Ch6.2 Pivoting Strategies

When performing Gaussian elimination, we construct multipliers m_{jk}, where

$$m_{jk} = \frac{a_{jk}^{(k)}}{a_{kk}^{(k)}}$$

Note that $|m_{jk}| >> 1$ whenever $|a_{kk}^{(k)}| << |a_{jk}^{(k)}|$.

Roundoff error in computing $a_{k\ell}^{(k)}$ is multiplied by m_{jk} when computing $a_{j\ell}^{(k+1)}$, which compounds the error.

Further, when backsolving,

$$x_k = \frac{a_{k,n+1}^{(n)} - \sum_{j=k+1}^{n} a_{kj}^{(n)}}{a_{kk}^{(n)}},$$

if $a_{kk}^{(n)} \approx 0$, then any error in numerator is greatly enlarged.

Example $0.003000x_1 + 59.14x_2 = 59.17$

$5.291 - 6.130x_2 = 46.78$

(Exact solution: $x_1 = 10.00$, $x_2 = 1.000$)

Augmented matrix:

$$\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

$$\xrightarrow{r_2 = r_2 - m_{21} r_1} \left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 0.0000 & -104300 & -104400 \end{array}\right]$$

Here, $m_{21} = \dfrac{5.291}{0.003000} \cong 1764$

(using four digit rounding.) Using precise values, we would obtain

$$\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 0.0000 & -10430\underline{9.37\overline{6}} & -104\underline{309.37\overline{6}} \end{array}\right]$$

Thus roundoff has been introduced, and will propogate in backsolving.

We have

$$\left[\begin{array}{cc:c} 0.003000 & 59.14 & 59.17 \\ 0.0000 & -104300 & -104400 \end{array}\right]$$

using backward substitution,

$$x_2 \approx \frac{-104400}{-104300} \approx 1.\underline{001} \quad \leftarrow \text{error } (x_2 = 1)$$

$$x_1 \approx \frac{59.17 - (59.14)(1.\underline{001})}{0.003000}$$

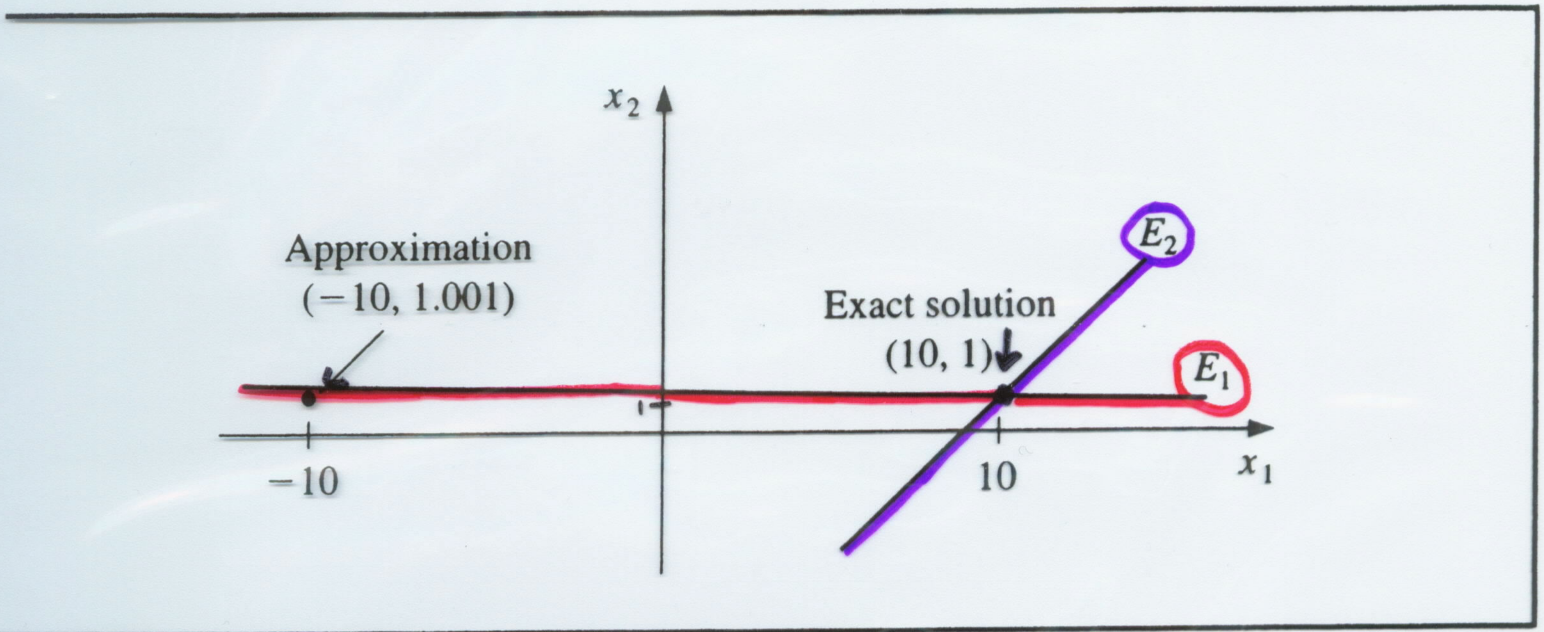
$$\approx \frac{59.17 - 59.20}{0.003000} = \frac{-0.03}{0.003000}$$

$$= -10.00$$

Note that the $0.\underline{001}$ error is multiplied by $\dfrac{59.14}{0.003000} \approx 20{,}000.$

Thus we obtain $x_1 = -10$ instead of the actual solution $x_1 = 10$. (Rel Error = 200%)

$$0.003000x_1 + 59.14x_2 = 59.17, \quad (E_1)$$
$$5.291x_1 - 6.130x_2 = 46.78. \quad (E_2)$$



Note: We found x_2 first when backsolving, obtaining $x_2 = 1.001$ instead of $x_2 = 1.000$. We then used $x_2 = 1.001$ to backsolve for x_1. As you can see from the graph, $x_2 = 1.001 \Rightarrow x_1 = -10$, instead of $x_1 = 10$.

To avoid the problem of a small pivot $a_{kk}^{(k)}$, we can search for a larger pivot in a different row, and perform a row interchange.

<u>Example</u> Reconsider our system

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$5.291x_1 - 6.130x_2 = 46.78$$

Augmented Matrix:

$$\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

Then

$$\max\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\} = 5.291$$

Thus perform $r_2 \leftrightarrow r_1$:

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0.003000 & +59.14 & 59.17 \end{array}\right]$$

(Called partial pivoting)

We have

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0.003000 & 59.14 & 59.17 \end{array}\right]$$

$$\xrightarrow{r_2 = r_2 - m_{21} r_1} \left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0 & 59.14 & 59.14 \\ & (1) & (2) \end{array}\right]$$

Here, $m_{21} = \dfrac{a_{21}^{(1)}}{a_{11}^{(1)}} = \dfrac{0.003000}{5.291} \cong \underline{0.0005670}$

Note that

(1) $59.14 - (\underline{0.0005670})(-6.130) \cong 59.14 + 0.0035$

$= 59.1435 \cong \underline{59.14}$

(2) $59.17 - (\underline{0.0005670})(46.78) \cong 59.17 - 0.02652$

$= 59.14348 \cong \underline{59.14}$

Backsolving, we obtain

$x_2 = 1, \quad x_1 = (46.78 + 6.130)/5.291$

$= (52.91)/5.291$

$= 10$

Example

$$30.00x_1 + 591400x_2 = 591700$$
$$5.291x_1 - 6.130x_2 = 46.78$$

Augmented matrix:

$$\left[\begin{array}{cc:c} 30.00 & 591400 & 591700 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

(Don't need partial pivoting)

$$\xrightarrow{r_2 = r_2 - m_{21} r_1} \left[\begin{array}{cc:c} 30.00 & 591400 & 591700 \\ 0 & \underset{①}{-104300} & \underset{②}{-104400} \end{array}\right]$$

Here,

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{5.291}{30.00} \doteq \underline{0.1764}$$

Note that

① $-6.130 - (0.\underline{1764})(591400) \doteq -6.130 - 104300$

$\doteq -104306.13 \doteq -\underline{104300}$

② $46.78 - (0.\underline{1764})(591700) \doteq 46.78 - 104400$

$= -104353.22 \doteq -\underline{104400}$

Backsolving: $x_2 \doteq 1.001$, $x_1 \doteq -10.00$

(using 4 digit rounding arithmetic)

<u>Scaled partial pivoting</u> can help avoid the inaccuracies of the previous example.

For a given row k, define

$$s_k = \max_{1 \le j \le n} |a_{kj}| \qquad \leftarrow \text{max element across row } k$$

We then select the first possible row p below row k for which

$$\frac{|a_{pk}|}{s_p} = \max_{1 \le j \le n} \frac{|a_{jk}|}{s_j}$$

← $|a_{jk}|$

← j

and perform $r_k \leftrightarrow r_p$.

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right]$$

Scaled partial pivoting places the element in the pivot position that is largest relative to the entries in its row.

Example Recall the system

$$\left[\begin{array}{cc|c} 30.00 & 591400 & 591700 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$$

Then

$$s_1 = \max_{1 \le j \le 2} |a_{1j}| = 591400$$

$$s_2 = \max_{1 \le j \le 2} |a_{2j}| = 6.130$$

We then have

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.0005073$$

$$\frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631$$

Conclusion: Switch r_1 & r_2:

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 30.00 & 591400 & 591700 \end{array}\right]$$

Thus

$$\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 30.00 & 591400 & 591700 \end{array}\right]$$

$$\xrightarrow{r_2 = r_2 - m_{21} r_1} \left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0 & \underset{1}{591435} & \underset{2}{591435} \end{array}\right]$$

Here,

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = \frac{30.00}{5.291} = 5.670$$

compare w/ p.6

Note that

1 $591400 - (\underline{5.670})(-6.130) \doteq 591400 + 34.76$

$= 591434.76 \doteq \underline{591435}$

2 $591700 - (\underline{5.670})(46.78) \doteq 591700 - 265.2$

$= 591434.8 \doteq \underline{591435}$

Backsolving: $x_2 = 1.000$, and

$$x_2 = \frac{46.78 + (6.130)(1.000)}{5.291} = \frac{52.91}{5.291} = 10.00$$