PHYS 396 HOMEWORK SET 6

1. $$ds^2 = -\left(1 - \frac{R_s}{r}\right)dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

- FOR θ = CONST., φ = CONST. FOR A RADIAL LINE ELEMENT $d\theta = d\varphi = 0$
- FOR A NULL CURVE, SET $ds^2 = 0$

THE LINE ELEMENT TAKES THE FORM...

$$0 = -\left(1 - \frac{R_s}{r}\right)dv^2 + 2\,dv\,dr = dv\left[-\left(1 - \frac{R_s}{r}\right)dv + 2dr\right]$$

THE ABOVE DIFFERENTIAL EQUATION HAS TWO SOLUTIONS..

1) $dv = 0$ OR $\left[\frac{dv}{dr} = 0\right.$

2) $-\left(1 - \frac{R_s}{r}\right)dv + 2dr = 0$ OR $\left.\frac{dv}{dr} = \frac{2}{\left(1 - \frac{R_s}{r}\right)}\right]$

- THESE ARE THE SLOPES OF THE LIGHT CONE AT POINT (v, r)

NOW

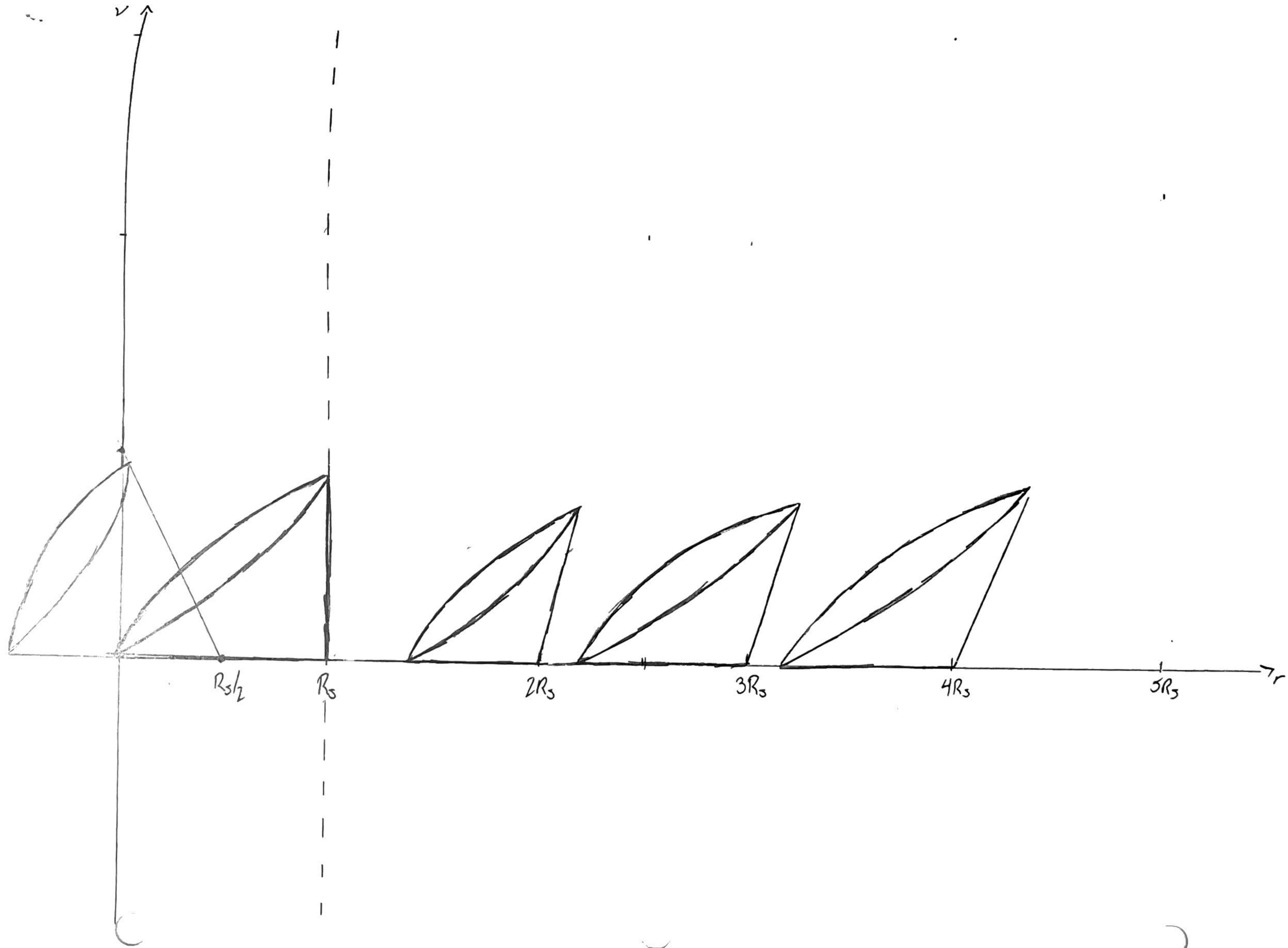
$$\left.\frac{dv}{dr}\right|_{r=R_s/2} = \frac{2}{1 - \frac{R_s}{R_s/2}} = \frac{2}{1-2} = -2$$

$$\left.\frac{dv}{dr}\right|_{r=R_s} = \frac{2}{1-1} = \infty$$

$$\left.\frac{dv}{dr}\right|_{r=2R_s} = \frac{2}{1 - \frac{R_s}{2R_s}} = \frac{2}{1-\frac{1}{2}} = 4$$

$$\left.\frac{dv}{dr}\right|_{r=3R_s} = \frac{2}{1 - \frac{R_s}{3R_s}} = \frac{2}{1-\frac{1}{3}} = 3$$

$$\left.\frac{dv}{dr}\right|_{r=4R_s} = \frac{2}{1 - \frac{R_s}{4R_s}} = \frac{2}{1-\frac{1}{4}} = \frac{8}{3}$$



v
r
Rs/2
Rs
2Rs
3Rs
4Rs
5Rs

2. $ds^2 = -c^2dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$

FOR A t = CONST. SLICE, $dt = 0$ AND THE LINE ELEMENT TAKES THE FORM..

$ds^2 = dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$

a) TO CALCULATE THE CIRCUMFERENCE, SET $r = R$; $\theta = \pi/2$ $\therefore$ $dr = d\theta = 0$.

THE LINE ELEMENT TAKES THE FORM..

$ds^2 = (b^2+R^2)\, d\varphi^2$ SO $ds = \sqrt{b^2+R^2}\, d\varphi$

so

$C = \oint ds = \int_0^{2\pi} \sqrt{b^2+R^2}\, d\varphi = \sqrt{b^2+R^2} \int_0^{2\pi} d\varphi$

$\left[C = 2\pi\sqrt{b^2+R^2} \right]$

b) TO CALCULATE THE DISTANCE FROM ONE SPHERE TO THE OTHER, SET θ = CONST. ; φ = CONST $\therefore$ $d\theta = d\varphi = 0$

THE LINE ELEMENT TAKES THE FORM...

$ds^2 = dr^2$ $\therefore$ $ds = dr$

SO

$S = \int_{-R}^{R} ds = \int_{-R}^{R} dr = 2R$

so

$\left[S = 2R \right]$

c) TO CALCULATE THE AREA OF THE TWO-SPHERE, SET $r = R$ $\therefore$ $dr = 0$

THE LINE ELEMENT TAKES THE FORM..

$ds^2 = (b^2+R^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$ SO $g_{22} = (b^2+R^2)$

$g_{33} = (b^2+R^2)\sin^2\theta$

$dx^2 = d\theta$

$dx^3 = d\varphi$

so

$dA = \sqrt{g_{22}g_{33}}\, dx^2 dx^3 = (b^2+R^2)\sin\theta\, d\theta\, d\varphi$

so

$$A = \int_0^{2\pi}\int_0^{\pi} (b^2+R^2)\sin\theta \, d\theta \, d\varphi = (b^2+R^2)\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta = 4\pi(b^2+R^2)$$

$$\boxed{A = 4\pi(b^2+R^2)}$$

d) $dV = \sqrt{g_{11}g_{22}g_{33}}\, dx^1 dx^2 dx^3$ where $g_{11} = 1$

$dx^1 = dr$

$$= (b^2+r^2)\sin\theta \, dr \, d\theta \, d\varphi$$

so

$$V = \int_{-R}^{R}\int_0^{\pi}\int_0^{2\pi} (b^2+r^2)\sin\theta \, dr \, d\theta \, d\varphi = \int_{-R}^{R} (b^2+r^2)\,dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi\left(b^2 r + \frac{1}{3}r^3\right)\Big|_{-R}^{R}$$

$$\boxed{V = 8\pi\left(b^2R + \frac{1}{3}R^3\right)}$$

⋮

3.

$$x = R\sin\chi\sin\theta\cos\varphi$$

$$y = R\sin\chi\sin\theta\sin\varphi$$

$$z = R\sin\chi\cos\theta$$

$$w = R\cos\chi$$

NOW

a) $$x^2+y^2+z^2+w^2 = R^2\left[\sin^2\chi\sin^2\theta\cos^2\varphi + \sin^2\chi\sin^2\theta\sin^2\varphi + \sin^2\chi\cos^2\theta + \cos^2\chi\right]$$

$$= R^2\left[\sin^2\chi\sin^2\theta\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} + \sin^2\chi\cos^2\theta + \cos^2\chi\right]$$

$$= R^2\left[\sin^2\chi\underbrace{(\sin^2\theta + \cos^2\theta)}_{1} + \cos^2\chi\right] = R^2\left[\sin^2\chi + \cos^2\chi\right] = R^2 \checkmark$$

b) $$dx = R\left[\cos\chi\sin\theta\cos\varphi\, d\chi + \sin\chi\cos\theta\cos\varphi\, d\theta - \sin\chi\sin\theta\sin\varphi\, d\varphi\right]$$

$$dy = R\left[\cos\chi\sin\theta\sin\varphi\, d\chi + \sin\chi\cos\theta\sin\varphi\, d\theta + \sin\chi\sin\theta\cos\varphi\, d\varphi\right]$$

$$dz = R\left[\cos\chi\cos\theta\, d\chi - \sin\chi\sin\theta\, d\theta\right]$$

$$dw = -R\sin\chi\, d\chi$$

c) $$dx^2 = R^2\left[\cos^2\chi\sin^2\theta\cos^2\varphi\, d\chi^2 + \sin^2\chi\cos^2\theta\cos^2\varphi\, d\theta^2 + \sin^2\chi\sin^2\theta\sin^2\varphi\, d\varphi^2 + 2\sin\chi\cos\chi\sin\theta\cos\theta\cos^2\varphi\, d\chi\, d\theta - 2\sin\chi\cos\chi\sin^2\theta\sin\varphi\cos\varphi\, d\chi\, d\varphi - 2\sin^2\chi\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi\right]$$

$$dy^2 = R^2\left[\cos^2\chi\sin^2\theta\sin^2\varphi\, d\chi^2 + \sin^2\chi\cos^2\theta\sin^2\varphi\, d\theta^2 + \sin^2\chi\sin^2\theta\cos^2\varphi\, d\varphi^2 + 2\sin\chi\cos\chi\sin\theta\cos\theta\sin^2\varphi\, d\chi\, d\theta + 2\sin\chi\cos\chi\sin^2\theta\sin\varphi\cos\varphi\, d\chi\, d\varphi + 2\sin^2\chi\sin\theta\cos\theta\sin\varphi\cos\varphi\, d\theta\, d\varphi\right]$$

NOW ADDING YIELDS...

$$dx^2+dy^2 = R^2\left[\cos^2\chi\sin^2\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}d\chi^2 + \sin^2\chi\cos^2\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}d\theta^2 + \sin^2\chi\sin^2\theta\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}d\varphi^2 + 2\sin\chi\cos\chi\sin\theta\cos\theta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}d\chi\, d\theta\right]$$

$$= R^2\left[\cos^2\chi\sin^2\theta\, d\chi^2 + \sin^2\chi\cos^2\theta\, d\theta^2 + \sin^2\chi\sin^2\theta\, d\varphi^2 + 2\sin\chi\cos\chi\sin\theta\cos\theta\, d\chi\, d\theta\right]$$

NOW

$$dz^2 = R^2\left[\cos^2\chi\cos^2\theta\, d\chi^2 + \sin^2\chi\sin^2\theta\, d\theta^2 - 2\sin\chi\cos\chi\sin\theta\cos\theta\, d\chi\, d\theta\right]$$

so

$$dx^2+dy^2+dz^2 = R^2\left[\cos^2\chi\,\underbrace{(\sin^2\theta+\cos^2\theta)}_{=1}d\chi^2 + \sin^2\chi\,\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}d\theta^2 + \sin^2\chi\sin^2\theta\, d\phi^2\right]$$

$$= R^2\left[\cos^2\chi\, d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi\sin^2\theta\, d\phi^2\right]$$

e) LASTLY

$dw = -R\sin\chi\, d\chi$ .. so $dw^2 = R^2\sin^2\chi\, d\chi^2$

so

$$ds^2 = dx^2+dy^2+dz^2+dw^2 = R^2\left[\underbrace{(\cos^2\chi+\sin^2\chi)}_{=1}d\chi^2 + \sin^2\chi\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right]$$

$$\Rightarrow \quad ds^2 = R^2\left[d\chi^2+\sin^2\chi\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right] \checkmark$$

f) let $r \equiv \sin\chi$

THEN $dr = \cos\chi\, d\chi$ so $d\chi = \dfrac{dr}{\cos\chi} = \dfrac{dr}{(1-\sin^2\chi)^{1/2}} = \dfrac{dr}{(1-r^2)^{1/2}}$

so

$$ds^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)\right]$$

g) $0 \le r \le 1$

4. SETTING $t = \text{const.}$ AND $\theta = \pi/2$, $dt = d\theta = 0$, THE LINE ELEMENT FOR THE 4D HOMOGENEOUS CLOSED UNIVERSE TAKES THE FORM...

(*)

a) $$d\Sigma^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right]$$

THE METRIC FOR 3D FLAT SPACE IN CYLINDRICAL COORDINATES TAKES THE FORM..

(**)

$$dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2$$

FOR OUR AXISYMMETRIC SURFACE, WE CAN TAKE

$$\psi = \varphi \quad ; \quad z = z(r) \quad ; \quad \rho = \rho(r)$$

$$\therefore dz = \frac{dz}{dr}dr \quad \therefore \quad dz^2 = \left(\frac{dz}{dr}\right)^2 dr^2$$

$$d\rho = \frac{d\rho}{dr}dr \quad \therefore \quad d\rho^2 = \left(\frac{d\rho}{dr}\right)^2 dr^2$$

PLUGGING THESE INTO (**) YIELDS...

$$dS^2 = \left[\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2$$

THIS TAKES THE SAME FORM AS (*) IF ...

1) $\rho^2 = r^2 R^2 \quad \therefore \quad \rho = rR \quad \therefore \quad \frac{d\rho}{dr} = R$

2) $\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2 = \frac{R^2}{1-r^2}$

$\Downarrow$

$$\left(\frac{dz}{dr}\right)^2 + R^2 = \frac{R^2}{1-r^2} \quad \therefore \quad \left(\frac{dz}{dr}\right)^2 = \frac{R^2}{1-r^2} - R^2 = \frac{R^2}{1-r^2}\left[1-(1-r^2)\right] = \frac{R^2 r^2}{1-r^2}$$

$$\therefore \boxed{\frac{dz}{dr} = \pm\frac{Rr}{\sqrt{1-r^2}}}$$

c) $$\int dz = \pm\int\frac{R r\,dr}{\sqrt{1-r^2}} = \mp\frac{1}{2}R\int\frac{du}{u^{1/2}}$$

let $u = 1 - r^2$

$du = -2r\,dr \quad \therefore \quad r\,dr = -\frac{1}{2}du$

SO

$$z(r) = C \mp R u^{1/2} = C \mp R\sqrt{1 - r^2}$$

SO

$$\pm R\sqrt{1 - r^2} = C - z(r)$$

SQUARING YIELDS

$$R^2(1 - r^2) = (C - z(r))^2$$

$$1 = \frac{1}{R^2}(C - z(r))^2 + r^2$$

NOW SETTING $C = 0$ YIELDS

$$\left[1 = \frac{z^2}{R^2} + r^2\right]$$