\theta)$$

$$= -\dot\theta\mathbf{e}_r + \dot\phi\cos\theta\,\mathbf{e}_\phi \tag{2}$$

Similarly,

$$\dot{\mathbf{e}}_\phi = (-\dot\phi\cos\phi, -\dot\phi\sin\phi, 0)$$

$$= -\dot\phi\cos\theta\,\mathbf{e}_\theta - \dot\phi\sin\theta\,\mathbf{e}_r \tag{3}$$

$$\dot{\mathbf{e}}_r = \dot\phi\sin\theta\,\mathbf{e}_\phi + \dot\theta\mathbf{e}_\theta \tag{4}$$

Now, let any position vector be $\mathbf{x}$. Then,

$$\mathbf{x} = r\mathbf{e}_r \tag{5}$$

$$\dot{\mathbf{x}} = r\dot{\mathbf{e}}_r + r\dot{\mathbf{e}}_r = r\left(\dot\phi\sin\theta\mathbf{e}_\phi + \dot\theta\mathbf{e}_\theta\right) + r\dot{\mathbf{e}}_r$$

$$= r\dot\phi\sin\theta\,\mathbf{e}_\phi + r\dot\theta\mathbf{e}_\theta + \dot r\mathbf{e}_r \tag{6}$$

$$\ddot{\mathbf{x}} = \left(\dot r\dot\phi\sin\theta + r\dot\theta\dot\phi\cos\theta + r\ddot\phi\sin\theta\right)\mathbf{e}_\phi + r\dot\phi\sin\theta\,\dot{\mathbf{e}}_\phi + \left(\dot r\dot\theta + r\ddot\theta\right)\mathbf{e}_\theta + r\dot\theta\dot{\mathbf{e}}_\theta + \ddot r\mathbf{e}_r + \dot r\dot{\mathbf{e}}_r$$

$$\begin{aligned}&= \left(2\dot r\dot\phi\sin\theta + 2r\dot\theta\dot\phi\cos\theta + r\ddot\phi\sin\theta\right)\mathbf{e}_\phi + \left(\ddot r - r\dot\phi^2\sin^2\theta - r\dot\theta^2\right)\mathbf{e}_r\\ &\qquad + \left(2\dot r\dot\theta + r\ddot\theta - r\dot\phi^2\sin\theta\cos\theta\right)\mathbf{e}_\theta\end{aligned} \tag{7}$$

or,

$$\boxed{\begin{aligned}\ddot{\mathbf{x}} = \mathbf{a} = \left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta\right]\mathbf{e}_r + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) - \dot{r}\dot{\phi}^2 \sin\theta\cos\theta\right]\mathbf{e}_\theta \\ + \left[\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\phi}\sin^2\theta\right)\right]\mathbf{e}_\phi\end{aligned}} \tag{8}$$

1-26. When a particle moves along the curve

$$r = k(1 + \cos\theta) \tag{1}$$

we have

$$\left.\begin{aligned}\dot{r} &= -k\dot{\theta}\sin\theta \\ \ddot{r} &= -k\left[\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right]\end{aligned}\right] \tag{2}$$

Now, the velocity vector in polar coordinates is [see Eq. (1.97)]

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta \tag{3}$$

so that

$$\begin{aligned}v^2 = |\mathbf{v}|^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ &= k^2\dot{\theta}^2\sin^2\theta + k^2\left(1 + 2\cos\theta + \cos^2\theta\right)\dot{\theta}^2 \\ &= k^2\dot{\theta}^2\left[2 + 2\cos\theta\right]\end{aligned} \tag{4}$$

and v^2 is, by hypothesis, constant. Therefore,

$$\dot{\theta} = \sqrt{\frac{v^2}{2k^2(1 + \cos\theta)}} \tag{5}$$

Using (1), we find

$$\boxed{\dot{\theta} = \frac{v}{\sqrt{2kr}}} \tag{6}$$

Differentiating (5) and using the expression for $\dot{r}$, we obtain

$$\ddot{\theta} = \frac{v^2\sin\theta}{4r^2} = \frac{v^2\sin\theta}{4k^2(1 + \cos\theta)^2} \tag{7}$$

The acceleration vector is [see Eq. (1.98)]

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta \tag{8}$$

so that

$$\begin{aligned}\mathbf{a}\cdot\mathbf{e}_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -k\left(\dot{\theta}^2 \cos\theta + \ddot{\theta}\sin\theta\right) - k\left(1+\cos\theta\right)\dot{\theta}^2 \\ &= -k\left[\dot{\theta}^2\cos\theta + \frac{\dot{\theta}^2 \sin^2\theta}{2\left(1+\cos\theta\right)} + \left(1+\cos\theta\right)\dot{\theta}^2\right] \\ &= -k\dot{\theta}^2\left[2\cos\theta + \frac{1-\cos^2\theta}{2\left(1+\cos\theta\right)} + 1\right] \\ &= -\frac{3}{2}k\dot{\theta}^2\left(1+\cos\theta\right) \qquad (9)\end{aligned}$$

or,

$$\boxed{\mathbf{a}\cdot\mathbf{e}_r = -\frac{3}{4}\frac{v^2}{k}} \qquad (10)$$

In a similar way, we find

$$\mathbf{a}\cdot\mathbf{e}_\theta = -\frac{3}{4}\frac{v^2}{k}\frac{\sin\theta}{1+\cos\theta} \qquad (11)$$

From (10) and (11), we have

$$|\mathbf{a}| = \sqrt{\left(\mathbf{a}\cdot\mathbf{e}_r\right)^2 + \left(\mathbf{a}\cdot\mathbf{e}_\theta\right)^2} \qquad (12)$$

or,

$$\boxed{|\mathbf{a}| = \frac{3}{4}\frac{v^2}{k}\sqrt{\frac{2}{1+\cos\theta}}} \qquad (13)$$

1-27. Since

$$\mathbf{r}\times\left(\mathbf{v}\times\mathbf{r}\right) = \left(\mathbf{r}\cdot\mathbf{r}\right)\mathbf{v} - \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r}$$

we have

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{r}\times\left(\mathbf{v}\times\mathbf{r}\right)\right] &= \frac{d}{dt}\left[\left(\mathbf{r}\cdot\mathbf{r}\right)\mathbf{v} - \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{r}\right] \\ &= \left(\mathbf{r}\cdot\mathbf{r}\right)\mathbf{a} + 2\left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v} - \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v} - \left(\mathbf{v}\cdot\mathbf{v}\right)\mathbf{r} - \left(\mathbf{r}\cdot\mathbf{a}\right)\mathbf{r} \\ &= r^2\mathbf{a} + \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v} - \mathbf{r}\left(v^2 + \mathbf{r}\cdot\mathbf{a}\right) \qquad (1)\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dt}\left[\mathbf{r}\times\left(\mathbf{v}\times\mathbf{r}\right)\right] = r^2\mathbf{a} + \left(\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v} - \mathbf{r}\left(\mathbf{r}\cdot\mathbf{a} + \mathbf{v}^2\right)} \qquad (2)$$

1-28.
$$\textbf{grad}(\ln|\mathbf{r}|) = \sum_i \frac{\partial}{\partial x_i}(\ln|\mathbf{r}|)\mathbf{e}_i \tag{1}$$

where

$$|\mathbf{r}| = \sqrt{\sum_i x_i^2} \tag{2}$$

Therefore,

$$\frac{\partial}{\partial x_i}(\ln|\mathbf{r}|) = \frac{1}{|\mathbf{r}|}\frac{x_i}{\sqrt{\sum_i x_i^2}}$$

$$= \frac{x_i}{|\mathbf{r}|^2} \tag{3}$$

so that

$$\textbf{grad}(\ln|\mathbf{r}|) = \frac{1}{|\mathbf{r}|^2}\left(\sum_i x_i\mathbf{e}_i\right) \tag{4}$$

or,

$$\boxed{\textbf{grad}(\ln|\mathbf{r}|) = \frac{\mathbf{r}}{r^2}} \tag{5}$$

1-29. Let $r^2 = 9$ describe the surface S_1 and $x + y + z^2 = 1$ describe the surface S_2. The angle θ between S_1 and S_2 at the point (2,–2,1) is the angle between the normals to these surfaces at the point. The normal to S_1 is

$$\textbf{grad}(S_1) = \textbf{grad}(r^2 - 9) = \textbf{grad}(x^2 + y^2 + z^2 - 9)$$

$$= (2x\mathbf{e}_1 + 2y\mathbf{e}_2 + 2z\mathbf{e}_3)\Big|_{x=2,y=2,z=1} \tag{1}$$

$$= 4\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3$$

In S_2, the normal is:

$$\textbf{grad}(S_2) = \textbf{grad}(x + y + z^2 - 1)$$

$$= (\mathbf{e}_1 + \mathbf{e}_2 + 2z\mathbf{e}_3)\Big|_{x=2,y=-2,z=1} \tag{2}$$

$$= \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3$$

Therefore,

$$\cos\theta = \frac{\mathbf{grad}(S_1)\cdot\mathbf{grad}(S_2)}{|\mathbf{grad}(S_1)||\mathbf{grad}(S_2)|}$$

$$= \frac{(4\mathbf{e}_1 - 4\mathbf{e}_2 + 2\mathbf{e}_3)\cdot(\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3)}{6\sqrt{6}} \tag{3}$$

or,

$$\cos\theta = \frac{4}{6\sqrt{6}} \tag{4}$$

from which

$$\boxed{\theta = \cos^{-1}\frac{\sqrt{6}}{9} = 74.2°} \tag{5}$$

1-30.

$$\mathbf{grad}(\phi\psi) = \sum_{i=1}^{3}\mathbf{e}_i\,\frac{\partial(\phi\psi)}{\partial x_i} = \sum_i \mathbf{e}_i\left[\phi\,\frac{\partial\psi}{\partial x_i} + \frac{\partial\phi}{\partial x_i}\,\psi\right]$$

$$= \sum_i \mathbf{e}_i\;\phi\,\frac{\partial\psi}{\partial x_i} + \sum_i \mathbf{e}_i\,\frac{\partial\phi}{\partial x_i}\,\psi$$

Thus,

$$\boxed{\mathbf{grad}(\phi\psi) = \phi\,\mathbf{grad}\;\psi + \psi\;\mathbf{grad}\;\phi}$$

1-31.

a)

$$\mathbf{grad}\;r^n = \sum_{i=1}^{3}\mathbf{e}_i\,\frac{\partial r_n}{\partial x_i} = \sum \mathbf{e}_i\,\frac{\partial}{\partial x_i}\left[\left(\sum_j x_j^2\right)^{1/2}\right]^n$$

$$= \sum_i \mathbf{e}_i\;2x_i\,\frac{n}{2}\left(\sum_j x_j^2\right)^{\frac{n}{2}-1}$$

$$= \sum_i \mathbf{e}_i\;x_i\;n\left(\sum_j x_j^2\right)^{\frac{n}{2}-1}$$

$$= \sum_i \mathbf{e}_i\;x_i\;n\,r^{(n-2)} \tag{1}$$

Therefore,

$$\boxed{\mathbf{grad}\;r^n = nr^{(n-2)}\,\mathbf{r}} \tag{2}$$

b)

$$\begin{aligned}
\mathbf{grad}\, f(r) &= \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial f(r)}{\partial x_i} = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial f(r)}{\partial r} \frac{\partial r}{\partial x_i} \\
&= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \left(\sum_j x_j^2 \right)^{1/2} \frac{\partial f(r)}{\partial r} \\
&= \sum_i \mathbf{e}_i \, x_i \left(\sum_j x_j^2 \right)^{-1/2} \frac{\partial f(r)}{\partial r} \\
&= \sum_i \mathbf{e}_i \frac{x_i}{r} \frac{\partial f}{dr}
\end{aligned} \tag{3}$$

Therefore,

$$\boxed{\mathbf{grad}\, f(r) = \frac{\mathbf{r}}{r} \frac{\partial f(r)}{\partial r}} \tag{4}$$

c)

$$\begin{aligned}
\nabla^2 (\ln r) &= \sum_i \frac{\partial^2 \ln r}{\partial x_i^2} = \sum \frac{\partial^2}{\partial x_i^2} \left[\ln \left(\sum_j x_j^2 \right)^{1/2} \right] \\
&= \sum_i \frac{\partial}{\partial x_i} \left[\frac{\frac{1}{2} \cdot 2x_i \left(\sum_j x_j^2 \right)^{-1/2}}{\left(\sum_j x_j^2 \right)^{1/2}} \right] \\
&= \sum_i \frac{\partial}{\partial x_i} \left[x_i \left(\sum_j x_j^2 \right)^{-1} \right] \\
&= \sum_i (-x_i)(2x_i) \left(\sum_j x_j^2 \right)^{-2} + \sum_i \frac{\partial x_i}{\partial x_i} \left(\sum_j x_j^2 \right)^{-1} \\
&= \sum_i \left(-2x_j^2 \right) \left(r^2 \right)^{-2} + 3 \left[\frac{1}{r^2} \right] \\
&= -\frac{2r^2}{r^4} + \frac{3}{r^2} = \frac{1}{r^2}
\end{aligned} \tag{5}$$

or,

$$\boxed{\nabla^2 (\ln r) = \frac{1}{r^2}} \tag{6}$$

1-32. Note that the integrand is a perfect differential:

$$2a\mathbf{r}\cdot\dot{\mathbf{r}}+2b\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}=a\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})+b\frac{d}{dt}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) \tag{1}$$

Clearly,

$$\boxed{\int(2a\mathbf{r}\cdot\dot{\mathbf{r}}+2b\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\,dt=ar^2+b\dot{r}^2+\text{const.}} \tag{2}$$

1-33. Since

$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right]=\frac{\dot{\mathbf{r}}r-\mathbf{r}\dot{r}}{r^2}=\frac{\dot{\mathbf{r}}}{r}-\frac{\mathbf{r}\dot{r}}{r^2} \tag{1}$$

we have

$$\int\left[\frac{\dot{\mathbf{r}}}{r}-\frac{\mathbf{r}\dot{r}}{r^2}\right]dt=\int\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right]dt \tag{2}$$

from which

$$\boxed{\int\left[\frac{\dot{\mathbf{r}}}{r}-\frac{\mathbf{r}\dot{r}}{r^2}\right]dt=\frac{\mathbf{r}}{r}+\mathbf{C}} \tag{3}$$

where **C** is the integration constant (a *vector*).

1-34. First, we note that

$$\frac{d}{dt}\left(\mathbf{A}\times\dot{\mathbf{A}}\right)=\dot{\mathbf{A}}\times\dot{\mathbf{A}}+\mathbf{A}\times\ddot{\mathbf{A}} \tag{1}$$

But the first term on the right-hand side vanishes. Thus,

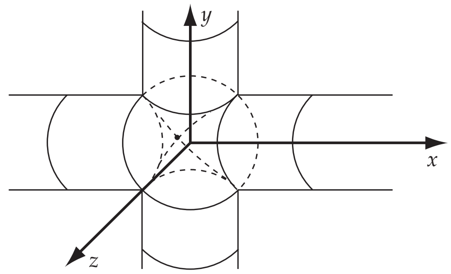
$$\int\left(\mathbf{A}\times\ddot{\mathbf{A}}\right)dt=\int\frac{d}{dt}\left(\mathbf{A}\times\dot{\mathbf{A}}\right)dt \tag{2}$$

so that

$$\boxed{\int\left(\mathbf{A}\times\ddot{\mathbf{A}}\right)dt=\mathbf{A}\times\dot{\mathbf{A}}+\mathbf{C}} \tag{3}$$

where **C** is a constant vector.

1-35.



We compute the volume of the intersection of the two cylinders by dividing the intersection volume into two parts. Part of the common volume is that of one of the cylinders, for example, the one along the y axis, between $y = -a$ and $y = a$:

$$V_1 = 2\left(\pi a^2\right)a = 2\pi a^3 \tag{1}$$

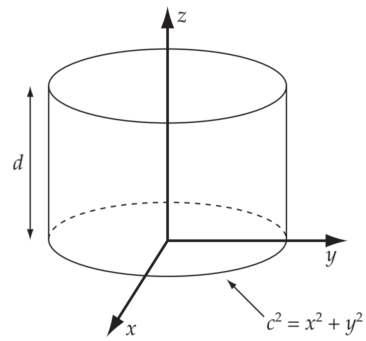
The rest of the common volume is formed by 8 equal parts from the other cylinder (the one along the x-axis). One of these parts extends from $x = 0$ to $x = a$, $y = 0$ to $y = \sqrt{a^2 - x^2}$, $z = a$ to $z = \sqrt{a^2 - x^2}$. The complementary volume is then

$$\begin{aligned} V_2 &= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_a^{\sqrt{a^2-x^2}} dz \\ &= 8\int_0^a dx\,\sqrt{a^2 - x^2}\left[\sqrt{a^2-x^2} - a\right] \\ &= 8\left[a^2 x - \frac{x^3}{3} - \frac{a^3}{2}\sin^{-1}\frac{x}{a}\right]_0^a \\ &= \frac{16}{3}a^3 - 2\pi a^3 \end{aligned} \tag{2}$$

Then, from (1) and (2):

$$\boxed{V = V_1 + V_2 = \frac{16a^3}{3}} \tag{3}$$

1-36.



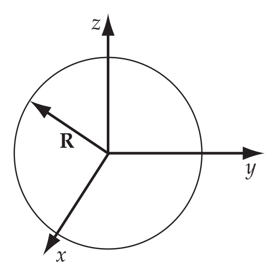
The form of the integral suggests the use of the divergence theorem.

$$\int_S \mathbf{A} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{A}\, dv \tag{1}$$

Since $\nabla \cdot \mathbf{A} = 1$, we only need to evaluate the total volume. Our cylinder has radius *c* and height *d*, and so the answer is

$$\int_V dv = \pi c^2 d \tag{2}$$

1-37.



To do the integral directly, note that $\mathbf{A} = R^3 \mathbf{e}_r$, on the surface, and that $d\mathbf{a} = da\mathbf{e}_r$.

$$\int_S \mathbf{A} \cdot d\mathbf{a} = R^3 \int_S da = R^3 \times 4\pi R^2 = 4\pi R^5 \tag{1}$$

To use the divergence theorem, we need to calculate $\nabla \cdot \mathbf{A}$. This is best done in spherical coordinates, where $\mathbf{A} = r^3 \mathbf{e}_r$. Using Appendix F, we see that

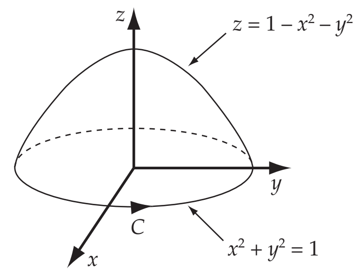
$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \mathbf{A}_r\right) = 5r^2 \tag{2}$$

Therefore,

$$\int_V \nabla \cdot \mathbf{A}\, dv = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^R r^2 \left(5r^2\right) dr = 4\pi R^5 \tag{3}$$

Alternatively, one may simply set $dv = 4\pi r^2\, dr$ in this case.

1-38.



By Stoke's theorem, we have

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_C \mathbf{A} \cdot d\mathbf{s} \tag{1}$$

The curve C that encloses our surface S is the unit circle that lies in the xy plane. Since the element of area on the surface $d\mathbf{a}$ is chosen to be outward from the origin, the curve is directed counterclockwise, as required by the right-hand rule. Now change to polar coordinates, so that we have $d\mathbf{s} = d\theta\, \mathbf{e}_\theta$ and $\mathbf{A} = \sin\theta\mathbf{i} + \cos\theta\mathbf{k}$ on the curve. Since $\mathbf{e}\theta \cdot \mathbf{i} = -\sin\theta$ and $\mathbf{e}\theta \cdot \mathbf{k} = 0$, we have

$$\int_C \mathbf{A} \cdot d\mathbf{s} = \int_0^{2\pi} \left(-\sin^2\theta\right) d\theta = -\pi \tag{2}$$

1-39.

a) Let's denote A = (1,0,0); B = (0,2,0); C = (0,0,3). Then $\overline{\mathbf{AB}} = (-1,2,0)$; $\overline{\mathbf{AC}} = (-1,0,3)$; and $\overline{\mathbf{AB}} \times \overline{\mathbf{AC}} = (6,3,2)$. Any vector perpendicular to plane (ABC) must be parallel to $\overline{\mathbf{AB}} \times \overline{\mathbf{AC}}$, so the unit vector perpendicular to plane (ABC) is $\mathbf{n} = (6/7, 3/7, 2/7)$

b) Let's denote D = (1,1,1) and H = (x,y,z) be the point on plane (ABC) closest to H. Then $\overline{\mathbf{DH}} = (x-1, y-1, z-1)$ is parallel to $\mathbf{n}$ given in a); this means

$$\frac{x-1}{y-1} = \frac{6}{3} = 2 \quad \text{and} \quad \frac{x-1}{z-1} = \frac{6}{2} = 3$$

Further, $\overline{\mathbf{AH}} = (x-1, y, z)$ is perpendicular to $\mathbf{n}$ so one has $6(x-1) + 3y + 2z = 0$.

Solving these 3 equations one finds

$$\mathrm{H} = (x,y,z) = (19/49, 34/49, 39/49) \text{ and } |DH| = \frac{5}{7}$$

1-40.

a) At the top of the hill, z is maximum;

$$0 = \frac{\partial z}{\partial x} = 2y - 6x - 18 \quad \text{and} \quad 0 = \frac{\partial z}{\partial y} = 2x - 8y + 28$$

so $x = -2$; $y = 3$, and the hill's height is max[z]= 72 m. Actually, this is the max value of z, because the given equation of z implies that, for each given value of x (or y), z describes an upside down parabola in term of y (or x) variable.

b) At point A: $x = y = 1$, $z = 13$. At this point, two of the tangent vectors to the surface of the hill are

$$t_1 = (1,0,\left.\frac{\partial z}{\partial x}\right|_{(1,1)}) = (1,0,-8) \quad \text{and} \quad t_2 = (0,1,\left.\frac{\partial z}{\partial y}\right|_{(1,1)}) = (0,1,22)$$

Evidently $t_1 \times t_2 = (8,-22,1)$ is perpendicular to the hill surface, and the angle θ between this and Oz axis is

$$\cos\theta = \frac{(0,0,1)\cdot(8,-22,1)}{\sqrt{8^2+22^2+1^2}} = \frac{1}{23.43} \quad \text{so} \quad \theta = 87.55 \text{ degrees.}$$

c) Suppose that in the α direction (with respect to W-E axis), at point A = (1,1,13) the hill is steepest. Evidently, $dy = (\tan\alpha)dx$ and

$$dz = 2xdy + 2ydx - 6xdx - 8ydy - 18dx + 28dy = 22(\tan\alpha - 1)dx$$

then

$$\tan\beta = \frac{\sqrt{dx^2+dy^2}}{dz} = \frac{dx/\cos\alpha}{22(\tan\alpha-1)dx} = \frac{-1}{22\sqrt{2}\cos(\alpha+45)}$$

The hill is steepest when $|\tan\beta|$ is minimum, and this happens when $\alpha = -45$ degrees with respect to W-E axis. (note that $\alpha = 135$ does not give a physical answer).

1-41.

$$\mathbf{A}\cdot\mathbf{B} = 2a(a-1)$$

then $\mathbf{A}\cdot\mathbf{B} = 0$ if only $a = 1$ or $a = 0$.

CHAPTER 2

Newtonian Mechanics—Single Particle

2-1. The basic equation is

$$F = m_i \ddot{x}_i \tag{1}$$

a) $F(x_i, t) = f(x_i)\, g(t) = m_i \ddot{x}_i$: *Not integrable* (2)

b) $F(\dot{x}_i, t) = f(\dot{x}_i)\, g(t) = m_i \ddot{x}_i$

$$m_i \frac{d\dot{x}_i}{dt} = f(\dot{x}_i)\, g(t)$$

$$\frac{d\dot{x}_i}{f(\dot{x}_i)} = \frac{g(t)}{m_i}\, dt : \textit{Integrable} \tag{3}$$

c) $F(x_i, \dot{x}_i) = f(x_i)\, g(\dot{x}_i) = m_i \ddot{x}_i$: *Not integrable* (4)

2-2. Using spherical coordinates, we can write the force applied to the particle as

$$\mathbf{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi \tag{1}$$

But since the particle is constrained to move on the surface of a sphere, there must exist a reaction force $-F_r \mathbf{e}_r$ that acts on the particle. Therefore, the total force acting on the particle is

$$\mathbf{F}_{total} = F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi = m\ddot{\mathbf{r}} \tag{2}$$

The position vector of the particle is

$$\mathbf{r} = R\mathbf{e}_r \tag{3}$$

where R is the radius of the sphere and is constant. The acceleration of the particle is

$$\mathbf{a} = \ddot{\mathbf{r}} = R\ddot{\mathbf{e}}_r \tag{4}$$

We must now express $\ddot{\mathbf{e}}_r$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$. Because the unit vectors in rectangular coordinates, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, do not change with time, it is convenient to make the calculation in terms of these quantities. Using Fig. F-3, Appendix F, we see that

$$\left.\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_3 \cos\theta \\
\mathbf{e}_\theta &= \mathbf{e}_1 \cos\theta\cos\phi + \mathbf{e}_2 \cos\theta\sin\phi - \mathbf{e}_3 \sin\theta \\
\mathbf{e}_\phi &= -\mathbf{e}_1 \sin\phi + \mathbf{e}_2 \cos\phi
\end{aligned}\right] \tag{5}$$

Then

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \mathbf{e}_1\left(-\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\cos\theta\cos\phi\right) + \mathbf{e}_2\left(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right) - \mathbf{e}_3\,\dot{\theta}\sin\theta \\
&= \mathbf{e}_\phi\,\dot{\phi}\sin\theta + \mathbf{e}_\theta\,\dot{\theta}
\end{aligned} \tag{6}$$

Similarly,

$$\dot{\mathbf{e}}_\theta = -\mathbf{e}_r\,\dot{\theta} + \mathbf{e}_\phi\,\dot{\phi}\cos\theta \tag{7}$$

$$\dot{\mathbf{e}}_\phi = -\mathbf{e}_r\,\dot{\phi}\sin\theta - \mathbf{e}_\theta\,\dot{\phi}\cos\theta \tag{8}$$

And, further,

$$\ddot{\mathbf{e}}_r = -\mathbf{e}_r\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \mathbf{e}_\theta\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) + \mathbf{e}_\phi\left(2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta\right) \tag{9}$$

which is the only second time derivative needed.

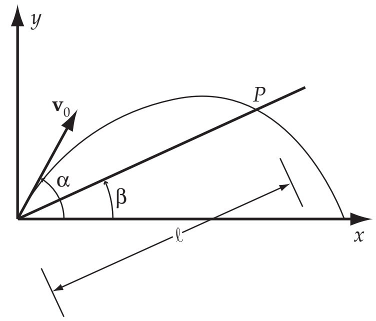
The total force acting on the particle is

$$\mathbf{F}_{total} = m\ddot{\mathbf{r}} = mR\ddot{\mathbf{e}}_r \tag{10}$$

and the components are

$$\boxed{\begin{aligned}
F_\theta &= mR\left(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta\right) \\
F_\phi &= mR\left(2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta\right)
\end{aligned}} \tag{11}$$

2-3.



The equation of motion is

$$\mathbf{F} = m\,\mathbf{a} \tag{1}$$

The gravitational force is the only applied force; therefore,

$$\left.\begin{aligned} F_x &= m\ddot{x} = 0 \\ F_y &= m\ddot{y} = -mg \end{aligned}\right] \tag{2}$$

Integrating these equations and using the initial conditions,

$$\left.\begin{aligned} \dot{x}(t=0) &= v_0 \cos\alpha \\ \dot{y}(t=0) &= v_0 \sin\alpha \end{aligned}\right] \tag{3}$$

We find

$$\left.\begin{aligned} \dot{x}(t) &= v_0 \cos\alpha \\ \dot{y}(t) &= v_0 \sin\alpha - gt \end{aligned}\right] \tag{4}$$

So the equations for x and y are

$$\left.\begin{aligned} x(t) &= v_0 t \cos\alpha \\ y(t) &= v_0 t \sin\alpha - \frac{1}{2} g t^2 \end{aligned}\right] \tag{5}$$

Suppose it takes a time t_0 to reach the point P. Then,

$$\left.\begin{aligned} \ell \cos\beta &= v_0 t_0 \cos\alpha \\ \ell \sin\beta &= v_0 t_0 \sin\alpha - \frac{1}{2} g t_0^2 \end{aligned}\right] \tag{6}$$

Eliminating ℓ between these equations,

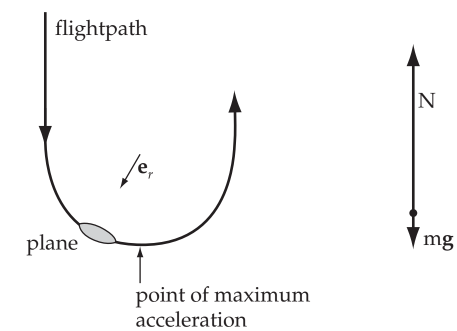
$$\frac{1}{2} g t_0 \left(t_0 - \frac{2v_0 \sin\alpha}{g} + \frac{2v_0}{g} \cos\alpha \tan\beta \right) = 0 \tag{7}$$

from which

$$\boxed{t_0 = \frac{2v_0}{g}\left(\sin\alpha - \cos\alpha\tan\beta\right)} \tag{8}$$

2-4. One of the balls' height can be described by $y = y_0 + v_0 t - gt^2/2$. The amount of time it takes to rise and fall to its initial height is therefore given by $2v_0/g$. If the time it takes to cycle the ball through the juggler's hands is $\tau = 0.9\ s$, then there must be 3 balls in the air during that time τ. A single ball must stay in the air for at least 3τ, so the condition is $2v_0/g \geq 3\tau$, or $v_0 \geq 13.2\ \mathrm{m\cdot s^{-1}}$.

2-5.



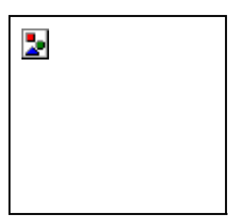
a) From the force diagram we have $\mathbf{N} - m\mathbf{g} = \left(mv^2/R\right)\mathbf{e}_r$. The acceleration that the pilot feels is $\mathbf{N}/m = \mathbf{g} + \left(mv^2/R\right)\mathbf{e}_r$, which has a maximum magnitude at the bottom of the maneuver.

b) If the acceleration felt by the pilot must be less than $9g$, then we have

$$R \geq \frac{v^2}{8g} = \frac{\left(3\cdot 330\ \mathrm{m\cdot s^{-1}}\right)}{8\cdot 9.8\ \mathrm{m\cdot s^{-2}}} \simeq 12.5\ \mathrm{km} \tag{1}$$

A circle smaller than this will result in pilot blackout.

2-6.



Let the origin of our coordinate system be at the tail end of the cattle (or the closest cow/bull).

a) The bales are moving initially at the speed of the plane when dropped. Describe one of these bales by the parametric equations

$$x = x_0 + v_0 t \tag{1}$$

$$y = y_0 - \frac{1}{2} g t^2 \tag{2}$$

where $y_0 = 80\ \text{m}$, and we need to solve for x_0. From (2), the time the bale hits the ground is $\tau = \sqrt{2y_0/g}$. If we want the bale to land at $x(\tau) = -30\ \text{m}$, then $x_0 = x(\tau) - v_0\tau$. Substituting $v_0 = 44.4\ \text{m}\cdot\text{s}^{-1}$ and the other values, this gives $x_0 \simeq -210\ \text{m}$. The rancher should drop the bales 210 m behind the cattle.

b) She could drop the bale earlier by any amount of time and not hit the cattle. If she were late by the amount of time it takes the bale (or the plane) to travel by 30 m in the x-direction, then she will strike cattle. This time is given by $(30\ \text{m})/v_0 \simeq 0.68\ \text{s}$.

2-7. Air resistance is always anti-parallel to the velocity. The vector expression is:

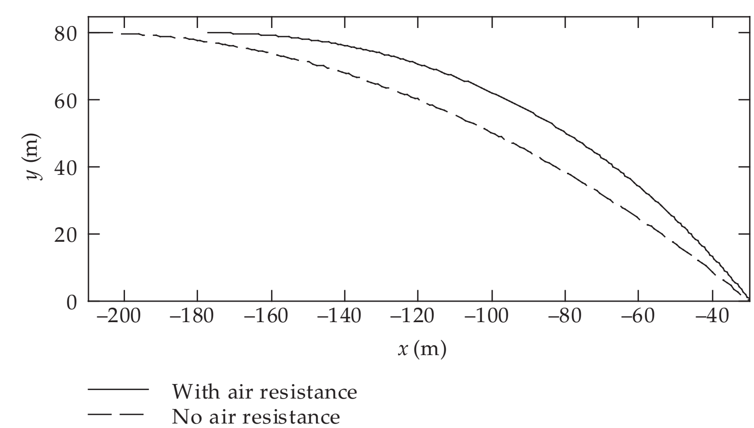
$$\mathbf{W} = \frac{1}{2} c_w \rho A v^2 \left[-\frac{\mathbf{v}}{v} \right] = -\frac{1}{2} c_w \rho A v \mathbf{v} \tag{1}$$

Including gravity and setting $\mathbf{F}_{net} = m\mathbf{a}$, we obtain the parametric equations

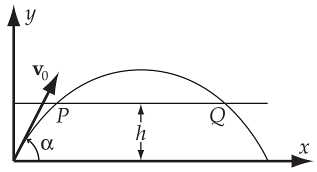
$$\ddot{x} = -b\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2} \tag{2}$$

$$\ddot{y} = -b\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g \tag{3}$$

where $b = c_w \rho A/2m$. Solving with a computer using the given values and $\rho = 1.3\ \text{kg}\cdot\text{m}^{-3}$, we find that if the rancher drops the bale 210 m behind the cattle (the answer from the previous problem), then it takes $\simeq 4.44$ s to land $\simeq 62.5$ m behind the cattle. This means that the bale should be dropped at $\simeq 178$ m behind the cattle to land 30 m behind. This solution is what is plotted in the figure. The time error she is allowed to make is the same as in the previous problem since it only depends on how fast the plane is moving.



2-8.



From problem 2-3 the equations for the coordinates are

$$x = v_0 t \cos\alpha \tag{1}$$

$$y = v_0 t \sin\alpha - \frac{1}{2} g t^2 \tag{2}$$

In order to calculate the time when a projective reaches the ground, we let $y = 0$ in (2):

$$v_0 t \sin\alpha - \frac{1}{2} g t^2 = 0 \tag{3}$$

$$t = \frac{2v_0}{g} \sin\alpha \tag{4}$$

Substituting (4) into (1) we find the relation between the range and the angle as

$$x = \frac{v_0^2}{g} \sin 2\alpha \tag{5}$$

The range is maximum when $2\alpha = \dfrac{\pi}{2}$, i.e., $\alpha = \dfrac{\pi}{4}$. For this value of α the coordinates become

$$\left.\begin{aligned} x &= \frac{v_0}{\sqrt{2}} t \\ x &= \frac{v_0}{\sqrt{2}} t - \frac{1}{2} g t^2 \end{aligned}\right] \tag{6}$$

Eliminating t between these equations yields

$$x^2 - \frac{v_0^2}{g} x + \frac{v_0^2}{g} y = 0 \tag{7}$$

We can find the x-coordinate of the projectile when it is at the height h by putting $y = h$ in (7):

$$x^2 - \frac{v_0^2}{g} x + \frac{v_0^2 h}{g} = 0 \tag{8}$$

This equation has two solutions:

$$\left.\begin{aligned} x_1 &= \frac{v_0^2}{2g} - \frac{v_0^2}{2g}\sqrt{v_0^2 - 4gh} \\ x_2 &= \frac{v_0^2}{2g} + \frac{v_0^2}{2g}\sqrt{v_0^2 - 4gh} \end{aligned}\right] \tag{9}$$

where x_1 corresponds to the point P and x_2 to Q in the diagram. Therefore,

$$\boxed{d = x_2 - x_1 = \frac{v_0}{g}\sqrt{v_0^2 - 4gh}} \tag{10}$$

2-9.

a) *Zero resisting force* ($F_r = 0$):

The equation of motion for the vertical motion is:

$$F = ma = m\frac{dv}{dt} = -mg \tag{1}$$

Integration of (1) yields

$$v = -gt + v_0 \tag{2}$$

where v_0 is the initial velocity of the projectile and $t = 0$ is the initial time.

The time t_m required for the projectile to reach its maximum height is obtained from (2). Since t_m corresponds to the point of zero velocity,

$$v(t_m) = 0 = v_0 - gt_m , \tag{3}$$

we obtain

$$\boxed{t_m = \frac{v_0}{g}} \tag{4}$$

b) *Resisting force proportional to the velocity* $(F_r = -kmv)$:

The equation of motion for this case is:

$$F = m\frac{dv}{dt} = -mg - kmv \tag{5}$$

where $-kmv$ is a *downward* force for $t < t'_m$ and is an *upward* force for $t > t'_m$. Integrating, we obtain

$$v(t) = -\frac{g}{k} + \frac{kv_0 + g}{k}e^{-kt} \tag{6}$$

For $t = t_m$, $v(t) = 0$, then from (6),

$$v_0 = \frac{g}{k}\left(e^{kt_m} - 1\right) \tag{7}$$

which can be rewritten as

$$kt_m = \ln\left[1 + \frac{kv_0}{g}\right] \tag{8}$$

Since, for small z ($z \ll 1$) the expansion

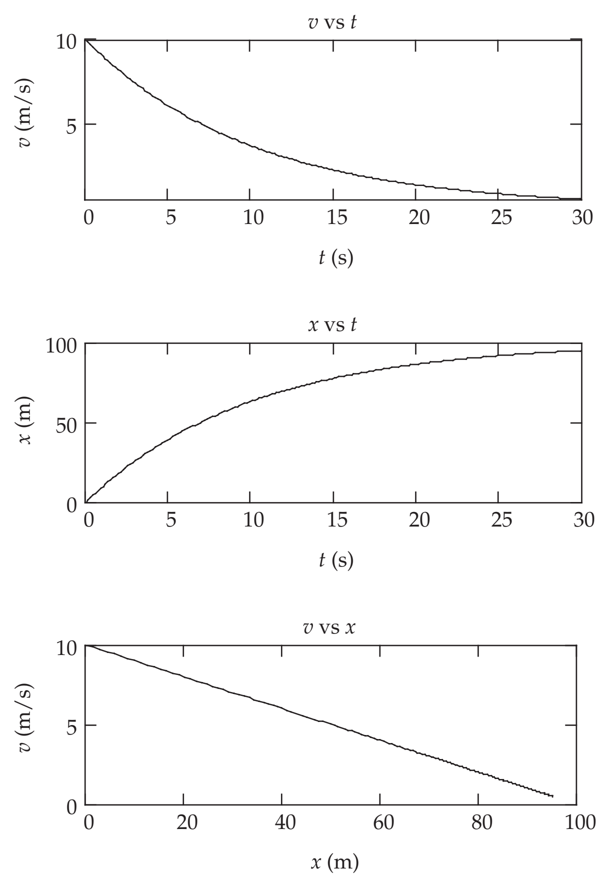
$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 \tag{9}$$

is valid, (8) can be expressed approximately as

$$\boxed{t_m = \frac{v_0}{g}\left[1 - \frac{kv_0}{2g} + \frac{1}{3}\left[\frac{kv_0}{2g}\right]^2 - \ldots\right]} \tag{10}$$

which gives the correct result, as in (4) for the limit $k \to 0$.

2-10. The differential equation we are asked to solve is Equation (2.22), which is $\ddot{x} = -k\dot{x}$. Using the given values, the plots are shown in the figure. Of course, the reader will not be able to distinguish between the results shown here and the analytical results. The reader will have to take the word of the author that the graphs were obtained using numerical methods on a computer. The results obtained were at most within 10^{-8} of the analytical solution.



2-11. The equation of motion is

$$m\frac{d^2x}{dt^2} = -kmv^2 + mg \tag{1}$$

This equation can be solved exactly in the same way as in problem 2-12 and we find

$$x = \frac{1}{2k} \log\left[\frac{g - kv_0^2}{g - kv^2}\right] \tag{2}$$

where the origin is taken to be the point at which $v = v_0$ so that the initial condition is $x(v = v_0) = 0$. Thus, the distance from the point $v = v_0$ to the point $v = v_1$ is

$$\boxed{s(v_0 \to v_1) = \frac{1}{2k} \log\left[\frac{g - kv_0^2}{g - kv_1^2}\right]} \tag{3}$$

2-12. The equation of motion for the upward motion is

$$m\frac{d^2x}{dt^2} = -mkv^2 - mg \tag{1}$$

Using the relation

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{2}$$

we can rewrite (1) as

$$\frac{v\,dv}{kv^2 + g} = -dx \tag{3}$$

Integrating (3), we find

$$\frac{1}{2k}\log\left(kv^2 + g\right) = -x + C \tag{4}$$

where the constant C can be computed by using the initial condition that $v = v_0$ when $x = 0$:

$$C = \frac{1}{2k}\log\left(kv_0^2 + g\right) \tag{5}$$

Therefore,

$$x = \frac{1}{2k}\log\frac{kv_0^2 + g}{kv^2 + g} \tag{6}$$

Now, the equation of downward motion is

$$m\frac{d^2x}{dt^2} = -mkv^2 + mg \tag{7}$$

This can be rewritten as

$$\frac{v\,dv}{-kv^2 + g} = dx \tag{8}$$

Integrating (8) and using the initial condition that $x = 0$ at $v = 0$ (w take the highest point as the origin for the downward motion), we find

$$x = \frac{1}{2k} \log \frac{g}{g - kv^2} \tag{9}$$

At the highest point the velocity of the particle must be zero. So we find the highest point by substituting $v = 0$ in (6):

$$x_h = \frac{1}{2k} \log \frac{kv_0^2 + g}{g} \tag{10}$$

Then, substituting (10) into (9),

$$\frac{1}{2k} \log \frac{kv_0^2 + g}{g} = \frac{1}{2k} \log \frac{g}{g - kv^2} \tag{11}$$

Solving for v,

$$v = \sqrt{\frac{\frac{g}{k} v_0^2}{v_0^2 + \frac{g}{k}}} \tag{12}$$

We can find the terminal velocity by putting $x \to \infty$ in(9). This gives

$$v_t = \sqrt{\frac{g}{k}} \tag{13}$$

Therefore,

$$\boxed{v = \frac{v_0 v_t}{\sqrt{v_0^2 + v_t^2}}} \tag{14}$$

2-13. The equation of motion of the particle is

$$m \frac{dv}{dt} = -mk\left(v^3 + a^2 v\right) \tag{1}$$

Integrating,

$$\int \frac{dv}{v\left(v^2 + a^2\right)} = -k \int dt \tag{2}$$

and using Eq. (E.3), Appendix E, we find

$$\frac{1}{2a^2} \ln\left[\frac{v^2}{a^2 + v^2}\right] = -kt + C \tag{3}$$

Therefore, we have

$$\frac{v^2}{a^2 + v^2} = C' \, e^{-At} \tag{4}$$

where $A \equiv 2a^2k$ and where C' is a new constant. We can evaluate C' by using the initial condition, $v = v_0$ at $t = 0$:

$$C' = \frac{v_0^2}{a^2 + v_0^2} \tag{5}$$

Substituting (5) into (4) and rearranging, we have

$$v = \left[\frac{a^2 C' e^{-At}}{1 - C' e^{-At}}\right]^{1/2} = \frac{dx}{dt} \tag{6}$$

Now, in order to integrate (6), we introduce $u \equiv e^{-At}$ so that $du = -Au\,dt$. Then,

$$x = \int \left[\frac{a^2 C' e^{-At}}{1 - C' e^{-At}}\right]^{1/2} dt = \frac{a}{A}\int \left[\frac{C'u}{1 - C'u}\right]^{1/2} \frac{du}{u}$$

$$= -\frac{a\sqrt{C'}}{A}\int \frac{du}{\sqrt{-C'u^2 + u}} \tag{7}$$

Using Eq. (E.8c), Appendix E, we find

$$x = \frac{a}{A}\sin^{-1}(1 - 2C'u) + C'' \tag{8}$$

Again, the constant C'' can be evaluated by setting $x = 0$ at $t = 0$; i.e., $x = 0$ at $u = 1$:

$$C'' = -\frac{a}{A}\sin^{-1}(1 - 2C') \tag{9}$$

Therefore, we have

$$x = \frac{a}{A}\left[\sin^{-1}\left(-2C' e^{-At} + 1\right) - \sin^{-1}(-2C' + 1)\right]$$

Using (4) and (5), we can write

$$x = \frac{1}{2ak}\left[\sin^{-1}\left[\frac{-v^2 + a^2}{v^2 + a^2}\right] - \sin^{-1}\left[\frac{-v_0^2 + a^2}{v_0^2 + a^2}\right]\right] \tag{10}$$

From (6) we see that $v \to 0$ as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} \sin^{-1}\left[\frac{-v^2 + a^2}{v^2 + a^2}\right] = \sin^{-1}(1) = \frac{\pi}{2} \tag{11}$$

Also, for very large initial velocities,

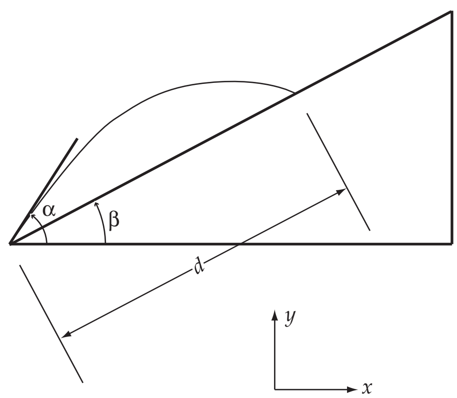
$$\lim_{v_0\to\infty} \sin^{-1}\left[\frac{-v_0^2 + a^2}{v_0^2 + a^2}\right] = \sin^{-1}(-1) = -\frac{\pi}{2} \tag{12}$$

Therefore, using (11) and (12) in (10), we have

$$\boxed{x(t\to\infty)=\frac{\pi}{2ka}} \tag{13}$$

and the particle can never move a distance greater than $\pi/2ka$ for any initial velocity.

2-14.



a) The equations for the projectile are

$$x=v_0\cos\alpha t$$

$$y=v_0\sin\alpha t-\frac{1}{2}gt^2$$

Solving the first for t and substituting into the second gives

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha}$$

Using $x = d\cos\beta$ and $y = d\sin\beta$ gives

$$d\sin\beta=d\cos\beta\tan\alpha-\frac{gd^2\cos^2\beta}{2v_0^2\cos^2\alpha}$$

$$0=d\left[\frac{gd\cos^2\beta}{2v_0^2\cos^2\alpha}-\cos\beta\tan\alpha+\sin\beta\right]$$

Since the root $d = 0$ is not of interest, we have

$$d=\frac{2(\cos\beta\tan\alpha-\sin\beta)v_0^2\cos^2\alpha}{g\cos^2\beta}$$

$$=\frac{2v_0^2\cos\alpha(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{g\cos^2\beta}$$

$$\boxed{d=\frac{2v_0^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}} \tag{1}$$

b) Maximize d with respect to α

$$\frac{d}{d\alpha}(d) = 0 = \frac{2v_0^2}{g\cos^2\beta}\left[-\sin\alpha\sin(\alpha-\beta) + \cos\alpha\cos(\alpha-\beta)\right]\cos(2\alpha-\beta)$$

$$\cos(2\alpha-\beta) = 0$$

$$2\alpha - \beta = \frac{\pi}{2}$$

$$\boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}}$$

c) Substitute (2) into (1)

$$d_{\max} = \frac{2v_0^2}{g\cos^2\beta}\left[\cos\left(\frac{\pi}{4}+\frac{\beta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right]$$

Using the identity

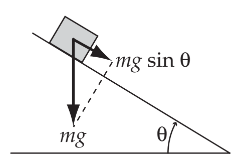
$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

we have

$$d_{\max} = \frac{2v_0^2}{g\cos^2\beta}\cdot\frac{\sin\frac{\pi}{2} - \sin\beta}{2} = \frac{v_0^2}{g}\left[\frac{1-\sin\beta}{1-\sin^2\beta}\right]$$

$$\boxed{d_{\max} = \frac{v_0^2}{g(1+\sin\beta)}}$$

2-15.



The equation of motion along the plane is

$$m\frac{dv}{dt} = mg\sin\theta - kmv^2 \tag{1}$$

Rewriting this equation in the form

$$\frac{1}{k}\frac{dv}{\frac{g}{k}\sin\theta - v^2} = dt \tag{2}$$

We know that the velocity of the particle continues to increase with time (i.e., $dv/dt > 0$), so that $(g/k)\sin\theta > v^2$. Therefore, we must use Eq. (E.5a), Appendix E, to perform the integration. We find

$$\frac{1}{k}\frac{1}{\sqrt{\frac{g}{k}\sin\theta}}\tanh^{-1}\left[\frac{v}{\sqrt{\frac{g}{k}\sin\theta}}\right] = t + C \tag{3}$$

The initial condition $v(t = 0) = 0$ implies $C = 0$. Therefore,

$$v = \sqrt{\frac{g}{k}\sin\theta}\,\tanh\left(\sqrt{gk\sin\theta}\,t\right) = \frac{dx}{dt} \tag{4}$$

We can integrate this equation to obtain the displacement x as a function of time:

$$x = \sqrt{\frac{g}{k}\sin\theta}\int\tanh\left(\sqrt{gk\sin\theta}\,t\right)dt$$

Using Eq. (E.17a), Appendix E, we obtain

$$x = \sqrt{\frac{g}{k}\sin\theta}\,\frac{\ln\cosh\left(\sqrt{gk\sin\theta}\,t\right)}{\sqrt{gk\sin\theta}} + C' \tag{5}$$

The initial condition $x(t = 0) = 0$ implies $C' = 0$. Therefore, the relation between d and t is

$$d = \frac{1}{k}\ln\cosh\left(\sqrt{gk\sin\theta}\,t\right) \tag{6}$$

From this equation, we can easily find

$$\boxed{t = \frac{\cosh^{-1}\left(e^{dk}\right)}{\sqrt{gk\sin\theta}}} \tag{7}$$

2-16. The only force which is applied to the article is the component of the gravitational force along the slope: $mg\sin\alpha$. So the acceleration is $g\sin\alpha$. Therefore the velocity and displacement along the slope for upward motion are described by:

$$v = v_0 - (g\sin\alpha)t \tag{1}$$

$$x = v_0 t - \frac{1}{2}(g\sin\alpha)t^2 \tag{2}$$

where the initial conditions $v(t = 0) = v_0$ and $x(t = 0) = 0$ have been used.

At the highest position the velocity becomes zero, so the time required to reach the highest position is, from (1),

$$t_0 = \frac{v_0}{g\sin\alpha} \tag{3}$$

At that time, the displacement is

$$x_0 = \frac{1}{2}\frac{v_0^2}{g \sin \alpha} \tag{4}$$

For downward motion, the velocity and the displacement are described by

$$v = (g \sin \alpha)t \tag{5}$$

$$x = \frac{1}{2}(g \sin \alpha)t^2 \tag{6}$$

where we take a new origin for x and t at the highest position so that the initial conditions are $v(t = 0) = 0$ and $x(t = 0) = 0$.

We find the time required to move from the highest position to the starting position by substituting (4) into (6):

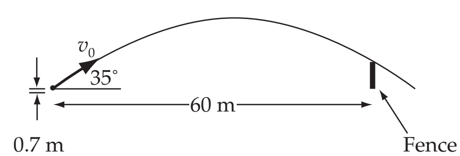
$$t' = \frac{v_0}{g \sin \alpha} \tag{7}$$

Adding (3) and (7), we find

$$\boxed{t = \frac{2v_0}{g \sin \alpha}} \tag{8}$$

for the total time required to return to the initial position.

2-17.



The setup for this problem is as follows:

$$x = v_0 t \cos \theta \tag{1}$$

$$y = y_0 + v_0 t \sin \theta - \frac{1}{2} g t^2 \tag{2}$$

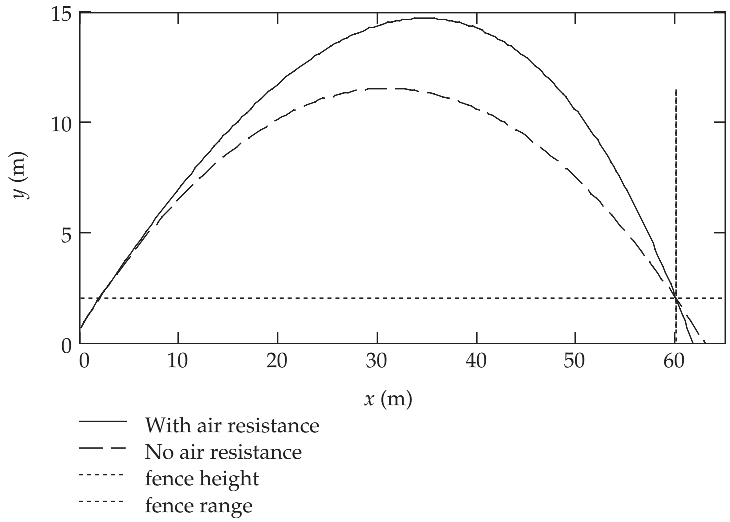
where $\theta = 35^\circ$ and $y_0 = 0.7$ m. The ball crosses the fence at a time $\tau = R/(v_0 \cos\theta)$, where $R = 60$ m. It must be at least $h = 2$ m high, so we also need $h - y_0 = v_0 \tau \sin \theta - g\tau^2/2$. Solving for v_0, we obtain

$$v_0^2 = \frac{gR^2}{2 \cos \theta\left[R \sin \theta - (h - y_0)\cos \theta\right]} \tag{3}$$

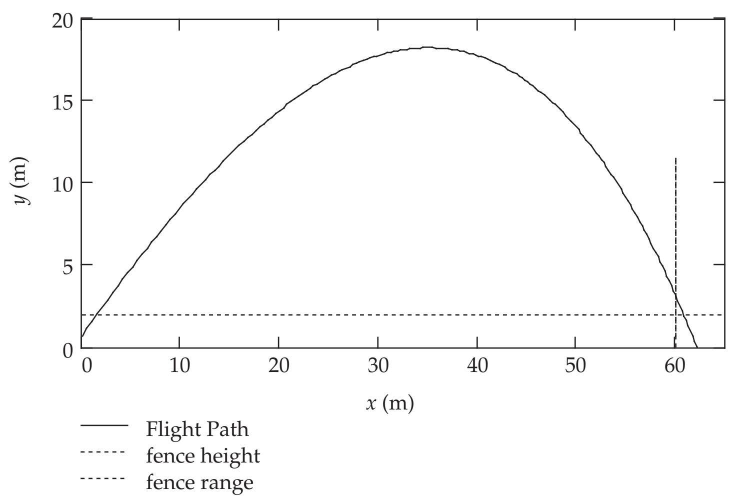
which gives $v_0 \simeq 25.4 \text{ m} \cdot \text{s}^{-1}$.

2-18.

a) The differential equation here is the same as that used in Problem 2-7. It must be solved for many different values of v_0 in order to find the minimum required to have the ball go over the fence. This can be a computer-intensive and time-consuming task, although if done correctly is easily tractable by a personal computer. This minimum v_0 is $\simeq 35.2\ \mathrm{m\cdot s^{-1}}$, and the trajectory is shown in Figure (a). (We take the density of air as $\rho = 1.3\ \mathrm{kg\cdot m^{-3}}$.)



b) The process here is the same as for part (a), but now we have v_0 fixed at the result just obtained, and the elevation angle θ must be varied to give the ball a maximum height at the fence. The angle that does this is $\simeq 0.71$ rad = 40.7°, and the ball now clears the fence by 1.1 m. This trajectory is shown in Figure (b).



2-19. The projectile's motion is described by

$$\left.\begin{aligned} x &= (v_0 \cos\alpha)t \\ y &= (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \end{aligned}\right] \tag{1}$$

where v_0 is the initial velocity. The distance from the point of projection is

$$r = \sqrt{x^2 + y^2} \tag{2}$$

Since r must always increase with time, we must have $\dot{r} > 0$:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} > 0 \tag{3}$$

Using (1), we have

$$x\dot{x} + y\dot{y} = \frac{1}{2}g^2t^3 - \frac{3}{2}g(v_0 \sin\alpha)t^2 + v_0^2 t \tag{4}$$

Let us now find the value of t which yields $x\dot{x} + y\dot{y} = 0$ (i.e., $\dot{r} = 0$):

$$t = \frac{3}{2}\frac{v_0 \sin\alpha}{g} \pm \frac{v_0}{2g}\sqrt{9\sin^2\alpha - 8} \tag{5}$$

For small values of α, the second term in (5) is imaginary. That is, $r = 0$ is never attained and the value of t resulting from the condition $\dot{r} = 0$ is unphysical.

Only for values of α greater than the value for which the radicand is zero does t become a physical time at which $\dot{r}$ does in fact vanish. Therefore, the maximum value of α that insures $\dot{r} > 0$ for all values of t is obtained from

$$9\sin^2\alpha_{\max} - 8 = 0 \tag{6}$$

or,

$$\sin\alpha_{\max} = \frac{2\sqrt{2}}{3} \tag{7}$$

so that

$$\boxed{\alpha_{\max} \cong 70.5°} \tag{8}$$

2-20. If there were no retardation, the range of the projectile would be given by Eq. (2.54):

$$R = \frac{v_0^2}{g}\sin 2\theta_0 \tag{1}$$

The angle of elevation is therefore obtained from

$$\sin 2\theta_0 = \frac{Rg}{v_0^2}$$

$$= \frac{(1000\text{ m})\times(9.8\text{ m/sec}^2)}{(140\text{ m/sec})^2}$$

$$= 0.50 \tag{2}$$

so that

$$\theta_0 = 15° \tag{3}$$

Now, the real range R', in the linear approximation, is given by Eq. (2.55):

$$R' = R\left[1 - \frac{4kV}{3g}\right]$$

$$= \frac{v_0^2 \sin 2\theta}{g}\left[1 - \frac{4kv_0 \sin\theta}{3g}\right] \tag{4}$$

Since we expect the real angle θ to be not too different from the angle θ_0 calculated above, we can solve (4) for θ by substituting θ_0 for θ in the correction term in the parentheses. Thus,

$$\sin 2\theta = \frac{g\,R'}{v_0^2\left[1 - \dfrac{4kv_0 \sin\theta_0}{3g}\right]} \tag{5}$$

Next, we need the value of k. From Fig. 2-3(c) we find the value of km by measuring the slope of the curve in the vicinity of v = 140 m/sec. We find $km \cong (110\text{ N})/(500\text{ m/s}) \cong 0.22\text{ kg/s}$. The curve is that appropriate for a projectile of mass 1 kg, so the value of k is

$$k \simeq 0.022\text{ sec}^{-1} \tag{6}$$

Substituting the values of the various quantities into (5) we find $\theta = 17.1°$. Since this angle is somewhat greater than θ_0, we should iterate our solution by using this new value for θ_0 in (5). We then find $\theta = 17.4°$. Further iteration does not substantially change the value, and so we conclude that

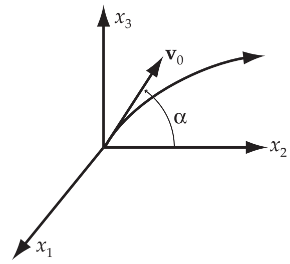
$$\boxed{\theta = 17.4°}$$

If there were no retardation, a projectile fired at an angle of 17.4° with an initial velocity of 140 m/sec would have a range of

$$R = \frac{(140\text{ m/sec})^2 \sin 34.8°}{9.8\text{ m/sec}^2}$$

$$\simeq 1140\text{ m}$$

2-21.



Assume a coordinate system in which the projectile moves in the $x_2 - x_3$ plane. Then,

$$\left.\begin{aligned} x_2 &= v_0 t \cos\alpha \\ x_3 &= v_0 t \sin\alpha - \frac{1}{2} g t^2 \end{aligned}\right] \tag{1}$$

or,

$$\begin{aligned} \mathbf{r} &= x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \\ &= (v_0 t \cos\alpha)\mathbf{e}_2 + \left(v_0 t \sin\alpha - \frac{1}{2} g t^2\right)\mathbf{e}_3 \end{aligned} \tag{2}$$

The linear momentum of the projectile is

$$\mathbf{p} = m\dot{\mathbf{r}} = m\left[(v_0 \cos\alpha)\mathbf{e}_2 + (v_0 \sin\alpha - gt)\mathbf{e}_3\right] \tag{3}$$

and the angular momentum is

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \left[(v_0 t \cos\alpha)\mathbf{e}_2 + (v_0 t \sin\alpha - g t^2)\mathbf{e}_3\right]\times m\left[(v_0 \cos\alpha)\mathbf{e}_2 + (v_0 \sin\alpha - gt)\mathbf{e}_3\right] \tag{4}$$

Using the property of the unit vectors that $\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_3\ \varepsilon_{ijk}$, we find

$$\mathbf{L} = \frac{1}{2}\left(mg\, v_0 t^2 \cos\alpha\right)\mathbf{e}_1 \tag{5}$$

This gives

$$\dot{\mathbf{L}} = -\left(mg\, v_0 t \cos\alpha\right)\mathbf{e}_1 \tag{6}$$

Now, the force acting on the projectile is

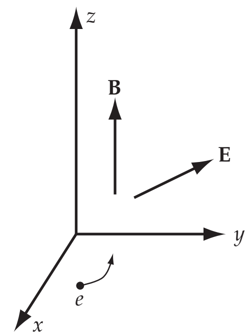
$$\mathbf{F} = -mg\,\mathbf{e}_3 \tag{7}$$

so that the torque is

$$\begin{aligned} \mathbf{N} = \mathbf{r}\times\mathbf{F} &= \left[(v_0 t \cos\alpha)\mathbf{e}_2 + \left(v_0 t \sin\alpha - \frac{1}{2} g t^2\right)\mathbf{e}_3\right](-mg)\,\mathbf{e}_3 \\ &= -\left(mg\, v_0 t \cos\alpha\right)\mathbf{e}_1 \end{aligned}$$

which is the same result as in (6).

2-22.



Our force equation is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1}$$

a) Note that when $\mathbf{E} = 0$, the force is always perpendicular to the velocity. This is a centripetal acceleration and may be analyzed by elementary means. In this case we have also $\mathbf{v} \perp \mathbf{B}$ so that $\left|\mathbf{v} \times \mathbf{B}\right| = vB$.

$$ma_{centripetal} = \frac{mv^2}{r} = qvB \tag{2}$$

Solving this for r

$$r = \frac{mv}{qB} = \frac{v}{\omega_c} \tag{3}$$

with $\omega_c \equiv qB/m$.

b) Here we don't make any assumptions about the relative orientations of **v** and **B**, i.e. the velocity may have a component in the z direction upon entering the field region. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, with $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \ddot{\mathbf{r}}$. Let us calculate first the $\mathbf{v} \times \mathbf{B}$ term.

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = B\left(\dot{y}\mathbf{i} - \dot{x}\mathbf{j}\right) \tag{4}$$

The Lorentz equation (1) becomes

$$\mathbf{F} = m\ddot{\mathbf{r}} = qB\dot{y}\mathbf{i} + q\left(E_y - B\dot{x}\right)\mathbf{j} + qE_z\mathbf{k} \tag{5}$$

Rewriting this as component equations:

$$\ddot{x} = \frac{qB}{m}\dot{y} = \omega_c\dot{y} \tag{6}$$

$$\ddot{y} = -\frac{qB}{m}\dot{x} + \frac{qE_y}{m} = -\omega_c\left(\dot{x} - \frac{E_y}{B}\right) \tag{7}$$

$$\ddot{z} = \frac{qE_z}{m} \tag{8}$$

The z-component equation of motion (8) is easily integrable, with the constants of integration given by the initial conditions in the problem statement.

$$z(t) = z_0 + \dot{z}_0 t + \frac{qE_z}{2m} t^2 \tag{9}$$

c) We are asked to find expressions for $\dot{x}$ and $\dot{y}$, which we will call v_x and v_y, respectively. Differentiate (6) once with respect to time, and substitute (7) for $\dot{v}_y$

$$\ddot{v}_x = \omega_c \dot{v}_y = -\omega_c^2 \left(v_x - \frac{E_y}{B} \right) \tag{10}$$

or

$$\ddot{v}_x + \omega_c^2 v_x = \omega_c^2 \frac{E_y}{B} \tag{11}$$

This is an inhomogeneous differential equation that has both a homogeneous solution (the solution for the above equation with the right side set to zero) and a particular solution. The most general solution is the sum of both, which in this case is

$$v_x = C_1 \cos(\omega_c t) + C_2 \sin(\omega_c t) + \frac{E_y}{B} \tag{12}$$

where C_1 and C_2 are constants of integration. This result may be substituted into (7) to get $\dot{v}_y$

$$\dot{v}_y = -C_1 \omega_c \cos(\omega_c t) - C_2 \omega_c \sin(\omega_c t) \tag{13}$$

$$v_y = -C_1 \sin(\omega_c t) + C_2 \cos(\omega_c t) + K \tag{14}$$

where K is yet another constant of integration. It is found upon substitution into (6), however, that we must have $K = 0$. To compute the time averages, note that both sine and cosine have an average of zero over one of their periods $T \equiv 2\pi/\omega_c$.

$$\langle \dot{x} \rangle = \frac{E_y}{B}, \quad \langle \dot{y} \rangle = 0 \tag{15}$$

d) We get the parametric equations by simply integrating the velocity equations.

$$x = \frac{C_1}{\omega_c} \sin(\omega_c t) - \frac{C_2}{\omega_c} \cos(\omega_c t) + \frac{E_y}{B} t + D_x \tag{16}$$

$$y = \frac{C_1}{\omega_c} \cos(\omega_c t) + \frac{C_2}{\omega_c} \sin(\omega_c t) + D_y \tag{17}$$

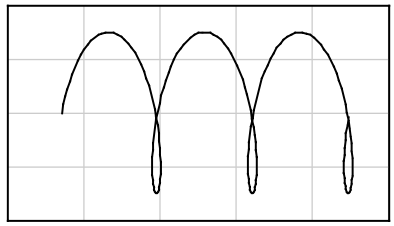
where, indeed, D_x and D_y are constants of integration. We may now evaluate all the C's and D's using our initial conditions $x(0) = -A/\omega_c$, $\dot{x}(0) = E_y/B$, $y(0) = 0$, $\dot{y}(0) = A$. This gives us $C_1 = D_x = D_y = 0$, $C_2 = A$ and gives the correct answer

$$x(t) = \frac{-A}{\omega_c} \cos(\omega_c t) + \frac{E_y}{B} t \tag{18}$$

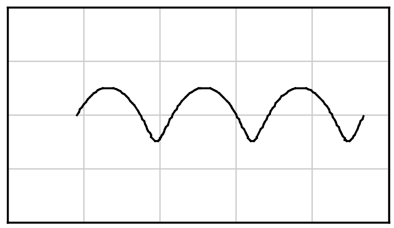
$$y(t) = \frac{A}{\omega_c} \sin\left(\omega_c t\right) \tag{19}$$

These cases are shown in the figure as (i) $A > E_y/B$, (ii) $A < E_y/B$, and (iii) $A = E_y/B$.

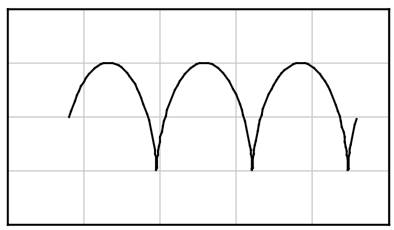
(i)



(ii)



(iii)



2-23.

$$F(t) = ma(t) = kte^{-at} \tag{1}$$

with the initial conditions $x(t) = v(t) = 0$. We integrate to get the velocity. Showing this explicitly,

$$\int_{v(0)}^{v(t)} a(t)\, dt = \frac{k}{m} \int_0^t te^{-\alpha t}\, dt \tag{2}$$

Integrating this by parts and using our initial conditions, we obtain

$$v(t) = \frac{k}{m}\left[\frac{1}{\alpha^2} - \frac{1}{\alpha}\left(t + \frac{1}{\alpha}\right)e^{-\alpha t}\right] \tag{3}$$

By similarly integrating $v(t)$, and using the integral (2) we can obtain x(t).

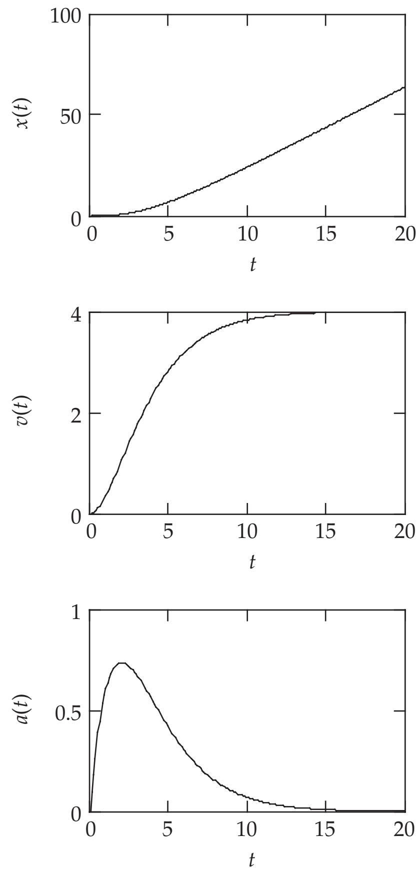
$$x(t) = \frac{k}{m}\left[-\frac{2}{\alpha^3} + \frac{1}{\alpha^2} + \frac{1}{\alpha^2}\left(t + \frac{2}{\alpha}\right)e^{-\alpha t}\right] \tag{4}$$

To make our graphs, substitute the given values of m = 1 kg, $k = 1\ \text{N}\cdot\text{s}^{-1}$, and $\alpha = 0.5\ \text{s}^{-1}$.

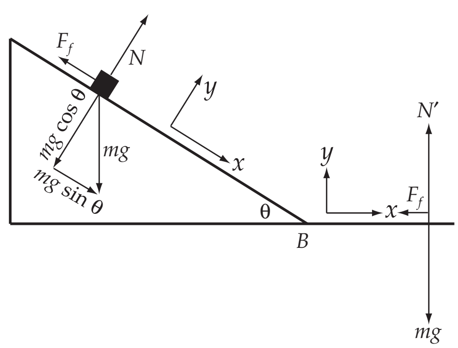
$$x(t) = te^{-t/2} \tag{5}$$

$$v(t) = 4 - 2(t+2)e^{-t/2} \tag{6}$$

$$\alpha(t) = -16 + 4t + 4(t+4)e^{-t/2} \tag{7}$$



2-24.



d = length of incline
s = distance skier travels along level ground

While on the plane:

$$\sum F_y = N - mg\cos\theta = m\ddot{y} = 0 \qquad \text{so } N = mg\cos\theta$$

$$\sum F_x = mg\sin\theta - F_f\,; \qquad F_f = \mu N = \mu mg\cos\theta$$

$$mg\sin\theta - \mu mg\cos\theta = m\ddot{x}$$

So the acceleration down the plane is:

$$a_1 = g\left(\sin\theta - \mu\cos\theta\right) = \text{ constant}$$

While on level ground: $N' = mg$; $F_f = -\mu mg$

So $\sum F_x = m\ddot{x}$ becomes $-\mu mg = m\ddot{x}$

The acceleration while on level ground is

$$a_2 = -\mu g = \text{ constant}$$

For motion with constant acceleration, we can get the velocity and position by simple integration:

$$\ddot{x} = a$$

$$v = \dot{x} = at + v_0 \tag{1}$$

$$x - x_0 = v_0 t + \frac{1}{2} at^2 \tag{2}$$

Solving (1) for t and substituting into (2) gives:

$$\frac{v - v_0}{a} = t$$

$$x - x_0 = \frac{v_0 \left(v - v_0\right)}{a} + \frac{1}{2} \cdot \frac{\left(v - v_0\right)^2}{a}$$

or

$$2a\left(x - x_0\right) = v^2 - v_0^2$$

Using this equation with the initial and final points being the top and bottom of the incline respectively, we get:

$$2a_1 d = V_B^2 \qquad V_B = \text{speed at bottom of incline}$$

Using the same equation for motion along the ground:

$$2a_2 s = -V_B^2 \tag{3}$$

Thus

$$a_1 d = -a_2 s \qquad a_1 = g\left(\sin\theta - \mu\cos\theta\right) \qquad a_2 = -\mu g$$

So

$$gd\left(\sin\theta - \mu\cos\theta\right) = \mu g s$$

Solving for μ gives

$$\mu = \frac{d\sin\theta}{d\cos\theta + s}$$

Substituting $\theta = 17°$, d = 100 m, s = 70 m gives

$$\boxed{\mu = 0.18}$$

Substituting this value into (3):

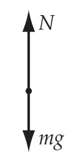
$$-2\mu g s = -V_B^2$$

$$V_B = \sqrt{2\mu g s}$$

$$\boxed{V_B = 15.6 \text{ m/sec}}$$

2-25.

a) At A, the forces on the ball are:



The track counters the gravitational force and provides centripetal acceleration

$$N - mg = mv^2/R$$

Get v by conservation of energy:

$$E_{top} = T_{top} + U_{top} = 0 + mgh$$

$$E_A = T_A + U_A = \frac{1}{2} mv^2 + 0$$

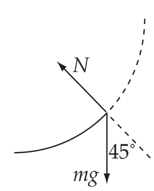
$$E_{top} = E_A \to v = \sqrt{2gh}$$

So

$$N = mg + m2gh/R$$

$$\boxed{N = mg\left(1 + \frac{2h}{R}\right)}$$

b) At B the forces are:

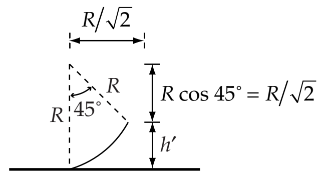


$$N = mv^2/R + mg \cos 45°$$

$$= mv^2/R + mg/\sqrt{2} \qquad (1)$$

Get v by conservation of energy. From a), $E_{total} = mgh$.

At B, $E = \dfrac{1}{2} mv^2 + mgh'$



$$R = \frac{R}{\sqrt{2}} + h' \quad \text{or} \quad h' = R\left(1 - \frac{1}{\sqrt{2}}\right)$$

So $E_{total} = T_B + U_B$ becomes:

$$mgh = mgR\left(1 - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}mv^2$$

Solving for v^2

$$2\left[gh - gR\left(1 - \frac{1}{\sqrt{2}}\right)\right] = v^2$$

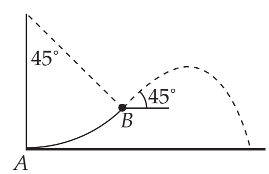
Substituting into (1):

$$N = mg\left[\frac{2h}{R} + \left(\frac{3}{\sqrt{2}} - 2\right)\right]$$

c) From b) $v_B^2 = 2g\left[h - R + R/\sqrt{2}\right]$

$$\boxed{v = \left[2g\left(h - R + R/\sqrt{2}\right)\right]^{1/2}}$$

d) This is a projectile motion problem



Put the origin at A.

The equations:

$$x = x_0 + v_{x0}t$$

$$y = y_0 + v_{y0}t - \frac{1}{2}gt^2$$

become

$$x = \frac{R}{\sqrt{2}} + \frac{v_B}{\sqrt{2}}t \tag{2}$$

$$y = h' + \frac{v_B}{\sqrt{2}}t - \frac{1}{2}gt^2 \tag{3}$$

Solve (3) for t when $y = 0$ (ball lands).

$$gt^2 - \sqrt{2}\, v_B t - 2h' = 0$$

$$t = \frac{\sqrt{2}\, v_B \pm \sqrt{2v_B^2 + 8gh'}}{2g}$$

We discard the negative root since it gives a negative time. Substituting into (2):

$$x = \frac{R}{\sqrt{2}} + \frac{v_B}{\sqrt{2}}\left[\frac{\sqrt{2}\, v_B \pm \sqrt{2v_B^2 + 8gh'}}{2g}\right]$$

Using the previous expressions for v_B and h' yields

$$\boxed{x = \left(\sqrt{2} - 1\right) R + h + \left[h^2 - \frac{3}{2} R^2 + \sqrt{2}\, R^2\right]^{1/2}}$$

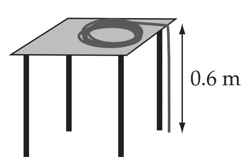
e) $U(x) = mgy(x)$, with $y(0) = h$, so $U(x)$ has the shape of the track.

2-26. All of the kinetic energy of the block goes into compressing the spring, so that $mv^2/2 = kx^2/2$, or $x = v\sqrt{m/k} \simeq 2.3$ m, where x is the maximum compression and the given values have been substituted. When there is a rough floor, it exerts a force $\mu_k mg$ in a direction that opposes the block's velocity. It therefore does an amount of work $\mu_k mgd$ in slowing the block down after traveling across the floor a distance d. After 2 m of floor, the block has energy $mv^2/2 - \mu_k mgd$, which now goes into compressing the spring and still overcoming the friction on the floor, which is $kx^2/2 + \mu_k mgx$. Use of the quadratic formula gives

$$x = -\frac{\mu mg}{k} + \sqrt{\left(\frac{\mu mg}{k}\right)^2 + \frac{mv^2}{k} - \frac{2\mu mgd}{k}} \qquad (1)$$

Upon substitution of the given values, the result is ≃ 1.12 m.

2-27.

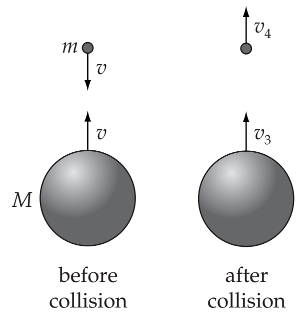


To lift a small mass dm of rope onto the table, an amount of work $dW = (dm)g(z_0 - z)$ must be done on it, where $z_0 = 0.6$ m is the height of the table. The total amount of work that needs to be done is the integration over all the small segments of rope, giving

$$W = \int_0^{z_0} (\mu\, dz) g (z_0 - z) = \frac{\mu g z_0^2}{2} \qquad (1)$$

When we substitute $\mu = m/L = (0.4 \text{ kg})/(4 \text{ m})$, we obtain $W \simeq 0.18$ J.

2-28.



The problem, as stated, is completely one-dimensional. We may therefore use the elementary result obtained from the use of our conservation theorems: energy (since the collision is elastic) and momentum. We can factor the momentum conservation equation

$$m_1v_1 + m_2v_2 = m_1v_3 + m_2v_4 \tag{1}$$

out of the energy conservation equation

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_3^2 + \frac{1}{2}m_2v_4^2 \tag{2}$$

and get

$$v_1 + v_3 = v_2 + v_4 \tag{3}$$

This is the "conservation" of relative velocities that motivates the definition of the coefficient of restitution. In this problem, we initially have the superball of mass M coming up from the ground with velocity $v = \sqrt{2gh}$, while the marble of mass m is falling at the same velocity. Conservation of momentum gives

$$Mv + m(-v) = Mv_3 + mv_4 \tag{4}$$

and our result for elastic collisions in one dimension gives

$$v + v_3 = (-v) + v_4 \tag{5}$$

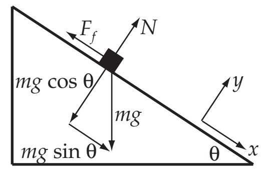
solving for v_3 and v_4 and setting them equal to $\sqrt{2gh_{item}}$, we obtain

$$h_{marble} = \left[\frac{3-\alpha}{1+\alpha}\right]^2 h \tag{6}$$

$$h_{superball} = \left[\frac{1-3\alpha}{1+\alpha}\right]^2 h \tag{7}$$

where $\alpha \equiv m/M$. Note that if $\alpha < 1/3$, the superball will bounce on the floor a second time after the collision.

2-29.



$$\theta = \tan^{-1} 0.08 = 4.6°$$

$$\sum F_y = N - mg \cos\theta$$

$$= m\ddot{y} = 0$$

$$N = mg \cos\theta$$

$$\sum F_x = mg \sin\theta - F_f$$

$$= m\ddot{x}$$

$$F_f = \mu N = \mu mg \cos\theta$$

so

$$m\ddot{x} = mg \sin\theta - \mu mg \cos\theta$$

$$\ddot{x} = g\left(\sin\theta - \mu\cos\theta\right)$$

Integrate with respect to time

$$\dot{x} = gt\left(\sin\theta - \mu\cos\theta\right) + \dot{x}_0 \tag{1}$$

Integrate again:

$$x = x_0 + \dot{x}_0 t + \frac{1}{2} g t^2 \left(\sin\theta - \mu\cos\theta\right) \tag{2}$$

Now we calculate the time required for the driver to stop for a given $\dot{x}_0$ (initial speed) by solving Eq. (1) for t with $\dot{x} = 0$.

$$t' = -\frac{\dot{x}_0}{g}\left(\sin\theta - \mu\cos\theta\right)^{-1}$$

Substituting this time into Eq. (2) gives us the distance traveled before coming to a stop.

$$\left(x' - x_0\right) = \dot{x}_0 t' + \frac{1}{2} g t'^2 \left(\sin\theta - \mu\cos\theta\right)$$

$$\Delta x = -\frac{\dot{x}_0^2}{g}\left(\sin\theta - \mu\cos\theta\right)^{-1} + \frac{1}{2} g \frac{\dot{x}_0^2}{g^2}\left(\sin\theta - \mu\cos\theta\right)^{-1}$$

$$\Delta x = \frac{\dot{x}_0^2}{2g}\left(\mu\cos\theta - \sin\theta\right)^{-1}$$

We have $\theta = 4.6°$, $\mu = 0.45$, $g = 9.8 \text{ m/sec}^2$.

For $\dot{x}_0 = 25 \text{ mph} = 11.2 \text{ m/sec}$, $\Delta x = 17.4 \text{ meters}$.

If the driver had been going at 25 mph, he could only have skidded 17.4 meters.

$$\boxed{\text{Therefore, he was speeding}}$$

How fast was he going?

$$\Delta x \geq 30 \text{ meters gives } \dot{x}_0 \geq 32.9 \text{ mph}.$$

2-30.
$$T = t_1 + t_2 \tag{1}$$

where T = total time = 4.021 sec.

t_1 = the time required for the balloon to reach the ground.

t_2 = the additional time required for the sound of the splash to reach the first student.

We can get t_1 from the equation

$$y = y_0 + \dot{y}_0 t - \frac{1}{2} g t^2; \qquad y_0 = \dot{y}_0 = 0$$

When $t = t_1$, $y = -h$; so (h = height of building)

$$-h = -\frac{1}{2} g t_1^2 \quad \text{or} \quad t_1 = \sqrt{\frac{2h}{g}}$$

$$t = \frac{\text{distance sound travels}}{\text{speed of sound}} = \frac{h}{v}$$

Substituting into (1):

$$T = \sqrt{\frac{2h}{g}} + \frac{h}{v} \quad \text{or} \quad \frac{h}{v} + \sqrt{\frac{2h}{g}} - T = 0$$

This is a quadratic equation in the variable $\sqrt{h}$. Using the quadratic formula, we get:

$$\sqrt{h} = \frac{-\sqrt{\frac{2}{g}} \pm \sqrt{\frac{2}{g} + \frac{4T}{v}}}{\frac{2}{v}} = \frac{v}{\sqrt{2g}} \left[-1 \pm \sqrt{1 + \frac{2gT}{V}} \right]$$

Substituting $V = 331 \text{ m/sec}$

$g = 9.8 \text{ m/sec}^2$

$T = 4.021 \text{ sec}$

and taking the positive root because it is the physically acceptable one, we get:

$$\sqrt{h} = 8.426\ \text{m}^{1/2}$$

$$\boxed{h = 71\ \text{meters}}$$

2-31. For $\dot{x}_0 \neq 0$, example 2.10 proceeds as is until the equations following Eq. (2.78). Proceeding from there we have

$$\alpha B = \dot{x}_0 \neq 0$$

$$\alpha A = \dot{z}_0$$

so

$$(x - x_0) = \frac{\dot{z}_0}{\alpha}\cos\alpha t + \frac{\dot{x}_0}{\alpha}\sin\alpha t$$

$$(y - y_0) = \dot{y}_0 t$$

$$(z - z_0) = -\frac{\dot{x}_0}{\alpha}\cos\alpha t + \frac{z_0}{\alpha}\sin\alpha t$$

Note that

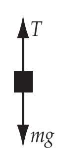
$$(x - x_0)^2 + (z - z_0)^2 = \frac{\dot{z}_0^2}{\alpha^2} + \frac{\dot{x}_0^2}{\alpha^2}$$

Thus the projection of the motion onto the x–z plane is a circle of radius $\frac{1}{\alpha}\left(\dot{x}_0^2 + \dot{z}_0^2\right)^{1/2}$.

$$\boxed{\text{So the motion is unchanged except for a change in the radius of the helix. The new radius is } \frac{m}{qB_0}\left(\dot{x}_0^2 + \dot{z}_0^2\right)^{1/2}.}$$

2-32.

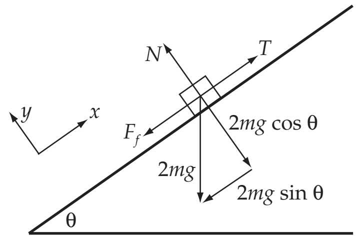
The forces on the hanging mass are



The equation of motion is (calling downward positive)

$$mg - T = ma \quad \text{or} \quad T = m(g - a) \tag{1}$$

The forces on the other mass are



The y equation of motion gives

$$N - 2mg\cos\theta = m\ddot{y} = 0$$

or

$$N = 2mg\cos\theta$$

The x equation of motion gives $\left(F_f = \mu_k N = 2\mu_k mg\cos\theta\right)$

$$T - 2mg\sin\theta - 2\mu_k mg\cos\theta = ma \tag{2}$$

Substituting from (1) into (2)

$$mg - 2mg\sin\theta - 2\mu_k mg\cos\theta = 2ma$$

When $\theta = \theta_0$, $a = 0$. So

$$g - 2g\sin\theta_0 - 2\mu_k g\cos\theta_0 = 0$$

$$\frac{1}{2} = \sin\theta_0 + \mu_k\cos\theta_0$$

$$= \sin\theta_0 + \mu_k\left(1 - \sin^2\theta_0\right)^{1/2}$$

Isolating the square root, squaring both sides and rearranging gives

$$\left(1 + \mu_k^2\right)\sin^2\theta_0 - \sin\theta_0\left(\frac{1}{4} - \mu_k^2\right) = 0$$

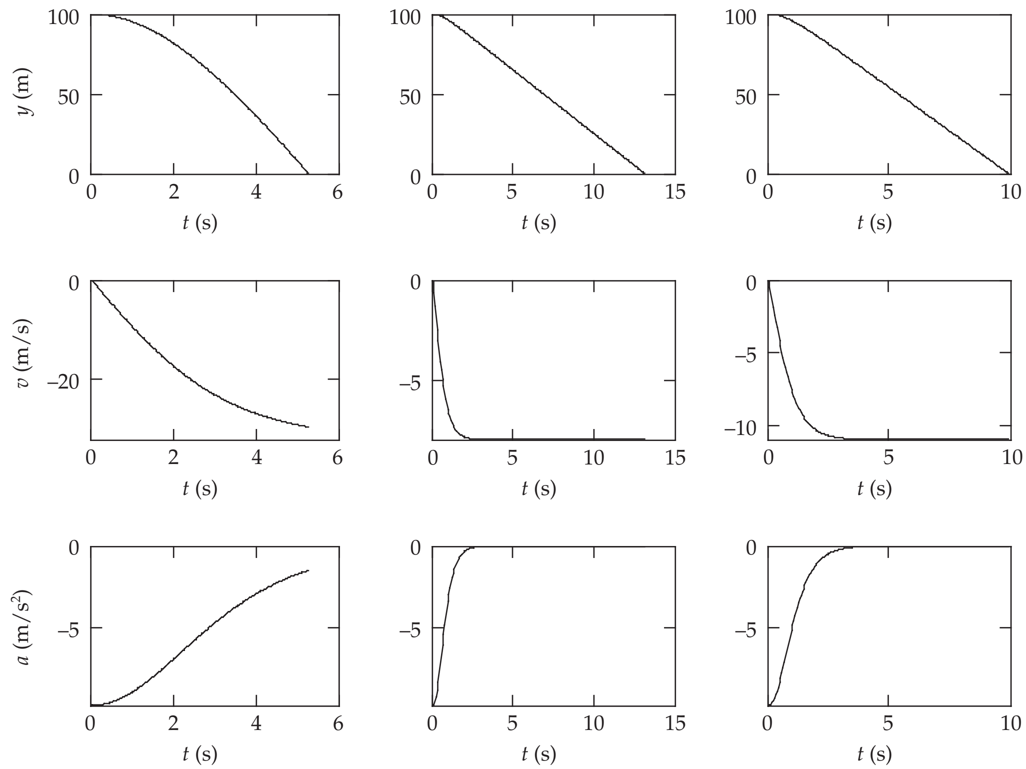
Using the quadratic formula gives

$$\boxed{\sin\theta_0 = \frac{1 \pm \mu_k\sqrt{3 + 4\mu_k^2}}{2\left(1 + \mu_k^2\right)}}$$

2-33. The differential equation to solve is

$$\ddot{y} = \frac{c_W\rho A v^2}{2m} - g = g\left[\left(\frac{v}{v_t}\right)^2 - 1\right] \tag{1}$$

where $v_t = \sqrt{2mg/c_w\rho A}$ is the terminal velocity. The initial conditions are $y_0 = 100\text{ m}$, and $v_0 = 0$. The computer integrations for parts (a), (b), and (c) are shown in the figure.

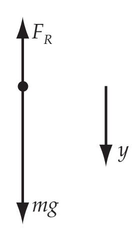


d) Taking $\rho = 1.3\ \text{kg}\cdot\text{m}^{-3}$ as the density of air, the terminal velocities are 32.2, 8.0, and 11.0 (all $\text{m}\cdot\text{s}^{-1}$) for the baseball, ping-pong ball, and raindrop, respectively. Both the ping-pong ball and the raindrop essentially reach their terminal velocities by the time they hit the ground. If we rewrite the mass as average density times volume, then we find that $v_t \propto \sqrt{\rho_{material} R}$. The differences in terminal velocities of the three objects can be explained in terms of their densities and sizes.

e) Our differential equation shows that the effect of air resistance is an acceleration that is inversely proportional to the square of the terminal velocity. Since the baseball has a higher terminal velocity than the ping-pong ball, the magnitude of its deceleration is smaller for a given speed. If a person throws the two objects with the same initial velocity, the baseball goes farther because it has less drag.

f) We have shown in part (d) that the terminal velocity of a raindrop of radius 0.004 m will be larger than for one with radius 0.002 m ($9.0\ \text{m}\cdot\text{s}^{-1}$) by a factor of $\sqrt{2}$.

2-34.



Take the y-axis to be positive downwards. The initial conditions are $y = \dot{y} = 0$ at $t = 0$.

a) $$F_R = \alpha v$$

The equation of motion is

$$m\ddot{y} = m\frac{dv}{dt} = mg - \alpha v$$

$$\frac{m\,dv}{mg - \alpha v} = dt$$

Integrating gives: $-\frac{m}{\alpha}\ln\left(mg - \alpha v\right) = t + C$

Evaluate C using the condition $v = 0$ at $t = 0$:

$$-\frac{m}{\alpha}\ln\left(mg\right) = C$$

So $$-\frac{m}{\alpha}\ln\left(mg - \alpha v\right) + \frac{m}{\alpha}\ln\left(mg\right) = t$$

or $-\frac{\alpha t}{m} = \ln\left[\frac{mg - \alpha v}{mg}\right] = \ln\left[1 - \frac{\alpha v}{mg}\right]$

Take the exponential of both sides and solve for v:

$$e^{-\alpha r/m} = 1 - \frac{\alpha v}{mg}$$

$$\frac{\alpha v}{mg} = 1 - e^{-\alpha t/m}$$

$$v = \frac{mg}{\alpha}\left(1 - e^{-\alpha t/m}\right) \tag{1}$$

$$dy = \frac{mg}{\alpha}\left(1 - e^{-\alpha t/m}\right)dt$$

Integrate again:

$$y + C = \frac{mg}{\alpha}\left(t + \frac{m}{\alpha}e^{-\alpha t/m}\right)$$

$y = 0$ at $t = 0$, so:

$$C = \frac{mg}{\alpha}\left[\frac{m}{\alpha}\right] = m^2 g/\alpha^2$$

$$y = \frac{mg}{\alpha}\left[-\frac{m}{\alpha} + t + \frac{m}{\alpha}e^{-\alpha t/m}\right] \tag{2}$$

Solve (3) for t and substitute into (4):

$$1 - \frac{\alpha v}{mg} = e^{-\alpha t/m} \tag{3}$$

$$t = -\frac{m}{\alpha}\ln\left[1-\frac{\alpha v}{mg}\right]$$

$$y = \frac{mg}{\alpha}\left[-\frac{m}{\alpha}-\frac{m}{\alpha}\ln\left[1-\frac{\alpha v}{mg}\right]+\frac{m}{\alpha}\left[1-\frac{\alpha v}{mg}\right]\right] = \frac{mg}{\alpha}\left[-\frac{v}{g}-\frac{m}{\alpha}\ln\left[1-\frac{\alpha v}{mg}\right]\right] \tag{4}$$

$$\boxed{y = -\frac{m}{\alpha}\left[v+\frac{mg}{\alpha}\ln\left[1-\frac{\alpha v}{mg}\right]\right]}$$

b) $F_R = \beta v^2$

The equation of motion becomes:

$$m\frac{dv}{dt} = mg - \beta v^2$$

$$\frac{m\,dv}{mg-\beta v^2} = dt$$

Integrate and apply the initial condition $v = 0$ at $t = 0$:

$$\int \frac{dv}{\frac{mg}{\beta}-v^2} = \frac{\beta}{m}\int dt$$

From integral tables $\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$; so

$$\frac{1}{a}\tanh^{-1}\frac{v}{a} = \frac{\beta}{m}t + C \quad \text{where } a \equiv \sqrt{\frac{mg}{\beta}}$$

$$\frac{1}{\alpha}\tanh^{-1}0 = 0 = 0 + C$$

so:

$$\frac{1}{a}\tanh^{-1}\frac{v}{a} = \frac{\beta}{m}t$$

Solving for v:

$$v = a\tanh\frac{a\beta t}{m} \tag{5}$$

$$\frac{dy}{dt} = a\tanh\frac{a\beta t}{m}$$

From integral tables $\int \tanh u\,du = \ln\cosh u$

So $y + C = \frac{m}{\beta}\ln\cosh\frac{a\beta t}{m}$

Apply the conditions at $y = 0$ and $t = 0$

$$C = \frac{m}{\beta}\ln\left(\cosh 0\right) = \frac{m}{\beta}\ln 1 = 0$$

So

$$y = \frac{m}{\beta}\ln\cosh\frac{\alpha\beta t}{m} \tag{6}$$

Solving (5) for t:

$$t = \frac{m}{\alpha\beta}\tanh^{-1}\frac{v}{a}$$

Substituting into (6):

$$y = \frac{m}{\beta}\ln\cosh\left[\tanh^{-1}\frac{v}{a}\right]$$

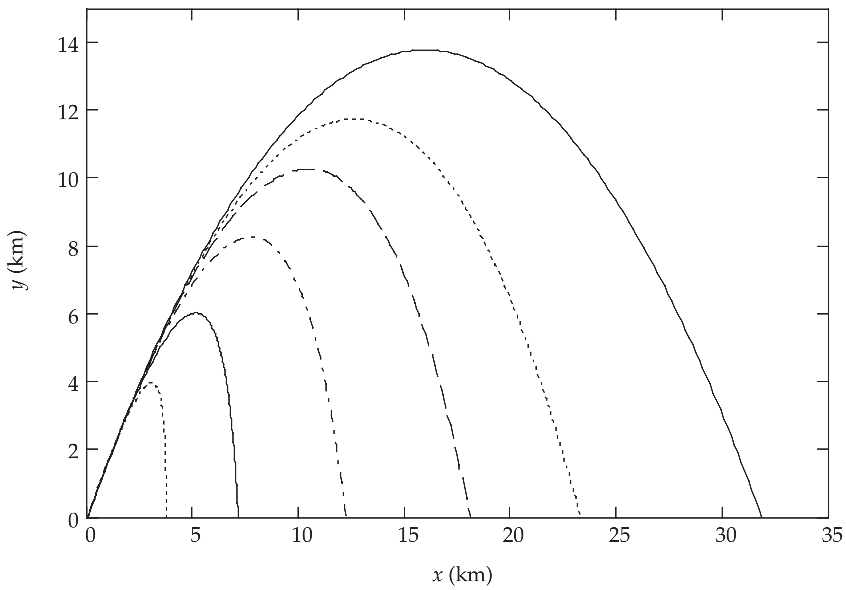
Use the identity: $\tanh^{-1} u = \cosh^{-1}\dfrac{1}{\sqrt{1-u^2}}$, where $|u| < 1$.

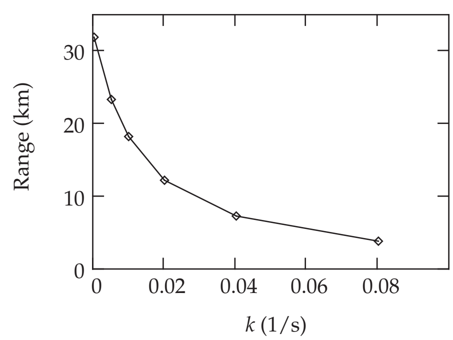
(In our case $|u| < 1$ as it should be because $\dfrac{v}{a} = \sqrt{\dfrac{\beta v^2}{mg}}$; and the condition that $|u| < 1$ just says that gravity is stronger than the retarding force, which it must be.) So

$$y = \frac{m}{\beta}\ln\cosh\left(\cosh^{-1}\frac{1}{\sqrt{1-\beta v^2/mg}}\right) = \frac{m}{\beta}\ln\left(1-\beta v^2/mg\right)^{-1/2}$$

$$\boxed{y = -\frac{m}{2\beta}\ln\left(1-\beta v^2/mg\right)}$$

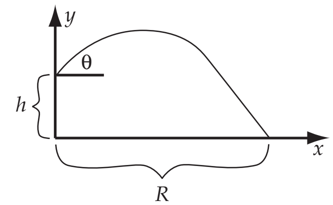
2-35.





We are asked to solve Equations (2.41) and (2.42), for the values k = 0, 0.005, 0.01, 0.02, 0.04, and 0.08 (all in s^{-1}), with initial speed $v_0 = 600\ \mathrm{m \cdot s^{-1}}$ and angle of elevation $\theta = 60°$. The first figure is produced by numerical solution of the differential equations, and agrees closely with Figure 2-8. Figure 2-9 can be most closely reproduced by finding the range for our values of k, and plotting them vs. k. A smooth curve could be drawn, or more ranges could be calculated with more values of k to fill in the plot, but we chose here to just connect the points with straight lines.

2-36.



Put the origin at the initial point. The equations for the x and y motion are then

$$x = v_0 \left(\cos\theta\right) t$$

$$y = v_0 \left(\sin\theta\right) t - \frac{1}{2} g t^2$$

Call τ the time when the projectile lands on the valley floor. The y equation then gives

$$-h = v_0 \left(\sin\theta\right) \tau - \frac{1}{2} g \tau^2$$

Using the quadratic formula, we may find τ

$$\tau = \frac{v_0 \sin\theta}{g} + \frac{\sqrt{v_0^2 \sin^2\theta + 2gh}}{g}$$

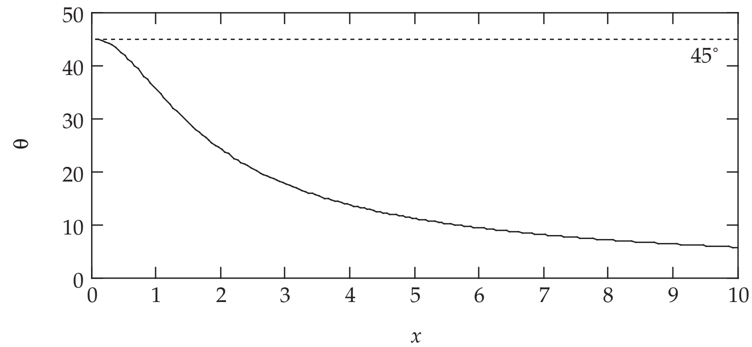
(We take the positive since $\tau > 0$.) Substituting τ into the x equation gives the range R as a function of θ.

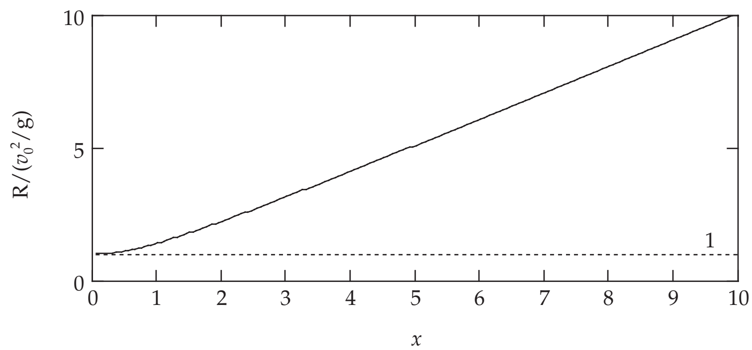
$$R = \frac{v_0^2}{g} \cos\theta \left(\sin\theta + \sqrt{\sin^2\theta + x^2} \right) \tag{1}$$

where we have defined $x^2 \equiv 2gh/v_0^2$. To maximize R for a given h and v_0, we set $dR/d\theta = 0$. The equation we obtain is

$$\cos^2\theta - \sin^2\theta - \sin\theta \sqrt{\sin^2\theta + x^2} + \frac{\sin\theta \cos^2\theta}{\sqrt{\sin^2\theta + x^2}} = 0 \tag{2}$$

Although it can give $x = x(\theta)$, the above equation cannot be solved to give $\theta = \theta(x)$ in terms of the elementary functions. The optimum θ for a given x is plotted in the figure, along with its respective range in units of v_0^2/g. Note that $x = 0$, which among other things corresponds to $h = 0$, gives the familiar result $\theta = 45°$ and $R = v_0^2/g$.





2-37. $v = \alpha/x \qquad dv/dx = -\alpha/x^2$

Since $\dfrac{dv}{dt} = \dfrac{dv}{dx}\dfrac{dx}{dt} = \dfrac{dv}{dx}v$ then

$$F = m\frac{dv}{dt} = mv\frac{dv}{dx} = m\left[\frac{\alpha}{x}\right]\left[-\frac{\alpha}{x^2}\right]$$

$$\boxed{F(x) = -m\alpha^2/x^3}$$

2-38. $v(x) = ax^{-n}$

a) $F = m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\dfrac{dx}{dt} = mv\dfrac{dv}{dx} = m\left(ax^{-n}\right)\left(-nax^{-n-1}\right)$

$$\boxed{F(x) = -mna^2\ x^{-(2n+1)}}$$

b) $v(x) = \dfrac{dx}{dt} = ax^{-n}$

$x^n dx = adt$

Integrate:

$$\frac{x^{n+1}}{n+1} = at + C \qquad C = 0 \text{ using given initial conditions}$$

$$x^{n+1} = (n+1)\ at$$

$$\boxed{x = \left[(n+1)at\right]^{1/(n+1)}}$$

c) Substitute $x(t)$ into $F(x)$:

$$F(t) = -mna^2\left[\left\{(n+1)at\right\}^{1/(n+1)}\right]^{-(2n+1)}$$

$$\boxed{F(t) = -mna^2\left[(n+1)at\right]^{-(2n+1)/(n+1)}}$$

2-39.

a) $F = -\alpha\, e^{\beta v}$

$$\frac{dv}{dt} = -\frac{\alpha}{m}\, e^{\beta v}$$

$$\int e^{-\beta v}\, dv = -\frac{\alpha}{m}\int dt$$

$$-\frac{1}{\beta}\, e^{-\beta v} = -\frac{\alpha}{m}\, t + C$$

$v = v_0$ at $t = 0$, so

$$-\frac{1}{\beta}\, e^{-\beta v_0} = C$$

$$-\frac{1}{\beta}\left(e^{-\beta v} - e^{-\beta v_0}\right) = -\frac{\alpha}{m}\, t$$

Solving for v gives

$$\boxed{v(t) = -\frac{1}{\beta}\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]}$$

b) Solve for t when $v = 0$

$$\frac{\alpha\beta t}{m} + e^{-\beta v_0} = 1$$

$$\boxed{t = \frac{m}{\alpha\beta}\left[1 - e^{-\beta v_0}\right]}$$

c) From a) we have

$$dx = -\frac{1}{\beta}\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right] dt$$

Using $\int \ln(ax+b)\,dx = \dfrac{ax+b}{a}\ln(ax+b) - x$ we obtain

$$x + C = -\frac{1}{\beta}\left[\frac{\left[\dfrac{\alpha\beta t}{m} + e^{-\beta v_0}\right]\ln\left[\dfrac{\alpha\beta t}{m} + e^{-\beta v_0}\right]}{\alpha\beta/m} - t\right]$$

Evaluating C using $x = 0$ at $t = 0$ gives

$$C = \frac{v_0 m}{\alpha\beta} e^{-\beta v_0}$$

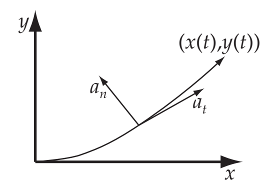
So

$$x = -\frac{m v_0}{\alpha\beta} e^{-\beta v_0} + \frac{t}{\beta} - \frac{m}{\alpha\beta^2}\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]\ln\left[\frac{\alpha\beta t}{m} + e^{-\beta v_0}\right]$$

Substituting the time required to stop from b) gives the distance required to stop

$$\boxed{x = \frac{m}{\alpha\beta}\left[\frac{1}{\beta} - e^{-\beta v_0}\left[v_0 + \frac{1}{\beta}\right]\right]}$$

2-40.



Write the velocity as $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. It follows that

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{dt} = a_t\mathbf{T} + a_n\mathbf{N} \tag{1}$$

where $\mathbf{N}$ is the unit vector in the direction of $d\mathbf{T}/dt$. That $\mathbf{N}$ is normal to $\mathbf{T}$ follows from $0 = d/dt(\mathbf{T}\cdot\mathbf{T})$. Note also a_n is positive definite.

a) We have $v = \sqrt{\dot{x}^2 + \dot{y}^2} = A\alpha\sqrt{5 - 4\cos\alpha t}$. Computing from the above equation,

$$a_t = \frac{dv}{dt} = \frac{2A\alpha^2 \sin\alpha t}{\sqrt{5 - 4\cos\alpha t}} \tag{2}$$

We can get a_n from knowing a in addition to a_t. Using $a = \sqrt{\ddot{x}^2 + \ddot{y}^2} = A\alpha^2$, we get

$$a_n = \sqrt{a^2 - a_t^2} = A\alpha^2\frac{|2\cos\alpha t - 1|}{\sqrt{5 - 4\cos\alpha t}} \tag{3}$$

b) Graphing a_n versus t shows that it has maxima at $\alpha t = n\pi$, where $a_n = A\alpha^2$.

2-41.

a) As measured on the train:

$$T_i = 0\,;\; T_f = \frac{1}{2}mv^2$$

$$\boxed{\Delta T = \frac{1}{2}mv^2}$$

b) As measured on the ground:

$$T_i = \frac{1}{2}mu^2\,;\; T_f = \frac{1}{2}m(v+u)^2$$

$$\boxed{\Delta T = \frac{1}{2}mv^2 + mvu}$$

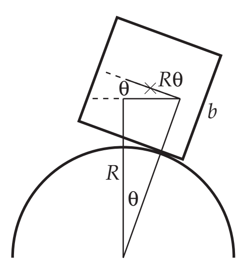
c) The woman does an amount of work equal to the kinetic energy gain of the ball as measured in her frame.

$$\boxed{W = \frac{1}{2}mv^2}$$

d) The train does work in order to keep moving at a constant speed u. (If the train did no work, its speed after the woman threw the ball would be slightly less than u, and the speed of the ball relative to the ground would not be $u + v$.) The term mvu is the work that must be supplied by the train.

$$\boxed{W = mvu}$$

2-42.



From the figure, we have $h(\theta) = (R + b/2)\cos\theta + R\theta\sin\theta$, and the potential is $U(\theta) = mgh(\theta)$. Now compute:

$$\frac{dU}{d\theta} = mg\left[-\frac{b}{2}\sin\theta + R\theta\cos\theta\right] \tag{1}$$

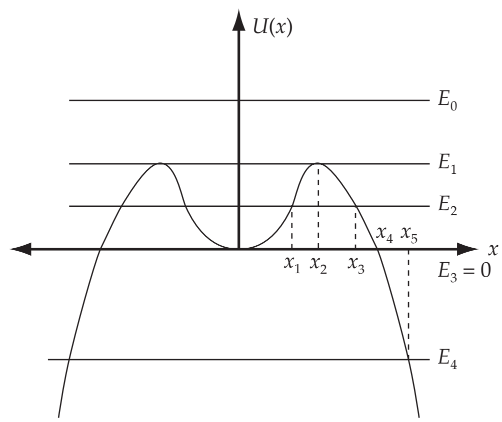
$$\frac{d^2U}{d\theta^2} = mg\left[\left(R - \frac{b}{2}\right)\cos\theta - R\theta\sin\theta\right] \tag{2}$$

The equilibrium point (where $dU/d\theta = 0$) that we wish to look at is clearly $\theta = 0$. At that point, we have $d^2U/d\theta^2 = mg\left(R - b/2\right)$, which is stable for $R > b/2$ and unstable for $R < b/2$. We can use the results of Problem 2-46 to obtain stability for the case $R = b/2$, where we will find that the first non-trivial result is in fourth order and is negative. We therefore have an equilibrium at $\theta = 0$ which is stable for $R > b/2$ and unstable for $R \leq b/2$.

2-43. $F = -kx + kx^3/\alpha^2$

$$U(x) = -\int F\,dx = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}$$

To sketch $U(x)$, we note that for small x, $U(x)$ behaves like the parabola $\frac{1}{2}kx^2$. For large x, the behavior is determined by $-\frac{1}{4}k\frac{x^4}{\alpha^2}$



$$E = \frac{1}{2}mv^2 + U(x)$$

For $E = E_0$, the motion is unbounded; the particle may be anywhere.

For $E = E_1$ (at the maxima in $U(x)$) the particle is at a point of unstable equilibrium. It may remain at rest where it is, but if perturbed slightly, it will move away from the equilibrium.

What is the value of E_1? We find the x values by setting $\frac{dU}{dx} = 0$.

$$0 = kx - kx^3/\alpha^2$$

$x = 0, \pm\,\alpha$ are the equilibrium points

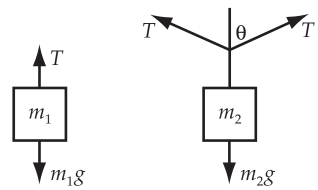
$$U(\pm\alpha) = E_1 = \frac{1}{2}k\alpha^2 - \frac{1}{4}k\alpha^2 = \frac{1}{4}k\alpha^2$$

For $E = E_2$, the particle is either bounded and oscillates between $-x_2$ and x_2; or the particle comes in from $\pm\infty$ to $\pm x_3$ and returns to $\pm\infty$.

For $E_3 = 0$, the particle is either at the stable equilibrium point $x = 0$, or beyond $x = \pm x_4$.

For E_4, the particle comes in from $\pm\infty$ to $\pm x_5$ and returns.

2-44.



From the figure, the forces acting on the masses give the equations of motion

$$m_1\ddot{x}_1 = m_1 g - T \tag{1}$$

$$m_2\ddot{x}_2 = m_2 g - 2T\cos\theta \tag{2}$$

where x_2 is related to x_1 by the relation

$$x_2 = \sqrt{\frac{\left(b - x_1\right)^2}{4} - d^2} \tag{3}$$

and $\cos\theta = d\big/\left[\left(b - x_1\right)/2\right]$. At equilibrium, $\ddot{x}_1 = \ddot{x}_2 = 0$ and $T = m_1 g$. This gives as the equilibrium values for the coordinates

$$x_{10} = b - \frac{4m_1 d}{\sqrt{4m_1^2 - m_2^2}} \tag{4}$$

$$x_{20} = \frac{m_2 d}{\sqrt{4m_1^2 - m_2^2}} \tag{5}$$

We recognize that our expression x_{10} is identical to Equation (2.105), and has the same requirement that $m_2/m_1 < 2$ for the equilibrium to exist. When the system is in motion, the descriptive equations are obtained from the force laws:

$$m_1(\ddot{x}_1 - g) = \frac{m_2\left(b - x_1\right)}{4x_2}(\ddot{x}_2 - g) \tag{6}$$

To examine stability, let us expand the coordinates about their equilibrium values and look at their behavior for small displacements. Let $\xi_1 \equiv x_1 - x_{10}$ and $\xi_2 \equiv x_2 - x_{20}$. In the calculations, take terms in ξ_1 and ξ_2, and their time derivatives, only up to first order. Equation (3) then becomes $\xi_2 \simeq -(m_1/m_2)\xi_1$. When written in terms of these new coordinates, the equation of motion becomes

$$\ddot{\xi}_1 = -\frac{g\left(4m_1^2 - m_2^2\right)^{3/2}}{4m_1 m_2\left(m_1 + m_2\right)d}\xi_1 \tag{7}$$

which is the equation for simple harmonic motion. The equilibrium is therefore stable, when it exists.

2-45. and **2-46.** Expand the potential about the equilibrium point

$$U(x) = \sum_{i=n+1}^{\infty} \frac{1}{i!} \left[\frac{d^i u}{dx^i} \right]_0 x^i \tag{1}$$

The leading term in the force is then

$$F(x) = -\frac{dU}{dx} = -\frac{1}{n!} \left[\frac{d^{(n+1)}U}{dx^{(n+1)}} \right]_0 x^n \tag{2}$$

The force is restoring for a stable point, so we need $F(x>0)<0$ and $F(x<0)>0$. This is never true when n is even (e.g., $U = kx^3$), and is only true for n odd when $\left(d^{(n+1)}U/dx^{(n+1)}\right)_0 < 0$.

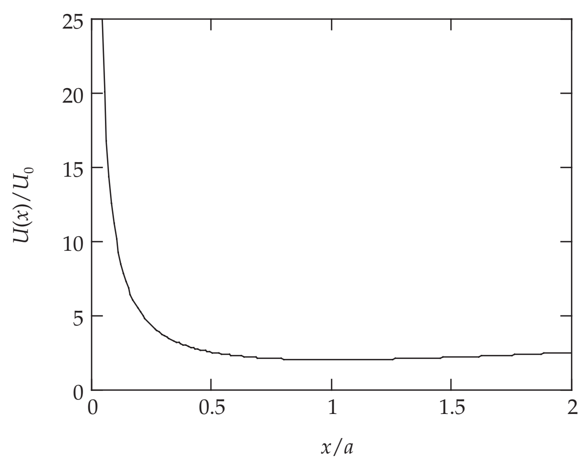
2-47. We are given $U(x) = U_0\left(a/x + x/a\right)$ for $x > 0$. Equilibrium points are defined by $dU/dx = 0$, with stability determined by d^2U/dx^2 at those points. Here we have

$$\frac{dU}{dx} = U_0 \left[-\frac{a}{x^2} + \frac{1}{a} \right] \tag{1}$$

which vanishes at $x = a$. Now evaluate

$$\left[\frac{d^2U}{dx^2} \right]_a = \left[\frac{2U_0}{a^3} \right] > 0 \tag{2}$$

indicating that the equilibrium point is stable.



2-48. In the equilibrium, the gravitational force and the eccentric force acting on each star must be equal

$$\frac{Gm^2}{d^2}=\frac{mv^2}{d/2}\Rightarrow v=\sqrt{\frac{mG}{2d}}\Rightarrow \tau=\frac{\pi d}{v}=\frac{\sqrt{2}\pi d^{3/2}}{\sqrt{mG}}$$

2-49. The distances from stars to the center of mass of the system are respectively

$$r_1=\frac{dm_2}{m_1+m_2} \quad \text{and} \quad r_2=\frac{dm_1}{m_1+m_2}$$

At equilibrium, like in previous problem, we have

$$\frac{Gm_1m_2}{d^2}=\frac{m_1{v_1}^2}{r_1}\Rightarrow v_1=\sqrt{\frac{G{m_2}^2}{d(m_1+m_2)}}\Rightarrow \tau=\frac{2\pi r_1}{v_1}=\frac{2\pi d^{3/2}}{\sqrt{G(m_1+m_2)}}$$

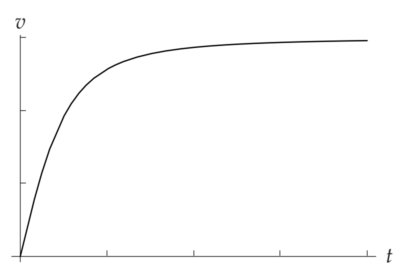
The result will be the same if we consider the equilibrium of forces acting on 2nd star.

2-50.

a) $$\frac{d}{dt}\left(\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\right)=F\Rightarrow\int_0^t d\left(\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{m_0v}{\sqrt{1-\frac{v^2}{c^2}}}=Ft\Rightarrow v(t)=\frac{Ft}{\sqrt{m_0^2+\frac{F^2t^2}{c^2}}}$$

$$\Rightarrow x(t)=\int_0^t v(t)dt=\frac{c^2}{F}\left(\sqrt{{m_0}^2+\frac{F^2t^2}{c^2}}-m_0\right)$$

b)



c) From a) we find

$$t=\frac{vm_0}{F\sqrt{1-\frac{v^2}{c^2}}}$$

Now if $\frac{F}{m_0}=10$, then

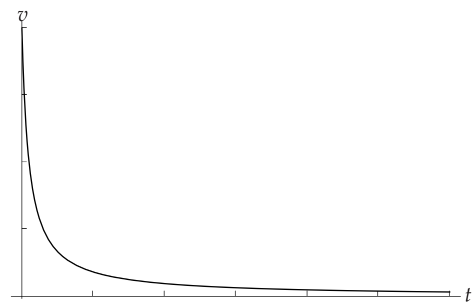
when $v = c/2$, we have $t = \dfrac{c}{10\sqrt{3}} = 0.55$ year

when v = 99% c, we have $t = \dfrac{99c}{10\sqrt{199}} = 6.67$ years

2-51.

a) $m\dfrac{dv}{dt} = -bv^2 \Rightarrow \int \dfrac{dv}{v^2} = -\int \dfrac{b}{m}dt \Rightarrow v(t) = \dfrac{mv_0}{btv_0 + m}$

Now let $v(t) = v_0/1000$, one finds $t = \dfrac{999m}{v_0 b} = 138.7$ hours.



b) $x(t) = \int_0^t v dt = \dfrac{m}{b}\ln\left(\dfrac{btv_0 + m}{m}\right)$

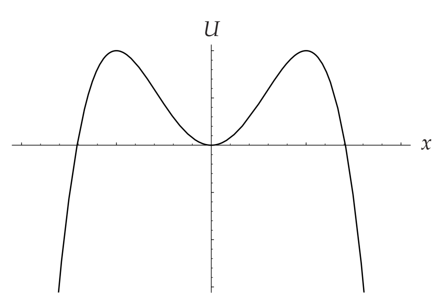
We use the value of t found in question a) to find the corresponding distance

$$x(t) = \frac{m}{b}\ln(1000) = 6.9 \text{ km}$$

2-52.

a) $F(x) = -\dfrac{dU}{dx} = -\dfrac{4U_0 x}{a^2}\left(1 - \dfrac{x^2}{a^2}\right)$

b)



When F = 0, there is equilibrium; further when U has a local minimum (i.e. $dF/dx < 0$) it is stable, and when U has a local maximum (i.e. $dF/dx > 0$) it is unstable.

So one can see that in this problem $x = a$ and $x = -a$ are unstable equilibrium positions, and $x = 0$ is a stable equilibrium position.

c) Around the origin, $F \approx -\dfrac{4U_0 x}{a^2} \equiv -kx \Rightarrow \omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4U_0}{ma^2}}$

d) To escape to infinity from $x = 0$, the particle needs to get at least to the peak of the potential,

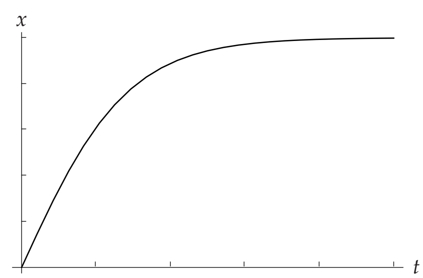
$$\frac{mv_{\min}^2}{2} = U_{\max} = U_0 \Rightarrow v_{\min} = \sqrt{\frac{2U_0}{m}}$$

e) From energy conservation, we have

$$\frac{mv^2}{2} + \frac{U_0 x^2}{a^2} = \frac{mv_{\min}^2}{2} \Rightarrow \frac{dx}{dt} = v = \sqrt{\frac{2U_0}{m}}\left(1 - \frac{x^2}{a^2}\right)$$

We note that, in the ideal case, because the initial velocity is the escape velocity found in d), ideally x is always smaller or equal to a, then from the above expression,

$$t = \sqrt{\frac{m}{2U_0}} \int_0^x \frac{dx}{\left(1 - \dfrac{x^2}{a^2}\right)} = \sqrt{\frac{ma^2}{8U_0}} \ln \frac{a + x}{a - x} \Rightarrow x(t) = \frac{a\left(\exp\left(t\sqrt{\dfrac{8U_0}{ma^2}}\right) - 1\right)}{\left(\exp\left(t\sqrt{\dfrac{8U_0}{ma^2}}\right) + 1\right)}$$



2-53.

F is a conservative force when there exists a non-singular potential function $U(x)$ satisfying $F(x) = -\mathbf{grad}(U(x))$. So if F is conservative, its components satisfy the following relations

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$

and so on.

a) In this case all relations above are satisfied, so F is indeed a conservative force.

$$F_x = -\frac{\partial U}{\partial x} = ayz + bx + c \Rightarrow U = -ayzx - \frac{bx^2}{2} - cx + f_1(y,z) \tag{1}$$

where $f_1(y,z)$ is a function of only y and z

$$F_y = -\frac{\partial U}{\partial y} = axz + bz \Rightarrow U = -ayzx - byz + f_2(x,z) \tag{2}$$

where $f_2(x,z)$ is a function of only x and z

$$F_z = -\frac{\partial U}{\partial z} = axy + by + c \Rightarrow U = -ayzx - byz + f_3(y,z) \tag{3}$$

then from (1), (2), (3) we find that

$$U = -axyz - byz - cx^2 - \frac{bx^2}{2} + C$$

where C is a arbitrary constant.

b) Using the same method we find that F in this case is a conservative force, and its potential is

$$U = -z\exp(-x) - y\ln z + C$$

c) Using the same method we find that F in this case is a conservative force, and its potential is (using the result of problem 1-31b):

$$U = -a\ln r$$

2-54.

a) Terminal velocity means final steady velocity (here we assume that the potato reaches this velocity before the impact with the Earth) when the total force acting on the potato is zero.

$$mg = kmv \quad \text{and consequently} \quad v = g/k = 1000 \text{ m/s}.$$

b)

$$\frac{dv}{dt} = \frac{F}{m} = -(g+kv) \Rightarrow \frac{dx}{v} = dt = -\frac{dv}{g+kv} \Rightarrow \int_0^x dx = -\int_{v_0}^0 \frac{vdv}{g+kv} \Rightarrow$$

$x_{\max} = \dfrac{v_0}{k} + \dfrac{g}{k^2}\ln\dfrac{g}{g+kv_0} = 679.7 \text{ m}$ where v_0 is the initial velocity of the potato.

2-55. Let's denote v_{x0} and v_{y0} the initial horizontal and vertical velocity of the pumpkin. Evidently, $v_{x0} = v_{y0}$ in this problem.

$$m\frac{dv_x}{dt} = F_x = -mkv_x \Rightarrow -\frac{dx}{v_x} = -dt = \frac{dv_x}{kv_x} \Rightarrow x_f = \frac{v_{x0} - v_{xf}}{k} \tag{1}$$

where the suffix *f* always denote the final value. From the second equality of (1), we have

$$-dt = \frac{dv_x}{kv_x} \Rightarrow v_{xf} = v_{x0}e^{-kt_f} \tag{2}$$

Combining (1) and (2) we have

$$x_f = \frac{v_{x0}}{k}\left(1 - e^{-kt_f}\right) \tag{3}$$

Do the same thing with the *y*-component, and we have

$$m\frac{dv_y}{dt}=F_y=-mg-mkv_y\Rightarrow -\frac{dy}{v_y}=-dt=\frac{dv_y}{g+kv_y}\Rightarrow 0=y_f=\frac{g}{k^2}\ln\frac{g+kv_{yf}}{g+kv_{y0}}+\frac{v_{y0}-v_{yf}}{k} \tag{4}$$

$$\text{and}\quad -dt=\frac{dv_y}{g+kv_y}\Rightarrow g+kv_{yf}=\left(g+kv_{0f}\right)e^{-kt_f} \tag{5}$$

From (4) and (5) with a little manipulation, we obtain

$$1-e^{-kt_f}=\frac{gkt_f}{g+kv_{y0}} \tag{6}$$

(3) and (6) are 2 equations with 2 unknowns, t_f and *k*. We can eliminate t_f, and obtain an equation of single variable *k.*

$$x_f=\frac{v_{x0}}{k}\left(1-e^{-kt_f\left(g+kv_{y0}\right)/\left(gv_{x0}\right)}\right)$$

Putting $x_f=142\text{ m}$ and $v_{x0}=v_{y0}=\frac{v_0}{\sqrt{2}}=38.2\text{ m/s}$ we can numerically solve for *k* and obtain

$$K = 0.00246\ \text{s}^{-1}.$$

CHAPTER 3

Oscillations

3-1.

a)

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10^4 \text{ dyne/cm}}{10^2 \text{ gram}}} = \frac{10}{2\pi}\sqrt{\frac{\frac{\text{gram}\cdot\text{cm}}{\text{sec}^2\cdot\text{cm}}}{\text{gram}}} = \frac{10}{2\pi} \text{ sec}^{-1}$$

or,

$$\boxed{\nu_0 \cong 1.6 \text{ Hz}} \tag{1}$$

$$\tau_0 = \frac{1}{\nu_0} = \frac{2\pi}{10} \text{ sec}$$

or,

$$\boxed{\tau_0 \cong 0.63 \text{ sec}} \tag{2}$$

b)

$$E = \frac{1}{2}kA^2 = \frac{1}{2}\times 10^4 \times 3^2 \text{ dyne-cm}$$

so that

$$\boxed{E = 4.5\times 10^4 \text{ erg}} \tag{3}$$

c) The maximum velocity is attained when the total energy of the oscillator is equal to the kinetic energy. Therefore,

$$\frac{1}{2}mv_{\max}^2 = 4.5\times 10^4 \text{ erg}$$

$$\text{v}_{\max} = \sqrt{\frac{2\times 4.5\times 10^4}{100}}$$

or,

$$\boxed{v_{\max} = 30 \text{ cm/sec}} \tag{4}$$

3-2.

a) The statement that at a certain time $t = t_1$ the maximum amplitude has decreased to one-half the initial value means that

$$\left|x_{en}\right| = A_0 e^{-\beta t_1} = \frac{1}{2} A_0 \tag{1}$$

or,

$$e^{-\beta t_1} = \frac{1}{2} \tag{2}$$

so that

$$\beta = \frac{\ln 2}{t_1} = \frac{0.69}{t_1} \tag{3}$$

Since $t_1 = 10$ sec ,

$$\boxed{\beta = 6.9 \times 10^{-2} \text{ sec}^{-1}} \tag{4}$$

b) According to Eq. (3.38), the angular frequency is

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2} \tag{5}$$

where, from Problem 3-1, $\omega_0 = 10 \text{ sec}^{-1}$. Therefore,

$$\begin{aligned} \omega_1 &= \sqrt{(10)^2 - \left(6.9 \times 10^{-2}\right)^2} \\ &\cong 10\left[1 - \frac{1}{2}(6.9)^2 \times 10^{-6}\right] \text{sec}^{-1} \end{aligned} \tag{6}$$

so that

$$\boxed{\nu_1 = \frac{10}{2\pi}(1 - 2.40 \times 10^{-5}) \text{ sec}^{-1}} \tag{7}$$

which can be written as

$$\nu_1 = \nu_0\left(1 - \delta\right) \tag{8}$$

where

$$\delta = 2.40 \times 10^{-5} \tag{9}$$

That is, ν_1 is only slightly different from ν_0.

c) The *decrement* of the motion is defined to be $e^{\beta\tau_1}$ where $\tau_1 = 1/\nu_1$. Then,

$$\boxed{e^{\beta\tau_1} \simeq 1.0445}$$

3-3. The initial kinetic energy (equal to the *total* energy) of the oscillator is $\frac{1}{2}mv_0^2$, where $m = 100$ g and $v_0 = 1$ cm/sec .

a) Maximum displacement is achieved when the total energy is equal to the potential energy. Therefore,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx_0^2$$

$$x_0 = \sqrt{\frac{m}{k}}\,v_0 = \sqrt{\frac{10^2}{10^4}} \times 1 = \frac{1}{10}\ \text{cm}$$

or,

$$\boxed{x_0 = \frac{1}{10}\ \text{cm}} \tag{1}$$

b) The maximum potential energy is

$$U_{\max} = \frac{1}{2}kx_0^2 = \frac{1}{2} \times 10^4 \times 10^{-2}$$

or,

$$\boxed{U_{\max} = 50\ \text{ergs}} \tag{2}$$

3-4.

a) *Time average:*

The position and velocity for a simple harmonic oscillator are given by

$$x = A\sin\omega_0 t \tag{1}$$

$$\dot{x} = \omega_0\, A\cos\omega_0 t \tag{2}$$

where $\omega_0 = \sqrt{k/m}$

The time average of the kinetic energy is

$$\langle T \rangle = \frac{1}{\tau}\int_t^{t+\tau} \frac{1}{2}m\dot{x}^2\, dt \tag{3}$$

where $\tau = \dfrac{2\pi}{\omega_0}$ is the period of oscillation.

By inserting (2) into (3), we obtain

$$\langle T \rangle = \frac{1}{2\tau} mA^2\omega_0^2 \int_t^{t+\tau} \cos^2 \omega_0 t \, dt \tag{4}$$

or,

$$\boxed{\langle T \rangle = \frac{mA^2\omega_0^2}{4}} \tag{5}$$

In the same way, the time average of the potential energy is

$$\begin{aligned}\langle U \rangle &= \frac{1}{\tau} \int_t^{t+\tau} \frac{1}{2} kx^2 \, dt \\ &= \frac{1}{2\tau} kA^2 \int_t^{t+\tau} \sin^2 \omega_0 t \, dt \\ &= \frac{kA^2}{4} \end{aligned} \tag{6}$$

and since $\omega_0^2 = k/m$, (6) reduces to

$$\boxed{\langle U \rangle = \frac{mA^2\omega_0^2}{4}} \tag{7}$$

From (5) and (7) we see that

$$\boxed{\langle T \rangle = \langle U \rangle} \tag{8}$$

The result stated in (8) is reasonable to expect from the conservation of the total energy.

$$E = T + U \tag{9}$$

This equality is valid instantaneously, as well as in the average. On the other hand, when *T* and *U* are expressed by (1) and (2), we notice that they are described by exactly the same function, displaced by a time $\tau/2$:

$$\left.\begin{aligned} T &= \frac{mA^2\omega_0^2}{2} \cos^2 \omega_0 t \\ U &= \frac{mA^2\omega_0 t}{2} \sin^2 \omega_0 t \end{aligned}\right] \tag{10}$$

Therefore, the time averages of *T* and *U* must be equal. Then, by taking time average of (9), we find

$$\langle T \rangle = \langle U \rangle = \frac{E}{2} \tag{11}$$

b) *Space average:*

The space averages of the kinetic and potential energies are

$$\overline{T} = \frac{1}{A}\int_0^A \frac{1}{2} m\dot{x}^2 \, dx \tag{12}$$

and

$$\overline{U} = \frac{1}{A}\int_0^A \frac{1}{2} kx^2 \, dx = \frac{m\omega_0^2}{2A}\int_0^A x^2 \, dx \tag{13}$$

(13) is readily integrated to give

$$\boxed{\overline{U} = \frac{m\omega_0^2 A^2}{6}} \tag{14}$$

To integrate (12), we notice that from (1) and (2) we can write

$$\dot{x}^2 = \omega_0^2 A^2 \cos^2 \omega_0 t = \omega_0^2 A^2 \left(1 - \sin^2 \omega_0 t\right)$$

$$= \omega_0^2 \left(A^2 - x^2\right) \tag{15}$$

Then, substituting (15) into (12), we find

$$\overline{T} = \frac{m\omega_0^2}{2A}\int_0^A \left[A^2 - x^2\right] dx$$

$$= \frac{m\omega_0^2}{2A}\left[A^3 - \frac{A^3}{3}\right] \tag{16}$$

or,

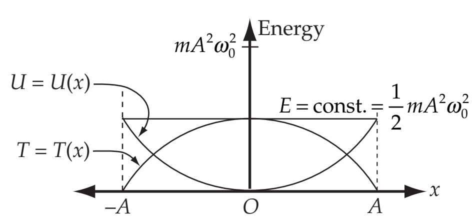
$$\boxed{\overline{T} = 2\,\frac{m\omega_0^2 A^2}{6}} \tag{17}$$

From the comparison of (14) and (17), we see that

$$\boxed{\overline{T} = 2\overline{U}} \tag{18}$$

To see that this result is reasonable, we plot $T = T(x)$ and $U = U(x)$:

$$\left.\begin{aligned} T &= \frac{1}{2} m\omega_0^2 A^2 \left[1 - \frac{x^2}{A^2}\right] \\ U &= \frac{1}{2} m\omega_0^2 x^2 \end{aligned}\right] \tag{19}$$



And the area between $T(x)$ and the x-axis is just twice that between $U(x)$ and the x-axis.

3-5. Differentiating the equation of motion for a simple harmonic oscillator,

$$x = A \sin \omega_0 t \tag{1}$$

we obtain

$$\Delta x = A\omega_0 \cos \omega_0 t \; \Delta t \tag{2}$$

But from (1)

$$\sin \omega_0 t = \frac{x}{A} \tag{3}$$

Therefore,

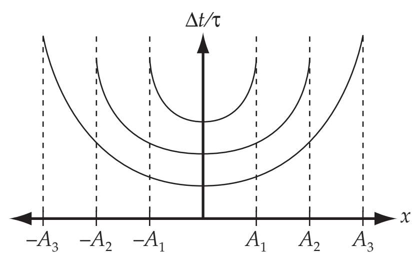
$$\cos \omega_0 t = \sqrt{1 - (x/A)^2} \tag{4}$$

and substitution into (2) yields

$$\Delta t = \frac{\Delta x}{\omega_0 \sqrt{A^2 - x^2}} \tag{5}$$

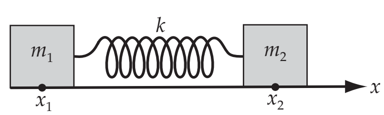
Then, the fraction of a complete period that a simple harmonic oscillator spends within a small interval Δx at position x is given by

$$\frac{\Delta t}{\tau} = \frac{\Delta x}{\omega_0 \tau \sqrt{A^2 - x^2}} = \frac{\Delta x}{2\pi \sqrt{A^2 - x^2}} \tag{6}$$



This result implies that the harmonic oscillator spends most of its time near $x = \pm A$, which is obviously true. On the other hand, we obtain a singularity for $\Delta t/\tau$ at $x = \pm A$. This occurs because at these points $\dot{x} = 0$, and (2) is not valid.

3-6.



Suppose the coordinates of m_1 and m_2 are x_1 and x_2 and the length of the spring at equilibrium is ℓ. Then the equations of motion for m_1 and m_2 are

$$m_1 \ddot{x}_1 = -k(x_1 - x_2 + \ell) \tag{1}$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1 + \ell) \tag{2}$$

From (2), we have

$$x_1 = \frac{1}{k}\left(m_2\ddot{x}_2 + kx_2 - k\ell\right) \tag{3}$$

Substituting this expression into (1), we find

$$\frac{d^2}{dt^2}\left[m_1 m_2 \ddot{x}_2 + \left(m_1 + m_2\right)kx_2\right] = 0 \tag{4}$$

from which

$$\ddot{x}_2 = -\frac{m_1 + m_2}{m_1 m_2} kx_2 \tag{5}$$

Therefore, x_2 oscillates with the frequency

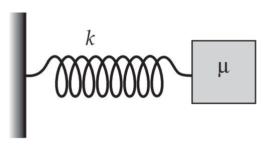
$$\boxed{\omega = \sqrt{\frac{m_1 + m_2}{m_1 m_2} k}} \tag{6}$$

We obtain the same result for x_1. If we notice that the reduced mass of the system is defined as

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{7}$$

we can rewrite (6) as

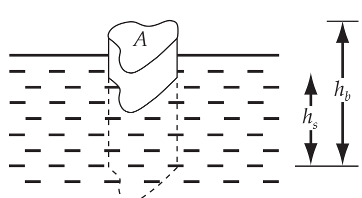
$$\omega = \sqrt{\frac{k}{\mu}} \tag{8}$$



This means the system oscillates in the same way as a system consisting of a single mass μ.

Inserting the given values, we obtain $\mu \simeq 66.7$ g and $\omega \simeq 2.74\ \mathrm{rad \cdot s^{-1}}$.

3-7.



Let A be the cross-sectional area of the floating body, h_b its height, h_s the height of its submerged part; and let ρ and ρ_0 denote the mass densities of the body and the fluid, respectively.

The volume of displaced fluid is therefore $V = Ah_s$. The mass of the body is $M = \rho A h_b$.

There are two forces acting on the body: that due to gravity (Mg), and that due to the fluid, pushing the body up ($-\rho_0 gV = -\rho_0 gh_s A$).

The equilibrium situation occurs when the total force vanishes:

$$\begin{aligned} 0 &= Mg - \rho_0 gV \\ &= \rho gAh_b - \rho_0 gh_s A \end{aligned} \tag{1}$$

which gives the relation between h_s and h_b:

$$h_s = h_b \frac{\rho}{\rho_0} \tag{2}$$

For a small displacement about the equilibrium position ($h_s \to h_s + x$), (1) becomes

$$M\ddot{x} = \rho Ah_b\ddot{x} = \rho gAh_b - \rho_0 g\left(h_s + x\right)A \tag{3}$$

Upon substitution of (1) into (3), we have

$$\rho Ah_b\ddot{x} = -\rho_0 gxA \tag{4}$$

or,

$$\ddot{x} + g\frac{\rho_0}{\rho h_b}x = 0 \tag{5}$$

Thus, the motion is oscillatory, with an angular frequency

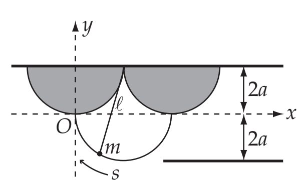
$$\omega^2 = g\frac{\rho_0}{\rho h_b} = \frac{g}{h_s} = \frac{gA}{V} \tag{6}$$

where use has been made of (2), and in the last step we have multiplied and divided by A. The period of the oscillations is, therefore,

$$\boxed{\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{V}{gA}}} \tag{7}$$

Substituting the given values, $\tau \simeq 0.18$ s.

3-8.



The force responsible for the motion of the pendulum bob is the component of the gravitational force on m that acts perpendicular to the straight portion of the suspension string. This component is seen, from the figure (a) below, to be

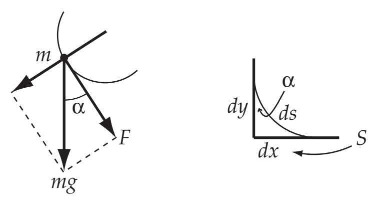
$$F = ma = m\dot{v} = -mg\cos\alpha \tag{1}$$

where α is the angle between the vertical and the tangent to the cycloidal path at the position of m. The cosine of α is expressed in terms of the differentials shown in the figure (b) as

$$\cos\alpha = \frac{dy}{ds} \tag{2}$$

where

$$ds = \sqrt{dx^2 + dy^2} \tag{3}$$



(a) **(b)**

The differentials, dx and dy, can be computed from the defining equations for $x(\phi)$ and $y(\phi)$ above:

$$\left.\begin{aligned} dx &= a(1-\cos\phi)\,d\phi \\ dy &= -a\sin\phi\,d\phi \end{aligned}\right] \tag{4}$$

Therefore,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= a^2\left[(1-\cos\phi)^2 + \sin^2\phi\right]d\phi^2 = 2a^2(1-\cos\phi)\,d\phi^2 \\ &= 4a^2\sin^2\frac{\phi}{2}\,d\phi^2 \end{aligned} \tag{5}$$

so that

$$ds = 2a\sin\frac{\phi}{2}\,d\phi \tag{6}$$

Thus,

$$\begin{aligned} \frac{dy}{ds} &= \frac{-a\sin\phi\,d\phi}{2a\sin\frac{\phi}{2}\,d\phi} \\ &= -\cos\frac{\phi}{2} = \cos\alpha \end{aligned} \tag{7}$$

The velocity of the pendulum bob is

$$v = \frac{ds}{dt} = 2a \sin\frac{\phi}{2}\frac{d\phi}{dt}$$

$$= -4a\frac{d}{dt}\left[\cos\frac{\phi}{2}\right] \tag{8}$$

from which

$$\dot{v} = -4a\frac{d^2}{dt^2}\left[\cos\frac{\phi}{2}\right] \tag{9}$$

Letting $z \equiv \cos\dfrac{\phi}{2}$ be the new variable, and substituting (7) and (9) into (1), we have

$$-4ma\ddot{z} = mgz \tag{10}$$

or,

$$\ddot{z} + \frac{g}{4a}z = 0 \tag{11}$$

which is the standard equation for simple harmonic motion,

$$\ddot{z} + \omega_0^2 z = 0 \tag{12}$$

If we identify

$$\boxed{\omega_0 = \sqrt{\frac{g}{\ell}}} \tag{13}$$

where we have used the fact that $\ell = 4a$.

Thus, the motion is exactly isochronous, independent of the amplitude of the oscillations. This fact was discovered by Christian Huygene (1673).

3-9. The equation of motion for $0 \le t \le t_0$ is

$$m\ddot{x} = -k\left(x - x_0\right) + F = -kx + \left(F + kx_0\right) \tag{1}$$

while for $t \ge t_0$, the equation is

$$m\ddot{x} = -k\left(x - x_0\right) = -kx + kx_0 \tag{2}$$

It is convenient to define

$$\xi = x - x_0$$

which transforms (1) and (2) into

$$m\ddot{\xi} = -k\xi + F\,; \quad 0 \le t \le t_0 \tag{3}$$

$$m\ddot{\xi} = -k\xi\,; \quad t \ge t_0 \tag{4}$$

The homogeneous solutions for both (3) and (4) are of familiar form $\xi(t) = Ae^{i\omega t} + Be^{-i\omega t}$, where $\omega = \sqrt{k/m}$. A particular solution for (3) is $\xi = F/k$. Then the general solutions for (3) and (4) are

$$\xi_- = \frac{F}{k} + Ae^{i\omega t} + Be^{-i\omega t}; \quad 0 \le t \le t_0 \tag{5}$$

$$\xi_+ = Ce^{i\omega t} + De^{-i\omega t}; \quad t \ge t_0 \tag{6}$$

To determine the constants, we use the initial conditions: $x(t=0) = x_0$ and $x(t = 0) = 0$. Thus,

$$\xi_-(t=0) = \dot{\xi}_-(t=0) = 0 \tag{7}$$

The conditions give two equations for A and B:

$$\left.\begin{aligned} 0 &= \frac{F}{k} + A + B \\ 0 &= i\omega(A - B) \end{aligned}\right] \tag{8}$$

Then

$$A = B = -\frac{F}{2k}$$

and, from (5), we have

$$\xi_- = x - x_0 = \frac{F}{k}(1 - \cos\omega t); \quad 0 \le t \le t_0 \tag{9}$$

Since for any physical motion, x and $\dot{x}$ must be continuous, the values of $\xi_-(t = t_0)$ and $\dot{\xi}_-(t = t_0)$ are the initial conditions for $\xi_+(t)$ which are needed to determine C and D:

$$\left.\begin{aligned} \xi_+(t=t_0) &= \frac{F}{k}(1-\cos\omega t_0) = Ce^{i\omega t_0} + De^{-i\omega t_0} \\ \dot{\xi}_+(t=t_0) &= \frac{F}{k}\omega\sin\omega t_0 = i\omega\left[Ce^{i\omega t_0} - De^{-i\omega t_0}\right] \end{aligned}\right] \tag{10}$$

The equations in (10) can be rewritten as:

$$\left.\begin{aligned} Ce^{i\omega t_0} + De^{-i\omega t_0} &= \frac{F}{k}(1-\cos\omega t_0) \\ Ce^{i\omega t_0} - De^{-i\omega t_0} &= \frac{-iF}{k}\sin\omega t_0 \end{aligned}\right] \tag{11}$$

Then, by adding and subtracting one from the other, we obtain

$$\left.\begin{aligned} C &= \frac{F}{2k}e^{-i\omega t_0}\left(1 - e^{i\omega t_0}\right) \\ D &= \frac{F}{2k}e^{i\omega t_0}\left(1 - e^{-i\omega t_0}\right) \end{aligned}\right] \tag{12}$$

Substitution of (12) into (6) yields

$$\xi_+ = \frac{F}{2k}\left[\left(e^{-i\omega t_0} - 1\right)e^{i\omega t} + \left(e^{i\omega t_0} - 1\right)e^{-i\omega t}\right]$$

$$= \frac{F}{2k}\left[e^{i\omega(t-t_0)} - e^{i\omega t} + e^{-i\omega(t-t_0)} - e^{-i\omega t}\right]$$

$$= \frac{F}{k}\left[\cos\omega\left(t-t_0\right) - \cos\omega t\right] \tag{13}$$

Thus,

$$\boxed{x - x_0 = \frac{F}{k}\left[\cos\omega\left(t-t_0\right) - \cos\omega t\right];\ t \geq t_0} \tag{14}$$

3-10. The amplitude of a damped oscillator is expressed by

$$x(t) = Ae^{-\beta t}\cos\left(\omega_1 t + \delta\right) \tag{1}$$

Since the amplitude decreases to $1/e$ after n periods, we have

$$\beta n T = \beta n\frac{2\pi}{\omega_1} = 1 \tag{2}$$

Substituting this relation into the equation connecting ω_1 and ω_0 (the frequency of undamped oscillations), $\omega_1^2 = \omega_0^2 - \beta^2$, we have

$$\omega_0^2 = \omega_1^2 + \left[\frac{\omega_1}{2\pi n}\right]^2 = \omega_1^2\left[1 + \frac{1}{4\pi^2 n^2}\right] \tag{3}$$

Therefore,

$$\frac{\omega_1}{\omega_0} = \left[1 + \frac{1}{4\pi^2 n^2}\right]^{-1/2} \tag{4}$$

so that

$$\boxed{\frac{\omega_1}{\omega_2} \cong 1 - \frac{1}{8\pi^2 n^2}}$$

3-11. The total energy of a damped oscillator is

$$E(t) = \frac{1}{2}m\dot{x}(t)^2 + \frac{1}{2}kx(t)^2 \tag{1}$$

where

$$x(t) = Ae^{-\beta t}\cos\left(\omega_1 t - \delta\right) \tag{2}$$

$$\dot{x}(t) = Ae^{-\beta t}\left[-\beta\cos\left(\omega_1 t - \delta\right) - \omega_1\sin\left(\omega_1 t - \delta\right)\right] \tag{3}$$

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2}\ , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

Substituting (2) and (3) into (1), we have

$$E(t) = \frac{A^2}{2} e^{-2\beta t} \Big[\left(m\beta^2 + k \right) \cos^2\left(\omega_1 t - \delta \right) + m\omega_1^2 \sin^2\left(\omega_1 t - \delta \right) + 2m\beta\omega_1 \sin\left(\omega_1 t - \delta \right) \cos\left(\omega_1 t - \delta \right) \Big] \tag{4}$$

Rewriting (4), we find the expression for $E(t)$:

$$\boxed{E(t) = \frac{mA^2}{2} e^{-2\beta t} \left[\beta^2 \cos 2\left(\omega_1 t - \delta \right) + \beta \sqrt{\omega_0^2 - \beta^2} \sin 2\left(\omega_1 t - \delta \right) + \omega_0^2 \right]} \tag{5}$$

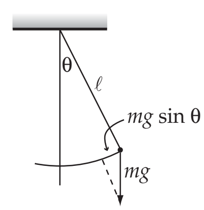
Taking the derivative of (5), we find the expression for $\dfrac{dE}{dt}$:

$$\boxed{\frac{dE}{dt} = \frac{mA^2}{2} e^{-2\beta t} \Big[\left(2\beta\omega_0^2 - 4\beta^3 \right) \cos 2\left(\omega_1 t - \delta \right) - 4\beta^2 \sqrt{\omega_0^2 - \beta^2} \sin 2\left(\omega_1 t - \delta \right)_0 - 2\beta\omega^2 \Big]} \tag{6}$$

The above formulas for E and dE/dt reproduce the curves shown in Figure 3-7 of the text. To find the average rate of energy loss for a lightly damped oscillator, let us take $\beta \ll \omega_0$. This means that the oscillator has time to complete some number of periods before its amplitude decreases considerably, i.e. the term $e^{-2\beta t}$ does not change much in the time it takes to complete one period. The cosine and sine terms will average to nearly zero compared to the constant term in dE/dt, and we obtain in this limit

$$\frac{dE}{dt} \simeq -m\beta\omega_0^2 A^2 e^{-2\beta t} \tag{7}$$

3-12.



The equation of motion is

$$-m\ell\ddot{\theta} = mg \sin \theta \tag{1}$$

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta \tag{2}$$

If θ is sufficiently small, we can approximate $\sin \theta \cong \theta$, and (2) becomes

$$\ddot{\theta} = -\frac{g}{\ell}\,\theta \tag{3}$$

which has the oscillatory solution

$$\theta(t) = \theta_0 \cos \omega_0 t \tag{4}$$

where $\omega_0 = \sqrt{g/\ell}$ and where θ_0 is the amplitude. If there is the retarding force $2m\sqrt{g\ell}\,\dot{\theta}$, the equation of motion becomes

$$-m\ell\ddot{\theta} = mg \sin\theta + 2m\sqrt{g\ell}\,\dot{\theta} \tag{5}$$

or setting $\sin\theta \cong \theta$ and rewriting, we have

$$\ddot{\theta} + 2\omega_0\dot{\theta} + \omega_0^2\theta = 0 \tag{6}$$

Comparing this equation with the standard equation for damped motion [Eq. (3.35)],

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{7}$$

we identify $\omega_0 = \beta$. This is just the case of *critical damping*, so the solution for $\theta(t)$ is [see Eq. (3.43)]

$$\theta(t) = (A + Bt)e^{-\omega_0 t} \tag{8}$$

For the initial conditions $\theta(0) = \theta_0$ and $\theta(0) = 0$, we find

$$\boxed{\theta(t) = \theta_0(1 + \omega_0 t)e^{-\omega_0 t}}$$

3-13. For the case of critical damping, $\beta = \omega_0$. Therefore, the equation of motion becomes

$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0 \tag{1}$$

If we assume a solution of the form

$$x(t) = y(t)e^{-\beta t} \tag{2}$$

we have

$$\left.\begin{aligned} \dot{x} &= \dot{y}e^{-\beta t} - \beta y e^{-\beta t} \\ \ddot{x} &= \ddot{y}e^{-\beta t} - 2\beta\dot{y}e^{-\beta t} + \beta^2 y e^{-\beta t} \end{aligned}\right] \tag{3}$$

Substituting (3) into (1), we find

$$\ddot{y}e^{-\beta t} - 2\beta\dot{y}e^{-\beta t} + \beta^2 y e^{-\beta t} + 2\beta\dot{y}e^{-\beta t} - 2\beta^2 y e^{-\beta t} + \beta^2 y e^{-\beta t} = 0 \tag{4}$$

or,

$$\ddot{y} = 0 \tag{5}$$

Therefore,

$$y(t) = A + Bt \tag{6}$$

and

$$\boxed{x(t) = (A + Bt)e^{-\beta t}} \tag{7}$$

which is just Eq. (3.43).

3-14. For the case of overdamped oscillations, $x(t)$ and $\dot{x}(t)$ are expressed by

$$x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] \tag{1}$$

$$\dot{x}(t)e^{-\beta t}\left[-\beta\left(A_1 e^{\omega_2 t} + +A_2 e^{-\omega_2 t}\right) + \left(A_1\omega_2 e^{\omega_2 t} - A_2\omega_2 e^{-\omega_2 t}\right)\right] \tag{2}$$

where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$. Hyperbolic functions are defined as

$$\cosh y = \frac{e^y + e^{-y}}{2}, \quad \sinh y = \frac{e^y - e^{-y}}{2} \tag{3}$$

or,

$$\left.\begin{aligned} e^y &= \cosh y + \sinh y \\ e^{-y} &= \cosh y - \sinh y \end{aligned}\right] \tag{4}$$

Using (4) to rewrite (1) and (2), we have

$$\boxed{x(t) = \left(\cosh \beta t - \sinh \beta t\right)\left[\left(A_1 + A_2\right)\cosh \omega_2 t + \left(A_1 - A_2\right)\sinh \omega_2 t\right]} \tag{5}$$

and

$$\boxed{\begin{aligned} \dot{x}(t) = \left(\cosh \beta t - \sinh \beta t\right)\Big[&\left(A_1\omega_2 - A_1\beta\right)\left(\cosh \omega_2 t + \sinh \omega_2 t\right) \\ &-\left(A_2\beta + A_2\omega_2\right)\left(\cosh \omega_2 t - \sinh \omega_2 t\right)\Big] \end{aligned}} \tag{6}$$

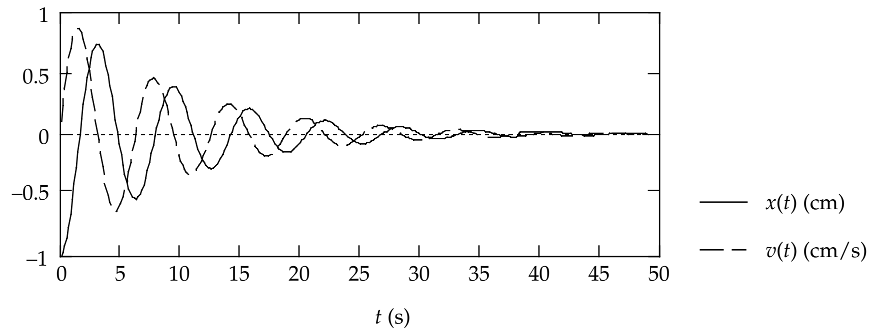
3-15. We are asked to simply plot the following equations from Example 3.2:

$$x(t) = Ae^{-\beta t}\cos\left(\omega_1 t - \delta\right) \tag{1}$$

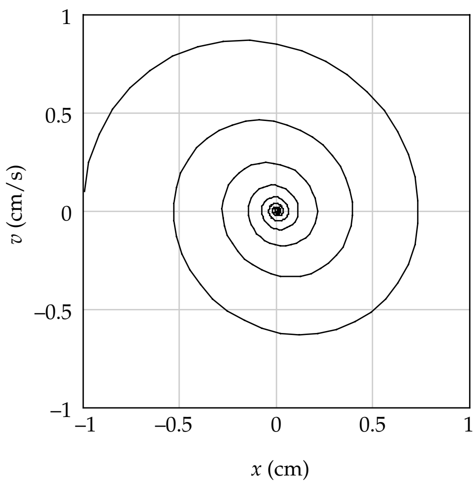
$$v(t) = -Ae^{-\beta t}\left[\beta\cos\left(\omega_1 t - \delta\right) + \omega_1 \sin\left(\omega_1 t - \delta\right)\right] \tag{2}$$

with the values $A = 1\text{ cm}$, $\omega_0 = 1\text{ rad}\cdot\text{s}^{-1}$, $\beta = 0.1\text{ s}^{-1}$, and $\delta = \pi$ rad. The position goes through $x = 0$ a total of 15 times before dropping to 0.01 of its initial amplitude. An exploded (or zoomed) view of figure (b), shown here as figure (B), is the best for determining this number, as is easily shown.

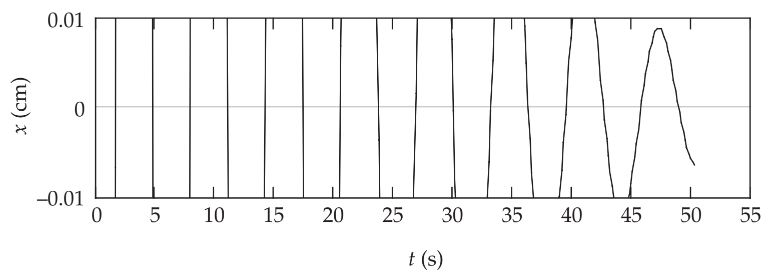
(b)



(c)



(B)



3-16. If the damping resistance b is negative, the equation of motion is

$$\ddot{x} - 2\beta\dot{x} + \omega_0^2 x = 0 \tag{1}$$

where $\beta \equiv -b/2m > 0$ because $b < 0$. The general solution is just Eq. (3.40) with β changed to $-\beta$:

$$x(t) = e^{\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}\ t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}\ t\right)\right] \tag{2}$$

From this equation, we see that the motion is not bounded, irrespective of the relative values of β^2 and ω_0^2.

The three cases distinguished in Section 3.5 now become:

a) *If* $\omega_0^2 > \beta^2$, the motion consists of an oscillatory solution of frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, multiplied by an ever-increasing exponential:

$$x(t) = e^{\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right] \tag{3}$$

b) *If* $\omega_0^2 = \beta^2$, the solution is

$$x(t) = (A + Bt)e^{\beta t} \tag{4}$$

which again is ever-increasing.

c) *If* $\omega_0^2 < \beta^2$, the solution is:

$$x(t) = e^{\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] \tag{5}$$

where

$$\omega_2 = \sqrt{\beta^2 - \omega_0^2} \le \beta \tag{6}$$

This solution also increases continuously with time.

The tree cases describe motions in which the particle is either always moving away from its initial position, as in cases b) or c), or it is oscillating around its initial position, but with an amplitude that grows with the time, as in a).

Because $b < 0$, the medium in which the particle moves continually gives energy to the particle and the motion grows without bound.

3-17. For a damped, driven oscillator, the equation of motion is

$$\ddot{x} = 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t \tag{1}$$

and the average kinetic energy is expressed as

$$\langle T\rangle = \frac{mA^2}{4}\frac{\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2} \tag{2}$$

Let the frequency n octaves above ω_0 be labeled ω_1 and let the frequency n octaves below ω_0 be labeled ω_2; that is

$$\left.\begin{aligned}\omega_1 &= 2^n\ \omega_0\\ \omega_2 &= 2^{-n}\ \omega_0\end{aligned}\right. \tag{3}$$

The average kinetic energy for each case is

$$\langle T\rangle_{\omega_1} = \frac{mA^2}{4}\frac{2^{2n}\omega_0^2}{\left(\omega_0^2 - 2^{2n}\omega_0^2\right)^2 + (4)2^{2n}\omega_0^2\beta^2} \tag{4}$$

$$\langle T\rangle_{\omega_2} = \frac{mA^2}{4}\frac{2^{-2n}\omega_0^2}{\left(\omega_0^2 - 2^{-2n}\omega_0^2\right)^2 + (4)2^{-2n}\omega_0^2\beta^2} \tag{5}$$

Multiplying the numerator and denominator of (5) by 2^{4n}, we have

$$\langle T\rangle_{\omega_2}=\frac{mA^2}{4}\frac{2^{2n}\omega_0^2}{\left(\omega_0^2-2^{2n}\omega_0^2\right)^2+(4)2^{2n}\omega_0^2\beta^2}$$

Hence, we find

$$\boxed{\langle T\rangle_{\omega_1}=\langle T\rangle_{\omega_2}} \tag{6}$$

and the proposition is proven.

3-18. Since we are near resonance and there is only light damping, we have $\omega_0 \simeq \omega_R \simeq \omega$, where ω is the driving frequency. This gives $Q \simeq \omega_0/2\beta$. To obtain the total energy, we use the solution to the driven oscillator, neglecting the transients:

$$x(t)=D\cos(\omega t-\delta) \tag{1}$$

We then have

$$E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2=\frac{mD^2}{2}\left[\omega^2\sin^2(\omega t-\delta)+\omega_0^2\cos^2(\omega t-\delta)\right]\simeq\frac{1}{2}m\omega_0^2D^2 \tag{2}$$

The energy lost over one period is

$$\int_0^T(2m\beta\dot{x})\cdot(\dot{x}dt)=2\pi m\omega\beta D^2 \tag{3}$$

where $T=2\pi/\omega$. Since $\omega\simeq\omega_0$, we have

$$\frac{E}{\text{energy lost over one period}}\simeq\frac{\omega_0}{4\pi\beta}\simeq\frac{Q}{2\pi} \tag{4}$$

which proves the assertion.

3-19. The amplitude of a damped oscillator is [Eq. (3.59)]

$$D=\frac{A}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\omega^2\beta^2}} \tag{1}$$

At the resonance frequency, $\omega=\omega_R=\sqrt{\omega_0^2-\beta^2}$, D becomes

$$D_R=\frac{A}{2\beta\sqrt{\omega_0^2-\beta^2}} \tag{2}$$

Let us find the frequency, $\omega=\omega'$, at which the amplitude is $\dfrac{1}{\sqrt{2}}D_R$:

$$\frac{1}{\sqrt{2}}D_R=\frac{1}{\sqrt{2}}\frac{A}{2\beta\sqrt{\omega_0^2-\beta^2}}=\frac{A}{\sqrt{\left(\omega_0^2-\omega'^2\right)^2+4\omega'^2\beta^2}} \tag{3}$$

Solving this equation for ω', we find

$$\omega'^2 = \omega_0^2 - 2\beta^2 \pm 2\beta\omega_0 \left[1 - \frac{\beta^2}{\omega_0^2}\right]^{1/2} \tag{4}$$

For a lightly damped oscillator, β is small and the terms in β^2 can be neglected. Therefore,

$$\omega'^2 \cong \omega_0^2 \pm 2\beta\omega_0 \tag{5}$$

or,

$$\omega' \cong \omega_0 \left[1 \pm \frac{\beta}{\omega_0}\right] \tag{6}$$

which gives

$$\Delta\omega = (\omega_0 + \beta) - (\omega_0 - \beta) = 2\beta \tag{7}$$

We also can approximate ω_R for a lightly damped oscillator:

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2} \cong \omega_0 \tag{8}$$

Therefore, Q for a lightly damped oscillator becomes

$$\boxed{Q \cong \frac{\omega_0}{2\beta} \cong \frac{\omega_0}{\Delta\omega}} \tag{9}$$

3-20. From Eq. (3.66),

$$\dot{x} = \frac{-A\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \sin(\omega t - \delta) \tag{1}$$

Therfore, the absolute value of the velocity amplitude v is given by

$$v_0 = \frac{A\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \tag{2}$$

The value of ω for v_0 a maximum, which is labeled ω_v, is obtained from

$$\left.\frac{\partial v_0}{\partial \omega}\right|_{\omega=\omega_v} = 0 \tag{3}$$

and the value is $\omega_v = \omega_0$.

Since the Q of the oscillator is equal to 6, we can use Eqs. (3.63) and (3.64) to express β in terms of ω_0:

$$\beta^2 = \frac{\omega_0^2}{146} \tag{4}$$

We need to find two frequencies, ω_1 and ω_2, for which $v_0 = v_{\max}/\sqrt{2}$, where $v_{\max} = v_0(\omega = \omega_0)$. We find

$$\frac{v_{\max}}{\sqrt{2}} = \frac{A}{2\sqrt{2}\beta} = \frac{A\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \tag{5}$$

Substituting for β in terms of ω_0 from (4), and by squaring and rearranging terms in (5), we obtain

$$\left(\omega_0^2 - \omega_{1,2}^2\right)^2 - \frac{2}{73}\left(\omega_{1,2}^2\ \omega_0^2\right) = 0 \tag{6}$$

from which

$$\omega_0^2 - \omega_{1,2}^2 = \pm\sqrt{\frac{2}{73}}\ \omega_{1,2}\omega_0 \cong \pm\frac{1}{6}\ \omega_{1,2}\omega_0 \tag{7}$$

Solving for ω_1, ω_2 we obtain

$$\omega_{1,2} \cong \left[\pm\frac{\omega_0}{12}\right] \pm \omega_0 \tag{8}$$

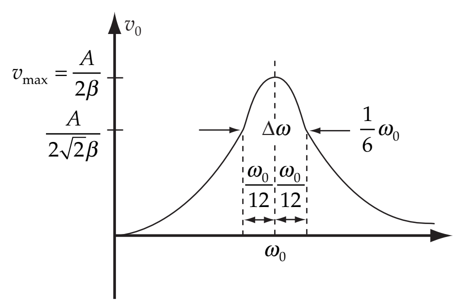
It is sufficient for our purposes to consider ω_1, ω_2 *positive*: then

$$\omega_1 \cong \frac{\omega_0}{12} + \omega_0\,; \qquad \omega_2 \cong -\frac{\omega_0}{12} + \omega_0 \tag{9}$$

so that

$$\Delta\omega = \omega_1 - \omega_2 = \frac{\omega_0}{6} \tag{10}$$

A graph of v_0 vs. ω for $Q = 6$ is shown.



3-21. We want to plot Equation (3.43), and its derivative:

$$x(t) = (A + Bt)e^{-\beta t} \tag{1}$$

$$v(t) = \left[B - \beta(A + Bt)\right]e^{-\beta t} \tag{2}$$

where A and B can be found in terms of the initial conditions

$$A = x_0 \tag{3}$$

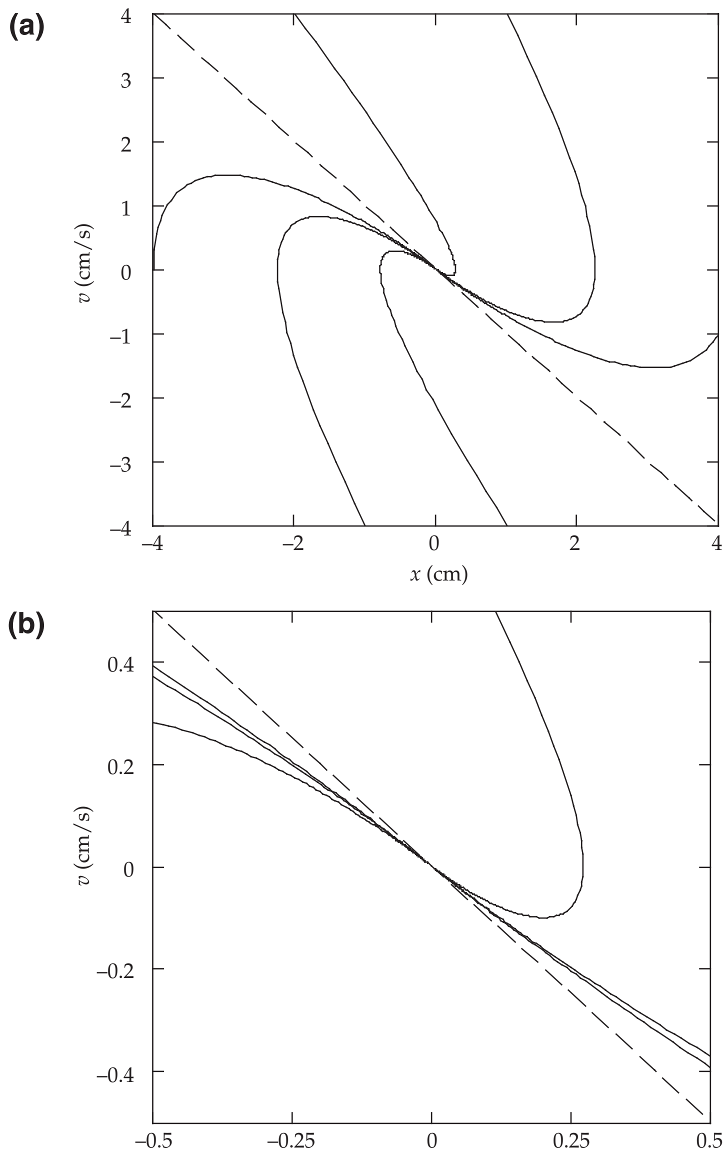
$$B = v_0 + \beta x_0 \tag{4}$$

The initial conditions used to produce figure (a) were $(x_0, v_0) = (-2, 4)$, $(1, 4)$, $(4, -1)$, $(1, -4)$, $(-1, -4)$, and $(-4, 0)$, where we take all x to be in cm, all v in $\text{cm} \cdot \text{s}^{-1}$, and $\beta = 1\ \text{s}^{-1}$. Figure (b) is a magnified view of figure (a). The dashed line is the path that all paths go to asymptotically as $t \to \infty$. This can be found by taking the limits.

$$\lim_{t\to\infty} v(t) = -\beta B t e^{-\beta t} \tag{5}$$

$$\lim_{t\to\infty} x(t) = B t e^{-\beta t} \tag{6}$$

so that in this limit, $v = -\beta x$, as required.



3-22. For overdamped motion, the position is given by Equation (3.44)

$$x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t} \tag{1}$$

The time derivative of the above equation is, of course, the velocity:

$$v(t) = -A_1\beta_1 e^{-\beta_1 t} - A_2\beta_2 e^{-\beta_2 t} \tag{2}$$

a) At $t = 0$:

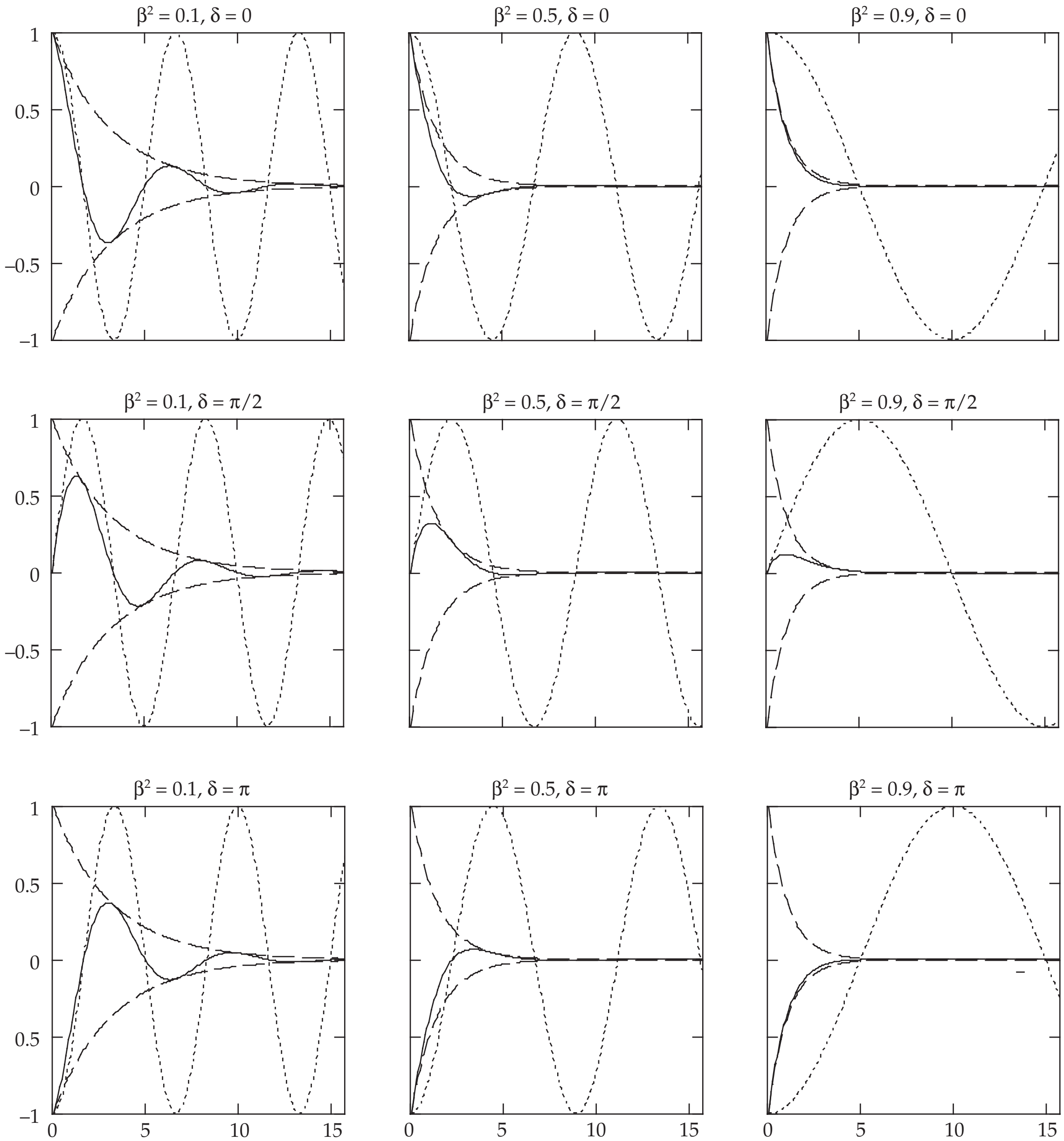
$$x_0 = A_1 + A_2 \tag{3}$$

$$v_0 = -A_1\beta_1 - A_2\beta_2 \tag{4}$$

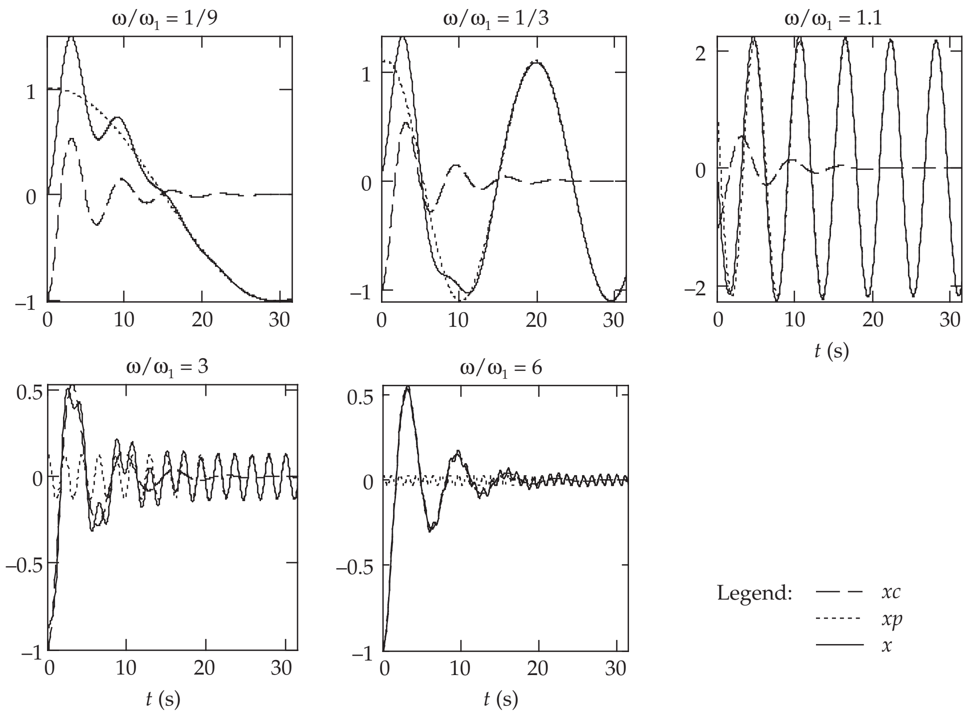
The initial conditions x_0 and v_0 can now be used to solve for the integration constants A_1 and A_2.

b) When $A_1 = 0$, we have $v_0 = -\beta_2 x_0$ and $v(t) = -\beta_2 x(t)$ quite easily. For $A_1 \neq 0$, however, we have $v(t) \to -\beta_1 A_1 e^{-\beta_1 t} = -\beta_1 x$ as $t \to \infty$ since $\beta_1 < \beta_2$.

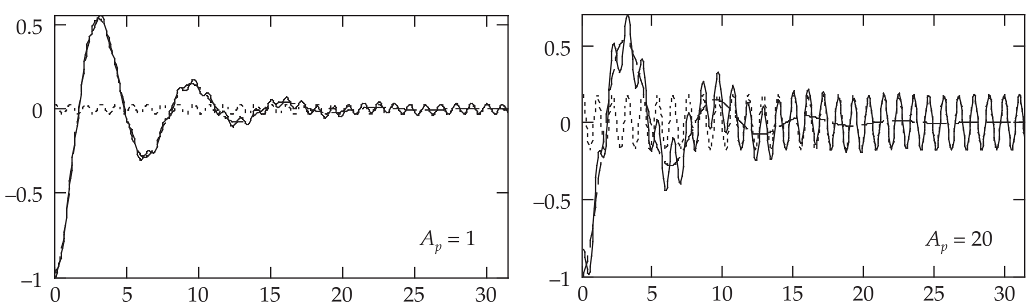
3-23. Firstly, we note that all the $\delta = \pi$ solutions are just the negative of the $\delta = 0$ solutions. The $\delta = \pi/2$ solutions don't make it all the way up to the initial "amplitude," A, due to the retarding force. Higher β means more damping, as one might expect. When damping is high, less oscillation is observable. In particular, $\beta^2 = 0.9$ would be much better for a kitchen door than a smaller β, e.g. the door closing ($\delta = 0$), or the closed door being bumped by someone who then changes his/her mind and does not go through the door ($\delta = \pi/2$).



3-24. As requested, we use Equations (3.40), (3.57), and (3.60) with the given values to evaluate the complementary and particular solutions to the driven oscillator. The amplitude of the complementary function is constant as we vary ω, but the amplitude of the particular solution becomes larger as ω goes through the resonance near $0.96 \text{ rad}\cdot\text{s}^{-1}$, and decreases as ω is increased further. The plot closest to resonance here has $\omega/\omega_1 = 1.1$, which shows the least distortion due to transients. These figures are shown in figure (a). In figure (b), the $\omega/\omega_1 = 6$ plot from figure (a) is reproduced along with a new plot with $A_p = 20 \text{ m}\cdot\text{s}^{-2}$.

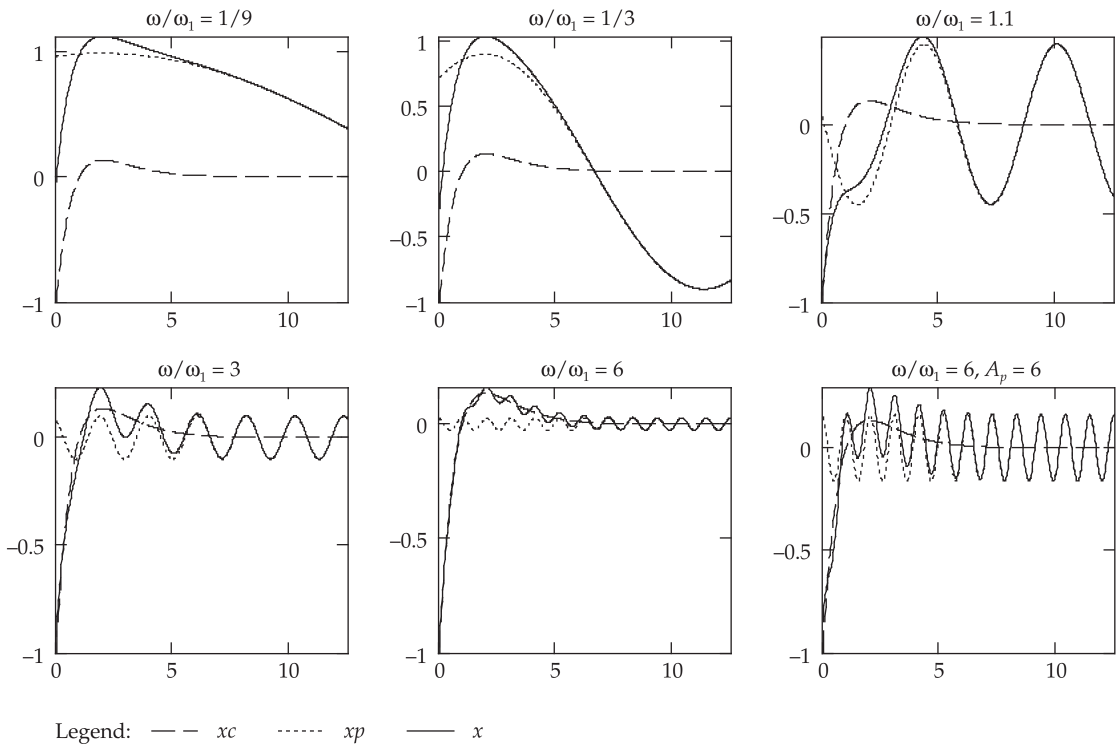


(a)



(b)

3-25. This problem is nearly identical to the previous problem, with the exception that now Equation (3.43) is used instead of (3.40) as the complementary solution. The distortion due to the transient increases as ω increases, mostly because the complementary solution has a fixed amplitude whereas the amplitude due to the particular solution only decreases as ω increases. The latter fact is because there is no resonance in this case.



3-26. The equations of motion of this system are

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -kx_1 - b_1\left(\dot{x}_1 - \dot{x}_2\right) + F\cos\omega t \\ m_2\ddot{x}_2 &= -b_2\dot{x}_2 - b_1\left(\dot{x}_2 - \dot{x}_1\right) \end{aligned}\right] \tag{1}$$

The electrical analog of this system can be constructed if we substitute in (1) the following equivalent quantities:

$$m_1 \to L_1; \quad k \to \frac{1}{C}; \quad b_1 \to R_1; \quad x \to q$$

$$m_2 \to L_2; \quad F \to \varepsilon_0; \quad b_2 \to R_2$$

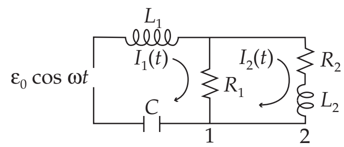
Then the equations of the equivalent electrical circuit are given by

$$\left.\begin{aligned} L_1\ddot{q}_1 + R_1\left(\dot{q}_1 - \dot{q}_2\right) + \frac{1}{C}q_1 &= \varepsilon_0\cos\omega t \\ L_2\ddot{q}_2 + R_2\dot{q}_2 + R_1\left(\dot{q}_2 - \dot{q}_1\right) &= 0 \end{aligned}\right] \tag{2}$$

Using the mathematical device of writing $\exp(i\omega t)$ instead of $\cos\,\omega t$ in (2), with the understanding that in the results only the real part is to be considered, and differentiating with respect to time, we have

$$\left.\begin{aligned} L_1\ddot{I}_1 + R_1\left(\dot{I}_1 - \dot{I}_2\right) + \frac{I_1}{C} &= i\omega\varepsilon_0 e^{i\omega t} \\ L_2\ddot{I}_2 + R_2\left(\dot{I}_2\right) + R_1\left(\dot{I}_2 - \dot{I}_1\right) &= 0 \end{aligned}\right] \tag{3}$$

Then, the equivalent electrical circuit is as shown in the figure:



The impedance of the system Z is

$$Z = i\omega L_1 - i\frac{1}{\omega C} + Z_1 \tag{4}$$

where Z_1 is given by

$$\frac{1}{Z_1} = \frac{1}{R_1} + \frac{1}{R_2 + i\omega L_2} \tag{5}$$

Then,

$$Z_1 = \frac{R_1\left[R_2\left(R_2 + R_1\right) + \omega^2 L_2^2 + i\omega L_2 R_1\right]}{\left(R_1 + R_2\right)^2 + \omega^2 L_2^2} \tag{6}$$

and substituting (6) into (4), we obtain

$$\boxed{Z = \frac{R_1\left[R_2\left(R_2 + R_1\right) + \omega^2 L_2^2\right] + i\left[R_1\omega L_2 + \left(\omega L_1 - \dfrac{1}{\omega C}\right)\left(\left(R_1 + R_2\right)^2 + \omega^2 L_2^2\right)\right]}{\left(R_1 + R_2\right)^2 + \omega^2 L_2^2}} \tag{7}$$

3-27. From Eq. (3.89),

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right) \tag{1}$$

We write

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos\left(n\omega t - \phi_n\right) \tag{2}$$

which can also be written using trigonometric relations as

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n\left[\cos n\omega t \cos\phi_n + \sin n\omega t \sin\phi_n\right] \tag{3}$$

Comparing (3) with (2), we notice that if there exists a set of coefficients c_n such that

$$\left.\begin{aligned} c_n \cos\phi_n &= a_n \\ c_n \sin\phi_n &= b_n \end{aligned}\right] \tag{4}$$

then (2) is equivalent to (1). In fact, from (4),

$$\left.\begin{aligned} c_n^2 &= a_n^2 + b_n^2 \\ \tan\phi_n &= \frac{b_n}{a_n} \end{aligned}\right] \tag{5}$$

with a_n and b_n as given by Eqs. (3.91).

3-28. Since $F(t)$ is an *odd* function, $F(-t) = -F(t)$, according to Eq. (3.91) all the coefficients a_n vanish identically, and the b_n are given by

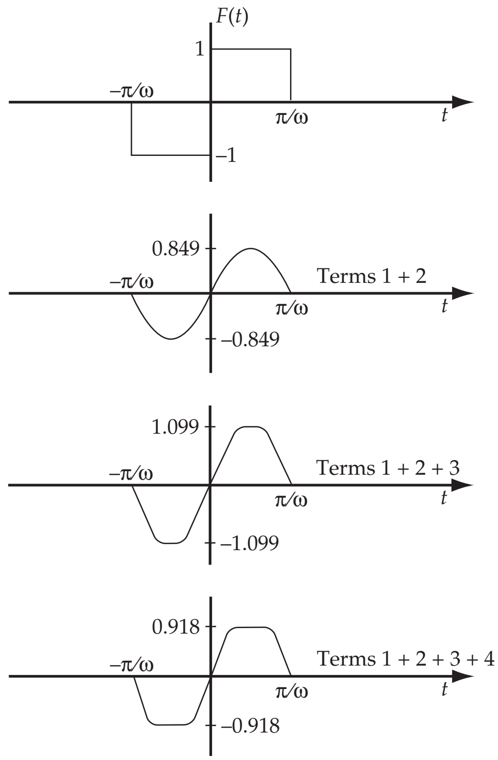
$$\begin{aligned} b_n &= \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} F(t')\sin n\omega t'\, dt' \\ &= \frac{\omega}{\pi}\left[-\int_{-\frac{\pi}{\omega}}^{0}\sin n\omega t'\, dt' + \int_0^{\frac{\pi}{\omega}}\sin n\omega t'\, dt'\right] \\ &= \frac{\omega}{\pi}\left[\left[\frac{1}{n\omega}\cos n\omega t'\right]_{-\frac{\pi}{\omega}}^{0} + \left[-\frac{1}{n\omega}\cos n\omega t'\right]_0^{\frac{\pi}{\omega}}\right] \\ &= \frac{2}{n\pi}\left(\cos 0 - \cos n\pi\right) \\ &= \left[\begin{array}{l} \dfrac{4}{n\pi} \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even} \end{array}\right. \end{aligned} \tag{1}$$

Thus,

$$\left.\begin{aligned} b_{(2n+1)} &= \frac{4}{(2n+1)\pi} \\ b_{(2n)} &= 0 \end{aligned}\right]\; n = 0, 1, 2, \ldots \tag{2}$$

Then, we have

$$\boxed{F(t) = \frac{4}{\pi}\sin\omega t + \frac{4}{3\pi}\sin 3\omega t + \frac{4}{5\pi}\sin 5\omega t + \ldots} \tag{3}$$



3-29. In order to Fourier analyze a function of arbitrary period, say $\tau = 2P/\omega$ instead of $2\pi/\omega$, proportional change of scale is necessary. Analytically, such a change of scale can be represented by the substitution

$$x = \frac{\pi t}{P} \quad \text{or} \quad t = \frac{Px}{\pi} \tag{1}$$

for when $t = 0$, then $x = 0$, and when $t = \tau = 2P/\omega$, then $x = 2\pi/\omega$.

Thus, when the substitution $t = Px/\pi$ is made in a function $F(t)$ of period $2P/\omega'$, we obtain the function

$$F\left[\frac{Px}{\pi}\right] = f(x) \tag{2}$$

and this, *as a function of x*, has a period of $2\pi/\omega$. Now, $f(x)$ can, of course, be expanded according to the standard formula, Eq. (3.91):

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega x + b_n \sin n\omega x\right) \tag{3}$$

where

$$\left.\begin{aligned} a_n &= \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x')\cos n\omega x'\,dx' \\ b_n &= \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x')\sin n\omega x'\,dx' \end{aligned}\right] \tag{4}$$

If, in the above expressions, we make the inverse substitutions

$$x = \frac{\pi t}{P} \quad \text{and} \quad dx = \frac{\pi}{P}\,dt \tag{5}$$

the expansion becomes

$$f\left[\frac{\pi t}{P}\right] = F\left[\frac{P}{\pi}\cdot\frac{\pi t}{P}\right] = F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left[\frac{n\omega\pi t}{P}\right] + b_n \sin\left[\frac{n\omega\pi t}{P}\right]\right] \tag{6}$$

and the coefficients in (4) become

$$\left.\begin{aligned} a_n &= \frac{\omega}{P}\int_0^{\frac{2P}{\omega}} F(t')\cos\left[\frac{n\omega\pi t'}{P}\right]dt' \\ b_n &= \frac{\omega}{P}\int_0^{\frac{2P}{\omega}} F(t')\sin\left[\frac{n\omega\pi t'}{P}\right]dt' \end{aligned}\right] \tag{7}$$

For the case corresponding to this problem, the period of $F(t)$ is $\dfrac{4\pi}{\omega}$, so that $P = 2\pi$. Then, substituting into (7) and replacing the integral limits 0 and τ by the limits $-\dfrac{\tau}{2}$ and $+\dfrac{\tau}{2}$, we obtain

$$\left.\begin{aligned} a_n &= \frac{\omega}{2\pi}\int_{-\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}} F(t')\cos\left[\frac{n\omega t'}{2}\right]dt' \\ b_n &= \frac{\omega}{2\pi}\int_{-\frac{2\pi}{\omega}}^{\frac{2\pi}{\omega}} F(t')\sin\left[\frac{n\omega t'}{2}\right]dt' \end{aligned}\right] \tag{8}$$

and substituting into (6), the expansion for $F(t)$ is

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left[\frac{n\omega t}{2}\right] + b_n \sin\left[\frac{n\omega t}{2}\right]\right] \tag{9}$$

Substituting $F(t)$ into (8) yields

$$\left.\begin{aligned} a_n &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin\omega t' \cos\left[\frac{n\omega t'}{2}\right]dt' \\ b_n &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin\omega t' \sin\left[\frac{n\omega t'}{2}\right]dt' \end{aligned}\right] \tag{10}$$

Evaluation of the integrals gives

$$\left.\begin{array}{l} b_2 = \dfrac{1}{2}\,;\ b_n = 0 \qquad \text{for } n \neq 2 \\[2ex] a_0 = a_1 = 0 \qquad a_n(n \geq 2) = \begin{bmatrix} 0 & n \text{ even} \\[1ex] \dfrac{-4}{\pi\left(n^2-4\right)} & n \text{ odd} \end{bmatrix} \end{array}\right] \tag{11}$$

and the resulting Fourier expansion is

$$\boxed{F(t) = \frac{1}{2}\sin\omega t + \frac{4}{3\pi}\cos\frac{\omega t}{2} - \frac{4}{5\pi}\cos\frac{3\omega t}{2} - \frac{4}{21\pi}\cos\frac{5\omega t}{2} - \frac{4}{45\pi}\cos\frac{7\omega t}{2} + \ldots} \tag{12}$$

3-30. The output of a full-wave rectifier is a periodic function $F(t)$ of the form

$$F(t) = \begin{bmatrix} -\sin\omega t; & -\dfrac{\pi}{\omega} < t \leq 0 \\[2ex] \sin\omega t; & 0 < t < \dfrac{\pi}{\omega} \end{bmatrix} \tag{1}$$

The coefficients in the Fourier representation are given by

$$\left.\begin{array}{l} a_n = \dfrac{\omega}{\pi}\left[\displaystyle\int_{-\frac{\pi}{\omega}}^{0}\left(-\sin\omega t'\right)\cos n\omega t'\,dt' + \int_0^{\frac{\pi}{\omega}}\sin\omega t'\cos n\omega t'\,dt'\right] \\[3ex] b_n = \dfrac{\omega}{\pi}\left[\displaystyle\int_{-\frac{\pi}{\omega}}^{0}\left(-\sin\omega t'\right)\sin n\omega t'\,dt' + \int_0^{\frac{\pi}{\omega}}\sin\omega t'\sin n\omega t'\,dt'\right] \end{array}\right] \tag{2}$$

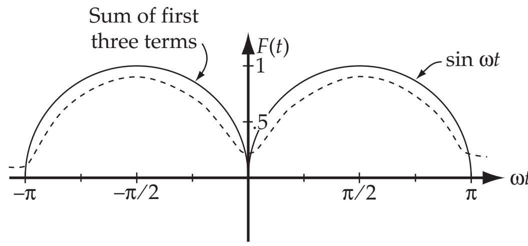
Performing the integrations, we obtain

$$\left.\begin{array}{l} a_n = \begin{bmatrix} \dfrac{4}{\pi\left(1-n^2\right)}\,; & \text{if } n \text{ even (or 0)} \\[2ex] 0\,; & \text{if } n \text{ odd} \end{bmatrix} \\[3ex] b_n = 0 \qquad \text{for all } n \end{array}\right] \tag{3}$$

The expansion for $F(t)$ is

$$\boxed{F(t) = \frac{2}{\pi} - \frac{4}{3\pi}\cos 2\omega t - \frac{4}{15\pi}\cos 4\omega t \ldots} \tag{4}$$

The exact function and the sum of the first three terms of (4) are shown below.



3-31. We can rewrite the forcing function so that it consists of two forcing functions for $t > \tau$:

$$\frac{F(t)}{m} = \begin{cases} 0 & t < 0 \\ a(t/\tau) & 0 < t < \tau \\ a(t/\tau) - \dfrac{a(t-\tau)}{\tau} & t > \tau \end{cases} \tag{1}$$

During the interval $0 < t < \tau$, the differential equation which describes the motion is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{at}{\tau} \tag{2}$$

The particular solution is $x_p = Ct + D$, and substituting this into (2), we find

$$2\beta C + \omega_0^2 Ct + \omega_0^2 D = \frac{at}{\tau} \tag{3}$$

from which

$$\left.\begin{aligned} &2\beta C + \omega_0^2 D = 0 \\ &C\omega_0^2 - \frac{a}{\tau} = 0 \end{aligned}\right] \tag{4}$$

Therefore, we have

$$D = -\frac{2\beta a}{\omega_0^4 \tau}, \qquad C = \frac{a}{\omega_0^2 \tau} \tag{5}$$

which gives

$$x_p = \frac{a}{\omega_0^2 \tau} t - \frac{2\beta a}{\omega_0^4 \tau} \tag{6}$$

Thus, the general solution for $0 < t < \tau$ is

$$x(t) = e^{-\beta t}\left[A \cos \omega_1 t + B \sin \omega_1 t\right] + \frac{a}{\omega_0^2 \tau} t - \frac{2\beta a}{\omega_0^4 \tau} \tag{7}$$

and then,

$$\dot{x}(t) = -\beta e^{-\beta t}\left[A\cos\omega_1 t + B\sin\omega_1 t\right] + \omega_1 e^{-\beta t}\left[-A\sin\omega_1 t + B\cos\omega_1 t\right] + \frac{a}{\omega_0^2\tau} \tag{8}$$

The initial conditions, $x(0) = 0$, $\dot{x}(0) = 0$, implies

$$\left.\begin{aligned} A &= \frac{2\beta a}{\omega_0^4\tau} \\ B &= \frac{a}{\omega_1\omega_0^2\tau}\left[\frac{2\beta^2}{\omega_0^2} - 1\right] \end{aligned}\right] \tag{9}$$

Therefore, the response function is

$$x(t) = \frac{a}{\omega_0^2\tau}\left[\frac{2\beta}{\omega_0^2}e^{-\beta t}\cos\omega_1 t + \frac{e^{-\beta t}}{\omega_1}\left[\frac{2\beta^2}{\omega_0^2} - 1\right]\sin\omega_1 t + t - \frac{2\beta}{\omega_0^2}\right] \tag{10}$$

For the forcing function $-\dfrac{a(t-\tau)}{\tau}$ in (1), we have a response similar to (10). Thus, we add these two equations to obtain the total response function:

$$\boxed{\begin{aligned} x(t) = \frac{a}{\omega_0^2\tau}\Bigg[\frac{2\beta}{\omega_0^2}e^{-\beta t}\left(\cos\omega_1 t - e^{\beta t}\cos\omega_1(t-\tau)\right) + \frac{e^{-\beta t}}{\omega_1}\left[\frac{2\beta^2}{\omega_0^2} - 1\right] \\ \times\left[\sin\omega_1 t + -e^{\beta t}\sin\omega_1(t-\tau)\right] + \tau\Bigg] \end{aligned}} \tag{11}$$

When $\tau \to 0$, we can approximate $e^{\beta\tau}$ as $1 + \beta\tau$, and also $\sin\omega_1\tau \cong \omega_1\tau$, $\cos\omega_1\tau \cong 1$. Then,

$$\begin{aligned} x(t) \xrightarrow[\tau\to 0]{} \frac{a}{\omega_0^2\tau}\Bigg[\frac{2\beta}{\omega_0^2}e^{-\beta t}\left[\cos\omega_1 t - (1+\beta\tau)(\cos\omega_1 t + \omega_1\tau\sin\omega_1 t)\right] + \frac{e^{-\beta t}}{\omega_1}\left[\frac{2\beta^2}{\omega_0^2} - 1\right] \\ \times\left[\sin\omega_1 t - (1+\beta\tau)(\sin\omega_1 t - \omega_1\tau\cos\omega_1 t)\right] + \tau\Bigg] \end{aligned}$$

$$= \frac{a}{\omega_0^2}\left[1 - e^{-\beta t}\cos\omega_1 t - e^{-\beta t}\left[\frac{2\beta\omega_1}{\omega_0^2} - \frac{\beta}{\omega_1} + \frac{2\beta^3}{\omega_1\omega_0^2}\right]\sin\omega_1 t\right] \tag{12}$$

If we use $\omega_1^2 = \omega_0^2 - \beta^2$, the coefficient of $e^{-\beta t}\sin\omega_1 t$ becomes β/ω_1. Therefore,

$$\boxed{x(t) \xrightarrow[\tau\to 0]{} \frac{a}{\omega_0^2}\left[1 - e^{-\beta t}\cos\omega_1 t - e^{-\beta t}\frac{\beta}{\omega_1}\sin\omega_1 t\right]} \tag{13}$$

This is just the response for a step function.

3-32.

a) *Response to a Step Function*:

From Eq. (3.100) $H(t_0)$ is defined as

$$H(t_0) = \begin{bmatrix} 0, & t < t_0 \\ a_1, & t > t_0 \end{bmatrix} \tag{1}$$

With initial conditions $x(t_0 = 0)$ and $\dot{x}(t_0 = 0)$, the general solution to Eq. (3.102) (equation of motion of a damped linear oscillator) is given by Eq. (3.105):

$$\left.\begin{array}{ll} x(t) = \dfrac{a}{\omega_0^2}\left[1 - e^{-\beta(t-t_0)} \cos \omega_1 (t - t_0) - \dfrac{\beta e^{-\beta(t-t_0)}}{\omega_1} \sin \omega_1 (t - t_0)\right] & \text{for } t > t_0 \\ x(t) = 0 & \text{for } t < t_0 \end{array}\right] \tag{2}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

For the case of overdamping, $\omega_0^2 < \beta^2$, and consequently $\omega_1 = i\sqrt{\beta^2 - \omega_0^2}$ is a pure imaginary number. Hence, $\cos \omega_1 (t - t_0)$ and $\sin \omega_1 (t - t_0)$ are no longer oscillatory functions; instead, they are transformed into hyperbolic functions. Thus, if we write $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$ (where ω_2 is *real*),

$$\left.\begin{array}{l} \cos \omega_1 (t - t_0) = \cos i\omega_2 (t - t_0) = \cosh \omega_2 (t - t_0) \\ \sin \omega_1 (t - t_0) = \sin i\omega_2 (t - t_0) = i \sinh \omega_2 (t - t_0) \end{array}\right] \tag{3}$$

The response is given by [see Eq. (3.105)]

$$\left.\begin{array}{ll} x(t) = \dfrac{a}{\omega_0^2}\left[1 - e^{-\beta(t-t_0)} \cosh \omega_2 (t - t_0) - \dfrac{\beta e^{-\beta(t-t_0)}}{\omega_2} \sinh \omega_1 (t - t_0)\right] & \text{for } t > t_0 \\ x(t) = 0 & \text{for } t < t_0 \end{array}\right] \tag{4}$$

For simplicity, we choose $t_0 = 0$, and the solution becomes

$$\boxed{x(t) = \frac{H(0)}{\omega_0^2}\left[1 - e^{-\beta t} \cosh \omega_2 t - \frac{\beta e^{-\beta t}}{\omega_2} \sinh \omega_2 t\right]} \tag{5}$$

This response is shown in (a) below for the case $\beta = \sqrt{5}\, \omega_0$.

b) *Response to **an** Impulse Function* (in the limit $\tau \to 0$):

From Eq. (3.101) the impulse function $I(t_0, t_1)$ is defined as

$$I(t_0, t_1) = \begin{bmatrix} 0 & t < t_0 \\ a & t_0 < t < t_1 \\ 0 & t > t_1 \end{bmatrix} \tag{6}$$

For $t_1 - t_2 = \tau \to 0$ in such a way that $a\tau$ is constant = *b*, the response function is given by Eq. (3.110):

$$x(t) = \frac{b}{\omega_1} e^{-\beta(t-t_0)} \sin\omega_1 (t - t_0) \quad \text{for } t > t_0 \tag{7}$$

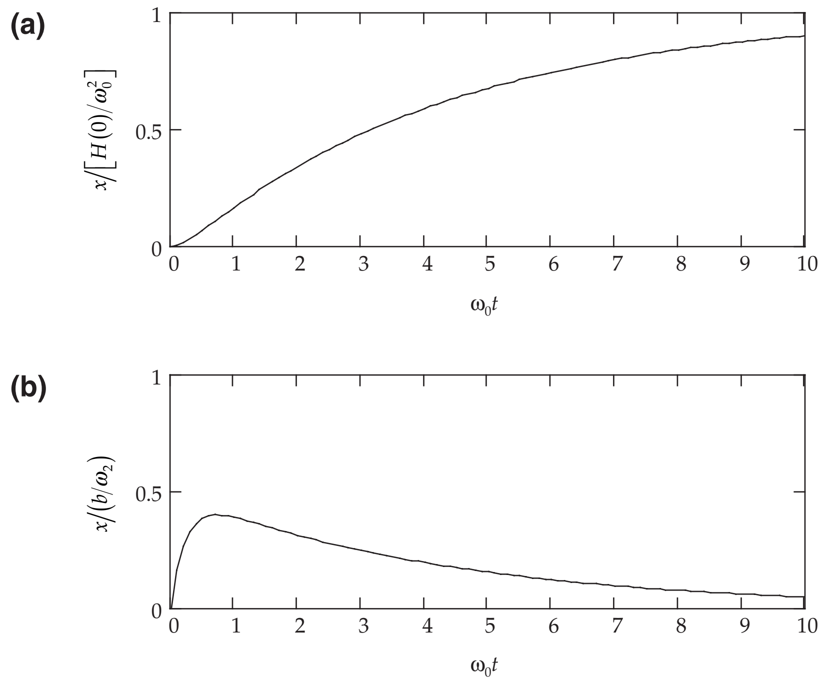
Again taking the "spike" to be at $t = 0$ for simplicity, we have

$$x(t) = \frac{b}{\omega_1} e^{-\beta t} \sin\omega_1 (t) \qquad \text{for } t > 0 \tag{8}$$

For $\omega_1 = i\omega_2 = i\sqrt{\beta^2 - \omega_0^2}$ (overdamped oscillator), the solution is

$$\boxed{x(t) = \frac{b}{\omega_2} e^{-\beta t} \sinh \omega_2 t; \;\; t > 0} \tag{9}$$

This response is shown in (b) below for the case $\beta = \sqrt{5}\,\omega_0$.



3-33.

a) In order to find the maximum amplitude of the response function shown in Fig. 3-22, we look for t_1 such that $|x(t)|$ given by Eq. (3.105) is maximum; that is,

$$\left.\frac{\partial(|x(t)|)}{\partial t}\right|_{t=t_1} = 0 \tag{1}$$

From Eq. (3.106) we have

$$\frac{\partial(|x(t)|)}{\partial t} = \frac{H(0)}{\omega_0^2} e^{-\beta t} \left| \left[\omega_1 + \frac{\beta^2}{\omega_1} \right] \sin \omega_1 t \right| \tag{2}$$

For $\beta = 0.2\omega_0$, $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = 0.98\omega_0$. Evidently, $t_1 = \pi/\omega_1$ makes (2) vanish. (This is the absolute maximum, as can be seen from Fig. 3-22.)

Then, substituting into Eq. (3.105), the maximum amplitude is given by

$$\left|x(t)\right|_{\max} = x(t_1) = \frac{a}{\omega_0^2}\left[1 + e^{-\frac{\beta\pi}{\omega_1}}\right] \tag{3}$$

or,

$$\boxed{x(t_1) \cong 1.53\,\frac{a}{\omega_0^2}} \tag{4}$$

b) In the same way we find the maximum amplitude of the response function shown in Fig. 3-24 by using $x(t)$ given in Eq. (3.110); then,

$$\left.\frac{\partial\left(\left|x(t)\right|\right)}{\partial t}\right|_{t=t_1} = be^{-\beta(t-t_0)}\left[\cos\omega_1(t-t_0) - \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\right]_{t=t_1} \tag{5}$$

If (5) is to vanish, t_1 is given by

$$t_1 - t_0 = \frac{1}{\omega_1}\tan^{-1}\left[\frac{\omega_1}{\beta}\right] = \frac{1}{\omega_1}\tan^{-1}(4.9) = \frac{1.37}{\omega_1} \tag{6}$$

Substituting (6) into Eq. (3.110), we obtain (for $\beta = 0.2\omega_0$)

$$\left|x(t)\right|_{\max} = x(t_1) = \frac{b}{0.98\omega_0}\,e^{\frac{-\beta}{\omega_1}\times 1.37}\,\sin(1.37) \tag{7}$$

or,

$$\boxed{x(t_1) \cong 0.76\,\frac{a\tau}{\omega_0}} \tag{8}$$

3-34. The response function of an undamped ($\beta = 0$) linear oscillator for an impulse function $I(0,\tau)$, with $\tau = \dfrac{2\pi}{\omega_0}$, can be obtained from Eqs. (3.105) and (3.108) if we make the following substitutions:

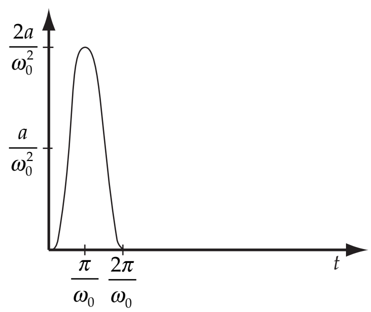
$$\left.\begin{array}{ll} \beta = 0\,; & \omega_1 = \omega_0 \\ t_0 = 0\,; & t_1 = \tau = \dfrac{2\pi}{\omega_0} \end{array}\right] \tag{1}$$

(For convenience we have assumed that the impulse forcing function is applied at $t = 0$.)

Hence, after substituting we have

$$\left.\begin{array}{ll} x(t)=0 & t<0 \\ x(t)=\dfrac{a}{\omega_0^2}\left[1-\cos\omega_0 t\right] & 0<t<\dfrac{2\pi}{\omega_0} \\ x(t)=\dfrac{a}{\omega_0^2}\left[\cos\left(w_0 t-2\pi\right)-\cos w_0 t\right]=0 & t>\tau=\dfrac{2\pi}{\omega_0} \end{array}\right] \tag{2}$$

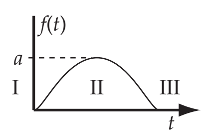
This response function is shown below. Since the oscillator is undamped, and since the impulse lasts exactly one period of the oscillator, the oscillator is returned to its equilibrium condition at the termination of the impulse.



3-35. The equation for a driven linear oscillator is

$$\ddot{x} + 2\beta\dot{x} + w_2^0 x = f(t)$$

where $f(t)$ is the sinusoid shown in the diagram.



Region I: $x = 0$ (1)

Region II: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = a\sin\omega t$ (2)

Region III: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ (3)

The solution of (2) is

$$x = e^{-\beta t}\left(A\sin\omega_1 t + B\cos\omega_1 t\right) + x_P \tag{4}$$

in which

$$x_P = Da\sin\left(\omega t - \delta\right) \tag{5}$$

where

$$D = \frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)+4\beta^2\omega^2}} \tag{6}$$

$$\delta = \tan^{-1} \frac{2\beta\omega}{\omega_0^2 - \omega^2} \tag{7}$$

Thus,

$$x = e^{-\beta t} \left(A \sin \omega_1 t + B \cos \omega_1 t \right) + D\, a \sin \left(\omega t - \delta \right) \tag{8}$$

The initial condition $x(0) = 0$ gives

$$B = aD \sin \delta \tag{9}$$

and $\dot{x}(0) = 0$

$$-\beta B + \omega_1 A + Da\omega \cos \delta = 0$$

or,

$$A = \left(\beta \sin \delta - \omega \cos \delta \right) \frac{aD}{\omega_1} \tag{10}$$

The solution of (3) is

$$x(t) = e^{-\beta t} \left[A' \sin \omega_1 t + B' \cos \omega_1 t \right] \tag{11}$$

We require that $x(t)$ and $\dot{x}(t)$ for regions II and III match at $t = \pi/\omega$. The condition that $x_{\text{II}} \left(\frac{\pi}{\omega} \right) = x_{\text{III}} \left(\frac{\pi}{\omega} \right)$ gives

$$e^{-\beta\pi/\omega} \left(A \sin \phi + B \cos \phi \right) + Da \sin \left(\pi - \delta \right) = e^{-\beta\pi/\omega} \left(A' \sin \phi + B' \cos \phi \right)$$

where $\phi = \frac{\omega_1}{\omega} \pi$ or,

$$A' + B' \cot \phi = A + B \cot \phi + \frac{aD \sin \delta}{\sin \phi} e^{\beta\pi/\omega} \tag{12}$$

The condition that $\dot{x}_{\text{II}} \left(\frac{\pi}{\omega} \right) = \dot{x}_{\text{III}} \left(\frac{\pi}{\omega} \right)$ gives

$$-\beta e^{-\beta\pi/\omega} \left(A \sin \phi + B \cos \phi \right) + aD\omega \cos \left(\pi - \delta \right) + e^{-\beta\pi/\omega} \left(A\omega_1 \cos \phi - B\omega_1 \sin \phi \right)$$
$$= -\beta e^{-\beta\pi/\omega} \left(A' \sin \phi + B' \cos \phi \right) + e^{-\beta\pi/\omega} \left(A'\omega_1 \cos \phi - B'\omega_1 \sin \phi \right)$$

or,

$$A' \left(\omega_1 \cos \phi - \beta \sin \phi \right) - B' \left(\omega_1 \sin \phi + \beta \cos \phi \right)$$
$$= A \left(-\beta \sin \phi + \omega_1 \cos \phi \right) - B \left(\omega_1 \sin \phi + \beta \cos \phi \right) - e^{\beta\pi/\omega} aD\omega \cos \delta$$

or,

$$A' - B' \left[\frac{\omega_1 \sin \phi + \beta \cos \phi}{\omega_1 \cos \phi - \beta \sin \phi} \right] = A - B \left[\frac{\omega_1 \sin \phi + \beta \cos \phi}{\omega_1 \cos \phi - \beta \sin \phi} \right] - e^{\beta\pi/\omega} \left[\frac{\omega Da \cos \delta}{\omega_1 \cos \phi - \beta \sin \phi} \right] \tag{13}$$

Substituting into (13) from (12), we have

$$B' \frac{(\omega_1 \cos\phi - \beta\sin\phi)\cos\phi + (\omega_1 \sin\phi + \beta\cos\phi)\sin\phi}{(\omega_1\cos\phi - \beta\sin\phi)\sin\phi}$$

$$= \frac{(\omega_1 \cos\phi - \beta\sin\phi)\cos\phi + (\omega_1 \sin\phi + \beta\cos\phi)\sin\phi}{(\omega_1\cos\phi - \beta\sin\phi)\sin\phi} + aDe^{\beta\pi/\omega}\left[\frac{\sin\delta}{\sin\phi} + \frac{\omega\cos\delta}{\omega_1\cos\phi - \beta\sin\phi}\right] \tag{14}$$

from which

$$B' = aD\sin\delta + \frac{a}{\omega_1}De^{\beta\pi/\omega}\left[\sin\delta(\omega_1\cos\phi - \beta\sin\phi) + \omega\cos\delta\sin\phi\right] \tag{15}$$

Using (12), we can find A':

$$A' = A + B\cot\phi + \frac{aD\sin\delta e^{\beta\pi/\omega}}{\sin\phi} - B'\cot\phi \tag{16}$$

Substituting for A, B, and B' from (10), (9), and (15), we have

$$A' = aD\sin\delta\left[\frac{\beta}{\omega_1} + e^{\beta\pi/\omega}\left(\frac{\beta}{\omega_1}\cos\phi + \sin\phi\right)\right] - \frac{aD\omega\cos\delta}{\omega_1}\left(1 + e^{\beta\pi/\omega}\cos\phi\right) \tag{17}$$

Thus, we obtained all constants giving us the response functions explicitly.

3-36. With the initial conditions, $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$, the solution for a step function for $t > t_0$ given by Eq. (3.103) yields

$$A_1 = x_0 - \frac{a}{\omega_0^2}; \quad A_2 = \frac{\dot{x}_0}{\omega_1} + \frac{\beta x_0}{\omega_1} - \frac{\beta a}{\omega_0^2\omega_1} \tag{1}$$

Therefore, the response to $H(t_0)$ for the initial conditions above can be expressed as

$$\begin{aligned} x(t) &= e^{-\beta(t-t_0)}\left[x_0\cos\omega_1(t-t_0) + \left[\frac{\dot{x}_0}{\omega_1} + \frac{\beta x_0}{\omega_1}\right]\sin\omega_1(t-t_0)\right] \\ &\quad + \frac{a}{\omega_0^2}\left[1 - e^{-\beta(t-t_0)}\cos\omega_1(t-t_0) - \frac{\beta}{\omega_1}e^{-\beta(t-t_0)}\sin\omega_1(t-t_0)\right] \qquad \text{for } t > t_0 \end{aligned} \tag{2}$$

The response to an impulse function $I(t_0, t_1) = H(t_1)$, for the above initial conditions will then be given by (2) for $t_0 < t < t_1$ and by a superposition of solutions for $H(t_0)$ and for $H(t_1)$ taken individually for $t > t_1$. We must be careful, however, because the solution for $t > t_1$ must be equal that given by (2) for $t = t_1$. This can be insured by using as a solution for $H(t_1)$ Eq. (3.103) with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, and using t_1 instead of t_0 in the expression.

The solution for $t > t_1$ is then

$$x(t) = e^{-\beta(t-t_0)}\left[x_0 \cos\omega_1(t-t_0) + \left[\frac{\dot{x}_0}{\omega_1} + \frac{\beta x_0}{\omega_1}\right]\sin\omega_1(t-t_0)\right] + x_1(t) \tag{3}$$

where

$$x_1(t) = \frac{ae^{-\beta(t-t_0)}}{\omega_0^2}\left[e^{\beta\tau}\cos\omega_1(t-t_0-\tau) - \cos\omega_1(t-t_0) + \right.$$
$$\left. + \frac{\beta e^{\beta\tau}}{\omega_1}\sin\omega_1(t-t_0-\tau) - \frac{\beta}{\omega_1}\sin\omega_1(t-t_0)\right] \qquad \text{for } t > t_1 \tag{4}$$

We now allow $a \to \infty$ as $\tau \to 0$ in such a way that $a\tau = b$ = constant; expanding (3) for this particular case, we obtain

$$\boxed{x(t) = e^{-\beta(t-t_0)}\left[x_0 \cos\omega_1(t-t_0) + \left[\frac{\dot{x}_0}{\omega_1} + \frac{\beta x_0}{\omega_1} + \frac{b}{\omega_1}\right]\sin\omega_1(t-t_0)\right] \quad t > t_0} \tag{5}$$

which is analogous to Eq. (3.119) but for initial conditions given above.

3-37. Any function $F(t)/m$ can be expanded in terms of step functions, as shown in the figure below where the curve is the sum of the various (positive and negative) step functions.

In general, we have

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \sum_{n=-\infty}^{\infty}\frac{F_n(t)}{m}$$
$$= \sum_{n=-\infty}^{\infty} H_n(t) \tag{1}$$

where

$$H_n(t) = \begin{cases} a_n(t) & t > t_n = n\tau \\ 0 & t < t_n = n\tau \end{cases} \tag{2}$$

Then, since (1) is a *linear* equation, the solution to a superposition of functions of the form given by (2) is the superposition of the solutions for each of those functions.

According to Eq. (3.105), the solution for $H_n(t)$ for $t > t_n$ is

$$x_n(t) = \frac{a_n}{\omega_0^2}\left[1 - e^{-\beta(t-t_n)}\cos\omega_1(t-t_n) - \frac{\beta e^{-\beta(t-t_n)}}{\omega_1}\sin\omega_1(t-t_n)\right] \tag{3}$$

then, for

$$\frac{F(t)}{m} = \sum_{n=-\infty}^{\infty} H_n(t) \tag{4}$$

the solution is

$$x(t) = \frac{1}{\omega_0^2} \sum_{n=-\infty}^{\infty} H_n(t) \left[1 - e^{-\beta(t-t_n)} \cos \omega_1 (t - t_n) - \frac{\beta e^{-\beta(t-t_n)}}{\omega_1} \sin \omega_1 (t - t_n) \right]$$

$$= \sum_{n=-\infty}^{\infty} m H_n(t) G_n(t) = \sum_{n=-\infty}^{\infty} F_n(t) G_n(t) \tag{5}$$

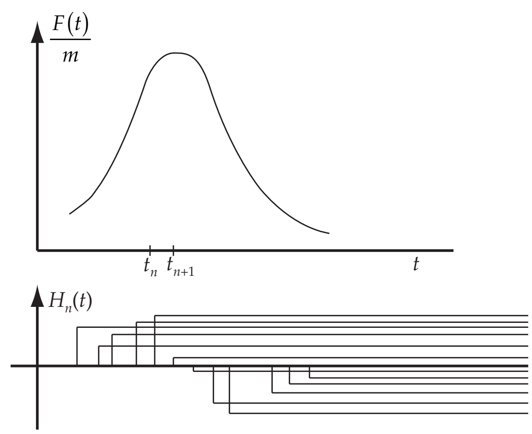
where

$$G_n(t) = \begin{bmatrix} \dfrac{1}{m\omega_0^2} \left[1 - e^{-\beta(t-t_n)} \cos \omega_1 (t - t_n) - \dfrac{\beta e^{-\beta(t-t_n)}}{\omega_1} \sin \omega_1 (t - t_n) \right]; & t \geq t_n \\ 0 & t < t_n \end{bmatrix} \tag{6}$$

or, comparing with (3)

$$G_n(t) = \begin{bmatrix} x_n(t)/ma_n\,, & t \geq t_n \\ 0 & t < t_n \end{bmatrix} \tag{7}$$

Therefore, the Green's function is the response to the unit step.



3-38. The solution for $x(t)$ according to Green's method is

$$x(t) = \int_{-\infty}^{t} F(t')\, G(t,t')\, dt'$$

$$= \frac{F_0}{m\omega_1} \int_0^t e^{-\gamma t'} \sin \omega t'\, e^{-\beta(t-t')} \sin \omega_1 (t - t') dt' \tag{1}$$

Using the trigonometric identity,

$$\sin \omega t' \sin \omega_1 (t - t') = \frac{1}{2} \Big[\cos \big[(\omega_1 + \omega) t' - \omega_1 t \big] - \cos \big[(\omega - \omega_1) t' + \omega_1 t \big] \Big] \tag{2}$$

we have

$$x(t) = \frac{F_0 e^{-\beta t}}{2m\omega_1}\left[\int_0^t dt' e^{(\beta-\gamma)t'} \cos\left[(\omega+\omega_1)t' - \omega_1 t\right] - \int_0^t dt' e^{(\beta-\gamma)t'} \cos\left[(\omega-\omega_1)t' + \omega_1 t\right]\right] \tag{3}$$

Making the change of variable, $z = (\omega+\omega_1)t' - \omega_1 t$, for the first integral and $y = (\omega-\omega_1)t' + \omega_1 t$ for the second integral, we find

$$x(t) = \frac{F_0 e^{-\beta t}}{2m\omega_1}\left[\frac{e^{\frac{(\beta-\gamma)\omega_1 t}{\omega+\omega_1}}}{\omega+\omega_1}\int_{-\omega_1 t}^{\omega t} dz\, e^{\frac{(\beta-\gamma)z}{\omega+\omega_1}} \cos z - \frac{e^{\frac{-(\beta-\gamma)\omega_1 t}{\omega-\omega_1}}}{\omega-\omega_1}\int_{\omega_1 t}^{\omega t} dy\, e^{\frac{(\beta-\gamma)y}{\omega-\omega_1}} \cos y\right] \tag{4}$$

After evaluating the integrals and rearranging terms, we obtain

$$\boxed{\begin{aligned} x(t) = \frac{F_0}{m}&\frac{\omega}{\left[(\beta-\gamma)^2 + (\omega+\omega_1)^2\right]\left[(\beta-\gamma)^2 + (\omega-\omega_1)^2\right]} \\ &\times\left[e^{-\gamma t}\left[2(\gamma-\beta)\cos\omega t + \left([\beta-\gamma]^2 + \omega_1^2 - \omega^2\right)\frac{\sin\omega t}{\omega}\right]\right. \\ &\left.\quad + e^{-\beta t}\left[2(\beta-\gamma)\cos\omega_1 t + \left([\beta-\gamma]^2 + \omega^2 - \omega_1^2\right)\frac{\sin\omega_1 t}{\omega}\right]\right]\end{aligned}} \tag{5}$$

3-39.

$$F(t) = \begin{cases} \sin\omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < 2\pi/\omega \end{cases}$$

From Equations 3.89, 3.90, and 3.91, we have

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

$$a_n = \frac{2}{\tau}\int_0^\tau F(t') \cos n\omega t'\, dt'$$

$$b_n = \frac{2}{\tau}\int_0^\tau F(t') \sin n\omega t'\, dt'$$

$$a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t' \cos n\omega t'\, dt'$$

$$a_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t'\, dt' = \frac{2}{\pi}$$

$$a_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin\omega t' \cos\omega t'\, dt' = 0$$

$$a_n(n \geq 2) = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin \omega t' \cos n\omega t' \, dt' = \frac{\omega}{\pi}\left[-\frac{\cos(1-n)\omega t'}{2(1-n)\omega} - \frac{\cos(1+n)\omega t}{2(1+n)\omega}\right]_0^{\pi/\omega}$$

Upon evaluating and simplifying, the result is

$$a_n = \begin{bmatrix} \dfrac{2}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{bmatrix} \qquad n = 0,1,2,\ldots$$

$b_0 = 0$ by inspection

$$b_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin^2 \omega t' \, dt' = \frac{1}{2}$$

$$b_n(n \geq 2) = \frac{\omega}{\pi}\int_0^{\pi/\omega} \sin \omega t' \sin n\omega t' \, dt' = \frac{\omega}{\pi}\left[\frac{\sin(1-n)\omega t'}{2(1-n)\omega} - \frac{\sin(1+n)\omega t'}{2(1+n)\omega}\right]_0^{\pi/\omega} = 0$$

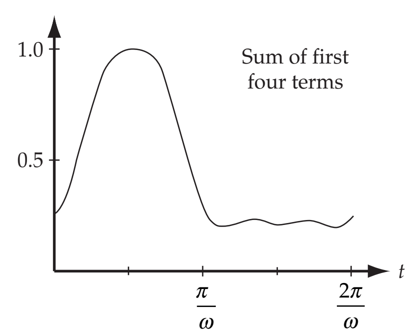
So

$$F(t) = \frac{1}{\pi} + \frac{1}{2}\sin^2 \omega t + \sum_{n=2,4,6,\ldots}^{\infty} \frac{2}{\pi(1-n^2)} \cos n\omega t$$

or, letting $n \to 2n$

$$\boxed{F(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t + \sum_{n=1,2,\ldots}^{\infty} \frac{2}{\pi(1-4n^2)} \cos 2n\omega t}$$

The following plot shows how well the first four terms in the series approximate the function.



3-40. The equation describing the car's motion is

$$m\frac{d^2y}{dt^2} = -k(y - a \sin \omega t)$$

where y is the vertical displacement of the car from its equilibrium position on a flat road, a is the amplitude of sine-curve road, and

$$k = \text{elastic coefficient} = \frac{dm \times g}{dy} = \frac{100 \times 9.8}{0.01} = 98000 \text{ N/m}$$

$\omega = \dfrac{2\pi v_0}{\lambda} = 174$ rad/s with v_0 and λ being the car's speed and wavelength of sine-curve road.
The solution of the motion equation can be cast in the form

$$y(t) = B\cos\left(\omega_0 t + \beta\right) + \frac{a{\omega_0}^2}{\omega_0^2 - \omega^2}\sin \omega t \quad \text{with} \quad \omega_0 = \sqrt{\frac{k}{m}} = 9.9 \text{ rad/s}$$

We see that the oscillation with angular frequency ω has amplitude

$$A = \frac{a\omega_0^2}{\omega_0^2 - \omega^2} = -0.16 \text{ mm}$$

The minus sign just implies that the spring is compressed.

3-41.

a) The general solution of the given differential equation is (see Equation (3.37))

$$x(t) = \exp(-\beta t)\left[A_1 \exp\left(t\sqrt{\beta^2 - \omega_0^2}\right) + A_2 \exp\left(-t\sqrt{\beta^2 - \omega_0^2}\right)\right]$$

and

$$v(t) = x'(t) = -\beta \exp(-\beta t)\left[A_1 \exp\left(t\sqrt{\beta^2 - \omega_0^2}\right) + A_2 \exp\left(-t\sqrt{\beta^2 - \omega_0^2}\right)\right]$$

$$+\exp(-\beta t)\sqrt{\beta^2 - \omega_0^2}\left[A_1 \exp\left(t\sqrt{\beta^2 - \omega_0^2}\right) - A_2 \exp\left(-t\sqrt{\beta^2 - \omega_0^2}\right)\right]$$

at $t = 0$, $x(t) = x_0$, $v(t) = v_0$ $\Rightarrow$

$$A_1 = \frac{1}{2}\left(x_0 + \frac{v_0 + \beta x_0}{\sqrt{\beta^2 - \omega_0^2}}\right) \quad \text{and} \quad A_1 = \frac{1}{2}\left(x_0 - \frac{v_0 + \beta x_0}{\sqrt{\beta^2 - \omega_0^2}}\right) \tag{1}$$

b)

i) Underdamped, $\beta = \dfrac{\omega_0}{2}$

In this case, instead of using above parameterization, it is more convenient to work with the following parameterization

$$x(t) = A\exp(-\beta t)\cos\left(t\sqrt{{\omega_0}^2 - \beta^2} - \delta\right) \tag{2}$$

$$v(t) = -A\exp(-\beta t)\left[\beta\cos\left(t\sqrt{\omega_0^2 - \beta^2} - \delta\right) + \sqrt{\omega_0^2 - \beta^2}\,\sin\left(t\sqrt{\omega_0^2 - \beta^2} - \delta\right)\right] \tag{3}$$

Using initial conditions of $x(t)$ and $v(t)$, we find

$$A = \frac{x_0}{\sqrt{\omega_0^2 - \beta^2}}\sqrt{\left(\frac{v_0}{x_0} + \beta\right)^2 + \omega_0^2 - \beta^2} \text{ and } \tan(\delta) = \frac{\dfrac{v_0}{x_0} + \beta}{\sqrt{\omega_0^2 - \beta^2}}$$

In the case $\beta = \dfrac{\omega_0}{2}$, and using (6) below we have

$$\tan\delta = \frac{2}{\sqrt{3}}\frac{v_0}{x_0\omega_0} + \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \delta = -30°$$

$$A = \frac{2}{\sqrt{3}}\frac{x_0}{\omega_0}\sqrt{\frac{v_0^2}{x_0^2} + \frac{v_0\omega_0}{x_0} + \omega_0^2} = \frac{2}{\sqrt{3}}x_0$$

so finally

$$x(t) = \frac{2}{\sqrt{3}}x_0 \exp\left(-\frac{1}{2}\omega_0 t\right)\cos\left(t\frac{\sqrt{3}}{2}\omega_0 + 30°\right) \tag{4}$$

ii) Critically damped, $\beta = \omega_0$, using the same parameterization as in i) we have from (2) and (3):

$$x(t) = A\exp(-\beta t) = x_0\exp(-\omega_0 t) \tag{5}$$

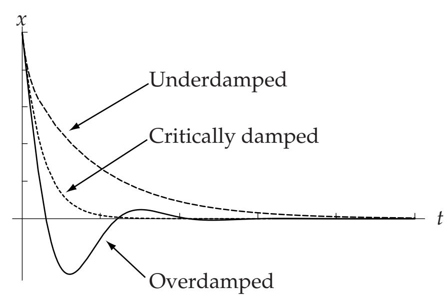
and $v(t) = x'(t) = -\omega_0 x_0 \exp(-\omega_0 t \quad \Rightarrow \quad v_0 = -\omega_0 x_0$ (6)

iii) Overdamped, $\beta = \omega_0$, returning to the original parameterization (1) we have (always using relation (6)),

$$x(t) = \exp(-\beta t)\left[A_1\exp\left(t\sqrt{\beta^2-\omega_0^2}\right) + A_2\exp\left(-t\sqrt{\beta^2-\omega_0^2}\right)\right]$$

$$= \frac{\left(\sqrt{3}+1\right)x_0}{2\sqrt{3}}\exp\left(\left(\sqrt{3}-2\right)\omega_0 t\right) + \frac{\left(\sqrt{3}-1\right)x_0}{2\sqrt{3}}\exp\left(-\left(\sqrt{3}+2\right)\omega_0 t\right) \tag{7}$$

Below we show sketches for equations (4), (5), (7)



3-42.

a)

$$m\left(x'' + \omega_0{}^2 x\right) - F_0 \sin\omega t = 0 \tag{1}$$

The most general solution is

$$x(t) = a\sin\omega_0 t + b\cos\omega_0 t + A\sin\omega t$$

where the last term is a particular solution.

To find A we put this particular solution (the last term) into (1) and find

$$A = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}$$

At $t = 0$, $x = 0$, so we find $b = 0$, and then we have

$$x(t) = a \sin \omega_0 t + A \sin \omega t \quad \Rightarrow \quad v(t) = a\omega_0 \cos \omega_0 t + A\omega \cos \omega t$$

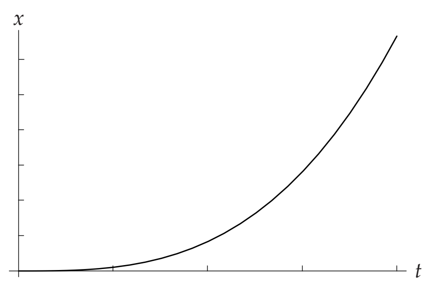
At $t = 0,\ v = 0 \quad \Rightarrow \quad A = \dfrac{-a\omega_0}{\omega} \quad \Rightarrow$

$$x(t) = \frac{F_0}{m\omega_0} \frac{1}{(\omega_0 + \omega)(\omega_0 - \omega)} \left(\omega_0 \sin \omega t - \omega \sin \omega_0 t\right)$$

b) In the limit $\omega \to \omega_0$ one can see that

$$x(t) \to \frac{F_0 t^3 \omega_0}{6m}$$

The sketch of this function is shown below.



3-43.

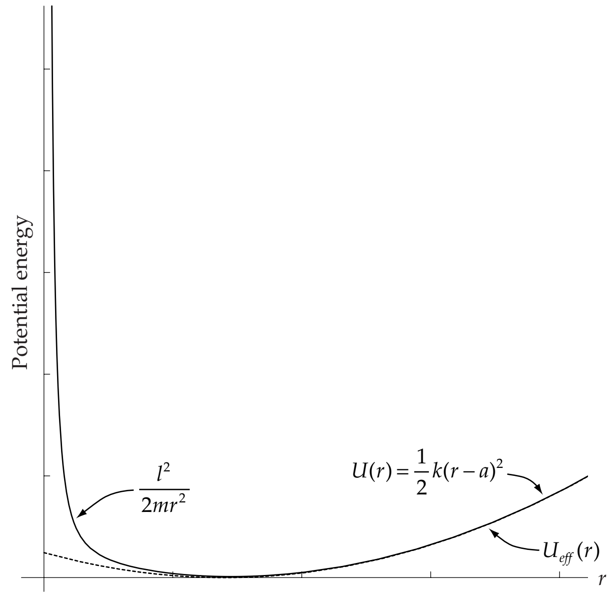
a) Potential energy is the elastic energy:

$$U(r) = \frac{1}{2}k(r-a)^2,$$

where m is moving in a central force field. Then the effective potential is (see for example, Chapter 2 and Equation (8.14)):

$$U_{eff}(r) = U(r) + \frac{l^2}{2mr^2} = \frac{1}{2}k(r-a)^2 + \frac{l^2}{2mr^2}$$

where $l = mvr = m\omega r^2$ is the angular momentum of m and is a conserved quantity in this problem. The solid line below is $U_{eff}(r)$; at low values of r, the dashed line represents $U(r) = \dfrac{1}{2}k(r-a)^2$, and the solid line is dominated by $\dfrac{l^2}{2mr^2}$. At large values of r, $U_{eff}(r) \cong U(r) = \dfrac{1}{2}k(r-a)^2$.



b) In equilibrium circular motion of radius r_0, we have

$$k(r_0-a)=m\omega_0^2 r_0 \quad \Rightarrow \quad \omega_0=\sqrt{\frac{k(r_0-a)}{mr_0}}$$

c) For given (and fixed) angular momentum *l*, $V(r)$ is minimal at r_0, because $V'(r)\big|_{r=r_0}=0$, so we make a Taylor expansion of $V(r)$ about r_0;

$$V(r)=V(r_0)+(r-r_0)V'(r_0)+\frac{1}{2}(r-r_0)^2V''(r_0)+\ldots\approx\frac{3m\omega_0^2(r-r_0)^2}{2}=\frac{K(r-r_0)^2}{2}$$

where $K=3m\omega_0^2$, so the frequency of oscillation is

$$\omega=\sqrt{\frac{K}{m}}=\sqrt{3}\omega_0=\sqrt{\frac{3k(r_0-a)}{mr_0}}$$

3-44. This oscillation must be underdamped oscillation (otherwise no period is present). From Equation (3.40) we have

$$x(t)=A\exp(-\beta t)\cos(\omega_1 t-\delta)$$

so the initial amplitude (at $t = 0$) is A.

Now at $t=4T=\frac{8\pi}{\omega_1}$

$$x(4T)=A\exp\left(-\beta\frac{8\pi}{\omega_1}\right)\cos(8\pi-\delta)$$

The amplitude now is $A\exp\left(-\beta\frac{8\pi}{\omega_1}\right)$, so we have

$$\frac{A\exp\left(-\beta\frac{8\pi}{\omega_1}\right)}{A}=\frac{1}{e}$$

and because $\beta=\sqrt{\omega_0^2-\omega_1^2}$, we finally find

$$\frac{\omega_1}{\omega_0}=\frac{8\pi}{\sqrt{64\pi^2+1}}$$

3-45. Energy of a simple pendulum is $\frac{mgl}{2}\theta^2$ where θ is the amplitude.

For a slightly damped oscillation $\theta(t)\approx\theta\exp(-\beta t)$.

Initial energy of pendulum is $\frac{mgl}{2}\theta^2$.

Energy of pendulum after one period, $T=2\pi\sqrt{\frac{l}{g}}$, is

$$\frac{mgl}{2}\theta(T)^2=\frac{mgl}{2}\theta^2\exp\left(-2\beta T\right)$$

So energy lost in one period is

$$\frac{mgl}{2}\theta^2\left(1-\exp\left(-2\beta T\right)\right)\approx\frac{mgl}{2}\theta^2 2\beta T=mgl\theta^2\beta T$$

So energy lost after 7 days is

$$mgl\theta^2\beta T\frac{(7\text{ days})}{T}=mgl\theta^2\beta\,(7\text{ days})$$

This energy must be compensated by potential energy of the mass M as it falls h meters:

$$Mgh=mgl\theta^2\beta(7\text{ days})\quad\Rightarrow\quad\beta=\frac{Mh}{ml\theta^2(7\text{ days})}=0.01\text{ s}^{-1}$$

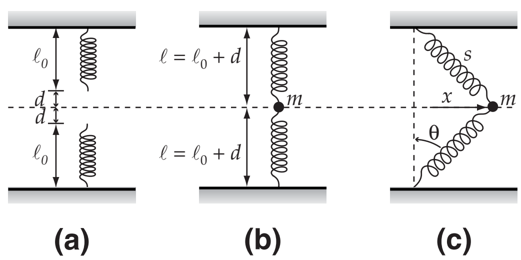
Knowing β we can easily find the coefficient Q (see Equation (3.64))

$$Q=\frac{\omega_R}{2\beta}=\frac{\sqrt{\omega_0{}^2-2\beta^2}}{2\beta}=\frac{\sqrt{\frac{g}{l}-2\beta^2}}{2\beta}=178$$

CHAPTER 4

Nonlinear Oscillations and Chaos

4-1.



The unextended length of each spring is ℓ_0, as shown in (a). In order to attach the mass m, each spring must be stretched a distance d, as indicated in (b). When the mass is moved a distance x, as in (c), the force acting on the mass (neglecting gravity) is

$$F = -2k\left(s - \ell_0\right)\sin\theta \tag{1}$$

where

$$s = \sqrt{\ell^2 + x^2} \tag{2}$$

and

$$\sin\theta = \frac{x}{\sqrt{\ell^2 + x^2}} \tag{3}$$

Then,

$$F(x) = -\frac{2kx}{\sqrt{\ell^2 + x^2}}\left[\sqrt{\ell^2 + x^2} - \ell_0\right] = -\frac{2kx}{\sqrt{\ell^2 + x^2}}\left[\sqrt{\ell^2 + x^2} - (\ell - d)\right]$$

$$= -2kx\left[1 - \frac{\ell - d}{\sqrt{\ell^2 + x^2}}\right] = -2kx\left[1 - \frac{\ell - d}{\ell}\left[1 + \frac{x^2}{\ell^2}\right]^{-1/2}\right] \tag{4}$$

Expanding the radical in powers of x^2/ℓ^2 and retaining only the first two terms, we have

$$F(x) \cong -2kx\left[1 - \frac{\ell - d}{\ell}\left[1 - \frac{1}{2}\frac{x^2}{\ell^2}\right]\right]$$

$$= -2kx\left[1 - \left[1 - \frac{d}{\ell}\right] + \frac{1}{2}\frac{\ell - d}{\ell}\frac{x^2}{\ell^2}\right]$$

$$= -\frac{2kd}{\ell}x - \frac{k(\ell - d)}{\ell^3}x^3 \tag{5}$$

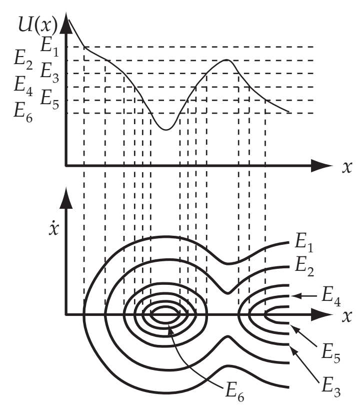
The potential is given by

$$U(x) = -\int F(x)\,dx \tag{6}$$

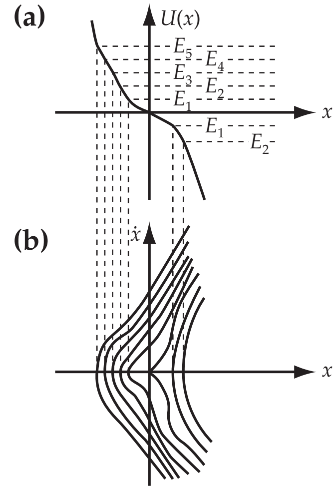
so that

$$\boxed{U(x) = \frac{kd}{\ell}x^2 + \frac{k(\ell - d)}{4\ell^3}x^4} \tag{7}$$

4-2. Using the general procedure explained in Section 4.3, the phase diagram is constructed as follows:



4-3. The potential $U(x) = -(\lambda/3x)^3$ has the form shown in (a) below. The corresponding phase diagram is given in (b):



4-4. Differentiation of Rayleigh's equation above yields

$$\dddot{x} - \left(a - 3b\dot{x}^2\right)\ddot{x} + \omega_0^2 \dot{x} = 0 \tag{1}$$

The substitution,

$$y = y_0 \sqrt{\frac{3b}{a}}\,\dot{x} \tag{2}$$

implies that

$$\left.\begin{aligned} \dot{x} &= \sqrt{\frac{a}{3b}}\,\frac{y}{y_0} \\ \ddot{x} &= \sqrt{\frac{a}{3b}}\,\frac{\dot{y}}{y_0} \\ \dddot{x} &= \sqrt{\frac{a}{3b}}\,\frac{\ddot{y}}{y_0} \end{aligned}\right\rbrack \tag{3}$$

When these expressions are substituted in (1), we find

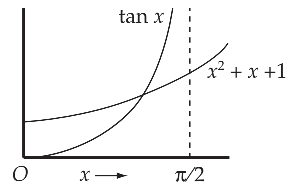
$$\sqrt{\frac{a}{3b}}\,\frac{\ddot{y}}{y_0} - \sqrt{\frac{a}{3b}}\left[a - \frac{3ba}{b}\,\frac{\dot{y}^2}{y_0^2}\right]\frac{\dot{y}}{y_0} + \omega_0^2\sqrt{\frac{a}{3b}}\,\frac{y}{y_0} = 0 \tag{4}$$

Multiplying by $y_0\sqrt{\dfrac{3b}{a}}$ and rearranging, we arrive at van der Pol's equation:

$$\boxed{\ddot{y} - \frac{a}{y_0^2}\left(y_0^2 - \dot{y}^2\right)\dot{y} + \omega_0^2 y = 0} \tag{5}$$

4-5.

a) A graph of the functions $f_1(x) = x^2 + x + 1$ and $f_2(x) = \tan x$ in the region $0 \le x \le \pi/2$ shows that there is an intersection (i.e., a solution) for $x_1 \cong 3\pi/8$.



The procedure is to use this approximate solution as a starting point and to substitute $x_1 = 3\pi/8$ into $f_1(x)$ and then solve for $x = \tan^{-1}[f_1(x_1)]$. If the result is within some specified amount, say 10^{-4}, of $3\pi/8$, then this is our solution. If the result is not within this amount of the starting value, then use the result as a new starting point and repeat the calculation. This procedure leads to the following values:

x_1	$f_1(x_1) = x_1^2 + x_1 + 1$	$\tan^{-1}[f_1(x_1)]$	Difference
1.1781	3.5660	1.2974	0.11930
1.2974	3.9806	1.3247	0.02728
1.3247	4.0794	1.3304	0.00573
1.3304	4.1004	1.3316	0.00118
1.3316	4.1047	1.3318	0.00024
1.3318	4.1056	1.3319	0.00005

Thus, the solution is $x = 1.3319$.

Parts b) and c) are solved in exactly the same way with the results:

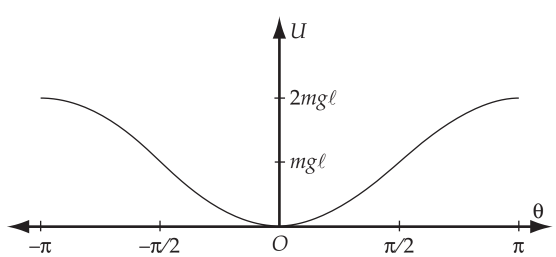
b) $x = 1.9151$

c) $x = 0.9271$

4-6. For the plane pendulum, the potential energy is

$$u = mg\ell[1 - \cos\theta] \tag{1}$$

If the total energy is *larger* than $2mg\ell$, all values of θ are allowed, and the pendulum revolves continuously in a circular path. The potential energy as a function of θ is shown in (a) below.



(a)

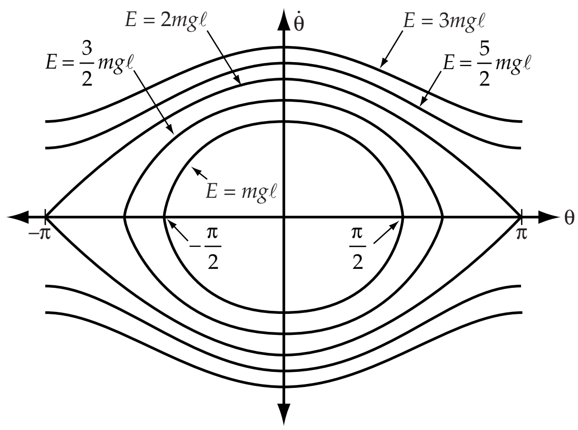
Since $T = E - U(\theta)$, we can write

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\dot{\theta}^2 = E - mg\ell\left(1 - \cos\theta\right) \tag{2}$$

and, therefore, the phase paths are constructed by plotting

$$\dot{\theta} = \sqrt{\frac{2}{m\ell^2}}\left[E - mg\ell\left(1 - \cos\theta\right)\right]^{1/2} \tag{3}$$

versus θ. The phase diagram is shown in (b) below.



(b)

4-7. Let us start with the equation of motion for the simple pendulum:

$$\ddot{\theta} = -\omega_0^2 \sin\theta \tag{1}$$

where $\omega^2 \equiv g/\ell$. Put this in terms of the horizontal component by setting $y \equiv x/\ell = \sin\theta$. Solving for θ and taking time derivatives, we obtain

$$\ddot{\theta} = \frac{y\dot{y}^2}{(1-y^2)^{3/2}} + \frac{\ddot{y}}{\sqrt{1-y^2}} \tag{2}$$

Since we are keeping terms to third order, we need to get a better handle on the $\dot{y}^2$ term. Help comes from the conservation of energy:

$$\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta = -mg\ell\cos\theta_0 \tag{3}$$

where θ_0 is the maximum angle the pendulum makes, and serves as a convenient parameter that describes the total energy. When written in terms of y, the above equation becomes (with the obvious definition for y_0)

$$\frac{\dot{y}^2}{1-y^2} = 2\omega_0^2\left(\sqrt{1-y^2} - \sqrt{1-y_0^2}\right) \tag{4}$$

Substituting (4) into (2), and the result into (1) gives

$$\ddot{y} + \omega_0^2 y \left(3\sqrt{1-y^2} - 2\sqrt{1-y_0^2} \right) = 0 \tag{5}$$

Using the binomial expansion of the square roots and keeping terms up to third order, we can obtain for the x equation of motion

$$\ddot{x} + \omega_0^2 x \left[1 + \frac{x_0^2}{\ell^2} \right] - \frac{3g}{2\ell^3} x^3 = 0 \tag{6}$$

4-8. For $x > 0$, the equation of motion is

$$m\ddot{x} = -F_0 \tag{1}$$

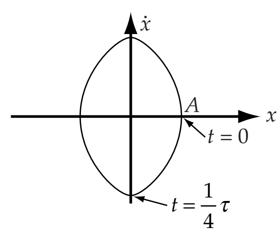
If the initial conditions are $x(0) = A$, $\dot{x}(0) = 0$, the solution is

$$x(t) = A - \frac{F_0}{2m} t^2 \tag{2}$$

For the phase path we need $\dot{x} = \dot{x}(x)$, so we calculate

$$\dot{x}(x) = \pm\sqrt{\frac{2F_0}{m}(A-x)} \tag{3}$$

Thus, the phase path is a parabola with a vertex on the x-axis at $x = A$ and symmetrical about both axes as shown below.



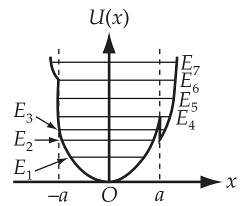
Because of the symmetry, the period τ is equal to 4 times the time required to move from $x = A$ to $x = 0$ (see diagram). Therefore, from (2) we have

$$\boxed{\tau = 4\sqrt{\frac{2mA}{F_0}}} \tag{4}$$

4-9. The proposed force derives from a potential of the form

$$U(x) = \begin{bmatrix} \dfrac{1}{2}kx^2 & |x| < a \\ \dfrac{1}{2}(x+\delta)x - \delta a x^2 & |x| > a \end{bmatrix}$$

which is plotted in (a) below.

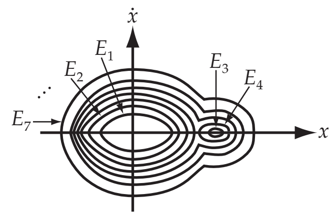


(a)

For small deviations from the equilibrium position ($x = 0$), the motion is just that of a harmonic oscillator.

For energies $E < E_6$, the particle cannot reach regions with $x < -a$, but it can reach regions of $x > a$ if $E > E_4$. For $E_2 < E < E_4$ the possibility exists that the particle can be trapped near $x = a$.

A phase diagram for the system is shown in (b) below.

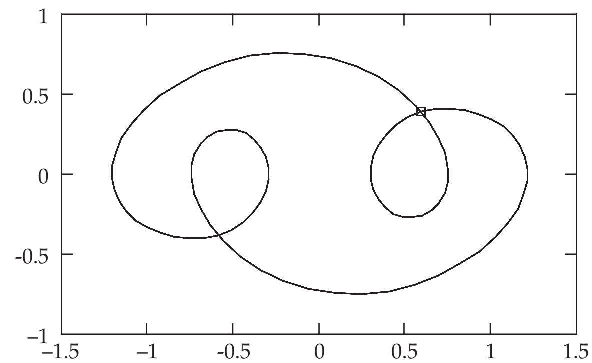


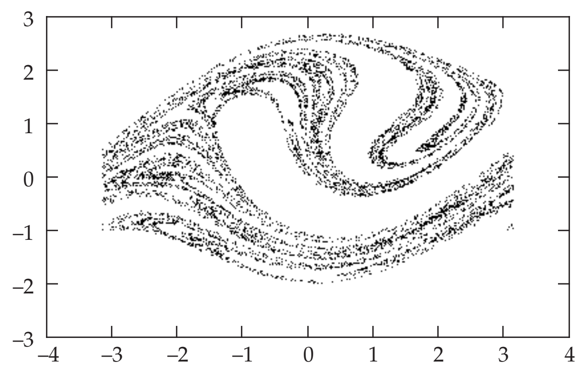
(b)

4-10. The system of equations that we need to solve are

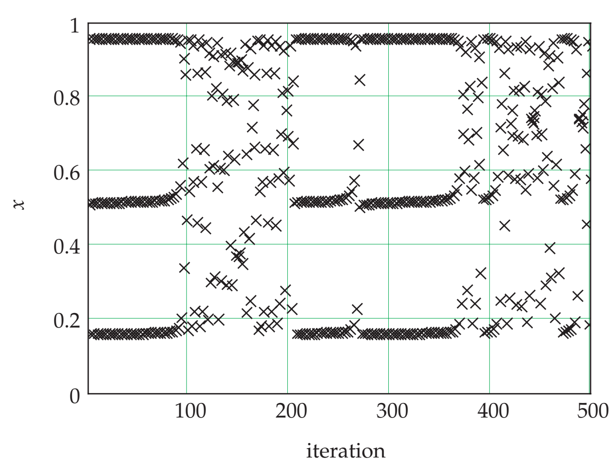
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -0.05\,y - \sin x + 0.7\cos \omega t \end{bmatrix} \tag{1}$$

The values of ω that give chaotic orbits are 0.6 and 0.7. Although we may appear to have chaos for other values, construction of a Poincaré plot that samples at the forcing frequency show that they all settle on a one period per drive cycle orbit. This occurs faster for some values of ω than others. In particular, when $\omega = 0.8$ the plot looks chaotic until it locks on to the point $(-2.50150, 0.236439)$. The phase plot for $\omega = 0.3$ shown in the figure was produced by numerical integration of the system of equations (1) with 100 points per drive cycle. The box encloses the point on the trajectory of the system at the start of a drive cycle. In addition, we also show Poincaré plot for the case $\omega = 0.6$ in figure, integrated over 8000 drive cycles with 100 points per cycle.

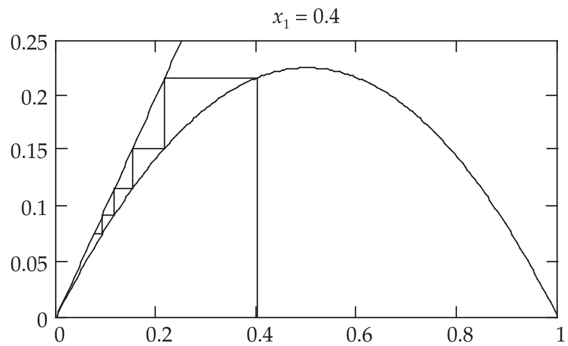


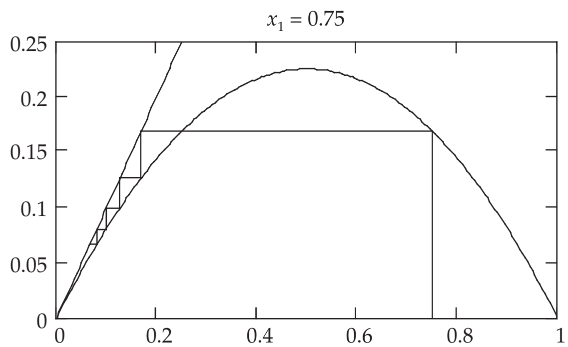


4-11. The three-cycle does indeed occur where indicated in the problem, and does turn chaotic near the 80th iteration. This value is approximate, however, and depends on the precision at which the calculations are performed. The behavior returns to a three-cycle near the 200th iteration, and stays that way until approximately the 270th iteration, although some may see it continue past the 300th.



4-12.



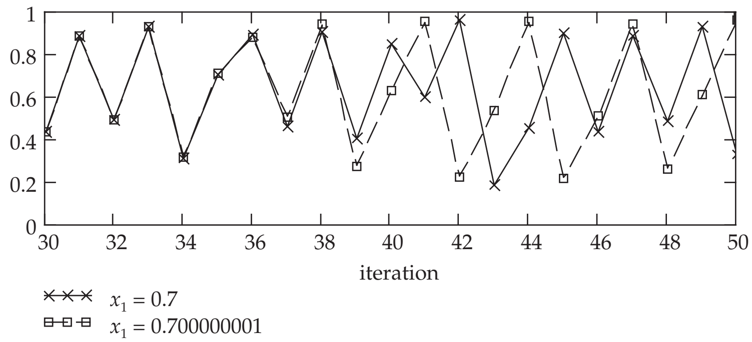


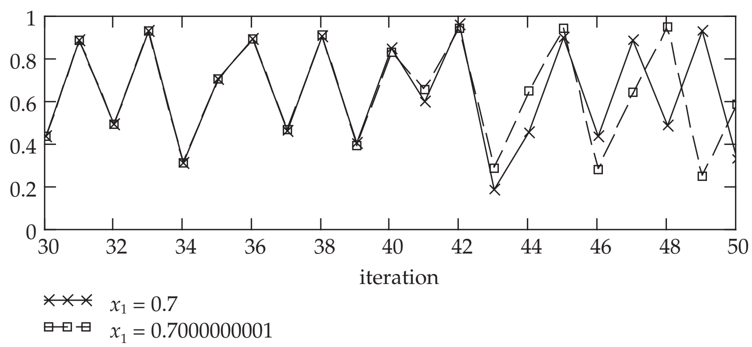
These plots are created in the manner described in the text. They are created with the logistic equation

$$x_{n+1} = 0.9 \cdot x_n \left(1 - x_n\right) \tag{1}$$

The first plot has the seed value $x_1 = 0.4$ as asked for in the text. Only one additional seed has been done here ($x_1 = 0.75$) as it is assumed that the reader could easily produce more of these plots after this small amount of practice.

4-13.



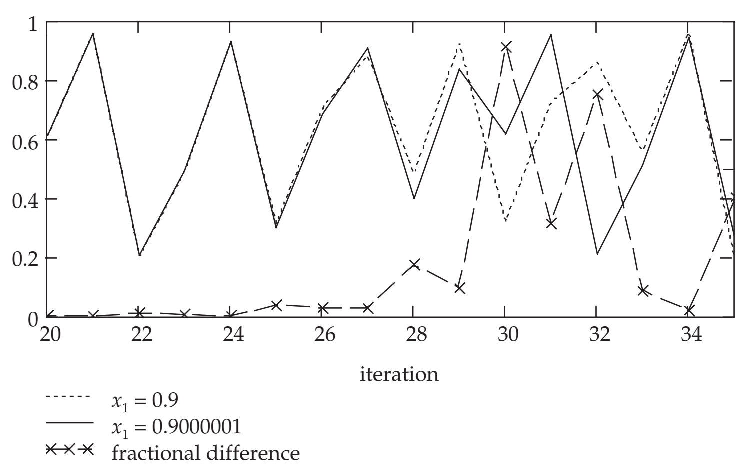


The plots are created by iteration on the initial values of (i) 0.7, (ii) 0.700000001, and (iii) 0.7000000001, using the equation

$$x_{n+1} = 2.5 \cdot x_n \left(1 - x_n^2\right) \tag{1}$$

A subset of the iterates from (i) and (ii) are plotted together, and clearly diverge by $n = 39$. The plot of (i) and (iii) clearly diverge by $n = 43$.

4-14.

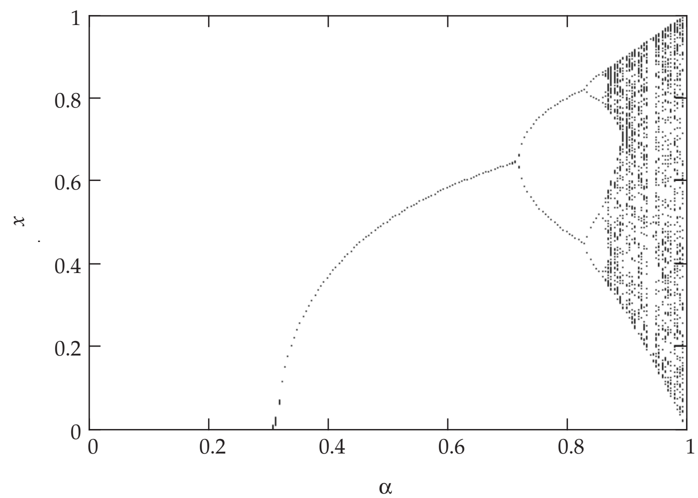


The given function with the given initial values are plotted in the figure. Here we use the notation $x_1 = 0.9$ and $y_1 = 0.9000001$, with $x_{n+1} = f(x_n)$ and $y_{n+1} = f(y_n)$ where the function is

$$f(x) = 2.5 \cdot x\left(1 - x^2\right) \tag{1}$$

The fractional difference is defined as $|x-y|/x$, and clearly exceeds 30% when n = 30.

4-15. A good way to start finding the bifurcations of the function $f(\alpha,x) = \alpha \sin \pi x$ is to plot its bifurcation diagram.



One can expand regions of the diagram to give a rough estimate of the location of a bifurcation. Its accuracy is limited by the fact that the map does not converge very rapidly near the bifurcation point, or more precisely, the Lyapunov exponent approaches zero. One may continue undaunted, however, with the help of a graphical fractal generating software application, to estimate quite a few of the period doublings α_n .Using Fractint for Windows, and Equation (4.47) to compute the Feigenbaum constant, we can obtain the following:

n	α	δ
1	0.71978	
2	0.83324	4.475
3	0.85859	4.611
4	0.86409	4.699
5	0.86526	4.680
6	0.86551	4.630
7	0.865564	4.463
8	0.8655761	

One can see that although we should obtain a better value of δ as n increases, numerical precision and human error quickly degrade the quality of the calculation. This is a perfectly acceptable answer to this question.

One may compute the α_n to higher accuracy by other means, all of which are a great deal more complicated. See, for example, *Exploring Mathematics with Mathematica*, which exploits the vanishing Lyapunov exponent. Using their algorithm, one obtains the following:

n	α	δ
1	0.719962	
2	0.833266	4.47089
3	0.858609	4.62871
4	0.864084	4.66198
5	0.865259	4.65633
6	0.865511	5.13450
7	0.865560	

Note that these are shown here only as reference, and the student may not necessarily be expected to perform to this degree of sophistication. The above values are only good to about 10^{-6}, but this time only limited by machine precision. Another alternative in computing the Feigenbaum constant, which is not requested in the problem, is to use the so-called "supercycles," or super-stable points R_n, which are defined by

$$f^{\left(2^{n-1}\right)}\left[R_n, \frac{1}{2}\right] = \frac{1}{2}$$

The values R_n obey the same scaling as the bifurcation points, and are much easier to compute since these points converge faster than for other α (the Lyapunov exponent goes to $-\infty$). See, for example, *Deterministic Chaos: An Introduction* by Heinz Georg Schuster or *Chaos and Fractals: New Frontiers of Science* by Peitgen, Jürgens and Saupe. As a result, the estimates for δ obtained in this way are more accurate than those obtained by calculating the bifurcation points.

4-16. The function $y = f(x)$ intersects the line $y = x$ at $x = x_0$, i.e. x_0 is defined as the point where $x_0 = f\left(x_0\right)$. Now expand $f(x)$ in a Taylor series, so that near x_0 we have

$$f(x) \simeq f\left(x_0\right) + \beta\left(x - x_0\right) = x_0 + \beta\left(x - x_0\right) \tag{1}$$

where

$$\beta \equiv \left.\frac{df}{dx}\right|_{x_0} \tag{2}$$

Now define $\varepsilon_n \equiv x_n - x_0$. If we have x_1 very close to x_0, then ε_1 should be very small, and we may use the Taylor expansion. The equation of iteration $x_{n+1} = f\left(x_n\right)$ becomes

$$\varepsilon_{n+1} \simeq \beta \varepsilon_n \tag{3}$$

If the approximation (1) remains valid from the initial value, we have $\varepsilon_{n+1} \simeq \beta^n \varepsilon_1$.

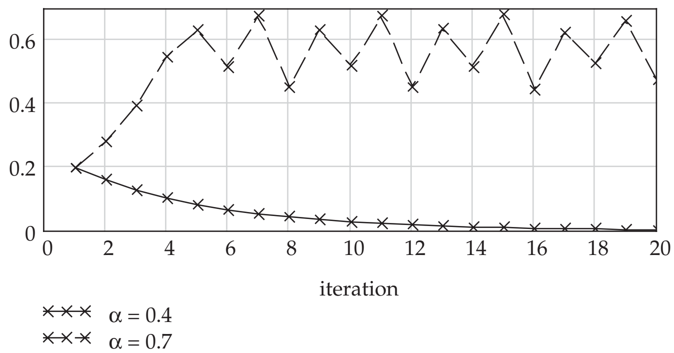
a) The values $x_n - x_0 = \varepsilon_n$ form the geometric sequence $\varepsilon_1,\ \beta\varepsilon_1,\ \beta^2\varepsilon_1,\ \ldots$.

b) Clearly, when $|\beta| < 1$ we have stability since

$$\lim_{n\to\infty} \varepsilon_n = 0$$

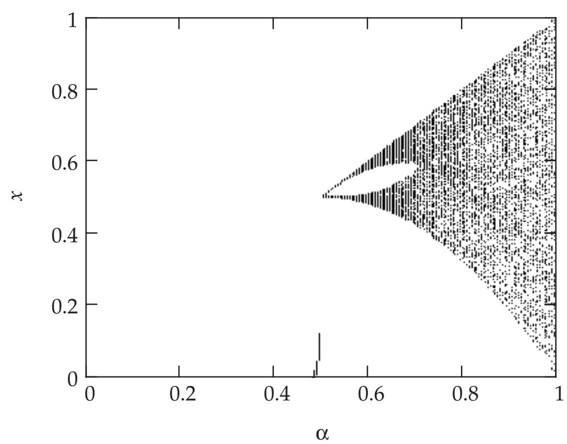
Similarly we have a divergent sequence when $|\beta| > 1$, although it will not really be exponentially divergent since the approximation (1) becomes invalid after some number of iterations, and normally the range of allowable x_n is restricted to some subset of the real numbers.

4-17.



The first plot (with $\alpha = 0.4$) converges rather rapidly to zero, but the second (with $\alpha = 0.7$) does appear to be chaotic.

4-18.



The tent map always converges to zero for $\alpha < 0.5$. Near $\alpha = 0.5$ it takes longer to converge, and that is the artifact seen in the figure. There exists a "hole" in the region $0.5 < \alpha < 0.7$ (0.7 is approximate), where the iterations are chaotic but oscillate between an upper and lower range of values. For $\alpha > 0.7$, there is only a single range of chaos, which becomes larger until it fills the range (0,1) at $\alpha = 1$.

4-19. From the definition in Equation (4.52) the Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \left. \frac{df}{dx} \right|_{x_i} \right| \tag{1}$$

The tent map is defined as

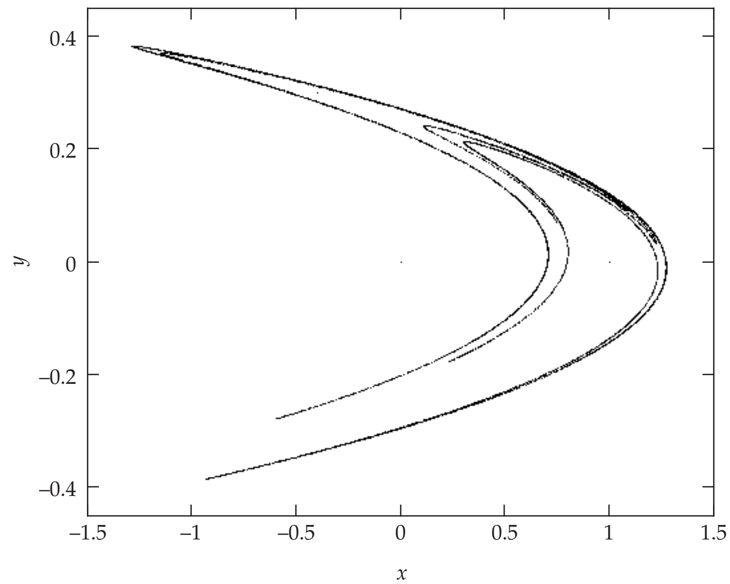
$$f(x) = \begin{cases} 2\alpha x & \text{for } 0 < x < 1/2 \\ 2\alpha(1-x) & \text{for } 1/2 < x < 1 \end{cases} \tag{2}$$

This gives $|df/dx| = 2\alpha$, so we have

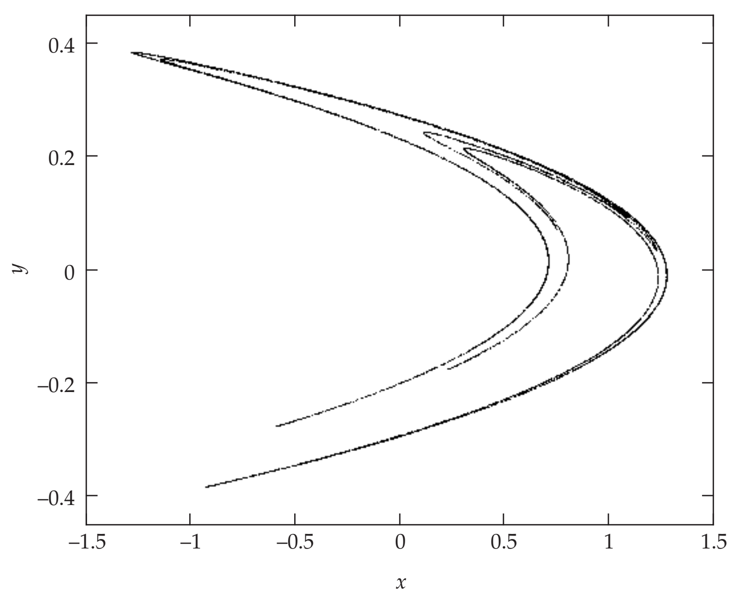
$$\lambda = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)\ln(2\alpha) = \ln(2\alpha) \tag{3}$$

As indicated in the discussion below Equation (4.52), chaos occurs when λ is positive: $\alpha > 1/2$ for the tent map.

4-20.



4-21.



The shape of this plot (the attractor) is nearly identical to that obtained in the previous problem. In Problem 4-20, however, we can clearly see the first few iterations (0,0), (1,0), (–0.4,0.3), whereas the next iteration (1.076,–0.12) is almost on the attractor. In this problem the initial value is taken to be on the attractor already, so we do not see any transient points.

4-22. The following. system of differential equations were integrated numerically

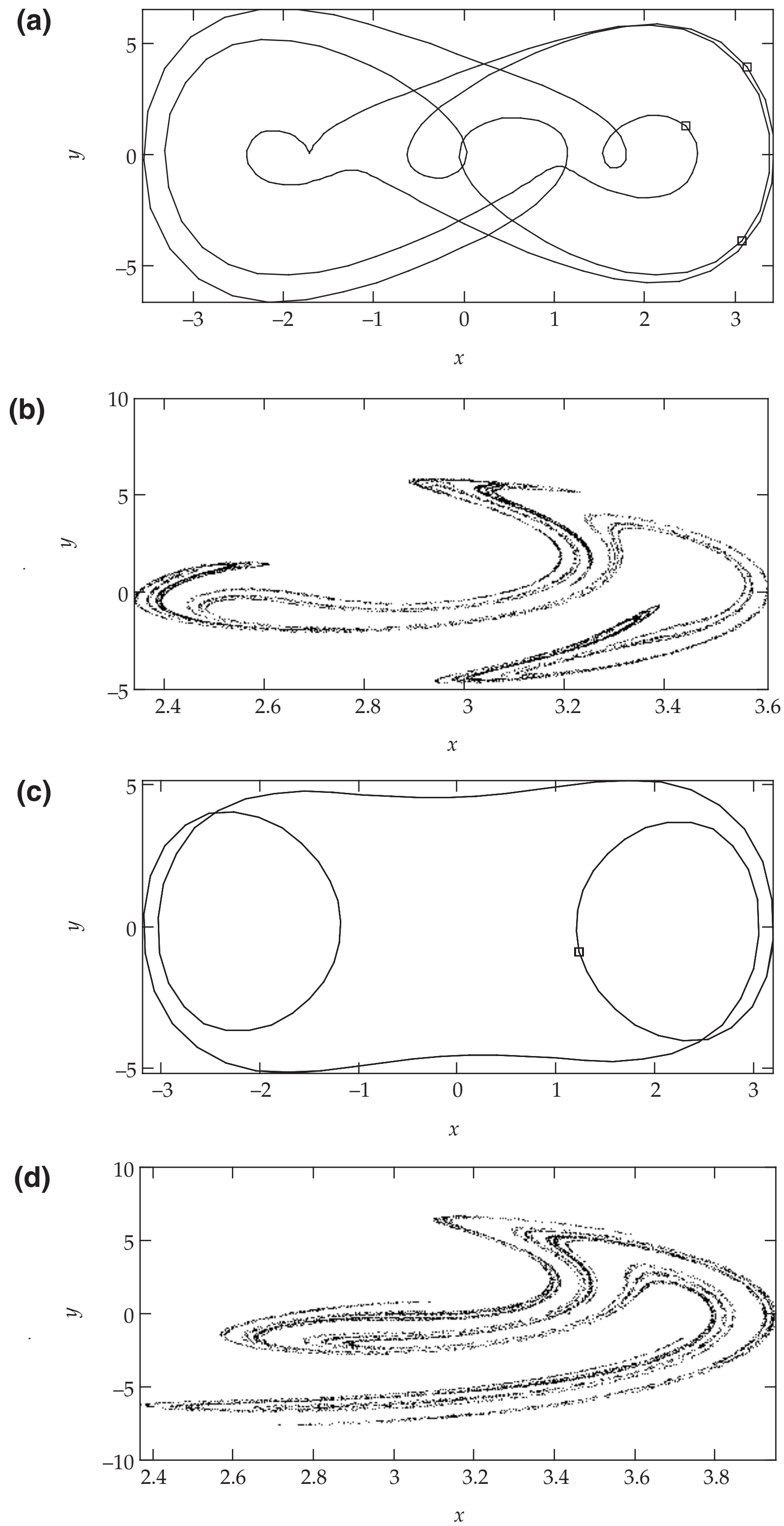
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -0.1\,y - x^3 + B\cos t \end{bmatrix} \tag{1}$$

using different values of B in the range [9.8,13.4], and with a variety of initial conditions. The integration range is over a large number of drive cycles, throwing away the first several before starting to store the data in order to reduce the effects of the transient response. For the case $B = 9.8$, we have a one period per three drive cycle orbit. The phase space plot (line) and Poincaré section (boxes) for this case are overlaid and shown in figure (a). All integrations are done here with 100 points per drive cycle. One can experiment with B and determine that the system becomes chaotic somewhere between 9.8 and 9.9. The section for $B = 10.0$, created by integrating over 8000 drive cycles, is shown in figure (b). If one further experiments with different values of B, and one is also lucky enough to have the right initial conditions, (0,0) is one that works, then a transition will be found for B in the range (11.6,11.7). As an example of the different results one can get depending on the initial conditions, we show two plots in figure (c). One is a phase plot, overlaid with its section, for $B = 12.0$ and the initial condition (0,0). Examination of the time evolution reveals that it has one period per cycle. The second plot is a Poincaré section for the same B but with the initial condition (10,0), clearly showing chaotic motion. Note that the section looks quite similar to the one for $B = 10.0$. Another transition is in the range (13.3,13.4), where the orbits become regular again, with one period per drive cycle, regardless of initial conditions. The phase plot for $B = 13.4$ looks similar to the one with $B = 12.0$ and initial condition (0,0).

To summarize, we may enumerate the above transition points by B_1, B_2, and B_3. Circumventing the actual task of computing where these transition points are, we do know that $9.8 < B_1 < 9.9$, $11.6 < B_2 < 11.7$, and $13.3 < B_3 < 13.4$. We can then describe the behavior of the system by region.

- $B < B_1$: one period per three drive cycles
- $B_1 < B < B_2$: chaotic
- $B_2 < B < B_3$: mixed chaotic/one period per drive cycle (depending on initial conditions)
- $B_3 < B$: one period per drive cycle

We should remind ourselves, though, that the above list only applies for B in the range we have examined here. We do not know the behavior when $B < 9.8$ and $B > 13.4$, without going beyond the scope of this problem.


(a)
y
5
0
−5
−3 −2 −1 0 1 2 3
x
(b)
y
10
5
0
−5
2.4 2.6 2.8 3 3.2 3.4 3.6
x
(c)
y
5
0
−5
−3 −2 −1 0 1 2 3
x
(d)
y
10
5
0
−5
−10
2.4 2.6 2.8 3 3.2 3.4 3.6 3.8
x

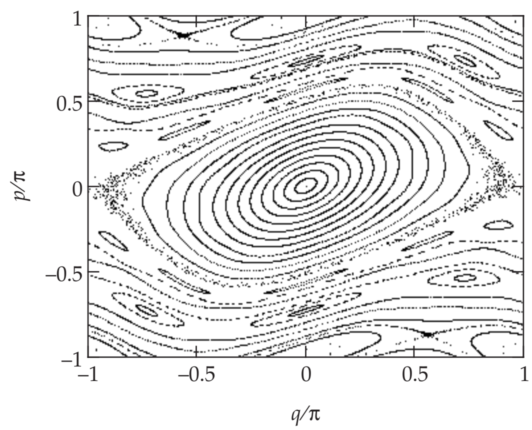
4-23. The Chirikov map is defined by

$$p_{n+1} = p_n - K \sin q_n \tag{1}$$

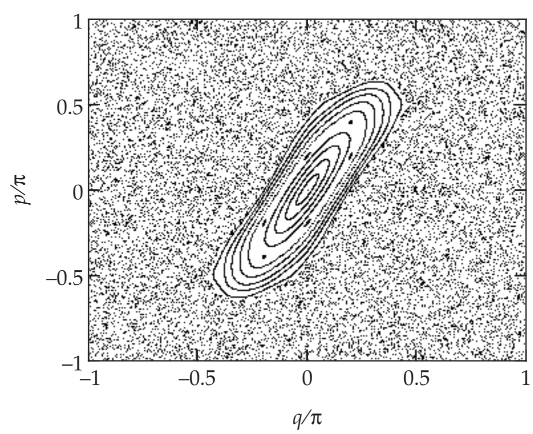
$$q_{n+1} = q_n - p_{n+1} \tag{2}$$

The results one should get from doing this problem should be some subset of the results shown in figures (a), (b), and (c) (for $K = 0.8$, 3.2, and 6.4, respectively). These were actually generated using some not-so-random initial points so that a reasonably complete picture could be made. What look to be phase paths in the figures are actually just different points that come from iterating on a single initial condition. For example, in figure (a), an ellipse about the origin (just pick one) comes from iterating on any one of the points on it. Above the ellipses is chaotic orbit, then a five ellipse orbit (all five come from a single initial condition), etc. The case for $K = 3.2$ is similar except that there is an orbit outside of which the system is always undergoing chaotic motion. Finally, for $K = 6.4$ the entire space is filled with chaotic orbits, with the exception of two small lobes. Inside of these lobes are regular orbits (the ones in the left are separate from the ones in the right).

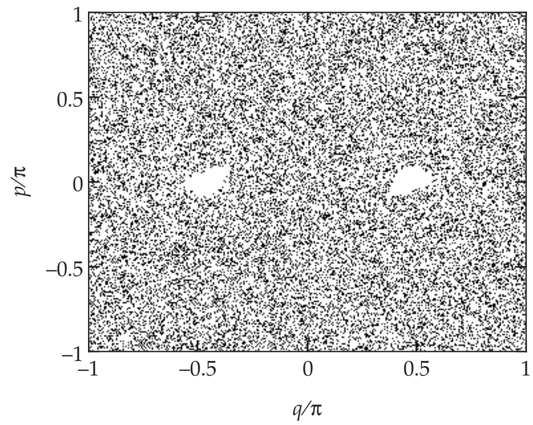
(a)



(b)



(c)



4-24.

a) The Van de Pol equation is

$$\frac{d^2x}{dt^2} + \omega_0{}^2 x = \mu\left(a^2 - x^2\right)\frac{dx}{dt}$$

Now look for solution in the form

$$x(t) = b\cos\omega_0 t + u(t) \tag{1}$$

we have

$$\frac{dx}{dt} = -b\omega_0 \sin\omega_0 t + \frac{du}{dt}$$

and

$$\frac{d^2x}{dt^2} = -b\omega_0{}^2\cos\omega_0 t + \frac{d^2u}{dt^2}$$

Putting these into the Van de Pol equation, we obtain

$$\frac{d^2u(t)}{dt^2} + \omega_0 u(t) = -\mu\left\{b^2\cos^2\omega_0 t + u^2(t) + 2bu(t)\cos\omega_0 t - a^2\right\}\left\{-b\omega_0\sin\omega_0 t + \frac{du(t)}{dt}\right\}$$

From this one can see that $u(t)$ is of order μ (i.e. $u \sim O(\mu)$), which is assumed to be small here. Keeping only terms up to order μ, the above equation reads

$$\frac{d^2u(t)}{dt^2} + \omega_0 u(t) = -\mu\left\{-b^3\omega_0\sin\omega_0 t\cos^2\omega_0 t + a^2 b\omega_0\sin\omega_0 t\right\}$$

$$= -\mu b\omega_0\left\{\left(a^2 - \frac{b^2}{4}\right)\sin\omega_0 t - \frac{b^2}{4}\sin 3\omega_0 t\right\}$$

(where we have used the identity $4\sin\omega_0 t\cos^2\omega_0 t = \sin\omega_0 t + \sin 3\omega_0 t$)

This equation has 2 frequencies (ω_0 and $3\omega_0$), and is complicated. However, if $b = 2a$ then the term $\sin\omega_0 t$ disappears and the above equation becomes

$$\frac{d^2u(t)}{dt^2} + \omega_0 u(t) = \mu\omega_0 \frac{b^3}{4} \sin 3\omega_0 t$$

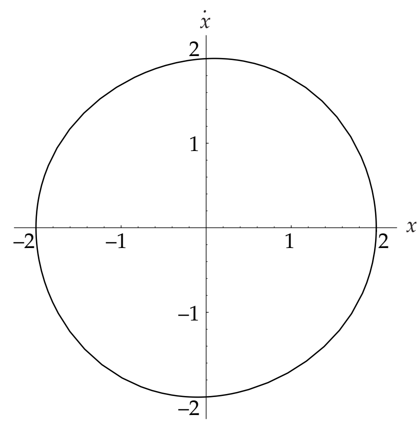
We let $b = 2a$, and the solution for this equation is

$$u(t) = -\frac{\mu b^3}{32\omega_0} \sin 3\omega_0 t = -\frac{\mu a^3}{4\omega_0} \sin 3\omega_0 t$$

So, finally putting this form of $u(t)$ into (1), we obtain one of the exact solutions of Van de Pol equation:

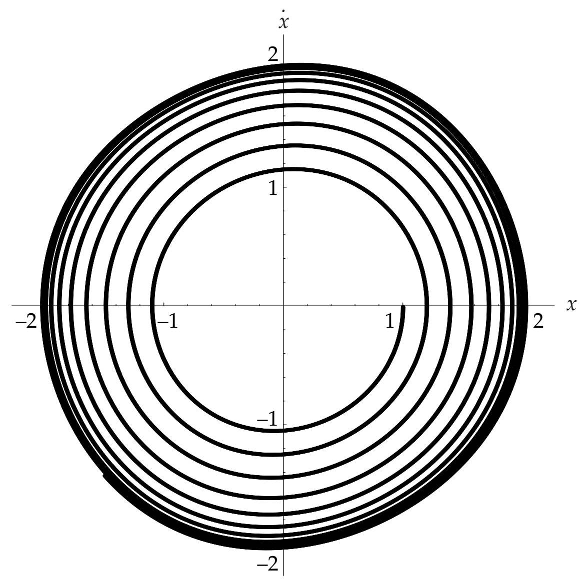
$$u(t) = 2a \cos \omega_0 t - \frac{\mu a^3}{4\omega_0} \sin 3\omega_0 t$$

b) See phase diagram below. Since $\mu = 0.05$ is very small, then actually the second term in the expression of $u(t)$ is negligible, and the phase diagram is very close to a circle of radius $b = 2a = 2$.

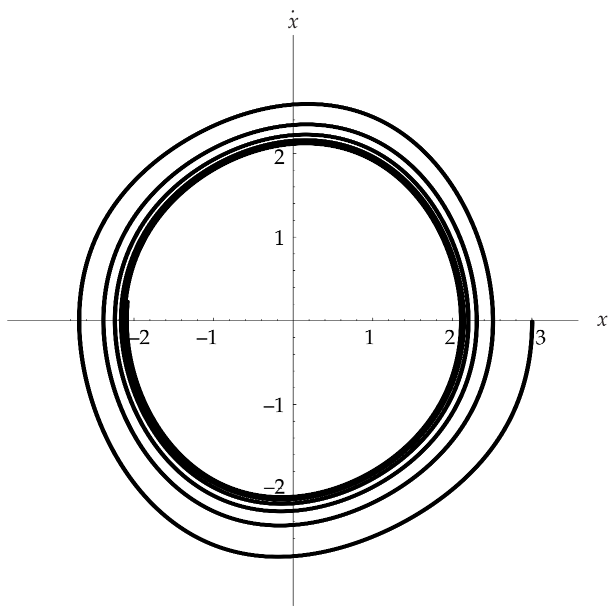


4-25. We have used *Mathematica* to numerically solve and plot the phase diagram for the van de Pol equation. Because $\mu = 0.07$ is a very small value, the limit cycle is very close to a circle of radius $b = 2a = 2$.

a) In this case, see figure a), the phase diagram starts at the point ($x = 1$, $x' = 0$) inside the limit cycle, so the phase diagram spirals outward to ultimately approach the stable solution presented by the limit cycle (see problem 4-24 for exact expression of stable solution).

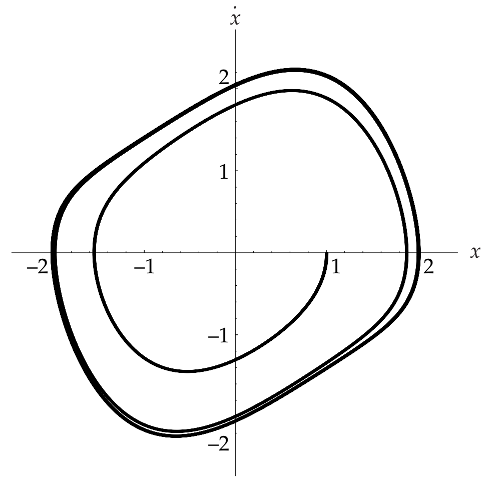


b) In this case, see figure b), the phase diagram starts at the point (x = 3, x' = 0) outside the limit cycle, so the phase diagram spirals inward to ultimately approaches the stable solution presented by the limit cycle (see problem 4-24 for exact expression of stable solution).

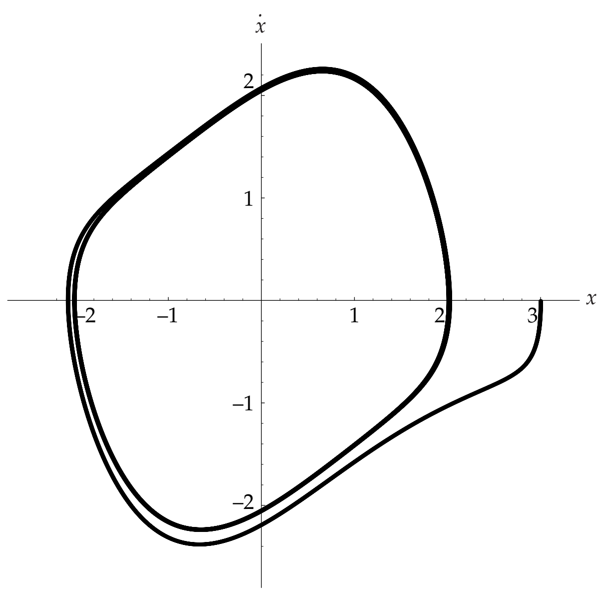


4-26. We have used *Mathematica* to numerically solve and plot the phase diagram for the van de Pol equation. Because $\mu = 0.5$ is not a small value, the limit cycle is NOT close to a circle (see problem 4-24 above).

a) In this case, see figure a), the phase diagram starts at the point (x = 1, x' = 0) inside the limit cycle, so the phase diagram spirals outward to ultimately approach the stable solution presented by the limit cycle (see problem 4-24 for exact expression of stable solution).



b) In this case (see figure below), the phase diagram starts at the point (x = 3, x' = 0) outside the limit cycle, so the phase diagram spirals inward to ultimately approaches the stable solution presented by the limit cycle (see problem 4-24 for exact expression of stable solution).



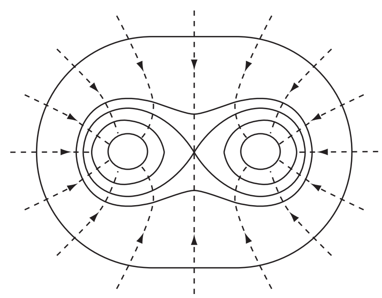
CHAPTER 5

Gravitation

5-1.

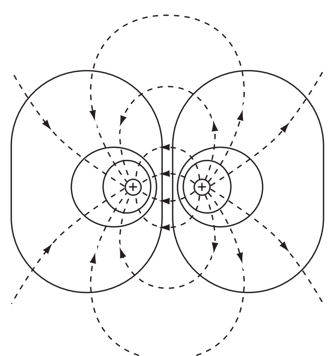
a) *Two identical masses*:

The lines of force (dashed lines) and the equipotential surfaces (solid lines) are as follows:



b) *Two masses, +M and –M*:

In this case the lines of force do not continue outward to infinity, as in a), but originate on the "negative" mass and terminate on the positive mass. This situation is similar to that for two electrical charges, $+q$ and $-q$; the difference is that the electrical lines of force run from $+q$ to $-q$.



5-2. Inside the sphere the gravitational potential satisfies

$$\nabla^2 \phi = 4\pi G\, \rho(r) \tag{1}$$

Since $\rho(r)$ is spherically symmetric, ϕ is also spherically symmetric. Thus,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \phi}{\partial r}\right] = 4\pi G\, \rho(r) \tag{2}$$

The field vector is independent of the radial distance. This fact implies

$$\frac{\partial \phi}{\partial r} = \text{constant} \equiv C \tag{3}$$

Therefore, (2) becomes

$$\frac{2C}{r} = 4\pi G \rho \tag{4}$$

or,

$$\boxed{\rho = \frac{C}{2\pi G r}} \tag{5}$$

5-3. In order to remove a particle from the surface of the Earth and transport it infinitely far away, the initial kinetic energy must equal the work required to move the particle from $r = R_e$ to $r = \infty$ against the attractive gravitational force:

$$\int_{R_e}^{\infty} G\,\frac{M_e m}{r^2}\, dr = \frac{1}{2} m v_0^2 \tag{1}$$

where M_e and R_e are the mass and the radius of the Earth, respectively, and v_0 is the initial velocity of the particle at $r = R_e$.

Solving (1), we have the expression for v_0:

$$v_0 = \sqrt{\frac{2G\, M_e}{R_e}} \tag{2}$$

Substituting $G = 6.67 \times 10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2$, $M_e = 5.98 \times 10^{24}\ \text{kg}$, $R_e = 6.38 \times 10^6\ \text{m}$, we have

$$\boxed{v_0 \cong 11.2\ \text{km/sec}} \tag{3}$$

5-4. The potential energy corresponding to the force is

$$U = -\int F\, dx = mk^2 \int \frac{dx}{x^3} = -\frac{mk^2}{2x^2} \tag{1}$$

The central force is conservative and so the total energy is constant and equal to the potential energy at the initial position, $x = d$:

$$E = \text{constant} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\frac{k^2}{x^2} = -\frac{1}{2}m\frac{k^2}{d^2} \tag{2}$$

Rewriting this equation in integrable form,

$$dt = -\int_d^0 \frac{dx}{\sqrt{k^2\left[\dfrac{1}{x^2} - \dfrac{1}{d^2}\right]}} = -\frac{d}{k}\int_d^0 \frac{x}{\sqrt{d^2 - x^2}}\,dx \tag{3}$$

where the choice of the negative sign for the radical insures that x decreases as t increases. Using Eq. (E.9), Appendix E, we find

$$t = \frac{d}{k}\sqrt{d^2 - x^2}\,\Big|_d^0 \tag{4}$$

or

$$\boxed{t = \frac{d^2}{k}}$$

5-5. The equation of motion is

$$m\ddot{x} = -G\frac{Mm}{x^2} \tag{1}$$

Using conservation of energy, we find

$$\frac{1}{2}\dot{x}^2 - G\,M\frac{1}{x} = E = -G\,M\frac{1}{x_\infty} \tag{2}$$

$$\frac{dx}{dt} = -\sqrt{2GM\left[\frac{1}{x} - \frac{1}{x_\infty}\right]} \tag{3}$$

where x_∞ is some fixed large distance. Therefore, the time for the particle to travel from x_∞ to x is

$$t = -\int_{x_\infty}^{x} \frac{dx}{\sqrt{2GM\left[\dfrac{1}{x} - \dfrac{1}{x_\infty}\right]}} = -\frac{1}{\sqrt{GM}}\int_{x_\infty}^{x}\sqrt{\frac{xx_\infty}{2(x_\infty - x)}}\,dx$$

Making the change of variable, $x \to y^2$, and using Eq. (E.7), Appendix E, we obtain

$$t = \sqrt{\frac{x_\infty}{2GM}}\left[\sqrt{x(x_\infty - x)} - x_\infty \sin^{-1}\sqrt{\frac{x}{x_\infty}}\right]_{x_\infty}^{x} \tag{4}$$

If we set $x = 0$ and $x = x_\infty/2$ in (4), we can obtain the time for the particle to travel the total distance and the first half of the distance.

$$T_0 = \int_{x_\infty}^{0} dt = \frac{1}{\sqrt{GM}}\left[\frac{x_\infty}{2}\right]^{3/2} \tag{5}$$

$$T_{1/2} = \int_{x_\infty}^{x_\infty/2} dt = \frac{1}{\sqrt{GM}}\left[\frac{x_\infty}{2}\right]^{3/2}\left[1+\frac{\pi}{2}\right] \tag{6}$$

Hence,

$$\frac{T_{1/2}}{T_0} = \frac{1+\dfrac{\pi}{2}}{\pi}$$

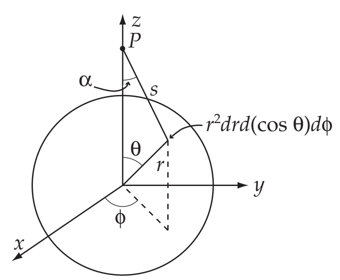
Evaluating the expression,

$$\frac{T_{1/2}}{T_0} = 0.818 \tag{7}$$

or

$$\boxed{\frac{T_{1/2}}{T_0} \cong \frac{9}{11}} \tag{8}$$

5-6.



Since the problem has symmetry around the z-axis, the force at the point P has only a z-component. The contribution to the force from a small volume element is

$$dg_z = -G\,\frac{\rho}{s^2}\,r^2 dr\, d(\cos\theta)\, d\phi \cos\alpha \tag{1}$$

where ρ is the density. Using $\cos\alpha = \dfrac{z - r\cos\theta}{s}$ and integrating over the entire sphere, we have

$$g_z = -G\rho \int_0^a r^2 dr \int_{-1}^{+1} d(\cos\theta) \int_0^{2\pi} d\phi \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2rz\cos\theta\right)^{3/2}} \tag{2}$$

Now, we can obtain the integral of $\cos\theta$ as follows:

$$I = \int_{-1}^{+1} \frac{z - r\cos\theta}{\left(r^2 + z^2 - 2rz\cos\theta\right)^{3/2}}\, d(\cos\theta)$$

$$= -\frac{\partial}{\partial z} \int_{-1}^{+1} \left(r^2 + z^2 - 2rz\cos\theta\right)^{1/2}\, d(\cos\theta)$$

Using Eq. (E.5), Appendix E, we find

$$I = -\frac{\partial}{\partial z}\left[-\frac{1}{rz}\left(r^2 + z^2 - 2rz\cos\theta\right)^{1/2} \Big|_{-1}^{+1} \right]$$

$$= -\frac{\partial}{\partial z}\left[\frac{2}{z}\right] = \frac{2}{z^2} \tag{3}$$

Therefore, substituting (3) into (2) and performing the integral with respect to r and ϕ, we have

$$g_z = -G\rho \frac{a^3}{3}\frac{2}{z^2}\cdot 2\pi$$

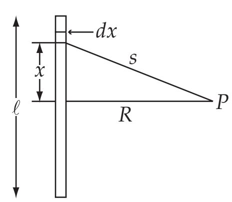
$$= -G\frac{4\pi}{3} a^3 \rho \frac{1}{z^2} \tag{4}$$

But $\frac{4\pi}{3} a^3 \rho$ is equal to the mass of the sphere. Thus,

$$\boxed{g_z = -GM\frac{1}{z^2}} \tag{5}$$

Thus, as we expect, the force is the same as that due to a point mass M located at the center of the sphere.

5-7.



The contribution to the potential at P from a small line element is

$$d\Phi = -G\frac{\rho_\ell}{s}\, dx \tag{1}$$

where $\rho_\ell = \frac{M}{\ell}$ is the linear mass density. Integrating over the whole rod, we find the potential

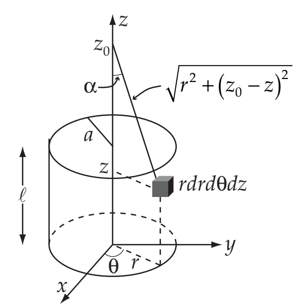
$$\Phi = -G\frac{M}{\ell}\int_{-\ell/2}^{\ell/2} \frac{1}{\sqrt{x^2 + R^2}}\, dx \tag{2}$$

Using Eq. (E.6), Appendix E, we have

$$\Phi = -G\frac{M}{\ell}\ln\left[x+\sqrt{x^2+R^2}\right]_{-\ell/2}^{\ell/2} = -\frac{GM}{\ell}\ln\left[\frac{\dfrac{\ell}{2}+\sqrt{\dfrac{\ell^2}{4}+R^2}}{-\dfrac{\ell}{2}+\sqrt{\dfrac{\ell^2}{4}+R^2}}\right]$$

$$\boxed{\Phi = -\frac{GM}{\ell}\ln\left[\frac{\sqrt{\ell^2+4R^2}+\ell}{\sqrt{\ell^2+4R^2}-\ell}\right]} \tag{3}$$

5-8.



Since the system is symmetric about the z-axis, the x and y components of the force vanish and we need to consider only the z-component of the force. The contribution to the force from a small element of volume at the point (r,θ,z) for a unit mass at $(0,0,z_0)$ is

$$dg_z = -G\rho\frac{rdrd\theta dz}{r^2+(z_0-z)^2}\cos\alpha$$

$$= -G\rho\frac{(z_0-z)\,rdrd\theta dz}{\left[r^2+(z_0-z)^2\right]^{3/2}} \tag{1}$$

where ρ is the density of the cylinder and where we have used $\cos\alpha = \dfrac{(z_0-z)}{\sqrt{r^2+(z_0-z)^2}}$. We can find the net gravitational force by integrating (1) over the entire volume of the cylinder. We find

$$g_z = -G\rho\int_0^a rdr\int_0^{2\pi}d\theta\int_0^{\ell}dz\frac{z_0-z}{\left[r^2+(z_0-z)^2\right]^{3/2}}$$

Changing the variable to $x = z_0 - z$, we have

$$g_z = 2\pi G\rho\int_0^a rdr\int_{z_0}^{z_0-\ell}\frac{xdx}{\left[r^2+x^2\right]^{3/2}} \tag{2}$$

Using the standard integral,

$$\int \frac{xdx}{\sqrt{\left(a^2 \pm x^2\right)^3}} = \frac{-1}{\sqrt{a^2 \pm x^2}} \tag{3}$$

we obtain

$$g_z = -2\pi G\rho \int_0^a dr \left[\frac{r}{\sqrt{r^2 + \left(z_0 - \ell\right)^2}} - \frac{r}{\sqrt{r^2 + z_0^2}} \right] \tag{4}$$

Next, using Eq. (E.9), Appendix E, we obtain

$$\boxed{g_z = -2\pi G\rho \left[\sqrt{a^2 + \left(z_0 - \ell\right)^2} - \sqrt{a^2 + z_0^2} + \ell \right]} \tag{5}$$

Now, let us find the force by first computing the potential. The contribution from a small element of volume is

$$d\Phi = -G\rho \frac{rdrd\theta dz}{\sqrt{r^2 + \left(z_0 - z\right)^2}} \tag{6}$$

Integrating over the entire volume, we have

$$d\Phi = -G\rho \int_0^\ell dz \int_0^{2\pi} d\theta \int_0^a dr \frac{r}{\sqrt{r^2 + \left(z_0 - z\right)^2}} \tag{7}$$

Using Eq. (E.9), Appendix E, again, we find

$$d\Phi = -2\pi G\rho \int_0^\ell dz \left[\sqrt{a^2 + \left(z_0 - z\right)^2} - \left(z_0 - z\right) \right] \tag{8}$$

Now, we use Eqs. (E.11) and (E.8a), Appendix E, and obtain

$$\Phi = -2\pi G\rho \left[-\frac{\left(z_0 - \ell\right)}{2} \sqrt{a^2 + \left(z_0 - \ell\right)^2} + \frac{a^2}{2} \ln\left[-2\left(z_0 - \ell\right) + 2\sqrt{\left(z_0 - \ell\right)^2 + a^2} \right] \right.$$
$$\left. + \frac{z_0}{2} \sqrt{a^2 + z_0^2} - \frac{a^2}{2} \ln\left[-2z_0 + 2\sqrt{z_0^2 + a^2} \right] - z_0 \ell + \frac{1}{2} \ell^2 \right]$$

Thus, the force is

$$g_z = -\frac{\partial \Phi}{\partial z_0} = -2\pi G\rho \left[\frac{1}{2}\sqrt{a^2 + (z_0 - \ell)^2} + \frac{1}{2}\frac{(z_0 - \ell)^2}{\sqrt{a^2 + (z_0 - \ell)^2}} + \frac{a^2}{2}\frac{1 - \dfrac{z_0 - \ell}{\sqrt{(z_0 - \ell)^2 + a^2}}}{-(z_0 - \ell) + \sqrt{(z_0 - \ell)^2 + a^2}} \right.$$

$$\left. -\frac{1}{2}\sqrt{a^2 + z_0^2} - \frac{1}{2}\frac{z_0^2}{\sqrt{a^2 + z_0^2}} - \frac{a^2}{2}\frac{1 - \dfrac{z_0}{\sqrt{z_0^2 + a^2}}}{-z_0 + \sqrt{z_0^2 + a^2}} + \ell \right] \tag{9}$$

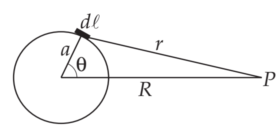
or,

$$\boxed{g_z = -2\pi G\rho \left[\sqrt{a^2 + (z_0 - \ell)^2} - \sqrt{a^2 + z_0^2} + \ell \right]} \qquad (10$$

and we obtain the same result as in (5).

In this case, it is clear that it is considerably easier to compute the force directly. (See the remarks in Section 5.4.)

5-9.



The contribution to the potential at the point P from a small line element $d\ell$ is

$$\Phi = -G\int \frac{\rho_\ell d\ell}{r} \tag{1}$$

where ρ_ℓ is the linear mass density which is expressed as $\rho_\ell = \dfrac{M}{2\pi a}$. Using $r = \sqrt{R^2 + a^2 - 2aR\cos\theta}$ and $d\ell = a\,d\theta$, we can write (1) as

$$\Phi = -\frac{GM}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} \tag{2}$$

This is the general expression for the potential.

If R is much greater than a, we can expand the integrand in (2) using the binomial expansion:

$$\frac{1}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} = \frac{1}{R}\left[1 - \left(2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right)\right]^{-1/2}$$

$$= \left[1 + \frac{1}{2}\left[2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right] + \frac{3}{8}\left[2\frac{a}{R}\cos\theta - \frac{a^2}{R^2}\right]^2\right]\frac{1}{R} + \ldots \tag{3}$$

If we neglect terms of order $\left(\frac{a}{R}\right)^3$ and higher in (3), the potential becomes

$$\Phi = -\frac{GM}{2\pi}\int_0^{2\pi}\left[1+\frac{a}{R}\cos\theta-\frac{a^2}{2R^2}+\frac{3}{2}\frac{a^2}{R^2}\cos^2\theta\right]d\theta$$

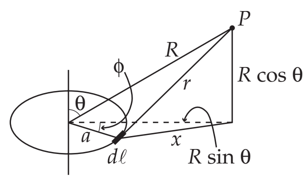
$$= -\frac{GM}{2\pi}\left[2\pi-\pi\frac{a^2}{R^2}+\frac{3}{2}\pi\frac{a^2}{R^2}\right] \tag{4}$$

or,

$$\boxed{\Phi(R)\cong -\frac{GM}{R}\left[1+\frac{1}{4}\frac{a^2}{R^2}\right]} \tag{5}$$

We notice that the first term in (5) is the potential when mass M is concentrated in the center of the ring. Of course this is a very rough approximation and the first correction term is $-\frac{GMa^2}{4R^3}$.

5-10.



Using the relations

$$x = \sqrt{(R\sin\theta)^2 + a^2 - 2aR\sin\theta\cos\phi} \tag{1}$$

$$r = \sqrt{x^2 + R^2\cos^2\theta} = \sqrt{R^2 + a^2 - 2aR\sin\theta\cos\phi} \tag{2}$$

$$\rho_\ell = \frac{M}{2\pi a} \text{ (the linear mass density),} \tag{3}$$

the potential is expressed by

$$\Phi = -G\int\frac{\rho_\ell d\ell}{r} = \frac{-GM}{2\pi R}\int_0^{2\pi}\frac{d\phi}{\sqrt{1-\left[2\frac{a}{R}\sin\theta\cos\phi-\frac{a^2}{R^2}\right]}} \tag{4}$$

If we expand the integrand and neglect terms of order $(a/R)^3$ and higher, we have

$$\left[1-\left[2\frac{a}{R}\sin\theta\cos\phi-\frac{a^2}{R^2}\right]\right]^{-1/2} \cong 1+\frac{a}{R}\sin\theta\cos\phi-\frac{1}{2}\frac{a^2}{R^2}+\frac{3}{2}\frac{a^2}{R^2}\sin^2\theta\cos^2\phi \tag{5}$$

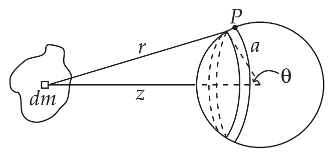
Then, (4) becomes

$$\Phi \cong -\frac{GM}{2\pi R}\left[2\pi - \frac{1}{2}\frac{a^2}{R^2}2\pi + \frac{3}{2}\frac{a^2}{R^2}\pi\sin^2\theta\right]$$

Thus,

$$\boxed{\Phi(R) \cong -\frac{GM}{R}\left[1 - \frac{1}{2}\frac{a^2}{R^2}\left[1 - \frac{3}{2}\sin^2\theta\right]\right]} \tag{6}$$

5-11.



The potential at P due to a small mass element dm inside the body is

$$d\Phi = -G\frac{dm}{r} = -G\frac{dm}{\sqrt{z^2 + a^2 - 2za\cos\theta}} \tag{1}$$

Integrating (1) over the entire volume and dividing the result by the surface area of the sphere, we can find the average field on the surface of the sphere due to dm:

$$d\Phi_{ave} = \frac{1}{4\pi a^2}\left[-G\,dm\int_0^{\pi}\frac{2\pi a^2\sin\theta\,d\theta}{\sqrt{z^2 + a^2 - 2za\cos\theta}}\right] \tag{2}$$

Making the variable change $\cos\theta = x$, we have

$$d\Phi_{ave} = -\frac{G}{2}dm\int_{-1}^{+1}\frac{dx}{\sqrt{\left(z^2 + a^2\right) - 2zax}} \tag{3}$$

Using Eq. (E.5), Appendix E, we find

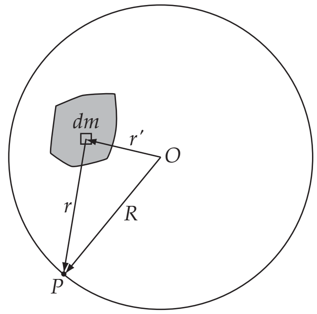
$$d\Phi_{ave} = -\frac{G}{2}dm\left[-\frac{1}{za}\sqrt{\left(z^2 + a^2\right) - 2za} + \frac{1}{za}\sqrt{\left(z^2 + a^2\right) + 2za}\right]$$

$$= -\frac{G}{2}dm\left[\frac{-(z-a) + (z+a)}{za}\right]$$

$$= -\frac{G}{z}dm \tag{4}$$

This is the same potential as at the center of the sphere. Since the average value of the potential is equal to the value at the center of the sphere at any arbitrary element dm, we have the same relation even if we integrate over the entire body.

5-12.



Let P be a point on the spherical surface. The potential $d\Phi$ due to a small amount of mass dm inside the surface at P is

$$d\Phi = -\frac{Gdm}{r} \tag{1}$$

The average value over the entire surface due to dm is the integral of (1) over $d\Omega$ divided by 4π. Writing this out with the help of the figure, we have

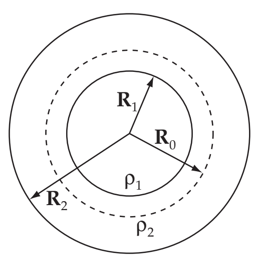
$$d\Phi_{ave} = -\frac{Gdm}{4\pi}\int_0^{\pi}\frac{2\pi \sin\theta\, d\theta}{\sqrt{r'^2 + R^2 - 2r'R\cos\theta}} \tag{2}$$

Making the obvious change of variable and performing the integration, we obtain

$$d\Phi_{ave} = -\frac{Gdm}{4\pi}\int_{-1}^{1}\frac{du}{\sqrt{r'^2 + R^2 - 2r'Ru}} = -\frac{Gdm}{R} \tag{3}$$

We can now integrate over all of the mass and get $\Phi_{ave} = -Gm/R$. This is a mathematical statement equivalent to the problem's assertion.

5-13.



$R_0 =$ position of particle. For $R_1 < R_0 < R_2$, we calculate the force by assuming that all mass for which $r < R_0$ is at $r = 0$, and neglect mass for which $r > R_0$. The force is in the radially inward direction ($-\mathbf{e}_r$).

The magnitude of the force is

$$F = \frac{GMm}{R_0^2}$$

where M = mass for which $r < R_0$

$$M = \frac{4}{3}\pi R_1^3 \rho_1 + \frac{4}{3}\pi\left(R_0^3 - R_1^3\right)\rho_2$$

So $\mathbf{F} = -\frac{4\pi G m}{3R_0^2}\left(\rho_1 R_1^3 + \rho_2 R_0^3 - \rho_2 R_1^3\right)\mathbf{e}_r$

$$\boxed{\mathbf{F} = -\frac{4}{3}\pi G m\left[\frac{\left(\rho_1 - \rho_2\right)R_1^3}{R_0^2} + \rho_2 R_0\right]\mathbf{e}_r}$$

5-14. Think of assembling the sphere a shell at a time ($r = 0$ to $r = R$).

For a shell of radius r, the incremental energy is $dU = dm\ \phi$ where ϕ is the potential due to the mass already assembled, and dm is the mass of the shell.

So

$$dm = \rho 4\pi r^2 dr = \left[\frac{3M}{4\pi R^3}\right] 4\pi r^2 dr = \frac{3Mr^2 dr}{R^3}$$

$$\phi = -\frac{Gm}{r} \text{ where } m = M\frac{r^3}{R^3}$$

So

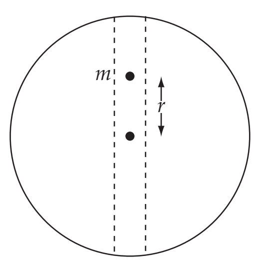
$$U = \int du$$

$$= \int_{r=0}^{R}\left[\frac{3Mr^2 dr}{R^3}\right]\left[-\frac{GMr^2}{R^3}\right]$$

$$= -\frac{3GM^2}{R^6}\int_0^R r^4 dr$$

$$\boxed{U = -\frac{3}{5}\frac{GM^2}{R}}$$

5-15. When the mass is at a distance r from the center of the Earth, the force is in the inward radial direction and has magnitude F_r:



$F_r = \frac{Gm}{r^2}\left[\frac{4}{3}\pi r^3 \rho\right]$ where ρ is the mass density of the Earth. The equation of motion is

$$F_r = m\ddot{r} = -\frac{Gm}{r^2}\left[\frac{4}{3}\pi r^3 \rho\right]$$

or

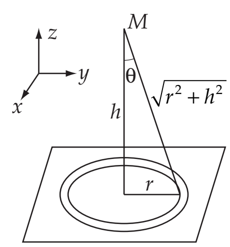
$$\ddot{r} + \omega^2 r = 0 \text{ where } \omega^2 = \frac{4\pi G\rho}{3}$$

This is the equation for simple harmonic motion. The period is

$$\boxed{T = \frac{2\pi}{\omega} = \sqrt{\frac{3\pi}{G\rho}}}$$

Substituting in values gives a period of about 84 minutes.

5-16.



For points external to the sphere, we may consider the sphere to be a point mass of mass M. Put the sheet in the x-y plane.

Consider force on M due to the sheet. By symmetry, $F_x = F_y = 0$

$$F_z = \int dF_z = \int_{r=0}^{\infty} \frac{GMdm}{\left(r^2+h^2\right)} \cos\theta$$

With $dm = \rho_s 2\pi r dr$ and $\cos\theta = \dfrac{h}{\sqrt{r^2+h^2}}$

we have

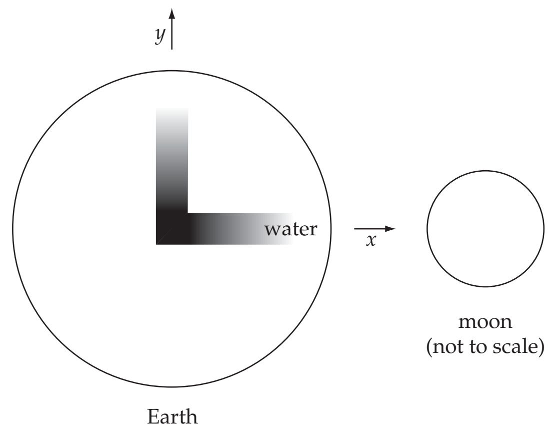
$$F_z = 2\pi\rho_s GMh \int_{r=0}^{\infty} \frac{rdr}{\left(r^2+h^2\right)^{3/2}}$$

$$F_z = -2\pi\rho_s GMh\left[\frac{1}{\left(r^2+h^2\right)^{1/2}}\right]_0^{\infty}$$

$$F_z = 2\pi\rho_s GM$$

$$\boxed{\begin{array}{c}\text{The sphere attracts the sheet in the } z\text{-direction} \\ \text{with a force of magnitude } 2\pi\rho_s GM\end{array}}$$

5-17.



Start with the hint given to us. The expression for g_x and g_y are given by

$$g_x = \frac{2GM_m x}{D^3} - \frac{GM_e x}{R^3}\ ;\ g_y = -\frac{GM_m y}{D^3} - \frac{GM_e y}{R^3} \tag{1}$$

where the first terms come from Equations (5.54) and the second terms come from the standard assumption of an Earth of uniform density. The origin of the coordinate system is at the center of the Earth. Evaluating the integrals:

$$\int_0^{x_{\max}} g_x\, dx = \left[\frac{2GM_m}{D^3} - \frac{GM_e}{R^3}\right]\frac{x_{\max}^2}{2}\ ;\ \int_0^{y_{\max}} g_y\, dy = \left[-\frac{GM_m}{D^3} - \frac{GM_e}{R^3}\right]\frac{y_{\max}^2}{2} \tag{2}$$

To connect this result with Example 5.5, let us write (1) in the following way

$$\frac{GM_m}{D^3}\left[x_{\max}^2 + \frac{y_{\max}^2}{2}\right] = \frac{GM_e}{2R^3}\left(x_{\max}^2 - y_{\max}^2\right) \tag{3}$$

The right-hand side can be factored as

$$\frac{GM_e}{2R^3}\left(x_{\max} + y_{\max}\right)\left(x_{\max} - y_{\max}\right) = \frac{GM_e}{2R^3}(2R)(h) = gh \tag{4}$$

If we make the approximation on the left-hand side of (3) that $x_{\max}^2 \simeq y_{\max}^2 \simeq R^2$, we get exactly Equation (5.55). Turning to the exact solution of (3), we obtain

$$h = 2R\,\frac{\sqrt{\dfrac{M_e}{R^3} + \dfrac{M_m}{D^3}} - \sqrt{\dfrac{M_e}{R^3} - \dfrac{2M_m}{D^3}}}{\sqrt{\dfrac{M_e}{R^3} + \dfrac{M_m}{D^3}} + \sqrt{\dfrac{M_e}{R^3} - \dfrac{2M_m}{D^3}}} \tag{5}$$

Upon substitution of the proper values, the answer is $\simeq 0.54$ m, the same as for Example 5.5. Inclusion of the centrifugal term in g_x does not change this answer significantly.

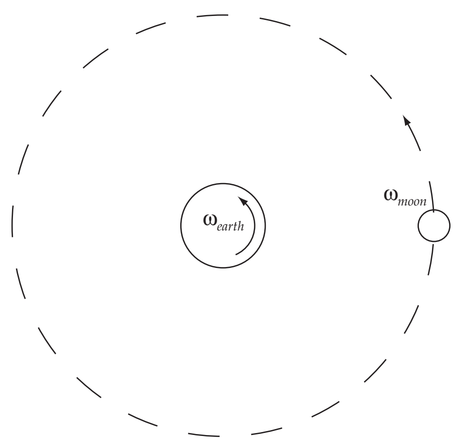
5-18. From Equation (5.55), we have with the appropriate substitutions

$$\frac{h_{\text{moon}}}{h_{\text{sun}}} = \frac{\dfrac{3GM_m r^2}{2gD^3}}{\dfrac{3GM_s r^2}{2gR_{es}^3}} = \frac{M_m}{M_s}\left[\frac{R_{es}}{D}\right]^3 \tag{1}$$

Substitution of the known values gives

$$\frac{h_{\text{moon}}}{h_{\text{sun}}} = \frac{7.350\times 10^{22}\ \text{kg}}{1.993\times 10^{30}\ \text{kg}}\left[\frac{1.495\times 10^{11}\ \text{m}}{3.84\times 10^{8}\ \text{m}}\right]^3 \simeq 2.2 \tag{2}$$

5-19.



Because the moon's orbit about the Earth is in the same sense as the Earth's rotation, the difference of their frequencies will be half the observed frequency at which we see high tides. Thus

$$\frac{1}{2T_{\text{tides}}} = \frac{1}{T_{\text{earth}}} - \frac{1}{T_{\text{moon}}} \tag{1}$$

which gives $T_{\text{tides}} \simeq$ 12 hours, 27 minutes.

5-20. The differential potential created by a thin loop of thickness dr at the point $(0,0,z)$ is

$$d\Phi(z) = \frac{-G}{\sqrt{z^2+r^2}}\frac{2\pi r dr M}{\pi R^2} = \frac{-GM}{R^2}\frac{d\left(r^2\right)}{\sqrt{z^2+r^2}} \Rightarrow \Phi(z) = \int d\Phi(z) = \frac{-2GM}{R^2}\left(\sqrt{z^2+R^2}-z\right)$$

Then one can find the gravity acceleration,

$$g(z) = -\hat{k}\frac{d\Phi}{dz} = -\hat{k}\frac{2GM}{R^2}\left(\frac{\sqrt{z^2+R^2}-z}{\sqrt{z^2+R^2}}\right)$$

where $\hat{k}$ is the unit vector in the z-direction.

5-21. (We assume the convention that $D > 0$ means m is not sitting on the rod.)

The differential force dF acting on point mass m from the element of thickness dx of the rod, which is situated at a distance x from m, is

$$dF = \frac{G(M/L)mdx}{x^2} \quad \Rightarrow \quad F = \int dF = \frac{GMm}{L}\int_D^{L+D}\frac{dx}{x^2} = \frac{GMm}{D(L+D)}$$

And that is the total gravitational force acting on m by the rod.

CHAPTER 6

Some Methods in the Calculus of Variations

6-1. If we use the varied function

$$y(\alpha, x) = x + \alpha \sin \pi(1-x) \tag{1}$$

Then

$$\frac{dy}{dx} = 1 - \alpha\pi \cos \pi(1-x) \tag{2}$$

Thus, the total length of the path is

$$S = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \left[2 - 2\alpha\pi \cos \pi(1-x) + \alpha^2\pi^2 \cos^2 \pi(1-x)\right]^{1/2} dx \tag{3}$$

Setting $\pi(1-x) \equiv u$, the expression for S becomes

$$S = \frac{1}{\pi} \int_0^\pi \sqrt{2} \left[1 - \alpha\pi \cos u + \frac{1}{2}\alpha^2\pi^2 \cos^2 u\right]^{1/2} du \tag{4}$$

The integral cannot be performed directly since it is, in fact, an *elliptic integral*. Because α is a small quantity, we can expand the integrand and obtain

$$S = \frac{\sqrt{2}}{\pi} \int_0^\pi \left[1 - \frac{1}{2}\left(\alpha\pi \cos u - \frac{1}{2}\alpha^2\pi^2 \cos^2 u\right) - \frac{1}{8}\left(\alpha\pi \cos u - \frac{1}{2}\alpha^2\pi^2 \cos^2 u\right)^2 + \ldots\right] du \tag{5}$$

If we keep the terms up to $\cos^2 u$ and perform the integration, we find

$$S = \sqrt{2} + \frac{\sqrt{2}}{16}\pi^2\alpha^2 \tag{6}$$

which gives

$$\frac{\partial S}{\partial \alpha} = \frac{\sqrt{2}}{8}\,\pi^2 \alpha \tag{7}$$

Therefore

$$\boxed{\left.\frac{\partial S}{\partial \alpha}\right|_{\alpha=0} = 0} \tag{8}$$

and S is a minimum when $\alpha = 0$.

6-2. The element of length on a plane is

$$dS = \sqrt{dx^2 + dy^2} \tag{1}$$

from which the total length is

$$S = \int_{(x_1,y_1)}^{(x_2,y_2)} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{2}$$

If S is to be minimum, f is identified as

$$f = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{3}$$

Then, the Euler equation becomes

$$\frac{d}{dx}\frac{d}{dy'}\left[\sqrt{1+y'^2}\right] = 0 \tag{4}$$

where $y' = \dfrac{dy}{dx}$. (4) becomes

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \tag{5}$$

or,

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} \equiv C \tag{6}$$

from which we have

$$y' = \sqrt{\frac{C^2}{1-C^2}} = \text{constant} \equiv a \tag{7}$$

Then,

$$\boxed{y = ax + b} \tag{8}$$

This is the equation of a straight line.

6-3. The element of distance in three-dimensional space is

$$dS = \sqrt{dx^2 + dy^2 + dz^2} \tag{1}$$

Suppose x, y, z depends on the parameter t and that the end points are expressed by $\left(x_1(t_1), y_1(t_1), z_1(t_1)\right)$, $\left(x_2(t_2), y_2(t_2), z_2(t_2)\right)$. Then the total distance is

$$S = \int_{t_1}^{t_2} \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\, dt \tag{2}$$

The function f is identified as

$$f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{3}$$

Since $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial z} = 0$, the Euler equations become

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} &= 0 \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} &= 0 \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{z}} &= 0 \end{aligned}\right] \tag{4}$$

from which we have

$$\left.\begin{aligned} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_1 \\ \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_2 \\ \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} &= \text{constant} \equiv C_3 \end{aligned}\right] \tag{5}$$

From the combination of these equations, we have

$$\left.\begin{aligned} \frac{\dot{x}}{C_1} &= \frac{\dot{y}}{C_2} \\ \frac{\dot{y}}{C_2} &= \frac{\dot{z}}{C_3} \end{aligned}\right] \tag{6}$$

If we integrate (6) from t_1 to the arbitrary t, we have

$$\left.\begin{aligned}\frac{x-x_1}{C_1}&=\frac{y-y_1}{C_2}\\[2ex]\frac{y-y_1}{C_2}&=\frac{z-z_1}{C_3}\end{aligned}\right] \tag{7}$$

On the other hand, the integration of (6) from t_1 to t_2 gives

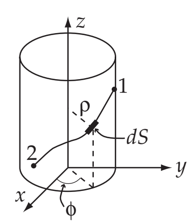
$$\left.\begin{aligned}\frac{x_2-x_1}{C_1}&=\frac{y_2-y_1}{C_2}\\[2ex]\frac{y_2-y_1}{C_2}&=\frac{z_2-z_1}{C_3}\end{aligned}\right] \tag{8}$$

from which we find the constants C_1, C_2, and C_3. Substituting these constants into (7), we find

$$\boxed{\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}} \tag{9}$$

This is the equation expressing a straight line in three-dimensional space passing through the two points (x_1, y_1, z_1), (x_2, y_2, z_2).

6-4.



The element of distance along the surface is

$$dS=\sqrt{dx^2+dy^2+dz^2} \tag{1}$$

In cylindrical coordinates (x,y,z) are related to (ρ,ϕ,z) by

$$\left.\begin{aligned}x&=\rho\cos\phi\\ y&=\rho\sin\phi\\ z&=z\end{aligned}\right] \tag{2}$$

from which

$$\left.\begin{aligned}dx&=-\rho\sin\phi\, d\phi\\ dy&=\rho\cos\phi\, d\phi\\ dz&=dz\end{aligned}\right] \tag{3}$$

Substituting (3) into (1) and integrating along the entire path, we find

$$S = \int_1^2 \sqrt{\rho^2 \, d\phi^2 + dz^2} = \int_{\phi_1}^{\phi_2} \sqrt{\rho^2 + \dot{z}^2} \, d\phi \tag{4}$$

where $\dot{z} = \dfrac{dz}{d\phi}$. If S is to be minimum, $f \equiv \sqrt{\rho^2 + \dot{z}^2}$ must satisfy the Euler equation:

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial \phi} \frac{\partial f}{\partial \dot{z}} = 0 \tag{5}$$

Since $\dfrac{\partial f}{\partial z} = 0$, the Euler equation becomes

$$\frac{\partial}{\partial \phi} \frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = 0 \tag{6}$$

from which

$$\frac{\dot{z}}{\sqrt{\rho^2 + \dot{z}^2}} = \text{constant} \equiv \text{C} \tag{7}$$

or,

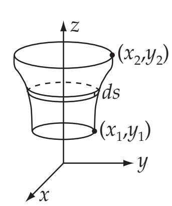
$$\dot{z} = \sqrt{\frac{\text{C}^2}{1 - \text{C}^2}} \, \rho \tag{8}$$

Since ρ is constant, (8) means

$$\boxed{\frac{dz}{d\phi} = \text{constant}}$$

and for any point along the path, z and ϕ change at the same rate. The curve described by this condition is a *helix*.

6-5.



The area of a strip of a surface of revolution is

$$dA = 2\pi \times ds = 2\pi \times \sqrt{dx^2 + dy^2} \tag{1}$$

Thus, the total area is

$$A = 2\pi \int_{x_1}^{x_2} x \sqrt{1+\dot{y}^2}\, dx \tag{2}$$

where $\dot{y} = \dfrac{dy}{dx}$. In order to make A a minimum, $f \equiv x\sqrt{1+\dot{y}^2}$ must satisfy equation (6.39). Now

$$\frac{\partial f}{\partial x} = \sqrt{1+\dot{y}^2}$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$$

Substituting into equation (6.39) gives

$$\sqrt{1+\dot{y}^2} = \frac{d}{dx}\left[x\sqrt{1+\dot{y}^2} - \frac{x\dot{y}^2}{\sqrt{1+\dot{y}^2}}\right] = \frac{d}{dx}\left[\frac{x}{\sqrt{1+\dot{y}^2}}\right]$$

$$= \frac{\sqrt{1+\dot{y}^2} - x\dot{y}\left(d\dot{y}/dx\right)\left(1+\dot{y}^2\right)^{-1/2}}{1+\dot{y}^2}$$

Multiplying by $\sqrt{1+\dot{y}^2}$ and rearranging gives

$$-\frac{dx}{x} = \frac{d\dot{y}}{\dot{y}\left(1+\dot{y}^2\right)}$$

Integration gives

$$-\ln x + \ln a = \frac{1}{2}\ln\frac{\dot{y}^2}{1+\dot{y}^2}$$

where $\ln a$ is a constant of integration. Rearranging gives

$$\dot{y}^2 = \frac{1}{\left(x^2/a^2\right)-1}$$

Integrating gives

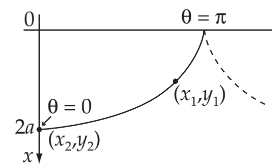
$$y = b + a\cosh^{-1}\frac{x}{a}$$

or

$$\boxed{x = a\cosh\frac{y-b}{a}}$$

which is the equation of a catenary.

6-6.



If we use coordinates with the same orientation as in Example 6.2 and if we place the minimum point of the cycloid at (2a,0) the parametric equations are

$$\left.\begin{aligned} x &= a(1+\cos\theta) \\ y &= a(\theta+\sin\theta) \end{aligned}\right] \tag{1}$$

Since the particle starts from rest at the point (x_1, y_1), the velocity at any elevation x is [cf. Eq. 6.19]

$$v = \sqrt{2g(x-x_1)} \tag{2}$$

Then, the time required to reach the point (x_2, y_2) is [cf. Eq. 6.20]

$$t = \int_{x_1}^{x_2} \left[\frac{1+y'^2}{2g(x-x_1)}\right]^{1/2} dx \tag{3}$$

Using (1) and the derivatives obtained therefrom, (3) can be written as

$$t = \sqrt{\frac{a}{g}} \int_{\theta_2=0}^{\theta_1} \left[\frac{1+\cos\theta}{\cos\theta - \cos\theta_1}\right]^{1/2} d\theta \tag{4}$$

Now, using the trigonometric identity, $1+\cos\theta = 2\cos^2\theta/2$, we have

$$t = \sqrt{\frac{a}{g}} \int_0^{\theta_1} \frac{\cos\frac{\theta}{2}\, d\theta}{\sqrt{\cos^2\frac{\theta}{2} - \cos^2\frac{\theta_1}{2}}}$$

$$= \sqrt{\frac{a}{g}} \int_0^{\theta_1} \frac{\cos\frac{\theta}{2}\, d\theta}{\sqrt{\sin^2\frac{\theta_1}{2} - \sin^2\frac{\theta}{2}}} \tag{5}$$

Making the change of variable, $z = \sin\theta/2$, the expression for t becomes

$$t = 2\sqrt{\frac{a}{g}} \int_0^{\sin\frac{\theta_1}{2}} \frac{dz}{\sqrt{\sin^2\frac{\theta_1}{2} - z^2}} \tag{6}$$

The integral is now in standard form:

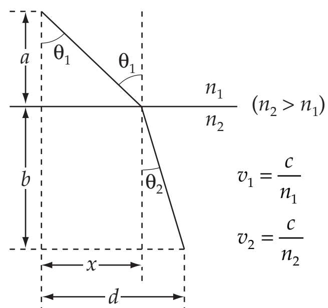
$$\int \frac{dx}{\sqrt{a^2 = x^2}} = \sin^{-1}\left[\frac{x}{a}\right] \tag{7}$$

Evaluating, we find

$$\boxed{t = \pi\sqrt{\frac{a}{g}}} \tag{8}$$

Thus, the time of transit from (x_1, y_1) to the minimum point does not depend on the position of the starting point.

6-7.



The time to travel the path shown is (cf. Example 6.2)

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{1+y'^2}}{v}\,dx \tag{1}$$

Although we have $v = v(y)$, we only have $dv/dy \neq 0$ when $y = 0$. The Euler equation tells us

$$\frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2}}\right] = 0 \tag{2}$$

Now use $v = c/n$ and $y' = -\tan\theta$ to obtain

$$n \sin\theta = \text{const.} \tag{3}$$

This proves the assertion. Alternatively, Fermat's principle can be proven by the method introduced in the solution of Problem 6-8.

6-8. To find the extremum of the following integral (cf. Equation 6.1)

$$J = \int f(y, x)\,dx$$

we know that we must have from Euler's equation

$$\frac{\partial f}{\partial y} = 0$$

This implies that we also have

$$\frac{\partial J}{\partial y} = \int \frac{\partial f}{\partial y}\, dx = 0$$

giving us a modified form of Euler's equation. This may be extended to several variables and to include the imposition of auxiliary conditions similar to the derivation in Sections 6.5 and 6.6. The result is

$$\frac{\partial J}{\partial y_i} + \sum_j \lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0$$

when there are constraint equations of the form

$$g_j(y_i, x) = 0$$

a) The volume of a parallelepiped with sides of lengths a_1, b_1, c_1 is given by

$$V = a_1 b_1 c_1 \tag{1}$$

We wish to *maximize* such a volume under the condition that the parallelepiped is circumscribed by a sphere of radius R; that is,

$$a_1^2 + b_1^2 + c_1^2 = 4R^2 \tag{2}$$

We consider a_1, b_1, c_1 as variables and V is the function that we want to maximize; (2) is the constraint condition:

$$g\{a_1, b_1, c_1\} = 0 \tag{3}$$

Then, the equations for the solution are

$$\left.\begin{aligned} \frac{\partial V}{\partial a_1} + \lambda\frac{\partial g}{\partial a_1} &= 0 \\ \frac{\partial V}{\partial b_1} + \lambda\frac{\partial g}{\partial b_1} &= 0 \\ \frac{\partial V}{\partial c_1} + \lambda\frac{\partial g}{\partial c_1} &= 0 \end{aligned}\right] \tag{4}$$

from which we obtain

$$\left.\begin{aligned} b_1 c_1 + 2\lambda a_1 &= 0 \\ a_1 c_1 + 2\lambda b_1 &= 0 \\ a_1 b_1 + 2\lambda c_1 &= 0 \end{aligned}\right] \tag{5}$$

Together with (2), these equations yield

$$a_1 = b_1 = c_1 = \frac{2}{\sqrt{3}} R \tag{6}$$

Thus, the inscribed parallelepiped is a cube with side $\frac{2}{\sqrt{3}} R$.

b) In the same way, if the parallelepiped is now circumscribed by an ellipsoid with semiaxes a, b, c, the constraint condition is given by

$$\frac{a_1^2}{4a^2}=\frac{b_1^2}{4b^2}=\frac{c_1^2}{4z^2}=1 \tag{7}$$

where a_1, b_1, c_1 are the lengths of the sides of the parallelepiped. Combining (7) with (1) and (4) gives

$$\frac{a_1^2}{a^2}=\frac{b_1^2}{b^2}=\frac{c_1^2}{c^2} \tag{8}$$

Then,

$$\boxed{a_1=a\frac{2}{\sqrt{3}},\, b_1=b\frac{2}{\sqrt{3}},\, c_1=c\frac{2}{\sqrt{3}}} \tag{9}$$

6-9. The average value of the square of the gradient of $\phi(x_1, x_2, x_3)$ within a certain volume V is expressed as

$$I=\frac{1}{V}\iiint(\nabla\phi)^2\, dx_1\, dx_2\, dx_3$$

$$v=\frac{1}{V}\iiint\left[\left[\frac{\partial\phi}{\partial x_1}\right]^2+\left[\frac{\partial\phi}{\partial x_2}\right]^2+\left[\frac{\partial\phi}{\partial x_3}\right]^2\right]dx_1dx_2dx_3 \tag{1}$$

In order to make I a minimum,

$$f=\left[\frac{\partial\phi}{\partial x_1}\right]^2+\left[\frac{\partial\phi}{\partial x_2}\right]^2+\left[\frac{\partial\phi}{\partial x_3}\right]^2$$

must satisfy the Euler equation:

$$\frac{\partial f}{\partial\phi}-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial\left[\dfrac{\partial\phi}{\partial x_i}\right]}\right]=0 \tag{2}$$

If we substitute f into (2), we have

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial x_i}=0 \tag{3}$$

which is just Laplace'sequation:

$$\boxed{\nabla^2\,\phi=0} \tag{4}$$

Therefore, ϕ must satisfy Laplace's equation in order that I have a minimum value.

6-10. This problem lends itself to the method of solution suggested in the solution of Problem 6-8. The volume of a right cylinder is given by

$$V = \pi R^2 H \tag{1}$$

The total surface area A of the cylinder is given by

$$A = A_{\text{bases}} + A_{\text{side}} = 2\pi R^2 + 2\pi RH = 2\pi R(R+H) \tag{2}$$

We wish A to be a minimum. (1) is the constraint condition, and the other equations are

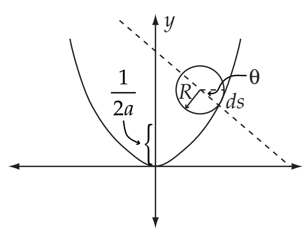
$$\left.\begin{aligned} \frac{\partial A}{\partial R} + \lambda\frac{\partial g}{\partial R} = 0 \\ \frac{\partial A}{\partial H} + \lambda\frac{\partial g}{\partial H} = 0 \end{aligned}\right] \tag{3}$$

where $g = V - \pi R^2 H = 0$.

The solution of these equations is

$$\boxed{R = \frac{1}{2}H} \tag{4}$$

6-11.



The constraint condition can be found from the relation $ds = Rd\theta$ (see the diagram), where ds is the differential arc length of the path:

$$ds = \left(dx^2 + dy^2\right)^{1/2} = Rd\theta \tag{1}$$

which, using $y = ax^2$, yields

$$\sqrt{1 + 4a^2x^2}\; dx = Rd\theta \tag{2}$$

If we want the equation of constraint in other than a differential form, (2) can be integrated to yield

$$A + R\theta = \frac{x}{2}\sqrt{4ax^2 + 1} + \frac{1}{4a}\ln\left(2ax + \sqrt{4a^2x^2 + 1}\right) \tag{3}$$

where A is a constant obtained from the initial conditions. The radius of curvature of a parabola, $y = ax^2$, is given at any point (x,y) by $r_0 \geq 1/2a$. The condition for the disk to roll with one and only one point of contact with the parabola is $R < r_0$; that is,

$$\boxed{R < \frac{1}{2a}} \tag{4}$$

6-12. The path length is given by

$$s = \int ds = \int \sqrt{1 + y'^2 + z'^2}\, dx \tag{1}$$

and our equation of constraint is

$$g(x, y, z) = x^2 + y^2 + z^2 - \rho^2 = 0 \tag{2}$$

The Euler equations with undetermined multipliers (6.69) tell us that

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right] = \lambda \frac{dg}{dy} = 2\lambda y \tag{3}$$

with a similar equation for z. Eliminating the factor λ, we obtain

$$\frac{1}{y}\frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\right] - \frac{1}{z}\frac{d}{dx}\left[\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\right] = 0 \tag{4}$$

This simplifies to

$$z\left[y''\left(1 + y'^2 + z'^2\right) - y'\left(y'y'' + z'z''\right)\right] - y\left[z''\left(1 + y'^2 + z'^2\right) - z'\left(y'y'' + z'z''\right)\right] = 0 \tag{5}$$

$$zy'' + \left(yy' + zz'\right)z'y'' - yz'' - \left(yy' + zz'\right)y'z'' = 0 \tag{6}$$

and using the derivative of (2),

$$\left(z - xz'\right)y'' = \left(y - xy'\right)z'' \tag{7}$$

This looks to be in the simplest form we can make it, but is it a plane? Take the equation of a plane passing through the origin:

$$Ax + By = z \tag{8}$$

and make it a differential equation by taking derivatives (giving $A + By' = z'$ and $By'' = z''$) and eliminating the constants. The substitution yields (7) exactly. This confirms that the path must be the intersection of the sphere with a plane passing through the origin, as required.

6-13. For the reason of convenience, without lost of generality, suppose that the closed curve passes through fixed points $A(-a,0)$ and $B(a,0)$ (which have been chosen to be on axis Ox). We denote the part of the closed curve above and below the Ox axis as $y_1(x)$ and $y_2(x)$ respectively. (note that $y_1 > 0$ and $y_2 < 0$)

The enclosed area is

$$J(y_1, y_2) = \int_{-a}^{a} y_1(x)dx - \int_{-a}^{a} y_2(x)dx = \int_{-a}^{a} \left(y_1(x) - y_2(x)\right)dx = \int_{-a}^{a} f(y_1, y_2)dx$$

The total length of closed curve is

$$K(y_1', y_2') = \int_{-a}^{a} \sqrt{1+(y_1')^2}\,dx + \int_{-a}^{a} \sqrt{1+(y_2')^2}\,dx = \int_{-a}^{a} \left\{ \sqrt{1+(y_1')^2} + \sqrt{1+(y_1')^2} \right\} dx = \int_{-a}^{a} g(y_1', y_2')\,dx$$

Then the generalized versions of Eq. (6.78) (see textbook) for this case are

$$\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} + \lambda\left\{ \frac{\partial g}{\partial y_1} - \frac{d}{dx}\frac{\partial g}{\partial y_1'} \right\} = 0 \quad \Rightarrow \quad 1 - \lambda \frac{d}{dx}\left(\frac{y_1'}{\sqrt{1+(y_1')^2}} \right) = 0 \tag{1}$$

$$\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} + \lambda\left\{ \frac{\partial g}{\partial y_2} - \frac{d}{dx}\frac{\partial g}{\partial y_2'} \right\} = 0 \quad \Rightarrow \quad 1 - \lambda \frac{d}{dx}\left(\frac{y_2'}{\sqrt{1+(y_2')^2}} \right) = 0 \tag{2}$$

Analogously to Eq. (6.85);

from (1) we obtain $$(x - A_1)^2 + (y_1 - A_2)^2 = \lambda^2 \tag{3}$$

from (2) we obtain $$(x - B_1)^2 + (y_2 - B_2)^2 = \lambda^2 \tag{4}$$

where constants A's, B's can be determined from 4 initial conditions

$$(x = \pm a, y_1 = 0) \quad \text{and} \quad (x = \pm a, y_2 = 0)$$

We note that $y_1 < 0$ and $y_2 > 0$, so actually (3) and (4) altogether describe a circular path of radius λ. And this is the sought configuration that renders maximum enclosed area for a given path length.

6-14. It is more convenient to work with cylindrical coordinates (r, ϕ, z) in this problem. The constraint here is $z = 1 - r$, then $dz = -dr$

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 = 2\left(dr^2 + r^2 d\beta^2\right)$$

where we have introduced a new angular coordinate $\beta = \dfrac{\phi}{\sqrt{2}}$

In this form of ds^2, we clearly see that the space is 2-dimensional Euclidean flat, so the shortest line connecting two given points is a straight line given by:

$$r = \frac{r_0}{\cos(\beta - \beta_0)} = \frac{r_0}{\cos\left(\dfrac{\phi - \phi_0}{\sqrt{2}}\right)}$$

this line passes through the endpoints ($r = 1$, $\phi = \pm\dfrac{\pi}{2}$), then we can determine unambiguously the shortest path equation

$$r(\phi) = \frac{\cos\dfrac{\pi}{2\sqrt{2}}}{\cos\left(\dfrac{\phi}{\sqrt{2}}\right)} \quad \text{and} \quad z = 1 - r$$

Accordingly, the shortest connecting length is

$$l = \int_{-\pi/2}^{\pi/2} d\phi \sqrt{2\left(\frac{dr}{d\phi}\right)^2 + r^2} = 2\sqrt{2}\,\sin\frac{\pi}{2\sqrt{2}}$$

6-15.

$$I[y] = \int_0^1 \left\{ \left(\frac{dy}{dx}\right)^2 - y^2 \right\} dx$$

a) Treating $I[y]$ as a mechanical action, we find the corresponding Euler-Lagrange equation

$$y(x) = -\frac{d^2y}{dx^2}$$

Combining with the boundary conditions ($x = 0$, $y = 0$) and ($x = 1$, $y = 1$), we can determine unambiguously the functional form of $y(x) = (\sin x)/(\sin 1)$.

b) The corresponding minimum value of the integral is

$$I[y] = \int_0^1 \left\{ \left(\frac{dy}{dx}\right)^2 - y^2 \right\} dx = \frac{1}{\sin^2 1}\int_0^1 dx \cos 2x = \cot(1) = 0.642$$

c) If $x = y$ then $I[y] = (2/3) = 0.667$.

6-16.

a) S is arc length

$$S = \int \sqrt{dx^2 + dy^2 + dz^2} = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} = \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \frac{9}{4}x} = \int L\, dx$$

Treating S and L like a mechanical action and Lagrangian respectively, we find the canonical momentum associated with coordinate y

$$p = \frac{\delta L}{\delta\left(\frac{dy}{dx}\right)} = \frac{\left(\frac{dy}{dx}\right)}{\sqrt{1 + \frac{9}{4}x + \left(\frac{dy}{dx}\right)^2}}$$

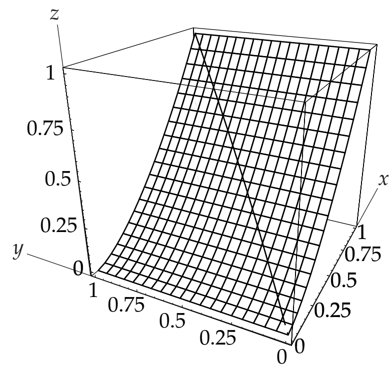
Because L does not depend on y explicitly, then E-L equation implies that p is constant (i.e. $dp/dx = 0$), then the above equation becomes

$$\frac{dy}{dx} = p\sqrt{\frac{1 + \frac{9}{4}x}{1 - p^2}} \Rightarrow y = \frac{p}{1 - p^2}\int dx \sqrt{1 + \frac{9}{4}x} = A\left(1 + \frac{9}{4}x\right)^{3/2} + B$$

where A and B are constants. Using boundary conditions ($x = 0$, $y = 0$) and ($x = 1$, $y = 1$) one can determine the arc equation unambiguously

$$y(x)=\frac{8}{13^{3/2}-8}\left\{\left(1+\frac{9}{4}x\right)^{3/2}-1\right\} \quad \text{and} \quad z=x^{3/2}$$

b)



6-17.

a) Equation of a ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which implies

$$xy \le \frac{ab}{2} \quad \text{because} \quad \frac{2xy}{ab} \le \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

so the maximal area of the rectangle, whose corners lie on that ellipse, is

$$\text{Max}[A] = \text{Max}[4xy] = 2\ \text{i}ab.$$

This happens when

$$x=\frac{a}{\sqrt{2}} \quad \text{and} \quad y=\frac{b}{\sqrt{2}}$$

b) The area of the ellipse is $A_0 = \pi ab$; so the fraction of rectangle area to ellipse area is then

$$\frac{Max[A]}{A_0}=\frac{2}{\pi}$$

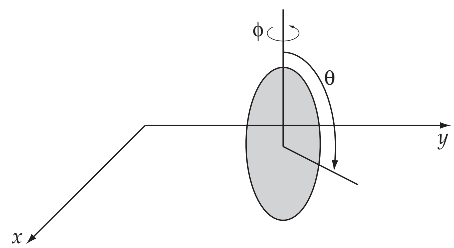
6-18. One can see that the surface $xy = z$ is "locally" symmetric with respect to the line $x=-y=\sqrt{-z}$ where $x > 0$, $y < 0$, $z < 0$. This line is a parabola. This implies that if the particle starts from point (1,-1,-1) (which belongs to the symmetry line) under gravity ideally will move downward along this line. Its velocity at altitude z ($z < -1$) can be found from the conservation of energy.

$$v(z)=\sqrt{-2g(z+1)}$$

CHAPTER 7

Hamilton's Principle— Lagrangian and Hamiltonian Dynamics

7-1. Four coordinates are necessary to completely describe the disk. These are the x and y coordinates, the angle θ that measures the rolling, and the angle ϕ that describes the spinning (see figure).



Since the disk may only roll in one direction, we must have the following conditions:

$$dx\cos\phi + dy\sin\phi = R\,d\theta \tag{1}$$

$$\frac{dy}{dx} = \tan\phi \tag{2}$$

These equations are not integrable, and because we cannot obtain an equation relating the coordinates, the constraints are nonholonomic. This means that although the constraints relate the infinitesimal displacements, they do not dictate the relations between the coordinates themselves, e.g. the values of x and y (position) in no way determine θ or ϕ (pitch and yaw), and vice versa.

7-2. Start with the Lagrangian

$$L = \frac{m}{2}\left[\left(v_0 + at + \ell\dot{\theta}\cos\theta\right)^2 + \left(\ell\dot{\theta}\sin\theta\right)^2\right] + mg\ell\cos\theta \tag{1}$$

$$= \frac{m}{2}\left[\left(v_0 + at\right)^2 + 2\left(v_0 + at\right)\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right] + mg\ell\cos\theta \tag{2}$$

Now let us just compute

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[m\left(v_0 + at\right)\ell \cos\theta + m\ell^2\dot{\theta}\right] \tag{3}$$

$$= ma\ell \cos\theta - m\left(v_0 - at\right)\ell\dot{\theta}\sin\theta + m\ell^2\ddot{\theta} \tag{4}$$

$$\frac{\partial L}{\partial \theta} = -m\left(v_0 + at\right)\ell\dot{\theta}\sin\theta - mg\ell\sin\theta \tag{5}$$

According to Lagrange's equations, (4) is equal to (5). This gives Equation (7.36)

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta + \frac{a}{\ell}\cos\theta = 0 \tag{6}$$

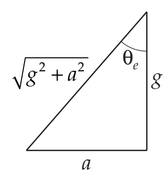
To get Equation (7.41), start with Equation (7.40)

$$\ddot{\eta} = -\frac{g\cos\theta_e - a\sin\theta_e}{\ell}\eta \tag{7}$$

and use Equation (7.38)

$$\tan\theta_e = -\frac{a}{g} \tag{8}$$

to obtain, either through a trigonometric identity or a figure such as the one shown here,



$$\cos\theta_e = \frac{g}{\sqrt{g^2+a^2}} \qquad \sin\theta_e = \frac{a}{\sqrt{g^2+a^2}} \tag{9}$$

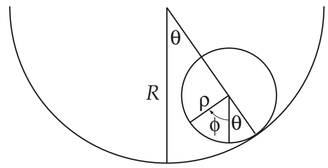
Inserting this into (7), we obtain

$$\ddot{\eta} = -\frac{\sqrt{a^2+g^2}}{\ell}\eta \tag{10}$$

as desired.

We know intuitively that the period of the pendulum cannot depend on whether the train is accelerating to the left or to the right, which implies that the sign of a cannot affect the frequency. From a Newtonian point of view, the pendulum will be in equilibrium when it is in line with the effective acceleration. Since the acceleration is sideways and gravity is down, and the period can only depend on the magnitude of the effective acceleration, the correct form is clearly $\sqrt{a^2+g^2}$.

7-3.



If we take angles θ and ϕ as our generalized coordinates, the kinetic energy and the potential energy of the system are

$$T = \frac{1}{2} m \left[(R-\rho)\dot{\theta} \right]^2 + \frac{1}{2} I \dot{\phi}^2 \tag{1}$$

$$U = \left[R - (R-\rho)\cos\theta \right] mg \tag{2}$$

where m is the mass of the sphere and where $U = 0$ at the lowest position of the sphere. I is the moment of inertia of sphere with respect to any diameter. Since $I = (2/5)\, m\rho^2$, the Lagrangian becomes

$$L = T - U = \frac{1}{2} m (R-\rho)^2 \dot{\theta}^2 + \frac{1}{5} m\rho^2 \dot{\phi}^2 - \left[R - (R-\rho)\cos\theta \right] mg \tag{3}$$

When the sphere is at its lowest position, the points A and B coincide. The condition $A0 = B0$ gives the equation of constraint:

$$f(\theta,\phi) = (R-\rho)\theta - \rho\phi = 0 \tag{4}$$

Therefore, we have two Lagrange's equations with one undetermined multiplier:

$$\left. \begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}} \right] + \lambda \frac{\partial f}{\partial \theta} = 0 \\ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}} \right] + \lambda \frac{\partial f}{\partial \phi} = 0 \end{aligned} \right] \tag{5}$$

After substituting (3) and $\partial f/\partial\theta = R - \rho$ and $\partial f/\partial\phi = -\rho$ into (5), we find

$$-(R-\rho) mg \sin\theta - m(R-\rho)^2 \ddot{\theta} + \lambda(R-\rho) = 0 \tag{6}$$

$$-\frac{2}{5} m\rho^2 \ddot{\phi} - \lambda\rho = 0 \tag{7}$$

From (7) we find λ:

$$\lambda = -\frac{2}{5} m\rho\ddot{\phi} \tag{8}$$

or, if we use (4), we have

$$\lambda = -\frac{2}{5} m(R-\rho)\ddot{\theta} \tag{9}$$

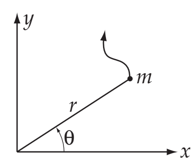
Substituting (9) into (6), we find the equation of motion with respect to θ:

$$\ddot{\theta} = -\omega^2 \sin\theta \tag{10}$$

where ω is the frequency of small oscillations, defined by

$$\omega = \sqrt{\frac{5g}{7(R-\rho)}} \tag{11}$$

7-4.



If we choose (r,θ) as the generalized coordinates, the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{1}$$

Since the force is related to the potential by

$$f = -\frac{\partial U}{\partial r} \tag{2}$$

we find

$$U = \frac{A}{\alpha}r^\alpha \tag{3}$$

where we let $U(r = 0) = 0$. Therefore, the Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{\alpha}r^\alpha \tag{4}$$

Lagrange's equation for the coordinate r leads to

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} = 0} \tag{5}$$

Lagrange's equation for the coordinate θ leads to

$$\boxed{\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0} \tag{6}$$

Since $mr^2\dot{\theta} = \ell$ is identified as the angular momentum, (6) implies that angular momentum is conserved. Now, if we use ℓ, we can write (5) as

$$m\ddot{r} - \frac{\ell^2}{mr^3} + Ar^{\alpha-1} = 0 \tag{7}$$

Multiplying (7) by $\dot{r}$, we have

$$m\dot{r}\ddot{r} - \frac{\dot{r}\ell^2}{mr^3} + A\dot{r}\,r^{\alpha-1} = 0 \tag{8}$$

which is equivalent to

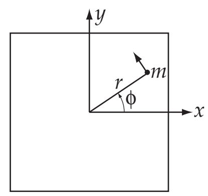
$$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right]+\frac{d}{dt}\left[\frac{\ell^2}{2mr^2}\right]+\frac{d}{dt}\left[\frac{A}{\alpha}r^\alpha\right]=0 \tag{9}$$

Therefore,

$$\boxed{\frac{d}{dt}(T+U)=0} \tag{10}$$

and the total energy is conserved.

7-5.



Let us choose the coordinate system so that the x-y plane lies on the vertical plane in a gravitational field and let the gravitational potential be zero along the x axis. Then the kinetic energy and the potential energy are expressed in terms of the generalized coordinates (r,ϕ) as

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right) \tag{1}$$

$$U=\frac{A}{\alpha}r^\alpha+mgr\sin\phi \tag{2}$$

from which the Lagrangian is

$$L=T-U=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-\frac{A}{\alpha}r^\alpha-mgr\sin\phi \tag{3}$$

Therefore, Lagrange's equation for the coordinate r is

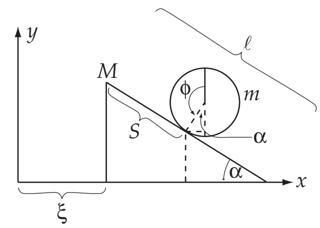
$$\boxed{m\ddot{r}-mr\dot{\phi}^2+Ar^{\alpha-1}+mg\sin\phi=0} \tag{4}$$

Lagrange's equation for the coordinate ϕ is

$$\boxed{\frac{d}{dt}\left(mr^2\dot{\phi}\right)+mgr\cos\phi=0} \tag{5}$$

Since $mr^2\dot{\phi}$ is the angular momentum along the z axis, (5) shows that the angular momentum is *not* conserved. The reason, of course, is that the particle is subject to a *torque* due to the gravitational force.

7-6.



Let us choose ξ, S as our generalized coordinates. The x,y coordinates of the center of the hoop are expressed by

$$\left.\begin{aligned} x &= \xi + S\cos\alpha + r\sin\alpha \\ y &= r\cos\alpha + (\ell - S)\sin\alpha \end{aligned}\right] \tag{1}$$

Therefore, the kinetic energy of the hoop is

$$\begin{aligned} T_{\text{hoop}} &= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+\frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}m\left[\left(\dot{\xi}+\dot{S}\cos\alpha\right)^2+\left(-\dot{S}\sin\alpha\right)^2\right]+\frac{1}{2}I\dot{\phi}^2 \end{aligned} \tag{2}$$

Using $I = mr^2$ and $S = r\phi$, (2) becomes

$$T_{\text{hoop}} = \frac{1}{2}m\left[2\dot{S}^2+\dot{\xi}^2+2\dot{\xi}\dot{S}\cos\alpha\right] \tag{3}$$

In order to find the total kinetic energy, we need to add the kinetic energy of the translational motion of the plane along the x-axis which is

$$T_{\text{plane}} = \frac{1}{2}M\,\dot{\xi}^2 \tag{4}$$

Therefore, the total kinetic energy becomes

$$T = m\dot{S}^2+\frac{1}{2}(m+M)\dot{\xi}^2+m\dot{\xi}\dot{S}\cos\alpha \tag{5}$$

The potential energy is

$$U = mgy = mg\left[r\cos\alpha+(\ell-S)\sin\alpha\right] \tag{6}$$

Hence, the Lagrangian is

$$l = m\dot{S}^2+\frac{1}{2}(m+M)\dot{\xi}^2+m\dot{\xi}\dot{S}\cos\alpha-mg\left[r\cos\alpha+(\ell-S)\sin\alpha\right] \tag{7}$$

from which the Lagrange equations for ξ and S are easily found to be

$$\boxed{2m\ddot{S}+m\ddot{\xi}\cos\alpha-mg\sin\alpha=0} \tag{8}$$

$$\boxed{(m+M)\ddot{\xi}+m\ddot{S}\cos\alpha=0} \tag{9}$$

or, if we rewrite these equations in the form of uncoupled equations by substituting for $\ddot{\xi}$ and $\ddot{S}$, we have

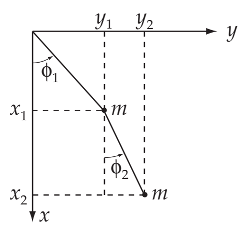
$$\left.\begin{aligned} &\left[2-\frac{m\cos^2\alpha}{m+M}\right]\ddot{S}-g\sin\alpha=0 \\ &\ddot{\xi}=-\frac{mg\sin\alpha\cos\alpha}{2(m+M)-m\cos^2\alpha} \end{aligned}\right] \tag{10}$$

Now, we can rewrite (9) as

$$\frac{d}{dt}\left[(m+M)\dot{\xi}+m\dot{S}\cos\alpha\right]=0 \tag{11}$$

where we can interpret $(m+M)\dot{\xi}$ as the x component of the linear momentum of the total system and $m\dot{S}\cos\alpha$ as the x component of the linear momentum of the hoop with respect to the plane. Therefore, (11) means that the x component of the total linear momentum is a constant of motion. This is the expected result because no external force is applied along the x-axis.

7-7.



If we take (ϕ_1,ϕ_2) as our generalized coordinates, the x,y coordinates of the two masses are

$$\left.\begin{aligned} x_1 &= \ell\cos\phi_1 \\ y_1 &= \ell\sin\phi_1 \end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned} x_2 &= \ell\cos\phi_1+\ell\cos\phi_2 \\ y_2 &= \ell\sin\phi_1+\ell\sin\phi_2 \end{aligned}\right] \tag{2}$$

Using (1) and (2), we find the kinetic energy of the system to be

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{m}{2}\left(\dot{x}_2^2+\dot{y}_2^2\right) \\ &= \frac{m}{2}\ell^2\left[\dot{\phi}_1^2+\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\left(\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2\right)\right] \\ &= \frac{m}{2}\ell^2\left[2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1-\phi_2\right)\right] \end{aligned} \tag{3}$$

The potential energy is

$$U = -mgx_1 - mgx_2 = -mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{4}$$

Therefore, the Lagrangian is

$$L = m\ell^2\left[\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + \dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right)\right] + mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{5}$$

from which

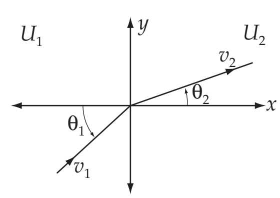
$$\left.\begin{aligned}
\frac{\partial L}{\partial \phi_1} &= m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - 2mg\ell\sin\phi_1 \\
\frac{\partial L}{\partial \dot{\phi}_1} &= 2m\ell^2\dot{\phi}_1 + m\ell^2\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right) \\
\frac{\partial L}{\partial \phi_2} &= -m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - mg\ell\sin\phi_2 \\
\frac{\partial L}{\partial \dot{\phi}_2} &= m\ell^2\dot{\phi}_2 + m\ell^2\dot{\phi}_1\cos\left(\phi_1 - \phi_2\right)
\end{aligned}\right] \tag{6}$$

The Lagrange equations for ϕ_1 and ϕ_2 are

$$\boxed{2\phi_1 + \ddot{\phi}_2\cos\left(\phi_1 - \phi_2\right) + \dot{\phi}_2^2\sin\left(\phi_1 - \phi_2\right) + 2\frac{g}{\ell}\sin\phi_1 = 0} \tag{7}$$

$$\boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos\left(\phi_1 - \phi_2\right) - \dot{\phi}_1^2\sin\left(\phi_1 - \phi_2\right) + \frac{g}{\ell}\sin\phi_2 = 0} \tag{8}$$

7-8.



Let us choose the x,y coordinates so that the two regions are divided by the y axis:

$$U(x) = \left[\begin{array}{ll} U_1 & x < 0 \\ U_2 & x > 0 \end{array}\right.$$

If we consider the potential energy as a function of x as above, the Lagrangian of the particle is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(x) \tag{1}$$

Therefore, Lagrange's equations for the coordinates x and y are

$$m\ddot{x} + \frac{dU(x)}{dx} = 0 \tag{2}$$

$$m\ddot{y} = 0 \tag{3}$$

Using the relation

$$m\ddot{x} = \frac{d}{dt} m\dot{x} = \frac{dP_x}{dt} = \frac{dP_x}{dx}\frac{dx}{dt} = \frac{P_x}{m}\frac{dp_x}{dx} \tag{4}$$

(2) becomes

$$\frac{P_x}{m}\frac{dP_x}{dx} + \frac{dU(x)}{dx} = 0 \tag{5}$$

Integrating (5) from any point in the region 1 to any point in the region 2, we find

$$\int_1^2 \frac{P_x}{m}\frac{dP_x}{dx}\,dx + \int_1^2 \frac{dU(x)}{dx}\,dx = 0 \tag{6}$$

$$\frac{P_{x_2}^2}{2m} - \frac{P_{x_1}^2}{2m} + U_2 - U_1 = 0 \tag{7}$$

or, equivalently,

$$\frac{1}{2} m\dot{x}_1^2 + U_1 = \frac{1}{2} m\dot{x}_2^2 + U_2 \tag{8}$$

Now, from (3) we have

$$\frac{d}{dt} m\dot{y} = 0$$

and $m\dot{y}$ is constant. Therefore,

$$m\dot{y}_1 = m\dot{y}_2 \tag{9}$$

From (9) we have

$$\frac{1}{2} m\dot{y}_1^2 = \frac{1}{2} m\dot{y}_2^2 \tag{10}$$

Adding (8) and (10), we have

$$\frac{1}{2} mv_1^2 + U_1 = \frac{1}{2} mv_2^2 + U_2 \tag{11}$$

From (9) we also have

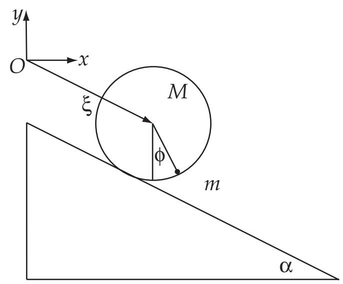
$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2 \tag{12}$$

Substituting (11) into (12), we find

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \left[1 + \frac{U_1 - U_2}{T_1}\right]^{1/2}} \tag{13}$$

This problem is the mechanical analog of the refraction of light upon passing from a medium of a certain optical density into a medium with a different optical density.

7-9.



Using the generalized coordinates given in the figure, the Cartesian coordinates for the disk are ($\xi \cos \alpha, -\xi \sin \alpha$), and for the bob they are ($\ell \sin \phi + \xi \cos \alpha, -\ell \cos \phi - \xi \sin \alpha$). The kinetic energy is given by

$$T = T_{\text{disk}} + T_{\text{bob}} = \left[\frac{1}{2} M\dot{\xi}^2 + \frac{1}{2} I\dot{\theta}^2\right] + \frac{1}{2} m\left(\dot{x}_{\text{bob}}^2 + \dot{y}_{\text{bob}}^2\right) \tag{1}$$

Substituting the coordinates for the bob, we obtain

$$T = \frac{1}{2}(M+m)\dot{\xi}^2 + \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\ell^2\dot{\phi}^2 + m\ell\dot{\phi}\dot{\xi}\cos(\phi + a) \tag{2}$$

The potential energy is given by

$$U = U_{\text{disk}} + U_{\text{bob}} = Mgy_{\text{disk}} + mgy_{\text{bob}} = -(M+m)g\xi \sin \alpha - mg\ell \cos \phi \tag{3}$$

Now let us use the relation $\xi = R\theta$ to reduce the degrees of freedom to two, and in addition substitute $I = MR^2/2$ for the disk. The Lagrangian becomes

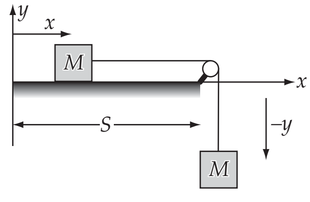
$$L = T - U = \left(\frac{3}{4} M + \frac{1}{2} m\right)\dot{\xi}^2 + \frac{1}{2} m\ell^2\dot{\phi}^2 + m\ell\dot{\phi}\dot{\xi}\cos(\phi + a) + (M+m)g\xi \sin \alpha + mg\ell \cos \phi \tag{4}$$

The resulting equations of motion for our two generalized coordinates are

$$\left(\frac{3}{2} M + m\right)\ddot{\xi} - (M+m)g \sin \alpha + m\ell\left[\ddot{\phi}\cos(\phi + \alpha) - \dot{\phi}^2 \sin(\phi + \alpha)\right] = 0 \tag{5}$$

$$\ddot{\phi} + \frac{1}{\ell}\ddot{\xi}\cos(\phi + \alpha) + \frac{g}{\ell}\sin \phi = 0 \tag{6}$$

7-10.



Let the length of the string be ℓ so that

$$(S-x)-y=\ell \tag{1}$$

Then,

$$\dot{x}=-\dot{y} \tag{2}$$

a) The Lagrangian of the system is

$$L=\frac{1}{2}M\dot{x}^2+\frac{1}{2}M\dot{y}^2-Mgy=M\dot{y}^2-Mgy \tag{3}$$

Therefore, Lagrange's equation for y is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}-\frac{\partial L}{\partial y}=2M\ddot{y}+Mg=0 \tag{4}$$

from which

$$\ddot{y}=-\frac{g}{2} \tag{5}$$

Then, the general solution for y becomes

$$y(t)=-\frac{g}{4}t^2+C_1t+C_2 \tag{6}$$

If we assign the initial conditions $y(t=0)=0$ and $\dot{y}(t=0)=0$, we find

$$\boxed{y(t)=-\frac{g}{4}t^2} \tag{7}$$

b) If the string has a mass m, we must consider its kinetic energy and potential energy. These are

$$T_{\text{string}}=\frac{1}{2}m\dot{y}^2 \tag{8}$$

$$U_{\text{string}}=-\frac{m}{\ell}yg\frac{y}{2}=-\frac{mg}{2\ell}y^2 \tag{9}$$

Adding (8) and (9) to (3), the total Lagrangian becomes

$$L=M\dot{y}^2-Mgy+\frac{1}{2}m\dot{y}^2+\frac{mg}{2\ell}y^2 \tag{10}$$

Therefore, Lagrange's equation for y now becomes

$$(2M+m)\ddot{y}-\frac{mg}{\ell}y+Mg=0 \tag{11}$$

In order to solve (11), we arrange this equation into the form

$$(2M+m)\ddot{y}=\frac{mg}{\ell}\left[y-\frac{M\ell}{m}\right] \tag{12}$$

Since $\frac{d^2}{dt^2}\left[y-\frac{M\ell}{m}\right]=\frac{d^2}{dt^2}y$, (12) is equivalent to

$$\frac{d^2}{dt^2}\left[y-\frac{M\ell}{m}\right]=\frac{mg}{\ell(2M+m)}\left[y-\frac{M\ell}{m}\right] \tag{13}$$

which is solved to give

$$y-\frac{M\ell}{m}=Ae^{\gamma t}+Be^{-\gamma t} \tag{14}$$

where

$$\gamma=\sqrt{\frac{mg}{\ell(2M+m)}} \tag{15}$$

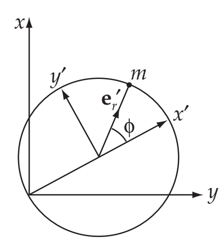
If we assign the initial condition $y(t=0)=0$; $\dot{y}(t=0)=0$, we have

$$A=+B=-\frac{M\ell}{2m}$$

Then,

$$\boxed{y(t)=\frac{M\ell}{m}(1-\cosh\gamma t)} \tag{16}$$

7-11.



The x,y coordinates of the particle are

$$\left.\begin{aligned} x &= R\cos\omega t+R\cos(\phi+\omega t) \\ y &= R\sin\omega t+R\sin(\phi+\omega t) \end{aligned}\right] \tag{1}$$

Then,

$$\left.\begin{aligned} \dot{x} &= -R\omega\sin\omega t-R(\dot{\phi}+\omega)\sin(\phi+\omega t) \\ \dot{y} &= R\omega\cos\omega t+R(\dot{\phi}+\omega)\cos(\phi+\omega t) \end{aligned}\right] \tag{2}$$

Since there is no external force, the potential energy is constant and can be set equal to zero. The Lagrangian becomes

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$= \frac{m}{2} \left[R^2 \omega^2 + R^2 \left(\phi + \omega \right)^2 + 2R^2 \omega \left(\phi + \omega \right) \cos \phi \right] \tag{3}$$

from which

$$\frac{\partial L}{\partial \phi} = -mR^2 \omega \left(\dot{\phi} + \omega \right) \sin \phi \tag{4}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} \left[mR^2 \left(\dot{\phi} + \omega + \omega \cos \phi \right) \right] \tag{5}$$

Therefore, Lagrange's equation for ϕ becomes

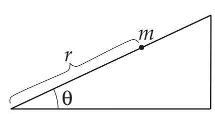
$$\boxed{\ddot{\phi} + \omega^2 \sin \phi = 0} \tag{6}$$

which is also the equation of motion for a simple pendulum. To make the result appear reasonable, note that we may write the acceleration felt by the particle in the rotating frame as

$$\mathbf{a} = \omega^2 R \left(\mathbf{i}' + \mathbf{e}_r' \right) \tag{7}$$

where the primed unit vectors are as indicated in the figure. The part proportional to $\mathbf{e}_r'$ does not affect the motion since it has no contribution to the torque, and the part proportional to $\mathbf{i}'$ is constant and does not contribute to the torque in the same way a constant gravitational field provides a torque to the simple pendulum.

7-12.



Put the origin at the bottom of the plane

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr \sin \theta$$

$$\theta = \alpha t; \quad \dot{\theta} = \alpha$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + \alpha^2 r^2 \right) - mgr \sin \alpha t$$

Lagrange's equation for r gives

$$m\ddot{r} = m\alpha^2 r - mg \sin \alpha t$$

or

$$\ddot{r} - \alpha^2 r = -g \sin \alpha t \tag{1}$$

The general solution is of the form $r = r_p + r_h$ where r_h is the general solution of the homogeneous equation $\ddot{r} - \alpha^2 r = 0$ and r_p is a particular solution of Eq. (1).

So

$$r_h = Ae^{\alpha t} + Be^{-\alpha t}$$

For r_p, try a solution of the form $r_p = C \sin \alpha t$. Then $\ddot{r}_p = -C\,\alpha^2 \sin \alpha t$. Substituting into (1) gives

$$-C\,\alpha^2 \sin \alpha t - C\ \alpha^2 \sin \alpha t = -g \sin \alpha t$$

$$C = \frac{g}{2\alpha^2}$$

So

$$r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

We can determine A and B from the initial conditions:

$$r(0) = r_0 \tag{2}$$

$$\dot{r}(0) = 0 \tag{3}$$

(2) implies $r_0 = A + B$

(3) implies $0 = A - B + \dfrac{g}{2\alpha^2}$

Solving for A and B gives:

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right] \qquad B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]$$

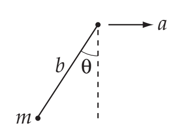
$$r(t) = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right]e^{\alpha t} + \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]e^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

or

$$\boxed{r(t) = r_0 \cosh \alpha t + \frac{g}{2\alpha^2}\left(\sin \alpha t - \sinh \alpha t\right)}$$

7-13.

a)



$$x = \frac{1}{2}at^2 - b\sin\theta$$

$$y = -b\cos\theta$$

$$\dot{x} = at - b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$= \frac{1}{2}m\left(a^2t^2 - 2at\,b\dot{\theta}\cos\theta + b^2\dot{\theta}^2\right) + mgb\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \quad \text{gives}$$

$$\frac{d}{dt}\left[-mat\,b\cos\theta + mb^2\dot{\theta}\right] = mat\,b\dot{\theta}\sin\theta - mgb\sin\theta$$

This gives the equation of motion

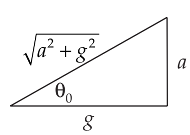
$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\cos\theta = 0}$$

b) To find the period for small oscillations, we must expand $\sin\theta$ and $\cos\theta$ about the equilibrium point θ_0. We find θ_0 by setting $\ddot{\theta} = 0$. For equilibrium,

$$g\sin\theta_0 = a\cos\theta_0$$

or

$$\tan\theta_0 = \frac{a}{g}$$



Using the first two terms in a Taylor series expansion for $\sin\theta$ and $\cos\theta$ gives

$$f(\theta) \simeq f(\theta_0) + f'(\theta)\Big]_{\theta=\theta_0}(\theta - \theta_0)$$

$$\sin\theta \simeq \sin\theta_0 + (\theta - \theta_0)\cos\theta_0$$

$$\cos\theta \simeq \cos\theta_0 - (\theta - \theta_0)\sin\theta_0$$

$$\tan\theta_0 = \frac{a}{g} \text{ implies } \sin\theta_0 = \frac{a}{\sqrt{a^2 + g^2}},$$

$$\cos\theta_0 = \frac{g}{\sqrt{a^2 + g^2}}$$

Thus

$$\sin\theta \simeq \frac{1}{\sqrt{a^2+g^2}}\left(a+g\theta-g\theta_0\right)$$

$$\cos\theta \simeq \frac{1}{\sqrt{a^2+g^2}}\left(g-a\theta+a\theta_0\right)$$

Substituting into the equation of motion gives

$$0=\ddot{\theta}+\frac{g}{b\sqrt{a^2+g^2}}\left(a+g\theta-g\theta_0\right)-\frac{a}{b\sqrt{a^2+g^2}}\left(g-a\theta+a\theta_0\right)$$

This reduces to

$$\ddot{\theta}+\frac{\sqrt{g^2+a^2}}{b}\,\theta=\frac{\sqrt{g^2+a^2}}{b}\,\theta_0$$

The solution to this inhomogeneous differential equation is

$$\theta=\theta_0+A\cos\omega\theta+B\sin\omega\theta$$

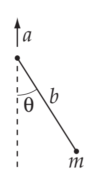
where

$$\omega=\frac{\left(g^2+a^2\right)^{1/4}}{b^{1/2}}$$

Thus

$$\boxed{T=\frac{2\pi}{\omega}=\frac{2\pi\, b^{1/2}}{\left(g^2+a^2\right)^{1/4}}}$$

7-14.



$$x=b\sin\theta$$

$$y=\frac{1}{2}at^2-b\cos\theta$$

$$\dot{x}=b\theta\cos\theta$$

$$\dot{y}=at+b\dot{\theta}\sin\theta$$

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}m\left(b^2\dot{\theta}^2+a^2t^2+2abt\dot{\theta}\sin\theta\right)$$

$$U = mgy = mg\left[\frac{1}{2}at^2 - b\cos\theta\right]$$

$$L = T - U = \frac{1}{2}m\left(b^2\dot{\theta}^2 + a^2t^2 + 2abt\dot{\theta}\sin\theta\right) + mg\left(b\cos\theta - \frac{1}{2}at^2\right)$$

Lagrange's equation for θ gives

$$\frac{d}{dt}\left[mb^2\dot{\theta} + mabt\sin\theta\right] = mabt\dot{\theta}\cos\theta - mgb\sin\theta$$

$$b^2\ddot{\theta} + ab\sin\theta + abt\dot{\theta}\cos\theta = abt\dot{\theta}\cos\theta - gb\sin\theta$$

$$\boxed{\ddot{\theta} + \frac{a+g}{b}\sin\theta = 0}$$

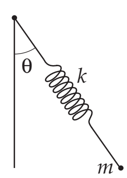
For small oscillations, $\sin\theta \simeq \theta$

$$\ddot{\theta} + \frac{a+g}{b}\theta = 0\ .$$

Comparing with $\ddot{\theta} + \omega^2\theta = 0$ gives

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{b}{a+g}}}$$

7-15.



b = unextended length of spring

ℓ = variable length of spring

$$T = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\,\dot{\theta}^2\right)$$

$$U = \frac{1}{2}k(\ell - b)^2 + mgy = \frac{1}{2}k(\ell - b)^2 - mg\,\ell\cos\theta$$

$$L = T - U = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\,\dot{\theta}^2\right) - \frac{1}{2}(\ell - b)^2 + mg\,\ell\cos\theta$$

Taking Lagrange's equations for ℓ and θ gives

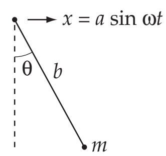
$$\ell : \frac{d}{dt}\left[m\dot{\ell}\right] = m\ell\dot{\theta}^2 - k(\ell - b) + mg\cos\theta$$

$$\theta: \frac{d}{dt}\left[m\ell^2\dot{\theta}\right] = -mg\,\ell\sin\theta$$

This reduces to

$$\boxed{\begin{aligned}&\ddot{\ell} - \ell\dot{\theta}^2 + \frac{k}{m}(\ell - b) - g\cos\theta = 0\\ &\ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{g}{\ell}\sin\theta = 0\end{aligned}}$$

7-16.



For mass m:

$$x = a\sin\omega t + b\sin\theta$$

$$y = -b\cos\theta$$

$$\dot{x} = a\omega\cos\omega t + b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

Substitute into

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$U = mgy$$

and the result is

$$L = T - U = \frac{1}{2}m\left(a^2\omega^2\cos^2\omega t + 2ab\omega\dot{\theta}\cos\omega t\cos\theta + b^2\dot{\theta}^2\right) + mgb\cos\theta$$

Lagrange's equation for θ gives

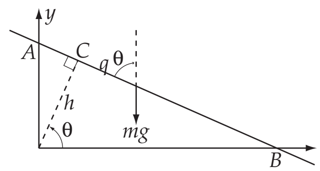
$$\frac{d}{dt}\left(mab\omega\cos\omega t\cos\theta + mb^2\dot{\theta}\right) = -mabw\dot{\theta}\cos\omega t\sin\theta - mgb\sin\theta$$

$$-ab\omega^2\sin\omega t\cos\theta - ab\omega\dot{\theta}\cos\omega t\sin\theta + b^2\ddot{\theta} = -\,ab\omega\dot{\theta}\cos\omega t\sin\theta - gb\sin\theta$$

or

$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\omega^2\sin\omega t\cos\theta = 0}$$

7-17.



Using q and $\theta\,(=\omega t$ since $\theta(0)=0)$, the x,y coordinates of the particle are expressed as

$$\left.\begin{aligned} x &= h\cos\theta + q\sin\theta = h\cos\omega t + q(t)\sin\omega t \\ y &= h\sin\theta - q\cos\theta = h\sin\omega t - q(t)\cos\omega t \end{aligned}\right] \tag{1}$$

from which

$$\left.\begin{aligned} \dot{x} &= -h\omega\sin\omega t + q\omega\cos\omega t + \dot{q}\sin\omega t \\ \dot{y} &= h\omega\cos\omega t + q\omega\sin\omega t - \dot{q}\cos\omega t \end{aligned}\right] \tag{2}$$

Therefore, the kinetic energy of the particle is

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}m\left(h^2\omega^2 + q^2\omega^2 + \dot{q}^2\right) - mh\omega\dot{q} \end{aligned} \tag{3}$$

The potential energy is

$$U = mgy = mg\left(h\sin\omega t - q\cos\omega t\right) \tag{4}$$

Then, the Lagrangian for the particle is

$$L = \frac{1}{2}mh^2\omega^2 + \frac{1}{2}mq^2\omega^2 + \frac{1}{2}m\dot{q}^2 - mgh\sin\omega t + mgq\cos\omega t - mh\omega\dot{q} \tag{5}$$

Lagrange's equation for the coordinate is

$$\ddot{q} - \omega^2 q = g\cos\omega t \tag{6}$$

The complementary solution and the particular solution for (6) are written as

$$\left.\begin{aligned} q_c(t) &= A\cos\left(i\omega t + \delta\right) \\ q_P(t) &= -\frac{g}{2\omega^2}\cos\omega t \end{aligned}\right] \tag{7}$$

so that the general solution is

$$q(t) = A\cos\left(i\omega t + \delta\right) - \frac{g}{2\omega^2}\cos\omega t \tag{8}$$

Using the initial conditions, we have

$$\left.\begin{aligned}q(0) &= A\cos\delta - \frac{g}{2\omega^2} = 0\\ \dot{q}(0) &= -i\omega A\sin\delta = 0\end{aligned}\right] \tag{9}$$

Therefore,

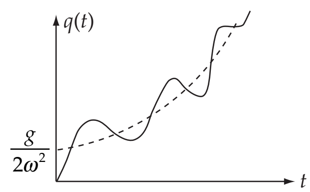
$$\delta = 0,\ A = \frac{g}{2\omega^2} \tag{10}$$

and

$$q(t) = \frac{g}{2\omega^2}\left(\cos i\omega t - \cos\omega t\right) \tag{11}$$

or,

$$\boxed{q(t) = \frac{g}{2\omega^2}\left(\cosh\omega t - \cos\omega t\right)} \tag{12}$$



In order to compute the Hamiltonian, we first find the canonical momentum of q. This is obtained by

$$p = \frac{\partial L}{\partial \dot{q}} = mq - m\omega h \tag{13}$$

Therefore, the Hamiltonian becomes

$$\begin{aligned}H &= p\dot{q} - L\\ &= m\dot{q}^2 - m\omega h\dot{q} - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 - \frac{1}{2}m\dot{q}^2 + mgh\sin\omega t - mgq\cos\omega t + m\omega\dot{q}h\end{aligned}$$

so that

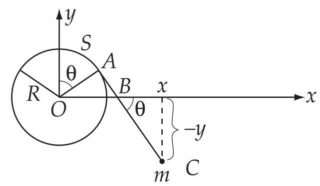
$$H = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t \tag{14}$$

Solving (13) for $\dot{q}$ and substituting gives

$$\boxed{H = \frac{p^2}{2m} + \omega hp - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t} \tag{15}$$

The Hamiltonian is therefore different from the total energy, $T + U$. The energy is not conserved in this problem since the Hamiltonian contains time explicitly. (The particle gains energy from the gravitational field.)

7-18.



From the figure, we have the following relation:

$$\overline{AC} = \ell - s = \ell - R\theta \tag{1}$$

where θ is the generalized coordinate. In terms of θ, the x,y coordinates of the mass are

$$\left.\begin{aligned} x &= \overline{AC}\cos\theta + R\sin\theta = (\ell - R\theta)\cos\theta + R\sin\theta \\ y &= R\cos\theta - \overline{AC}\sin\theta = R\cos\theta - (\ell - R\theta)\sin\theta \end{aligned}\right] \tag{2}$$

from which

$$\left.\begin{aligned} \dot{x} &= R\theta\dot{\theta}\sin\theta - \ell\dot{\theta}\sin\theta \\ \dot{y} &= R\theta\dot{\theta}\cos\theta - \ell\dot{\theta}\cos\theta \end{aligned}\right] \tag{3}$$

Therefore, the kinetic energy becomes

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\ell^2\dot{\theta}^2 + R^2\theta^2\dot{\theta}^2 - 2R\ell\theta\dot{\theta}^2\right] \tag{4}$$

The potential energy is

$$U = mgy = mg\left[R\cos\theta - (\ell - R\theta)\sin\theta\right] \tag{5}$$

Then, the Lagrangian is

$$L = T - U = \frac{1}{2}m\left[\ell^2\dot{\theta}^2 + R^2\theta^2\dot{\theta}^2 - 2R\ell\theta\dot{\theta}^2\right] - mg\left[R\cos\theta - (\ell - R\theta)\sin\theta\right] \tag{6}$$

Lagrange's equation for θ is

$$(\ell - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0 \tag{7}$$

Now let us expand about some angle θ_0, and assume the deviations are small. Defining $\varepsilon \equiv \theta - \theta_0$, we obtain

$$\ddot{\varepsilon} + \frac{g\sin\theta_0}{\ell - R\theta_0}\varepsilon = \frac{g\cos\theta_0}{\ell - R\theta_0} \tag{8}$$

The solution to this differential equation is

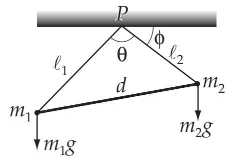
$$\varepsilon = A\sin(\omega t + \delta) + \frac{\cos\theta_0}{\sin\theta_0} \tag{9}$$

where A and δ are constants of integration and

$$\omega \equiv \sqrt{\frac{g \sin \theta_0}{\ell - R\theta_0}} \tag{10}$$

is the frequency of small oscillations. It is clear from (9) that θ extends equally about θ_0 when $\theta_0 = \pi/2$.

7-19.



Because of the various constraints, only one generalized coordinate is needed to describe the system. We will use ϕ, the angle between a plane through P perpendicular to the direction of the gravitational force vector, and one of the extensionless strings, e.g., ℓ_2, as our generalized coordinate.

The, the kinetic energy of the system is

$$T = \frac{1}{2} m_1 \left(\ell_1 \dot{\phi}\right)^2 + \frac{1}{2} m_2 \left(\ell_2 \dot{\phi}\right)^2 \tag{1}$$

The potential energy is given by

$$U = -m_1 g \ell_1 \sin\left(\pi - (\phi + \theta)\right) - m_2 g \ell_2 \sin \phi \tag{2}$$

from which the Lagrangian has the form

$$L = T - U = \frac{1}{2}\left(m_1 \ell_1^2 + m_2 \ell_2^2\right)\dot{\phi}^2 + m_1 g \ell_1 \sin(\phi + \theta) + m_2 g \ell_2 \sin \phi \tag{3}$$

The Lagrangian equation for ϕ is

$$m_2 g \ell_2 \cos \phi + m_1 g \ell_1 \cos(\phi + \theta) - \left(m_1 \ell_1^2 + m_2 \ell_2^2\right)\ddot{\phi} = 0 \tag{4}$$

This is the equation which describes the motion in the plane m_1, m_2, P.

To find the frequency of small oscillations around the equilibrium position (defined by $\phi = \phi_0$), we expand the potential energy U about ϕ_0:

$$U(\phi) = U(\phi_0) + U'(\phi_0)\phi + \frac{1}{2} U''(\phi_0)\phi^2 + \ldots$$

$$= \frac{1}{2} U''(\phi_0)\phi^2 \tag{5}$$

where the last equality follows because we can take $U(\phi_0) = 0$ and because $U'(\phi_0) = 0$.

From (4) and (5), the frequency of small oscillations around the equilibrium position is

$$\omega^2 = \frac{U''(\phi_0)}{m_1\ell_1^2 + m_2\ell_2^2} \tag{6}$$

The condition $U'(\phi_0) = 0$ gives

$$\tan\phi_0 = \frac{m_2\ell_2 + m_1\ell_1\cos\theta}{m_1\ell_1\sin\theta} \tag{7}$$

or,

$$\sin\phi_0 = \frac{m_2\ell_2 + m_1\ell_1\cos\theta}{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_1m_2\ell_1\ell_2\cos\theta\right)^{1/2}} \tag{8}$$

Then from (2), (7), and (8), $U''(\phi_0)$ is found to be

$$U''(\phi_0) = g\sin\phi_0\left(m_2\ell_2 + m_1\ell_1\cos\theta + m_1\ell_1\sin\theta\cot\phi_0\right)$$

$$= \frac{g\left(m_2\ell_2 + m_1\ell_1\cos\theta\right)}{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_2m_1\ell_2\ell_1\cos\theta\right)^{1/2}}\left[m_2\ell_2 + m_1\ell_1\cos\theta + \frac{m_1^2\ell_1^2\sin^2\theta}{m_2\ell_2 + m_1\ell_1\cos\theta}\right]$$

$$= g\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_2m_1\ell_2\ell_1\cos\theta\right)^{1/2} \tag{9}$$

Finally, from (6) and (9), we have

$$\omega^2 = g\frac{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_1m_2\ell_1\ell_2\cos\theta\right)^{1/2}}{\left(m_1\ell_1^2 + m_2\ell_2^2\right)} \tag{10}$$

which, using the relation,

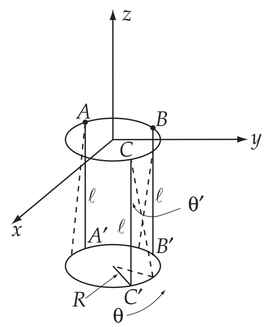
$$\cos\theta = \frac{\ell_1^2 + \ell_2^2 - d^2}{2\ell_1\ell_2} \tag{11}$$

can be written as

$$\boxed{\omega^2 = \frac{g\left[\left(m_1 + m_2\right)\left(m_1\ell_1^2 + m_2\ell_2^2\right) - d^2m_1m_2\right]^{1/2}}{\left(m_1\ell_1^2 + m_2\ell_2^2\right)}} \tag{12}$$

Notice that ω^2 degenerates to the value g/ℓ appropriate for a simple pendulum when d $\to 0$ (so that $\ell_1 = \ell_2$).

7-20. The x-y plane is horizontal, and A, B, C are the fixed points lying in a plane above the hoop. The hoop rotates about the vertical through its center.



The kinetic energy of the system is given by

$$T = \frac{1}{2} I\omega^2 + \frac{1}{2} M\dot{z}^2 = \frac{MR^2}{2}\dot{\theta}^2 + \frac{1}{2} M\left[\frac{\partial z}{\partial \theta}\right]^2 \dot{\theta}^2 \tag{1}$$

For small θ, the second term can be neglected since $\left(\partial z/\partial \theta\right)\big|_{\theta=0} = 0$

The potential energy is given by

$$U = Mgz \tag{2}$$

where we take $U = 0$ at $z = -\ell$.

Since the system has only one degree of freedom we can write z in terms of θ. When $\theta = 0$, $z = -\ell$. When the hoop is rotated thorough an angle θ, then

$$z^2 = \ell^2 - \left(R - R\cos\theta\right)^2 - \left(R\sin\theta\right)^2 \tag{3}$$

so that

$$z = -\left[\ell^2 + 2R^2\left(\cos\theta - 1\right)\right]^{1/2} \tag{4}$$

and the potential energy is given by

$$U = -Mg\left[\ell^2 + 2R^2\left(\cos\theta - 1\right)\right]^{1/2} \tag{5}$$

for small θ, $\cos\theta - 1 \cong -\theta^2/2$; then,

$$U \cong -Mg\ell\left[1 - \frac{R^2\theta^2}{\ell^2}\right]^{1/2}$$

$$\cong -Mg\ell\left[1 - \frac{R^2\theta^2}{2\ell^2}\right] \tag{6}$$

From (1) and (6), the Lagrangian is

$$L = T - U = \frac{1}{2} MR^2\dot{\theta}^2 + Mg\ell\left[1 - \frac{R^2\theta^2}{2\ell^2}\right], \tag{7}$$

for small θ. The Lagrange equation for θ gives

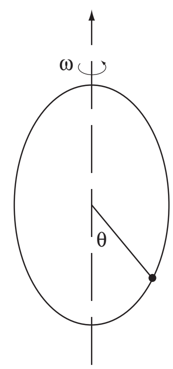
$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \tag{8}$$

where

$$\boxed{\omega = \sqrt{\frac{g}{\ell}}} \tag{9}$$

which is the frequency of small rotational oscillations about the vertical through the center of the hoop and is the same as that for a simple pendulum of length ℓ.

7-21.



From the figure, we can easily write down the Lagrangian for this system.

$$T = \frac{mR^2}{2}\left(\dot{\theta}^2 + \omega^2 \sin^2 \theta\right) \tag{1}$$

$$U = -mgR\cos\theta \tag{2}$$

The resulting equation of motion for θ is

$$\ddot{\theta} - \omega^2 \sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0 \tag{3}$$

The equilibrium positions are found by finding the values of θ for which

$$0 = \ddot{\theta}\Big|_{\theta=\theta_0} = \left(\omega^2 \cos\theta_0 - \frac{g}{R}\right)\sin\theta_0 \tag{4}$$

Note first that 0 and π are equilibrium, and a third is defined by the condition

$$\cos\theta_0 = \frac{g}{\omega^2 R} \tag{5}$$

To investigate the stability of each of these, expand using $\varepsilon = \theta - \theta_0$

$$\ddot{\varepsilon} = \omega^2\left(\cos\theta_0 - \frac{g}{\omega^2 R} - \varepsilon\sin\theta_0\right)\left(\sin\theta_0 + \varepsilon\cos\theta_0\right) \tag{6}$$

For $\theta_0 = \pi$, we have

$$\ddot{\varepsilon} = \omega^2\left(1 + \frac{g}{\omega^2 R}\right)\varepsilon \tag{7}$$

indicating that it is unstable. For $\theta_0 = 0$, we have

$$\ddot{\varepsilon} = \omega^2\left(1 - \frac{g}{\omega^2 R}\right)\varepsilon \tag{8}$$

which is stable if $\omega^2 < g/R$ and unstable if $\omega^2 > g/R$. When stable, the frequency of small oscillations is $\sqrt{\omega^2 - g/R}$. For the final candidate,

$$\ddot{\varepsilon} = -\omega^2 \sin^2\theta_0 \varepsilon \tag{9}$$

with a frequency of oscillations of $\sqrt{\omega^2 - (g/\omega R)^2}$, when it exists. Defining a critical frequency $\omega_c^2 \equiv g/R$, we have a stable equilibrium at $\theta_0 = 0$ when $\omega < \omega_c$, and a stable equilibrium at $\theta_0 = \cos^{-1}\left(\omega_c^2/\omega^2\right)$ when $\omega \geq \omega_c$. The frequencies of small oscillations are then $\omega\sqrt{1-(\omega_c/\omega)^2}$ and $\omega\sqrt{1-(\omega_c/\omega)^4}$, respectively.

To construct the phase diagram, we need the Hamiltonian

$$H \equiv \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \tag{10}$$

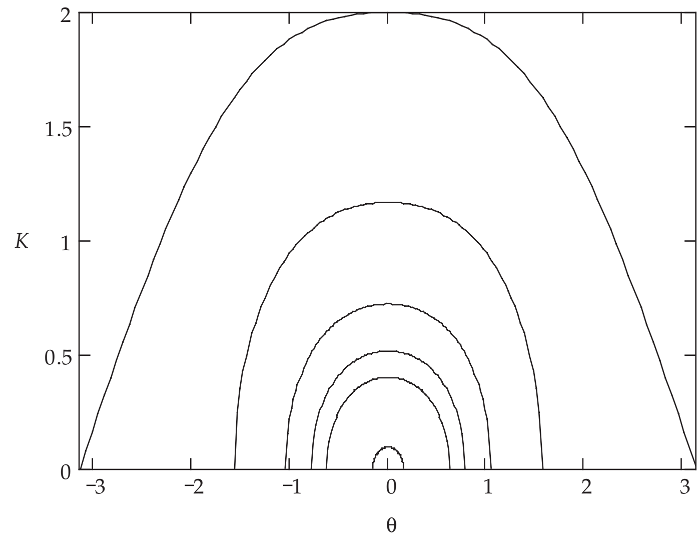
which is not the total energy in this case. A convenient parameter that describes the trajectory for a particular value of *H* is

$$K \equiv \frac{H}{m\omega_c^2 R^2} = \frac{1}{2}\left[\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\right)^2 \sin^2\theta\right] - \cos\theta \tag{11}$$

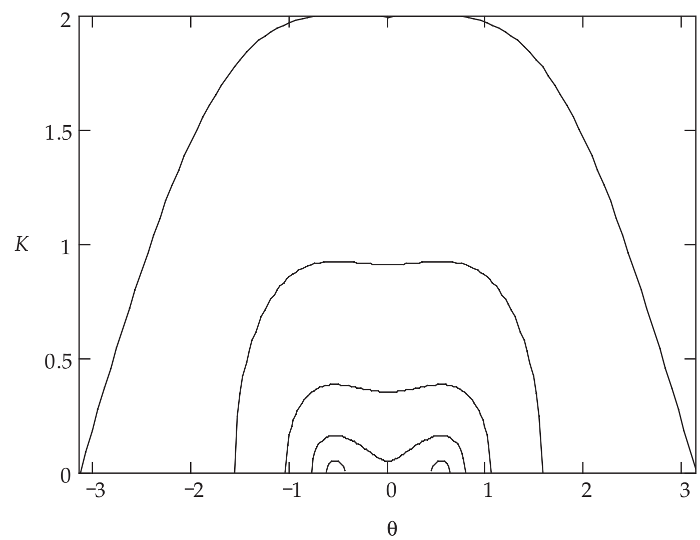
so that we'll end up plotting

$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2(K + \cos\theta) + \left(\frac{\omega}{\omega_c}\right)^2 \sin^2\theta \tag{12}$$

for a particular value of ω and for various values of *K*. The results for $\omega < \omega_c$ are shown in figure (b), and those for $\omega > \omega_c$ are shown in figure (c). Note how the origin turns from an attractor into a separatrix as ω increases through ω_c. As such, the system could exhibit chaotic behavior in the presence of damping.



(b)



(c)

7-22. The potential energy U which gives the force

$$F(x,t) = \frac{k}{x^2}\, e^{-(t/\tau)} \tag{1}$$

must satisfy the relation

$$F = -\frac{\partial U}{\partial x} \tag{2}$$

we find

$$U = \frac{k}{x}\, e^{-t/\tau} \tag{3}$$

Therefore, the Lagrangian is

$$\boxed{L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{k}{x}e^{-t/\tau}} \tag{4}$$

The Hamiltonian is given by

$$H = p_x\dot{x} - L = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \tag{5}$$

so that

$$\boxed{H = \frac{p_x^2}{2m} + \frac{k}{x}e^{-t/\tau}} \tag{6}$$

The Hamiltonian is equal to the total energy, $T + U$, because the potential does not depend on velocity, but the total energy of the system is not conserved because H contains the time explicitly.

7-23. The Hamiltonian function can be written as [see Eq. (7.153)]

$$H = \sum_j p_j\dot{q}_j - L \tag{1}$$

For a particle which moves freely in a conservative field with potential U, the Lagrangian in rectangular coordinates is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U$$

and the linear momentum components in rectangular coordinates are

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= m\dot{y} \\ p_z &= m\dot{z} \end{aligned}\right] \tag{2}$$

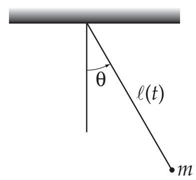
$$\begin{aligned} H &= \left[m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2\right] - \left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\right] \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \end{aligned} \tag{3}$$

which is just the total energy of the particle. The canonical equations are [from Eqs. (7.160) and (7.161)]

$$\boxed{\begin{aligned} \dot{p}_x &= m\ddot{x} = -\frac{\partial U}{\partial x} = F_x \\ \dot{p}_y &= m\ddot{y} = -\frac{\partial U}{\partial y} = F_y \\ \dot{p}_z &= m\ddot{z} = -\frac{\partial U}{\partial z} = F_z \end{aligned}} \tag{4}$$

These are simply Newton's equations.

7-24.



The kinetic energy and the potential energy of the system are expressed as

$$\left.\begin{aligned} T &= \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) = \frac{1}{2}m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) \\ U &= -mg\ell\cos\theta \end{aligned}\right] \tag{1}$$

so that the Lagrangian is

$$L = T - U = \frac{1}{2}m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) + mg\ell\cos\theta \tag{2}$$

The Hamiltonian is

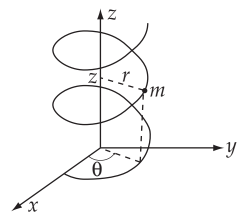
$$H = p_\theta\dot{\theta} - L = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L$$

$$= \boxed{\frac{p_\theta^2}{2m\ell^2} - \frac{1}{2}m\alpha^2 - mg\ell\cos\theta} \tag{3}$$

which is different from the total energy, $T + U$. The total energy is not conserved in this system because work is done on the system and we have

$$\frac{d}{dt}(T + U) \neq 0 \tag{4}$$

7-25.



In cylindrical coordinates the kinetic energy and the potential energy of the spiraling particle are expressed by

$$\left.\begin{aligned} T &= \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right] \\ U &= mgz \end{aligned}\right] \tag{1}$$

Therefore, if we use the relations,

$$\left.\begin{aligned} &z = k\theta \qquad \text{i.e., } \dot{z} = k\dot{\theta} \\ &r = \text{const.} \end{aligned}\right] \tag{2}$$

the Lagrangian becomes

$$L = \frac{1}{2} m\left[\frac{r^2}{k^2}\dot{z}^2 + \dot{z}^2\right] - mgz \tag{3}$$

Then the canonical momentum is

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\left[\frac{r^2}{k^2} + 1\right]\dot{z} \tag{4}$$

or,

$$\dot{z} = \frac{p_z}{m\left[\dfrac{r^2}{k^2} + 1\right]} \tag{5}$$

The Hamiltonian is

$$H = p_z\, \dot{z} - L = p_z \frac{p_z}{m\left[\dfrac{r^2}{k^2} + 1\right]} - \frac{p_z^2}{2m\left[\dfrac{r^2}{k^2} + 1\right]} + mgz \tag{6}$$

or,

$$H = \frac{1}{2} \frac{p_z^2}{m\left[\dfrac{r^2}{k^2} + 1\right]} + mgz \tag{7}$$

Now, Hamilton's equations of motion are

$$-\frac{\partial H}{\partial z} = \dot{p}_z\,; \qquad \frac{\partial H}{\partial p_z} = \dot{z} \tag{8}$$

so that

$$-\frac{\partial H}{\partial z} = -mg = \dot{p}_z \tag{9}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m\left[\dfrac{r^2}{k^2}+1\right]} = \dot{z} \tag{10}$$

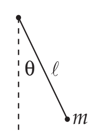
Taking the time derivative of (10) and substituting (9) into that equation, we find the equation of motion of the particle:

$$\boxed{\ddot{z} = \frac{g}{\left[\dfrac{r^2}{k^2}+1\right]}} \tag{11}$$

It can be easily shown that Lagrange's equation, computed from (3), gives the same result as (11).

7-26.

a)



$$L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 - mgy$$

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$$

so

$$\dot{\theta} = \frac{p_\theta}{m\ell^2}$$

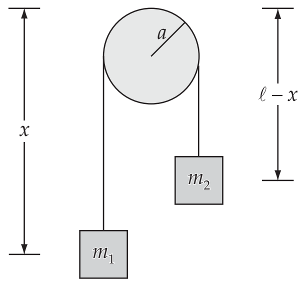
Since U is velocity-independent and the coordinate transformations are time-independent, the Hamiltonian is the total energy

$$\boxed{H = T + U = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta}$$

The equations of motion are

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell \sin\theta}$$

b)



$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{a^2}$$

where I = moment of inertia of the pulley

$$U = -m_1 g x - m_2 g(\ell - x)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = \left[m_1 + m_2 + \frac{I}{a^2}\right]\dot{x}$$

So

$$\dot{x} = \frac{p_x}{\left[m_1 + m_2 + \frac{I}{a^2}\right]}$$

$$H = T + U$$

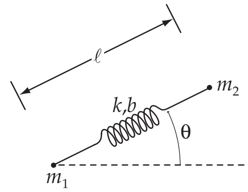
$$\boxed{H = \frac{p_x^2}{2\left[m_1 + m_2 + \frac{I}{a^2}\right]} - m_1 g x - m_2 g(\ell - x)}$$

The equations of motion are

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{2\left[m_1 + m_2 + \frac{I}{a^2}\right]} \\ \dot{p}_x &= -\frac{\partial H}{\partial p_x} = m_1 g - m_2 g = g(m_1 - m_2) \end{aligned}}$$

7-27.

a)



$$x_i, y_i = \text{coordinates of } m_i$$

Using ℓ, θ as polar coordinates

$$x_2 = x_1 + \ell \cos\theta$$

$$y_2 = y_1 + \ell \sin\theta$$

$$\dot{x}_2 = \dot{x}_1 + \dot{\ell}\cos\theta - \ell\dot{\theta}\sin\theta \tag{1}$$

$$\dot{y}_2 = \dot{y}_1 + \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta \tag{2}$$

If we substitute (1) and (2) into

$$L = T - U = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) - \frac{1}{2}k(\ell - b)^2$$

the result is

$$L = \frac{1}{2}(m_1 + m_2)\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right)$$
$$+ m_2\dot{\ell}\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right) + m_2\ell\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right) - \frac{1}{2}k(\ell - b)^2$$

The equations of motion are

$$x_1 : \frac{d}{dt}\left[(m_1 + m_2)\dot{x}_1 + m_2\dot{\ell}\cos\theta - m_2\ell\dot{\theta}\sin\theta\right] = 0$$

$$= m_1\dot{x}_1 + m_2\dot{x}_2 = p_x$$

So p_x = constant

$$y_1 : \frac{d}{dt}\left[(m_1 + m_2)\dot{y}_1 + m_2\dot{\ell}\sin\theta + m_2\ell\dot{\theta}\cos\theta\right] = 0$$

$$= m_1\dot{y}_1 + m_2\dot{y}_2 = p_y$$

So p_y = constant

$$\ell : \frac{d}{dt}\left[m_2\dot{\ell} + m_2\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right)\right] = m_2\ell\dot{\theta}^2 - k(\ell - b) + m_2\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right)$$

which reduces to

$$\boxed{\ddot{\ell}-\ell\dot{\theta}^2+\ddot{x}_1\cos\theta+\ddot{y}_1\sin\theta+\frac{k}{m_2}(\ell-b)=0}$$

$$\theta:\frac{d}{dt}\left[m_2\ell^2\dot{\theta}+m_2\ell\left(\dot{y}_1\cos\theta-\dot{x}_1\sin\theta\right)\right]$$
$$=-m_2\dot{\ell}\left(\dot{x}_1\sin\theta-\dot{y}_1\cos\theta\right)+m_2\ell\dot{\theta}\left(-\dot{x}_1\cos\theta-\dot{y}_1\sin\theta\right)$$

which reduces to

$$\boxed{\ddot{\theta}+\frac{2}{\ell}\dot{\ell}\dot{\theta}+\frac{\cos\theta}{\ell}\ddot{y}_1-\frac{\sin\theta}{\ell}\ddot{x}_1=0}$$

b) As was shown in (a)

$$\frac{\partial L}{\partial \dot{x}_1}=p_x=\text{constant}$$

$$\frac{\partial L}{\partial \dot{y}_1}=p_y=\text{constant}\qquad\text{(total linear momentum)}$$

c) Using L from part (a)

$$p_{x_1}=\frac{\partial L}{\partial \dot{x}_1}=\left(m_1+m_2\right)\dot{x}_1+m_2\dot{\ell}\cos\theta-m_2\ell\dot{\theta}\sin\theta$$

$$p_{y_1}=\frac{\partial L}{\partial \dot{y}_1}=\left(m_1+m_2\right)\dot{y}_1+m_2\dot{\ell}\sin\theta-m_2\ell\dot{\theta}\cos\theta$$

$$p_\ell=\frac{\partial L}{\partial \dot{\ell}}=m_2\dot{x}_1\cos\theta+m_2\dot{y}_1\sin\theta+m_2\dot{\ell}$$

$$p_\theta=-m_2\ell\dot{x}_1\sin\theta+m_2\ell\dot{y}_1\cos\theta+m_2\ell^2\dot{\theta}$$

Inverting these equations gives (after much algebra)

$$\dot{x}_1=\frac{1}{m_1}\left[p_{x_1}-p_\ell\cos\theta+\frac{\sin\theta}{\ell}p_\theta\right]$$

$$\dot{y}_1=\frac{1}{m_1}\left[p_{y_1}-p_\ell\sin\theta-\frac{\cos\theta}{\ell}p_\theta\right]$$

$$\dot{\ell}=\frac{1}{m_1}\left[-p_{x_1}\cos\theta-p_{y_1}\sin\theta+\frac{m_1+m_2}{m_2}p_\ell\right]$$

$$\dot{\theta}=\frac{1}{m_1\ell}\left[p_{x_1}\sin\theta-p_{y_1}\cos\theta+\frac{m_1+m_2}{m_2\ell}p_\theta\right]$$

Since the coordinate transformations are time independent, and U is velocity independent,

$$H=T+U$$

Substituting using the above equations for $\dot{q}_i$ in terms of p_i gives

$$H = \frac{1}{2m_1}\left[p_{x_1}^2 + p_{y_1}^2 + \frac{m_1+m_2}{m_2}\left[p_\ell^2 + \frac{p_\theta^2}{\ell^2}\right] - 2p_\ell\left(p_{x_1}\cos\theta + p_{y_1}\sin\theta\right)\right.$$
$$\left. + 2\frac{p_\theta}{\ell}\left(p_{x_1}\sin\theta - p_{y_1}\cos\theta\right)\right] + \frac{1}{2}k(\ell-b)^2$$

The equations of motion are

$$\dot{x}_1 = \frac{\partial H}{\partial p_{x_1}} = \frac{1}{m_1}\left[p_{x_1} - p_\ell\cos\theta + \frac{\sin\theta}{\ell}p_\theta\right]$$

$$\dot{y}_1 = \frac{\partial H}{\partial p_{y_1}} = \frac{1}{m_1}\left[p_{y_1} - p_\ell\sin\theta - \frac{\cos\theta}{\ell}p_\theta\right]$$

$$\dot{\ell} = \frac{\partial H}{\partial p_\ell} = \frac{1}{m_1}\left[\frac{m_1+m_2}{m_2}p_\ell - p_{x_1}\cos\theta - p_{y_1}\sin\theta\right]$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{m_1\ell}\left[\frac{m_1+m_2}{m_2\ell}p_\theta + p_{x_1}\sin\theta - p_{y_1}\cos\theta\right]$$

$$\dot{p}_{x_1} = -\frac{\partial H}{\partial x_1} = 0 \qquad \dot{p}_{y_1} = -\frac{\partial H}{\partial y_1} = 0$$

$$\dot{p}_\ell = -\frac{\partial H}{\partial \ell} = \frac{(m_1+m_2)p_\theta^2}{m_1 m_2 \ell^3} + \frac{p_\theta}{m_1\ell^2}\left(p_{x_1}\sin\theta - p_{y_1}\cos\theta\right) - k(\ell-b)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\ell}{m_1}\left(-p_{x_1}\sin\theta + p_{y_1}\cos\theta\right) - \frac{p_\theta}{m_1\ell}\left[p_{x_1}\cos\theta + p_{y_1}\sin\theta\right]$$

Note: This solution chooses as its generalized coordinates what the student would most likely choose at this point in the text. If one looks ahead to Section 8.2 and 8.3, however, it would show another choice of generalized coordinates that lead to three cyclic coordinates (x_{CM}, y_{CM}, and θ), as shown in those sections.

7-28. $F = -kr^{-2}$ so $U = -kr^{-1}$

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{so} \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{so} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Since the coordinate transformations are independent of t, and the potential energy is velocity-independent, the Hamiltonian is the total energy.

$$H = T + U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r}$$

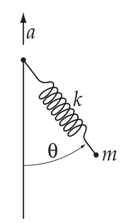
$$= \frac{1}{2}m\left[\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right] - \frac{k}{r}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

Hamilton's equations of motion are

$$\boxed{\begin{array}{ll} \dot{r} = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m} & \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2} \\ \dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{mr^3} - \dfrac{k}{r^2} & \\ \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0 & \end{array}}$$

7-29.



b = unextended length of spring

ℓ = variable length of spring

a) $x = \ell\sin\theta \qquad \dot{x} = \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta$

$$y = \frac{1}{2}at^2 - \ell\cos\theta \qquad \dot{y} = at - \dot{\ell}\cos\theta + \ell\dot{\theta}\sin\theta$$

Substituting into $T = \dfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

$$U = mgy + \frac{1}{2}k\left(\ell - b\right)^2$$

gives

$$L = T - U = \frac{1}{2}m\left[\dot{\ell}^2 + \ell^2\dot{\theta}^2 + a^2t^2 + 2at\left(\ell\dot{\theta}\sin\theta - \dot{\ell}\cos\theta\right)\right] + mg\left(\ell\cos\theta - \frac{at^2}{2}\right) - \frac{k}{2}\left(\ell - b\right)^2$$

Lagrange's equations give:

$$\ell : \frac{d}{dt}\left[m\dot{\ell} - amt\cos\theta\right] = m\ell\dot{\theta}^2 + mat\dot{\theta}\sin\theta + mg\cos\theta - k\left(\ell - b\right)$$

$$\theta : \frac{d}{dt}\left[m\ell^2\dot{\theta} + mat\ell\sin\theta\right] = mat\dot{\ell}\sin\theta - mg\ell\sin\theta + mat\ell\dot{\theta}\cos\theta$$

Upon simplifying, the equations of motion reduce to:

$$\boxed{\begin{aligned} \ddot{\ell} - \ell\dot{\theta}^2 - \left(a+g\right)\cos\theta + \frac{k}{m}\left(\ell - b\right) = 0 \\ \ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{a+g}{\ell}\sin\theta = 0 \end{aligned}}$$

b)

$$p_\ell = \frac{\partial L}{\partial \dot{\ell}} = m\dot{\ell} - mat\cos\theta \quad \text{or} \quad \dot{\ell} = \frac{p_\ell}{m} + at\cos\theta \qquad (1)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} + mat\ell\sin\theta$$

$$\text{or} \quad \dot{\theta} = \frac{p_\theta}{m\ell^2} - \frac{at\sin\theta}{\ell} \qquad (2)$$

Since the transformation equations relating the generalized coordinates to rectangular coordinates are not time-independent, the Hamiltonian is not the total energy.

$$H = \sum p_i\dot{q}_i - L = p_\ell\dot{\ell} + p_\theta\dot{\theta} - L$$

Substituting (1) and (2) for $\dot{\ell}$ and $\dot{\theta}$ and simplifying gives

$$H = \frac{p_\ell^2}{2m} + \frac{p_\theta^2}{2m\ell^2} - \frac{at}{\ell}p_\theta\sin\theta + atp_\ell\cos\theta + \frac{1}{2}k\left(\ell - b\right)^2 + \frac{1}{2}mgat^2 - mg\ell\cos\theta$$

The equations for $\dot{\theta}$ and $\dot{\ell}$ are

$$\boxed{\begin{aligned} &\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} - \frac{at}{\ell}\sin\theta \\ &\dot{\ell} = \frac{\partial H}{\partial p_\ell} = \frac{p_\ell}{m} + at\cos\theta \qquad \text{agreeing with (1) and (2)} \end{aligned}}$$

The equations for $\dot{p}_\ell$ and $\dot{p}_\theta$ are

$$\dot{p}_\ell = -\frac{\partial H}{\partial \ell} = -\frac{at}{\ell^2}p_\theta\sin\theta - k\left(\ell - b\right) + mg\cos\theta + \frac{p_\theta^2}{m\ell^3}$$

or

$$\boxed{\dot{p}_\ell + \frac{at}{\ell^2}p_\theta\sin\theta + k\left(\ell - b\right) - mg\cos\theta + \frac{p_\theta^2}{m\ell^3} = 0}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{at}{\ell}p_\theta\cos\theta + at\,p_\ell\sin\theta - mg\ell\sin\theta$$

or

$$\boxed{\dot{p}_\theta - \frac{at}{\ell} p_\theta \cos\theta - at\, p_\ell \sin\theta + mg\ell \sin\theta = 0}$$

c) $\sin\theta \simeq \theta, \cos\theta \simeq 1 - \dfrac{\theta^2}{2}$

Substitute into Lagrange's equations of motion

$$\ddot{\ell} - \ell\dot{\theta}^2 - (a+g)\left[1 - \frac{\theta^2}{2}\right] + \frac{k}{m}(\ell - b) = 0$$

$$\ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{a+g}{\ell}\theta - \frac{at\dot{\ell}\theta}{\ell} = 0$$

For small oscillations, $\theta \ll 1, \dot{\theta} \ll 1, \dot{\ell} \ll 1$. Dropping all second-order terms gives

$$\ddot{\ell} + \frac{k}{m}\ell = a + g + \frac{k}{m}b$$

$$\ddot{\theta} + \frac{a+g}{\ell}\theta = 0$$

For θ,

$$\boxed{T_\theta = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{a+g}}}$$

The solution to the equation for ℓ is

$$\ell = \ell_{\text{homogeneous}} + \ell_{\text{particular}}$$

$$= A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t + \frac{m}{k}(a+g) + b$$

So for ℓ,

$$\boxed{T_\ell = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

7-30.

a) From the definition of a total derivative, we can write

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k}\frac{\partial p_k}{\partial t}\right] \tag{1}$$

Using the canonical equations

$$\left.\begin{aligned}\frac{\partial q_k}{\partial t} &= \dot{q}_k = \frac{\partial H}{\partial p_k} \\ \frac{\partial p_k}{\partial t} &= \dot{p}_k = -\frac{\partial H}{\partial q_k}\end{aligned}\right] \tag{2}$$

we can write (1) as

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right] \tag{3}$$

or

$$\boxed{\frac{dg}{dt} = \frac{\partial g}{\partial t} + [g, H]} \tag{4}$$

b)

$$\dot{q}_j = \frac{\partial q_j}{\partial t} = \frac{\partial H}{\partial p_j} \tag{5}$$

According to the definition of the Poisson brackets,

$$[q_j, H] = \sum_k \left[\frac{\partial q_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right] \tag{6}$$

but

$$\frac{\partial q_j}{\partial q_k} = \delta_{jk} \text{ and } \frac{\partial q_j}{\partial p_k} = 0 \text{ for any } j,k \tag{7}$$

then (6) can be expressed as

$$\boxed{[q_j, H] = \frac{\partial H}{\partial p_j} = \dot{q}_j} \tag{8}$$

In the same way, from the canonical equations,

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{9}$$

so that

$$[p_j, H] = \sum_k \left[\frac{\partial p_j}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial p_j}{\partial p_k} \frac{\partial H}{\partial q_k} \right] \tag{10}$$

but

$$\frac{\partial p_j}{\partial p_k} = \delta_{jk} \text{ and } \frac{\partial p_j}{\partial q_k} = 0 \text{ for any } j,k \tag{11}$$

then,

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j} = \left[p_j, H\right]} \tag{12}$$

c)

$$\left[p_k, p_j\right] = \sum_\ell \left[\frac{\partial p_k}{\partial q_\ell}\frac{\partial p_j}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell}\frac{\partial p_j}{\partial q_\ell}\right] \tag{13}$$

since,

$$\frac{\partial p_k}{\partial q_\ell} = 0 \text{ for any } k,\ell \tag{14}$$

the right-hand side of (13) vanishes, and

$$\boxed{\left[p_k, p_j\right] = 0} \tag{15}$$

In the same way,

$$\left[q_k, q_j\right] = \sum_\ell \left[\frac{\partial q_k}{\partial q_\ell}\frac{\partial q_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell}\frac{\partial q_j}{\partial q_\ell}\right] \tag{16}$$

since

$$\frac{\partial q_j}{\partial p_\ell} = 0 \text{ for any } j,\ell \tag{17}$$

the right-hand side of (16) vanishes and

$$\boxed{\left[q_k, q_j\right] = 0} \tag{18}$$

d)

$$\left[q_k, p_j\right] = \sum_\ell \left[\frac{\partial q_k}{\partial q_\ell}\frac{\partial p_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell}\frac{\partial p_j}{\partial q_\ell}\right]$$

$$= \sum_\ell \delta_{k\ell}\,\delta_{j\ell} \tag{19}$$

or,

$$\boxed{\left[q_k, p_j\right] = \delta_{kj}} \tag{20}$$

e) Let $g(p_k, q_k)$ be a quantity that does not depend explicitly on the time. If $g(p_k, q_k)$ commutes with the Hamiltonian, i.e., if

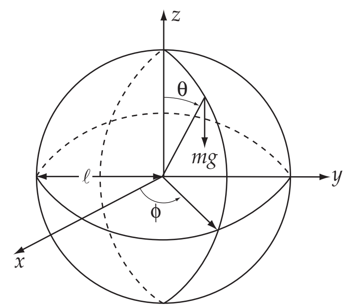
$$\left[g, H\right] = 0 \tag{21}$$

then, according to the result in a) above,

$$\boxed{\frac{dg}{dt} = 0} \tag{22}$$

and g is a constant of motion.

7-31. A spherical pendulum can be described in terms of the motion of a point mass m on the surface of a sphere of radius ℓ, where ℓ corresponds to the length of the pendulum support rod. The coordinates are as indicated below.



The kinetic energy of the pendulum is

$$T = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 \dot{\theta}^2 = \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) \tag{1}$$

and the potential energy is

$$U = mg\ell \cos \theta \tag{2}$$

The Lagrangian is

$$L = \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) - mg\ell \cos \theta \tag{3}$$

so that the momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} \tag{4}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\ell^2 \dot{\phi} \sin^2 \theta \tag{5}$$

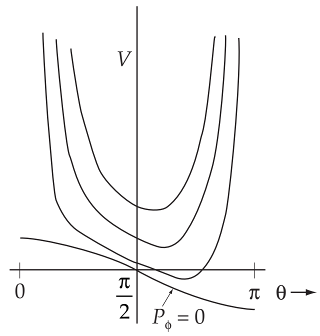
The Hamiltonian then becomes

$$\begin{aligned} H &= p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + mg\ell \cos \theta \\ &= \frac{p_\theta^2}{2m\ell^2} + V\left(\theta, p_\phi\right) \end{aligned} \tag{6}$$

which is just the total energy of the system and where the effective potential is

$$V\left(\theta, p_\phi\right) = \frac{p_\theta^2}{2m\ell^2 \sin^2 \theta} + mg\ell \cos \theta \tag{7}$$

When $p_\phi = 0$, $V(\theta,0)$ is finite for all θ, with a maximum at $\theta = 0$ (top of the sphere) and a minimum at $\theta = \pi$ (bottom of the sphere); this is just the case of the ordinary pendulum. For different values of p_ϕ, the V–θ diagram has the appearance below:



When $p_\phi > 0$, the pendulum never reaches $\theta = 0$ or $\theta = \pi$ because V is infinite at these points. The V–θ curve has a single minimum and the motion is oscillatory about this point. If the total energy (and therefore V) is a minimum for a given p_ϕ, θ is a constant, and we have the case of a conical pendulum.

For further details, see J. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, New York, 1947, pp. 79–86.

7-32. The Lagrangian for this case is

$$L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \dot{\theta}\dot{\phi}^2\right) + \frac{k}{r} \tag{1}$$

where spherical coordinates have been used due to the symmetry of U.

The generalized coordinates are r, θ, and ϕ, and the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{2}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{3}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2 \theta \tag{4}$$

The Hamiltonian can be constructed as in Eq. (7.155):

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\ddot{\phi} - L \\ &= \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta\right) - \frac{k}{r} \\ &= \frac{1}{2}\left[\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2 \sin^2 \theta}\right] - \frac{k}{r} \end{aligned} \tag{5}$$

Eqs. (7.160) applied to H as given in (5) reproduce equations (2), (3), and (4). The canonical equations of motion are obtained applying Eq. (7.161) to H:

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{k}{r^2} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2 \theta} \tag{6}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cot\theta}{mr^2 \sin^2\theta} \tag{7}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \tag{8}$$

The last equation implies that $p_\phi = \text{const}$, which reduces the number of variables on which H depends to four: r, θ, p_r, p_θ:

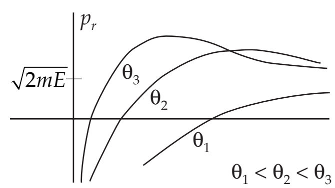
$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{\text{const}}{r^2 \sin^2\theta}\right] - \frac{k}{r} \tag{9}$$

For motion with constant energy, (9) fixes the value of any of the four variables when the other three are given.

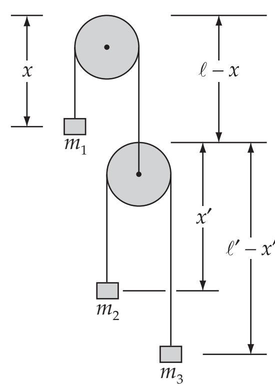
From (9), for a given constant value of $H = E$, we obtain

$$\boxed{p_r = \left[2mE - \frac{p_\theta^2 \sin^2\theta + const}{r^2 \sin^2\theta} + \frac{2mk}{r}\right]^{1/2}} \tag{10}$$

and so the projection of the phase path on the $r - p_r$ plane are as shown below.



7-33.



Neglect the masses of the pulleys

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{x}' - \dot{x})^2 + \frac{1}{2} m_3 (-\dot{x} - \dot{x}')^2$$

$$U = -m_1 g x - m_2 g(\ell - x + x') - m_3 g(\ell - x + \ell' - x')$$

$$L = \frac{1}{2}\left(m_1 + m_2 + m_3\right)\dot{x}^2 + \frac{1}{2}\left(m_2 + m_3\right)\dot{x}'^2 + \dot{x}\dot{x}'\left(m_3 - m_2\right)$$
$$+ g\left(m_1 - m_2 - m_3\right)x + g\left(m_2 - m_3\right)x' + \text{constant}$$

We redefine the zero in U such that the constant in $L = 0$.

$$p_x = \frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2 + m_3\right)\dot{x} + \left(m_3 - m_2\right)\dot{x}' \tag{1}$$

$$p_{x'} = \frac{\partial L}{\partial \dot{x}'} = \left(m_3 - m_2\right)\dot{x} + \left(m_2 + m_3\right)\dot{x}' \tag{2}$$

Solving (1) and (2) for p_x and $p_{x'}$ gives

$$\dot{x} = D^{-1}\left[\left(m_2 + m_3\right)p_x + \left(m_2 - m_3\right)p_{x'}\right]$$

$$\dot{x}' = D^{-1}\left[\left(m_2 + m_3\right)p_x + \left(m_1 + m_2 + m_3\right)p_{x'}\right]$$

where $D = m_1 m_3 + m_1 m_2 + 4m_2 m_3$

$$H = T + U$$

$$= \frac{1}{2}\left(m_1 + m_2 + m_3\right)\dot{x}^2 + \frac{1}{2}\left(m_2 + m_3\right)\dot{x}'^2 + \left(m_3 - m_2\right)x\dot{x}$$
$$-g\left(m_1 - m_2 - m_3\right)x - g\left(m_2 - m_3\right)x'$$

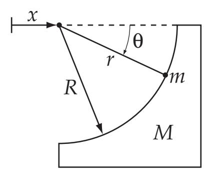
Substituting for $\dot{x}$ and $\dot{x}'$ and simplifying gives

$$\boxed{\begin{aligned} H = {} & \frac{1}{2}\left(m_2 + m_3\right)D^{-1}\, p_x^2 + \frac{1}{2}\left(m_1 + m_2 + m_3\right)D^{-1}\, p_{x'}^2 \\ & +\left(m_2 - m_3\right)D^{-1}\, p_x p_{x'} - g\left(m_1 - m_2 + m_3\right)x - g\left(m_2 - m_3\right)x' \\ & \text{where } D = m_1 m_3 + m_1 m_2 + 4m_2 m_3 \end{aligned}}$$

The equations of motion are

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \left(m_2 + m_3\right)D^{-1}\, p_x + \left(m_2 - m_3\right)D^{-1}\, p_{x'} \\ \dot{x}' &= \frac{\partial H}{\partial p_{x'}} = \left(m_2 - m_3\right)D^{-1}\, p_{x'} + \left(m_1 + m_2 + m_3\right)D^{-1}\, p_{x'} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = g\left(m_1 - m_2 - m_3\right) \\ \dot{p}_{x'} &= -\frac{\partial H}{\partial x'} = g\left(m_2 - m_3\right) \end{aligned}}$$

7-34.



The coordinates of the wedge and the particle are

$$x_M = x \qquad x_m = r\cos\theta + x$$
$$y_M = 0 \qquad y_m = -r\sin\theta \tag{1}$$

The Lagrangian is then

$$L = \frac{M+m}{2}\dot{x}^2 + \frac{m}{r}\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\cos\theta - 2\dot{x}r\dot{\theta}\sin\theta\right) + mgr\sin\theta \tag{2}$$

Note that we do not take r to be constant since we want the reaction of the wedge on the particle. The constraint equation is $f(x,\theta,r) = r - R = 0$.

a) Right now, however, we may take $r = R$ and $\dot{r} = \ddot{r} = 0$ to get the equations of motion for x and θ. Using Lagrange's equations,

$$\ddot{x} = aR\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \tag{3}$$

$$\ddot{\theta} = \frac{\ddot{x}\sin\theta + g\cos\theta}{R} \tag{4}$$

where $a \equiv m/(M+m)$.

b) We can get the reaction of the wedge from the Lagrange equation for r

$$\lambda = m\ddot{x}\cos\theta - mR\dot{\theta}^2 - mg\sin\theta \tag{5}$$

We can use equations (3) and (4) to express $\ddot{x}$ in terms of θ and $\dot{\theta}$, and substitute the resulting expression into (5) to obtain

$$\lambda = \left[\frac{a-1}{1-a\sin^2\theta}\right]\left(R\dot{\theta}^2 + g\sin\theta\right) \tag{6}$$

To get an expression for $\dot{\theta}$, let us use the conservation of energy

$$H = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right) - mgR\sin\theta = -mgR\sin\theta_0 \tag{7}$$

where θ_0 is defined by the initial position of the particle, and $-mgR\sin\theta_0$ is the total energy of the system (assuming we start at rest). We may integrate the expression (3) to obtain $\dot{x} = aR\dot{\theta}\sin\theta$, and substitute this into the energy equation to obtain an expression for $\dot{\theta}$

$$\dot{\theta}^2 = \frac{2g\left(\sin\theta - \sin\theta_0\right)}{R\left(1-a\sin^2\theta\right)} \tag{8}$$

Finally, we can solve for the reaction in terms of only θ and θ_0

$$\lambda = -\frac{mMg\left(3\sin\theta - a\sin^3\theta - 2\sin\theta_0\right)}{(M+m)\left(1 - a\sin^2\theta\right)^2} \tag{9}$$

7-35. We use z_i and p_i as our generalized coordinates, the subscript i corresponding to the ith particle. For a uniform field in the z direction the trajectories $z = z(t)$ and momenta $p = p(t)$ are given by

$$\left.\begin{aligned} z_i &= z_{i0} + v_{i0}t - \frac{1}{2}gt^2 \\ p_i &= p_{i0} - mgt \end{aligned}\right] \tag{1}$$

where z_{i0}, p_{i0}, and $v_{i0} = p_{i0}/m$ are the initial displacement, momentum, and velocity of the ith particle.

Using the initial conditions given, we have

$$z_1 = z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2a}$$

$$p_1 = p_0 - mgt \tag{2b}$$

$$z_2 = z_0 + \Delta z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2c}$$

$$p_2 = p_0 - mgt \tag{2d}$$

$$z_3 = z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2e}$$

$$p_3 = p_0 + \Delta p_0 - mgt \tag{2f}$$

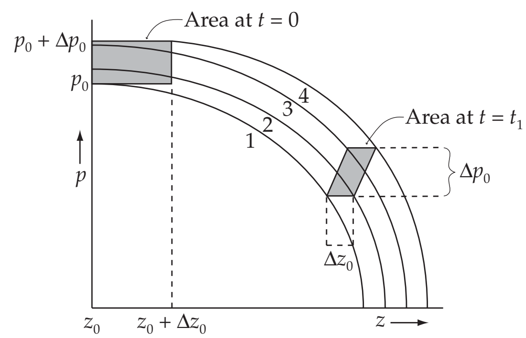
$$z_4 = z_0 + \Delta Z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2g}$$

$$p_4 = p_0 + \Delta p_0 - mgt \tag{2h}$$

The Hamiltonian function corresponding to the ith particle is

$$H_i = T_i + V_i = \frac{m\dot{z}_i^2}{2} + mgz_i = \frac{p_i^2}{2m} + mgz_i = \text{const.} \tag{3}$$

From (3) the phase space diagram for any of the four particles is a parabola as shown below.

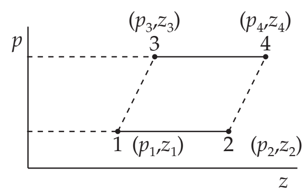


From this diagram (as well as from 2b, 2d, 2f, and 2h) it can be seen that *for any time t*,

$$p_1 = p_2 \tag{4}$$

$$p_3 = p_4 \tag{5}$$

Then for a certain time t the shape of the area described by the representative points will be of the general form



where the base $\overline{1\ 2}$ must parallel to the top $\overline{3\ 4}$. At time $t = 0$ the area is given by $\Delta z_0 \Delta p_0$, since it corresponds to a rectangle of base Δz_0 and height Δp_0. At any other time the area will be given by

$$A = \left\{ \text{base of parallelogram}\Big|_{t=t_1} = \left(z_2 - z_1\right)\Big|_{t=t_1} \right.$$

$$\left. = \left(z_4 - z_3\right)\Big|_{t=t_1} = \Delta z_0 \right\}$$

$$x \left\{ \text{height of parallelogram}\Big|_{t=t_1} = \left(p_3 - p_1\right)\Big|_{t=t_1} \right.$$

$$\left. = \left(p_4 - p_2\right)\Big|_{t=t_1} = \Delta p_0 \right\}$$

$$= \Delta p_0 \; \Delta z_0 \tag{6}$$

Thus, the area occupied in the phase plane is constant in time.

7-36. The initial volume of phase space accessible to the beam is

$$V_0 = \pi R_0^2 \pi p_0^2 \tag{1}$$

After focusing, the volume in phase space is

$$V_1 = \pi R_1^2 \pi p_1^2 \tag{2}$$

where now p_1 is the resulting radius of the distribution of transverse momentum components of the beam with a circular cross section of radius R_1. From Liouville's theorem the phase space accessible to the ensemble is invariant; hence,

$$V_0 = \pi R_0^2 \pi p_0^2 = V_1 = \pi R_1^2 \pi p_1^2 \tag{3}$$

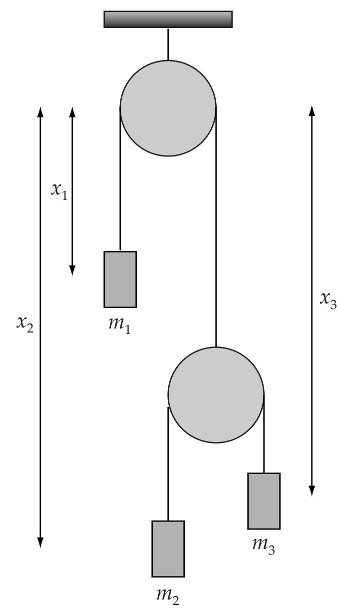
from which

$$p_1 = \frac{R_0 p_0}{R_1} \tag{4}$$

If $R_1 < R_0$, then $p_1 > p_0$, which means that the resulting spread in the momentum distribution has *increased*.

This result means that when the beam is better focused, the transverse momentum components are increased and there is a subsequent divergence of the beam past the point of focus.

7-37. Let's choose the coordinate system as shown:



The Lagrangian of the system is

$$L = T - U = \frac{1}{2}\left(m_1 \left(\frac{dx_1}{dt}\right)^2 + m_2 \left(\frac{dx_2}{dt}\right)^2 + m_3 \left(\frac{dx_3}{dt}\right)^2 \right) + g\left(m_1 x_1 + m_2 x_2 + m_3 x_3\right)$$

with the constraints

$$x_1 + y = l_1 \quad \text{and} \quad x_2 - y + x_3 - y = l_2$$

which imply $2x_1 + x_2 + x_3 - (2l_1 + l_2) = 0 \Rightarrow 2\dfrac{d^2 x_1}{dt^2} + \dfrac{d^2 x_2}{dt^2} + \dfrac{d^2 x_3}{dt^2} = 0$ (1)

The motion equations (with Lagrange multiplier λ) are

$$m_1 g - m_1 \frac{d^2 x_1}{dt^2} + 2\lambda = 0 \tag{2}$$

$$m_2 g - m_2 \frac{d^2 x_2}{dt^2} + \lambda = 0 \tag{3}$$

$$m_3 g - m_3 \frac{d^2 x_3}{dt^2} + \lambda = 0 \tag{4}$$

Combining (1)–(4) we find

$$\lambda = \frac{-4g}{\dfrac{4}{m_1} + \dfrac{1}{m_2} + \dfrac{1}{m_3}}$$

Finally, the string tension that acts on m_1 is (see Eq. (2))

$$T_1 = m_1 g - m_1 \frac{d^2 x_1}{dt^2} = -2\lambda = \frac{8g}{\dfrac{4}{m_1} + \dfrac{1}{m_2} + \dfrac{1}{m_3}}$$

7-38. The Hamiltonian of the system is

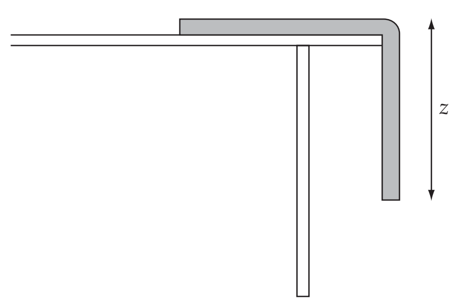
$$H = T + U = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 + \frac{kx^2}{2} + \frac{bx^4}{4} = \frac{p^2}{2m} + \frac{kx^2}{2} + \frac{bx^4}{4}$$

The Hamiltonian motion equations that follow this Hamiltonian are

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -(kx + bx^3)$$

7-39.



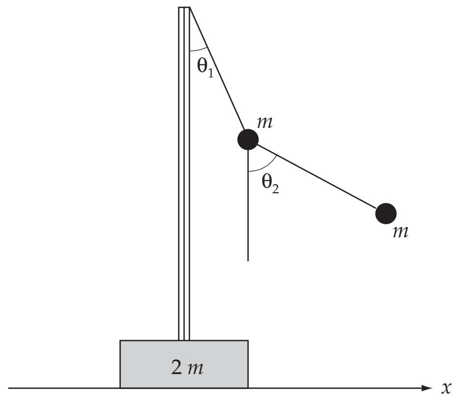
The Lagrangian of the rope is

$$L = T - U = \frac{1}{2} m \left(\frac{dz}{dt} \right)^2 - \left(-\frac{mz}{b} g \frac{z}{2} \right) = \frac{1}{2} m \left(\frac{dz}{dt} \right)^2 + \frac{mgz^2}{2b}$$

from which follows the equation of motion

$$\frac{\partial L}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} \Rightarrow \frac{mgz}{b} = m\frac{d^2 z}{dt^2}$$

7-40.



We choose the coordinates for the system as shown in the figure.

The kinetic energy is

$$T = \frac{1}{2}2m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\left[b^2\left(\frac{d\theta_1}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + 2b\frac{d\theta_1}{dt}\frac{dx}{dt}\cos\theta_1\right]$$

$$+\frac{1}{2}m\left[\left(\frac{dx}{dt} + b\frac{d\theta_1}{dt}\cos\theta_1 + b\frac{d\theta_2}{dt}\cos\theta_2\right)^2 + \left(b\frac{d\theta_1}{dt}\sin\theta_1 + b\frac{d\theta_2}{dt}\sin\theta_2\right)^2\right]$$

The potential energy is

$$U = -mgb\cos\theta_1 - mg(b\cos\theta_1 + b\cos\theta_2)$$

And the Lagrangian is

$$L = T - U = 2m\left(\frac{dx}{dt}\right)^2 + mb^2\left(\frac{d\theta_1}{dt}\right)^2 + 2mb\frac{dx}{dt}\frac{d\theta_1}{dt}\cos\theta_1 + \frac{1}{2}mb^2\left(\frac{d\theta_2}{dt}\right)^2$$

$$+mb\frac{dx}{dt}\frac{d\theta_2}{dt}\cos\theta_2 + mb^2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\cos(\theta_1 - \theta_2) + 2mgb\cos\theta_1 + mgb\cos\theta_2$$

From this follow 3 equations of motion

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \quad \Rightarrow \quad 0 = 4\frac{d^2 x}{dt^2} + b\left[2\frac{d^2\theta_1}{dt^2}\cos\theta_1 + \frac{d^2\theta_2}{dt^2}\cos\theta_2\right]$$

$$-b\left[2\left(\frac{d\theta_1}{dt}\right)^2\sin\theta_1 + \left(\frac{d\theta_2}{dt}\right)^2\sin\theta_2\right]$$

$$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} \quad \Rightarrow \quad -2g\sin\theta_1 = 2b\frac{d^2\theta_1}{dt^2} + 2\frac{d^2x}{dt^2}\cos\theta_1 + b\frac{d^2\theta_2}{dt^2}\cos(\theta_1 - \theta_2)$$

$$+b\left(\frac{d\theta_2}{dt}\right)^2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} \quad \Rightarrow$$

$$-g\sin\theta_2 = b\frac{d^2\theta_2}{dt^2} + \frac{d^2x}{dt^2}\cos\theta_2 + b\frac{d^2\theta_1}{dt^2}\cos(\theta_1 - \theta_2) - b\left(\frac{d\theta_1}{dt}\right)^2 \sin(\theta_1 - \theta_2)$$

7-41. For small angle of oscillation θ we have

$$T = \frac{1}{2}mb^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}m\left(\frac{db}{dt}\right)^2 \quad \text{and} \quad U = -mgb\cos\theta$$

So the Lagrangian reads

$$L = T - U = \frac{1}{2}mb^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}m\left(\frac{db}{dt}\right)^2 + mgb\cos\theta$$

from which follow 2 equations of motion

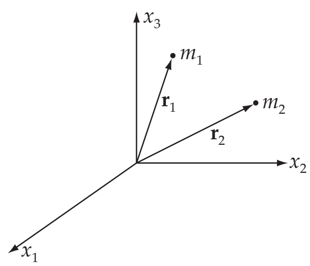
$$\frac{\partial L}{\partial b} = \frac{d}{dt}\frac{\partial L}{\partial \dot{b}} \quad \Rightarrow \quad b\left(\frac{d\theta}{dt}\right)^2 + g\cos\theta = \frac{d^2b}{dt^2} = -\frac{d\alpha}{dt}$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \quad \Rightarrow \quad -mgb\sin\theta = 2mb\frac{db}{dt}\frac{d\theta}{dt} + mb^2\frac{d^2\theta}{dt^2} = -2mb\alpha\frac{d\theta}{dt} + mb^2\frac{d^2\theta}{dt^2}$$

CHAPTER 8

Central-Force Motion

8-1.



In a uniform gravitational field, the gravitational acceleration is everywhere constant. Suppose the gravitational field vector is in the x_1 direction; then the masses m_1 and m_2 have the gravitational potential energies:

$$\left.\begin{aligned} U_g^{(1)} &= -F^{(1)}\,x_1^{(1)} = -m_1\alpha x_1^{(1)} \\ U_g^{(2)} &= -F^{(2)}\,x_1^{(2)} = -m_2\alpha x_1^{(2)} \end{aligned}\right] \tag{1}$$

where $\mathbf{r}_1 = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right)$ and where α is the constant gravitational acceleration. Therefore, introducting the relative coordinate $\mathbf{r}$ and the center of mass coordinate $\mathbf{R}$ according to

$$\left.\begin{aligned} &\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \\ &m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \left(m_1 + m_2\right)\mathbf{R} \end{aligned}\right] \tag{2}$$

we can express $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{r}$ and $\mathbf{R}$ by

$$\left.\begin{aligned} \mathbf{r}_1 &= \frac{m_2}{m_1 + m_2}\,\mathbf{r} + \mathbf{R} \\ \mathbf{r}_2 &= -\frac{m_1}{m_1 + m_2}\,\mathbf{r} + \mathbf{R} \end{aligned}\right] \tag{3}$$

which differ from Eq. (8.3) in the text by R. The Lagrangian of the two-particle system can now be expressed in terms of **r** and **R**:

$$L = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 - U(|\mathbf{r}|) - U_g^{(1)} - U_g^{(2)}$$

$$= \frac{1}{2} m_1 \left| \frac{m_2}{m_1+m_2} \dot{\mathbf{r}} + \dot{\mathbf{R}} \right|^2 + \frac{1}{2} m_2 \left| -\frac{m_1}{m_1+m_2} \dot{\mathbf{r}} + \dot{\mathbf{R}} \right|^2 - U(r) \tag{4}$$

$$+ m_1 \alpha \left[\frac{m_2}{m_1+m_2} x + X \right] + m_2 \alpha \left[-\frac{m_1}{m_1+m_2} x + X \right]$$

where x and X are the x_1 components of **r** and **R**, respectively. Then, (4) becomes

$$L = \frac{1}{2} m_1 \left[\frac{m_2}{m_1+m_2} \right]^2 |\dot{\mathbf{r}}|^2 + \frac{1}{2} m_2 \left[\frac{m_1}{m_1+m_2} \right] |\dot{\mathbf{r}}|^2 + \frac{1}{2}(m_1+m_2)|\dot{\mathbf{R}}|^2$$
$$- U(r) + \alpha \frac{m_1 m_2 - m_1 m_2}{m_1+m_2} x + (m_1+m_2)\alpha X \tag{5}$$

Hence, we can write the Lagrangian in the form

$$\boxed{L = \frac{1}{2}\mu |\dot{\mathbf{r}}|^2 - U(r) + \frac{1}{2}(m_1+m_2)|\dot{\mathbf{R}}|^2 + (m_1+m_2)\alpha X} \tag{6}$$

where μ is the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1+m_2} \tag{7}$$

Therefore, this case is reducible to an equivalent one-body problem.

8-2. Setting $u = 1/r$, Eq. (8.38) can be rewritten as

$$\theta = -\int \frac{du}{\sqrt{\dfrac{2\mu E}{\ell^2} + \dfrac{2\mu k}{\ell^2} u - u^2}} \tag{1}$$

where we have used the relation $du = -(1/r^2)\,dr$. Using the standard form of the integral [see Eq. (E.8c), Appendix E]:

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{-1}{\sqrt{-a}} \sin^{-1}\left[\frac{2ax+b}{\sqrt{b^2-4ac}} \right] + \text{const.} \tag{2}$$

we have

$$\theta + \text{const.} = \sin^{-1}\left[\frac{-\dfrac{2}{r} + \dfrac{2\mu k}{\ell^2}}{\sqrt{\left[\dfrac{2\mu k}{\ell^2}\right]^2 + 8\dfrac{\mu E}{\ell^2}}} \right] \tag{3}$$

or, equivalently,

$$\sin(\theta + \text{const.}) = \frac{-\dfrac{2}{r} + \dfrac{2\mu k}{\ell^2}}{\sqrt{\left[\dfrac{2\mu k}{\ell^2}\right]^2 + 8\,\dfrac{\mu E}{\ell^2}}} \tag{4}$$

We can choose the point from which θ is measured so that the constant in (4) is $-\pi/2$. Then,

$$\boxed{\cos\theta = \frac{\dfrac{\ell^2}{\mu k}\dfrac{1}{r} - 1}{\sqrt{1 + \dfrac{2E\ell^2}{\mu k^2}}}} \tag{5}$$

which is the desired expression.

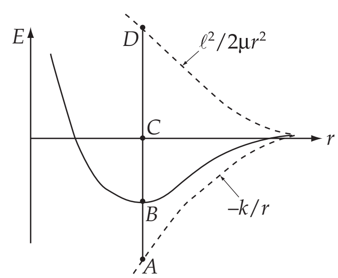
8-3. When $k \to k/2$, the potential energy will decrease to half its former value; but the kinetic energy will remain the same. Since the original orbit is circular, the instantaneous values of T and U are equal to the average values, $\langle T\rangle$ and $\langle U\rangle$. For a $1/r^2$ force, the virial theorem states

$$\langle T\rangle = -\frac{1}{2}\langle U\rangle \tag{1}$$

Hence,

$$E = T + U = -\frac{1}{2}U + U = \frac{1}{2}U \tag{2}$$

Now, consider the energy diagram



where

$$\overline{CB} = E \qquad \text{original total energy}$$

$$\overline{CA} = U \qquad \text{original potential energy}$$

$$\overline{CD} = U_C \qquad \text{original centrifugal energy}$$

The point B is obtained from $\overline{CB} = \overline{CA} - \overline{CD}$. According to the virial theorem, $E = (1/2)U$ or $\overline{CB} = (1/2)\overline{CA}$. Therefore,

$$\overline{CD} = \overline{CB} = \overline{BA}$$

Hence, if U suddenly is halved, the total energy is raised from B by an amount equal to $(1/2)\overline{CA}$ or by $\overline{CB}$. Thus, the total energy point is raised from B to C; i.e., E(final) = 0 and the orbit is *parabolic*.

8-4. Since the particle moves in a central, inverse-square law force field, the potential energy is

$$U = -\frac{k}{r} \tag{1}$$

so that the time average is

$$\langle U \rangle = -\frac{1}{\tau}\int_0^\tau \frac{k}{r}\, dt \tag{2}$$

Since this motion is a central motion, the angular momentum is a constant of motion. Then,

$$\mu r^2\, \dot{\theta} \equiv \ell = const. \tag{3}$$

from which

$$dt = \frac{\mu r^2}{\ell}\, d\theta \tag{4}$$

Therefore, (2) becomes

$$\langle U \rangle = -\frac{1}{\tau}\int_0^{2\pi} \frac{k}{r}\frac{\mu r^2}{\ell}\, d\theta = -\frac{k\mu}{\tau\ell}\int_0^{2\pi} r d\theta \tag{5}$$

Now, substituting $\alpha/r = 1 + \varepsilon\cos\theta$ and $\tau = (2\mu/\ell)\pi\sqrt{\alpha}\, a^{3/2}$, (5) becomes

$$\langle U \rangle = -\frac{k\sqrt{\alpha}}{2\pi a^{3/2}}\int_0^{2\pi} \frac{1}{1+\varepsilon\cos\theta}\, d\theta \tag{6}$$

where a is the semimajor axis of the ellipse. Using the standard integral [see Eq. (E.15), Appendix E],

$$\int_0^{2\pi} \frac{1}{1+\varepsilon\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{1-\varepsilon^2}} \tag{7}$$

and the relation,

$$\alpha = a\left(1-\varepsilon^2\right) \tag{8}$$

(6) becomes

$$\langle U \rangle = -\frac{k}{a} \tag{9}$$

The kinetic energy is

$$T = -\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} \tag{10}$$

and the time average is

$$\langle T \rangle = -\frac{1}{\tau}\int_0^\tau T\,dt = \frac{1}{2\pi\ell}\sqrt{\frac{k\mu}{a^3}}\int_0^{2\pi} Tr^2\,d\theta \tag{11}$$

Part of this integral is trivial,

$$\langle T \rangle = \frac{1}{2\pi\ell}\sqrt{\frac{k\mu}{a^3}}\left[\frac{\mu}{2}\int_0^{2\pi}(\dot{r}r)^2\,d\theta + \frac{\pi\ell^2}{\mu}\right] \tag{12}$$

To evaluate the integral above, substitute the expression for r and make a change of variable

$$\frac{1}{2}\int_0^{2\pi}(\dot{r}r)^2\,d\theta = \frac{1}{2}\left(\frac{\ell\varepsilon}{\mu}\right)^2\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{(1+\varepsilon\cos\theta)^2} = \left(\frac{\ell\varepsilon}{\mu}\right)^2\int_{-1}^{1}\frac{\sqrt{1-x^2}\,dx}{(1+\varepsilon x)^2} \tag{13}$$

The reader is invited to evaluate this integral in either form. The solution presented here is to integrate by parts twice, which gives a third integral that can be looked up in a table:

$$\int_{-1}^{1}\frac{\sqrt{1-x^2}\,dx}{(1+\varepsilon x)^2} = -\left.\frac{\sqrt{1-x^2}}{\varepsilon(1+\varepsilon x)}\right|_{-1}^{1} - \frac{1}{\varepsilon}\int_{-1}^{1}\frac{x\,dx}{\sqrt{1-x^2}\,(1+\varepsilon x)} \tag{14}$$

$$= -\frac{1}{\varepsilon}\left[\left.\frac{x\sin^{-1}x}{1+\varepsilon x}\right|_{-1}^{1} - \int_{-1}^{1}\frac{\sin^{-1}x\,dx}{(1+\varepsilon x)^2}\right] \tag{15}$$

$$= -\frac{1}{\varepsilon^2}\left[\sin^{-1}x + \frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\left.\sqrt{\frac{(1-\varepsilon)(1-x)}{(1+\varepsilon)(1+x)}}\right|_{-1}^{1}\right] \tag{16}$$

$$= \frac{\pi}{\varepsilon^2}\left[-1 + \frac{1}{\sqrt{1-\varepsilon^2}}\right] \tag{17}$$

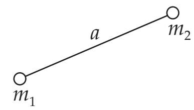
Substituting this into (13) and then into (12), we obtain the desired answer,

$$\langle T \rangle = \frac{k}{2a} \tag{18}$$

This explicitly verifies the virial theorem, which states that for an inverse-square law force,

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle \tag{19}$$

8-5.



Suppose two particles with masses m_1 and m_2 move around one another in a circular orbit with radius a. We can consider this motion as the motion of one particle with the reduced mass μ moving under the influence of a central force $G\, m_1 m_2 / a^2$. Therefore, the equation of motion before the particles are stopped is

$$\mu a \omega^2 = G \frac{m_1 m_2}{a^2} \tag{1}$$

where

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \qquad \omega = \frac{2\pi}{\tau} \tag{2}$$

The radius of circular motion is

$$a = \left[\frac{G\, m_1 m_2\; \tau^2}{4\pi^2 \mu} \right]^{1/3} \tag{3}$$

After the circular motion is stopped, the particle with reduced mass μ starts to move toward the force center. We can find the equation of motion from the conservation of energy:

$$-G \frac{m_1 m_2}{a} = \frac{1}{2} \mu \dot{x}^2 - G \frac{m_1 m_2}{x} \tag{4}$$

or,

$$\dot{x} = \left[\frac{2G\, m_1 m_2}{\mu} \left(\frac{1}{x} - \frac{1}{a} \right) \right]^{1/2} \tag{5}$$

Therefore, the time elapsed before the collision is

$$t = \int dt = -\int_a^0 \frac{dx}{\sqrt{\dfrac{2G\, m_1 m_2}{\mu} \left[\dfrac{1}{x} - \dfrac{1}{a} \right]}} \tag{6}$$

where the negative sign is due to the fact that the time increases as the distance decreases. Rearranging the integrand, we can write

$$t = -\sqrt{\frac{a\mu}{2G\, m_1 m_2}} \int_a^0 \frac{\sqrt{x}}{\sqrt{a - x}} dx \tag{7}$$

Setting $x \equiv y^2 \; (dx = 2y\, dy)$, the integral in (7) becomes

$$I = \int_a^0 \sqrt{\frac{x}{a - x}}\, dx = 2 \int_{\sqrt{a}}^0 \frac{y^2}{\sqrt{a - y^2}}\, dy$$

Using Eq. (E.7), Appendix E, we find

$$I = 2\left[-\frac{y\sqrt{a-y^2}}{2} + \frac{a}{2}\sin^{-1}\left[\frac{y}{\sqrt{a}}\right]\right]_{\sqrt{a}}^{0} = -\frac{\pi a}{2} \tag{8}$$

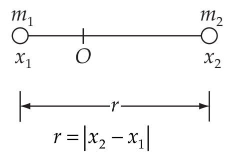
Therefore,

$$t = \sqrt{\frac{a\mu}{2G\, m_1 m_2}}\,\frac{a\pi}{2}$$

or,

$$\boxed{t = \frac{\tau}{4\sqrt{2}}} \tag{9}$$

8-6.



When two particles are initially at rest separated by a distance r_0, the system has the total energy

$$E_0 = -G\,\frac{m_1 m_2}{r_0} \tag{1}$$

The coordinates of the particles, x_1 and x_2, are measured from the position of the center of mass. At any time the total energy is

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - G\,\frac{m_1 m_2}{r} \tag{2}$$

and the linear momentum, at any time, is

$$p = m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \tag{3}$$

From the conservation of energy we have $E = E_0$, or

$$-G\,\frac{m_1 m_2}{r_0} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - G\,\frac{m_1 m_2}{r} \tag{4}$$

Using (3) in (4), we find

$$\boxed{\begin{aligned} \dot{x}_1 = v_1 &= m_2\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \\ \dot{x}_1 = v_2 &= -m_1\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \end{aligned}} \tag{5}$$

8-7. Since $F(r) = -kr$ is a *central* force, angular momentum is conserved and the areal velocity, $dA/dt = \ell/2\mu$, is trivially constant (see Section 8.3). In order to compute $\langle U \rangle$, we start with

$$dt = \frac{dr}{\sqrt{\dfrac{2}{\mu}\left[E - U - \dfrac{\ell^2}{2\mu r^2}\right]}} \tag{1}$$

and

$$U = \frac{kr^2}{2} \tag{2}$$

The time average of the potential energy becomes

$$\begin{aligned}\langle U \rangle &= \frac{1}{\tau}\int_0^\tau U\,dt \\ &= \frac{2}{\tau}\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{kr^3}{2} \frac{dr}{\sqrt{\dfrac{2}{\mu}\left[Er^2 - \dfrac{kr^4}{2} - \dfrac{\ell^2}{2\mu}\right]}}\end{aligned} \tag{3}$$

Substituting

$$r^2 = x \qquad dr = \frac{1}{2r}\,dx \tag{4}$$

(4) becomes

$$\langle U \rangle = \frac{k}{2\tau}\sqrt{\frac{\mu}{2}}\int_{r_{\text{min}}^2}^{r_{\text{max}}^2} \frac{x\,dx}{\sqrt{-\dfrac{\ell^2}{2\mu} + Ex - \dfrac{k}{2}x^2}} \tag{5}$$

Using the integrals in Eqs. (E.9) and (E.8c), Appendix E,

$$\int \frac{x\,dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{a}\sqrt{ax^2 + bx + c} + \frac{b}{2a\sqrt{-a}}\sin^{-1}\left[\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right] \tag{6}$$

(5) becomes

$$\langle U \rangle = \frac{k}{2\tau}\sqrt{\frac{\mu}{2}}\left[-\frac{E}{k}\left(\frac{2}{k}\right)^{1/2}\sin^{-1}\left[\frac{E - kr^2}{\left[E^2 - \dfrac{k\ell^2}{\mu}\right]^{1/2}}\right] - \frac{2}{k}\sqrt{-\frac{k}{2}r^4 + Er^2 - \frac{\ell^2}{2\mu}}\right]_{r_{\text{min}}}^{r_{\text{max}}} \tag{7}$$

But r_{max} and r_{min} were originally defined as the roots of $\sqrt{E - U - \dfrac{\ell^2}{2\mu r^2}}$. Hence, the second term vanishes at both limits of integration. On the other hand,

$$\tau = 2\int_{r_{\min}}^{r_{\max}} \frac{dr}{\sqrt{\dfrac{2}{\mu}\left(E - U - \dfrac{\ell^2}{2\mu r^2}\right)}}$$

$$= \sqrt{2\mu}\int_{r_{\min}}^{r_{\max}} \frac{r\,dr}{\sqrt{-\dfrac{k}{2}r^4 + Er^2 - \dfrac{\ell^2}{2\mu}}} \tag{8}$$

or, using (5),

$$\tau = \sqrt{\frac{\mu}{2}}\int_{r_{\min}^2}^{r_{\max}^2} \frac{dx}{\sqrt{-\dfrac{k}{2}x^2 + Ex - \dfrac{\ell^2}{2\mu}}}$$

$$= -\sqrt{\frac{\mu}{2}}\left[\frac{2}{k}\right]^{1/2} \sin^{-1}\left[\frac{E - kr^2}{\left[E^2 - \dfrac{k\ell^2}{\mu}\right]^{1/2}}\right]\Bigg|_{r_{\min}}^{r_{\max}} \tag{9}$$

Using (9) to substitute for τ in (7), we have

$$\langle U\rangle = \frac{E}{2} \tag{10}$$

Now,

$$\langle T\rangle = E - \langle U\rangle = \frac{E}{2} \tag{11}$$

The virial theorem states:

$$\langle T\rangle = \frac{n+1}{2}\langle U\rangle \quad \text{when } U = kr^{n+1} \tag{12}$$

In our case $n = 1$, therefore,

$$\boxed{\langle T\rangle = \langle U\rangle = \frac{E}{2}} \tag{13}$$

8-8. The general expression for $\theta(r)$ is [see Eq. (8.17)]

$$\theta(r) = \int \frac{\left(\ell/r^2\right)dr}{\sqrt{2\mu\left[E - U - \dfrac{\ell^2}{2\mu r^2}\right]}} \tag{1}$$

where $U = -\int kr\,dr = -kr^2/2$ in the present case. Substituting $x = r^2$ and $dx = 2r\,dr$ into (1), we have

$$\theta(r) = \frac{1}{2}\int \frac{dx}{x\sqrt{\frac{\mu k}{\ell^2}x^2 + \frac{2\mu E}{\ell^2}x - 1}} \tag{2}$$

Using Eq. (E.10b), Appendix E,

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left[\frac{bx+2c}{|x|\sqrt{b^2-4ac}}\right] \tag{3}$$

and expressing again in terms of r, we find

$$\theta(r) = \frac{1}{2}\sin^{-1}\left[\frac{\left[\frac{\mu E}{\ell^2}r^2 - 1\right]}{r^2\sqrt{\frac{\mu^2 E^2}{\ell^4} + \frac{\mu k}{\ell^2}}}\right] + \theta_0 \tag{4}$$

or,

$$\sin 2(\theta - \theta_0) = \frac{1}{\sqrt{1 + \frac{\ell^2 k}{\mu E^2}}} - \frac{1}{r^2}\frac{\ell^2/\mu E}{\sqrt{1 + \frac{\ell^2 k}{\mu E^2}}} \tag{5}$$

In order to interpret this result, we set

$$\left.\begin{aligned} \sqrt{1+\frac{\ell^2 k}{\mu E^2}} &\equiv \varepsilon' \\ \frac{\ell^2}{\mu E} &\equiv \alpha' \end{aligned}\right] \tag{6}$$

and specifying $\theta_0 = \pi/4$, (5) becomes

$$\frac{\alpha'}{r^2} = 1 + \varepsilon' \cos 2\theta \tag{7}$$

or,

$$\alpha' = r^2 + \varepsilon' r^2\left(\cos^2\theta - \sin^2\theta\right) \tag{8}$$

Rewriting (8) in x-y coordinates, we find

$$\alpha' = x^2 + y^2 + \varepsilon'\left(x^2 - y^2\right) \tag{9}$$

or,

$$1 = \frac{x^2}{\frac{\alpha'}{1+\varepsilon'}} + \frac{y^2}{\frac{\alpha'}{1-\varepsilon'}} \tag{10}$$

Since $a' > 0$, $\varepsilon' > 1$ from the definition, (10) is equivalent to

$$\boxed{1 = \frac{x^2}{\dfrac{\alpha'}{1+\varepsilon'}} + \frac{y^2}{\dfrac{\alpha'}{|1-\varepsilon'|}}} \tag{11}$$

which is the equation of a hyperbola.

8-9.

(a) By the virial theorem, $T = -U/2$ for a circular orbit.

The firing of the rocket doesn't change U, so $U_f = U_i$

But

$$T_f = \frac{1}{2} m\left(v^2 + v^2\right) = 2T_i$$

So

$$E_f = 2T_i + U_i = -U_i + U_i = 0$$

$$\boxed{\frac{E_f}{E_i} = 0}$$

The firing of the rocket doesn't change the angular momentum since it fires in a radial direction.

$$\boxed{\frac{\ell_f}{\ell_i} = 1}$$

(b) $E = 0$ means the orbit is parabolic. The satellite will be lost.

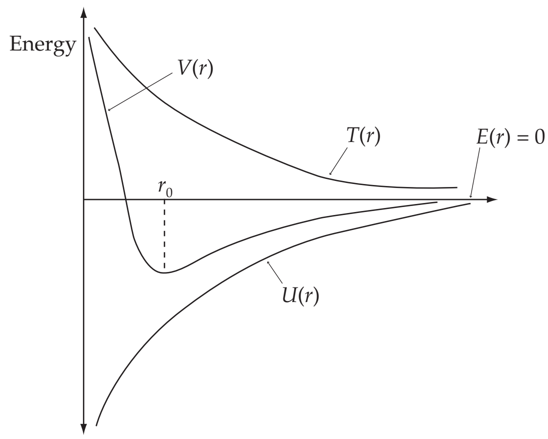
$$E(r) = 0 \qquad U(r) = -\frac{GM_e\, m_s}{r}$$

$$T(r) = E - U = \frac{GM_e\, m_s}{r}$$

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2} = -\frac{GM_e\, m_s}{r} + \frac{\ell^2}{2\mu r^2}$$

Behavior of $V(r)$ is determined by

$$\left[\begin{array}{ll} \ell^2/2\mu r^2 & \text{for small } r \\ -GM_e m_s/r & \text{for large } r \end{array}\right.$$



Minimum in $V(r)$ is found by setting $\frac{dV}{dr} = 0$ at $r = r_0$

$$0 = -\frac{GM_e\, m_s}{r_0^2} + \frac{\ell^2}{\mu r_0^3}$$

$$r_0 = -\frac{\ell^2}{\mu GM_e\, m_s}$$

8-10. For circular motion

$$T = \frac{1}{2} m_e \omega^2\, r_e^2$$

$$U = -\frac{GM_s\, m_e}{r_e}$$

We can get ω^2 by equating the gravitational force to the centripetal force

$$\frac{GM_s\, m_e}{r_e^2} = m_e \omega^2\, r_e$$

or

$$\omega^2 = \frac{GM_s}{r_e^3}$$

So

$$T = \frac{1}{2} m_e r_e^2 \cdot \frac{GM_s}{r_e^3} = \frac{GM_s\, m_e}{2r_e} = -\frac{1}{2} U$$

$$E = T + U = \frac{1}{2} U$$

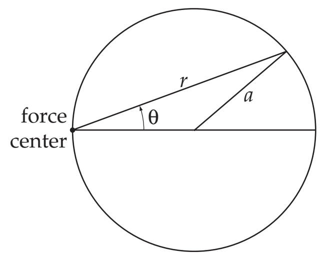
If the sun's mass suddenly goes to $\frac{1}{2}$ its original value, T remains unchanged but U is halved.

$$E' = T' + U' = T + \frac{1}{2}U = -\frac{1}{2}U + \frac{1}{2}U = 0$$

$$\boxed{\text{The energy is 0, so the orbit is a parabola. For a parabolic orbic, the earth will escape the solar system.}}$$

8-11. For central-force motion the equation of orbit is [Eq. (8.21)]

$$\frac{d^2}{d\theta^2}\left[\frac{1}{r}\right] + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r) \tag{1}$$



In our case the equation of orbit is

$$r = 2a \cos\theta \tag{2}$$

Therefore, (1) becomes

$$\frac{1}{2a}\frac{d^2}{d\theta^2}\left[(\cos\theta)^{-1}\right] + \frac{1}{2a}(\cos\theta)^{-1} = -\frac{4a^2\mu}{\ell^2} F(r)\cos^2\theta \tag{3}$$

But we have

$$\frac{d^2}{d\theta^2}\left[(\cos\theta)^{-1}\right] = \frac{d}{d\theta}\left[\frac{\sin\theta}{\cos^2\theta}\right]$$

$$= \frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} \tag{4}$$

Therefore, we have

$$\frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} + \frac{1}{\cos\theta} = -\frac{8a^3\,\mu}{\ell^2} F(r)\cos^2\theta \tag{5}$$

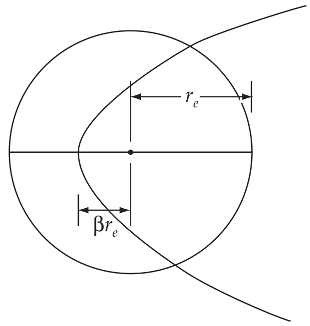
or,

$$F(r) = -\frac{\ell^2}{8a^3\mu}\frac{2}{\cos^5\theta} = -\frac{8a^2\ell^2}{\mu}\frac{1}{r^5} \tag{6}$$

so that

$$\boxed{F(r) = -\frac{k}{r^5}} \tag{7}$$

8-12.



The orbit of the comet is a parabola ($\varepsilon = 1$), so that the equation of the orbit is

$$\frac{\alpha}{r} = 1 + \cos\theta \tag{1}$$

We choose to measure θ from perihelion; hence

$$r(\theta = 0) = \beta r_E \tag{2}$$

Therefore,

$$\alpha = \frac{\ell^2}{\mu k} = 2\beta r_E \tag{3}$$

Since the total energy is zero (the orbit is parabolic) and the potential energy is $U = -k/r$, the time spent within the orbit of the Earth is

$$\begin{aligned} T &= 2\int_{\mathrm{B}r_E}^{r_E} \frac{dr}{\sqrt{\frac{2}{\mu}\left[\frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right]}} \\ &= \sqrt{\frac{2\mu}{k}} \int_{\beta r_E}^{r_E} \frac{r\,dr}{\sqrt{r - \beta r_E}} \\ &= \sqrt{\frac{2\mu}{k}} \left[-\frac{2(-2\beta r_E - r)}{3}\sqrt{r - \beta r_E}\right]_{\beta r_E}^{r_E} \end{aligned} \tag{4}$$

from which

$$T = \sqrt{\frac{2\mu}{k}} \left[\frac{2}{3} r_E^{3/2} (2\beta + 1)\sqrt{1 - \beta}\right] \tag{5}$$

Now, the period and the radius of the Earth are related by

$$\tau_E^2 = \frac{4\pi^2 \mu_E}{k'} r_E^3 \tag{6}$$

or,

$$r_E^{3/2} = \sqrt{\frac{k'}{\mu_E}}\frac{\tau_E}{2\pi} \tag{7}$$

Substituting (7) into (5), we find

$$T = \sqrt{\frac{2\mu}{k}}\frac{2}{3}\sqrt{\frac{k'}{\mu_E}}\frac{\tau_E}{2\pi}(2\beta+1)\sqrt{1-\beta} \tag{8}$$

where $k = GM_s\mu$ and $k' = GM_s\mu_E$. Therefore,

$$\boxed{T = \frac{1}{3\pi}\sqrt{2(1-\beta)}\,(1+2\beta)\,\tau_E} \tag{9}$$

where $\tau_E = 1$ year . Now, $\beta = r_{\text{Mercury}}/r_E = 0.387$. Therefore,

$$T = \frac{1}{3\pi}\sqrt{2(1-0.387)}\,(1+2\times 0.387)\times 365 \text{ days}$$

so that

$$\boxed{T = 76 \text{ days}} \tag{10}$$

8-13. Setting $u \equiv 1/r$ we can write the force as

$$F = -\frac{k}{r^2} - \frac{\lambda}{r^3} = -ku^2 - \lambda u^3 \tag{1}$$

Then, the equation of orbit becomes [cf. Eq. (8.20)]

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}\left(-ku^2 - \lambda u^3\right) \tag{2}$$

from which

$$\frac{d^2u}{d\theta^2} + u\left[1 - \frac{\mu\lambda}{\ell^2}\right] = \frac{\mu k}{\ell^2} \tag{3}$$

or,

$$\frac{d^2u}{d\theta^2} + \left[1 - \frac{\mu\lambda}{\ell^2}\right]\left[u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}}\right] = 0 \tag{4}$$

If we make the change of variable,

$$v = u - \frac{\mu k}{\ell^2}\frac{1}{1 - \frac{\mu\lambda}{\ell^2}} \tag{5}$$

we have

$$\frac{d^2v}{d\theta^2} + \left[1 - \frac{\mu\lambda}{\ell^2}\right]v = 0 \tag{6}$$

or,

$$\frac{d^2v}{d\theta^2} + \beta^2 v = 0 \tag{7}$$

where $\beta^2 = 1 - \mu\lambda/\ell^2$. This equation gives different solutions according to the value of λ. Let us consider the following three cases:

i) $\lambda < \ell^2/\mu$:

For this case $\beta^2 > 0$ and the solution of (7) is

$$v = A\cos(\beta\theta - \delta)$$

By proper choice of the position $\theta = 0$, the integration constant δ can be made to equal zero. Therefore, we can write

$$\boxed{\frac{1}{r} = A\cos\beta\theta + \frac{\mu k}{\ell^2 - \mu\lambda}} \tag{9}$$

When $\beta = 1$ ($\lambda = 0$), this equation describes a conic section. Since we do not know the value of the constant A, we need to use what we have learned from Kepler's problem to describe the motion. We know that for $\lambda = 0$,

$$\frac{1}{r} = \frac{\mu k}{\ell^2}(1 + \varepsilon\cos\theta)$$

and that we have an ellipse or circle ($0 \le \varepsilon < 1$) when $E < 1$, a parabola ($\varepsilon = 1$) when $E = 0$, and a hyperbola otherwise. It is clear that for this problem, if $E \ge 0$, we will have some sort of parabolic or hyperbolic orbit. An ellipse should result when $E < 0$, this being the only bound orbit. When $\beta \ne 1$, the orbit, whatever it is, precesses. This is most easily seen in the case of the ellipse, where the two turning points do not have an angular separation of π. One may obtain most constants of integration (in particular A) by using Equation (8.17) as a starting point, a more formal approach that confirms the statements made here.

ii) $\lambda = \ell^2/\mu$

For this case $\beta^2 = 0$ and (3) becomes

$$\frac{d^2u}{d\theta^2} = \frac{\mu k}{\ell^2} \tag{10}$$

so that

$$\boxed{u = \frac{1}{r} = \frac{\mu k}{2\ell^2}\theta^2 + A\theta + B} \tag{11}$$

from which we see that r continuously decreases as θ increases; that is, the particle spirals in toward the force center.

iii) $\lambda > \ell^2/\mu$

For this case $\beta^2 < 0$ and the solution (7) is

$$v = A\cosh\left(\sqrt{-\beta^2}\ \theta - \delta\right) \tag{12}$$

δ may be set equal to zero by the proper choice of the position at which $\theta = 0$. Then,

$$\boxed{\frac{1}{r} = A\cosh\left(\sqrt{-\beta^2}\ \theta\right) + \frac{\mu k}{\ell^2 - \mu\lambda}} \tag{13}$$

Again, the particle spirals in toward the force center.

8-14. The orbit equation for the central-force field is [see Eq. (8.17)]

$$\left[\frac{dr}{d\theta}\right]^2 = \frac{2\mu r^4}{\ell^2}\left[E - U - \frac{\ell^2}{2\mu r^2}\right] \tag{1}$$

But we are given the orbit equation:

$$r = k\theta^2 \tag{2}$$

from which

$$\left[\frac{dr}{d\theta}\right]^2 = 4k^2\theta^2 \tag{3}$$

Substituting (2) into (3), we have

$$\left[\frac{dr}{d\theta}\right]^2 = 4k^2\,\frac{r}{k} = 4kr \tag{4}$$

From (1) and (4), we find the equation for the potential U:

$$4kr = \frac{2\mu r^4}{\ell^2}\left[E - U - \frac{\ell^2}{2\mu r^2}\right] \tag{5}$$

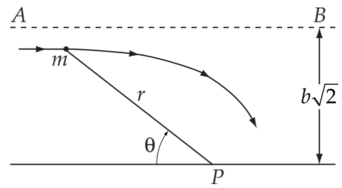
from which

$$U = E - \frac{2k\ell^2}{\mu}\frac{1}{r^3} - \frac{\ell^2}{2\mu}\frac{1}{r^2} \tag{6}$$

and $F(r) = -\partial U/\partial r$. Therefore,

$$\boxed{F(r) = -\frac{\ell^2}{\mu}\left[\frac{6k}{r^4} + \frac{1}{r^3}\right]} \tag{7}$$

8-15.



Let us denote by v the velocity of the particle when it is infinitely far from P and traveling along the line AB. The angular momentum is

$$\ell = \frac{\sqrt{k}}{b} = mvb\sqrt{2} = vb\sqrt{2} \tag{1}$$

where we have used $m = 1$. Therefore,

$$v = \frac{\sqrt{k}}{\sqrt{2}\,b^2} \tag{2}$$

The total energy E of the particle is equal to the initial kinetic energy:

$$E = \frac{1}{2}v^2 = \frac{k}{4b^4} \tag{3}$$

The general orbit equation for a force, $F(r) = -k/r^5$, is

$$d\theta = \frac{\ell}{r^2}\frac{dr}{\sqrt{2\left[E + \dfrac{k}{4r^4} - \dfrac{\ell^2}{2r^2}\right]}} \tag{4}$$

Substituting for ℓ and E from (1) and (3), we have

$$\begin{aligned} d\theta &= \frac{\sqrt{k}}{b}\frac{1}{r^2}\frac{dr}{\sqrt{\dfrac{k}{2b^4} + \dfrac{k}{2r^4} - \dfrac{k}{b^2r^2}}} \\ &= b\sqrt{2}\,\frac{dr}{\sqrt{r^4 - 2b^2r^2 + b^4}} \\ &= b\sqrt{2}\,\frac{dr}{\sqrt{\left(r^2 - b^2\right)^2}} \end{aligned}$$

or,

$$d\theta = -b\sqrt{2}\,\frac{dr}{r^2 - b^2} \tag{5}$$

where we have taken the *negative* square root because r *decreases* as θ *increases* (see the diagram).

We can now use the integral [see Eq. (E.4b), Appendix E]

$$\int \frac{dx}{a^2x^2 - b^2} = -\frac{1}{ab}\coth^{-1}\left[\frac{ax}{b}\right] \tag{6}$$

from which we obtain

$$\theta = \sqrt{2}\coth^{-1}\left[\frac{r}{b}\right] + \theta_0 \tag{7}$$

or,

$$r = b\coth\left[\frac{\theta - \theta_0}{\sqrt{2}}\right] \tag{8}$$

Now, coth $\phi \to \infty$ as $\phi \to 0$, since $r \to \infty$ as $\theta \to 0$, we must have $\theta_0 = 0$. Thus,

$$\boxed{r = b\coth\left(\theta/\sqrt{2}\right)} \tag{9}$$

Notice that r is always greater than b (because coth $\phi \to 1$ as $\phi \to \infty$), so that the denominator in (5) never equals zero nor changes sign. Thus, r always decreases as θ increases. This is, the particle spirals in toward P but never approaches closer than a distance b.

8-16. The total energy of the particle is

$$E = T + U \tag{1}$$

a principle that by no means pushes the philosophical envelope of physical interpretation. The impulse that causes $v \to v + \delta v$ changes the kinetic energy, not the potential energy. We therefore have

$$\delta E = \delta T = \delta\left(\frac{1}{2}mv^2\right) = mv\,\delta v \tag{2}$$

By the virial theorem, for a nearly circular orbit we have

$$E = -\frac{1}{2}mv^2 \tag{3}$$

so that

$$\frac{\delta E}{-E} = \frac{2\delta v}{v} \tag{4}$$

where we have written $-E$ since $E < 0$. The major and minor axes of the orbit are given by

$$a = -\frac{k}{2E} \qquad b = \frac{\ell}{\sqrt{-2\mu E}} \tag{5}$$

Now let us compute the changes in these quantities. For a we have

$$\delta a = -\delta\left(\frac{k}{2E}\right) = \frac{k\delta E}{2E^2} = a\left(\frac{\delta E}{-E}\right) \tag{6}$$

and for b we have

$$\delta b = \delta\left[\frac{\ell}{\sqrt{-2\mu E}}\right] = \frac{\delta\ell}{\sqrt{-2\mu E}} + \frac{\ell}{\sqrt{-2\mu E^3}}\left[-\frac{1}{2}\delta E\right] = b\left[\frac{\delta\ell}{\ell} - \frac{\delta E}{2E}\right] \tag{7}$$

Easily enough, we can show that $\delta\ell/\ell = \delta v/v$ and therefore

$$\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{\delta E}{-E} = \frac{2\delta v}{v} \tag{8}$$

8-17. The equation of the orbit is

$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta \tag{1}$$

from which

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta} \tag{2}$$

where $\alpha = \ell^2/\mu k$ and $\varepsilon = \sqrt{1 + \dfrac{2E\ell^2}{mk^2}}$. Therefore, the radial distance r can vary from the maximum value $\alpha/(1-\varepsilon)$ to the minimum value $\alpha/(1+\varepsilon)$. Now, the angular velocity of the particle is given by

$$\omega = \frac{\ell}{\mu r^2} \tag{3}$$

Thus, the maximum and minimum values of ω become

$$\left.\begin{aligned} \omega_{\max} &= \frac{\ell}{\mu\, r_{\min}^2} = \frac{\ell}{\mu\left[\dfrac{\alpha}{1+\varepsilon}\right]^2} \\ \omega_{\min} &= \frac{\ell}{\mu\, r_{\max}^2} = \frac{\ell}{\mu\left[\dfrac{\alpha}{1-\varepsilon}\right]^2} \end{aligned}\right] \tag{4}$$

Thus,

$$\frac{\omega_{\max}}{\omega_{\min}} = \left[\frac{1+\varepsilon}{1-\varepsilon}\right]^2 = n \tag{5}$$

from which we find

$$\boxed{\varepsilon = \frac{\sqrt{n}-1}{\sqrt{n}+1}} \tag{6}$$

8-18. Kepler's second law states that the areal velocity is constant, and this implies that the angular momentum L is conserved. If a body is acted upon by a force and if the angular momentum of the body is not altered, then the force has imparted no torque to the body; thus, the force must have acted only along the line connecting the force center and the body. That is, the force is central.

Kepler's first law states that planets move in elliptical orbits with the sun at one focus. This means the orbit can be described by Eq. (8.41):

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \qquad \text{with } 0 < \varepsilon < 1 \tag{1}$$

On the other hand, for central forces, Eq. (8.21) holds:

$$\frac{d^2}{d\theta^2}\left[\frac{1}{r}\right] + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r) \tag{2}$$

Substituting $1/r$ from (1) into the left-hand side of (2), we find

$$\frac{1}{a} = -\frac{\mu}{\ell^2} r^2 F(r) \tag{3}$$

which implies, that

$$\boxed{F(r) = -\frac{\ell^2}{\alpha\mu r^2}} \tag{4}$$

8-19. The semimajor axis of an orbit is defined as one-half the sum of the two apsidal distances, $r_{\max}$ and $r_{\min}$ [see Eq. (8.44)], so

$$\frac{1}{2}\left[r_{\max} + r_{\min}\right] = \frac{1}{2}\left[\frac{\alpha}{1+\varepsilon} + \frac{\alpha}{1-\varepsilon}\right] = \frac{\alpha}{1-\varepsilon^2} \tag{1}$$

This is the same as the semimajor axis defined by Eq. (8.42). Therefore, by using Kepler's Third Law, we can find the semimajor axis of Ceres in astronomical units:

$$\frac{a_C}{a_E} = \left[\frac{\dfrac{k_C}{4\pi^2\mu_C}\,\tau_C^2}{\dfrac{k_E}{4\pi^2\mu_E}\,\tau_E^2}\right] \tag{2}$$

where $k_c = \gamma M_s m_c$, and

$$\frac{1}{\mu_c} = \frac{1}{M_s} + \frac{1}{m_c}$$

Here, M_s and m_c are the masses of the sun and Ceres, respectively. Therefore, (2) becomes

$$\frac{a_C}{a_E} = \left[\frac{M_s + m_c}{M_s + m_e}\left[\frac{\tau_c}{\tau_E}\right]^2\right]^{1/3} \tag{3}$$

from which

$$\frac{a_C}{a_E} = \left[\frac{333,480 + \dfrac{1}{8,000}}{333,480 + 1} (4.6035)^2 \right]^{1/3} \tag{4}$$

so that

$$\boxed{\frac{a_C}{a_E} \cong 2.767} \tag{5}$$

The period of Jupiter can also be calculated using Kepler's Third Law:

$$\frac{\tau_J}{\tau_E} = \left[\frac{\dfrac{4\pi^2 \mu_J}{k_J} a_J^3}{\dfrac{4\pi^2 \mu_E}{k_E} a_E^3} \right]^{1/2} = \left[\frac{M_s + m_E}{M_s + m_J} \left[\frac{a_J}{a_E} \right]^3 \right]^{1/2} \tag{6}$$

from which

$$\frac{\tau_J}{\tau_E} = \left[\frac{333,480 + 1}{333,480 + 318.35} (5.2028)^3 \right]^{1/2} \tag{7}$$

Therefore,

$$\boxed{\frac{\tau_J}{\tau_E} \cong 11.862} \tag{8}$$

The mass of Saturn can also be calculated from Kepler's Third law, with the result

$$\boxed{\frac{m_s}{m_e} \cong 95.3} \tag{9}$$

8-20. Using Eqs. (8.42) and (8.41) for a and r, we have

$$\left\langle \left(\frac{a}{r} \right)^4 \cos\theta \right\rangle = \frac{1}{\tau} \int_0^\tau dt \left[\frac{1 + \varepsilon \cos\theta}{1 - \varepsilon^2} \right]^4 \cos\theta \tag{1}$$

From Kepler's Second Law, we can find the relation between t and θ:

$$dt = \frac{\tau}{\pi ab} dA = \frac{\tau}{\pi ab} \frac{1}{2} \frac{\alpha^2}{(1 + \varepsilon \cos\theta)^2} d\theta \tag{2}$$

since $dA = (1/2) r^2 \, d\theta$. Therefore, (1) becomes

$$\left\langle \left(\frac{a}{r} \right)^4 \cos\theta \right\rangle = \frac{1}{\tau} \frac{1}{(1 - \varepsilon^2)^4} \frac{\tau}{\pi ab} \frac{a^2}{2} \int_0^{2\pi} \cos\theta (1 + \varepsilon \cos\theta)^2 \, d\theta \tag{3}$$

It is easily shown that the value of the integral is $2\pi\varepsilon$. Therefore,

$$\left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{1}{\left(1-\varepsilon^2\right)^4} \frac{1}{ab} \alpha^2 \varepsilon \tag{4}$$

After substituting a and b in terms of ε and α [see Eqs. (8.42) and (8.43)], we obtain

$$\boxed{\left\langle \left(\frac{a}{r}\right)^4 \cos\theta \right\rangle = \frac{\varepsilon}{\left(1-\varepsilon^2\right)^{5/2}}} \tag{5}$$

8-21. If we denote the total energy and the potential of the family of orbits by E and $U(r)$, we have the relation

$$\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = E \tag{1}$$

from which

$$\ell^2 = 2\mu r^2\left(E - U(r) - \frac{1}{2}\mu\dot{r}^2\right) \tag{2}$$

Here, E and $U(r)$ are same for all orbits, and the different values of ℓ result from different values of $(1/2)\mu r^2$. For stable circular motion, $\dot{r} = 0$, but for all other motions, $\dot{r} \neq 0$. Therefore, for non-circular motions, $r^2 > 0$ and ℓ is smaller than for the circular case. That is, the angular momentum of the circular orbit is the largest among the family.

8-22. For the given force, $F(r) = -k/r^3$, the potential is

$$U(r) = -\frac{k}{2r^2} \tag{1}$$

and the effective potential is

$$V(r) = \frac{1}{2}\left[\frac{\ell^2}{\mu} - k\right]\frac{1}{r^2} \tag{2}$$

The equation of the orbit is [cf. Eq. (8.20)]

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2}\left(-ku^3\right) \tag{3}$$

or,

$$\frac{d^2u}{d\theta^2} + \left[1 - \frac{\mu k}{\ell^2}\right]u = 0 \tag{4}$$

Let us consider the motion for various values of ℓ.

i) $\ell^2 = \mu k$:

In this case the effective potential $V(r)$ vanishes and the orbit equation is

$$\frac{d^2u}{d\theta^2} = 0 \tag{5}$$

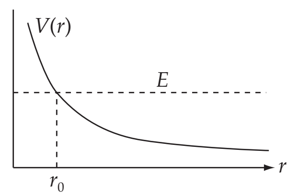
with the solution

$$\boxed{u = \frac{1}{r} = A\theta + B} \tag{6}$$

and the particle spirals toward the force center.

ii) $\ell^2 > \mu k$:

In this case the effective potential is positive and decreases monotonically with increasing r. For any value of the total energy E, the particle will approach the force center and will undergo a reversal of its motion at $r = r_0$; the particle will then proceed again to an infinite distance.



Setting $1 - \mu k/\ell^2 \equiv \beta^2 > 0$, (4) becomes

$$\frac{d^2u}{d\theta^2} + \beta^2 u = 0 \tag{7}$$

with the solution

$$\boxed{u = \frac{1}{r} = A\cos(\beta\theta - \delta)} \tag{8}$$

Since the minimum value of u is zero, this solution corresponds to unbounded motion, as expected from the form of the effective potential $V(r)$.

iii) $\ell^2 < \mu k$:

For this case we set $\mu k/\ell^2 - 1 \equiv G^2 > 0$, and the orbit equation becomes

$$\frac{d^2u}{d\theta^2} - G^2 u = 0 \tag{9}$$

with the solution

$$\boxed{u = \frac{1}{r} = A\cosh(\beta\theta - \delta)} \tag{10}$$

so that the particle spirals in toward the force center.

In order to investigate the stability of a circular orbit in a $1/r^3$ force field, we return to Eq. (8.83) and use $g(r) = k/\mu r^3$. Then, we have

$$\ddot{x} - \frac{\ell^2}{\mu^2 \rho^3 \left[1 + (x/\rho)\right]^3} = -\frac{k}{\mu \rho^3 \left[1 + (x/\rho)\right]^3} \tag{11}$$

or,

$$\ddot{x} + \left[k - \frac{\ell^2}{\mu}\right] \cdot \frac{1}{\mu \rho^3 \left[1 + (x/\rho)\right]^3} = 0 \tag{12}$$

Since $\ddot{r}\big|_{r=p} = 0$, Eq. (8.87) shows that $k = \ell^2/\mu$. Therefore, (12) reduces to

$$\ddot{x} = 0 \tag{13}$$

so that the perturbation x increases uniformly with the time. The circular orbit is therefore not stable.

We can also reach the same conclusion by examining the basic criterion for stability, namely, that

$$\left.\frac{\partial V}{\partial r}\right|_{r=\rho} = 0 \text{ and } \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=\rho} > 0$$

The first of these relations requires $k = \ell^2/\mu$ while the second requires $\ell^2/\mu > k$. Since these requirements cannot be met simultaneously, no stable circular orbits are allowed.

8-23. Start with the equation of the orbit:

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \tag{1}$$

and take its time derivative

$$\frac{\dot{r}}{r^2} = \frac{\varepsilon}{\alpha}\dot{\theta}\sin\theta = \frac{\varepsilon \ell}{\alpha \mu r^2}\sin\theta \tag{2}$$

Now from Equation (8.45) and (8.43) we have

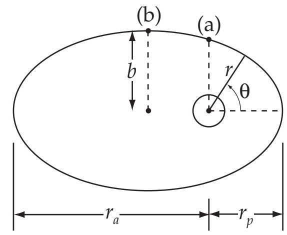
$$\tau = \frac{2\mu}{\ell} \cdot \pi ab = \frac{2\pi \mu a \alpha}{\ell \sqrt{1 - \varepsilon^2}} \tag{3}$$

so that from (2)

$$\left|\dot{r}\right|_{\max} = \frac{\varepsilon}{\mu} \cdot \frac{\ell}{\alpha} = \frac{2\pi a \varepsilon}{\tau \sqrt{1 - \varepsilon^2}} \tag{4}$$

as desired.

8-24.



a) With the center of the earth as the origin, the equation for the orbit is

$$\frac{\alpha}{r} = 1 + \varepsilon\, cox\, \theta \tag{1}$$

Also we know

$$r_{\min} = a(1-\varepsilon) \tag{2}$$

$$r_{\max} = a(1+\varepsilon)$$

$$r_{\min} = r_p = 300 \text{ km} + r_e = 6.67 \times 10^6 \text{ m}$$

$$r_{\max} = r_a = 3500 \text{ km} + r_e = 9.87 \times 10^6 \text{ m}$$

$$a = \frac{1}{2}\left(r_a + r_p\right) = 8.27 \times 10^6 \text{ m}$$

Substituting (2) gives $\varepsilon = 0.193$. When $\theta = 0$,

$$\frac{a}{r_{\min}} = 1 + \varepsilon$$

which gives $\alpha = 7.96 \times 10^6$ m. So the equation of the orbit is

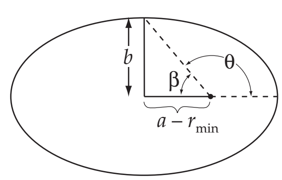
$$\frac{7.96 \times 10^6 \text{ m}}{r} = 1 + 0.193 \cos\theta$$

When $\theta = 90°$,

$$\boxed{r = \alpha = 7.96 \times 10^6 \text{ m}}$$

$$\boxed{\text{The satellite is 1590 km above the earth.}}$$

b)



$$\theta = \pi - \beta$$

$$= \pi - \tan^{-1}\frac{b}{a - r_{\text{min}}}$$

Using $b = \sqrt{\alpha a}$

$$\theta = \pi - \tan^{-1}\frac{\sqrt{\alpha a}}{a - r_{\text{min}}} \simeq 101°$$

Substituting into (1) gives

$$r = 8.27 \times 10^6 \text{ m} \text{; which is}$$

$$\boxed{\text{1900 km above the earth}}$$

8-25. Let us obtain the major axis a by exploiting its relationship to the total energy. In the following, let M be the mass of the Earth and m be the mass of the satellite.

$$E = -\frac{GMm}{2a} = \frac{1}{2}mv_p^2 = \frac{GMm}{r_p} \tag{1}$$

where r_p and v_p are the radius and velocity of the satellite's orbit at perigee. We can solve for a and use it to determine the radius at apogee by

$$r_a = 2a - r_p = r_p\left[\frac{2GM}{r_p v_p^2} - 1\right]^{-1} \tag{2}$$

Inserting the values

$$G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2} \tag{3}$$

$$M = 5.976 \times 10^{24} \text{ kg} \tag{4}$$

$$r_p = 6.59 \times 10^6 \text{ m} \tag{5}$$

$$v_p = 7.797 \times 10^3 \text{ m} \cdot \text{s}^{-1} \tag{6}$$

we obtain $r_a \simeq 1.010\, r_p = 6.658 \times 10^6 \text{ m}$, or 288 km above the earth's surface. We may get the speed at apogee from the conservation of angular momentum,

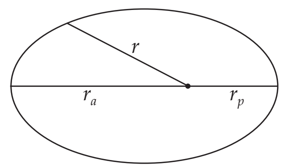
$$mr_a v_a = mr_p v_p \tag{7}$$

giving $v_a = 27{,}780 \text{ km} \cdot \text{hr}^{-1}$. The period can be found from Kepler's third law

$$t^2 = \frac{4\pi^2 a^3}{GM} \tag{8}$$

Substitution of the value of a found from (1) gives $\tau = 1.49$ hours.

8-26.



First, consider a velocity kick Δv applied along the direction of travel at an arbitrary place in the orbit. We seek the optimum location to apply the kick.

$$E_1 = \text{initial energy}$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$E_2 = \text{final energy}$$

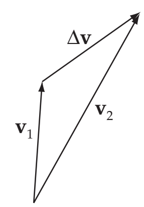
$$= \frac{1}{2}m(v + \Delta v)^2 - \frac{GMm}{r}$$

We seek to maximize the energy gain $E_2 - E_1$:

$$E_2 - E_1 = \frac{1}{2}m\left(2v\,\Delta v + \Delta v^2\right)$$

For a given Δv, this quantity is clearly a maximum when v is a maximum; i.e., at perigee.

Now consider a velocity kick ΔV applied at perigee in an arbitrary direction:



The final energy is

$$\frac{1}{2}mv_2^2 = \frac{GMm}{r_p}$$

This will be a maximum for a maximum $|\mathbf{v}_2|$; which clearly occurs when $\mathbf{v}_1$ and $\Delta\mathbf{v}$ are along the same direction.

Thus, the most efficient way to change the energy of an elliptical orbit (for a single engine thrust) is by firing along the direction of travel at perigee.

8-27. By conservation of angular momentum

$$mr_a v_a = mr_p v_p$$

$$\text{or} \qquad v_a = \frac{r_p v_p}{r_a}$$

Substituting gives

$$\boxed{v_a = 1608 \text{ m/s}}$$

8-28. Use the conservation of energy for a spacecraft leaving the surface of the moon with just enough velocity v_{esc} to reach $r = \infty$:

$$T_i + U_i = T_f + U_f$$

$$\frac{1}{2} m v_{\text{esc}}^2 - \frac{\text{GM}_\text{m} m}{r_m} = 0 + 0$$

$$v_{\text{esc}} = \sqrt{\frac{2GM_m}{r_m}}$$

where

$$M_m = \text{mass of the moon} = 7.36 \times 10^{22} \text{ kg}$$

$$r_m = \text{radius of the moon} = 1.74 \times 10^6 \text{ m}$$

Substituting gives

$$\boxed{v_{\text{esc}} = 2380 \text{ m/s}}$$

8-29. $$v_{\max} = v + v_0, \quad v_{\min} = v - v_0$$

From conservation of angular momentum we know

$$mv_a r_a = mv_b r_b$$

or

$$v_{\max} r_{\min} = v_{\min} r_{\max}; \quad \frac{r_{\max}}{r_{\min}} = \frac{v_{\max}}{v_{\min}} \tag{1}$$

Also we know

$$r_{\min} = a(1-e) \tag{2}$$

$$r_{\max} = a(1+e) \tag{3}$$

Dividing (3) by (2) and setting the result equal to (1) gives

$$\frac{r_{\max}}{r_{\min}} = \frac{1+e}{1-e} = \frac{v_{\max}}{v_{\min}}$$

$$v_{\min}\,(1+e) = v_{\max}\,(1-e)$$

$$e(v_{\min} + v_{\max}) = v_{\max} - v_{\min}$$

$$e(2v) = 2v_0$$

$$\boxed{e = \frac{v_0}{v}}$$

8-30. To just escape from Earth, a velocity kick must be applied such that the total energy E is zero. Thus

$$\frac{1}{2}mv_2^2 - \frac{GM_e m}{r} = 0 \tag{1}$$

where

$$v_2 = \text{velocity after kick}$$

$$M_e = 5.98 \times 10^{24}\ \text{kg}$$

$$G = 6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2$$

$$\begin{aligned} r &= 200\ \text{km} + r_e \\ &= 200\ \text{km} + 6.37 \times 10^6\ \text{m} \\ &= 6.57 \times 10^6\ \text{m} \end{aligned}$$

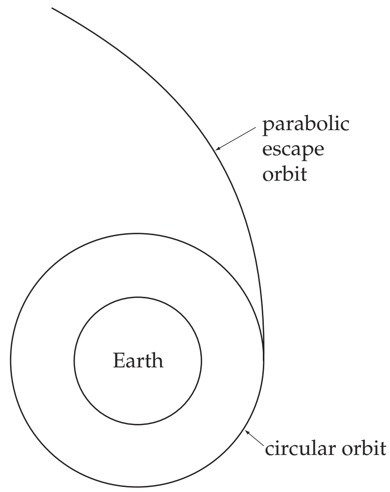
Substituting into (1) gives $v_2 = 11.02\ \text{km/sec}$.

For a circular orbit, the initial velocity v_1 is given by Eq. (8.51)

$$v_1 = \sqrt{\frac{GM_e}{r}} = 7.79\ \text{km/sec}$$

Thus, to escape from the earth, a velocity kick of 3.23 km/sec must be applied.

Since $E = 0$, the trajectory is a parabola.



8-31. From the given force, we find

$$\frac{dF(r)}{dr} = F'(r) = \frac{2k}{r^3} + \frac{4k'}{r^5} \tag{1}$$

Therefore, the condition of stability becomes [see Eq. (8.93)]

$$\frac{F'(\rho)}{F(\rho)} + \frac{3}{\rho} = \frac{\dfrac{2}{\rho^5}\left(k\rho^2 + 2k'\right)}{-\dfrac{1}{\rho^4}\left(k\rho^2 + k'\right)} + \frac{3}{\rho} > 0 \tag{2}$$

or,

$$\frac{k\rho^2 - k'}{\rho\left(k\rho^2 + k'\right)} > 0 \tag{3}$$

Therefore, if $\rho^2 k > k'$, the orbit is stable.

8-32. For this force, we have

$$\begin{aligned} \frac{dF(r)}{dr} = F'(r) &= \frac{2k}{r^3}\, e^{-r/a} + \frac{k}{ar^2}\, e^{-r/a} \\ &= \frac{k}{r^3}\, e^{-r/a}\left[2 + \frac{r}{a}\right] \end{aligned} \tag{1}$$

Therefore, the condition of stability [see Eq. (8.93)] becomes

$$\frac{F'(r)}{F(r)} + \frac{3}{r} = \frac{-\left[2 + \dfrac{r}{a}\right] + 3}{r} > 0 \tag{2}$$

This condition is satisfied if $r < a$.

8-33. The Lagrangian of the particle subject to a gravitational force is written in terms of the cylindrical coordinates as

$$L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz \tag{1}$$

From the constraint $r^2 = 4az$, we have

$$\dot{z} = \frac{r\dot{r}}{2a} \tag{2}$$

Therefore, (1) becomes

$$L = \frac{1}{2} m\left[\left[1 + \frac{r^2}{4a^2}\right]\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{mg}{4a} r^2 \tag{3}$$

Lagrange's equation for θ is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \tag{4}$$

This equation shows that the angular momentum of the system is constant (as expected):

$$mr^2\dot{\theta} = \ell = \text{const.} \tag{5}$$

Lagrange's equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{4a^2} r\dot{r}^2 + mr\dot{\theta}^2 - \frac{mg}{2a} r - \frac{d}{dt}\left[m\left[1 + \frac{r^2}{4a^2}\right]\dot{r}\right] = 0 \tag{6}$$

from which

$$\frac{m}{4a^2} r\dot{r}^2 + mr\dot{\theta}^2 - \frac{mg}{2a} r - m\left[1 + \frac{r^2}{4a^2}\right]\ddot{r} - \frac{m}{2a^2} r\dot{r}^2 = 0 \tag{7}$$

After rearranging, this equation becomes

$$m\left[1 + \frac{r^2}{4a^2}\right]\ddot{r} + \frac{m}{4a^2} r\dot{r}^2 + \frac{mg}{2a} r - \frac{\ell^2}{m}\frac{1}{r^3} = 0 \tag{8}$$

For a circular orbit, we must have $\dot{r} = \ddot{r} = 0$ or, $r = \rho$ = constant. Then,

$$\frac{mg\rho}{2a} = \frac{\ell^2}{m\rho^3} \tag{9}$$

or,

$$\ell^2 = \frac{m^2 g}{2a}\rho^4 \tag{10}$$

Equating this with $\ell^2 = m^2\rho^4\dot{\theta}^2$, we have

$$m^2\rho^4\dot{\theta}^2 = \frac{m^2 g}{2a}\rho^4 \tag{11}$$

or,

$$\dot{\theta}^2 = \frac{g}{2a} \tag{12}$$

Applying a perturbation to the circular orbit, we can write

$$r \to \rho + x \quad \text{where} \quad \frac{x}{\rho} \ll 1 \tag{13}$$

This causes the following changes:

$$\left.\begin{aligned} & r^2 \to \rho^2 + 2\rho x \\ & \frac{1}{r^3} \to \frac{1 - 3\dfrac{x}{\rho}}{\rho^3} \\ & \dot{r} \to \dot{x} \\ & \ddot{r} \to \ddot{x} \end{aligned}\right] \tag{14}$$

from which, we have

$$\left.\begin{aligned} & r\dot{r}^2 \to (\rho + x)\dot{x}^2 \cong 0, \text{ in lowest order} \\ & r^2\ddot{r} \to \left(\rho^2 + 2\rho x\right)\ddot{x} \cong \rho^2\ddot{x}, \text{ in lowest order} \end{aligned}\right] \tag{15}$$

Thus, (8) becomes

$$m\left[1 + \frac{1}{4a^2}\rho^2\right]\ddot{x} + \frac{mg}{2a}(\rho + x) - \frac{\ell^2}{m\rho^3}\left(1 - 3\frac{x}{\rho}\right) = 0 \tag{16}$$

But

$$\frac{mg\rho}{2a} = \frac{\ell^2}{m\rho^3} \tag{17}$$

so that (16) becomes

$$m\left[1 + \frac{\rho^2}{4a^2}\right]\ddot{x} + \frac{mg}{2a}x + \frac{3\ell^2}{m\rho^4}x = 0 \tag{18}$$

Substituting (17) into (18), we find

$$m\left[1 + \frac{\rho^2}{4a^2}\right]\ddot{x} + \frac{2mg}{a}x = 0 \tag{19}$$

or,

$$\ddot{x} + \frac{2g}{a + \dfrac{\rho^2}{4a}}x = 0 \tag{20}$$

Therefore, the frequency of small oscillations is

$$\boxed{\omega = \sqrt{\frac{2g}{a + z_0}}} \tag{21}$$

where

$$z_0 = \frac{\rho^2}{4a}$$

8-34. The total energy of the system is

$$E = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2 \alpha\right) + mgr \cot \alpha \tag{1}$$

or,

$$E = \frac{1}{2} m\left(1 + \cot^2 \alpha\right)\dot{r}^2 + \frac{1}{2} mr^2\dot{\theta}^2 + mgr \cot \alpha \tag{2}$$

Substituting $\ell = mr^2\dot{\theta}$, we have

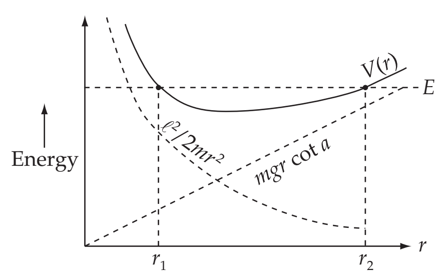
$$E = \frac{1}{2} m\left(1 + \cot^2 \alpha\right)\dot{r}^2 + \frac{\ell^2}{2mr^2} + mgr \cot \alpha \tag{3}$$

Therefore, the effective potential is

$$V(r) = \frac{\ell^2}{2mr^2} + mgr \cot \alpha \tag{4}$$

At the turning point we have $\dot{r} = 0$, and (3) becomes a cubic equation in r:

$$mgr^3 \cot \alpha - Er^2 + \frac{\ell^2}{2m} = 0 \tag{5}$$



This cubic equation has three roots. If we attempt to find these roots graphically from the intersections of E = const. and $V(r) = \ell^2/2mr^2 + mgr \cot \alpha$, we discover that only two of the roots are real. (The third root is imaginary.) These two roots specify the planes between which the motion takes place.

8-35. If we write the radial distance r as

$$r = \rho + x, \qquad \rho = \text{const.} \tag{1}$$

then x obeys the oscillatory equation [see Eqs. (8.88) and (8.89)]

$$\ddot{x} + \omega_0^2 x = 0 \tag{2}$$

where

$$\omega_0 = \sqrt{\frac{3g(\rho)}{\rho} + g'(\rho)} \tag{3}$$

The time required for the radius vector to go from any maximum value to the succeeding minimum value is

$$\Delta t = \frac{\tau_0}{2} \tag{4}$$

where $\tau_0 = \dfrac{2\pi}{\omega_0}$, the period of x. Thus,

$$\Delta t = \frac{\pi}{\omega_0} \tag{5}$$

The angle through which the particle moves during this time interval is

$$\phi = \omega\Delta t = \frac{\pi\omega}{\omega_0} \tag{6}$$

where ω is the angular velocity of the orbital motion which we approximate by a circular motion. Now, under the force $F(r) = -\mu g(r)$, ω satisfies the equation

$$\mu\rho\omega^2 = -F(r) = \mu g(\rho) \tag{7}$$

Substituting (3) and (7) into (6), we find for the apsidal angle

$$\phi = \frac{\pi\omega}{\omega_0} = \frac{\pi\sqrt{\dfrac{g(\rho)}{\rho}}}{\sqrt{\dfrac{3g(\rho)}{\rho} + g'(\rho)}} = \frac{\pi}{\sqrt{3 + \rho\dfrac{g'(\rho)}{g(\rho)}}} \tag{8}$$

Using $g(r) = \dfrac{k}{\mu}\dfrac{1}{r^n}$, we have

$$\frac{g'(\rho)}{g(\rho)} = -\frac{n}{\rho} \tag{9}$$

Therefore, (8) becomes

$$\boxed{\phi = \pi/\sqrt{3-n}} \tag{10}$$

In order to have the closed orbits, the apsidal angle must be a rational fraction of 2π. Thus, n must be

$$n = 2, -1, -6, \ldots$$

$n = 2$ corresponds to the inverse-square-force and $n = -1$ corresponds to the harmonic oscillator force.

8-36. The radius of a circular orbit in a force field described by

$$F(r) = -\frac{k}{r^2}\, e^{-r/a} \tag{1}$$

is determined by equating $F(r)$ to the centrifugal force:

$$\frac{k}{r^2}\, e^{-\rho/a} = \frac{\ell^2}{m\rho^3} \tag{2}$$

Hence, the radius ρ of the circular orbit must satisfy the relation

$$\rho e^{-\rho/a} = \frac{\ell^2}{mk} \tag{3}$$

Since the orbit in which we are interested is almost circular, we write

$$r(\theta) = \rho\left[1 + \delta(\theta)\right] \tag{4}$$

where $\delta(\theta) \ll 1$ for all values of θ. (With this description, the apsides correspond to the maximum and minimum values of δ.)

We can express the following quantities in terms of δ by using (4):

$$u = \frac{1}{r} = \frac{1}{\rho}(1 - \delta) \tag{5a}$$

$$\frac{d^2}{d\theta^2}\left[\frac{1}{r}\right] = -\frac{1}{\rho}\frac{d^2\delta}{d\theta^2} \tag{5b}$$

$$\begin{aligned} F(u) &= -ku^2 e^{-1/au} \\ &\cong -\frac{k\, e^{-\rho/a}}{\rho^2 (1+\delta)^2}\left(1 - \rho\delta/a\right) \end{aligned} \tag{5c}$$

Then, substitution into Eq. (8.20) yields

$$-\frac{1}{\rho}\frac{d^2\delta}{d\theta^2} + \frac{1}{\rho}(1 - \delta) = \frac{mke^{-\rho/a}}{\ell^2}\left(1 - p\delta/a\right) \tag{6}$$

Multiplying by ρ, using (3) and simplifying, (6) reduces to

$$\frac{d^2\delta}{d\theta^2} + \left(1 - \rho/a\right)\delta = 0 \tag{7}$$

This equation obviously has two types of solution depending on whether ρ/a is larger than or smaller than 1; we consider only $\rho < a$. (In fact, there is no stable circular orbit for $\rho > a$.)

For the initial condition, we choose $\delta = \delta_0$ to be a maximum (i.e., an apside) at $\theta = 0$. Then, we have

$$\delta = \delta_0 \cos\left(1 - \rho/a\right)^{1/2} \theta, \text{ for } \rho < a \tag{8}$$

This solution describes an orbit with well-defined apsides. The advance of the apsides can be found from (8) by computing for what value of θ is δ again a maximum. Thus,

$$\theta = \frac{2\pi}{\sqrt{1 - \rho/a}} \tag{9}$$

The advance of the apside is given by

$$\Delta = \theta - 2\pi = 2\pi\left[1 - \left(1 - \rho/a\right)^{-1/2}\right] \tag{10}$$

In the particular case in which $\rho \ll a$ we obtain, by extending (10),

$$\Delta \cong 2\pi - 2\pi\left[1 + \frac{\rho}{2a}\right] \tag{11}$$

so that

$$\boxed{\Delta \cong \frac{\pi\rho}{a}} \tag{12}$$

8-37. From the equations in Section 8.8 regarding Hohmann transfers:

$$\Delta v = \Delta v_1 + \Delta v_2$$

$$\Delta v = v_{t_1} - v_1 + v_2 - v_{t_2}$$

$$\Delta v = \sqrt{\frac{2k}{mr_1}\left[\frac{r_2}{r_1 + r_2}\right]} - \sqrt{\frac{k}{mr_1}} + \sqrt{\frac{k}{mr_2}} - \sqrt{\frac{2k}{mr_2}\left[\frac{r_1}{r_1 + r_2}\right]} \tag{1}$$

Substituting

$$\frac{k}{m} = GM_e = \left(6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2\right)\left(5.98 \times 10^{24}\ \text{kg}\right)$$

$$r_1 = \text{initial height above center of Earth} = 2r_e$$

$$r_2 = \text{final height above center of Earth} = 3r_e$$

$$r_e = \text{radius of the Earth} = 6.37 \times 10^6\ \text{m}$$

gives

$$\boxed{\Delta v \simeq 1020\ \text{m/s}}$$

8-38. Substitute the following into Eq. (1) of problem 8-37:

$$\frac{k}{m} = GM_s = \left(6.67\times10^{-11}\ \text{Nm}^2/\text{kg}^2\right)\left(1.99\times10^{30}\ \text{kg}\right)$$

$$r_1 = \text{mean Earth-sun distance} \simeq 1.50\times10^{11}\ \text{m}$$

$$r_2 = \text{mean Venus-sun distance} \simeq 1.08\times10^{11}\ \text{m}$$

The result is $\Delta v = -5275\ \text{m/s}$. The answer is negative because $r_2 < r_1$; so the rocket must be fired in the direction opposite to the motion (the satellite must be slowed down).

$$\boxed{\Delta v = 5275\ \text{m/s; opposite to direction of motion.}}$$

From Eq. (8.58), the time is given by

$$T = \pi\sqrt{\frac{m}{k}}\,a_t^{3/2} = \pi\sqrt{\frac{m}{k}}\left[\frac{r_1+r_2}{2}\right]^{3/2} \tag{1}$$

Substituting gives

$$\boxed{\tau \simeq 148\ \text{days}}$$

8-39. We must calculate the quantity Δv_1 for transfers to Venus and Mars. From Eqs. (8.54), (8.53), and (8.51):

$$\begin{aligned}\Delta v_1 &= v_{t_1} - v_1\\ &= \sqrt{\frac{2k}{mr_1}\left[\frac{r_2}{r_1+r_2}\right]} - \sqrt{\frac{k}{mr_1}}\end{aligned}$$

where

$$\frac{k}{m} = GM_s = \left(6.67\times10^{-11}\ \text{m}^3/\text{s}^2\text{-kg}\right)\left(1.99\times10^{30}\ \text{kg}\right)$$

$$r_1 = \text{mean Earth-sun distance} = 150\times10^{9}\ \text{m}$$

$$r_2 = \text{mean}\begin{bmatrix}\text{Venus}\\ \text{Mars}\end{bmatrix} - \text{sun distance} = \begin{bmatrix}108\\ 228\end{bmatrix}\times10^{9}\ \text{m}$$

Substituting gives

$$\Delta v_{\text{Venus}} = -2.53\ \text{km/sec}$$

$$\Delta v_{\text{Mars}} = 2.92\ \text{km/sec}$$

where the negative sign for Venus means the velocity kick is opposite to the Earth's orbital motion.

$$\boxed{\text{Thus, a Mars flyby requires a larger } \Delta v \text{ than a Venus flyby.}}$$

8-40. To crash into the sun, we calculate Δv_1 from Eq. (8.54) with r_1 = mean distance from sun to Earth, and r_2 = radius of the sun. Using Eqs. (8.54), (8.53), and (8.51) we have

$$(\Delta v_1)_{\text{sun}} = \sqrt{\frac{2GM_s}{r_1}\left[\frac{r_2}{r_1+r_2}\right]} - \sqrt{\frac{GM_s}{r_1}}$$

Substituting

$$G = 6.67 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$$

$$M_s = 1.99 \times 10^{30}\ \text{kg}$$

$$r_1 = r_{se} = 1.5 \times 10^{11}\ \text{m}$$

$$r_2 = r_{\text{sun}} = 6.96 \times 10^{8}\ \text{m}$$

gives

$$(\Delta v_1)_{\text{sun}} = -26.9\ \text{km/sec}$$

To escape from the solar system, we must overcome the gravitational pull of both the sun and Earth. From conservation of energy ($E_{\text{final}} = 0$) we have:

$$-\frac{GM_s m}{r_{se}} - \frac{GM_e m}{r_e} + \frac{1}{2}mv^2 = 0$$

Substituting values gives

$$v = 43500\ \text{m/s}$$

Now

$$\begin{aligned}\Delta v &= v - v_i \\ &= v - \sqrt{\frac{GM_s}{r_{se}}} \\ &= (43500 - 29700)\ \text{m/s}\end{aligned}$$

$$(\Delta v)_{\text{escape}} = 13.8\ \text{km/s}$$

To send the waste out of the solar system requires less energy than crashing it into the sun.

8-41. From the equations in Section 8.8 regarding Hohmann transfers

$$\begin{aligned}\Delta v &= \Delta v_1 + \Delta v_2 \\ &= v_{t_1} - v_1 + v_2 - v_{t_2}\end{aligned}$$

where

$$v_{t_1} = \sqrt{\frac{2k}{mr_1}\left[\frac{r_2}{r_1 + r_2}\right]}; \quad v_1 = \sqrt{\frac{k}{mr_1}}$$

$$v_{t_2} = \sqrt{\frac{2k}{mr_2}\left[\frac{r_1}{r_1 + r_2}\right]}; \quad v_1 = \sqrt{\frac{k}{mr_2}}$$

Substituting

$$\frac{k}{m} = GM_e = \left(6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2\right)\left(5.98 \times 10^{24}\ \text{kg}\right)$$

$$r_1 = 200\ \text{km} + r_e = 6.37 \times 10^6\ \text{m} + 2 \times 10^5\ \text{m}$$

$$r_2 = \text{mean Earth-moon distance} = 3.84 \times 10^8\ \text{m}$$

gives

$$\boxed{\Delta v = 3966\ \text{m/s}}$$

From Eq. (8.58), the time of transfer is given by

$$T = \tau \sqrt{\frac{m}{k}}\, a_t^{3/2} = \pi \sqrt{\frac{m}{k}} \left[\frac{r_1 + r_2}{2}\right]^{3/2}$$

Substituting gives

$$\boxed{\tau = 429{,}000\ \text{sec.} \simeq 5\ \text{days}}$$

8-42.

$$G = 6.67 \times 10^{-11}\ \text{Nm}^2/\text{kg}^2$$

$$M_e = 5.98 \times 10^{24}\ \text{kg}$$

$$r_1 = 2 \times 10^5\ \text{m} + 6.37 \times 10^6\ \text{m}$$

$$r_2 = ?$$

$$r_3 = \text{mean Earth-moon distance} = 3.84 \times 10^8\ \text{m}$$

We can get r_2 from Kepler's Third Law (with $\tau = 1$ day)

$$r_2 = \left[\frac{GM_e\ \tau^2}{4\pi^2}\right]^{1/3} = 4.225 \times 10^7\ \text{m}$$

We know $E = -GMm/2r$

So

$$E(r_1) = -\frac{GM_e m}{2r_1} = -3.04 \times 10^{11} \text{ J}$$

$$E(r_2) = -4.72 \times 10^{10} \text{ J}$$

$$E(r_3) = -5.19 \times 10^{9} \text{ J}$$

To place the satellite in a synchronous orbit would require a minimum energy of $E(r_2) - E(r_1) =$

$$\boxed{2.57 \times 10^{11} \text{ J}}$$

8-43. In a circular orbit, the velocity v_0 of satellite is given by

$$\frac{mv_0^2}{R} = \frac{GMm}{R^2} \Rightarrow v_0 = \sqrt{\frac{GM}{R}}$$

where M is the Earth's mass.

Conservation of energy implies

$$\frac{mv_1^2}{2} - \frac{GMm}{R} = \frac{mv_2^2}{2} - \frac{GMm}{2R}$$

Conservation of angular momentum gives

$$mRv_1 = m2Rv_2$$

From these equations, we find

$$v_1 = \sqrt{\frac{4GM}{3R}}$$

so the velocity need to be increased by a factor $\sqrt{4/3}$ to change the orbit.

8-44. The bound motion means that $E = \frac{mv^2}{2} + V < 0$

where $V = -\frac{k}{r} e^{-r/a}$.

The orbit of particle moving in this central force potential is given by

$$\theta(r) = \int_{r_{\min}}^{r} \frac{(\ell / r^2)\, dr}{\sqrt{2\mu\left(E - V\dfrac{\ell^2}{2\mu r^2}\right)}}$$

$$= \frac{\ell}{\sqrt{2\mu}} \int_{r_{\min}}^{r} \frac{1}{r^2} \frac{dr}{\sqrt{E + \dfrac{ke^{-ra}}{r} - \dfrac{\ell^2}{2\mu r^2}}}$$

In first order of (r / a), this is

$$\theta_{(r)} \approx \frac{\ell}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{E + \dfrac{k}{r} - \dfrac{\ell^2}{2\mu r^2} - \dfrac{k}{a}}} = \frac{\ell}{\sqrt{2\mu}} \int \frac{dr}{r^2 \sqrt{\left(E - \dfrac{k}{a}\right) + \dfrac{k}{r} - \dfrac{\ell^2}{2\mu r^2}}}$$

Now effectively, this is the orbit of particle of total energy $\left(E - \dfrac{k}{a}\right)$ moving in potential $-\dfrac{k}{r}$. It is well known that this orbit is given by (see Chapter 8)

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$$

where $\alpha = \dfrac{\ell^2}{\mu k}$ and $\varepsilon = \sqrt{1 + \dfrac{2\ell^2}{\mu k^2}\left(E - \dfrac{k}{a}\right)}$

If $0 < \varepsilon < 1$, the orbit is ellipsoid; if $\varepsilon = 0$, the orbit is circular.

8-45.

a) In equilibrium, for a circular orbit of radius r_0,

$$F_0 = m\omega_\phi^2 r_0 \;\Rightarrow\; \omega_\phi = \sqrt{\frac{F_0}{mr_0}}$$

b) The angular momentum (which is conserved) of a particle in circular orbit is

$$L = mr_0^2\omega_\phi = \sqrt{mFr_0^3}$$

The force acting on a particle, which is placed a distance *r* (*r* is very close to equilibrium position r_o) from the center of force is

$$F = m\omega_\phi^2 r - F_0 = \frac{L^3}{mr^3} - F_0$$

$$\approx \frac{L^3}{mr_0^3} - \frac{3L^2}{mr_0^2}(r - r_0) - F_0 = -\frac{3L^2}{mr_0^4}(r - r_0) = -k(r - r_0)$$

where $k \equiv 3L^2/mr_0^4$. So the frequency of oscillation is

$$\omega_r = \sqrt{\frac{k}{m}} = \sqrt{\frac{3L^2}{m^2 r_0^4}} = \sqrt{\frac{3F_0}{mr_0}}$$

8-46. In equilibrium circular orbit,

$$\frac{Mv^2}{R} = \frac{GM^2}{4R^2} \quad \Rightarrow \quad R = \frac{GM}{4v^2}$$

where M is the Sun's mass.

The period is

$$T = \frac{2\pi R}{v} = \frac{4\pi R\sqrt{R}}{\sqrt{GM}} = \frac{\sqrt{2}\pi D^{3/2}}{\sqrt{GM}} \approx 9 \times 10^7 \text{ yr}$$

where $D = 2R$ is the separation distance of 2 stars.

8-47. In equilibrium circular orbit of 1st star

$\dfrac{M_1 v_1^2}{L_1} = \dfrac{GM_1M_2}{L_2}$ where $L_1 = \dfrac{LM_2}{M_1 + M_2}$ is the distance from 1st star to the common center of mass.

The corresponding velocity is

$$v_1 = \sqrt{\frac{GM_2L_1}{L^2}} = \sqrt{\frac{GM_2^2}{L(M_1 + M_2)}}$$

Finally, the period is

$$T = \frac{2\pi L_1}{v_1} = \frac{2\pi L^{3/2}}{\sqrt{G(M_1 + M_2)}} = 1.2 \times 10^8 \text{ yr.}$$

CHAPTER 9

Dynamics of a System of Particles

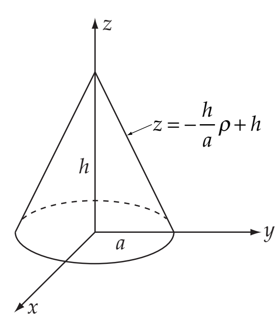
9-1. Put the shell in the $z > 0$ region, with the base in the x-y plane. By symmetry, $\overline{x} = \overline{y} = 0$.

$$\overline{z} = \frac{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=r_1}^{r_2} \rho\, z r^2\, dr \sin\theta\, d\theta\, d\phi}{\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=r_1}^{r_2} \rho\, r^2\, dr \sin\theta\, d\theta\, d\phi}$$

Using $z = r\cos\theta$ and doing the integrals gives

$$\boxed{\overline{z} = \frac{3\left(r_2^4 - r_1^4\right)}{8\left(r_2^3 - r_1^3\right)}}$$

9-2.



By symmetry, $\overline{x} = \overline{y} = 0$.

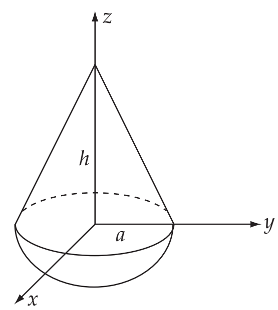
Use cylindrical coordinates ρ, ϕ, z.

$$\rho_0 = \text{ mass density}$$

$$\bar{z} = \frac{\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\int_{z=0}^{-\frac{h}{a}\rho+h} \rho_0 z \, \rho \, d\rho \, d\phi \, dz}{\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\int_{z=0}^{-\frac{h}{4}\rho+h} \rho_0 \, \rho \, d\rho \, d\phi \, dz} = \frac{h}{4}$$

$$\boxed{\text{The center of mass is on the axis of the cone } \frac{3}{4}\, h \text{ from the vertex.}}$$

9-3.



By symmetry, $\bar{x} = \bar{y} = 0$.

From problem 9-2, the center of mass of the cone is at $z = \frac{1}{4} h$.

From problem 9-1, the center of mass of the hemisphere is at

$$z = -\frac{3}{8} a \left(r_2 = a, r_1 = 0 \right)$$

So the problem reduces to

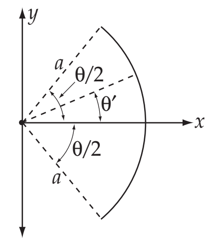
- $z_1 = \frac{1}{4} h; \; m_1 = \rho_1 \frac{1}{3} \pi a^2 \, h$
- $z_2 = -\frac{3}{8} a; \; m_2 = \rho_2 \frac{2}{3} \pi a^3$

$$\bar{z} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} = \frac{\rho_1 h^2 - 3\rho_2 a^2}{4\left(\rho_1 h + 2\rho_2 a\right)}$$

for $\rho_1 = \rho_2$

$$\boxed{\bar{z} = \frac{h^2 - 3a^3}{4\left(2a + h\right)}}$$

9-4.



By symmetry, $\bar{y} = 0$.

If $\sigma = \text{mass/length}$ then $M = \sigma a \theta$

So

$$\bar{x} = \frac{1}{M} \int_{\theta' = -\frac{\theta}{2}}^{\theta/2} x dm$$

$$\bar{x} = \frac{1}{M} \int_{\theta' = -\frac{\theta}{2}}^{\theta/2} x \sigma a d\theta'$$

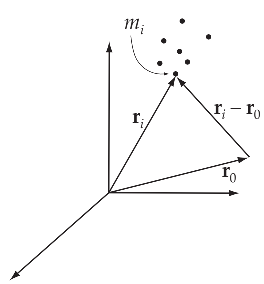
Using $M = \sigma a \theta$ and $x = a \cos \theta'$,

$$\bar{x} = \frac{1}{\theta} \int_{-\theta/2}^{\theta/2} a \cos \theta' d\theta' = \frac{a}{\theta} \left[\sin \frac{\theta}{2} - \sin\left(-\frac{\theta}{2} \right) \right]$$

$$= \frac{a}{\theta} 2 \sin \frac{\theta}{2}$$

$$\boxed{\begin{aligned} \bar{x} &= \frac{2a}{\theta} \sin \frac{\theta}{2} \\ \bar{y} &= 0 \end{aligned}}$$

9-5.



$$\mathbf{r}_i = \text{position of the } i^{\text{th}} \text{ particle}$$

$$m_1 = \text{mass of the } i^{\text{th}} \text{ particle}$$

$$M = \sum m_i = \text{total mass}$$

$$\mathbf{g} = \text{constant gravitational field}$$

Calculate the torque about $\mathbf{r}_0$

$$\begin{aligned}
\tau &= \sum \tau_i \\
&= \sum (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i \\
&= \sum (\mathbf{r}_i - \mathbf{r}_0) \times m_i\, \mathbf{g} \\
&= \sum \mathbf{r}_1 \times m_i\, \mathbf{g} - \sum \mathbf{r}_0 \times m_i\, \mathbf{g} \\
&= \left(\sum m_i \mathbf{r}_i\right) \times \mathbf{g} - \left(\sum m_i\right) \mathbf{r}_0 \times \mathbf{g} \\
&= \left(\sum m_i \mathbf{r}_i\right) \times \mathbf{g} - M\mathbf{r}_0 \times \mathbf{g}
\end{aligned}$$

Now if the total torque is zero, we must have

$$\sum m_i \mathbf{r}_i = M\mathbf{r}_0$$

or

$$\mathbf{r}_0 = \frac{1}{M} \sum m_i \mathbf{r}_i$$

which is the definition of the center of mass. So

$$\boxed{\begin{array}{c} \tau = 0 \text{ about } \mathbf{r}_0 = \mathbf{r}_{\text{CM}} \\ \text{or center of gravity} = \text{center of mass.} \end{array}}$$

9-6. Since particle 1 has $\mathbf{F} = 0$, $\mathbf{r}_0 = \mathbf{v}_0 = 0$, then $\mathbf{r}_1 = 0$. For particle 2

$$\mathbf{F}_2 = F_0\, \hat{\mathbf{x}} \qquad \text{then } \ddot{r} = \frac{F_0}{m}\, \hat{\mathbf{x}}$$

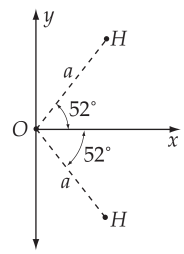
Integrating twice with $\mathbf{r}_0 = \mathbf{v}_0 = 0$ gives

$$r_2 = \frac{F_0}{2m} t^2\, \hat{\mathbf{x}}$$

$$\mathbf{r}_{\text{CM}} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{F_0}{4m} t^2\, \hat{\mathbf{x}}$$

$$\boxed{\begin{aligned} \mathbf{r}_{CM} &= \frac{F_0}{4m} t^2\, \hat{\mathbf{x}} \\ \mathbf{v}_{CM} &= \frac{F_0}{2m} t\, \hat{\mathbf{x}} \\ \mathbf{a}_{CM} &= \frac{F_0}{2m} \hat{\mathbf{x}} \end{aligned}}$$

9-7.



By symmetry $\boxed{\bar{y} = 0}$

$m_0 = 16\, m_H$

Let $m_H = m,\, m_0 = 16\, m$

Then

$$\bar{x} = \frac{1}{M} \sum_{1}^{3} m_i\, x_i$$

$$\bar{x} = \frac{1}{18m}\left(2ma \cos 52°\right) = \frac{a \cos 52°}{9}$$

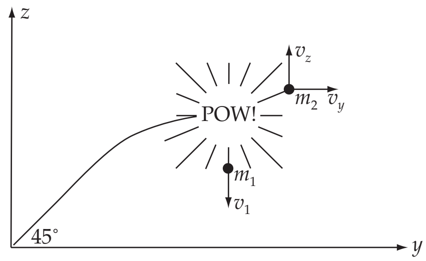
$$\boxed{\bar{x} = 0.068\, a}$$

9-8. By symmetry, $\boxed{\bar{x} = 0}$. Also, by symmetry, we may integrate over the $x > 0$ half of the triangle to get $\bar{y}$. σ = mass/area

$$\bar{y} = \frac{\int_{x=0}^{\frac{a}{\sqrt{2}}} \int_{y=0}^{\frac{a}{\sqrt{2}}-x} \sigma\, y\, dy\, dx}{\int_{x=0}^{\frac{a}{\sqrt{2}}} \int_{y=0}^{\frac{a}{\sqrt{2}}-x} \sigma\, dy\, dx} = \frac{a}{3\sqrt{2}}$$

$$\boxed{\bar{y} = \frac{a}{3\sqrt{2}}}$$

9-9.



Let the axes be as shown with the projectile in the y-z plane. At the top just before the explosion, the velocity is in the y direction and has magnitude $v_{0y} = \frac{v_0}{\sqrt{2}}$.

$$v_{0y} = \frac{v_0}{\sqrt{2}} = \frac{\sqrt{\frac{2E_0}{m_1+m_2}}}{\sqrt{2}} = \sqrt{\frac{E_0}{m_1+m_2}}$$

where m_1 and m_2 are the masses of the fragments. The initial momentum is

$$p_i = (m_1+m_2)\left[0, \sqrt{\frac{E_0}{m_1+m_2}}, 0\right]$$

The final momentum is

$$p_f = p_1 + p_2$$

$$p_1 = m_1(0,0,v_1)$$

$$p_2 = m_2(v_x, v_y, v_z)$$

The conservation of momentum equations are

$$p_x: \qquad 0 = m_2 v_x \quad or \quad \boxed{v_x = 0}$$

$$p_y: \qquad \sqrt{E_0(m_1+m_2)} = m_2 v_y \quad or \quad \boxed{v_y = \frac{1}{m_2}\sqrt{E_0(m_1+m_2)}}$$

$$p_z: \qquad 0 = m_1 v_1 + m_2 v_z \text{ or } v_1 = -\frac{m_2}{m_1} v_z$$

The energy equation is

$$\frac{1}{2}(m_1+m_2)\frac{E_0}{m_1+m_2} + E_0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2\left(v_y^2 + v_z^2\right)$$

or

$$3E_0 = m_1 v_1^2 + m_2\left(v_y^2 + v_z^2\right)$$

Substituting for v_y and v_1 gives

$$\boxed{v_z = \sqrt{\frac{E_0 m_1 (2m_2 - m_1)}{m_2^2 (m_1 + m_2)}}}$$

$$v_1 = -\frac{m_2}{m_1} v_z \text{ gives}$$

$$\boxed{v_1 = -\sqrt{\frac{E_0 (2m_2 - m_1)}{m_1 (m_1 + m_2)}}}$$

So

m_1 travels straight down with speed $= |v_1|$

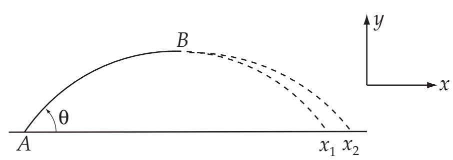
m_2 travels in the y-z plane

$$\boxed{\begin{aligned} v_2 &= \left(v_y^2 + v_z^2\right)^{1/2} = \sqrt{\frac{E_0 (4m_1 + m_2)}{m_2 (m_1 + m_2)}} \\ \theta &= \tan^{-1} \frac{v_z}{v_y} = \tan^{-1} \sqrt{\frac{m_1 (2m_2 - m_1)}{(m_1 + m_2)}} \end{aligned}}$$

The mass m_1 is the largest it can be when $v_1 = 0$, meaning $2m_2 = m_1$ and the mass ratio is

$$\frac{m_1}{m_2} = \frac{1}{2}$$

9-10.



First, we find the time required to go from A to B by examining the motion. The equation for the y-component of velocity is

$$v_y = v_0 \sin\theta - gt \tag{1}$$

At B, $v_y = 0$; thus $t_B = v_0 \sin\theta / g$. The shell explodes giving m_1 and m_2 horizontal velocities v_1 and v_2 (in the original direction). We solve for v_1 and v_2 using conservation of momentum and energy.

$$p_x: \qquad (m_1 + m_2) v_0 \cos\theta = m_1 v_1 + m_2 v_2 \tag{2}$$

$$E: \qquad \frac{1}{2}(m_1 + m_2) v_0^2 \cos^2\theta + E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{3}$$

Solving for v_2 in (2) and substituting into (3) gives an equation quadratic in v_1. The solution is

$$v_1 = v_0 \cos\theta \pm \sqrt{\frac{2m_2 E}{m_1(m_1+m_2)}} \tag{4}$$

and therefore we also must have

$$v_2 = v_0 \cos\theta \mp \sqrt{\frac{2m_1 E}{m_2(m_1+m_2)}} \tag{5}$$

Now we need the positions where m_1 and m_2 land. The time to fall to the ocean is the same as the time it took to go from A to B. Calling the location where the shell explodes $x = 0$ gives for the positions of m_1 and m_2 upon landing:

$$x_1 = v_1 t_B; \qquad x_2 = v_2 t_B \tag{6}$$

Thus

$$|x_1 - x_2| = \frac{v_0 \sin\theta}{g}|v_1 - v_2| \tag{7}$$

Using (4) and (5) and simplifying gives

$$|x_1 - x_2| = \frac{v_0 \sin\theta}{g}\sqrt{\frac{2E}{m_1+m_2}}\left[\sqrt{\frac{m_1}{m_2}} + \sqrt{\frac{m_2}{m_1}}\right] \tag{8}$$

9-11. The term in question is

$$\sum_{\alpha}\sum_{\substack{\beta \\ \alpha\neq b}} \mathbf{f}_{\alpha\beta}$$

For n = 3, this becomes

$$\mathbf{f}_{12} + \mathbf{f}_{13} + \mathbf{f}_{21} + \mathbf{f}_{23} + \mathbf{f}_{31} + \mathbf{f}_{32} = (\mathbf{f}_{12} + \mathbf{f}_{21}) + (\mathbf{f}_{13} + \mathbf{f}_{31}) + (\mathbf{f}_{23} + \mathbf{f}_{32})$$

But by Eq. (9.1), each quantity in parentheses is zero. Thus

$$\boxed{\sum_{\alpha=1}^{3}\sum_{\substack{\beta=1 \\ \alpha\neq\beta}}^{3} \mathbf{f}_{\alpha\beta} = 0}$$

9-12.

a)
$$v = v_0 + u \ln\frac{m_0}{m}$$

Assuming $v_0 = 0$, we have

$$v = \left[100\ \frac{\text{m}}{\text{s}}\right]\ln\frac{100}{98}$$

$$\boxed{v = 2.02\ \text{m/s; yes, he runs out of gas.}}$$

b) Relative to Stumblebum's original frame of reference we have:

Before throwing tank

$$\underset{\boxed{}}{98\text{ kg}} \rightarrow 2.02\text{ m/s}$$

After throwing the tank we want Stumblebum's velocity to be slightly greater than 3 m/s (so that he will catch up to the orbiter).

$$V \leftarrow \underset{\boxed{}}{8\text{ kg}} \qquad \underset{\boxed{}}{90\text{ kg}} \rightarrow 3\text{m/s}$$

Conservation of momentum gives

$$(98\text{ kg})(2.02\text{ m/s}) = (90\text{ kg})(3\text{ m/s}) - (8\text{ kg})v$$

$$v = 9\text{ m/s}$$

(This velocity is relative to Stumblebum's original reference frame; i.e., before he fires his pressurized tank.) Since Stumblebum is traveling towards the orbiter at 2.02 m/s, he must throw the tank at v = 9 m/s + 2.02 m/s

$$\boxed{v = 11\text{ m/s}}$$

9-13. From Eq. (9.9), the total force is given by

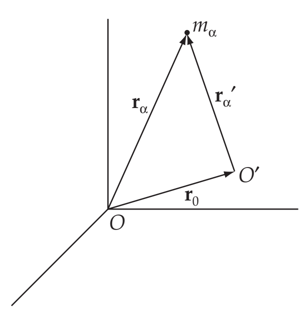
$$\sum_{\alpha} \mathbf{F}_{\alpha}^{(e)} + \sum_{\substack{\alpha \\ \alpha \neq \beta}} \sum_{\beta} \mathbf{f}_{\alpha\beta}$$

As shown in Section 9.3, the second term is zero. So the total force is

$$\sum_{\alpha} \mathbf{F}_{\alpha}^{(e)}$$

It is given that this quantity is zero.

Now consider two coordinate systems with origins at 0 and 0′



where

$\mathbf{r}_0$ is a vector from 0 to 0′

$\mathbf{r}_\alpha$ is the position vector of m_α in 0

$$\mathbf{r}'_\alpha \text{ is the position vector of } m_\alpha \text{ in } 0'$$

We see that $\mathbf{r}_a = \mathbf{r}_0 + \mathbf{r}'_\alpha$

The torque in 0 is given by

$$\tau = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)}$$

The torque in 0′ is

$$\begin{aligned}\tau' &= \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{F}_\alpha^{(e)} \\ &= \sum_\alpha \left(\mathbf{r}_\alpha - \mathbf{r}_0\right) \times \mathbf{F}_\alpha^{(e)} \\ &= \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)} - \sum_\alpha \mathbf{r}_0 \times \mathbf{F}_\alpha^{(e)} \\ &= \tau - \mathbf{r}_0 \times \sum_\alpha \mathbf{F}_\alpha^{(e)}\end{aligned}$$

But it is given that $\sum_\alpha \mathbf{F}_\alpha^{(e)} = 0$

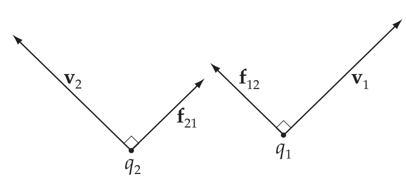
Thus

$$\boxed{\tau = \tau'}$$

9-14. Neither Eq. (9.11) or Eq. (9.31) is valid for a system of particles interacting by magnetic forces. The derivations leading to both of these equations assumes the weak statement of Newton's Third Law [Eq. (9.31) assumes the strong statement of the Third Law also], which is

$$\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$$

That this is not valid for a system with magnetic interactions can be seen by considering two particles of charge q_1 and q_2 moving with velocities v_1 and v_2:



Now

$$\mathbf{f}_{ij} = q_i \mathbf{v}_i \times \mathbf{B}_{ij}$$

where $\mathbf{B}_{ij}$ is the magnetic field at q_i due to the motion of q_j.

Since $\mathbf{f}_{ij}$ is perpendicular to both $\mathbf{v}_i$ and $\mathbf{B}_{ij}$ (which is either in or out of the paper), $\mathbf{f}_{ij}$ can only be parallel to $\mathbf{f}_{ji}$ if $\mathbf{v}_i$ and $\mathbf{v}_j$ are parallel, which is not true in general.

$$\boxed{\text{Thus, equations (9.11) and (9.31) are not valid for a system of particles with magnetic interactions.}}$$

9-15.

$$\sigma = \text{mass/length}$$

$$F = \frac{dp}{dt} \text{ becomes}$$

$$mg = m\dot{v} + m\dot{v}$$

where m is the mass of length x of the rope. So

$$m = \sigma x;\ \dot{m} = \sigma\dot{x}$$

$$\sigma\, x\, g = \sigma\, x \frac{dv}{dt} + \sigma\, \dot{x}\, v$$

$$x\, g = x \frac{dv}{dx}\frac{dx}{dt} + v^2$$

$$x\, g = xv \frac{dv}{dx} + v^2$$

Try a power law solution:

$$v = ax^n;\ \frac{dv}{dx} = nax^{n-1}$$

Substituting,

$$x\, g = x\left(ax^n\right)\left(nax^{n-1}\right) + a^2\ x^{2n}$$

or

$$x\, g = a^2\,(n+1)\,x^{2n}$$

Since this must be true for all x, the exponent and coefficient of x must be the same on both sides of the equation.

Thus we have: $1 = 2n$ or $n = \dfrac{1}{2}$

$$g = a^2\,(n+1) \text{ or } a = \sqrt{\frac{2g}{3}}$$

So

$$\boxed{v = \sqrt{\frac{2gx}{3}}}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \left[\frac{2gx}{3}\right]^{1/2} \frac{g}{3}\left[\frac{2gx}{3}\right]^{-1/2}$$

$$\boxed{a = \frac{g}{3}}$$

$$T_i = 0 \qquad U_i = 0 \qquad (y = 0 \text{ on table})$$

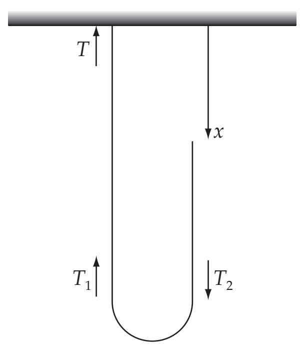
$$T_f = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{2gL}{3}\right] = \frac{mgL}{3}$$

$$U_f = mgh = -mg\frac{L}{2}$$

So $E_i = 0$; $E_f = -\dfrac{mgL}{6}$

$$\boxed{\text{Energy lost} = \frac{mgL}{6}}$$

9-16.



The equation of motion for the falling side of the chain is, from the figure,

$$\frac{\rho(b-x)}{2}\ddot{x} = \frac{\rho(b-x)}{2}g + T_2 \tag{1}$$

From Example 9.2, we have for the energy conservation case

$$\ddot{x} = g - \frac{g\left(2bx - x^2\right)}{2(b-x)^2} = g + \frac{\dot{x}^2}{2(b-x)} \tag{2}$$

Substitution gives us

$$T_2 = \frac{\rho\dot{x}^2}{4} \tag{3}$$

To find the tension on the other side of the bend, change to a moving coordinate system in which the bend is instantaneously at rest. This frame moves downward at a speed $u = \dot{x}/2$ with respect to the fixed frame. The change in momentum at the bend is

$$\Delta p = (\rho \Delta x)\cdot(2u) = 2\rho u^2 \ \Delta t = \frac{\rho \dot{x}^2}{2}\Delta t \tag{4}$$

Equating this with the net force gives

$$T_1 + T_2 = \frac{\rho \dot{x}^2}{2} \tag{5}$$

Using equation (3), we obtain

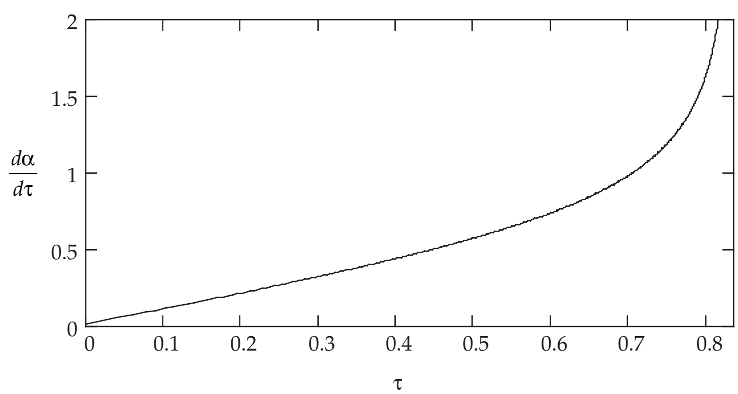
$$T_1 = \frac{\rho \dot{x}^2}{4} \tag{6}$$

as required. Note that equation (5) holds true for both the free fall and energy conservation cases.

9-17. As the problem states, we need to perform the following integral

$$\tau = \int_{\varepsilon}^{1/2} \sqrt{\frac{1-2\alpha}{2\alpha(1-\alpha)}}\, d\alpha \tag{1}$$

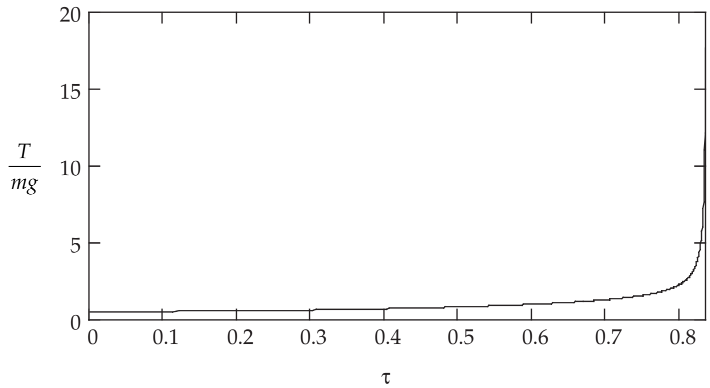
Our choice of ε is 10^{-4} for this calculation, and the results are shown in the figure. We plot the natural velocity $d\alpha/d\tau = \dot{x}/\sqrt{2gb}$ vs. the natural time τ.



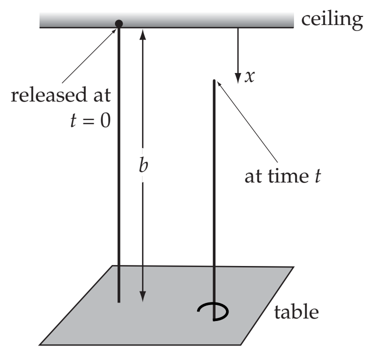
9-18. Once we have solved Problem 9-17, it becomes an easy matter to write the expression for the tension (Equation 9.18):

$$\frac{T}{mg} = \frac{1+2\alpha-6\alpha^2}{2(1-2\alpha)} \tag{1}$$

This is plotted vs. the natural time using the solution of Problem 9-17.



9-19.



The force that the tabletop exerts on the chain counteracts the force due to gravity, so that we may write the change in momentum of the center of the mass of the chain as

$$\frac{dp}{dt} = \rho bg - F \tag{1}$$

We can write out what the momentum is, though:

$$p = \rho(b - x)\dot{x} \tag{2}$$

which has a time derivative

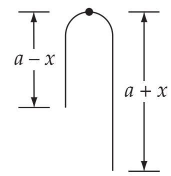
$$\frac{dp}{dt} = \rho\left[-\dot{x}^2 + (b - x)\ddot{x}\right] = \rho(bg - 3gx) \tag{3}$$

where we have used $\ddot{x} = g$ and $\dot{x} = \sqrt{2gx}$. Setting this last expression equal to (1) gives us

$$F = 3\rho gx \tag{4}$$

Although M. G. Calkin (personal communication) has found that experimentally the time of fall for this problem is consistently less than the value one would obtain in the above treatment by about 1.5%, he also finds evidence that suggests the free fall treatment is more valid if the table is energy absorbing.

9-20.



$$\text{Let } \rho = \text{mass/length}$$

The force on the rope is due to gravity

$$\begin{aligned} F &= (a+x)\rho g - (a-x)\rho g \\ &= 2x\rho g \end{aligned}$$

$$\frac{dp}{dt} = m\frac{dv}{dt} = 2a\rho\frac{dv}{dt}$$

So $F = \dfrac{dp}{dt}$ becomes

$$xg = a\frac{dv}{dt}$$

Now

$$\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

So

$$xg = av\frac{dv}{dx}$$

or

$$vdv = \frac{g}{a}xdx$$

Integrating yields

$$\frac{1}{2}v^2 = \frac{g}{2a}x^2 + c$$

Since $v = 0$ when $x = 0$, $c = 0$.
Thus

$$v^2 = \frac{g}{a}x^2$$

When the rope clears the nail, $x = a$. Thus

$$\boxed{v = \sqrt{ga}}$$

9-21. Let us call x the length of rope hanging over the edge of the table, and L the total length of the rope. The equation of motion is

$$\frac{mgx}{L} = m\frac{dx^2}{dt^2} \;\Rightarrow\; \ddot{x} = \frac{gx}{L}$$

Let us look for solution of the form

$$x = Ae^{\omega t} + Be^{-\omega t}$$

Putting this into equation of motion, we find

$$\omega = \sqrt{\frac{g}{L}}$$

Initial conditions are $x_{(t=0)} = x_0 = 0.3\text{ m}$; $v_{(t=0)} = 0\text{ m/s}$.

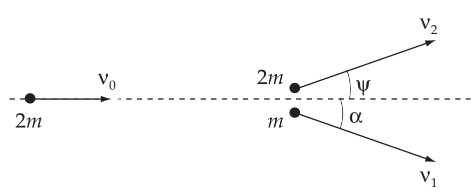
From these we find $A = B = x_0/2$.

Finally $x = x_0 \cosh(\omega t)$. When $x = L$, the corresponding time is

$$t = \frac{1}{\omega}\cosh^{-1}\left(\frac{L}{x_0}\right) = 0.59\text{ s.}$$

9-22. Let us denote (see figure)

m and $2m$	mass of neutron and deuteron respectively
v_0	velocity of deuteron before collision
v_1 and v_2	velocity of neutron and deuteron, respectively after collision



a) Conservation of energy:

$$\frac{2mv_0^2}{2} = \frac{mv_1^2}{2} + \frac{2mv_2^2}{2} \;\Rightarrow\; v_0^2 = \frac{v_1^2}{2} + \frac{v_2^2}{2}$$

Conservation of momentum is

$$2m\vec{v}_0 = 2m\vec{v}_2 + \vec{v}_1 \;\Rightarrow\; 4v_2^2 + 4v_0^2 = v_1^2 + 8v_0 v_2 \cos\psi$$

Solving these equations, we obtain 2 sets of solutions

$$v_1 = \frac{2v_0}{3}\sqrt{6 - 4\cos^2\psi \mp 2\cos\psi\sqrt{4\cos^2\psi - 3}}$$

$$v_2 = \frac{2v_0\cos\psi \pm \sqrt{4v_0^2\cos^2\psi - 3v_0^2}}{3}$$

or numerically

$$v_1 = 5.18 \text{ km/s} \qquad v_2 = 14.44 \text{ km/s} \text{ and}$$

$$v_1 = 19.79 \text{ km/s} \qquad v_2 = 5.12 \text{ km/s}$$

b) Let us call α the lab scattering angle of the neutron, then from the sine theorem we have

$$\frac{mv_1}{\sin\psi} = \frac{2mv_2}{\sin\alpha} \quad \Rightarrow \quad \sin\alpha = 2\frac{v_2}{v_1}\sin\psi$$

$\Rightarrow \alpha = 74.84°$ and $\alpha = 5.16°$

c) From a) we see that $\cos\psi = \dfrac{4v_0^2 + 4v_2^2 - v_1^2}{8v_0 v_2}$

$$= \frac{2v_0^2 + 6v_2^2}{8v_0 v_2} \geq \frac{\sqrt{3}}{2} \quad \Rightarrow \quad |\psi| \leq 30^0 \Rightarrow \psi_{\max} = 30°$$

9-23. Conservation of momentum requires v_f to be in the same direction as u_1 (component of v_f $\perp$ to u_1 must be zero).

$$p_i = m_1 u_1$$

$$p_f = (m_1 + m_2) v_f$$

$$p_i = p_f \rightarrow v_f = \frac{m_1}{m_1 + m_2} u_1$$

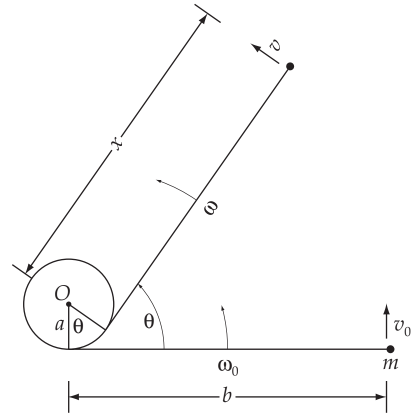
The fraction of original kinetic energy lost is

$$\frac{K_i - K_f}{K_i} = \frac{\dfrac{1}{2} m_1 u_1^2 - \dfrac{1}{2}(m_1 + m_2)\dfrac{m_1^2 u_1^2}{(m_1 + m_2)^2}}{\dfrac{1}{2} m_1 u_1^2}$$

$$= \frac{m_1 - \dfrac{m_1^2}{m_1 + m_2}}{m_1}$$

$$= \boxed{\frac{m_2}{m_1 + m_2}}$$

9-24.



The energy of the system is, of course, conserved, and so we have the following relation involving the instantaneous velocity of the particle:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 \tag{1}$$

The angular momentum about the center of the cylinder is not conserved since the tension in the string causes a torque. Note that although the velocity of the particle has both radial and angular components, there is only one independent variable, which we chose to be θ. Here $\omega = \dot{\theta}$ is the angular velocity of the particle about the point of contact, which also happens to be the rate at which the point of contact is rotating about the center of the cylinder. Hence we may write

$$v_0 = \omega_0 b; \qquad v = \omega_0 (b - a\theta) \tag{2}$$

From (1) and (2), we can solve for the angular velocity after turning through an angle θ

$$\omega = \frac{\omega_0}{1 - \dfrac{a}{b}\theta} \tag{3}$$

The tension will then be (look at the point of contact)

$$T = m\omega^2 (b - a\theta) = m\omega_0 \omega b \tag{4}$$

9-25. The best elements are those that will slow down the neutrons as much as possible. In a collision between m_1 (the neutron) and m_2 (moderator atom), we would thus want to minimize T_1 (kinetic energy of the neutron after the collision); or alternatively, maximize T_2 (kinetic energy of the moderator atom after the collision). From Eq. (9.88)

$$\frac{T_2}{T_0} = \frac{4m_1 m_2}{(m_1 + m_2)^2} \cos^2 \zeta$$

Since one cannot control the angle ζ, we want to maximize the function

$$f = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

with respect to m_2. (m_1 = constant)

$$\frac{df}{dm_2} = \frac{m_1\left(m_1^2 - m_2^2\right)}{\left(m_1 + m_2\right)^4} = 0 \text{ when } m_1 = m_2$$

By evaluating $\left.\frac{d^2 f}{dm_2^2}\right]_{m_1=m_2}$ one can show that the equilibrium point is a maximum. Thus, T_2 is a maximum for $m_1 = m_2$. Back to reactors, one would want elements whose mass is as close as possible to the neutron mass (thus, as light as possible). Naturally, there are many other factors to consider besides mass, but in general, the lower the mass of the moderator, the more energy is lost per collision by the neutrons.

9-26. The internal torque for the system is

$$\mathbf{N} = \mathbf{r}_1 \times \mathbf{f}_{12} + \mathbf{r}_2 \times \mathbf{f}_{21} \tag{1}$$

where $\mathbf{f}_{12}$ is the force acting on the first particle due to the second particle. Now

$$\mathbf{f}_{21} = -\mathbf{f}_{12} \tag{2}$$

Then,

$$\begin{aligned}\mathbf{N} &= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \mathbf{f}_{12} \\ &= k\left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \left[\left(\mathbf{r}_2 - \mathbf{r}_1\right) - \frac{r}{v_0}\left(\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1\right)\right] \\ &= \frac{kr}{v_0}\left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \left(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2\right) \end{aligned} \tag{3}$$

This is not zero in general because $\left(\mathbf{r}_1 - \mathbf{r}_2\right)$ and $\left(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2\right)$ are not necessarily parallel. The internal torque vanishes only if the internal force is directed along the line joining two particles. The system is not conservative.

9-27. The equation for conservation of p_y in the lab system is (see fig. 9-10c):

$$0 = m_1 v_1 \sin\psi - m_2 v_2 \sin\zeta$$

Thus

$$\sin\zeta = \frac{m_1 v_1}{m_2 v_2} \sin\psi$$

or

$$\boxed{\sin\zeta = \sqrt{\frac{m_1 T_1}{m_2 T_2}} \sin\psi}$$

9-28. Using the notation from the chapter:

$$m_1: \quad T_i = T_0, \quad T_f = T_1$$

$$m_2: \quad T_i = 0; \quad T_f = T_2$$

Thus

$$T_0 = T_1 + T_2 \quad or \quad 1 = \frac{T_1}{T_0} + \frac{T_2}{T_0} \tag{1}$$

If we want the kinetic energy loss for m_1 to be a maximum, we must minimize $\dfrac{T_1}{T_0}$ or, equivalently, maximize $\dfrac{T_2}{T_0}$.

From Eq. (9.88):

$$\frac{T_2}{T_0} = \frac{4m_1 m_2}{\left(m_1 + m_2\right)^2} \cos^2 \zeta$$

To maximize this, $\zeta = 0$ (it can't $= 180°$).

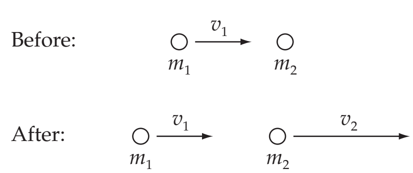
$$\frac{T_2}{T_0} = \frac{4m_1 m_2}{\left(m_1 + m_2\right)^2}$$

The kinetic energy loss for m_1 is $T_0 - T_1$. The fraction of kinetic energy loss is thus

$$\frac{T_0 - T_1}{T_0} = 1 - \frac{T_1}{T_0} = \frac{T_2}{T_0} \quad \text{(from (1))}$$

$$\boxed{\left.\frac{T_0 - T_1}{T_0}\right|_{\max} = \frac{4m_1 m_2}{\left(m_1 + m_2\right)^2}}$$

$\zeta = 0$ implies $\psi = 0, 180°$ (conservation of p_y). So the reaction is as follows



$$p_x: \quad m_1 v = m_1 v_1 + m_2 v_2$$

$$E: \quad \frac{1}{2} m_1 v^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Solving for v_1 gives $v_1 = \dfrac{m_1 - m_2}{m_1 + m_2} v$

So

m_2 travels in $+\,x$ direction

$$m_1 \text{ travels in } \left[\begin{array}{l} +x \text{ direction if } m_1 > m_2 \\ -x \text{ direction if } m_1 < m_2 \end{array} \right.$$

9-29. From Eq. (9.69)

$$\tan \psi = \frac{\sin \theta}{\cos \theta + (m_1/m_2)}$$

From Eq. (9.74)

$$\theta = \pi - 2\zeta$$

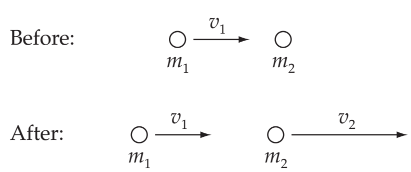
Substituting gives

$$\tan \psi = \frac{\sin(\pi - 2\zeta)}{(m_1/m_2) + \cos(\pi - 2\zeta)}$$

or

$$\boxed{\tan \psi = \frac{\sin(2\zeta)}{(m_1/m_2) - \cos(2\zeta)}}$$

9-30.



a)

$$\Delta p_y = (0.06 \text{ kg})\left[16 \text{ m/s } \cos 15° - (-8 \text{ m/s}) \cos 45°\right]$$

$$= 1.27 \text{ N} \cdot \text{sec}$$

$$\Delta p_x = (0.06 \text{ kg})\left[16 \text{ m/s } \cos 15° (-8 \text{ m/s}) \sin 45°\right]$$

$$= -0.09 \text{ N} \cdot \text{sec}$$

The impulse **P** is the change in momentum.

So

$$\boxed{\mathbf{P} = (-0.09\mathbf{x} + 1.27\mathbf{y}) \text{ N} \cdot \text{sec}}$$

b)

$$\mathbf{P} = \int \overline{\mathbf{F}}\, dt = \mathbf{F}\, \Delta t$$

So

$$\boxed{\overline{\mathbf{F}} = -(9\mathbf{x} + 127\mathbf{y}) \text{ N}}$$

9-31. From Eq. (9.69)

$$\tan \psi = \frac{\sin \theta}{\dfrac{m_1}{m_2} - \cos \theta}$$

From Eq. (9.74)

$$\theta = \pi - \phi$$

Substituting gives

$$\boxed{\tan \psi = \frac{\sin \phi}{\dfrac{m_1}{m_2} - \cos \phi}}$$

9-32.

$$p_i = mu_1$$

$$p_f = mv_1 + 2mv_2$$

Conservation of momentum gives

$$u_1 = v_1 + 2v_2 \quad or \quad v_1 = u_1 - 2v_2$$

$$\begin{aligned}\Delta T &= \frac{1}{2} mu_1^2 - \frac{1}{2} mv_1^2 - mv_2^2 \\ &= \frac{1}{2} mu_1^2 - \frac{1}{2} m\left(u_1^2 - 4u_1 v_2 + 4v_2^2\right) - mv_2^2 \\ &= 2mu_1 v_2 - 3mv_2^2\end{aligned}$$

$$\frac{d(\Delta T)}{dv_2} = 0 \text{ implies } 2u_1 = 6v_2 \quad \text{or} \quad v_2 = \frac{u_1}{3}$$

$$\left[\frac{d^2(\Delta T)}{dv_2^2} < 0, \text{ so this is a maximum}\right]$$

$$v_1 = u_1 - 2v_2 = \frac{u_1}{3}$$

$$\boxed{v_1 = v_2 = \frac{u_1}{3}}$$

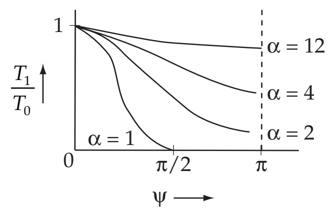
9-33. From Eq. (9.87b) in the text, we have

$$\frac{T_1}{T_0} = \frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi + \sqrt{\left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi}\right]^2$$

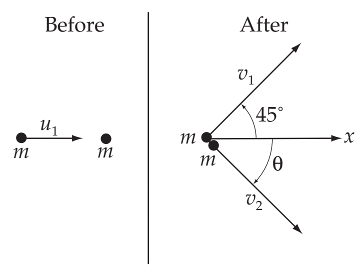
$$= \frac{1}{\left[1+\dfrac{m_2}{m_1}\right]^2}\left[\cos^2\psi + \left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi + 2\cos\psi\sqrt{\left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi}\right]$$

Substituting $m_2/m_1 \equiv \alpha$ and $\cos\psi \equiv y$ we have

$$\boxed{\frac{T_1}{T_0} = (1+\alpha)^{-2}\left[2y^2+\alpha^2-1+2y\sqrt{\alpha^2+y^2-1}\right]} \qquad (1)$$



9-34.



Cons. of p_z: $\quad mu_1 = mv_1\cos 45° + mv_2\cos\theta \qquad (1)$

Cons. of p_y: $\quad 0 = mv_1\sin 45° - mv_2\sin\theta \qquad (2)$

Cons. of energy (elastic collision)

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 \qquad (3)$$

Solve (1) for $\cos\theta$:

$$\cos\theta = \frac{u_1 - v_1/\sqrt{2}}{v_2}$$

Solve (2) for $\sin\theta$:

$$\sin\theta = \frac{v_1}{\sqrt{2}\,v_2}$$

Substitute into $\cos^2\theta + \sin^2\theta = 1$, simplify, and the result is

$$u_1^2 = v_2^2 - v_1^2 + \sqrt{2}\,u_1 v_1$$

Combining this with (3) gives

$$2v_1^2 = \sqrt{2}\,u_1 v_1$$

We are told $v_1 \neq 0$, hence

$$\boxed{v_1 = u_1/\sqrt{2}}$$

Substitute into (3) and the result is

$$\boxed{v_2 = u_1/\sqrt{2}}$$

Since $v_1 = v_2$, (2) implies

$$\boxed{0 = 45°}$$

9-35. From the following two expressions for T_1/T_0,

$$\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2} \qquad \text{Eq. (9.82)}$$

$$\frac{T_1}{T} = \frac{m_1^2}{(m_1 + m_2)^2}\left[\cos\psi \pm \sqrt{\left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi}\,\right]^2 \qquad \text{Eq. (9.87b)}$$

we can find the expression for the final velocity v_1 of m_1 in the lab system in terms of the scattering angle ψ:

$$v_1 = \frac{m_1 u_1}{m_1 + m_2}\left[\cos\psi \pm \sqrt{\left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi}\,\right] \tag{1}$$

If time is to be constant on a certain surface that is a distance *r* from the point of collision, we have

$$r = v_1 t_0 \tag{2}$$

Thus,

$$r = \frac{m_1 u_1 t_0}{m_1 + m_2}\left[\cos\psi \pm \sqrt{\left[\frac{m_2}{m_1}\right]^2 - \sin^2\psi}\,\right] \tag{3}$$

This is the equation of the required surface. Let us consider the following cases:

i) $m_2 = m_1$:

$$r = \frac{u_1 t_0}{2}\left[\cos\psi \pm \sqrt{1-\sin^2\psi}\right] = u_1 t_0 \cos\psi \tag{4}$$

(The possibility $r = 0$ is uninteresting.)

ii) $m_2 = 2m_1$:

$$r = \frac{u_1 t_0}{3}\left[\cos\psi \pm \sqrt{4-\sin^2\psi}\right] \tag{5}$$

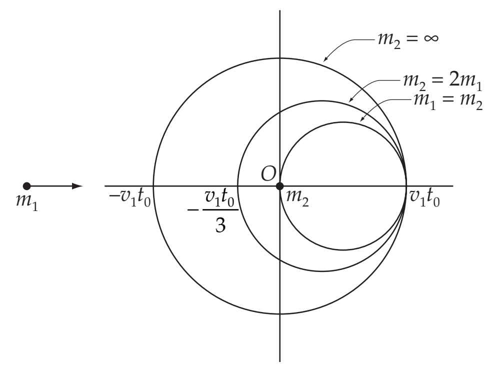
iii) $m_2 = \infty$: Rewriting (3) as

$$r = \frac{m_1 u_1 t_0}{1+\dfrac{m_1}{m_2}}\left[\frac{\cos\psi}{m_2} \pm \sqrt{\left[\frac{1}{m_1}\right]^2 - \frac{\sin^2\psi}{m_2^2}}\right] \tag{6}$$

and taking the limit $m_2 \to \infty$, we find

$$r = u_1 t_0 \tag{7}$$

All three cases yield spherical surfaces, but with the centers displaced:



This result is useful in the design of a certain type of nuclear detector. If a hydrogenous material is placed at 0 then for neutrons incident on the material, we have the case $m_1 = m_2$. Therefore, neutrons scattered from the hydrogenous target will arrive on the surface A with the same time delay between scattering and arrival, independent of the scattering angle. Therefore, a coincidence experiment in which the time delay is measured can determine the energies of the incident neutrons. Since the entire surface A can be used, a very efficient detector can be constructed.

9-36. Since the initial kinetic energies of the two particles are equal, we have

$$\frac{1}{2}m_1 u_1^2 = \frac{1}{2}m_2 u_2^2 = \frac{1}{2}\alpha^2 m_2 u_1^2 \tag{1}$$

or,

$$\frac{m_1}{m_2} = \alpha^2 \tag{2}$$

Now, the kinetic energy of the system is conserved because the collision is *elastic*. Therefore,

$$\frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_2 u_2^2 = m_1 u_1^2 = \frac{1}{2} m_2 v_2^2 \tag{3}$$

since $v_1 = 0$. Momentum is also conserved, so we can write

$$m_1 u_1 + m_2 u_2 = \left(m_1 + \alpha m_2\right) u_1 = m_2 v_2 \tag{4}$$

Substituting the second equality in (4) into (3), we find

$$m_1 u_1^2 = \frac{1}{2} m_2 \left[\frac{m_1 + \alpha m_2}{m_2}\right]^2 u_1^2 \tag{5}$$

or,

$$m_1 = \frac{1}{2} m_2 \left[\frac{m_1}{m_2} + \alpha\right]^2 \tag{6}$$

Using $m_1/m_2 = \alpha^2$, (6) becomes

$$2\alpha^2 = \left(\alpha^2 + \alpha\right)^2 \tag{7}$$

solving for α, we obtain

$$\alpha = -1 \mp \sqrt{2}; \qquad \alpha^2 = 3 \mp 2\sqrt{2} \tag{8}$$

This gives us

$$\frac{m_1}{m_2} = 3 \pm 2\sqrt{2}; \quad \frac{u_2}{u_1} = -\left(1 \pm \sqrt{2}\right) \text{ with } \left[\begin{array}{l} + : \alpha < 0 \\ - : \alpha > 0 \end{array}\right. \tag{9}$$

9-37.

$$\text{Impulse} = \int F\, dt$$

$$= \int_{t=0}^{6\times10^{-3}} \left(360 - 10^7\, t^2\right) dt$$

$$= \left(360 \cdot 6 \times 10^{-3} - \frac{10^7}{3} \cdot \left(6 \times 10^{-3}\right)^3\right) \text{ N} \cdot \text{s}$$

$$\boxed{\text{Impulse} = 1.44 \, \frac{\text{kg m}}{\text{s}}}$$

Since the initial velocity is zero, $v_f = \Delta v$

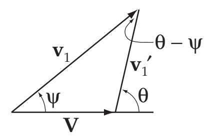
$$\text{Impulse} = \Delta P = m\Delta v$$

So

$$v_f = \frac{1.44\ \dfrac{\text{kg m}}{\text{s}}}{0.003\ \text{kg}}$$

$$\boxed{v_{\text{muzzle}} = 480\ \frac{\text{m}}{\text{s}}}$$

9-38.



$$\left.\begin{aligned} T_0 &= \frac{1}{2} m_1 u_1^2 \\ T_1 &= \frac{1}{2} m_1 v_1^2 \end{aligned}\right] \tag{1}$$

Thus,

$$\frac{T_1}{T_0} = \frac{v_1^2}{u_1^2} \tag{2}$$

Now, from the diagram above, we have

$$v_1 = V \cos\psi + v_1' \cos(\theta - \psi) \tag{3}$$

Using Eq. (9.68) in the text, this becomes

$$v_1 = V \left[\cos\psi + \frac{m_2}{m_1} \cos(\theta - \psi) \right] \tag{4}$$

Thus,

$$\frac{v_1^2}{u_1^2} = \frac{V^2}{u_1^2} \left[\cos\psi + \frac{m_2}{m_1} (\theta - \psi) \right]^2 \tag{5}$$

Using Eq. (9.84) in the text,

$$\frac{V}{u_1} = \frac{m_1}{m_1 + m_2} \tag{6}$$

Therefore, we find

$$\frac{T_1}{T_0} = \frac{m_1^2}{(m_1 + m_2)^2} \left[\cos\psi + \frac{\cos(\theta - \psi)}{(m_1/m_2)} \right]^2 \tag{7}$$

If we define

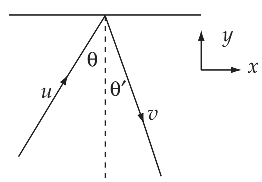
$$S \equiv \cos\psi + \frac{\cos(\theta - \psi)}{(m_1/m_2)} \tag{8}$$

we have

$$\boxed{\frac{T_1}{T_0} = \frac{m_1^2}{(m_1 + m_2)^2} \times S^2} \tag{9}$$

as desired.

9-39.



As explained in Section 9.8, the component of velocity parallel to the wall is unchanged. So

$$v_x = u\sin\theta$$

v_y is given by

$$\varepsilon = \frac{|v_y|}{|u_y|} = \frac{|v_y|}{u\cos\theta}$$

or

$$|v_y| = \varepsilon\, u\cos\theta$$

Thus

$$v = \left[u^2\sin^2\theta + \varepsilon^2 u^2\cos^2\theta\right]^{1/2}$$

$$\boxed{v = u\left[\sin^2\theta + \varepsilon^2\cos^2\theta\right]^{1/2}}$$

$$\tan\theta' = \frac{u\sin\theta}{\varepsilon\, u\cos\theta}$$

or

$$\boxed{\theta' = \tan^{-1}\left[\frac{1}{\varepsilon}\tan\theta\right]}$$

9-40. Because of the string, m_2 is constrained to move in a circle of radius a. Thus, initially, m_2 will move straight up (taken to be the y direction). Newton's rule applies to the velocity component along u_1. The perpendicular component of velocity (which is zero) is unchanged. Thus m_1 will move in the original direction after the collision.

From conservation of p_y we have

$$m_1 u_1 \sin\alpha = m_1 v_1 \sin\alpha + m_2 v_2 \tag{1}$$

From Newton's rule we have

$$\varepsilon = \frac{v_2 \cos(90° - \alpha) - v_1}{u_1}$$

or

$$v_1 = v_2 \sin\alpha - \varepsilon u_1 \tag{2}$$

Substituting (2) into (1) and solving for v_2 gives

$$\boxed{v_2 = \frac{(\varepsilon + 1) m_1 u_1 \sin\alpha}{m_1 \sin^2\alpha + m_2} \qquad \text{straight up}}$$

(2) then gives

$$\boxed{v_1 = \frac{u_1\left(m_1 \sin^2\alpha - \varepsilon\, m_2\right)}{m_1 \sin^2\alpha + m_2} \qquad \text{along } \mathbf{u}_1}$$

9-41. Using $y = v_0 t - \frac{1}{2} g t^2$ and $v = v_0 - gt$, we can get the velocities before and after the collision:

Before: $\quad u_1 = -g t_1 \quad$ where $\quad h_1 = \frac{1}{2} g t_1^2$

So $$u_1 = -g\sqrt{\frac{2h_1}{g}} = -\sqrt{2gh_1}$$

After: $\quad 0 = v_0 - g t_2 \quad$ or $\quad t_2 = v_0/g$

$$h_2 = v_0 t_2 - \frac{1}{2} g t_2^2$$

$$= \frac{v_0^2}{g} - \frac{1}{2}\frac{v_0^2}{g} \quad \text{or} \quad v_0 = \sqrt{2gh_2}$$

So $$v_1 = \sqrt{2gh_2}$$

Thus

$$\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|} = \frac{\sqrt{2gh_2}}{\sqrt{2gh_1}}$$

$$\boxed{\varepsilon = \sqrt{\frac{h_2}{h_1}}}$$

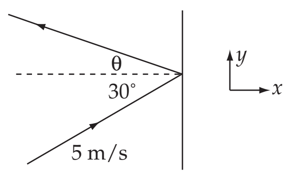
$$T_{\text{lost}} = T_i - T_f$$

$$\text{Fraction lost} = \frac{T_i - T_f}{T_i}$$

$$= \frac{u_1^2 - v_1^2}{u_1^2} = \frac{h_1 - h_2}{h_1} = 1 - \frac{h_2}{h_1}$$

$$\boxed{\frac{T_i - T_f}{T_i} = 1 - \varepsilon^2}$$

9-42.



As explained in Section 9.8, the velocity component in the y-direction is unchanged.

$$v_y = u_y = \left[5\,\frac{\text{m}}{\text{s}}\right] \sin 30° = 2.5\ \text{m/s}$$

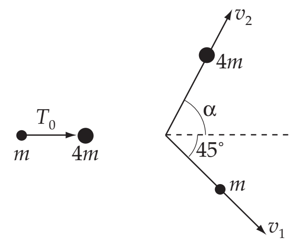
For the x component we have

$$0.8 = \frac{|v_x|}{|u_x|} = \frac{v_x}{\left[5\,\frac{\text{m}}{\text{s}}\right]\cos 30°} = \frac{v_x}{\frac{\sqrt{3}}{2}\,5\,\frac{\text{m}}{\text{s}}}$$

$$|v_x| = 2\sqrt{3}\,\frac{\text{m}}{\text{s}}$$

$$\boxed{\begin{array}{l} v_f = \frac{1}{2}\sqrt{73} \simeq 4.3\ \text{m/s} \\[2ex] \theta = \tan^{-1}\frac{2.5}{2\sqrt{3}} \simeq 36° \end{array}}$$

9-43.



Conservation of p_x gives

$$\sqrt{2mT_0} = 4mv_2 \cos\alpha + \frac{1}{\sqrt{2}} mv_1$$

or

$$\cos\alpha = \frac{\sqrt{2mT_0} - \frac{1}{\sqrt{2}} mv_1}{4mv_2}$$

Conservation of p_y gives

$$0 = \frac{1}{\sqrt{2}} mv_1 - 4mv_2 \sin\alpha$$

or

$$\sin\alpha = \frac{v_1}{4\sqrt{2}\, v_2}$$

Substituting into $\sin^2\alpha + \cos^2\alpha = 1$ gives

$$1 = \frac{v_1^2}{32v_2^2} + \frac{2mT_0 + \frac{1}{2} m^2 v_1^2 - 2\sqrt{mT_0}\, mv_1}{16\, m^2\, v_2^2}$$

Simplifying gives

$$v_2^2 = \frac{v_1^2}{16} + \frac{T_0}{8m} - \frac{v_1\sqrt{T_0 m}}{8m} \tag{1}$$

The equation for conservation of energy is

$$T_0 - \frac{T_0}{6} = \frac{1}{2} mv_1^2 + \frac{1}{2}(4m)v_2^2$$

or

$$5T_0 = 3mv_1^2 + 12mv_2^2 \tag{2}$$

Substituting (1) into (2) gives a quadratic in v_1:

$$15mv_1^2 - 6\sqrt{T_0 m}\, v_1 - 14T_0 = 0$$

Using the quadratic formula (taking the positive sign since $v_1 > 0$) gives:

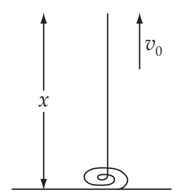
$$v_1 = 1.19\sqrt{\frac{T_0}{m}}$$

Substituting this into the previous expressions for cos α and sin α and dividing gives

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = 1.47$$

$$\boxed{\text{Thus } \alpha\text{, the recoil angle of the helium, is } 55.8°.}$$

9-44.



$$F_{\text{grav}} = mg = \mu xg \text{ where } \mu = \text{mass/length}$$

$$F_{\text{impulse}} = \dot{m}v + m\dot{v} + \dot{m}v_0\text{, since } v = v_0, \dot{v} = 0.$$

We have

$$\dot{m} = \frac{d}{dt}(\mu x) = \mu\frac{dx}{dt} = \mu v_0$$

So the total force is

$$F(x) = \mu xg + \mu v_0^2$$

We want $F(x = a)$

$$F(a) = \mu ag + \mu v_0^2$$

or

$$\boxed{F = \mu ag\left[1 + \frac{v_0^2}{ag}\right]}$$

9-45. Since the total number of particles scattered into a unit solid angle must be the same in the lab system as in the CM system [cf. Eq. (9.124) in the text],

$$\sigma(\theta)\,2\pi\sin\theta\,d\theta = \sigma(\psi)\cdot 2\pi\sin\psi\,d\psi \qquad (1)$$

Thus,

$$\sigma(\theta) = \sigma(\psi)\frac{\sin\psi}{\sin\theta}\frac{d\psi}{d\theta} \qquad (2)$$

The relation between θ and ψ is given by Eq. (9.69), which is

$$\tan\psi = \frac{\sin\theta}{\cos\theta + x} \tag{3}$$

where $x = m_1/m_2$. Using this relation, we can eliminate ψ from (2):

$$\sin\psi = \frac{1}{\sqrt{1+\dfrac{1}{\tan^2\psi}}} = \frac{1}{\sqrt{1+\dfrac{(\cos\theta + x)^2}{\sin^2\theta}}} = \frac{\sin\theta}{\sqrt{1+2x\cos\theta + x^2}} \tag{4}$$

$$\frac{d\psi}{d\theta} = \frac{d\psi}{d(\tan\psi)}\frac{d(\tan\psi)}{d\theta} = \cos^2\psi\,\frac{\cos\theta(\cos\theta + x) + \sin^2\theta}{(\cos\theta + x)^2} \tag{5}$$

Since $\cos^2\psi = \dfrac{1}{1+\tan^2\psi}$, (5) becomes

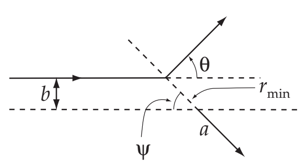
$$\frac{d\psi}{d\theta} = \frac{1}{1+\dfrac{\sin^2\theta}{(\cos\theta + x)^2}}\,\frac{1+x\cos\theta}{(\cos\theta + x)^2} = \frac{1+x\cos\theta}{1+2x\cos\theta + x^2} \tag{6}$$

Substituting (4) and (6) into (2), we find

$$\boxed{\sigma(\theta) = \sigma(\psi)\,\frac{1+x\cos\theta}{\left(1+2x\cos\theta + x^2\right)^{3/2}}} \tag{7}$$

9-46. The change in angle for a particle of mass μ moving in a central-force field is [cf. Eq. (9.121)]. Let ψ = capital θ here.

$$\Delta\psi = \int_{r_{\min}}^{r_{\max}} \frac{\left(\ell/r^2\right)dr}{\sqrt{2\mu\left(E - U - \ell^2/2\mu r^2\right)}} \tag{1}$$



In the scattering from an impenetrable sphere, $r_{\min}$ is the radius of that sphere. Also, we can see from the figure that $\theta = \pi - 2\psi$.

For $r > r_{\min}$, $U = 0$. Thus (1) becomes

$$\Delta\psi = \int_a^{\infty} \frac{\left(\ell/r^2\right)dr}{\sqrt{2\mu E - \ell^2/r^2}} \tag{2}$$

Substituting

$$b\sqrt{2\mu T_0'} = \ell; \qquad E = T_0' \tag{3}$$

(2) becomes

$$\Delta\psi = \int_{\alpha}^{\infty} \frac{dr}{r\sqrt{\frac{r^2}{b^2}-1}} \tag{4}$$

This integral can be solved by using Eq. (E. 10b), Appendix E:

$$\psi = \sin^{-1}\left[\frac{-2}{|r|\sqrt{\frac{4}{b^2}}}\right]_a^{\infty} \tag{5}$$

Thus,

$$\sin\psi = \frac{b}{a} \tag{6}$$

Therefore, we can find the relation between θ and b by substituting $(\theta = \pi - 2\psi)$ into (6). We have

$$b = a\cos\frac{\theta}{2} \tag{7}$$

Now, the differential cross section is given by Eq. (9.120):

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{8}$$

From (7), we have

$$\sigma(\theta) = \frac{1}{\sin\theta}a\cos\frac{\theta}{2}\times\frac{a}{2}\sin\frac{\theta}{2} = \frac{a^2}{4} \tag{9}$$

Total cross section is given by

$$\sigma_t = \int \sigma(\theta)\, d\Omega = \frac{a^2}{4}\cdot 4\pi \tag{10}$$

so that

$$\boxed{\sigma_t = \pi a^2} \tag{11}$$

9-47. The number of recoil particles scattered into unit solid angle in each of the two systems, lab and CM, are the same. Therefore,

$$\sigma(\phi)\sin\phi\, d\phi = \sigma(\zeta)\sin\zeta\, d\zeta \tag{1}$$

where ϕ and ζ are the CM and lab angles, respectively, of the recoil particle. From (1) we can write [cf. Eq. (9.125) in the text]

$$\frac{\sigma(\phi)}{\sigma(\zeta)} = \frac{\sin\zeta}{\sin\phi}\frac{d\zeta}{d\phi} \tag{2}$$

Now, in general, $\phi = 2\zeta$ [see Eq. (9.74)]. Hence,

$$\frac{\sin\zeta}{\sin\phi} = \frac{\sin\zeta}{\sin 2\zeta} = \frac{1}{2\cos\zeta} \tag{3}$$

and

$$\frac{d\zeta}{d\phi} = \frac{1}{2} \tag{4}$$

Using (3) and (4) in (2), we have

$$\frac{\sigma(\phi)}{\sigma(\zeta)} = \frac{1}{4\cos\zeta} \tag{5}$$

For $m_1 = m_2$, the Rutherford scattering cross section is [Eq. (9.141)]

$$\sigma(\theta) = \frac{k^2}{4T_0^2} \times \frac{1}{\sin^4(\theta/2)} \tag{6}$$

Also for this case, we have [Eqs. (9.71) and (9.75)]

$$\left.\begin{aligned} \psi &= \frac{\theta}{2} \\ \psi &= \frac{\pi}{2} - \zeta \end{aligned}\right] \tag{7}$$

Hence,

$$\sin\frac{\theta}{2} = \sin\psi = \sin\left(\frac{\pi}{2} - \zeta\right) = \cos\zeta \tag{8}$$

and since the CM recoil cross section $\sigma(\phi)$ is the same as the CM scattering cross section $\sigma(\theta)$, (6) becomes

$$\sigma(\phi) = \frac{k^2}{4T_0^2} \times \frac{1}{\cos^4\zeta} \tag{9}$$

Using (5) to express $\sigma(\zeta)$, we obtain

$$\sigma(\zeta) = \sigma(\phi) \times 4\cos\zeta \tag{10}$$

or,

$$\boxed{\sigma(\zeta) = \frac{k^2}{T_0^2} \times \frac{1}{\cos^3\zeta}} \tag{11}$$

9-48. In the case $m_1 \gg m_2$, the scattering angle ψ for the incident particle measured in the lab system is very small for all energies. We can then anticipate that $\sigma(\psi)$ will rapidly approach zero as ψ increases.

Eq. (9.140) gives the Rutherford cross section in terms of the scattering angle in the CM system:

$$\sigma_{\text{CM}}(\theta) = \frac{k^2}{(4T_0')^2} \frac{1}{\sin^4(\theta/2)} \tag{1}$$

From Eq. (9.79) we see that for $m_1 \gg m_2$,

$$T_0' = \frac{m_2}{m_1 + m_2} T_0 \cong \frac{m_2}{m_1} T_0 \tag{2}$$

Furthermore, from Eq. (9.69),

$$\tan\psi = \frac{\sin\theta}{\dfrac{m_1}{m_2} + \cos\theta} \cong \frac{m_2}{m_1} \sin\theta \tag{3}$$

and therefore, since ψ is expected to be small for all cases of interest,

$$\sin\theta \cong \frac{m_1}{m_2} \tan\psi \cong \frac{m_1}{m_2}\psi \tag{4}$$

Then,

$$\cos\theta = \sqrt{1 - \left[\frac{m_1}{m_2}\psi\right]^2} \tag{5}$$

and

$$\sin^2(\theta/2) = \frac{1}{2}\left[1 - \sqrt{1 - \left[\frac{m_1}{m_2}\psi\right]^2}\,\right] \tag{6}$$

(Notice that $\psi \ll 1$, but since $m_1 \gg m_2$, the quantity $m_1\psi/m_2$ is not necessarily small compared to unity.)

With the help of (2) and (6), we can rewrite the CM cross section in terms of ψ as

$$\sigma_{\text{CM}}(\psi) = \left[\frac{m_1 k}{2m_2 T_0}\right]^2 \frac{1}{\left[1 - \sqrt{1 - \left[\dfrac{m_1}{m_2}\psi\right]^2}\,\right]^2} \tag{7}$$

According to Eq. (9.129),

$$\sigma_{\text{LAB}}(\psi) = \sigma_{\text{CM}}(\theta)\frac{\left[\dfrac{m_1}{m_2}\cos\psi + \sqrt{1-\left[\dfrac{m_1}{m_2}\sin\psi\right]^2}\,\right]^2}{\sqrt{1-\left[\dfrac{m_1}{m_2}\sin\psi\right]^2}} \tag{8}$$

We can compute $\sigma_{\text{LAB}}(\psi)$ with the help of (7) and the simplifications introduced in the right-hand side of (8) by the fact that $\psi \ll 1$:

$$\sigma_{\text{LAB}}(\psi) \cong \left[\frac{m_1 k}{2m_2 T_0}\right]^2 \frac{\left[\dfrac{m_1}{m_2} + \sqrt{1-\left[\dfrac{m_1}{m_2}\psi\right]^2}\,\right]^2}{\left[1-\sqrt{1-\left[\dfrac{m_1}{m_2}\psi\right]^2}\,\right]^2 \sqrt{1-\left[\dfrac{m_1}{m_2}\psi\right]^2}} \tag{9}$$

and so,

$$\boxed{\sigma_{\text{LAB}}(\psi) \cong \frac{\left(m_1^2 k/2m_2^2 T_0\right)^2}{\left[1-\sqrt{1-\left[\dfrac{m_1}{m_2}\psi\right]^2}\,\right]^2 \sqrt{1-\left[\dfrac{m_1}{m_2}\psi\right]^2}}} \tag{10}$$

This expression shows that the cross section has a second-order divergence at $\psi = 0$. For values of $\psi > m_2/m_1$, (9) gives complex values for σ_{lab}. This result is due to the approximations involved in its derivation, making our result invalid for angles larger than m_2/m_1.

9-49. The differential cross section for Rutherford scattering in the CM system is [cf. Eq. (9.140) in the text]

$$\sigma(\theta) = \frac{k^2}{16T_0'^2}\frac{1}{\sin^4\dfrac{\theta}{2}} \tag{1}$$

where [cf. Eq. (9.79)]

$$T_0' = \frac{m_2}{m_1+m_2}T_0 \tag{2}$$

Thus,

$$\sigma(\theta) = \frac{k^2}{16T_0^2} \frac{1}{\sin^4 \dfrac{\theta}{2}} \left[\frac{m_1 + m_2}{m_2} \right]^2$$

$$= \frac{k^2}{16T_0^2} \frac{1}{\sin^4 \dfrac{\theta}{2}} \left[1 + \frac{m_1}{m_2} \right]^2 \tag{3}$$

Since $m_1/m_2 \ll 1$, we expand

$$\left[1 + \frac{m_1}{m_2} \right]^2 \cong 1 + 2\frac{m_1}{m_2} + \ldots \tag{4}$$

Thus, to the first order in m_1/m_2, we have

$$\boxed{\sigma(\theta) = \frac{k^2}{16T_0^2} \frac{1}{\sin^4 \dfrac{\theta}{2}} \left[1 + 2\frac{m_1}{m_2} \right]} \tag{5}$$

This result is the same as Eq. (9.140) except for the correction term proportional to m_1/m_2.

9-50. The potential for the given force law is

$$U(r) = \frac{k}{2r^2} \tag{1}$$

First, we make a change of variable, $z = 1/r$. Then, from Eq. (9.123), we can write

$$\theta = \int_0^{z_{\max}} \frac{b\, dz}{\sqrt{1 - \left[b^2 + \dfrac{k}{mu_0^2} \right] z^2}}$$

$$= \frac{b}{2\sqrt{b^2 + \dfrac{k}{mu_0^2}}} \sin^{-1} \frac{z}{z_{\max}} \Bigg|_0^{z_{\max} = \left[b^2 + \frac{k}{mu_0^2} \right]^{-1/2}} \tag{2}$$

$$= \frac{\pi b}{2\sqrt{b^2 + \dfrac{k}{mu_0^2}}}$$

Solving (2) for $b = b(\theta)$,

$$b\theta = \sqrt{\frac{k}{mu_0^2}} \frac{2\theta}{\sqrt{\pi^2 - 4\theta^2}} \tag{3}$$

According to Fig. 9-22 and Eq. (9.122),

$$\theta = \frac{1}{2}(\pi - \theta) \tag{4}$$

so that $b(\theta)$ can be rewritten as (θ):

$$b(\theta) = \sqrt{\frac{k}{mu_0^2}}\frac{\pi - \theta}{\sqrt{\theta(2\pi - \theta)}} \tag{5}$$

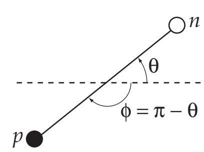
The differential cross section can now be computed from Eq. (9.120):

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{6}$$

with the result

$$\boxed{\sigma(\theta) = \frac{k\pi^2(\pi - \theta)}{mu_0^2\theta^2(2\pi - \theta)^2 \sin\theta}} \tag{7}$$

9-51.



In the CM system, whenever the neutron is scattered through the angle θ, the proton recoils at the angle $\phi = \pi - \theta$. Thus, the neutron scattering cross section is equal to the recoil cross section at the corresponding angles:

$$\frac{dN_n}{d\Omega(\theta)} = \frac{dN_p}{d\Omega(\phi)} \tag{1}$$

Thus,

$$\frac{dN_n}{d\Omega(\theta)} = \frac{dN_p}{dT_p}\left|\frac{dT_p}{d\Omega(\phi)}\right| \tag{2}$$

where dN_p/dT_p is the energy distribution of the recoil protons. According to experiment, dN_p/dT_p = const. Since $m_p \cong m_n$, T_p is expressed in terms of the angle ψ by using Eq. (9.89b):

$$T_p = T_0 \sin^2\psi \tag{3}$$

We also have $\psi = \dfrac{\theta}{2}$ for the case $m_p \cong m_n$. Thus,

$$\frac{dT_p}{d\Omega(\phi)} = \frac{1}{2\pi\sin\phi}\frac{d}{d\phi}\left(T_0\sin^2\psi\right) \tag{4}$$

or,

$$\left|\frac{dT_p}{d\Omega(\phi)}\right| = \left|\frac{T_0}{2\pi \sin\phi}\frac{d}{d\phi}\sin^2\left[\frac{\pi-\phi}{2}\right]\right|$$

$$= \frac{T_0}{2\pi \sin\phi}\sin\frac{\phi}{2}\cos\frac{\phi}{2} = \frac{T_0}{4\pi} \tag{5}$$

Therefore, we find for the angular distribution of the scattered neutron,

$$\boxed{\frac{dN_n}{d\Omega(\theta)} = \frac{dN_p}{dT_p}\cdot\frac{T_0}{4\pi}} \tag{6}$$

Since $dN_p/dT_p = \text{const.}$, $dN_n/d\Omega$ is also constant. That is, the scattering of neutrons by protons is *isotropic* in the CM system.

9-52. Defining the differential cross section $\sigma(\theta)$ in the CM system as in Eq. (9.116), the number of particles scattered into the interval from θ to $\theta + d\theta$ is proportional to

$$dN \propto \sigma(\theta)\sin\theta\, d\theta = -\sigma(\theta)\, d(\cos\theta) \tag{1}$$

From Eq. (9.87a) and the assumption of elastic collisions (i.e., $T_0 = T_1 + T_2$), we obtain

$$\frac{T_2}{T_0} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \tag{2}$$

or, solving for cos θ,

$$\cos\theta = \frac{T_m - 2T_2}{T_m} \tag{3}$$

where $T_m = \dfrac{4m_1m_2}{(m_1+m_2)^2}T_0$ is the maximum energy attainable by the recoil particle in the lab system. Then, (1) can be rewritten as

$$dN \propto 2\sigma(\theta)\frac{dT_2}{T_m} \tag{4}$$

and consequently, we obtain the desired result for the energy distribution:

$$\boxed{\frac{dN}{dT_2} \propto \sigma(\theta)} \tag{5}$$

9-53.

m_1 = mass of particle α_1 , m_2 = mass of ^{238}U

u_1, u_1': velocity of particle α in LAB and CM before collision

v_1, v_1': velocity of particle $\propto$ in LAB and CM after collision

u_2, u_2': velocity of ^{238}U in LAB and CM before collision

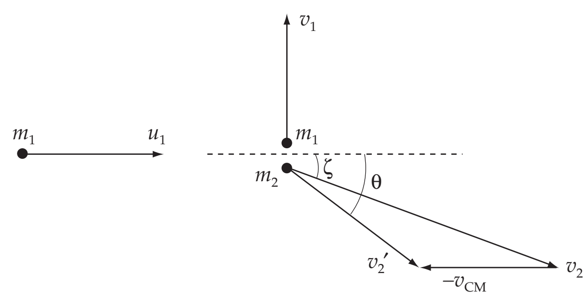
v_2, v_2'' velocity of ^{238}U in LAB and CM after collision

$u_2 = 0$, $T_1 = \dfrac{m_1 u_1^2}{2} = 7.7\text{MeV}$

$\psi = 90°$ is angle through which particle α is deflected in LAB

θ is angle through which particle α and ^{238}U are deflected in CM

ζ is recoil angle of ^{238}U in LAB



a) Conservation of momentum in LAB:

$$\left.\begin{aligned} m_1\, u_1 &= m_2\, v_2 \cos\zeta \\ m_1\, v_1 &= m_2\, v_2 \sin\zeta \end{aligned}\right\} \Rightarrow \quad v_1 = u_1 \tan\zeta$$

Conservation of energy in LAB:

$$\frac{m_1 u_1^2}{2} = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}$$

From these equations we obtain the recoil scattering angle of ^{238}U

$$\tan\,\zeta = \sqrt{\frac{m_2 - m_1}{m_2 + m_1}} \quad \Rightarrow \quad \zeta = 44.52°$$

b) The velocity of CM of system is

$$\vec{v}_{\text{CM}} = \frac{m_1 \vec{u}_1}{m_1 + m_2}$$

The velocity of ^{238}U in CM after collision is $\vec{v}_2' = \vec{v}_2 - \vec{v}_{\text{CM}}$. From the above figure we can obtain the scattering angle of particle ^{238}U in CM to be

$$\tan\theta = \frac{v_2 \sin\zeta}{v_2\cos\zeta - v_{cm}} = \sqrt{\frac{m_2^2 - m_1^2}{m_1^2}} \quad \Rightarrow \quad \theta = 89.04°$$

In CM, clearly after collision, particle α moves in opposite direction of that of ^{238}U.

c) The kinetic energy of particle ^{238}U after collision in LAB is

$$\frac{m_2 v_2^2}{2} = \frac{m_2}{2}\left(\frac{m_1 v_0}{m_2\cos\zeta}\right)^2 = \frac{m_1^2 u_1^2}{m_1 + m_2} = \frac{2m_1}{m_1+m_2}T_1 = 0.25\text{ MeV}$$

Evidently, conservation of energy is satisfied.

d) The impact parameter in CM is given in Section 9.10.

$$b = \frac{k}{2T_0'}\cot\left(\frac{\theta}{2}\right)$$

where $k = \dfrac{q_1 q_2}{9\pi\varepsilon_0}$ and $T_0' = \dfrac{1}{2}\left(m_1 u_1'^2 + m_2 u_2'^2\right)$ is the total energy of system in CM,

so
$$b = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{m_1+m_2}{m_1 m_2 u_1^2}\cot\left(\frac{\theta}{2}\right) = 1.8\times10^{-14}\text{ m}$$

We note that b is the impact parameter of particle α with respect to CM, so the impact parameter of particle α with respect to ^{238}U is $\dfrac{(m_1+m_2)b}{m_2} = 1.83\times10^{-14}$ m.

e) In CM system, the orbit equation of particle α is

$$\frac{\alpha}{r'} = (1+\varepsilon)\cos\theta \quad \text{where } \theta = 0 \quad \text{corresponds to } r = r_{\min}$$

$\Rightarrow r'_{\min} = \dfrac{\alpha}{1+\varepsilon}$ is closest distance from particle α to the center of mass, and

$$\alpha = \frac{\ell^2}{m_1 k} = \frac{4\pi\varepsilon_0}{q_1 q_2}\frac{(m_1 u_1' b)^2}{m_1}$$

and

$$\varepsilon = \sqrt{1 + 2\frac{E\ell^2}{m_1 k}} = \sqrt{1 + 2E\frac{4\pi\varepsilon_0}{q_1 q_2}\frac{(m_1 u_1' b)^2}{m_1}}$$

$$= \sqrt{1 + m_1 u_1'^2\,\frac{4\pi\varepsilon_0}{q_1 q_2}\frac{(m_1 u_1' b)^2}{m_1}}$$

But the actual minimum distance between particles is

$$r_{\min} = \frac{m_1+m_2}{m_2}r'_{\min} = 0.93\times10^{-14}\text{ m.}$$

f) Using formula

$$\sigma_{\text{LAB}}(\psi) = \sigma_{\text{CM}}(\theta)\frac{\left(x\cos\psi + \sqrt{1-x^2\sin^2\psi}\right)^2}{\sqrt{1-x^2\sin^2\psi}}$$

where $x = \dfrac{m_1}{m_2}$, $\psi = 90°$, $\sigma_{\text{CM}}(\theta) = \dfrac{k^2}{(4T_0')^2}\dfrac{1}{\sin^4\left(\dfrac{\theta}{2}\right)}$

We find this differential cross section in LAB at $\psi = 90°$:

$$\sigma_{\text{LAB}}(\psi = 90°) = 3.16\times 10^{-28}\ \text{m}^2$$

g) Since $\dfrac{\text{d}N}{N} = \sigma(\theta)\sin\psi\,\text{d}\psi\,\text{d}\phi$ we see that the ratio of probability is

$$\frac{\sigma(\psi)\sin\psi}{\sigma(\psi')\sin\psi'} = 11.1$$

9-54. Equation 9.152 gives the velocity of the rocket as a function of mass:

$$v = v_0 + u\ln\frac{m_0}{m} = \mu\ln\frac{m_0}{m} \qquad (v_0 = 0)$$

$$p = mv = mu\ln\frac{m_0}{m}$$

To maximize p, set $\dfrac{dp}{dm} = 0$

$$0 = \frac{dp}{dm} = u\left[\ln\frac{m_0}{m} - 1\right]$$

$$\ln\frac{m_0}{m} = 1 \qquad \frac{m_0}{m} = e \qquad or \qquad \frac{m}{m_0} = e^{-1}$$

To check that we have a maximum, examine

$$\left.\frac{d^2p}{dm^2}\right]_{m=m_0e^{-1}} \qquad\qquad \frac{d^2p}{dm^2} = -\frac{u}{m}$$

$$\left.\frac{d^2p}{dm^2}\right]_{m=m_0e^{-1}} = -\frac{u}{m_0}e < 0\text{, so we have a maximum.}$$

$$\boxed{\frac{m}{m_0} = e^{-1}}$$

9-55. The velocity equation (9.165) gives us:

$$v(t) = -gt + u \ln\left[\frac{m_0}{m(t)}\right] \tag{1}$$

where $m(t) = m_0 - \alpha t$, the burn rate $\alpha = 9m_0/10\tau$, the burn time $\tau = 300$ s, and the exhaust velocity $u = 4500\ \text{m}\cdot\text{s}^{-1}$. These equations are good only from $t = 0$ to $t = \tau$. First, let us check that the rocket does indeed lift off at $t = 0$: the thrust $\alpha u = 9um_0/10\tau = 13.5\ \text{m}\cdot\text{s}^{-2}\cdot m_0 > m_0 g$, as required. To find the maximum velocity of the rocket, we need to check it at the times $t = 0$ and $t = \tau$, and also check for the presence of any extrema in the region $0 < t < \tau$. We have $v(0) = 0$, $v(\tau) = -g\tau + u \ln 10 = 7400\ \text{m}\cdot\text{s}^{-1}$, and calculate

$$\frac{dv}{dt} = -g + \frac{\alpha u}{m(t)} = g\left[\frac{\alpha u}{m(t)g} - 1\right] > 0 \tag{2}$$

The inequality follows since $\alpha u > m_0 g > m(t) g$. Therefore the maximum velocity occurs at $t = \tau$, where $v = -g\tau + u \ln 10 = 7400\ \text{m}\cdot\text{s}^{-1}$. A similar single-stage rocket cannot reach the moon since $v(t) < u \ln\left(m_0/m_{final}\right) = u \ln 10 \simeq 10.4\ \text{m}\cdot\text{s}^{-1}$, which is less than escape velocity and independent of fuel burn rate.

9-56.

a) Since the rate of change of mass of the droplet is proportional to its cross-sectional area, we have

$$\frac{dm}{dt} = k\pi r^2 \tag{1}$$

If the density of the droplet is ρ,

$$m = \frac{4\pi}{3}\rho r^3 \tag{2}$$

so that

$$\frac{dm}{dt} = \frac{dm}{dr}\frac{dr}{dt} = 4\pi\rho r^2 \frac{dr}{dt} = \pi k r^2 \tag{3}$$

Therefore,

$$\frac{dr}{dt} = \frac{k}{4\rho} \tag{4}$$

or,

$$\boxed{r = r_0 + \frac{k}{4\rho}t} \tag{5}$$

as required.

b) The mass changes with time, so the equation of motion is

$$F = \frac{d}{dt}(mv) = m\,\frac{dv}{dt} + v\,\frac{dm}{dt} = mg \tag{6}$$

Using (1) and (2) this becomes

$$\frac{4\pi}{3}\rho r^3\,\frac{dv}{dt} + \pi k r^2 v = \frac{4\pi}{3}\rho r^3 g \tag{7}$$

or,

$$\frac{dv}{dt} + \frac{3k}{4\rho r}\,v = g \tag{8}$$

Using (5) this becomes

$$\frac{dv}{dt} + \frac{3k}{4\rho}\,\frac{v}{r_0 + \dfrac{k}{4\rho}t} = g \tag{9}$$

If we set $A = \dfrac{3k}{4\rho}$ and $B = \dfrac{k}{4\rho}$, this equation becomes

$$\frac{dv}{dt} + \frac{A}{r_0 + Bt}\,v = g \tag{10}$$

and we recognize a standard form for a first-order differential equation:

$$\frac{dv}{dt} + P(t)v = Q(t) \tag{11}$$

in which we identify

$$P(t) = \frac{A}{r_0 + Bt}; \quad Q(t) = g \tag{12}$$

The solution of (11) is

$$v(t) = e^{-\int P(t)\,dt}\left[\int e^{\int P(t)\,dt}\,Q\,dt + \text{constant}\right] \tag{13}$$

Now,

$$\int P(t)\,dt = \int \frac{A}{r_0 + Bt}\,dt = \frac{A}{B}\ln\left(r_0 + Bt\right)$$

$$= \ln\left(r_0 + Bt\right)^3 \tag{14}$$

since $\dfrac{A}{B} = 3$. Therefore,

$$e^{\int P dt} = \left(r_0 + Bt\right)^3 \tag{15}$$

Thus,

$$v(t) = (r_0 + Bt)^{-3} \left[\int (r_0 + Bt)^3 g \, dt + \text{ constant} \right]$$

$$= (r_0 + Bt)^{-3} \left[\frac{g}{4B} (r_0 + Bt)^4 + C \right] \tag{16}$$

The constant C can be evaluated by setting $v(t=0) = v_0$:

$$v_0 = \frac{1}{r_0^3} \left[\frac{g}{4B} \left(r_0^4 + C \right) \right] \tag{17}$$

so that

$$C = v_0 r_0^3 - \frac{g}{4B} r_0^4 \tag{18}$$

We then have

$$v(t) = \frac{1}{(r_0 + Bt)^3} \left[\frac{g}{4B} (r_0 + Bt)^4 + v_0 r_0^3 - \frac{g}{4B} r_0^4 \right] \tag{19}$$

or,

$$v(t) = \frac{1}{(Bt)^3} \left[\frac{g}{4B} (Bt)^4 + 0\left(r_0^3 \right) \right] \tag{20}$$

where $0\left(r_0^3\right)$ means "terms of order r_0^3 and higher." If r_0 is sufficiently small so that we can neglect these terms, we have

$$\boxed{v(t) \propto t} \tag{21}$$

as required.

9-57. Start from our definition of work:

$$W = \int F \, dx = \int \frac{dp}{dt} dx = \int v \, dp \tag{1}$$

We know that for constant acceleration we must have $v = at$ (zero initial velocity). From Equation (9.152) this means

$$m = m_0 e^{-at/u} \tag{2}$$

We can then compute dp:

$$dp = d(mv) = d(mat) = ma \, dt + at \, dm = m_0 \, ae^{-at/u} \left[1 - \frac{at}{u} \right] dt \tag{3}$$

This makes our expression for the work done on the rocket

$$W_r = \frac{m_0 a}{u} \int_0^t (at)(u - at) e^{-at/u} \, dt \tag{4}$$

The work done on the exhaust, on the other hand, is given with $v \to (v - u)$ and $dp \to dm_{exhaust}\,(v-u)$, so that

$$W_e = \frac{m_0 a}{u}\int_0^t (at-u)^2\, e^{-at/u}\, dt \tag{5}$$

The upper limit on the integrals is the burnout time, which we can take to be the final velocity divided by the acceleration. The total work done by the rocket engines is the sum of these two quantities, so that

$$W = \frac{m_0 a}{u}\int_0^{v/a} \left(u^2 - uat\right) e^{-at/u}\, dt = m_0 u^2 \int_0^{v/u} (1-x)\, e^{-x}\, dx \tag{6}$$

where the obvious substitution was made in the last expression. Upon evaluating the integral we find

$$W = m_0 u v e^{-v/u} = muv \tag{7}$$

where m is the mass of the rocket after its engines have turned off and v is its final velocity.

9-58. From Eq. (9.165) the velocity is

$$v = \frac{dy}{dt} = -gt + u \ln \frac{m_0}{m}$$

$$\int dy = \int \left[-gt + u \ln \frac{m_0}{m}\right] dt$$

Since $\dfrac{m}{dt} = -\alpha$, $dt = -dm/\alpha$

$$y + C = -\frac{1}{2} g t^2 - \frac{u}{\alpha}\int \ln \frac{m_0}{m}\, dm$$

$\displaystyle\int \ln \frac{a}{x}\, dx = x\left(1 + \ln \frac{a}{x}\right)$, so we have

$$y + C = -\frac{1}{2} g t^2 - \frac{u}{\alpha}\left[m + m \ln \frac{m_0}{m}\right]$$

Evaluate C using $y = 0$ when $t = 0$, $m = m_0$

$$C = -\frac{u m_0}{\alpha}$$

$$y = \frac{u\left(m_0 - m\right)}{\alpha} - \frac{1}{2} g t^2 - \frac{mu}{\alpha} \ln \frac{m_0}{m}\,;\ m_0 - m = \alpha t$$

$$y = ut - \frac{1}{2} g t^2 - \frac{mu}{\alpha} \ln \frac{m_0}{m}$$

At burnout, $y = y_B$, $t = t_B$

$$\boxed{y_B = ut_B - \frac{1}{2}gt_B^2 - \frac{mu}{\alpha}\ln\frac{m_0}{m}}$$

After burnout, the equations are

$$y = y_0 + v_0 t - \frac{1}{2}gt^2 \text{ and } v = v_0 - gt$$

Calling the top of the path the final point

$$v_f = 0 = v_B - gt_f \quad \text{or} \quad t_f = v_B/g$$

$$y - y_0 = \frac{v_B^2}{g} - \frac{v_B^2}{2g} = \frac{v_B^2}{2g}; \quad y_0 = 0$$

$$\boxed{y = \frac{v_B^2}{2g}}$$

9-59. In order to immediately lift off, the thrust must be equal in magnitude to the weight of the rocket. From Eq. (9.157):

$$\text{Thrust} = v_0\alpha \qquad v_0 = \text{velocity of fuel}$$

So

$$v_0\alpha = mg$$

or

$$\boxed{v_0 = mg/\alpha}$$

9-60. The rocket will lift off when the thrust just exceeds the weight of the rocket.

$$\text{Thrust} = -u\frac{dm}{dt} = u\alpha$$

$$\text{Weight} = mg = (m_0 - \alpha t)g$$

Set thrust = weight and solve for t:

$$u\alpha = (m_0 - \alpha t)g\,;\ t = \frac{m_0}{\alpha} - \frac{u}{g}$$

With $m_0 = 70000$ kg, $\alpha = 250$ kg/s, $u = 2500$ m/s, $g = 9.8$ m/s^2

$$\boxed{t \simeq 25 \text{ sec}}$$

The design problem is that there is too much fuel on board. The rocket sits on the ground burning off fuel until the thrust equals the weight. This is not what happens in an actual launch. A real rocket will lift off as soon as the engines reach full thrust. The time the rocket sits on the ground with the engines on is spent building up to full thrust, not burning off excess fuel.

9-61. From Eq. (9.153), the velocity after the first state is:

$$v_1 = v_0 + u \ln k$$

After the second stage:

$$v_2 = v_1 + u \ln k = v_0 + 2u \ln k$$

After the third stage:

$$v_3 = v_2 + u \ln k = v_0 + 3u \ln k$$

After the n stages:

$$v_n = v_{n-1} + u \ln k = v_0 + nu \ln k$$

$$\boxed{v_n = v_0 + nu \ln k}$$

9-62. To hover above the surface requires the thrust to counteract the gravitational force of the moon. Thus:

$$-u \frac{dm}{dt} = \frac{1}{6} mg$$

$$-\frac{6u}{g} \frac{dm}{m} = dt$$

Integrate from $m = m_0$ to $0.8\, m_0$ and $t = 0$ to T:

$$T = -\frac{6u}{g} \ln 0.8 = -\frac{6(2000 \text{ m/s})}{9.8 \text{ m/s}^2} \ln 0.8$$

$$\boxed{T = 273 \text{ sec}}$$

9-63.

a) With no air resistance and constant gravity, the problem is simple:

$$\frac{1}{2} m v_0^2 = mgh \tag{1}$$

giving the maximum height of the object as $h = v_0^2/2g \simeq 1800 \text{ km}$. The time it takes to do this is $v_0/g \simeq 610 \text{ s}$.

b) When we add the expression for air resistance, the differential equation that describes the projectile's ascent is

$$F = m \frac{dv}{dt} = -mg - \frac{1}{2} c_W \rho A v^2 = -mg \left[1 + \left(\frac{v}{v_t} \right)^2 \right] \tag{2}$$

where $v_t = \sqrt{2mg/c_W \rho A} \simeq 2.5\ \mathrm{km \cdot s^{-1}}$ would be the terminal velocity if the object were falling from a sufficient height (using $1.3\ \mathrm{kg \cdot m^{-3}}$ as the density of air). Solution of this differential equation gives

$$v(t) = v_t \tan\left[\tan^{-1}\left(\frac{v_0}{v_t}\right) - \frac{gt}{v_t}\right] \tag{3}$$

This gives $v = 0$ at time $\tau = (v_t/g)\tan^{-1}(v_0/v_t) \simeq 300\ \mathrm{s}$. The velocity can in turn be integrated to give the y-coordinate of the projectile on the ascent. The height it reaches is the y-coordinate at time τ:

$$h = \frac{v_t^2}{2g}\ln\left[1+\left(\frac{v_0}{v_t}\right)^2\right] \tag{4}$$

which is $\simeq$ 600 km.

c) Changing the acceleration due to gravity from $-g$ to $-GM_e/(R_e+y)^2 = -g\left[R_e/(R_e+y)\right]^2$ changes our differential equation for y to

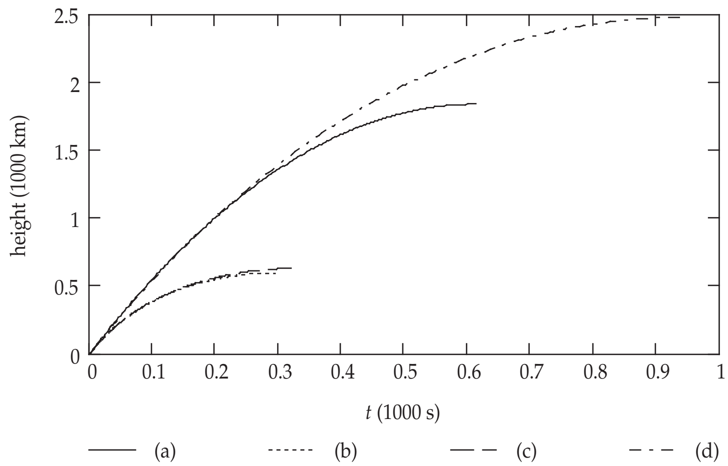
$$\ddot{y} = -g\left[\left(\frac{\dot{y}}{v_t}\right)^2 + \left(\frac{R_e}{R_e+y}\right)^2\right] \tag{5}$$

Using the usual numerical techniques, we find that the projectile reaches a height of $\simeq$ 630 km in a flight time of $\simeq$ 330 s.

d) Now we must replace the ρ in the air resistance equation with $\rho(y)$. Given the dependence of v_t on ρ, we may write the differential equation

$$\ddot{y} = -g\left[\frac{\rho(y)}{\rho_0}\left(\frac{\dot{y}}{v_t}\right)^2 + \left(\frac{R_e}{R_e+y}\right)^2\right] \tag{6}$$

where we use $\log_{10}\rho(y) = 0.11 - (5\times 10^{-5})y$ and $\rho_0 = 1.3$, with the ρ's in $\mathrm{kg \cdot m^{-3}}$ and y in meters. The projectile then reaches a height of $\simeq$ 2500 km in a flight time of $\simeq$ 940 s. This is close to the height to which the projectile rises when there is no air resistance, which is $\simeq$ 2600 km.



9-64. We start with the equation of motion for a rocket influenced by an external force, Eq. (9.160), with F_{ext} including gravity, and later, air resistance.

a) There is only constant acceleration due to gravity to worry about, so the problem can be solved analytically. From Eq. (9.166), we can obtain the rocket's height at burnout

$$y_b = ut_b - \frac{1}{2}gt_b^2 - \frac{mu}{\alpha}\ln\left[\frac{m_0}{m_b}\right] \tag{1}$$

where m_b is the mass of the rocket at burnout and $\alpha = (m_0 - m_b)/t_b$. Substitution of the given values gives $y_b \simeq 250$ km. After burnout, the rocket travels an additional $v_b^2/2g$, where v_b is the rocket velocity at burnout. The final height the rocket ends up being ≃ 3700 km, after everything is taken into account.

b) The situation, and hence the differential equation, becomes more complicated when air resistance is added. Substituting $F_{ext} = -mg - c_W\rho Av^2/2$ (with $\rho = 1.3\ \text{kg}\cdot\text{m}^{-3}$) into Equation (9.160), we obtain

$$\frac{dv}{dt} = \frac{u\alpha}{m} - g - \frac{c_W\rho Av^2}{2m} \tag{2}$$

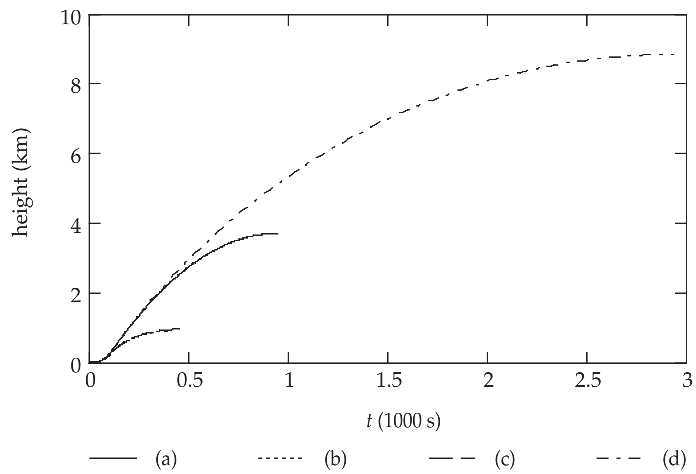
We must remember that the mass *m* is also a function of time, and we must therefore include it also in the system of equations. To be specific, the system of equations we must use to do this by computer are

$$\begin{bmatrix} \dot{y} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} v \\ \dfrac{u\alpha}{m} - g - \dfrac{c_W\rho Av^2}{2m} \\ -\alpha \end{bmatrix} \tag{3}$$

These must be integrated from the beginning until the burnout time, and therefore must be integrated with the substitution α = 0. Firstly, we get the velocity and height at burnout to be $v_b \simeq 7000\ \text{m}\cdot\text{s}^{-1}$ and $y_b \simeq 230$ km. We can numerically integrate to get the second part of the journey, or use the results of Problem 9-63(b) to help us get the additional distance travelled with air resistance, analytically. The total height to which the rocket rises is ≃ 890 km in a total flight time of ≃ 410 s.

c) The variation in the acceleration of gravity is taken into account by substituting $GM_e/(R_e+y)^2 = g\left[R_e/(R_e+y)\right]^2$ for g in the differential equation in part (b). This gives $v_b \simeq 6900\ \mathrm{m\cdot s^{-1}}$, $y_b \simeq 230\ \mathrm{km}$, with total height ≃ 950 km and time-of-flight ≃ 460 s.

d) Now one simply substitutes the given expression for the air density, $\rho(y)$ for ρ, into the differential equation from part (c). This gives $v_b \simeq 8200\ \mathrm{m\cdot s^{-1}}$, $y_b \simeq 250\ \mathrm{km}$, and total height ≃ 8900 km with time-of-flight ≃ 2900 s.



9-65.

Total impulse	$P = 8.5\ \mathrm{N\cdot s}$
Total mass	$m_0 = 0.054\ \mathrm{kg}$
Burn time	$t_f = 1.5\ \mathrm{s}$
Rocket cross section area	$S = \dfrac{\pi d^2}{4} = 4.5\times 10^{-4}\ \mathrm{m^2}$
Drag coefficient	$c_w = 0.75$
Drag force	$D = \dfrac{1}{2}\rho c_w S v^2 = K v^2 = 2\times 10^{-4}\ v^2\ \mathrm{N}$
	where $\rho = 1.2\ \mathrm{kg/m^3}$ is density of air
	and v is rocket's speed

Rocket exhaust speed u = 800 m/s

a) The total mass of propellant is

$$\Delta m = \frac{P}{u} = 0.0106\ \mathrm{kg}$$

Since $\vartriangle m \sim 20\%\ m_0$, we will assume that the rocket's mass is approximately constant in this problem. The equation of motion of rocket is

$$m_0 \frac{dv}{dt} = -m_o g + u\alpha - Kv^2$$

(where $\alpha = \dfrac{\Delta m}{t_f} = 7.1 \times 10^{-3}\ \text{kg/s}$ is fuel burn rate)

$$\Rightarrow \frac{m_0 dv}{(u\alpha - m_0 g) - Kv^2} = dt$$

Using the initial condition at t = 0, v = 0, we find

$$v(t) = \sqrt{\frac{(u\alpha - m_0 g)}{K}} \tanh\left[\sqrt{\frac{K(u\alpha - m_0 g)}{m_0{}^2}}\right]$$

At burn-out, $t = t_f = 1.5\ \text{s}$, we find

$$v_f = v(t_f) = 114.3\ \text{m/s}$$

The height accordingly is given by

$$h(t) = \int_0 v(t)\,\mathrm{d}t = \frac{m_0}{k} \ln\left[\cosh\left(t\sqrt{\frac{k(u\alpha - m_0 g)}{m_0^2}}\right)\right]$$

At burn out, $t = t_f$, we find the burn-out height

$$h_f = h(t_f) = 95.53\ \text{m}$$

b) After the burn-out, the equation of motion is

$$m_0 \frac{\mathrm{d}v}{\mathrm{d}t} = -m_0 g - Kv^2 \quad \Rightarrow \quad \frac{m_0 \mathrm{d}v}{m_0 g + Kv^2} = -\mathrm{d}t$$

with solution

$$v(t) = -\sqrt{\frac{m_0 g}{K}} \tan\left(t\sqrt{\frac{Kg}{m_0}} - C\right)$$

Using the initial condition at $t = t_f$, $v_{(t)} = v_f$, we find the constant C = –1.43 rad, so

$$v(t) = -\sqrt{\frac{m_0 g}{K}} \tan\left(t\sqrt{\frac{Kg}{m_0}} - 1.43\right)$$

and the corresponding height

$$h(t) = h_f + \int_{t_f}^{t} v(t)\,\mathrm{d}t = h_f + \frac{m_0}{K}\left\{0.88 + \ln\left[\cos\left(t\sqrt{\frac{Kg}{m_0}} - 1.43\right)\right]\right\}$$

When the rocket reaches its maximum height (at $t = t_{\max}$) the time $t_{\max}$ can be found by setting $v(t_{\max}) = 0$. We then find $t_{\max} = 7.52$ s. And the maximum height the rocket can reach is

$$h_{\max} = h(t_{\max}) = 334 \text{ m}$$

c) Acceleration in burn-out process is (see $v(t)$ in a))

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{u\alpha - m_0 g}{m_0} \frac{1}{\cosh^2\left[t\sqrt{\frac{k(u\alpha - m_0 g)}{m_0^2}}\right]}$$

Evidently, the acceleration is maximum when $t = 0$ and

$$a_{max} = a(t=0) = \frac{u\alpha - m_0 g}{m_0} = 95.4 \text{ m/s}^2$$

d) In the fall-down process, the equation of motion is

$$\frac{m_0 \mathrm{d}v}{\mathrm{d}t} = m_0 g - Kv^2,$$

With the initial condition $t = t_{\max}$, $v = 0$, we find $(t \geq t_{\max})$

$$v(t) = -\sqrt{\frac{m_0 g}{K}} \tanh\left[(t - t_{\max})\sqrt{\frac{Kg}{m_0}}\right]$$

($v(t)$ is negative for $t \geq t_{\max}$, because then the rocket falls downward)

The height of the rocket is

$$h = h_{\max} + \int_{t_{\max}}^{t} v(t)\, dt = h_{\max} - \frac{m_0}{K} \ln\left\{\cosh\left[(t - t_{\max})\sqrt{\frac{Kg}{m_0}}\right]\right\}$$

To find the total flight-time, we set $h = 0$ and solve for t. We find $t_{\text{total}} = 17.56$ s

e) Putting $t = t_{\text{total}}$ into the expression of $V_{(t)}$ in part d), we find the speed at ground impact to be

$$v_g = -\sqrt{\frac{m_0 g}{K}} \tanh\left[(t_{total} - t_{\max})\sqrt{\frac{Kg}{m_0}}\right] = -49.2 \text{ m/s}$$

9-66. If we take into account the change of the rocket's mass with time $m = m_0 - \alpha t$, where α is the fuel burn rate,

$$\alpha = 7.1 \times 10^{-3} \text{ kg/s}$$

as calculated in problem 9-65.

The equation of motion for the rocket during boost phase is

$$\left(m_0-\alpha t\right)\frac{\mathrm{d}v}{\mathrm{d}t}=u\alpha-Kv^2 \quad\Rightarrow\quad \frac{\mathrm{d}v}{Kv^2-u\alpha}=\frac{\mathrm{d}t}{\alpha t-m_0}$$

Integrating both sides we obtain finally

$$v(t)=\left(\frac{1+C\left(\alpha t-m_0\right)^{2\sqrt{Ku/\alpha}}}{1-C\left(\alpha t-m_0\right)^{2\sqrt{Ku/\alpha}}}\right)\sqrt{\frac{u\alpha}{K}}$$

where C is a constant. Using the initial condition $v(t)=0$ at $t=0$, we can find C and the velocity is

$$v(t)=\sqrt{\frac{u\alpha}{K}}\left(\frac{1-\left(1-\dfrac{\alpha t}{m_0}\right)^{2\sqrt{\frac{Ku}{\propto}}}}{1+\left(1-\dfrac{\alpha t}{m_0}\right)^{2\sqrt{\frac{Ku}{\propto}}}}\right)$$

a) The rocket speed at burn-out is (note $t_f=1.5$ s).

$$v_f=v(t_f)=131.3\ \text{m/s}$$

b) The distance the rocket has traveled to the burn-out is

$$h_f=\int_{0}^{t_f}v(t)\,dt=108.5\ \text{m}$$

9-67. From Equation (9.167) we have

$$H_{bo}=-\frac{g\left(m_0-m_f^2\right)}{2\alpha^2}+\frac{u}{\alpha}\left[m_f\ln\left(\frac{m_f}{m_0}\right)+m_0-m_f\right]$$

Using numerical values from Example 9.12

$\alpha=-1.42\times10^4\ \text{kg/s}$

$m_0=2.8\times10^6\ \text{kg}$

$m_f=0.7\times10^6\ \text{kg}$

$u=2.600\ \text{m/s}$

we find $H_{bo}=97.47\ \text{km}$.

From Equation (9.168) we find

$$v_{bo}=-\frac{g(m_0-m_f)}{\alpha}+u\ln\left(\frac{m_o}{m_f}\right)$$

$$v_{bo}=2125\ \text{m/s}$$

CHAPTER 10

Motion in a Noninertial Reference Frame

10-1. The accelerations which we feel at the surface of the Earth are the following:

(1) Gravitational : $\boxed{980\ \text{cm/sec}^2}$

(2) Due to the Earth's rotation on its own axis:

$$r\omega^2 = \left(6.4\times 10^8\ \text{cm}\right)\times\left[\frac{2\pi\ \text{rad/day}}{86400\ \text{sec/day}}\right]^2$$

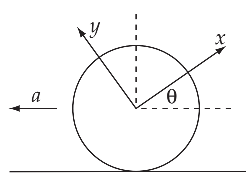
$$= \left(6.4\times 10^8\right)\times\left(7.3\times 10^{-5}\right)^2 = \boxed{3.4\ \text{cm/sec}^2}$$

(3) Due to the rotation about the sun:

$$r\omega^2 = \left(1.5\times 10^{13}\ \text{cm}\right)\times\left[\frac{2\pi\ \text{rad/year}}{86400\times 365\ \text{sec/day}}\right]^2$$

$$= \left(1.5\times 10^{13}\right)\times\left[\frac{7.3\times 10^{-5}}{365}\right]^2 = \boxed{0.6\ \text{cm/sec}^2}$$

10-2. The fixed frame is the ground.



The rotating frame has the origin at the center of the tire and is the frame in which the tire is at rest.

From Eqs. (10.24), (10.25):

$$\mathbf{a}_f = \ddot{\mathbf{R}}_f + \mathbf{a}_r + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right) + 2\boldsymbol{\omega}\times\mathbf{v}_r$$

Now we have

$$\ddot{\mathbf{R}}_f = -a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j}$$

$$\mathbf{r} = r_0\mathbf{i} \qquad \mathbf{v}_r = \mathbf{a}_r = 0$$

$$\omega = \frac{V}{r_0}\mathbf{k} \qquad \dot{\omega} = \frac{a}{r_0}\mathbf{k}$$

Substituting gives

$$\mathbf{a}_f = -a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + a\,\mathbf{j} - \frac{v^2}{r_0}\mathbf{i}$$

$$\mathbf{a}_f = -\mathbf{i}\left[\frac{v^2}{r_0} + a\cos\theta\right] + \mathbf{j}\left(\sin\theta + 1\right)a \tag{1}$$

We want to maximize $\left|\mathbf{a}_f\right|$, or alternatively, we maximize $\left|\mathbf{a}_f\right|^2$:

$$\left|\mathbf{a}_f\right|^2 = \frac{v^4}{r_0^2} + a^2\cos^2\theta + \frac{2av^2}{r_0}\cos\theta + a^2 + 2a^2\sin\theta + a^2\sin^2\theta$$

$$= \frac{v^4}{r_0^2} + 2a^2 + \frac{2av^2}{r_0}\cos\theta + a^2\sin^2\theta$$

$$\frac{d\left|\mathbf{a}_f\right|^2}{d\theta} = -\frac{2av^2}{r_0}\cos\theta + 2a^2\cos\theta$$

$$= 0 \text{ when } \tan\theta = \frac{ar_0}{v^2}$$

(Taking a second derivative shows this point to be a maximum.)

$$\tan\theta = \frac{ar_0}{v^2} \text{ implies } \cos\theta = \frac{v^2}{\sqrt{a^2r_0^2 + v^4}}$$

and

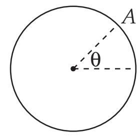
$$\sin\theta = \frac{ar_0}{\sqrt{a^2r_0^2 + v^4}}$$

Substituting into (1)

$$\boxed{\mathbf{a}_f = -\mathbf{i}\left[\frac{v^2}{r_0} + \frac{av^2}{\sqrt{a^2r_0^2 + v^4}}\right] + \mathbf{j}\left[\frac{ar_0}{\sqrt{a^2r_0^2 + v^4}} + 1\right]a}$$

This may be written as

$$\boxed{\left|\mathbf{a}_f\right| = a + \sqrt{a^2 + v^4/r_0^2}}$$

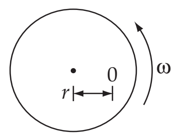


This is the maximum acceleration. The point which experiences this acceleration is at A:

$$\text{where } \tan\theta = \frac{ar_0}{v^2}$$

10-3. We desire $\mathbf{F}_{\text{eff}} = 0$. From Eq. (10.25) we have

$$\mathbf{F}_{\text{eff}} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\omega}\times\mathbf{r} - m\omega\times(\omega\times\mathbf{r}) - 2m\omega\times\mathbf{v}_r$$



The only forces acting are centrifugal and friction, thus $\mu_s mg = m\omega^2 r$, or

$$\boxed{r = \frac{\mu_s g}{\omega^2}}$$

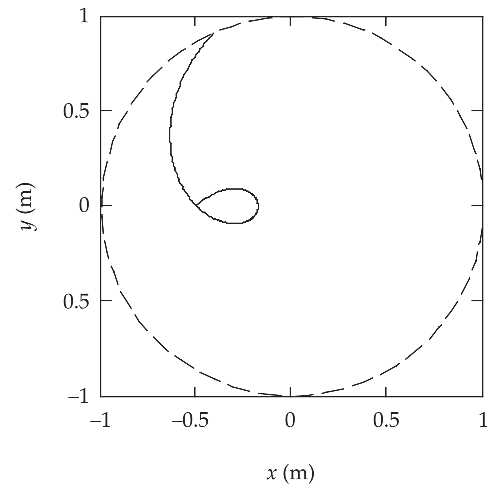
10-4. Given an initial position of (–0.5R,0) the initial velocity (0,0.5ωR) will make the puck motionless in the fixed system. In the rotating system, the puck will appear to travel clockwise in a circle of radius 0.5R. Although a numerical calculation of the trajectory in the rotating system is a great aid in understanding the problem, we will forgo such a solution here.

10-5. The effective acceleration in the merry-go-round is given by Equation 10.27:

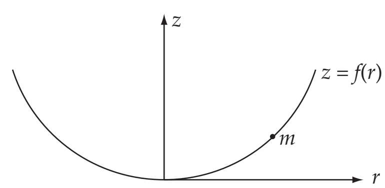
$$\ddot{x} = \omega^2 x + 2\omega\dot{y} \tag{1}$$

$$\ddot{y} = \omega^2 y - 2\omega\dot{x} \tag{2}$$

These coupled differential equations must be solved with the initial conditions $x_0 \equiv x(0) = -0.5\text{ m}$, $y_0 \equiv y(0) = 0\text{ m}$, and $\dot{x}(0) = \dot{y}(0) = v_0/\sqrt{2}\ \text{m}\cdot\text{s}^{-1}$, since we are given in the problem that the initial velocity is at an angle of 45° to the x-axis. We will vary v_0 over some range that we know satisfies the condition that the path cross over (x_0, y_0) . We can start by looking at Figures 10-4e and 10-4f, which indicate that we want $v_0 > 0.47\ \text{m}\cdot\text{s}^{-1}$. Trial and error can find a trajectory that does loop but doesn't cross its path at all, such as $v_0 = 0.53\ \text{m}\cdot\text{s}^{-1}$. From here, one may continue to solve for different values of v_0 until the wanted crossing is eyeball-suitable. This may be an entirely satisfactory answer, depending on the inclinations of the instructor. An interpolation over several trajectories would show that an accurate answer to the problem is $v_0 = 0.512\ \text{m}\cdot\text{s}^{-1}$, which exits the merry-go-round at 3.746 s. The figure shows this solution, which was numerically integrated with 200 steps over the time interval.



10-6.



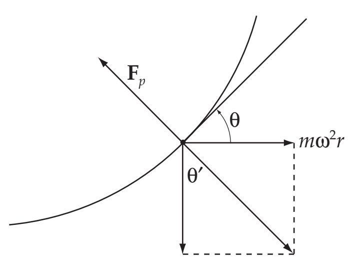
Consider a small mass m on the surface of the water. From Eq. (10.25)

$$\mathbf{F}_{\text{eff}} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times r) - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

In the rotating frame, the mass is at rest; thus, $\mathbf{F}_{\text{eff}} = 0$. The force **F** will consist of gravity and the force due to the pressure gradient, which is normal to the surface in equilibrium. Since $\ddot{\mathbf{R}}_f = \dot{\boldsymbol{\omega}} = \mathbf{v}_r = 0$, we now have

$$0 = m\mathbf{g} + \mathbf{F}_p - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{F}_p$ is due to the pressure gradient.



Since $\mathbf{F}_{\text{eff}} = 0$, the sum of the gravitational and centrifugal forces must also be normal to the surface.

Thus $\theta' = \theta$.

$$\tan\theta' = \tan\theta = \frac{\omega^2 r}{g}$$

but

$$\tan\theta = \frac{dz}{dr}$$

Thus

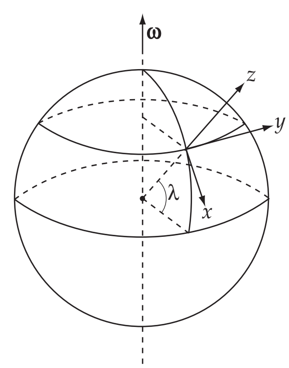
$$\boxed{\begin{array}{c} z = \dfrac{\omega^2}{2g} r^2 + \text{constant} \\ \\ \text{The shape is a circular paraboloid.} \end{array}}$$

10-7. For a spherical Earth, the difference in the gravitational field strength between the poles and the equator is only the centrifugal term:

$$g_{\text{poles}} - g_{\text{equator}} = \omega^2 R$$

For $\omega = 7.3\times 10^{-5}\,\text{rad}\cdot\text{s}^{-1}$ and R = 6370 km, this difference is only $34\ \text{mm}\cdot\text{s}^{-2}$. The disagreement with the true result can be explained by the fact that the Earth is really an oblate spheroid, another consequence of rotation. To qualitatively describe this effect, approximate the real Earth as a somewhat smaller sphere with a massive belt about the equator. It can be shown with more detailed analysis that the belt pulls inward at the poles more than it does at the equator. The next level of analysis for the undaunted is the "quadrupole" correction to the gravitational potential of the Earth, which is beyond the scope of the text.

10-8.



Choose the coordinates x, y, z as in the diagram. Then, the velocity of the particle and the rotation frequency of the Earth are expressed as

$$\left.\begin{aligned} \mathbf{v} &= (0,0,\dot{z}) \\ \boldsymbol{\omega} &= (-\omega\cos\lambda, 0, \omega\sin\lambda) \end{aligned}\right] \tag{1}$$

so that the acceleration due to the Coriolis force is

$$\mathbf{a} = -2\boldsymbol{\omega}\times\dot{\mathbf{r}} = 2\omega(0, -\dot{z}\cos\lambda, 0) \tag{2}$$

This acceleration is directed along the y axis. Hence, as the particle moves along the z axis, it will be accelerated along the y axis:

$$\ddot{y} = -2\omega\dot{z}\cos\lambda \tag{3}$$

Now, the equation of motion for the particle along the z axis is

$$\dot{z} = v_0 - gt \tag{4}$$

$$z = v_0 t - \frac{1}{2}gt^2 \tag{5}$$

where v_0 is the initial velocity and is equal to $\sqrt{2gh}$ if the highest point the particle can reach is h:

$$v_0 = \sqrt{2gh} \tag{6}$$

From (3), we have

$$\dot{y} = -2\omega z\cos\lambda + c \tag{7}$$

but the initial condition $\dot{y}(z=0)=0$ implies $c = 0$. Substituting (5) into (7) we find

$$\dot{y} = -2\omega\cos\lambda\left(v_0 t - \frac{1}{2}gt^2\right)$$

$$= \omega\cos\lambda\left(gt^2 - 2v_0 t^2\right) \tag{8}$$

Integrating (8) and using the initial condition $y(t = 0) = 0$, we find

$$y = \omega\cos\lambda\left[\frac{1}{3}gt^2 - v_0 t^2\right] \tag{9}$$

From (5), the time the particle strikes the ground ($z = 0$) is

$$0 = \left(v_0 - \frac{1}{2}gt\right)t$$

so that

$$t = \frac{2v_0}{g} \tag{10}$$

Substituting this value into (9), we have

$$y = \omega\cos\lambda\left[\frac{1}{3}g\frac{8v_0^3}{g^3} - v_0\frac{4v_0^2}{g^2}\right]$$

$$= -\frac{4}{3}\omega\cos\lambda\frac{v_0^3}{g^2} \tag{11}$$

If we use (6), (11) becomes

$$\boxed{y = -\frac{4}{3}\,\omega \cos\lambda \sqrt{\frac{8h^3}{g}}} \tag{12}$$

The negative sign of the displacement shows that the particle is displaced to the *west*.

10-9. Choosing the same coordinate system as in Example 10.3 (see Fig. 10-9), we see that the lateral deflection of the projectile is in the x direction and that the acceleration is

$$a_x = \ddot{x} = 2\,\omega_z\, v_y = 2(\omega \sin\lambda)(V_0 \cos\alpha) \tag{1}$$

Integrating this expression twice and using the initial conditions, $\dot{x}(0) = 0$ and $x(0) = 0$, we obtain

$$x(t) = \omega\, V_0 t^2 \cos\alpha \sin\lambda \tag{2}$$

Now, we treat the z motion of the projectile as if it were undisturbed by the Coriolis force. In this approximation, we have

$$z(t) = V_0 t \sin\alpha - \frac{1}{2} g t^2 \tag{3}$$

from which the time T of impact is obtained by setting $z = 0$:

$$T = \frac{2V_0 \sin\alpha}{g} \tag{4}$$

Substituting this value for T into (2), we find the lateral deflection at impact to be

$$\boxed{x(T) = \frac{4\omega V_0^3}{g^2} \sin\lambda \cos\alpha \sin^2\alpha} \tag{5}$$

10-10. In the previous problem we assumed the z motion to be unaffected by the Coriolis force. Actually, of course, there is an upward acceleration given by $-2\omega_x v_y$ so that

$$\ddot{z} = 2\omega\, V_0 \cos\alpha \cos\lambda - g \tag{1}$$

from which the time of flight is obtained by integrating twice, using the initial conditions, and then setting $z = 0$:

$$T' = \frac{2V_0 \sin\alpha}{g - 2\omega V_0 \cos\alpha \cos\lambda} \tag{2}$$

Now, the acceleration in the y direction is

$$\begin{aligned} a_y = \ddot{y} &= 2\omega_x v_z \\ &= 2(-\omega\cos\lambda)(V_0 \sin\alpha - gt) \end{aligned} \tag{3}$$

Integrating twice and using the initial conditions, $\dot{y}(0) = V_0 \cos\alpha$ and $y(0) = 0$, we have

$$y(t) = \frac{1}{3}\,\omega g t^3 \cos\lambda - \omega V_0 t^2 \cos\lambda \sin\alpha + V_0 t \cos\alpha \tag{4}$$

Substituting (2) into (4), the range R′ is

$$R' = \frac{8}{3}\frac{\omega V_0^3\, g \sin^3\alpha \cos\lambda}{\left(g - 2\omega V_0 \cos\alpha \cos\lambda\right)^3} - \frac{4\omega V_0^3 \sin^3\alpha \cos\lambda}{\left(g - 2\omega V_0 \cos\alpha \cos\lambda\right)^2} + \frac{2V_0^2 \cos\alpha \cos\lambda}{g - 2\omega V_0 \cos\alpha \cos\lambda} \tag{5}$$

We now expand each of these three terms, retaining quantities up to order ω but neglecting all quantities proportional to ω^2 and higher powers of ω. In the first two terms, this amounts to neglecting $2\omega V_0 \cos\alpha \cos\lambda$ compared to g in the denominator. But in the third term we must use

$$\frac{2V_0^2 \cos\alpha \sin\alpha}{g\left[1 - \dfrac{2\omega V_0}{g}\cos\alpha\cos\lambda\right]} \cong \frac{2V_0^2}{g}\cos\alpha\sin\alpha\left[1 + \frac{2\omega V_0}{g}\cos\alpha\cos\lambda\right]$$

$$= R_0' + \frac{4\omega V_0^3}{g^2}\sin\alpha\cos^2\alpha\cos\lambda \tag{6}$$

where R_0' is the range when Coriolis effects are neglected [see Example 2.7]:

$$R_0' = \frac{2V_0^2}{g}\cos\alpha\sin\alpha \tag{7}$$

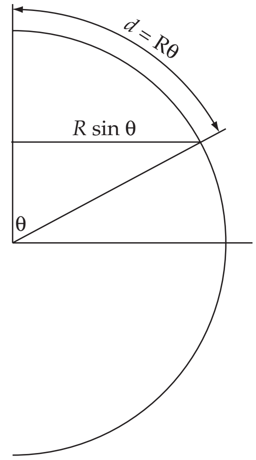
The range difference, $\Delta R' = R' - R_0'$, now becomes

$$\Delta R' = \frac{4\omega V_0^3}{g^2}\cos\lambda\left(\sin\alpha\cos^2\alpha - \frac{1}{3}\sin^3\alpha\right) \tag{8}$$

Substituting for V_0 in terms of R_0' from (7), we have, finally,

$$\boxed{\Delta R' = \sqrt{\frac{2R_0'}{g}}\,\omega\cos\lambda\left(\cot^{1/2}\alpha - \frac{1}{3}\tan^{3/2}\alpha\right)} \tag{9}$$

10-11.



This problem is most easily done in the fixed frame, not the rotating frame. Here we take the Earth to be fixed in space but rotating about its axis. The missile is fired from the North Pole at some point on the Earth's surface, a direction that will always be due south. As the missile travels towards its intended destination, the Earth will rotate underneath it, thus causing it to miss. This distance is:

$$\Delta = (\text{transverse velocity of Earth at current latitude}) \times (\text{missile's time of flight})$$

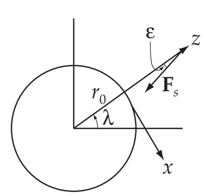
$$= \omega R \sin \theta \times T \tag{1}$$

$$= \frac{d\omega R}{v} \sin\left(\frac{d}{R}\right) \tag{2}$$

Note that the actual distance d traveled by the missile (that distance measured in the fixed frame) is less than the flight distance one would measure from the Earth. The error this causes in Δ will be small as long as the miss distance is small. Using $R = 6370$ km, $\omega = 7.27 \times 10^{-5}$ $\text{rad} \cdot \text{s}^{-1}$, we obtain for the 4800 km, T = 600 s flight a miss distance of 190 km. For a 19300 km flight the missile misses by only 125 km because there isn't enough Earth to get around, or rather there is less of the Earth to miss. For a fixed velocity, the miss distance actually peaks somewhere around $d = 12900$ km.

Doing this problem in the rotating frame is tricky because the missile is constrained to be in a path that lies close to the Earth. Although a perturbative treatment would yield an order of magnitude estimate on the first part, it is entirely wrong on the second part. Correct treatment in the rotating frame would at minimum require numerical methods.

10-12.



Using the formula

$$\mathbf{F}_{\text{eff}} = m\mathbf{a}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \tag{1}$$

we try to find the direction of $\mathbf{F}_{\text{eff}}$ when $m\mathbf{a}_f$ (which is the true force) is in the direction of the z axis. Choosing the coordinate system as in the diagram, we can express each of the quantities in (1) as

$$\left.\begin{aligned} \mathbf{v}_r &= 0 \\ \boldsymbol{\omega} &= (-\omega\cos\lambda, 0, \omega\sin\lambda) \\ \mathbf{r} &= (0,0,R) \\ m\mathbf{a}_f &= (0,0,-mg_0) \end{aligned}\right\} \tag{2}$$

Hence, we have

$$\boldsymbol{\omega} \times \mathbf{r} = R\omega\cos\lambda\,\mathbf{e}_y \tag{3}$$

and (1) becomes

$$\mathbf{F}_{\text{eff}} = -mg_0\mathbf{e}_z - m\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & R\omega\cos\lambda & 0 \end{vmatrix} \tag{4}$$

from which, we have

$$\mathbf{F}_{\text{eff}} = -mg_0\mathbf{e}_z + mR\omega^2\sin\lambda\cos\lambda\,\mathbf{e}_x + mR\omega^2\cos^2\lambda\,\mathbf{e}_z \tag{5}$$

Therefore,

$$\left.\begin{aligned} (F_f)_x &= mR\omega^2\sin\lambda\cos\lambda \\ (F_f)_z &= -mg_0 + mR\omega^2\cos^2\lambda \end{aligned}\right\} \tag{6}$$

The angular deviation is given by

$$\tan\varepsilon = \frac{\left|(F_f)_x\right|}{\left|(F_f)_z\right|} = \frac{R\omega^2\sin\lambda\cos\lambda}{g_0 - R\omega^2\cos^2\lambda} \tag{7}$$

Since ε is very small, we can put $\varepsilon \cong \varepsilon$. Then, we have

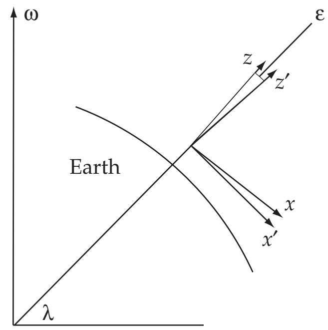
$$\boxed{\varepsilon = \frac{R\omega^2\sin\lambda\cos\lambda}{g_0 - R\omega^2\cos^2\lambda}} \tag{8}$$

It is easily shown that ε is a maximum for $\lambda \simeq 45°$.

Using $R = 6.4 \times 10^8$ cm, $\omega = 7.3 \times 10^{-5}$ sec^{-1}, $g = 980$ cm/sec^2, the maximum deviation is

$$\varepsilon \cong \frac{1.7}{980} \cong 0.002 \text{ rad} \tag{9}$$

10-13.



The small parameters which govern the approximations that need to be made to find the southerly deflection of a falling particle are:

$$\delta \equiv \frac{h}{R} = \frac{\text{height of fall}}{\text{radius of Earth}} \tag{1}$$

and

$$\alpha \equiv \frac{R\omega^2}{g_0} = \frac{\text{centrifugal force}}{\text{purely gravitational force}} \tag{2}$$

The purely gravitational component is defined the same as in Problem 10-12. Note that although both δ and α are small, the product $\delta\alpha = h\omega^2/g_0$ is still of order ω^2 and therefore expected to contribute to the final answer.

Since the plumb line, which defines our vertical direction, is not in the same direction as the outward radial from the Earth, we will use two coordinate systems to facilitate our analysis. The unprimed coordinates for the Northern Hemisphere-centric will have its x-axis towards the south, its y-axis towards the east, and its z-axis in the direction of the plumb line. The primed coordinates will share both its origin and its y'-axis with its unprimed counterpart, with the z'- and x'-axes rotated to make the z'-axis an outward radial (see figure). The rotation can be described mathematically by the transformation

$$x = x' \cos\varepsilon + z' \sin\varepsilon \tag{3}$$

$$y = y' \tag{4}$$

$$z = -x' \sin\varepsilon + z' \cos\varepsilon \tag{5}$$

where

$$\varepsilon \equiv \frac{R\omega^2}{g} \sin\lambda \cos\lambda \tag{6}$$

as found from Problem 10-12.

a) The acceleration due to the Coriolis force is given by

$$\mathbf{a}_X \equiv -2\boldsymbol{\omega} \times \mathbf{v}' \tag{7}$$

Since the angle between ω and the z'-axis is $\pi - \lambda$, (7) is most appropriately calculated in the primed coordinates:

$$\ddot{x}' = 2\omega\dot{y}' \sin\lambda \tag{8}$$

$$\ddot{y}' = -2\omega\left(\dot{z}' \cos\lambda + \dot{x}' \sin\lambda\right) \tag{9}$$

$$\ddot{z}' = 2\omega\dot{y}' \cos\lambda \tag{10}$$

In the unprimed coordinates, the interesting component is

$$\ddot{x} = 2\omega\dot{y}\left(\sin\lambda \cos\varepsilon + \cos\lambda \sin\varepsilon\right) \tag{11}$$

At our level approximation this becomes

$$\ddot{x} \simeq 2\omega\dot{y} \sin\lambda \tag{12}$$

Using the results for $\dot{y}$ and $\dot{z}$, which is correct to order ω (also found from Example 10.3),

$$\ddot{x} \simeq 2\omega^2 g t^2 \sin\lambda \cos\lambda \tag{13}$$

Integrating twice and using the zeroth order result for the time-of-fall, $t = \sqrt{2h/g}$, we obtain for the deflection

$$d_X = \frac{2}{3}\frac{h^2}{g}\omega^2 \sin\lambda \cos\lambda \tag{14}$$

b) The centrifugal force gives us an acceleration of

$$\mathbf{a}_c \equiv -\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{r}'\right) \tag{15}$$

The component equations are then

$$\ddot{x}' = \omega^2 \sin\lambda\left[x' \sin\lambda + \left(R + z'\right)\cos\lambda\right] \tag{16}$$

$$\ddot{y}' = \omega^2 y' \tag{17}$$

$$\ddot{z}' = \omega^2 \cos\lambda\left[x' \sin\lambda + \left(R + z'\right)\cos\lambda\right] - g_0 \tag{18}$$

where we have included the pure gravitational component of force as well. Now transform to the unprimed coordinates and approximate

$$\ddot{x} \simeq \omega^2\left(R + z\right)\sin\lambda \cos\lambda - g_0 \sin\varepsilon \tag{19}$$

We can use Problem 10-12 to obtain $\sin\varepsilon$ to our level of approximation

$$\sin\varepsilon \simeq \varepsilon \simeq \frac{R\omega^2}{g_0}\sin\lambda \cos\lambda \tag{20}$$

The prompts a cancellation in equation (19), which becomes simply

$$x \simeq \omega^2 z \sin\lambda \cos\lambda \tag{21}$$

Using the zeroth order result for the height, $z = h - gt^2/2$, and for the time-of-fall estimates the deflection due to the centrifugal force

$$d_c \simeq \frac{5}{6}\frac{h^2}{g}\omega^2 \sin\lambda \cos\lambda \tag{22}$$

c) Variation in gravity causes the acceleration

$$\mathbf{a}_g \equiv -\frac{GM}{r^3}\,\mathbf{r} + g_0\mathbf{k} \tag{23}$$

where $\mathbf{r} = x'\mathbf{i} + y'\mathbf{j} + (R + z')\mathbf{k}$ is the vector pointing to the particle from the center of the spherical Earth. Near the surface

$$r^2 = x'^2 + y'^2 + (z' + R)^2 \simeq R^2 + 2Rz' \tag{24}$$

so that (23) becomes, with the help of the binomial theorem,

$$\mathbf{a}_g \simeq -\frac{g_0}{R}\left(x'\mathbf{i} + y'\mathbf{j} - 2z'\mathbf{k}\right) \tag{25}$$

Transform and get the x component

$$\ddot{x} \simeq \frac{g_0}{R}\left(-x' \cos\varepsilon + 2z' \sin\varepsilon\right) \tag{26}$$

$$= \frac{g_0}{R}\left[-\left(x\cos\varepsilon - z\sin\varepsilon\right)\cos\varepsilon + 2\left(x\sin\varepsilon + z\cos\varepsilon\right)\sin\varepsilon\right] \tag{27}$$

$$\simeq \frac{g_0}{R}\left(-x + 3z\sin\varepsilon\right) \tag{28}$$

Using (20),

$$\ddot{x} \simeq 3\omega^2 z \sin\lambda \cos\lambda \tag{29}$$

where we have neglected the x/R term. This is just thrice the part (b) result,

$$d_g \simeq \frac{5}{2}\frac{h^2}{g}\,\omega^2 \sin\lambda \cos\lambda \tag{30}$$

Thus the total deflection, correct to order ω^2, is

$$d \simeq 4\frac{h^2}{g}\,\omega^2 \sin\lambda \cos\lambda \tag{31}$$

(The solution to this and the next problem follow a personal communication of Paul Stevenson, Rice University.)

10-14. The solution to part (c) of the Problem 10-13 is modified when the particle is dropped down a mineshaft. The force due to the variation of gravity is now

$$\mathbf{a}_g \equiv -\frac{g_0}{R}\,\mathbf{r} + g_0\mathbf{k} \tag{1}$$

As before, we approximate $\mathbf{r}$ for near the surface and (1) becomes

$$\mathbf{a}_g \simeq -\frac{g_0}{R}\left(x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}\right) \tag{2}$$

In the unprimed coordinates,

$$\ddot{x} \simeq -g_0 \frac{x}{R} \tag{3}$$

To estimate the order of this term, as we probably should have done in part (c) of Problem 10-13, we can take $x \sim h^2\omega^2/g$, so that

$$\ddot{x} \sim \omega^2 h \times \frac{h}{R} \tag{4}$$

which is reduced by a factor h/R from the accelerations obtained previously. We therefore have no southerly deflection in this order due to the variation of gravity. The Coriolis and centrifugal forces still deflect the particle, however, so that the total deflection in this approximation is

$$d \simeq \frac{3}{2}\frac{h^2}{g}\omega^2 \sin\lambda \cos\lambda \tag{5}$$

10-15. The Lagrangian in the fixed frame is

$$L = \frac{1}{2} m v_f^2 - U\left(r_f\right) \tag{1}$$

where v_f and r_f are the velocity and the position, respectively, in the fixed frame. Assuming we have common origins, we have the following relation

$$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}_r \tag{2}$$

where v_r and r_r are measured in the rotating frame. The Lagrangian becomes

$$L = \frac{m}{2}\left[v_r^2 + 2\mathbf{v}_r \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_r\right) + \left(\boldsymbol{\omega} \times \mathbf{r}_r\right)^2\right] - U\left(r_r\right) \tag{3}$$

The canonical momentum is

$$\mathbf{p}_r \equiv \frac{\partial L}{\partial \mathbf{v}_r} = m\mathbf{v}_r + m\left(\boldsymbol{\omega} \times \mathbf{r}_r\right) \tag{4}$$

The Hamiltonian is then

$$H \equiv \mathbf{v}_r \cdot \mathbf{p}_r - L = \frac{1}{2} m v_r^2 - U\left(r_r\right) - \frac{1}{2} m\left(\boldsymbol{\omega} \times \mathbf{r}_r\right)^2 \tag{5}$$

H is a constant of the motion since $\partial L/\partial t = 0$, but $H \neq E$ since the coordinate transformation equations depend on time (see Section 7.9). We can identify

$$U_c = -\frac{1}{2} m\left(\boldsymbol{\omega} \times \mathbf{r}_r\right)^2 \tag{6}$$

as the centrifugal potential energy because we may find, with the use of some vector identities,

$$-\nabla U_c = \frac{m}{2}\nabla\left[\omega^2 r_r^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}_r\right)^2\right] \tag{7}$$

$$= m\left[\omega^2 \mathbf{r}_r - \left(\boldsymbol{\omega} \cdot \mathbf{r}_r\right)\boldsymbol{\omega}\right] \tag{8}$$

$$= -m\omega \times (\omega \cdot \mathbf{r}_r) \tag{9}$$

which is the centrifugal force. Computing the derivatives of (3) required in Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathbf{L}}{\partial \mathbf{v}_r} = m\mathbf{a}_r + m\omega \times \mathbf{v}_r \tag{10}$$

$$\frac{\partial \mathbf{L}}{\partial \mathbf{r}_r} = m\nabla\left[(\mathbf{v}_r \times \omega)\cdot \mathbf{r}_r\right] - \nabla(U_c + U) \tag{11}$$

$$= -m(\omega \times \mathbf{v}_r) - m\omega \times (\omega \times \mathbf{r}_r) - \nabla U \tag{12}$$

The equation of motion we obtain is then

$$m\mathbf{a}_r = -\nabla U - m\omega \times (\omega \times \mathbf{r}_r) - 2m(\omega \times \mathbf{v}_r) \tag{13}$$

If we identify $\mathbf{F}_{\text{eff}} = m\mathbf{a}_r$ and $\mathbf{F} = -\nabla U$, then we do indeed reproduce the equations of motion given in Equation 10.25, without the second and third terms.

10-16. The details of the forces involved, save the Coriolis force, and numerical integrations in the solution of this problem are best explained in the solution to Problem 9-63. The only thing we do here is add an acceleration caused by the Coriolis force, and re-work every part of the problem over again. This is conceptually simple but in practice makes the computation three times more difficult, since we now also must include the transverse coordinates in our integrations. The acceleration we add is

$$\mathbf{a}_c = 2\omega\left[v_y \sin\lambda \mathbf{i} - (v_x \sin\lambda + v_z \cos\lambda)\mathbf{j} + v_y \cos\lambda \mathbf{k}\right] \tag{1}$$

where we have chosen the usual coordinates as shown in Figure 10-9 of the text.

a) Our acceleration is

$$\mathbf{a} = -g\mathbf{k} + \mathbf{a}_C \tag{2}$$

As a check, we find that the height reached is $\simeq 1800$ km, in good agreement with the result of Problem 9-63(a). The deflection at this height is found to be $\simeq 77$ km, to the west.

b) This is mildly tricky. The correct treatment says that the equation of motion with air resistance is (cf. equation (2) of Problem 9-63 solution)

$$\mathbf{a} = -g\left[\mathbf{k} + \frac{v}{v_t^2}\mathbf{v}\right] + \mathbf{a}_C \tag{3}$$

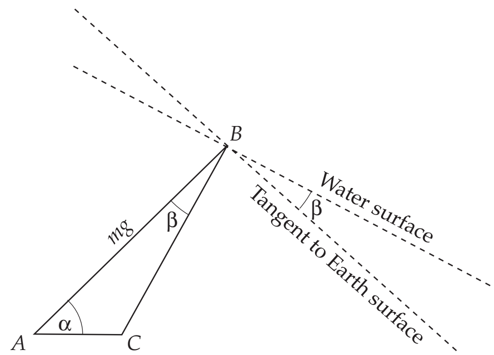
The deflection is calculated to be $\simeq 8.9$ km.

c) Adding the vaiation due to gravity gives us a deflection of $\simeq 10$ km.

d) Adding the variation of air density gives us a deflection of $\simeq 160$ km.

Of general note is that the deflection in all cases was essentially westward. The usual small deflection to the north did not contribute significantly to the total transverse deflection at this precision. All of the heights obtained agreed well with the answers from Problem 9-63. Inclusion of the centrifugal force also does not change the deflections to a significant degree at our precision.

10-17. Due to the centrifugal force, the water surface of the lake is not exactly perpendicular to the Earth's radius (see figure).



The length BC is (using cosine theorem)

$$BC = \sqrt{AC^2 + (mg)^2 - 2ACmg \cos \alpha}$$

where AC is the centrifugal force $AC = m\omega^2 R \cos \alpha$ with α = 47° and Earth's radius $R \cong 6400 \text{ km}$,

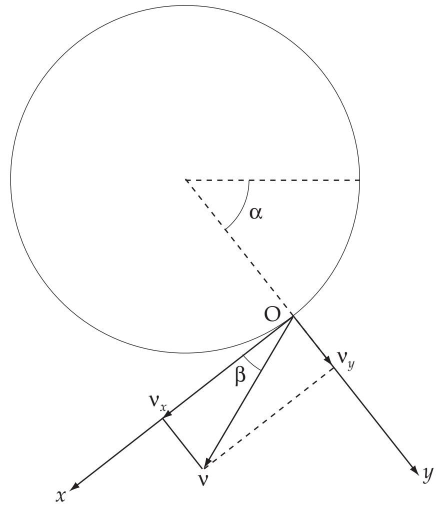
The angle β that the water surface is deviated from the direction tangential to the Earth's surface is

$$\frac{BC}{\sin \alpha} = \frac{AC}{\sin \beta} \quad \Rightarrow \quad \sin \beta = \frac{AC \sin \alpha}{BC} = 4.3 \times 10^{-5}$$

So the distance the lake falls at its center is $h = r \sin \beta$ where r = 162 km is the lake's radius.

So finally we find h = 7 m.

10-18. Let us choose the coordinate system Oxyz as shown in the figure.



The projectile's velocity is

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \begin{pmatrix} v_0 \cos\beta \\ v_0 \sin\beta - gt \\ 0 \end{pmatrix} \quad \text{where } \beta = 37°$$

The Earth's angular velocity is

$$\vec{\omega} = \begin{pmatrix} -\omega\cos\alpha \\ -\omega\sin\alpha \\ 0 \end{pmatrix} \quad \text{where} \quad \alpha = 50°$$

So the Coriolis acceleration is

$$\vec{a}_c = 2\vec{v}\times\vec{\omega} = \left(-2v_0\,\omega\cos\beta\sin\alpha + 2\left(v_0\sin\beta - gt\right)\omega\cos\alpha\right)\mathbf{e}_z$$

The velocity generated by Coriolis force is

$$v_c = \int_0^t a_c\,dt = 2v_0\,\omega t\left(\cos\beta\sin\alpha - \sin\beta\cos\alpha\right) - gt^2\omega\cos\alpha$$

And the distance of deviation due to the Coriolis force is

$$z_c = \int_0^t v_c\,dt = -v_0\,\omega t^2\sin\left(\alpha-\beta\right) - \frac{gt^3\omega\cos\alpha}{3}$$

The flight time of the projectile is $t = \dfrac{2v_0\sin\beta}{2}$. If we put this into z_c, we find the deviation distance due to Coriolis force to be

$$z_c \sim 260\text{ m}$$

10-19. The Coriolis force acting on the car is

$$\vec{F}_c = 2m\,\vec{v}\times\vec{\omega} \Rightarrow \left|\vec{F}_c\right| = 2mv\omega\sin\alpha$$

where α = 65°, m = 1300 kg, v = 100 km/hr.

So $\left|\vec{F}_c\right| = 4.76$ N.

10-20. Given the Earth's mass, $M = 5.976\times10^{24}$ kg, the magnitude of the gravitational field vector at the poles is

$$g_{pole} = \frac{GM}{R_{pole}^2} = 9.866\text{ m/s}^2$$

The magnitude of the gravitational field vector at the equator is

$$g_{eq} = \frac{GM}{R_{eq}^2} - \omega^2\,\mathrm{R}_{eq} = 9.768\text{ m/s}^2$$

where ω is the angular velocity of the Earth about itself.

If one use the book's formula, we have

$$g(\lambda = 90°) = 9.832\text{ m/s}^2\text{ at the poles}$$

and

$$g(\lambda = 0°) = 9.780\text{ m/s}^2\text{ at the equator}$$

10-21. The Coriolis acceleration acting on flowing water is

$$\vec{a}_c = 2\vec{v}\times\vec{\omega} \quad\Rightarrow\quad \left|\vec{a}_c\right| = 2v\omega\sin\alpha$$

Due to this force, the water is higher on the west bank. As in problem 10-17, the angle β that the water surface is deviated from the direction tangential to Earth's surface is

$$\sin\beta = \frac{a_c}{\sqrt{g^2 + a_c^2}} = \frac{2v\omega\sin\alpha}{\sqrt{g^2 + 4v^2\omega^2\sin^2\alpha\propto}} = 2.5\times10^{-5}$$

The difference in heights of the two banks is

$$\Delta h = \ell\sin\beta = 1.2\times10^{-3}\text{ m}$$

where $\ell = 47$ m is the river's width.

10-22. The Coriolis acceleration is $\vec{a}_c = 2\vec{v}\times\vec{\omega}$. This acceleration $\vec{a}_c$ pushes lead bullets eastward with the magnitude $\left|\vec{a}_c\right| = 2v\omega\cos\alpha = 2gt\,\omega\cos\alpha$, where α = 42°.

The velocity generated by the Coriolis force is

$$v_c(t) = \int a\, dt = gt^2 \omega \cos \alpha$$

and the deviation distance is

$$\Delta x_c = \int v_c(t)\, dt = \frac{gt^3}{3} \omega \cos \alpha$$

The falling time of the bullet is $t = \sqrt{2h/g}$. So finally

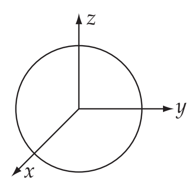
$$\Delta x_c = \frac{\omega}{3} \sqrt{\frac{8h^3}{g}} \cos \alpha = 2.26 \times 10^{-3} \text{ m}$$

CHAPTER 11

Dynamics of Rigid Bodies

11-1. The calculation will be simplified if we use spherical coordinates:

$$\left.\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}\right] \tag{1}$$



Using the definition of the moment of inertia,

$$I_{ij} = \int \rho(r) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j \right] dv \tag{2}$$

we have

$$\begin{aligned} I_{33} &= \rho \int \left(r^2 - z^2 \right) dv \\ &= \rho \int \left(r^2 - r^2 \cos^2\theta \right) r^2 dr\, d(\cos\theta)\, d\phi \end{aligned} \tag{3}$$

or,

$$I_{33} = \rho \int_0^R r^4\, dr \int_{-1}^{+1} \left(1 - \cos^2\theta\right) d(\cos\theta) \int_0^{2\pi} d\phi$$

$$= 2\pi\rho \frac{R^5}{5} \cdot \frac{4}{3} \tag{4}$$

The mass of the sphere is

$$M = \frac{4\pi}{3}\rho R^3 \tag{5}$$

Therefore,

$$I_{33} = \frac{2}{5}MR^2 \tag{6}$$

Since the sphere is symmetrical around the origin, the diagonal elements of $\{I\}$ are equal:

$$I_{11} = I_{22} = I_{33} = \frac{2}{5}MR^2 \tag{7}$$

A typical off-diagonal element is

$$\begin{aligned} I_{12} &= \rho\int(-xy)\,dv \\ &= -\rho\int r^2\sin^2\theta\sin\phi\cos\phi\, r^2 dr\, d(\cos\theta)\, d\phi \end{aligned} \tag{8}$$

This vanishes because the integral with respect to ϕ is zero. In the same way, we can show that all terms except the diagonal terms vanish. Therefore, the secular equation is

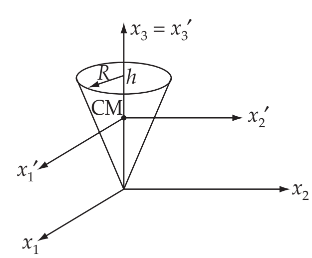
$$\begin{vmatrix} I_{11}-I & 0 & 0 \\ 0 & I_{22}-I & 0 \\ 0 & 0 & I_{33}-I \end{vmatrix} = 0 \tag{9}$$

From (9) and (7), we have

$$\boxed{I_1 = I_2 = I_3 = \frac{2}{5}MR^2} \tag{10}$$

11-2.

a) Moments of inertia with respect to the x_i axes:



It is easily seen that $I_{ij} = 0$ for $i \neq j$. Then the diagonal elements I_{ii} become the principal moments I_i, which we now calculate.

The computation can be simplified by noting that because of the symmetry, $I_1 = I_2 \neq I_3$. Then,

$$I_1 = I_2 = \frac{I_1 + I_2}{2} = \frac{\rho}{2}\int\left(2x_3^2 + x_1^2 + x_2^2\right)dv \tag{1}$$

which, in cylindrical coordinates, can be written as

$$I_1 = I_2 = \frac{\rho}{2}\int_0^{2\pi} d\phi \int_0^h dz \int_0^{Rz/h} \left(r^2 + 2z^2\right) r dr \tag{2}$$

where

$$\rho = \frac{M}{V} = \frac{3M}{\pi R^2 h} \tag{3}$$

Performing the integration and substituting for ρ, we find

$$I_1 = I_2 = \frac{3}{20} M\left(R^2 + 4h^2\right) \tag{4}$$

I_3 is given by

$$I_3 = \rho\int\left(x_1^2 + x_2^2\right) dv = \rho\int r^2 \cdot rdr\, d\phi\, dz \tag{5}$$

from which

$$\boxed{I_3 = \frac{3}{10} MR^2} \tag{6}$$

b) Moments of inertia with respect to the x_i' axes:

Because of the symmetry of the body, the center of mass lies on the x_3' axis. The coordinates of the center of mass are $(0,0,z_0)$, where

$$z_0 = \frac{\int x_3'\, dv}{\int dv} = \frac{3}{4} h \tag{7}$$

Then, using Eq. (11.49),

$$I_{ij}' = I_{ij} - M\left[a^2\delta_{ij} - a_i a_j\right] \tag{8}$$

In the present case, $a_1 = a_2 = 0$ and $a_3 = (3/4)h$, so that

$$\boxed{\begin{aligned} I_1' &= I_1 - \frac{9}{16} Mh^2 = \frac{3}{20} M\left(R^2 + \frac{1}{4} h^2\right) \\ I_2' &= I_2 - \frac{9}{16} Mh^2 = \frac{3}{20} M\left(R^2 + \frac{1}{4} h^2\right) \\ I_3' &= I_3 - \frac{3}{10} MR^2 \end{aligned}}$$

11-3. The equation of an ellipsoid is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \tag{1}$$

which can be written in normalized form if we make the following substitutions:

$$x_1 = a\xi, \quad x_2 = b\eta, \quad x_3 = c\zeta \tag{2}$$

Then, Eq. (1) reduces to

$$\xi^2 + \eta^2 + \zeta^2 = 1 \tag{3}$$

This is the equation of a sphere in the (ξ,η,ζ) system.

If we denote by dv the volume element in the x_i system and by $d\tau$ the volume element in the (ξ,η,ζ) system, we notice that the volume of the ellipsoid is

$$\begin{aligned} V &= \int dv = \int dx_1\, dx_2\, dx_3 = abc \int d\xi\, d\eta\, d\zeta \\ &= abc \int d\tau = \frac{4}{3}\pi\, abc \end{aligned} \tag{4}$$

because $\int d\tau$ is just the volume of a sphere of unit radius.

The rotational inertia with respect to the x_3-axis passing through the center of mass of the ellipsoid (we assume the ellipsoid to be homogeneous), is given by

$$\begin{aligned} I_3 &= \frac{M}{V} \int \left(x_1^2 + x_2^2 \right) dv \\ &= \frac{M}{V}\, abc \int \left(a^2\xi^2 + b^2\eta^2 \right) d\tau \end{aligned} \tag{5}$$

In order to evaluate this integral, consider the following equivalent integral in which $z = r\cos\theta$:

$$\begin{aligned} \int a^2 z^2\, dv &= \int a^2 z^2 \left(r\, dr\, r\sin\theta\, d\theta\, d\phi \right) \\ &= a^2 \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \int_0^{R=1} r^4\, dr \\ &= a^2 \times 2\pi \times \frac{2}{3} \times \frac{1}{5} \\ &= \frac{4\pi a^2}{15} \end{aligned} \tag{6}$$

Therefore,

$$\int \left(a^2\xi^2 + b^2\eta^2 \right) d\tau = \frac{4\pi}{15}\left(a^2 + b^2 \right) \tag{7}$$

and

$$\boxed{I_3 = \frac{1}{5} M\left(a^2 + b^2 \right)} \tag{8}$$

Since the same analysis can be applied for any axis, the other moments of inertia are

$$\boxed{\begin{aligned} I_1 &= \frac{1}{5} M\left(b^2 + c^2\right) \\ I_2 &= \frac{1}{5} M\left(a^2 + c^2\right) \end{aligned}} \tag{9}$$

11-4.

The linear density of the rod is

$$\rho_\ell = \frac{m}{\ell} \tag{1}$$

For the origin at one end of the rod, the moment of inertia is

$$I = \int_0^\ell \rho_\ell\, x^2\, dx = \frac{m}{\ell}\frac{\ell^3}{3} = \frac{m}{3}\ell^2 \tag{2}$$

If all of the mass were concentrated at the point which is at a distance *a* from the origin, the moment of inertia would be

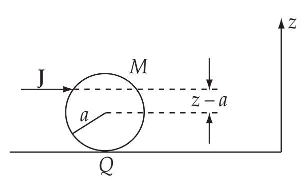
$$I = ma^2 \tag{3}$$

Equating (2) and (3), we find

$$\boxed{a = \frac{\ell}{\sqrt{3}}} \tag{4}$$

This is the *radius of gyration*.

11-5.



a) The solid ball receives an impulse **J**; that is, a force **F**(t) is applied during a short interval of time τ so that

$$\mathbf{J} = \int \mathbf{F}(t')\, dt' \tag{1}$$

The equations of motion are

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{2}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} \tag{3}$$

which, for this case, yield

$$\Delta\mathbf{p} = \int \mathbf{F}(t')\,dt' = \mathbf{J} \tag{4}$$

$$\Delta\mathbf{L} = \int \mathbf{r} \times \mathbf{F}(t')\,dt' = \mathbf{r} \times \mathbf{J} \tag{5}$$

Since $\mathbf{p}(t = 0) = 0$ and $\mathbf{L}(t = 0) = 0$, after the application of the impulse, we have

$$\mathbf{p} = \mathbf{MV}_{\text{CM}} = \mathbf{J}; \quad \mathbf{L} = I_0\,\boldsymbol{\omega} = \mathbf{r} \times \mathbf{J} = (z-a)\,J\,\frac{\boldsymbol{\omega}}{\omega} \tag{6}$$

so that

$$\mathbf{V}_{\text{CM}} = \frac{\mathbf{J}}{\mathbf{M}} \tag{7}$$

and

$$\boldsymbol{\omega} = \frac{J}{I_0}(z-a)\,\frac{\boldsymbol{\omega}}{\omega} \tag{8}$$

where $I_0 = (2/5)Ma^2$.

The velocity of any point a on the ball is given by Eq. (11.1):

$$\mathbf{v}_\alpha = \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \mathbf{r}_\alpha \tag{9}$$

For the point of contact Q, this becomes

$$\begin{aligned} \mathbf{v}_Q &= \mathbf{V}_{\text{CM}} - \omega a\,\frac{\mathbf{J}}{J} \\ &= \frac{J}{M}\left[1 - \frac{5(z-a)}{2a}\right] \end{aligned} \tag{10}$$

Then, for rolling without slipping, $\mathbf{v}_Q = 0$, and we have

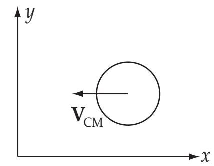
$$2a = 5(z-a) \tag{11}$$

so that

$$\boxed{z = \frac{7}{5}a} \tag{12}$$

b) Many billiard tricks are performed by striking the ball at different heights and at different angles in order to impart slipping and spinning motion (“English”). For the table not to introduce spurious effects, the rail must be at such a height that the ball will be “reflected” upon collision.

Consider the case in which the ball is incident normally on the rail, as in the diagram. We have the following relationships:



	Before Collision	After Collision
Linear Momentum	$p_x = -MV_{CM}$ $p_y = 0$	$p'_x = +MV_{CM}$ $p'_y = 0$
Angular Momentum	$L_x = 0$ $L_y = *$ $L_z = 0$	$L'_x = 0$ $L'_y = -Ly$ $L'_z = 0$

* The relation between L_y and V_{CM} depends on whether or not slipping occurs.

Then, we have

$$\Delta p = -2p_x = J = 2MV_{CM} \tag{13}$$

$$\Delta L = -2L_y = 2I_0\ \omega = J(z-a) \tag{14}$$

so that

$$2I_0\ \omega = 2MV_{CM}\,(z-a) \tag{15}$$

from which

$$z - a = \frac{I_0\ \omega}{MV_{CM}} = \frac{2}{5}\frac{Ma^2\omega}{MV_{CM}} = \frac{2}{5}\frac{a^2\omega}{V_{CM}} \tag{16}$$

If we assume that the ball rolls without slipping before it contacts the rail, then $V_{CM} = \omega a$, and we obtain the same result as before, namely,

$$z - a = \frac{2}{5}a \tag{17}$$

or,

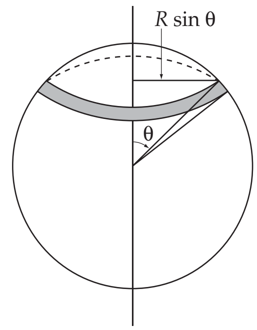
$$\boxed{z = \frac{7}{5}a} \tag{18}$$

Thus, the height of the rail must be at a height of $(2/5)a$ above the center of the ball.

11-6. Let us compare the moments of inertia for the two spheres for axes through the centers of each. For the solid sphere, we have

$$I_s = \frac{2}{5}MR^2 \qquad \text{(see Problem 11-1)} \tag{1}$$

For the hollow sphere,



$$I_h = \sigma \int_0^{2\pi} d\phi \int_0^{\pi} \left(R\sin\theta\right)^2 R^2 \sin\theta \, d\theta$$

$$= 2\pi\sigma R^4 \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{8}{3}\pi\sigma R^4$$

or, using $4\pi\sigma R^2 = M$, we have

$$I_h = \frac{2}{3} MR^2 \tag{2}$$

Let us now roll each ball down an inclined plane. [Refer to Example 7.9.] The kinetic energy is

$$T = \frac{1}{2} M\,\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2 \tag{3}$$

where y is the measure of the distance along the plane. The potential energy is

$$U = Mg\left(\ell - y\right)\sin\alpha \tag{4}$$

where ℓ is the length of the plane and α is the angle of inclination of the plane. Now, $y = R\theta$, so that the Lagrangian can be expressed as

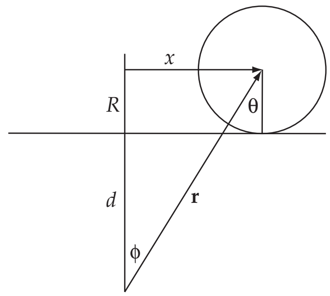
$$L = \frac{1}{2} M\,y^2 + \frac{1}{2}\frac{I}{R^2}\,y^2 + Mgy\sin\alpha \tag{5}$$

where the constant term in U has been suppressed. The equation of motion for y is obtained in the usual way and we find

$$\ddot{y} = \frac{gMR^2\sin\alpha}{MR^2 + I} \tag{6}$$

Therefore, the sphere with the *smaller* moment of inertia (the solid sphere) will have the *greater* acceleration down the plane.

11-7.



The force between the force center and the disk is, from the figure

$$\mathbf{F} = -k\mathbf{r} \tag{1}$$

Only the component along x does any work, so that the effective force is $F_x = -kr \sin\phi = -kx$. This corresponds to a potential $U = kx^2/2$. The kinetic energy of the disk is

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{3}{4}M\dot{x}^2 \tag{2}$$

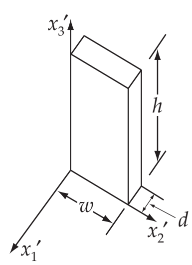
where we use the result $I = MR^2/2$ for a disk and $dx = R\,d\theta$. Lagrange's equations give us

$$\frac{3}{2}M\ddot{x} + kx = 0 \tag{3}$$

This is simple harmonic motion about $x = 0$ with an angular frequency of oscillations

$$\omega = \sqrt{\frac{2k}{3m}} \tag{4}$$

11-8.



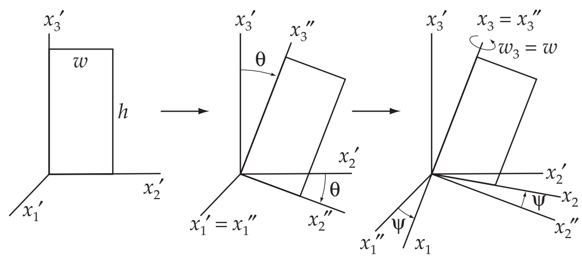
We let x_3' be the vertical axis in the fixed system. This would be the axis (i.e., the hinge line) of the door if it were properly hung (no self-rotation), as indicated in the diagram. The mass of the door is $M=\rho whd$.

The moment of inertia of the door around the x_3' axis is

$$I_3 = \frac{m}{whd}\int_0^h dh' \int_0^w w'^2\, d\, dw' = \frac{1}{3}Mw^2 \tag{1}$$

where the door is considered to be a thin plate, i.e., $d \ll w,h$.

The initial position of the self-closing door can be expressed as a two-step transformation, starting with the position in the diagram above. The first rotation is around the x_1'-axis through an angle θ and the second rotation is around the x_1''-axis through an angle ψ:



The x_1'-axes are the fixed-system axes and the x_i-axes are the body system (or rotating) axes which are attached to the door. Here, the Euler angle ϕ is zero.

The rotation matrix that transforms the fixed axes into the body axes $\left(x_i' \to x_i\right)$ is just Eq. (11.99) with $\phi = 0$ and $\theta \to -\theta$ since this rotation is performed clockwise rather than counterclockwise as in the derivation of Eq. (11.99):

$$\lambda = \begin{bmatrix} \cos\psi & \cos\theta\sin\psi & -\sin\theta\sin\psi \\ -\sin\psi & \cos\theta\cos\psi & -\sin\theta\cos\psi \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

The procedure is to find the torque acting on the door expressed in the fixed coordinate system and then to obtain the x_3 component, i.e., the component in the body system. Notice that when the door is released from rest at some initial angle ψ_0, the rotation is in the direction to *decrease* ψ. According to Eq. (11.119),

$$I_3\dot{\omega}_3 = N_3 = I_3\,\ddot{\psi} \tag{3}$$

where $\omega_1 = \omega_2 = 0$ since $\dot{\phi} = \dot{\theta} = 0$.

In the body (x_i) system the coordinates of the center of mass of the door are

$$R = \frac{1}{2}\begin{bmatrix} 0 \\ w \\ h \end{bmatrix} \tag{4}$$

where we have set the thickness equal to zero. In the fixed (x_i') system, these coordinates are obtained by applying the inverse transformation λ^{-1} to R; but $\lambda^{-1} = \lambda^t$, so that

$$\mathbf{R}' = \lambda^t\mathbf{R} = \frac{1}{2}\begin{bmatrix} -w\sin\psi \\ w\cos\theta\cos\psi + h\sin\theta \\ -w\sin\theta\cos\psi + h\cos\theta \end{bmatrix} \tag{5}$$

Now, the gravitational force acting on the door is downward, and in the x_i' coordinate system is

$$\mathbf{F}' = -Mg\,\mathbf{e}_3' \tag{6}$$

There the torque on the door, expressed in the fixed system, is

$$\begin{aligned}\mathbf{N}' &= \mathbf{R}' \times \mathbf{F}' \\ &= -\frac{1}{2} Mg \begin{bmatrix} \mathbf{e}_1' & \mathbf{e}_2' & \mathbf{e}_3' \\ -w \sin\psi & w\cos\theta\cos\psi + h\sin\theta & -w\sin\theta\cos\psi + h\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \\ &= -\frac{1}{2} Mg \begin{bmatrix} w\cos\theta\cos\psi + h\sin\theta \\ w\sin\psi \\ 0 \end{bmatrix}\end{aligned} \tag{7}$$

so that in the body system we have

$$\mathbf{N}' = \lambda\mathbf{N}' = -\frac{1}{2} Mg \begin{bmatrix} w\cos\theta\cos^2\psi + h\sin\theta\cos\psi + w\cos\theta\sin^2\psi \\ -h\sin\theta\sin\psi \\ w\sin\theta\sin\psi \end{bmatrix} \tag{8}$$

Thus,

$$-N_3 = \frac{1}{2} Mgw\sin\theta\sin\psi \tag{9}$$

and substituting this expression into Eq. (3), we have

$$-\frac{1}{2} Mgw\sin\theta\sin\psi = I_3\,\ddot{\psi} = \frac{1}{3} Mw^2\,\ddot{\psi} \tag{10}$$

where we have used Eq. (1) for I_3. Solving for $\ddot{\psi}$,

$$\ddot{\psi} = -\frac{3}{2}\frac{g}{w}\sin\theta\sin\psi \tag{11}$$

This equation can be integrated by first multiplying by $\dot{\psi}$:

$$\begin{aligned}\int \ddot{\psi}\,\dot{\psi}\,dt = \frac{1}{2}\dot{\psi}^2 &= -\frac{3}{2}\frac{g}{w}\sin\theta\int\sin\psi\,\dot{\psi}\,dt \\ &= \frac{3}{2}\frac{g}{w}\sin\theta\cos\psi\end{aligned} \tag{12}$$

where the integration constant is zero since $\cos\psi = 0$ when $\dot{\psi} = 0$. Thus,

$$\dot{\psi} = \pm\sqrt{\frac{3g}{w}\sin\theta\cos\psi} \tag{13}$$

We must choose the negative sign for the radical since $\dot{\psi} < 0$ when $\cos\psi > 0$. Integrating again, from $\psi = 90°$ to $\psi = 0°$,

$$\int_{\frac{\pi}{2}}^{0}\frac{d\psi}{\sqrt{\cos\psi}} = -\sqrt{\frac{3g}{w}\sin\theta}\int_0^T dt \tag{14}$$

where T = 2 sec. Rewriting Eq. (14),

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\cos\psi}} = T\sqrt{\frac{3g}{w}\sin\theta} \tag{15}$$

Using Eq. (E. 27a), Appendix E, we find

$$\int_0^{\frac{\pi}{2}} \cos^{-1/2}\psi\, d\psi = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left[\frac{1}{4}\right]}{\Gamma\left[\frac{3}{4}\right]} \tag{16}$$

From Eqs. (E.20) and (E.23),

$$\frac{1}{4}\Gamma\left[\frac{1}{4}\right] = \Gamma\left[1\frac{1}{4}\right] = 0.906$$

$$\Gamma\left[\frac{1}{4}\right] = 3.624 \tag{17}$$

And from Eqs. (E.20) and (E.24),

$$\frac{3}{4}\Gamma\left[\frac{3}{4}\right] = \Gamma\left[1\frac{3}{4}\right] = 0.919$$

$$\Gamma\left[\frac{3}{4}\right] = 1.225 \tag{18}$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{\cos\psi}} = \frac{\sqrt{\pi}}{2}\frac{3.624}{1.225} = 2.62 \tag{19}$$

Returning to Eq. (15) and solving for $\sin\theta$,

$$\sin\theta = \frac{w}{3gT^2} \times (2.62)^2 \tag{20}$$

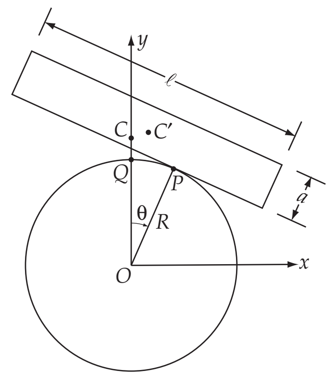
Inserting the values for g, w(= 1m), and T(= 2 sec), we find

$$\theta = \sin^{-1}(0.058)$$

or,

$$\boxed{\theta \cong 3.33°} \tag{21}$$

11-9.



The diagram shows the slab rotated through an angle θ from its equilibrium position. At equilibrium the contact point is Q and after rotation the contact point is P. At equilibrium the position of the center of mass of the slab is C and after rotation the position is C'.

Because we are considering only small departures from $\theta = 0$, we can write

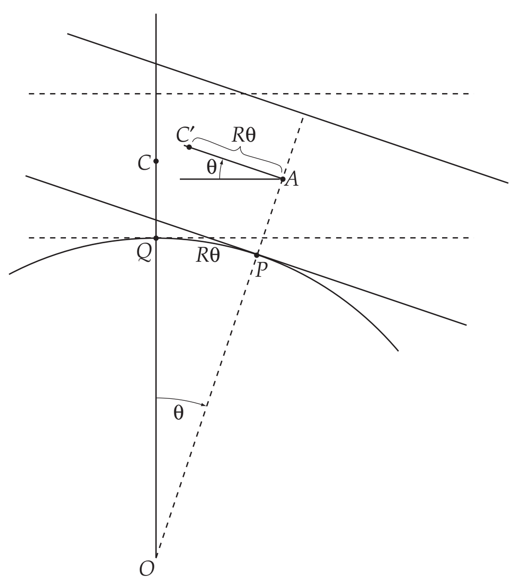
$$\overline{QP} \cong R\theta \tag{1}$$

Therefore, the coordinates of C′ are (see enlarged diagram below)

$$\mathbf{r} = \mathbf{OA} + \mathbf{AC}' \tag{2}$$

so that

$$\left.\begin{aligned} x &= \left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta \\ y &= \left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta \end{aligned}\right] \tag{3}$$



Consequently,

$$\dot{x} = \left[\left(R + \frac{a}{2}\right)\cos\theta - R\cos\theta + R\theta\sin\theta\right]\dot{\theta}$$

$$= \left(\frac{a}{2}\cos\theta + R\theta\sin\theta\right)\dot{\theta}$$

$$\dot{y} = \left[-\left(R + \frac{a}{2}\right)\sin\theta + R\theta\cos\theta + R\sin\theta\right]\dot{\theta}$$

$$= \left(-\frac{a}{2}\sin\theta + R\theta\cos\theta\right)\dot{\theta}$$

from which

$$\dot{x}^2 + \dot{y}^2 = \left(\frac{a^2}{4} + R^2\theta^2\right)\dot{\theta}^2 \tag{4}$$

The kinetic energy is

$$T = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 \tag{5}$$

where I is the moment of inertia of the slab with respect to an axis passing through the center of mass and parallel to the z-axis:

$$I = \frac{1}{12}M\left(\ell^2 + a^2\right) \tag{6}$$

Therefore,

$$T = \frac{1}{2}f_1(\theta)\dot{\theta}^2 \tag{7}$$

where

$$f_1(\theta) = M\left(\frac{a^2}{4} + R^2\theta^2\right) + I \tag{8}$$

The potential energy is

$$U = Mgy = -f_2(\theta) \tag{9}$$

where

$$f_2(\theta) = -Mg\left[\left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta\right] \tag{10}$$

and where Eq. (3) has been used for y.

The Lagrangian is

$$L = \frac{1}{2}f_1(\theta)\dot{\theta}^2 + f_2(\theta) \tag{11}$$

The Lagrange equation for θ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{12}$$

Now,

$$\frac{\partial L}{\partial \dot{\theta}} = f_1(\theta)\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = f_1(\theta)\ddot{\theta} + \dot{f}_1(\theta)\dot{\theta}$$

$$= \left[M\left(\frac{a^2}{4} + R^2\theta^2\right) + I\right]\ddot{\theta} + 2MR^2\,\theta\,\dot{\theta}^2 \tag{13}$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}f_1'(\theta)\dot{\theta}^2 + f_2'(\theta)$$

$$= MR^2\,\theta\,\dot{\theta}^2 + Mg\left[\left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta - R\sin\theta\right] \tag{14}$$

Combining, we find

$$\left[M\left(\frac{a^2}{4} + R^2\theta^2\right) + I\right]\ddot{\theta} + MR^2\,\theta\,\dot{\theta}^2 - Mg\left[\left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta - R\sin\theta\right] = 0 \tag{15}$$

For the case of small oscillations, $\theta^2 \ll \theta$ and $\dot{\theta}^2 \ll \dot{\theta}$, so that Eq. (15) reduces to

$$\ddot{\theta} + \frac{Mg\left(R - \frac{a}{2}\right)}{\frac{Ma^2}{4} + I}\theta = 0 \tag{16}$$

The system is stable for oscillations around $\theta = 0$ only if

$$\frac{Mg\left(R - \frac{a}{2}\right)}{\frac{Ma^2}{4} + I} = \omega^2 > 0 \tag{17}$$

This condition is satisfied if $R - a/2 > 0$, i.e.,

$$\boxed{R > \frac{a}{2}} \tag{18}$$

Then, the frequency is

$$\omega = \sqrt{\frac{Mg\left(R - \frac{a}{2}\right)}{\frac{Ma^2}{4} + \frac{1}{12}M\left(\ell^2 + a^2\right)}} \tag{19}$$

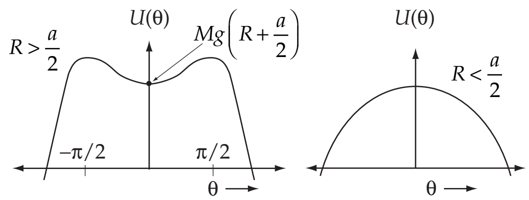
Simplifying, we have

$$\boxed{\omega = \sqrt{\frac{12g\left(R - \dfrac{a}{2}\right)}{\left(\ell^2 + 4a^2\right)}}} \tag{20}$$

According to Eqs. (9) and (10), the potential energy is

$$U(\theta) = Mg\left[\left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta\right] \tag{21}$$

This function has the following forms for $R > a/2$ and $R < a/2$:



To verify that a stable condition exists only for $R > a/2$, we need to evaluate $\partial^2 U/\partial\theta^2$ at $\theta = 0$:

$$\frac{\partial U}{\partial\theta} = Mg\left[-\frac{a}{2}\sin\theta + R\theta\cos\theta\right] \tag{22}$$

$$\frac{\partial^2 U}{\partial\theta^2} = Mg\left[-\frac{a}{2}\cos\theta + R\cos\theta - R\theta\sin\theta\right] \tag{23}$$

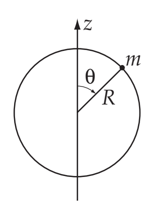
and

$$\left.\frac{\partial^2 U}{\partial\theta^2}\right|_{\theta=0} = Mg\left(R - \frac{a}{2}\right) \tag{24}$$

so that

$$\frac{\partial^2 U}{\partial\theta^2} > 0 \text{ if } R > \frac{a}{2} \tag{25}$$

11-10.



When the mass m is at one pole, the z component of the angular momentum of the system is

$$L_z = I\omega = \frac{2}{5}MR^2\omega \tag{1}$$

After the mass has moved a distance $vt = R\theta$ along a great circle on the surface of the sphere, the z component of the angular momentum of the system is

$$L_z = \left[\frac{2}{5} MR^2 + mR^2 \sin^2 \theta \right] \dot{\phi} \tag{2}$$

where $\dot{\phi}$ is the new angular velocity. Since there is no external force acting on the system, angular momentum must be conserved. Therefore, equating (1) and (2), we have

$$\dot{\phi} = \frac{\frac{2}{5} MR^2 \omega}{\frac{2}{5} MR^2 + mR^2 \sin^2 \theta} \tag{3}$$

Substituting $\theta = vt/R$ and integrating over the time interval during which the mass travels from one pole to the other, we have

$$\phi = \int_{t=0}^{t=\frac{\pi R}{V}} \frac{\frac{2}{5} MR^2 \omega}{\frac{2}{5} MR^2 + mR^2 \sin^2 (vt/R)} dt \tag{4}$$

Making the substitutions,

$$vt/R \equiv u, \quad dt = (R/v)\, du \tag{5}$$

we can rewrite (4) as

$$\phi = \int_0^{\pi} \frac{\frac{2}{5} MR^2 \omega}{\frac{2}{5} MR^2 + mR^2 \sin^2 u} \frac{R}{v} du$$

$$= \frac{2R\omega}{v} \int_0^{\pi/2} \frac{du}{1 + \beta \sin^2 u} \tag{6}$$

where $\beta \equiv 5m/2M$ and where we have used the fact that the integrand is symmetric around $u = \pi/2$ to write ϕ as twice the value of the integral over half the range. Using the identity

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \tag{7}$$

we express (6) as

$$\phi = \frac{2R\omega}{v} \int_0^{\pi/2} \frac{du}{\left(1 + \frac{1}{2}\beta\right) - \frac{1}{2}\beta \cos 2u} \tag{8}$$

or, changing the variable to $x = 2u$,

$$\phi = \frac{R\omega}{v} \int_0^{\pi} \frac{dx}{\left(1 + \frac{1}{2}\beta\right) - \frac{1}{2}\beta \cos x} \tag{9}$$

Now, we can use Eq. (E.15), Appendix E, to obtain

$$\phi = \frac{2R\omega}{v\sqrt{1+\beta}} \tan^{-1}\left[\frac{(1+\beta)\tan(x/2)}{\sqrt{1+\beta}}\right]\Bigg|_0^\pi$$

$$= \frac{\pi R\omega}{v}\sqrt{\frac{2M}{2M+5m}}$$

$$= \omega T\sqrt{\frac{2M}{2M+5m}} \tag{10}$$

where $T = \pi R/v$ is the time required for the particle to move from one pole to the other.

If $m = 0$, (10) becomes

$$\phi(m=0) = \omega T \tag{11}$$

Therefore, the angle of retardation is

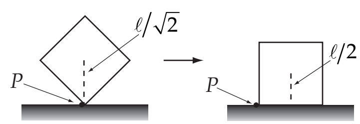
$$\alpha = \phi(m=0) - \phi(m) \tag{12}$$

or,

$$\boxed{\alpha = \omega T\left[1 - \sqrt{\frac{2M}{2M+5m}}\right]} \tag{13}$$

11-11.

a) No sliding:



From energy conservation, we have

$$mg\frac{\ell}{\sqrt{2}} = mg\frac{\ell}{2} + \frac{1}{2}mv_{\text{C.M.}}^2 + \frac{1}{2}I\omega^2 \tag{1}$$

where v_{CM} is the velocity of the center of mass when one face strikes the plane; $v_{\text{C.M.}}$ is related to ω by

$$v_{\text{CM}} = \frac{\ell}{\sqrt{2}}\omega \tag{2}$$

I is the moment of inertia of the cube with respect to the axis which is perpendicular to one face and passes the center:

$$I = \frac{1}{6}m\ell^2 \tag{3}$$

Then, (1) becomes

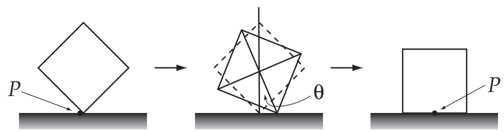
$$\frac{mg\ell}{2}\left(\sqrt{2}-1\right)=\frac{1}{2}m\left[\frac{\ell\omega}{\sqrt{2}}\right]^2+\frac{1}{2}\left[\frac{m\ell^2}{6}\right]\omega^2=\frac{1}{3}m\ell^2\omega^2 \tag{4}$$

from which, we have

$$\boxed{\omega^2=\frac{3}{2}\frac{g}{\ell}\left(\sqrt{2}-1\right)} \tag{5}$$

b) Sliding without friction:

In this case there is no external force along the horizontal direction; therefore, the cube slides so that the center of mass falls directly downward along a vertical line.



While the cube is falling, the distance between the center of mass and the plane is given by

$$y=\frac{\ell}{\sqrt{2}}\cos\theta \tag{6}$$

Therefore, the velocity of center of mass when one face strikes the plane is

$$\dot{y}\Big|_{0=\pi/4}=-\frac{\ell}{\sqrt{2}}\sin\theta\,\dot{\theta}\Big|_{\theta=\pi/4}=-\frac{1}{2}\ell\dot{\theta}=-\frac{1}{2}\ell\omega \tag{7}$$

From conservation of energy, we have

$$mg\frac{\ell}{\sqrt{2}}=mg\frac{\ell}{2}+\frac{1}{2}m\left(-\frac{1}{2}\ell\omega\right)^2+\frac{1}{2}\left(\frac{1}{6}m\ell^2\right)\omega^2 \tag{8}$$

from which we have

$$\boxed{\omega^2=\frac{12}{5}\frac{g}{\ell}\left(\sqrt{2}-1\right)} \tag{9}$$

11-12. According to the definition of the principal moments of inertia,

$$\begin{aligned}I_j+I_k&=\int\rho\left(x_i^2+x_k^2\right)dv+\int\rho\left(x_i^2+x_j^2\right)dv\\&=\int\rho\left(x_j^2+x_k^2\right)dv+2\int\rho\,x_i^2\,dv\\&=I_i+2\int\rho\,x_i^2\,dv\end{aligned} \tag{1}$$

since

$$\int\rho\,x_i^2\,dv>0$$

we have

$$\boxed{I_j + I_k \geq I_i} \tag{2}$$

11-13. We get the elements of the inertia tensor from Eq. 11.13a:

$$I_{11} = \sum_\alpha m_\alpha \left(x_{\alpha,2}^2 + x_{\alpha,3}^2 \right)$$

$$= 3m\left(b^2\right) + 4m\left(2b^2\right) + 2m\left(b^2\right) = 13mb^2$$

Likewise $I_{22} = 16mb^2$ and $I_{33} = 15mb^2$

$$I_{12} = I_{21} = -\sum_\alpha m_\alpha\, x_{\alpha,1}\, x_{\alpha,2}$$

$$= -4m\left(b^2\right) - 2m\left(-b^2\right) = -2mb^2$$

Likewise $I_{13} = I_{31} = mb^2$

and $I_{23} = I_{32} = 4mb^2$

Thus the inertia tensor is

$$\boxed{\{I\} = mb^2 \begin{bmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{bmatrix}}$$

The principal moments of inertia are gotten by solving

$$mb^2 \begin{bmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{bmatrix} = 0$$

Expanding the determinant gives a cubic equation in λ:

$$\lambda^3 - 44\lambda^2 + 622\lambda - 2820 = 0$$

Solving numerically gives

$$\lambda_1 = 10.00$$

$$\lambda_2 = 14.35$$

$$\lambda_3 = 19.65$$

$$\boxed{\begin{array}{c} \text{Thus the principal moments of inertia are } I_1 = 10\ mb^2 \\ I_2 = 14.35\ mb^2 \\ I_3 = 19.65\ mb^2 \end{array}}$$

To find the principal axes, we substitute into (see example 11.3):

$$\left(13-\lambda_i\right)\omega_{1i}-2\omega_{2i}+\omega_{3i}=0$$

$$-2\,\omega_{1i}+\left(16-\lambda_i\right)\omega_{2i}+4\omega_{3i}=0$$

$$\omega_{1i}+4\omega_{2i}+\left(15-\lambda_i\right)\omega_{3i}=0$$

For i = 1, we have $\left(\lambda_1=10\right)$

$$3\omega_{11}-2\omega_{21}+\omega_{31}=0$$

$$-2\omega_{11}-6\omega_{21}+4\omega_{31}=0$$

$$\omega_{11}-4\omega_{21}+5\omega_{31}=0$$

Solving the first for ω_{31} and substituting into the second gives

$$\omega_{11}=\omega_{21}$$

Substituting into the third now gives

$$\omega_{31}=-\omega_{21}$$

or

$$\omega_{11}:\omega_{21}:\omega_{31}=1:1:-1$$

So, the principal axis associated with I_1 is

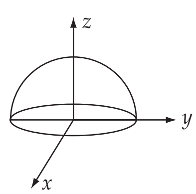
$$\boxed{\frac{1}{\sqrt{3}}\left(x+y-z\right)}$$

Proceeding in the same way gives the other two principal axes:

$$\boxed{\begin{array}{ll} i=2: & -.81\mathbf{x}+.29\mathbf{y}-.52\mathbf{z} \\ i=3: & -.14\mathbf{x}+.77\mathbf{y}+.63\mathbf{z} \end{array}}$$

We note that the principal axes are mutually orthogonal, as they must be.

11-14.



Let the surface of the hemisphere lie in the x-y plane as shown. The mass density is given by

$$\rho=\frac{M}{V}=\frac{M}{\dfrac{2}{3}\pi b^3}=\frac{3M}{2\pi b^3}$$

First, we calculate the center of mass of the hemisphere. By symmetry

$$x_{\text{CM}} = y_{\text{CM}} = 0$$

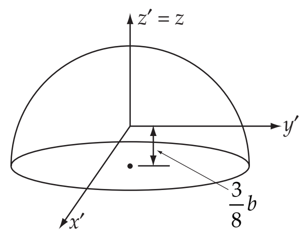
$$z_{\text{CM}} = \frac{1}{M} \int_v \rho z \, dv$$

Using spherical coordinates $(z = r \cos\theta, dv = r^2 \sin\theta \, dr \, d\theta \, d\phi)$ we have

$$z_{\text{CM}} = \frac{\rho}{M} \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} \sin\theta \cos\theta \, d\theta \int_{r=0}^{b} r^3 \, dr$$

$$= \left[\frac{3}{2\pi b^3}\right](2\pi)\left[\frac{1}{2}\right]\left[\frac{1}{4} b^4\right] = \frac{3}{8} b$$

We now calculate the inertia tensor with respect to axes passing through the center of mass:



By symmetry, $I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = 0$. Thus the axes shown are the principal axes.

Also, by symmetry $I_{11} = I_{22}$. We calculate I_{11} using Eq. 11.49:

$$I_{11} = J_{11} - M\left[\frac{3}{8} v\right]^2 \tag{1}$$

where J_{11} = the moment of inertia with respect to the original axes

$$J_{11} = \rho \int_v \left(y^2 + z^2\right) dv$$

$$= \int_v \left(r^2 \sin^2\theta \sin^2\phi + r^2 \cos^2\theta\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{3M}{2\pi b^3} \int_{r=0}^{b} r^4 \, dr \int_{\theta=0}^{\pi/2} \left[\int_{\phi=0}^{2\pi} \left(\sin^2\theta \sin^2\phi + \cos^2\theta\right) d\phi\right] \sin\theta \, d\theta$$

$$= \frac{3Mb^2}{10\pi} \int_{\theta=0}^{\pi/2} \left(\pi \sin^3\theta + 2\pi \cos^2\theta \sin\theta\right) d\theta$$

$$= \frac{2}{5} Mb^2$$

Thus, from (1)

$$I_{11} = I_{22} = \frac{2}{5} Mb^2 - \frac{9}{64} Mb^2 = \frac{83}{320} Mb^2$$

Also, from Eq. 11.49

$$I_{33} = J_{33} - M(0) = J_{33}$$

($I_{33} = J_{33}$ should be obvious physically)

So

$$I_{33} = \rho \int_v \left(x^2 + y^2\right) dv$$

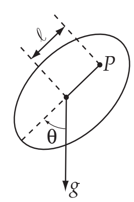
$$= \rho \int_v r^4 \sin^3 \theta \, dr \, d\theta \, d\phi = \frac{2}{5} Mb^2$$

Thus, the principal axes are the primed axes shown in the figure. The principal moments of inertia are

$$\mathrm{I}_{11} = I_{22} = \frac{83}{320} Mb^2$$

$$\mathrm{I}_{33} = \frac{2}{5} Mb^2$$

11-15.



We suspend the pendulum from a point P which is a distance ℓ from the center of mass. The rotational inertia with respect to an axis through P is

$$I = MR_0^2 + M\ell^2 \tag{1}$$

where R_0 is the radius of gyration about the center of mass. Then, the Lagrangian of the system is

$$L = T - U = \frac{I\dot{\theta}^2}{2} - Mg\ell\left(1 - \cos\theta\right) \tag{2}$$

Lagrange's equation for θ gives

$$I\ddot{\theta} + Mg\ell \sin\theta = 0 \tag{3}$$

For small oscillation, $\sin\theta \cong \theta$. Then,

$$\ddot{\theta} + \frac{Mg\ell}{I}\theta = 0 \tag{4}$$

or,

$$\ddot{\theta} + \frac{g\ell}{R_0^2 + \ell^2}\theta = 0 \tag{5}$$

from which the period of oscillation is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_0^2 + \ell^2}{g\ell}} \tag{6}$$

If we locate another point P' which is a distance ℓ' from the center of mass such that the period of oscillation is also τ, we can write

$$\frac{R_0^2 + \ell^2}{g\ell} = \frac{R_0^2 + \ell'^2}{g\ell'} \tag{7}$$

from which $R_0^2 = \ell\ell'$. Then, the period must be

$$\tau = 2\pi\sqrt{\frac{\ell\ell' + \ell^2}{g\ell}} \tag{8}$$

or,

$$\boxed{\tau = 2\pi\sqrt{\frac{\ell + \ell'}{g}}} \tag{9}$$

This is the same as the period of a simple pendulum of the length $\ell + \ell'$. Using this method, one does not have to measure the rotational inertia of the pendulum used; nor is one faced with the problem of approximating a simple pendulum physically. On the other hand, it is necessary to locate the two points for which τ is the same.

11-16. The rotation matrix is

$$(\boldsymbol{\lambda}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The moment of inertia tensor transforms according to

$$(\mathbf{I}') = (\boldsymbol{\lambda})(\mathbf{I})(\boldsymbol{\lambda}^t) \tag{2}$$

That is

$$(I') = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(A+B)\cos\theta + \frac{1}{2}(A-B)\sin\theta & -\frac{1}{2}(A+B)\sin\theta + \frac{1}{2}(A-B)\cos\theta & 0 \\ \frac{1}{2}(A-B)\cos\theta + \frac{1}{2}(A+B)\sin\theta & -\frac{1}{2}(A-B)\sin\theta + \frac{1}{2}(A+B)\cos\theta & 0 \\ 0 & 0 & C \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(A+B)\cos^2\theta + (A-B)\cos\theta\sin\theta + \frac{1}{2}(A+B)\sin^2\theta & \frac{1}{2}(A-B)\cos^2\theta - \frac{1}{2}(A-B)\sin^2\theta & 0 \\ -\frac{1}{2}(A-B)\sin^2\theta + \frac{1}{2}(A-B)\cos^2\theta & \frac{1}{2}(A+B)\sin^2\theta - (A-B)\sin\theta\cos\theta + \frac{1}{2}(A+B)\cos^2\theta & 0 \\ 0 & 0 & C \end{bmatrix}$$

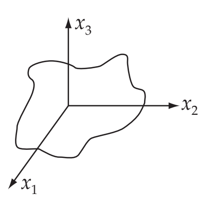
or

$$(\mathbf{I}') = \begin{bmatrix} \frac{1}{2}(A+B) + (A-B)\cos\theta\sin\theta & \frac{1}{2}(A-B)\cos^2\theta - \frac{1}{2}(A-B)\sin^2\theta & 0 \\ -\frac{1}{2}(A-B)\sin^2\theta + \frac{1}{2}(A-B)\cos^2\theta & \frac{1}{2}(A+B) - (A-B)\cos\theta\sin\theta & 0 \\ 0 & 0 & C \end{bmatrix} \tag{3}$$

If $\theta = \pi/4$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Then,

$$\boxed{(\mathbf{I}') = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}} \tag{4}$$

11-17.



The plate is assumed to have negligible thickness and the mass per unit area is ρ_s. Then, the inertia tensor elements are

$$I_{11} = \rho_s \int \left(r^2 - x_1^2\right) dx_1\, dx_2$$

$$= \rho_s \int \left(x_2^2 + x_3^2\right) dx_1 dx_2 = \rho_s \int x_2^2\, dx_1\, dx_2 \equiv A \tag{1}$$

$$I_{22} = \rho_s \int \left(r^2 - x_2^2\right) dx_1\, dx_2 = \rho_s \int x_1^2\, dx_1\, dx_2 \equiv B \tag{2}$$

$$I_{33} = \rho_s \int \left(r^2 - x_3^2\right) dx_1\, dx_2 = \rho_s \int \left(x_1^2 + x_2^2\right) dx_1\, dx_2 \tag{3}$$

Defining A and B as above, I_{33} becomes

$$I_{33} = A + B \tag{4}$$

Also,

$$I_{12} = \rho_s \int \left(-x_1 x_2\right) dx_1\, dx_2 \equiv -C \tag{5}$$

$$I_{21} = \rho_s \int \left(-x_2 x_1\right) dx_1\, dx_2 = -C \tag{6}$$

$$I_{13} = \rho_s \int \left(-x_1 x_3\right) dx_1\, dx_2 = 0 = I_{31} \tag{7}$$

$$I_{23} = \rho_s \int \left(-x_2 x_3\right) dx_1\, dx_2 = 0 = I_{32} \tag{8}$$

Therefore, the inertia tensor has the form

$$\boxed{\{\mathbf{I}\} = \begin{bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{bmatrix}} \tag{9}$$

11-18. The new inertia tensor $\{\mathbf{I}'\}$ is obtained from $\{\mathbf{I}\}$ by a similarity transformation [see Eq. (11.63)]. Since we are concerned only with a rotation around the x_3-axis, the transformation matrix is just $\boldsymbol{\lambda}_\phi$, as defined in Eq. (11.91). Then,

$$\mathbf{I}' = \boldsymbol{\lambda}_\phi\, \mathbf{I}\, \boldsymbol{\lambda}_\phi^{-1} \tag{1}$$

where

$$\boldsymbol{\lambda}_\phi^{-1} = \boldsymbol{\lambda}_\phi^t \tag{2}$$

Therefore, the similarity transformation is

$$\mathbf{I}' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Carrying out the operations and simplifying, we find

$$\{\mathbf{I}'\} = \begin{bmatrix} A\cos^2\theta - C\sin 2\theta + B\sin^2\theta & -C\cos 2\theta + \frac{1}{2}(B-A)\sin 2\theta & 0 \\ -C\cos 2\theta + \frac{1}{2}(B-A)\sin 2\theta & A\sin^2\theta + C\sin 2\theta + B\cos^2\theta & 0 \\ 0 & 0 & A+B \end{bmatrix} \tag{3}$$

Making the identifications stipulated in the statement of the problem, we see that

$$\left.\begin{aligned} I'_{11} &= A', \quad I'_{22} = B' \\ I'_{12} &= I'_{21} = -C' \end{aligned}\right] \tag{4}$$

and

$$I'_{33} = A + B = A' + B' \tag{5}$$

Therefore

$$\boxed{\{\mathbf{I}'\} = \begin{bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A'+B' \end{bmatrix}} \tag{6}$$

In order that x_1 and x_2 be principal axes, we require $C' = 0$:

$$C\cos 2\theta - \frac{1}{2}(B-A)\sin 2\theta = 0 \tag{7}$$

or,

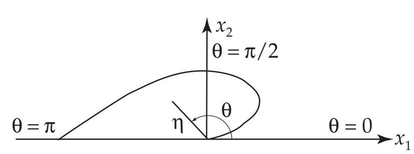
$$\tan 2\theta = \frac{2C}{B-A} \tag{8}$$

from which

$$\boxed{\theta = \frac{1}{2}\tan^{-1}\left[\frac{2C}{B-A}\right]} \tag{9}$$

Notice that this result is still valid if $A = B$. Why? (What does $A = B$ mean?)

11-19.



The boundary of the plate is given by $r = ke^{\alpha\theta}$. Any point (η,θ) has the components

$$\left.\begin{aligned} x_1 &= \eta\cos\theta \\ x_2 &= \eta\sin\theta \end{aligned}\right] \tag{1}$$

The moments of inertia are

$$I_1 = A = \rho \int_0^{\pi} \int_0^{ke^{\alpha\theta}} x_2^2 \ \eta \, d\eta \, d\theta$$

$$= \rho \int_0^{\pi} \sin^2 \theta \, d\theta \int_0^{ke^{\alpha\theta}} \eta^3 \ d\eta$$

The integral over θ can be performed by using Eq. (E.18a), Appendix E, with the result

$$I_1 = A = \frac{\rho k^4}{2\alpha} P \tag{2}$$

where

$$P = \frac{e^{4\pi\alpha} - 1}{16\left(1 + 4\alpha^2\right)} \tag{3}$$

In the same way,

$$I_2 = B = \rho \int_0^{\pi} \int_0^{ke^{\alpha\theta}} x_1^2 \ \eta \, d\eta \, d\theta$$

$$= \rho \int_0^{\pi} \cos^2 \theta \, d\theta \int_0^{ke^{\alpha\theta}} \eta^3 \ d\eta \tag{4}$$

Again, we use Eq. (E.18a) by writing $\cos^2 \theta = 1 - \sin^2 \theta$, and we find

$$I_2 = B = \frac{\rho k^4}{2\alpha} P \left(1 + 8\alpha^2\right) \tag{5}$$

Also

$$\begin{aligned} I_{12} = -C &= -\rho \int_0^{\pi} \int_0^{ke^{\alpha\theta}} x_1 x_2 \ \eta \, d\eta \, d\theta \\ &= -\rho \int_0^{\pi} \cos\theta \sin \theta \, d\theta \int_0^{ke^{\alpha\theta}} \eta^3 \ d\eta \end{aligned} \tag{6}$$

In order to evaluate the integral over θ in this case we write $\cos \theta \sin \theta = \left(1/2\right) \sin 2\theta$ and use Eq. (E.18), Appendix E. We find

$$I_{12} = -C = \rho k^4 P \tag{7}$$

Using the results of problem 11-17, the entire inertia tensor is now known.

According to the result of Problem 11-18, the angle through which the coordinates must be rotated in order to make $\{\mathbf{I}\}$ diagonal is

$$\theta = \frac{1}{2} \tan^{-1} \left[\frac{2C}{B - A} \right] \tag{8}$$

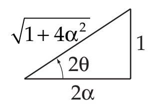
Using Eqs. (2), (5), and (7) for A, B, C, we find

$$\frac{2C}{B - A} = \frac{1}{2\alpha} \tag{9}$$

so that

$$\tan 2\theta = \frac{1}{2\alpha} \tag{10}$$

Therefore, we also have



$$\left.\begin{aligned} \sin 2\theta &= \frac{1}{\sqrt{1+4\alpha^2}} \\ \cos 2\theta &= \frac{2\alpha}{\sqrt{1+4\alpha^2}} \end{aligned}\right] \tag{11}$$

Then, according to the relations specified in Problem 11-18,

$$I_1' = A' = A\cos^2\theta - C\sin 2\theta + B\sin^2\theta \tag{12}$$

Using $\cos^2\theta = (1/2)(1+\cos 2\theta)$ and $\sin^2\theta = (1/2)(1-\cos 2\theta)$, we have

$$I_1' = A' = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos 2\theta - C\sin 2\theta \tag{13}$$

Now,

$$\left.\begin{aligned} A+B &= \frac{\rho k^4 P}{\alpha}\left(1+4\alpha^2\right) \\ A-B &= -4\alpha\rho k^4 P \end{aligned}\right] \tag{14}$$

Thus,

$$I_1' = A' = \frac{\rho k^4 P}{2\alpha}\left(1+4\alpha^2\right) - 2\alpha\rho k^4 P \times \frac{2\alpha}{\sqrt{1+4\alpha^2}} - \rho k^4 P \times \frac{1}{\sqrt{1+4\alpha^2}} \tag{15}$$

or,

$$\boxed{I_1' = A' = \rho k^4 P(Q-R)} \tag{16}$$

where

$$\left.\begin{aligned} Q &= \frac{1+4\alpha^2}{2\alpha} \\ R &= \sqrt{1+4a^2} \end{aligned}\right] \tag{17}$$

Similarly,

$$\boxed{I_2' = B' = \rho k^4 P(Q+R)} \tag{18}$$

and, of course,

$$I_3' = A' + B' = I_1' + I_2' \tag{19}$$

We can also easily verify, for example, that $I_{12}' = -C' = 0$.

11-20. We use conservation of energy. When standing upright, the kinetic energy is zero. Thus, the total energy is the potential energy

$$E = U_1 = mg\,\frac{b}{2}$$

($\frac{b}{2}$ is the height of the center of mass above the floor.)

When the rod hits the floor, the potential energy is zero. Thus

$$E = T_2 = \frac{1}{2} I\omega^2$$

where I is the rotational inertia of a uniform rod about an end. For a rod of length b, mass/length σ:

$$I_{\text{end}} = \int_0^b \sigma x^2\, dx = \frac{1}{3}\sigma b^3 = \frac{1}{3} mb^2$$

Thus

$$T_2 = \frac{1}{6} mb^2\,\omega^2$$

By conservation of energy

$$U_1 = T_2$$

$$mg\,\frac{b}{2} = \frac{1}{6} mb^2\,\omega^2$$

$$\boxed{\omega = \sqrt{\frac{3g}{b}}}$$

11-21. Using I to denote the matrix whose elements are those of $\{\mathbf{I}\}$, we can write

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} \tag{11.54}$$

$$\mathbf{L}' = \mathbf{I}'\boldsymbol{\omega}' \tag{11.54a}$$

We also have $\mathbf{x}' = \boldsymbol{\lambda}\,\mathbf{x}$ and $\mathbf{x}' = \boldsymbol{\lambda}^t\,\mathbf{x}'$ and therefore we can express $\mathbf{L}$ and $\boldsymbol{\omega}$ as

$$\mathbf{L} = \boldsymbol{\lambda}^t\mathbf{L}' \tag{11.55a}$$

$$\boldsymbol{\omega} = \boldsymbol{\lambda}^t\boldsymbol{\omega}' \tag{11.55b}$$

substituting these expressions into Eq. (11.54), we have

$$\boldsymbol{\lambda}^t \mathbf{L}' = \mathbf{I}\boldsymbol{\lambda}^t \boldsymbol{\omega}'$$

and multiplying on the left by λ,

$$\boldsymbol{\lambda}\boldsymbol{\lambda}^t \mathbf{L}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^t \boldsymbol{\omega}'$$

or

$$\mathbf{L}' = \left(\boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^t\right)\boldsymbol{\omega}'$$

by virtue of Eq. (11.54a), we identify

$$\boxed{\mathbf{I}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^t} \tag{11.61}$$

11-22. According to Eq. (11.61),

$$I'_{ij} = \sum_{k,l} \lambda_{ik}\ I_{k\ell}\ \lambda^{-1}_{\ell j} \tag{1}$$

Then,

$$tr\{\mathbf{I}'\} = \sum_i I'_{ii} = \sum_i \sum_{k,\ell} \lambda_{ik}\ I_{k\ell}\ \lambda^{-1}_{\ell i}$$

$$= \sum_{k,\ell} I_{k\ell} \sum_i \lambda^{-1}_{\ell i}\ \lambda_{ik}$$

$$= \sum_{k,\ell} I_{k\ell}\ \delta_{\ell k} = \sum_k I_{kk} \tag{2}$$

so that

$$\boxed{tr\{\mathbf{I}'\} = tr\{\mathbf{I}\}} \tag{3}$$

This relation can be verified for the examples in the text by straightforward calculations.

Note: A *translational* transformation is *not* a *similarity* transformation and, in general, $tr\{\mathbf{I}\}$ is not invariant under translation. (For example, $tr\{\mathbf{I}\}$ will be different for inertia tensors expressed in coordinate system with different origins.)

11-23. We have

$$\mathbf{I}' = \boldsymbol{\lambda}\mathbf{I}\boldsymbol{\lambda}^{-1} \tag{1}$$

Then,

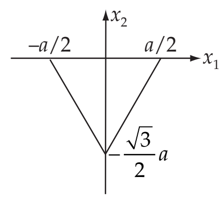
$$\begin{aligned}|\mathbf{I}'| &= |\lambda \mathbf{I} \lambda^{-1}| \\ &= |\lambda| \times |\mathbf{I}| \times |\lambda^{-1}| \\ &= |\lambda \lambda^{-1}| \times |\mathbf{I}|\end{aligned}$$

so that,

$$\boxed{|\mathbf{I}'| = |\mathbf{I}|} \tag{2}$$

This result is easy to verify for the various examples involving the cube.

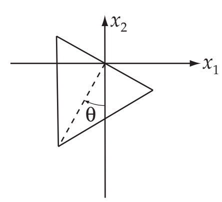
11-24.



The area of the triangle is $A = \sqrt{3}\, a^2/4$, so that the density is

$$\rho = \frac{M}{A} = \frac{4M}{\sqrt{3}\, a^2} \tag{1}$$

a) The rotational inertia with respect to an axis through the point of suspension (the origin) is



$$\begin{aligned}I_3 &= \rho \int \left(x_1^2 + x_2^2\right) dx_1\, dx_2 \\ &= 2\rho \int_0^{a/2} dx_1 \int_{-\frac{\sqrt{3}}{2}(a-2x_1)}^{0} \left(x_1^2 + x_2^2\right) dx_2 \\ &= \frac{\sqrt{3}}{24} \rho a^4 = \frac{1}{6} M a^2\end{aligned} \tag{2}$$

When the triangle is suspended as shown and when $\theta = 0$, the coordinates of the center of mass are $(0, \bar{x}_2, 0)$, where

$$\overline{x}_2 = \frac{1}{M} \int \rho x_2 \, dx_1 \, dx_2$$

$$= \frac{2\rho}{M} \int_0^{a/2} dx_1 \int_{-\sqrt{3}(a-2x_1)}^{0} x_2 \, dx_2$$

$$= -\frac{a}{2\sqrt{3}} \tag{3}$$

The kinetic energy is

$$T = \frac{1}{2} I_3 \dot{\theta}^2 = \frac{1}{12} Ma^2 \dot{\theta}^2 \tag{4}$$

and the potential energy is

$$U = \frac{Mga}{2\sqrt{3}} \left(1 - \cos\theta\right) \tag{5}$$

Therefore,

$$L = \frac{1}{12} Ma^2 \dot{\theta}^2 + \frac{Mga}{2\sqrt{3}} \cos\theta \tag{6}$$

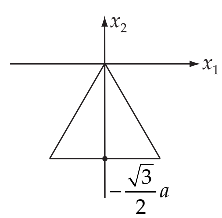
where the constant term has been suppressed. The Lagrange equation for θ is

$$\ddot{\theta} + \sqrt{3} \frac{g}{a} \sin\theta = 0 \tag{7}$$

and for oscillations with small amplitude, the frequency is

$$\boxed{\omega = \sqrt{\sqrt{3} \frac{g}{a}}} \tag{8}$$

b) The rotational inertia for an axis through the point of suspension for this case is



$$I_3' = 2\rho \int_{-\frac{\sqrt{3}a}{2}}^{0} dx_2 \int_0^{-x_2/\sqrt{3}} \left(x_1^2 + x_2^2\right) dx_1$$

$$= \frac{5}{12} Ma^2 \tag{9}$$

The Lagrangian is now

$$L = \frac{5}{24} Ma^2 \dot{\theta}^2 + \frac{Mga}{\sqrt{3}} \cos\theta \tag{10}$$

and the equation of motion is

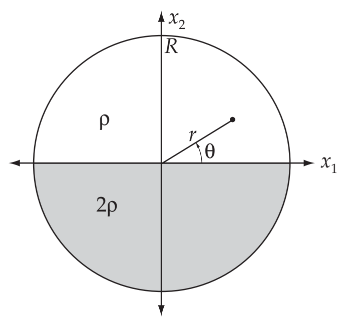
$$\ddot{\theta}+\frac{12}{5\sqrt{3}}\frac{g}{a}\sin\theta=0 \tag{11}$$

so that the frequency of small oscillations is

$$\boxed{\omega=\sqrt{\frac{12}{5\sqrt{3}}\frac{g}{a}}} \tag{12}$$

which is slightly smaller than the previous result.

11-25.



The center of mass of the disk is $(0,\bar{x}_2)$, where

$$\begin{aligned}\bar{x}_2&=\frac{\rho}{M}\left[2\int\limits_{\substack{\text{lower}\\\text{semicircle}}}x_2\,dx_1\,dx_2+\int\limits_{\substack{\text{upper}\\\text{semicircle}}}x_2\,dx_1\,dx_2\right]\\&=\frac{\rho}{M}\left[\int_0^R\int_0^\pi(r\sin\theta)\cdot rdrd\theta+2\int_0^R\int_\pi^{2\pi}(r\sin\theta)\cdot rdrd\theta\right]\\&=-\frac{2}{3}\frac{\rho R^3}{M}\end{aligned} \tag{1}$$

Now, the mass of the disk is

$$\begin{aligned}M&=\rho\cdot\frac{1}{2}\pi R^2+2\rho\cdot\frac{1}{2}\pi R^2\\&=\frac{3}{2}\rho\pi R^2\end{aligned} \tag{2}$$

so that

$$\bar{x}_2=-\frac{4}{9\pi}R \tag{3}$$

The direct calculation of the rotational inertia with respect to an axis through the center of mass is tedious, so we first compute I with respect to the x_3-axis and then use Steiner's theorem.

$$I_3 = \rho\left[\int_0^R \int_0^\pi r^2 \cdot rdr\, d\theta + 2\int_0^R \int_\pi^{2\pi} r^2 \cdot rdr\, d\theta\right]$$

$$= \frac{3}{4}\pi\rho R^4 = \frac{1}{2}MR^2 \tag{4}$$

Then,

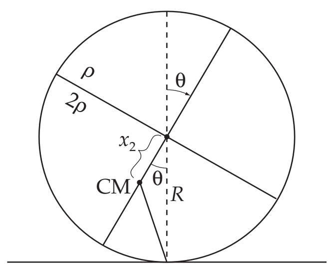
$$I_0 = I_3 - M\overline{x}_2^2$$

$$= \frac{1}{2}MR^2 - M\cdot\frac{16}{81\pi^2}R^2$$

$$= \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right] \tag{5}$$

When the disk rolls without slipping, the velocity of the center of mass can be obtained as follows:

Thus



$$x_{\mathrm{CM}} = R\theta - |\overline{x}_2|\sin\theta$$

$$y_{\mathrm{CM}} = R - |\overline{x}_2|\cos\theta$$

$$\dot{x}_{\mathrm{CM}} = R\dot{\theta} - |\overline{x}_2|\,\dot{\theta}\cos\theta$$

$$\dot{y}_{\mathrm{CM}} = |\overline{x}_2|\,\dot{\theta}\sin\theta$$

$$\left(\dot{x}_{\mathrm{CM}}^2 + \dot{y}_{\mathrm{CM}}^2\right) = V^2 = R^2\dot{\theta}^2 + \overline{x}_2^2\,\dot{\theta}^2 - 2\dot{\theta}^2\,R|\overline{x}_2|\cos\theta$$

$$V^2 = a^2\dot{\theta}^2 \tag{6}$$

where

$$a = \sqrt{R^2 + \overline{x}_2^2 - 2R\,|\overline{x}_2|\cos\theta} \tag{7}$$

Using (3), a can be written as

$$a = R\sqrt{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta} \tag{8}$$

The kinetic energy is

$$T = T_{\text{trans}} + T_{\text{rot}}$$

$$= \frac{1}{2} Mv^2 + \frac{1}{2} I_0 \dot{\theta}^2 \tag{9}$$

Substituting and simplifying yields

$$T = \frac{1}{2} MR^2 \dot{\theta}^2 \left[\frac{3}{2} - \frac{8}{9\pi} \cos\theta \right] \tag{10}$$

The potential energy is

$$U = Mg \left[\frac{1}{2} R + \bar{x}_2 \cos\theta \right]$$

$$= \frac{1}{2} MgR \left[1 - \frac{8}{9\pi} \cos\theta \right] \tag{11}$$

Thus the Lagrangian is

$$\boxed{L = \frac{1}{2} MR \left[R\dot{\theta}^2 \left[\frac{3}{2} - \frac{8}{9\pi} \cos\theta \right] - g \left[1 - \frac{8}{9\pi} \cos\theta \right] \right]} \tag{12}$$

11-26. Since $\boldsymbol{\omega}_\phi = \dot{\phi}$ lies along the fixed x_3'-axis, the components of ω_ϕ along the body axes (x_i) are given by the application of the transformation matrix λ [Eqs. (11.98) and (11.99)]:

$$\begin{bmatrix} (\omega_\phi)_1 \\ (\omega_\phi)_2 \\ (\omega_\phi)_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{bmatrix} = \boldsymbol{\lambda} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \tag{1}$$

Carrying out the matrix multiplication, we find

$$\begin{bmatrix} (\omega_\phi)_1 \\ (\omega_\phi)_2 \\ (\omega_\phi)_3 \end{bmatrix} = \dot{\phi} \begin{bmatrix} \sin\psi \sin\theta \\ \cos\psi \sin\theta \\ \cos\theta \end{bmatrix} \tag{2}$$

which is just Eq. (11.101a).

The direction of $\boldsymbol{\omega}_\theta = \dot{\theta}$ coincides with the line of nodes and lies along the x_1''' axis. The components of ω_θ along the body axes are therefore obtained by the application of the transformation matrix $\boldsymbol{\lambda}_\psi$ which carries the x_i''' system into the x_i system:

$$\begin{bmatrix} (\omega_\theta)_1 \\ (\omega_\theta)_2 \\ (\omega_\theta)_3 \end{bmatrix} = \boldsymbol{\lambda}_\psi \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \dot{\theta} \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix} \tag{3}$$

which is just Eq. (11.101b).

Finally, since $\boldsymbol{\omega}_\psi$ lies along the body x_3-axis, no transformation is required:

$$\begin{bmatrix} (\omega_\psi)_1 \\ (\omega_\psi)_2 \\ (\omega_\psi)_3 \end{bmatrix} = \dot{\psi} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

which is just Eq. (11.101c).

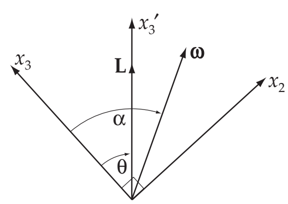
Combining these results, we obtain

$$\boldsymbol{\omega} = \begin{bmatrix} (\omega_\phi)_1 + (\omega_\theta)_1 + (\omega_\psi)_1 \\ (\omega_\phi)_2 + (\omega_\theta)_2 + (\omega_\psi)_2 \\ (\omega_\phi)_3 + (\omega_\theta)_3 + (\omega_\psi)_3 \end{bmatrix}$$

$$= \dot{\phi} \begin{bmatrix} \dot{\phi} \sin\psi \sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi \sin\theta - \dot{\theta}\sin\psi \\ \dot{\theta}\cos\theta + \dot{\psi} \end{bmatrix} \tag{5}$$

which is just Eq. (11.02).

11-27.



Initially:

$$L_1 = 0 = I_1\omega_1$$

$$L_2 = L\sin\theta = I_1\omega_2 = I_1\omega\sin\alpha$$

$$L_3 = L\ xos\ \theta = I_3\omega_3 = I_3\omega\cos\alpha$$

Thus

$$\tan\theta = \frac{L_2}{L_3} = \frac{I_1}{I_3}\tan\alpha \tag{1}$$

From Eq. (11.102)

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Since $\omega_3 = \omega\cos\alpha$, we have

$$\dot{\phi}\cos\theta = \omega\cos\alpha - \dot{\psi} \tag{2}$$

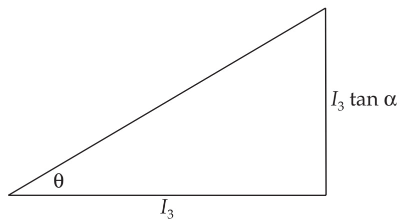
From Eq. (11.131)

$$\dot{\psi} = -\Omega = -\frac{I_3 - I_1}{I_1}\,\omega_3$$

(2) becomes

$$\dot{\phi}\cos\theta = \frac{I_3}{I_1}\,\omega\cos\alpha \tag{3}$$

From (1), we may construct the following triangle



from which $\cos\theta = \dfrac{I_3}{\left[I_3^2 + I_1^2\tan^2\alpha\right]^{1/2}}$

Substituting into (3) gives

$$\dot{\phi} = \frac{\omega}{I_1}\sqrt{I_1^2\sin^2\alpha + I_3^2\cos^2\alpha}$$

11-28. From Fig. 11-7c we see that $\boldsymbol{\omega}_\phi = \dot{\phi}$ is along the x_3'-axis, $\boldsymbol{\omega}_\theta = \dot{\theta}$ is along the line of nodes, and $\boldsymbol{\omega}_\psi = \dot{\psi}$ is along the x_3-axis. Then,

$$\boldsymbol{\omega}_\phi' = \dot{\phi}\,\mathbf{e}_3' \tag{1}$$

whee $\mathbf{e}_3'$ is the unit vector in the x_3' direction.

Projecting the lines of nodes into the x_1'- and x_2'-axes, we obtain

$$\boldsymbol{\omega}_\theta' = \dot{\theta}\left(\mathbf{e}_1'\cos\phi + \mathbf{e}_2'\sin\phi\right) \tag{2}$$

ω_ψ' has components along all three of the x_i' axes. First, we write $\boldsymbol{\omega}_\psi'$ in terms of a component along the x_3'-axis and a component normal to this axis:

$$\boldsymbol{\omega}_\psi' = \dot{\psi}\left(\mathbf{e}_{12}'\sin\theta + \mathbf{e}_3'\cos\theta\right) \tag{3}$$

where

$$\mathbf{e}_{12}' = \mathbf{e}_1'\sin\phi - \mathbf{e}_2'\cos\phi \tag{4}$$

Then,

$$\boldsymbol{\omega}_\psi' = \dot{\psi}\left(\mathbf{e}_1'\sin\theta\sin\phi - \mathbf{e}_2'\sin\theta\cos\phi + \mathbf{e}_3'\cos\theta\right) \tag{5}$$

Collecting the various components, we have

$$\boxed{\begin{aligned}\omega_1' &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \omega_2' &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \omega_3' &= \dot{\psi}\cos\theta + \dot{\phi}\end{aligned}} \tag{6}$$

11-29. When the motion is vertical $\theta = 0$. Then, according to Eqs. (11.153) and (11.154),

$$P_\phi = I_3\left(\dot{\phi} + \dot{\psi}\right) = P_\psi \tag{1}$$

and using Eq. (11.159), we see that

$$P_\psi = P_\phi = I_3\,\omega_3 \tag{2}$$

Also, when $\theta = 0$ (and $\dot{\theta} = 0$), the energy is [see Eq. (11.158)]

$$E = \frac{1}{2}I_3\,\omega_3^2 + Mgh \tag{3}$$

Furthermore, referring to Eq. (11.160),

$$E' = E - \frac{1}{2}I_3\,\omega_3^2 = Mgh \tag{3}$$

If we wish to examine the behavior of the system near $\theta = 0$ in order to determine the conditions for stability, we can use the values of P_ψ, P_ϕ, and E' for $\theta = 0$ in Eq. (11.161). Thus,

$$Mgh = \frac{1}{2}I_{12}\dot{\theta}^2 + \frac{I_3^2\,\omega_3^2\left(1-\cos\theta\right)^2}{2I_{12}\sin^2\theta} + Mgh\cos\theta \tag{5}$$

Changing the variable to $z = \cos\theta$ and rearranging, Eq. (5) becomes

$$\dot{z}^2 = \frac{(1-z)^2}{I_{12}^2}\left[2Mgh\,I_{12}\,(1+z) - I_3^2\,w_3^2\right] \tag{6}$$

The questions concerning stability can be answered by examining this expression. First, we note that for physically real motion we must have $\dot{z}^2 \geq 0$. Now, suppose that the top is spinning very rapidly, i.e., that ω_3 is large. Then, the term in the square brackets will be negative. In such a case, the only way to maintain the condition $\dot{z}^2 \geq 0$ is to have $z = 1$, i.e., $\theta = 0$. Thus, the motion at $\theta = 0$ will be stable as long as

$$4Mgh\,I_{12} - I_3^2\,\omega_3^2 < 0 \tag{7}$$

or,

$$\boxed{\frac{4Mgh\,I_{12}}{I_3^2\,\omega_3^2} < 1} \tag{8}$$

Suppose now that the top is set spinning with $\theta = 0$ but with ω_3 sufficiently small that the condition in Eq. (8) is not met. Any small disturbance away from $\theta = 0$ will then give $\dot{z}$ a negative value and θ will continue to increase; i.e., the motion is unstable. In fact, θ will continue until z reaches a value z_0 that again makes the square brackets equal to zero. This is a turning point for the motion and nutation between $z = 1$ and $z = z_0$ will result.

From this discussion it is evident that there exists a critical value for the angular velocity, ω_c, such that for $\omega_3 > \omega_c$ the motion is stable and for $\omega < \omega_c$ there is nutation:

$$\omega_c = \frac{2\sqrt{Mgh\, I_{12}}}{I_3} \tag{9}$$

If the top is set spinning with $\omega_3 > \omega_c$ and $\theta = 0$, the motion will be stable. But as friction slows the top, the critical angular velocity will eventually be reached and nutation will set in. This is the case of the "sleeping top."

11-30. If we set $\dot{\theta} = 0$, Eq. (1.162) becomes

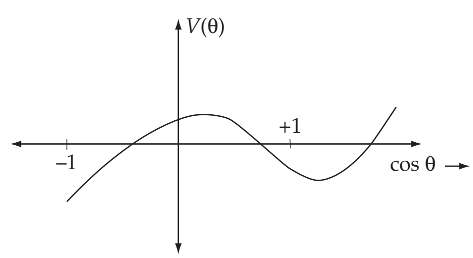
$$E' = V(\theta) = \frac{\left(P_\phi - P_\psi \cos\theta\right)^2}{2\, I_{12}\left(1 - \cos^2\theta\right)} + Mgh\cos\theta \tag{1}$$

Re-arranging, this equation can be written as

$$\left(2Mgh\, I_{12}\right)\cos^3\theta - \left(2E'\, I_{12} + P_\psi^2\right)\cos^2\theta + 2\left(P_\phi P_\psi - Mgh\, I_{12}\right)\cos\theta + \left(2E'\, I_{12} - P_\phi^2\right) = 0 \tag{2}$$

which is cubic in cos θ.

$V(\theta)$ has the form shown in the diagram. Two of the roots occur in the region $-1 \le \cos\theta \le 1$, and one root lies outside this range and is therefore imaginary.



11-31. The moments of inertia of the plate are

$$\left.\begin{aligned} &I_1 = I_2 \cos 2\alpha \\ &I_2 \\ &I_3 = I_1 + I_2 \\ &\quad = I_2 (1 + \cos 2\alpha) \\ &\quad = 2I_2 \cos^2 \alpha \end{aligned}\right\rbrack \tag{1}$$

We also note that

$$\begin{aligned} I_1 - I_2 &= -I_2 (1 - \cos 2\alpha) \\ &= -2I_2 \sin^2 \alpha \end{aligned} \tag{2}$$

Since the plate moves in a force-free manner, the Euler equations are [see Eq. (11.114)]

$$\left.\begin{aligned} (I_1 - I_2)\, \omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \\ (I_2 - I_3)\, \omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \\ (I_3 - I_1)\, \omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \end{aligned}\right\rbrack \tag{3}$$

Substituting (1) and (2) into (3), we find

$$\left.\begin{aligned} (-2I_2 \sin^2 \alpha)\, \omega_1\omega_2 - (2I_2 \cos^2 \alpha)\, \dot{\omega}_3 = 0 \\ (-I_2 \cos 2\alpha)\, \omega_2\omega_3 - (I_2 \cos 2\alpha)\, \dot{\omega}_1 = 0 \\ I_2\, \omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \end{aligned}\right\rbrack \tag{4}$$

These equations simplify to

$$\left.\begin{aligned} &\dot{\omega}_3 = -\omega_1\omega_2 \tan^2 \alpha \\ &\dot{\omega}_1 = -\omega_2\omega_3 \\ &\dot{\omega}_2 = \omega_3\omega_1 \end{aligned}\right\rbrack \tag{5}$$

From which we can write

$$\omega_1\omega_2\omega_3 = \omega_2\dot{\omega}_2 = -\omega_1\dot{\omega}_1 = -\omega_3\dot{\omega}_3 \cot^2 \alpha \tag{6}$$

Integrating, we find

$$\omega_2^2 - \omega_2^2(0) = -\omega_1^2 + \omega_1^2(0) = -\omega_3^2 \cot^2 \alpha + \omega_3^2(0) \cot^2 \alpha \tag{7}$$

Now, the initial conditions are

$$\left.\begin{aligned}\omega_1(0) &= \Omega\cos\alpha \\ \omega_2(0) &= 0 \\ \omega_3(0) &= \Omega\sin\alpha\end{aligned}\right] \tag{8}$$

Therefore, the equations in (7) become

$$\omega_2^2 = -\omega_1^2 + \Omega^2\cos^2\alpha = -\omega_3^2\cot^2\alpha + \Omega^2\cos^{2\alpha} \tag{9}$$

From (5), we can write

$$\dot{\omega}_2^2 = \omega_3^2\,\omega_1^2 \tag{10}$$

and from (9), we have $\omega_1^2 = \omega_3^2\cot^2\alpha$. Therefore, (10) becomes

$$\dot{\omega}_2 = \omega_3^2\cot\alpha \tag{11}$$

and using $\omega_3^2 = \Omega^2\sin^2\alpha - \omega_2^2\tan^2\alpha$ from (9), we can write (11) as

$$\frac{\dot{\omega}_2}{\omega_2^2\tan^2\alpha - \Omega^2\sin^2\alpha} = -\cot\alpha \tag{12}$$

Since $\dot{\omega}_2 = d\omega_2/dt$, we can express this equation in terms of integrals as

$$\int\frac{d\omega_2}{\omega_2^2\tan^2\alpha - \Omega^2\sin^2\alpha} = -\cot\alpha\int dt \tag{13}$$

Using Eq. (E.4c), Appendix E, we find

$$-\frac{1}{(\tan\alpha)(\Omega\sin\alpha)}\tanh^{-1}\left[\frac{\omega_2\tan\alpha}{\Omega\sin\alpha}\right] = -t\cot\alpha \tag{14}$$

Solving for ω_2,

$$\boxed{\omega_2(t) = \Omega\cos\alpha\tanh(\Omega t\sin\alpha)} \tag{15}$$

11-32.

a) The exact equation of motion of the physical pendulum is

$$I\ddot{\theta} + MgL\sin\theta = 0$$

where $I = Mk^2$, so we have

$$\ddot{\theta} = -\frac{gL}{k^2}\sin\theta$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\theta}) = \frac{gL}{k^2}\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\theta}$$

or

$$\dot{\theta}\,\mathrm{d}\left(\dot{\theta}\right) = \frac{gL}{k^2}\,\mathrm{d}\left(\cos\theta\right)$$

so

$$\dot{\theta}^2 = \frac{2gL}{k^2}\cos\theta + a$$

where a is a constant determined by the initial conditions. Suppose that at $t = 0$, $\theta = \theta_0$ and at that initial position the angular velocity of the pendulum is zero, we find $a = \dfrac{-2gl}{k^2}\cos\theta_0$. So finally

$$\dot{\theta} = \sqrt{\frac{2gl}{k^2}\left(\cos\theta - \cos\theta_0\right)}$$

b) One could use the conservation of energy to find the angular velocity of the pendulum at any angle θ , but it is exactly the result we obtained in a), so at $\theta = 1^0$, we have

$$\omega = \dot{\theta} = \sqrt{\frac{2gL}{k^2}\left(\cos\theta - \cos\theta_0\right)} = 53.7\ \mathrm{s}^{-1}$$

11-33. Cats are known to have a very flexible body that they can manage to twist around to a feet-first descent while falling with conserved zero angular momentum. First they thrust their back legs straight out behind their body and at the same time they tuck their front legs in. Extending their back legs helps to resist spinning, since rotation velocity evidently is inversely proportional to inertia momentum. This allows the cat to twist their body differently to preserve zero angular momentum: the front part of the body twisting more than the back. Tucking the front legs encourages spinning to a downward direction preparing for touchdown and as this happens, cats can easily twist the rear half of their body around to catch up with the front.

However, whether or not cats land on their feet depends on several factors, notably the distance they fall, because the twist maneuver takes a certain time, apparently around 0.3 sec. Thus the minimum height required for cats falling is about 0.5m.

11-34. The Euler equation, which describes the rotation of an object about its symmetry axis, say *0x*, is

$$I_x\,\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z = N_x$$

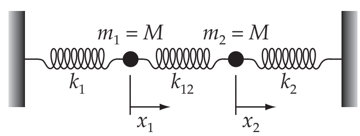
where $N_x = -b\,\omega_x$ is the component of torque along Ox. Because the object is symmetric about Ox, we have $I_y = I_z$, and the above equation becomes

$$I_x\,\frac{\mathrm{d}\omega_x}{\mathrm{d}t} = -b\,\omega_x \quad \Rightarrow \quad \omega_x = e^{-\frac{b}{I_x}t}\,\omega_{x0}$$

CHAPTER 12

Coupled Oscillations

12-1.



The equations of motion are

$$\left.\begin{aligned} M\ddot{x}_1 + (\kappa_1 + \kappa_{12})x_1 - \kappa_{12}\, x_2 = 0 \\ M\ddot{x}_2 + (\kappa_2 + \kappa_{12})x_2 - \kappa_{12}\, x_1 = 0 \end{aligned}\right] \tag{1}$$

We attempt a solution of the form

$$\left.\begin{aligned} x_1(t) = B_1\, e^{i\omega t} \\ x_2(t) = B_2\, e^{i\omega t} \end{aligned}\right] \tag{2}$$

Substitution of (2) into (1) yields

$$\left.\begin{aligned} (\kappa_1 + \kappa_{12} - M\omega^2)\, B_1 - \kappa_{12}\, B_2 = 0 \\ -\kappa_{12}\, B_1 + (\kappa_2 + \kappa_{12} - M\omega^2)\, B_2 = 0 \end{aligned}\right] \tag{3}$$

In order for a non-trivial solution to exist, the determinant of coefficients of B_1 and B_2 must vanish. This yields

$$\left[\kappa_1 + (\kappa_{12} - M\omega^2)\right]\left[\kappa_2 + (\kappa_{12} - M\omega^2)\right] = \kappa_{12}^2 \tag{4}$$

from which we obtain

$$\boxed{\omega^2 = \frac{\kappa_1 + \kappa_2 + 2\kappa_{12}}{2M} \pm \frac{1}{2M}\sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}^2}} \tag{5}$$

This result reduces to $\omega^2 = (\kappa + \kappa_{12} \pm \kappa_{12})/M$ for the case $\kappa_1 = \kappa_2 = \kappa$ (compare Eq. (12.7)].

If m_2 were held fixed, the frequency of oscillation of m_1 would be

$$\omega_{01}^2 = \frac{1}{M}(\kappa_1 + \kappa_{12}) \tag{6}$$

while in the reverse case, m_2 would oscillate with the frequency

$$\omega_{02}^2 = \frac{1}{M}(\kappa_2 + \kappa_{12}) \tag{7}$$

Comparing (6) and (7) with the two frequencies, ω_+ and ω_-, given by (5), we find

$$\begin{aligned}\omega_+^2 &= \frac{1}{2M}\left[\kappa_1 + \kappa_2 + 2\kappa_{12} + \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}^2}\,\right] \\ &> \frac{1}{2M}\left[\kappa_1 + \kappa_2 + 2\kappa_{12} + (\kappa_1 - \kappa_2)\right] = \frac{1}{M}(\kappa_1 + \kappa_{12}) = \omega_{01}^2\end{aligned} \tag{8}$$

so that

$$\omega_+ > \omega_{01} \tag{9}$$

Similarly,

$$\begin{aligned}\omega_-^2 &= \frac{1}{2M}\left[\kappa_1 + \kappa_2 + 2\kappa_{12} - \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}^2}\,\right] \\ &< \frac{1}{2M}\left[\kappa_1 + \kappa_2 + 2\kappa_{12} - (\kappa_1 - \kappa_2)\right] = \frac{1}{M}(\kappa_2 + \kappa_{12}) = \omega_{02}^2\end{aligned} \tag{10}$$

so that

$$\omega_- < \omega_{02} \tag{11}$$

If $\kappa_1 > \kappa_2$, then the ordering of the frequencies is

$$\omega_+ > \omega_{01} > \omega_{02} > \omega_- \tag{12}$$

12-2. From the preceding problem we find that for $\kappa_{12} \ll \kappa_1, \kappa_2$

$$\omega_1 \cong \sqrt{\frac{\kappa_1 + \kappa_{12}}{M}}; \quad \omega_2 \cong \sqrt{\frac{\kappa_2 + \kappa_{12}}{M}} \tag{1}$$

If we use

$$\omega_{01} = \sqrt{\frac{\kappa_1}{M}}; \quad \omega_{02} = \sqrt{\frac{\kappa_2}{M}} \tag{2}$$

then the frequencies in (1) can be expressed as

$$\left.\begin{aligned}\omega_1 &= \omega_{01}\sqrt{1+\frac{\kappa_{12}}{\kappa_1}} \cong \omega_{01}\left(1+\varepsilon_1\right)\\ \omega_2 &= \omega_{02}\sqrt{1+\frac{\kappa_{12}}{\kappa_2}} \cong \omega_{02}\left(1+\varepsilon_2\right)\end{aligned}\right] \tag{3}$$

where

$$\varepsilon_1 = \frac{\kappa_{12}}{2\kappa_1}; \quad \varepsilon_2 = \frac{\kappa_{12}}{2\kappa_2} \tag{4}$$

For the initial conditions [Eq. 12.22)],

$$x_1(0) = D,\ x_2(0) = 0,\ \dot{x}_1(0) = 0,\ \dot{x}_2(0) = 0, \tag{5}$$

the solution for $x_1(t)$ is just Eq. (12.24):

$$x_1(t) = D\cos\left[\frac{\omega_1+\omega_2}{2}t\right]\cos\left[\frac{\omega_1-\omega_2}{2}t\right] \tag{6}$$

Using (3), we can write

$$\begin{aligned}\omega_1+\omega_2 &= \left(\omega_{01}+\omega_{02}\right)+\left(\varepsilon_1\omega_{01}+\varepsilon_2\omega_{02}\right)\\ &\equiv 2\Omega_+ + 2\varepsilon_+\end{aligned} \tag{7}$$

$$\begin{aligned}\omega_1-\omega_2 &= \left(\omega_{01}-\omega_{02}\right)+\left(\varepsilon_1\omega_{01}-\varepsilon_2\omega_{02}\right)\\ &\equiv 2\Omega_- + 2\varepsilon_-\end{aligned} \tag{8}$$

Then,

$$x_1(t) = D\cos\left(\Omega_+ t + \varepsilon_+ t\right)\cos\left(\Omega_- t + \varepsilon_- t\right) \tag{9}$$

Similarly,

$$\begin{aligned}x_2(t) &= D\sin\left[\frac{\omega_1+\omega_2}{2}t\right]\sin\left[\frac{\omega_1-\omega_2}{2}t\right]\\ &= D\sin\left(\Omega_+ t + \varepsilon_+ t\right)\sin\left(\Omega_- t + \varepsilon_- t\right)\end{aligned} \tag{10}$$

Expanding the cosine and sine functions in (9) and (10) and taking account of the fact that ε_+ and ε_- are small quantities, we find, to first order in the ε's,

$$x_1(t) \cong D\left[\cos\Omega_+ t\cos\Omega_- t - \varepsilon_+ t\sin\Omega_+ t\cos\Omega_- t - \varepsilon_- t\cos\Omega_+ t\sin\Omega_- t\right] \tag{11}$$

$$x_2(t) \cong D\left[\sin\Omega_+ t\sin\Omega_- t + \varepsilon_+ t\cos\Omega_+ t\sin\Omega_- t + \varepsilon_- t\sin\Omega_+ t\cos\Omega_- t\right] \tag{12}$$

When either $x_1(t)$ or $x_2(t)$ reaches a maximum, the other is at a minimum which is greater than zero. Thus, the energy is never transferred completely to one of the oscillators.

12-3. The equations of motion are

$$\left.\begin{aligned} \ddot{x}_1 + \frac{m}{M}\ddot{x}_2 + \omega_0^2\, x_1 = 0 \\ \ddot{x}_2 + \frac{m}{M}\ddot{x}_2 + \omega_0^2\, x_2 = 0 \end{aligned}\right] \tag{1}$$

We try solutions of the form

$$x_1(t) = B_1\, e^{i\omega t};\quad x_2(t) = B_2\, e^{i\omega t} \tag{2}$$

We require a non-trivial solution (i.e., the determinant of the coefficients of B_1 and B_2 equal to zero), and obtain

$$\left(\omega_0^2 - \omega^2\right)^2 - \omega^4\left[\frac{m}{M}\right]^2 = 0 \tag{3}$$

so that

$$\omega_0^2 - \omega^2 = \pm\omega^2\,\frac{m}{M} \tag{4}$$

and then

$$\omega^2 = \frac{\omega_0^2}{1 \pm \dfrac{m}{M}} \tag{5}$$

Therefore, the frequencies of the normal modes are

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{\omega_0^2}{1+\dfrac{m}{M}}} \\ \omega_2 &= \sqrt{\frac{\omega_0^2}{1-\dfrac{m}{M}}} \end{aligned}} \tag{6}$$

where ω_1 corresponds to the symmetric mode and ω_2 to the antisymmetric mode.

By inspection, one can see that the normal coordinates for this problem are the same as those for the example of Section 12.2 [i.e., Eq. (12.11)].

12-4. The total energy of the system is given by

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}M\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{1}{2}\kappa\left(x_1^2 + x_2^2\right) + \frac{1}{2}\kappa_{12}\left(x_2 - x_1\right)^2 \end{aligned} \tag{1}$$

Therefore,

$$\frac{dE}{dt} = M\left(\dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2\right) + \kappa\left(x_1\dot{x}_1 + x_2\dot{x}_2\right) + \kappa_{12}\left(\dot{x}_2 - \dot{x}_1\right)\left(x_2 - x_1\right)$$

$$= \left[M\ddot{x}_1 + \left[\kappa x_1 - \kappa_{12}\left(x_2 - x_1\right)\right]\right]\dot{x}_1 + \left[M\ddot{x}_2 + \left[\kappa x_2 + \kappa_{12}\left(x_2 - x_1\right)\right]\right]\dot{x}_2$$

$$= \left[M\ddot{x}_1 + \left(\kappa + \kappa_{12}\right)x_1 - \kappa_{12}x_2\right]\dot{x}_1 + \left[M\ddot{x}_2 - \kappa_{12}x_1 + \left(\kappa + \kappa_{12}\right)x_2\right]\dot{x}_2 \tag{2}$$

which exactly vanishes because the coefficients of $\dot{x}_1$ and $\dot{x}_2$ are the left-hand sides of Eqs. (12.1a) and (12.1b).

An analogous result is obtained when T and U are expressed in terms of the generalized coordinates η_1 and η_2 defined by Eq. (12.11):

$$T = \frac{1}{4}M\left(\dot{\eta}_1^2 + \dot{\eta}_2^2\right) \tag{3}$$

$$U = \frac{1}{4}\kappa\left(\eta_1^2 + \eta_2^2\right) + \frac{1}{2}\kappa_{12}\eta_1^2 \tag{4}$$

Therefore,

$$2\cdot\frac{dE}{dt} = \left[M\ddot{\eta}_1 + \left(\kappa + 2\kappa_{12}\right)\eta_1\right]\dot{\eta}_1 + \left[M\ddot{\eta}_2 + \kappa\eta_2\right]\dot{\eta}_2 \tag{5}$$

which exactly vanishes by virtue of Eqs. (12.14).

When expressed explicitly in terms of the generalized coordinates, it is evident that there is only one term in the energy that has κ_{12} as a coefficient (namely, $\kappa_{12}\eta_1^2$), and through Eq. (12.15) we see that this implies that such a term depends on the C_1's and ω_1, but not on the C_2's and ω_2.

To understand why this is so, it is sufficient to recall that η_1 is associated with the anitsymmetrical mode of oscillation, which obviously must have κ_{12} as a parameter. On the other hand, η_2 is associated to the symmetric mode, $x_1(t) = x_2(t)$, $\dot{x}_1(t) = \dot{x}_2(t)$, in which both masses move as if linked together with a rigid, massless rod. For this mode, therefore, if the spring connecting the masses is changed, the motion is not affected.

12-5. We set $\kappa_1 = \kappa_2 = \kappa_{12} \equiv \kappa$. Then, the equations of motion are

$$\left.\begin{aligned} m_1\ddot{x}_1 + 2\kappa x_1 - \kappa x_2 &= 0 \\ m_2\ddot{x}_2 + 2\kappa x_2 - \kappa x_1 &= 0 \end{aligned}\right] \tag{1}$$

Assuming solutions of the form

$$\left.\begin{aligned} x_1(t) &= B_1 e^{i\omega t} \\ x_2(t) &= B_2 e^{i\omega t} \end{aligned}\right] \tag{2}$$

we find that the equations in (1) become

$$\left.\begin{aligned}\left(2\kappa - m_1\omega^2\right)B_1 - \kappa B_2 = 0\\ -\kappa B_1 + \left(2\kappa - m_2\omega^2\right)B_2 = 0\end{aligned}\right] \tag{3}$$

which lead to the secular equation for ω^2:

$$\left(2\kappa - m_1\omega^2\right)\left(2\kappa - m_2\omega^2\right) = \kappa^2 \tag{4}$$

Therefore,

$$\omega^2 = \frac{\kappa}{\mu}\left[1 \pm \sqrt{1 - \frac{3\mu}{m_1 + m_2}}\right] \tag{5}$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system. Notice that (5) agrees with Eq. (12.8) for the case $m_1 = m_2 = M$ and $\kappa_{12} = \kappa$. Notice also that ω^2 is always real and positive since the maximum value of $3\mu/(m_1 + m_2)$ is $3/4$. (Show this.)

Inserting the values for ω_1 and ω_2 into either of the equations in (3), we find[1]

$$\left[2 - \frac{m_1}{\mu}\left[1 + \sqrt{1 - \frac{3\mu}{m_1 + m_2}}\right]\right]a_{11} = a_{21} \tag{6}$$

and

$$a_{12} = \left[2 - \frac{m_1}{\mu}\left[1 - \sqrt{1 - \frac{3\mu}{m_1 + m_2}}\right]\right]a_{22} \tag{7}$$

Using the orthonormality condition produces

$$a_{11} = \frac{1}{\sqrt{D_1}} \tag{8}$$

$$a_{21} = \frac{2 - \dfrac{m_1 + m_2}{m_2}\left[1 + \sqrt{1 - \dfrac{3m_1 m_2}{(m_1 + m_2)^2}}\right]}{\sqrt{D_1}} \tag{9}$$

where

$$D_1 \equiv 2\left(m_2 - m_1\right) + 2\frac{m_1^2}{m_2} + \frac{2}{m_2}\left(m_1^2 - m_2^2\right)\sqrt{1 - \frac{3m_1 m_2}{(m_1 + m_2)^2}} \tag{10}$$

The second eigenvector has the components

$$a_{12} = \frac{2 - \dfrac{m_1 + m_2}{m_2}\left[1 - \sqrt{1 - \dfrac{3m_1 m_2}{(m_1 + m_2)^2}}\right]}{\sqrt{D_2}} \tag{11}$$

[1] Recall that when we use $\omega = \omega_1$, we call the coefficients $\beta_1(\omega = \omega_1) = a_{11}$ and $\beta_2(\omega = \omega_1) = a_{21}$, etc.

$$a_{22} = \frac{1}{\sqrt{D_2}} \tag{12}$$

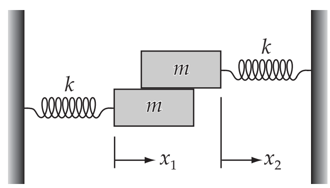
where

$$D_2 \equiv 2m_1\left[1+\frac{m_1^2}{m_2^2}\right]+m_2\left[1-\frac{3m_1^2}{m_2^2}\right]+2m_1\left[1-\frac{m_1^2}{m_2^2}\right]\sqrt{1-\frac{3m_1m_2}{\left(m_1+m_2\right)^2}} \tag{13}$$

The normal coordinates for the case in which $\dot{q}_j(0)=0$ are

$$\boxed{\begin{aligned}\eta_1(t) &= \left(m_1a_{11}x_{10}+m_2a_{21}x_{20}\right)\cos\omega_1 t\\ \eta_2(t) &= \left(m_1a_{12}x_{10}+m_2a_{22}x_{20}\right)\cos\omega_2 t\end{aligned}} \tag{14}$$

12-6.



If the frictional force acting on mass 1 due to mass 2 is

$$f = -\beta\left(\dot{x}_1-\dot{x}_2\right) \tag{1}$$

then the equations of motion are

$$\left.\begin{aligned} m\ddot{x}_1+\beta\left(\dot{x}_1-\dot{x}_2\right)+\kappa x_1 &= 0\\ m\ddot{x}_2+\beta\left(\dot{x}_2-\dot{x}_1\right)+\kappa x_2 &= 0\end{aligned}\right] \tag{2}$$

Since the system is not conservative, the eigenfrequencies will not be entirely real as in the previous cases. Therefore, we attempt a solution of the form

$$x_1(t) = B_1e^{\alpha t};\qquad x_2(t)=B_2e^{\alpha t} \tag{3}$$

where $\alpha=\lambda+i\omega$ is a complex quantity to be determined. Substituting (3) into (1), we obtain the following secular equation by setting the determinant of the coefficients of the B's equal to zero:

$$\left(m\alpha^2+\beta\alpha+\kappa\right)^2 = \beta^2\alpha^2 \tag{4}$$

from which we find the two solutions

$$\left.\begin{aligned} \alpha_1 &= \pm i\sqrt{\frac{\kappa}{m}};\qquad \omega_1=\pm\sqrt{\frac{\kappa}{m}}\\ \alpha_2 &= \frac{1}{m}\left(-\beta\pm\sqrt{\beta^2-m\kappa}\right)\end{aligned}\right] \tag{5}$$

The general solution is therefore

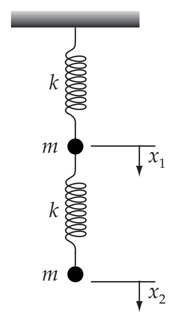
$$\boxed{x_1(t) = B_{11}^{+}\, e^{i\sqrt{\kappa/m}\, t} + B_{11}^{-}\, e^{i\sqrt{\kappa/m}\, t} + e^{-\beta t/m}\left(B_{12}^{+}\, e^{\sqrt{\beta^2 - m\kappa}\; t/m} + B_{12}^{-}\, e^{\sqrt{\beta^2 - m\kappa}\; t/m}\right)} \tag{6}$$

and similarly for $x_2(t)$.

The first two terms in the expression for $x_1(t)$ are purely oscillatory, whereas the last two terms contain the damping factor $e^{-\beta t}$. (Notice that the term $B_{12}^{+}\exp\left(\sqrt{\beta^2 - m\kappa}\; t\right)$ *increases* with time if $\beta^2 > m\kappa$, but B_{12}^{+} is not required to vanish in order to produce physically realizable motion because the damping term, exp(–βt), *decreases* with time at a more rapid rate; that is $-\beta + \sqrt{\beta^2 - m\kappa} < 0$.)

To what modes do α_1 and α_2 apply? In Mode 1 there is purely oscillating motion without friction. This can happen only if the two masses have no relative motion. Thus, Mode 1 is the *symmetric* mode in which the masses move *in phase*. Mode 2 is the *antisymmetric* mode in which the masses move *out of phase* and produce frictional damping. If $\beta^2 < m\kappa$, the motion is one of damped oscillations, whereas if $\beta^2 > m\kappa$, the motion proceeds monotonically to zero amplitude.

12-7.



We define the coordinates x_1 and x_2 as in the diagram. Including the constant downward gravitational force on the masses results only in a displacement of the equilibrium positions and does not affect the eigenfrequencies or the normal modes. Therefore, we write the equations of motion without the gravitational terms:

$$\left.\begin{aligned} m\ddot{x}_1 + 2\kappa x_1 - \kappa x_2 &= 0 \\ m\ddot{x}_2 + \kappa x_2 - \kappa x_1 &= 0 \end{aligned}\right] \tag{1}$$

Assuming a harmonic time dependence for $x_1(t)$ and $x_2(t)$ in the usual way, we obtain

$$\left.\begin{aligned} \left(2\kappa - m\omega^2\right)B_1 - \kappa B_2 &= 0 \\ -\kappa B_1 + \left(\kappa - m\omega^2\right)B_2 &= 0 \end{aligned}\right] \tag{2}$$

Solving the secular equation, we find the eigenfrequencies to be

$$\boxed{\begin{aligned} \omega_1^2 &= \frac{3+\sqrt{5}}{2}\frac{\kappa}{m} \\ \omega_2^2 &= \frac{3-\sqrt{5}}{2}\frac{\kappa}{m} \end{aligned}} \tag{3}$$

Substituting these frequencies into (2), we obtain for the eigenvector components

$$\left.\begin{aligned} \frac{1-\sqrt{5}}{2}a_{11} &= a_{21} \\ \frac{1+\sqrt{5}}{2}a_{12} &= a_{22} \end{aligned}\right] \tag{4}$$

For the initial conditions $\dot{x}_1(0) = \dot{x}_2(0) = 0$, the normal coordinates are

$$\boxed{\begin{aligned} \eta_1(t) &= ma_{11}\left(x_{10} + \frac{1-\sqrt{5}}{2} x_{20} \right) \cos \omega_1 t \\ \eta_2(t) &= ma_{12}\left(x_{10} + \frac{1+\sqrt{5}}{2} x_{20} \right) \cos \omega_2 t \end{aligned}} \tag{5}$$

Therefore, when $x_{10} = -1.6180\, x_{20}$, $\eta_2(t) = 0$ and the system oscillates in Mode 1, the antisymmetrical mode. When $x_{10} = 0.6180\, x_{20}$, $\eta_1(t) = 0$ and the system oscillates in Mode 2, the symmetrical mode.

When mass 2 is held fixed, the equation of motion of mass 1 is

$$m\ddot{x}_1 + 2\kappa x_1 = 0 \tag{6}$$

and the frequency of oscillation is

$$\omega_{10} = \sqrt{\frac{2\kappa}{m}} \tag{7}$$

When mass 1 is held fixed, the equation of motion of mass 2 is

$$m\ddot{x}_2 + \kappa x_2 = 0 \tag{8}$$

and the frequency of oscillation is

$$\omega_{20} = \sqrt{\frac{\kappa}{m}} \tag{9}$$

Comparing these frequencies with ω_1 and ω_2 we find

$$\left.\begin{aligned} \omega_1 &= \sqrt{\frac{3+\sqrt{5}}{4}}\sqrt{\frac{2\kappa}{m}} = 1.1441\sqrt{\frac{2\kappa}{m}} > \omega_{10} \\ \omega_2 &= \sqrt{\frac{3-\sqrt{5}}{4}}\sqrt{\frac{\kappa}{m}} = 0.6180\sqrt{\frac{\kappa}{m}} < \omega_{20} \end{aligned}\right]$$

Thus, the coupling of the oscillators produces a shift of the frequencies away from the uncoupled frequencies, in agreement with the discussion at the end of Section 12.2.

12-8. The kinetic and potential energies for the double pendulum are given in Problem 7-7. If we specialize these results to the case of small oscillations, we have

$$T = \frac{1}{2} m\ell^2 \left(2\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\, \dot{\phi}_2\right) \tag{1}$$

$$U = \frac{1}{2} mg\ell\left(2\phi_1^2 + \phi_2^2\right) \tag{2}$$

where ϕ_1 refers to the angular displacement of the upper pendulum and ϕ_2 to the lower pendulum, as in Problem 7-7. (We have also discarded the constant term in the expression for the potential energy.)

Now, according to Eqs. (12.34),

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \tag{3}$$

$$U = \frac{1}{2}\sum_{j,k} A_{jk}q_jq_k \tag{4}$$

Therefore, identifying the elements of $\{\mathbf{m}\}$ and $\{\mathbf{A}\}$, we find

$$\{\mathbf{m}\} = m\ell^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{5}$$

$$\{\mathbf{A}\} = mg\ell \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

and the secular determinant is

$$\begin{vmatrix} 2\dfrac{g}{\ell} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \dfrac{g}{\ell} - \omega^2 \end{vmatrix} = 0 \tag{7}$$

or,

$$\left(2\frac{g}{\ell} - 2\omega^2\right)\left[\frac{g}{\ell} - \omega^2\right] - \omega^4 = 0 \tag{8}$$

Expanding, we find

$$\omega^4 - 4\frac{g}{\ell}\omega^2 + 2\left[\frac{g}{\ell}\right]^2 = 0 \tag{9}$$

which yields

$$\omega^2 = \left(2 \pm \sqrt{2}\right) \frac{g}{\ell} \tag{10}$$

and the eigenfrequencies are

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{2+\sqrt{2}}\,\sqrt{\frac{g}{\ell}} = 1.848\,\sqrt{\frac{g}{\ell}} \\ \omega_2 &= \sqrt{2-\sqrt{2}}\,\sqrt{\frac{g}{\ell}} = 0.765\,\sqrt{\frac{g}{\ell}} \end{aligned}} \tag{11}$$

To get the normal modes, we must solve

$$\sum_j \left(A_{jk} - \omega_r^2\, m_{jk}\right) a_{jr} = 0$$

For $k = 1$, this becomes:

$$\left(A_{11} - \omega_r^2\, m_{11}\right) a_{1r} + \left(A_{21} - \omega_r^2\, m_{21}\right) a_{2r} = 0$$

For $r = 1$:

$$\left[2mg\ell - \left(2+\sqrt{2}\right)\frac{g}{\ell}\,2m\ell^2\right] a_{11} - \left(2+\sqrt{2}\right)\frac{g}{\ell}\,m\ell^2\, a_{21} = 0$$

Upon simplifying, the result is

$$a_{21} = -\sqrt{2}\, a_{11}$$

Similarly, for r = 2, the result is

$$a_{22} = \sqrt{2}\, a_{12}$$

The equations

$$x_1 = a_{11}\,\eta_1 + a_{12}\,\eta_2$$

$$x_2 = a_{21}\,\eta_1 + a_{22}\,\eta_2$$

can thus be written as

$$x_1 = a_{11}\,\eta_1 + \frac{1}{\sqrt{2}}\, a_{22}\,\eta_2$$

$$x_1 = -\sqrt{2}\, a_{11}\,\eta_1 + a_{22}\,\eta_2$$

Solving for η_1 and η_2:

$$\eta_1 = \frac{\sqrt{2}\,x_1 - x_2}{2\sqrt{2}\,a_{11}}; \quad \eta_2 = \frac{\sqrt{2}\,x_1 + x_2}{2a_{22}}$$

$$\eta_1 \text{ occurs when } \eta_2 = 0\text{; i.e. when } x_1 = -\frac{x_2}{\sqrt{2}}$$

$$\eta_2 \text{ occurs when } \eta_1 = 0\text{; i.e. when } x_1 = \frac{x_2}{\sqrt{2}}$$

Mode 2 is therefore the symmetrical mode in which both pendula are always deflected in the *same* direction; and Mode 1 is the antisymmetrical mode in which the pendula are always deflected in *opposite* directions. Notice that Mode 1 (the antisymmetrical mode), has the higher frequency, in agreement with the discussion in Section 12.2.

12-9. The general solutions for $x_1(t)$ and $x_2(t)$ are given by Eqs. (12.10). For the initial conditions we choose oscillator 1 to be displaced a distance *D* from its equilibrium position, while oscillator 2 is held at $x_2 = 0$, and both are released from rest:

$$x_1(0) = D,\ x_2(0) = 0,\ \dot{x}_1(0) = 0,\ \dot{x}_2(0) = 0 \tag{1}$$

Substitution of (1) into Eq. (12.10) determines the constants, and we obtain

$$x_1(t) = \frac{D}{2}\left(\cos \omega_1 t + \cos \omega_2 t\right) \tag{2}$$

$$x_2(t) = \frac{D}{2}\left(\cos \omega_2 t - \cos \omega_1 t\right) \tag{3}$$

where

$$\omega_1\sqrt{\frac{\kappa + 2\kappa_{12}}{M}} > \omega_2 = \sqrt{\frac{\kappa}{M}} \tag{4}$$

As an example, take $\omega_1 = 1.2\,\omega_2$; $x_1(t)$ vs. $x_2(t)$ is plotted below for this case.

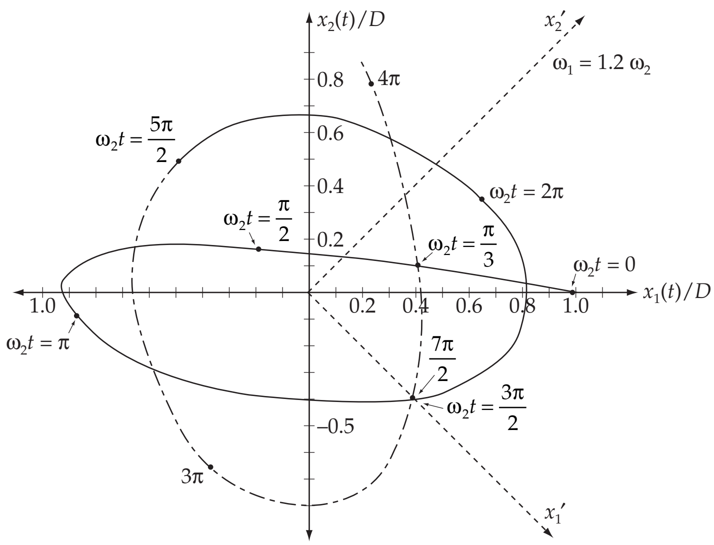
It is possible to find a rotation in configuration space such that the projection of the system point onto each of the new axes is simple harmonic.

By inspection, from (2) and (3), the new coordinates must be

$$x_1' \equiv x_1 - x_2 = D\cos \omega_1 t \tag{5}$$

$$x_2' \equiv x_1 + x_2 = D\cos \omega_2 t \tag{6}$$

These new *normal axes* correspond to the description by the normal modes. They are represented by dashed lines in the graph of the figure.



12-10. The equations of motion are

$$\left.\begin{aligned} m\ddot{x}_1 + b\dot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}\,x_2 &= F_0 \cos \omega t \\ m\ddot{x}_2 + b\dot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}\,x_1 &= 0 \end{aligned}\right] \tag{1}$$

The normal coordinates are the same as those for the undamped case [see Eqs. (12.11)]:

$$\eta_1 = x_1 - x_2\,; \qquad \eta_2 = x_1 + x_2 \tag{2}$$

Expressed in terms of these coordinates, the equations of motion become

$$\left.\begin{aligned} m(\ddot{\eta}_2 + \ddot{\eta}_1) + b(\dot{\eta}_2 + \dot{\eta}_1) + (\kappa + \kappa_{12})(\eta_2 + \eta_1) - \kappa_{12}(\eta_2 - \eta_1) &= 2F_0 \cos \omega t \\ m(\ddot{\eta}_2 - \ddot{\eta}_1) + b(\dot{\eta}_2 - \dot{\eta}_1) + (\kappa + \kappa_{12})(\eta_2 - \eta_1) - \kappa_{12}(\eta_2 + \eta_1) &= 0 \end{aligned}\right] \tag{3}$$

By adding and subtracting these equations, we obtain the uncoupled equations:

$$\left.\begin{aligned} \ddot{\eta}_1 + \frac{b}{m}\,\dot{\eta}_1 + \frac{\kappa + 2\kappa_{12}}{m}\,\eta_1 &= \frac{F_0}{m} \cos \omega t \\ \ddot{\eta}_2 + \frac{b}{m}\,\dot{\eta}_2 + \frac{\kappa}{m}\,\eta_2 &= \frac{F_0}{m} \cos \omega t \end{aligned}\right] \tag{4}$$

With the following definitions,

$$\left.\begin{aligned} 2\beta &= \frac{b}{m} \\ \omega_1^2 &= \frac{\kappa + 2\kappa_{12}}{m} \\ \omega_2^2 &= \frac{\kappa}{m} \\ A &= \frac{F_0}{m} \end{aligned}\right] \tag{5}$$

the equations become

$$\boxed{\begin{aligned} \ddot{\eta}_1 + 2\beta\dot{\eta}_1 + \omega_1^2\eta_1 = A\cos\omega t \\ \ddot{\eta}_2 + 2\beta\dot{\eta}_2 + \omega_2^2\eta_2 = A\cos\omega t \end{aligned}} \tag{6}$$

Referring to Section 3.6, we see that the solutions for $\eta_1(t)$ and $\eta_2(t)$ are exactly the same as that given for $x(t)$ in Eq. (3.62). As a result $\eta_1(t)$ exhibits a resonance at $\omega = \omega_1$ and $\eta_2(t)$ exhibits a resonance at $\omega = \omega_2$.

12-11. Taking a time derivative of the equations gives $(\dot{q} = I)$

$$L\ddot{I}_1 + \frac{I_1}{C} + M\ddot{I}_2 = 0$$

$$L\ddot{I}_2 + \frac{I_2}{C} + M\ddot{I}_1 = 0$$

Assume $I_1 = B_1 e^{i\omega t}$, $I_2 = B_2 e^{i\omega t}$; and substitute into the previous equations. The result is

$$-\omega^2 L B_1 e^{i\omega t} + \frac{1}{C} B_1 e^{i\omega t} - M\omega^2 B_2 e^{i\omega t} = 0$$

$$-\omega^2 L B_2 e^{i\omega t} + \frac{1}{C} B_2 e^{i\omega t} - M\omega^2 B_1 e^{i\omega t} = 0$$

These reduce to

$$B_1\left[\frac{1}{C} - \omega^2 L\right] + B_2\left(-M\omega^2\right) = 0$$

$$B_1\left(-M\omega^2\right) + B_2\left[\frac{1}{C} - \omega^2 L\right] = 0$$

This implies that the determinant of coefficients of B_1 and B_2 must vanish (for a non-trivial solution). Thus

$$\begin{vmatrix} \dfrac{1}{C}-\omega^2 L & -M\omega^2 \\ -M\omega^2 & \dfrac{1}{C}-\omega^2 L \end{vmatrix} = 0$$

$$\left[\frac{1}{C}-\omega^2 L\right]^2 - \left(M\omega^2\right)^2 = 0$$

$$\frac{1}{C}-\omega^2 L = \pm M\omega^2$$

or

$$\omega^2 = \frac{1}{C\left(L \pm M\right)}$$

Thus

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{1}{C\left(L+M\right)}} \\ \omega_2 &= \sqrt{\frac{1}{C\left(L-M\right)}} \end{aligned}}$$

12-12. From problem 12-11:

$$L\ddot{I}_1 + \frac{1}{C} I_1 + M\ddot{I}_2 = 0 \tag{1}$$

$$L\ddot{I}_2 + \frac{1}{C} I_2 + M\ddot{I}_1 = 0 \tag{2}$$

Solving for $\ddot{I}_1$ in (1) and substituting into (2) and similarly for $\ddot{I}_2$, we have

$$\left.\begin{aligned} \left(L-\frac{M^2}{L}\right)\ddot{I}_1 + \frac{1}{C} I_1 - \frac{M}{CL} I_2 = 0 \\ \left(L-\frac{M^2}{L}\right)\ddot{I}_2 + \frac{1}{C} I_2 - \frac{M}{CL} I_1 = 0 \end{aligned}\right] \tag{3}$$

If we identify

$$\left.\begin{aligned} m &= L - \frac{M^2}{L} \\ \kappa_{12} &= \frac{M}{LC} \\ \kappa &= \frac{1}{C}\left(1-\frac{M}{L}\right) \end{aligned}\right] \tag{4}$$

then the equations in (3) become

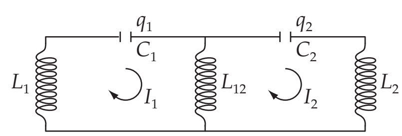
$$\left.\begin{aligned} m\ddot{I}_1 + (\kappa + \kappa_{12}) I_1 - \kappa_{12} I_2 = 0 \\ m\ddot{I}_2 + (\kappa + \kappa_{12}) I_2 - \kappa_{12} I_1 = 0 \end{aligned}\right] \tag{5}$$

which are identical in form to Eqs. (12.1). Then, using Eqs. (12.8) for the characteristic frequencies, we can write

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{1+\dfrac{M}{L}}{C\left(L-\dfrac{M^2}{L}\right)}} = \frac{1}{\sqrt{C(L-M)}} \\ \omega_2 &= \sqrt{\frac{1-\dfrac{M}{L}}{C\left(L-\dfrac{M^2}{L}\right)}} = \frac{1}{\sqrt{C(L+M)}} \end{aligned}} \tag{6}$$

which agree with the results of the previous problem.

12-13.



The Kirchhoff circuit equations are

$$\left.\begin{aligned} L_1\dot{I}_1 + \frac{q_1}{C_1} + L_{12}\left(\dot{I}_1 - \dot{I}_2\right) = 0 \\ L_2\dot{I}_2 + \frac{q_2}{C_2} + L_{12}\left(\dot{I}_2 - \dot{I}_1\right) = 0 \end{aligned}\right] \tag{1}$$

Differentiating these equations using $\dot{q} = I$, we can write

$$\left.\begin{aligned} (L_1 + L_{12})\ddot{I}_1 + \frac{1}{C_1} I_1 - L_{12}\ddot{I}_2 = 0 \\ (L_2 + L_{12})\ddot{I}_2 + \frac{1}{C_2} I_2 - L_{12}\ddot{I}_1 = 0 \end{aligned}\right] \tag{2}$$

As usual, we try solutions of the form

$$I_1(t) = B_1 e^{i\omega t}; \qquad I_2(t) = B_2 e^{i\omega t} \tag{3}$$

which lead to

$$\left.\begin{aligned}\left[\omega^2(L_1+L_{12})-\frac{1}{C_1}\right]B_1 - L_{12}\,\omega^2 B_2 = 0\\ -L_{12}\,\omega^2 B_1 + \left[\omega^2(L_2+L_{12})-\frac{1}{C_1}\right]B_2 = 0\end{aligned}\right] \tag{4}$$

Setting the determinant of the coefficients of the B's equal to zero, we obtain

$$\left[\omega^2(L_1+L_{12})-\frac{1}{C_1}\right]\left[\omega^2(L_2+L_{12})-\frac{1}{C_2}\right] = \omega^4 L_{12}^2 \tag{5}$$

with the solution

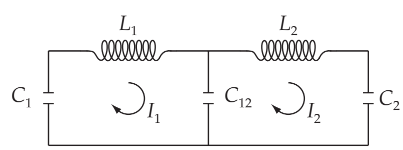
$$\boxed{\omega^2 = \frac{(L_1+L_{12})C_1+(L_2+L_{12})C_2 \pm \sqrt{\left[(L_1+L_{12})C_1-(L_2+L_{12})C_2\right]^2+4L_{12}^2C_1C_2}}{2C_1C_2\left[(L_1+L_{12})(L_2+L_{12})-L_{12}^2\right]}} \tag{6}$$

We observe that in the limit of weak coupling $(L_{12}\to 0)$ and $L_1 = L_2 = L$, $C_1 = C_2 = C$, the frequency reduces to

$$\omega = \frac{1}{\sqrt{LC}} \tag{7}$$

which is just the frequency of uncoupled oscillations [Eq. (3.78)].

12-14.



The Kirchhoff circuit equations are (after differentiating and using $\dot{q} = I$)

$$\left.\begin{aligned}L_1\ddot{I}_1 + \left[\frac{1}{C_1}+\frac{1}{C_{12}}\right]I_1 - \frac{1}{C_{12}}I_2 = 0\\ L_2\ddot{I}_2 + \left[\frac{1}{C_2}+\frac{1}{C_{12}}\right]I_2 - \frac{1}{C_{12}}I_1 = 0\end{aligned}\right] \tag{1}$$

Using a harmonic time dependence for $I_1(t)$ and $I_2(t)$, the secular equation is found to be

$$\left[L_1\omega^2 - \frac{C_1+C_{12}}{C_1\,C_{12}}\right]\left[L_2\omega^2 - \frac{C_2+C_{12}}{C_2\,C_{12}}\right] = \frac{1}{C_{12}^2} \tag{2}$$

Solving for the frequency,

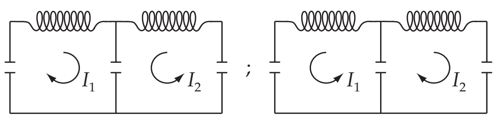
$$\boxed{\omega^2 = \frac{C_1L_1(C_2+C_{12})+C_2L_2(C_1+C_{12}) \pm \sqrt{\left[C_1L_1(C_2+C_{12})-C_2L_2(C_1+C_{12})\right]^2+4C_1^2C_2^2L_1L_2}}{2L_1L_2C_1C_2C_{12}}} \tag{3}$$

Because the characteristic frequencies are given by this complicated expression, we examine the normal modes for the special case in which $L_1 = L_2 = L$ and $C_1 = C_2 = C$. Then,

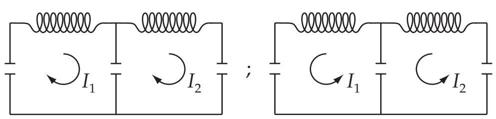
$$\boxed{\begin{aligned}\omega_1^2 &= \frac{2C + C_{12}}{LCC_{12}}\\ \omega_2^2 &= \frac{1}{LC}\end{aligned}} \tag{4}$$

Observe that ω_2 corresponds to the case of uncoupled oscillations. The equations for this simplified circuit can be set in the same form as Eq. (12.1), and consequently the normal modes can be found in the same way as in Section 12.2. There will be two possible modes of oscillation: (1) *out of phase*, with frequency ω_1, and (2) *in phase*, with frequency ω_2.

Mode 1 corresponds to the currents I_1 and I_2 oscillating always *out of phase*:

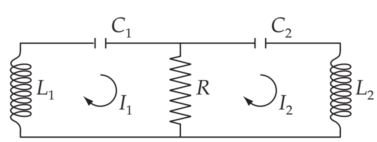


Mode 2 corresponds to the currents I_1 and I_2 oscillating always *in phase*:



(The analogy with two oscillators coupled by a spring can be seen by associating case 1 with Fig. 12-2 for $\omega = \omega_1$ and case 2 with Fig. 12-2 for $\omega = \omega_2$.) If we now let $L_1 \neq L_2$ and $C_1 \neq C_2$, we do not have pure symmetrical and antisymmetrical symmetrical modes, but we can associate ω_2 with the mode of highest degree of symmetry and ω_1 with that of lowest degree of symmetry.

12-15.



Setting up the Kirchhoff circuit equations, differentiating, and using $\dot{q} = I$, we find

$$\left.\begin{aligned} L_1\ddot{I}_1 + R\left(\dot{I}_1 - \dot{I}_2\right) + \frac{1}{C_1} I_1 = 0 \\ L_2\ddot{I}_2 + R\left(\dot{I}_2 - \dot{I}_1\right) + \frac{1}{C_2} I_2 = 0 \end{aligned}\right\} \tag{1}$$

Using a harmonic time dependence for $I_1(t)$ and $I_2(t)$, the secular equation is

$$\left(\omega^2 L_1 - \frac{1}{C_1} - i\omega R\right)\left(\omega^2 L_2 - \frac{1}{C_2} - i\omega R\right) + \omega^2 R^2 = 0 \tag{2}$$

From this expression it is clear that the oscillations will be damped because ω will have an imaginary part. (The resistor in the circuit dissipates energy.) In order to simplify the analysis, we choose the special case in which $L_1 = L_2 = L$ and $C_1 = C_2 = C$. Then, (2) reduces to

$$\left(\omega^2 L - \frac{1}{C} - i\omega R\right)^2 + \omega^2 R^2 = 0 \tag{3}$$

which can be solved as in Problem 12-6. We find

$$\boxed{\begin{aligned} \omega_1 &= \pm\frac{1}{\sqrt{LC}} \\ \omega_2 &= \frac{i}{L}\left[R \pm \sqrt{R^2 - \frac{L}{C}}\right] \end{aligned}} \tag{4}$$

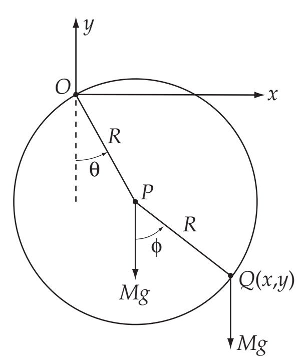
The general solution for $I_1(t)$ is

$$\boxed{I_1(t) = B_{11}^+ e^{i\sqrt{1/LC}\,t} + B_{11}^- e^{-i\sqrt{1/LC}\,t} + e^{-Rt/L}\left[B_{12}^+ e^{\sqrt{R^2 - L/C}\,t/L} + B_{12}^- e^{-\sqrt{R^2 - L/C}\,t/L}\right]} \tag{5}$$

and similarly for $I_2(t)$. The implications of these results follow closely the arguments presented in Problem 12-6.

Mode 1 is purely oscillatory with no damping. Since there is a resistor in the circuit, this means that I_1 and I_2 flow in *opposite* senses in the two parts of the circuit and *cancel* in *R*. Mode 2 is the mode in which both currents flow in the *same* direction through *R* and energy is dissipated. If $R^2 < L/C$, there will be damped oscillations of I_1 and I_2, whereas if $R^2 > L/C$, the currents will decrease monotonically without oscillation.

12-16.



Let *O* be the fixed point on the hoop and the origin of the coordinate system. *P* is the center of mass of the hoop and *Q*(*x*,*y*) is the position of the mass *M*. The coordinates of *Q* are

$$\left.\begin{aligned} x &= R(\sin\theta + \sin\phi) \\ y &= -R(\cos\theta + \cos\phi) \end{aligned}\right] \tag{1}$$

The rotational inertia of the hoop through *O* is

$$I_{\mathrm{O}} = I_{\mathrm{CM}} + MR^2 = 2MR^2 \tag{2}$$

The potential energy of the system is

$$\begin{aligned} U &= U_{\text{hoop}} + U_{\text{mass}} \\ &= -MgR\left(2\cos\theta + \cos\phi\right) \end{aligned} \tag{3}$$

Since θ and ϕ are small angles, we can use $\cos x \cong 1 - x^2/2$. Then, discarding the constant term in U, we have

$$U = \frac{1}{2} MgR\left(2\theta^2 + \phi^2\right) \tag{4}$$

The kinetic energy of the system is

$$\begin{aligned} T &= T_{\text{hoop}} + T_{\text{mass}} \\ &= \frac{1}{2} I_{\mathrm{O}}\dot{\theta}^2 + \frac{1}{2} M\left(\dot{x}^2 + \dot{y}^2\right) \\ &= MR^2\dot{\theta}^2 + \frac{1}{2} MR^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right] \end{aligned} \tag{5}$$

where we have again used the small-angle approximations for θ and ϕ. Thus,

$$T = \frac{1}{2} MR^2\left[3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right] \tag{6}$$

Using Eqs. (12.34),

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\, \dot{q}_j\, \dot{q}_k \tag{7}$$

$$U = \frac{1}{2}\sum_{j,k} A_{jk}\, q_j\, q_k \tag{8}$$

we identify the elements of $\{\mathbf{m}\}$ and $\{\mathbf{A}\}$:

$$\{\mathbf{m}\} = MR^2\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \tag{9}$$

$$\{\mathbf{A}\} = MgR\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{10}$$

The secular determinant is

$$\begin{vmatrix} 2\dfrac{g}{R} - 3\omega^2 & -\omega^2 \\ -\omega^2 & \dfrac{g}{R} - \omega^2 \end{vmatrix} = 0 \tag{11}$$

from which

$$\left(2\frac{g}{R}-3\omega^2\right)\left(\frac{g}{R}-\omega^2\right)-\omega^4=0 \tag{12}$$

Solving for the eigenfrequencies, we find

$$\boxed{\begin{aligned}\omega_1&=\sqrt{2}\sqrt{\frac{g}{R}}\\ \omega_2&=\frac{\sqrt{2}}{2}\sqrt{\frac{g}{R}}\end{aligned}} \tag{13}$$

To get the normal modes, we must solve:

$$\sum_j\left(A_{jk}-\omega_r^2\, m_{jk}\right)a_{jr}=0$$

For $k = r = 1$, this becomes:

$$\left(2mgR-2\frac{g}{R}3mR^2\right)a_{11}-2\frac{g}{R}mR^2\, a_{21}=0$$

or

$$a_{21}=-2a_{11}$$

For $k = 1$, $r = 2$, the result is

$$a_{12}=a_{22}$$

Thus the equations

$$x_1=a_{11}\,\eta_1+a_{12}\,\eta_2$$

$$x_2=a_{21}\,\eta_1+a_{22}\,\eta_2$$

can be written as

$$x_1=a_{11}\,\eta_1+a_{22}\,\eta_2$$

$$x_2=-2a_{11}\,\eta_1+a_{22}\,\eta_2$$

Solving for η_1, η_2

$$\eta_1=\frac{x_1-x_2}{3a_{11}};\qquad \eta_1=\frac{2x_1+x_2}{3a_{22}}$$

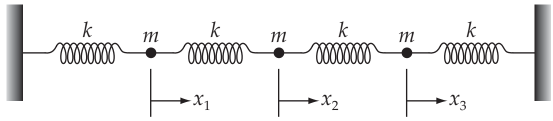
η_1 occurs when the initial conditions are such that $\eta_2=0$; i.e., $x_{10}=-\frac{1}{2}x_{20}$

This is the antisymmetrical mode in which the CM of the hoop and the mass are on opposite sides of the vertical through the pivot point.

η_2 occurs when the initial conditions are such that $\eta_1=0$; i.e., $x_{10}=x_{20}$

This is the symmetrical mode in which the pivot point, the CM of the hoop, and the mass always lie on a straight line.

12-17.



Following the procedure outlined in section 12.6:

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2 + \frac{1}{2}kx_3^2$$

$$= k\left[x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3\right]$$

Thus

$$\mathbf{m} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

Thus we must solve

$$\begin{vmatrix} 2k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 m & -k \\ 0 & -k & 2k - \omega^2 m \end{vmatrix} = 0$$

This reduces to

$$\left(2k - \omega^2 m\right)^3 - 2k^2\left(2k - \omega^2 m\right) = 0$$

or

$$\left(2k - \omega^2 m\right)\left[\left(2k - \omega^2 m\right)^2 - 2k^2\right] = 0$$

If the first term is zero, then we have

$$\boxed{\omega_1 = \sqrt{\frac{2k}{m}}}$$

If the second term is zero, then

$$2k - \omega^2 m = \pm\sqrt{2}\,k$$

which leads to

$$\boxed{\omega_2 = \sqrt{\frac{\left(2+\sqrt{2}\right)k}{m}}; \qquad \omega_3 = \sqrt{\frac{\left(2-\sqrt{2}\right)k}{m}}}$$

To get the normal modes, we must solve

$$\sum_j \left(A_{jk} - \omega_r^2\ m_{jk}\right) a_{jr} = 0$$

For $k = 1$ this gives:

$$\left(2k - \omega_r^2 m\right) a_{1r} + \left(-k\right) a_{2r} = 0$$

Substituting for each value of r gives

$$r = 1: \quad (2k - 2k) a_{11} - k a_{21} = 0 \rightarrow a_{21} = 0$$

$$r = 2: \quad \left(-\sqrt{2}\ k\right) a_{12} - k a_{22} = 0 \rightarrow a_{22} = -\sqrt{2}\ a_{12}$$

$$r = 3: \quad \left(\sqrt{2}\ k\right) a_{13} - k a_{23} = 0 \rightarrow a_{23} = \sqrt{2}\ a_{13}$$

Doing the same for $k = 2$ and 3 yields

$$a_{11} = -a_{31} \qquad a_{21} = 0$$

$$a_{12} = a_{32} \qquad a_{22} = -\sqrt{2}\ a_{32}$$

$$a_{13} = a_{33} \qquad a_{23} = \sqrt{2}\ a_{33}$$

The equations

$$x_1 = a_{11}\ \eta_1 + a_{12}\ \eta_2 + a_{13}\ \eta_3$$

$$x_2 = a_{21}\ \eta_1 + a_{22}\ \eta_2 + a_{23}\ \eta_3$$

$$x_3 = a_{31}\ \eta_1 + a_{32}\ \eta_2 + a_{33}\ \eta_3$$

can thus be written as

$$x_1 = a_{11}\ \eta_1 - \frac{1}{\sqrt{2}} a_{22}\ \eta_2 + a_{33}\ \eta_3$$

$$x_2 = a_{22}\ \eta_2 + \sqrt{2}\ a_{33}\ \eta_3$$

$$x_3 = -a_{11}\ \eta_1 - \frac{1}{\sqrt{2}} a_{22}\ \eta_2 + a_{33}\ \eta_3$$

We get the normal modes by solving these three equations for η_1, η_2, η_3:

$$\eta_1 = \frac{x_1 - x_3}{2a_{11}}$$

$$\eta_2 = \frac{-x_1 + \sqrt{2}\, x_2 - x_3}{2\sqrt{2}\, a_{22}}$$

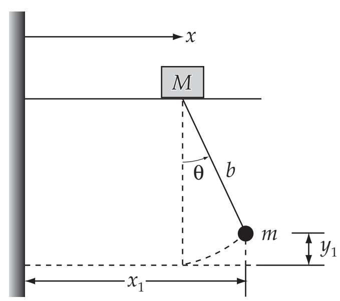
and

$$\eta_3 = \frac{x_1 + \sqrt{2}\, x_2 + x_3}{4a_{33}}$$

The normal mode motion is as follows

η_1 :		•	→	•		←	•		$x_1 = -x_3$
η_2 :	←	•		•	→	←	•		$x_2 = -\sqrt{2}\, x_1 = -\sqrt{2}\, x_3$
η_3 :		•	→	•	→		•	→	$x_2 = \sqrt{2}\, x_1 = \sqrt{2}\, x_3$

12-18.



$$x_1 = x + b \sin\theta; \quad \dot{x}_1 = \dot{x} + b\dot{\theta}\cos\theta$$

$$y_1 = b - b\cos\theta; \quad \dot{y}_1 = b\dot{\theta}\sin\theta$$

Thus

$$T = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\left(\dot{x}^2 + b^2\,\dot{\theta}^2 + 2b\,\dot{x}\dot{\theta}\cos\theta\right)$$

$$U = mgy_1 = mgb\left(1 - \cos\theta\right)$$

For small θ, $\cos\theta \simeq 1 - \dfrac{\theta^2}{2}$. Substituting and neglecting the term of order $\theta^2\dot{\theta}$ gives

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left(b^2\dot{\theta}^2 + 2b\,\dot{x}\dot{\theta}\right)$$

$$U = \frac{mgb}{2}\,\theta^2$$

Thus

$$\mathbf{m} = \begin{bmatrix} M+m & mb \\ mb & mb^2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & mgb \end{bmatrix}$$

We must solve

$$\begin{vmatrix} -\omega^2(M+m) & -\omega^2 mb \\ -\omega^2 mb & mgb - \omega^2 mb^2 \end{vmatrix} = 0$$

which gives

$$\omega^2(M+m)\left(\omega^2 mb^2 - mgb\right) - \omega^4 m^2 b^2 = 0$$

$$\omega^2\left[\omega^2 Mb^2 - mgb(m+M)\right] = 0$$

Thus

$$\boxed{\begin{aligned} &\omega_1 = 0 \\ &\omega_2 = \sqrt{\frac{g}{mb}(M+m)} \end{aligned}}$$

$$\sum_j \left(A_{jk} - \omega_r^2\, m_{jk}\right) a_{jr} = 0$$

Substituting into this equation gives

$$a_{21} = 0 \qquad (k=2,\ r=1)$$

$$a_{12} = -\frac{bm}{(m+M)}\,a_{22} \qquad (k=2,\ r=2)$$

Thus the equations

$$x = a_{11}\,\eta_1 + a_{12}\,\eta_2$$

$$\theta = a_{21}\,\eta_1 + a_{22}\,\eta_2$$

become

$$x = a_{11}\,\eta_1 - \frac{mb}{(m+M)}\,a_{22}\,\eta_2$$

$$\theta = a_{22}\ \eta_2$$

Solving for η_1, η_2:

$$\eta_2 = \frac{\theta}{a_{22}}$$

$$n_1 = \frac{x + \dfrac{bm}{(m+M)}\theta}{a_{11}}$$

$$\boxed{\begin{array}{l} n_1 \text{ occurs when } n_2 = 0\text{; or } \theta = 0 \\ n_2 \text{ occurs when } n_1 = 0\text{; or } x = -\dfrac{bm}{(m+M)}\theta \end{array}}$$

12-19. With the given expression for U, we see that $\{A\}$ has the form

$$\{\mathbf{A}\} = \begin{bmatrix} 1 & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{12} & 1 & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & 1 \end{bmatrix} \tag{1}$$

The kinetic energy is

$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) \tag{2}$$

so that $\{\mathbf{m}\}$ is

$$\{\mathbf{m}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The secular determinant is

$$\begin{vmatrix} 1-\omega^2 & -\varepsilon_{12} & -\varepsilon_{13} \\ -\varepsilon_{12} & 1-\omega^2 & -\varepsilon_{23} \\ -\varepsilon_{13} & -\varepsilon_{23} & 1-\omega^2 \end{vmatrix} = 0 \tag{4}$$

Thus,

$$\left(1-\omega^2\right)^3 - \left(1-\omega^2\right)\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right) - 2\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} = 0 \tag{5}$$

This equation is of the form (with $1-\omega^2 \equiv x$)

$$x^3 - 3\alpha^2 x - 2\beta^2 = 0 \tag{6}$$

which has a double root *if and only if*

$$\left(\alpha^2\right)^{3/2} = \beta^2 \tag{7}$$

Therefore, (5) will have a double root *if and only if*

$$\left[\frac{\varepsilon_{12}^2+\varepsilon_{13}^2+\varepsilon_{23}^2}{3}\right]^{3/2}=\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} \tag{8}$$

This equation is satisfied only if

$$\boxed{\varepsilon_{12}=\varepsilon_{13}=\varepsilon_{23}} \tag{9}$$

Consequently, there will be no degeneracy unless the three coupling coefficients are identical.

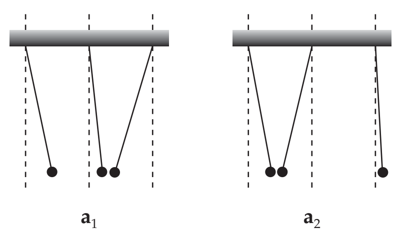
12-20. If we require $a_{11}=2a_{21}$, then Eq. (12.122) gives $a_{31}=-3a_{21}$, and from Eq. (12.126) we obtain $a_{21}=1/\sqrt{14}$. Therefore,

$$\boxed{\mathbf{a}_1=\left[\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}},-\frac{3}{\sqrt{14}}\right]} \tag{1}$$

The components of a_2 can be readily found by substituting the components of a_1 above into Eq. (12.125) and using Eqs. (12.123) and (12.127):

$$\boxed{\mathbf{a}_2=\left[\frac{4}{\sqrt{42}},\frac{-5}{\sqrt{42}},\frac{1}{\sqrt{42}}\right]} \tag{2}$$

These eigenvectors correspond to the following cases:



12-21. The tensors $\{\mathbf{A}\}$ and $\{\mathbf{m}\}$ are:

$$\{\mathbf{A}\}=\begin{bmatrix}\kappa_1 & \frac{1}{2}\kappa_3 & 0\\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3\\ 0 & \frac{1}{2}\kappa_3 & \kappa_1\end{bmatrix} \tag{1}$$

$$\{\mathbf{m}\}=\begin{bmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{bmatrix} \tag{2}$$

thus, the secular determinant is

$$\begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0 \tag{3}$$

from which

$$\left(\kappa_1 - m\omega^2\right)^2 \left(\kappa_2 - m\omega^2\right) - \frac{1}{2}\kappa_3^2 \left(\kappa_1 - m\omega^2\right) = 0 \tag{4}$$

In order to find the roots of this equation, we first set $(1/2)\kappa_3^2 = \kappa_1\kappa_2$ and then factor:

$$\begin{aligned} &\left(\kappa_1 - m\omega^2\right)\left[\left(\kappa_1 - m\omega^2\right)\left(\kappa_2 - m\omega^2\right) - \kappa_1\kappa_2\right] = 0 \\ &\left(\kappa_1 - m\omega^2\right)\left[m^2\omega^4 - \left(\kappa_1 + \kappa_2\right)m\omega^2\right] = 0 \\ &\left(\kappa_1 - m\omega^2\right)m\omega^2\left[m\omega^2 - \left(\kappa_1 + \kappa_2\right)\right] = 0 \end{aligned} \tag{5}$$

Therefore, the roots are

$$\boxed{\begin{aligned} &\omega_1 = \sqrt{\frac{\kappa_1}{m}} \\ &\omega_2 = \sqrt{\frac{\kappa_1 + \kappa_2}{m}} \\ &\omega_3 = 0 \end{aligned}} \tag{6}$$

Consider the case $\omega_3 = 0$. The equation of motion is

$$\ddot{\eta}_3 + \omega_3^2 \eta_3 = 0 \tag{7}$$

so that

$$\ddot{\eta}_3 = 0 \tag{8}$$

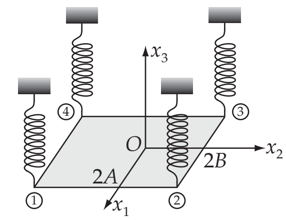
with the solution

$$\eta_3(t) = at + b \tag{9}$$

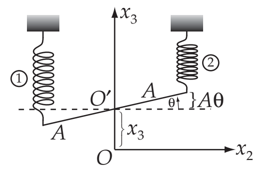
That is, the zero-frequency mode corresponds to a *translation* of the system with oscillation.

12-22. The equilibrium configuration is shown in diagram (a) below, and the non-equilibrium configurations are shown in diagrams (b) and (c).

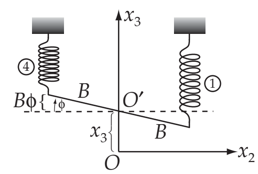
(a)



(b)



(c)



The kinetic energy of the system is

$$T = \frac{1}{2}M\dot{x}_3^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2 \tag{1}$$

where $I_1 = (1/3)MA^2$ and $I_2 = (1/3)MB^2$.

The potential energy is

$$U = \frac{1}{2}\kappa\left[(x_3 - A\theta - B\phi)^2 + (x_3 + A\theta - B\phi)^2 + (x_3 + A\theta + B\phi)^2 + (x_3 - A\theta + B\phi)^2\right]$$

$$= \frac{1}{2}\kappa\left(4x_3^2 + 4A^2\theta^2 + 4B^2\phi^2\right) \tag{2}$$

Therefore, the tensors $\{\mathbf{m}\}$ and $\{\mathbf{A}\}$ are

$$\{\mathbf{m}\} = \begin{bmatrix} M & 0 & 0 \\ 0 & \frac{1}{3}MA^2 & 0 \\ 0 & 0 & \frac{1}{3}MB^2 \end{bmatrix} \tag{3}$$

$$\{\mathbf{A}\} = \begin{bmatrix} 4\kappa & 0 & 0 \\ 0 & 4\kappa A^2 & 0 \\ 0 & 0 & 4\kappa B^2 \end{bmatrix} \tag{4}$$

The secular equation is

$$\left(4\kappa - M\omega^2\right)\left(4\kappa A^2 - \frac{1}{3}MA^2\omega^2\right)\left(4\kappa B^2 - \frac{1}{3}MB^2\omega^2\right) = 0 \tag{5}$$

Hence, the characteristic frequencies are

$$\boxed{\begin{aligned} \omega_1 &= 2\sqrt{\frac{\kappa}{M}} \\ \omega_2 &= 2\sqrt{\frac{3\kappa}{M}} \\ \omega_3 &= 2\sqrt{\frac{3\kappa}{M}} = \omega_2 \end{aligned}} \tag{6}$$

We see that $\omega_2 = \omega_3$, so the system is *degenerate*.

The eigenvector components are found from the equation

$$\sum_j \left(A_{jk} - \omega_r^2\, m_{jk}\right) a_{jr} = 0 \tag{7}$$

Setting $a_{32} = 0$ to remove the indeterminacy, we find

$$\boxed{\mathbf{a}_1 = \begin{bmatrix} 1\sqrt{M} \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ \sqrt{3/MA^2} \\ 0 \end{bmatrix}; \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{3/MB^2} \end{bmatrix}} \tag{8}$$

The normal coordinates are (for $\dot{x}_3(0) = \dot{\theta}(0) = \dot{\phi}(0) = 0$)

$$\boxed{\begin{aligned} \eta_1(t) &= x_{30}\sqrt{M}\cos\omega_1 t \\ \eta_2(t) &= \frac{\theta_0 A\sqrt{M}}{\sqrt{3}}\cos\omega_2 t \\ \eta_3(t) &= \frac{\phi_0 B\sqrt{M}}{\sqrt{3}}\cos\omega_3 t \end{aligned}} \tag{9}$$

Mode 1 corresponds to the simple vertical oscillations of the plate (without tipping). Mode 2 corresponds to rotational oscillations around the x_1 axis, and Mode 3 corresponds to rotational oscillations around the x_2-axis.

The degeneracy of the system can be removed if the symmetry is broken. For example, if we place a bar of mass m and length $2A$ along the x_2-axis of the plate, then the moment of inertia around the x_1-axis is changed:

$$I_1' = \frac{1}{3}(M+m)A^2 \tag{10}$$

The new eigenfrequencies are

$$\boxed{\begin{aligned} \omega_1 &= 2\sqrt{\frac{\kappa}{M}} \\ \omega_2 &= 2\sqrt{\frac{3\kappa}{M+m}} \\ \omega_3 &= 2\sqrt{\frac{3\kappa}{M}} \end{aligned}} \tag{11}$$

and there is no longer any degeneracy.

12-23. The total energy of the r-th normal mode is

$$\begin{aligned} E_r &= T_r + U_r \\ &= \frac{1}{2}\dot{\eta}_r^2 + \frac{1}{2}\omega_r^2\eta_r^2 \end{aligned} \tag{1}$$

where

$$\eta_r = \beta_r e^{i\omega_r t} \tag{2}$$

Thus,

$$\dot{\eta}_r = i\omega_r\beta_r e^{i\omega_r t} \tag{3}$$

In order to calculate T_r and U_r, we must take the squares of the real parts of $\dot{\eta}_r$ and η_r:

$$\begin{aligned} \dot{\eta}_r^2 &= (\mathrm{Re}\,\dot{\eta}_r)^2 = \left[\mathrm{Re}\, i\,\omega_r(\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\right]^2 \\ &= \left[-\omega_r\,\nu_r\cos\omega_r t - \omega_r\,\mu_r\sin\omega_r t\right]^2 \end{aligned} \tag{4}$$

so that

$$T_r = \frac{1}{2}\omega_r^2\left[\nu_r\cos\omega_r t + \mu_r\sin\omega_r t\right]^2 \tag{5}$$

Also

$$\begin{aligned} \eta_r^2 &= (\mathrm{Re}\,\eta_r)^2 = \left[\mathrm{Re}(\mu_r + i\nu_r)(\cos\omega_r t + i\sin\omega_r t)\right]^2 \\ &= \left[\mu_r\cos\omega_r t - \nu_r\sin\omega_r t\right]^2 \end{aligned} \tag{6}$$

so that

$$U_r = \frac{1}{2}\omega_r^2\left[\mu_r\cos\omega_r t - \nu_r\,xin\,\omega_r t\right]^2 \tag{7}$$

Expanding the squares in T_r and U_r, and then adding, we find

$$E_r = T_r + U_r$$

$$= \frac{1}{2}\,\omega_r^2\left(\mu_r^2 + \nu_r^2\right)$$

Thus,

$$\boxed{E_r = \frac{1}{2}\,\omega_r^2\left|\beta_r\right|^2} \tag{8}$$

So that the total energy associated with each normal mode is separately conserved.

For the case of Example 12.3, we have for Mode 1

$$\eta_1 = \sqrt{\frac{M}{2}}\left(x_{10} - x_{20}\right)\cos\omega_1 t \tag{9}$$

Thus,

$$\dot{\eta}_1 = -\omega_1\sqrt{\frac{M}{2}}\left(x_{10} - x_{20}\right)\sin\omega_1 t \tag{10}$$

Therefore,

$$E_1 = \frac{1}{2}\,\dot{\eta}_1^2 + \frac{1}{2}\,\omega_1^2\eta_1^2 \tag{11}$$

But

$$\omega_1^2 = \frac{\kappa + 2\kappa_{12}}{M} \tag{12}$$

so that

$$E_1 = \frac{1}{2}\,\frac{\kappa + 2\kappa_{12}}{M}\,\frac{M}{2}\left(x_{10} - x_{20}\right)^2\sin^2\omega_1 t + \frac{1}{2}\,\frac{\kappa + 2\kappa_{12}}{M}\,\frac{M}{2}\left(x_{10} - x_{20}\right)^2\cos^2\omega_1 t$$

$$= \frac{1}{4}\left(\kappa + 2\kappa_{12}\right)\left(x_{10} - x_{20}\right)^2 \tag{13}$$

which is recognized as the value of the potential energy at $t = 0$. [At $t = 0$, $\dot{x}_1 = \dot{x}_2 = 0$, so that the total energy is $U_1\left(t = 0\right)$.]

12-24. Refer to Fig. 12-9. If the particles move *along* the line of the string, the equation of motion of the j-th particle is

$$m\ddot{x}_j = -\kappa\left(x_j - x_{j-1}\right) - \kappa\left(x_j - x_{j+1}\right) \tag{1}$$

Rearranging, we find

$$\ddot{x}_j = \frac{\kappa}{m}\left(x_{j-1} - 2x_j + x_{j+1}\right) \tag{2}$$

which is just Eq. (12.131) if we identify τ/md with κ/m.

12-25. The initial conditions are

$$\left.\begin{array}{l} q_1(0) = q_2(0) = q_3(0) = a \\ \dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0 \end{array}\right] \tag{1}$$

Since the initial velocities are zero, all of the ν_r [see Eq. (12.161b)] vanish, and the μ_r are given by [see Eq. (12.161a)]

$$\mu_r = \frac{a}{2}\left[\sin\frac{\pi r}{4} + \sin\frac{\pi r}{2} + \sin\frac{3\pi r}{4}\right] \tag{2}$$

so that

$$\left.\begin{array}{l} \mu_1 = \dfrac{\sqrt{2}+1}{2}a \\ \mu_2 = 0 \\ \mu_3 = \dfrac{\sqrt{2}-1}{2}a \end{array}\right] \tag{3}$$

The quantities $\sin\left[j r\pi/(n+1)\right]$ are the same as in Example 12.7 and are given in Eq. (12.165). The displacements of the particles are

$$\boxed{\begin{array}{l} q_1(t) = \dfrac{1}{2}a\left(\cos\omega_1 t + \cos\omega_3 t\right) + \dfrac{\sqrt{2}}{4}a\left(\cos\omega_1 t - \cos\omega_3 t\right) \\ q_2(t) = \dfrac{\sqrt{2}}{2}a\left(\cos\omega_1 t - \cos\omega_3 t\right) + \dfrac{1}{2}a\left(\cos\omega_1 t + \cos\omega_3 t\right) \\ q_3(t) = \dfrac{1}{2}a\left(\cos\omega_1 t - \cos\omega_3 t\right) + \dfrac{\sqrt{2}}{4}a\left(\cos\omega_1 t + \cos\omega_3 t\right) \end{array}} \tag{4}$$

where the characteristic frequencies are [see Eq. (12.152)]

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\left[\frac{r\pi}{8}\right], \quad r = 1,2,3 \tag{5}$$

Because all three particles were initially displaced, there can exist no normal modes in which any one of the particles is located at a node. For three particles on a string, there is only *one* normal mode in which a particle is located at a node. This is the mode $\omega = \omega_2$ (see Figure 12-11) and so this mode is absent.

12-26. Kinetic energy $T = \dfrac{mb^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) \Rightarrow [m] = \begin{bmatrix} mb^2 & & \\ & mb^2 & \\ & & +mb^2 \end{bmatrix}$

Potential energy

$$U = mgb\left[(1-\cos\theta_1)+(1-\cos\theta_2)+(1-\cos\theta_3)\right]+\frac{k}{2}b^2\left[(\sin\theta_2-\sin\theta_1)^2+(\sin\theta_3-\sin\theta_2)^2\right]$$

$$\approx \frac{mgb}{2}\left(\theta_1^2+\theta_2^2+\theta_3^2\right)+\frac{kb^2}{2}\left(\theta_1^2+2\theta_2^2+\theta_3^2-2\theta_1\theta_2-2\theta_2\theta_3\right)$$

$$\Rightarrow \quad [A]=\begin{bmatrix} mgb+kb^2 & -kb^2 & 0 \\ -kb^2 & mgb+kb^2 & -kb^2 \\ 0 & -kb^2 & mgb+kb^2 \end{bmatrix}$$

The proper frequencies are solutions of the equation

$$0=\mathrm{Det}\left([A]-\omega^2[m]\right)=\mathrm{Det}\begin{bmatrix} \left(mgb+kb^2-mb^2\omega^2\right) & -kb^2 & 0 \\ -kb^2 & \left(mgb+2kb^2-mb^2\omega^2\right) & -kb^2 \\ 0 & -kb^2 & \left(mgb+kb^2-mb^2\omega^2\right) \end{bmatrix}$$

We obtain 3 different proper frequencies

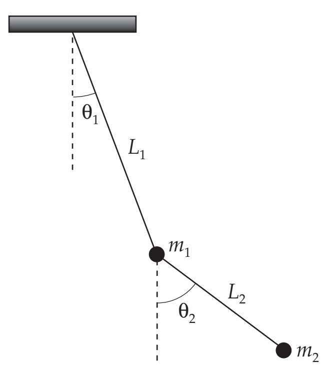
$$\omega_1^2=\frac{mg+kb}{mb} \quad \Rightarrow \quad \omega_1=\sqrt{\frac{mg+kb}{mb}}=4.64 \text{ rad/s}$$

$$\omega_2^2=\frac{mg+3kb}{mb} \quad \Rightarrow \quad \omega_2=\sqrt{\frac{mg+3kb}{mb}}=4.81 \text{ rad/s}$$

$$\omega_3^2=\frac{mg}{mb} \quad \Rightarrow \quad \omega_3=\sqrt{\frac{g}{f}}=4.57 \text{ rad/s}$$

Actually those values are very close to one another, because k is very small.

12-27. The coordinates of the system are given in the figure:



Kinetic energy:

$$T=\frac{1}{2}m_1^2\,\dot{\theta}_1^2\,L_1^2+\frac{1}{2}m_2\left(\dot{L}_1^2\,\dot{\theta}_1^2+L_2^2\,\dot{\theta}_2^2-2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$

$$\approx\frac{1}{2}\dot{\theta}_1^2\left(m_1L_1^2+m_2L_1^2\right)+\frac{1}{2}m_2L_2^2\dot{\theta}_2^2-m_2L_1L_2\dot{\theta}_1\dot{\theta}_2=\frac{1}{2}\sum_{jk}m_{jk}\,\dot{\theta}_j\,\dot{\theta}_k$$

$$\Rightarrow \quad \left[m_{jk}\right]=\begin{bmatrix}(m_1+m_2)L_1^2 & -\ m_2L_1L_2\\ -m_2L_1L_2 & m_2L_2^2\end{bmatrix}$$

Potential energy:

$$U=m_1gL_1\left(1-\cos\theta_1\right)+m_2g\left[L_1\left(1-\cos\theta_1\right)+L_2\left(1-\cos\theta_2\right)\right]$$

$$\approx\left(m_1+m_2\right)gL_1\frac{\theta_1^2}{2}+m_2gL_2\frac{\theta_2^2}{2}=\frac{1}{2}\sum_{jk}A_{jk}\theta_j\theta_k$$

$$\Rightarrow\left[A_{jk}\right]=\begin{bmatrix}(m_1+m_2)gL_1 & 0\\ 0 & m_2gL_2\end{bmatrix}$$

Proper oscillation frequencies are solutions of the equation

$$\mathrm{Det}\left(\left[A\right]-\omega^2\left[m\right]\right)=0$$

$$\Rightarrow \quad \omega_{1,2}=\frac{\left(m_1+m_2\right)g\left(L_1+L_2\right)\pm\sqrt{\left(m_1+m_2\right)g^2\left[m_1\left(L_1-L_2\right)^2+m_2\left(L_1+L_2\right)^2\right]}}{2m_1L_1L_2}$$

The eigenstate corresponding to ω_1 is $\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}$ where

$$a_{21}=\frac{\left(m_1+m_2\right)L_1}{m_1L_2}\left(1-\frac{2gm_1L_2}{\left(m_1+m_2\right)g\left(L_1+L_2\right)+\sqrt{\left(m_1+m_2\right)g^2\left[m_1\left(L_1-L_2\right)^2+m_2\left(L_1+L_2\right)^2\right]}}\right)\times a_{11}$$

The eigenstate corresponding to ω_2 is $\begin{pmatrix}a_{12}\\ a_{22}\end{pmatrix}$ where

$$a_{22}=\frac{\left(m_1+m_2\right)L_1}{m_1L_2}\left(1-\frac{2gm_1L_2}{\left(m_1+m_2\right)g\left(L_1+L_2\right)-\sqrt{\left(m_1+m_2\right)g^2\left[m_1\left(L_1-L_2\right)^2+m_2\left(L_1+L_2\right)^2\right]}}\right)\times a_{12}$$

These expressions are rather complicated; we just need to note that a_{11} and a_{21} have the same sign $\left(\frac{a_{11}}{a_{21}}>0\right)$ while a_{12} and a_{22} have opposite sign $\left(\frac{a_{11}}{a_{21}}<0\right)$.

The relationship between coordinates $\left(\theta_1,\theta_2\right)$ and normal coordinates η_1,η_2 are

$$\left.\begin{aligned}\theta_1&=a_{11}\ \eta_1+a_{12}\ \eta_2\\ \theta_2&=a_{21}\ \eta_1+a_{22}\ \eta_2\end{aligned}\right\} \quad\Leftrightarrow\quad \left\{\begin{aligned}\eta_1&\sim\theta_1-\frac{a_{12}}{a_{22}}\theta_2\\ \eta_2&\sim\theta_1-\frac{a_{11}}{a_{21}}\theta_2\end{aligned}\right.$$

To visualize the normal coordinate η_1, let $\eta_2 = 0$. Then to visualize the normal coordinate η_2, we let $\eta_1 = 0$. Because $\dfrac{a_{11}}{a_{21}} > 0$ and $\dfrac{a_{12}}{a_{22}} < 0$, we see that these normal coordinates describe two oscillation modes. In the first one, the two bobs move in opposite directions and in the second, the two bobs move in the same direction.

12-28. Kinetic energy: $T = \dfrac{1}{2} m_1 b^2 \dot{\theta}_1^2 + \dfrac{1}{2} m_2 b^2 \dot{\theta}_2^2 \quad \Rightarrow \quad [m] = \begin{bmatrix} m_2 b^2 & 0 \\ 0 & m_2 b^2 \end{bmatrix}$

Potential energy: $U = m_1 g b \left(1 - \cos\theta_1\right) + m_2 g b \left(1 - \cos\theta_2\right) + \dfrac{k}{2}\left(b \sin\theta_1 - b \sin\theta_2\right)$

$$\Rightarrow \quad [A] \approx \begin{bmatrix} m_1 g b + k b^2 & -k b^2 \\ -k b^2 & m_2 g b + k b^2 \end{bmatrix}$$

Solving the equation, $\mathrm{Det}\left([A] - \omega^2 [m]\right) = 0$, gives us the proper frequencies of oscillation,

$$\omega_1^2 = \frac{g}{b} = 25\,(\mathrm{rad/s})^2 \qquad \omega_2^2 = \frac{g}{b} + \frac{k}{m_1} + \frac{k}{m_2} = 25.11\,(\mathrm{rad/s})^2$$

The eigenstate corresponding to ω_1 is $\begin{pmatrix} a_{11} \\ a_{22} \end{pmatrix}$ with $a_{21} = 7.44\, a_{11}$

The eigenstate corresponding to ω_2 is $\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$ with $a_{22} = 8.55\, a_{12}$

From the solution of problem 12-27 above, we see that the normal coordinates are

$$\eta_1 \sim \theta_1 - \frac{a_{12}}{a_{22}} \theta_2 = \theta_1 + 0.12\, \theta_2$$

$$\eta_2 \sim \theta_1 - \frac{a_{11}}{a_{21}} \theta_2 = \theta_1 + 0.13\, \theta_2$$

Evidently η_1 then characterizes the in-phase oscillation of two bobs, and η_2 characterizes the out-of-phase oscillation of two bobs.

Now to incorporate the initial conditions, let us write the most general oscillation form:

$$\theta_1 = \mathrm{Re}\left(\alpha\, a_{11}\, e^{i\omega_1 t - i\delta_1} + \alpha\, a_{12}\, e^{i\omega_2 t - i\delta_2}\right)$$

$$\theta_2 = \mathrm{Re}\left(\alpha\, a_{21}\, e^{i\omega_1 t - i\delta_1} + \alpha\, a_{22}\, e^{i\omega_2 t - i\delta_2}\right)$$

$$= \mathrm{Re}\left(7.44 \alpha a_{11}\, e^{i\omega_1 t - i\delta_1} - 8.35 \alpha a_{12}\, e^{i\omega_2 t - i\delta_2}\right)$$

where α is a real normalization constant. The initial conditions helps to determine parameters α's, a's, δ's.

$$\begin{cases} \operatorname{Re}\left(\theta_1\left(t=0\right)=-7^{\circ} \quad \Rightarrow \quad \alpha a_{11}\cos\delta_1+\alpha a_{12}\cos\delta_2=-0.122\text{ rad}\right) \\ \operatorname{Re}\left(\theta_2\left(t=0\right)=0^{\circ} \quad \Rightarrow \quad 7.44\alpha a_{11}\cos\delta_1+8.35\alpha a_{12}\cos\delta_2=0\right) \end{cases}$$

$\Rightarrow \quad \sin\delta_1=\sin\delta_1=0$. Then

$$\theta_1=\alpha a_{11}\cos\omega_1 t+\alpha a_{12}\cos\omega_2 t=-0.065\cos\omega_1 t-0.057\cos\omega_2$$

$$\theta_2=7.44\alpha a_{11}\cos\omega_1 t+8.35\alpha a_{12}\cos\omega_2 t=0.48\left(\cos\omega_2 t-\cos\omega_1 t\right)$$

where $\omega_1=5.03\text{ rad/s}$, $\omega_2=4.98\text{ rad/s}$ (found earlier)

Approximately, the maximum angle θ_2 is 0.096 rad and it happens when

$$\begin{cases} \cos\omega_2 t=1 \\ \cos\omega_1 t=-1 \end{cases}$$

which gives

$$\left.\begin{aligned} \omega_2 t=2n\,\pi \\ \omega_1 t=\left(2k+1\right)\pi \end{aligned}\right\} \quad \Rightarrow \quad \frac{\omega_1}{\omega_2}=\frac{2k+1}{2n}$$

because $\dfrac{\omega_1}{\omega_2}=\dfrac{101}{100}$ we finally find $k=n=50$ and $t=\dfrac{100\pi}{\omega_2}=63\text{ s}$.

Note: $\theta_{2\max}=0.96\text{ rad}$ and at this value the small-angle approximation breaks down, and the value $\theta_{2\max}$ we found is just a rough estimate.

CHAPTER 13

Continuous Systems; Waves

13-1. The initial velocities are zero and so all of the ν_r vanish [see Eq. (13.8b)]. The μ_r are given by [see Eq. (13.8a)]

$$\begin{aligned}\mu_r &= \frac{2A}{L}\int_0^L \sin\frac{3\pi x}{L}\sin\frac{r\pi x}{L}\,dx \\ &= A\delta_{3r}\end{aligned} \tag{1}$$

so that

$$\left.\begin{aligned}&\mu_3 = A \\ &\mu_r = 0, \quad r \neq 3\end{aligned}\right] \tag{2}$$

The characteristic frequency ω_3 is [see Eq. (13.11)]

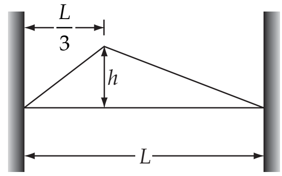
$$\omega_3 = \frac{3\pi}{L}\sqrt{\frac{\tau}{\rho}} \tag{3}$$

and therefore,

$$\boxed{q(x,t) = A\cos\left[\frac{3\pi}{L}\sqrt{\frac{\tau}{\rho}}\,t\right]\sin\left[\frac{3\pi x}{L}\right]} \tag{4}$$

For the particular set of initial conditions used, only one normal mode is excited. Why?

13-2.



The initial conditions are

$$q(x,0) = \begin{bmatrix} \dfrac{3h}{L}x, & 0 \le x \le \dfrac{L}{3} \\ \dfrac{3h}{2L}(L-x), & \dfrac{L}{3} \le x \le L \end{bmatrix} \tag{1}$$

$$\dot{q}(x,0) = 0 \tag{2}$$

Because $\dot{q}(x,0) = 0$, all of the ν_r vanish. The μ_r are given by

$$\mu_r = \frac{6h}{L^2}\int_0^{L/3} x \sin\frac{r\pi x}{L}\,dx + \frac{3h}{L^2}\int_{L/3}^{L}(L-x)\sin\frac{r\pi x}{L}\,dx$$

$$= \frac{9h}{r^2\pi^2}\sin\frac{r\pi}{3} \tag{3}$$

We see that $\mu_r = 0$ for r = 3, 6, 9, etc. The displacement function is

$$\boxed{q(x,t) = \frac{9\sqrt{3}\,h}{2\pi^2}\left[\sin\frac{\pi x}{L}\cos\omega_1 t + \frac{1}{4}\sin\frac{2\pi x}{L}\times\cos\omega_2 t - \frac{1}{16}\sin\frac{4\pi x}{L}\cos\omega_4 t - \ldots\right]} \tag{4}$$

where

$$\omega_r = \frac{r\pi}{L}\sqrt{\frac{\tau}{\rho}} \tag{5}$$

The frequencies ω_3, ω_6, ω_9, etc. are absent because the initial displacement at $L/3$ prevents that point from being a node. Thus, none of the harmonics with a node at $L/3$ are excited.

13-3. The displacement function is

$$\frac{q(x,t)}{8h/\pi^2} = \sin\frac{\pi x}{L}\cos\omega_1 t - \frac{1}{9}\sin\frac{3\pi x}{L}\cos\omega_3 t + \frac{1}{25}\sin\frac{5\pi x}{L}\cos\omega_5 t + \ldots \tag{1}$$

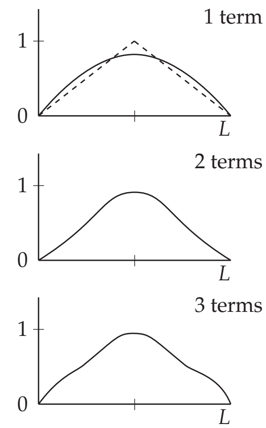
where

$$\left.\begin{aligned} \omega_1 &= \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}} \\ \omega_r &= r\omega_1 \end{aligned}\right] \tag{2}$$

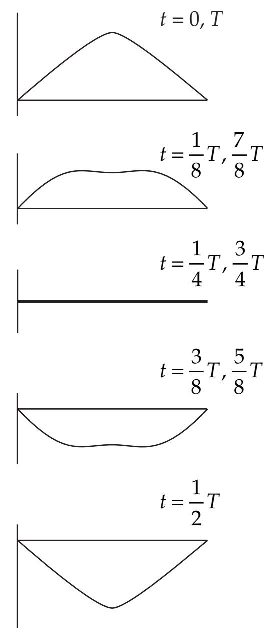
For t = 0,

$$\frac{q(x,0)}{8h/\pi^2} = \sin\frac{\pi x}{L} - \frac{1}{9}\sin\frac{3\pi x}{L} + \frac{1}{25}\sin\frac{5\pi x}{L} + \ldots \tag{3}$$

The figure below shows the first term, the first two terms, and the first three terms of this function. It is evident that the triangular shape is well represented by the first three terms.



The time development of $q(x,t)$ is shown below at intervals of $1/8$ of the fundamental period.



13-4. The coefficients ν_r are all zero and the μ_r are given by Eq. (13.8a):

$$\mu_r = \frac{8}{L^2}\int_0^L x(L-x)\sin\frac{r\pi x}{L}\,dx$$

$$= \frac{16}{r^3\pi^3}\left[1-(-1)^r\right] \tag{1}$$

so that

$$\boxed{\mu_r = \begin{cases} 0, & r \text{ even} \\ \dfrac{32}{r^3\pi\sqrt{3}}, & r \text{ odd} \end{cases}} \tag{2}$$

Since

$$q(x,t)=\sum_r \mu_r \sin\frac{r\pi x}{L}\cos\omega_r t \tag{3}$$

the amplitude of the n-th mode is just μ_n.

The characteristic frequencies are given by Eq. (13.11):

$$\omega_n=\frac{n\pi}{L}\sqrt{\frac{\tau}{\rho}} \tag{4}$$

13-5. The initial conditions are

$$q(x,0)=0$$

$$\dot{q}(x,t)=\begin{bmatrix} v_0, & \left|x-\dfrac{1}{2}\right|\le s \\ 0, & \text{otherwise}\end{bmatrix} \tag{1}$$

The μ_r are all zero and the ν_r are given by [see Eq. (13.8b)]

$$\begin{aligned}\nu_r&=-\frac{2}{\omega_r L}\int_{(L/2)-s}^{(L/2)+s} v_0\sin\frac{r\pi x}{L}\,dx\\ &=\frac{4v_0}{r\pi\omega_r}\sin\frac{r\pi}{2}\sin\frac{r\pi s}{L}\end{aligned} \tag{2}$$

from which

$$\nu_r=\begin{bmatrix} 0, & r\text{ even}\\ -\dfrac{4v_0}{r\pi\omega_r}(-1)^{(r-1)/2}\sin\dfrac{r\pi s}{L}, & r\text{ odd}\end{bmatrix} \tag{3}$$

(Notice that the even modes are all missing, as expected from the symmetrical nature of the initial conditions.)

Now, from Eq. (13.11),

$$\omega_1=\frac{\pi}{L}\sqrt{\frac{\tau}{\rho}} \tag{4}$$

and $\omega_r=r\omega_1$. Therefore,

$$\nu_r=\frac{-4v_0}{r^2\pi\omega_1}(-1)^{(r-1)/2}\sin\frac{r\pi s}{L},\quad r\text{ odd} \tag{5}$$

According to Eq. (13.5),

$$q(x,t) = \sum_r \beta_r e^{i\omega_r t} \sin\frac{r\pi x}{L}$$

$$= -\sum_r \nu_r \sin\omega_r t \sin\frac{r\pi x}{L} \tag{6}$$

Therefore,

$$\boxed{q(x,t) = \frac{4v_0}{\pi\omega_1}\left(\sin\omega_1 t \sin\frac{\pi s}{L}\sin\frac{\pi x}{L} - \frac{1}{9}\sin\omega_3 t\sin\frac{3\pi s}{L}\sin\frac{3\pi x}{L} + \ldots\right)} \tag{7}$$

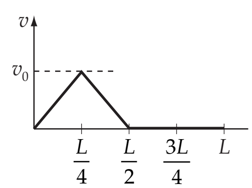
Notice that some of the odd modes—those for which $\sin(3\pi s/L) = 0$—are absent.

13-6. The initial conditions are

$$q(x,0) = 0$$

$$\dot{q}(x,0) = \begin{bmatrix} \dfrac{4v_0}{L}x & 0 \le x \le \dfrac{L}{4} \\ \dfrac{4v_0}{L}\left[\dfrac{L}{2} - x\right] & \dfrac{L}{4} \le x \le \dfrac{L}{2} \\ 0 & \dfrac{L}{2} \le x \le L \end{bmatrix} \tag{1}$$

The velocity at $t = 0$ along the string, $\dot{q}(x,0)$, is shown in the diagram.



The μ_r are identically zero and the ν_r are given by:

$$\nu_r = -\frac{2}{rL\omega_1}\int_0^L \dot{q}(x,0)\sin\frac{r\pi x}{L}\,dx$$

$$= \frac{8v_0}{r^3\pi^2\omega_1}\left[\sin\frac{r\pi}{2} - 2\sin\frac{r\pi}{4}\right] \tag{2}$$

Observe that for $r = 4n$, ν_r is zero. This happens because at $t = 0$ the string was struck at $L/4$, and none of the harmonics with modes at that point can be excited.

Evaluation of the first few ν_r gives

$$\left.\begin{array}{ll} \nu_1 = -0.414 \cdot \dfrac{8v_0}{\pi^2 \omega_1} & \nu_4 = 0 \\[2ex] \nu_2 = -\dfrac{1}{4} \cdot \dfrac{8v_0}{\pi^2 \omega_1} & \nu_5 = \dfrac{2.414}{125} \cdot \dfrac{8v_0}{\pi^2 \omega_1} \\[2ex] \nu_3 = -\dfrac{2.414}{27} \cdot \dfrac{8v_0}{\pi^2 \omega_1} & \nu_6 = \dfrac{2}{216} \cdot \dfrac{8v_0}{\pi^2 \omega_1} \end{array}\right] \tag{3}$$

and so,

$$\boxed{\begin{aligned} q(x,t) = \frac{8v_0}{\pi^2 \omega_1} \bigg(& 0.414 \sin \omega_1 t \sin \frac{\pi x}{L} + \frac{1}{4} \sin \omega_2 t \sin \frac{2\pi x}{L} \\ & + \frac{2.414}{27} \sin \omega_3 t \sin \frac{3\pi x}{L} - \frac{2.414}{135} \sin \omega_5 t \sin \frac{5\pi x}{L} - \ldots \bigg) \end{aligned}} \tag{4}$$

From these amplitudes we can find how many db down the fundamental are the various harmonics:

Second harmonic:

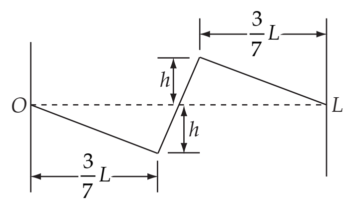
$$10 \log \left[\frac{0.250}{0.414} \right]^2 = -4.4 \text{ db} \tag{5}$$

Third harmonic:

$$10 \log \left[\frac{2.414/27}{0.414} \right]^2 = -13.3 \text{ db} \tag{6}$$

These values are much smaller than those found for the case of example (13.1). Why is this so? (Compare the degree of symmetry of the initial conditions in each problem.)

13-7.



Since $\dot{q}(x,0) = 0$, we know that all of the ν_r are zero and the μ_r are given by Eq. (13.8a):

$$\mu_r = \frac{2}{L} \int_0^L q(x,0) \sin \frac{r\pi x}{L}\, dx \tag{1}$$

The initial condition on $q(x,t)$ is

$$q(x,0) = \begin{bmatrix} -\dfrac{7h}{3L}x, & 0 \le x \le \dfrac{3}{7}L \\ \dfrac{7h}{L}(2x - L), & \dfrac{3}{7}L \le x \le \dfrac{4}{7}L \\ \dfrac{7h}{3L}(L - x), & \dfrac{4}{7}L \le x \le L \end{bmatrix} \tag{2}$$

Evaluating the μ_r we find

$$\boxed{\mu_r = \frac{98}{3}\frac{h}{r^2\pi^2}\left[\sin\frac{4r\pi}{7} - \sin\frac{3r\pi}{7}\right]} \tag{3}$$

Obviously, $\mu_r = 0$ when $4r/7$ and $3r/7$ simultaneously are integers. This will occur when r is any multiple of 7 and so we conclude that the modes with frequencies that are multiples of $7\omega_1$ will be absent.

13-8. For the loaded string, we have [see Eq. (12.152)]

$$\omega_r = 2\sqrt{\frac{\tau}{md}}\sin\frac{r\pi}{2(n+1)} \tag{1}$$

Using $\rho = m/d$ and $L = (n+1)d$, we have

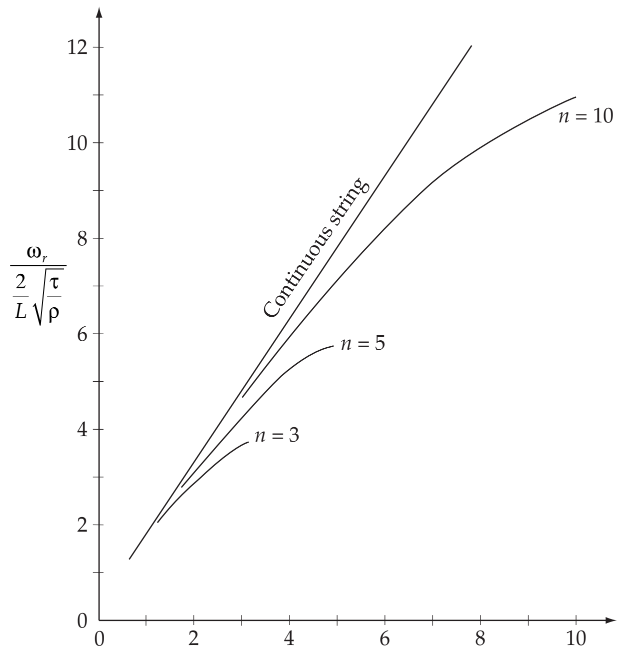
$$\begin{aligned} \omega_r &= \frac{2}{d}\sqrt{\frac{\tau}{\rho}}\sin\frac{r\pi}{2(n+1)} \\ &= \frac{2(n+1)}{L}\sqrt{\frac{\tau}{\rho}}\sin\frac{r\pi}{2(n+1)} \end{aligned} \tag{2}$$

The function

$$\boxed{\frac{\omega_r}{\dfrac{2}{L}\sqrt{\dfrac{\tau}{\rho}}} = (n+1)\sin\frac{r\pi}{2(n+1)}} \tag{3}$$

is plotted in the figure for n = 3, 5, and 10. For comparison, the characteristic frequency for a continuous string is also plotted:

$$\frac{\omega_r}{\dfrac{2}{L}\sqrt{\dfrac{\tau}{\rho}}} = \frac{r\pi}{2} \tag{4}$$



Of course, the curves have meaning only at the points for which r is an integer.

13-9. From Eq. (13.49), we have:

$$\beta = \frac{D}{2\rho}; \quad \omega_0^2 = \frac{s^2\pi^2\tau}{\rho b} \tag{1}$$

From section 3.5, we know that underdamped motion requires:

$$\beta^2 < \omega_0^2$$

Using (1) this becomes

$$\frac{D^2}{4\rho^2} < \frac{s^2\pi^2\tau}{\rho b}$$

or	$D^2 < \dfrac{4\rho s^2\pi^2\tau}{b}$	underdamped
Likewise	$D^2 = \dfrac{4\rho s^2\pi^2\tau}{b}$	critically damped
	$D^2 > \dfrac{4\rho s^2\pi^2\tau}{b}$	overdamped

The complementary solution to Eq. (13.48) for underdamped motion can be written down using Eq. (3.40). The result is:

$$\eta_s(t) = C_s e^{-\beta t} \cos(\omega_1 t - \phi_s)$$

where $\omega_1^2 = \omega_0^2 - \beta^2$, ω_0 and β are as defined in (1), and C_s and ϕ_s are arbitrary constants depending on the initial conditions. The complete solution to Eq. (13.48) is the sum of the particular and complementary solutions (analogous to Eq. (13.50)):

$$\eta_s(t) = C_s e^{-\beta t}\cos(\omega_1 t - \phi_s) + \frac{2F_0 \sin\left[\dfrac{s\pi}{2}\right]\cos(\omega t - \delta_s)}{\rho b\sqrt{\left[\dfrac{s^2\pi^2\tau}{\rho b} - \omega^2\right] + \dfrac{D}{\rho}\,\omega^2}}$$

where

$$\delta_s = \tan^{-1}\left[\frac{D\omega}{\rho\left[\dfrac{s^2\pi^2\tau}{\rho b}\right] - \omega^2}\right]$$

From Eq. (13.40):

$$q(x,t) = \sum_r \eta_r(t)\sin\frac{r\pi x}{b}$$

Thus

$$\boxed{q(x,t) = \sum_r\left[C_r \exp\left[-\frac{Dt}{2\rho}\right]\cos\left[\sqrt{\frac{s^2\pi^2\tau}{\rho b} - \frac{D^2}{4\rho^2}}\,t - \phi_r\right] + \frac{2F_0\sin\left[\dfrac{r\pi}{2}\right]\cos(\omega t - \delta_r)}{\rho b\sqrt{\left[\dfrac{r^2\pi^2\tau}{\rho b} - \omega^2\right] + \dfrac{D}{\rho}\,\omega^2}}\right]\sin\frac{r\pi x}{b}}$$

(underdamped)

13-10. From Eq. (13.44) the equation for the driving Fourier coefficient is:

$$f_s(t) = \int_0^b F(x,t)\sin\frac{s\pi x}{b}\,dx$$

If the point x is a node for normal coordinate s, then

$$\frac{x}{b} = \frac{n}{s} \quad \text{where } n \text{ is an integer} \le s$$

(This comes from the fact that normal mode s has s-half wavelengths in length b.)

For $\dfrac{x}{b} = \dfrac{n}{s}$,

$$\sin\frac{s\pi x}{b} = \sin n\pi = 0;\ \text{hence}\ f_s(t) = 0$$

$$\boxed{\begin{array}{l}\text{Thus, if the string is driven at an arbitrary point,}\\ \text{none of the normal modes with nodes at the driving}\\ \text{point will be excited.}\end{array}}$$

13-11. From Eq. (13.44)

$$f_s(t) = \int_0^b F(x,t) \sin \frac{s\pi x}{b} \, dx \tag{1}$$

where $F(x,t)$ is the driving force, and $f_s(t)$ is the Fourier coefficient of the Fourier expansion of $F(x,t)$. Eq. (13.45) shows that $f_s(t)$ is the component of $F(x,t)$ effective in driving normal coordinate s. Thus, we desire $F(x,t)$ such that

$$\begin{aligned} f_s(t) &= 0 \quad \text{for } s \neq n \\ &\neq 0 \quad \text{for } s = n \end{aligned}$$

From the form of (1), we are led to try a solution of the form

$$F(x,t) = g(t) \sin \frac{n\pi x}{b}$$

where $g(t)$ is a function of t only.

Thus

$$f_s(t) = \int_0^b g(t) \sin \frac{n\pi x}{b} \sin \frac{s\pi x}{b} \, dx$$

For $n \neq s$, the integral is proportional to $\sin \left.\dfrac{(n \pm s)\pi x}{b}\right]_{x=0}^{b}$; hence $f_s(t) = 0$ for $s \neq n$.

For $n = s$, we have

$$f_s(t) = g(t) \int_0^b \sin^2 \frac{n\pi x}{b} \, dx = g(t) \frac{b}{2} \neq 0$$

Only the n^{th} normal coordinate will be driven.

$$\boxed{\begin{array}{c}\text{Thus, to drive the } n^{\text{th}} \text{ harmonic only,}\\ F(x,t) = g(t) \sin \dfrac{n\pi x}{b}\end{array}}$$

13-12. The equation to be solved is

$$\ddot{\eta}_s + \frac{D}{\rho}\,\dot{\eta}_s + \frac{s^2\pi^2\tau}{\rho b}\,\eta_s = 0 \tag{1}$$

Compare this equation to Eq. (3.35):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2\,x = 0$$

The solution to Eq. (3.35) is Eq. (3.37):

$$x(t) = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2 - \omega_0^2}\;t\right) + A_2 \exp\left(-\sqrt{\beta^2 - \omega_0^2}\;t\right)\right]$$

Thus, by analogy, the solution to (1) is

$$\boxed{n_s(t) = e^{-Dt/2\rho}\left[A_1 \exp\left[\sqrt{\frac{D^2}{4\rho^2} - \frac{s^2\pi^2\tau}{\rho b}}\;t\right] + A_2 \exp\left[-\sqrt{\frac{D^2}{4\rho^2} - \frac{s^2\pi^2\tau}{\rho b}}\;t\right]\right]}$$

13-13. Assuming k is real, while ω and v are complex, the wave function becomes

$$\psi(x,t) = Ae^{i(\alpha t + i\beta t - kx)}$$

$$= Ae^{(\alpha t - kx)}\;e^{-\beta t} \tag{1}$$

whose real part is

$$\psi(x,t) = Ae^{-\beta t}\cos(\alpha t - kx) \tag{2}$$

and the wave is damped in time, with damping coefficient β.

From the relation

$$k^2 = \frac{\omega^2}{v^2} \tag{3}$$

we obtain

$$(\alpha + i\beta)^2 = k^2(u + iw)^2 \tag{4}$$

By equating the real and imaginary part of this equation we can solve for α and β in terms of u and w:

$$\alpha = \frac{k^2uw}{\beta} \tag{5}$$

and

$$\beta = \left[\begin{array}{l} kw \\ iku \end{array}\right. \tag{6}$$

Since we have assumed β to be real, we choose the solution

$$\boxed{\beta = kw} \tag{7}$$

Substituting this into (5), we have

$$\boxed{\alpha = ku} \tag{8}$$

as expected.

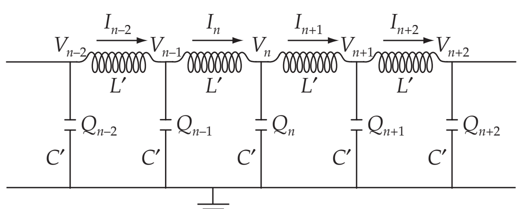
Then, the phase velocity is obtained from the oscillatory factor in (2) by its definition:

$$V = \frac{\mathrm{Re}\,\omega}{k} = \frac{\alpha}{k} \tag{9}$$

That is,

$$\boxed{V = u}$$

13-14.



Consider the above circuit. The circuit in the n^{th} inductor is I_n, and the voltage above ground at the point between the n^{th} elements is V_n. Thus we have

$$V_n = \frac{Q_n}{C'}$$

and

$$L'\frac{dI_n}{dt} = V_{n-1} - V_n$$

$$= \frac{Q_{n-1}}{C'} - \frac{Q_n}{C'} \tag{1}$$

We may also write

$$\frac{dQ_n}{dt} = I_n - I_{n+1} \tag{2}$$

Differentiating (1) with respect to time and using (2) gives

$$L'\frac{d^2 I_n}{dt^2} = \frac{1}{C'}\left[I_{n-1} - 2I_n + I_{n+1}\right] \tag{3}$$

or

$$\frac{d^2 I_n}{dt^2} = \frac{1}{L'C'}\left[I_{n-1} - 2I_n + I_{n+1}\right] \tag{4}$$

Let us define a parameter x which increases by Δx in going from one loop to the next (this will become the coordinate x in the continuous case), and let us also define

$$L \equiv \frac{L'}{\Delta x}; \qquad C \equiv \frac{C'}{\Delta x} \tag{5}$$

which will become the inductance and the capacitance, respectively, per unit length in the limit $\Delta x \to 0$.

From the above definitions and

$$\Delta I_r = I_{r+1} - I_r \tag{6}$$

(4) becomes

$$\frac{d^2 I_n}{dt^2} + \frac{1}{L'C'}\left(\Delta I_{n-1} - \Delta I_n\right) = 0 \tag{7}$$

or,

$$\frac{d^2 I_n}{dt^2} - \frac{\Delta\left(\Delta I_n\right)}{L'C'} = 0 \tag{8}$$

Dividing by $(\Delta x)^2$, and multiplying by $(-L'C')$, we find

$$\frac{\Delta\left(\Delta I_n\right)}{(\Delta x)^2} - LC\frac{d^2 I_n}{dt^2} = 0 \tag{9}$$

But by virtue of the above definitions, we can now pass to the continuous limit expressed by

$$I_n(t) \to I(x,t) \tag{10}$$

Then,

$$\frac{\Delta\left(\Delta I(x,t)\right)}{\Delta x^2} - LC\frac{\partial^2 I(x,t)}{\partial t^2} = 0 \tag{11}$$

and for $\Delta x \to 0$, we obtain

$$\boxed{\frac{\partial^2 I}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 I}{\partial t^2} = 0} \tag{12}$$

where

$$v = \frac{1}{\sqrt{LC}} \tag{13}$$

13-15. Consider the wave functions

$$\left.\begin{aligned} \psi_1 &= A\exp\left[i(\omega t - kx)\right] \\ \psi_2 &= B\exp\left[i\left((\omega + \Delta\omega)t - (k + \Delta k)x\right)\right] \end{aligned}\right] \tag{1}$$

where $\Delta\omega \ll \omega; \Delta k \ll k$. A and B are complex constants:

$$\left.\begin{aligned} A &= |A| \exp\left(i\phi_a\right) \\ B &= |B| \exp\left(i\phi_b\right) \end{aligned}\right] \tag{2}$$

The superposition of ψ_1 and ψ_2 is given by

$$\psi = \psi_1 + \psi_2$$

$$= \exp\left(i\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right]\right) \times \left[|A| \exp\left(i\phi_a\right) e^{-i\left[\frac{\Delta\omega t - \Delta k x}{2}\right]} + |B| \exp\left(i\phi_b\right) e^{i\left[\frac{\Delta\omega t - \Delta k x}{2}\right]}\right] \tag{3}$$

which can be rewritten as

$$\psi = \left[\exp\left[i\left[\left(w + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x + \frac{\phi_a + \phi_b}{2}\right]\right]\right] \times \left[|A|\, e^{-i\left[\frac{\Delta\omega t - \Delta k x - \phi_a + \phi_b}{2}\right]} + |B|\, e^{i\left[\frac{\Delta\omega t - \Delta k x + \phi_b - \phi_a}{2}\right]}\right] \tag{4}$$

Define

$$\left.\begin{aligned} t\Delta\omega - x\Delta k &\equiv \delta \\ \phi_b - \phi_a &\equiv \alpha \end{aligned}\right] \tag{5}$$

and

$$|A|\, e^{-i(\delta+\alpha)/2} + |B|\, e^{i(\delta+\alpha)/2} = |\Gamma|\, e^{i\theta} \tag{6}$$

Therefore,

$$|\Gamma|^2 = 2\left(|A|^2 + |B|^2\right) \tag{7}$$

$$\cos\theta = \frac{|A| + |B|}{\left[2\left(|A|^2 + |B|^2\right)\right]^{1/2}} \cos\frac{(\delta + a)}{2} \tag{8}$$

$$\sin\theta = \frac{|B| - |A|}{\left[2\left(|A|^2 + |B|^2\right)\right]^{1/2}} \sin\frac{(\delta + a)}{2} \tag{9}$$

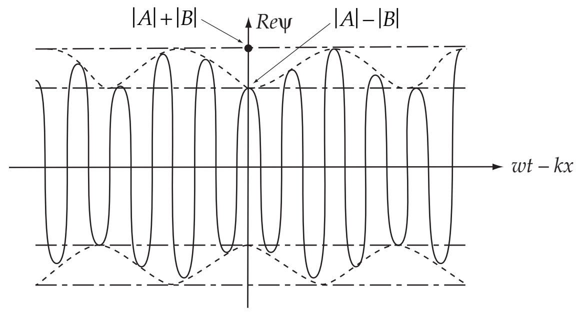
That is, θ is a function of $(\Delta\omega)t - (\Delta k)x$. Using (6) and (7) – (9), we can rewrite (4) as

$$\psi = |\Gamma| \exp\left[i\left\{\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x + \frac{\phi_a + \phi_b}{2}\right\}\right] e^{i\theta} \tag{10}$$

and then,

$$\boxed{\begin{aligned} \operatorname{Re}\psi = |\Gamma| \Bigg\{ &\cos\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right] \cos\left(\theta + \frac{\phi_a + \phi_b}{2}\right) \\ &- \sin\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right] \sin\left(\theta + \frac{\phi_a + \phi_b}{2}\right)\Bigg\} \end{aligned}} \tag{11}$$

From this expression we see that the wave function is modulated and that the phenomenon of beats occurs, but for A ≠ B, the waves never beat to zero amplitude; the minimum amplitude is, from Eq. (11), $\|A|-|B\|$, and the maximum amplitude is $|A|+|B|$. The wave function has the form shown in the figure.



13-16. As explained at the end of section 13.6, the wave will be reflected at $x = x_0$ and will then propagate in the –x direction.

13-17. We let

$$m_j = \begin{cases} m', & j = 2n \\ m'', & j = 2n+1 \end{cases} \tag{1}$$

where n is an integer.

Following the procedure in Section 12.9, we write

$$F_{2n} = m'\ddot{q}_{2n} = \frac{\tau}{d}\left(q_{2n-1} - 2q_{2n} + q_{2n+1}\right) \tag{2a}$$

$$F_{2n+1} = m''\ddot{q}_{2n+1} = \frac{\tau}{d}\left(q_{2n} - 2q_{2n+1} + q_{2n+2}\right) \tag{2b}$$

Assume solutions of the form

$$q_{2n} = Ae^{i(\omega t - 2nkd)} \tag{3a}$$

$$q_{2n+1} = Be^{i[\omega t-(2n+1)kd]} \tag{3b}$$

Substituting (3a,b) into (2a,b), we obtain

$$\left.\begin{aligned} -\omega^2 A &= \frac{\tau}{m'd}\left(Be^{ikd} - 2A + Be^{-ikd}\right) \\ -\omega^2 B &= \frac{\tau}{m''d}\left(Ae^{ikd} - 2B + Ae^{-ikd}\right) \end{aligned}\right] \tag{4}$$

from which we can write

$$\left.\begin{aligned} A\left[\frac{2\tau}{m'd}-\omega^2\right]-B\frac{2\tau}{m'd}\cos kd=0 \\ -A\frac{2\tau}{m''d}\cos kd+B\left[\frac{2\tau}{m''d}-\omega^2\right]=0 \end{aligned}\right] \tag{5}$$

The solution to this set of coupled equations is obtained by setting the determinant of the coefficients equal to zero. We then obtain the secular equation

$$\left[\frac{2\tau}{m'd}-\omega^2\right]\left[\frac{2\tau}{m''d}-\omega^2\right]-\frac{1}{m'm''}\left[\frac{2\tau}{d}\cos kd\right]^2=0 \tag{6}$$

Solving for ω, we find

$$\omega^2=\frac{\tau}{d}\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)\pm\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)^2-\frac{4}{m'm''}\sin^2 kd\right]^{1/2}\right] \tag{7}$$

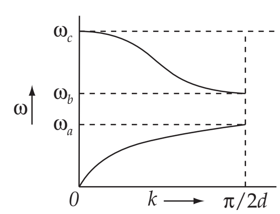
from which we find the two solutions

$$\boxed{\begin{aligned} \omega_1^2=\frac{\tau}{d}\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)+\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)^2-\frac{4}{m'm''}\sin^2 kd\right]^{1/2}\right] \\ \omega_2^2=\frac{\tau}{d}\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)-\left[\left(\frac{1}{m'}+\frac{1}{m''}\right)^2-\frac{4}{m'm''}\sin^2 kd\right]^{1/2}\right] \end{aligned}} \tag{8}$$

If $m' < m''$, and if we define

$$\omega_a \equiv \sqrt{\frac{2\tau}{m''d}},\quad \omega_b \equiv \sqrt{\frac{2\tau}{m'd}},\quad \omega_c = \sqrt{\omega_a^2+\omega_b^2} \tag{9}$$

Then the ω vs. k curve has the form shown below in which two branches appear, the lower branch being similar to that for $m' = m''$ (see Fig. 13-5).



Using (9) we can write (6) as

$$\sin^2 kd = \frac{\omega^2}{\omega_a^2\omega_b^2}\left(\omega_a^2+\omega_b^2-\omega^2\right)\equiv W(\omega) \tag{10}$$

From this expression and the figure above we see that for $\omega > \omega_c$ and for $\omega_a < \omega < \omega_b$, the wave number k is complex. If we let $k=\kappa+i\beta$, we then obtain from (10)

$$\sin^2(\kappa+i\beta)d=\sin^2\kappa d\cosh^2\beta d-\cos^2\kappa d\sinh^2\beta d+2i\sin\kappa d\cos\kappa d\sinh\beta d\cosh\beta d=W(\omega) \tag{11}$$

Equating the real and imaginary parts, we find

$$\left.\begin{aligned}\sin\kappa d\cos\kappa d\sinh\beta d\cosh\beta d &= 0\\ \sin^2\kappa d\cosh^2\beta d-\cos^2\kappa d\sinh^2\beta d &= W(\omega)\end{aligned}\right] \tag{12}$$

We have the following possibilities that will satisfy the first of these equations:

a) **sin κd = 0, which gives κ = 0.** This condition also means that cos κd = 1; then β is determined from the second equation in (12):

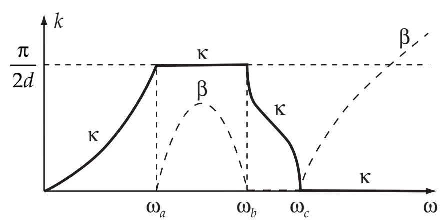
$$-\sinh^2\beta d = W(\omega) \tag{13}$$

Thus, $\omega > \omega_c$, and κ is purely imaginary in this region.

b) **cos κd = 0, which gives $\kappa = \pi/2d$.** Then, sin κd = 1, and $\cosh^2\beta d = W(\omega)$. Thus, $\omega_a < \omega < \omega_b$, and κ is constant at the value $\pi/2d$ in this region.

c) **sinh βd = 0, which gives β = 0.** Then, $\sin^2\kappa d = W(\omega)$. Thus, $\omega < \omega_a$ or $\omega_b < \omega < \omega_c$, and κ is real in this region.

Altogether we have the situation illustrated in the diagram.



13-18. The phase and group velocities for the propagation of waves along a loaded string are

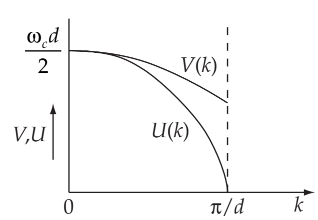
$$V(k)=\frac{\omega}{k}=\frac{\omega_c d}{2}\frac{\left|\sin(kd/2)\right|}{kd/2} \tag{1}$$

$$U(k)=\frac{d\omega}{dk}=\frac{\omega_c d}{2}\left|\cos(kd/2)\right| \tag{2}$$

where

$$\omega=\omega_c\left|\sin(kd/2)\right| \tag{3}$$

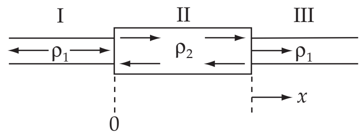
The phase and group velocities have the form shown below.



When $k=\pi/d$, $U=0$ but $V=\omega_c d/\pi$. In this situation, the group (i.e., the wave envelope) is stationary, but the wavelets (i.e., the wave structure inside the envelope) move forward with the velocity V.

13-19. The linear mass density of the string is described by

$$\rho = \begin{bmatrix} \rho_1 & \text{if } x < 0;\, x > L \\ \\ \rho_2 > \rho_1 & \text{if } 0 < x < L \end{bmatrix}$$



Consider the string to be divided in three different parts: I for $x < 0$, II for $0 < x < L$, and III for $x > L$.

Let $\phi = A\, e^{i(\omega t - k_1 x)}$ be a wave train, oscillating with frequency ω, incident from the left on II. We can write for the different zones the corresponding wave functions as follows:

$$\left.\begin{aligned} \psi_I &= Ae^{i(\omega t - k_1 x)} + Be^{i(\omega t + k_1 x)} = e^{i\omega t}\left[Ae^{-ik_1 x} + Be^{ik_1 x}\right] \\ \psi_{II} &= Ce^{i(\omega t - k_2 x)} + De^{i(\omega t + k_2 x)} = e^{i\omega t}\left[Ce^{-ik_2 x} + De^{ik_2 x}\right] \\ \psi_{III} &= Ee^{i(\omega t - k_1 x)} \end{aligned}\right] \tag{1}$$

Where

$$k_1 = \frac{\omega}{V_1}, \quad k_2 = \frac{\omega}{V_2}, \quad V_1 = \sqrt{\frac{\tau}{\rho_1}}, \quad V_2 = \sqrt{\frac{\tau}{\rho_2}} \tag{2}$$

and where τ is the tension in the string (constant throughout). To solve the problem we need to state first the boundary conditions; these will be given by the continuity of the wave function and its derivative at the boundaries $x = 0$ and $x = L$. For $x = 0$, we have

$$\left.\begin{aligned} \psi_{\rm I}(x=0) &= \psi_{\rm II}(x=0) \\ \left.\frac{\partial \psi_{\rm I}}{\partial x}\right|_{x=0} &= \left.\frac{\partial \psi_{\rm II}}{\partial x}\right|_{x=0} \end{aligned}\right] \tag{3}$$

and for $x = L$, the conditions are

$$\left.\begin{aligned} \psi_{\rm II}(x=0) &= \psi_{\rm III}(x=L) \\ \left.\frac{\partial \psi_{\rm II}}{\partial x}\right|_{x=L} &= \left.\frac{\partial \psi_{\rm III}}{\partial x}\right|_{x=L} \end{aligned}\right] \tag{4}$$

Substituting ψ as given by (1) into (3) and (4), we have

$$\left.\begin{aligned} A + B &= C + D \\ -A + B &= \frac{k_2}{k_1}(-C + D) \end{aligned}\right] \tag{5}$$

and

$$\left.\begin{aligned} C\,e^{-ik_2L} + D\,e^{ik_2L} &= E\,e^{-ik_1L} \\ -C\,e^{-ik_2L} + D\,e^{ik_2L} &= -\frac{k_1}{k_2}\,E\,e^{-ik_1L} \end{aligned}\right] \tag{6}$$

From (6) we obtain

$$\left.\begin{aligned} C &= \frac{1}{2}\left(1+\frac{k_1}{k_2}\right)E\,e^{i(k_2-k_1)L} \\ D &= \frac{1}{2}\left(1-\frac{k_1}{k_2}\right)E\,e^{-i(k_2+k_1)L} \end{aligned}\right] \tag{7}$$

Hence,

$$C = \frac{k_2+k_1}{k_2-k_1}\,e^{i2k_2L}\,D \tag{8}$$

From (5) we have

$$A = \frac{1}{2}\left(1+\frac{k_2}{k_1}\right)C + \frac{1}{2}\left(1-\frac{k_2}{k_1}\right)D \tag{9}$$

Using (7) and rearranging the above equation

$$A = \frac{1}{2k_1}\left[\frac{(k_1+k_2)^2}{(k_2-k_1)}\,e^{i2k_2L} - (k_2-k_1)\right]D \tag{10}$$

In the same way

$$B = \frac{1}{2k_1}\left[-(k_2+k_1)\,e^{i2k_2L} + (k_1+k_2)\right]D \tag{11}$$

From (10) and (11) we obtain

$$\boxed{\frac{B}{A} = \frac{(k_1^2-k_2^2)\left[e^{i2k_2L}-1\right]}{(k_1+k_2)^2\,e^{i2k_2L} - (k_1-k_2)^2}} \tag{12}$$

On the other hand, from (6) and (8) we have

$$E = \frac{2k_2D}{k_2-k_1}\,e^{i(k_2+k_1)L} \tag{13}$$

which, together with (10) gives

$$\boxed{\frac{E}{A} = \frac{4k_1k_2\,e^{i(k_1+k_2)L}}{(k_1+k_2)^2\,e^{i2k_2L} - (k_1-k_2)^2}} \tag{14}$$

Since the *incident intensity* I_0 is proportional to $|A|^2$, the *reflected intensity* is $I_r = |B|^2$, and the total transmitted intensity is $I_t = |E|^2$, we can write

$$I_r = I_0 \frac{|B|^2}{|A|^2}, \quad I_t = I_0 \frac{|E|^2}{|A|^2} \tag{15}$$

Substituting (12) and (14) into (15), we have, for the reflected intensity,

$$I_r = I_0 \left| \frac{\left(k_1^2 - k_2^2\right)\left[e^{i2k_2L} - 1\right]}{\left(k_1 + k_2\right)^2 e^{i2k_2L} - \left(k_1 - k_2\right)^2} \right|^2 \tag{16}$$

From which

$$\boxed{I_r = I_0 \left[\frac{\left(k_1^2 - k_2^2\right)^2 \left(1 - \cos 2k_2L\right)}{k_1^4 + k_2^4 + 6k_1^2k_2^2 - \left(k_1^2 - k_2^2\right)^2 \cos 2k_2L} \right]} \tag{17}$$

and for the transmitted intensity, we have

$$I_t = I_0 \left| \frac{4k_1k_2\, e^{i(k_1+k_2)L}}{\left(k_1 + k_2\right)^2 e^{i2k_2L} - \left(k_1 - k_2\right)^2} \right|^2 \tag{18}$$

so that

$$\boxed{I_t = I_0 \frac{8\, k_1^2k_2^2}{k_1^4 + k_2^4 + 6k_1^2k_2^2 - 2\left(k_1^2 - k_2^2\right)^2 \cos 2k_2L}} \tag{19}$$

We observe that $I_r + I_t = I_0$, as it must.

For maximum transmission we need minimum reflection; that is, the case of best possible transmission is that in which

$$\left.\begin{aligned} I_t &= I_0 \\ I_r &= 0 \end{aligned}\right] \tag{20}$$

In order that $I_r = 0$, (17) shows that L must satisfy the requirement

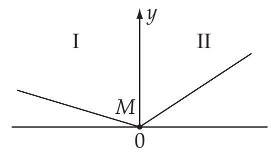
$$1 - \cos 2k_2L = 0 \tag{21}$$

so that we have

$$\boxed{L = \frac{m\pi}{k_2} = \frac{m\pi}{\omega}\sqrt{\frac{\tau}{\rho_2}}, \quad m = 0, 1, 2, \ldots} \tag{22}$$

The optical analog to the reflection and transmission of waves on a string is the behavior of light waves which are incident on a medium that consists of two parts of different optical densities (i.e., different indices of refraction). If a lens is given a coating of precisely the correct thickness of a material with the proper index of refraction, there will be almost no reflected wave.

13-20.



We divide the string into two zones:

$$\text{I:} \quad x < 0$$

$$\text{II:} \quad x > 0$$

Then,

$$\left.\begin{array}{l} \psi_{\text{I}} = A_1 e^{i(\omega t - kx)} + B_1 e^{i(\omega t + kx)} \\ \psi_{\text{II}} = A_2 e^{i(\omega t - kx)} \end{array}\right] \tag{1}$$

The boundary condition is

$$\psi_{\text{I}}(x=0) = \psi_{\text{II}}(x=0) \tag{2}$$

That is, the string is continuous at $x = 0$. But because the mass M is attached at $x = 0$, the derivative of the wave function will not be continuous at this point. The condition on the derivative is obtained by integrating the wave equation from $x = -\varepsilon$ to $x = +\varepsilon$ and then taking the limit $\varepsilon \to 0$.

Thus,

$$M \left.\frac{\partial^2 \psi}{\partial t^2}\right|_{x=0} = \tau \left.\left[\frac{\partial \psi_{\text{II}}}{\partial x} - \frac{\partial \psi_{\text{I}}}{\partial x}\right]\right|_{x=0} \tag{3}$$

Substituting the wave functions from (1), we find

$$A_1 + B_1 = A_2 \tag{4}$$

$$ik\tau\left(-A_2 + A_1 - B_1\right) = -\omega^2 M A_2 \tag{5}$$

which can be rewritten as

$$A_1 - B_1 = A_2 \frac{\left(ik\tau - \omega^2 M\right)}{ik\tau} \tag{6}$$

From (4) and (6) we obtain

$$\frac{A_1 + B_1}{A_1 - B_1} = \frac{ik\tau}{ik\tau - \omega^2 M} \tag{7}$$

from which we write

$$\frac{B_1}{A_1} = \frac{\omega^2 M}{2ik\tau - \omega^2 M} = \frac{\omega^2 M/2ik\tau}{1 - \omega^2 M/2ik\tau} \tag{8}$$

Define

$$\frac{\omega^2 M}{2k\tau} = P = \tan\theta \tag{9}$$

Then, we can rewrite (8) as

$$\frac{B_1}{A_1} = \frac{-iP}{1+iP} \tag{10}$$

And if we substitute this result in (4), we obtain a relation between A_1 and A_2:

$$\frac{A_2}{A_1} = \frac{1}{1+iP} \tag{11}$$

The *reflection coefficient*, $R = \left|\frac{B_1}{A_1}\right|^2$, will be, from (10),

$$R = \left|\frac{B_1}{A_1}\right|^2 = \frac{P^2}{1+P^2} = \frac{\tan^2\theta}{1+\tan^2\theta} \tag{12}$$

or,

$$\boxed{R = \sin^2\theta} \tag{13}$$

and the transmission coefficient, $T = \left|\frac{A_2}{A_1}\right|^2$, will be from (10)

$$T = \left|\frac{A_2}{A_1}\right|^2 = \frac{1}{1+P^2} = \frac{1}{1+\tan^2\theta} \tag{14}$$

or,

$$\boxed{T = \cos^2\theta} \tag{15}$$

The phase changes for the reflected and transmitted waves can be calculated directly from (10) and (11) if we substitute

$$\left.\begin{aligned} B_1 &= |B_1|\, e^{i\phi}B_1 \\ A_1 &= |A_1|\, e^{i\phi}A_1 \\ A_2 &= |A_2|\, e^{i\phi}A_2 \end{aligned}\right] \tag{16}$$

Then,

$$\frac{B_1}{A_1} = \frac{|B_1|}{|A_1|}\, e^{i\left(\phi_{B_1} - \phi_{A_1}\right)} = \frac{P}{\sqrt{1+P^2}}\, e^{i\tan^{-1}(1/P)} \tag{17}$$

and

$$\frac{A_2}{A_1} = \frac{|A_2|}{|A_1|}\, e^{i\left(\phi_{A_2} - \phi_{A_1}\right)} = \frac{1}{\sqrt{1+P^2}}\, e^{i\tan^{-1}(-P)} \tag{18}$$

Hence, the phase changes are

$$\boxed{\begin{aligned} \phi_{B_1} - \phi_{A_1} &= \tan^{-1}\left[\frac{1}{P}\right] = \tan^{-1}(\cot\theta) \\ \phi_{A_2} - \phi_{A_1} &= -\tan^{-1}(P) = -\tan^{-1}(\tan\theta) = -\theta \end{aligned}} \tag{19}$$

13-21. The wave function can be written as {see Eq. (13.111a)]

$$\psi(x,t) = \int_{-\infty}^{+\infty} A(k) e^{i(\omega t - kx)}\, dk \tag{1}$$

Since $A(k)$ has a non-vanishing value only in the vicinity of $k = k_0$, (1) becomes

$$\psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(\omega t - kx)}\, dk \tag{2}$$

According to Eq. (13.113),

$$\omega = \omega_0 + \omega_0'(k - k_0) \tag{3}$$

Therefore, (2) can now be expressed as

$$\begin{aligned} \psi(x,t) &= e^{i(\omega_0 - w_0' k_0)t} \int_{k_0-\Delta k}^{k_0+\Delta k} e^{i(\omega_0' t - x)k}\, dk \\ &= e^{i(\omega_0 - \omega_0' k_0)t} \left[\frac{e^{i(k_0+\Delta k)(\omega_0' t - x)} - e^{i(k_0-\Delta k)(\omega_0' t - x)}}{i(\omega_0' t - x)} \right] \\ &= \frac{2e^{i(\omega_0 - \omega_0' k_0)t}}{\omega_0' t - x} \left[\frac{e^{i(\omega_0' t - x)\Delta k} - e^{i(\omega_0' t - x)\Delta k}}{2i} \right] \end{aligned} \tag{4}$$

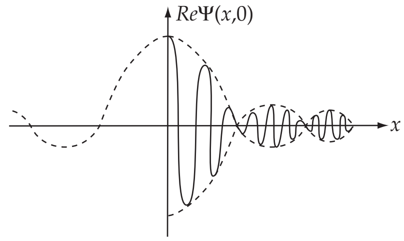
and writing the term in the brackets as a sine, we have

$$\boxed{\psi(x,t) = \frac{2\sin\left[(\omega_0' t - x)\Delta k\right]}{\omega_0' t - x} e^{i(\omega_0 t - k_0 x)}} \tag{5}$$

The real part of the wave function at $t = 0$ is

$$\boxed{\mathrm{Re}\,\psi(x,0) = \frac{2\sin(x\Delta k)}{x} \cos k_0 x} \tag{6}$$

If $\Delta k \ll k_0$, the cosine term will undergo many oscillations in one period of the sine term. That is, the sine term plays the role of a slowly varying amplitude and we have the situation in the figure below.



13-22.

a) Using Eq. (13.111a), we can write (for $t = 0$)

$$\psi(x,0) = \int_{-\infty}^{+\infty} A(k)\, e^{-ikx}\, dk$$

$$= B\int_{-\infty}^{+\infty} e^{-\sigma(k-k_0)^2}\, e^{-ikx}\, dk$$

$$= Be^{-ik_0x}\int_{-\infty}^{+\infty} e^{-\sigma(k-k_0)^2}\, e^{-i(k-k_0)x}\, dk$$

$$= Be^{-ik_0x}\int_{-\infty}^{+\infty} e^{-\sigma u^2}\, e^{-iux}\, du \tag{1}$$

This integral can be evaluated by completing the square in the exponent:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, e^{bx}\, dx = \int_{-\infty}^{+\infty} e^{-a\left(x^2-\frac{b}{a}x\right)}\, dx$$

$$= \int_{-\infty}^{+\infty} e^{-a\left[x^2-\frac{b}{a}x+\frac{b^2}{4a^2}\right]}\, e^{\frac{b^2}{4a}}\, dx$$

$$= e^{\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-a\left[x-\frac{b}{2a}\right]^2}\, dx \tag{2}$$

and letting $y = x - b/2a$, we have

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, e^{bx}\, dx = e^{\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-ay^2}\, dy \tag{3}$$

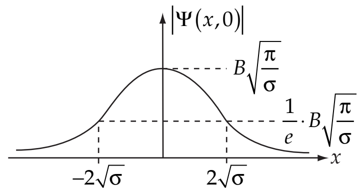
Using Eq. (E.18c) in Appendix E, we have

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, e^{bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}} \tag{4}$$

Therefore,

$$\boxed{\psi(x,0) = B\sqrt{\frac{\pi}{\sigma}}\, e^{-ik_0x}\, e^{-x^2/4\sigma}} \tag{5}$$

The form of $|\psi(x,0)|$ (the wave packet) is Gaussian with a $1/e$ width of $4\sqrt{\sigma}$, as indicated in the diagram below.



b) The frequency can be expressed as in Eq. (13.113a):

$$\omega(k) = \omega_0 + \omega_0'(k-k_0) + \ldots \tag{6}$$

and so,

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{+\infty} A(k)\, e^{i(\omega t - kx)}\, dk \\
&= \int_{-\infty}^{+\infty} A(k)\, e^{i[\omega_0 t + \omega_0'(k-k_0)t - kx]}\, dk \\
&= Be^{i(\omega_0 t - k_0 x)} \int_{-\infty}^{+\infty} e^{-\sigma(k-k_0)^2}\, e^{i[\omega_0'(k-k_0)t - (k-k_0)x]}\, dk \\
&= Be^{i(\omega_0 t - k_0 x)} \int_{-\infty}^{+\infty} e^{-\sigma u^2}\, e^{i(\omega_0' t - x)u}\, du
\end{aligned} \tag{7}$$

Using the same integral as before, we find

$$\boxed{\psi(x,t) = B\sqrt{\frac{\pi}{\sigma}}\, e^{i(\omega_0 t - k_0 x)}\, e^{-(\omega_0' t - x)^2/4\sigma}} \tag{8}$$

c) Retaining the second-order term in the Taylor expansion of $\omega(k)$, we have

$$\omega(k) = \omega_0 + \omega_0'(k-k_0) + \frac{1}{2}\omega_0''(k-k_0)^2 + \ldots \tag{9}$$

Then,

$$\begin{aligned}
\psi(x,t) &= e^{i(\omega_0 t - k_0 x)} \int_{-\infty}^{+\infty} A(k)\, e^{i\left[\omega_0'(k-k_0)t + \frac{1}{2}\omega_0''(k-k_0)^2 t - (k-k_0)x\right]}\, dk \\
&= Be^{i(\omega_0 t - k_0 x)} \int_{-\infty}^{+\infty} e^{-\left(\sigma - i\frac{\omega_0'' t}{2}\right)u^2}\, e^{i(w_0' t - x)u}\, du
\end{aligned} \tag{10}$$

We notice that if we make the change $\sigma - i\omega_0'' t/2 \to \sigma$, then (10) becomes identical to (7). Therefore,

$$\boxed{\psi(x,t) = B\sqrt{\frac{2\pi}{2\sigma - iw_0''t}}\ i^{i(\omega_0 t - k_0 x)}\ e^{-\alpha(x,t)}} \tag{11}$$

where

$$\alpha(x,t) = \frac{\left(\omega_0' t - x\right)^2 \left(\sigma + \frac{1}{2} i\omega_0'' t\right)}{4\sigma^2 + \left(\omega_0''\right)^2 t^2} \tag{12}$$

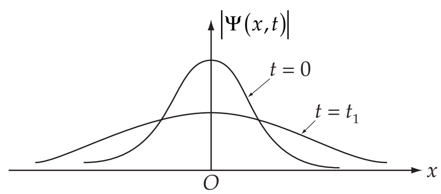
The $1/e$ width of the wave packet will now be

$$w_{1/e}(t) = 2\sqrt{\frac{4\sigma^2 + \left(\omega_0''\right)^2 t^2}{\sigma}} \tag{13}$$

or,

$$\boxed{W_{1/e}(t) = 4\sqrt{\sigma}\ \sqrt{1 + \left[\frac{\omega_0'' t}{2\sigma}\right]^2}} \tag{14}$$

In first order, $W_{1/e}$, shown in the figure above, does not depend upon the time, but in second order, $W_{1/e}$ depends upon t through the expression (14). But, as can be seen from (8) and (11), the group velocity is ω_0', and is the same in both cases. Thus, the wave packet propagates with velocity ω_0' but it spreads out as a function of time, as illustrated below.



CHAPTER 14

The Special Theory of Relativity

14-1. Substitute Eq. (14.12) into Eqs. (14.9) and (14.10):

$$x_1' = \gamma\left(x_1 - \frac{v}{c}x_1\right) \tag{1}$$

$$x_1 = \gamma\left(x_1' + \frac{v}{c}x_1'\right) \qquad (\gamma = \gamma') \tag{2}$$

From (1)

$$\frac{x_1'}{x_1} = \gamma\left[1 - \frac{v}{c}\right]$$

From (2)

$$\frac{x_1'}{x_1} = \frac{1}{\gamma\left[1 + \dfrac{v}{c}\right]}$$

So

$$\gamma\left[1 - \frac{v}{c}\right] = \frac{1}{\gamma\left[1 + \dfrac{v}{c}\right]}$$

or

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

14-2. We introduce $\cosh\alpha \cong \gamma$, $\sinh\alpha \cong \gamma\, v/c$ and substitute these expressions into Eqs. (14.14); then

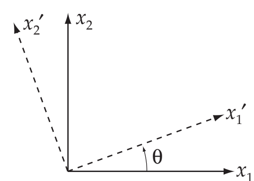
$$\left.\begin{aligned} x_1' &= x_1\cosh\alpha - ct\sinh\alpha \\ t' &= t\cosh a - \frac{x_1}{c}\sinh\alpha \\ x_2' &= x_2; \qquad x_3' = x_3 \end{aligned}\right] \tag{1}$$

Now, if we use $\cosh\alpha = \cos(i\alpha)$ and $i\sinh\alpha = \sin(i\alpha)$, we can rewrite (1) as

$$\left.\begin{aligned} x_1' &= x_1\cos(i\alpha) + ict\sin(i\alpha) \\ ict' &= -x_1\sin(i\alpha) + ict\cos(i\alpha) \end{aligned}\right] \tag{2}$$

Comparing these equations with the relation between the rotated system and the original system in ordinary three-dimensional space,

$$\left.\begin{aligned} x_1' &= x_1\cos\theta + x_2\sin\theta \\ x_2' &= -x_1\sin\theta + x_2\cos\theta \\ x_3' &= x_3 \end{aligned}\right] \tag{3}$$



We can see that (2) corresponds to a rotation of the $x_1 - ict$ plane through the angle $i\alpha$.

14-3. If the equation

$$\nabla^2\psi(x,ict) - \frac{1}{c^2}\frac{\partial^2\,\psi(x,ict)}{\partial t^2} = 0 \tag{1}$$

is Lorentz invariant, then in the transformed system we must have

$$\nabla'^2\psi(x',ict') - \frac{1}{c^2}\frac{\partial^2\,\psi(x',ict')}{\partial t'^2} = 0 \tag{2}$$

where

$$\nabla'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} \tag{3}$$

We can rewrite (2) as

$$\sum_{\mu=1}^{4}\frac{\partial^2\psi(x',ict')}{\partial x_\mu'^2} = 0 \tag{4}$$

Now, we first determine how the operator $\sum_{\mu}\frac{\partial^2}{\partial x_{\mu}'^2}$ is related to the original operator $\sum_{\mu}\frac{\partial^2}{\partial x_{\mu}^2}$.

We know the following relations:

$$x_{\mu}' = \sum_{\nu}\lambda_{\mu\nu}\, x_{\nu} \tag{5}$$

$$x_{\nu} = \sum_{\mu}\lambda_{\mu\nu}\, x_{\mu}' \tag{6}$$

$$\sum_{\mu}\lambda_{\mu\nu}\,\lambda_{\mu\lambda} = \delta_{\nu\lambda} \tag{7}$$

Then,

$$\frac{\partial}{\partial x_{\mu}'} = \sum_{\nu}\frac{\partial}{\partial x_{\nu}}\frac{\partial x_{\nu}}{\partial x_{\mu}'} = \sum_{\nu}\lambda_{\mu\nu}\frac{\partial}{\partial x_{\nu}} \tag{8}$$

$$\frac{\partial^2}{\partial x_{\mu}'^2} = \sum_{\nu}\lambda_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\sum_{\lambda}\lambda_{\mu\lambda}\frac{\partial}{\partial x_{\lambda}} = \sum_{\nu}\sum_{\lambda}\lambda_{\mu\nu}\,\lambda_{\mu\lambda}\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x_{\lambda}} \tag{9}$$

Therefore,

$$\begin{aligned}\sum_{\mu}\frac{\partial^2}{\partial x_{\mu}'^2} &= \sum_{\nu}\sum_{\lambda}\sum_{\mu}\lambda_{\mu\nu}\,\lambda_{\mu\lambda}\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x_{\lambda}}\\ &= \sum_{\nu}\sum_{\lambda}\delta_{\nu\lambda}\frac{\partial}{\partial x_{\nu}}\frac{\partial}{\partial x_{\lambda}}\\ &= \sum\frac{\partial^2}{\partial x_{\lambda}^2}\end{aligned} \tag{10}$$

Since μ and λ are dummy indices, we see that the operator $\sum\partial^2/\partial x_{\mu}^2$ is invariant under a Lorentz transformation. So we have

$$\sum_{\mu}\frac{\partial^2\psi(x',ict')}{\partial x_{\mu}'^2} = 0 \tag{11}$$

This equation means that the function ψ taken at the transformed point (x',ict') satisfies the same equation as the original function $\psi(x,ict)$ and therefore the equation is invariant. In a Galilean transformation, the coordinates become

$$\left.\begin{aligned} x' &= x - v_x t\\ y' &= y - v_y t\\ z' &= z - v_z t\\ t' &= t\end{aligned}\right] \tag{12}$$

Using these relations, we have

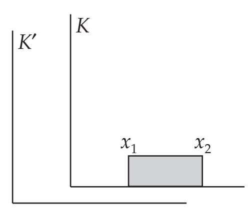
$$\left.\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} = \frac{\partial}{\partial x} - \frac{1}{v_x}\frac{\partial}{\partial t} \\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial y} - \frac{1}{v_y}\frac{\partial}{\partial t} \\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial y} - \frac{1}{v_z}\frac{\partial}{\partial t} \\
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t}
\end{aligned}\right] \tag{13}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2}{\partial x'2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] + \left[\frac{1}{v_x^2} + \frac{1}{v_y^2} + \frac{1}{v_z^2}\right]\frac{\partial^2}{\partial t^2} \\
-2\left[\frac{1}{v_x}\frac{\partial^2}{\partial x \partial t} + \frac{1}{v_y}\frac{\partial^2}{\partial y \partial t} + \frac{1}{v_z}\frac{\partial^2}{\partial z \partial t}\right]
\end{aligned} \tag{14}$$

This means that the function $\psi(x',ict')$ does not satisfy the same form of equation as does $\psi(x,ict)$, and the equation is not invariant under a Galilean transformation.

14-4. In the *K* system the rod is at rest with its ends at x_1 and x_2. The *K′* system moves with a velocity *v* (along the *x* axis) relative to *K*.



If the observer measures the time for the ends of the rod to pass over a fixed point in the *K′* system, we have

$$\left.\begin{aligned}
t_1' &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t_1 - \frac{v}{c^2}x_1\right) \\
t_2' &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t_2 - \frac{v}{c^2}x_2\right)
\end{aligned}\right] \tag{1}$$

where t_1' and t_2' are measured in the *K′* system. From (1), we have

$$t_1' - t_2' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[\left(t_1 - t_2\right) - \frac{v}{c^2}\left(x_1 - x_2\right)\right] \tag{2}$$

We also have

$$x_1 - x_2 = \ell \tag{3}$$

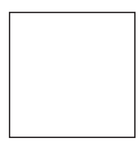
$$v\left(t_1 - t_2\right) = \ell \tag{4}$$

$$v\left(t_1' - t_2'\right) = \ell' \tag{5}$$

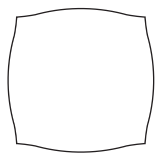
Multiplying (2) by v and using (3), (4), and (5), we obtain the FitzGerald-Lorentz contraction:

$$\boxed{\ell' = \ell\sqrt{1 - \frac{v^2}{c^2}}} \tag{6}$$

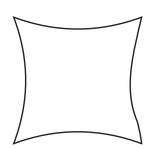
14-5. The "apparent shape" of the cube is that shape which would be recorded at a certain instant by the eye or by a camera (with an infinitesimally short shutter speed!). That is, we must find the positions that the various points of the cube occupy such that light emitted from these points arrives simultaneously at the eye of the observer. Those parts of the cube that are *farther* from the observer must then emit light *earlier* than those parts that are closer to the observer. An observer, looking directly at a cube at rest, would see just the front face, i.e., a square.



When in motion, the edges of the cube are distorted, as indicated in the figures below, where the observer is assumed to be on the line passing through the center of the cube. We also note that the face of the cube in (a) is actually bowed *toward* the observer (i.e., the face appears convex), and conversely in (b).

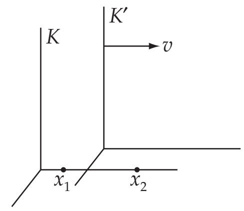


(a) Cube moving *toward* the observer.



(a) Cube moving *away from* the observer.

14-6.



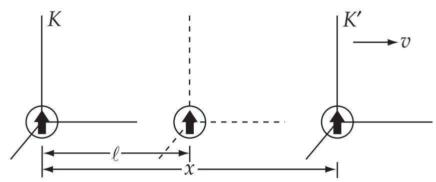
We transform the time t at the points x_1 and x_2 in the K system into the K' system. Then,

$$\left.\begin{aligned} t_1' &= \gamma\left[t - \frac{vx_1}{c^2}\right] \\ t_2' &= \gamma\left[t - \frac{vx_2}{c^2}\right] \end{aligned}\right] \tag{1}$$

From these equations, we have

$$\boxed{\Delta t' = t_1' - t_2' = -\gamma v\,\frac{(x_1 - x_2)}{c^2} = -\gamma v\Delta x\,\frac{1}{c^2}} \tag{2}$$

14-7.



Suppose the origin of the K' system is at a distance x from the origin of the K system after a time t measured in the K system. When the observer sees the clock in the K' system at that time, he actually sees the clock as it was located at an earlier time because it takes a certain time for a light signal to travel to 0. Suppose we see the clock when it is a distance ℓ from the origin of the K system and the time is t_1 in K and t_1' in K'. Then we have

$$\left.\begin{aligned} t_1' &= \gamma\left(t_1 - \frac{v\ell}{c^2}\right) \\ c(t - t_1) &= \ell \\ tv &= x \\ t_1 v &= \ell \end{aligned}\right] \tag{1}$$

We eliminate ℓ, t_1, and x from these equations and we find

$$\boxed{t_1' = \gamma\left[1-\frac{v}{c}\right]t} \tag{2}$$

This is the time the observer reads by means of a telescope.

14-8. The velocity of a point on the surface of the Earth at the equator is

$$v = \frac{2\pi R_e}{\tau} = \frac{2\pi\times\left(6.38\times10^8\ \text{cm}\right)}{8.64\times10^4\ \text{sec}} \tag{1}$$

$$= 4.65\times10^4\ \text{cm/sec}$$

which gives

$$\beta = \frac{v}{c} = \frac{4.65\times10^4\ \text{cm/sec}}{3\times10^{10}\ \text{cm/sec}} = 1.55\times10^{-6} \tag{2}$$

According to Eq. (14.20), the relationship between the polar and equatorial time intervals is

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}} \cong \Delta t\left(1+\frac{1}{2}\beta^2\right) \tag{3}$$

so that the accumulated time difference is

$$\Delta = \Delta t' - \Delta t = \frac{1}{2}\beta^2\ \Delta t \tag{4}$$

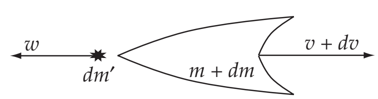
Supplying the values, we find

$$\Delta = \frac{1}{2}\times\left(1.55\times10^{-6}\right)\times\left(3.156\times10^7\ \text{sec/yr}\right)\times\left(10^2\ \text{yr}\right) \tag{5}$$

Thus,

$$\boxed{\Delta = 0.0038\ \text{sec}} \tag{6}$$

14-9.



The unsurprising part of the solution to the problem of the relativistic rocket requires that we apply conservation of momentum, as was done for the nonrelativistic case. The surprising, and key, part of the solution is that we not assume the mass of the ejected fuel is the same as the mass lost from the rocket. Hence

$$p = \gamma m v = \left(\gamma + d\gamma\right)\left(m+dm\right)\left(v+dv\right) + \gamma_w dm' w \tag{1}$$

where $-dm$ is the mass lost from the rocket, dm' is the mass of the ejected fuel, $w \equiv (v-V)\big/\left(1-vV/c^2\right)$ is the velocity of the exhaust with respect to the inertial frame, and $\gamma_w \equiv 1\big/\sqrt{1-w^2/c^2}$. One can easily calculate $d\gamma = \gamma^3\beta\, d\beta$, ad after some algebra one obtains

$$\gamma^2 m\, dv + v\, dm + \frac{\gamma_w\, dm'}{\gamma} w \tag{2}$$

where we of course keep infinitesimals only to first order. The additional unknown dm' is unalarming because of another conservation law

$$E = \gamma mc^2 = (\gamma + d\gamma)(m + dm)c^2 + \gamma_w\, dm'\, c^2 \tag{3}$$

Subsequent substitution of dm' into (2) gives, in one of its many intermediate forms

$$\gamma^2 m\, dv\left(1 - \frac{\beta w}{c}\right) + dm(v - w) = 0 \tag{4}$$

and will finally come to its desired form after dividing by dt

$$m\frac{dv}{dt} + V\frac{dm}{dt}\left(1 - \beta^2\right) = 0 \tag{5}$$

The quantity dt can be measured in any inertial frame, but would presumably only make sense for the particular one in which we measure v. Interestingly, it is not important for the ejected fuel to have an especially large kinetic energy but rather that it be near light speed, a nontrivial distinction. For such a case, a rocket can reach 0.6c by ejecting half its mass.

14-10. From Eq. (14.14)

$$x_1' = \gamma(x_1 - vt) \tag{1}$$

$$t' = \gamma\left(t - \frac{v}{c^2} x_1\right) \tag{2}$$

Solving (1) for x_1 and substituting into (2) gives

$$t' = \gamma\left[t - \frac{v}{c^2}\left[\frac{x_1'}{\gamma} + vt\right]\right]$$

$$t' + \frac{v}{c^2} x_1' = \gamma t - \gamma \frac{v^2}{c^2} t = \frac{t}{\gamma}$$

$$\boxed{t = \gamma\left[t' + \frac{v}{c^2} x_1'\right]}$$

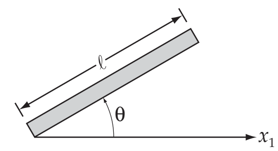
Solving (2) for t and substituting into (1) gives

$$x_1' = \gamma\left[x_1 - v\left[\frac{t'}{\gamma} + \frac{v}{c^2} x_1\right]\right]$$

or

$$\boxed{x_1 = \gamma(x_1' + vt')}$$

14-11.



From example 14.1 we know that, to an observer in motion relative to an object, the dimensions of objects are contracted by a factor of $\sqrt{1-v^2/c^2}$ in the direction of motion. Thus, the x_1' component of the stick will be

$$\ell \cos\theta \sqrt{1-v^2/c^2}$$

while the perpendicular component will be unchanged:

$$\ell \sin\theta$$

So, to the observer in K', the length and orientation of the stick are

$$\ell' = \ell\left[\sin^2\theta + \left(1-v^2/c^2\right)\cos^2\theta\right]^{1/2}$$

$$\theta' = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta\sqrt{1-v^2/c^2}}\right]$$

or

$$\boxed{\begin{array}{l} \ell' = \ell\left[\sin^2\theta + \dfrac{\cos^2\theta}{\gamma^2}\right]^{1/2} \\ \tan\theta' = \gamma\tan\theta \end{array}}$$

14-12. The ground observer measures the speed to be

$$v = \frac{100\ \text{m}}{.4\ \mu\text{sec}} = \boxed{2.5\times 10^8\ \text{m/s}}$$

The length between the markers as measured by the racer is

$$\ell' = \ell\sqrt{1-v^2/c^2}$$

$$= 100\ \text{m}\sqrt{1-\left[\frac{2.5}{3}\right]^2} = \boxed{55.3\ \text{meters}}$$

The time measured in the racer's frame is given by

$$t' = \gamma\left(t - \frac{v}{c^2}x_1\right)$$

$$= \frac{\left(.4\ \mu\text{sec} - \dfrac{\left(2.5\times10^8\ \text{m/s}\right)\left(100\ \text{m}\right)}{c^2}\right)}{\sqrt{1-\left(2.5/3\right)^2}}$$

$$= \boxed{.22\ \mu\text{sec}}$$

The speed observed by the racer is

$$v = \frac{\ell'}{t'} = \frac{\ell}{t} = \boxed{2.5\times10^8\ \text{m/s}}$$

14-13.

$$\Delta t' = \gamma\ \Delta t$$

$$\Delta t = 1.5\ \mu\text{s}$$

$$\gamma = \left(1-0.999^2\right)^{-1/2} \simeq 22.4$$

Therefore $\boxed{\Delta t' \simeq 34\mu\text{s}}$.

14-14.

In K, the energy and momentum of each photon emitted are

$$E = h\nu_0 \quad \text{and} \quad p = \frac{h\nu_0}{c}$$

Using Eq. (14.92) to transform to K′:

$$E' = h\nu = \gamma\left(E - vp_1\right); \quad \left(p_1 = -\frac{h\nu_0}{c}\right)$$

$$= \gamma\left(h\nu_0 + \frac{v}{c}h\nu_0\right)$$

So

$$\nu = \nu_0\, \gamma\left(1 + \frac{v}{c}\right)$$

$$= \nu_0\, \frac{1+\beta}{\sqrt{1-\beta^2}} = \nu_0 \sqrt{\frac{1+\beta}{1-\beta}}$$

which agrees with Eq. (14.31).

14-15. From Eq. (14.33)

$$\nu = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\, \nu_0$$

Since $\lambda = c/\nu$

$$\lambda_0 = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\, \lambda$$

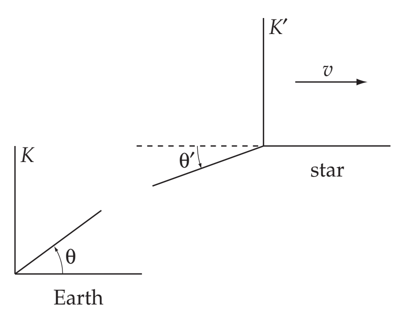
or

$$\lambda = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}}\, \lambda_0$$

With $\lambda_0 = 656.3$ nm and $\beta = \dfrac{4\times 10^4}{3\times 10^8}$, $\lambda = 656.4$ nm.

$\boxed{\text{So the shift is 0.1 nm toward the red (longer wavelength).}}$

14-16.



Consider a photon sent from the star to the Earth. From Eq. (14.92)

$$E' = \gamma\left(E - vp_1\right)$$

also

$$E = \gamma\left(E' + vp_1'\right)$$

Now

$$E = h\nu, \quad E' = h\nu_0, \quad p_1 = -\frac{h\nu}{c}\cos\theta, \quad p_1' = -\frac{h\nu_0}{c}\cos\theta'$$

Substituting yields

$$\nu_0 = \nu\gamma\left(1 + \beta\cos\theta\right)$$

and

$$\nu = \gamma\ \nu_0\left(1 - \beta\cos\theta'\right)$$

Thus

$$\left(1 + \beta\cos\theta\right)\left(1 - \beta\cos\theta'\right) = \gamma^{-2}$$

$$1 + \beta\cos\theta - \beta\cos\theta' - \beta^2\cos\theta\cos\theta' = 1 - \beta^2$$

$$\cos\theta - \cos\theta' - \beta\cos\theta\cos\theta' = -\beta$$

Solving for $\cos\theta$ yields

$$\boxed{\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}}$$

where

$\beta = v/c$

$\theta =$ angle in earth's frame

$\theta' =$ angle in star's frame

14-17. From Eq. (14.33)

$$\nu = \sqrt{\frac{1-\beta}{1+\beta}}\ \nu_0$$

Since

$$\nu = c/\lambda\,,$$

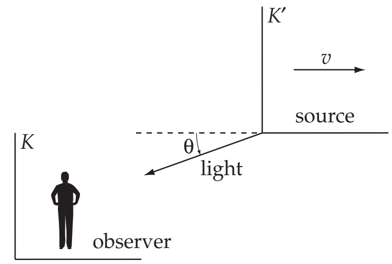
$$\lambda = \sqrt{\frac{1+\beta}{1-\beta}}\ \lambda_0$$

We have $\lambda = 1.5\,\lambda_0$. This gives $\beta = \dfrac{5}{13}$

or

$$\boxed{v = 1.2 \times 10^8\ \text{m/sec}}$$

14-18.



Proceeding as in example 14.11, we treat the light as a photon of energy $h\nu$.

$$\text{In } K': \quad E' = h\nu_0 \,,\, p' = \frac{h\nu_0}{c}$$

$$\text{In } K: \quad E = h\nu = \gamma\left(E' + vp_1\right)$$

For the source approaching the observer at an early time we have

$$p_1 = \frac{h\nu_0}{c}$$

Thus

$$\nu = \gamma\left(\nu_0 + \frac{v}{c}\nu_0\right) = \nu_0\sqrt{\frac{1+\beta}{1-\beta}}$$

For the source receding from the observer (at a much later time) we have

$$p_1 = -\frac{h\nu_0}{c}$$

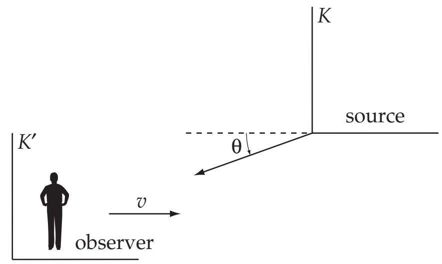
and

$$\nu = \nu_0\sqrt{\frac{1-\beta}{1+\beta}}$$

So

$$\boxed{\begin{array}{l} \nu = \nu_0\sqrt{\dfrac{1+\beta}{1-\beta}} \text{ source approaching observer} \\[2ex] \nu = \nu_0\sqrt{\dfrac{1-\beta}{1+\beta}} \text{ source receding from observer} \end{array}}$$

14-19.



Proceeding as in the previous problem, we have

In K': $E' = h\nu$

$$p_1' = -\frac{h\nu}{c}\cos\theta = -\frac{h\nu}{c}\frac{\beta_r}{\sqrt{\beta_r^2+\beta_t^2}}$$

In K: $E = \gamma\left(E' + vp_1'\right) = h\nu_0$

So

$$h\nu_0 = \frac{1}{\sqrt{1-\beta_r^2-\beta_t^2}}\left[h\nu - \left[c\sqrt{\beta_r^2+\beta_t^2}\right]\left[\frac{h\nu\beta_r}{c\sqrt{\beta_r^2+\beta_t^2}}\right]\right]$$

or

$$\nu_0 = \frac{\nu(1-\beta_r)}{\sqrt{1-\beta_r^2-\beta_t^2}}$$

$$\boxed{\frac{\nu}{\nu_0} = \frac{\lambda_0}{\lambda} = \frac{\sqrt{1-\beta_r^2-\beta_t^2}}{1-\beta_r}}$$

For $\lambda > \lambda_0$, we have

$$(1-\beta_r)^2 > 1-\beta_r^2-\beta_t^2$$

$$\beta_t^2 > 2\beta_r - 2\beta_r^2$$

$$\boxed{\beta_t^2 > 2\beta_r(1-\beta_r)}$$

14-20. As measured by observers on Earth, the entire trip takes

$$2\left[\frac{4 \text{ lightyears}}{0.3\text{ c}}\right] = \frac{80}{3}\text{ years}$$

The people on earth age $\frac{80}{3}$ years. The astronaut's clock is ticking slower by a factor of γ. Thus, the astronaut ages

$$\frac{80}{3}\sqrt{1-0.3^2} = 0.95\left[\frac{80}{3}\right] \text{ years}$$

So

$$\boxed{\begin{array}{l}\text{Those on Earth age 26.7 years.}\\ \text{The astronaut ages 25.4 years.}\end{array}}$$

14-21.

$$F = \frac{d}{dt}\left[\frac{m_0 \mathbf{v}}{\sqrt{1-\beta^2}}\right] = m_0\left[\frac{\dot{\mathbf{v}}}{\sqrt{1-\beta^2}} + \mathbf{v}\,\frac{\left(-\frac{1}{2}\right)(-2\beta)\dot{\beta}}{\left(1-\beta^2\right)^{3/2}}\right] \tag{1}$$

$$= m_0\left[\frac{\dot{\mathbf{v}}}{\sqrt{1-\beta^2}} + \frac{\mathbf{v}\beta\dot{\beta}}{\left(1-\beta^2\right)^{3/2}}\right]$$

If we take $\mathbf{v} = v_1\mathbf{e}_1$ (this does not mean $\dot{v}_2 = \dot{v}_3 = 0$), we have

$$\boxed{F_1 = m_0\left[\frac{v_1}{\sqrt{1-\beta^2}} + \frac{v_1\,\frac{v_1}{c}\,\frac{\dot{v}_1}{c}}{\left(1-\beta^2\right)^{3/2}}\right] = \frac{m_0}{\left(1-\beta^2\right)^{3/2}}\,\dot{v}_1 = m_\ell\,\dot{v}_1} \tag{2}$$

$$\boxed{F_2 = \frac{m_0}{\sqrt{1-\beta^2}}\,\dot{v}_2 = m_t\,\dot{v}_2} \tag{3}$$

$$\boxed{F_3 = \frac{m_0}{\sqrt{1-\beta^2}}\,\dot{v}_3 = m_t\,\dot{v}_3} \tag{4}$$

14-22. The total energy output of the sun is

$$\frac{dE}{dt} = \left(1.4\times10^3 \text{ W}\cdot\text{m}^{-2}\right)\times 4\pi R^2 \tag{1}$$

where $R = 1.50\times10^{11}$ m is the mean radius of the Earth's orbit around the sun. Therefore,

$$\frac{dE}{dt} \simeq 3.96\times10^{26} \text{ W} \tag{2}$$

The corresponding rate of mass decrease is

$$\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt} \simeq 4.4\times10^9 \text{ kg}\cdot\text{s}^{-1} \tag{3}$$

The mass of the sun is approximately 1.99×10^{30} kg, so this rate of mass decrease can continue for a time

$$T = \frac{1.99 \times 10^{30}\ \text{yr}}{4.4 \times 10^{9}\ \text{kg} \cdot \text{s}^{-1}} \simeq 1.4 \times 10^{13}\ \text{yr} \tag{4}$$

Actually, the lifetime of the sun is limited by other factors and the sun is expected to expire about 4.5×10^9 years from now.

14-23. From Eq. (14.67)

$$\begin{aligned} p^2c^2 &= E^2 - E_0^2 \\ &= \left(E_0 + T\right)^2 - E_0^2 \\ &= 2E_0T + T^2 \end{aligned}$$

$$\boxed{p^2c^2 = 2T\ mc^2 + T^2}$$

14-24. The minimum energy will occur when the four particles are all at rest in the center of the mass system after the collision.

Conservation of energy gives (in the CM system)

$$2E_p = 4m_pc^2$$

or

$$E_{p,\text{CM}} = 2m_pc^2 = 2E_0$$

which implies $\gamma = 2$ or $\beta = \sqrt{3}/2$

To find the energy required in the lab system (one proton at rest initially), we transform back to the lab

$$E = \gamma\left(E' + vp_1'\right) \tag{1}$$

The velocity of K'(CM) with respect to K(lab) is just the velocity of the proton in the K' system. So $u = v$.

Then

$$vp_1' = v\left(p_{\text{CM}}\right) = v\left(\gamma mu\right) = \gamma mv^2 = \gamma mc^2\beta^2$$

Since $\gamma = 2,\ \beta = \sqrt{3}/2$,

$$vp_1' = \frac{3}{2}E_0$$

Substituting into (1)

$$E_{\text{lab}} = \gamma\left(2E_0 + \frac{3}{2}E_0\right) = 2\left[\frac{7}{2}E_0\right] = 7E_0$$

$$\boxed{\begin{array}{l}\text{The minimum proton energy in the lab system} \\ \text{is } 7\, m_p c^2 \text{, of which } 6\, m_p c^2 \text{ is kinetic energy.}\end{array}}$$

14-25. Let $\mathbf{B} = B_0 \mathbf{z}$

$$\mathbf{v} = v_x\, \mathbf{i} + v_y\, \mathbf{j}$$

Then

$$q\mathbf{v} \times \mathbf{B} = q\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & 0 \\ 0 & 0 & B_0 \end{vmatrix}$$

$$= q\left[v_y\, B_0\, \mathbf{i} - v_x\, B_0\, \mathbf{j}\right]$$

$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = \dfrac{d}{dt}(\mathbf{p}) = \gamma m \dfrac{d}{dt}(\mathbf{v})$ gives

$$\frac{d}{dt}\mathbf{v} = \frac{qB_0}{\gamma m}\left(v_y\, \mathbf{i} - v_x\, \mathbf{j}\right)$$

Define $\omega \equiv q\, B_0 / \gamma m$

Thus

$$\dot{v}_x = \omega v_y \text{ and } \dot{v}_y = -\omega v_x$$

or

$$\ddot{v}_x = \omega \dot{v}_y = -\omega^2 v_x$$

and

$$\ddot{v}_y = -\omega \dot{v}_x = -\omega^2 v_y$$

So

$$v_x = A \cos \omega t + B \sin \omega t$$

$$v_y = C \cos \omega t + D \sin \omega t$$

Take $v_x(0) = v$, $v_y(0) = 0$. Then $A = v$, $C = 0$. Then $\dot{v}_x(0) = \omega v_y(0) = 0$

$$\dot{v}_y(0) = -\omega v_x(0) = -\omega v$$

$$\rightarrow B = 0,\, D = -v$$

Thus

$$\mathbf{v} = \mathbf{i}\, v \cos \omega t - \mathbf{j}\, v \sin \omega t$$

Then

$$\mathbf{r} = \mathbf{i}\, \frac{v}{\omega} \sin \omega t + \mathbf{j}\, \frac{v}{\omega} \cos \omega t$$

The path is a circle of radius $\frac{v}{\omega}$

$$r = \frac{v}{q\, B_0 / \gamma m} = \frac{\gamma m v}{q\, B_0} = \frac{p}{q\, B_0}$$

From problem 14-22

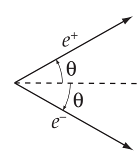
$$p = \left[2Tm + \frac{T^2}{c^2} \right]^{1/2}$$

So

$$\boxed{r = \frac{\left[2Tm + \frac{T^2}{c^2} \right]^{1/2}}{q\, B_0}}$$

14-26. Suppose a photon traveling in the x-direction is converted into an e^- and e^+ as shown below

before



after

Cons. of energy gives

$$p_p c = 2E_e$$

where

$$p_p = \text{momentum of the photon}$$

$$E_e = \text{energy of } e^+ = \text{energy of } e^-$$

Cons. of p_x gives

$$p_p = 2p_e \cos \theta \qquad \left(p_e = \text{momentum of } e^+, e^- \right)$$

Dividing gives

$$\frac{p_p c}{p_p} = c = \frac{E_e}{p_e \cos\theta}$$

or

$$p_e^2 c^2 \cos^2\theta = E_e^2 \tag{1}$$

But $E_e^2 > p_e^2 c^2$, so (1) cannot be satisfied for $\cos^2\theta \leq 1$.

$$\boxed{\text{An isolated photon cannot be converted into an electron-positron pair.}}$$

This result can also be seen by transforming to a frame where $p_x = 0$ after the collision. But, before the collision, $p_x = p_p c \neq 0$ in any frame moving along the x-axis. So, without another object nearby, momentum cannot be conserved; thus, the process cannot take place.

14-27. The minimum energy required occurs when the p and $\overline{p}$ are at rest after the collision. By conservation of energy

$$2E_e = 2(938 \text{ MeV})$$

$$E_e = 938 \text{ MeV} = T + E_0$$

Since $E_e = 0.5 \text{ MeV}$,

$$\boxed{T_{e^+} = T_{e^-} = 937.5 \text{ MeV}}$$

14-28. $T_{\text{classical}} = \frac{1}{2} mv^2$

$$T_{\text{rel}} = (\gamma - 1)\, mc^2 \geq T_{\text{classical}}$$

We desire

$$\frac{T_{\text{rel}} - T_{\text{classical}}}{T_{\text{rel}}} \leq 0.01$$

$$1 - \frac{\frac{1}{2} mv^2}{(\gamma - 1)\, mc^2} \leq 0.01$$

$$\frac{\frac{1}{2} v^2}{(\gamma - 1)\, c^2} \geq 0.99$$

$$\frac{\beta^2}{\gamma - 1} \geq 1.98$$

Putting $\gamma = \left(1 - \beta^2\right)^{-1/2}$ and solving gives

$$v \le 0.115\,c$$

$$\boxed{\text{The classical kinetic energy will be within 1\% of the correct value for } 0 \le v \le 3.5\times 10^{7}\ \text{m/sec, independent of mass.}}$$

14-29. $$E = \gamma E_0$$

For

$$E = 30\times 10^{9}\ \text{eV}$$

$$E_0 \simeq 0.51\times 10^{6}\ \text{eV},$$

$$\gamma \simeq 5.88\times 10^{4}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{or} \quad \beta = \left(1-\gamma^{-2}\right)^{1/2}$$

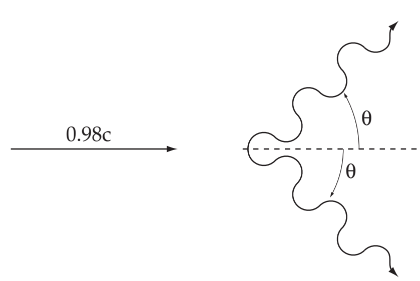
$$\beta \simeq 1 - \frac{1}{2\gamma^2} = 1 - 1.4\times 10^{-10}$$

$$\boxed{\begin{aligned} v &= \left(1-1.4\times 10^{-10}\right)\ \text{c} \\ &= 0.99999999986\ \text{c} \end{aligned}}$$

14-30. A neutron at rest has an energy of 939.6 MeV. Subtracting the rest energies of the proton (938.3 MeV) and the electron (0.5 MeV) leaves 0.8 MeV.

$$\boxed{\text{Other than rest energies 0.8 MeV is available.}}$$

14-31.



Conservation of energy gives

$$E_\pi = 2E_p$$

where $E_p =$ energy of each photon (Cons. of p_y implies that the photons have the same energy).

Thus

$$\gamma E_0 = 2E_p$$

$$E_p = \frac{\gamma E_0}{2} = \frac{135\text{ MeV}}{2\sqrt{1-0.98^2}} = 339\text{ MeV}$$

$$\boxed{\text{The energy of each photon is 339 MeV.}}$$

Conservation of p_x gives

$$\gamma m v = 2p_p \cos\theta \text{ where } p_p = \text{momentum of each photon}$$

$$\cos\theta = \frac{\left(135\text{ Mev/c}^2\right)(0.98\text{ c})}{2\sqrt{1-0.98^2}\,(339\text{ MeV/c})} = 0.98$$

$$\boxed{\theta = \cos^{-1} 0.98 = 11.3°}$$

14-32. From Eq. (14.67) we have

$$E^2 - E_0^2 = p^2c^2$$

With $E = E_0 + T$, this reduces to

$$2E_0T + T^2 = p^2c^2$$

Using the quadratic formula (taking the + root since $T \geq 0$) gives

$$T = \sqrt{E_0^2 + p^2c^2} - E_0$$

Substituting pc = 1000 MeV

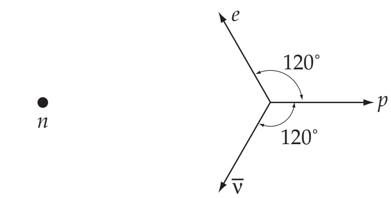
$$E_0(\text{electron}) = 0.5\text{ MeV}$$

$$E_0(\text{proton}) = 938\text{ MeV}$$

gives

$$\boxed{\begin{array}{l} T_{\text{electron}} = 999.5\text{ MeV} \\ T_{\text{proton}} = 433\text{ MeV} \end{array}}$$

14-33.



Conservation of p_y gives

$$p_e \sin 60° = p_{\bar{\nu}} \sin 60° \text{ or } p_e = p_{\bar{\nu}}$$

Conservation of p_x gives

$$p_p = p_e \cos 60° + p_{\bar{\nu}} \cos 60° = p_e$$

So

$$p_e = p_p = p_{\bar{\nu}} \equiv p$$

Conservation of energy gives

$$E_{0n} = E_e + E_p + E_{\bar{\nu}}$$

$$E_{0n} = \sqrt{E_{0e}^2 + p^2c^2} + \sqrt{E_{0p}^2 + p^2c^2} + pc \tag{1}$$

Substituting

$$E_{0n} = 939.6 \text{ MeV}$$

$$E_{0p} = 938.3 \text{ MeV}$$

$$E_{0e} = 0.5 \text{ MeV}$$

and solving for pc gives

$$\text{p} = 0.554 \text{ MeV/c}$$

$$\boxed{p_p = p_e = p_{\bar{\nu}} = 0.554 \text{ MeV/c}}$$

Substituting into

$$T = E - E_0$$

$$= \sqrt{E_0^2 + p^2c^2} - E_0$$

gives $\left(E_{0\nu} = 0\right)$

$$\boxed{\begin{array}{l} T_{\bar{\nu}} = 0.554 \text{ MeV} \\ T_p = 2 \times 10^{-4} \text{ MeV, or } 200 \text{ eV} \\ T_e = 0.25 \text{ MeV} \end{array}}$$

14-34.
$$\Delta s'^2 = -c^2 t'^2 + x_1'^2 + x_2'^2 + x_3'^2$$

Using the Lorentz transformation this becomes

$$\Delta s'^2 = \frac{-c^2 t^2 - \dfrac{v^2 x_1^2}{c^2} + 2x_1 vt}{1 - v^2/c^2} + \frac{x_1^2 + v^2 t^2 - 2x_1 vt}{1 - v^2/c^2} + x_2^2 + x_3^2$$

$$= \frac{\left[x_1^2 - \dfrac{v^2 x_1^2}{c^2}\right] - c^2\left[t^2 - \dfrac{v^2}{c^2}t^2\right]}{1 - v^2/c^2} + x_2^2 + x_3^2$$

$$= -c^2 t^2 + x_1^2 + x_2^2 + x_3^2$$

So

$$\boxed{\Delta s'^2 = \Delta s^2}$$

14-35. Let the frame of Saturn be the unprimed frame, and let the frame of the first spacecraft be the primed frame. From Eq. (14.17a) (switch primed and unprimed variables and change the sign of v)

$$u_1 = \frac{u_1' + v}{1 + \dfrac{u_1' v}{c^2}}$$

Substituting $v = 0.9\,c$

$$u_1' = 0.2\,c$$

gives

$$\boxed{u_1 = 0.93\,c}$$

14-36. Since

$$F_\mu = \frac{d}{d\tau}\left[m\,\frac{dX\mu}{d\tau}\right] \text{ and } X_\mu = (x_1, x_2, x_3, ict)$$

we have

$$F_1 = \frac{d}{d\tau}\left[m\,\frac{dx_1}{d\tau}\right] = m\,\frac{d^2 x_1}{d\tau^2}$$

$$F_2 = m\,\frac{d^2 x_2}{d\tau^2} \qquad F_3 = m\,\frac{d^2 x_3}{d\tau^2}$$

$$F_4 = \frac{d}{d\tau}\left[m\,\frac{d(ict)}{d\tau}\right] = icm\,\frac{d^2 t}{d\tau^2}$$

Thus

$$F_1' = m\frac{d^2x_1}{d\tau^2} - m\frac{d^2}{d\tau^2}\left[\gamma\left(x_1 - vt\right)\right]$$

$$= \gamma m\frac{d^2x_1'}{d\tau^2} = \gamma mv\frac{d^2t}{d\tau^2} = \gamma\left(F_1 + i\beta F_4\right)$$

$$F_2' = m\frac{d^2x_2'}{d\tau^2} = m\frac{d^2x_2}{d\tau^2} = F_2\ ; \quad F_3' = F_3$$

$$F_4' = icm\frac{d}{d\tau^2}\left[\gamma\left(t - \frac{vx_1}{c^2}\right)\right]$$

$$= \gamma\ icm\frac{d^2t}{d\tau^2} - \gamma\ i\beta m\frac{d^2x_1}{d\tau^2}$$

$$= \gamma\left(F_4 - i\beta F_1\right)$$

Thus the required transformation equations are shown.

14-37. From the Lagrangian

$$L = mc^2\left(1 - \sqrt{1-\beta^2}\right) - \frac{1}{2}kx^2 \tag{1}$$

we compute

$$\frac{\partial L}{\partial x} = -kx \tag{2}$$

$$\frac{\partial L}{\partial v} = \frac{\partial \beta}{\partial v}\frac{\partial L}{\partial \beta} = mc\frac{\beta}{\sqrt{1-\beta^2}} \tag{3}$$

Then, from (2) and (3), the Lagrange equation of motion is

$$\frac{d}{dt}\left[\frac{mc\beta}{\sqrt{1-\beta^2}}\right] + kx = 0 \tag{4}$$

from which

$$\frac{mc\dot{\beta}}{\left(1-\beta^2\right)^{3/2}} + kx = 0 \tag{5}$$

Using the relation

$$c\dot{\beta} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{6}$$

we can rewrite (4) as

$$\frac{mc^2\beta}{\left(1-\beta^2\right)^{3/2}}\frac{d\beta}{dx}+kx=0 \tag{7}$$

This is easily integrated to give

$$\frac{mc^2}{\sqrt{1-\beta^2}}+\frac{1}{2}kx^2=E \tag{8}$$

where E is the constant of integration.

The value of E is evaluated for some particular point in phase space, the easiest being $x = a$; $\beta = 0$:

$$\boxed{E=mc^2+\frac{1}{2}ka^2} \tag{9}$$

From (8) and (9),

$$\frac{mc^2}{\sqrt{1-\beta^2}}+\frac{1}{2}kx^2=mc^2+\frac{1}{2}ka^2 \tag{10}$$

Eliminating β^2 from (10), we have

$$\beta^2=1-\frac{m^2c^4}{\left[mc^2+\frac{1}{2}k\left(a^2-x^2\right)\right]^2}$$

$$=k\left(a^2-x^2\right)\frac{\left[mc^2+\frac{k}{4}\left(a^2-x^2\right)\right]}{\left[mc^2+\frac{k}{2}\left(a^2-x^2\right)\right]^2} \tag{11}$$

and, therefore,

$$\beta=\frac{1}{c}\frac{dx}{dt}=\frac{\sqrt{k\left(a^2-x^2\right)}\sqrt{mc^2+k\left(a^2-x^2\right)/4}}{mc^2+k\left(a^2-x^2\right)/2} \tag{12}$$

The period will then be four times the integral of $dt = dt(x)$ from $x = 0$ to $x = a$:

$$\tau=4\sqrt{\frac{m}{k}}\int_0^a\frac{\left[1+\frac{k}{2mc^2}\left(a^2-x^2\right)\right]}{\sqrt{a^2-x^2}\sqrt{1+\frac{k}{4mc^2}\left(a^2-x^2\right)}}dx \tag{13}$$

Since x varies between 0 and a, the variable x/a takes on values in the interval 0 to 1, and therefore, we can define

$$\sin\phi=\frac{x}{a} \tag{14}$$

from which

$$\cos\phi = \frac{\sqrt{a^2 - x^2}}{a} \tag{15}$$

and

$$dx = \sqrt{a^2 - x^2}\; d\phi \tag{16}$$

We also define the dimensionless parameter,

$$\kappa \equiv \frac{a}{2}\sqrt{\frac{k}{mc^2}} \tag{17}$$

Using (14) – (17), (13) transforms into

$$\boxed{\tau = \frac{2a}{\kappa c}\int_0^{\pi/2} \frac{\left(1 + 2\kappa^2 \cos^2\phi\right)}{\sqrt{1 + \kappa^2 \cos^2\phi}}\, d\phi} \tag{18}$$

Since $ka^2/mc^2 \ll 1$ for the weakly relativistic case, we can expand the integrand of (18) in a series of powers of κ:

$$\begin{aligned}
\frac{\left(1 + 2\kappa^2 \cos^2\phi\right)}{\left(1 + \kappa^2 \cos^2\phi\right)^{1/2}} &\cong \left(1 + 2\kappa^2 \cos^2\phi\right)\left(1 - \frac{\kappa^2}{2}\cos^2\phi\right) \\
&\cong 1 + \left[2 - \frac{1}{2}\right]\kappa^2 \cos^2\phi \\
&= 1 + \frac{3}{2}\kappa^2 \cos^2\phi
\end{aligned} \tag{19}$$

Substitution of (19) into (18) yields

$$\begin{aligned}
\tau &\cong \frac{2a}{\kappa c}\int_0^{\pi/2}\left[1 + \frac{3}{2}\kappa^2 \cos^2\phi\right] d\phi \\
&= \frac{a\pi}{\kappa c} + \frac{3\kappa a}{2c}\left[\phi + \frac{1}{2}\sin 2\phi\right]_0^{\pi/2}
\end{aligned} \tag{20}$$

Evaluating (20) and substituting the expression for κ from (17), we obtain

$$\tau = 2\pi\sqrt{\frac{m}{k}} + \frac{3\pi a^2}{8c^2}\sqrt{\frac{k}{m}} \tag{21}$$

or,

$$\boxed{\tau = \tau_0\left[1 + \frac{3}{16}\frac{ka^2}{mc^2}\right]} \tag{22}$$

14-38.

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m u)$$

$$= m\frac{d}{dt}(\gamma u) \qquad \text{(for } m = \text{constant)}$$

$$= m\frac{d}{dt}\left[\frac{u}{\sqrt{1-u^2/c^2}}\right]$$

$$= m\left[\frac{\left(1-u^2/c^2\right)^{1/2} - u\left[-\frac{u}{c^2}\right]\left(1-u^2/c^2\right)^{-1/2}}{\left(1-u^2/c^2\right)}\right]\frac{du}{dt}$$

$$= m\left(1-u^2/c^2\right)^{-3/2}\frac{du}{dt}$$

Thus

$$\boxed{F = m\frac{du}{dt}\left(1-u^2/c^2\right)^{-3/2}}$$

14-39. The kinetic energy is

$$T = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 \tag{1}$$

For a momentum of 100 MeV/c,

$$T_{\text{proton}} = \sqrt{10^4 + (931)^2} - 931 \cong 936 - 931 = 5 \text{ MeV} \tag{2}$$

$$T_{\text{electron}} = \sqrt{10^4 + (0.51)^2} - 0.51 \cong 100 - 0.5 = 99.5 \text{ MeV} \tag{3}$$

In order to obtain γ and β, we use the relation

$$E = mc^2 = \gamma m_0c^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}} \tag{4}$$

so that

$$\gamma = \frac{E}{m_0c^2} \tag{5}$$

and

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \tag{6}$$

$$\gamma_{\text{electron}} = \frac{100}{0.51} \cong 200 \tag{7}$$

$$\beta_{\text{electron}} = \sqrt{1-\left[\frac{1}{200}\right]^2} \cong 0.999988 \tag{8}$$

This is a relativistic velocity.

$$\gamma_{\text{proton}} = \frac{936}{931} \cong 1.0054 \tag{9}$$

$$\beta_{\text{proton}} = \sqrt{1-\left[\frac{1}{1.0053}\right]^2} \cong 0.1 \tag{10}$$

This is a nonrelativistic velocity.

14-40. If we write the velocity components of the center-of-mass system as v_j, the transformation of $p_{\alpha,j}$ into the center-of-mass system becomes

$$p'_{\alpha,j} = \gamma\left(p_{\alpha,j} - \frac{v_j E_\alpha}{c^2}\right) \tag{1}$$

where $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v_j^2}{c^2}}}$. Since in the center-of-mass system, $\sum_\alpha p'_{\alpha,j} = 0$ must be satisfied, we have

$$\sum_\alpha p'_{\alpha,j} = \sum_\alpha \gamma\left[p_{\alpha,j} - \frac{v_j E_\alpha}{c^2}\right] = 0 \tag{2}$$

or,

$$\boxed{\frac{v_j}{c} = \frac{\sum_\alpha p_{\alpha,j}\, c}{\sum_\alpha E_\alpha}} \tag{3}$$

14-41. We want to compute

$$\frac{T_1}{T_0} = \frac{E_1 - m_0 c^2}{E_0 - m_0 c^2} \tag{1}$$

where T and E represent the kinetic and total energy in the laboratory system, respectively, the subscripts 0 and 1 indicate the initial and final states, and m_0 is the rest mass of the incident particle.

The expression for E_0 in terms of γ_1 is

$$E_0 - m_0 c^2 \gamma_1 \tag{2}$$

E_1 can be related to E'_1 (total energy of particle 1 in the center of momentum reference frame after the collision) through the Lorentz transformation [cf. Eq. (14.92)] (remembering that for the inverse transformation we switch the primed and unprimed variables and change the sign of v):

$$E_1 = \gamma_1' \left(E_1' + c\beta_1' p_1' \cos\theta \right) \tag{3}$$

where $p_1' = m_0 c \beta_1 \gamma_1'$ and $E_1' = m_0 c^2 \gamma_1'$:

$$E_1 = m_0 c^2 \gamma_1'^2 \left(1 + \beta_1'^2 \cos\theta \right) \tag{4}$$

Then, from (1), (2), and (4),

$$\frac{T_1}{T_0} = \frac{\gamma_1'^2 + \gamma_1'^2 \beta_1'^2 \cos\theta - 1}{\gamma_1 - 1} \tag{5}$$

For the case of collision between two particles of equal mass, we have, from Eq. (14.127),

$$\gamma_1'^2 = \frac{1 + \gamma_1}{2} \tag{6}$$

and, consequently,

$$\gamma_1'^2 \beta_1'^2 = \gamma_1'^2 - 1 = \frac{\gamma_1 - 1}{2} \tag{7}$$

Thus, with the help of (6) and (7), (5) becomes

$$\frac{T_1}{T_0} = \frac{\gamma_1 - 1 + (\gamma_1 - 1)\cos\theta}{2(\gamma_1 - 1)}$$

$$= \frac{1 + \cos\theta}{2} \tag{8}$$

We must now relate the scattering angle θ in the center of momentum system to the angle ψ in the lab system.

Squaring Eq. (14.128), which is valid only for $m_1 = m_2$, we obtain an equation quadratic in cos θ. Solving for cos θ in terms of $\tan^2 \psi$, we obtain

$$\cos\theta = \frac{-\dfrac{\gamma_1 + 1}{2}\tan^2\psi \pm 1}{1 + \dfrac{\gamma_1 + 1}{2}\tan^2\psi} \tag{9}$$

One of the roots given in (9) corresponds to $\theta = \pi$, i.e., the incident particle reverses its path and is projected back along the incident direction. Substitution of the other root into (8) gives

$$\frac{T_1}{T_0} = \frac{1}{1 + \dfrac{\gamma_1 + 1}{2}\tan^2\psi} = \frac{2\cos^2\psi}{2\cos^2\psi + (\gamma_1 + 1)\sin^2\psi} \tag{10}$$

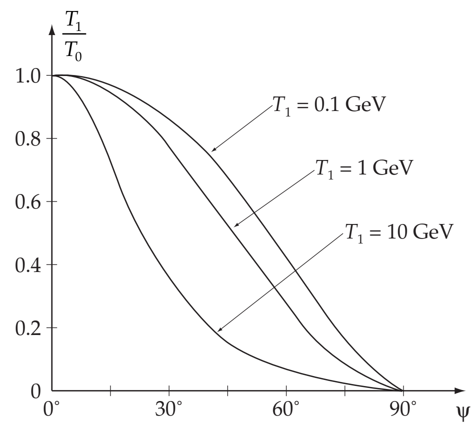
An elementary manipulation with the denominator of (10), namely,

$$\begin{aligned}2\cos^2\psi+(\gamma_1+1)\sin^2\psi &= 2\cos^2\psi+\gamma_1\left(1-\cos^2\psi\right)+\sin^2\psi\\ &=\gamma_1+\sin^2\psi+\cos^2\psi-\gamma_1\cos^2\psi+\cos^2\psi\\ &=\gamma_1+1-\gamma_1\cos^2\psi+\cos^2\psi\\ &=(\gamma_1+1)-(\gamma_1-1)\cos^2\psi\end{aligned}\tag{11}$$

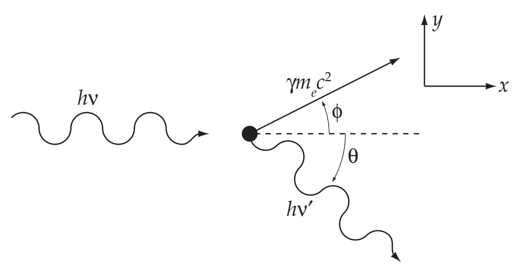
provides us with the desired result:

$$\boxed{\frac{T_1}{T_0}=\frac{2\cos^2\psi}{(\gamma_1+1)-(\gamma_1-1)\cos^2\psi}}\tag{12}$$

Notice that the shape of the curve changes when $T_1 > m_0c^2$, i.e., when $\gamma_1 > 2$.



14-42.



From conservation of energy, we have

$$h_\nu + m_e c^2 = \gamma m_e c^2 + h\nu' \tag{1}$$

Momentum conservation along the x axis gives

$$\frac{h\nu}{c}=\frac{h\nu'}{c}\cos\theta+\gamma m_e\, v\cos\phi \tag{2}$$

Momentum conservation along the y axis gives

$$\gamma m_e\, v\sin\phi=\frac{h\nu'}{c}\sin\theta \tag{3}$$

In order to eliminate ϕ, we use (2) and (3) to obtain

$$\left.\begin{aligned} \cos\phi &= \frac{1}{\gamma m_e v}\left[\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta\right] \\ \sin\phi &= \frac{h\nu'}{\gamma m_e v}\sin\theta \end{aligned}\right] \tag{4}$$

Then,

$$\cos^2\phi + \sin^2\phi = 1 = \frac{1}{\gamma^2 m_e^2 v^2}\left[\left[\frac{h\nu}{c}\right]^2 + \left[\frac{h\nu'}{c}\right]^2 - 2\left[\frac{h\nu}{c}\right] + \left[\frac{h\nu'}{c}\right]\cos\theta\right] \tag{5}$$

Since $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ and $v = \dfrac{c}{\gamma}\sqrt{\gamma^2 - 1}$ we have

$$\gamma^2 v^2 = c^2\left(\gamma^2 - 1\right) \tag{6}$$

Substituting γ from (1) into (6), we have

$$\gamma^2 v^2 = \frac{2h}{m_e}(\nu - \nu') + \frac{h^2}{m_e^2 c^2}(\nu - \nu')^2 \tag{7}$$

From (5) and (7), we can find the equation for ν':

$$\left[\frac{h\nu}{c}\right]^2 + \left[\frac{h\nu'}{c}\right]^2 - 2\left[\frac{h\nu}{c}\right]\left[\frac{h\nu'}{c}\right]\cos\theta = 2\,hm_e\,(\nu - \nu') + \frac{h^2}{c^2}(\nu - \nu')^2 \tag{8}$$

or,

$$\left[\frac{2m_e c^2}{h} + 2\nu\left(1 - \cos\theta\right)\right]\nu' = \frac{2m_e c^2}{h}\nu \tag{9}$$

Then,

$$\nu' = \left[\frac{1}{1 + \dfrac{h\nu}{m_e c^2}\left(1 - \cos\theta\right)}\right]\nu \tag{10}$$

or,

$$\boxed{E' = E\left[1 + \frac{E}{m_e c^2}\left(1 - \cos\theta\right)\right]^{-1}} \tag{11}$$

The kinetic energy of the electron is

$$T = \gamma m_e c^2 - m_e c^2 = h\nu - h\nu' = E\left[1 - \frac{1}{1 + \frac{E}{m_e c^2}\left(1 - \cos\theta\right)}\right]$$

$$\boxed{T = \frac{E^2}{m_e c^2}\left[\frac{1 - \cos\theta}{1 + \frac{E}{m_e c^2}\left(1 - \cos\theta\right)}\right]} \tag{12}$$