Tues: 9.2.3, 9.3.3.

## Electromagnetic waves: energy

Consider plane sinusoidal electromagnetic waves. These can be specified entirely in terms of the electric field, which in complex form is:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\vec{k}$ = wavenumber vector , $\omega = kv = kc$

$\tilde{\vec{E}}_0$ = complex amplitude vector

Specifying these two vectors completely specifies the sinusoidal wave since the magnetic field is:

$$\vec{B} = \frac{1}{\omega}(\vec{k} \times \vec{E})$$

We then found that the electromagnetic energy density is

$$u = \epsilon_0 E^2 = \epsilon_0 \vec{E}\cdot\vec{E}$$

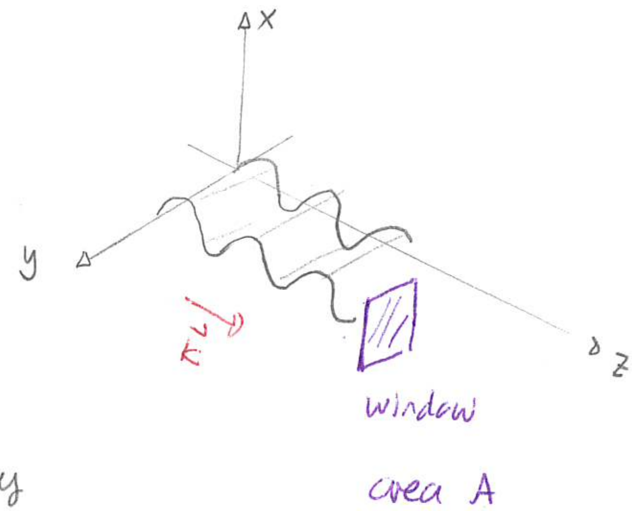
and the Poynting vector is

$$\vec{S} = \epsilon_0 c E^2 \hat{k}$$

where $\hat{k} = \vec{k}/k$ is the direction of propagation of the wave.

# Intensity of electromagnetic radiation

The sinusoidal plane waves that we have described extend through all space. In order to interpret the Poynting vector we imagine a window that is perpendicular to the direction of propagation




Then the rate at which energy flows through this window is

$$P = \int \vec{S} \cdot d\vec{a}$$

Here $d\vec{a} = da\,\hat{k}$ $\Rightarrow$ $\vec{S} \cdot d\vec{a} = \epsilon_0 c E^2$ and

$$P = \epsilon_0 c E^2 A$$

where $E$ is the amplitude of the real wave. Consider a wave traveling along the $z$ direction. Then

$$\tilde{\vec{E}}_0 = E_{0x} e^{i\delta_x} \hat{x} + E_{0y} e^{i\delta_y} \hat{y}$$

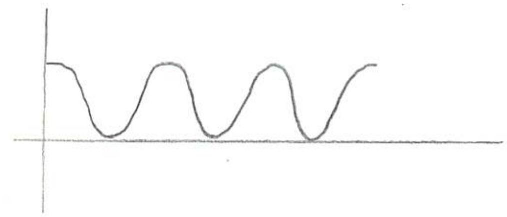
and

$$\vec{E} = E_{0x} \cos(kz - \omega t + \delta_x)\hat{x} + E_{0y} \cos(kz - \omega t + \delta_y)\hat{y}$$

So

$$P = \epsilon_0 c A \left[ E_{0x}^2 \cos^2(kz - \omega t + \delta_x) + E_{0y}^2 \cos^2(kz - \omega t + \delta_y) \right]$$

This clearly produces a power that fluctuates with respect to time but is always positive.



In many situations these fluctuations are very rapid compared to the time resolution of the measuring device. We really only observe an average rate at which energy flows. For any sinusoidally oscillating function with period $T = 2\pi/\omega$ we can compute a time average as:

> Given a function $f$ with period $T = 2\pi/\omega$ the time average of $f$ is:
>
> $$\langle f \rangle := \frac{1}{T}\int_0^T f(t)\,dt.$$

We then define

> The intensity of an electromagnetic wave is the time averaged power per unit area:
>
> $$I = \langle S \rangle$$
>
> where $S$ is the magnitude of the Poynting vector.

So here

$$I = \epsilon_0 c \langle E^2 \rangle$$

and then

$$P = IA$$

1 **Intensity for sinusoidal plane waves**

Show that the intensity of any linearly polarized sinusoidal wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

is

$$I = \frac{1}{2} c\epsilon_0 E_0^2.$$

Answer: Here

$$S = \epsilon_0 c\, \vec{E}\cdot\vec{E}$$

$$= \epsilon_0 c\, \vec{E}_0\cdot\vec{E}_0 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$= \epsilon_0 c\, E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\langle S\rangle = \epsilon_0 c E_0^2 \langle \cos^2(\vec{k}\cdot\vec{r} - \omega t)\rangle$$

Now

$$\langle \cos^2(\vec{k}\cdot\vec{r} - \omega t)\rangle = \frac{1}{T}\int_0^T \cos^2(\vec{k}\cdot\vec{r} - \omega t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left(1 + \cos\left[2(\vec{k}\cdot\vec{r} - \omega t)\right]\right) dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\,dt + \frac{1}{T}\int_0^T \cos(2\vec{k}\cdot\vec{r} - 2\omega t)\,dt$$

$$= \frac{1}{2} + \frac{1}{T}\,\frac{-1}{2\omega}\sin(2\vec{k}\cdot\vec{r} - 2\omega t)\Big|_0^T$$

$$= \frac{1}{2} - \frac{1}{2\omega T}\Big[\underbrace{\sin(2\vec{k}\cdot\vec{r} - \underbrace{2\omega T}_{4\pi})}_{\sin(2\vec{k}\cdot\vec{r})} - \sin(2\vec{k}\cdot\vec{r})\Big]$$

$$= \frac{1}{2} \qquad \Longrightarrow \qquad \langle S\rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

# Electromagnetic radiation pressure

We know that electromagnetic fields exert forces and that when these are analyzed we find that in a closed system momentum can only be conserved if the electromagnetic fields themselves carry momentum. We saw that the electromagnetic momentum density is

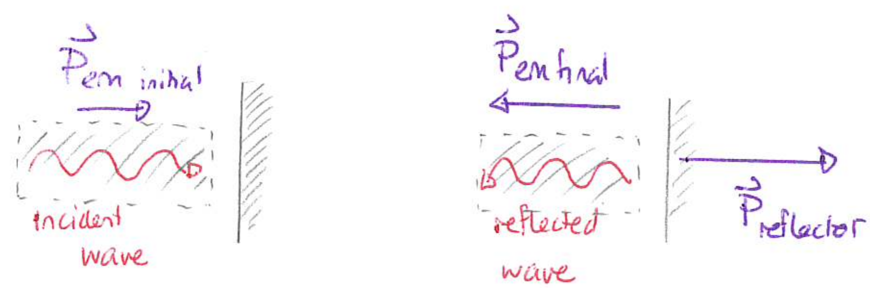
$$\vec{g} = \mu_0 \epsilon_0 \vec{S}$$

Thus in any region the total electromagnetic momentum is

$$\vec{P}_{em} = \int_{\text{region}} \mu_0 \epsilon_0 \vec{S}\, d\tau$$

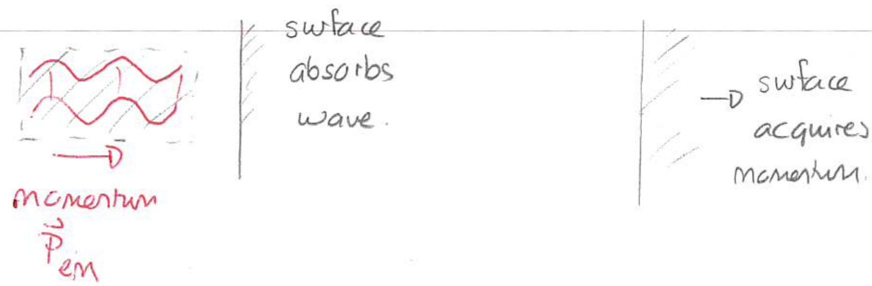
This will imply that electromagnetic waves can exert forces when they are reflected off any surface.

Conservation of momentum implies that the reflector will recoil.




We can account for this via a force on the reflector. This will depend on the surface area of the reflector and thus we will be able to obtain a recoil force per area. In effect this is a pressure exerted by the electromagnetic radiation.

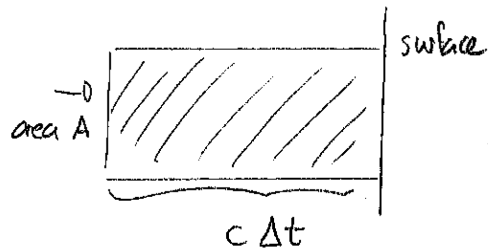
The simplest case is that where electromagnetic radiation is absorbed by the surface



Then the force exerted on the surface is

$$\vec{F} = \frac{d\vec{P}_{em}}{dt}$$

We consider a wave incident perpendicularly. Then in time $\Delta t$ all of the shaded area reaches the surface. The total momentum delivered is then




$$\Delta \vec{P}_{em} = \int \vec{g}\, d\tau$$

$$= \mu_0 \epsilon_0 \vec{S} A c \Delta t.$$

Thus

$$\frac{\Delta \vec{P}_{em}}{\Delta t} = A \underbrace{\mu_0 \epsilon_0 c}_{\frac{1}{c^2}} \vec{S} = \frac{A}{c}\vec{S}$$

$$\Rightarrow \vec{F} = \frac{A}{c}\vec{S}$$

Then the force per unit area is called the <u>electromagnetic radiation pressure</u>:

$$P = F/A$$

We get that $P = \frac{1}{c} S$ and this fluctuates with time.

The time averaged electromagnetic radiation pressure is: $P = \frac{1}{c}\langle S \rangle$

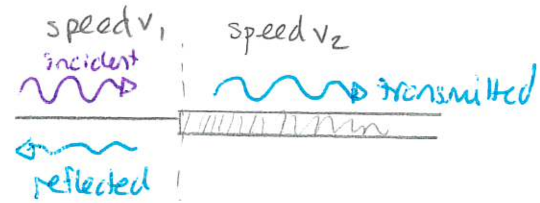
We then get

$$\boxed{P = I/c}$$

where $I$ is the intensity of the radiation.

## Reflection and transmission of electromagnetic waves

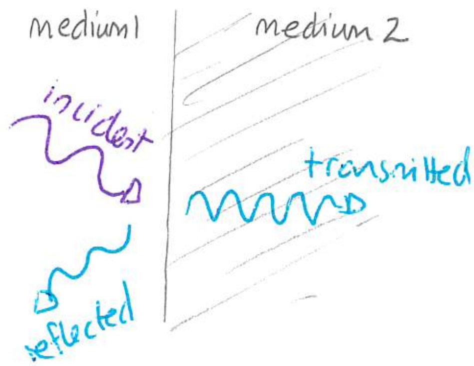
In general waves that travel through one homogeneous medium to another with different physical properties will be partly transmitted and partly reflected. This occurs for waves on a string and also for electromagnetic waves




The main distincts will involve the fact that electromagnetic waves:

1) travel in three dimensions (directions of propagation)

2) are vectors (polarization)




We will find that there will be a partly transmitted and a partly reflected wave.

We will analyze the situation for plane sinusoidal waves and aim to:

1) relate the electric fields of the transmitted and reflected waves to those of the incident wave

2) relate the directions of the waves

3) " " intensities " " "

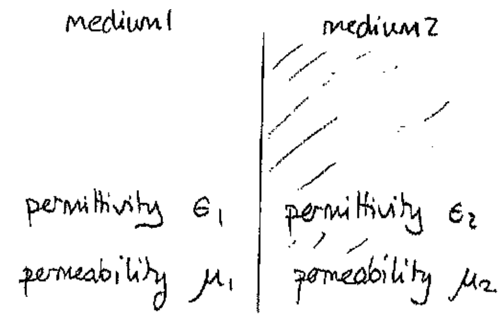
This analysis is important for understanding

1) reflection + transmission in optics

2) electromagnetic waves and conducting / partly conducting surfaces.

# Boundary conditions for waves in dielectric media

Initially we will consider waves in two dielectric media separated by a plane boundary. We will assume that there are no free charges in these media. Then Maxwell's equations become:




$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

and for each medium

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

where $\epsilon$ and $\mu$ are the permittivity and permeability of the medium. For such linear media the structure of Maxwell's equations mimics that of those in a vacuum and we can derive wave equations for electric and magnetic fields. These reveal that waves travel with speed

$$\boxed{V = \frac{1}{\sqrt{\epsilon\mu}}}$$

The general boundary conditions are derived as before (various infinitesimal pillboxes + loops). They give

$$D_1^{\perp} = D_2^{\perp} \quad \Rightarrow \quad \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$$ (component perpendicular to surface)

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \quad \quad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$ (component parallel to surface)

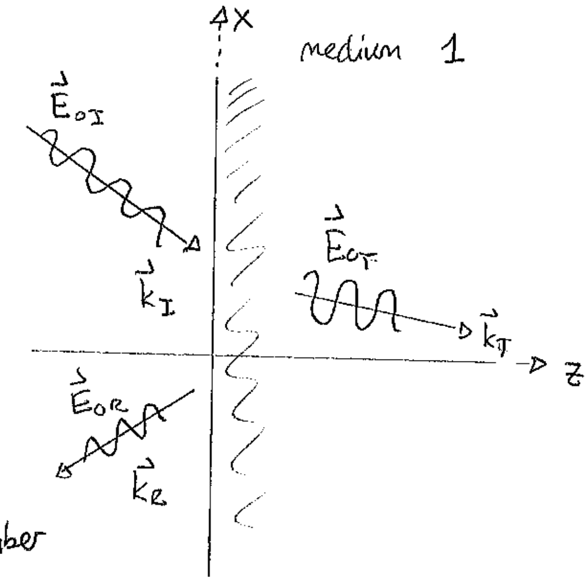
$$B_1^{\perp} = B_2^{\perp} \quad \quad B_1^{\perp} = B_2^{\perp}$$

$$\vec{H}_1^{\parallel} = \vec{H}_2^{\parallel} \quad \Rightarrow \quad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

# Reflection + transmission of plane sinusoidal waves

Consider the possibility that the incident, transmitted and reflected waves are all plane sinusoidal waves. Each can be described by

* a real wavenumber vector, $\vec{k}$
* a complex amplitude. $\vec{E}_0$




The set up will be that the bandary between the two media is the $z=0$ plane. We then arrange the other axes so that the wavenumber vector for the incident wave is in the x-z plane. Then we use the generic form for the electric fields

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \omega = kv$$

and for the magnetic fields

$$\tilde{\vec{B}} = \frac{1}{\omega} \vec{k} \times \vec{E} = \frac{1}{\omega} \vec{k} \times \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

The waves that we will have to match are:

1) incident (medium 1)

$$\tilde{\vec{E}}_I = \tilde{\vec{E}}_{0I} e^{i(\vec{k}_I\cdot\vec{r} - \omega_I t)} \qquad \tilde{\vec{B}}_I = \tilde{\vec{B}}_{0I} e^{i(\vec{k}_I\cdot\vec{r} - \omega_I t)} = \frac{1}{\omega_I} \vec{k}_I \times \tilde{\vec{E}}_I$$

2) reflected (medium 1)

$$\tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R} e^{i(\vec{k}_R\cdot\vec{r} - \omega_R t)} \qquad \tilde{\vec{B}}_R = \tilde{\vec{B}}_{0R} e^{i(\vec{k}_R\cdot\vec{r} - \omega_R t)}$$

3) transmitted

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T} e^{i(\vec{k}_T\cdot\vec{r} - \omega_T t)} \qquad \tilde{\vec{B}}_T = \tilde{\vec{B}}_{0T} e^{i(\vec{k}_T\cdot\vec{r} - \omega_T t)}$$

Then the total electric field in medium 1 is (dropping the tilde)

$$\vec{E}_1 = \vec{E}_I + \vec{E}_R = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)} + \vec{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$$

The magnetic field is

$$\vec{B}_1 = \vec{B}_I + \vec{B}_R = \frac{1}{\omega_I}(\vec{k}_I \times \vec{E}_I) + \frac{1}{\omega_R}(\vec{k}_R \times \vec{E}_R)$$

The total electric field in medium 2 is

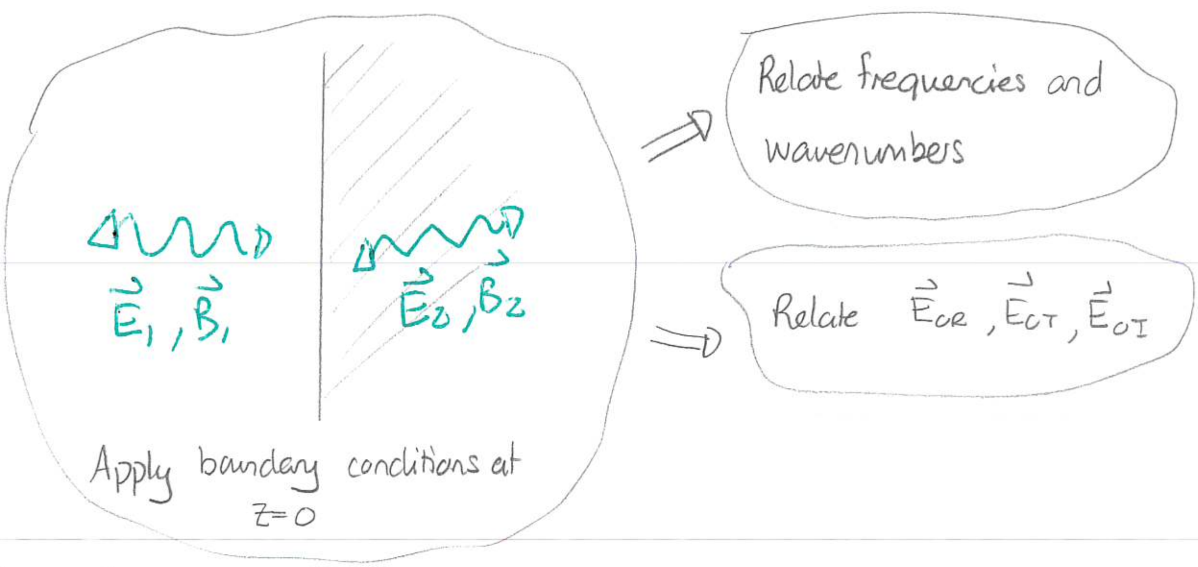
$$\vec{E}_2 = \vec{E}_T = \vec{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$$

and the total magnetic field in medium 2 is

$$\vec{B}_2 = \vec{B}_T = \frac{1}{\omega_T}(\vec{k}_T \times \vec{E}_T)$$

We need to impose the boundary matching conditions on $\vec{E}_1, \vec{E}_2, \vec{B}_1$ and $\vec{B}_2$ (and not the individual incident and reflected parts at $z=0$.

What will occur is:




We will need

1) to show boundary matching conditions apply to complex representations

2) a lemma regarding exponential functions.

**Lemma:** Suppose that for all $x$ $(A, B, C \neq 0)$

$$Ae^{\alpha x} + Be^{\beta x} = Ce^{\gamma x}$$

where $\alpha, \beta, \gamma$ are independent of $x$. Then:

i) $A + B = C$

ii) $\alpha = \beta = \gamma$

**Proof:** i) Set $x = 0$ and this follows trivially

ii) $$Ae^{\alpha x} + Be^{\beta x} = Ce^{\gamma x}$$

$$\Rightarrow Ae^{(\alpha-\gamma)x} + Be^{(\beta-\gamma)x} = C$$

Differentiate w.r.t $x$ $\Rightarrow$ $A(\alpha-\gamma)e^{(\alpha-\gamma)x} + B(\beta-\gamma)e^{(\beta-\gamma)x} = 0$

$$\Rightarrow A(\alpha-\gamma)e^{(\alpha-\beta)x} + B(\beta-\gamma) = 0$$

Differentiate w.r.t $x$ $\Rightarrow$ $A(\alpha-\gamma)(\alpha-\beta) = 0$.

Assuming $A \neq 0$ $\alpha = \gamma$ or $\alpha = \beta$.

Substituting one gives either $Ae^{\alpha x} + Be^{\alpha x} = Ce^{\gamma x}$ OR...

and repeating the argument gives $\alpha = \gamma$ AND $\alpha = \beta$ ∎

**2 Boundary conditions for the electric field.**

Consider an electric field incident on a plane at $z = 0$. Suppose that the wavenumber vector for the incident field is in the $xz$ plane.

a) Apply the boundary conditions to the electric fields at *one location on the boundary* and the lemma in class to relate the frequencies of the three waves.

b) Apply the boundary conditions to the electric fields at all locations along the boundary to show that all wavenumber vectors lie in the $xz$ plane.

c) In the special case where the incident wave propagates perpendicular to the boundary, show that the wavenumber vectors of the reflected and transmitted waves are also perpendicular to to the boundary.

Answer: a) $\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

$$\Rightarrow \epsilon_1 E_{0I}^\perp e^{i(\vec{k}_I\cdot\vec{r}-\omega_I t)} + \epsilon_1 E_{0R}^\perp e^{i(\vec{k}_R\cdot\vec{r}-\omega_R t)} = \epsilon_2 E_{0T}^\perp e^{i(\vec{k}_T\cdot\vec{r}-\omega_T t)}$$

$$\Rightarrow \underbrace{\epsilon_1 E_{0I}^\perp e^{i(\vec{k}_I\cdot\vec{r})}}_{A} e^{-i\omega_I t} + \underbrace{\epsilon_1 E_{0R}^\perp e^{i(\vec{k}_R\cdot\vec{r})}}_{B} e^{-i\omega_R t} = \underbrace{\epsilon_2 E_{0T}^\perp e^{i(\vec{k}_T\cdot\vec{r})}}_{C} e^{-i\omega_T t}$$

Fixing $\vec{r}$ provides constants $A, B, C$ as indicated

By the lemma this is only true for all $t$ if

$$\omega_I = \omega_R = \omega_T$$

Thus there is only one angular frequency present

$$\omega = \omega_I = \omega_R = \omega_T.$$

b) Use

$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$$

$$\Rightarrow \epsilon_1 \vec{E}_{0I}^{\parallel} e^{i(\vec{k}_I\cdot\vec{r}-\omega t)} + \epsilon_1 \vec{E}_{0R}^{\parallel} e^{i(\vec{k}_R\cdot\vec{r}-\omega t)} = \epsilon_2 \vec{E}_{0T}^{\parallel} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

The $e^{-i\omega t}$ terms cancel.

So

$$\epsilon_1 \vec{E}_{0I}^{\parallel} e^{i(\vec{k}_I \cdot \vec{r})} + \epsilon_1 \vec{E}_{0R}^{\parallel} e^{i\vec{k}_R \cdot \vec{r}} = \epsilon_2 \vec{E}_{0T}^{\parallel} e^{i\vec{k}_T \cdot \vec{r}}$$

Now along the interface $z=0$ and thus $\vec{r} = x\hat{x} + y\hat{y}$. This gives

$$\epsilon_1 \vec{E}_{0I}^{\parallel} e^{i(k_{Ix}x + k_{Iy}y)} + \epsilon_1 \vec{E}_{0R}^{\parallel} e^{i(k_{Rx}x + k_{Ry}y)} = \epsilon_2 \vec{E}_{0T}^{\parallel} e^{i(k_{Tx}x + k_{Ty}y)}$$

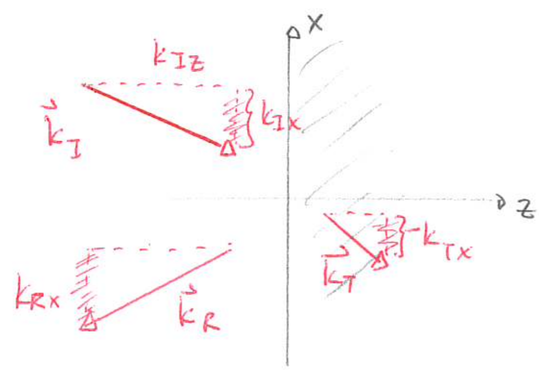
We can now vary $x$ and $y$ independently. The result is that the equality can only be satisfied if

$$k_{Ix} = k_{Rx} = k_{Tx}$$

$$k_{Iy} = k_{Ry} = k_{Ty}$$

But $k_{Iy} = 0 \Rightarrow \vec{k}_R$ and $\vec{k}_T$ have no $y$ component. Then the wavenumber vectors are:

$$\left.\begin{aligned} \vec{k}_I &= k_{Ix}\hat{x} + k_{Iz}\hat{z} \\ \vec{k}_R &= k_{Rx}\hat{x} + k_{Rz}\hat{z} \\ \vec{k}_T &= k_{Tx}\hat{x} + k_{Tz}\hat{z} \end{aligned}\right\} \text{all lie in } x\text{-}z \text{ plane}$$




all shaded components are the same

c) Here $k_{Ix} = 0 \Rightarrow k_{Rx} = k_{Tx} = 0$

wavenumbers all point along $\hat{z}$

Thus derivations directly from electromagnetic theory give:

For sinusoidal plane waves:

1) the frequencies of the incident, reflected and transmitted waves are identical

2) the wavenumber vectors all lie in the same plane

The angular frequency will be denoted $\omega$. The plane in which the wavenumber vectors lie is:

1) perpendicular to the surface that forms the interface between the media

2) aligned so that $\vec{k}_I$ lies in the plane.

These facts uniquely define the plane. This is called the <u>plane of incidence</u>.