Fri: HW

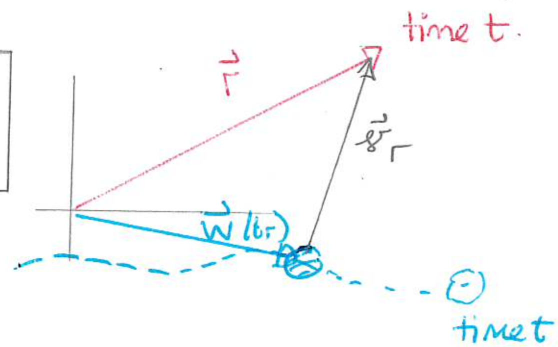
Tues: Read 11.1.2, 11.2.1

Fields produced by a moving point charge

We consider a point particle with charge q and trajectory $\vec{w}(t)$.
The fields produced are:

1) Electric field

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}_r\times(\vec{u}\times\vec{a})\right]}$$



and the ingredients are:

a) retarded time t_r that satisfies

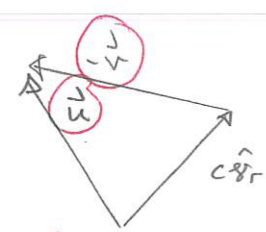
$$c(t-t_r) = |\vec{r}-\vec{w}(t_r)|$$

b) retarded separation vector

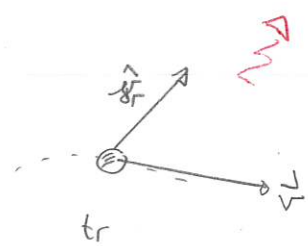
$$\vec{\mathscr{r}}_r = \vec{r}-\vec{w}(t_r) \qquad \text{magnitude} \quad \mathscr{r}_r = c(t-t_r)$$

c) velocity and acceleration

$$\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r} \qquad \vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$$



d) vector $\vec{u} = c\hat{\mathscr{r}}_r - \vec{v}$



2) Magnetic field

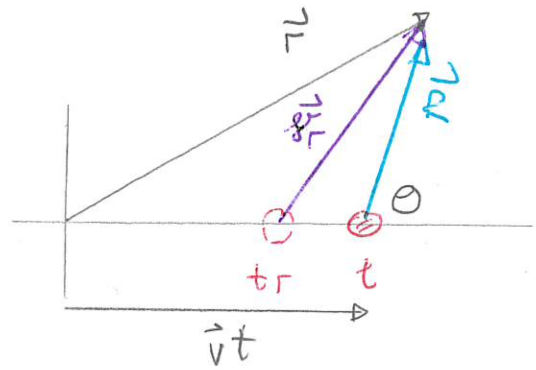
$$\boxed{\vec{B} = \frac{1}{c}\,\hat{\mathscr{r}}_r\times\vec{E}(\vec{r},t)}$$

Constant velocity point source

We showed that, if the particle velocity is constant

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}_r^{\,2}}\frac{1}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3}\left(1-\frac{v^2}{c^2}\right)\frac{(\vec{r}-\vec{v}t)}{\mathscr{r}_r}$$

This refers to the retarded separation vector $\vec{\mathscr{r}}$. It will be more convenient to refer to the current position of the particle. Then the current separation vector is



$$\vec{R} = \vec{r} - \vec{v}t.$$

In an exercise we will see that

$$1-\hat{\mathscr{r}}\cdot\vec{v}/c = \frac{R}{\mathscr{r}_r}\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}$$

where θ is the illustrated angle, between $\vec{v}$ and $\vec{R}$. Thus:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{1-v^2/c^2}{(1-v^2/c^2\sin^2\theta)^{3/2}}\frac{\vec{R}}{R} \Rightarrow \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\hat{R}}$$

Separately using

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}}_r\times\vec{E}$$

we get

$$\boxed{\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}}$$

Derivation:

$$\vec{B} = \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{\mathscr{r}}_r \times \vec{E}\right)$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{r} - \vec{v}t_r\right) \times \vec{E}$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}\left(\vec{R} + \vec{v}(t-t_r)\right) \times \vec{E}$$

$$= \frac{1}{c}\frac{1}{\mathscr{r}_r}(t-t_r)\ \vec{v} \times \vec{E} \qquad \text{since} \qquad \vec{R} \times \vec{E} = 0$$

$$= \frac{1}{c^2}\frac{\cancel{(t-t_r)}}{\cancel{(t-t_r)}}\ \vec{v} \times \vec{E} \qquad \boxtimes$$

1 Energy flow for a charge moving with constant velocity

A charged particle moves with constant velocity $\mathbf{v}$ to the right along the x axis.

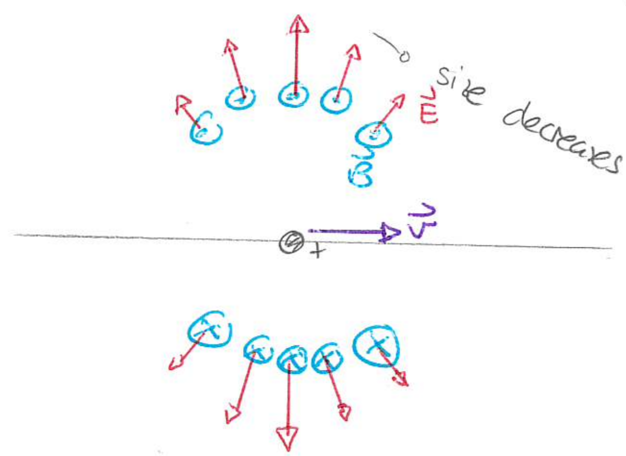
a) Sketch the fields, relative to the particle's location, at any instant. Indicate the dependence of the magnitude of the field on the angle from the x axis.

b) Determine an expression for the Poynting vector at any instant. Express this in terms of the component of $\mathbf{v}$ perpendicular to $\mathbf{R}$.

c) Sketch the direction of energy flow at any instant.

Answer: a) The crucial term is

$$\frac{1-\frac{v^2}{c^2}}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \quad \begin{array}{l} \to \theta = 0 \to 1-\frac{v^2}{c^2} \quad \text{smaller.} \\ \to \theta = \pi/2 \to \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \quad \text{larger} \end{array}$$



b) $$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

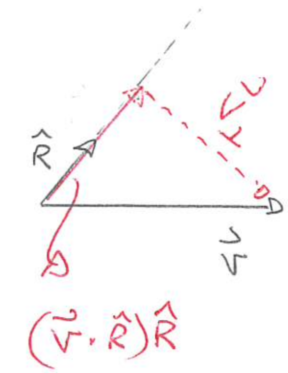
$$= \frac{1}{\mu_0}\frac{1}{c^2}\vec{E}\times(\vec{v}\times\vec{E}) = \frac{1}{\mu_0}\frac{1}{c^2}\left[\vec{v}(\vec{E}\cdot\vec{E}) - \vec{E}(\vec{v}\cdot\vec{E})\right]$$

$$= \frac{1}{\mu_0}\frac{1}{c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{R^4}\frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^3}\left[\vec{v}(\hat{R}\cdot\hat{R}) - \hat{R}(\vec{v}\cdot\hat{R})\right]$$

$$= \frac{1}{\mu_0}\mu_0\epsilon_0\frac{q^2}{16\pi^2\epsilon_0^2}\frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^3}\frac{1}{R^4}\left[\vec{v} - (\vec{v}\cdot\hat{R})\hat{R}\right]$$

Note that

$$(\vec{v} \cdot \hat{R})\,\hat{R}$$



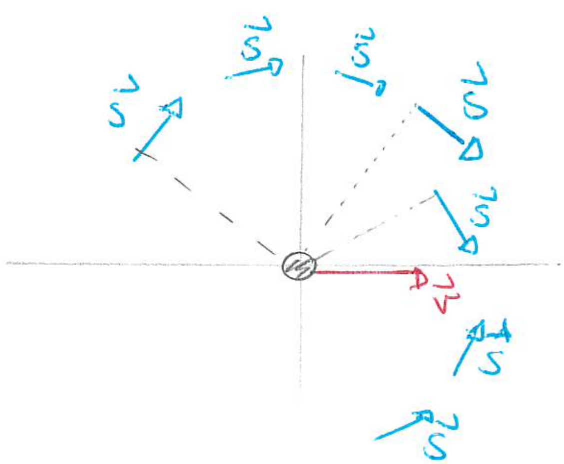
is the component of $\vec{v}$ parallel to $\vec{R}$

Thus

$$\vec{v} - (\vec{v} \cdot \hat{R})\,\hat{R} = \vec{v}_\perp$$

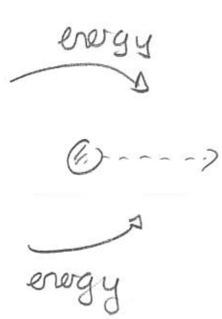
$$\Rightarrow \quad \vec{S} = \frac{q^2}{16\pi^2 \epsilon_0} \frac{1 - v^2/c^2}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^3} \frac{1}{R^4}\,\vec{v}_\perp$$

c)

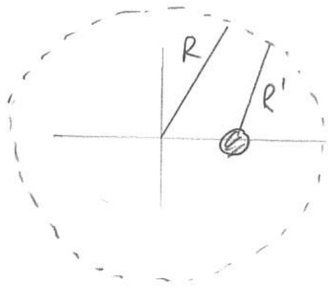


We see that the energy is flowing in the particles forwards direction

It does not appear that the particle radiates energy outwards



We can investigate radiation in the following way. Consider a sphere of radius R centered at the origin. Suppose that the particle is within the sphere. Then the total energy radiated out of the sphere is:



$$\oint \vec{S} \cdot d\vec{a}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \vec{S} \cdot \hat{r} \; R^2 \sin\theta' \, d\theta' \, d\phi'$$

Then $\vec{S} \cdot \hat{r} = \dfrac{q^2}{16\pi^2 \varepsilon_0} \dfrac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2\theta')^3} \dfrac{1}{R'^4} \vec{V}_\perp \cdot \hat{r}$. Thus the rate at which energy is radiated is:

$$\oint \vec{S} \cdot d\vec{a} = \frac{q^2}{16\pi^2 \varepsilon_0} (1 - v^2/c^2) \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \frac{\sin\theta'}{(1 - \frac{v^2}{c^2} \sin^2\theta')^3} \frac{R^2}{R'^4} \vec{V}_\perp \cdot \hat{r}$$

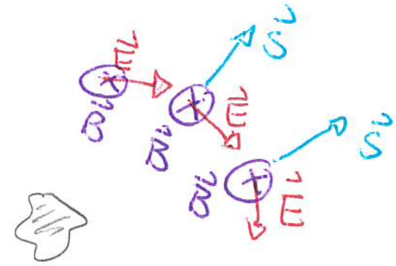
depend on θ, ϕ

Now as $R \to \infty$ $\dfrac{R^2}{R'^4} \to 0$ and the flux is zero. So

$\boxed{\text{A particle moving with constant velocity does not radiate electromagnetic energy.}}$

Radiation

We want to consider situations where charge distributions produce electromagnetic waves. Examples that eventually yield an outward radiating Poynting vector are:



distrib
depends on t

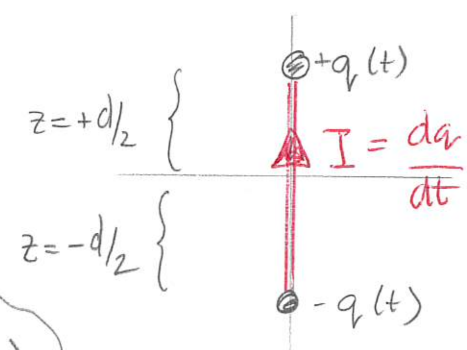
1) oscillating electric dipoles

2) oscillating magnetic dipoles

3) certain moving point charges. (accelerating).

Oscillating electric dipole

Consider a point dipole with two equal oppositely charged particles separated by distance d.
There must, for charge conservation purposes, be a current connecting these.
Then the charge distribution is:



$$\rho(\vec{r}';t) = q(t)\,\delta(\vec{r}' - \tfrac{d}{2}\hat{z}) - q(t)\,\delta(\vec{r}' - \tfrac{d}{2}\hat{z})$$

and the current density satisfies

$$\vec{J}(\vec{r}';t)\,d\tau' \begin{cases} \frac{dq}{dt}\,dz\;\hat{z} & -d/2 \leq z \leq d/2 \\ 0 & \text{otherwise.} \end{cases}$$

We will need to compute a scalar potential

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'}$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$t_r = t - \mathscr{r}/c$

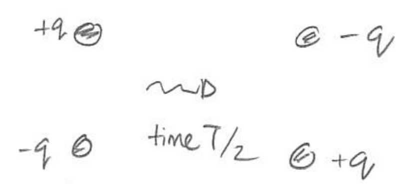
There will also be a vector potential:

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\mathscr{r}} \, d\tau'}$$

We consider an approximate solution that uses:

1) the observation point is distant from the charges (dipole approx)

2) the oscillation rate is such that the inversion of the charge distribution would be non-relativistic



$$\boxed{\frac{d}{T/2} \ll c}$$

3) the observation point is very distant compared to the wavelength

We then reach a first approximation:

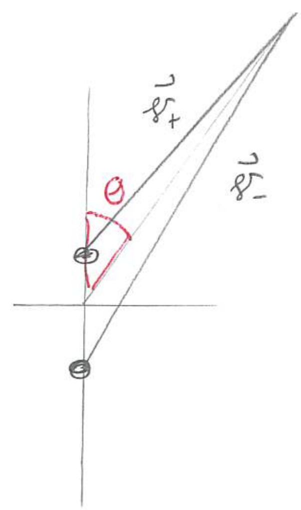
If $r \gg d$ then

$$V(\vec{r},t) \approx \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{ q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right) - q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{d}{2r^2}\cos\theta\left\{ q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right) + q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\}$$

Proof: Use the illustrated geometrical scheme.

Then



$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q(t-\frac{\mathscr{r}_+}{c})}{\mathscr{r}_+} - \frac{q(t-\frac{\mathscr{r}_-}{c})}{\mathscr{r}_-}\right]$$

Then $\vec{\mathscr{r}}_\pm = \vec{r} \mp \frac{d}{2}\hat{z}$. So

$$\mathscr{r}_\pm = \sqrt{\vec{\mathscr{r}}_\pm\cdot\vec{\mathscr{r}}_\pm} = \Big[r^2 + \frac{d^2}{4} \mp d\,\underbrace{\vec{r}\cdot\hat{z}}_{r\cos\theta}\Big]^{1/2}$$

$$= r\left[1 + \frac{d^2}{4r^2} \mp \frac{d}{r}\cos\theta\right]^{1/2}$$

$$\approx r\left[1 \mp \frac{d}{2r}\cos\theta\right]$$

So

$$V = \frac{1}{4\pi\epsilon_0}\left\{\frac{q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right)}{r\left(1-\frac{d}{2r}\cos\theta\right)} - \frac{q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)}{r\left(1+\frac{d}{2r}\cos\theta\right)}\right\}.$$

Now $\frac{1}{1\pm x} \approx 1 \mp x$ for x small. So

$$V \approx \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right)\left[1+\frac{d}{2r}\cos\theta\right]\right.$$
$$\left. - q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\left[1-\frac{d}{2r}\cos\theta\right]\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left\{q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos 2\theta\right)\right) - q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{d}{2r^2}\cos\theta\left\{q\left(t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right) + q\left(t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right)\right\} \qquad \square$$

Suppose that the dipole oscillates via

$$\boxed{q(t) = q_0 \cos(\omega t)}$$

We now invoke another aspect of the approximation – the inversion speed must be non-relativistic. Then

time 0   time $T/2$ (half period

$+q$ $-q$

$-q$ inverts $+q$

$$\Rightarrow \quad \text{speed} = \frac{d}{(T/2)} = \frac{2d}{T} \ll c.$$

Now $\quad T = \frac{1}{f} \Rightarrow \quad 2fd \ll c \quad \Rightarrow \quad 2\pi f\, d \ll c\pi$

$$\Rightarrow \quad \omega d \ll c \quad \Rightarrow \quad \boxed{d \ll \frac{c}{\omega}}$$

This results in:

If $\quad d \ll r \quad$ and $\quad d \ll \frac{c}{\omega} \quad$ and $\quad q(t) = q_0 \sin(\omega t)$

$$V(\vec{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{p_0 \cos\theta}{r} \left\{ -\frac{\omega}{c} \sin[\omega(t - r/c)] + \frac{1}{r} \cos[\omega(t - r/c)] \right\}$$

where $\quad p_0 = q_0 d \quad$ is the magnitude of the dipole.

We prove this in an exercise.

2 Oscillating dipole

If $r \ll d$ then the general time varying dipole yields the following approximate potential:

$$V(\mathbf{r},t) \approx \frac{1}{4\pi\epsilon_0 r}\left\{q\left[t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right]-q\left[t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right]\right\}$$
$$+\frac{d\cos\theta}{8\pi\epsilon_0 r^2}\left\{q\left[t-\frac{r}{c}\left(1-\frac{d}{2r}\cos\theta\right)\right]+q\left[t-\frac{r}{c}\left(1+\frac{d}{2r}\cos\theta\right)\right]\right\}$$

Suppose that

$$q = q_0\cos(\omega t)$$

and $d \ll c/\omega$.

a) Rewrite the terms in the potential in terms of

$$\cos[\omega(t-r/c)] \qquad \sin[\omega(t-r/c)] \qquad \cos[\omega d\cos\theta/2c] \qquad \sin[\omega d\cos\theta/2c]$$

b) Substitute, use the approximation and simplify to get

$$V(\mathbf{r},t) \approx \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]+\frac{1}{r}\cos[\omega(t-r/c)]\right\}.$$

Answer: a) $q\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right) = q_0\cos\left[\omega t-\frac{\omega r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right]$

$$= q_0\cos\left[\omega t-\frac{\omega r}{c}\pm\frac{\omega d}{2c}\cos\theta\right]$$

$$= q_0\cos\left[\omega(t-r/c)\pm\omega d\cos\theta/2c\right]$$

$$= q_0\left\{\cos[\omega(t-r/c)]\cos[\omega d\cos\theta/2c] \mp \sin[\omega(t-r/c)]\sin[\omega d\cos\theta/2c]\right\}.$$

b) If $d \ll c/\omega$ then $\frac{\omega d}{2c} \ll 1$ so that argument is small

$$\cos[\omega d\cos\theta/2c] \approx 1-\frac{1}{2}\left(\frac{\omega d}{2c}\right)^2\cos^2\theta$$

$$\sin[\omega d\cos\theta/2c] \approx \frac{\omega d}{2c}\cos\theta$$

Thus

$$q\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right) \approx q_0\left[\cos[\omega(t-r/c)]\left(1-\frac{1}{2}\left(\frac{\omega d}{2c}\right)^2\cos\theta\right) \mp \frac{\omega d}{2c}\sin[\omega(t-r/c)]\cos\theta\right]$$

So

$$V \approx \frac{q_0}{4\pi\epsilon_0 r}\left\{-\cancel{2}\frac{\omega d}{\cancel{2}c}\cos\theta\,\sin[\omega(t-r/c)]\right\}$$

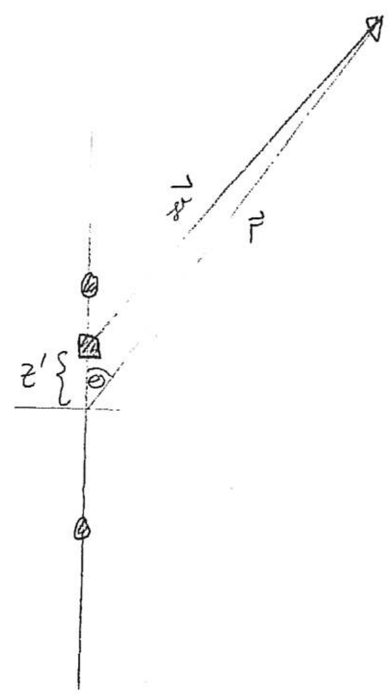
$$+\frac{q_0 d\cos\theta}{8\pi\epsilon_0 r^2}\left\{2\cos[\omega(t-r/c)]\right\}.$$

$$= \frac{q_0 d\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)] + \frac{1}{r}\cos[\omega(t-r/c)]\right\}$$

Vector Potential

Similar calculations give the result for the vector potential.

$$\boxed{\begin{array}{l} \text{If} \quad r \gg d \text{ and } \quad d \ll c/\omega \text{ then} \\[1em] \vec{A}(\vec{r},t) \approx \dfrac{-\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega(t - r/c)\right] \hat{z} \end{array}}$$



Proof.

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{I(\vec{r}', t_r)}{\mathscr{r}} dz' \, \hat{z}$$

and

$$\mathscr{r} = r\left[1 - 2\frac{z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2\right]^{1/2}$$

Then if $d \ll r$ and thus $z' \ll r$

$$\frac{1}{\mathscr{r}} \approx \frac{1}{r}\left[1 + \frac{z'}{r}\cos\theta\right]$$

Thus:

$$\vec{A} \approx \frac{\mu_0}{4\pi}\frac{1}{r} \int_{-d/2}^{d/2} \left(1 + \frac{z'}{r}\cos\theta\right) I(\vec{r}', t_r) \, dz'$$

Now $\quad I = \dfrac{dq}{dt} = -\omega q_0 \sin(\omega t)$. Thus

$$I(\vec{r}', t_r) = -\omega q_0 \sin(\omega t_r)$$
$$= -\omega q_0 \sin\left[\omega(t - \mathscr{r}/c)\right]$$

$\Rightarrow \quad I = -\omega q_0 \sin\left[\omega\left(t - \frac{r}{c}\sqrt{1 - \frac{2z'}{r}\cos\theta + \left(\frac{z'}{r}\right)^2}\right)\right]$

Then with $\quad z' \ll r \quad$ we get.

$$I \approx -\omega q_0 \sin\left[\omega\left(t - \tfrac{r}{c}\left(1 - \tfrac{z'}{r}\cos\theta\right)\right)\right]$$

$$= -\omega q_0 \sin\left[\omega(t - r/c) + \frac{\omega z'}{c}\cos\theta\right]$$

$$= -q_0\omega \sin\left[\omega(t - r/c)\right]\cos\left[\frac{\omega z'}{c}\cos\theta\right]$$

$$- q_0\omega\cos\left[\omega(t - r/c)\right]\sin\left[\frac{\omega z'}{c}\cos\theta\right]$$

Then $\quad d \ll c/\omega \quad \Rightarrow \quad z' \ll c/\omega \quad$ and thus

$$I \approx -q_0\omega \sin\left[\omega(t - r/c)\right] - q_0\frac{\omega^2 z'}{c}\cos\theta\cos\left[\omega(t - r/c)\right]$$

Then $\left(1 + \frac{z'}{r}\cos\theta\right) I(\vec{r}', t_r)$ contains terms with $(z')^0, (z')^1, (z')^2, \ldots$

Then z' terms integrate to zero. Thus:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\left\{-q_0\omega\sin\left[\omega(t - r/c)\right]\right\}\int_{-d/2}^{d/2} dz' \quad \hat{z}$$

$$+ \frac{\mu_0}{4\pi}\frac{1}{r}\left\{-\frac{q_0\omega^2}{c}\cos\theta\cos\left[\omega(t - r/c)\right]\right\}\frac{1}{r}\int_{-d/2}^{d/2} z'^2\, dz' \quad \hat{z}$$

$$= -\frac{\mu_0 q_0 d\omega}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z} - \underbrace{\frac{\mu_0 q_0}{4\pi r^2}\frac{\omega^2}{c}\cos\theta\cos\left[\omega(t - r/c)\right]\frac{2}{3}\left(\frac{d}{2}\right)^3\hat{z}}_{\text{contains } \frac{\omega d}{c} \ll 1 \;\; \approx 0}$$

$$\vec{A}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z} \qquad \blacksquare$$