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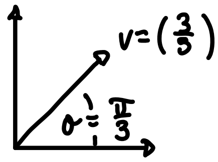
PHYS 396 HW 4

Problem 1



$\cos(\pi/3) = 1/2$
$\sin(\pi/3) = \frac{\sqrt{3}}{2}$

a.)

$$\boxed{\underset{\sim}{u}^{\alpha} = \left(\gamma,\ \gamma \cdot \tfrac{3}{10},\ \gamma \tfrac{3\sqrt{3}}{10},\ 0\right)}$$

$\gamma = 5/4$ $\qquad \frac{5}{4} \cdot \frac{3\sqrt{3}}{10} = \frac{15\sqrt{3}}{40} = \frac{3\sqrt{3}}{8}$

b.)

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = (\gamma, \gamma V_x, \gamma V_y, \gamma V_z) \cdot (\gamma, \gamma V_x, \gamma V_y, \gamma V_z)$$

$$= -\gamma^2 + \gamma^2 V_x^2 + \gamma^2 V_y^2 + \cancel{\gamma^2 V_z^2}^{\,0}$$

$$= -\gamma^2 + \gamma^2\left(\frac{9}{100}\right) + \gamma^2\left(\frac{27}{100}\right) = -\gamma^2 + \gamma^2\left(\frac{36}{100}\right)$$

$$= -\frac{25}{16} + \frac{25}{16}\left(\frac{36}{100}\right) = -\frac{25}{16}\left(1 - \left(\frac{36}{100}\right)\right) = -\frac{25}{16}\left(\frac{64}{100}\right) = -\frac{1}{4} \cdot \frac{4}{1} = -1$$

$\gamma^2 = (1 - v^2)^{-1}$
$= (1 - 9/25)^{-1}$
$= (16/25) = 5/4$

$$\boxed{\underset{\sim}{u} \cdot \underset{\sim}{u} = -1}$$

c.)

$\underset{\sim}{u}^{t'} = \gamma(u^t - v u^x)$ $\quad : \quad \gamma = 5/3,\ v = 4/5,\ u^t = 5/4,\ u^x = 3/8,\ u^y = (3\sqrt{3})/8$

$\underset{\sim}{u}^{x'} = \gamma(u^x - v u^t)$

$\underset{\sim}{u}^{y'} = \gamma(u^y - v u^t)$

$$\underset{\sim}{u}^{t'} = \frac{5}{3}\left(5/4 - 4/5(3/8)\right) = \frac{5}{3}\left(50/40 - 12/40\right) = \frac{5}{3}\left(38/40\right) = \frac{190}{120} = \frac{19}{12}$$

$$\underset{\sim}{u}^{x'} = \frac{5}{3}\left(3/8 - 4/5(5/4)\right) = \frac{5}{3}\left(3/8 - 1\right) = \frac{5}{3}\left(-5/8\right) = \frac{-25}{24}$$

$$\underset{\sim}{u}^{y'} = \frac{3\sqrt{3}}{8}$$

$$\boxed{\underset{\sim}{u}' = \left[\frac{19}{12}, -\frac{25}{24}, \frac{3\sqrt{3}}{8}, 0\right]}$$

d.)

$$\underset{\sim}{u}' \cdot \underset{\sim}{u}' = -\left(\frac{19}{12}\right)^2 + \left(\frac{-25}{24}\right)^2 + \left(\frac{3\sqrt{3}}{8}\right)^2 = -1 \quad \checkmark$$

$$\boxed{\underset{\sim}{u}' \cdot \underset{\sim}{u}' = -1}$$

Problem 2

$$u^\alpha(t) = \left(\sqrt{1+f^2(t)},\ f(t), 0, 0\right)$$

a.) $$u^\alpha(t)\cdot u^\alpha(t) = -\sqrt{1+f^2(t)}\cdot\sqrt{1+f^2(t)} + f(t)^2 = -(1+f^2(t)) + f^2(t) = -1$$

$$\boxed{\underset{\sim}{u}^\alpha(t)\cdot\underset{\sim}{u}^\alpha(t) = -1}$$

b.) $$\frac{d}{d\tau}\left(\sqrt{1+f^2(t)}\right) = \frac{dt}{d\tau}\frac{d}{dt}\left(\sqrt{1+f^2(t)}\right) = \sqrt{1+f^2(t)}\cdot\frac{1}{2}(1+f^2(t))^{-\frac{1}{2}}\cdot 2f(t)\cdot\dot{f} = f(t)\dot{f}(t)$$

$$\frac{d}{d\tau}(f(t)) = \frac{d}{d\tau}\underset{\sim}{u}' = \frac{d}{d\tau}f(t) = \frac{dt}{d\tau}\frac{d}{dt}f(t) = \sqrt{1+f^2(t)}\cdot\dot{f}(t)$$

$$\boxed{\underset{\sim}{a}^\alpha(t) = \left(f\dot{f},\ \sqrt{1+f^2(t)}\,\dot{f}, 0, 0\right)}$$

c.) $\underset{\sim}{u}^\alpha\cdot\underset{\sim}{a}^\alpha = 0$

$$\left(\sqrt{1+f^2(t)},\ f(t), 0, 0\right)\cdot\left(f\dot{f},\ \sqrt{1+f^2(t)}\,\dot{f}, 0, 0\right) = -f\dot{f}\sqrt{1+f^2(t)} + f\dot{f}\sqrt{1+f^2(t)} = 0 \ \checkmark$$

$$\boxed{\underset{\sim}{u}^\alpha(t)\cdot\underset{\sim}{a}^\alpha(t) = 0}$$

d.) $V(t) = dx/dt$ $\qquad \underset{\sim}{u}^\alpha = (\gamma, \gamma V)$, $\quad \underset{\sim}{u}^\alpha = \left(\sqrt{1+f^2(t)},\ f(t), 0, 0\right)$

$$\underset{\sim}{u}^\alpha = (\gamma, \gamma V_x, \gamma V_y, \gamma V_z) = \left(\sqrt{1+f^2(t)},\ f(t), 0, 0\right)$$

$$:\ \gamma = \sqrt{1+f^2(t)},\quad \gamma V_x = f(t),\quad V_x = \frac{f(t)}{\gamma}$$

$$\boxed{\vec{V} = \left[\frac{f(t)}{\sqrt{1+f^2(t)}}\hat{x},\ 0\hat{y},\ 0\hat{z}\right]}$$

e.) $\tau(t)$ and $x(t)$

$\tau(t)$: $\quad x^\alpha = x^\alpha(\tau) \qquad u^t = \frac{dt}{d\tau}\ :\ \int d\tau = \int\frac{1}{u^t}dt\ :\ \rightarrow\ \tau(t) = \int\frac{dt}{\sqrt{1+f^2(t)}}$

$$u^x = \frac{dx}{d\tau} = \frac{dx}{dt}\cdot\frac{dt}{d\tau} = f(t)\ :\ \frac{dx}{dt}\cdot\sqrt{1+f^2(t)} = f(t)$$

$$\int\frac{x}{\sqrt{1+x^2}}dx \qquad u = 1+x^2,\quad du = 2x\,dx,\quad dx = \frac{du}{2x}$$

$$\frac{dx}{dt} = \frac{f(t)}{\sqrt{1+f^2(t)}}\ :\ \int dx = \int\frac{f(t)}{\sqrt{1+f^2(t)}}dt\ :\ x = \sqrt{1+f^2(t)}$$

$$\frac{1}{2}\int u^{-\frac{1}{2}}du = u^{\frac{1}{2}} = \sqrt{1+x^2}$$

Problem 2) Continued

$$\boxed{\begin{aligned} \tau(t) &= \int \frac{dt}{\sqrt{1+f^2(t)}} \\ x(t) &= \int \frac{f(t)}{\sqrt{1+f^2(t)}}\, dt \end{aligned}}$$

f.) $\underset{\sim}{f} = m \cdot \frac{d\underset{\sim}{u}}{d\tau} = m \cdot \underset{\sim}{a} \quad : \quad \vec{F} = m\vec{a}$

$$\underset{\sim}{a} = \left(f\dot{f}, \sqrt{1+f^2(t)}\,\dot{f}, 0, 0 \right) \quad , \quad \vec{a} = \left(\dot{f}, 0, 0 \right)$$

$$\boxed{\begin{aligned} \underset{\sim}{f} &= \left(mf\dot{f}, m\dot{f}\sqrt{1+f^2(t)}, 0, 0 \right) \\ \vec{F} &= \left(m\dot{f}, 0, 0 \right) \end{aligned}}$$

Problem 3

$$t(\tau) = \frac{1}{\alpha} \ln\left(\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau)\right), \quad x(\tau) = \frac{-1}{a} \ln(\cos(a\tau))$$

a.) From the above two equations the only parameter that they depend upon is τ. If we look at the line element: $ds^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2$, $dx=dy=dz=0$ so the line element simplifies to $ds^2 = -(cdt)^2$ where dt is the time that the observer sees, or formally known as the proper time.

b.) $\frac{dt}{d\tau}\left(\frac{1}{\alpha}\ln\left(\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau)\right)\right)$: $\frac{d}{dx}\ln(x) = \frac{x'}{x}$

$$= \frac{1}{\alpha}\cdot\left(\frac{\alpha\cdot\sec^2(\alpha\tau) + \alpha\cdot\operatorname{Tan}(\alpha\tau)\cdot\sec(\alpha\tau)}{\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau)}\right) = \frac{\sec(\alpha\tau)\cancel{(\sec(\alpha\tau) + \operatorname{Tan}(\alpha\tau))}}{\cancel{(\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau))}} = \sec(\alpha\tau)$$

$$\frac{dt}{d\tau} = \sec(\alpha t)$$

$$\frac{dx}{d\tau}\left(\frac{-1}{\alpha}\ln(\cos(\alpha\tau))\right) = \frac{-1}{\alpha}\cdot\frac{-\alpha\cdot\sin(\alpha t)}{\cos(\alpha t)} = \operatorname{Tan}(\alpha t)$$

$$\boxed{\underset{\sim}{u}^{\alpha} = \left[\sec(\alpha\tau), \operatorname{Tan}(\alpha\tau), 0, 0\right]}$$

$$\frac{d^2t}{d\tau^2} = \alpha\cdot\sec(\alpha\tau)\cdot\operatorname{Tan}(\alpha\tau)$$

$$\frac{d^2x}{d\tau^2} = \alpha\cdot\sec^2(\alpha\tau)$$

$$\boxed{\underset{\sim}{a}^{\alpha} = \left[\alpha\cdot\sec(\alpha\tau)\cdot\operatorname{Tan}(\alpha\tau), \alpha\cdot\sec^2(\alpha\tau), 0, 0\right]}$$

c.) $t' = \gamma(t - Vx/c^2)$, $x' = \gamma(x - Vt)$

$$t' = \gamma\left(\frac{1}{a}\cdot\ln(\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau)) - V\cdot\left(\frac{-1}{\alpha}\right)\ln(\cos(\alpha t))\right)$$

$$x' = \gamma\left(-\frac{1}{a}\ln(\cos(\alpha\tau)) - V\left(\frac{1}{a}\cdot\ln(\operatorname{Tan}(\alpha\tau) + \sec(\alpha\tau))\right)\right)$$

$$\frac{dt'}{d\tau} = \gamma\left(\frac{1}{\alpha}\cdot\alpha\cdot\sec(\alpha\tau) - V\cdot\frac{1}{\alpha}\cdot\alpha\operatorname{Tan}(\alpha\tau)\right) = \gamma\left(\sec(\alpha\tau) + V\cdot\operatorname{Tan}(\alpha\tau)\right)$$

$$\frac{dx'}{d\tau} = \gamma\left(\frac{1}{\alpha}\cdot\alpha\operatorname{Tan}(\alpha\tau) - V\cdot\frac{1}{\alpha}\cdot\alpha\sec(\alpha\tau)\right) = \gamma\left(\operatorname{Tan}(\alpha\tau) - V\cdot\sec(\alpha\tau)\right)$$

$$\boxed{\underset{\sim}{u}^{\alpha} = \left[\gamma(\sec(\alpha\tau) - V\operatorname{Tan}(\alpha\tau)), \gamma(\operatorname{Tan}(\alpha\tau) - V\cdot\sec(\alpha\tau)), 0, 0\right]}$$

Problem 3 | Continued

d.) $\underset{\sim}{u}^{\alpha} = \left[-\gamma(-\sec(\alpha\tau) + V\tan(\alpha\tau)),\ \gamma(\tan(\alpha\tau) - V\cdot\sec(\alpha\tau)),\ 0,\ 0\right]$

$$\underset{\sim}{u}^{\alpha}\cdot\underset{\sim}{u}^{\alpha} = \left(-\gamma^2(-\sec(\alpha\tau) + V\cdot\tan(\alpha\tau))^2 + \gamma^2(\tan(\alpha\tau) - V\cdot\sec(\alpha\tau))^2\right)$$

$$= \gamma^2\left(-(-\sec(\alpha\tau) + V\cdot\tan(\alpha\tau))^2 + (\tan(\alpha\tau) - V\cdot\sec(\alpha\tau))^2\right)$$

$$= \gamma^2\left(-(\sec^2(\alpha\tau) - 2v\sec(\alpha\tau)\tan(\alpha\tau) + v^2\cdot\tan^2(\alpha\tau))\right) + \gamma^2\left(\tan^2(\alpha\tau) - 2v\sec(\alpha\tau)\tan(\alpha\tau) + v^2\sec^2(\alpha\tau)\right)$$

$$= \gamma^2\left[(-\sec^2(\alpha\tau) + \overset{0}{\cancel{2v\sec(\alpha\tau)\tan(\alpha\tau)}} - v^2\cdot\tan^2(\alpha\tau)) + \tan^2(\alpha\tau) - \overset{0}{\cancel{2v\sec(\alpha\tau)\tan(\alpha\tau)}} + v^2\sec^2(\alpha\tau)\right]$$

$$= \gamma^2\left[-\sec^2(\alpha\tau) + v^2\sec^2(\alpha\tau) + \tan^2(\alpha\tau) - v^2\tan^2(\alpha\tau)\right] \qquad \sec^2 x - \tan^2 x = 1$$

$$= \gamma^2\left[\sec^2(\alpha\tau)(v^2-1) - \tan^2(\alpha\tau)(v^2-1)\right] = \gamma^2\left[(\sec^2(\alpha\tau) - \tan^2(\alpha\tau))(v^2-1)\right]$$

$$= \gamma^2(v^2-1) = -1 \quad ✓ \quad \therefore \quad \boxed{\underset{\sim}{u}^{\alpha}\cdot\underset{\sim}{u}^{\alpha} = -1}$$

$$\underset{\sim}{u}' = \left[\gamma(\sec(\alpha\tau) - V\tan(\alpha\tau)),\ \gamma(\tan(\alpha\tau) - V\sec(\alpha\tau)),\ 0,\ 0\right]$$

$$\frac{d\underset{\sim}{u}^{\alpha'}}{d\tau} = \left[\gamma(\alpha\tan(\alpha\tau)\sec(\alpha\tau) - V\cdot\alpha\cdot\sec^2(\alpha\tau)),\ \gamma(\alpha\cdot\sec^2(\alpha\tau) - V\cdot\alpha\cdot\tan(\alpha\tau)\sec(\alpha\tau)),\ 0,\ 0\right] = \underset{\sim}{a}'$$

$$\underset{\sim}{u}'\cdot\underset{\sim}{a}' = -\gamma^2(\alpha\tan(\alpha\tau)\sec(\alpha\tau) - V\cdot\alpha\cdot\sec^2(\alpha\tau))(\sec(\alpha\tau) - V\cdot\tan(\alpha\tau)) + \gamma^2(\tan(\alpha\tau) - V\sec(\alpha\tau))\cdot(\alpha\cdot\sec^2(\alpha\tau) - V\cdot\alpha\tan(\alpha\tau)\sec(\alpha\tau))$$

$\rightarrow$ ① + ②

$$① = -\gamma^2(\alpha\tan(\alpha\tau)\sec^2(\alpha\tau) - V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau) - V\alpha\sec^3(\alpha\tau) + V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau))$$
$$= \gamma^2(V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau) - V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau) + V\alpha\sec^3(\alpha\tau) - \alpha\tan(\alpha\tau)\sec^2(\alpha\tau))$$

$$② = \gamma^2(\alpha\tan(\alpha\tau)\sec^2(\alpha\tau) - V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau) - V\alpha\sec^3(\alpha\tau) + V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau))$$
$$= \gamma^2(-V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau) + V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau) - V\alpha\sec^3(\alpha\tau) + \alpha\tan(\alpha\tau)\sec^2(\alpha\tau))$$

$$① + ②:\ \gamma^2(\cancel{V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau)} - \cancel{V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau)} + \cancel{V\alpha\sec^3(\alpha\tau)} - \cancel{\alpha\tan(\alpha\tau)\sec^2(\alpha\tau)})$$
$$+$$
$$\gamma^2(\cancel{-V\alpha\tan^2(\alpha\tau)\sec(\alpha\tau)} + \cancel{V^2\alpha\tan(\alpha\tau)\sec^2(\alpha\tau)} - \cancel{V\alpha\sec^3(\alpha\tau)} + \cancel{\alpha\tan(\alpha\tau)\sec^2(\alpha\tau)})$$

$$= 0 \quad \therefore \quad \boxed{\underset{\sim}{u}'\cdot\underset{\sim}{a}' = 0} \quad ✓$$

Problem 4

a.) $v = \frac{3}{5}c$, $E = 12{,}000$ MeV : $E_i = E_f$ $\quad \gamma M c^2 = M_0 c^2 + E \rightarrow \gamma M v = \frac{E}{c}$

$\gamma = \frac{5}{4}$: $\frac{5}{4} \cdot M \cdot \frac{3}{5} c = 12{,}000 \frac{\text{MeV}}{c}$: $\frac{3}{4} M c = 12{,}000 \frac{\text{MeV}}{c}$ $\therefore$ $M = 16{,}000 \frac{\text{MeV}}{c^2}$

$$\boxed{M = 16{,}000 \frac{\text{MeV}}{c^2}}$$

b.) $P_i = P_f$: $E_i = E_f$: $\vec{P} = \gamma m \vec{v}$: $E = \gamma m c^2$: $E = \gamma M$: $E_i = E_f$ $\quad \gamma = 5/4$, $v = \frac{3}{5}c$

$E_i = \gamma_i M c^2$, $E_f = \gamma_f M_0 c^2 + 12{,}000$ MeV $\qquad P_i = \gamma M v_i$, $P_f = 12{,}000 \frac{\text{MeV}}{c}$

$\gamma_i M c^2 = M_0 c^2 + 12{,}000$ MeV $\qquad \frac{5}{4} M \cdot \frac{3}{5} c = 12{,}000 \frac{\text{MeV}}{c}$

$\gamma_i = 5/4$

$M_0 = \gamma_i M - \frac{12{,}000}{c^2}$ MeV $\qquad \frac{3}{4} c \cdot M = 12{,}000 \frac{\text{MeV}}{c}$

$M = 16{,}000 \frac{\text{MeV}}{c^2}$

$M_0 = \frac{5}{4}\left(16{,}000 \frac{\text{MeV}}{c^2}\right) - 12{,}000 \frac{\text{MeV}}{c^2}$

$M_0 = 20{,}000 \frac{\text{MeV}}{c^2} - 12{,}000 \frac{\text{MeV}}{c^2} = 8{,}000 \frac{\text{MeV}}{c^2}$

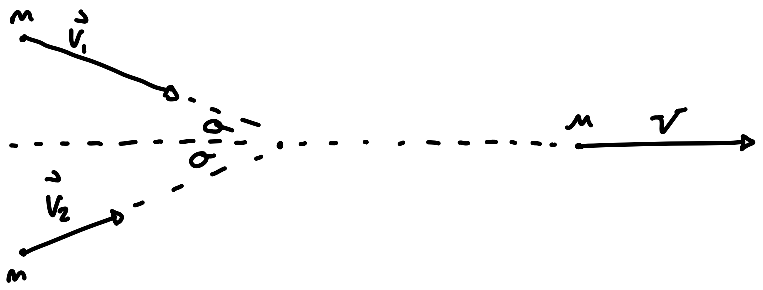
$$\boxed{M_0 = 8{,}000 \frac{\text{MeV}}{c^2}}$$

c.) $\vec{P} = \vec{P}_m$:

$$\boxed{\vec{P} = 12{,}000 \frac{\text{MeV}}{c}}$$

Problem 5

$V = \frac{4}{5}c \quad , \quad m = \frac{1}{3}M$



$P_i = P_f$

$\gamma_v m v\cos\theta + \gamma_v m v\cos\theta = \gamma_V M V \qquad \gamma_V = \frac{5}{3}$

$2\gamma_v m v\cos\theta = \frac{5}{3}M \cdot \frac{4}{5}c \;:\; \frac{2mv\cos\theta}{\sqrt{1-(v/c)^2}} = \frac{4}{3}Mc$

$\frac{2\cdot \frac{1}{3}M\, v\cos\theta}{\sqrt{1-(v/c)^2}} = \frac{4}{3}Mc \;:\; \frac{2v\cos\theta}{\sqrt{1-(v/c)^2}} = 4c$

$\frac{v/c}{\sqrt{1-(v/c)^2}} = \frac{2}{\cos\theta} \;:\; \frac{\sqrt{21}/5}{\frac{2}{5}} = \frac{2}{\cos\theta} \;:\; \cos\theta = \frac{4}{\sqrt{21}} \;:\; \theta = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right)$

$\theta = 29.2^\circ$

$E_i = E_f$

$\gamma_v mc^2 + \gamma_v mc^2 = \gamma_V MC^2 \;:\; \frac{2}{3}\gamma_v MC^2 = \gamma_V MC^2 \;:\; \frac{2}{3}\gamma_v = \frac{5}{3}$

$\gamma_v = \frac{5}{2} \;:\; \frac{1}{\sqrt{1-(v/c)^2}} = \frac{5}{2} \;:\; \sqrt{1-(v/c)^2} = 2/5 \;:\; 1-(v/c)^2 = 4/25$

$(v/c)^2 - 1 = -4/25 \;:\; (v/c)^2 = 1 - 4/25 = (v/c)^2 = \sqrt{21}/25 \;:\; v/c = \sqrt{21}/5$

$v_x = v\cos\theta = \frac{\sqrt{21}}{5}c \cdot \frac{4}{\sqrt{21}} = \frac{4}{5}c \;:\; v_y = v\sin\theta = \frac{\sqrt{21}}{5}c \cdot \frac{\sqrt{5}}{\sqrt{21}} = \frac{1}{\sqrt{5}}c$

$$\boxed{\begin{aligned}\vec{v}_1 &= (4/5\,c,\ -1/\sqrt{5}\,c,\ 0)\\ \vec{v}_2 &= (4/5\,c,\ 1/\sqrt{5}\,c,\ 0)\end{aligned}}$$