6.3 HW Linear Systems of Nonreal Eigenvalues

1-16

6.3.1 $\bar{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$

God $= i = \sqrt{-1}$ $\therefore (\sqrt{-1})^2 = -1$
$i^2 = -1$

$|A - \lambda I| = 0$

$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$

$(\lambda)^2 - 1(-1)$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$\lambda^2 + 1$

$\alpha = 0$ $\beta = : \dfrac{\sqrt{4(1)(1) - 0^2}}{2(1)} = \dfrac{\sqrt{4}}{2} = 1$

$\alpha = 0$ $\beta = 1$

$(A - \lambda I)\bar{v}$ $\lambda = -i$

$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$ $-i \cdot R' = \begin{bmatrix} 1 & -i \\ -1 & i \end{bmatrix}$ $R_2^* = R_1 + R_2 = \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x = iy$ $\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda = i$

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ $R_1^* = i \cdot R_1 = \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$ $R_2^* = R_1 + R_2 \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$ $y_i = -x$ $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\bar{q} = \begin{bmatrix} 0 \\ -i \end{bmatrix}$

$x_{RE} = \cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ i \end{bmatrix}$

$x_{Im} = \sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ i \end{bmatrix}$

$x(t) = c_1\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_1\sin(t)\begin{bmatrix} 0 \\ i \end{bmatrix} + c_2\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\cos(t)\begin{bmatrix} 0 \\ i \end{bmatrix}$

$$\boxed{x(t) = c_1 \begin{bmatrix} \cos(t) \\ -i\sin(t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(t) \\ i\cos(t) \end{bmatrix}}$$

6.3.3 $\bar{x}' = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \bar{x}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} (1-\lambda) & 2 \\ -2 & (1-\lambda) \end{bmatrix} \right| = 0$

$(1-\lambda)(1-\lambda) - 2(-2) = 0$

$1 - 2\lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$\alpha = \dfrac{+2}{2(1)} = +1$ $\lambda = \alpha + i\beta$
$= 1 + 2i, 1 - 2i$

$\beta = \dfrac{\sqrt{4(1)(5) - (-2)^2}}{2(1)} = 2$

$p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\bar{q} = \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(A - \lambda I)\bar{v}$

$\lambda = 1 \pm 2i$

$\begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$ $R_1^* = 2iR_1$ $\begin{bmatrix} 4 & 4i \\ -2 & -2i \end{bmatrix}$ $R_2^* = 2R_2$

$\begin{bmatrix} 4 & 4i \\ -4 & -4i \end{bmatrix}$ $R_2^* = R_2 + R_1 = \begin{bmatrix} 4 & 4i \\ 0 & 0 \end{bmatrix}$ $R_1^* = R_1/4 = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $-x = iy$

$\bar{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\lambda = 1 - 2i$

$(A - \lambda I)\bar{v}$

$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}$ $R_1^* = -2iR_1 = \begin{bmatrix} 4 & -4i \\ -2 & 2i \end{bmatrix}$ $R_2^* = 2R_2 = \begin{bmatrix} 4 & -4i \\ -4 & 4i \end{bmatrix}$

$R_2^* = R_1 + R_2 = \begin{bmatrix} 4 & -4i \\ 0 & 0 \end{bmatrix}$ $R_1^* = R_1/4 = \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $x - iy = 0$
$x = iy$
$\bar{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$x(t)_R = e^t\left(c_1\cos(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - c_2\sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$x(t)_I = e^t\left(c_2\sin(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$\boxed{x(t) = c_1 e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}$$

6.3.5) $\bar{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\bar{x}$

$\bar{x}_{Re} = e^{\alpha t}(\cos\beta t\,\bar{p} - \sin\beta t\,\bar{q})$

$\bar{x}_{Im} = e^{\alpha t}(\sin\beta t\,\bar{p} + \cos\beta t\,\bar{q})$

$|A - \lambda I| = 0$

$\left|\begin{bmatrix} 1-\lambda & 1 \\ -2 & -1-\lambda \end{bmatrix}\right| = 0$

$(1-\lambda)(-1-\lambda)$

$-1+\lambda^2 - 1(-2)$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$\alpha = 0 \quad \beta = \dfrac{\sqrt{4ac - b^2}}{2a}$

$\beta = 1$

$\beta = 1$

$(A - \lambda I) = 0$

$\lambda = i$

$\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}$ $R_1^* = (1+i)R_1 = \begin{bmatrix} 2 & 1+i \\ -2 & -i-i \end{bmatrix}$ $R_2^* = R_2 + R_1 = \begin{bmatrix} 2 & 1+i \\ 0 & 0 \end{bmatrix}$

$(1-i)(1+i)$

$1 - i^2 = 2$

$R_1^* = R_2/2 = \begin{bmatrix} 1 & \frac{1+i}{2} \\ 0 & 0 \end{bmatrix}$ $x = -\dfrac{(1+i)}{2}y$ $\bar{v}_1 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$

$\lambda = -i$

$\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}$ $R_1^* = R_1(1-i)\begin{bmatrix} 2 & 1-i \\ -2 & -1+i \end{bmatrix}$ $R_2^* = R_1 + R_2 = \begin{bmatrix} 2 & 1-i \\ 0 & 0 \end{bmatrix}$

$R_1^* = R_1/2 = \begin{bmatrix} 1 & \frac{1-i}{2} \\ 0 & 0 \end{bmatrix}$ $x = -\dfrac{(1-i)}{2}y$ $\bar{v}_2 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$ $\bar{p} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\bar{q} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\bar{x}_R = c_1\cos(t)\begin{bmatrix} -1 \\ 2 \end{bmatrix} - c_1\sin(t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\bar{x}_I = c_2\sin(t)\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\cos(t)\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\boxed{x(t) = c_1\begin{bmatrix} -\cos(t) + \sin(t) \\ 2\cos(t) \end{bmatrix} + c_2\begin{bmatrix} -\sin(t) - \cos(t) \\ 2\sin(t) \end{bmatrix}}$$

6.3.7) $\bar{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\bar{x}$

$\lambda = 1 \pm 2i$

$\lambda_1 = 1 + 2i$

$\lambda_2 = 1 - 2i$

$|A - \lambda I| = 0$

$\left|\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right| = 0$ $\quad \alpha = 1 \quad \beta = 2$

$(3-\lambda)(-1-\lambda) + 2(4)$

$-3 - 2\lambda + \lambda^2 + 8$

$\lambda^2 - 2\lambda + 5$

$\dfrac{\sqrt{4(1)(5) - 2^2}}{2} = \dfrac{\sqrt{16}}{2} = 2$

$(A - \lambda I)\bar{v}_1 \quad \lambda_1$

$\begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}$

$(2-2i)(2+2i)$

$4 - 4i^2 = 8$

$(2+2i)(2-2i)$

$4 - 4i^2 = 8$

$\begin{bmatrix} 8 & -4-4i \\ 8 & -4-4i \end{bmatrix} = \begin{bmatrix} 8 & -4-4i \\ -8 & +4+4i \end{bmatrix} = \begin{bmatrix} 8 & -4-4i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}-\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$

$x = (\frac{1}{2} + \frac{1}{2}i)y$

$v_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$-2(1-2i)$

$-2+4i$

$(A - \lambda I)\bar{v}_2$

$(2+2i)(2-2i) \quad -2(2-2i)$

$4 - 4i^2 \quad -4+4i$

$\begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} = \begin{bmatrix} 8 & -4+4i \\ 4 & -2+2i \end{bmatrix} = \begin{bmatrix} 8 & -4+4i \\ -8 & 4-4i \end{bmatrix}$

$\begin{bmatrix} 8 & -4+4i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}+\frac{1}{2}i \\ 0 & 0 \end{bmatrix}$ $x = (\frac{1}{2} - \frac{1}{2}i)y$

$v_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

$x(t)_R = c_1e^t(\cos\beta t\,\bar{p} - \sin\beta t\,\bar{q})$

$x(t)_I = c_2e^t(\sin\beta t\,\bar{p} + \cos\beta t\,\bar{q})$

$\bar{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{q} = \pm i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x_R(t) = c_1 e^{t}\left(\cos 2t \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$x_I(t) = c_2 e^{t}\left(\sin 2t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\boxed{x(t) = c_1 e^{t} \begin{bmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}}$$

6.3.13) $\bar{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \bar{x}$ $\qquad \bar{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\left| \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} \right| = 0$

$(1-\lambda)(1-\lambda) + 1(1)$

$1 - 2\lambda + \lambda^2 + 1$

$\lambda^2 - 2\lambda + 2$

$\lambda = \alpha \pm i\beta$

$\alpha = \frac{-b}{2a} = 1$

$\beta = \frac{\sqrt{4}}{2} = 1$

$\lambda_1 = 1 + i$

$\lambda_2 = 1 - i$

$(A - \lambda I)\bar{v}_1$

$\lambda = 1 + i$

$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$x = y(i)$

$1 = y(i)$

$y = \frac{1}{i} = -i$

$v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$(A - \lambda I)\bar{v}_2$

$\lambda = 1 - i$

$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}$ $\qquad x = -y(i)$

$1 = -y(i)$

$\frac{1}{i} = -y, \; -i = -y$

$i = y$

$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ $\quad x = -iy$

$v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$x_R(t) = c_1 e^{t}\left(\cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$x_I(t) = c_2 e^{t}\left(\sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$x(t) = c_1 e^{t} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^{t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$t = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right] \qquad c_1 = -1, \; c_2 = 1$$

$$x(t) = -e^{t} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + e^{t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\boxed{x = e^{t} \begin{bmatrix} \sin(t) - \cos(t) \\ \sin(t) + \cos(t) \end{bmatrix}}$$