HOMEWORK SET 8 SOLUTIONS

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

a) FIRST, CALCULATE THE COORDINATES R_1 & R_2 CORRESPONDING TO THE CIRCUMFERENCES $6\pi M$ & $20\pi M$...

SO, SETTING $t = \text{CONST.}$, $r = \text{CONST.}$, & $\theta = \pi/2$, THE LINE ELEMENT TAKES THE FORM...

$$\therefore ds^2 = R^2 d\varphi^2 \quad \therefore \quad ds = R\,d\varphi$$

SO

$$C = \oint ds = \int_0^{2\pi} R\,d\varphi = 2\pi R \quad \text{SO} \quad 6\pi M = 2\pi R_1 \;\therefore\; R_1 = 3M$$
$$20\pi M = 2\pi R_2 \;\therefore\; R_2 = 10M$$

SECOND, CALCULATE THE PHYSICAL RADIAL DISTANCE...

$t = \text{CONST}$, $\theta = \text{CONST.}$, $\varphi = \text{CONST.}$

$$\therefore ds^2 = \left(1-\frac{2M}{r}\right)^{-1}dr^2 \quad \therefore \quad ds = \frac{dr}{\sqrt{1-2M/r}}$$

SO

$$S = \int_{R_1}^{R_2} ds = \int_{3M}^{10M} \frac{dr}{\sqrt{1-2M/r}} = 2M\int_{3/2}^{5} \frac{dx}{\sqrt{1-1/x}} = 2M\int_{3/2}^{5}\sqrt{\frac{x}{x-1}}\,dx \overset{\text{WOLFRAMALPHA}}{=} 2M \cdot 4.39127 \simeq 8.78M$$

LET $r = 2Mx$, $dr = 2M\,dx$

OR

$$\int \frac{dr}{\sqrt{1-\frac{2M}{r}}} = r\sqrt{1-\frac{2M}{r}} + \frac{1}{2}\cdot 2M \ln\left[2r\left(\sqrt{1-\frac{2M}{r}}+1\right)\right] \quad \text{— WOLFRAMALPHA}$$

SO

$$\int_{3M}^{10M} \frac{dr}{\sqrt{1-\frac{2M}{r}}} = 10M\sqrt{1-\frac{2M}{10M}} + M\ln\left[20M\left(\sqrt{1-\frac{2M}{10M}}+1\right)\right] - \left[3M\sqrt{1-\frac{2M}{3M}} + M\ln\left[6M\left(\sqrt{1-\frac{2M}{3M}}+1\right)\right]\right]$$

$$= 10M\sqrt{\frac{4}{5}} + M\ln\left[20M\left(\sqrt{\frac{4}{5}}+1\right)\right] - 3M\sqrt{\frac{1}{3}} - M\ln\left[6M\left(\sqrt{\frac{1}{3}}+1\right)\right]$$

$$= \frac{20}{\sqrt{5}}M - \sqrt{3}M + M\ln\left[\frac{20M(1+\sqrt{4/5})}{6M(1+\sqrt{1/3})}\right] = M\left[\left(\frac{20}{\sqrt{5}} - \sqrt{3}\right) + \ln\left(\frac{10}{3}\left(\frac{1+\sqrt{4/5}}{1+\sqrt{1/3}}\right)\right)\right]$$

$$\simeq M\left[(8.944 - 1.732) + \ln\left(\frac{18.944}{4.732}\right)\right] \simeq M\left[7.212 + 1.387\right] \simeq 8.599M$$

SO

$$\left[S \simeq 8.78M\right]$$

$$dV = \sqrt{g_{11}g_{22}g_{33}}\,dx^1dx^2dx^3 = \left[\frac{1}{\left(1-\frac{2M}{r}\right)}\cdot r^2\cdot r^2\sin^2\theta\right]^{1/2} dr\,d\theta\,d\varphi = \frac{r^2\sin\theta}{\sqrt{1-2M/r}}\,dr\,d\theta\,d\varphi$$

so

$$V = \int dV = \int_0^{2\pi}\int_0^{\pi}\int_{3M}^{10M}\frac{r^2\sin\theta}{\sqrt{1-2M/r}}\,dr\,d\theta\,d\varphi = \int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\int_{3M}^{10M}\frac{r^2\,dr}{\sqrt{1-2M/r}} = 4\pi\int_{3M}^{10M}\frac{r^2\,dr}{\sqrt{1-2M/r}}$$

let $r = 2Mx$
$dr = 2M\,dx$

$$= 4\pi(2M)^3\int_{1.5}^{5}\frac{x^2\,dx}{\sqrt{1-1/x}} = 4\pi(2M)^3\int_{1.5}^{5}\frac{x^{5/2}}{\sqrt{x-1}}\,dx = 4\pi(2M)^3\int_{1.5}^{5}\sqrt{\frac{x^5}{x-1}}\,dx$$

$$\simeq 4\pi(2M)^3\cdot 48.1385$$

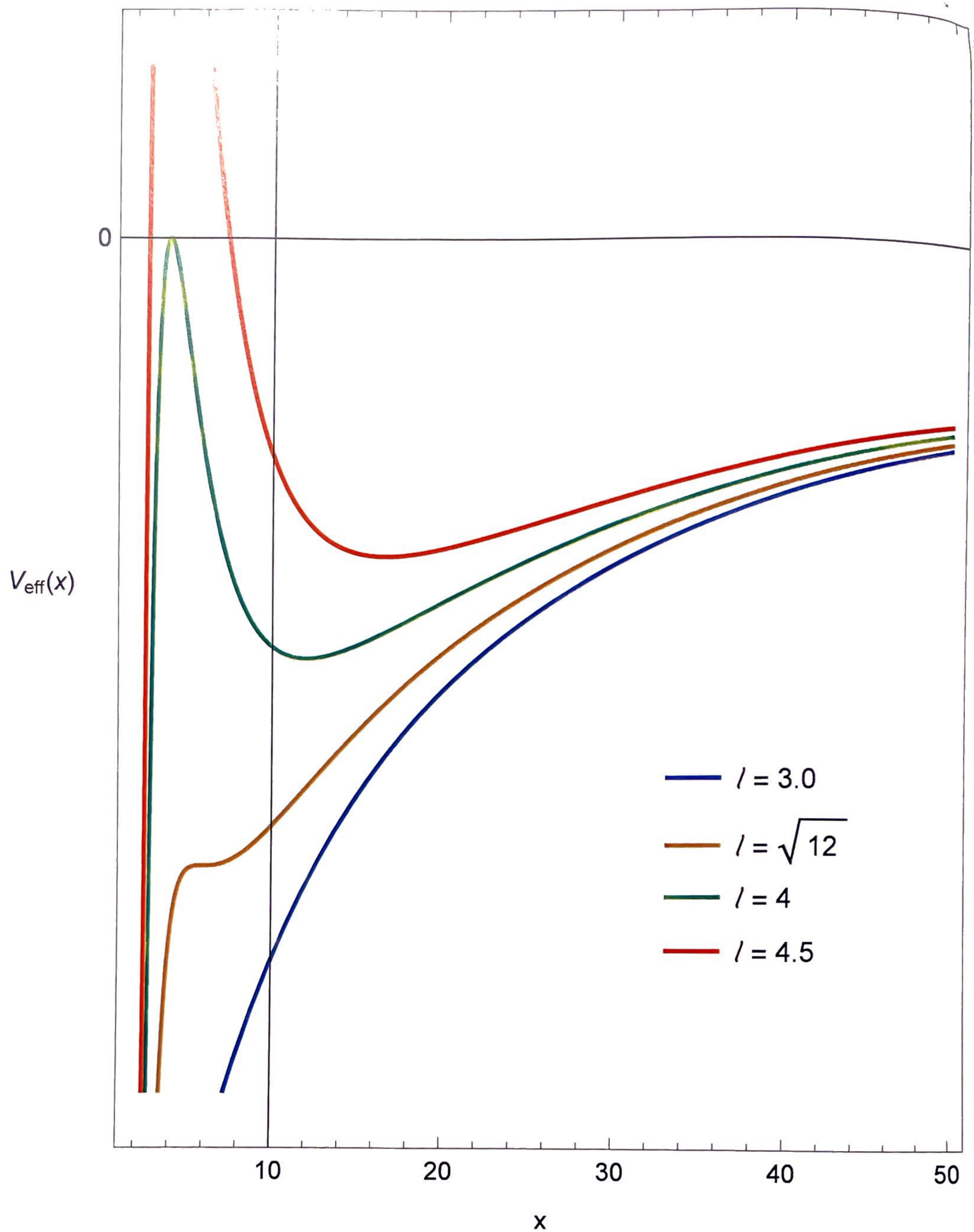
$$\therefore \boxed{V \simeq 4840M^3}$$

2a) $V_{eff}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}$

Setting $r \equiv Mx$ & $l = M\tilde{l}$, WE OBTAIN

$$V_{eff}(x) = -\frac{M}{Mx} + \frac{M^2\tilde{l}^2}{2M^2x^2} - \frac{M \cdot M^2\tilde{l}^2}{M^3x^3} = -\frac{1}{x} + \frac{\tilde{l}^2}{2x^2} - \frac{\tilde{l}^2}{x^3}$$ ✓

b)



0
$V_{\text{eff}}(x)$
10
20
30
40
50
x
$l = 3.0$
$l = \sqrt{12}$
$l = 4$
$l = 4.5$

3. $$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

a) SINCE $t \to t + \text{CONST.}$ LEAVES ds^2 UNCHANGED, WE HAVE AN ASSOCIATED KILLING VECTOR $\underset{\sim}{\xi}$ w/ $\xi^\alpha = (1, 0, 0, 0)$,

WHICH YIELDS A FIRST INTEGRAL OF THE FORM...

$$-e \equiv \underset{\sim}{\xi} \cdot \underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{tt}\xi^t u^t = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau}$$

$$\therefore \left[e = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\frac{dt}{d\tau}\right]$$

SINCE $\phi \to \phi + \text{CONST.}$ LEAVES ds^2 UNCHANGED, WE HAVE AN ASSOCIATED KILLING VECTOR $\underset{\sim}{\eta}$ w/ $\eta^\alpha = (0, 0, 0, 1)$,

WHICH YIELDS A FIRST INTEGRAL OF THE FORM...

$$l \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{\phi\phi}\eta^\phi u^\phi = r^2\sin^2\theta\frac{d\phi}{d\tau}$$

$$\therefore \left[l = r^2\sin^2\theta\frac{d\phi}{d\tau}\right]$$

b) ALSO

$$-1 = \underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)(u^t)^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}(u^r)^2 + r^2(u^\theta)^2 + r^2\sin^2\theta(u^\phi)^2$$

OR

$$-1 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2$$

c) SETTING $\theta = \pi/2$, $\frac{d\theta}{d\tau} = 0$. THE ABOVE FIRST INTEGRAL TAKES THE FORM...

(*)
$$-1 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

NOW

$$\frac{dt}{d\tau} = e\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}$$

$$\frac{d\phi}{d\tau} = \frac{l}{r^2}$$

PLUGGING THESE INTO (*) YIELDS...

$$-1 = -e^2\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1} + \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2}$$

OR

$$-\left(1 + \frac{l^2}{r^2}\right)\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) = -e^2 + \left(\frac{dr}{d\tau}\right)^2$$

NOW SOLVING FOR e^2...

$$e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)$$

NOW DEFINE THE CONSTANT

$$\mathcal{E} = \frac{1}{2}(e^2-1) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right) - 1\right]$$

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\cancel{1} - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{\ell^2}{r^2} - \frac{2M\ell^2}{r^3} - \frac{\ell^2}{r^2}\cdot\frac{\Lambda}{3}r^2 \cancel{-1}\right]$$

$$= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \left[-\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}\left(\ell^2 + r^2\right)\right]$$

So

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \quad \text{WHERE} \quad V_{eff}(r) \equiv -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} - \frac{\Lambda}{6}(\ell^2 + r^2)\Big]$$

#4. THE EXTREMA OF THE EFFECTIVE POTENTIAL IS FOUND VIA

a) $$\frac{dV_{eff}}{dr} = 0$$

SO

$$0 = \frac{d}{dr}\left[-\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} - \frac{\Lambda}{6}(l^2 + r^2)\right] = \frac{M}{r^2} - \frac{l^2}{r^3} + \frac{3Ml^2}{r^4} - \frac{\Lambda}{3}r$$

$$= \frac{1}{r^4}\left[Mr^2 - l^2 r + 3Ml^2 - \frac{\Lambda}{3}r^5\right] \quad \therefore \quad -\frac{\Lambda}{3}r^5 + Mr^2 - l^2 r + 3Ml^2 = 0,$$

WHICH IS A 5^{TH} ORDER POLYNOMIAL.

b) THE TURNING POINTS ARE DETERMINED BY...

$$\mathcal{E} = V_{eff}(r_{TP})$$

SO

$$\left[\mathcal{E} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} - \frac{\Lambda}{6}(l^2 + r^2)\right] r^3$$

$$\mathcal{E}r^3 = -Mr^2 + \frac{l^2}{2}r - Ml^2 - \frac{\Lambda}{6}l^2 r^3 - \frac{\Lambda}{6}r^5$$

OR

$$-\frac{\Lambda}{6}r^5 - \left(\mathcal{E} + \frac{\Lambda}{6}l^2\right)r^3 - Mr^2 + \frac{1}{2}l^2 r - Ml^2 = 0$$

, WHICH IS A 5^{TH} ORDER POLYNOMIAL.

c) $$V_{eff}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} - \frac{\Lambda}{6}(l^2 + r^2)$$

NOW SET $r \equiv Mx$

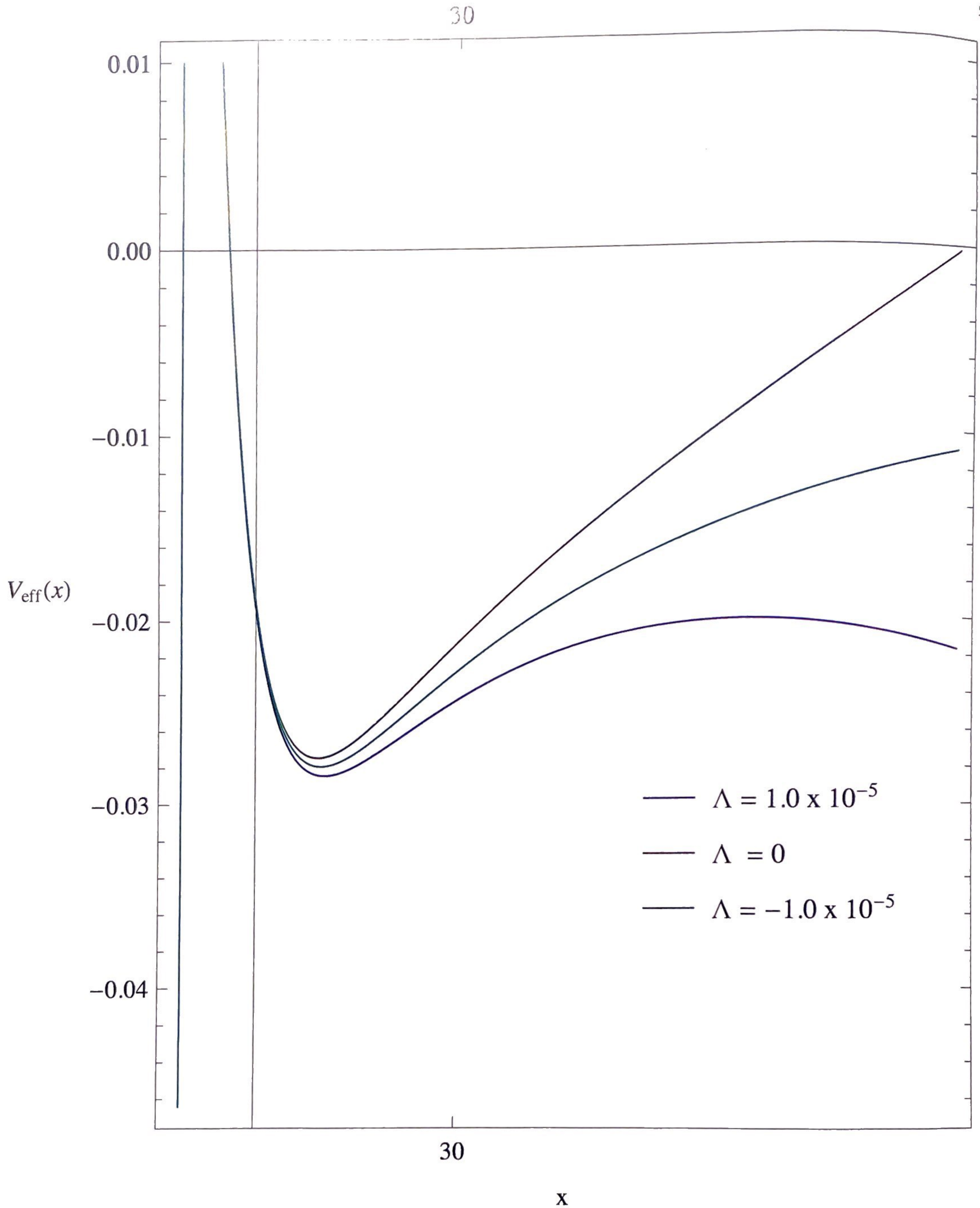
$l = M\tilde{l}$

$\Lambda \equiv \frac{\tilde{\Lambda}}{M^2}$

YIELDS

$$V_{eff}(x) = \frac{-M}{Mx} + \frac{M^2\tilde{l}^2}{2M^2x^2} - \frac{M \cdot M^2\tilde{l}^2}{M^3x^3} - \frac{\tilde{\Lambda}}{6M^2}(M^2\tilde{l}^2 + M^2x^2)$$

$$= -\frac{1}{x} + \frac{\tilde{l}^2}{2x^2} - \frac{\tilde{l}^2}{x^3} - \frac{\tilde{\Lambda}}{6}(l^2 + x^2) \checkmark$$


30
0.01
0.00
−0.01
−0.02
−0.03
−0.04
$V_{\mathrm{eff}}(x)$
$\Lambda = 1.0 \times 10^{-5}$
$\Lambda = 0$
$\Lambda = -1.0 \times 10^{-5}$
30
x

5. $ds^2 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)dt^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)$

a) $e = \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\frac{dt}{d\lambda}$

$\ell = r^2\sin^2\theta\, \frac{d\phi}{d\lambda}$

b) $0 = \mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}u^\alpha u^\beta = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)(u^t)^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}(u^r)^2 + r^2(u^\theta)^2 + r^2\sin^2\theta\,(u^\phi)^2$

or

$$0 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2$$

c) SETTING $\theta = \pi/2$, $\frac{d\theta}{d\lambda} = 0$, THE ABOVE FIRST INTEGRAL TAKES THE FORM...

(*)
$$0 = -\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

NOW

$$\frac{dt}{d\lambda} = e\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}$$

$$\frac{d\phi}{d\lambda} = \frac{\ell}{r^2}$$

NOW PLUGGING THESE INTO (*) YIELDS...

$$0 = -e^2\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1} + \left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}$$

or

$$0 = -e^2 + \left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)$$

or

$$\frac{e^2}{\ell^2} = +\frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right)$$

$= \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$ WHERE $W_{eff}(r) \equiv \frac{1}{r^2}\left(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2\right) = \frac{1}{r^2} - \frac{2M}{r^3} - \frac{\Lambda}{3}$

d) THE EXTREMA OF THIS EFFECTIVE POTENTIAL OCCUR WHEN

$\frac{dW_{eff}}{dr} = 0$ or $0 = -\frac{2}{r^3} + \frac{6M}{r^4} = \frac{2}{r^4}(3M - r)$ so $r_e = 3M$

$$W_{eff}(r_c) = \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)\Bigg|_{r=3M} = \frac{1}{9M^2}\left(1 - \frac{2M}{3M} - \frac{\Lambda}{3}\cdot 9M^2\right) = \frac{1}{9M^2}\left(\frac{1}{3} - \frac{\Lambda}{3}9M^2\right)$$

$$= \frac{1}{27M^2}\left(1 - 9\Lambda M^2\right)$$

e) THE TURNING POINTS ARE DETERMINED BY ...

$$\frac{e^2}{\ell^2} = V_{eff}(r_{TP})$$

so

$$\left[\frac{e^2}{\ell^2} = \frac{1}{r^2} - \frac{2M}{r^3} - \frac{\Lambda}{3}\right]r^3$$

$$\frac{e^2}{\ell^2}r^3 = r - 2M - \frac{\Lambda}{3}r^3 \quad \text{or} \quad \left[0 = -\left(\frac{e^2}{\ell^2} + \frac{\Lambda}{3}\right)r^3 + r - 2M\right]$$