Ch. 6 : Gravity as Geometry

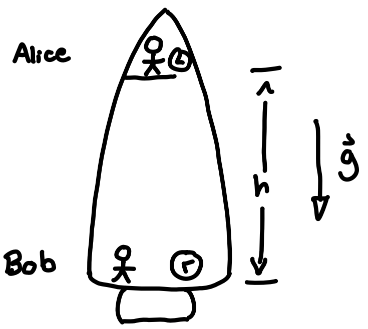
2-22-19

$$|\vec{F}_{12}| = \frac{G m_1 m_2}{|\vec{r}_1(t) - \vec{r}_2(t)|^2}$$

The Equivalence Principle

- The gravitational field has only a relative existence.
- There is no experiment that can distinguish a uniform acceleration from a uniform gravitational field.

Clocks In A Gravitational Field

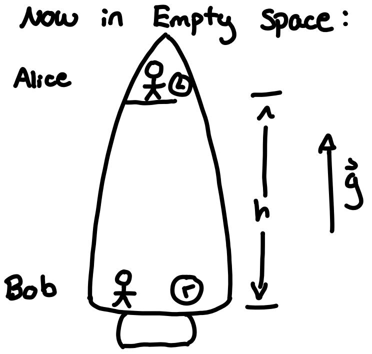


- Alice emits light signals at equal intervals $\Delta\tau_A$
- Bob receives light signals at equal intervals $\Delta\tau_B$

$$\overset{\times}{\Delta\tau_A = \Delta\tau_B}\ ,\quad \overset{\times}{\Delta\tau_A < \Delta\tau_B}\ ,\quad \overset{\checkmark}{\Delta\tau_A > \Delta\tau_B}$$

Use Equivalence principle .....

Now in Empty space:



Four events

1. 1^{st} Pulse emitted by Alice @ $t=0$
2. 1^{st} Pulse is received by Bob @ $t=t_1$
3. 2^{nd} Pulse emitted by Alice @ $t=\Delta\tau_A$
4. 2^{nd} Pulse received by Bob @ $t=t_1+\Delta\tau_B$

$$z_B(t) = \tfrac{1}{2}gt^2 \sim \text{Bob's position}$$

$$z_A(t) = h + \tfrac{1}{2}gt^2 \sim \text{Alice's position}$$

The distance traveled by the first pulse ...

$$z_A(0) - z_B(t_1) = ct_1 \quad \therefore\ h - \tfrac{1}{2}gt_1^2 = ct_1$$

The distance traveled by the second pulse (is shorter....)

$$z_A(\Delta\tau_A) - z_B(t_1+\Delta\tau_B) = c(t_1+\Delta\tau_B-\Delta\tau_A) \ \therefore\ h+\tfrac{1}{2}g\Delta\tau_A^2 - \tfrac{1}{2}g(t_1+\Delta\tau_B)^2$$

$$= c(t_1+\Delta\tau_B-\Delta\tau_A)$$

$$= h+\tfrac{1}{2}g\Delta\tau_A^2 - \tfrac{1}{2}g(t_1^2+2t_1\Delta\tau_B+\Delta\tau_B^2)$$ Since $\Delta\tau_B$ and $\Delta\tau_A$ are small : $\Delta\tau_A^2 \,\&\, \Delta\tau_B^2 \approx 0$

① $\to$ $= h-\tfrac{1}{2}gt_1^2 - gt_1\Delta\tau_B = c(t_1+\Delta\tau_B-\Delta\tau_A)$: $h-\tfrac{1}{2}gt_1^2 = ct_1 \leftarrow$ ②

Subtracting

$$-gt_1\Delta\tau_B = c(\Delta\tau_B - \Delta\tau_A) \quad : \quad c\Delta\tau_A = c\Delta\tau_B + gt_1\Delta\tau_B = c\left(1+\frac{gt_1}{c}\right)\Delta\tau_B$$

$$\Delta\tau_A = \left(1+\frac{gt_1}{c}\right)\Delta\tau_B = \left(1+\frac{gh}{c^2}\right)\Delta\tau_B \quad : \quad \Delta\tau_B = \frac{1}{(1+gh/c^2)}\Delta\tau_A \qquad (1+x)^{-1} \approx 1-x$$

(*) $$\left[\Delta\tau_B = (1-gh/c^2)\Delta\tau_A\right]$$

From (*)

$$\frac{1}{\Delta\tau_B} = \left(1+\frac{gh}{c^2}\right)\frac{1}{\Delta\tau_A}$$

- $\frac{1}{\Delta\tau's}$ are both rates of emission & reception
- Also $gh = \Phi_A - \Phi_B$ - Gravitational potential

(*) $$\left[\left(\text{The rate signals received @ B}\right) = \left(1+\frac{\Phi_A-\Phi_B}{c^2}\right)\left(\begin{matrix}\text{Rate Signals}\\ \text{emitted @ A}\end{matrix}\right)\right] \quad : \quad \Delta\tau_B < \Delta\tau_A$$

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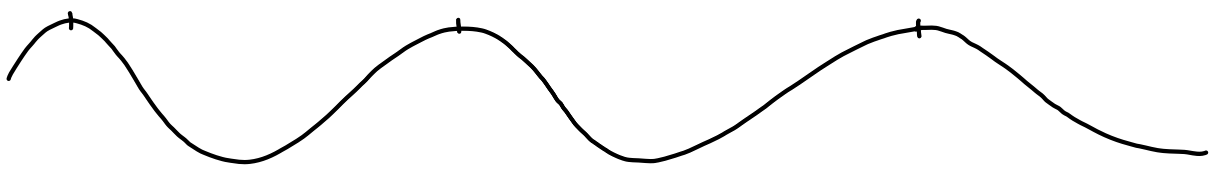
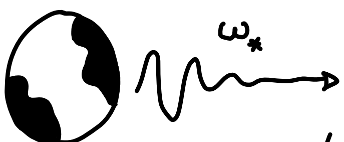
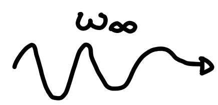
(*) Holds for a non-uniform gravitational fields as well....

$$\Phi_A = \Phi(x_A), \quad \Phi_B = \Phi(x_B)$$

The Gravitational Redshift







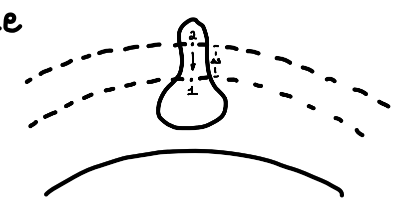
- Light emitted from surface of a star, ω_*, $\Phi(R) = -\frac{GM}{R}$
- Light received far from the star, ω_∞, $\Phi = 0$

$$\omega_\infty = \left(1 - \frac{\frac{GM}{R} - 0}{c^2}\right)\omega_*, \quad \left[\omega_\infty = \left(1-\frac{GM}{Rc^2}\right)\omega_*\right] \text{ - Gravitational Redshift}$$

Equivalence Principle

Experiments in a sufficiently small, freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those in an inertial reference frame in empty space.

i.e



- The nearer ball referred w/ $\vec{v}$ such that circular orbit
- The farther ball referred w/ $\vec{v}$ such that elliptical orbit

$$a_1 = \frac{v^2}{R} = \frac{GM_\oplus}{R^2} \equiv g, \quad a_2 = \frac{GM_\oplus}{(R+s)^2} = \frac{GM_\oplus}{R^2}\cdot\frac{1}{(1+s/R)^2} = \frac{GM_\oplus}{R^2}\left(1+\frac{s}{R}\right)^{-2}$$

$$\frac{GM_\oplus}{R^2}\left(1+\frac{s}{R}\right)^{-2} \approx \frac{GM_\oplus}{R^2}\left(1-\frac{2s}{R}\right) \quad : \quad \vec{a}_{Rel} = a_1 - a_2 = \frac{GM_\oplus}{R^2}\cdot\frac{2s}{R} = \frac{2gs}{R} \text{ for } s \ll R$$

Separation of balls in δt ....

$$\delta s \simeq \frac{1}{2} a_{Rel}\, \delta t^2 = \frac{gs\,\delta t^2}{R} \quad , \quad V = \frac{2\pi R}{T} \;\therefore\; T = \frac{2\pi R}{V} \quad : \quad T^2 = \frac{4\pi^2 R^2}{V^2} = \frac{4\pi^2 R}{g} \quad : \quad \frac{g}{R} = \frac{4\pi^2}{T^2}$$

$$\frac{4\pi^2}{T^2}\cdot s \cdot \delta t^2 = \left(\frac{2\pi\,\delta t}{T}\right)^2 s \quad : \quad \delta s = \left(\frac{2\pi\,\delta t}{T}\right)^2 s$$

As $s : \delta t \to 0$, $\delta s \to 0$

Spacetime Is Curved

Newtonian Gravity in Spacetime Terms

Newtonian Gravity can be expressed Completely in Geometric terms in the Curved Spacetime

$$ds^2 = -\left(1+\frac{2\Phi(x)}{c^2}\right)c^2dt^2 + \left(1-\frac{2\Phi(x_i)}{c^2}\right)(dx^2+dy^2+dz^2) \quad : \quad \Phi(x_i) \leftarrow \text{Gravitational potential}$$

Function of position

Satisfies $\nabla^2\Phi(x) = 4\pi G\mu(\vec{x})$

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Newtonian Gravity in Spacetime Terms

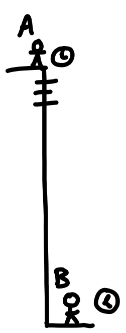
$$\left[ds^2 = -\left(1+\frac{2\Phi(x_i)}{c^2}\right)c^2dt^2 + \left(1-\frac{2\Phi(x_i)}{c^2}\right)(dx^2+dy^2+dz^2) \quad \text{where} \quad \nabla^2\Phi(x) = -4\pi G\mu(x) \right]$$

Line Element for a Static, weak Field

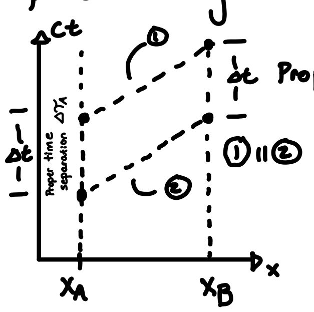
Outside a spherically symmetric body ...

$$\Phi(r) = \frac{-GM}{r}$$

Rates of Emission : Reception



Draw a spacetime diagram of my emitter, receiver of 2 light signals



$ds^2 = 0$: Determines the ct vs. x curve for a light beam

- But worldlines of Successive signals will have the Same Shape because the geometry is not time-Dependent

$$t \to t + c \;\therefore\; ds^2 \to ds^2$$

- Same coordinate separation Δt for A & B
- Different proper time intervals $\Delta\tau_A : \Delta\tau_B$

Two Emissions @ x_A

- Coordinate separation are Δt : $\Delta x = \Delta y = \Delta z = 0$

so

$$\Delta s^2 = -\left(1 + \frac{2\Phi_A}{c^2}\right) c^2 \Delta t^2 = -c^2 \Delta \tau_A^2 \longleftarrow \text{Since } d\tau^2 = -\frac{ds^2}{c^2}$$

$$\Delta \tau_A = \left(1 + \frac{2\Phi_A}{c^2}\right)^{1/2} \Delta t \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t \quad : \quad ^{(*)}\left[\Delta \tau_A = \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t\right] \quad \text{so } \Delta t = \frac{\Delta \tau_A}{(1 + \Phi_A/c^2)}$$

Two Receptions @ x_B

- Coordinate separations Δt : $\Delta x = \Delta y = \Delta z = 0$

$$\Delta s^2 = -\left(1 + \frac{2\Phi_B}{c^2}\right) c^2 \Delta t^2 = -c^2 \Delta \tau_B^2$$

Likewise : $^{(*)(*)}\left[\Delta \tau_B = \left(1 + \Phi_B/c^2\right)\Delta t\right]$

Plug (*) into (*)(*) ....

$$\Delta \tau_B = \frac{(1 + \Phi_B/c^2)}{(1 + \Phi_A/c^2)} \Delta \tau_A \simeq (1 + \Phi_B/c^2)(1 - \Phi_A/c^2)\Delta \tau_A = \left(1 + \Phi_B/c^2 - \Phi_A/c^2 - \frac{\Phi_A \Phi_B}{c^4}\right)\Delta \tau_A$$

$$\left[\Delta \tau_B = \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta \tau_A\right]$$

Newtonian Description of Gravity

- Mass produces a gravitational potential Φ
- Particles accelerate by $\vec{a} = -\vec{\nabla}\Phi$

GR Description of Gravity

- Mass produces a spacetime curvature
- Particles move in "straight lines" in this curved geometry.