11-12-18

## Ch. 7 Electrodynamics

### Ohm's Law

For most substances....

$\vec{J} = \sigma \vec{f}$

Current Density

Conductivity

Force per charge

Note $\rho = \frac{1}{\sigma}$ - Resistivity, for perfect conductors : $\sigma \to \infty$

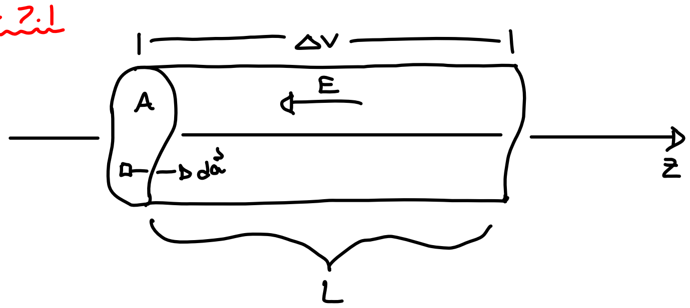
Now the lorentz force law takes the form.....

$\vec{f} = \frac{\vec{F}}{q} = \vec{E} + V \times \vec{B} = \frac{\vec{J}}{\sigma} \quad \therefore \quad \vec{J} = \sigma [\vec{E} + \vec{v} \times \vec{B}]$

now, when $\vec{v}$ is small......

$[\vec{J} = \sigma \vec{E}]$ - Ohm's law

i.e 7.1



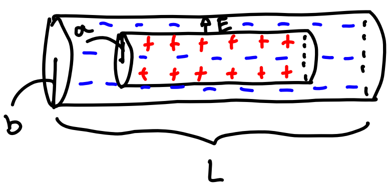

J & E is uniform w/ in the wire

$I = \int \vec{J} \cdot d\vec{a} = \int J da = J \int da = J \cdot A = \sigma E A$

$\Delta V = -\int \vec{E} \cdot d\vec{l} = -\int E dl = -E \int_0^L dl = -EL$

$\therefore E = \frac{\Delta V}{L} \quad \therefore \quad \left[ I = \frac{\sigma \cdot \Delta V \cdot A}{L} \right]$

i.e 7.2




From before.....

$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0} \cdot \frac{1}{s} \hat{s} \quad : \lambda = \frac{Q}{L}$

The current can be calculated from....

$I = \int \vec{J} \cdot d\vec{a} = \frac{\sigma \lambda}{2\pi\varepsilon_0} \int_0^L \int_0^{2\pi} \frac{1}{s} \cdot s \, d\phi dz = \frac{\sigma\lambda}{2\pi\varepsilon_0} \cdot 2\pi \cdot L = \frac{\sigma\lambda}{\varepsilon_0} \cdot L = \frac{\sigma Q}{\varepsilon_0}$

$d\vec{a} = s \, d\phi dz \hat{s}$

Now, the potential difference between the cylinder is....

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = -\frac{\lambda}{2\pi\varepsilon_0}\int_b^a \frac{1}{s}\, ds = \frac{\lambda}{2\pi\varepsilon_0}\cdot \ln(s)\Big|_a^b = \frac{\lambda}{2\pi\varepsilon_0}\ln(b)-\ln(a) = \frac{\lambda}{2\pi\varepsilon_0}\cdot \ln(b/a)$$

$$d\vec{l} = ds\,\hat{s}$$

$$\therefore\quad \lambda = \frac{2\pi\varepsilon_0\Delta V}{\ln(b/a)}\ ,\quad \text{So}\quad I = \frac{\sigma L}{\cancel{\varepsilon_0}}\cdot\frac{2\pi\cancel{\varepsilon_0}\cdot\Delta V}{\ln(b/a)} = \left(\frac{2\pi\sigma L}{\ln(b/a)}\right)\Delta V$$

$\boxed{\Delta V = IR}$ - Also Ohm's law

Resistance From i.e 7.1 ..... $R = \frac{L}{\sigma A}$ , From i.e 7.2 ...... $R = \frac{\ln(b/a)}{2\pi L\sigma}$

$\Rightarrow$ Depends on
- Conductivity of the material
- Geometry of alignment

$$[R] = \frac{[\Delta V]}{[I]} = \frac{V}{A} \equiv \Omega$$

For steady currents and uniform conductivity

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\sigma}\vec{\nabla}\cdot\vec{J}\quad \therefore\ \rho = 0 \Rightarrow \text{any unbalanced charge resides on the surface}$$

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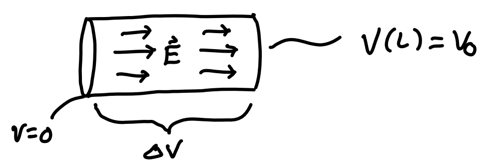
$$\vec{J} = \sigma\vec{f}$$
$$\vec{J} = \sigma\vec{E} \quad \text{- Ohms law}$$
$$\Delta V = IR$$

i.e 7.1 Revisited.....

Prove $\vec{E}$ = uniform in cylindrical resistor of i.e 7.1

As $\rho = 0$




In 1D, the Solution to Laplace's equation....

$$\frac{d^2V}{dt^2} = 0 \quad \therefore \quad V(z) = b + mz$$

Now applying B.C.'s

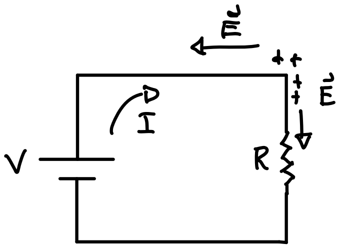
$$V(z=0) = b = 0$$
$$V(z=L) = mL = V_0$$

$$m = \frac{V_0}{L} \quad \therefore \quad \left[V(z) = V_0\cdot\frac{z}{L}\right]$$ - The uniqueness theorem garunteed that this is the Solution

Now

$$\vec{E} = -\vec{\nabla} V = -\hat{z} \cdot \frac{d}{dz}\left(\frac{V_0}{L} z\right) = -\frac{V_0}{L}(\hat{z})$$

## Electromotive Force




Two forces per charge involved in driving a current around a circuit.

$$\vec{f} = \vec{f}_s + \vec{E}$$

$\vec{f}$ : Force per charge driving current

$\vec{f}_s$ : Source Force/Charge

$\vec{E}$ : Electrostatic Force

The electromotive force (EMF) is defined as ....

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint (\vec{f}_s + \vec{E}) \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} \quad \left(\text{As } \oint \vec{E} \cdot d\vec{l} = 0 \text{ for electrostatic fields}\right)$$

- Line integral of force per unit charge (work done per unit charge by the source)
- Not a force at all!

Consider an ideal source of EMF (Resistanceless Battery)
- As $\sigma \to \infty$, $\vec{f} \to 0$, so $\vec{E} = -\vec{f}_s$

The potential difference of between the terminals (a : b)

$$\Delta V = -\int_a^b \vec{E} \cdot d\vec{l} = \int_a^b \vec{f}_s \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} = \mathcal{E}$$

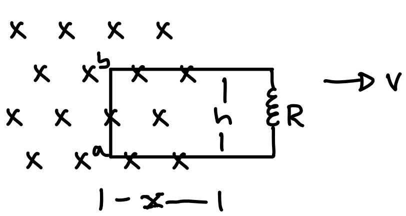
As $\vec{f}_s = 0$ outside the source

∴ A battery establishes : maintains a potential difference equal to the EMF

## Motional EMF

- Arises when you move a wire through a magnetic field.

Consider a circuit (wire Loop : Resistor) in the presence of a magnetic field.




- Pull the entire loop to the right w/ speed V

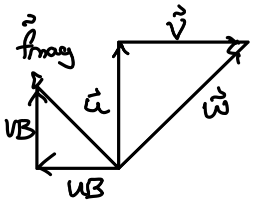
$$F_{mag,q} = qvB = q f_{mag}$$ which drives the current around the loop.

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} = \int_a^b vB\,dy = vBh$$

Note: Line integral that calculates $\mathcal{E}$ is done @ one instant in time

$\therefore$ $d\vec{l}$ Points straight up, even though the loop is moving to the right.

Calculate the work done by the pull .....

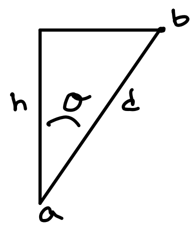



$$f_{mag} = wB = \frac{F_{mag}}{q}$$

$$f_{mag_x} = uB$$
$$f_{mag_y} = vB$$

If the wire moves w/ constant velocity, then

$f_{pull} = uB$ – to counteract, $f_{mag,x}$




Now, the work done per unit charge is....

$$\frac{W}{q} = \int \vec{f}_{pull} \cdot d\vec{l} = \int_a^b f_{pull}\, dl \sin\theta = \int_a^b uB\sin\theta\, dl$$

$$\frac{W}{q} = uB\sin\theta \cdot \frac{h}{\cos\theta} = uBh \cdot \tan\theta = vBh$$

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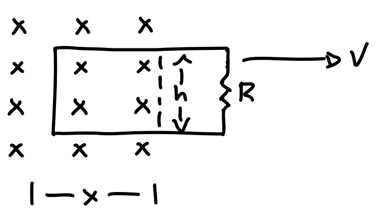
From last time.....

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$$




Theres another way to express $\mathcal{E}$.....

$\Phi_B = \int \vec{B} \cdot d\vec{a}$ – Flux of B through the loop

For the above rectangular loop .....

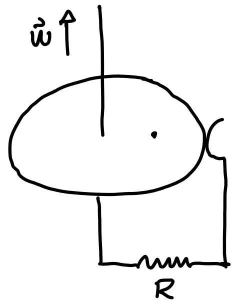
$$\Phi = BA = Bhx$$

As the loop moves to the right, Flux decreases ......

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -BhV \quad \left(\text{as } \frac{dx}{dt} \text{ is negative}\right)$$

$\left[\mathcal{E} = -\frac{d\Phi}{dt}\right]$ – EmF generated in the loop is minus the time rate of change of the flux
– Flux rule

i.e 7.4




$$\vec{v} = s\omega\,\hat{\phi}$$

$$f_{mag} = \vec{v}\times\vec{B} = s\omega B\,\hat{s}$$

$$\mathcal{E} = \oint \vec{f}\cdot d\vec{l} = \int_0^a s\omega B\,\hat{s}\cdot ds\,\hat{s} = \omega B\int_0^a s\,ds = \omega B\frac{s^2}{2}\Big|_0^a = \frac{1}{2}\omega B a^2$$

∴ The current in the wire .....

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$$

## Faraday's Law

Faraday realized that .....

A changing $\vec{B}$-Field induces an $\vec{E}$-Field

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = \int (\vec{\nabla}\times\vec{E})\,d\vec{a}$$

so

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ – Faraday's law in differential form

Note: For constant $\vec{B}$-Fields, Faraday's law is of the form .....

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \text{or} \quad \vec{\nabla}\times\vec{E} = 0$$

The universal flux rule is .....

Whenever the magnetic flux through a loop changes, an EMF

$$\mathcal{E} = -\frac{d\Phi}{dt}$$ will appear in the loop.

## Lenz's Law

The induced current will flow in such a direction that the flux it produces tends to cancel the change.

## The Induced $\vec{E}$-Field

Faraday's discovery tells us that there are two distinct kinds of $\vec{E}$-Fields.

- Those attribute direction to changes.
- Those associated with changing $\vec{B}$-fields $\left( \vec{E}(\vec{r}) = \frac{\mu_0}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,d\tau'}{\eta^2}\hat{\eta} \right)$

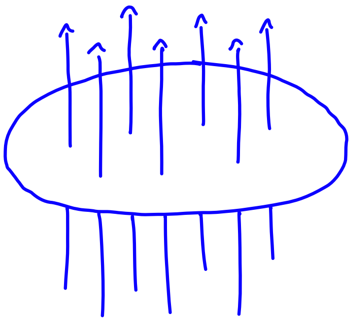
For,

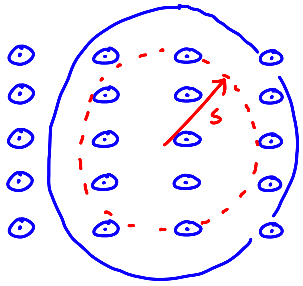
$$\vec{\nabla}\cdot\vec{B} = 0,\quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\times\frac{\hat{\eta}}{\eta^2}\,d\tau'$$

$$\vec{\nabla}\cdot\vec{E} = \cancel{\frac{\rho}{\varepsilon_0}}^{\,0},\quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \rightarrow \text{Replace } \mu_0\vec{J} \text{ w/ } -\frac{\partial\vec{B}}{\partial t}$$

when no charge density

To determine the Faraday induced $\vec{E}$-field i.e






The EmF is given by .....

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

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$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$, $\vec{E}$ - Field is circumferential.

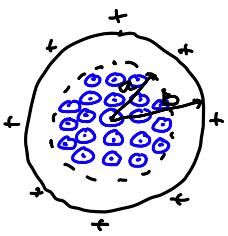
$\oint \vec{E}\cdot d\vec{l} = \oint E\,dl = E\oint dl = E2\pi s$ , $\vec{E} \parallel d\vec{l}$ , $\int \vec{B}\cdot d\vec{a} = \int B\,da = B\int da = B\pi s^2$

$\vec{B} \parallel d\vec{a}$ , $d\vec{a} = s\,ds\,d\phi\,\hat{z}$

(*) Becomes .....

$$E2\pi s = -\frac{d}{dt}B\pi s^2 = -\pi s^2\frac{dB}{dt} \quad , \quad E = -\frac{1}{2}s\frac{dB}{dt} \quad , \quad \boxed{\vec{E} = -\frac{1}{2}s\frac{dB}{dt}\hat{\phi}}$$

i.e




$\lambda = \frac{q}{2\pi b}$ $\therefore$ $q = 2\pi b\lambda$

now

$\vec{B} = B_0\hat{z}$ when $s < a$

$= 0$ $a < s < b$

The EMF is given by

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}, \qquad \int \vec{B}\cdot d\vec{a} = \int B\,da = B\int da = B\pi a^2$$

Now the torque ....

$$\vec{\tau} = \vec{r}\times\vec{F} = bF = bqE = b\cdot 2\pi\cdot b\cdot\lambda\cdot E = b\cdot\lambda\cdot 2\pi bE = b\lambda\cdot 2\pi bE = b\lambda\oint \vec{E}\cdot d\vec{l}$$

$$\tau = b\lambda\cdot -\pi a^2\frac{dB}{dt} = \frac{dL}{dt}$$

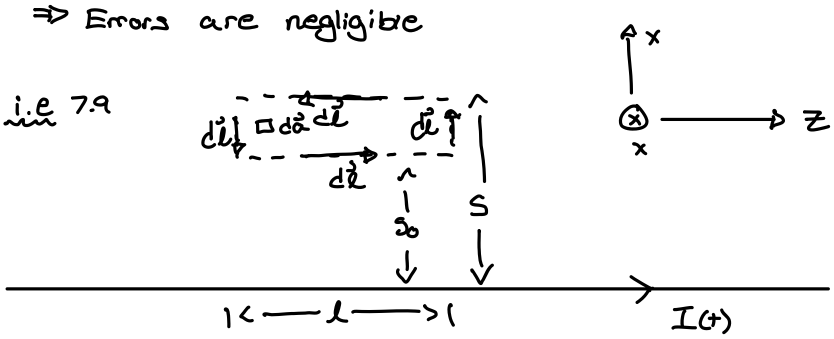
$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = -\frac{d}{dt}(B\pi a^2) = -\pi a^2\cdot\frac{dB}{dt}$$

$$\therefore \int_0^L dL = -\pi b a^2\lambda\int_{B_0}^0 dB \quad , \quad \boxed{L = \pi a^2 b\lambda B}$$

- EM induction occurs when the magnetic fields are changing
  - we'd like to use magnetostatics to calculate our $\vec{B}$-Fields
  - $\Rightarrow$ Results are only approximately correct

$\Rightarrow$ Errors are negligible

i.e 7.9




$\longrightarrow$ Quasistatic Approximation .....

$$B(t) = \frac{\mu_0 I(t)}{2\pi y}$$

E(s)? $\oint \vec{E}\cdot d\vec{l} = E(s_0)l - E(s)l$

$$\int \vec{B}\cdot d\vec{a} = \int_0^l \int_0^s \frac{\mu_0 I(t)}{2\pi y}\, dy\, dz = \frac{\mu_0 I(t)}{2\pi}\int_0^l dz \int_{s_0}^{s} \frac{dy}{y} = \frac{\mu_0 I(t)}{2\pi} l \cdot \ln y \Big|_{s_0}^{s}$$

$d\vec{a} = dy\, dz\ (\hat{x})$

$$= \frac{\mu_0 I(t) l}{2\pi}\left(\ln(s) - \ln(s_0)\right) = \frac{\mu_0 I(t) l}{2\pi} \ln\left(\frac{s}{s_0}\right)$$

$$E(s_0)l - E(s)l = -\frac{\mu_0}{2\pi} l \ln\left(\frac{s}{s_0}\right)\frac{dI(t)}{dt}, \quad \vec{E}(s) = \left[\frac{\mu_0}{2\pi}\ln(s/s_0)\frac{dI(t)}{dt} + E(s_0)\right]\hat{z}$$

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## Electrodynamics Before Maxwell

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ - Gauss' Law, $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ - Faradays Law, $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ - Ampere's law

However, theres a problem!

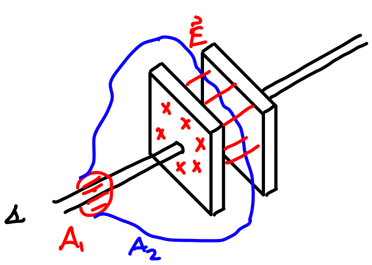
$1^{st}$, take the divergence of Faradays law.....

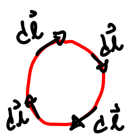
$$\vec{\nabla}\cdot(\underbrace{\vec{\nabla}\times\vec{E}}_{0}) = \vec{\nabla}\cdot -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{0})$$

Now, the divergence of Ampere's Law .....

$$\vec{\nabla}\cdot(\underbrace{\vec{\nabla}\times\vec{B}}_{0}) = \mu_0(\underbrace{\vec{\nabla}\cdot\vec{J}}_{-\frac{\partial\rho}{\partial t}})$$

note: $\vec{\nabla}\cdot\vec{J} = 0$ for magnetostatics but not in general! An inconsistency!




Ampere's law cannot be the whole truth. Ampere's law doesn't specify the area A used, provided it's Bounded by C.

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{Enc}$$

- If $A_1$ is used, a net current flows through it. $\therefore$ $\vec{B}$-Field
- If $A_2$ is used, no net current flows through it $\therefore$ $\vec{B}=0$

While charging (or discharging) a capacitor, a $\vec{B}$-Field is generated between the plates.
Between the plates....

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\cdot\frac{Q}{A}$$

Area of the plate

thus

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\cdot\frac{dQ}{dt} = \frac{1}{\varepsilon_0}\cdot\frac{I_D}{A} = \frac{J_D}{\varepsilon_0} \quad \therefore \left[\vec{J}_D = \varepsilon_0\cdot\frac{\partial \vec{E}}{\partial t}\right]$$

- Displacement current density

$$\text{Let } \vec{J} \to \vec{J} + \vec{J}_D = \vec{J} + \varepsilon_0\cdot\frac{\partial \vec{E}}{\partial t}$$

The maxwell's Equations in differential form.....

$$\left[\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}, & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t} \end{array}\right]$$

These plus the Lorentz force law

$$\vec{F} = Q[\vec{E} + \vec{v}\times\vec{B}]$$

Completely describe classical electromagnetic theory

Lets know take the divergence of Ampere's law.....

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot\vec{J} + \varepsilon_0\mu_0\vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$= \mu_0\left[\vec{\nabla}\cdot\vec{J} + \varepsilon_0\mu_0(\vec{\nabla}\cdot\vec{E})\right] = \mu_0\left[\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right] = 0$$

continuity Equation

Consider Maxwell's equations in a vacuum.... ($\rho=0$, $\vec{J}=0$)

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$$

Taking the curl of Ampere's law : Faraday's law yields.....

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$= \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0}) - \nabla^2\vec{E}$$

using amperes law .....

$$-\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}\right) = -\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$0 = \nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{E} = \left(\nabla^2 - \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\vec{E}$$

Likewise .....

using Faraday's law

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \varepsilon_0\mu_0\cdot\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t} = \varepsilon_0\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$= \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B}$$

a wave equation in 3D....

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\vec{E}}{\vec{B}}\right) = 0$$

$$0 = \nabla^2\vec{B} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t}\vec{B} = \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{B}$$

$$\mu_0\varepsilon_0 = \frac{1}{v^2}$$

$$V = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = \frac{1}{\sqrt{(8.85\times10^{-12}\ C^2/Nm^2)(4\pi\times10^{-7}\ N/A^2)}} = 2.99863\times10^{8}\ m/s$$

$$\underline{V = 2.99863\times10^{8}\ m/s}$$