MAT 201
Larson/Edwards – Section 4.2
Sigma Notation

In section 4.1 we introduced the idea of integration and formed basic rules of integration. This section is the beginning of our investigation of integration as it relates to area. In order for us to develop a formal definition for definite integration and the Fundamental Theorem of Calculus we need to fully understand sigma notation and its properties. We begin today by introducing ***sigma notation***.

<u>Sigma Notation</u>:
The sum of n terms in $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

i is known as the ***index of the summation***.
a_i is the ***i^{th} term*** of the sum.
n and 1 are the ***upper and lower bounds of summation***, respectively.

Ex: Explain why $\sum_{i=1}^{5} i = \sum_{k=5}^{9} (k-4)$.

Ex: In second grade, Carl Friedrich Gauss (1777—1855) was asked to find the sum of the first 100 natural numbers. Minutes later he had the correct answer. Use sigma notation to write the sum of the first 100 natural numbers.

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Ex: Explain why $\sum_{i=1}^{5} i = \sum_{k=5}^{9} (k-4)$.

$1+2+3+4+5$

$5 \quad 9$

$(5-4)+(6-4)+(7-4)+(8-4)+(9-4)$

$= 1+2+3+4+5$

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101

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

101

101

$$\begin{array}{r} 101 \\ \times 50 \\ \hline 5050 \end{array}$$

$$\boxed{\sum_{i=1}^{100} i}$$

Ex: Use sigma notation to write the sum:

$$\left(\frac{3}{n}\right)\left(\frac{1+1}{n}\right)^2 + \left(\frac{3}{n}\right)\left(\frac{2+1}{n}\right)^2 + \cdots + \left(\frac{3}{n}\right)\left(\frac{n+1}{n}\right)^2$$

Ex: Expand the following as a sum of n terms: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x_i$

Using the commutative and associative properties of addition and the distributive properties of multiplication over addition we can derive the following summation properties.

<u>Properties of Summation</u>: k is a constant

1. $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$
2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

<u>Summation Formulas (Theorem 4.2)</u>: c is a constant

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Ex: Use sigma notation to write the sum:

$$\underbrace{\left(\frac{3}{n}\right)\left(\frac{1+1}{n}\right)^2}_{1^{st}\text{ term}} + \underbrace{\left(\frac{3}{n}\right)\left(\frac{2+1}{n}\right)^2}_{2^{nd}\text{ term}} + \cdots + \underbrace{\left(\frac{3}{n}\right)\left(\frac{n+1}{n}\right)^2}_{n^{th}\text{ term}}$$

$$\sum_{i=1}^{n}\left(\frac{3}{n}\right)\left(\frac{i+1}{n}\right)^2$$

Ex: Expand the following as a sum of n terms:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$= \boxed{f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n}$$

Summation Formulas (Theorem 4.2): c is a constant

1. $\sum_{i=1}^{n} c = cn$ $= \underbrace{c+c+c+c+\cdots+c}_{n \text{ times}}$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof by induction:

$n=1$? ✓

Assume true for n

Show true for $n+1$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

$$1+4+9+\cdots+n^2 + (n+1)^2 = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}$$

Assuming true

$$\therefore \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{1\cdot 6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6}$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(2n+3)(n+2)}{6}$$

Ex: Use the properties of summation and formulas (Theorem 4.2) to evaluate the sum. Check your answer with a calculator.

$$\sum_{i=1}^{20} (i + 2)^2$$

Ex: Use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10$.

$$\sum_{k=1}^{n} \frac{6k(k-1)}{n^3}$$

Ex: Use the properties of summation and formulas (Theorem 4.2) to evaluate the sum. Check your answer with a calculator.

$$\sum_{i=1}^{20}(i+2)^2 = \sum_{i=1}^{20} i^2 + 4i + 4$$

$$= \sum_{i=1}^{20} i^2 + 4\sum_{i=1}^{20} i + \sum_{i=1}^{20} 4$$

$$= \frac{20(20+1)(2(20)+1)}{6} + 4\cdot\frac{20(20+1)}{2} + 4\cdot 20$$

$$= \boxed{3790}$$

Ex: Use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10$.

$$\sum_{k=1}^{n} \frac{6k(k-1)}{n^3} = \sum_{k=1}^{n} \frac{6k^2 - 6k}{n^3} = \sum_{k=1}^{n} \left(\frac{6k^2}{n^3} - \frac{6k}{n^3}\right)$$

$$= \sum_{k=1}^{n} \left(\frac{6k^2}{n^3}\right) - \sum_{k=1}^{n} \left(\frac{6k}{n^3}\right)$$

$$= \frac{6}{n^3}\left(\sum_{k=1}^{n} k^2\right) - \frac{6}{n^3}\left(\sum_{k=1}^{n} k\right)$$

$$= \frac{\cancel{6}}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}^{1}(n+1)(2n+1)}{\cancel{6}}\right) - \frac{\cancel{6}^{3}}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}_{1}}\right)$$

$$= \frac{(n+1)(2n+1)}{n^2} - \frac{3(n+1)}{n^2}$$

$$(n+1)(\cancel{2}n+1) - 3)$$

$$= \frac{(n+1)(2n+1) - 3(n+1)}{n^2}$$

$$= \frac{(n+1)(2n-2)}{n^2}$$

$$n = 10 \quad \frac{(10+1)(20-2)}{(100)} = \boxed{1.98}$$

Ex: Find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2\right] = \lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n}(i^2-2i+1)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - \frac{\cancel{2}n(n+1)}{\cancel{2}} + n\right)\right]$$

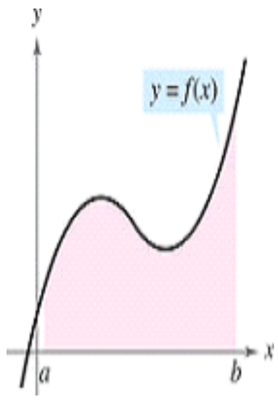
$$= \lim_{n\to\infty}\left[\frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6} - (n^2+n) + n\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{2n^3}{6n^3} + \frac{3n^2}{6n^3} + \frac{n}{6n^3} - \frac{n^2}{n^3} \cancel{- \frac{n}{n^3} + \frac{n}{n^3}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{3} + \underbrace{\cancel{\frac{1}{2n}}}_{0} + \underbrace{\cancel{\frac{1}{6n^2}}}_{0} - \underbrace{\cancel{\frac{1}{n}}}_{0}\right]$$

$$= \frac{1}{3}$$

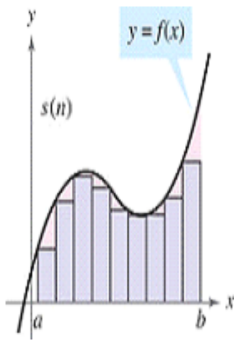
In geometry we have many formulas for finding the areas of different polygons. Calculating areas of regions with "curved" edges like the region given in the figure below isn't as easy.

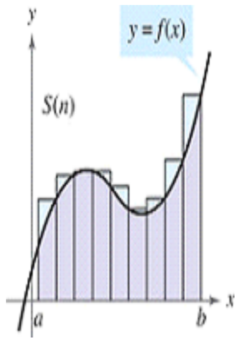


How would we find the area of the region bounded by the x-axis, $y = f(x)$, and the lines $x = a$, and $x = b$?

Add an infinite # of rectangles.

By adding rectangles of equal width we can obtain some good estimates for the area:





Which of these is a better estimate?

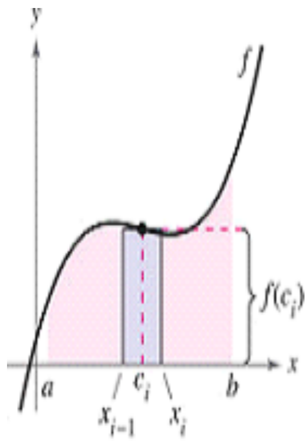
Area is between $s(n)$ and $S(n)$

<u>Definition of the Area of a Region in the Plane</u>:
Let f be continuous and nonnegative on the interval [a, b]. The area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is

$$Area = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

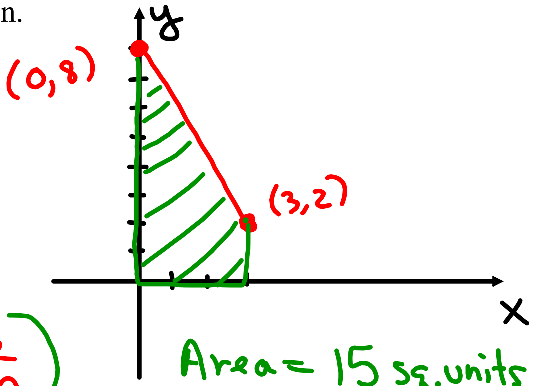
where $$\Delta x = \frac{b-a}{n}$$

Note: You have flexibility in choosing right or left endpoints. The book is choosing the right endpoints of the rectangle by letting $c_i = a + i\Delta x$.



Ex: Use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval. Sketch the region.

a) $y = 8 - 2x, [\overset{a}{0}, \overset{b}{3}]$



Area = 15 sq.units

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

$$c_i = a + i\Delta x = 0 + i \cdot \frac{3}{n} = \frac{3i}{n}$$

$$f(c_i) = 8 - 2\left(\frac{3i}{n}\right) = 8 - \frac{6i}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(8 - \frac{6i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{24}{n} - \frac{18i}{n^2}\right)$$

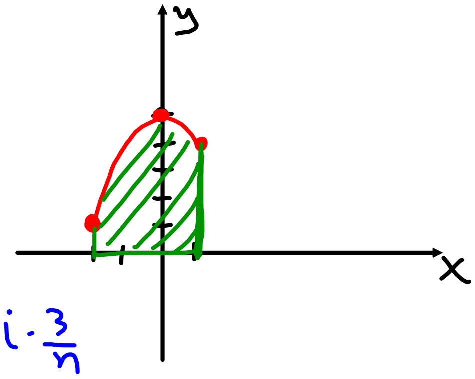
$$= \lim_{n\to\infty} \left(\frac{24}{\cancel{n}} \cdot \cancel{n} - \frac{\overset{9}{\cancel{18}}}{n^2} \cdot \frac{n^2+n}{\underset{1}{\cancel{2}}}\right)$$

$$= \lim_{n\to\infty} \left(24 - 9 - \frac{\cancel{9}}{n}^{\,0}\right)$$

$$= 15 \text{ sq.units}$$

b) $y = 5 - x^2, [-2, 1]$

$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}$



$c_i = a + i\Delta x = -2 + i \cdot \frac{3}{n}$
$= -2 + \frac{3i}{n}$

$f(c_i) = 5 - \left(-2 + \frac{3i}{n}\right)^2 = 5 - \left(4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right)$
$= 1 + \frac{12i}{n} - \frac{9i^2}{n^2}$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{12i}{n} - \frac{9i^2}{n^2}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} + \frac{36i}{n^2} - \frac{27i^2}{n^3}$$

$$= \lim_{n\to\infty} \left[3 + \frac{\cancel{36}^{18}}{n^2} \cdot \frac{n^2+n}{\cancel{2}_1} - \frac{\cancel{27}^{9}}{n^3} \cdot \frac{2n^3+3n^2+n}{\cancel{6}_2}\right]$$

$$= \lim_{n\to\infty} \left[3 + 18 + \cancel{\frac{18}{n}}^{0} - 9 - \cancel{\frac{27}{2n}}^{0} - \cancel{\frac{9}{2n^2}}^{0}\right]$$

$= 12$ sq. units