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PHYS 396
HW 8

Problem 1

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

a.) Inner Shell : $C_1 = 6\pi M$ $\quad dt^2 = d\theta^2 = d\varphi^2 = 0$

Outer Shell : $C_2 = 20\pi M$

$C = 2\pi r$: $C_1 = C$: $6\pi M = 2\pi r_1 \rightarrow r_1 = 3M$

$C_2 = C$: $20\pi M = 2\pi r_2 \rightarrow r_2 = 10M$

$$ds^2 = \left(1-\frac{2M}{r}\right)^{-1}dr^2 \rightarrow S = \int_{r_1}^{r_2}\left(1-\frac{2M}{r}\right)^{-1/2}dr \quad \text{using TI-nspire}$$

$$S = \int_{3M}^{10M}\left(1-\frac{2M}{r}\right)^{-1/2}dr = -\left(M\cdot\ln|r| + 2M\ln\left(\sqrt{\frac{r-2M}{r}}-1\right) - r\sqrt{\frac{r-2M}{r}}\right)\Bigg|_{3M}^{10M}$$

$$S = M\left[\ln\left(-(4\sqrt{5}+9)\cdot(\sqrt{3}-2)\right) + (4\sqrt{5}-3)\right] = 8.78\,M$$

$$\boxed{S = 8.78\,M}$$

b.) $dV = \sqrt{g_{11}g_{22}g_{33}}\,dx^1dx^2dx^3$

$g_{11} = (1-2M/r)^{-1}$ $dx^1 = dr$, $g_{22} = r^2$ $dx^2 = d\theta$, $g_{33} = r^2\sin^2\theta$ $dx^3 = d\varphi$

$$dV = \left((1-2M/r)^{-1}\cdot r^2\cdot r^2\sin^2\theta\right)^{1/2}dr\,d\theta\,d\varphi \; : \; V = \int_0^{2\pi}\int_0^{\pi}\int_{3M}^{10M}\frac{r^2}{\sqrt{1-2M/r}}\sin\theta\,dr\,d\theta\,d\varphi$$

$$V = 4\pi\cdot\int_{3M}^{10M}\frac{r^2}{\sqrt{1-2M/r}}dr \rightarrow r = 2Mx \quad r = 3M,\ x = 3/2 \quad dr = 2M\,dx$$

$r = 10M$, $x = 5$

$$= 4\pi\int_{\frac{3}{2}}^{5}\frac{8M^3x^2}{\sqrt{1-1/x}}dx = 32M^3\pi\int_{\frac{3}{2}}^{5}\frac{x^2}{\sqrt{1-1/x}}dx = 32M^3\pi\cdot 48.1385 = 4840\,M^3$$

$$\boxed{V = 4840\,M^3}$$

Problem 2

$$\mathcal{E} = \tfrac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

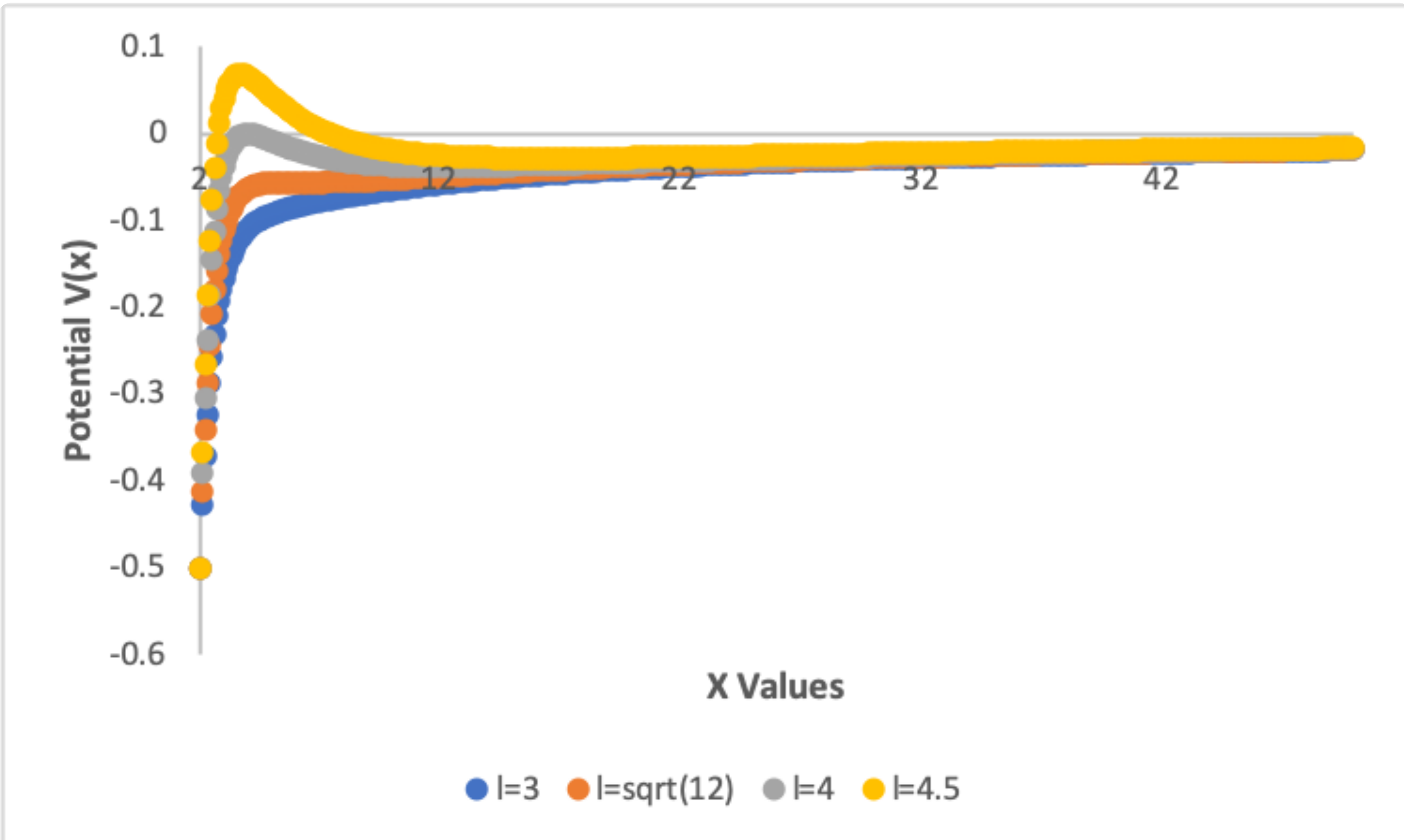
a.) $V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$

$r = Mx \ , \ \ell = M\tilde{\ell}$

$$V_{eff}(Mx) = -\frac{M}{Mx} + \frac{M^2\tilde{\ell}^2}{2\cdot M^2x^2} - \frac{M\cdot M^2\tilde{\ell}^2}{M^3x^3} = -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3}$$

$$\boxed{V_{eff}(x) = -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3}}$$

b.)



Problem 3

$$ds^2 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$

a.)

$$\underset{\sim}{\xi} = (1,0,0,0)\ , \quad \underset{\sim}{\eta} = (0,0,0,1)$$

$$\underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{00}\xi^0 u^0 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau} = -e$$

$$\boxed{-e = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau}}$$

$$\underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{33}\eta^3 u^3 = r^2\sin^2\theta\,\frac{d\varphi}{d\tau} = \ell \qquad \frac{dt}{d\tau} = e\left(1-2M/r-\Lambda/3\, r^2\right)^{-1}$$

$$\boxed{\ell = r^2\sin^2\theta\,\frac{d\varphi}{d\tau}} \qquad \frac{d\varphi}{d\tau} = \ell\,(r^2\sin^2\theta)^{-1}$$

b.) $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = g_{00}u^0u^0 + g_{11}u^1u^1 + g_{22}u^2u^2 + g_{33}u^3u^3 = -1$$

$$\boxed{\underset{\sim}{u}\cdot\underset{\sim}{u} = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + (r^2)\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 = -1}$$

c.) $\theta = \pi/2$

$$-\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

$$-\left(1-(2M/r)-(\Lambda/3)r^2\right)e^2\left(1-(2M/r)-(\Lambda/3)r^2\right)^{-2} + \left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\ell^2(r^2)^{-2} = -1$$

$$-e^2\left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1} + \left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -1$$

$$-e^2\left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1} + \left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = -1-\frac{\ell^2}{r^2}$$

$$-\left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}e^2 + \left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} = -1$$

$$-\left(1-(2M/r)-(\Lambda/3)r^2\right)^{-1}\left(e^2-\left(\frac{dr}{d\tau}\right)^2\right) = -1-\frac{\ell^2}{r^2}$$

$$e^2-\left(\frac{dr}{d\tau}\right)^2 = \left(1+\frac{\ell^2}{r^2}\right)\left(1-(2M/r)-(\Lambda/3)r^2\right)$$

$$e^2-\left(\frac{dr}{d\tau}\right)^2 = 1-\frac{2M}{r}+\frac{\ell^2}{r^2}-\frac{2M\ell^2}{r^3}-\frac{\Lambda}{3}(r^2+\ell^2)$$

Problem 3 Continued

$$\frac{1}{2}\left[e^2-1-\left(\frac{dr}{d\tau}\right)^2=\frac{-2M}{r}+\frac{\ell^2}{r^2}-\frac{2M\ell^2}{r^3}-\frac{\Lambda}{3}(r^2+\ell^2)\right]$$

$$=\frac{e^2}{2}-\frac{1}{2}-\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2=\frac{-M}{r}+\frac{\ell^2}{2r^2}-\frac{M\ell^2}{r^3}-\frac{\Lambda}{6}(r^2+\ell^2)$$

$$V_{eff}(r)=\left(-\frac{M}{r}+\frac{\ell^2}{2r^2}-\frac{M\ell^2}{r^3}-\frac{\Lambda}{6}(r^2+\ell^2)\right)$$

$$=\frac{1}{2}(e^2-1)=V_{eff}(r)+\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \quad : \quad \mathcal{E}=\frac{1}{2}(e^2-1)$$

$$\boxed{\mathcal{E}=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+V_{eff}(r)}$$

Problem 4)

a.) $V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2)$

$$\frac{dV_{eff}(r)}{dr} = \frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4} - \frac{\Lambda}{6}(2r) = \frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4} - \frac{\Lambda r}{3} = 0$$

$$\boxed{\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4} - \frac{\Lambda r}{3} = 0}$$

5th degree polynomial

b.) Turning points occur when there is no radial displacement. Simply put

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2) \qquad \mathcal{E} = \frac{1}{2}\Big(\overset{0}{\cancel{dr/d\tau}}\Big)^2 + V_{eff}(r) \qquad : \mathcal{E} = (e^2 - 1)/2$$

$$\boxed{\frac{(e^2-1)}{2} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2)}$$

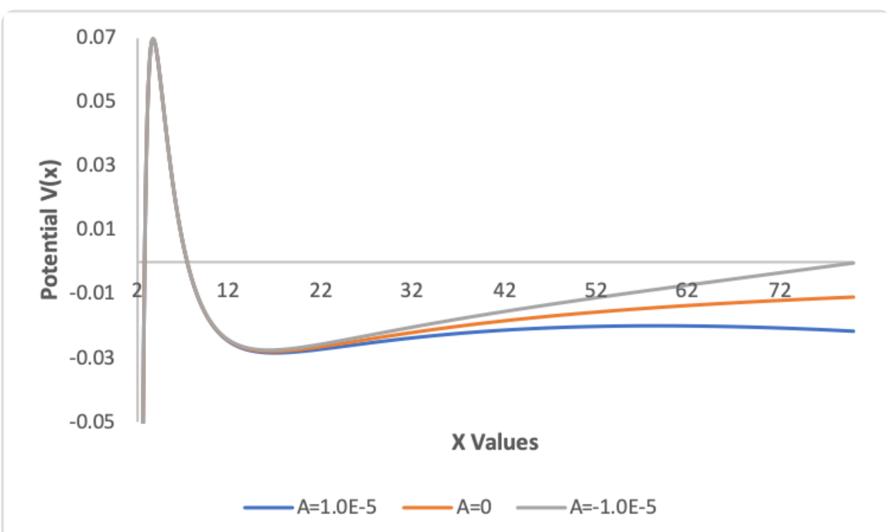
c.) $V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2) \quad : r \equiv Mx, \; \ell \equiv M\tilde{\ell}, \; \Lambda \equiv \hat{\Lambda}/M^2$

$$V_{eff}(Mx) = \frac{-M}{Mx} + \frac{(M\tilde{\ell})^2}{2(Mx)^2} - \frac{M(M\tilde{\ell})^2}{(Mx)^3} - \frac{1}{6}\frac{\hat{\Lambda}}{M^2}\left((M\tilde{\ell})^2 + (Mx)^2\right)$$

$$= -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3} - \frac{\tilde{\Lambda}}{6}(\tilde{\ell}^2 + x^2)$$

$$\boxed{V_{eff}(x) = -\frac{1}{x} + \frac{\tilde{\ell}^2}{2x^2} - \frac{\tilde{\ell}^2}{x^3} - \frac{\tilde{\Lambda}}{6}(\tilde{\ell}^2 + x^2)}$$

d.)



Problem 5

$$ds^2 = -\left(1 - 2m/r - \Lambda/3\, r^2\right)dt^2 + \left(1 - 2m/r - \Lambda/3\, r^2\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

a.) $-e = \underset{\sim}{\xi} \cdot \underset{\sim}{u}$ $\qquad \xi = (1,0,0,0) \quad \eta = (0,0,0,1)$

$\ell = \underset{\sim}{\eta} \cdot \underset{\sim}{u}$

$$\underset{\sim}{\xi} \cdot \underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{00}\xi^0 u^0 = -\left(1 - 2m/r - \Lambda/3\, r^2\right)\frac{dt}{d\lambda} = -e$$

$$\underset{\sim}{\eta} \cdot \underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{33}\eta^3 u^3 = (r^2\sin^2\theta)\frac{d\varphi}{d\lambda} = \ell$$

$$\boxed{\begin{aligned} -e &= -\left(1 - (2m/r) - (\Lambda/3)r^2\right)\frac{dt}{d\lambda} \\ \ell &= r^2\sin^2\theta\,\frac{d\varphi}{d\lambda} \end{aligned}}$$

b.) $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = g_{00}u^0u^0 + g_{11}u^1u^1 + g_{22}u^2u^2 + g_{33}u^3u^3$$

$$= -\left(1-(2m/r)-(\Lambda/3)r^2\right)\left(dt/d\lambda\right)^2 + \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(dr/d\lambda\right)^2 + (r^2)\left(d\theta/d\lambda\right)^2 + (r^2\sin^2\theta)\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\boxed{\underset{\sim}{u} \cdot \underset{\sim}{u} = -\left(1-(2m/r)-(\Lambda/3)r^2\right)\left(dt/d\lambda\right)^2 + \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(dr/d\lambda\right)^2 + (r^2)\left(d\theta/d\lambda\right)^2 + (r^2\sin^2\theta)\left(\frac{d\varphi}{d\lambda}\right)^2}$$

c.) $\theta = \pi/2 \qquad \dfrac{dt}{d\lambda} = e\left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1} \;:\; \dfrac{d\varphi}{d\lambda} = \ell\left(r^2\sin^2\theta\right)^{-1}$

$$= -\left(1-(2m/r)-(\Lambda/3)r^2\right)e^2\left(1-(2m/r)-(\Lambda/3)r^2\right)^{-2} + \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(dr/d\lambda\right)^2 + (r^2)\cdot\ell^2(r^2)^{-2}$$

$$= -e^2\left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1} + \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(dr/d\lambda\right)^2 + \frac{\ell^2}{r^2}$$

$$-\frac{\ell^2}{r^2} = -e^2\left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1} + \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(dr/d\lambda\right)^2$$

$$\frac{\ell^2}{r^2} = \left(1-(2m/r)-(\Lambda/3)r^2\right)^{-1}\left(e^2 - \left(dr/d\lambda\right)^2\right) \;:\; \frac{\ell^2}{r^2}\left(1-(2m/r)-(\Lambda/3)r^2\right) = e^2 - \left(dr/d\lambda\right)^2$$

$$\frac{\left(1-(2m/r)-(\Lambda/3)r^2\right)}{r^2} = \frac{e^2}{\ell^2} - \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 \;:\; W_{eff}(r) = \frac{1}{r^2}\left(1-(2m/r)-(\Lambda/3)r^2\right)$$

$$\boxed{\frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)}$$

d.) $W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) = \frac{1}{r^2} - \frac{2M}{r^3} - \frac{\Lambda}{3}$

$\frac{dW_{eff}(r)}{dr} = \frac{-2}{r^3} + \frac{6M}{r^4}$: $\frac{2}{r^3}(-1 + 3M/r) = 0$: $-1 + 3M/r = 0$ $\quad -r + 3M = 0$

$r = 3M$

$$W_{eff}(3M) = \frac{1}{9M^2}\left(1 - \frac{2}{3} - \frac{\Lambda}{3}(9M^2)\right) = \frac{1}{9M^2}\left(\frac{1}{3} - 3\Lambda M^2\right) = \frac{1}{M^2}\left(\frac{1}{27} - \frac{1}{3}\Lambda M^2\right)$$

$$\boxed{W_{eff}(3M) = \frac{1}{M^2}\left(\frac{1}{27} - \frac{1}{3}\Lambda M^2\right)}$$

e.) Turning points occur when there is no radial velocity : i.e. $\frac{dr}{d\lambda} = 0$

$$\frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\cancelto{0}{\frac{dr}{d\lambda}}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)$$

$$\boxed{\frac{e^2}{\ell^2} = \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)}$$