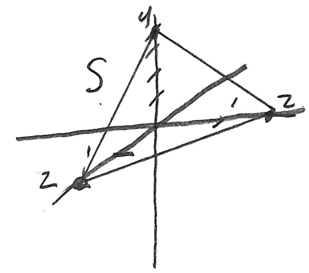
Math 360 Ch10.6 Surface Integrals: In-Class Examples

① $\vec{F} = [\cosh x, 1, \sinh y]$, $S: 4x+4y+2z=8$, $\underbrace{x \geq 0, y \geq 0, z \geq 0}_{1^{st} \text{ octant}}$

Find $\iint_S \vec{F} \cdot \vec{n}\, dA$

Soln For surface S, observe $z = 4-2x-2y$

$\therefore \vec{r}(u,v) = [u, v, 4-2u-2v]$, $u \geq 0, v \geq 0, z \geq 0$



$\vec{F}(\vec{r}) = [\cosh u, 1, \sinh v]$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = 2\vec{i} + 2\vec{j} + \vec{k}$$

$$\vec{F}(\vec{r}) \cdot \vec{N} = 2\cosh u + 2 + \sinh v$$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_S \vec{F}(\vec{r}) \cdot \vec{N}\, dv\, du = \int_0^2 \int_0^{2-u} (2\cosh u + 2 + \sinh v)\, dv\, du$$

$$= \int_0^2 (2v\cosh u + 2v + \cosh v)\Big|_0^{2-u} du$$

$$= \int_0^2 [2(2-u)\cosh u + 2(2-u) + \cosh(2-u) - 1]\, du$$

$$= \int_0^2 4\cosh u\, du - 2\int_0^2 u\cosh u\, du + \int_0^2 (4-2u)\, du - \sinh(2-u)\Big|_0^2 - u\Big|_0^2$$

⊛

$\boxed{\begin{matrix} w = u & dv = \cosh u\, du \\ dw = du & v = \sinh u \end{matrix}}$

$$= 4\sinh u\Big|_0^2 - 2\Big[u\sinh u\Big|_0^2 - \int_0^2 \sinh u\, du\Big]^{⊛} + (4u - u^2)\Big|_0^2 + \sinh(2) - 2$$

Recall

(1) $\sinh(0) = 0$

(2) $\cosh(0) = 1$

$$= 4\sinh(2) - 2[2\sinh(2) - \cosh(2) + 1] + (8-4) + \sinh(2) - 2$$

$$= 4\sinh(2) - 4\sinh(2) + 2\cosh(2) - 2 + 4 + \sinh(2) - 2$$

$$= 2\cosh(2) + \sinh(2)$$

$$\cong 11.1513$$

(2) $\vec{F} = [0, x, -z]$, S: $x^2+y^2=4$, $x \ge 0$, $y \ge 0$, $0 \le z \le 1$

Find $\iint_S \vec{F} \cdot \vec{n}\, dA$

Soln $\vec{r}(u,v) = [2\cos u, 2\sin u, v]$, $0 \le u \le \pi/2$, $0 \le v \le 2$ (See Ch.10.5)

$\vec{F}(\vec{r}(u,v)) = [0, 2\cos u, -v]$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos u\,\vec{i} + 2\sin u\,\vec{j} + 0\vec{k}$$

$\vec{F}(\vec{r}) \cdot \vec{N} = 4\sin u\cos u$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \int_0^2\int_0^{\pi/2} 4\sin u\cos u\, du\, dv = 4\int_0^2\int_0^1 u\, du\, dv = 4\int_0^2 \tfrac{1}{2}u^2\Big|_0^1 dv$$
$$= 2\int_0^2 1\, dv = 2v\Big|_0^2 = 4$$

$u = \sin u$
$du = \cos u\, du$

(3) $G(x,y,z) = x - y + z$, S: $x^2+y^2=4$, 1st octant, $0 \le z \le 1$

Find $\iint_S G(\vec{r})\, dA$

Soln $\vec{r}(u,v) = [2\cos u, 2\sin u, v]$, $0 \le u \le \pi/2$, $0 \le v \le 1$ (See #2 above, or Ch10.5)

$G(\vec{r}) = 2\cos u - 2\sin u + v$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [2\cos u, 2\sin u, 0]\ ;\ |\vec{N}| = \sqrt{4} = 2$$

$$\iint_S G(\vec{r})\, dA = \iint_S G(\vec{r}(u,v))\, |\vec{N}|\, du\, dv = \int_0^1\int_0^{\pi/2} (2\cos u - 2\sin u + v) \cdot 2\, du\, dv$$
$$= \int_0^1 4\sin u + 4\cos u + uv\Big|_0^{\pi/2} dv = \int_0^1 \left(4 - 0 + \tfrac{\pi}{2}v\right) - (0 + 4 + 0)\, dv$$
$$= \int_0^1 \tfrac{\pi}{2} v\, dv = \tfrac{\pi}{4}v^2\Big|_0^1 = \frac{\pi}{4}$$