MAT 202
Larson – Section 8.7
Indeterminate Forms and L'Hôpital's Rule

We will now briefly shift our focus to limits. The next section will ask us to evaluate integrals that require a technique that utilizes limits, so it is important that we understand how to use ***L'Hopital's rule***.

Recall in Calculus I we first learned how to manage limits that, after direct substitution, simplified to ***indeterminate forms***. For example:

$$\lim_{x \to -1} \frac{2x^2 - 2}{x + 1} = \frac{0}{0} \quad \text{and} \quad \lim_{x \to \infty} \frac{3x^2 - 1}{2x^2 + 1} = \frac{\infty}{\infty}$$

In each case we were able to apply an algebraic technique to evaluate. Not all indeterminate forms, however, can be evaluated by algebraic manipulation. This is often true when both algebraic and transcendental functions are involved:

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \frac{0}{0} \quad \text{or by rewriting,} \quad \lim_{x \to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x} \right) = \infty - \infty$$

Either way, we get indeterminate forms in which we cannot apply algebraic manipulation. We need another tool to help us solve this problem!

<u>L'Hôpital's Rule</u>:
Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\dfrac{0}{0}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c produces any one of the indeterminate forms $\dfrac{\infty}{\infty}, \dfrac{-\infty}{\infty}, \dfrac{\infty}{-\infty}, \dfrac{-\infty}{-\infty}$.
L'Hopital's rule may also be applied on ***limits at infinity***:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. The proof of this utilizes "**The Extended Mean Value Theorem**" given on page 558 of your text.

What L'Hopital's rule says is that the limit of a quotient is the same as the limit of the quotient of their respective derivatives, provided the limit on the right exists.

Note: L'Hopital's rule involves the derivative of *f(x)* and *g(x)* separately, not the derivative of *f(x)/g(x)* using the quotient rule.

Ex: Find the following limits (i) by algebraic manipulation, and (ii) by using L'Hopital's rule:

a) $\lim_{x \to -1} \frac{2x^2 - 2}{x + 1}$

b) $\lim_{x \to \infty} \frac{3x^2 - 1}{2x^2 + 1}$

Ex: Evaluate the following:

a) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$

b) $\lim_{x \to \infty} \frac{\ln x}{x}$

c) $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$

Other Indeterminate Forms Where L'Hopital's Rule Can Be Applied:

1. $0 \cdot \infty$ (Note: $\lim_{x \to 0}\left(\frac{1}{x}\right)x = 1,\ \lim_{x \to 0}\left(\frac{2}{x}\right)x = 2,\ \lim_{x \to \infty}\left(\frac{1}{e^x}\right)x = 0,\ \lim_{x \to \infty}\left(\frac{1}{x}\right)e^x = \infty$)
2. 1^∞
3. ∞^0
4. 0^0
5. $\infty - \infty$

Don't Use L'Hopital's Rule On Determinate Forms:

1. $\infty + \infty \to \infty$
2. $-\infty - \infty \to -\infty$
3. $0^\infty \to 0$
4. $0^{-\infty} \to \infty$
5. $\frac{1}{0} \to \infty$

Ex: Find the following limits (i) by algebraic manipulation, and (ii) by using L'Hopital's rule:

a) $\lim_{x \to -1} \frac{2x^2 - 2}{x + 1} = \frac{0}{0}$ Indeterminate Form

Algebraic Manipulation

$\lim_{x \to -1} \frac{2(x+1)(x-1)}{x+1} = \lim_{x \to -1} 2(x-1) = 2(-1-1)$

$2(-2)$

-4

L'Hopital's Rule:

$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

$\lim_{x \to -1} \frac{2x^2 - 2}{x+1} = \lim_{x \to -1} \frac{4x}{1} = 4(-1) = -4$

b) $\lim_{x \to \infty} \frac{3x^2 - 1}{2x^2 + 1} = \frac{\infty}{\infty}$ Indeterminate Form

Algebraically:

$$\lim_{x \to \infty} \frac{\frac{3x^2}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{3 - \cancel{\frac{1}{x^2}}^{\,0}}{2 + \cancel{\frac{1}{x^2}}^{\,0}} = \frac{3}{2}$$

L'Hopital's:

$$\lim_{x \to \infty} \frac{6x}{4x} = \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{6}{4} = \frac{3}{2}$$

Ex: Evaluate the following:

a) $\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \frac{0}{0}$ Ind. form

$$\lim_{x \to 0} \frac{2e^{2x}}{1} = \frac{2e^{2(0)}}{1} = 2e^{0} = 2$$

b) $\lim_{x \to \infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

c) $\lim_{x \to -\infty} \dfrac{x^2}{e^{-x}} = \dfrac{(-\infty)^2}{e^{(-(-\infty))}} = \dfrac{\infty^2}{e^{\infty}} = \dfrac{\infty}{\infty}$

$\lim_{x \to -\infty} \dfrac{2x}{-e^{-x}} = \dfrac{-\infty}{-\infty}$ (Do this again)

$\lim_{x \to -\infty} \dfrac{2}{e^{-x}} = 0$

$\dfrac{2}{\frac{1}{e^x}} = 2e^x$

$2e^{-\infty}$

$\dfrac{2}{e^{\infty}}$

Ex: For the following, determine if L'Hopital's rule can be applied, then find the limit.

a) $\lim_{x \to \infty} e^{-x}\sqrt{x}$

b) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

c) $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x - 1}\right)$

d) $\lim_{x \to 0} \left(\frac{e^{x}}{x}\right)$

Ex: For the following, determine if L'Hopital's rule can be applied, then find the limit.

a) $\lim_{x \to \infty} e^{-x}\sqrt{x} = 0 \cdot \infty$ ← Indeterminate form

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^x}$$

∴ Yes!

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty} \frac{1}{2e^x\sqrt{x}} = 0$$

b) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \left(1 + \cancel{\frac{1}{\infty}}^{\,0}\right)^{\infty} = 1^{\infty}$

Ind. form
Yes!

$\ln y = \ln \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

$= \lim_{x \to \infty} \ln\left(1 + \frac{1}{x}\right)^x$

$= \lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right)$

$= \lim_{x \to \infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{x^{-1}}$

$= \lim_{x \to \infty} \frac{\frac{1}{\left(1 + \frac{1}{x}\right)} \cdot \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}}$

$= \lim_{x \to \infty} \frac{1}{\left(1 + \frac{1}{x}\right)}$

$= 1$

$e^{\ln y} = e^1$

$y = \boxed{e}$

c) $\lim_{x \to 1^+} \left(\frac{1^{(x-1)}}{(\ln x)^{(x-1)}} - \frac{1^{\ln x}}{(x-1)_{\ln x}} \right) = (\infty - \infty)$

Indeterminate form

Yes!

$$\lim_{x \to 1^+} \left(\frac{x-1-\ln x}{(x-1)\ln x} \right)$$

$$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{1 \cdot \ln x + \frac{1}{x}(x-1)}$$

$$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\ln x + 1 - \frac{1}{x}} = \frac{0}{0}$$

Do it again!

$$= \lim_{x \to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{\frac{1}{1}}{\frac{1}{1} + \frac{1}{1}} = \frac{1}{2}$$

d) $\lim_{x \to 0}\left(\frac{e^x}{x}\right)$ $= \frac{e^0}{0} = \frac{1}{0} = \pm\infty$

Determinate!

Cannot use L'Hopital

Limit D.N.E.