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PHYS 311
HW 3

Problem 1 $T=(x,y,z)=3z^2+4xz+yz^3$ $\qquad \vec{a}=(0,0,0)\ ,\ \vec{b}=(1,1,1)$

FTOG:

$$\int_a^b (\nabla \vec{T})\cdot d\vec{l} \equiv T(b)-T(a)$$

(a) $(0,0,0)\to(1,0,0)\to(1,1,0)\to(1,1,1)$

(i) $(0,0,0)\to(1,0,0)$

$$\vec{\nabla}T=[4z,z^3,6z+4x+3yz^2] \qquad d\vec{l}=dx\,\hat{x},\ \underset{dy=0}{dy\,\hat{y}},\ \underset{dz=0}{dz\,\hat{z}}$$

$$\int \vec{\nabla}T\cdot d\vec{l}=\int 4z\,dx\Big|_{z=0}=\int_0^1 0\,dx=0$$

$$T(b)-T(a)=[3(0)^2+4(1)(0)+0(0)^3]-[3(0)^2+4(0)(0)+0(0)^3]=0-0$$

$$T(b)-T(a)=0 \qquad \therefore \boxed{\int \vec{\nabla}T\cdot d\vec{l}=T(b)-T(a)}$$

(ii) $(1,0,0)\to(1,1,0)$

$$\vec{\nabla}=[4z,z^3,6z+4x+3yz^2] \qquad d\vec{l}=\underset{dx=0}{dx\,\hat{x}},\ dy\,\hat{y}\ ,\ \underset{dz=0}{dz\,\hat{z}}$$

$$\int \vec{\nabla}T\cdot d\vec{l}=\int z^3\,dy\Big|_{z=0}=\int 0\,dy=0$$

$$T(b)-T(a)=[3(0)^2+4(1)(0)+1(0)^3]-[3(0)^2+4(1)(0)+0(0)^3]$$

$$T(b)-T(a)=0 \qquad 0-0=0$$

$$\therefore \boxed{\int \vec{\nabla}T\cdot d\vec{l}\equiv T(b)-T(a)}$$

(iii) $(1,1,0)\to(1,1,1)$

$$\vec{\nabla}T=[4z,z^3,6z+4x+3yz^2] \qquad d\vec{l}=\underset{\substack{dx=0\\x=1}}{dx\,\hat{x}}+\underset{\substack{dy=0\\y=1}}{dy\,\hat{y}}+dz\,\hat{z}$$

$$\int \vec{\nabla}T\cdot d\vec{l}=\int_0^1 6z+4x+3yz^2\,dz\Big|_{x=1,y=1}\equiv\int_0^1 4+6z+3z^2\,dz=4z+3z^2+z^3\Big|_0^1$$

$$4(1)+3(1)^2+(1)^3-0=8 \qquad \int \vec{\nabla}T\cdot d\vec{l}=8$$

$$T(b)-T(a)=[3(1)^2+4(1)(1)+1(1)^3]-[0]=8 \quad : \quad T(b)-T(b)=8$$

$$\boxed{\text{Paths}: (i)+(ii)+(iii)=8} \qquad \therefore \boxed{\int \vec{\nabla}T\cdot d\vec{l}=T(b)-T(a)}$$

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

(i) $(0,0,0) \to (0,0,1)$

$dx = dy = 0$

$\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$

$\vec{\nabla} T = [4z, z^3, 6z + 4x + 3yz^2] \quad d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

$\int \vec{\nabla} T \cdot d\vec{l} = \int 3yz^2 \, dz \Big|_{y=0} = \int_0^1 0 \, dz = 0$

$T(b) - T(a) = [3(1)^2 + 4(0)(1) + (0)(1)^3] - [3(0)^2 + 4(0)(0) + (0)(0)^3] = 0$

$\boxed{\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)}$

(ii) $(0,0,1) \to (0,1,1)$

$dx = dz = 0$

$\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$

$\vec{\nabla} T = [4z, z^3, 6z + 4x + 3yz^2] \quad d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

$\int \vec{\nabla} T \cdot d\vec{l} = \int_0^1 z^3 \, dy \Big|_{z=1} = \int_0^1 dy = 1$

$T(b) - T(a) = [3(1)^2 + 4(0)(1) + 1(1)^3] - [3(1)^2 + 4(0)(1) + (0)(1)^3]$

$3 + 1 - 3 = 1$

$T(b) - T(a) = 1$

$\boxed{\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)}$

(iii) $(0,1,1) \to (1,1,1)$

$dy = dz = 0$

$\vec{\nabla} T = [4z, z^3, 6z + 4x + 3yz^2] \quad d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

$\int \vec{\nabla} T \cdot d\vec{l} = \int_0^1 4z \, dx \Big|_{z=1} = \int_0^1 4 \, dx = 4x\Big|_0^1 = 4$

$T(b) - T(a) = [3(1)^2 + 4(1)(1) + 1(1)^3] - [3(1)^2 + 4(0)(1) + 1(1)^3]$

$3 + 4 + 1 - 3 - 1 = 4$

$\boxed{\text{Paths (i)+(ii)+(iii)} = 4}$

$\boxed{\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)}$

(c) $z = x^2 \quad y = x \quad ; \; x = y \qquad T(x,y,z) = 3z^2 + 4xz + yz^3$

$(0,0,0) \to (y, x, x^2)$

$dy = dx \quad dz = 2x\,dx$

$$T(b) - T(a) = [3x^4 + 4x^3 + x^7] - [0] = x^7 + 3x^4 + 4x^3$$

$$T(b) - T(a) = x^7 + 3x^4 + 4x^3$$

$$\vec{\nabla} T = [4z, z^3, 6z + 4x + 3yz^2] \qquad d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\int \vec{\nabla} T \cdot d\vec{l} = \int 4z\,dx + z^3 dy + (6z + 4x + 3yz^2)\,dz$$

$$\int 4x^2 dx + x^6 dx + 2x(6x^2 + 4x + 3x^5)\,dx$$

$$\int_0^y 4x^2 + x^6 + 12x^3 + 8x^2 + 6x^6 \; dx$$

$$\int_0^y 7x^6 + 12x^3 + 12x^2 \; dx$$

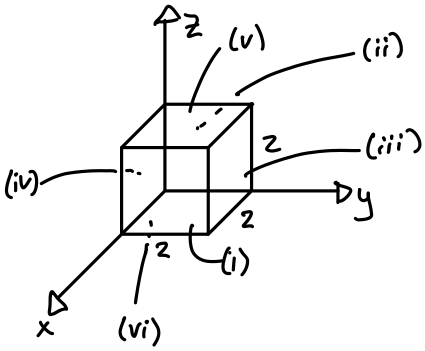
$$x^7 + 3x^4 + 4x^3 \Big|_0^y = y^7 + 3y^4 + 4y^3$$

$$y \equiv x \quad \therefore \quad y^7 + 3y^4 + 4y^3 \equiv x^7 + 3x^4 + 4x^3 = \int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

$$\boxed{\int \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)}$$

Problem 2]

$$\vec{v} = (y^2z^2)\hat{x} + (3z^2x^2)\hat{y} + (4x^2y^2)\hat{z}$$



$$\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

$$\vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

$$\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$$

$$\frac{\partial}{\partial x}(y^2z^2) = 0 \qquad \frac{\partial}{\partial y}(3z^2x^2) = 0 \qquad \frac{\partial}{\partial z}(4x^2y^2) = 0$$

$$\int_{\mathcal{V}} 0\, d\tau = 0$$

(i) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dy\,dz\,\hat{x}$

$$\oint_S \vec{v}\cdot d\vec{a} = \int_0^2\int_0^2 y^2z^2\,dy\,dz$$
$$= \int_0^2 \tfrac{1}{3}y^3\Big|_0^2 z^2\,dz$$
$$= \frac{8}{3}\int_0^2 z^2\,dz$$
$$= \frac{8}{3}\left[\frac{z^3}{3}\Big|_0^2\right] = \frac{8}{3}\left[\frac{8}{3}\right] = \frac{64}{9}$$

(ii) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dy\,dz\,-\hat{x}$

$$\oint_S \vec{v}\cdot d\vec{a} = -\int_0^2\int_0^2 y^2z^2\,dy\,dz$$
$$= -\int_0^2 z^2\,dz\left(\frac{y^3}{3}\right)\Big|_0^2$$
$$= -\frac{8}{3}\left[\frac{z^3}{3}\Big|_0^2\right] = -\frac{8}{3}\left(\frac{8}{3}\right) = -\frac{64}{9}$$

(iii) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dx\,dz\,\hat{y}$

$$\oint_S \vec{v}\cdot d\vec{a} = \int_0^2\int_0^2 3z^2x^2\,dx\,dz = 3\int_0^2\int_0^2 x^2z^2\,dx\,dz$$
$$= 3\int_0^2 z^2\,dz\left[\frac{x^3}{3}\Big|_0^2\right]$$
$$= 3\left(\frac{8}{3}\right)\left[\frac{z^3}{3}\Big|_0^2\right] = 8\left[8/3\right] = 64/3$$

(iv) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dx\,dz\,-\hat{y}$

$$\oint_S \vec{v}\cdot d\vec{a} = -3\int_0^2\int_0^2 z^2x^2\,dx\,dz$$
$$= -3\int_0^2 z^2\,dz\left[\frac{x^3}{3}\Big|_0^2\right]$$
$$= -3\left(\frac{8}{3}\right)\left[\frac{z^3}{3}\Big|_0^2\right] = -8\left[8/3\right] = -64/3$$

(v) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dx\,dy\,\hat{z}$ $\quad \oint_S \vec{v}\cdot d\vec{a} = 4\int_0^2\int_0^2 x^2y^2\,dx\,dy$

$$4\int_0^2 y^2\,dy\left[\frac{x^3}{3}\Big|_0^2\right]$$

$$4\left(\frac{8}{3}\right)\left[\frac{y^3}{3}\Big|_0^2\right] = 4\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = \frac{256}{9}$$

(vi) $\vec{v} = [y^2z^2, 3z^2x^2, 4x^2y^2]$

$d\vec{a} = dx\,dy\,-\hat{z}$ $\quad \oint_S \vec{v}\cdot d\vec{a} = -4\int_0^2\int_0^2 x^2y^2\,dx\,dy$

$$-4\int_0^2 y^2\,dy\left[\frac{x^3}{3}\Big|_0^2\right]$$

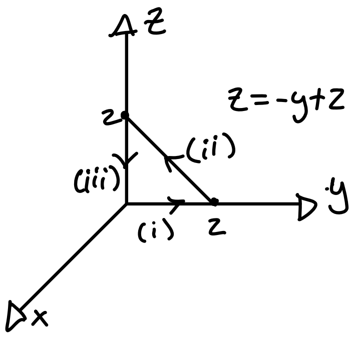
$$-4\left(\frac{8}{3}\right)\left(\frac{y^3}{3}\Big|_0^2\right) = -4\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = -\frac{256}{9}$$

(i) + (ii) + (iii) + (iv) + (v) + (vi)

$\frac{64}{9} - \frac{64}{9} + \frac{64}{3} - \frac{64}{3} + \frac{256}{9} - \frac{256}{9} = 0 \quad \vdots \quad \therefore \quad \boxed{\int_V (\vec{\nabla}\cdot\vec{v})\,d\tau = \oint_S \vec{v}\cdot d\vec{a}}$

Problem 3

$$\vec{v} = (y^2z^2)\hat{x} + (3z^2x^2)\hat{y} + (4x^2y^2)\hat{z}$$



$z = -y + 2 \;:\; y = 2 - z$

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$$

$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$: $\quad d\vec{a} = dy\,dz\,\hat{x}$

$$\vec{\nabla}\times\vec{v} : \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^2 & 3z^2x^2 & 4x^2y^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(4x^2y^2) - \frac{\partial}{\partial z}(3z^2x^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(y^2z^2) - \frac{\partial}{\partial x}(4x^2y^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(3z^2x^2) - \frac{\partial}{\partial y}(y^2z^2)\right)$$

$$= \hat{x}(8x^2y - 6zx^2) + \hat{y}(2y^2z - 8xy^2) + \hat{z}(6z^2x - 2yz^2)$$

$$(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 8x^2y - 6zx^2\,dy\,dz$$

$$\int_S (\nabla\times v)\cdot d\vec{a} = \int_0^2\int_0^{2-z} 8x^2y - 6zx^2\,dy\,dz\,\Big|_{x=0} = \int_0^2\int_0^{2-z} 0\,dy\,dz = 0$$

(i) $(0,0,0) \to (0,2,0)$

$dx = dz = 0 \qquad d\vec{l} = dy\,\hat{y}$

$$\oint_{(i)} \vec{v}\cdot d\vec{l} = \int_0^2 3z^2x^2\,dy\,\Big|_{x=0,\,z=0} = \int_0^2 0\,dy = 0$$

(ii) $(0,2,0) \to (0,0,2)$

$dx = 0 \qquad d\vec{l} = dy\,\hat{y} + dz\,\hat{z}$

$$\oint_{(ii)} \vec{v}\cdot d\vec{l} = \int_{(ii)} 3z^2x^2\,dy + 4x^2y^2\,dz\,\Big|_{x=0} = \int_{(ii)} 0 = 0$$

(iii) $(0,0,2) \to (0,0,0)$

$dx = dy = 0 \qquad d\vec{l} = dz\,\hat{z}$

$$\oint_{(iii)} \vec{v}\cdot d\vec{l} = \int_{(iii)} 4x^2y^2\,dz\,\Big|_{x=0,\,y=0} = \int_{(iii)} 0 = 0$$

$$(i) + (ii) + (iii) = 0 + 0 + 0 = \int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$$

$\therefore$

$$\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}}$$

Problem 4) $\vec{v} = (r^2\sin\theta)\,\hat{r} + (r^2\cos\theta)\,\hat{\theta} + (r^2\sin\phi\cos\theta)\,\hat{\phi}$

a.) $$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

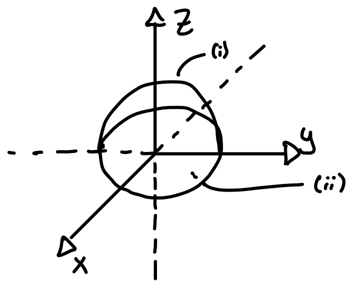
$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r^2\sin\phi\cos\theta)$$

$$= \frac{1}{r^2}\left[4r^3\sin\theta\right] + \frac{r^2}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\cos^2\theta - \sin^2\theta\right] + \frac{r^2\cos\theta}{r\sin\theta}\left[\cos\phi\right]$$

$$= 4r\sin\theta + \frac{r}{\sin\theta}\left[\cos(2\theta)\right] + \frac{r\cos\theta}{\sin\theta}\cos\phi$$

$$\boxed{\vec{\nabla}\cdot\vec{v} = 4r\sin\theta + \frac{r\cos 2\theta}{\sin\theta} + r\cot\theta\cos\phi}$$

b.) $$\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a} \qquad d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi$$



$$\int_V \left(4r\sin\theta + \frac{r\cos(2\theta)}{\sin\theta} + r\frac{\cos\theta}{\sin\theta}\cos\phi\right) d\tau$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R 4r^3\sin^2\theta + r^3\cos(2\theta) + r^3\cos\theta\cos\phi \; dr\, d\theta\, d\phi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[r^4\sin^2\theta + \frac{r^4}{4}\cos(2\theta) + \frac{r^4}{4}\cos\theta\cos\phi\right]_0^R d\theta\, d\phi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} R^4\sin^2\theta + R^4/4\cos(2\theta) + R^4/4\cos\theta\cos\phi \; d\theta\, d\phi$$

$$\int_0^{2\pi}\left[R^4\int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta + R^4/4\int_0^{\frac{\pi}{2}}\cos(2\theta)\, d\theta + R^4/4\cos\phi\int_0^{\pi/2}\cos\theta\, d\theta\right] d\phi$$

$$\int_0^{2\pi}\left[\frac{R^4}{2}\int_0^{\frac{\pi}{2}} 1-\cos(2\theta)\, d\theta + \frac{R^4}{4}\left[\frac{\sin(2\theta)}{2}\Big|_0^{\frac{\pi}{2}}\right] + R^4/4\cos\phi\left[\sin\theta\Big|_0^{\frac{\pi}{2}}\right]\right] d\phi$$

$$\int_0^{2\pi}\left[\frac{R^4}{2}\left(\theta - \frac{\sin(2\theta)}{2}\Big|_0^{\frac{\pi}{2}}\right) + \frac{R^4}{4}(0) + \frac{R^4}{4}\cos\phi\,(1)\right] d\phi$$

$$\int_0^{2\pi}\left(\frac{R^4}{2}\left(\frac{\pi}{2}\right)\right) + \frac{R^4}{4}\cos\phi\, d\phi = \int_0^{2\pi}\frac{\pi R^4}{4} + \frac{R^4}{4}\cos\phi\, d\phi$$

$$\frac{R^4}{4}\int_0^{2\pi}\pi + \cos\phi\, d\phi = \frac{R^4}{4}\left[\pi\phi + \sin\phi\Big|_0^{2\pi}\right] = \frac{R^4}{4}\left[2\pi^2\right] = \boxed{\frac{R^4\pi^2}{2}}$$

$$\oint_{(i)}\vec{v}\cdot d\vec{a} = [r^2\sin\theta,\ r^2\cos\theta,\ r^2\sin\phi\cos\theta]\cdot[r^2\sin\theta, 0, 0]\, d\phi\, d\theta$$

$d\vec{a} = r^2\sin\theta\, d\phi\, d\theta\,\hat{r}$

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^4\sin^2\theta\, d\phi\, d\theta \qquad \sin^2\theta = \tfrac{1}{2}(1-\cos(2\theta))$$

$$r^4\int_0^{\pi/2}\int_0^{2\pi}\tfrac{1}{2}(1-\cos(2\theta))\, d\phi\, d\theta$$

$$\frac{r^4}{2}\int_0^{\pi/2}\phi(1-\cos(2\theta))\Big|_0^{2\pi} d\theta$$

$$\frac{r^4}{2}\int_0^{\pi/2} 2\pi(1-\cos(2\theta))\, d\theta$$

$$r^4\pi\left[\theta - \sin(2\theta)/2\right]\Big|_0^{\pi/2} = r^4\pi\left[\left(\tfrac{\pi}{2}-0\right) - (0-0)\right] = \frac{\pi^2 r^4}{2}$$

$$\boxed{\oint_{(i)}\vec{v}\cdot d\vec{a} = \frac{\pi^2 r^4}{2}}$$

$$\oint_{(i)} \vec{v}\cdot d\vec{a}\Big|_{\theta=\pi/2} \therefore \quad \vec{v} = [r^2, 0, 0], \quad d\vec{a} = -\hat{z}\, r\,dr\,d\phi = -\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\; r\,dr\,d\phi$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\oint_{(i)} \vec{v}\cdot d\vec{a}\Big|_{\theta=\pi/2} = \oint_{(i)} r^3\cos\theta\; dr\,d\phi\Big|_{\theta=\pi/2} = \oint_{(i)} 0\; dr\,d\phi = 0 \;\therefore\; \boxed{\oint_{(i)} \vec{v}\cdot d\vec{a} = 0}$$

$$\boxed{\oint_{(i)} \vec{v}\cdot d\vec{a} + \oint_{(ii)} \vec{v}\cdot d\vec{a} = \frac{\pi^2 r^4}{2} = \int_V (\vec{\nabla}\cdot\vec{v})\, dv}$$

Problem 5

a.) Divergence = $\vec{\nabla}\cdot\vec{v}$ $\quad \vec{v}=s^2(3+\sin^2\phi)\hat{s} + s^2\sin\phi\cos\phi\,\hat{\phi} + 5z^2\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$s v_s = s^3(3+\sin^2\phi)$

$\frac{\partial}{\partial s} s^3(3+\sin^2\phi) = 3s^2(3+\sin^2\phi)$

$\frac{1}{s}\cdot 3s^2(3+\sin^2\phi) = 3s(3+\sin^2\phi)$

$3s(3+\frac{1}{2}-\frac{1}{2}\cos(2\phi)) = 3s(\frac{7}{2}-\frac{1}{2}\cos(2\phi)) = \frac{21}{2}s - \frac{3}{2}s\cos(2\phi)$

$\frac{\partial}{\partial\phi}(s^2\sin\phi\cos\phi) = s^2(\cos^2\phi-\sin^2\phi) = s^2\cos(2\phi)$

$\frac{1}{s}\cdot s^2\cos(2\phi) = s\cos(2\phi)$

$\sin^2\phi = \frac{1}{2}(1-\cos(2\phi))$

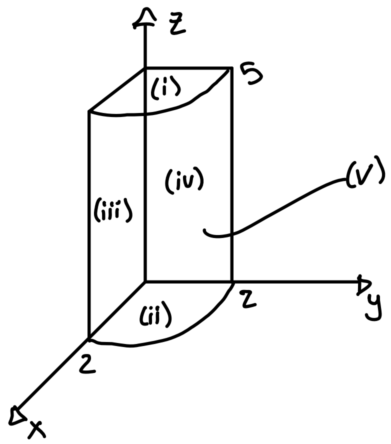
$\cos^2\phi-\sin^2\phi = \cos(2\phi)$

$\frac{\partial}{\partial z}(5z^2) = 10z$

$$\vec{\nabla}\cdot\vec{v} = \frac{21}{2}s - \frac{3}{2}s\cos(2\phi) + s\cos(2\phi) + 10z = \frac{21}{2}s - \frac{1}{2}s\cos(2\phi) + 10z$$

$$\boxed{\nabla\cdot\vec{v} = \frac{1}{2}s(21-\cos(2\phi)) + 10z}$$

b.) $\int_v(\vec{\nabla}\cdot\vec{v})\,d\tau = \oint_s \vec{v}\cdot d\vec{a}$



$\int_v(\vec{\nabla}\cdot\vec{v})\,d\tau$: $(\nabla\cdot\vec{v}) = \frac{s(21-\cos(2\phi))}{2} + 10z$

$d\tau = s\,ds\,d\phi\,dz$

$$\int_0^5\int_0^{\pi/2}\int_0^2 \frac{s^2(21-\cos(2\phi))}{2} + 10zs\,ds\,d\phi\,dz$$

$$\int_0^5\int_0^{\pi/2}\left[\frac{s^3(21-\cos(2\phi))}{6} + 5zs^2\right]_0^2 d\phi\,dz$$

$$\int_0^5\int_0^{\pi/2} 28 - 4/3(\cos(2\phi)) + 20z\,d\phi\,dz$$

$$\int_0^5\left[28\phi - 4/6\sin(2\phi) + 20z\phi\right]_0^{\pi/2} dz$$

$$\int_0^5 (14\pi - 0 + 10\pi z) - (0-0+0)\,dz$$

$$\int_0^5 14\pi + 10\pi z\,dz = 14\pi z + 5\pi z^2\Big|_0^5$$

$70\pi + 125\pi = 195\pi\ m^3$

$$\boxed{\int_v(\nabla\cdot v)\,d\tau = 195\pi\ m^3}$$

$\oint_{(i)}\vec{v}\cdot d\vec{a}\Big|_{z=5}$ $\quad \vec{v} = s^2(3+\sin^2\phi)\hat{s} + s^2\sin\phi\cos\phi\,\hat{\phi} + 125\hat{z}$

$d\vec{a} = s\,ds\,d\phi\,\hat{z}$

$$\oint_{(i)}\vec{v}\cdot d\vec{a} = \oint_{(i)}[s^2(3+\sin^2\phi), s^2\sin\phi\cos\phi, 125]\cdot[0,0,1]\,s\,ds\,d\phi$$

$$\int_0^{\pi/2}\int_0^2 125s\,ds\,d\phi = \frac{1}{2}\int_0^{\pi/2}(125)s^2\Big|_0^2 d\phi = \frac{1}{2}\int_0^{\pi/2} 500\,d\phi$$

$$250\int_0^{\pi/2} d\phi = 250\left(\phi\Big|_0^{\pi/2}\right) = 125\pi$$

$$\oint_{(i)}\vec{v}\cdot d\vec{a} = 125\pi$$

$$\oint_{(ii)} \vec{v}\cdot d\vec{a}\Big|_{z=0} \quad \vec{v} = s^2(3+\sin^2\phi)\,\hat{s} + s^2\sin\phi\cos\phi\,\hat{\phi} + 0\,\hat{z}$$
$$d\vec{a} = -s\,ds\,d\phi\,\hat{z}$$

$$\oint_{(ii)} \vec{v}\cdot d\vec{a} = \oint_{(ii)} [s^2(3+\sin^2\phi), s^2\sin\phi\cos\phi, 0]\cdot[0,0,-1]\,s\,ds\,dz = \oint_{(ii)} 0\,s\,ds\,dz = 0$$

$$\oint_{(ii)} \vec{v}\cdot d\vec{a} = 0$$

$$\oint_{(iii)} \vec{v}\cdot d\vec{a}\Big|_{\phi=0} \quad \vec{v} = 3s^2\hat{s} + 0\hat{\phi} + 5z^2\hat{z}$$
$$d\vec{a} = -ds\,dz\,\hat{y}$$
$$\hat{y} = \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}$$
$$d\vec{a} = -\sin\phi\,\hat{s} - \cos\phi\,\hat{\phi}\,ds\,dz\Big|_{\phi=0}$$
$$d\vec{a} = -\hat{\phi}\,ds\,dz$$

$$\oint_{(iii)} \vec{v}\cdot d\vec{a} = [3s^2, 0, 5z^2]\cdot[0,-1,0]\,ds\,dz = \oint_{(iii)} 0\,ds\,dz = 0$$

$$\oint_{(iii)} \vec{v}\cdot d\vec{a} = 0$$

$$\oint_{(iv)} \vec{v}\cdot d\vec{a}\Big|_{\phi=\pi/2} \quad \vec{v} = 4s^2\hat{s} + 0\hat{\phi} + 5z^2\hat{z}$$
$$d\vec{a} = -ds\,dz\,\hat{x}$$
$$\hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}$$
$$d\vec{a} = -\cos\phi\,\hat{s} + \sin\phi\,\hat{\phi}\,ds\,dz\Big|_{\phi=\pi/2}$$
$$d\vec{a} = \hat{\phi}\,ds\,dz$$

$$\oint_{(iv)} \vec{v}\cdot d\vec{a} = [4s^2, 0, 5z^2]\cdot[0,1,0]\,ds\,dz = \oint_{(iv)} 0\,ds\,dz = 0$$

$$\oint_{(iv)} \vec{v}\cdot d\vec{a} = 0$$

$$\oint_{(v)} \vec{v}\cdot d\vec{a}\Big|_{s=2} \quad \vec{v} = 4(3+\sin^2\phi)\hat{s} + 4\sin\phi\cos\phi\,\hat{\phi} + 5z^2\hat{z}$$
$$d\vec{a} = s\,d\phi\,dz\,\hat{s}\Big|_{s=2} \quad d\vec{a} = 2\,d\phi\,dz$$

$$\oint_{(v)} \vec{v}\cdot d\vec{a} = [4(3+\sin^2\phi), 4\sin\phi\cos\phi, 5z^2]\cdot[1,0,0]\,2\,d\phi\,dz = 8\oint_{(v)} 3+\sin^2\phi\,d\phi\,dz$$

$$8\int_0^5\int_0^{\frac{\pi}{2}} \tfrac{7}{2} - \tfrac{1}{2}\cos(2\phi)\,d\phi\,dz = 8\int_0^5\left[\tfrac{7}{2}\phi - \frac{\sin(2\phi)}{4}\right]_0^{\frac{\pi}{2}} dz = 8\int_0^5 \tfrac{7}{4}\pi\,dz$$
$$14\pi\int_0^5 dz = 14\pi\cdot z\Big|_0^5 = 70\pi$$

$$\boxed{\oint_{(i)} \vec{v}\cdot d\vec{a} + \oint_{(ii)} \vec{v}\cdot d\vec{a} + \oint_{(iii)} \vec{v}\cdot d\vec{a} + \oint_{(iv)} \vec{v}\cdot d\vec{a} + \oint_{(v)} \vec{v}\cdot d\vec{a} = 195\pi = \int_V (\vec{\nabla}\cdot\vec{v})\,d\tau}$$

Part C next page

c.) curl of $\vec{v} = s^2(3+\sin^2\phi)\,\hat{s} + s^2\sin\phi\cos\phi\,\hat{\phi} + 5z^2\hat{z}$

$$\vec{\nabla}\times\vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

$\hat{s}$: $\frac{\partial v_z}{\partial\phi} = \frac{\partial}{\partial\phi}(5z^2) = 0$

$\frac{\partial v_\phi}{\partial z} = \frac{\partial}{\partial z}(s^3\sin\phi\cos\phi) = 0$

$\hat{\phi}$: $\frac{\partial v_s}{\partial z} = \frac{\partial}{\partial z}(s^2(3+\sin^2\phi)) = 0$

$\frac{\partial v_z}{\partial s} = \frac{\partial}{\partial s}(5z^2) = 0$

$\hat{z}$: $\frac{\partial}{\partial s}(s^3\sin\phi\cos\phi) = 3s^2\sin\phi\cos\phi$

$\frac{\partial}{\partial\phi}(s^2(3+\sin^2\phi)) = 2s^2\sin(\phi)\cos(\phi)$

$$\vec{\nabla}\times\vec{v} = \left[\tfrac{1}{s}(0) - 0\right]\hat{s} + \left[0-0\right]\hat{\phi} + \tfrac{1}{s}\left[3s^2\sin\phi\cos\phi - 2s^2\sin\phi\cos\phi\right]\hat{z}$$

$$\vec{\nabla}\times\vec{v} = 0\hat{s} + 0\hat{\phi} + \tfrac{1}{s}\left[s^2\sin\phi\cos\phi\right]\hat{z}$$

$$\boxed{\vec{\nabla}\times\vec{v} = 0\hat{s} + 0\hat{\phi} + s\cdot\sin\phi\cos\phi\,\hat{z}}$$

Problem 6]

$$\vec{F}_1 = y^2\hat{x} + z^2\hat{y} + x^2\hat{z}$$
$$\vec{F}_2 = x^2\hat{x} + y^2\hat{y} + z^2\hat{z}$$

(a)

$$\vec{\nabla}\cdot\vec{F} = \left[\frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z\right]$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) + \hat{y}\left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right) + \hat{z}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

$$\vec{F}_1 = [y^2, z^2, x^2]$$

$$\vec{\nabla}\cdot\vec{F}_1 = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(x^2) = 0$$
$$\vec{\nabla}\times\vec{F}_1 = \hat{x}\left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(z^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(y^2) - \frac{\partial}{\partial x}(x^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial y}(y^2)\right)$$
$$= -2z\hat{x} - 2x\hat{y} - 2y\hat{z}$$

$$\boxed{\vec{\nabla}\cdot\vec{F}_1 = 0 \quad , \quad \vec{\nabla}\times\vec{F}_1 = -2z\hat{x} - 2x\hat{y} - 2y\hat{z}}$$

$$\vec{F}_2 = [x^2, y^2, z^2]$$

$$\vec{\nabla}\cdot\vec{F}_2 = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z = 2(x+y+z)$$
$$\vec{\nabla}\times\vec{F}_2 = \hat{x}\left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right)$$
$$= 0\hat{x} + 0\hat{y} + 0\hat{z} = 0$$

$$\boxed{\vec{\nabla}\cdot\vec{F}_2 = 2(x+y+z) \quad , \quad \vec{\nabla}\times\vec{F}_2 = \vec{0}}$$

Gradient of Scalar : $\to \vec{F}_2$

$$V(x,y,z) = \left(-\frac{x^3}{3} - \frac{y^3}{3} - \frac{z^3}{3}\right) : -\vec{\nabla}V = -\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\left(-\frac{x^3}{3} - \frac{y^3}{3} - \frac{z^3}{3}\right) = -\left[-x^2\hat{i} - y^2\hat{j} - z^2\hat{k}\right]$$
$$-\vec{\nabla}V = [x^2\hat{i} + y^2\hat{j} + z^2\hat{k}] = \vec{F}_2 \quad \checkmark$$

$$\therefore \boxed{V = \left[-\frac{x^3}{3} - \frac{y^3}{3} - \frac{z^3}{3}\right]}$$

Curl of a Vector : $\to \vec{F}_1$

$$(\vec{\nabla}\times\vec{F}) = \hat{x}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) + \hat{y}\left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right) + \hat{z}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

$y^2 \qquad z^2 \qquad x^2$

$$F_z = \frac{y^3}{3} \qquad F_x = \frac{z^3}{3} \qquad F_y = \frac{x^3}{3}$$

$$\frac{\partial}{\partial y}F_z = y^2 \qquad \frac{\partial}{\partial z}F_y = 0 \qquad \frac{\partial}{\partial z}F_x = z^2 \quad \frac{\partial}{\partial x}F_z = 0 \qquad \frac{\partial}{\partial x}F_y = x^2 \quad \frac{\partial}{\partial y}F_x = 0$$

$$(\vec{\nabla}\times\vec{F}) = \hat{x}(y^2 - 0) + \hat{y}(z^2 - 0) + \hat{z}(x^2 - 0) = \hat{x}(y^2) + \hat{y}(z^2) + \hat{z}(x^2) = \vec{F}_1 \quad \checkmark$$

$$\therefore \boxed{\vec{v} = (y^2)\hat{x} + (z^2)\hat{y} + (x^2)\hat{z}}$$

(b) $\vec{F}_3 = (2yz)\hat{x} + (2zx)\hat{y} + (2xy)\hat{z}$

$$\vec{\nabla} \times \vec{F}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & 2zx & 2xy \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(2xy) - \frac{\partial}{\partial z}(2zx)\right) + \hat{y}\left(\frac{\partial}{\partial z}(2yz) - \frac{\partial}{\partial x}(2xy)\right) + \hat{z}\left(\frac{\partial}{\partial x}(2zx) - \frac{\partial}{\partial y}(2yz)\right)$$

$$= \hat{x}(2x - 2x) + \hat{y}(2y - 2y) + \hat{z}(2z - 2z)$$

$$= \hat{x}(0) + \hat{y}(0) + \hat{z}(0) = 0$$

$\vec{\nabla} \times \vec{F}_3 = 0$ ✓

$$\vec{\nabla} \cdot \vec{F}_3 = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] = \frac{\partial}{\partial x}(2yz) + \frac{\partial}{\partial y}(2zx) + \frac{\partial}{\partial z}(2xy)$$

$$= 0 + 0 + 0$$

$$= 0$$

$\vec{\nabla} \cdot \vec{F}_3 = 0$ ✓

<u>Gradient of a Scalar</u>

$-\vec{\nabla} V$ must $= [2yz, 2zx, 2xy]$

$V(x,y,z) = -2xyz$

$\frac{\partial V}{\partial x} = -2yz : \frac{\partial V}{\partial y} = -2xz : \frac{\partial V}{\partial z} = -2xy$

$V(x,y,z) : -2xyz$

$$-\vec{\nabla} V = -\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](-2xyz)$$

$$= -[-2yz, -2xz, -2xy] = [2yz, 2xz, 2xy] = \vec{F}_3 \quad ✓$$

$\therefore$ $\boxed{V(x,y,z) = -2xyz}$

<u>Curl of a vector</u> $(\vec{\nabla} \times \vec{F}) = \hat{x}\left(\underbrace{\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y}_{2yz}\right) + \hat{y}\left(\underbrace{\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z}_{2xz}\right) + \hat{z}\left(\underbrace{\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x}_{2xy}\right)$

$F_z = y^2 z$	$F_x = xz^2$	$F_y = x^2 y$
$\frac{\partial}{\partial y}(F_z) = 2yz$	$\frac{\partial}{\partial z}(F_x) = 2xz$	$\frac{\partial}{\partial x}(F_y) = 2xy$
$\frac{\partial}{\partial z}(F_y) = 0$	$\frac{\partial}{\partial x}(F_z) = 0$	$\frac{\partial}{\partial y}(F_x) = 0$

$$(\vec{\nabla} \times \vec{F}) = \hat{x}(2yz - 0) + \hat{y}(2xz - 0) + \hat{z}(2xy - 0) = [2yz, 2xz, 2xy] = \vec{F}_3 \quad ✓$$

$\therefore$ $\boxed{\vec{v} = (xz^2)\hat{x} + (x^2 y)\hat{y} + (y^2 z)\hat{z}}$