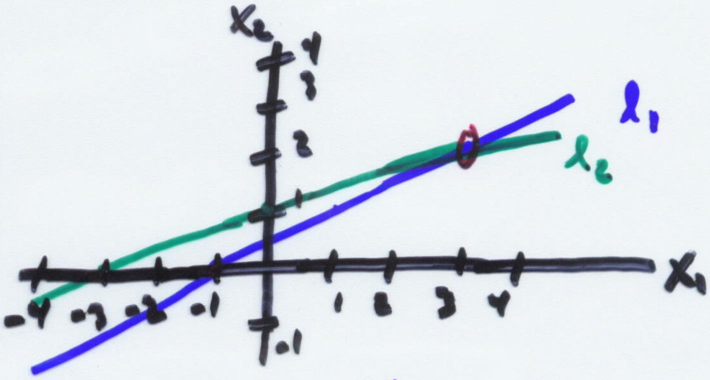
Ch6.1 : Linear Systems of Equations

Example

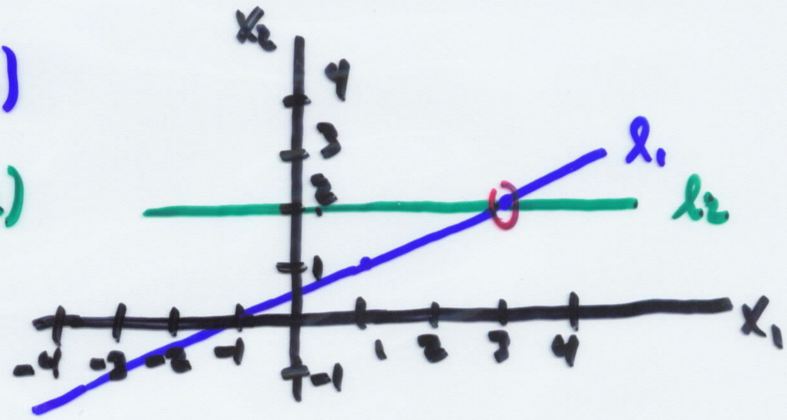
$$x_1 - 2x_2 = -1 \quad (\ell_1)$$
$$-x_1 + 3x_2 = 3 \quad (\ell_2)$$



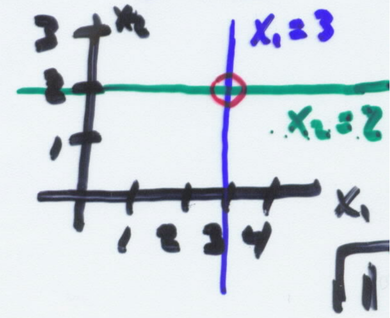
One way to solve this system is to add a multiple of one equation to the other, creating a new but equivalent system (same solution).

$$x_1 - 2x_2 = -1 \quad (\ell_1)$$
$$x_2 = 2 \quad (\ell_2)$$

(Row 2 = Row 2 + Row 1)



Now that we know $\underline{x_2 = 2}$, we can back solve (backward substitution) to find x_1:



$$x_1 = -1 + 2x_2 = -1 + 2(2) = 3$$

Solution: $x_1 = 3$, $x_2 = 2$

Matrix Representation

Example On previous slide, we had

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \iff \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

$r_2 = r_2 + r_1$

and

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \iff \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

← upper triangular matrix

Coefficient Matrix: $\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = A$

Augmented Matrix: $\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] = [A \ \bar{b}]$

Right-hand side vector: $\bar{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The coefficient matrix A is 2×2, the augmented matrix $[A \ \bar{b}]$ is 2×3, and $\bar{b}$ is 2×1.

When reducing a system of linear equations to an equivalent system that is easier to solve (back subs.), we use **elementary row operations** that must satisfy the following:

(1) Replace one row by the sum of itself and a multiple of another row.

(2) Interchange two rows.

(3) Multiply all entries in a row by a nonzero constant.

Example (from previous slide)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \iff \left[\begin{array}{cc:c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \xrightarrow{r_2 = r_2 + r_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

Solution: $x_1 = 3, \quad x_2 = 2$

$(x_1 = -1 + 2x_2 = -1 + 2(2) = 3)$

General Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= a_{1,n+1} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= a_{2,n+1} \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad &\quad\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= a_{n,n+1} \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right] = \tilde{A}^{(1)}$$

Provided the pivot $a_{11} \neq 0$, we perform:

$$\xrightarrow[\substack{\vdots \\ r_n = r_n - \frac{a_{n1}}{a_{11}} r_1}]{r_2 = r_2 - \frac{a_{21}}{a_{11}} r_1 \\ \vdots} \left[\begin{array}{cccc:c} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} & a_{1,n+1}^{(2)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & a_{2,n+1}^{(2)} \\ \vdots & \vdots & & & \\ 0 & a_{n,1}^{(2)} & \cdots & a_{n,n}^{(2)} & a_{n,n+1}^{(2)} \end{array}\right] = \tilde{A}^{(2)}$$

where $a_{1k}^{(2)} = a_{1k}^{(1)}, \quad k = 1, \ldots, n+1$

$a_{k1}^{(2)} = 0, \quad k = 2, 3, \ldots, n$

$a_{kj}^{(2)} = a_{kj}^{(1)} - m_{k1} \cdot a_{1j}^{(1)}, \quad m_{k1} = \dfrac{a_{k1}^{(1)}}{a_{11}^{(1)}} \quad \Bigg\} \begin{array}{l} k = 2, \ldots, n \\ j = 2, \ldots, n+1 \end{array}$

k-th step:

$$
\left[\begin{array}{ccccccc|c}
a_{11}^{(k)} & a_{12}^{(k)} & \cdots & a_{1,k-1}^{(k)} & a_{1,k}^{(k)} & \cdots & a_{1n}^{(k)} & a_{1,n+1}^{(k)} \\
0 & a_{22}^{(k)} & \cdots & a_{2,k-1}^{(k)} & a_{2,k}^{(k)} & \cdots & a_{2n}^{(k)} & a_{2,n+1}^{(k)} \\
0 & 0 & \ddots & \vdots & \vdots & & \vdots & \vdots \\
\vdots & \vdots & \ddots & a_{k-1,k-1}^{(k)} & a_{k-1,k}^{(k)} & \cdots & a_{k-1,n}^{(k)} & a_{k-1,n+1}^{(k)} \\
\vdots & \vdots & \ddots & 0 & a_{k,k}^{(k)} & \cdots & a_{k,n}^{(k)} & a_{k,n+1}^{(k)} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_{n,k}^{(k)} & \cdots & a_{n,n}^{(k)} & a_{n,n+1}^{(k)}
\end{array}\right]
$$

where:

$$
a_{ij}^{(k)} = \begin{cases}
a_{ij}^{(k-1)} , & i \le k-1 \text{ and } j \le n+1 \\
0 , & k \le i \le n \text{ and } j \le k-1 \\
a_{ij}^{(k-1)} - m_{i,k-1} \cdot a_{k-1,j}^{(k-1)} , & k \le i \le n,\ k \le j \le n+1 ,
\end{cases}
$$

where $m_{i,k-1} = \dfrac{a_{i,k-1}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}}$, $k \le i \le n$

(need row switch ...)

The procedure will fail if one of the pivots $a_{11}^{(1)}, a_{22}^{(2)}, \ldots, a_{nn}^{(n)}$ is zero.

Otherwise, our solution will be:

$$x_n = \frac{a_{n,n+1}^{(n)}}{a_{nn}^{(n)}}$$

$$x_i = \frac{a_{i,n+1}^{(n)} - a_{i,n}^{(n)} x_n - a_{i,n-1}^{(n)} x_{n-1} - \cdots - a_{i,i+1}^{(n)} x_{i+1}}{a_{ii}^{(n)}}$$

$$= \frac{a_{i,n+1}^{(n)} - \sum_{j=i+1}^{n} a_{ij}^{(n)} x_j}{a_{ii}^{(n)}}, \quad i = n-1, n-2, \ldots, 2, 1$$

(Backward substitution)

$$\tilde{A}^{(n)} = \left[\begin{array}{cccc:c} a_{11}^{(n)} & a_{12}^{(n)} & \cdots & a_{1n}^{(n)} & a_{1,n+1}^{(n)} \\ 0 & a_{22}^{(n)} & \cdots & a_{2n}^{(n)} & a_{2,n+1}^{(n)} \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)} & a_{n,n+1}^{(n)} \end{array}\right]$$

Example

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Aug. Matrix:

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \xrightarrow{r_3 = r_3 + 4r_1} \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{r_2 = \frac{1}{2} r_2} \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{r_3 = r_3 + 3r_2} \left[\begin{array}{ccc:c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Back substitution:

$x_3 = 3,$ $\quad x_2 = 4 + 4x_3 = 4 + 4(3) = 16$

$x_1 = 0 + 2x_2 - x_3 = 2(16) - 3 = 29$

Solution $\quad x_1 = 29, \; x_2 = 16, \; x_3 = 3$

This method of solution is called **Gaussian elimination with backward substitution.**

Example

$$\begin{aligned} x_1 + x_2 \qquad\quad + 3x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

Augmented matrix:

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right] \xrightarrow[\substack{r_3 = r_3 - 3r_1 \\ r_4 = r_4 + r_1}]{r_2 = r_2 - 2r_1} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right]$$

$$\xrightarrow{r_2 = -r_2} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 5 & 7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \xrightarrow[r_4 = r_4 - 3r_2]{r_3 = r_3 + 4r_2} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

$$\xrightarrow{r_4 = -\frac{1}{13} r_4} \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 5 & 7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= 4 - x_2 - 3x_4 = -1 \\ x_2 &= 7 - x_3 - 5x_4 = 2 \\ x_3 &= \tfrac{1}{3}(13 - 13x_4) = 0 \\ x_4 &= 1 \end{aligned}$$

Solution: $x_1 = -1, \quad x_2 = 2, \quad x_3 = 0, \quad x_4 = 1$

Math 361 Ch6.1 Gaussian Elimination

Examples Use elementary row operations to transform the following matrices into upper triangular form. Be sure to label your row operations as discussed in class.

(1)

$$A = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ -2 & 7 & -8 & -19 \\ 3 & -5 & -9 & -9 \end{array}\right] \xrightarrow[R_3 =]{R_2 =}$$

(Ans to system of eqns: $x_1 = -5$, $x_2 = -3$, $x_3 = 1$)

(2)

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right] \xrightarrow[R_2 = -R_2]{R_1 \leftrightarrow R_3}$$

(Ans to system of eqns: $x_1 = x_2 = x_3 = 0$)

Example

$$
\begin{aligned}
x_1 - x_2 + 2x_3 - x_4 &= -8 \\
2x_1 - 2x_2 + 3x_3 - 3x_4 &= -20 \\
x_1 + x_2 + x_3 \phantom{{}-3x_4} &= -2 \\
x_1 - x_2 + 4x_3 + 3x_4 &= 4
\end{aligned}
$$

Augmented matrix :

$$
\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{cccc|c}
1 & -1 & 2 & -1 & -8 \\
2 & -2 & 3 & -3 & -20 \\
1 & 1 & 1 & 0 & -2 \\
1 & -1 & 4 & 3 & 4
\end{array}\right]
$$

$$
\xrightarrow[r_4 = r_4 - r_1]{\substack{r_2 = r_2 - 2r_1 \\ r_3 = r_3 - r_1}}
\left[\begin{array}{cccc|c}
1 & -1 & 2 & -1 & -8 \\
0 & 0 & -1 & -1 & -4 \\
0 & 2 & -1 & 1 & 6 \\
0 & 0 & 2 & 4 & 12
\end{array}\right] = \tilde{A}^{(2)}
$$

$$
\xrightarrow{r_2 \leftrightarrow r_3}
\left[\begin{array}{cccc|c}
1 & -1 & 2 & -1 & -8 \\
0 & 2 & -1 & 1 & 6 \\
0 & 0 & -1 & -1 & -4 \\
0 & 0 & 2 & 4 & 12
\end{array}\right] = \tilde{A}^{(2)'} = \tilde{A}^{(3)}
$$

$$
\xrightarrow{r_4 = r_4 + 2r_3}
\left[\begin{array}{cccc|c}
1 & -1 & 2 & -1 & -8 \\
0 & 2 & -1 & 1 & 6 \\
0 & 0 & -1 & -1 & -4 \\
0 & 0 & 0 & 2 & 4
\end{array}\right] = \tilde{A}^{(4)}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}
\begin{array}{l}
\textbf{Backsolve:} \\
x_4 = 2 \\
x_3 = 2 \\
x_2 = 3 \\
x_1 = 7
\end{array}
$$

Example (Problem 26)

$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \end{aligned}$$

Solve using G.E. and 2-digit rounding. Do not reorder eqns. (no row switches)

Augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 2 & 4 & -1 & -5 \\ 1 & 1 & -3 & -9 \end{array}\right] \xrightarrow[r_3 = r_3 - \frac{1}{4} r_1]{r_2 = r_2 - \frac{2}{4} r_1} \left[\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0.75 & -3.5 & -11 \end{array}\right]$$

Third row Sample Calculation

$$\left[-9 - \left(\tfrac{1}{4}\right)(9) = -9 - \overset{9/4}{2.25} = \cancel{-11.25} \cong -11 \right]$$

$$-9 - 0.25(9) = -9 - 2.25 \cong -9 - 2.3 = -11.3 \cong -11$$

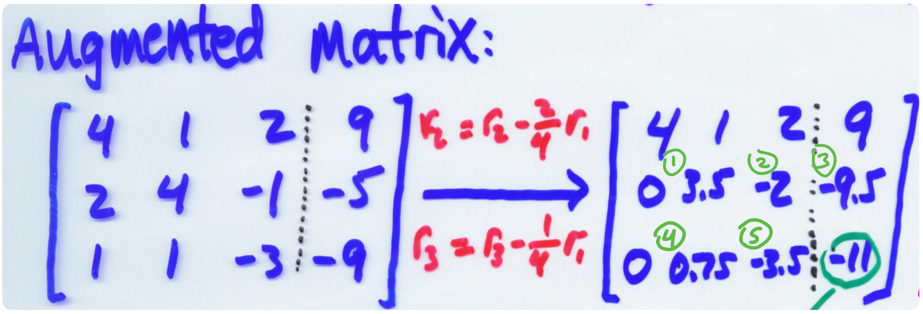
Updated Calculation

$$\xrightarrow{r_3 = r_3 - \frac{0.75}{3.5} r_2} \left[\begin{array}{ccc|c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0 & -3.1 & -9 \end{array}\right]$$

Third row Calculations:

$$\left[-3.5 - \left(\tfrac{.75}{3.5}\right)(-2) = -3.5 + (.21)(2) = -3.5 + .42 \right.$$
$$\left. = -3.08 = -3.1 \right]$$

$$\left[-11 - \left(\tfrac{.75}{3.5}\right)(-9.5) = -11 - (.21)(-9.5) = -11 + 1.995 \right.$$
$$\left. = -11 + 2.0 = -9 \right]$$



Second row calculations

1

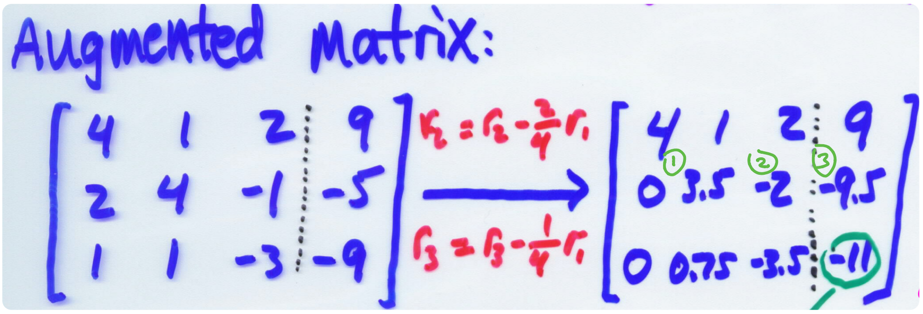
2

3

Third row calculations

4

5



Second row calculations

division first (left to right)
multiplication next
subtraction last

① $4 - \frac{2}{4}(1) = 4 - 0.5(1) = 4 - 0.5 = 3.5$

② $-1 - \frac{2}{4}(2) = -1 - 0.5(2) = -1 - (1) = -2$

③ $-5 - \frac{2}{4}(9) = -5 - 0.5(9) = -5 - 4.5 = -9.5$

Third row calculations

④ $1 - \frac{1}{4}(1) = 1 - 0.25(1) = 1 - 0.25 = 0.75$

⑤ $-3 - \frac{1}{4}(2) = -3 - 0.25(2) = -3 - 0.5 = -3.5$

From the previous slide, we have

$$\left[\begin{array}{ccc:c} 4 & 1 & 2 & 9 \\ 0 & 3.5 & -2 & -9.5 \\ 0 & 0 & -3.1 & -9 \end{array}\right]$$

Backward substitution

$$-3.1\,x_3 = -9 \Rightarrow x_3 = \frac{9}{3.1} = 2.903\ldots \cong \underline{2.9}$$

$$3.5x_2 = -9.5 + 2x_3$$

$$= -9.5 + 2(2.9) = -9.5 + 5.8 = -3.7$$

$$\Rightarrow x_2 = \frac{1}{3.5}(-3.7) = -1.057\ldots \cong \underline{-1.1}$$

$$4x_1 = 9 - x_2 - 2x_3$$

$$= 9 - (-1.1) - 2(2.9) = 9 + 1.1 - 5.8$$

$$= 3.2 + 1.1 = 4.3$$

$$\Rightarrow x_1 = \frac{4.3}{4} = 1.075 = \underline{1.1}$$

Solution $x_1 = 1.1$, $x_2 = -1.1$, $x_3 = 2.9$

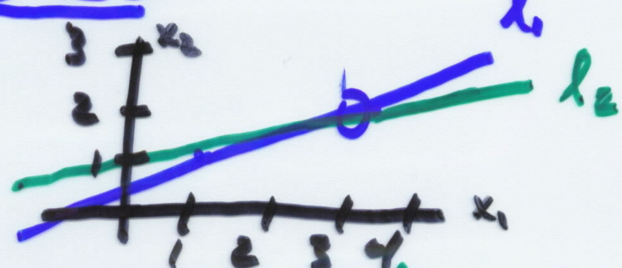
In general, a linear system of equations has either

1. No solution (Inconsistent Sys.)
2. Exactly one solution (Existence/Unique)
3. Infinitely many solutions. (non unique)

Consider 2x2 case:

$x_1 - 2x_2 = -1$ (ℓ_1)

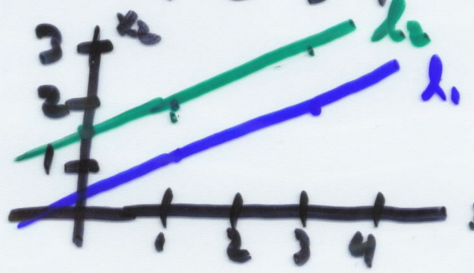
$-x_1 + 3x_2 = 3$ (ℓ_2)



unique soln.

$x_1 - 2x_2 = -1$ (ℓ_1)

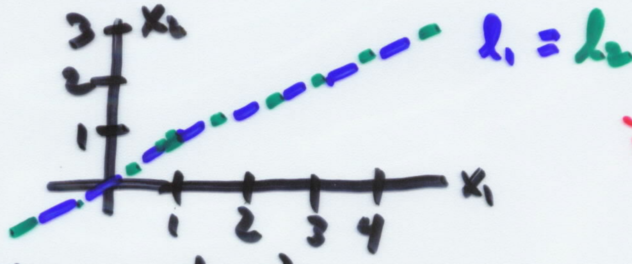
$-x_1 + 2x_2 = 3$



no soln.

$x_1 - 2x_2 = -1$ (ℓ_1)

$-x_1 + 2x_2 = 1$ (ℓ_2)



infinitely many solns.

Reduced Matrices (in order):

$$\left[\begin{array}{cc:c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right], \quad \left[\begin{array}{cc:c} 1 & -2 & -1 \\ 0 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{cc:c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

unique soln — no soln. — infinitely many solns.

Example: Two similar systems

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 2 & 6 \end{array}\right], \quad \tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \end{array}\right]$$

$$\Big\downarrow \begin{array}{l} r_2 = r_2 - 2r_1 \\ r_3 = r_3 - r_1 \end{array} \qquad\qquad \Big\downarrow \begin{array}{l} r_2 = r_2 - 2r_1 \\ r_3 = r_3 - r_1 \end{array}$$

$$\tilde{A}^{(2)} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \tilde{A}^{(2)} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\Downarrow$

$x_1 = 4 - x_2 - x_3$

$-x_3 = -2$

$x_3 = 2$

$(x_2 = \text{free})$

$\Downarrow$

$x_1 = 4 - x_2 - 2$

$\quad = 2 - x_2$

$x_2 = \text{free}$

$x_3 = 2$

$\Downarrow$

$x_1 = 4 - x_2 - x_3$

$\left.\begin{array}{l} -x_3 = -4 \\ x_3 = 2 \end{array}\right\}$ contradiction

$(x_2 = \text{free})$

$\Downarrow$

no solution

In both cases, our Gaussian Elimination algorithm fails.
(no pivot in column 2)

Math 361 Ch6.1 Gaussian Elimination

Examples Use elementary row operations to transform the following matrices into upper triangular form. Be sure to label your row operations as discussed in class.

(1)

$$A = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ -2 & 7 & -8 & -19 \\ 3 & -5 & -9 & -9 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{array}\right] \xrightarrow{R_3 = R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(Ans to system of eqns: $x_1 = -5,\ x_2 = -3,\ x_3 = 1$)

(2)

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right] \xrightarrow[R_2 = -R_2]{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right]$$

(Ans to system of eqns: $x_1 = x_2 = x_3 = 0$)

$$\xrightarrow[R_4 = R_4 - 5R_1]{R_3 = R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 = R_4 - 4R_2]{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 3x_3 = 0 \Rightarrow x_1 = -3x_3 = 0 \\ x_2 - 4x_3 = 0 \Rightarrow x_2 = 4x_3 = 0 \\ 7x_3 = 0 \Rightarrow x_3 = 0 \end{array}$$