Ch. 3 : Special Techniques

In General : Calculate $\vec{E}$·Field of a given stationary charge distribution.

Option 1 : Coulombs law of Gauss' Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau \quad \text{or} \quad \oint \vec{E}\cdot d\vec{a} = \frac{Q_{Enc}}{\varepsilon_0}$$

Need symmetry

Difficult except for simple charge configurations

Option 2 : Calculate Potential : Then $\vec{E}$ . . . . . .

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}} \quad \& \quad \vec{E} = -\vec{\nabla}V$$

Really difficult

Alternatively . . . .

Solve Poisson's Equation

$$\left[\nabla^2 V = -\frac{\rho}{\varepsilon_0}\right]$$

& Apply appropriate B.C's
- Equivalent to option 2

In Regions where the charge density is zero . . . . .

$$\left[\nabla^2 V = 0\right]$$ – Laplaces equation

or

$$\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V = 0$$ in Cartesian coordinates

Laplaces Equation in 1D . . . . .
in 1D, Laplaces equation reconstructed...

$$\frac{d^2V}{dx^2} = 0$$

w/ general solutions . . . .

$$V(x) = mx + b$$

where m & b are determined by boundary conditions.

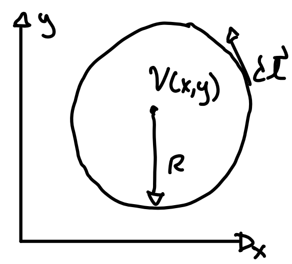
Two features of the 1D solution (which carry over to 2D : 3D)
1. V(x) is the average of V(x+a) + V(x-a), for any a

Proof :

$$V(x+a) = m(x+a) + b \qquad \therefore\ V(x+a) + V(x-a) = 2mx + 2b = 2(mx+b)$$
$$V(x-a) = m(x-a) + b \qquad \therefore\ V(x) = mx+b = \frac{1}{2}\left[V(x+a) - V(x-a)\right]$$

2. Laplace's equation tolerates no local minimum's or maximum's, extrema occur at endpoints.

Laplace's Equation In 2D



$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{where} \quad V = V(x,y)$$

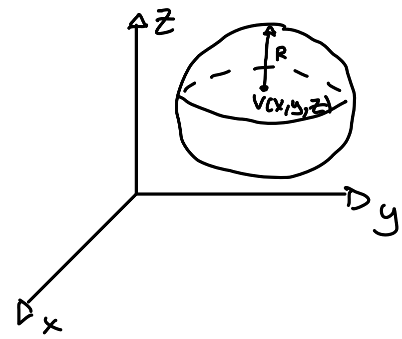
2^{nd} order PDE

Two Features of The 2D Solution ...

1. $V(x,y) = \frac{1}{2\pi R} \oint_{\text{Circle}} V \, dl$

2. V has no local maxima or minima, all extrema occur at the boundary

Laplace's Equation in 3D



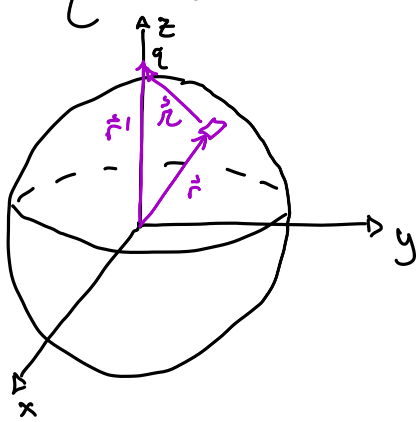
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{where} \quad V = V(x,y,z)$$

1. $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0 R^2} \oint_{\text{Sphere}} V \, da$

2. See 2. From Laplace's equation in 2D

Proof: Calculate the average potential over a spherical surface of radius R due to a point charge q located outside the sphere.

Place q on Z-axis : Center of sphere at the origin of the coordinate System.



The potential over the surface charge is ....

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\mathscr{r}} \qquad \vec{r}' = z\hat{z} \;:\; \vec{r} = R\hat{r}$$

$$\vec{\mathscr{r}} = R\hat{r} - z\hat{z}$$

$$\mathscr{r}^2 = (\vec{\mathscr{r}} \cdot \vec{\mathscr{r}})^2 = (R\hat{r} - z\hat{z}) \cdot (R\hat{r} - z\hat{z})$$

$$= (R + z^2 \cdot 2zR(\hat{r} \cdot \hat{z}))$$

$$= R^2 + z^2 - 2zR\cos\theta$$

$$V_{Avg} = \frac{1}{4\pi R^2} \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\mathscr{r}} R^2 \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi \cancel{R^2}} \cdot \frac{q\cancel{R^2}}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{R^2 + z^2 - 2zR\cos(\theta)}}$$

$$\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{R^2 + z^2 - 2zR\cos(\theta)}} \qquad u = R^2 + z^2 - 2zR\cos(\theta) \quad du = 2zR\sin(\theta)\,d\theta \quad \Rightarrow \quad \frac{q}{(4\pi)^2\varepsilon_0} \cdot 2\pi \cdot \frac{1}{2Rz} \int_{(z-R)^2}^{(z+R)^2} \frac{du}{u^{1/2}}$$

$$= \frac{q}{8\pi\varepsilon_0} \cdot \frac{1}{2Rz} \, 2u^{1/2} \Big|_{(z-R)^2}^{(z+R)^2} = \frac{q}{8\pi\varepsilon_0} \cdot \frac{1}{Rz}\left[(z+R) - (z-R)\right] = \frac{q}{\underset{4}{\cancel{8}}\pi\varepsilon_0} \cdot \frac{1}{\cancel{R}z} \cdot \cancel{2R}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{q}{z} \quad \therefore \quad V = \frac{1}{4\pi\varepsilon_0} \frac{q}{z} \longleftarrow @ \text{ center of sphere}$$

For A collection of charges by the principle of superposition, their average potential over the sphere is equal to the net potential at the center.

10-8-18

Boundary Conditions: Uniqueness Theorems

To uniquely determine V:

- Solve Laplaces Equation $\nabla^2 V = 0$
- Apply boundary Conditions

Suppose you do this. How do you know you have the unique solution?

1st uniqueness theorem

The Solution to Laplaces in some volume $\mathcal{V}$ is uniquely determined if the potential V is specified on the boundary surface

Proof: Suppose two solutions to Laplace's equation V_1 : V_2

$$\nabla^2 V_1 = 0 \quad : \quad \nabla^2 V_2 = 0$$ ← Subjected to the same specificed Volume on the surface

- Look at the difference: $V_3 \equiv V_2 - V_1$

Now
$$\nabla^2 V_3 = \nabla^2 (V_2 - V_1) = \nabla^2 V_2 - \nabla^2 V_1 = 0$$ — Satisfies Laplaces equation

- On the Boundary $V_1 = V_2 \Big|_{\substack{\text{on}\\\text{Boundary}}} \therefore V_3 = 0 \Big|_{\substack{\text{on}\\\text{Boundary}}}$

- Laplace's equation have no local maxima or minima
  $V_3 = 0$ everywhere $\therefore$ $V_1 = V_2$

- Allow for a charge density ρ
  Suppose two solutions to Poisson's equation
  V_1 : V_2

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon_0} \qquad \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0}$$

- Look at the difference $V_3 = V_2 - V_1$

$$\nabla^2 V_3 = \nabla^2 (V_2 - V_1) = 0$$

Proof follows as before.

Corollary:

The potential in a Volume $\mathcal{V}$ is uniquely determined if

a) The charge density throughout a region is specified.

b) The value of V on the boundaries is specified.

2^{nd} Uniqueness Theorem:

In a volume $\mathcal{V}$ surrounded by conductors & Containing a specified charge density ρ
The $\vec{E}$-Field is uniquely determined if the total charge on each conductor is given.

Proof:

- Suppose $\vec{E}_1$ & $\vec{E}_2$, satisfy the conditions of the problem.
  - Both of these solutions obey Gauss' Law in differential form (In space between conductors)

(*) $\vec{\nabla}\cdot\vec{E}_1 = \frac{\rho}{\varepsilon_0}$, $\vec{\nabla}\cdot\vec{E}_2 = \frac{\rho}{\varepsilon_0}$

- Both obey Gauss' Law in Integral form (For a Gaussian Surface covering each Cond.)

(**)

$$\oint_{i^{th}\text{ conducting Surface}} \vec{E}_1\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} \quad \& \quad \oint_{i^{th}\text{ conducting Surface}} \vec{E}_2\cdot d\vec{a} = \frac{Q_i}{\varepsilon_0}$$

For the outer boundary...

(***)

$$\oint_{\text{Outer boundary}} \vec{E}_1\cdot d\vec{a} = \frac{Q_{Enc}}{\varepsilon_0} \quad \& \quad \oint_{\text{Outer boundary}} \vec{E}_2\cdot d\vec{a} = \frac{Q_{Enc}}{\varepsilon_0}$$

Look at the difference & $\vec{E}_3 = \vec{E}_2 - \vec{E}_1$

Now

$\vec{\nabla}\cdot\vec{E}_3 = 0$ From (*) in between Conductors

$\oint \vec{E}_3\cdot d\vec{a} = 0$, From (**) & (***) over each boundary surface.

We don't know how Q_i will distribute itself over each Conductor,
we do know that each V is a constant over each conducting surface.

Consider $\vec{\nabla}\cdot(V_3\vec{E}_3)$....

$$\vec{\nabla}\cdot(V\vec{E}_3) = V_3(\cancelto{0}{\vec{\nabla}\cdot\vec{E}_3}) + \vec{E}_3\cdot\vec{\nabla}V_3 = -E_3^2$$

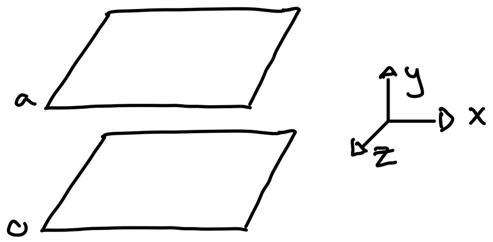
Now integrate over the entire region between Conductors

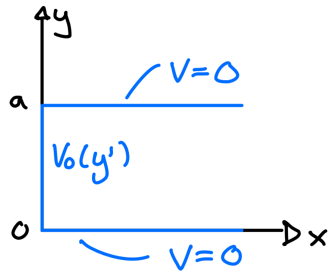
$$\int_{\mathcal{V}}\vec{\nabla}\cdot(V_3\vec{E}_3)\,d\tau = -\int_{\mathcal{V}}E_3^2\,d\tau = \oint_S V_3\vec{E}_3\cdot d\vec{a} = V_3\oint_S\vec{E}_3\cdot d\vec{a} = 0 \quad \& \quad \vec{E}_3 = 0$$

By divergence theorem

10-10-18

Seperation of Variables





Configuration is independent of z $\therefore$ $V = V(x,y)$

Laplaces equation
$$\nabla^2 V = 0$$
$$(*)\quad \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V = 0$$
where $V = V(x,y)$

Boundary Conditions

(1) $V(x, y=0) = 0$
(2) $V(x, y=a) = 0$
(3) $V(x=0, y) = V_0(y)$
(4) $\lim_{x \to \infty} V(x,y) = 0$

Choose a product solution of the form....

$$V(x,y) = X(x)\,Y(y)$$

Although seemingly restrictive, we'll get the general solution!

(*) Becomes....

$$\left[Y \frac{d^2}{dx^2} X + X \frac{d^2}{dy^2} Y = 0 \right] \cdot \frac{1}{XY}$$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{f(x) = k^2} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{g(y) = -k^2} = 0$$

$$\frac{1}{X}\frac{d^2}{dx^2} X = k^2 \quad , \quad \frac{1}{Y}\frac{d^2}{dy^2} Y = -k^2$$

$$\frac{d^2 X}{dx^2} = k^2 X \qquad \frac{d^2 Y}{dy^2} = -k^2 Y$$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$$\therefore \quad V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin(ky) + D\cos(ky))$$

Applying B.C's

1) $V(x, y=0) = (Ae^{kx} + Be^{-kx})D = 0 \longrightarrow D \text{ must } = 0 \quad \therefore \quad D = 0$

4) $\lim_{x\to\infty} V(x,y) = \lim_{x\to\infty} Ae^{kx} \cdot C\sin(ky) = 0 \longrightarrow A \text{ must } = 0 \quad \therefore \quad A = 0$

$$\therefore \quad V(x,y) = \underbrace{(BC)}_{\to C' \to C} e^{-kx}\sin(ky) = Ce^{-kx}\sin(ky)$$

2) $V(x, y=a) = Ce^{-kx}\sin(ka) = 0 \quad \therefore \quad ka = n\pi$ where $n = \pm1, \pm2, \pm3 \ldots\ldots$ $\quad k_n = \frac{n\pi}{a}$

$$\left[V_n(x,y) = C_n e^{-n\pi x/a} \sin(n\pi y/a) \right] \Rightarrow$$ Seperation of variables yields an infinite set of solutions (one for each n)

If $V_1, V_2, V_3, \ldots$ Satisfy Laplaces equation seperately, so does a linear combination of them.

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \ldots$$

Proof:

$$\nabla^2 V = \nabla^2(\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3) = \alpha_1 \underbrace{\nabla^2 V_1}_{0} + \alpha_2 \underbrace{\nabla^2 V_2}_{0} + \alpha_3 \underbrace{\nabla^2 V_3}_{0} \ldots = 0$$

So a more general solution to Laplaces equation is of the form....

$$\left[V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \right] = C_1 e^{-\pi x/a} \sin\left(\frac{\pi y}{a}\right) + C_2 e^{-\pi x/a} \sin\left(\frac{2\pi y}{a}\right) + \ldots$$

Now, the last boundary condition

$$V(x=0,y) = V_0(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)$$

Fourier's Trick : Multiply both sides by $\sin\left(\frac{n'\pi y}{a}\right) dy$: Integrate from 0 to a

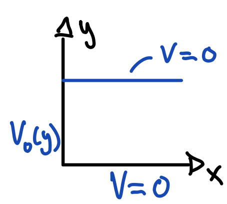
$$\int_0^a \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy}_{\substack{0 \text{ if } n \neq n' \\ \frac{a}{2} \text{ if } n = n'}} = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$C_{n'} \frac{a}{2} = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy \quad : \quad C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

So $$\underbrace{V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \quad w/ \quad C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy}_{\text{General Solution}}$$

10-15-18



$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \qquad n = 1, 2, 3, \ldots$$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a} \cdot \frac{a}{n\pi} \cos\left(\frac{n\pi y}{a}\right) \Big|_0^a = \frac{2V_0}{n\pi}\left(1 - (-1)^n\right)$$

$C_n = \frac{4V_0}{n(\pi)}$ for n odd

$C_n = 0$ for n even

$$\left[V(x,y) = \frac{4V_0}{n\pi} \cdot \sum \frac{1}{n} e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \right]$$

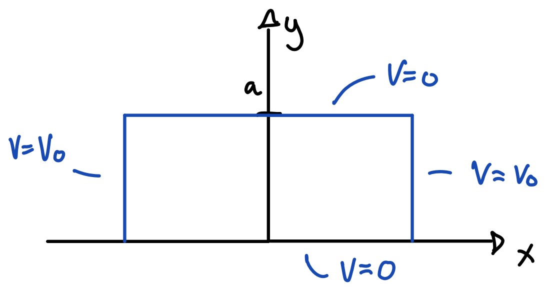
A set of functions is complete if any other function can be expressed as a linear combination of them.

$$f(y) = \sum_{n=1}^{\infty} C_n f_n(y)$$

A set of functions is said to be orthogonal if the integral of the product of any two different numbers of the set is zero....

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \quad \text{if } n \neq n'$$

i.e 3.4



Laplaces equation becomes of the form....

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Let $V(x,y) = X(x)Y(y)$

Becomes....

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)=K^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)=-K^2} = 0 \qquad \frac{d^2X}{dx^2} = K^2 X \,, \quad \frac{d^2Y}{dy^2} = -K^2 Y$$

$$\therefore \; X(x) = Ae^{Kx} + Be^{-Kx}$$
$$Y(y) = C\sin(Ky) + D\cos(Ky)$$

Evaluate B.C.'s

i) $V(x, y=0) = 0 \quad \therefore \; (Ae^{Kx} + Be^{-Kx})D = 0$

ii) $V(x, y=a) = 0 \quad \therefore \; (Ae^{Kx} + Be^{-Kx})C\sin(Ka) = 0$ — $K_n = \frac{n\pi}{a}$ $n = 1, 2, 3 \ldots.. \quad \therefore \; (Ae^{K_n x} + Be^{-K_n x})C\sin(K_n y)$

iii) $V(x=-b, y) = V_0 \quad \therefore \; (Ae^{-Kb} + Be^{Kb})C\sin(Ky) = V_0$

iv) $V(x=b, y) = V_0 \quad \therefore \; (Ae^{Kb} + Be^{-Kb})C\sin(Ky) = V_0$ — must have $A=B$

$$V(x,y) = AC(e^{Kx} + e^{-Kx})\sin(Ky) = \underbrace{\underbrace{2AC}_{C'}}_{DC}\frac{(e^{Kx}+e^{-Kx})}{2}\sin(Ky) = C\cdot\cosh(Kx)\sin(Ky)$$

The general solution is of the form....

$$V(x,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y)\Big|_{x=b} = V_0 = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

Multiply both sides by $\sin\left(\frac{n'\pi y}{a}\right) dy$: integrate from 0 to a....

$$\int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\Big|_0^b \int_0^a \sin\left(\cancel{\frac{n\pi y}{a}}\right) \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$= C_{n'} \cosh\left(\frac{n'\pi b}{a}\right)\frac{a}{2} \qquad C_{n'} = \frac{2}{a} \cdot \frac{1}{\cosh\left(\frac{n'\pi b}{a}\right)} \int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy = \frac{2V_0}{a \cdot \cosh\left(\frac{n'\pi b}{a}\right)} \cdot \frac{a}{n\pi} \cos\left(\frac{n\pi b}{a}\right)\Big|_0^a$$

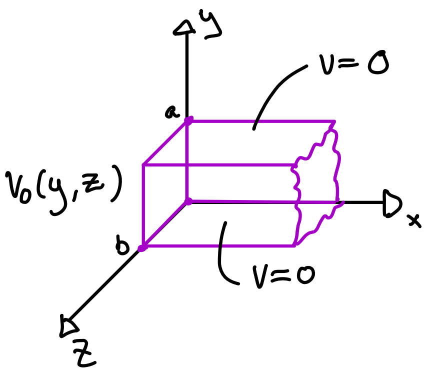
$$= \frac{2V_0}{n'\pi \cosh\left(\frac{n'\pi b}{a}\right)} (1 - \cos(n'\pi)) \qquad C_{n'} = \frac{4V_0}{n'\pi \cosh\left(\frac{n'\pi b}{a}\right)} \quad n' \text{ odd}$$

$$C_{n'} = 0 \qquad n' \text{ even}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi y/a)} \sin(n\pi y/a)$$

10-17-18

i.e 3.5



$$\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right] - V(x,y,z) = 0$$

$$V(x,y,z) = X(x)\, Y(y)\, Z(z)$$

$$YZ \frac{d^2}{dx^2} X + XZ \frac{d^2}{dy^2} + XY \frac{d^2}{dz^2} Z$$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{f(x) = k^2 + l^2} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{g(y) = -k^2} + \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{h(z) = -l^2} = 0 \qquad \frac{d^2 X}{dx^2} = (k^2 + l^2) X, \quad \frac{d^2 Y}{dy^2} = -k^2 Y, \quad \frac{d^2 Z}{dz^2} = -l^2 Z$$

$$Y(y) = \sin(ky) \qquad \frac{d^2 Y}{dy^2} = -k^2 \sin(ky) \quad \therefore \quad Y(y) = C\sin(ky) + D\cos(ky)$$

$$Z(z) = \sin(ly) \qquad \frac{d^2 Z}{dz^2} = -l^2 \sin(ly) \quad \therefore \quad Z(z) = E\sin(lz) + F\cos(lz)$$

$$X(x) = Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x}$$

Boundary Conditions

i) $V(y=0) = 0$

ii) $V(y=a) = 0$

iii) $V(z=0) = 0$

iv) $V(z=b) = 0$

v) $\lim_{x \to \infty} V(x,y,z) \to 0$

vi) $V(x=0) = V_0(y,z)$

Apply B.C's

(i) $V(y=0)=0 = X Z \cdot D = 0 \therefore D=0$

(iii) $V(z=0) = X Y \cdot F = 0 \therefore F=0$

(ii) $V(y=a) = X(C\sin(ka))Z = 0 \therefore ka = n\pi \, ; \, k_n = \frac{n\pi}{a}$

(iv) $V(z=b) = X Y(E\sin(lb)) = 0 \therefore lb = m\pi \therefore l_m = \frac{m\pi}{b}$

(v) $V(x \to \infty) = Ae^{\sqrt{k^2+l^2}} Y Z = 0 \therefore A = 0$

$$V(x,y,z) = (BCE)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)e^{-\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}\,x} \qquad BCE \to \beta$$

we have 2 unspecified integers, n : m

The most general linear combinations is a double sum...

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_{nm} e^{-\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi x} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

$$V(x=0,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_{nm} \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) = V_0(y,z)$$

multiply both sides by $\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right) dy dz$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \beta_{nm} \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) dy}_{n=n' \to \frac{a}{2} \quad n\neq n' \to 0} \underbrace{\int_0^b \sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{m'\pi z}{b}\right) dz}_{m=m' \to \frac{b}{2} \quad m\neq m' \to 0}$$

$$\beta_{nm} = \int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right) dy dz$$

$$\left[\ \beta_{nm} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right) dy dz\ \right]$$