Ch2.6 Zeros of Polynomials, Müller's Method

A polynomial of degree n has the form
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
where $a_n \neq 0$. The zero function, $P(x) = 0$, $\forall x \in \mathbb{R}$, is considered a polynomial but is assigned no degree.

Fundamental Theorem of Algebra:
If $P(x)$ is a polynomial of degree $n \geq 1$, with real or complex coeffs, then $P(x) = 0$ has at least one (possibly complex) solution p.

Corollary For $P(x)$ as above, $\exists!\ x_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$ s.t.
$$P(x) = a_n (x - x_1)(x - x_2) \cdots (x - x_n)$$
$$= a_n (x - x_1)^{m_1} (x - x_2)^{m_2} \cdots (x - x_k)^{m_k}$$

Corollary Let $P(x)$ and $Q(x)$ be polynomials of degree at most n. If $x_1, x_2, \ldots, x_k$, with $k > n$, are distinct numbers with $P(x_i) = Q(x_i)$, $i = 1, \ldots, k$, then $P(x) = Q(x)$ for all x.

Recall Newton's Method for polynomial $P(x)$:

$$g(x) = x - \frac{P(x)}{P'(x)}$$

$$p_n = g(p_{n-1})$$

The function evaluations $P(p_{n-1})$ and $P'(p_{n-1})$ are best done in a nested manner.

Horner's Method incorporates this nesting technique, and hence requires only n multiplications and n additions to evaluate an arbitrary nth-degree polynomial.

Theorem (Horner's Method)

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

If $b_n = a_n$ and

$$b_k = a_k + b_{k+1} x_0 , \quad k = n-1, n-2, \ldots, 1, 0$$

then

$$b_0 = P(x_0)$$

Moreover, if

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$$

then

$$P(x) = (x - x_0) Q(x) + b_0$$

Proof By definition of $Q(x)$,

$(x - x_0) Q(x) + b_0$ = *algebra & simplification* =

$$b_n x^n + (b_{n-1} - b_n x_0) x^{n-1} + \cdots + (b_1 - b_2 x_0) x + (b_0 - b_1 x_0)$$

By hypothesis, $b_n = a_n$ & $b_k - b_{k+1} x_0 = a_k$, hence

$$(x - x_0) Q(x) + b_0 = P(x)$$

and $b_0 = P(x_0)$. ∎

Ex $\dfrac{x^3-2x^2-4x+1}{x-3} = x^2+x-1 - \dfrac{2}{x-3}$

$$
\begin{array}{r l}
 & 1x^2 + 1x - 1 \\
x-3 & \overline{)\,x^3 - 2x^2 - 4x + 1} \\
 & \underline{1x^3 - 3x^2} \\
 & \quad 1x^2 - 4x \\
 & \quad \underline{x^2 - 3x} \\
 & \qquad -1x + 1 \\
 & \qquad \underline{-x + 3} \\
 & \qquad\quad \boxed{-2}
\end{array}
$$

Synthetic Divison:

$$
\begin{array}{r|rrrr}
3 & 1 & -2 & -4 & 1 \\
 & & 3 & 3 & -3 \\
\hline
 & 1 & 1 & -1 & \boxed{-2}
\end{array}
$$

$x^2 + x - 1 - \dfrac{2}{x-3}$

Ex $\dfrac{2x^4 - x^2 + 5x - 3}{x-4}$

$$
\begin{array}{r|rrrrr}
4 & 2 & 0 & -1 & 5 & -3 \\
 & & 8 & 32 & 124 & 516 \\
\hline
 & 2 & 8 & 31 & 129 & \boxed{513}
\end{array}
$$

$2x^3 + 8x^2 + 31x + 129 + \dfrac{513}{x-4} = \dfrac{2x^4 - x^2 + 5x - 3}{x-4}$ (Ans)

Or $\underbrace{(2x^3+8x^2+31x+129)}_{q(x)}\underbrace{(x-4)}_{x-c} + \underbrace{513}_{r} = \underbrace{2x^4-x^2+5x-3}_{f(x)}$

Notes ① $f(4) = 513$

② Does $x-4$ divide $f(x)$? No; $r \neq 0$

③ 4 is not a root of f; $x-4$ not a factor of f

<u>Example</u> Use Horner's Method to evaluate $P(x) = 2x^4 - 3x^2 + 3x - 4$ at $x_0 = -2$.

When we use hand calculation in Horner's Method, we first construct a table, also used in synthetic division:

	Coeff of x^4	Coeff of x^3	Coeff x^2	Coeff x	Coeff x^0
$x_0 = -2$	$a_4 = 2$	$a_3 = 0$	$a_2 = -3$	$a_1 = 3$	$a_0 = -4$
		$b_4 x_0 = -4$	$b_3 x_0 = 8$	$b_2 x_0 = -10$	$b_1 x_0 = 14$
	$b_4 = 2$	$b_3 = -4$	$b_2 = 5$	$b_1 = -7$	$b_0 = 10$

Here, we recall $b_n = a_n$ and

$$Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \cdots + b_2 x + b_1$$

where $b_k = a_k + b_{k+1} x_0, \quad k = n-1, n-2, \ldots, 1, 0$

and $P(x) = (x - x_0) Q(x) + b_0$

Thus

$$P(x) = (x+2)(2x^3 - 4x^2 + 5x - 7) + 10$$

and hence

$$P(-2) = 10.$$

Note that using Horner's Method (or synthetic division) to obtain

$$P(x) = (x - x_0)Q(x) + b_0,$$

it follows that

$$P'(x) = Q(x) + (x - x_0)Q'(x)$$

and hence

$$P'(x_0) = Q(x_0)$$

Thus for Newton's Method,

$$x_1 = g(x_0) = x_0 - \frac{P(x_0)}{P'(x_0)}$$

$$= x_0 - \frac{b_0}{Q(x_0)}$$

where $\deg Q(x_0) = n-1$, and coeffs of Q are found recursively via synthetic division table.

Example Find an approximation to one of the zeros of $P(x) = 2x^4 - 3x^2 + 3x - 4$, using Newton's Method and synthetic division to evaluate $P(x_n)$ and $P'(x_n)$ for each iterate x_n.

With $x_0 = -2$ as an initial approx, we obtained $P(-2)$ in previous example by

$$
\begin{array}{r|rrrrrl}
 & 2 & 0 & -3 & 3 & -4 & \leftarrow P(x) \\
x_0 = -2 & & -4 & 8 & -10 & 14 & \\
\hline
Q(x) \rightarrow & 2 & -4 & 5 & -7 & \boxed{10} & = P(-2)
\end{array}
$$

Then

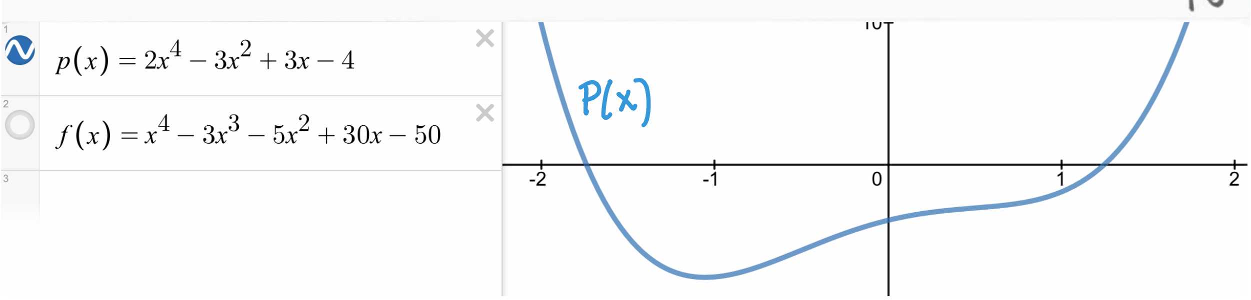
$$Q(x) = 2x^3 - 4x^2 + 5x - 7$$

and $P'(-2) = Q(-2)$. It follows that

$$
\begin{array}{r|rrrrl}
 & 2 & -4 & 5 & -7 & \leftarrow Q(x) \\
x_0 = -2 & & -4 & 16 & -42 & \\
\hline
 & 2 & -8 & 21 & \boxed{-49} & = Q(-2) = P'(-2)
\end{array}
$$

Thus

$$x_1 = x_0 - \frac{P(x_0)}{P'(x_0)} = -2 - \frac{10}{(-49)} \cong -1.796$$



Repeating the procedure to find x_2,

$$
\begin{array}{r|rrrrrl}
-1.796 & 2 & 0 & -3 & 3 & -4 & \leftarrow P(x) \\
 & & -3.592 & 6.451 & -6.197 & 5.742 & \\
\hline
Q(x) \rightarrow & 2 & -3.592 & 3.451 & -3.197 & \boxed{1.742} & = P(x_1) \\
 & & -3.592 & 12.902 & -29.368 & & \\
\hline
 & 2 & -7.184 & 16.353 & \boxed{-32.565} & & = Q(x_1) = P'(x_1)
\end{array}
$$

Thus

$$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)} = -1.796 - \frac{1.742}{-32.565}$$

$$\cong -1.7425$$

Similarly, it can be shown that

$x_3 = -1.73897$ (see next page)

The actual zero to five decimal places is

$p = -1.73896$

Note above that $Q(x)$ changes for each iteration, and depends on the approximation x_k being used.

Example Recall $P(x) = 2x^4 - 3x^2 + 3x - 4$

Horner's method uses nested computations to perform the polynomial evaluations needed for Newton's method: $g(x) = x - \dfrac{P(x)}{P'(x)}$ Use nested method, as in notes

$x_0 = -2$

$x_1 = g(-2) \cong -1.796$

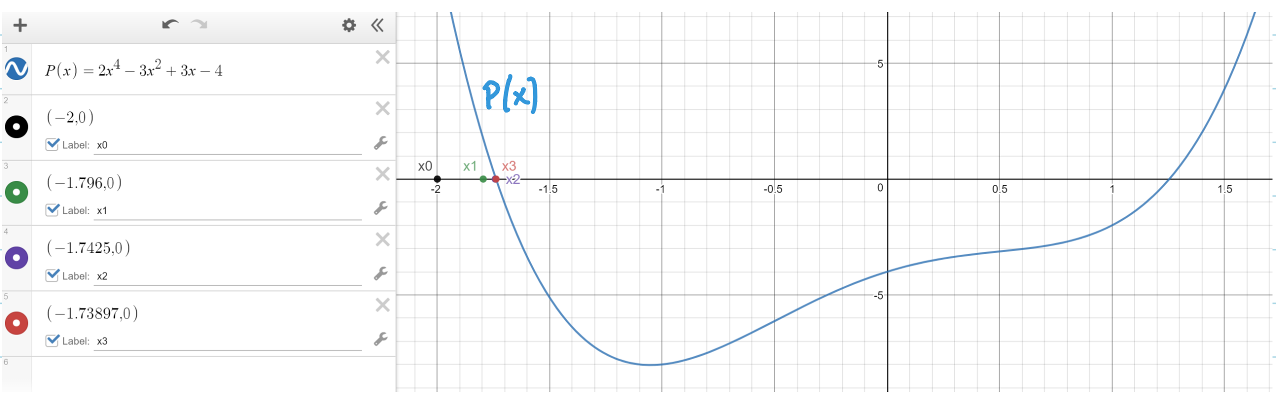
$x_2 = g(-1.796) \cong -1.7425$

} See steps shown previously

Now find x_3, using synthetic division as in notes for x_2.

-1.7425	2	0	-3	3	-4
		-3.485	6.0726125	-5.354027281	4.101892538
	2	-3.485	3.0726125	-2.354027281	$\boxed{0.1018925376}$
		-3.485	12.145225	-26.51708184	
	2	-6.97	15.2178375	$\boxed{-28.87110912}$	

$$x_3 = g(x_2) = -1.7425 - \frac{0.1018925376}{-28.87110912} \cong -1.7425 + 0.0035292214 \cong -1.73897$$



<u>Example</u> Find approximations to within 10^{-5} to all zeros of f by first finding the real zeros using Newton's Method and then reducing to polynomials of lower degree to find any complex zeros.

$$f(x) = x^4 - 3x^3 - 5.0x^2 + 30x - 50$$

See next slide for graph, and real roots.

$$f'(x) = 4x^3 - 9x^2 - 10x + 30$$

$$g(x) = x - \frac{f(x)}{f'(x)} \quad \text{(Iterating fcn.)}$$

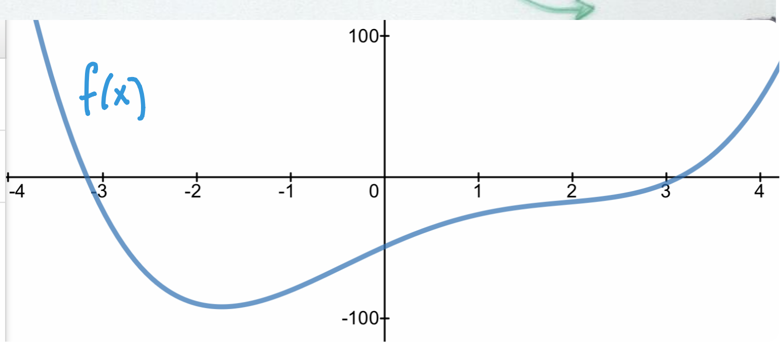
<u>Newton's Method:</u>

$p_0 = -3$	$p_0 = 3$
$p_1 = g(p_0) = -3.178294574$	$p_1 = g(p_1) = 3.185185185$
$p_2 = g(p_1) = -3.162414375$	$p_2 = g(p_1) = 3.162669897$
$p_3 = g(p_2) = -3.16227767$	$p_3 = g(p_2) = 3.16227777$
$\lvert p - p_3 \rvert = 1 \times 10^{-8} < 10^{-5}$	$\lvert p - p_3 \rvert = -1.1 \times 10^{-7} < 10^{-5}$

<u>Deflate:</u> $f(x) = (x + 3.16228)(x - 3.16228)\,Q(x)$

$p(x) = 2x^4 - 3x^2 + 3x - 4$

$f(x) = x^4 - 3x^3 - 5x^2 + 30x - 50$



We have

$$f(x) = x^4 - 3x^3 - 5x^2 + 30x - 50$$

$$= (x+3.16228)\underbrace{(x-3.16228)Q(x)}_{Q_1(x)}$$

To find $Q_1(x)$, use synthetic division:

-3.16228	1	-3	-5	30	-50
		-3.16228	19.49005	-45.82160	50.03233
3.16228	1	-6.16228	14.49005	-15.82160	$\boxed{0.03233}$
		3.16228	-9.48684	15.82155	
	1	-3	5.00321	$\boxed{0.00005}$	

$Q_1(x) \cong x^3 - 6.16228x^2 + 14.49005x - 15.82160$

$Q(x) \cong x^2 - 3x + 5$

Thus $f(x) \cong (x+3.16228)(x-3.16228)(x^2-3x+5)$

Since $b^2 - 4ac = 9 - 20 = -11 < 0$, the remaining two roots are complex:

$$p = \frac{3 \pm \sqrt{11}\, i}{2} = 1.5 \pm 1.65831\, i$$

Horner's

To evaluate the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - x_0)Q(x) + b_0$$

and its derivative at x_0:

INPUT degree n; coefficients $a_0, a_1, \ldots, a_n$; x_0.

OUTPUT $y = P(x_0)$; $z = P'(x_0)$.

Step 1 Set $y = a_n$; (*Compute* b_n *for* P.)
$z = a_n$. (*Compute* b_{n-1} *for* Q.)

Step 2 For $j = n-1, n-2, \ldots, 1$
set $y = x_0 y + a_j$; (*Compute* b_j *for* P.)
$z = x_0 z + y$. (*Compute* b_{j-1} *for* Q.)

Step 3 Set $y = x_0 y + a_0$. (*Compute* b_0 *for* P.)

Step 4 OUTPUT (y, z);
STOP.