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PHYS 396
HW 3

Problem 1

$$\begin{bmatrix} t' = \gamma(t - vx/c^2) \\ x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{bmatrix}$$

$$ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2$$
$$= -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

a.)

$$dt' = \frac{dt'}{dt}.dt = \gamma \cdot dt \quad : \quad dt' = \frac{dt'}{dx}.dx = -\gamma \cdot \frac{v}{c^2}dx \quad \therefore \; dt' = \gamma dt - \gamma \frac{v}{c^2}dx$$

$$dx' = \frac{dx'}{dt}.dt = -\gamma v\,dt \quad : \quad dx' = \frac{dx'}{dx}.dx = \gamma\,dx \quad \therefore \; dx' = -\gamma v\,dt + \gamma dx$$

$$dy' = \frac{dy'}{dt}.dt = 0 \quad : \quad dy' = \frac{dy'}{dy}.dy = dy \quad \therefore \; dy' = dy$$

$$dz' = \frac{dz'}{dt}.dt = 0 \quad : \quad dz' = \frac{dz'}{dz}.dz = dz \quad \therefore \; dz' = dz$$

$$ds^2 = -c^2\left(\gamma dt - \gamma(v/c^2)dx\right)^2 + (-\gamma v\,dt + \gamma dx)^2 + (dy)^2 + (dz)^2$$

$$= -c^2\left(\gamma^2dt^2 - 2\gamma^2(v/c^2)dxdt + \gamma^2(v/c^2)^2dx^2\right) + \left(\gamma^2v^2dt^2 - 2\gamma^2v\,dxdt + \gamma^2dx^2\right) + dy^2 + dz^2$$

$$= \left(-c^2\gamma^2dt^2 + \cancel{2\gamma^2v\,dxdt} - \gamma^2(v/c)^2dx^2\right) + \left(\gamma^2v^2dt^2 - \cancel{2\gamma^2v\,dxdt} + \gamma^2dx^2\right) + dy^2 + dz^2$$

$$= \gamma^2dt^2(v^2 - c^2) + \gamma^2dx^2(1 - (v/c)^2) + dy^2 + dz^2$$

$$= -\gamma^2dt^2(c^2 - v^2) + \gamma^2dx^2(1 - (v/c)^2) + dy^2 + dz^2 \qquad \gamma^2 = \frac{1}{1 - v^2/c^2}$$

$$= -c^2\gamma^2dt^2(1 - v^2/c^2) + \gamma^2dx^2(1 - v^2/c^2) + dy^2 + dz^2$$

$$= -c^2dt^2\frac{(1 - v^2/c^2)}{1 - v^2/c^2} + dx^2\frac{(1 - v^2/c^2)}{1 - v^2/c^2} + dy^2 + dz^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \quad \checkmark$$

$$\boxed{ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2}$$

b.) $t' = \gamma(t - vx/c^2)$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ $\qquad x = v \cdot t$

$$t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} = \frac{(t - vx/c^2)\sqrt{1 - v^2/c^2}}{(1 - v^2/c^2)} = \frac{(t - v(v \cdot t)/c^2)\sqrt{1 - v^2/c^2}}{(1 - v^2/c^2)}$$

$$= \frac{(t - v^2t/c^2)\sqrt{1 - v^2/c^2}}{(1 - v^2/c^2)} = \frac{t\,\cancel{(1 - v^2/c^2)}\sqrt{1 - v^2/c^2}}{\cancel{(1 - v^2/c^2)}} = t\sqrt{1 - v^2/c^2}$$

$$\boxed{t' = t\sqrt{1 - v^2/c^2}}$$

3

c.) $\Delta s^2 = \Delta s'^2$

$$-c^2\Delta t^2 + \underbrace{\Delta x^2}_{L_*^2} = \cancel{c^2\Delta t'^2} + \underbrace{\Delta x'^2}_{L^2}$$

$$-c^2\Delta t^2 + L_*^2 = L^2 \qquad \Delta t' = 0 = \gamma(\Delta t - v\Delta x/c^2) \;:\; \Delta t - v\Delta x/c^2 = 0 \rightarrow \Delta t = \frac{v\Delta x}{c^2} = \frac{vL_*}{c^2}$$

$$-c^2 \cdot \frac{v^2}{c^4} L_*^2 + L_*^2 = L^2 \rightarrow -\frac{v^2}{c^2} L_*^2 + L_*^2 = L^2 \rightarrow L^2 = L_*^2(1 - v^2/c^2)$$

$$L^2 = L_*^2(1 - v^2/c^2) \rightarrow L = L_*\sqrt{1 - v^2/c^2}$$

$$\Delta x' = \Delta x\sqrt{1 - v^2/c^2}$$

$$\boxed{\Delta x' = \Delta x\sqrt{1 - v^2/c^2}}$$

d.) $t' = \gamma(t - vx/c^2) \;:\; t' = \gamma(t_i - vx_i/c^2)$

$$t' = \gamma(t_f - vx_f/c^2)$$

$$\gamma(t_i - vx_i/c^2) = \gamma(t_f - vx_f/c^2)$$

$$t_i - \frac{vx_i}{c^2} = t_f - \frac{vx_f}{c^2}$$

$$\frac{vx_f}{c^2} - \frac{vx_i}{c^2} = t_f - t_i \qquad x_f - x_i = D$$

$$t_f - t_i = \Delta t$$

$$\frac{v}{c^2}(x_f - x_i) = \Delta t$$

$$\boxed{\Delta t = \frac{vD}{c^2}}$$

Problem 2

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}=0\ ,\ \nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}=0$$

$$\begin{bmatrix} t'=\gamma(t-vx/c^2) \\ x'=\gamma(x-vt) \\ y'=y \\ z'=z \end{bmatrix}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \qquad \begin{matrix}\vec{E}=(E_x,E_y,E_z,t)\\ \vec{B}=(B_x,B_y,B_z,t)\end{matrix}$$

$$\frac{\partial}{\partial x}=\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \ :\ \frac{\partial}{\partial t}=\frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}+\frac{\partial x'}{\partial t}\frac{\partial}{\partial x'}\ ,\ \frac{\partial^2}{\partial y^2}=0\ ,\ \frac{\partial^2}{\partial z^2}=0$$

$$\frac{\partial x'}{\partial x}=\gamma\ ,\ \frac{\partial t'}{\partial x}=-\gamma\cdot\frac{v}{c^2}\ :\ \frac{\partial t'}{\partial t}=\gamma\ ,\ \frac{\partial x'}{\partial t}=-\gamma v$$

$$\frac{\partial}{\partial x}=\gamma\frac{\partial}{\partial x'}-\gamma\cdot\frac{v}{c^2}\frac{\partial}{\partial t'}\ ,\ \frac{\partial}{\partial t}=\gamma\frac{\partial}{\partial t'}-\gamma v\frac{\partial}{\partial x'}$$

$$\frac{\partial^2}{\partial x^2}=\left(\gamma\frac{\partial}{\partial x'}-\gamma\cdot\frac{v}{c^2}\frac{\partial}{\partial t'}\right)^2 \quad \frac{\partial^2}{\partial t^2}=\left(\gamma\frac{\partial}{\partial t'}-\gamma v\frac{\partial}{\partial x'}\right)^2 \ :\ \nabla^2\vec{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}$$

$$\frac{\partial^2}{\partial x^2}=\gamma^2\frac{\partial^2}{\partial x'^2}-2\gamma^2\cdot\frac{v}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}$$

$$\frac{\partial^2}{\partial t^2}=\gamma^2\cdot\frac{\partial^2}{\partial t'^2}-2\gamma^2 v\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\gamma^2 v^2\frac{\partial^2}{\partial x'^2}\ :\ \frac{1}{c^2}\left(\gamma^2\frac{\partial^2}{\partial t'^2}-2\gamma^2 v\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\gamma^2v^2\frac{\partial^2}{\partial x'^2}\right)$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\frac{\gamma^2\partial^2}{c^2\partial t'^2}-2\gamma^2\frac{v}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\frac{\gamma^2v^2\partial^2}{c^2\partial x'^2}$$

$$\gamma^2\frac{\partial^2}{\partial x'^2}-2\gamma^2\cdot\frac{v}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}=\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2}-2\gamma^2\frac{v}{c^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'}+\gamma^2\frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2}$$

$$\gamma^2\frac{\partial^2}{\partial x'^2}+\gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}=\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2}+\gamma^2\frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2}$$

$$\gamma^2\frac{\partial^2}{\partial x'^2}-\gamma^2\frac{v^2}{c^2}\frac{\partial^2}{\partial x'^2}=\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2}-\gamma^2\frac{v^2}{c^4}\frac{\partial^2}{\partial t'^2}$$

$$\gamma^2\frac{\partial^2}{\partial x'^2}\left(1-\frac{v^2}{c^2}\right)=\frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2}\left(1-\frac{v^2}{c^2}\right)$$

$$\frac{\partial^2}{\partial x'^2}=\frac{1}{c^2}\frac{\partial}{\partial t'^2}\longrightarrow\frac{\partial^2}{\partial x'^2}\vec{E}=\frac{1}{c^2}\frac{\partial}{\partial t'^2}\vec{E}\ \text{ or }\ \frac{\partial^2}{\partial x'^2}\vec{B}=\frac{1}{c^2}\frac{\partial}{\partial t'^2}\vec{B}$$

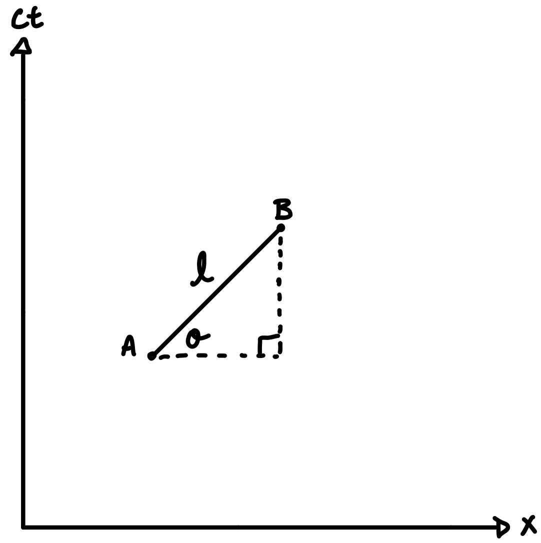
I omitted the ($\vec{E}$ & $\vec{B}$) to help simplify the algebra, since $\partial^2/\partial y'^2$, $\partial^2/\partial z'^2$ are both 0, the above equations simplify to,

$$\boxed{\nabla'^2\vec{E}=\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\vec{E}\ ,\ \nabla'^2\vec{B}=\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\vec{B}}$$

– Hence proving this invariance

Problem 3)

a.)



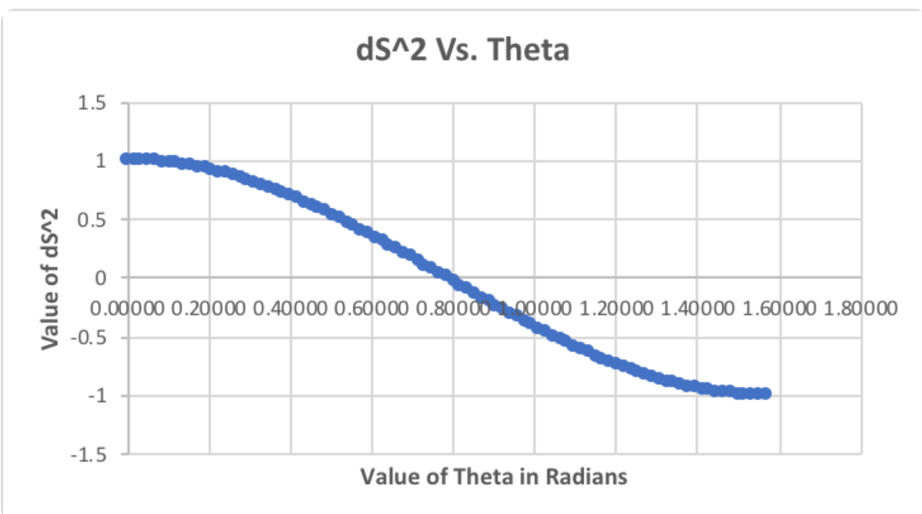
b.) $$ds^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\cancel{\Delta y})^2 + (\cancel{\Delta z})^2$$

$$= -(c\Delta t)^2 + (\Delta x)^2$$
$$= -(\ell \cdot \sin\theta)^2 + (\ell \cdot \cos\theta)^2$$
$$= \ell^2 \cdot \cos\theta^2 - \ell^2 \cdot \sin^2\theta$$
$$= \ell^2(\cos^2\theta - \sin^2\theta)$$

$$c\Delta t = \ell \cdot \sin\theta$$
$$\Delta x = \ell \cdot \cos\theta$$

$$\boxed{\Delta s^2 = \ell^2(\cos^2\theta - \sin^2\theta)}$$

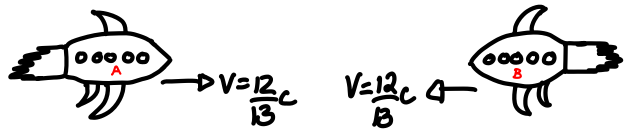
c.)



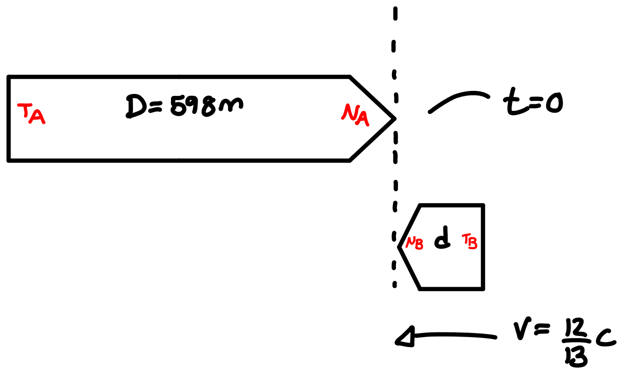
d.)

Δs^2 ranges from -1 to 1. The angle that has the smallest magnitude for this distance is $\pi/4$, since the magnitude here is 0. The maximum value for Δs^2 occurs at $\theta = 0$, and the minimum occurs at $\theta = \pi/2$.

Problem 4



a.)



A sees : $A = 598m$

$B = 115m \equiv d$

$\Delta t = \frac{VD}{c^2}$

b.) Time of clocks?

$N_A = N_B = 0$ — This is because the clocks are synchronized at the initial time $t=0$.

$T_A = 0$ - Since we are analyzing this with the A space ship at rest.

T_B : $\Delta t' = \frac{VD}{c^2}$: $V = \frac{12c}{13}$, $D = 299m$: $\Delta t' = \frac{1}{c^2} \cdot \frac{12}{13} c \cdot 299m = 276 \frac{m}{c}$

$\Delta t_2' = \gamma(\Delta t_2 - \frac{v}{c^2}\Delta x_2)$

$\Delta x_2' = \gamma(\Delta x_2^0 - v\cancel{\Delta t_2})$ 0

$\Delta t_2' = \gamma\left(0 - \frac{v}{c^2}\frac{\Delta x_2'}{\gamma}\right)$

$\Delta t_2' = -\frac{v}{c^2}\Delta x_2'$

$$\boxed{\begin{array}{l} N_A = 0,\ T_A = 0 \\ N_B = 0,\ T_B = 276\ \frac{m}{c} \end{array}}$$

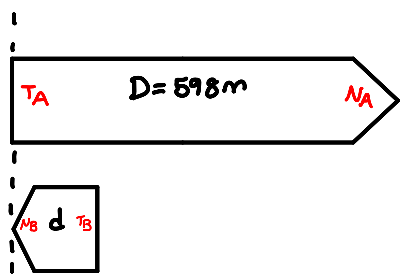
$\Delta t' = -\frac{VD}{c^2}$

c.) How long for N_B to reach T_A?

$D = V \cdot \Delta t'$: $\Delta t' = \frac{D}{V} \rightarrow$ $D = 598m$, $V = (12/13)c$: $\Delta t' = \frac{598m}{(12/13)c} = \frac{3887}{6} \approx 648 \frac{m}{c}$

$$\boxed{\Delta t' = 648\ \frac{m}{c}}$$

d.)



e.) Time of clocks? $t' = t\sqrt{1 - v^2/c^2}$: $\Delta t' = (648\ m/c)\sqrt{1 - (12/13)^2} = (648\ m/c)(5/13) = \frac{3240}{13} \approx 249 \frac{m}{c}$

$\Delta x' = 0$: $\Delta x = vt$

$\Delta t' = \gamma(\Delta t - v^2 \cdot \Delta t / c^2)$

$\Delta t' = \frac{\Delta t \cdot (1 - v^2/c^2)}{\sqrt{1 - v^2/c^2}}$

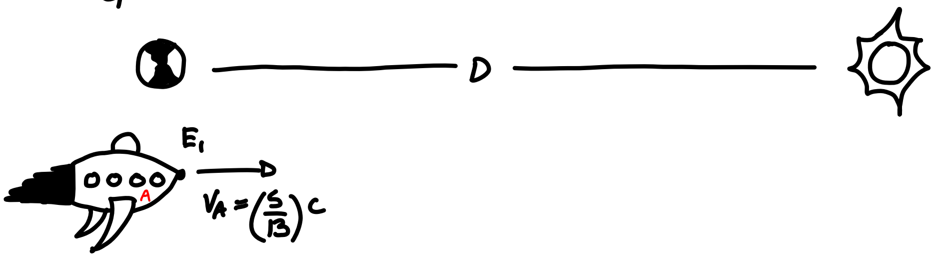
$\Delta t' = \Delta t\sqrt{1 - v^2/c^2}$

$$\boxed{\begin{array}{l} N_A = 648\ m/c,\ T_A = 648\ m/c \\ N_B = 249\ m/c,\ T_B = 525\ m/c \end{array}}$$

Problem 5

a.)

Spaceship A Departs Earth



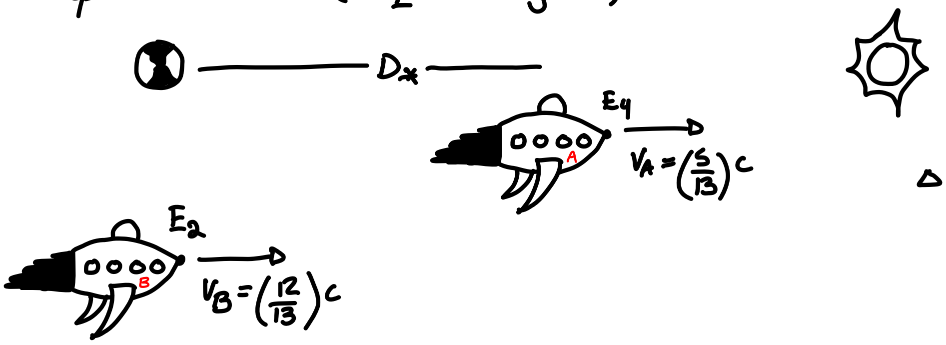
$D = V_a \cdot t_a$

$D = V_b \cdot t_b$

$D = D_* + V_a t_b$

$D_* + V_a t_b = V_b t_b$

Spaceship B Departs Earth $(\Delta t_E = 7.00 \text{ years})$



$D_* = V_a \cdot 7\text{yrs}$

$\Delta x_{13} = \Delta x_{23}$

Spaceship A & B arrive at star

b.)

$$V_a \cdot 7\text{yrs} + V_a \cdot t_b = V_b t_b \rightarrow V_a \cdot 7\text{yrs} = V_b t_b - V_a t_b \rightarrow V_a \cdot 7\text{yrs} = t_b(V_b - V_a)$$

$$t_b = \frac{V_a \cdot 7\text{yrs}}{(V_b - V_a)} = \frac{(\frac{5}{13}c) \cdot 7\text{yrs}}{(\frac{12}{13}c - \frac{5}{13}c)} = \frac{(\frac{5}{13})c \cdot 7\text{yrs}}{(\frac{7}{13})c} = \frac{5}{7} \cdot 7\text{yrs} = 5\text{yrs} \quad \therefore \quad t_b = 5\text{yrs}$$

$$\Delta t = 7\text{yrs} + t_b = 7\text{yrs} + 5\text{yrs} = 12\text{yrs}$$

$$\boxed{\Delta t = 12\text{yrs}}$$

c.) D? $\quad V = \frac{D}{t} \quad \therefore \quad D = V \cdot t = V_A \cdot \Delta t = \left(\frac{5}{13}\right)c\left(12\text{yrs}\right) = \frac{60}{13} c \cdot \text{yrs}$

$$\boxed{D = 4.6\ c \cdot \text{yrs}}$$

Problem 5) continued

d.) $\Delta t' = \Delta t\sqrt{1-v^2/c^2}$

$V = \left(\frac{5}{13}\right)c$ $\quad$ $\Delta t' = 7\text{ yrs}\cdot\sqrt{1-(5/13)^2} = 7\text{yrs}\cdot\sqrt{1-\frac{25}{169}} = 7\text{yrs}\cdot\sqrt{144/169}$

$t = \gamma(t' + Vx'/c^2)$ $\quad$ $7\text{yrs}\cdot\frac{12}{13} = 6.5\text{ yrs}$

$\Delta x' = 0$

$$\boxed{\Delta t'_A = 6.5\text{ yrs}}$$

e.) $\Delta t' = \Delta t\sqrt{1-v^2/c^2}$

$V = \left(\frac{5}{13}\right)c$ $\quad$ $\Delta t' = 12\text{ yrs}\sqrt{1-(5/13)^2} = 12\text{ yrs}\cdot(12/13) = \frac{144}{13}\text{ yrs}$

$\Delta x' = 0$

$$\boxed{\Delta t'_A = 11.1\text{ yrs}}$$

f.) $D = \frac{60}{13}\text{ c}\cdot\text{yrs}$, $\Delta t = \frac{D}{V}$, $\Delta t = \frac{60/13\text{ c}\cdot\text{yrs}}{(12/13)c} = \frac{60}{12}\text{ yrs} = 5\text{ yrs}$ $\leftarrow$ Earth time

$\Delta t' = \Delta t\sqrt{1-v^2/c^2} = 5\text{ yrs}\sqrt{1-(12/13)^2} = 5\text{ yrs}\cdot(5/13) = \frac{25}{13}\text{ yrs}$

$$\boxed{\Delta t'_B = 1.9\text{ yrs}}$$

g.) $d = D\sqrt{1-v^2/c^2}$: $D = \frac{60}{13}\text{ c}\cdot\text{yrs}$, $V = \frac{5}{13}c$

$$d = \frac{60}{13}\text{ c}\cdot\text{yrs}\sqrt{1-(5/13)^2} = \frac{60}{13}\text{ c}\cdot\text{yrs}\cdot\frac{12}{13} = \frac{720}{169}\text{ c}\cdot\text{yrs}$$

$\Delta x = \gamma(\Delta x' + vt')$ $\quad \Delta t' = 0$

$\Delta x' = \gamma^{-1}\Delta x \rightarrow d = D\sqrt{1-v^2/c^2}$

$$\boxed{d = 4.3\text{ c}\cdot\text{yrs}}$$

h.) $d = D\sqrt{1-v^2/c^2}$: $D = \frac{60}{13}\text{ c}\cdot\text{yrs}$, $V = \frac{12}{13}c$

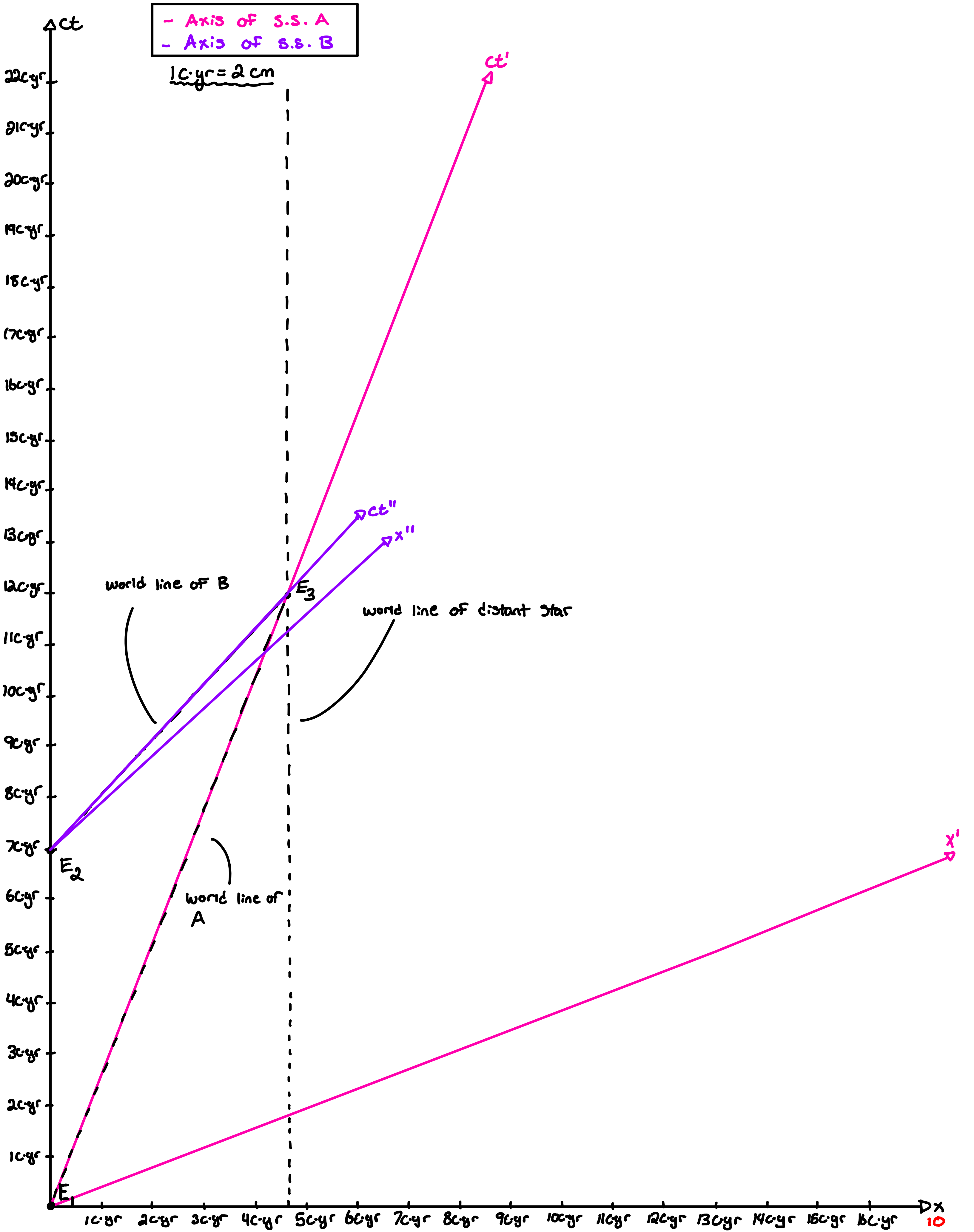
$$d = \frac{60}{13}\text{ c}\cdot\text{yrs}\sqrt{1-(12/13)^2} = \frac{60}{13}\text{ c}\cdot\text{yrs}\cdot\left(\frac{5}{13}\right) = \frac{300}{169}\text{ c}\cdot\text{yrs}$$

$\Delta x = \gamma(\Delta x' + vt')$ $\quad \Delta t' = 0$

$\Delta x' = \gamma^{-1}\Delta x \rightarrow d = D\sqrt{1-v^2/c^2}$

$$\boxed{d = 1.8\text{ c}\cdot\text{yrs}}$$

Problem 6



- Axis of S.S. A
- Axis of S.S. B
1c·yr = 2 cm
ct
ct'
x'
ct''
x''
x
E
E2
E3
world line of B
world line of distant star
world line of A
22c·yr
21c·yr
20c·yr
19c·yr
18c·yr
17c·yr
16c·yr
15c·yr
14c·yr
13c·yr
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