Taylor Larrochea　　Ch. 7, #(4, 6, 8, 11, 14, 21, 27, 31, 34, 39, 61)

Dr. Schiffbauer

PHYS 231 HW 7

Problem 4

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x,y,z)}\left[\frac{\partial^2\psi(x,y,z)}{\partial x^2}+\frac{\partial^2\psi(x,y,z)}{\partial y^2}+\frac{\partial^2\psi(x,y,z)}{\partial z^2}\right]=E-V(r)$$

$x = r\sin\theta\cos\phi$　　$r=\sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi$　　$\theta=\cos^{-1}\left(\frac{z}{r}\right)$ (Polar Angle)

$z = r\cos\theta$　　$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ (Azimuthal Angle)

Laplacian in Spherical:
$$\nabla^2\psi(r,\theta,\phi)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r,\theta,\phi)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi(r,\theta,\phi)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi(r,\theta,\phi)}{\partial\phi^2}\right)$$

$E\equiv\mu$

$$\frac{-\hbar^2}{2\mu}\frac{1}{\psi(r,\theta,\phi)}\left[\nabla^2\psi(r,\theta,\phi)\right]=E-V(r)$$

$$\nabla^2\psi(r,\theta,\phi)=-\frac{2\mu}{\hbar^2}\left(E-V(r)\right)\psi(r,\theta,\phi)$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r,\theta,\phi)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi(r,\theta,\phi)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi(r,\theta,\phi)}{\partial^2\phi^2}\right)=-\frac{2\mu}{\hbar^2}\left(E-V(r)\right)\psi(r,\theta,\phi)}$$

Problem 6

Equation 7.10 - $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{\hbar^2}\left[E-V-\frac{\hbar^2}{2\mu}\frac{\ell(\ell+1)}{r^2}\right]R=0$

$n=2 \quad \ell=1$

$R_{21}=\frac{r}{a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}$

$$\frac{dR_{21}}{dr}=\frac{1}{a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}+\frac{r}{a_0}\left(-\frac{1}{2a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}\right)$$

$$=\frac{1}{a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}+\frac{r}{a_0}\left(\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}\right)\left(-\frac{1}{2a_0}\right)$$

$$=\left(\frac{1}{r}-\frac{1}{2a_0}\right)\left(\frac{r}{a_0}\frac{e^{-r/2a_0}}{\sqrt{3}(2a_0)^{3/2}}\right)=\left(\frac{1}{r}-\frac{1}{2a_0}\right)R_{21}$$

$$\frac{dR_{21}}{dr}=\left(\frac{1}{r}-\frac{1}{2a_0}\right)R_{21}$$

$\ell=1$: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{\hbar^2}\left[E-V-\frac{\hbar^2}{2\mu}\frac{1(1+1)}{r^2}\right]R=0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R=0 \qquad V=\frac{-e^2}{4\pi\varepsilon_0 r} \qquad \frac{dR}{dr}\Leftrightarrow\frac{dR_{21}}{dr}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{1}{r}-\frac{1}{2a_0}\right)R_{21}\right)+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R_{21}=0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\left(r-\frac{r^2}{2a_0}\right)R_{21}\right)+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R_{21}=0$$

$$\frac{1}{r^2}\left(\left(1-\frac{r}{a_0}\right)R_{21}+\left(r-\frac{r^2}{2a_0}\right)\frac{dR_{21}}{dr}\right)+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R_{21}=0$$

$$\frac{1}{r^2}\left(\left(1-\frac{r}{a_0}\right)R_{21}+\left(r-\frac{r^2}{2a_0}\right)\left(\frac{1}{r}-\frac{1}{2a_0}\right)R_{21}\right)+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R_{21}=0$$

$$\frac{1}{r^2}\left[1-\frac{r}{a_0}+\left(1-\frac{r}{2a_0}-\frac{r}{2a_0}+\frac{r^2}{4a_0^2}\right)\right]R_{21}+\frac{2\mu}{\hbar^2}\left[E+\frac{e^2}{4\pi\varepsilon_0 r}-\frac{\hbar^2}{\mu r^2}\right]R_{21}=0$$

$$\frac{1}{r^2}\left[2-\frac{2r}{a_0}+\frac{r^2}{4a_0^2}\right]R_{21}+\left[\frac{2\mu E}{\hbar^2}+\frac{2\mu e^2}{4\pi\varepsilon_0 r\hbar^2}-\frac{2}{r^2}\right]R_{21}=0$$

$$\left[\frac{2}{r^2}-\frac{2}{ra_0}+\frac{1}{4a_0^2}+\frac{2\mu E}{\hbar^2}+\frac{2\mu e^2}{4\pi\varepsilon_0 r\hbar^2}-\frac{2}{r^2}\right]R_{21}=0 \qquad a_0=\frac{4\pi\varepsilon_0\hbar^2}{\mu e^2}$$

$$\left[-\frac{2}{ra_0}+\frac{1}{4a_0^2}+\frac{2\mu E}{\hbar^2}+\frac{2}{ra_0}\right]R_{21}=0$$

$$\frac{1}{4a_0^2}+\frac{2\mu E}{\hbar^2}=0 \ : \ \frac{2\mu E}{\hbar^2}=-\frac{1}{4a_0^2} \ : \ E=-\frac{\hbar^2}{8\mu a_0^2} \qquad E_0=\frac{\hbar^2}{2\mu a_0^2}$$

$E=\frac{-1}{4}E_0$: Bohr Model Says $E_n=-\frac{E_0}{n^2}$, Since $n=2$, $E_2=-\frac{E_0}{4}=E$ ✓

$\boxed{\text{The resulting energy is } E=-\tfrac{1}{4}E_0\text{, This is consistent with the Bohr Model}}$

Problem 8

$\psi(r,\theta,\phi) = Ae^{-r/a_0}$

$\psi^*(r,\theta,\phi) = Ae^{-r/a_0}$

Spherical : $r^2\sin\theta \, dr \, d\theta \, d\phi$

1D Normalizing : $\int_{-\infty}^{\infty} \psi^*\psi \, dx = 1$

3D Normalizing : $\int_0^{\infty} \psi^*\psi \, r^2 \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 1$ $\quad \psi^*\psi = A^2e^{-2r/a_0}$

$$\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} A^2e^{-2r/a_0} r^2\sin\theta \, d\phi \, d\theta \, dr = 1$$

$$\int_0^{\infty}\int_0^{\pi} A^2e^{-2r/a_0} r^2\sin\theta \cdot \phi \Big|_0^{2\pi} d\theta \, dr = 1$$

$$\int_0^{\infty}\int_0^{\pi} 2\pi A^2e^{-2r/a_0} r^2\sin\theta \, d\theta \, dr = 1$$

$-\cos(\pi) + \cos(0) = 1+1 = 2$

$$\int_0^{\infty} 2\pi A^2e^{-2r/a_0} r^2\left(-\cos\theta\Big|_0^{\pi}\right) dr = 1$$

$$2\pi A^2\int_0^{\infty} r^2e^{-2r/a_0} (2) \, dr = 1$$

$$4\pi A^2\int_0^{\infty} r^2e^{-2r/a_0} \, dr = 1$$

$\int_0^{\infty} x^ne^{-x} = n!$

$$4\pi A^2\int_0^{\infty}\left(\frac{a_0}{2}\right)^2 x^2e^{-x} \, dx = 1$$

$x = 2r/a_0$ $\quad r = -\frac{x a_0}{2}$

$$4\pi A^2\left(\frac{a_0}{2}\right)^2\int_0^{\infty} x^2e^{-x}\left(\frac{a_0}{2}\right)dx = 1$$

$\frac{dx}{dr} = \frac{2}{a_0}$ $\quad r^2 = \left(\frac{a_0}{2}\right)^2x^2$

$$4\pi A^2\left(\frac{a_0}{2}\right)^3(2!) = 1$$

$dr = \frac{a_0}{2}dx$

$$4\pi A^2\left(\frac{a_0^3}{8}\right)(2) = 1$$

$$4\pi A^2\left(\frac{a_0^3}{4}\right) = 1$$

$$A^2a_0^3\pi = 1 \quad ; \quad A = \sqrt{\frac{1}{\pi a_0^3}}$$

$$\boxed{A = \sqrt{\frac{1}{\pi a_0^3}}}$$

Problem 11

$\ell =$	0	1	2	3	4	5
Letter:	S	P	D	F	G	H

3P : $n=3$, $\ell=1$, $|m_\ell| \leq 1$

$n=3$, $\ell=1$ $m_\ell=0$: $\frac{1}{a_0^{3/2}} \frac{4}{81\sqrt{6}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0} \cdot \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$ $\quad \psi_{310} = \frac{1}{81}\sqrt{\frac{2}{\pi}}\, a_0^{-3/2}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\cos\theta$

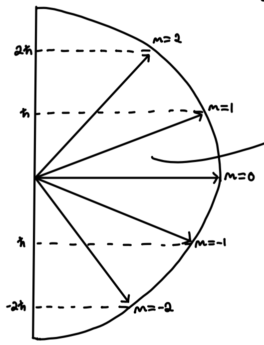
$n=3$, $\ell=1$ $m_\ell=\pm1$: $\frac{1}{a_0^{3/2}} \frac{4}{81\sqrt{6}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0} \cdot \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\phi}$ $\quad \psi_{31\pm1} = \mp\frac{1}{81\sqrt{\pi}}\, a_0^{-3/2}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\sin\theta e^{\pm i\phi}$

$$\boxed{\begin{aligned}\psi_{310} &= \frac{1}{81}\sqrt{\frac{2}{\pi}}\, a_0^{-3/2}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\cos\theta \\ \psi_{31\pm1} &= \mp\frac{1}{81\sqrt{\pi}}\, a_0^{-3/2}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\sin\theta e^{\pm i\phi}\end{aligned}}$$

Problem 14

$3d: \quad n=3,\ \ell=2,\ |m_\ell| = 0, \pm1, \pm2$

a.)



$L = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{2(2+1)}\,\hbar = \sqrt{6}\,\hbar$

$L = \sqrt{6}\,\hbar$

b.)

$L^2 = L_x^2 + L_y^2 + L_z^2$

$L = \sqrt{6}\hbar\ ,\ L_z = (-\hbar)$

$(\sqrt{6}\hbar)^2 = L_x^2 + L_y^2 + (-\hbar)^2$

$6\hbar^2 = L_x^2 + L_y^2 + \hbar^2$

$5\hbar^2 = L_x^2 + L_y^2$

$\boxed{L_x^2 + L_y^2 = 5\hbar}$

Problem 21)

$$\Delta\lambda = \frac{\lambda_0^2 \mu_B B}{hc}$$

$\Delta E = \mu_B B$ $\quad E = \dfrac{hc}{\lambda_0}$ $\quad \dfrac{dE}{d\lambda_0} = -\dfrac{hc}{\lambda_0^2}$ $\quad : \quad |\Delta E| = \dfrac{hc}{\lambda_0^2}\Delta\lambda_0$

$$\frac{hc}{\lambda_0^2}\Delta\lambda_0 = \mu_B B$$

$$\boxed{\Delta\lambda_0 = \frac{\lambda_0^2 \mu_B B}{hc}} \checkmark$$

Problem 27

$\Delta x = 0.01\ m$, $T = 1273\ K$

a.) $d = \frac{1}{2} a_z t^2 = \frac{1}{2}\left(\frac{F_z}{m}\right)t^2 = \frac{1}{2m}\left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x}{v_x}\right)^2$

$a_z = F_z / m$ $\quad F_z = -\frac{dU}{dz} = \mu_z \frac{dB}{dz}$

$t = \Delta x / v_x$

$E = K$

$\frac{1}{2} m v_x^2 = \frac{3}{2} kT$

$m v_x^2 = 3kT$

$v_x = \sqrt{\frac{3kT}{m}}$

$d = \frac{1}{2m}\left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x}{v_x}\right)^2$

$d = \frac{1}{2m}\left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x^2 m}{3kT}\right)$

$d \to \frac{d}{2}$ $\quad d = \frac{1}{2}\left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x^2}{3kT}\right)$

$\frac{d}{2} = \frac{1}{2}\left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x^2}{3kT}\right)$

$d = \left(\mu_z \frac{dB}{dz}\right)\left(\frac{\Delta x^2}{3kT}\right)$

$\frac{3dkT}{\mu_z} = \Delta x^2 \frac{dB}{dz}$

$d = 1.0\times10^{-2} m$, $k = 1.3807\times10^{-23}\ J/K$

$T = 1273\ K$, $\mu_z = 9.27\times10^{-24}\ J/T$

$$\Delta x^2 \frac{dB}{dz} = \frac{3(1.0\times10^{-2} m)(1.3807\times10^{-23} J/K)(1273\ K)}{9.27\times10^{-24} J/T} = 56.88\ T{\cdot}m$$

$$\frac{m \cdot \cancel{J}/\cancel{K} \cdot \cancel{K}}{\cancel{J}/T} = T{\cdot}m$$

The magnet should be made so that the vertical length squared and its vertical magnetic field gradient as 57 T.m

Problem 31

$d = 0.21\ m$, $n = 1$, $E = 5.9 \times 10^{-6}\ eV$: $E = 9.452 \times 10^{-25}\ J/k$

$$E = \frac{3}{2}kT \quad : \quad T = \frac{2E}{3k} = \frac{2(9.452 \times 10^{-25}\ J)}{3(1.3807 \times 10^{-23}\ J/k)} = 0.0456\ k$$

$K = 1.3807 \times 10^{-23}\ J/k$

$E = 9.452 \times 10^{-25}\ J$

$$\boxed{T = 0.0456\ k}$$

Problem 34

(n, ℓ, m_ℓ, m_s) $\Delta n = \text{anything}$, $\Delta \ell = \pm 1$, $\Delta m_\ell = 0, \pm 1$

(a) $(5,2,1,\frac{1}{2}) \longrightarrow (5,2,1,-\frac{1}{2})$ $\Delta \ell = 0$, $\Delta m_\ell = 0$

$\boxed{\text{This transition is not allowed, } \Delta \ell \neq \pm 1}$

(b) $(4,3,0,\frac{1}{2}) \longrightarrow (4,2,1,-\frac{1}{2})$ $\Delta \ell = -1$, $\Delta m_\ell = 1$

$$\Delta E = -13.6\,\text{ev}\left(\frac{1}{4^2} - \frac{1}{4^2}\right) + \mu_B B(\Delta m_\ell + 2\Delta m_s) = 0 + \mu_B B(1 + 2(-1)) = -\mu_B B$$

$\boxed{\text{This transistion is allowed, } \Delta E = -\mu_B B \text{, This corresponds to emission}}$

(c) $(5,2,-2,-\frac{1}{2}) \longrightarrow (1,0,0,-\frac{1}{2})$ $\Delta \ell = 2$, $\Delta m_\ell = 2$

$\boxed{\text{This transition is not allowed, } \Delta \ell \neq 1}$

(d) $(2,1,1,\frac{1}{2}) \longrightarrow (4,2,1,\frac{1}{2})$ $\Delta \ell = 1$, $\Delta m_\ell = 0$

$$\Delta E = -13.6\,\text{ev}\left(\frac{1}{4^2} - \frac{1}{2^2}\right) + \mu_B B(0 + 2(0)) = -13.6\,\text{ev}\left(\frac{1}{16} - \frac{4}{16}\right) = \left(-\frac{3}{16}\right)(-13.6\,\text{ev}) = 2.65\text{ ev}$$

$\boxed{\text{This transition is allowed, } \Delta E = 2.55\text{ev} \text{, This corresponds to absorption}}$

Problem 39 $\quad P(r)\,dr = r^2 |R_{n\ell}(r)|^2 dr \quad$ From $0.95\,a_0$ to $1.05\,a_0$

Ground state: $n=1$, $\ell=0$ $\therefore$ $\quad R_{10} = \frac{2}{(a_0)^{3/2}} e^{-r/a_0}$

$$P(r)\,dr = r^2 \left(\frac{4}{a_0^3} e^{-2r/a_0}\right) dr = \frac{4r^2 e^{-2r/a_0}}{a_0^3}\,dr$$

$$P(r)\,dr = \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr \qquad P = \int_a^b P(r)\,dr$$

$$P = \int_{0.95a_0}^{1.05a_0} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3}\int_{0.95a_0}^{1.05a_0} r^2 e^{-2r/a_0}\,dr$$

$u = r^2 \quad dv = e^{-2r/a_0} \qquad u = r \quad dv = e^{-2r/a_0}$

$du = 2r\,dr \quad v = -\frac{a_0}{2} e^{-2r/a_0} \qquad du = dr \quad v = -\frac{a_0}{2} e^{-2r/a_0}$

$$\frac{4}{a_0^3}\left[-\frac{1}{2} r^2 a_0 e^{-2r/a_0} + \int_{0.95a_0}^{1.05a_0} a_0 r e^{-2r/a_0}\,dr\right]$$

$$\frac{4}{a_0^3}\left[-\frac{1}{2} r^2 a_0 e^{-2r/a_0} + a_0\left[-\frac{1}{2} a_0 r + \int_{0.95a_0}^{1.05a_0} \frac{1}{2} a_0 e^{-2r/a_0}\,dr\right]\right]$$

$$\frac{4}{a_0^3}\left[-\frac{1}{2} r^2 a_0 e^{-2r/a_0} + a_0\left[-\frac{1}{2} a_0 r + \frac{1}{2} a_0\left(-\frac{a_0}{2} e^{-2r/a_0}\right)\right]\right]\Bigg|_{0.95a_0}^{1.05a_0}$$

$$\frac{4}{a_0^3}\left[-\frac{1}{2} r^2 a_0 e^{-2r/a_0} + a_0\left[-\frac{1}{2} a_0 r + \frac{1}{2} a_0\left(-\frac{a_0}{2} e^{-2r/a_0}\right)\right]\right]\Bigg|_{0.95a_0}^{1.05a_0} = 0.0541$$

$$\boxed{P = 0.054}$$

Problem 51

$$\int \psi_f^* \psi_i \, d\tau \Rightarrow \int_{-\theta}^{\theta}\int_{-\phi}^{\phi}\int_{-r}^{r} \psi^* \psi \, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$x = r\sin\theta\cos\phi$ $\quad r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi$ $\quad \theta = \cos^{-1}\left(\frac{z}{r}\right)$ (Polar Angle)

$z = r\cos\theta$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$ (Azimuthal Angle)

$\psi_i = \frac{2}{a_0^{3/2}} e^{-r/a_0}$: R_{10}

$\psi_f^* = \left(2 - \frac{r}{a_0}\right)\frac{e^{-r/2a_0}}{(2a_0)^{1/2}}$: R_{20}

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{x^2+y^2+z^2}$

$$\boxed{\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\infty} \frac{2}{a_0^{3/2}} e^{-r/a_0}\left(2 - \frac{r}{a_0}\right)\frac{e^{-r/2a_0}}{(2a_0)^{1/2}} \, r^2 \sin\theta \, dr\, d\theta\, d\phi}$$

I cannot for the life of me figure out how to change this to cartesian. I know the above integral will evaluate to be zero.