MAT 201
Larson/Edwards – Section 3.5
Limits at Infinity

Up to this point, when given a function, we have discussed ways to find the functions vertical asymptotes (if any), relative extrema (if any), intervals of increase or decrease, concavity, and inflection points. All of these can be used to help us graph the function. What we haven't discussed yet is the end-behavior of the graph. That is, how does the graph behave when x approaches very large numbers both positive and negative? In order for us to determine end-behavior we must understand the notion of limits at infinity.

<u>Limit At Infinity</u>: The limit at infinity specifies a finite value approached by a function as x increases (or decreases) without bound.

- $\lim\limits_{x \to \infty} \dfrac{c}{x^r} = 0, \quad r > 0,\ c$ is a real constant
- $\lim\limits_{x \to -\infty} \dfrac{c}{x^r} = 0, \quad r > 0,\ c$ is a real constant

Note: for the "formal" definition of limits at infinity, see page 198 in your text.

Ex: Find the limit for the following:

a) $\lim_{x \to \infty} \frac{7}{1 - x}$

b) $\lim_{x \to -\infty} \frac{5x^2 - 2}{x^2 - x - 2}$

c) $\lim_{x \to \infty} \frac{x^2 + 4x + 3}{x^2 - 9}$

c) $\lim_{x \to \infty} \frac{2x^2 - x + 1}{x + 3}$

<u>Tools for Finding Limits at ±∞ of Rational Functions</u>:

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function at ±∞ is 0.
2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function at ±∞ is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function at ±∞ does not exist.

Ex: Find the limit for the following:

a) $\lim_{x \to \infty} \frac{7}{1-x} = \lim_{x \to \infty} \frac{\frac{7}{x}}{\frac{1}{x} - \frac{x}{x}}$

H.A.

$y=0$

$= \lim_{x \to \infty} \frac{\cancel{\frac{7}{x}} \to 0}{\cancel{\frac{1}{x}} \to 0 - 1} = \frac{0}{0-1} = \frac{0}{-1} = 0$

b) $\lim_{x \to -\infty} \frac{5x^2 - 2}{x^2 - x - 2} = \lim_{x \to -\infty} \frac{\frac{5\cancel{x^2}}{\cancel{x^2}} - \frac{2}{x^2}}{\frac{1\cancel{x^2}}{1\cancel{x^2}} - \frac{x}{x^2} - \frac{2}{x^2}}$

H.A.

$y=5$

$= \lim_{x \to -\infty} \frac{5 - \cancel{\frac{2}{x^2}} \to 0}{1 - \cancel{\frac{1}{x}} \to 0 - \cancel{\frac{2}{x^2}} \to 0} = \frac{5}{1} = \boxed{5}$

c) $\lim_{x \to \infty} \frac{x^2 + 4x + 3}{x^2 - 9} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{4x}{x^2} \to 0 + \frac{3}{x^2} \to 0}{\frac{x^2}{x^2} - \frac{9}{x^2} \to 0}$

H.A.
$y = 1$

$= 1$

d) $\lim_{x \to \infty} \frac{2x^2 - x + 1}{x + 3} = \lim_{x \to \infty} \frac{\frac{2x^2}{x^2} \to 2 - \frac{x}{x^2} \to 0 + \frac{1}{x^2} \to 0}{\frac{x}{x^2} \to 0 + \frac{3}{x^2} \to 0}$

No H.A.
It does have a slant (or oblique) asymptote.

D.N.E. $\boxed{\infty}$

<u>Infinite Limit</u>: The type of limit in which *f(x)* approaches infinity (or negative infinity) as *x* approaches *c* from the left or from the right is an infinite limit.

<u>Horizontal Asymptotes</u>: If *f* is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x \to \infty} f(x) = L_1 \quad and \quad \lim_{x \to -\infty} f(x) = L_2$$

Denote limits at infinity. The lines $y = L_1$ and $y = L_2$ are horizontal asymptotes of the graph of *f*. Note that this definition strays a little from the text, but it supports the fact that the function *f* can have at most two horizontal asymptotes – one to the right and one to the left.

Ex: What would the horizontal asymptotes be for each of the previous examples (a) – (d)?

Ex: Find the following limits:

a) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x - 1}$ b) $\lim_{x \to -\infty} \frac{\sqrt{x^2 - 1}}{2x - 1}$

Ex: Find the following limits:

a) $\lim_{x \to \infty} \frac{\sqrt{x^2 - 1}}{2x - 1} = \lim_{x \to \infty} \frac{\frac{\sqrt{x^2-1}}{\sqrt{x^2}}}{\frac{2x-1}{x}}$

Since $x \to \infty$ $x > 0$

$x = \sqrt{x^2}$

$= \lim_{x \to \infty} \frac{\sqrt{1 - \cancel{\frac{1}{x^2}} \to 0}}{2 - \cancel{\frac{1}{x}} \to 0} = \frac{\sqrt{1}}{2} = \frac{1}{2}$

H.A.

$y = \frac{1}{2}$

b) $\lim_{x \to -\infty} \frac{\sqrt{x^2 - 1}}{2x - 1} = \lim_{x \to -\infty} \frac{\frac{\sqrt{x^2-1}}{-\sqrt{x^2}}}{\frac{2x-1}{x}}$

$x \to -\infty$ $x < 0$

$x = -\sqrt{x^2}$

$= \lim_{x \to -\infty} \frac{-\sqrt{1 - \cancel{\frac{1}{x^2}} \to 0}}{2 - \cancel{\frac{1}{x}} \to 0} = \frac{-\sqrt{1}}{2} = \frac{-1}{2}$

H.A.

$y = \frac{-1}{2}$

What do problems (a) and (b) above indicate in terms of the horizontal asymptotes of the function $f(x) = \dfrac{\sqrt{x^2 - 1}}{2x - 1}$?

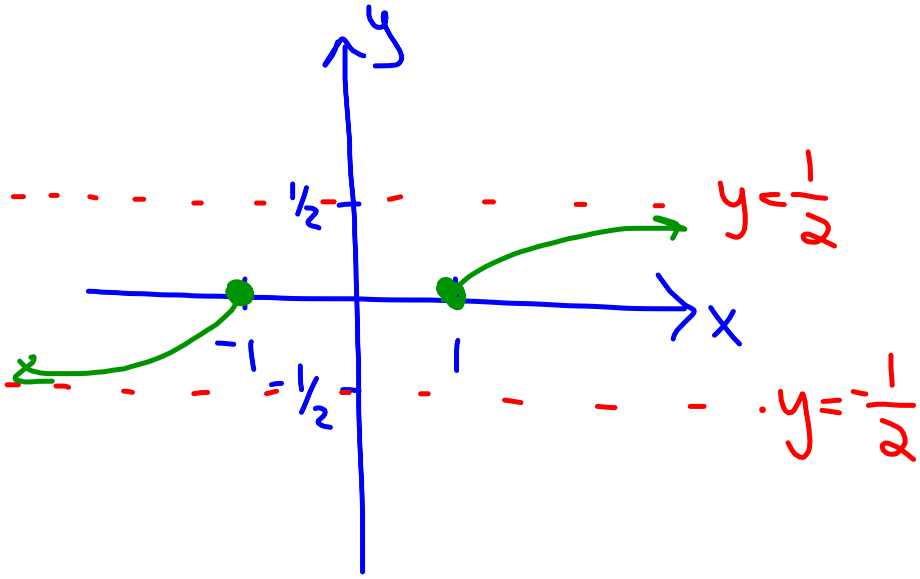
Ex: Find the following limits:

a) $\lim\limits_{x \to \infty} \cos x$

b) $\lim\limits_{x \to -\infty} \dfrac{\cos x}{x}$

c) $\lim\limits_{x \to \infty} \dfrac{1}{2x + \sin x}$

$$y = \frac{\sqrt{x^2 - 1}}{2x - 1}$$



What do problems (a) and (b) above indicate in terms of the horizontal asymptotes of the function

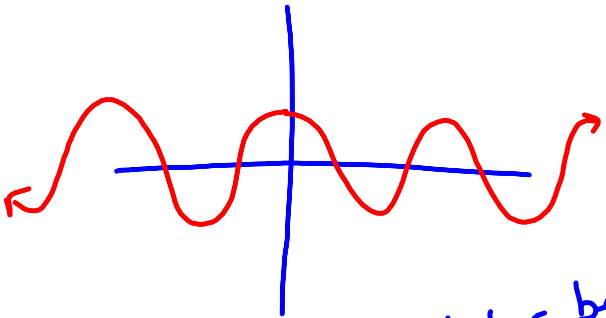
$f(x) = \dfrac{\sqrt{x^2 - 1}}{2x - 1}$? Two H.A.'s.

H.A.: $y = \frac{1}{2}$

$y = \frac{-1}{2}$

Ex: Find the following limits:

a) $\lim_{x \to \infty} \cos x$ D.N.E.



Oscillates between 1 & -1

b) $\lim_{x \to -\infty} \frac{\cos x}{x} = \boxed{0}$

$-\infty$

c) $\lim_{x \to \infty} \frac{1}{2x + \sin x}$ $= \lim_{x \to \infty} \frac{\frac{1}{x} \to 0}{\cancel{\frac{2x}{x}} + \frac{\sin x}{x} \to 0} = \frac{0}{2+0} = \boxed{0}$