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MATH 360
CP Ch. 10.6

Equations 1-3*

Eq (1): $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

Eq (2): $\vec{N} = \vec{r}_u \times \vec{r}_v$ and unit normal vector $\hat{n} = \frac{1}{|\vec{N}|}\vec{N}$

Eq (3): For a given vector function $\vec{F}$ we can define the surface integral over S by

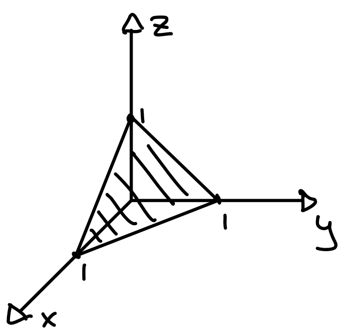
$$\iint_S \vec{F}\cdot\vec{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v)\, du dv$$

Eq (3*): $\vec{n} dA = \vec{n}|\vec{N}|\, du dv = N\, du dv$

Example 2 Surface Integral

Evaluate (3) when $\vec{F} = [x^3, 0, 3y^2]$ & S is the portion of the plane $x+y+z=1$ in the first octant.

Solution. Writing $x=u$ & $y=v$, we have $z = 1-x-y = 1-u-v$. Hence we can represent the plane $x+y+z=1$ in the form $\vec{r}(u,v) = [u, v, 1-u-v]$. We obtain the first-octant portion S of this plane by restricting $x=u$ and $y=u$ to the projection R of S in the xy-plane. R is the triangle bounded by the two coordinate axes and the straight line $x+y=1$, obtained from $x+y+z=1$ by setting $z=0$. Thus $0 \le x \le 1-y$, $0 \le y \le 1$



By inspection or by differentiation,

$$\vec{N} = \vec{r}_u \times \vec{r}_v = [1,0,-1]\times[0,1,-1] = [1,1,1]$$

Hence $\vec{F}(S)\cdot\vec{N} = [u^2, 0, 3v^2]\cdot[1,1,1] = u^2 + 3v^2$ by (3)

$$\iint_S \vec{F}\cdot\vec{n}\, du = \iint_R (u^2+3v^2)\, du dv = \int_0^1\int_0^{1-v} u^2+3v^2\, du dv$$

$$\iint_S \vec{F}\cdot\vec{n}\, du = \int_0^1 \left[\tfrac{1}{3}(1-v)^3 + 3v^2(1-v)\right] dv = \frac{1}{3}$$

Theorem 1 Change of Orientation in a Surface Integral

The replacement of $\vec{n}$ by $-\vec{n}$ (hence of $\vec{N}$ by $-\vec{N}$) corresponds to the multiplication of the integral in (3) or (4) by -1.

Equation 6

Another type of surface integral is

$$\iint_S G(\vec{r})\, dA = \iint_R G(\vec{r}(u,v))\,|\vec{N}(u,v)|\, du dv$$

Equation 8

When $G=1$ the area $A(S)$ of S is,

$$A(S) = \iint_S dA = \iint_R |r_u \times r_v| \, du dv$$

Discussion

Reprentations $z = f(x,y)$. If a surface S is given by $z=f(x,y)$, then setting $u=x$, $v=y$, $\vec{r} = [u, v, f]$ gives

$$|\vec{N}| = |r_u \times r_v| = |[1, 0, f_u] \times [0, 1, f_v]| = |[-f_u, -f_v, 1]| = \sqrt{1 + f_u^2 + f_v^2}$$

and, since $f_u = f_x$, $f_v = f_y$, formula (6) becomes

Eq (11) $$\iint_S G(\vec{r})\, dA = \iint_{R^*} G(x, y, f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx dy$$

When $G=1$ we obtain for the area $A(S)$ of $S: z = f(x,y)$ the formula

Eq (12) $$A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx dy$$