Physics 311

Exam 2

1. a) Draw a plot of the three charge configuration below

$$q_1 = q \quad @ \quad x = y = 0$$

$$q_2 = 3q \quad @ \quad x = 0\ ,\ y = 3\ell$$

$$q_3 = -2q \quad @ \quad x = 4\ell\ ,\ y = 0\ .$$

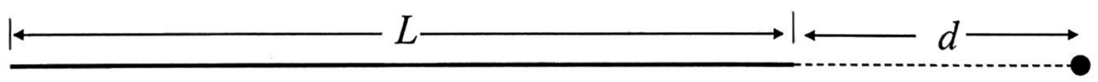
b) How much work does it take to assemble this charge configuration?

c) With the configuration assembled as stated above, how much work does it take to bring in a charge $q_4 = 5q$ from infinity and place it at the location $x = 4\ell\ ,\ y = 3\ell$?

2. Consider a *thin* rod of length L, which carries a uniform line charge λ.

a) Calculate the potential a distance d from the nearest end of the rod along the axis of the rod.

b) Calculate the electric field a distance d from the nearest end of the rod along the axis of the rod.



3. A *very* long coaxial cable carries a charge density

$$\rho(s) = \rho_0 \frac{s}{a} \tag{1}$$

in the region $a \leq s \leq b$ *only*.

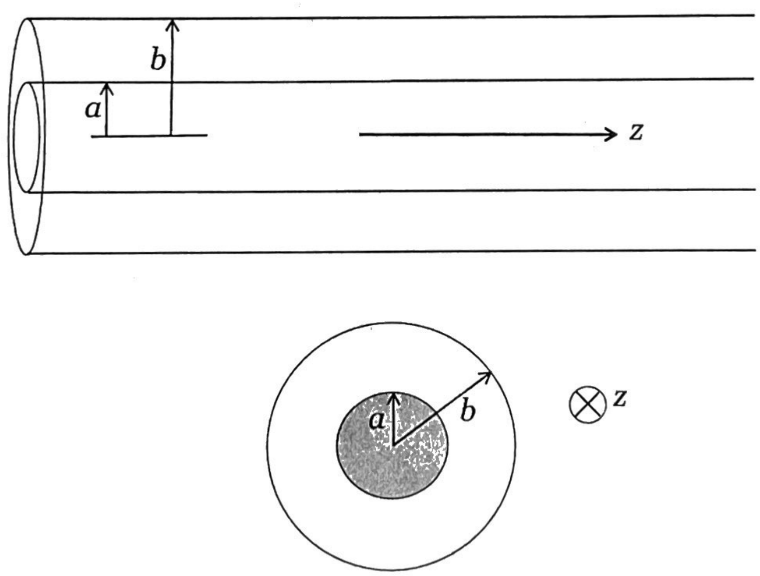
Find the electric field in each of the three regions:

a) $s < a$.

b) $a < s < b$.

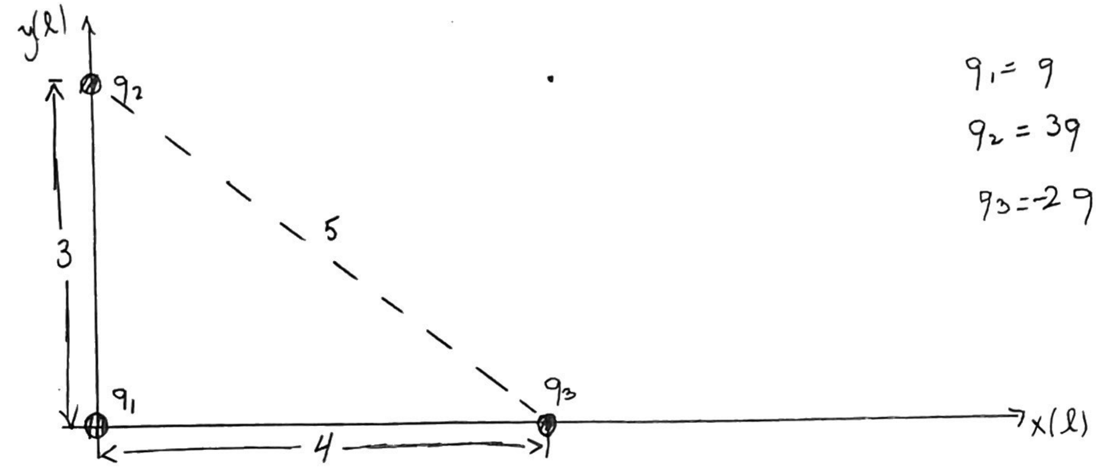
c) $b < s$.

d) Using this electric field in the three regions, calculate the potential difference between a point on the axis and a point on the outer cylinder.



PHYS 311 EXAM 2 SOLUTIONS

1a)



$q_1 = q$

$q_2 = 3q$

$q_3 = -2q$

b) $V(\vec{r}_1) = V_2(\vec{r}_1) + V_3(\vec{r}_1) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3q}{3l} + \frac{-2q}{4l}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(1 - \frac{1}{2}\right) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{2l}$ ✓

$V(\vec{r}_2) = V_1(\vec{r}_2) + V_3(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{3l} + \frac{-2q}{5l}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(\frac{1}{3} - \frac{2}{5}\right) = -\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{15l}$ ✓

$V(\vec{r}_3) = V_1(\vec{r}_3) + V_2(\vec{r}_3) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{4l} + \frac{3q}{5l}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(\frac{1}{4} + \frac{3}{5}\right) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{17q}{20l}$ ✓

NO

$W = \frac{1}{2}\sum_{i=1}^{3} q_i V(\vec{r}_i) = \frac{1}{2}\left[q_1 V(\vec{r}_1) + q_2 V(\vec{r}_2) + q_3 V(\vec{r}_3)\right]$

$= \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{2l} + 3q\cdot\frac{-q}{15l} + -2q\cdot\frac{17q}{20l}\right] = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left(\frac{1}{2} - \frac{1}{5} - \frac{17}{10}\right)$

$= \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left(\frac{5-2-17}{10}\right) = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{l}\cdot\frac{-14}{10} = -\frac{7}{10}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{l}$

c) $V(\vec{r}_4) = V_1(\vec{r}_4) + V_2(\vec{r}_4) + V_3(\vec{r}_4) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{5l} + \frac{3q}{4l} - \frac{2q}{3l}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(\frac{1}{5} + \frac{3}{4} - \frac{2}{3}\right)$

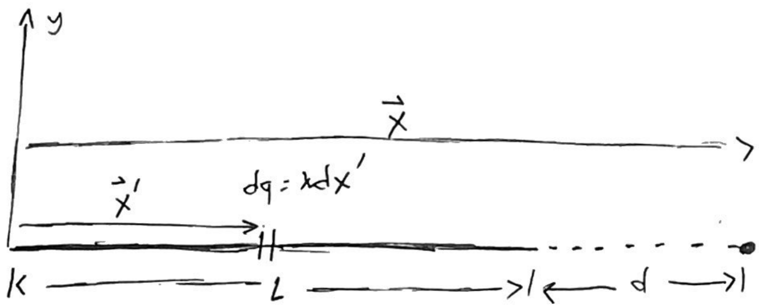
$= \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(\frac{12+45-40}{60}\right) = \frac{17}{60}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{l}$

so

$W = q_4 V(\vec{r}_4) = 5q\cdot\frac{17}{60}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{l}$

$= \frac{17}{12}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}$

2.



now

$\vec{x} = (L+d)\hat{x}$

$\vec{x}' = x'\hat{x}$

so

$\vec{\imath} = (L+d-x')\hat{x}$

$\therefore \imath = (L+d-x') \quad ; \quad \hat{\imath} = \hat{x}$

now $\lambda = \frac{q}{L} \therefore q = \lambda L$

now

a) $V(x=L+d) = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dx'}{(L+d-x')} = -\frac{\lambda}{4\pi\varepsilon_0}\int_{L+d}^{d}\frac{du}{u} = \frac{\lambda}{4\pi\varepsilon_0}\ln u\Big|_d^{L+d} = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln(L+d) - \ln(d)\right)$

let $u = L+d-x'$

$du = -dx'$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{L+d}{d}\right)$$

$$\boxed{V(x=L+d) = \frac{1}{4\pi\varepsilon_0}\frac{q}{L}\ln\left(1+\frac{L}{d}\right)} \checkmark$$

b) $\vec{E}(x=L+d) = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda dx'\,\hat{x}}{(L+d-x')^2} = -\frac{\lambda}{4\pi\varepsilon_0}\hat{x}\int_{L+d}^{d}\frac{du}{u^2} = +\frac{\lambda}{4\pi\varepsilon_0}\hat{x}\,\frac{1}{u}\Big|_{L+d}^{d}$

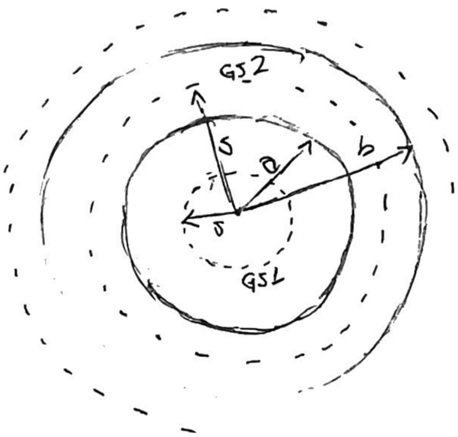
let $u = L+d-x'$

$du = -dx'$

$$= \hat{x}\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{d} - \frac{1}{(L+d)}\right] = \hat{x}\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{L+d-d}{d(L+d)}\right] = \hat{x}\frac{\lambda}{4\pi\varepsilon_0}\frac{L}{d(d+L)}$$

$$\therefore \boxed{\vec{E}(x=L+d) = \frac{1}{4\pi\varepsilon_0}\frac{q}{d(d+L)}\hat{x}}$$

3. For $s<a$...



a) For $s<a$...

For Gaussian surface 1...

$$Q_{enc} = 0 \quad \therefore \quad \vec{E} = 0$$

b) For $a<s<b$...

$$Q_{enc} = \int \rho \, d\tau = \int_0^{l}\int_a^{s}\int_0^{2\pi} \rho_0 \frac{s}{a} \cdot s \, d\phi \, ds \, dz$$

$$= \frac{\rho_0}{a}\int_0^{l} dz \int_0^{2\pi} d\phi \int_a^{s} s^2 \, ds = \frac{\rho_0}{a} \cdot l \cdot 2\pi \cdot \frac{1}{3} s^3 \Big|_a^{s}$$

so

$$Q_{enc} = \frac{2\pi}{3} \frac{\rho_0 l}{a}\left(s^3 - a^3\right) = \frac{2\pi}{3}\rho_0 l\left(\frac{s^3}{a} - a^2\right) ✓$$

now

$$\oint \vec{E} \cdot d\vec{a} = \int_{\text{tube}} \vec{E} \cdot d\vec{a} = E 2\pi s l \quad ✓$$

so Gauss' law yields...

$$E 2\pi s l = \frac{2\pi}{3\varepsilon_0}\rho_0 l\left(\frac{s^3}{a} - a^2\right) \quad \therefore \quad E = \frac{\rho_0}{3\varepsilon_0}\left(\frac{s^2}{a} - \frac{a^2}{s}\right)$$

$$\therefore \quad \boxed{\vec{E} = \frac{\rho_0}{3\varepsilon_0}\left(\frac{s^2}{a} - \frac{a^2}{s}\right)\hat{s}}$$

For $s>b$..

$$Q_{enc} = \frac{\rho_0}{a}\int_0^{l} dz \int_0^{2\pi} d\phi \int_a^{b} s^2 \, ds = \frac{\rho_0}{a} \cdot l \cdot 2\pi \cdot \frac{1}{3} s^3 \Big|_a^{b} = \frac{2\pi}{3}\frac{\rho_0 l}{a}\left(b^3 - a^3\right) = \frac{2\pi}{3}\rho_0 l\left(\frac{b^3}{a} - a^2\right)$$

so Gauss' law yields..

$$E 2\pi s l = \frac{2\pi}{3\varepsilon_0}\rho_0 l\left(\frac{b^3}{a} - a^2\right) \quad \therefore \quad E = \frac{\rho_0}{3\varepsilon_0}\left(\frac{b^3}{a} - a^2\right) \cdot \frac{1}{s}$$

$$\therefore \quad \boxed{\vec{E} = \frac{\rho_0}{3\varepsilon_0}\left(\frac{b^3}{a} - a^2\right)\frac{\hat{s}}{s}}$$

d) $$\Delta V = V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{l} - \int \vec{E}\cdot d\vec{l}$$

$$= -\int_a^b \frac{\rho_0}{3\varepsilon_0}\left(\frac{s^2}{a} - \frac{a^2}{s}\right)\hat{s}\cdot ds\,\hat{s} = -\frac{\rho_0}{3\varepsilon_0}\int_a^b \left(\frac{s^2}{a} - \frac{a^2}{s}\right) ds$$

$$= -\frac{\rho_0}{3\varepsilon_0}\left[\frac{1}{a}\frac{s^3}{3}\Big|_a^b - a^2\int_a^b \frac{ds}{s}\right] = \frac{\rho_0}{3\varepsilon_0}\left[\frac{1}{3a}\left(a^3 - b^3\right) + a^2 \ln s\Big|_a^b\right]$$

$$= \frac{\rho_0}{3\varepsilon_0}\left[\frac{a^2}{3}\left(1 - \frac{b^3}{a^3}\right) + a^2\left[\ln(b) - \ln a\right]\right]$$

$$\Delta V = \frac{\rho_0}{3\varepsilon_0} a^2\left[\frac{1}{3}\left(1 - \frac{b^3}{a^3}\right) + \ln(b/a)\right] \checkmark$$