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PHYS 311 HW 8

Problem 1

a.) $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

V must depend only on r!

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \cancelto{0}{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)} + \cancelto{0}{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}} = 0$$

$$\nabla^2 V = \underbrace{\frac{1}{r^2}}_{\text{Never can be zero}}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2 \frac{dV}{dr} = 0 \quad \vdots \quad r^2 \frac{dV}{dr} = C_1$$

$$\int dV = C_1 \int \frac{dr}{r^2} \quad \vdots \quad V = -\frac{C_1}{r} + C_2$$

$$\boxed{V(r) = C_2 - \frac{C_1}{r}}$$

b.) $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

V must only depend on S!

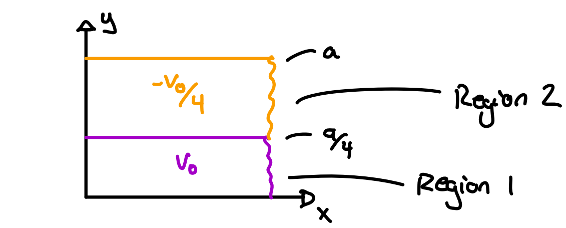
$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) + \cancelto{0}{\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}} + \cancelto{0}{\frac{\partial^2 V}{\partial z^2}} = 0$$

$$\nabla^2 V = \underbrace{\frac{1}{s}}_{\text{Never can be zero}}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) = 0$$

$$s \cdot \frac{dV}{ds} = 0 \quad \vdots \quad s \cdot \frac{dV}{ds} = C_1 \quad \therefore \quad \int dV = C_1 \int \frac{ds}{s}, \quad V = C_1 \ln(s) + C_2$$

$$\boxed{V(s) = C_1 \ln(s) + C_2}$$

Problem 2



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$X = (Ae^{kx} + Be^{-kx})(C\cos(ky) + D\sin(ky))$$

Region 1 & 2 B.C's

(i) $V=0,\ y=0$: $V(x,0) = (Ae^{kx}+Be^{-kx})(C\cancel{\cos}(0)^{1} + D\cancel{\sin}(0)^{0}) = 0 \quad \therefore\ C=0$

(ii) $V=0,\ y=a$: $V(x,a) = (Ae^{kx}+Be^{-kx})(D\sin(ka)) = 0 \quad \therefore\ k = \frac{n\pi}{a}$

(iii) $V=0,\ x\to\infty$: $V(\infty,y) = (A)(D\sin(ky)) = 0 \quad \therefore\ A=0$

(iv) $V=V_0(y),\ x=0$ $\quad V(x,y) = Be^{-kx}\cdot D\sin(ky)$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right)$$

BD → C $\quad k = \frac{n\pi}{a}$

$$\text{(iv)}\quad V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}y\right) = V_0(y)$$

$$= \sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy = \int_0^a V_0(y)\sin\left(\frac{n'\pi}{a}y\right)dy \quad \text{iff } n'=n$$

$$= \sum_{n=1}^{\infty} C_n \cdot \frac{a}{2} = \int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy$$

$$= \sum_{n=1}^{\infty} C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy$$

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy \ :\ C_n = \frac{2}{a}\left[\int_0^{a/4} V_0 \sin\left(\frac{n\pi}{a}y\right)dy + \int_{a/4}^{a} \frac{-V_0}{4}\sin\left(\frac{n\pi}{a}y\right)dy\right]$$

$$V_0\int_0^{a/4}\sin\left(\frac{n\pi}{a}y\right)dy = V_0\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}y\right)\right]_0^{a/4} = -\frac{V_0 a}{n\pi}\left[\cos\left(\frac{n\pi}{4}\right) - \cos(0)\right]$$

$$= \frac{V_0 a}{n\pi}\left[1 - \cos\left(\frac{n\pi}{4}\right)\right] \quad (1^*)$$

$$-\frac{V_0}{4}\int_{a/4}^{a}\sin\left(\frac{n\pi}{a}y\right)dy = \frac{V_0}{4}\left[\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}y\right)\right]_{a/4}^{a} = \frac{V_0 a}{4n\pi}\left[\cos(n\pi) - \cos(n\pi/4)\right]$$

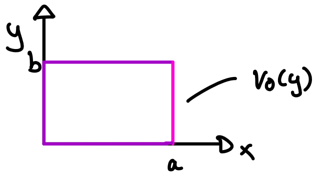
$$= \frac{V_0 a}{4n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{4}\right)\right] \quad (2^*)$$

$$\frac{2}{a}\left[(1^*) + (2^*)\right] = \frac{2}{a}\left[\frac{V_0 a}{n\pi}\left[1-\cos\left(\frac{n\pi}{4}\right)\right] + \frac{V_0 a}{4n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{4}\right)\right]\right]$$

$$= \frac{2V_0}{n\pi}\left[\left(1-\cos\left(\frac{n\pi}{4}\right)\right) + \frac{1}{4}\left((-1)^n - \cos\left(\frac{n\pi}{4}\right)\right)\right] = \frac{2V_0}{n\pi}\left[1 + \frac{(-1)^n}{4} - \frac{5}{4}\cos(n\pi/4)\right]$$

$$\therefore\ \boxed{V(x,y) = \frac{2V_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + \frac{(-1)^n}{4} - \frac{5}{4}\cos(n\pi/4)\right] e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)}$$

Problem 3



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad X(x) = Ae^{kx} + Be^{-kx}, \quad Y(y) = C\cos(ky) + D\sin(ky)$$

$$V(x,y) = X(x)\,Y(y)$$

a.) <u>B.C's</u>

(i) $V=0,\ y=0$

(ii) $V=0,\ y=b$

(iii) $V=V_0(y),\ x=a$

(iv) $V=0,\ x=0$

(i) $V(x,0) = (Ae^{kx}+Be^{-kx})(C\overset{1}{\cancel{\cos}}(0) + D\overset{0}{\cancel{\sin}}(0)) = 0 \quad \therefore \quad C=0$

(ii) $V(x,b) = (Ae^{kx}+Be^{-kx})(D\sin(kb)) = 0 \quad \therefore \quad k = \frac{n\pi}{b}$

(iv) $V(0,y) = (A+B)(D\sin(ky)) = 0 \quad \therefore \quad B=-A$

$$V(x,y) = \sum_{n=1}^{\infty} \underbrace{2\cdot A\cdot D}_{2AD \to C}\cdot\sinh(kx)\sin(ky) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right) \; : \; V(a,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right) = V_0(y)$$

$$\sum_{n=1}^{\infty} C_n\cdot\sinh\left(\frac{n\pi}{b}a\right)\cdot\int_0^b \sin\left(\frac{n'\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)dy = \int_0^b V_0(y)\sin\left(\frac{n'\pi}{b}y\right)dy$$

iff $n=n'$

$$\sum_{n=1}^{\infty} C_n\cdot\sinh\left(\frac{n\pi}{b}a\right)\cdot\frac{b}{2} = \int_0^b V_0(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

$$\boxed{C_n = \frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\cdot\frac{2}{b}\int_0^b V_0(y)\sin\left(\frac{n\pi}{b}y\right)dy}$$

b.) $V_0(y) \to V_0$

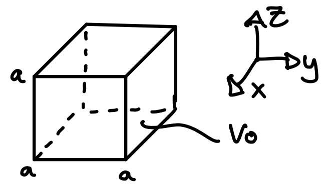
$$C_n = \frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\cdot\frac{2}{b}\cdot V_0\int_0^b \sin\left(\frac{n\pi}{b}\cdot y\right)dy = \frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\cdot\frac{2V_0}{b}\left[-\frac{b}{n\pi}\cos\left(\frac{n\pi}{b}y\right)\right]_0^b$$

$$C_n = \frac{-1}{\sinh\left(\frac{n\pi}{b}a\right)}\cdot\frac{2V_0}{n\pi}\left[\cos(n\pi) - 1\right] = \frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\frac{2V_0}{n\pi}\left[1-(-1)^n\right]$$

$$C_n = \begin{cases} \frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\frac{4V_0}{n\pi} & \text{iff } n=2k+1 \\ 0 & \text{iff } n=2k \end{cases}$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n\ \text{odd}}^{\infty}\frac{1}{n}\,\frac{\sinh\left(\frac{n\pi}{b}x\right)}{\sinh\left(\frac{n\pi}{b}a\right)}\sin\left(\frac{n\pi}{b}y\right)}$$

Problem 4



$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$X(x) = A\cos(kx) + B\sin(kx)$$
$$Y(y) = Ce^{\sqrt{k^2+\ell^2}\,y} + De^{-\sqrt{k^2+\ell^2}\,y}$$
$$Z(z) = E\cos(\ell z) + F\sin(\ell z)$$

B.C's

(i) $V=0,\ y=0$
(ii) $V=0,\ z=0$
(iii) $V=0,\ x=0$
(iv) $V=V_0,\ y=a$
(v) $V=0,\ z=a$
(vi) $V=0,\ x=a$

B.C's

(i) $V(x,0,z) = X(C+D)Z = 0 \quad \therefore \quad D=-C$

(ii) $V(x,y,0) = XY(E) = 0 \quad \therefore \quad E=0$ $\qquad 2\cdot ACE - DC$

(iii) $V(0,y,z) = (A)\,YZ = 0 \quad \therefore \quad A=0$

(v) $V(x,y,a) = XY(F\sin(\ell a)) = 0 \quad \therefore \quad \ell_m = \frac{m\pi}{a}$

(vi) $V(a,y,z) = B\sin(ka)\,YZ = 0 \quad \therefore \quad k_n = \frac{n\pi}{a}$

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\sinh\left(\sqrt{n^2+m^2}\,\tfrac{\pi}{a}y\right)\sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{m\pi}{a}z\right)$$

(iv): $$V(x,a,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\sinh\left(\sqrt{n^2+m^2}\cdot\pi\right)\sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{m\pi}{a}z\right) = V_0(x,z)$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\sinh\left(\sqrt{n^2+m^2}\,\pi\right)\int_0^a \sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{n'\pi}{a}x\right)dx\int_0^a \sin\left(\tfrac{m'\pi}{a}z\right)\sin\left(\tfrac{m\pi}{a}z\right)dz$$
$$= \int_0^a\int_0^a V_0(x,z)\sin\left(\tfrac{n'\pi}{a}x\right)\sin\left(\tfrac{m'\pi}{a}z\right)dx\,dz$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\sinh\left(\sqrt{n^2+m^2}\,\pi\right)\cdot\frac{a}{2}\cdot\frac{a}{2} = \int_0^a\int_0^a V_0\sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{n\pi}{a}z\right)dx\,dz$$

$$C_{mn}\cdot\sinh\left(\sqrt{n^2+m^2}\,\pi\right)\cdot\frac{a^2}{4} = V_0\int_0^a\int_0^a \sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{m\pi}{a}z\right)dx\,dz$$

$$C_{mn} = \frac{4V_0}{a^2}\cdot\frac{1}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)}\int_0^a\int_0^a \sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{m\pi}{a}z\right)dx\,dz$$

$$C_{mn} = \frac{4V_0}{a^2}\cdot\frac{1}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)}\left[\frac{a^2}{mn\pi^2}\cos\left(\tfrac{n\pi}{a}x\right)\Big|_0^a\cos\left(\tfrac{m\pi}{a}z\right)\Big|_0^a\right]$$

$$C_{mn} = \frac{4V_0}{mn\pi^2}\,\frac{1}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)}\Big[(\cos(n\pi)-1)(\cos(m\pi)-1)\Big]$$

$$C_{mn} = \frac{4V_0}{mn\pi^2}\,\frac{1}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)}\Big[((-1)^n-1)((-1)^m-1)\Big]$$

$$C_{mn} = \begin{cases} \dfrac{16V_0}{mn\pi^2}\,\dfrac{1}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)} & n,m = 2k+1 \\ 0 & n,m = 2k \end{cases}$$

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n\,odd}^{\infty}\sum_{m\,odd}^{\infty}\frac{1}{mn}\,\frac{\sinh\left(\sqrt{n^2+m^2}\,\frac{\pi}{a}y\right)}{\sinh\left(\sqrt{n^2+m^2}\,\pi\right)}\sin\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{n\pi}{a}z\right)}$$