Ch. 18 Cosmological Models

(4-26-19)

Homogeneous : Isotropic in Space <u>not</u> spacetime

The line element for a flat, Homogeneous, Isotropic Cosmological geometry is....

$$ds^2 = -dt^2 + \underbrace{a^2(t)}_{\text{Scale factor}}(dx^2 + dy^2 + dz^2) \text{ - Flat Robertson - Walker metric}$$

- For $t = \text{const.}$ slice, 3D flat Euclidean Space
- The coordinates (x, y, z) are combining - Individual galaxy has the <u>same</u> coordinates x^2, for all time

$d_{coord} = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$ – coordinate distance (i.e Between Galaxies)

$d(t) \equiv a(t)\, d_{coord}$ - Physical Distance Intersects w/ time

The energy of a particle will change as it moves through this geometry

For a photon, $\Delta E \neq 0 \therefore \Delta\omega \neq 0$ - cosmological red shift

In polar coordinates.....

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

Choose $r = 0$ as our location
$r = R$ as Galaxy's location (r is comoving $\therefore$ $r = R$ labels same galaxy for all time)

Consider a radial light pulse emitted by distant observer @ $r = R$ $(ds^2 = d\theta^2 = d\varphi^2)$

$$0 = -dt^2 + a^2(t)\, dr^2$$

$$\int_0^R dr = \int_{t_e}^{t_o} \frac{dt}{a(t)} \quad \therefore \quad R = \int_{t_e}^{t_o} \frac{dt}{a(t)}$$

- Suppose emitter emits a series of pulses spaced by <u>Equal</u> Short-time intervals δt_e

$$\omega_e = \frac{2\pi}{\delta t_e} \text{ – Frequency @ emission} \quad : \quad \omega_o = \frac{2\pi}{\delta t_o} \text{ – Frequency @ reception}$$

$$R = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{dt}{a(t)}$$

- If δt_e & δt_o are small such $a(t_o) = a(t_o + \delta t_o)$
$a(t_e) = a(t_e + \delta t_e)$

$\Rightarrow$ Net change must vanish. $\quad \dfrac{\delta t_o}{a(t_o)} - \dfrac{\delta t_e}{a(t_e)} = 0 \quad \therefore \quad \dfrac{a(t_e)}{a(t_o)} = \dfrac{\delta t_e}{\delta t_o} = \dfrac{\delta t_e}{2\pi} \cdot \dfrac{2\pi}{\delta t_o} = \dfrac{\omega_o}{\omega_e}$

$\therefore \left[\dfrac{\omega_0}{\omega_e} = \dfrac{a(t_e)}{a(t_0)} \right] \Rightarrow$ Holds for all nonflat homogeneous isotropic models

For an expanding universe .....

$\dfrac{a(t_e)}{a(t_0)} < 1 \therefore \dfrac{\omega_0}{\omega_e} < 1 \Rightarrow$ The received frequency is less than the emitted frequency

The redshift

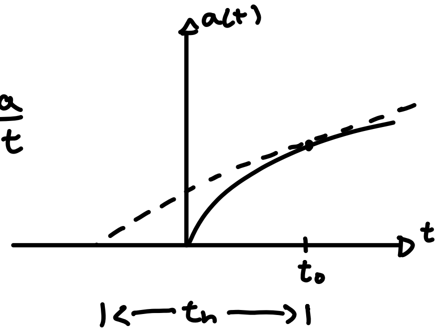
$\Rightarrow$ Cosmological redshift

$$z \equiv \frac{\Delta\lambda}{\lambda_e} = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1 \quad \therefore \left[1 + z = \frac{\lambda_0}{\lambda_e} = \frac{\omega_e}{\omega_0} = \frac{a(t_0)}{a(t_e)} \right]$$

__Hubble Constant__

$\left[H_0 = \dfrac{\dot{a}(t)}{a(t_0)} \right]$ where $\dot{a} = \dfrac{da}{dt}$

Today



$$t_H = \frac{1}{H_0} \; : \; H_0 = 72 \frac{km}{sec \cdot MPC} \times \frac{1000\,m}{1\,km} \times \frac{1\,MPC}{1 \times 10^6 PC} \times \frac{1\,PC}{3.09 \times 10^{16} m} = 2.33 \times 10^{-18} s^{-1}$$

$$t_H = \frac{1}{2.33 \times 10^{-18} s} = 4.3 \times 10^{17} s \times \frac{1\,hr}{3600\,s} \cdot \frac{1\,Day}{24\,hr} \cdot \frac{1\,year}{365\,Days} = 13.6 \times 10^9 \text{ years}$$

(4-29-19)

__Matter, Radiation : Vacuum__

1^{st} Law of Thermodynamics : For any change in a volume $d(\Delta V)$ containing a fixed # of particles, the change in the total volume is

- The work done on it minus the heat that flows out of it

The 1^{st} Law of Thermodynamics for the cosmological fluid in a volume ΔV then connects infinitesimal changes in that volume $d(\Delta V)$ to the infinitesimal changes in its total energy $d(\Delta E)$

(*) $\left[d(\Delta E) = -p\,d(\Delta V) \right]$ where the energy in the volume : $\Delta E = \rho \Delta V$

Total energy

$\Delta V_{coord} = \Delta x \Delta y \Delta z$ Define a volume in which the # of particles remain fixed

$\Delta V = a^3 \Delta x \Delta y \Delta z = a^3(t) \Delta V_{coord}$ — Physical size of the volume

(*) Becomes .....

$$\frac{d}{dt}\left(\rho a^3 \Delta V_{coord}\right) = -p\frac{d}{dt}\left(a^3 \Delta V_{coord}\right)$$

(**)

$$\left[\frac{d}{dt}\left(\rho(t)a^3(t)\right) = -p(t)\frac{d}{dt}\left(a^3(t)\right)\right]$$ - 1st Law of thermodynamics for cosmology

This can also be written as ......

$$\left[\dot{\rho}a^3 + \rho\cdot 3a^2\dot{a} = -p3a^2\dot{a}\right]\frac{1}{a^3} \longrightarrow \dot{\rho} + 3\rho\frac{\dot{a}}{a} = -3p\frac{\dot{a}}{a}$$

$(\widetilde{**})$

$$\left[\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p) = 0\right]$$

Pressure-less Matter $p_m = 0$

$\therefore$ (**) becomes ....

$$\frac{d}{dt}\left(\rho_m a^3\right) = 0$$

Rest energy in density of the galaxies

$$\therefore \rho_m(t) \propto \frac{1}{a^3(t)} \quad : \quad \rho_m(t) = \rho_m(t_0)\cdot\frac{a^3(t_0)}{a^3(t)} \quad : \quad \left[\rho_m(t) = \rho_m(t_0)\left[\frac{a(t_0)}{a(t)}\right]^3\right]$$

Radiation for a gas of blackbody @ temp T

$$\left[p_r = \frac{1}{3}\rho_r\right] \text{———} \rho_r \sim T^4$$

Inserting this into $(\widetilde{**})$ equation....

$$0 = \dot{\rho}_r + 3\left(\frac{\dot{a}}{a}\right)(\rho_r + p_r) = \dot{\rho}_r + 4\rho_r\left(\frac{\dot{a}}{a}\right)$$

$$0 = \frac{d\rho_r}{dt} + \frac{4\rho_r}{a}\frac{da}{dt}$$

$$\frac{d\rho_r}{dt} = \frac{-4\rho_r}{a}\frac{da}{dt} \longrightarrow \int\frac{d\rho_r}{\rho_r} = -4\int\frac{da}{a} \longrightarrow \ln(\rho_r(t)) = \ln(\rho_{r0}a_0^4) - 4\ln(a(t))$$

$$= \ln(\rho_{r0}a_0^4 + \ln a^{-4}(t))$$

$$= \ln\left(\rho_{r0}\frac{a_0^4}{a^4(t)}\right)$$

$$\left[\rho_r(t) = \rho_r(t_0)\left(\frac{a(t_0)}{a(t)}\right)^4\right]$$

This yields

$$\left[T(t) = T(t_0)\frac{a(t_0)}{a(t)}\right] \cdots \quad \lim_{t\to 0} a(t) \longrightarrow 0 \; : \; \lim_{t\to 0} \rho(t) \longrightarrow \infty \; : \; \lim_{t\to 0} T(t) \longrightarrow \infty$$

$\rho_{vm}(t_0) \sim 10^{-31}\, g/cm^3 \sim 10^3 \rho_r(t_0)$: $\rho_r(t_0) \sim 10^{-34}\, g/cm^3$

$\Rightarrow$ Matter dominates radiation now but the converse was true in the early universe

$$1 \approx \frac{\rho_r(t)}{\rho_m(t)} = \frac{\rho_{ro}\, a_0^4/a^4(t)}{\rho_{mo}\, a_0^3/a^3(t)} \approx 10^{-3}\frac{a(t_0)}{a(t)} \Rightarrow \frac{a(t)}{a(t_0)} \sim 10^{-3}$$

(5-1-19)

(*)
$$\frac{d}{dt}\left[\rho a^3\right] = -p\frac{d}{dt}(a^3)$$

(**)
$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad : \quad \rho_m \sim \frac{1}{a^3}, \quad \rho_r \sim \frac{1}{a^4}, \quad \rho_v \sim \rho_{V_{10}}, \quad p_m \sim o\rho, \quad p_r = \frac{1}{3}\rho, \quad p_v = -\rho$$

Vacuum energy is....

(i) Constant in Space : time $\Rightarrow \dot{\rho} = 0$

(**) Becomes.....

$\left[p_v \equiv -\rho_v\right] \Rightarrow$ Positive energy density imparts a negative pressure : $\rho_v = \frac{c^4 \Lambda}{8\pi G}$ Cosmological constant

Evolution of the Flat FRW Models

(*)
$$\left[\dot{a}^2 - \frac{8\pi}{3}\rho a^3 = 0\right]$$ - Friedman equation for a flat FRW model

$$\frac{8\pi}{3}\rho = \frac{\dot{a}^2}{a^2} \quad \therefore \quad 8\pi\rho = 3\frac{\dot{a}^2}{a^2}$$

Evaluating @ the present time....

$$\left[\frac{\dot{a}(t_0)}{a(t_0)}\right]^2 - \frac{8\pi}{3}\rho(t_0) = 0 \longrightarrow H_0^2 = \frac{8\pi}{3}\rho(t_0)$$

$$\rho(t_0) = \rho_{crit} = \frac{3H_0^2}{8\pi}$$

In SI units : $\rho_{crit} = \frac{3H_0^2}{8\pi G}$: $H_0 = 72\, \frac{km}{s \cdot Mpc} = 2.3 \times 10^{-18} s^{-1}$

$$\rho_{crit} = \frac{3}{8\pi} \cdot \frac{(2.3\times10^{-18} s^{-1})^2}{(6.67\times10^{-11} Nm^2/kg^2)} = 9.5\times10^{-27}\frac{kg}{m^3} \times \frac{1000g}{1kg} \times \left(\frac{1m}{100cm}\right)^3 = 9.5\times10^{-30}\frac{g}{cm^3}$$

$$\rho(t_0) = \rho_{crit}(t_0) = \rho_r(t_0) + \rho_m(t_0) + \rho_v(t_0) \quad : \quad 1 = \underbrace{\frac{\rho_r(t_0)}{\rho_{crit}(t_0)}}_{\Omega_r} + \underbrace{\frac{\rho_m(t_0)}{\rho_{crit}(t_0)}}_{\Omega_m} + \underbrace{\frac{\rho_v(t_0)}{\rho_{crit}(t_0)}}_{\Omega_v}$$

$$\boxed{1 = \Omega_{r_0} + \Omega_{m_0} + \Omega_{v_0}}$$ — Relative Fractions at the present.

For general time t ....

$$\rho(t) = \rho_v(t) + \rho_m(t) + \rho_r(t)$$
$$= \rho_v(t_0) + \frac{\rho_m(t_0)}{a^3} + \frac{\rho_r(t)}{a^4} = \rho_{crit}\left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)$$

$\therefore \quad \rho(t) = \rho_{crit}\left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)$: Plugging this into the Friedman equation$^{(*)}$ yields

$$\dot{a}^2 - \underbrace{\frac{8\pi}{3}\rho_{crit}}_{H_0^2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right) = 0$$

Dividing by $2H_0^2$.....

$$\left[\frac{1}{2H_0^2}\dot{a}^2 - \frac{1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right) = 0, \text{ where } V_{eff}(a) = \frac{-1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right)\right]$$

For a matter dominated universe.....

$$\Omega_m = 1, \; \Omega_r = \Omega_v = 0$$

(*) becomes .....

$$\frac{1}{2H_0^2}\dot{a}^2 - \frac{1}{2a} = 0 \; : \; \dot{a}^2 = \frac{H_0^2}{a} \; : \; \dot{a} = \frac{H_0}{a^{1/2}} \; : \; a^{1/2}\frac{da}{dt} = H_0 \; : \; \int a^{1/2}da = \int H_0\, dt$$

$$\frac{2}{3}a^{3/2} = H_0 t \quad \therefore \quad a = \left(\tfrac{3}{2}H_0 t\right)^{2/3} \quad \text{define } t_0 = \frac{2}{3H_0} \quad \therefore \boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3}}$$

Decelerated expansion

For a radiation dominated universe ....

$$\Omega_{r0} = 1 \; : \; \Omega_{m0} = \Omega_{v0} = 0$$

(*) Becomes .... $\frac{1}{2H_0^2}\dot{a}^2 - \frac{1}{2a^2} = 0 \; : \; \dot{a}^2 = \frac{H_0^2}{a^2} \; \therefore \; \dot{a} = \frac{H_0}{a} \; : \; \dot{a}a = H_0 \; : \; \frac{da}{dt}a = H_0$

$$\int a\, da = \int H_0\, dt \; : \; \frac{1}{2}a^2 = H_0 t \; : \; a(t) = (2H_0 t)^{1/2} \; \therefore \; \boxed{a(t) = \left(\frac{t}{t_0}\right)^{1/2}}$$
$$t_0 = \frac{1}{2H_0}$$

Decelerated expansion

For a vacuum dominated universe .....

$$\Omega_v = 1, \; \Omega_m = \Omega_r = 0$$

(*) Becomes ......

$$\frac{1}{2H_0^2}\dot{a}^2 - \frac{1}{2}a^2 = 0 \;:\; \dot{a}^2 = H_0^2 a^2 \;:\; \frac{da}{dt} = H_0 a \;:\; \int \frac{da}{a} = \int H_0 t \;\therefore\; \ln(a) = H_0(t-t_0)$$

$$\therefore \frac{\dot{a}}{a} = H_0$$

(5-3-19)

Flat FRW Cosmology

$$\dot{a}^2 - \frac{8\pi}{3}\rho a^2 = 0 \;:\; 1 = \Omega_r + \Omega_m + \Omega_v \;:\; \rho(a) = \rho_{crit}\left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right)$$

$$\frac{1}{2H_0^2}\dot{a}^2 = -\frac{1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_v}{a^2}\right) = 0$$

$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$ - Radiation - dominated

$a(t) = \left(\frac{t}{t_0}\right)^{2/3}$ - Matter - dominated

$a(t) = e^{H_0(t-t_0)}$ - Vacuum dominated

$\lim_{t\to 0} a(t) \longrightarrow 0 \;\therefore\; \lim_{t\to 0} \rho(a) \longrightarrow \infty \longleftrightarrow$ Physical Singularity

If the universe is flat, homogeneous, and isotropic then it is infinite in spacial extent. How big is the observable universe?

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Consider a radial light ray ....

$$ds^2 = 0 \;,\; d\theta = d\varphi = 0 \;:\; 0 = -dt^2 + a^2(t)dr^2$$

$$\therefore \int dr = \int_0^{t_0} \frac{dt}{a(t)} \;:\; \left[r_{Horizon}(t_0) = \int_0^{t_0}\frac{dt}{a(t)}\right]$$ — observer's particle horizon

The physical distance to the horizon @ time t is

$$\left[d_{Horizon}(t) = a(t)\,r_{Horizon}(t) = a(t)\int_0^t \frac{dt'}{a(t')}\right]$$

Physical radius not physical diameter

Spacial Curvature Robertson - Walker Metrics

In general ......

$$ds^2 = -dt^2 + a^2(t)\,d\mathcal{L}^2$$ Line element of a homogeneous, isotropic 3D space

Closed FRW Models

Suppose a unit 3-Sphere in a ficticious 4D Flat Euclidean Space

$$^{*}ds^2 = dw^2 + dx^2 + dy^2 + dz^2$$

the 3-Sphere Surface is given by .....

$$w^2 + x^2 + y^2 + z^2 = 1$$

A point on this surface is labeled (in Polar Coordinates in 4D)

$$\begin{aligned} W &= R\cos\chi & & 0 \le \chi \le \pi \\ X &= R\sin\theta\cos\varphi\sin\chi & &: 0 \le \theta \le \pi \\ Y &= R\sin\theta\sin\varphi\sin\chi & &: 0 \le \varphi \le 2\pi \\ Z &= R\cos\theta\sin\chi \end{aligned}$$

Plugging this into (*) yields

$$d\mathcal{L}^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\left[ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]\right]$$

The Volume of a spacial slice of this FRW model

$$dV = \sqrt{-g_{\chi\chi}\, g_{\theta\theta}\, g_{\varphi\varphi}}\; d\chi\, d\theta\, d\varphi = a^3(t)\sin^3\chi\,\sin\theta\, d\chi\, d\theta\, d\varphi$$

$$\begin{aligned} g_{\chi\chi} &= a^2(t) \\ g_{\theta\theta} &= a^2(t)\sin^2\chi \\ g_{\varphi\varphi} &= a^2(t)\sin^2\chi\sin^2\theta \end{aligned}$$

$$V(t) = a^3(t) = \int_0^{\pi} \sin^2\chi\, d\chi \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 2\pi^2 a^3(t) \quad : \quad V(t) = 2\pi^2 a^3(t)$$

Open FRW Model

Geometry of a Lorentz Hyperboloid

- 3 surface in a 4D Flat Spacetime : $ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$

The 3-surface is given by ...... : $-T^2 + X^2 + Y^2 + Z^2 = 1$

A point on this surface is labeled by ....

$$\begin{aligned} T &= \cosh\chi \\ X &= \sinh\chi\sin\theta\cos\varphi \\ Y &= \sinh\chi\sin\theta\sin\varphi \\ Z &= \sinh\chi\cos\theta \end{aligned}$$

$$d\mathcal{L}^2 = dx^2 + \sinh^2\chi\,(d\sigma^2 + \sin^2\sigma\, d\varphi^2)$$

This yields.....

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + \sinh^2\chi\,(d\sigma^2 + \sin^2\sigma\, d\varphi^2)\right]$$

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The General FRW Model

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \begin{Bmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{Bmatrix}(d\sigma^2 + \sin^2\sigma\, d\varphi^2)\right]$$

Another representation ..... Replace $\{\ \}$ w/ r^2

The line element becomes.....

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\sigma^2 + \sin^2\sigma\, d\varphi^2)\right] \quad \text{w/ } K = -1, 0, 1$$

Dynamics of the Universe

The Friedmann Equation (In General) is .....

(*) $$\left[\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = -k\right]$$ GR connects the time evolution of the universe to its Spacial Curvature and the Energy Density

Dividing (*) by a^2: Evaluating today $t=t_0$ yields ..... i.e $(a(t=t_0)=a_0)$

$$H_0^2 - \frac{8\pi}{3}\rho_0 = \frac{-K}{a^2} = \frac{8\pi}{3}(\rho_{crit} - \rho_0) \quad \text{where we defined } \frac{8\pi}{3}\rho_{crit} = H_0^2 \left(\rho_{crit} = \frac{3H_0^2}{8\pi}\right)$$

$\Omega > 1$ If $\rho_0 > \rho_{CRIT}$, universe is positively curved $(K=+1)$: Closed

$\Omega < 1$ If $\rho_0 < \rho_{CRIT}$, Universe is negatively curved $(K=-1)$: Open

$\Omega = 1$ If $\rho_0 = \rho_{CRIT}$, Universe is Spatially flat where $\Omega = \Omega_r + \Omega_m + \Omega_v$

(*) Becomes.....

$$\frac{\dot{a}^2}{a^2} - \frac{8\pi}{3}\rho = \frac{-k}{a^2} \quad \text{where } \rho(a) = \rho_v + \rho_m + \rho_r = \rho_{v,0}(t_0) + \rho_{m,0}\left[\frac{a(t_0)}{a(t)}\right]^3 + \rho_{r,0}\left[\frac{a(t_0)}{a(t)}\right]^4$$

$$\frac{\dot{a}^2}{a^2} - \frac{8\pi}{3}\left[\rho_v(t_0) + \rho_m(t_0)\cdot\frac{a_0^3}{a^3(t)} + \rho_r(t_0)\cdot\frac{a_0^4}{a^4(t)}\right] = \frac{-k}{a^2}$$

Evaluating now, $t=t_0$, yields

$$H_0^2 - \frac{8\pi}{3}\left[\rho_v(t_0) + \rho_m(t_0) + \rho_r(t_0)\right] = \frac{-k}{a_0^2}, \quad \text{w/ } \Omega_0 = \frac{\rho(t_0)}{\rho_{crit}}, \quad H_0^2 - \frac{8\pi}{3}\rho_{CRIT}\left[\Omega_v + \Omega_m + \Omega_r\right] = \frac{-k}{a_0^2}$$

Dividing by H_0^2.....

$$1 - \underbrace{\left[\Omega_v + \Omega_m + \Omega_r\right]}_{\Omega} = \frac{-k}{a_0^2 H_0^2} \qquad \text{Now define } \Omega_c = \frac{-k}{a_0^2 H_0^2}$$

So the Friedmann equation, evaluated at the present time, becomes .....

$$\Omega_r + \Omega_m + \Omega_v + \Omega_c = 1$$

Now, the fundamental equation @ time t .....

$$\frac{\dot{a}^2}{a^2} - \underbrace{\frac{8\pi}{3}\rho_{crit}}_{H_0^2}\left[\Omega_v + \Omega_m \frac{a_0^3}{a^3} + \Omega_r \frac{a_0^4}{a^4}\right] = \frac{-k}{a^2}$$

Dividing by $2H_0^2$ yields ......

$$\frac{1}{2H_0^2}\frac{\dot{a}^2}{a^2} - \frac{1}{2}\left[\Omega_v + \Omega_m \frac{a_0^3}{a^3} + \Omega_r \frac{a_0^4}{a^4}\right] = \frac{-k}{a^2 \cdot 2H_0^2} = \frac{-k}{a_0^2 H_0^2} \cdot \frac{a_0^2}{a^2} \cdot \frac{1}{2} = \frac{1}{2}\Omega_c \frac{a_0^2}{a^2}$$

Now let

$$\tilde{a} = \frac{a(t)}{a_0} \quad ; \quad \tilde{t} = H_0 t \quad \therefore \quad \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = \frac{a_0}{a}\frac{d}{dt}\left(\frac{a}{a_0}\right) = \frac{a_0}{a}\frac{d\tilde{t}}{dt}\cdot\frac{d}{d\tilde{t}}\left(\frac{a}{a_0}\right) = \frac{1}{\tilde{a}} H_0 \frac{d\tilde{a}}{d\tilde{t}}$$

So $\displaystyle \frac{\dot{a}^2}{a^2} = \frac{1}{\tilde{a}^2} H_0^2 \left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2$

(**) Becomes .....

$$\frac{1}{2\tilde{a}^2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 - \frac{1}{2}\left[\Omega_v + \frac{\Omega_m}{\tilde{a}^3} + \frac{\Omega_r}{\tilde{a}^4}\right] = \frac{1}{2}\frac{\Omega_c}{\tilde{a}^2}$$

Multiplying by $\tilde{a}^2$ yields .....

$$\frac{1}{2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 - \frac{1}{2}\left[\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} + \frac{\Omega_r}{\tilde{a}^2}\right] = \frac{1}{2}\Omega_c$$

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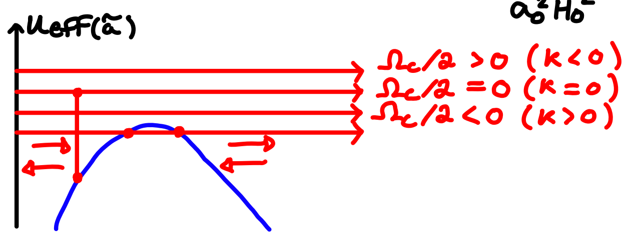
$$\frac{1}{2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 + U_{eff}(\tilde{a}) = \frac{1}{2}\Omega_c \quad \text{where } U_{eff}(\tilde{a}) = -\frac{1}{2}\left[\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} + \frac{\Omega_r}{\tilde{a}^2}\right]$$

$\Omega_v, \Omega_m, \Omega_r = 0$ Notice: $\Omega_c = \dfrac{-k}{a_0^2 H_0^2}$ where $\Omega_v + \Omega_m + \Omega_r + \Omega_c = 1$



if $\Omega_c = -1$: $\Omega_r + \Omega_m + \Omega_v = 2$

if $\Omega_c = \frac{1}{2}$: $\Omega_r + \Omega_m + \Omega_v = \frac{1}{2}$

$-1 \leq \Omega_c \leq 1$