Phys 311 Homework Set 6

1) From Homework Set 5, we found the $\vec{E}$-field in each region...

Now electric potential, in terms of electric field, is..

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$\vec{E} = 0 \quad \text{for} \quad r<a$$
$$= \frac{k}{(\pi+3)\varepsilon_0}\left[r^{\pi+2} - \frac{a^{\pi+3}}{r^2}\right]\hat{r} \quad \text{for} \quad a\le r\le b$$
$$= \frac{k}{(\pi+3)\varepsilon_0}\left[b^{\pi+3} - a^{\pi+3}\right]\frac{\hat{r}}{r^2} \quad \text{for} \quad r>b$$

$$V(r=0) = -\int_{\infty}^{b}\frac{k}{(\pi+3)\varepsilon_0}\left[b^{\pi+3}-a^{\pi+3}\right]\frac{\hat{r}}{r^2}\cdot dr\,\hat{r} - \int_{b}^{a}\frac{k}{(\pi+3)\varepsilon_0}\left[r^{\pi+1}-\frac{a^{\pi+3}}{r^2}\right]\hat{r}\cdot dr\,\hat{r} - \int_{a}^{0} 0\, dr$$

$$= -\frac{k}{(\pi+3)\varepsilon_0}\left[\left[b^{\pi+3}-a^{\pi+3}\right]\int_{\infty}^{b}\frac{dr}{r^2} + \int_{b}^{a}\left[r^{\pi+1}-\frac{a^{\pi+3}}{r^2}\right]dr\right]$$

$$= \frac{-k}{(\pi+3)\varepsilon_0}\left[\left[b^{\pi+3}-a^{\pi+3}\right]\cdot -\frac{1}{r}\Big|_{\infty}^{b} + \frac{r^{\pi+2}}{(\pi+2)}\Big|_{b}^{a} - a^{\pi+3}\cdot -\frac{1}{r}\Big|_{b}^{a}\right]$$

$$= \frac{-k}{(\pi+3)\varepsilon_0}\left[-\left[b^{\pi+3}-a^{\pi+3}\right]\cdot\frac{1}{b} + \frac{1}{(\pi+2)}\left[a^{\pi+2}-b^{\pi+2}\right] + a^{\pi+3}\left[\frac{1}{a}-\frac{1}{b}\right]\right]$$

$$= -\frac{k}{(\pi+3)\varepsilon_0}\left[-b^{\pi+2} + \cancel{\frac{a^{\pi+3}}{b}} + \frac{1}{(\pi+2)}\left[a^{\pi+2}-b^{\pi+2}\right] + a^{\pi+2} - \cancel{\frac{a^{\pi+3}}{b}}\right]$$

$$= \frac{-k}{(\pi+3)\varepsilon_0}\left[-b^{\pi+2}\left(1+\frac{1}{\pi+2}\right) + a^{\pi+2}\left(\frac{1}{\pi+2}+1\right)\right] = \frac{-k}{(\pi+3)\varepsilon_0}\cdot\left(a^{\pi+2}-b^{\pi+2}\right)\cdot\frac{(\pi+3)}{(\pi+2)}$$

$$= -\frac{k}{(\pi+2)\varepsilon_0}\left(a^{\pi+2}-b^{\pi+2}\right)$$

$$\therefore \boxed{V(r=0) = \frac{k}{(\pi+2)\varepsilon_0}\left(b^{\pi+2}-a^{\pi+2}\right)}$$

2 FROM HOMEWORK SET 5, WE FOUND THE $\vec{E}$-FIELD IN EACH REGION...

$$\vec{E} = \frac{\rho_0}{3\varepsilon_0}\frac{s^2}{a}\hat{s} \quad \text{for} \quad s<a$$

$$= \frac{\rho_0}{3\varepsilon_0}\frac{a^2}{s}\hat{s} \quad \text{for} \quad a<s<b$$

$$= 0 \quad \text{for} \quad s>b$$

THE POTENTIAL DIFFERENCE BETWEEN A POINT ON AXIS AND A POINT ON THE OUTER CYLINDER IS FOUND VIA...

$$V(s=0) - V(s=b) = -\int_b^0 \vec{E}\cdot d\vec{l} = -\int_b^a \vec{E}\cdot d\vec{l} - \int_a^0 \vec{E}\cdot d\vec{l}$$

$$= -\int_b^a \frac{\rho_0}{3\varepsilon_0}\frac{a^2}{s}\hat{s}\cdot ds\,\hat{s} - \int_a^0 \frac{\rho_0}{3\varepsilon_0}\frac{s^2}{a}\hat{s}\cdot ds\,\hat{s}$$

$$= -\frac{\rho_0}{3\varepsilon_0}\left[a^2\int_b^a \frac{ds}{s} + \frac{1}{a}\int_a^0 s^2 ds\right] = -\frac{\rho_0}{3\varepsilon_0}\left[a^2 \ln s\Big|_b^a + \frac{1}{a}\cdot\frac{s^3}{3}\Big|_a^0\right]$$

$$= -\frac{\rho_0}{3\varepsilon_0}\left[a^2(\ln a - \ln b) - \frac{1}{a}\cdot\frac{a^3}{3}\right] = -\frac{\rho_0}{3\varepsilon_0}\left[a^2\ln\left(\frac{a}{b}\right) - \frac{1}{3}a^2\right]$$

$$= \frac{\rho_0}{3\varepsilon_0}a^2\left[\frac{1}{3} - \ln\left(\frac{a}{b}\right)\right] = \frac{\rho_0}{3\varepsilon_0}a^2\left[\frac{1}{3} + \ln\left(\frac{b}{a}\right)\right]$$

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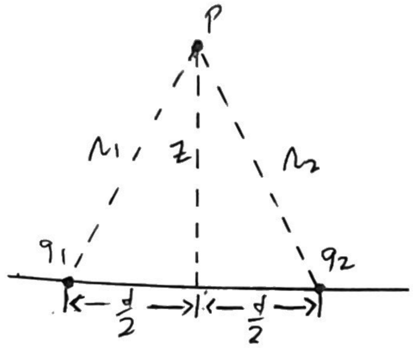


$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{2}\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r_1}+\frac{q_2}{r_2}\right] = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2q}{r}$$

now $q_1 = q_2 = q$

$r_1 = r_2 = \sqrt{z^2 + d^2/4}$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2q}{\sqrt{z^2+d^2/4}}}$$

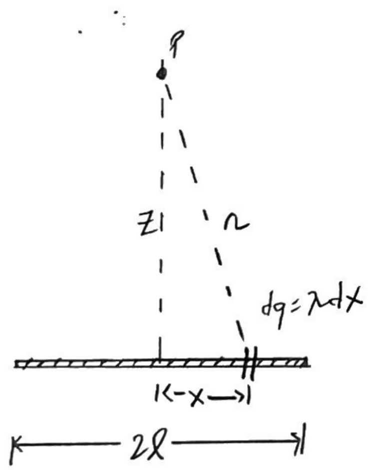
now

$$\vec{E} = -\vec{\nabla}V = -\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\cdot\frac{1}{4\pi\varepsilon_0}2q\left(z^2+d^2/4\right)^{-1/2}$$

$$= -\frac{2q}{4\pi\varepsilon_0}\hat{z}\frac{\partial}{\partial z}\left(z^2+d^2/4\right)^{-1/2}$$

$$= \frac{-2q}{4\pi\varepsilon_0}\hat{z}\cdot -\frac{1}{2}\left(z^2+d^2/4\right)^{-3/2}\cdot 2z$$

$$\boxed{\vec{E} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{2z}{\left(z^2+d^2/4\right)^{3/2}}\hat{z}}$$



now $r = \sqrt{x^2+z^2}$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{+L}\frac{\lambda(\vec{r}')\,dl'}{r}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-L}\frac{\lambda\,dx}{\sqrt{x^2+z^2}} = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dx}{\sqrt{x^2+z^2}}$$

let $x = z\sinh\theta$

$dx = z\cosh\theta\,d\theta$

even integral

$$= \frac{2\lambda}{4\pi\varepsilon_0}\int_0^{\sinh^{-1}(L/z)}\frac{z\cosh\theta\,d\theta}{\sqrt{z^2(1+\sinh^2\theta)}} = \frac{\lambda}{2\pi\varepsilon_0}\theta\Big|_0^{\sinh^{-1}(L/z)}$$

$$\boxed{V(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\sinh^{-1}(L/z)}$$

now

$$\vec{E} = -\vec{\nabla}V = -\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\frac{\lambda}{2\pi\varepsilon_0}\sinh^{-1}(L/z) = \frac{-\lambda}{2\pi\varepsilon_0}\hat{z}\frac{\partial}{\partial z}\sinh^{-1}\left(\frac{L}{z}\right)$$

now

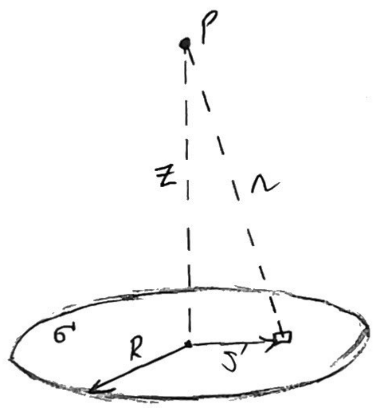
$$\frac{\partial}{\partial z}\sinh^{-1}\left(\frac{L}{z}\right) = -\frac{L}{z^2\sqrt{\frac{L^2}{z^2}+1}}$$ from WOLFRAMALPHA

$$= -\frac{L}{z\sqrt{L^2+z^2}}$$

so

$$\vec{E} = \frac{-\lambda}{2\pi\varepsilon_0}\hat{z}\cdot -\frac{L}{z\sqrt{L^2+z^2}} = \frac{\lambda L}{2\pi\varepsilon_0}\cdot\frac{1}{z\sqrt{L^2+z^2}}\hat{z}$$ now $Q=\lambda(2L)$

$$\left[\vec{E} = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{z\sqrt{L^2+z^2}}\hat{z}\right]$$



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\vec{r}')}{ɹ}da'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{\sigma\, s'd\varphi'ds'}{\sqrt{s'^2+z^2}} = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}d\varphi'\int_0^R\frac{s'ds'}{\sqrt{s'^2+z^2}}$$

let $u = s'^2+z^2$

$du = 2s'ds'$

$$= \frac{\sigma}{4\pi\varepsilon_0}\cdot 2\pi\cdot\frac{1}{2}\int_{z^2}^{R^2+z^2}\frac{du}{u^{1/2}}$$

$$= \frac{\sigma}{4\varepsilon_0}\cdot\frac{u^{1/2}}{\frac{1}{2}}\Bigg|_{z^2}^{R^2+z^2} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-z\right]$$

so $\left[V(\vec{r}) = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-z\right]\right]$

now

$$\vec{E} = -\vec{\nabla}V = -\left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-z\right] = -\frac{\sigma}{2\varepsilon_0}\hat{z}\frac{\partial}{\partial z}\left[\sqrt{R^2+z^2}-z\right]$$

$$= -\frac{\sigma}{2\varepsilon_0}\hat{z}\left[\frac{1}{2}(R^2+z^2)^{-1/2}\cdot 2z - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}$$

$$\left[\vec{E}(z) = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{\sqrt{z^2+R^2}}\right]\hat{z}\right]$$

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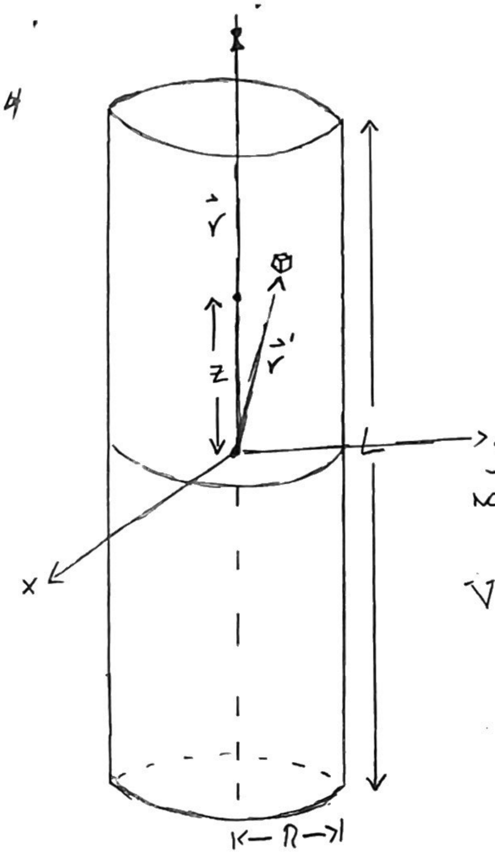


now $\vec{r} = z\hat{z}$
$\vec{r}\,' = s'\hat{s}' + z'\hat{z}'$

so $\vec{\boldsymbol{\mathscr{r}}} = z\hat{z} - (s'\hat{s}' + z'\hat{z}')$
$= (z-z')\hat{z} - s'\hat{s}'$

so

$$\mathscr{r}^2 = \vec{\mathscr{r}}\cdot\vec{\mathscr{r}} = (z-z')^2 + s'^2 \quad \therefore \quad \mathscr{r} = \sqrt{(z-z')^2 + s'^2}$$

now the electric potential is... uniform charge density

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\rho_0 \int_{-\frac{L}{2}}^{\frac{L}{2}}\int_0^R\int_0^{2\pi} \frac{s'\,d\varphi'\,ds'\,dz'}{\sqrt{(z-z')^2 + s'^2}}$$

$$= \frac{\rho_0}{4\pi\varepsilon_0}\int_{-L/2}^{+L/2} dz' \int_0^R \frac{s'\,ds'}{\sqrt{(z-z')^2+s'^2}} \int_0^{2\pi} d\varphi'$$

let $u = (z-z')^2 + s'^2$

$du = 2s'\,ds'$

so

$$V(z) = \frac{\rho_0}{4\pi\varepsilon_0}\cdot 2\pi \int_{-L/2}^{L/2} dz' \cdot \frac{1}{2}\int_{(z-z')^2}^{(z-z')^2+R^2} \frac{du}{u^{1/2}} = \frac{\rho_0}{4\varepsilon_0}\int_{-L/2}^{L/2} dz' \cdot \frac{u^{1/2}}{1/2}\Bigg|_{(z-z')^2}^{(z-z')^2+R^2}$$

$$= \frac{\rho_0}{2\varepsilon_0}\int_{-L/2}^{L/2} dz' \left[\sqrt{(z-z')^2+R^2} - (z-z')\right] = -\frac{\rho_0}{2\varepsilon_0}\int_{z+L/2}^{z-L/2} du \left[\sqrt{u^2+R^2} - u\right]$$

let $u = z - z'$

$du = -dz'$

from WolframAlpha..

$$\int du\sqrt{u^2+R^2} = \frac{1}{2}\left[u\sqrt{R^2+u^2} + R^2\ln\left(\sqrt{R^2+u^2}+u\right)\right]$$

so

$$V(z) = -\frac{\rho_0}{2\varepsilon_0}\left[\cdot\frac{1}{2}\left[u\sqrt{R^2+u^2} + R^2\ln\left(\sqrt{R^2+u^2}+u\right)\right]\right]\Bigg|_{z+L/2}^{z-L/2} - \frac{u^2}{2}\Bigg|_{z+L/2}^{z-L/2}$$

$$V(z) = -\frac{\rho_0}{4\varepsilon_0}\Bigg[\Big[(z-L/2)\sqrt{R^2+(z-L/2)^2} + R^2\ln\left(\sqrt{R^2+(z-L/2)^2} + (z-L/2)\right)$$
$$-(z+L/2)\sqrt{R^2+(z+L/2)^2} - R^2\ln\left(\sqrt{R^2+(z+L/2)^2} + (z+L/2)\right)\Big]$$
$$-(z-L/2)^2 + (z+L/2)^2\Bigg]$$

$$= -\frac{\rho_0}{4\varepsilon_0}\Bigg[(z-L/2)\sqrt{R^2+(z-L/2)^2} - (z+L/2)\sqrt{R^2+(z+L/2)^2}$$
$$+R^2\ln\left(\frac{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}\right) + \cancel{z^2} + zL + \cancel{L^2/4} - (\cancel{z^2} - zL + \cancel{L^2/4})\Bigg]$$

$$= -\frac{\rho_0}{4\varepsilon_0}\Bigg[z\left(\sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2}\right) - \frac{L}{2}\left(\sqrt{R^2+(z-L/2)^2} + \sqrt{R^2+(z+L/2)^2}\right)$$
$$+R^2\ln\left(\frac{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}\right) + 2zL\Bigg]$$

Now

$$\vec{E} = -\vec{\nabla}V = -\hat{z}\frac{\partial}{\partial z}V$$

$$= \frac{\rho_0}{4\varepsilon_0}\hat{z}\frac{\partial}{\partial z}\Bigg[z\left(\sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2}\right) - \frac{L}{2}\left(\sqrt{R^2+(z-L/2)^2} + \sqrt{R^2+(z+L/2)^2}\right)$$
$$+R^2\ln\left(\frac{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}\right) + 2zL\Bigg]$$

Calculating the derivatives in pieces..

$$\frac{\partial}{\partial z}\left[z\left(\sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2}\right)\right] = \sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2}$$
$$+z\left[\frac{1}{2}\frac{2(z-L/2)}{\sqrt{R^2+(z-L/2)^2}} - \frac{1}{2}\cdot\frac{2(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}\right]$$

$$= \sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2} + \frac{z(z-L/2)}{\sqrt{R^2+(z-L/2)^2}} - \frac{z(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}$$

Whereas

$$\frac{\partial}{\partial z}\left(\sqrt{R^2+(z-L/2)^2}+\sqrt{R^2+(z+L/2)^2}\right) = \frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}} + \frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}$$

Whereas

$$\frac{\partial}{\partial z} \ln f(z) = \frac{f'(z)}{f(z)}$$

where $f(z) = \dfrac{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}$

now

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left(\left(\sqrt{R^2+(z-L/2)^2}+(z-L/2)\right)\left(\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right)^{-1}\right)$$

$$= \left[\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}}+1\right]\left(\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right)^{-1}$$

$$-\left(\sqrt{R^2+(z-L/2)^2}+(z-L/2)\right)\left(\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right)^{-2}\left[\frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}+1\right]$$

so

$$\frac{\partial}{\partial z}\ln f(z) = \frac{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}{\sqrt{R^2+(z-L/2)^2}+(z-L/2)} \cdot \left(\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right)^{-2} \cdot$$

$$\left[\left(\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}}+1\right)\left(\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right)-\left(\frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}+1\right)\left(\sqrt{R^2+(z-L/2)^2}+(z-L/2)\right)\right]$$

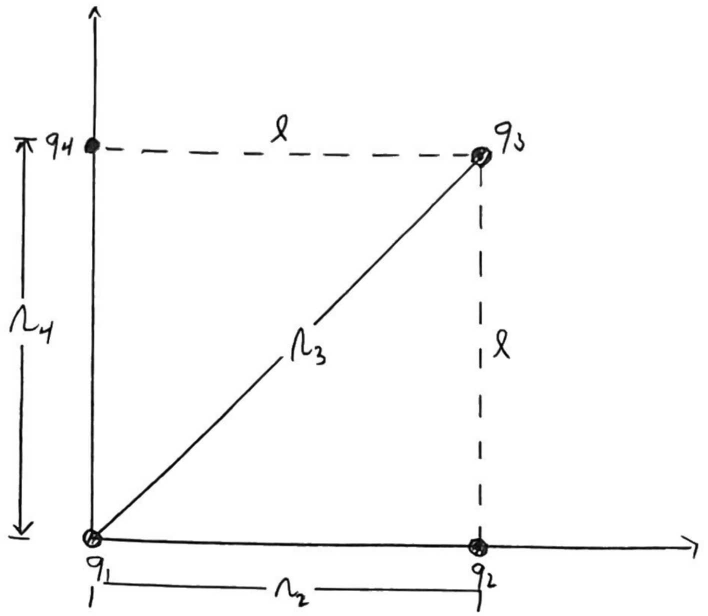
$$= \left(\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}}+1\right)\cdot\frac{1}{\left[\sqrt{R^2+(z-L/2)^2}+(z-L/2)\right]} - \left(\frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}+1\right)\cdot\frac{1}{\left[\sqrt{R^2+(z+L/2)^2}+(z+L/2)\right]}$$

SO PUTTING IT ALL TOGETHER...

$$
\begin{aligned}
\vec{E} = \frac{\rho_0}{4\varepsilon_0}\hat{z}\Bigg[& \sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2} + \frac{z(z-L/2)}{\sqrt{R^2+(z-L/2)^2}} - \frac{z(z+L/2)}{\sqrt{R^2+(z+L/2)^2}} \\
& - \frac{L}{2}\left[\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}} + \frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}\right] + \frac{R^2}{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}\left[\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}}+1\right] \\
& - \frac{R^2}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}\left[\frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}+1\right] + 2L \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{\rho_0}{4\varepsilon_0}\hat{z}\Bigg[& 2L + \sqrt{R^2+(z-L/2)^2} - \sqrt{R^2+(z+L/2)^2} + \frac{(z-L/2)^2}{\sqrt{R^2+(z-L/2)^2}} - \frac{(z+L/2)^2}{\sqrt{R^2+(z+L/2)^2}} \\
& + \frac{R^2}{\sqrt{R^2+(z-L/2)^2}+(z-L/2)}\left[\frac{(z-L/2)}{\sqrt{R^2+(z-L/2)^2}}+1\right] - \frac{R^2}{\sqrt{R^2+(z+L/2)^2}+(z+L/2)}\left[\frac{(z+L/2)}{\sqrt{R^2+(z+L/2)^2}}+1\right]\Bigg]
\end{aligned}
$$

5.



$q_1 = q$
$q_2 = 2q$
$q_3 = -q$
$q_4 = 3q$

a) FIRST, CALCULATE THE POTENTIAL AT THE ORIGIN, WHERE CHARGE q_1 IS, DUE TO CHARGES $q_2, q_3, \because q_4$

$$V(0,0) = \frac{1}{4\pi\varepsilon_0} \sum_{i=2}^{4} \frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_2}{r_2} + \frac{q_3}{r_3} + \frac{q_4}{r_4}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{2q}{l} - \frac{q}{\sqrt{2}\,l} + \frac{3q}{l}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left[2 - \frac{1}{\sqrt{2}} + 3\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left[5 - \frac{1}{\sqrt{2}}\right]$$

NOW, THE WORK TO BRING q_1 IN FROM INFINITY TO THE ORIGIN IS..

$$W = q_1 V(0,0) = q\,V(0,0)$$

$$\boxed{W = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left(5 - \frac{1}{\sqrt{2}}\right)}$$

b) TO CALCULATE THE TOTAL WORK TO ASSEMBLE THE FOUR CHARGES IS CALCULATED VIA...

$$W = \frac{1}{2}\sum_{i=1}^{4} q_i V(\vec{r}_i) = \frac{1}{2}\left[q_1 V(\vec{r}_1) + q_2 V(\vec{r}_2) + q_3 V(\vec{r}_3) + q_4 V(\vec{r}_4)\right]$$

WHERE

$$V(\vec{r}_1) = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left[5 - \frac{1}{\sqrt{2}}\right]$$

$$V(\vec{r}_2) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{l} + \frac{q_3}{l} + \frac{q_4}{\sqrt{2}\,l}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{l} - \frac{q}{l} + \frac{3q}{\sqrt{2}\,l}\right] = \frac{1}{4\pi\varepsilon_0}\cdot\frac{3q}{\sqrt{2}\,l}$$

$$V(\vec{r}_3) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{\sqrt{2}\,l} + \frac{q_2}{l} + \frac{q_4}{l}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{2}\,l} + \frac{2q}{l} + \frac{3q}{l}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(5 + \frac{1}{\sqrt{2}}\right)$$

$$V(\vec{r}_4) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_3}{l} + \frac{q_1}{l} + \frac{q_2}{\sqrt{2}l}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{l} + \frac{q}{l} + \frac{2q}{\sqrt{2}l}\right] = \frac{1}{4\pi\varepsilon_0}\sqrt{2}\frac{q}{l}$$

so

$$W = \frac{1}{2}\left[q_1 \cdot \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\left(5 - \frac{1}{\sqrt{2}}\right) + q_2 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{3q}{\sqrt{2}l} + q_3 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{l}\left(5 + \frac{1}{\sqrt{2}}\right) + q_4 \cdot \frac{1}{4\pi\varepsilon_0}\sqrt{2}\frac{q}{l}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0}\left[q \cdot \frac{q}{l}\left(5 - \frac{1}{\sqrt{2}}\right) + 2q \cdot \frac{3q}{\sqrt{2}l} - q \cdot \frac{q}{l}\left(5 + \frac{1}{\sqrt{2}}\right) + 3q \cdot \sqrt{2}\frac{q}{l}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{q^2}{l}\left[\left(\cancel{5} - \frac{1}{\sqrt{2}}\right) + \frac{6}{\sqrt{2}} - \left(\cancel{5} + \frac{1}{\sqrt{2}}\right) + 3\sqrt{2}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left[-\frac{2}{\sqrt{2}} + \frac{6}{\sqrt{2}} + 3\sqrt{2}\right] = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0}\frac{q^2}{l}\left[-\sqrt{2} + 3\sqrt{2} + 3\sqrt{2}\right]$$

$$W = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0}\frac{q^2}{l} \cdot 5\sqrt{2} \;]$$

6.

CONSIDER THE UNIFORMLY CHARGED SPHERE OF CHARGE q : RADIUS R

$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$

OUTSIDE THE SPHERE, THE $\hat{E}$-FIELD : ELECTRIC POTENTIAL ARE SAME AS THAT OF A POINT CHARGE...

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{for } r > R$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \qquad \text{for } r > R$$

TO FIND THE $\vec{E}$-FIELD : ELECTRIC POTENTIAL INSIDE THE SPHERE, CHOOSE A CONCENTRIC GAUSSIAN SPHERE W/ $r<R$

$$Q_{enc} = \int \rho\, d\tau = \rho\int d\tau = \rho\cdot\frac{4}{3}\pi r^3 = \frac{q}{\frac{4}{3}\pi R^3}\cdot\frac{4}{3}\pi r^3 = q\frac{r^3}{R^3}$$

$$\oint \vec{E}\cdot d\vec{a} = E\,4\pi r^2$$

so

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{YIELDS} \quad E\,4\pi r^2 = \frac{1}{\varepsilon_0}\frac{q r^3}{R^3} \quad \therefore \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q r}{R^3}\hat{r} \qquad \text{for } r<R$$

WHEREAS THE ELECTRIC POTENTIAL IS..

$$V(r<R) = -\int_{\infty}^{R}\vec{E}\cdot d\vec{l} - \int_{R}^{r}\vec{E}\cdot d\vec{l} = \frac{-q}{4\pi\varepsilon_0}\left[\int_{\infty}^{R}\frac{dr}{r^2} + \frac{1}{R^3}\int_{R}^{r} r\,dr\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\Big|_{\infty}^{R} + \frac{1}{R^3}\frac{r^2}{2}\Big|_{R}^{r}\right] = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{R} + \frac{1}{2R^3}(r^2-R^2)\right] = -\frac{q}{4\pi\varepsilon_0}\left[-\frac{3}{2R} + \frac{r^2}{2R^3}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{2R}\left[3 - \frac{r^2}{R^2}\right]$$

so

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q r}{R^3}\hat{r} \qquad \text{for } r<R$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left[3 - \frac{r^2}{R^2}\right] \qquad \text{for } r<R$$

NOW $\quad$ ← UNIFORMLY CHARGED $\therefore \rho$ IS CONSTANT

a) $W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\rho \int V d\tau$: NOTICE THAT I NEED TO INTEGRATE OVER THE REGION OF SPACE

$$= \frac{1}{2}\frac{q}{\frac{4}{3}\pi R^3}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left[3-\frac{r^2}{R^2}\right]r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{1}{2}\cdot\frac{q}{\frac{4}{3}\pi R^3}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{2R}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_0^{R}\left[3r^2-\frac{1}{R^2}r^4\right]dr$$

$$= \frac{1}{2}\cdot\frac{q}{\frac{4}{3}\pi R^3}\cdot\frac{1}{\cancel{4\pi}\varepsilon_0}\cdot\frac{q}{2R}\cdot\cancel{4\pi}\left[r^3-\frac{r^5}{5R^2}\right]\Big|_0^R = \frac{3}{16\pi\varepsilon_0}\frac{q^2}{R^4}\left[R^3-\frac{1}{5}R^3\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3}{4}\frac{q^2}{R^4}\cdot\frac{4}{5}R^3 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{3}{5}\frac{q^2}{R}$$

b) $W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\left[\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{R^6}r^2 r^2\sin\theta\, dr\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{r^4}r^2\sin\theta\, dr\, d\theta\, d\phi\right]$

$$= \frac{\varepsilon_0}{2}\cdot\frac{1}{16\pi^2\varepsilon_0^2}q^2\left[\frac{1}{R^6}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_0^{R}r^4 dr + \int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int_R^{\infty}\frac{dr}{r^2}\right]$$

$$= \frac{1}{32\pi^2\varepsilon_0}q^2\cdot 4\pi\left[\frac{1}{R^6}\cdot\frac{r^5}{5}\Big|_0^R - \frac{1}{r}\Big|_R^{\infty}\right] = \frac{1}{8\pi\varepsilon_0}q^2\left[\frac{1}{5R}+\frac{1}{R}\right] = \frac{1}{8\pi\varepsilon_0}q^2\cdot\frac{6}{5R}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{3}{5}\frac{q^2}{R}$$

c) $W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{a}\right)$

NOW

$$\int_V E^2 d\tau = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 q^2\left[\frac{1}{R^6}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}r^2\cdot r^2\sin\theta\, dr\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}\int_R^{a}\frac{1}{r^4}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi\right]$$

$$= \frac{q^2}{16\pi^2\varepsilon_0^2}\cdot 4\pi\left[\frac{1}{R^6}\cdot\frac{r^5}{5}\Big|_0^R - \frac{1}{r}\Big|_R^a\right] = \frac{q^2}{4\pi\varepsilon_0^2}\left[\frac{1}{5R}-\left(\frac{1}{a}-\frac{1}{R}\right)\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0^2}\left[\frac{6}{5R}-\frac{1}{a}\right]$$

$$V(r=a,\ r>R) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{a}$$

$$\vec{E}(r=a,\ r>R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\hat{r}$$

$$\oint V\vec{E}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\varepsilon_0}\frac{q}{a}\cdot\frac{1}{4\pi\varepsilon_0}\frac{q}{a^2}\hat{r}\cdot a^2\sin\theta\, d\theta\, d\varphi\, \hat{r} = \frac{q^2}{16\pi^2\varepsilon_0^2}\frac{1}{a}\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\, d\theta$$

$$= \frac{q^2}{16\pi^2\varepsilon_0^2}\cdot\frac{1}{a}\,4\pi = \frac{q^2}{4\pi\varepsilon_0^2}\cdot\frac{1}{a}$$

$$W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S V\vec{E}\cdot d\vec{a}\right) = \frac{\varepsilon_0}{2}\left[\frac{q^2}{4\pi\varepsilon_0^2}\left[\frac{6}{5R} - \cancel{\frac{1}{a}}\right] + \frac{q^2}{4\pi\varepsilon_0^2}\cdot\cancel{\frac{1}{a}}\right]$$

$$= \frac{\varepsilon_0}{2}\cdot\frac{q^2}{4\pi\varepsilon_0^2}\cdot\frac{6}{5R} = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\cdot\frac{q^2}{R}\ \checkmark$$