4.3 Complex characteristic Roots

4.3 # 14, 21-28, 42, 50, 75

Characteristic Equation with $\Delta < 0$

$ar^2 + br + c = 0$

$\Delta = b^2 - 4ac < 0$ $\quad y(t) = c_1 e^{(\alpha+\beta)t} + c_2 e^{(\alpha - i\beta)t}$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a} i \qquad r_1 = \alpha + \beta i$$

$$r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} i \qquad r_2 = \alpha - \beta i$$

Eulers Identity

$e^{i\theta} = \cos\theta + i\sin\theta$

$= c_1(\cos\theta(\alpha+\beta i) + i\sin(\alpha+\beta t))$

$= c_2(\cos\theta(\alpha-\beta i) + i\sin(\alpha-\beta t))$

$= e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$

$$y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

Ex:) $y'' + 2y' + 4y = 0$ $\qquad y(0) = 0$, $y'(0) = 1$

$r^2 + 2r + 4 = 0$ $\qquad a = 1$, $b = 2$, $c = 4$ $\qquad \Delta = 4 - 4(1)(4)$, $4 - 16 = 12$

$r_1: \alpha + i\beta$

$\alpha = \frac{-b}{2a} = \frac{-2}{2(1)} = -1$ $\qquad \beta = \frac{\sqrt{-12}}{2}$ $\qquad c_1 = 0$

$\alpha = -1$ $\qquad \beta = \sqrt{3}$ $\qquad c_2 = -\frac{1}{\sqrt{3}}$

$r = -1 \pm i\sqrt{3}$

$y(t) = e^{-t}(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t))$

$0 = e^0(c_1\cos(0) + c_2\sin(0))$

$0 = c_1$

$y'(t) = -e^{-t}(c_1\cos(\sqrt{3}t) + c_2\sin(\sqrt{3}t)) + e^{-t}(-\sqrt{3}c_1\sin(\sqrt{3}t) + \sqrt{3}c_2\cos(\sqrt{3}t))$

$-1 = -e^0(c_1\cos(0) + c_2\sin(0)) + e^0(-\sqrt{3}c_1\sin 0 + \sqrt{3}c_2\cos(0))$

$-1 = -1(c_1) + 1(\sqrt{3}c_2)$

$-1 = -c_1 + c_2\sqrt{3}$

$-1 = c_2\sqrt{3}$

$\frac{-1}{\sqrt{3}} = c_2$

$y(t) = e^{-t}\left(-\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right)$

$y(t) = \frac{-1}{\sqrt{3}} e^{-t}(\sin(\sqrt{3}t))$

undamped Mass spring system

$\Delta < 0$, $y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$

Alternate Form

$x(t) = A(t)\cos(\omega_0 t - \delta)$

$A(t) = Ae^{-(b/2m)t}$

Phase Angle (δ) measured in radians

Phase shift $\varphi = \delta/\omega_d$

Circular Frequency $\omega_d = \frac{\sqrt{4mk - b^2}}{2m}$ (rad/s)

Natural Quasi Frequency $f_d = \omega_d/2\pi$ (in hertz)

Quasi - Period $T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{4\pi m}{\sqrt{4mk - b^2}}$ (In seconds)

Time constant $\tau = 2m/b$

$y''' + y'' - 5y' + 3y = 0$

$r^3 + r^2 - 5r + 3 = 0$

Rational Roots:

Factors of last term: $3 <^{1\,\pm}_{3\,\pm}$

$$\frac{r^3 + r^2 - 5r + 3}{x - 1}$$

$$\begin{array}{c|cccc} 1 & 1 & 1 & -5 & 3 \\ & & 1 & 2 & -3 \\ \hline & 1 & 2 & -3 & \boxed{0} \end{array}$$

$x^2 + 2x - 3$

$(x+3)(x-1)$

$x = -3$

$x = 1$

$x = 1$

$y(t) = c_1 e^{-3t} + c_2 e^{t} + c_2 t e^{t}$ ⇐ repeated roots

$y^{(5)} + 3y^{(4)} + 3y''' + y'' = 0$

$r^5 + 3r^4 + 3r^3 + r^2 = 0$

$r^2(r^3 + 3r^2 + 3r + 1) = 0$

$r^2 (r+1)^3 = 0$

$y(t) = c_1 e^0 + c_2 t e^0 + C_3 e^{-t} + C_4 t e^{-t} + C_5 t^2 e^{-t}$

$y(t) = c_1 + c_2 + c_3 e^{-t} + c_4 t e^{-t} + c_5 t^2 e^{-t}$

$y'' + y = 0$ $\quad y(0) = 0,\ y(\pi/2) = 0$

$y = \sin(t)$ $\quad -\sin(t) + \sin(t) = 0$ ✓

$y' = \cos(t)$

$y'' = -\sin(t)$ $\quad \alpha = 0$ $\quad y(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

$(r^2 + 1) = 0$ $\quad \beta = 1$ $\quad 0 = e^0 (c_1 \cos(0) + c_2 \sin 0)$

$r = \pm i$ $\quad 0 = c_1$

$y(\pi/2) = c^{\frac{\pi}{2}t}(c_1 \cos\frac{\pi}{2} + c_2 \sin\frac{\pi}{2})$

only solution is the trivial solution $\quad 0 = c_2$