Lecture 5 Exercises

Exercise 1 : Measurement Probabilities

$|\psi\rangle = 3i|+\hat{z}\rangle + 4|-\hat{z}\rangle$

a.) Normalize and denote by $|\psi\rangle$

$\langle\psi|\psi\rangle = |3i|^2 + |4|^2 = |9i^2| + |16| = |-9| + |16| = 9+16 = 25 \quad : \quad A^2 \cdot 25 = 1$

$A^2 = \frac{1}{25} \quad \therefore \quad A = \pm\frac{1}{5} \quad \therefore \quad \frac{1}{5}\cdot|\psi\rangle \; : \; \boxed{|\psi\rangle = \frac{3i}{5}|+\hat{z}\rangle + \frac{4}{5}|-\hat{z}\rangle}$

b.) Determine probabilities of $|\psi\rangle$ Note : probability $\equiv \rho$

i.) probability of $(S_z = +\hbar/2)$: $\rho(\langle +z|\psi\rangle) = |\langle +z|\psi\rangle|^2$

using $\langle +\hat{z}|\psi\rangle = a_+\underbrace{\langle +\hat{z}|+\hat{z}\rangle}_{=1} + a_-\underbrace{\langle +\hat{z}|-\hat{z}\rangle}_{=0} = a_+(1) + a_-(0) = a_+$

$\rho(\langle +z|\psi\rangle) = |\langle +\hat{z}|\psi\rangle|^2 = \left|\frac{3i}{5}\right|^2 = \left|\frac{-9}{25}\right| = \frac{9}{25}$

$$\boxed{\text{Prob}(S_z = +\hbar/2) = \frac{9}{25}}$$

ii.) probability of $(S_z = -\hbar/2)$: $\rho(\langle -\hat{z}|\psi\rangle) = |\langle -\hat{z}|\psi\rangle|^2$

using $\langle -\hat{z}|\psi\rangle = a_+\underbrace{\langle -\hat{z}|+\hat{z}\rangle}_{=0} + a_-\underbrace{\langle -\hat{z}|-\hat{z}\rangle}_{=1} = a_+(0) + a_-(1) = a_-$

$\rho(\langle -\hat{z}|\psi\rangle) = |\langle -\hat{z}|\psi\rangle|^2 = \left|\frac{4}{5}\right|^2 = \left|\frac{16}{25}\right| = \frac{16}{25}$

$$\boxed{\text{Prob}(S_z = -\hbar/2) = \frac{16}{25}}$$

c.) $|\varphi\rangle = e^{i\varphi}|\psi\rangle = \frac{3i\cdot e^{i\varphi}}{5}|+\hat{z}\rangle + \frac{4e^{i\varphi}}{5}|-\hat{z}\rangle$

i.) $\rho(\langle +\hat{z}|\psi\rangle) = |\langle +\hat{z}|\psi\rangle|^2 = \left|\frac{3i}{5}\right|^2\cdot\left|e^{i\varphi}\right|^2 = \left|\frac{-9}{25}\right|\cdot\left|e^{i\varphi}\cdot e^{-i\varphi}\right| = \frac{9}{25}\left|e^{0}\right| = \frac{9}{25}$

$$\boxed{\text{Prob}(S_z = +\hbar/2) = \frac{9}{25}}$$

ii.) $\rho(\langle -\hat{z}|\psi\rangle) = |\langle -\hat{z}|\psi\rangle|^2 = \left|\frac{4}{5}\right|^2\cdot\left|e^{i\varphi}\right|^2 = \left|\frac{16}{25}\right|\cdot\left|e^{i\varphi}\cdot e^{-i\varphi}\right| = \frac{16}{25}\left|e^{0}\right| = \frac{16}{25}$

$$\boxed{\text{prob}(S_z = -\hbar/2) = \frac{16}{25}}$$

Exercise 2: Kets in Terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$

$$|+\hat{n}\rangle = \cos(\theta/2)|+\hat{z}\rangle + e^{i\varphi}\sin(\theta/2)|-\hat{z}\rangle \quad , \quad |-\hat{n}\rangle = \sin(\theta/2)|+\hat{z}\rangle - e^{i\varphi}\cos(\theta/2)|-\hat{z}\rangle$$

a.) Express $|+\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ $\quad x: \theta = \pi/2 \, , \, \varphi = 0$

$$|+\hat{x}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{0}\sin(\pi/4)|-\hat{z}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}|-\hat{z}\rangle$$

$$\boxed{|+\hat{x}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}|-\hat{z}\rangle}$$

b.) Express $|-\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ $\quad x: \theta = \pi/2 \, , \, \varphi = 0$

$$|-\hat{x}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{0}\cos(\pi/4)|-\hat{z}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{\sqrt{2}}{2}|-\hat{z}\rangle$$

$$\boxed{|-\hat{x}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{\sqrt{2}}{2}|-\hat{z}\rangle}$$

c.) Express $|+\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ $\quad y: \theta = \pi/2 \, , \, \varphi = \pi/2$

$$|+\hat{y}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{i(\pi/2)}\sin(\pi/2)|-\hat{z}\rangle \quad : \quad e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i(\pi/2)} = \underbrace{\cancel{\cos(\pi/2)}}_{0} + i\sin(\pi/2) = i$$

$$\boxed{|+\hat{y}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{i\sqrt{2}}{2}|-\hat{z}\rangle}$$

d.) Express $|-\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$ $\quad y: \theta = \pi/2 \, , \, \varphi = \pi/2$

$$|-\hat{y}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{i(\pi/2)}\cos(\pi/2)|-\hat{z}\rangle$$

$$\boxed{|-\hat{y}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{i\sqrt{2}}{2}|-\hat{z}\rangle}$$

Exercise 3 : Measurement Probabilities

a.) Determine probabilities of a particle in state $|+\hat{x}\rangle$ subjected to SG $\hat{z}$ measurement.

$$\text{Prob}(S_z = +\hbar/2) = |\langle +\hat{z}|+\hat{x}\rangle|^2$$

$$\langle +\hat{z}|+\hat{x}\rangle = \langle +\hat{z}|\,\frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}|-\hat{z}\rangle\rangle = \frac{\sqrt{2}}{2}\cancelto{1}{\langle +\hat{z}|+\hat{z}\rangle} + \frac{\sqrt{2}}{2}\cancelto{0}{\langle +\hat{z}|-\hat{z}\rangle} = \frac{\sqrt{2}}{2}$$

$$|\langle +\hat{z}|+\hat{x}\rangle|^2 = \left|\frac{\sqrt{2}}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{\text{Prob}(S_z = +\hbar/2) = \frac{1}{2}}$$

$$\text{Prob}(S_z = -\hbar/2) = |\langle -\hat{z}|+\hat{x}\rangle|^2$$

$$\langle -\hat{z}|+\hat{x}\rangle = \langle -\hat{z}|\,\frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}|-z\rangle\rangle = \frac{\sqrt{2}}{2}\cancelto{0}{\langle -\hat{z}|+\hat{z}\rangle} + \frac{\sqrt{2}}{2}\cancelto{1}{\langle -\hat{z}|-\hat{z}\rangle} = \frac{\sqrt{2}}{2}$$

$$|\langle -\hat{z}|+\hat{x}\rangle|^2 = \left|\frac{\sqrt{2}}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{\text{Prob}(S_z = -\hbar/2) = \frac{1}{2}}$$

b.) Determine probabilities of a particle in state $|-\hat{y}\rangle$ subjected to SG $\hat{x}$ measurement.

$$\text{Prob}(S_x = +\hbar/2) = |\langle +\hat{x}|-\hat{y}\rangle|^2$$

$$|+\hat{x}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}|-\hat{z}\rangle \;:\; |-\hat{y}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{i\sqrt{2}}{2}|-\hat{z}\rangle$$

$$|\langle +\hat{x}|-\hat{y}\rangle|^2 = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\langle +\hat{z}|+\hat{z}\rangle - \cancelto{0}{\frac{\sqrt{2}}{2}\cdot\frac{i\sqrt{2}}{2}\langle +\hat{z}|-\hat{z}\rangle} + \cancelto{0}{\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\langle -\hat{z}|+\hat{z}\rangle} - \frac{i\sqrt{2}}{2}\frac{\sqrt{2}}{2}\langle -\hat{z}|-\hat{z}\rangle$$

$$= \frac{2}{4} - \frac{i\cdot 2}{4} = \frac{1}{2} - \frac{i}{2} = \frac{1}{2}(1-i)$$

$$\boxed{\text{Prob}(S_x = +\hbar/2) = \frac{1}{2}(1-i)}$$

$\text{Prob}(S_x = -\hbar/2) = |\langle -\hat{x}|-\hat{y}\rangle|^2$ ⟶ Need clarification on this

$$\boxed{\text{Prob}(S_x = -\hbar/2) = \frac{1}{2}(i-1)}$$

Solving for Rate of Dipole Axis Rotation

$$\frac{d\vec{\mu}}{dt} = \gamma(\vec{\mu}\times\vec{B}) \;:\; \frac{d\vec{\mu}_x}{dt} = \gamma(\vec{\mu}\times\vec{B})_x \;:\; \frac{d\vec{\mu}_y}{dt} = \gamma(\vec{\mu}\times\vec{B})_y \;:\; \frac{d\vec{\mu}_z}{dt} = \frac{\mu_0-\mu_z}{T_1}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \mu_x & \mu_y & \mu_z \\ B_x & B_y & B_z \end{vmatrix} = (\mu_y B_z - \overset{0}{\cancel{\mu_z B_y}})\hat{\imath} + (\overset{0}{\cancel{\mu_z B_x}} - \mu_x B_z)\hat{\jmath} + (\mu_x B_y - \mu_y B_x)\hat{k}$$

$$\frac{d\mu_x}{dt} = \gamma(\mu_y B_z - \overset{0}{\cancel{\mu_z B_y}}) \longrightarrow (1) \quad \frac{d\mu_y}{dt} = \gamma(\overset{0}{\cancel{\mu_z B_x}} - \mu_x B_z) \longrightarrow (2) \quad \frac{d\mu_z}{dt} = \frac{\mu_0-\mu_z}{T_1} \longrightarrow (3)$$

Solving Equation (3): $\frac{d\vec{\mu}_z}{dt} = \frac{\mu_0-\mu_z}{T_1} \;:\; \frac{d(\mu_z-\mu_0)}{\mu_z-\mu_0} = -\frac{1}{T_1}dt \;:\; \ln(\mu_z-\mu_0) = -\frac{t}{T_1}+k$

$$\mu_z-\mu_0 = e^{-t/T_1+k} \;\therefore\; \mu_z = \mu_0 + e^{-t/T_1+k} \quad @\; t=0 \quad \mu_z = \mu_0 + k \;\therefore\; k = \mu_z - \mu_0$$

$$\mu_z = \mu_0 + (\mu_{z_i}-\mu_0)e^{-t/T_1} \quad\therefore\quad \mu_z = \mu_{z_i}e^{-t/T_1} + \mu_0(1-e^{-t/T_1})$$

$$\boxed{\mu_z = \mu_{z_i}e^{-t/T_1} + \mu_0(1-e^{-t/T_1})}$$

Solving Equation (2): $\frac{d\mu_y}{dt} = -\gamma\mu_x B_z \;:\; \frac{d}{dt}\cdot\frac{d\mu_y}{dt} = -\frac{d}{dt}\cdot\gamma\frac{\mu_x}{dt}\cdot B_z$

$$\frac{d^2\mu_y}{dt^2} = -\gamma B_z\cdot\frac{d\mu_x}{dt} \longrightarrow \text{Subbing in } \frac{d\mu_x}{dt} = \gamma B_z \mu_y \text{ from (1)}$$

$$\frac{d^2\mu_y}{dt} = -\gamma^2 B_z^2\mu_y \equiv \ddot{\mu}_y + \gamma^2 B_z^2 \mu_y = 0 \;:\; \Delta = b^2-4ac = 0-4(1)\gamma^2 B_z^2 < 0$$

$$\Delta<0 \;\therefore\; \mu_y = e^{\alpha t}(A\cdot\cos(\beta t) + B\cdot\sin(\beta t)) \;:\; \alpha = \frac{-b}{2a} = \frac{-0}{2(1)} = 0$$

$$\beta = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{4(1)(\gamma^2)(B_z)^2-0}}{2(1)} = \frac{\sqrt{4(\gamma^2)(B_z^2)}}{2} = \pm\frac{2\gamma B_z}{2} = \pm\gamma\cdot B_z$$

$$\boxed{\mu_y = A\cos(\gamma B_z t) + B\sin(\gamma B_z t)}$$

Solving Equation (1) $\qquad \mu_x = -\frac{1}{\gamma B_z}\frac{d\mu_y}{dt}$

$$\frac{d\mu_y}{dt} = -\gamma B_z\cdot A\sin(\gamma B_z t) + \gamma B_z\cdot B\cos(\gamma B_z t) \;\therefore\; \mu_x = \frac{-1}{\gamma B_z}(-\gamma B_z\cdot A\sin(\gamma B_z t) + \gamma B_z\cdot B\cos(\gamma B_z t))$$

$$\boxed{\mu_x = A\sin(\gamma B_z t) - B\cos(\gamma B_z t)}$$

More ket Algebra

$|\Psi\rangle = \cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}$

$|+\hat{n}\rangle = \cos(\theta/2)|+\hat{z}\rangle + e^{i\varphi}\sin(\theta/2)|-\hat{z}\rangle$

$|-\hat{n}\rangle = \sin(\theta/2)|+\hat{z}\rangle - e^{i\varphi}\cos(\theta/2)|-\hat{z}\rangle$

a.) We want $|\Psi\rangle$ in terms of $|+\hat{z}\rangle$ and $|-\hat{z}\rangle$, in X,Y Plane, $\theta = \pi/2$, $\varphi = \omega t$

$|+\hat{n}\rangle = \cos(\pi/4)|+\hat{z}\rangle + e^{i\omega t}\sin(\pi/4)|-\hat{z}\rangle$, $|-\hat{n}\rangle = \sin(\pi/4)|+\hat{z}\rangle - e^{i\omega t}\cos(\pi/4)|-\hat{z}\rangle$

$= \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}e^{i\omega t}|-\hat{z}\rangle$ $\qquad = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{\sqrt{2}}{2}e^{i\omega t}|-\hat{z}\rangle$

$|+\hat{n}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}e^{i\omega t}|-\hat{z}\rangle$, $|-\hat{n}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle - \frac{\sqrt{2}}{2}e^{i\omega t}|-\hat{z}\rangle$

$$\boxed{|+\hat{n}\rangle = \frac{\sqrt{2}}{2}|+\hat{z}\rangle + \frac{\sqrt{2}}{2}e^{i\omega t}|-\hat{z}\rangle}$$

b.) Measure SG $\hat{z}$ probabilities.

$$|\langle+\hat{z}|+\hat{n}\rangle|^2 = \frac{\sqrt{2}}{2}\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} + \frac{\sqrt{2}}{2}e^{i\omega t}\overset{0}{\cancel{\langle+\hat{z}|-\hat{z}\rangle}} = \left|\frac{\sqrt{2}}{2}\right|^2 = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{|\langle+\hat{z}|+\hat{n}\rangle|^2 = \frac{1}{2}}$$

c.) Measure SG $\hat{x}$ probabilities. $\hat{x}$: $\theta = \frac{\pi}{2}$, $\varphi = 0$: $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$

$$\langle+\hat{x}|+\hat{n}\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}e^{i\omega t}\overset{1}{\cancel{\langle-\hat{z}|-\hat{z}\rangle}} = \frac{1}{2} + \frac{1}{2}(e^{i\omega t}) = \frac{1}{2}(1+e^{i\omega t})$$

$$|\langle+\hat{x}|+\hat{n}\rangle|^2 = \frac{1}{4}(1+e^{i\omega t})(1+e^{-i\omega t}) = \frac{1}{4}(1+2\cos(\omega t)+1) = \frac{1}{4}(2\cos(\omega t)+2)$$

$$\boxed{|\langle+\hat{x}|+\hat{n}\rangle|^2 = \frac{1}{2}(\cos(\omega t)+1)}$$

d.) Measure SG $\hat{y}$ Probabilities. $\hat{y}$: $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$: $e^{i(\frac{\pi}{2})} = \overset{0}{\cancel{\cos(\pi/2)}} + i\sin(\pi/2) = i$

$$|\langle+\hat{y}|+\hat{n}\rangle|^2 = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} - \frac{1}{\sqrt{2}}\cdot\frac{i}{\sqrt{2}}e^{i\omega t}\overset{1}{\cancel{\langle-\hat{z}|-\hat{z}\rangle}} = \frac{1}{2}(1-ie^{i\omega t})$$

$$|\langle+\hat{y}|+\hat{n}\rangle|^2 = \frac{1}{4}(1-ie^{i\omega t})(1+ie^{-i\omega t}) = \frac{1}{4}(1+ie^{-i\omega t}-ie^{i\omega t}-i^2) = \frac{1}{4}(2+i(e^{-i\omega t}-e^{i\omega t}))$$

$$= \frac{1}{4}(2+i(\cos(\omega t)-i\sin(\omega t)-\cos(\omega t)-i\sin(\omega t)) = \frac{1}{4}(2+i(-2i\sin(\omega t))$$

$$= \frac{1}{4}(2-2i^2\sin(\omega t)) = \frac{1}{2}(\sin(\omega t)+1)$$

$$\boxed{|\langle+\hat{y}|+\hat{n}\rangle|^2 = \frac{1}{2}(\sin(\omega t)+1)}$$