Physics 396

Exam 3

Show all correct work for credit!

1. Consider the line element of the form

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2, \tag{1}$$

which corresponds to a 2D spacetime slice of a maximally-symmetric space with positive curvature called *de Sitter spacetime.*

a) Write down all unique combinations of $g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}$ for the line element given in Eq. (1).

b) Explicitly calculate all unique Christoffel symbols.

2. Reconsider the *de Sitter spacetime* line element given by Eq. (1). Using your Christoffel symbols from the previous problem and the *geodesic equation*, construct the equations of motion.

3. Reconsider the *de Sitter spacetime* line element given by Eq. (1).

a) The line element of Eq. (1) contains a symmetry. Identify this symmetry and *explicitly* show that by shifting the coordinate associated with this symmetry by a constant, the line element remains unchanged.

b) Construct the *first integral* associated with this symmetry.

c) Another *first integral* can be constructed from the fact that freely-falling massive particles follow *timelike* four-velocities. Using Eq. (1), construct this first integral.

d) Using the first integrals from b) and c), show that these equate to an expression of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r), \tag{2}$$

where $\mathcal{E} \equiv (e^2 - 1)/2$. Explicitly state the functional form of $V_{\text{eff}}(r)$.

4. Consider *de Sitter spacetime* in static coordinates of the form

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

a) Consider a generic coordinate transformation to new coordinates (v, r, θ, ϕ) of the form

$$t = v + f(r). \tag{4}$$

Generate the differential equation that determines $f(r)$ by setting $g_{rr} = 0$ and brings Eq. (3) to form

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5}$$

b) Solve this equation for $f(r)$.

c) Consider *radial light rays* in this de Sitter spacetime in the new coordinates (v, r, θ, ϕ). Calculate the slopes of the light cone at a point (v, r).

1. $ds^2 = -(1-\frac{\Lambda}{3}r^2)dt^2 + (1-\frac{\Lambda}{3}r^2)^{-1}dr^2$

$g_{tt} = -(1-\frac{\Lambda}{3}r^2)$ ✓

$g_{rr} = (1-\frac{\Lambda}{3}r^2)^{-1}$ ✓

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

$g_{t\delta}\Gamma^{\delta}_{tt}$ ✓, $g_{t\delta}\Gamma^{\delta}_{tr}$ ✓, $g_{r\delta}\Gamma^{\delta}_{tt}$ ✓, $g_{t\delta}\Gamma^{\delta}_{rr}$ ✓, $g_{r\delta}\Gamma^{\delta}_{tr}$ ✓, $g_{r\delta}\Gamma^{\delta}_{rr}$ ✓

1) $g_{t\delta}\Gamma^{\delta}_{tt} = \frac{1}{2}\left(\cancel{\frac{\partial g_{tt}}{\partial t}} + \cancel{\frac{\partial g_{tt}}{\partial t}} - \cancel{\frac{\partial g_{tt}}{\partial t}}\right) = 0$ ✓

2) $g_{t\delta}\Gamma^{\delta}_{tr} = \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial r} + \cancel{\frac{\partial g_{tr}}{\partial t}} - \cancel{\frac{\partial g_{tr}}{\partial t}}\right) = -\frac{1}{2}\frac{\partial}{\partial r}(1-\frac{\Lambda}{3}r^2) = -\frac{1}{2}\cdot -\frac{2}{3}\Lambda r = \frac{1}{3}\Lambda r$ ✓

3) $g_{r\delta}\Gamma^{\delta}_{tt} = \frac{1}{2}\left(\cancel{\frac{\partial g_{rt}}{\partial t}} + \cancel{\frac{\partial g_{rt}}{\partial t}} - \frac{\partial g_{tt}}{\partial r}\right) = +\frac{1}{2}\frac{\partial}{\partial r}(1-\frac{\Lambda}{3}r^2) = \frac{1}{2}\cdot -\frac{2}{3}\Lambda r = -\frac{1}{3}\Lambda r$ ✓

) $g_{t\delta}\Gamma^{\delta}_{rr} = \frac{1}{2}\left(\cancel{\frac{\partial g_{tr}}{\partial r}} + \cancel{\frac{\partial g_{tr}}{\partial r}} - \cancel{\frac{\partial g_{rr}}{\partial t}}\right) = 0$ ✓

) $g_{r\delta}\Gamma^{\delta}_{tr} = \frac{1}{2}\left(\cancel{\frac{\partial g_{rt}}{\partial r}} + \cancel{\frac{\partial g_{rr}}{\partial t}} - \cancel{\frac{\partial g_{tr}}{\partial r}}\right) = 0$ ✓

$g_{r\delta}\Gamma^{\delta}_{rr} = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial r} + \cancel{\frac{\partial g_{rr}}{\partial r}} - \cancel{\frac{\partial g_{rr}}{\partial r}}\right) = \frac{1}{2}\frac{\partial}{\partial r}(1-\frac{\Lambda}{3}r^2)^{-1} = -\frac{1}{2}(1-\frac{\Lambda}{3}r^2)^{-2}\cdot -\frac{2}{3}\Lambda r$

$= \frac{1}{3}\frac{\Lambda r}{(1-\frac{\Lambda}{3}r^2)^2}$ ✓

so

$g_{t\delta}\Gamma^{\delta}_{tr} = g_{tt}\Gamma^{t}_{tr} = -(1-\frac{\Lambda}{3}r^2)\Gamma^{t}_{tr} = \frac{1}{3}\Lambda r \quad \therefore \left[\Gamma^{t}_{tr} = -\frac{1}{3}\frac{\Lambda r}{(1-\frac{\Lambda}{3}r^2)}\right]$ ✓

$g_{r\delta}\Gamma^{\delta}_{tt} = g_{rr}\Gamma^{r}_{tt} = (1-\frac{\Lambda}{3}r^2)^{-1}\Gamma^{r}_{tt} = -\frac{1}{3}\Lambda r \quad \therefore \left[\Gamma^{r}_{tt} = -\frac{1}{3}\Lambda r(1-\frac{\Lambda}{3}r^2)\right]$ ✓

$g_{r\delta}\Gamma^{\delta}_{rr} = g_{rr}\Gamma^{r}_{rr} = (1-\frac{\Lambda}{3}r^2)^{-1}\Gamma^{r}_{rr} = \frac{1}{3}\frac{\Lambda r}{(1-\frac{\Lambda}{3}r^2)^2} \quad \therefore \left[\Gamma^{r}_{rr} = \frac{1}{3}\frac{\Lambda r}{(1-\frac{\Lambda}{3}r^2)}\right]$ ✓

2. $ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2$ where $g_{tt} = -\left(1-\frac{\Lambda}{3}r^2\right)$

$g_{rr} = \left(1-\frac{\Lambda}{3}r^2\right)^{-1}$

THE GEODESIC EQN IS...

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$$

so

$$\frac{d^2t}{d\tau^2} = -\Gamma^t_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^t_{tr}\frac{dt}{d\tau}\frac{dr}{d\tau} - \Gamma^t_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} = -2\cdot -\frac{1}{3}\frac{\Lambda r}{\left(1-\frac{\Lambda}{3}r^2\right)}\frac{dr}{d\tau}\frac{dt}{d\tau}$$

so $$\left[\frac{d^2t}{d\tau^2} = \frac{2}{3}\frac{\Lambda r}{\left(1-\frac{\Lambda}{3}r^2\right)}\frac{dr}{d\tau}\frac{dt}{d\tau}\right] \checkmark$$

$$\frac{d^2r}{d\tau^2} = -\Gamma^r_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^r_{rr}\left(\frac{dr}{d\tau}\right)^2$$

$$= +\frac{1}{3}\Lambda r\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{3}\frac{\Lambda r}{\left(1-\frac{\Lambda}{3}r^2\right)}\left(\frac{dr}{d\tau}\right)^2$$

so $$\left[\frac{d^2r}{d\tau^2} = \frac{1}{3}\Lambda r\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{3}\frac{\Lambda r}{\left(1-\frac{\Lambda}{3}r^2\right)}\left(\frac{dr}{d\tau}\right)^2\right] \checkmark$$

3. $ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2$ WHERE $g_{tt} = -\left(1-\frac{\Lambda}{3}r^2\right)$

$g_{rr} = \left(1-\frac{\Lambda}{3}r^2\right)^{-1}$

SINCE $t \to t + \text{CONST}$ LEAVES THE LINE ELEMENT INVARIANT, THERE IS AN ASSOCIATED KILLING VECTOR $\underset{\sim}{\xi}$

$dt \to dt \therefore ds^2 \to ds^2$ w/ $\xi^\alpha = (1, 0)$

so

$$-e = \underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{tt}\xi^t u^t = -\left(1-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau}$$ so $e = \left(1-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau}$ ✓

PROOF

TAKING A TIME DERIVATIVE..

$$① : \frac{d}{d\tau}\left(1-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau} = -\frac{2}{3}\Lambda r\frac{dr}{d\tau}\frac{dt}{d\tau} + \left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{d^2t}{d\tau^2}\right)$$

or

$$\frac{d^2t}{d\tau^2} = +\frac{2}{3}\frac{\Lambda r}{\left(1-\frac{\Lambda}{3}r^2\right)}\frac{dr}{d\tau}\frac{dt}{d\tau}$$ ✓

THERE IS ALSO A FIRST INTEGRAL OF THE FORM...

$$-1 = \underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = +g_{tt}(u^t)^2 + g_{rr}(u^r)^2$$

$$= -\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

NOW

$$\frac{dt}{d\tau} = e\left(1-\frac{\Lambda}{3}r^2\right)^{-1}$$

$$-1 = -e^2\left(1-\frac{\Lambda}{3}r^2\right)^{-1} + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

$$-\left(1-\frac{\Lambda}{3}r^2\right) = e^2 + \left(\frac{dr}{d\tau}\right)^2$$ so $e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{\Lambda}{3}r^2\right)$

so

$$\mathcal{E} = \frac{1}{2}(e^2-1) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{\Lambda}{3}r^2\right) - 1\right] = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{\Lambda}{6}r^2$$ ✓

4. $ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2$ WHERE $g_{tt} = -\left(1-\frac{\Lambda}{3}r^2\right)$

$g_{rr} = \left(1-\frac{\Lambda}{3}r^2\right)^{-1}$

Let $t = v + f(r)$

so $dt = dv + \frac{df}{dr}dr$

so

$$dt^2 = dv^2 + 2\frac{df}{dr}dv\,dr + \left(\frac{df}{dr}\right)^2 dr^2$$

so

$$ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)\left(dv^2 + 2\frac{df}{dr}dv\,dr + \left(\frac{df}{dr}\right)^2 dr^2\right) + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2$$

(*)

$$ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)\left(dv^2 + 2\frac{df}{dr}dv\,dr\right) + \left(-\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{df}{dr}\right)^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}\right)dr^2$$

TO MAKE THE g_{rr} TERM ZERO, SET ...

$$-\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{df}{dr}\right)^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1} = 0$$

so

$$\left(\frac{df}{dr}\right)^2 = \left(1-\frac{\Lambda}{3}r^2\right)^{-2} \quad \text{so} \quad \left[\frac{df}{dr} = \pm\frac{1}{\left(1-\frac{\Lambda}{3}r^2\right)}\right]$$

so

$$\int df = \pm\int\frac{dr}{\left(1-\frac{\Lambda}{3}r^2\right)}$$

NOW

$$\int\frac{dr}{\left(1-\frac{\Lambda}{3}r^2\right)} = \sqrt{\frac{3}{\Lambda}}\int\frac{\operatorname{sech}^2\theta\,d\theta}{1-\tanh^2\theta} = \sqrt{\frac{3}{\Lambda}}\,\theta = \sqrt{\frac{3}{\Lambda}}\tanh^{-1}\left(\sqrt{\frac{\Lambda}{3}}r\right)$$

$\sqrt{\frac{\Lambda}{3}}r = \tanh\theta \quad \therefore \theta = \tanh^{-1}\left(\sqrt{\frac{\Lambda}{3}}r\right)$

$\sqrt{\frac{\Lambda}{3}}dr = \operatorname{sech}^2\theta\,d\theta$ so $f(r) = \pm\sqrt{\frac{3}{\Lambda}}\tanh^{-1}\left(\sqrt{\frac{\Lambda}{3}}r\right) + c$

$\therefore dr = \sqrt{\frac{3}{\Lambda}}\operatorname{sech}^2\theta\,d\theta$

WHERE

$\cosh^2\theta - \sinh^2\theta = 1$

so

$1 - \tanh^2\theta = \operatorname{sech}^2\theta$

NOW PLUGGING THESE INTO (*) YIELDS...

$$ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)\left(dv^2 \pm \frac{2}{\left(1-\frac{\Lambda}{3}r^2\right)}dv\,dr\right)$$

SO, CHOOSING THE NEGATIVE SIGN...

$$ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)dv^2 + 2\,dv\,dr$$

SO THE SLOPES OF RADIAL LIGHT RAYS IS FOUND BY SETTING $ds^2 = 0$

$$0 = -\left(1-\frac{\Lambda}{3}r^2\right)dv^2 + 2\,dv\,dr = dv\,dr\left[-\left(1-\frac{\Lambda}{3}r^2\right)\frac{dv}{dr} + 2\right]$$

SO

$$\frac{dv}{dr} = \frac{2}{\left(1-\frac{\Lambda}{3}r^2\right)}$$ ✓