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PHYS 396
HW 4

Problem 1

$$ds^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$$

a.) $x' = h$: $ds^2 = -\left(1 + \frac{gh}{c^2}\right)^2 c^2 dt'^2$: $d\tau^2 = -\frac{ds^2}{c^2}$ $\therefore$ $ds^2 = -c^2 d\tau^2$

$$-\cancel{c^2} d\tau^2 = -\left(1 + \frac{gh}{c^2}\right)^2 \cancel{c^2} dt'^2 \rightarrow d\tau = \left(1 + \frac{gh}{c^2}\right) dt'$$

$d\tau \rightarrow \Delta\tau$
$dt' \rightarrow \Delta t'$

$$\boxed{\Delta\tau_A = \left(1 + \frac{gh}{c^2}\right)\Delta t'}$$

b.) $x' = 0$: $ds^2 = -\left(1 + \frac{g(0)}{c^2}\right)^2 c^2 dt'^2$: $d\tau^2 = -\frac{ds^2}{c^2}$ $\therefore$ $ds^2 = -c^2 d\tau^2$

$$-\cancel{c^2} d\tau^2 = -\left(1 + 0\right)^2 \cancel{c^2} dt'^2 \rightarrow d\tau = dt'$$

$$\boxed{\Delta\tau_B = \Delta t'}$$

c.)

$$\Delta\tau_A = \left(1 + \frac{gh}{c^2}\right)\Delta\tau_B$$

Alice measures a longer time interval

Problem 2)

$$ct' = \frac{c^2}{g}\tanh^{-1}\left(\frac{ct}{x+c^2/g}\right), \quad x' = \left[\left(x+\frac{c^2}{g}\right)^2 - c^2t^2\right]^{1/2} - \frac{c^2}{g}$$

$$c\,dt' = \frac{c^2}{g}\frac{d}{dt}\left(\tanh^{-1}\left(\frac{ct}{x+c^2/g}\right)\right) = \frac{c^2}{g}\cdot\frac{1}{1-\frac{c^2t^2}{(x+c^2/g)^2}}\cdot\frac{d}{dt}\left(\frac{ct}{x+c^2/g}\right)$$

now

$$d\left(\frac{ct}{x+c^2/g}\right) = \frac{c\,dt}{x+c^2/g} - \frac{ct\,dx}{x+c^2/g} = \frac{c}{(x+c^2/g)^2}\Big[(x+c^2/g)\,dt - t\,dx\Big]$$

$$c\,dt' = \frac{c^2}{g}\cdot\frac{1}{1-\frac{c^2t^2}{(x+c^2/g)^2}}\cdot\frac{c}{(x+c^2/g)^2}\Big[(x+c^2/g)\,dt - t\,dx\Big] = \frac{c^3}{g}\cdot\frac{1}{(x+c^2/g)^2-c^2t^2}\Big[(x+c^2/g)\,dt - t\,dx\Big]$$

$$x' = \Big[(x+c^2/g)^2 - c^2t^2\Big]^{1/2} - \frac{c^2}{g} \quad \text{so} \quad dx' = \frac{1}{2}\left[\left(x+\frac{c^2}{g}\right) - c^2t^2\right]^{-1/2}\cdot\left[2\left(x+\frac{c^2}{g}\right)dx - 2c^2t\,dt\right]$$

$$dx' = \Big[(x+c^2/g)^2 - c^2t^2\Big]^{-1/2}\Big[(x+c^2/g)\,dx - c^2t\,dt\Big]$$

$$c^2dt'^2 = \frac{c^6}{g^2}\cdot\frac{1}{[(x+c^2/g)-c^2t^2]^2}\Big[(x+c^2/g)^2dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\Big]$$

$$dx'^2 = \frac{1}{[(x+c^2/g)^2-c^2t^2]}\Big[(x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\Big]$$

$$ds^2 = -\left[1+\frac{g}{c^2}\Big[(x+c^2/g)^2 - c^2t^2\Big]^{1/2} - 1\right]^2\cdot\frac{c^2}{g^2}\cdot\frac{1}{[(x+c^2/g)^2-c^2t^2]}\Big[(x+c^2/g)^2dt + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\Big]$$

$$+ \frac{1}{[(x+c^2/g)^2-c^2t^2]}\Big[(x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\Big] + dy^2 + dz^2$$

$$= -\frac{g^2}{c^4}\Big[(x+c^2/g)^2 - c^2t^2\Big]\frac{c^6}{g^2}\cdot\frac{1}{[(x+c^2/g)^2-c^2t^2]^2}\Big[(x+c^2/g)^2dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\Big]$$

$$+ \frac{1}{[(x+c^2/g)^2-c^2t^2]}\Big[(x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\Big] + dy^2 + dz^2$$

$$= \frac{1}{[(x+c^2/g)-c^2t^2]}\Big[-c^2\Big[(x+c^2/g)^2dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx + (x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\Big]$$

$$+ dy^2 + dz^2\Big]$$

$$ds^2 = \frac{1}{[(x+c^2/g)^2-c^2t^2]}\Big[-c^2\big((x+c^2/g)^2-c^2t^2\big)dt^2 + \big((x+c^2/g)^2-c^2t^2\big)dx^2\Big] + dy^2 + dz^2$$

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

Problem 3)

$$ct=\left(\frac{c^2}{g}+x'\right)\sinh\left(\frac{gt'}{c}\right)=\frac{c^2}{g}\sinh\left(\frac{gt'}{c}\right)+x'\sinh\left(\frac{gt'}{c}\right)$$

$$x=\left(\frac{c^2}{g}+x'\right)\cosh\left(\frac{gt'}{c}\right)-\frac{c^2}{g}=\frac{c^2}{g}\cosh\left(\frac{gt'}{c}\right)+x'\cosh\left(\frac{gt'}{c}\right)$$

$$y=y' \quad , \quad z=z'$$

$$c\frac{dt}{dt'}:\ \frac{c^2}{g}\sinh\left(\frac{gt'}{c}\right)+x'\sinh\left(\frac{gt'}{c}\right)=\left(c\cdot\cosh\left(\frac{gt'}{c}\right)+\frac{g}{c}x'\cosh\left(\frac{gt'}{c}\right)\right)$$

$$c\frac{dt}{dx'}:\ \frac{c^2}{g}\sinh\left(\frac{gt'}{c}\right)+x'\sinh\left(\frac{gt'}{c}\right)=\left(\sinh\left(\frac{gt'}{c}\right)\right)$$

$$\left[c\,dt=(c+g/c\,x')\cosh(gt'/c)\,dt'+\sinh(gt'/c)\,dx'\right]^{(*)}$$

$$\frac{dx}{dt'}:\ \left(\frac{c^2}{g}+x'\right)\cosh\left(\frac{gt'}{c}\right)-\frac{c^2}{g}=\left(c\cdot\sinh\left(\frac{gt'}{c}\right)+\frac{g}{c}x'\sinh\left(\frac{gt'}{c}\right)\right)$$

$$\frac{dx}{dx'}:\ \left(\frac{c^2}{g}+x'\right)\cosh\left(\frac{gt'}{c}\right)-\frac{c^2}{g}=\left(\cosh\left(\frac{gt'}{c}\right)\right)$$

$$\left[dx=(c+g/c\,x')\sinh(gt'/c)\,dt'+\cosh(gt'/c)\,dx'\right]^{(**)}$$

$$\left[c^2dt^2=(c+g/c\,x')^2\cosh^2(gt'/c)\,dt'^2+2(c+g/c\,x')\sinh(gt'/c)\cosh(gt'/c)\,dx'dt'+\sinh^2(gt'/c)\,dx'^2\right]$$

$$\left[dx^2=(c+g/c\,x')^2\sinh^2(gt'/c)\,dt'^2+2(c+g/c\,x')\sinh(gt'/c)\cosh(gt'/c)\,dx'dt'+\cosh^2(gt'/c)\,dx'^2\right]$$

$$\left[-c^2dt^2=-(c+g/c\,x')^2\cosh^2(gt'/c)\,dt'^2-2(c+g/c\,x')\sinh(gt'/c)\cosh(gt'/c)\,dx'dt'-\sinh^2(gt'/c)\,dx'^2\right]$$

$dx^2 \to ②$, $-c^2dt^2 \to ①$

$$①+②:\ -(c+g/c\,x')^2\cosh^2(gt'/c)\,dt'^2\cancel{-2(c+g/c\,x')\sinh(gt'/c)\cosh(gt'/c)\,dx'dt'}-\sinh^2(gt'/c)\,dx'^2$$
$$+$$
$$(c+g/c\,x')^2\sinh^2(gt'/c)\,dt'^2\cancel{+2(c+g/c\,x')\sinh(gt'/c)\cosh(gt'/c)\,dx'dt'}+\cosh^2(gt'/c)\,dx'^2$$

$$①+②\quad (c+g/c\,x')^2\left(\sinh^2(gt'/c)-\cosh^2(gt'/c)\right)dt'^2+\left(\cosh^2(gt'/c)-\sinh^2(gt'/c)\right)dx'^2$$

$$=(c+g/c\,x')^2(-1)\,dt'^2+dx'^2+dy'^2+dz'^2$$: [dy' & dz' components held off till end to simplify math]

$$=-(1+g/c^2\,x')^2c^2dt^2+dx'^2+dy'^2+dz'^2$$

$$\therefore\ \boxed{ds^2=-(1+g/c^2\,x')^2c^2dt^2+dx'^2+dy'^2+dz'^2}$$

Problem 4

a.) $ct = \left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right)$, $x = \left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$, $y = y'$, $z = z'$

$ct' = \frac{c^2}{g}\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)$, $x' = \left[\left(x + \frac{c^2}{g}\right)^2 - c^2t^2\right]^{\frac{1}{2}} - \frac{c^2}{g}$, $y' = y$, $z' = z$

③ $(ct')\cdot\frac{g}{c^2} = \tanh^{-1}\left(\frac{ct}{x + c^2/g}\right) \rightarrow \tanh\left((ct')\cdot\frac{g}{c^2}\right) = \frac{ct}{x + c^2/g}\tanh^{-1}\left(\frac{ct}{x+c^2/g}\right)$

$ct = (x + c^2/g)\tanh\left((ct')\cdot\frac{g}{c^2}\right)$: $c^2t^2 = (x + c^2/g)^2\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)$

④ $x' + c^2/g = \sqrt{\left(x + \frac{c^2}{g}\right)^2 - c^2t^2} \rightarrow (x' + c^2/g)^2 = (x + c^2/g)^2 - c^2t^2$

$(c^2t^2) = (x + c^2/g)^2 - (x' + c^2/g)^2$: $c^2t^2 = (x + c^2/g)^2\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)^2$

$$(x + c^2/g)^2 - (x' + c^2/g)^2 = (x + c^2/g)^2\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)^2$$

$$-(x' + c^2/g)^2 = (x + c^2/g)^2\tanh^{-1^2}\left(\frac{ct}{x + c^2/g}\right) - (x + c^2/g)^2$$

$$(x' + c^2/g)^2 = (x + c^2/g)^2 - (x + c^2/g)\tanh^{-1^2}\left(\frac{ct}{x + c^2/g}\right)$$

$$(x' + c^2/g)^2 = (x + c^2/g)^2\left(1 - \tanh^{-1^2}\left(\frac{ct}{x + c^2/g}\right)\right)$$

$$(x + c^2/g)^2\,\mathrm{sech}^2\left(\frac{ct}{x + c^2/g}\right) = (x' + c^2/g)^2$$

$$(x + c^2/g) = \frac{(x' + c^2/g)}{\mathrm{sech}\left(\frac{ct}{x + c^2/g}\right)}$$

$$\boxed{x = (x' + c^2/g)\cosh\left(\frac{ct}{x + c^2/g}\right) - c^2/g}$$

③ $ct' = \frac{c^2}{g}\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right) \rightarrow \frac{gt'}{c} = \tanh^{-1}\left(\frac{ct}{x + c^2/g}\right) \rightarrow \tanh\left(\frac{gt'}{c}\right) = \frac{ct}{x + c^2/g}$

$(x + c^2/g) = \frac{ct}{\tanh(gt'/c)} \rightarrow x = \frac{ct}{\tanh(gt'/c)} - \frac{c^2}{g}$

④ $x' + c^2/g = \sqrt{\left(x + \frac{c^2}{g}\right)^2 - c^2t^2} \rightarrow (x' + c^2/g)^2 = (x + c^2/g)^2 - c^2t^2$

$(x' + c^2/g)^2 + c^2t^2 = (x + c^2/g)^2 \rightarrow (x + c^2/g) = \sqrt{(x' + c^2/g)^2 + c^2t^2}$

$x = \sqrt{(x' + c^2/g)^2 + c^2t^2} - c^2/g$

Problem 4 Continued

$$x = \sqrt{(x' + c^2/g)^2 + c^2t^2} - c^2/g \quad : \quad x = \frac{ct}{\tanh(gt'/c)} - \frac{c^2}{g}$$

$$\sqrt{(x' + c^2/g)^2 + c^2t^2} - c^2/g = \frac{ct}{\tanh(gt'/c)} - \frac{c^2}{g}$$

$$\sqrt{(x' + c^2/g)^2 + c^2t^2} = \frac{ct}{\tanh(gt'/c)}$$

$$(x' + c^2/g)^2 + c^2t^2 = \frac{c^2t^2}{\tanh^2(gt'/c)}$$

$$\tanh^2(gt'/c)(x' + c^2/g)^2 + \tanh^2(gt'/c)\,c^2t^2 = c^2t^2$$

$\tanh = \frac{\sinh}{\cosh}$

$$\tanh^2(gt'/c)(x' + c^2/g)^2 = c^2t^2(1 - \tanh^2(gt'/c))$$

$\operatorname{sech} = \frac{1}{\cosh}$

$$\tanh^2(gt'/c)(x' + c^2/g)^2 = c^2t^2(\operatorname{sech}^2(gt'/c))$$

$$c^2t^2 = \frac{(x' + c^2/g)^2 \tanh^2(gt'/c)}{\operatorname{sech}^2(gt'/c)}$$

$$c^2t^2 = (x' + c^2/g)^2 \sinh^2(gt'/c)$$

$$\boxed{ct = (x' + c^2/g)\sinh(gt'/c)}$$

b.) $ct = \left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right)$, $x = \left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$

$\text{Taylor}(\sinh(\theta)) = \theta \quad : \quad \text{Taylor}(\cosh(\theta)) = 1 + \frac{\theta^2}{2}$

$\theta \equiv \frac{gt'}{c}$

$$\left(\frac{c^2}{g} + x'\right)\left(\frac{gt'}{c}\right) = ct' + \frac{gt'}{c}x' = \boxed{ct'\left(1 + \frac{gx'}{c^2}\right)}$$ ✓

$$\left(\frac{c^2}{g} + x'\right)\left(1 + \frac{1}{2}\left(\frac{g^2t'^2}{c^2}\right)\right) - \frac{c^2}{g} = \frac{c^2}{g} + \frac{1}{2}gt'^2 + x' + \frac{1}{2}\frac{g^2t'^2}{c^2}x' - \frac{c^2}{g}$$

$$= \frac{1}{2}gt'^2 + x' + \frac{1}{2}\frac{g^2t'^2}{c^2}x' = \boxed{\frac{1}{2}gt'^2\left(1 + \frac{gx'}{c^2}\right) + x'}$$ ✓

c.) $c^2/g \rightarrow c \cdot \frac{c}{g} \rightarrow c\,(3.0581 \times 10^7\,s) \cdot \frac{1\,\text{year}}{3.15576 \times 10^7\,s} = 0.96\ (c \cdot yr)$

$3.15576 \times 10^7\,s = 1$ year

$$\boxed{c^2/g = 0.96\ (c \cdot yr)}$$

Problem 5

a.) $u^\alpha = \frac{dx^\alpha}{d\tau}$: $\frac{dx^1}{d\tau} = \frac{dx}{dt'} \cdot \frac{dt'}{d\tau}$: $\frac{dx^0}{d\tau} = \frac{dt}{dt'} \cdot \frac{dt'}{d\tau}$

$$\frac{dx}{dt'} = \left(\frac{c^2}{g} + x'\right)\frac{g}{c}\sinh\left(\frac{gt'}{c}\right) = \left(c + \frac{gx'}{c}\right)\sinh\left(\frac{gt'}{c}\right)$$

$$d\tau = \left(1 + \frac{gh}{c^2}\right)dt' \to d\tau = \left(1 + \frac{gx'}{c^2}\right)dt' \to \frac{dt'}{d\tau} = \frac{1}{\left(1 + \frac{gx'}{c^2}\right)}$$

$$c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right) \cdot \frac{1}{\left(1 + \frac{gx'}{c^2}\right)} = c\sinh\left(\frac{gt'}{c}\right) \quad : \quad \frac{dx^1}{d\tau} = c\sinh\left(\frac{gt'}{c}\right)$$

$$ct = \left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right) \quad : \quad \frac{dx^0}{d\tau} = \frac{dt}{dt'} \cdot \frac{dt'}{d\tau} \quad : \quad \frac{dt'}{d\tau} = (1 + gx'/c^2)$$

$$\frac{dct}{dt'} = \left(c + \frac{gx'}{c}\right)\cosh\left(\frac{gt'}{c}\right) \quad : \quad \frac{dt}{dt'} \cdot \frac{dt'}{d\tau} = \left(1 + \frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right) * \left(1 + \frac{gx'}{c^2}\right)^{-1}$$

$$\frac{dx^0}{d\tau} = \cosh\left(\frac{gt'}{c}\right)$$

$$\boxed{u^\alpha = \left(\cosh\left(\frac{gt'}{c}\right), c\sinh\left(\frac{gt'}{c}\right), 0, 0\right)}$$

b.) $a^\alpha = \frac{d^2x^\alpha}{d\tau^2}$: $a^\alpha = \left(\frac{du^0}{d\tau}, \frac{du^1}{d\tau}, \frac{du^2}{d\tau}, \frac{du^3}{d\tau}\right)$

$$\frac{du^0}{d\tau} = \frac{du^0}{dt'} \cdot \frac{dt'}{d\tau} = \frac{g}{c}\sinh\left(\frac{gt'}{c}\right) \cdot \left(1 + \frac{gx'}{c^2}\right)^{-1}$$

$$\frac{du^1}{d\tau} = \frac{du^1}{dt'} \cdot \frac{dt'}{d\tau} = g\cosh\left(\frac{gt'}{c}\right) \cdot \left(1 + \frac{gx'}{c^2}\right)^{-1}$$

$$\boxed{a^\alpha = \left(\frac{g}{c}\sinh\left(\frac{gt'}{c}\right) \cdot \left(1 + \frac{gx'}{c^2}\right)^{-1}, g\cosh\left(\frac{gt'}{c}\right) \cdot \left(1 + \frac{gx'}{c^2}\right)^{-1}, 0, 0\right)}$$

c.) $a \equiv (\eta_{\alpha\beta} a^\alpha a^\beta)^{1/2}$

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} a^\alpha a^\beta = (-1)a^0 a^0 + (1)a^1 a^1$$

$$a^0 = \frac{g}{c}\sinh\left(\frac{gt'}{c}\right)\left(1 + \frac{gx'}{c^2}\right)^{-1}, \quad a^1 = g\cosh\left(\frac{gt'}{c}\right)\left(1 + \frac{gx'}{c^2}\right)^{-1}$$

Problem 5 Continued

$$= -\left(\frac{g}{c}\right)^2 \sinh^2\left(\frac{gt'}{c}\right)\left(1+\frac{gx'}{c^2}\right)^{-2} + \left(g\right)^2\cosh^2\left(\frac{gt'}{c}\right)\left(1+\frac{gx'}{c^2}\right)^{-2}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\eta_{\alpha\beta}a^\alpha a^\beta = g^2\left(1+\frac{gx'}{c^2}\right)^{-2} \; : \; \left(\eta_{\alpha\beta}a^\alpha a^\beta\right)^{1/2} = \frac{g}{\left(1+gx'/c^2\right)}$$

$$\boxed{a = \frac{g}{\left(1+gx'/c^2\right)}}$$

d.)

$$\boxed{\begin{array}{ll} x'=0: & a = g \\ x'=h: & a = \dfrac{g}{1+gh/c^2} \end{array}}$$