Physics 396

Homework Set 6

1. Consider the 4D spacetime, spanned by coordinates (v, r, θ, ϕ), with a line element

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $R_s \equiv 2GM/c^2$ is called the *event horizon* (a.k.a. *Schwarzschild radius*). It is noted that the above line element describes the 4D spacetime outside a static, spherically symmetric object of mass M in Eddington-Finkelstein coordinates.

Consider a *radial null curve* ($\theta =$ const., $\phi =$ const., $ds^2 = 0$).

a) Calculate the slopes of the light cone at a point (v, r).

b) Evaluate the slopes of the light cones at points $(v, r) = (0, R_s/2),\ (0, R_s),\ (0, 2R_s),$ $(0, 3R_s)$, and $(0, 4R_s)$. Draw the corresponding (v, r) spacetime diagram with the apex of the light cones positioned at these locations. This diagram should fill a standard piece of paper. Use a ruler to make sure that your drawn curves are consistent with your calculated values.

c) How do the slopes of the light cones change with v for a given value of r?

2. In lecture, we studied the 4D wormhole spacetime with a line element of the form

$$ds^2 = -c^2dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

We found the *embedding diagram* for a 2D equatorial slice of the 4D wormhole and that this surface has two asymptotically flat regions connected by a throat of circumference $2\pi b$. We also found that this spacetime is spherically symmetric since a surface with constant r and t has the geometry of a sphere.

Consider a $t =$ const. slice of the wormhole geometry bounded by two spheres of *coordinate radius* R centered on $r = 0$, which reside on each side of the throat.

a) Calculate the circumference of the equator of these spheres.

b) Calculate the distance S from one sphere to the other, through the throat, along a $\theta =$ const., $\phi =$ const. line.

c) Calculate the area of the two-sphere of coordinate radius $r = R$.

d) Calculate the 3D volume bounded by the two spheres of coordinate radius R.

3. In lecture, we arrived at the 2D *non-Euclidean* line element of a two-sphere of radius R from 3D Euclidean space by performing the coordinate transformation

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\phi, \end{aligned} \tag{3}$$

where $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$. It is noted that the above transformation obeys the equation of constraint

$$x^2 + y^2 + z^2 = R^2, \tag{4}$$

which effectively constrains the radial coordinate to take on a constant value, $r = R$.

a) Here we wish to arrive at a 3D *non-Euclidean* line element of a three-sphere of radius R from a fictitious 4D Euclidean space. Consider the coordinate transformation

$$\begin{aligned} x &= R\sin\chi\sin\theta\cos\phi \\ y &= R\sin\chi\sin\theta\sin\phi \\ z &= R\sin\chi\cos\theta \\ w &= R\cos\chi, \end{aligned} \tag{5}$$

where $0 \leq \chi < \pi$, $0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. Show that the above transformation obeys the equation of constraint

$$x^2 + y^2 + z^2 + w^2 = R^2, \tag{6}$$

which effectively constrains the radial coordinate to take on a constant value, $r = R$.

b) Calculate $dx,\ dy,\ dz,\ dw$.

c) Calculate $dx^2 + dy^2$.

d) Calculate $dx^2 + dy^2 + dz^2$.

e) Show that the 4D Euclidean line element

$$dS^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{7}$$

becomes that of a 3D *non-Euclidean* line element of a three-sphere of radius R of the form

$$dS^2 = R^2 \left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{8}$$

under the above coordinate transformation.

f) By defining $r \equiv \sin\chi$, show that the above line element takes the form

$$dS^2 = R^2 \left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{9}$$

Notice that the line element of Eqs. (8) and (9) equate to a $t =$ const. slice of a *homogeneous closed universe*, which was analyzed in Example 7.6.

g) What is the range of this r coordinate?

4. In lecture, we found an *embedding diagram* for the 4D wormhole. This equated to finding a curved 2D surface in 3D Euclidean space with the *same* intrinsic geometry as a 2D equatorial slice ($t =$ const., $\theta = \pi/2$) of the 4D wormhole line element.

a) Consider the $t =$ const. slice of a *homogeneous closed universe* given by Eq. (9). Construct the corresponding 2D equatorial slice ($\theta = \pi/2$) for this homogenous closed universe, analogous to Eq. (7.41) of your text.

b) Following a procedure similar to that of Section 7.7 of your text, show that the curved 2D surface obeys the differential equation

$$\frac{dz}{dr} = \pm R\frac{r}{\sqrt{1-r^2}}. \tag{10}$$

c) By integrating Eq. (10), show that the solution obeys the expression

$$\frac{z^2}{R^2} + r^2 = 1. \tag{11}$$

d) Plot Eq. (11) on a $z(r)/R$ vs. r set of axes.