# Physics 311

## Exam 1

1. Consider the vector functions

$$\begin{aligned} \vec{A} &= (3x^2)\hat{x} + (2z^3)\hat{z} \\ \vec{B} &= (2x^2 z)\hat{y} + (xy^2)\hat{z}. \end{aligned} \tag{1}$$

a) Calculate the quantity $\vec{\nabla}(\vec{A} \cdot \vec{B})$. Is this a scalar or a vector quantity?

b) Calculate the quantity $\vec{A} \cdot (\vec{\nabla} \times \vec{B})$. Is this a scalar or a vector quantity?

2. Consider the vector function

$$\vec{v}(x,y,z) = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}. \tag{2}$$

a) Calculate the *divergence* of the above vector function.

b) Calculate the *curl* of the above vector function.

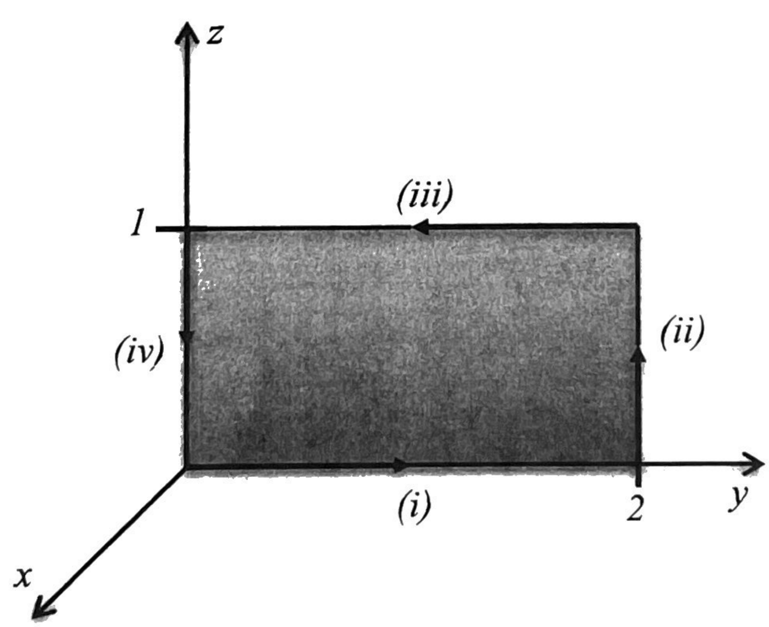
Consider the rectangular shaded area of the figure below.

c) Compute the flux of the curl of the above vector function through the rectangular shaded region. Namely, compute

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} \tag{3}$$

d) Compute the closed line integral of the above vector function around the perimeter of the rectangular shaded region in the counterclockwise direction. Namely, compute

$$\oint_{\mathcal{P}} \vec{v} \cdot d\vec{l} \tag{4}$$



3. Consider the vector function in spherical-polar coordinates

$$\vec{v}(r,\theta,\phi) = (r^2\cos\theta)\hat{r} + (r^2\cos\phi)\hat{\theta} - (r^2\cos\theta\sin\phi)\hat{\phi}. \tag{5}$$

a) Calculate the *divergence* of the above vector function.

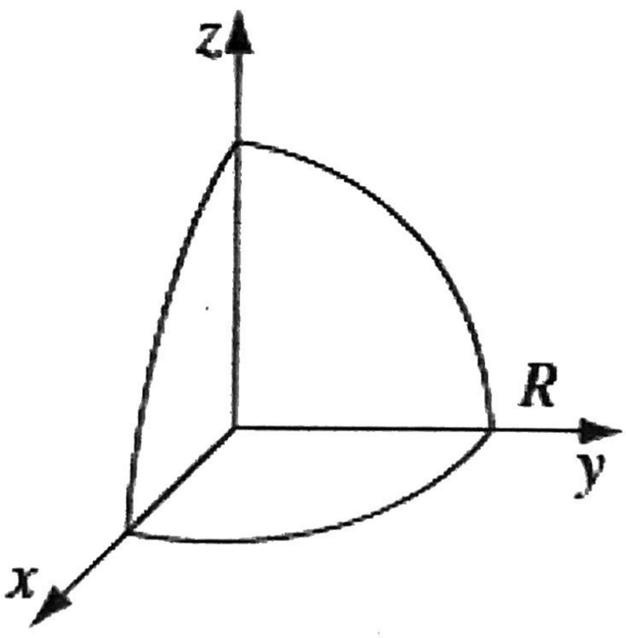
Consider an octant of a sphere of radius $R$ as shown in the figure below.

b) Compute the volume integral of the divergence of the above vector function through the octant of the sphere. Namely, compute

$$\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{v})\, d\tau. \tag{6}$$

c) Compute the flux of the above vector function through the surface of the octant of the sphere, as shown in the figure below. Namely, compute

$$\int_{S} \vec{v}\cdot d\vec{a}. \tag{7}$$



# PHYS 311 EXAM 1 SOLUTIONS

$\vec{A} = (3x^2)\hat{x} + (2z^3)\hat{z}$

$\vec{B} = (2x^2z)\hat{y} + (xy^2)\hat{z}$

a) $\vec{A}\cdot\vec{B} = \left[(3x^2)\hat{x} + (2z^3)\hat{z}\right]\cdot\left[(2x^2z)\hat{y} + (xy^2)\hat{z}\right] = 2xy^2z^3$ ✓

so

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right](2xy^2z^3) =$$

$$= \hat{x}(2y^2z^3) + \hat{y}(4xyz^3) + \hat{z}(6xy^2z^2)$$ – A VECTOR QUANTITY

b) $$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2x^2z & xy^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(xy^2) - \frac{\partial}{\partial z}(2x^2z)\right) + \hat{y}\left(0 - \frac{\partial}{\partial x}(xy^2)\right) + \hat{z}\left(\frac{\partial}{\partial x}(2x^2z) - 0\right)$$

$$= \hat{x}(2xy - 2x^2) - y^2\hat{y} + 4xz\hat{z}$$

so

$$\vec{\nabla}\times\vec{B} = 2x(y-x)\hat{x} - y^2\hat{y} + (4xz)\hat{z}$$

now

$$\vec{A}\cdot(\vec{\nabla}\times\vec{B}) = \left[(3x^2)\hat{x} + (2z^3)\hat{z}\right]\cdot\left[2x(y-x)\hat{x} - y^2\hat{y} + 4xz\hat{z}\right]$$

$$= 3x^2\cdot 2x(y-x) + 2z^3\cdot 4xz = 6x^3(y-x) + 8xz^4$$ – A SCALAR QUANTITY

2. $\vec{v}(x,y,z) = (2xz+3y^2)\hat{y} + (4yz^2)\hat{z}$

a) $\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial y}(2xz+3y^2) + \frac{\partial}{\partial z}(4yz^2) = 6y + 8yz$ ✓

b) $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & (2xz+3y^2) & 4yz^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xz+3y^2)\right) + \hat{y}\left(0 - \cancel{\frac{\partial}{\partial x}(4yz^2)}^{\to 0}\right)$

$+\hat{z}\left(\frac{\partial}{\partial x}(2xz+3y^2) - 0\right)$

$= (4z^2-2x)\hat{x} + (2z)\hat{z}$ ✓

c) NOW $d\vec{a} = dy\,dz\,\hat{x}$ ✓ so $(\vec{\nabla}\times\vec{v})\cdot d\vec{a}\Big|_{x=0} = (4z^2-2x)\,dy\,dz\Big|_{x=0}$ ✓ $= 4z^2\,dy\,dz$

so $\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_0^1\int_0^2 4z^2\,dy\,dz = 4\int_0^2 dy \int_0^1 z^2\,dz = 4y\Big|_0^2 \frac{z^3}{3}\Big|_0^1$

$= 4\cdot 2\cdot \frac{1}{3} = \frac{8}{3}$ ✓

d) ALONG (i)

$d\vec{l} = dy\,\hat{y}$ so $\vec{v}\cdot d\vec{l}\Big|_{x=z=0}$ ✓ $= (2xz+3y^2)\,dy = 3y^2\,dy$ ✓ so $\int_{(i)} \vec{v}\cdot d\vec{l} = \int_0^2 3y^2\,dy = y^3\Big|_0^2 = 8$ ✓

ALONG (ii)

$d\vec{l} = dz\,\hat{z}$ so $\vec{v}\cdot d\vec{l}\Big|_{\substack{x=0\\y=2}} = (4yz^2)\,dz\Big|_{\substack{x=0\\y=2}}$ ✓ $= 8z^2\,dz$ so $\int_{(ii)} \vec{v}\cdot d\vec{l} = \int_0^1 8z^2\,dz = \frac{8z^3}{3}\Big|_0^1 = \frac{8}{3}$ ✓

ALONG (iii)

$d\vec{l} = dy\,\hat{y}$ so $\vec{v}\cdot d\vec{l}\Big|_{\substack{x=0\\z=1}} = (2xz+3y^2)\,dy\Big|_{\substack{x=0\\z=1}} = 3y^2\,dy$ so $\int_{(iii)} \vec{v}\cdot d\vec{l} = \int_2^0 3y^2\,dy = -8$ ✓

Along (iv)

$$d\vec{l} = dz\,\hat{z} \quad \text{so} \quad \vec{v}\cdot d\vec{l}\Big|_{x=0=y} = 4yz^2\,dz\Big|_{x=y=0} = 0 \quad \therefore \int_{(iv)} \vec{v}\cdot d\vec{l} = 0 \checkmark$$

so

$$\oint \vec{v}\cdot d\vec{l} = 8 + \frac{8}{3} - 8 + 0 = \frac{8}{3}$$

3. $\vec{v}(r,\theta,\varphi) = (r^2\cos\theta)\hat{r} + (r^2\cos\varphi)\hat{\theta} - (r^2\cos\theta\sin\varphi)\hat{\varphi}$

a) $\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r^2\cos\varphi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(-r^2\cos\theta\sin\varphi)$

$= \frac{1}{r^2}\cdot 4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\varphi + \frac{1}{r\sin\theta}\cdot -r^2\cos\theta\cos\varphi \quad = 4r\cos\theta$

b) now

$\int(\vec{\nabla}\cdot\vec{v})d\tau = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^R 4r\cos\theta \cdot r^2\sin\theta\, dr\, d\theta\, d\varphi = 4\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{\pi/2} d\varphi$

let
$u = \sin\theta$
$du = \cos\theta\, d\theta$

$= r^4\Big|_0^R \int_0^1 u\,du \cdot \frac{\pi}{2} = \frac{\pi R^4}{2}\cdot \frac{u^2}{2}\Big|_0^1$

$= \frac{\pi}{4}R^4$

c) now

$d\vec{a} = R^2\sin\theta\, d\theta\, d\varphi\, \hat{r}$ so $\vec{v}\cdot d\vec{a}\Big|_{r=R} = R^2\cos\theta \cdot R^2\sin\theta\, d\theta\, d\varphi = R^4\sin\theta\cos\theta\, d\theta\, d\varphi$

so

$\int_{r=R} \vec{v}\cdot d\vec{a} = \int_0^{\pi/2}\int_0^{\pi/2} R^4\sin\theta\cos\theta\, d\theta\, d\varphi = R^4\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta \int_0^{\pi/2} d\varphi = R^4\int_0^1 u\,du \cdot \frac{\pi}{2} = R^4\frac{u^2}{2}\Big|_0^1 \cdot \frac{\pi}{2}$

let
$u = \sin\theta$
$du = \cos\theta\, d\theta$

$= \frac{\pi}{4}R^4$