MAT 201
Larson/Edwards – Section 7.3 (Part 1)
Volume: The Shell Method

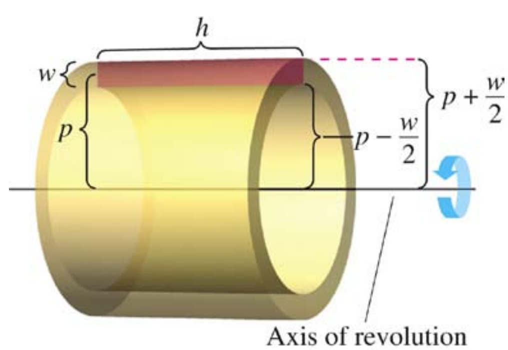
In this section we will further our investigation into solving volume problems. So far we have discussed how to find volumes by using the disk, washer, and familiar cross sections. We will now discuss the ***shell method*** for finding volume.

Why is it called the shell method?

Consider the rectangle given in the figure below that has a width, w, and a height, h. If we were to revolve this rectangle around the x-axis we would obtain a cylindrical shell. If we let p be the distance from the x-axis to the center of the rectangle, then we may obtain expressions for the outer and inner radius of the shell.

$$\text{Outer radius: } \quad p + \frac{w}{2}$$

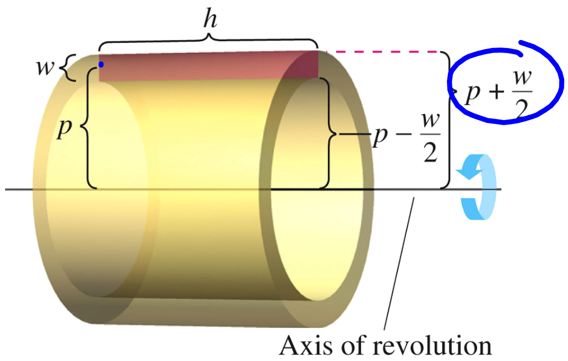
$$\text{Inner radius: } \quad p - \frac{w}{2}$$



How would we find the volume of this shell?

Outer radius: $p + \dfrac{w}{2}$

Inner radius: $p - \dfrac{w}{2}$



How would we find the volume of this shell?

Volume of the cylinder - volume of the hole.

$$\pi\left(p+\frac{w}{2}\right)^2 h - \pi\left(p-\frac{w}{2}\right)^2 h$$

$$\pi h\left[\left(p+\frac{w}{2}\right)^2 - \left(p-\frac{w}{2}\right)^2\right]$$

$$\pi h\left[p^2 + pw + \frac{w^2}{4} - \left(p^2 - pw + \frac{w^2}{4}\right)\right]$$

$$\pi h\left[\cancel{p^2} + pw + \cancel{\frac{w^2}{4}} \cancel{-p^2} + pw \cancel{-\frac{w^2}{4}}\right]$$

$$\pi h(2pw)$$

$$V = 2\pi hpw$$

Our volume equation for a horizontal axis of revolution will then take on the following transformation:

$$V = 2\pi phw$$

$$V = 2\pi(average\ radius)(height)(thickness)$$

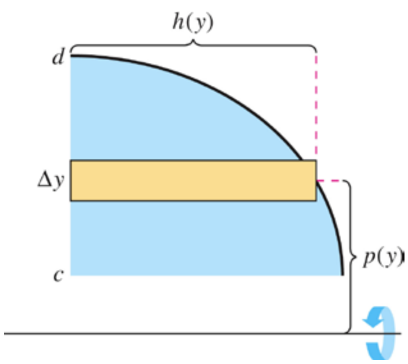
$$\Delta V = 2\pi\left[p(y)h(y)\right]\Delta y$$

$$V = 2\pi\int_c^d \left[p(y)h(y)\right]dy$$

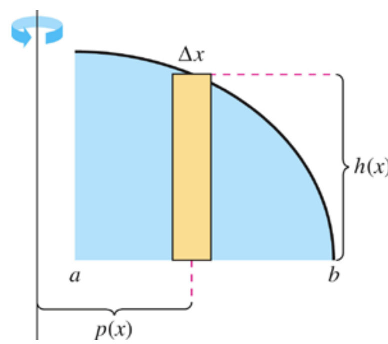
<u>The Shell Method</u>:

To find the volume of a solid of revolution with the shell method, use one of the following (see figure below):

1. Horizontal Axis of Revolution: $V = 2\pi\int_c^d \left[p(y)h(y)\right]dy$
2. Vertical Axis of Revolution: $V = 2\pi\int_a^b \left[p(x)h(x)\right]dx$



Horizontal Axis of Revolution



Vertical Axis of Revolution

$$V = 2\pi phw$$

$$V = 2\pi(average\ radius)(height)(thickness)$$

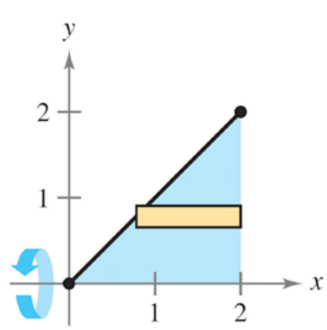
$$\Delta V = 2\pi\left[p(y)h(y)\right]\Delta y$$

$$V = 2\pi\int_c^d\left[p(y)h(y)\right]dy$$ ← Axis of Revolution is horizontal

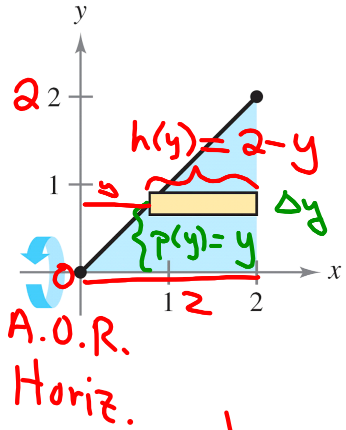
$$V = 2\pi\int_a^b\left[p(x)\,h(x)\right]dx$$ ← A.O.R. is vertical.

Ex: Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-axis: $y = \sqrt{x-2}$, $y = 0$, $x = 4$.

Ex: Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region given below about the x-axis. The line is: $y = x$



Ex: Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region given below about the x-axis. The line is: $y = x$



Disc

$$V = \pi \int_0^2 x^2 \, dx = \pi \left[\frac{x^3}{3} \Big|_0^2 \right] = \frac{8\pi}{3}$$

$$V = 2\pi \int_c^d [p(y)\, h(y)] \, dy$$

$$V = 2\pi \int_0^2 [y(2-y)] \, dy$$

$$= 2\pi \int_0^2 2y - y^2 \, dy$$

$$= 2\pi \left[y^2 - \frac{y^3}{3} \Big|_0^2 \right]$$

$$= 2\pi \left[4 - \frac{8}{3} \right]$$

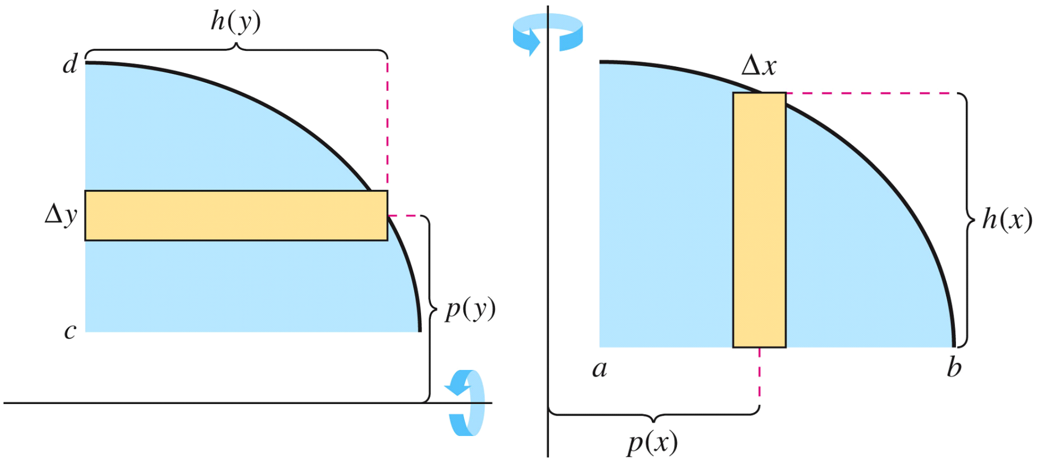
$$= 2\pi \left[\frac{4}{3} \right]$$

$$= \frac{8\pi}{3}$$

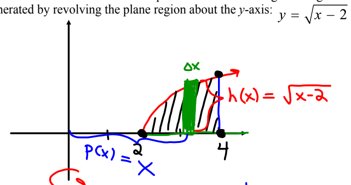
<u>The Shell Method</u>:
To find the volume of a solid of revolution with the shell method, use one of the following (see figure below):

Horizontal Axis of Revolution: $V = 2\pi\int_c^d \left[p(y)h(y)\right]dy$

Vertical Axis of Revolution: $V = 2\pi\int_a^b \left[p(x)h(x)\right]dx$



Ex: Use the shell method to set up and evaluate the integral that gives the volume of the solid generated by revolving the plane region about the y-axis: $y=\sqrt{x-2}$, $y=0$, $x=4$.



Vertical A.O.R.

$$V=2\pi\int_a^b\left[p(x)\,h(x)\right]dx$$

$x=u+2$

$*\ u=x-2$

$du=dx$

$x=2\to u=0$

$x=4\to u=2$

$$V=2\pi\int_2^4\left[x\sqrt{x-2}\right]dx$$

$$V=2\pi\int_0^2(u+2)\sqrt{u}\;du$$

$$V=2\pi\int_0^2 u^{3/2}+2u^{1/2}\,du$$

$$V=2\pi\left[\frac{2}{5}u^{5/2}+\frac{4}{3}u^{3/2}\Big|_0^2\right]$$

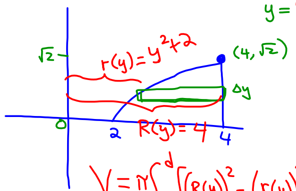
$$V=2\pi\left[\frac{2}{5}(2)^{5/2}+\frac{4}{3}(2)^{3/2}-(0)\right]$$

$$V=2\pi\left[\frac{3\cdot 8\sqrt{2}}{5\cdot 3}+\frac{5\cdot 8\sqrt{2}}{3\cdot 5}\right]$$

$$V=2\pi\left[\frac{64\sqrt{2}}{15}\right]$$

$$\boxed{V=\frac{128\pi\sqrt{2}}{15}\approx 37.9\text{ units}^3}$$

Washer Method



$y^2=(\sqrt{x-2})^2$

$y^2=x-2$

$y^2+2=x$

$$V=\pi\int_c^d\left[(R(y))^2-(r(y))^2\right]dy$$

$$V=\pi\int_0^{\sqrt{2}}\left[16-(y^2+2)^2\right]dy$$

$$V=\pi\int_0^{\sqrt{2}}\left(16-(y^4+4y^2+4)\right)dy$$

$$V=\pi\int_0^{\sqrt{2}}12-y^4-4y^2\;dy$$

$$V=\pi\left[12y-\frac{y^5}{5}-\frac{4y^3}{3}\Big|_0^{\sqrt{2}}\right]$$

$$V=\pi\left[12\sqrt{2}-\frac{4\sqrt{2}}{5}-\frac{8\sqrt{2}}{3}\right]$$

$$V\approx 37.9\text{ units}^3$$