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MATH 360
CP Ch. 5.1

Example 1 Familiar Power Series are the Maclaurin Series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \dots \quad (|x|<1, \text{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$\sin(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$$

Example 2 Power Series Solution. Solve $y' - y = 0$.

Solution. In the first step we insert

(2) $$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{m=0}^{\infty} a_m x^m$$

and the series obtained by termwise differentiation

(3) $$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

into the ODE:

$$(a_1 + 2a_2 x + 3a_3 x^2 + \dots) - (a_0 + a_1 x + a_2 x^2 + \dots) = 0.$$

Then we collect like powers of x, finding

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \dots = 0.$$

Equating the coefficient of each power of x to zero, we have

$$a_1 - a_0 = 0, \quad 2a_2 - a_1 = 0, \quad 3a_3 - a_2 = 0, \dots$$

Solving these equations, we may express $a_1, a_2, \dots$ in terms of a_0, which remains arbitrary:

$$a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2!}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \dots$$

With these values of the coefficients, the series solution becomes the familiar general solution

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \dots = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) = a_0 e^x.$$

Test your comprehension by solving $y'' + y = 0$ by power series. You should get the result $y = a_0 \cos(x) + a_1 \sin(x)$.

Example 3 A Special Legendre Equation. The ODE

$$(1-x^2)y'' - 2xy' + 2y = 0$$

occurs in models exhibiting spherical symmetry. Solve it.

Solution. Substitute (2), (3), and (5) into the ODE. $(1-x^2)y''$ gives two series, one for y'' and one for $-x^2y''$. In the term $-2xy'$ use (3) and in $2y$ use (2). Write like powers of x vertically aligned. This gives

$$\begin{aligned}
y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots\cdots \\
-x^2y'' &= \qquad\qquad\qquad\quad - 2a_2 x^2 - 6a_3 x^3 - 12a_4 x^4 - \cdots\cdots \\
-2xy' &= \qquad\quad - 2a_1 x - 4a_2 x^2 - 6a_3 x^3 - 8a_4 x^4 - \cdots\cdots \\
2y &= 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots\cdots
\end{aligned}$$

Add terms of like powers of x. For each power $x^0, x, x^2, \ldots\ldots$ equate the sum obtained to zero. Denote these sums by [0] (constant terms), [1] (first power of x), and so on:

Sum	Power	Equations
[0]	$[x^0]$	$a_2 = -a_0$
[1]	$[x]$	$a_3 = 0$
[2]	$[x^2]$	$12a_4 = 4a_2, \quad a_4 = \frac{4}{12}a_2 = -\frac{1}{3}a_0$
[3]	$[x^3]$	$a_5 = 0 \quad \text{Since} \quad a_3 = 0$
[4]	$[x^4]$	$30a_6 = 18a_4, \quad a_6 = \frac{18}{30}a_4 = \frac{18}{30}\left(-\frac{1}{3}\right)a_0 = -\frac{1}{5}a_0.$

This gives the solution

$$y = a_1 x + a_0\left(1 - x^2 - \tfrac{1}{3}x^4 - \tfrac{1}{5}x^6 - \cdots\right).$$

a_0 and a_1 remain arbitrary. Hence, this is a general solution that consists of two solutions: x and $1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^6-\cdots\cdots$. These two solutions are members of families of functions called Legendre polynomials $P_n(x)$ and Legendre functions $Q_n(x)$: here we have $x = P_1(x)$ and $1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^6-\cdots = -Q_1(x)$. The minus is by convention. The index 1 is called the order of these two functions and here the order is 1. More on Legendre polynomilas in the next section.

Example 4 Convergence Radius $R = \infty, 1, 0$

For all these series let $m \to \infty$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots, \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \to 0, \qquad R = \infty$$

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{1} = 1 \qquad R = 1$$

$$\sum_{m=0}^{\infty} m!\,x^m = 1 + x + 2x^2 + \cdots \qquad \left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+1)!}{m!} = m+1 \to \infty \qquad R = 0$$

Convergence for all x ($R=\infty$) is the best possible case, convergence in some finite interval the usual, and convergence only at the center ($R=0$) is useless.