Statistical and Thermal Physics: Homework 10

Due: 25 February 2020

1 Gould and Tobochnik, *Statistical and Thermal Physics*, 2.63, page 106.

2 Heat engine with a finite reservoir

Consider a heat engine that absorbs heat Q_h from a reservoir that is initially at temperature T_{hi} and ejects heat Q_c to a reservoir that is initially at temperature T_{ci}. In class we analyzed the efficiency of such an engine with reservoirs which remained at a constant temperature. Now suppose that the reservoirs each have a finite but constant heat capacity and that these are the same. Thus the temperatures of the reservoirs will change. In one cycle the high temperature reservoir will drop to temperature T_{hf} and the low temperature reservoir will rise to T_{cf}. If c is the heat capacity of the higher temperature reservoir, then

$$Q_h = -c(T_{hf} - T_{hi})$$

and similarly, if the low temperature reservoir has the same heat capacity,

$$Q_c = c(T_{cf} - T_{ci}).$$

a) Starting with

$$\mathrm{d}S_h = \frac{c\,\mathrm{d}T}{T}$$

determine an expression for the change in entropy for the hot reservoir in terms of T_{hi}, T_{hf} and c. Rewrite T_{hf} in terms of Q_h and c to obtain expressions for the change in entropy in terms of T_{hi}, c and Q_h. Repeat this for the cold reservoir.

b) Use the second law to show that this implies that

$$\frac{Q_c}{Q_h} \geqslant \frac{T_{ci}}{T_{hi}} + \frac{Q_c}{cT_{hi}} = \frac{T_{cf}}{T_{hi}}$$

and use this to determine a bound on the efficiency of the heat engine.

c) Check that when the heat capacity becomes infinite, that one obtains the original efficiency that we obtained in class. How does this compare to the efficiency when the heat capacity is finite?

d) As time passes and the engine runs through many cycles, what will happen to its efficiency?

3 Carnot engine efficiency: bath temperatures

Consider a Carnot engine using an ideal gas and operating between heat baths at temperatures T_{high} and T_{low}.

a) Assume that the temperature of the hotter reservoir is fixed but the temperature of the cooler reservoir can vary. Which of the following is true?

   i) The efficiency of the engine increases as the temperature difference between the baths increases.

   ii) The efficiency of the engine decreases as the temperature difference between the baths increases.

   iii) The efficiency of the engine stays the same as the temperature difference between the baths increases.

   Explain your answer.

b) Consider two Carnot engines, denoted A and B. The temperatures of the heat baths which they access are different but they are such that for each heat engine the temperature *difference* for the baths is the same. Which of the following is true?

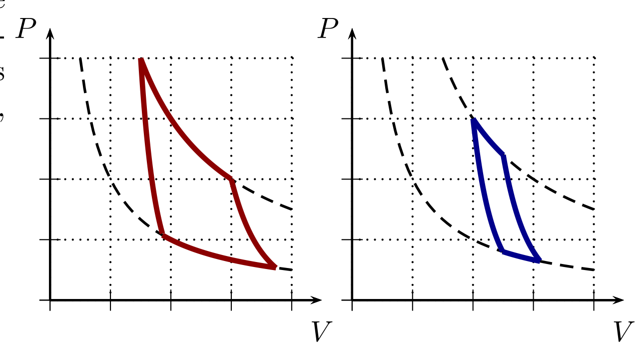
   i) The efficiency of the engine A will be the same as that of engine B regardless of any other facts.

   ii) The efficiency of the engine A might be larger than that of engine B depending on the circumstances.

   Explain your answer.

4 Carnot engine and efficiency: cycle sizes

Consider two Carnot engines which operate between the same two isothermals. The cycles are as illustrated. How do the efficiencies of the two engines compare (i.e. is one larger, smaller, etc.)? Explain your answer.



5 Carnot engine and entropy

During the Carnot cycle, the engine extracts heat from a high temperature bath and loses heat to a low temperature bath. How does the entropy change of the high temperature bath compare to that of the low temperature bath?

6 Combining heat engines

One can imagine using the work output of one heat engine as the input to another heat engine. By connecting abstract engines together in this way, you can arrive at conclusions about thermodynamic processes. Suppose that an engine exists which takes heat from a high temperature reservoir and converts all of this into work with no waste heat. Show that by connecting this to a refrigerator, one could produce a device whose sole effect is to transfer heat from a low temperature bath to a high temperature bath. What does this say about the possible existence of an engine exists which takes heat from a high temperature reservoir and converts all of this into work with no waste heat?

2.63

The efficiency is

$$\eta = \frac{W_{total}}{Q_1}$$

where W_{total} is the work done by the gas over the cycle and Q_1 the heat supplied. We can ignore $1 \to 5$ and $5 \to 1$ as the works done here will cancel. In general we will use

$$\Delta E = Q + W \qquad \text{and } E = fNkT = fPV \qquad f = 3/2, 5/2, \ldots$$

and we can do the following:

1) const volume $\quad W = 0 \quad \Rightarrow \quad Q = \Delta E$

$$\Delta E = \Delta(fPV) = fV\Delta P$$

2) adiabatic $\quad Q = 0 \quad \Rightarrow \quad W = \Delta E = f\Delta(PV)$

$$\Rightarrow W = f(P_fV_f - P_iV_i)$$

Here $\quad P_fV_f^{\gamma} = P_iV_i^{\gamma} \quad \Rightarrow \quad P_f = P_i V_i^{\gamma}/V_f^{\gamma}$

So along adiabatic $\quad W = fP_i\left(\frac{V_i^{\gamma}}{V_f^{\gamma}} V_f - V_i\right)$

$$= fP_iV_i\left(\frac{V_i^{\gamma-1}}{V_f^{\gamma-1}} - 1\right)$$

We apply this to the stages.

	W	Q	ΔE
$1\to 2$	$f P_1V_1\left(\left(\frac{V_1}{V_2}\right)^{\gamma-1}-1\right)$	0	$f P_1V_1\left(\left(\frac{V_1}{V_2}\right)^{\gamma-1}-1\right)$
$2\to 3$	0	$f(P_3V_3-P_2V_2)$	$f(P_3V_3-P_1V_1)$
$3\to 4$	$f P_3V_3\left(\left(\frac{V_3}{V_4}\right)^{\gamma-1}-1\right)$	0	$f P_3V_3\left(\left(\frac{V_3}{V_4}\right)^{\gamma-1}-1\right)$
$4\to 1$	0	$f(P_4V_4-P_1V_1)$	$f(P_4V_4-P_1V_1)$

We aim to rewrite these in terms of V_1 and V_2. First the heat needed is

$$Q_1 = f(P_3V_3-P_2V_2)$$

But $V_3=V_2 \Rightarrow$ $Q_1 = f V_2(P_3-P_2)$

Then

$$W_{3\to 4} = f P_3V_3\left[\left(\frac{V_3}{V_4}\right)^{\gamma-1}-1\right]$$

But $V_3=V_2$ and $V_4=V_1 \Rightarrow$

$$W_{3\to 4} = f P_3V_2\left[\left(\frac{V_2}{V_1}\right)^{\gamma-1}-1\right]$$

So

$$-W_{total} = W_{1\to 2}+W_{3\to 4} = f\left[P_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}\left(1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right) - P_3V_2\left(1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right)\right]$$

So $\quad -W_{total} = f\left[1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right]\left[P_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} - P_3V_2\right]$

Now $\quad P_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = \frac{P_1V_1^{\gamma}}{V_2^{\gamma-1}}$ and $P_1V_1^{\gamma} = P_2V_2^{\gamma}$ gives.

$$P_1V_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = P_2V_2^{\gamma}V_2^{1-\gamma} = P_2V_2$$

So

$$-W_{total} = f\left[1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right]V_2\,(P_2-P_3)$$

Thus

$$\eta = \frac{-f\left[1-\left(V_2/V_1\right)^{\gamma-1}\right]V_2(P_2-P_3)}{fV_2(P_3-P_2)}$$

$$\Rightarrow \quad \eta = 1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}$$

For 10 : 1 ratio $\quad \eta = 1-\left(\frac{1}{10}\right)^{2/3} = 0.79$ for monoatomic

$\eta = 1-\left(\frac{1}{10}\right)^{2/5} = 0.60$ for diatomic

a) $$\Delta S_h = \int \frac{c\,dT}{T} = c\int_{T_{hi}}^{T_{hf}} \frac{dT}{T}$$

$$\Rightarrow \Delta S_h = c\ln\left(\frac{T_{hf}}{T_{hi}}\right)$$

Similarly $$\Delta S_c = c\ln\left(\frac{T_{cf}}{T_{ci}}\right)$$

Now $$-Q_h = c\,\Delta T_h \Rightarrow -Q_h = c(T_{hf} - T_{hi})$$

$$\Rightarrow T_{hf} = T_{hi} - \frac{Q_h}{c}.$$

$$\Rightarrow \Delta S_h = c\ln\left(\frac{T_{hi} - Q_h/c}{T_{hi}}\right) = c\ln\left(1 - \frac{Q_h}{T_{hi}c}\right)$$

Then $$Q_c = c\,\Delta T_c \Rightarrow Q_c = c(T_{cf} - T_{ci})$$

$$\Rightarrow T_{cf} = T_{ci} + \frac{Q_c}{c}$$

So $$\Delta S_c = c\ln\left(\frac{T_{ci} + \frac{Q_c}{c}}{T_{ci}}\right) = c\ln\left(1 + \frac{Q_c}{T_{ci}c}\right)$$

Summary: $$\Delta S_h = c\ln\left(1 - \frac{Q_h}{T_{hi}c}\right)$$

$$\Delta S_c = c\ln\left(1 + \frac{Q_c}{T_{ci}c}\right)$$

b) $$\Delta S \geq 0 \Rightarrow \Delta S_h + \Delta S_c + \cancel{\Delta S_{gas}}^{\,0} \geq 0$$

$$\Rightarrow \cancel{c}\ln\left(1 - \frac{Q_h}{T_{hi}c}\right) + \cancel{c}\ln\left(1 + \frac{Q_c}{T_{ci}c}\right) \geq 0$$

$$\Rightarrow \ln\left[\left(1 - \frac{Q_h}{T_{hi}c}\right)\left(1 + \frac{Q_c}{T_{ci}c}\right)\right] \geq 0$$

$$\Rightarrow \left(1 - \frac{Q_h}{T_{hi}c}\right)\left(1 + \frac{Q_c}{T_{ci}c}\right) \geq 1$$

$$\Rightarrow \cancel{1} - \frac{Q_h}{T_{hi}C} + \frac{Q_c}{T_{ci}C} - \frac{Q_hQ_c}{T_{ci}T_{hi}C^2} \geq \cancel{1}$$

$$\Rightarrow \frac{Q_c}{T_{ci}C} \geq \frac{Q_hQ_c}{T_{ci}T_{hi}C^2} + \frac{Q_h}{T_{hi}C}$$

$$\Rightarrow \frac{Q_c}{Q_h} \geq \frac{Q_c}{T_{hi}C} + \frac{T_{ci}}{T_{hi}} = \frac{1}{T_{hi}}\left(\frac{Q_c}{C} + T_{ci}\right)$$

Now $Q_c = C(T_{cf} - T_{ci})$

$$\Rightarrow \frac{Q_c}{C} = T_{cf} - T_{ci}$$

$$\Rightarrow \frac{Q_c}{Q_h} \geq \frac{1}{T_{hi}} T_{cf}$$

$$e = 1 - \frac{Q_c}{Q_h} \leq 1 - \frac{T_{cf}}{T_{hi}} \Rightarrow e \leq 1 - \frac{T_{cf}}{T_{hi}}$$

$$\text{or} \quad e \leq 1 - \frac{T_{ci}}{T_{hi}} - \frac{Q_c}{CT_{hi}}$$

c) As $C \to \infty$ $\quad e \leq 1 - \frac{T_{ci}}{T_{hi}}$ obtained in class.

When C is finite $1 - \frac{T_{ci}}{T_{hi}} - \frac{Q_c}{CT_{hi}}$ is lower than $1 - \frac{T_{ci}}{T_{hi}}$

So the bound is lower

d) T_{cf} rises and T_{hi} drops. So the efficiency $\to 0$

a) $$\eta = 1 - \frac{T_{low}}{T_{high}}$$

$$= \frac{T_{high} - T_{low}}{T_{high}} = \frac{\Delta T}{T_{high}} \Rightarrow \eta = \frac{\Delta T}{T_{high}}$$

So the efficiency depends on ΔT and T_{high}. One can increase ΔT by increasing T_{high}, lowering T_{low} or doing a combination of these. So it really depends on both. Changing $\Delta T \to 2\Delta T$ but simultaneously $T_{high} \to 3\,T_{high}$ will lower efficiency.

Note if T_h is fixed then η increases as ΔT increases

If T_l is fixed $T_h = T_l + \Delta T \Rightarrow \eta = \frac{\Delta T}{T_l + \Delta T} = \frac{1}{1 + T_l/\Delta T}$

and as ΔT increases this also increases $\boxed{i}$

b) $$\eta = \frac{\Delta T}{T_{high}} \Rightarrow \text{efficiency depends on } T_{high} \Rightarrow \boxed{ii}$$

2020 HW 10 Q4

For a Carnot engine $\eta = 1 - \frac{T_{low}}{T_{high}}$

The temps T_L are same for both

" " T_h " " " "

$\Rightarrow$ η = same both.

2020 HW 10 Q5

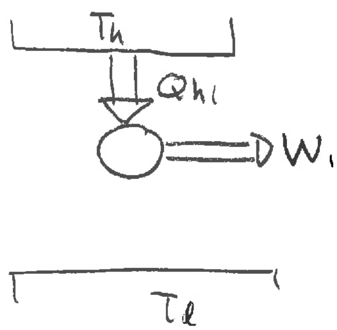
$\Delta S_{tot} = 0$

$\Rightarrow \Delta S_{engine} + \Delta S_h + \Delta S_c = 0$

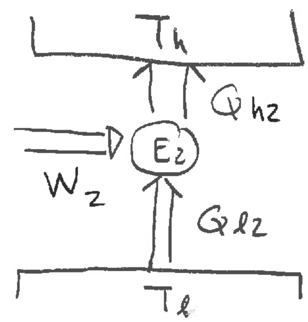
$\Rightarrow \Delta S_h = -\Delta S_c$

The entropy changes are exactly opposite.

hypothetical



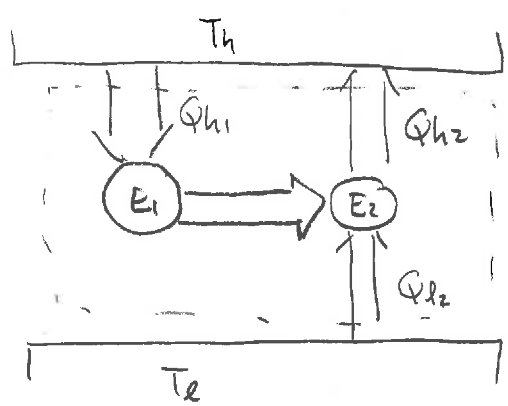
refrigerator

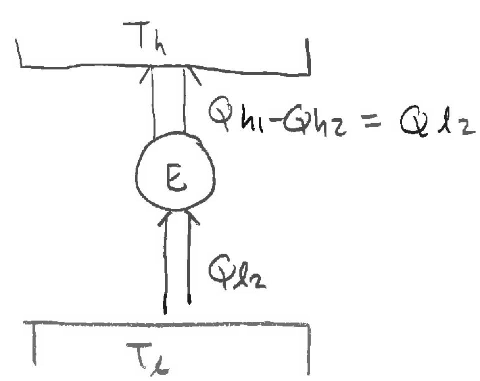


$W_1 = Q_{h1}$

$W_2 + Q_{l2} = Q_{h2}$

We can adjust so $W_1 = W_2 \Rightarrow Q_{h1} - Q_{h2} = Q_{l2}$





The last device is impossible.