5.4 Coordinates and Diagonalization

5.4 12, 14, 40, 42, 53, 54

$\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, in $\mathbb{R}^2$ the basis vectors are
$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For any vector $\vec{a}$ in $\mathbb{R}^2$ we can write
$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$$

Are $\vec{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

1.) $\text{Det}\begin{bmatrix} 2 & 2 \\ 5 & 6 \end{bmatrix} \neq 0$,

2.) $\text{RREF}\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

Choose a different basis
$$\vec{a} = \delta_1\vec{b}_1 + \delta_2\vec{b}_2$$

Ex) $\vec{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\vec{a} = 3\vec{e}_1 + 1\vec{e}_2$$

Coordinates

Let $\vec{v}$ be a vector in the finite vector with basis $B = \{\vec{B}_1, \vec{B}_2, \ldots, \vec{B}\}$ for $\mathbb{V}$. Then the coordinates of $\vec{v}$ relative to B are the unique real numbers $\beta_1, \beta_2, \ldots, \beta_n$ such that
$$\vec{v} = \beta_1\vec{b}_1 + \beta_2\vec{b}_2 + \ldots + \beta_n\vec{b}_n$$

Ex 1) $x^2 - 2xy + 2y^2 = 9$

$x = u + v$ $\quad (u+v)^2 - 2(u+v)(v) + 2(v)^2 = 9$

$y = v$ $\quad u^2 + 2uv + v^2 - 2uv - 2v^2 + 2v^2 = 9$

$$u^2 + v^2 = 9$$

Ex 2) $\vec{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$\vec{u} = 6\vec{e}_1 + 4\vec{e}_2$ $\qquad B = \{\vec{b}_1, \vec{b}_2\}$

$\vec{u} = \beta_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Change Bases In $\mathbb{R}^n$

Let $\vec{u}_B$ be the coordinate vector relative to basis $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, and $\vec{u}_S$ be the coordinate vector relative to the standard basis

- To change from standard basis to basis B:

$$M_S\vec{u}_S = \vec{u}_B \qquad M_S = M_B^{-1} = [\vec{b}_1 \mid \vec{b}_2 \mid \ldots \mid \vec{b}_n]^{-1}$$

- To change from basis B to the standard basis:

$$M_B\vec{u}_B = \vec{u}_S \qquad M_B = [\vec{b}_1 \mid \vec{b}_2 \mid \ldots \mid \vec{b}_n]$$

System of DE's

$x_1' = x_1 + x_2$

$x_2' = 4x_1 + x_2$

$\vec{x}' = A\vec{x}$

$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\quad x = P\vec{u} \Rightarrow P\vec{u}' = AP\vec{u}$

$\vec{u}' = P^{-1}AP\vec{u}$

$\lambda_1 = 3 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad P = [\vec{v}_1 | \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

$\lambda_2 = -1 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\quad u_1' = 3u_1 \quad \frac{dy}{dt} = 3y$

$u_2' = -1u_2 \quad \frac{dy}{dt} = -y$

$u_1 = c_1 e^{3t} \quad u_2 = c_2 e^{-t}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{bmatrix}$

Diagonalize a matrix

Find eigenvectors and eigenvalues of A.

- Construct an nxn diagonal matrix D with the eigenvalues on the diagonal (Eigenvalues with multiplicity m appear m times)
- construct another nxn matrix P with the columns being the eigenvectors

$AP = PD$

$A = PDP^{-1}$

$P^{-1}AP = D$

Ex 8)

$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad \lambda_1 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\lambda_3 = -1 \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$