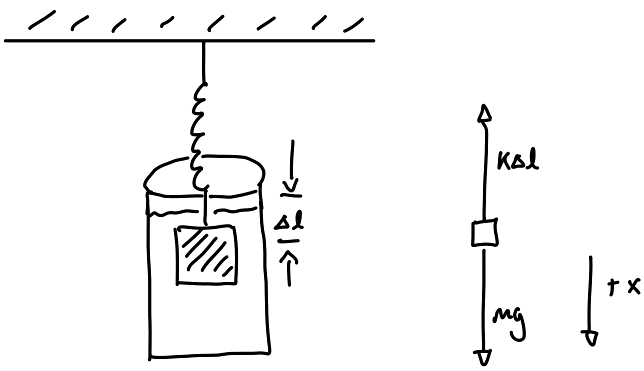
Damped Harmonic Oscillator

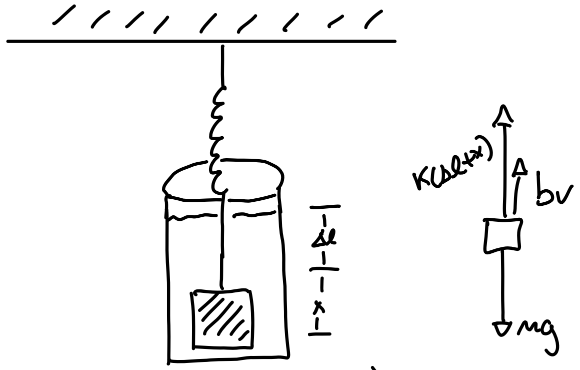
- All physical oscillators have some damping from the medium, unless in vacuum
- Sometimes damping is desired... i.e shocks, screen doors, bridges etc

Consider a damping force of the form
$F_d = -bv$ $b \equiv$ Damping coefficient
$\equiv$ Dependent upon
- Shape
- Vicosity of fluid

- Force is linearly proportional to the speed
- Valid for low speeds, $F_d \propto -v^2$ for high speeds



$\vec{F}_{net} = m\vec{a}_x$
$mg - K\Delta l = 0$



$\gamma = b/m$
$\omega_0 = \sqrt{k/m}$
natural frequency w/ no damping
- Frequency of solution when $b, \gamma = 0$

$\vec{F}_{net_x} = m\vec{a}_x$
$mg - K(\Delta l + x) - bv = ma_x$
$-Kx - bv = ma_x$
$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

$$\boxed{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0}$$

3 Diff Cases of Damping
1. Light Damping
2. Overdamping
3. Critical damping

Light Damping

$\omega \neq \omega_0$

$x(t) = A_0 e^{-\beta t}\cos(\omega t)$

Plug into $\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$: Show its a solution

- $\dot{x} = -\beta A_0 e^{-\beta t}\cos(\omega t) - \omega A_0 e^{-\beta t}\sin(\omega t)$
  $= -A_0 e^{-\beta t}(\beta\cos(\omega t) + \omega\sin(\omega t))$
- $\ddot{x} = A_0 \beta e^{-\beta t}(\beta\cos(\omega t) + \omega\sin(\omega t)) - A_0 e^{-\beta t}(-\omega\beta\sin(\omega t) + \omega^2\cos(\omega t))$
  $= A_0 e^{-\beta t}((\beta^2 - \omega^2)\cos(\omega t)) + 2\omega\beta\sin(\omega t))$

- $A_0 e^{-\beta t}((\beta^2 - \omega^2)\cos(\omega t)) + 2\omega\beta\sin(\omega t)) - \gamma A_0 e^{-\beta t}(\beta\cos(\omega t) + \omega\sin(\omega t)) + \omega_0^2 A e^{-\beta t}\cos(\omega t)$
- $A_0 e^{-\beta t}(\underbrace{(\beta^2 - \omega^2 - \gamma\beta + \omega_0^2)}_{0}\cos(\omega t) + \omega\underbrace{(2\beta - \gamma)}_{0}\sin(\omega t)) = 0$

note
- This eqn must hold for all time
- $\omega \neq 0$

1. $2\beta - \gamma = 0$
2. $\beta^2 - \omega^2 - \gamma\beta + \omega_0^2 = 0$

- $\frac{\gamma^2}{4} - \omega^2 - \frac{\gamma^2}{2} + \omega_0^2 = 0$
- $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$: $\omega = \sqrt{\omega_0^2 - (\gamma/2)^2}$
- $x(t) = A_0 e^{-\gamma t/2}\cos\left(\sqrt{\omega_0^2 - (\gamma/2)^2}\, t\right)$

This type of motion is representative of oscillatory motion

- when w is real i.e when $\gamma^2/4 < w_0^2$

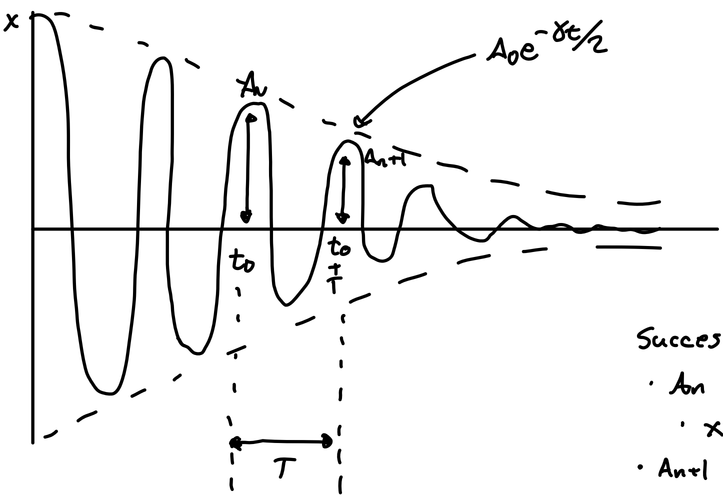
$$\left[\begin{array}{l} x(t) = A_0 e^{-\gamma t/2}\cos(wt+\varnothing) \\ w = \sqrt{w_0^2 - \left(\frac{\gamma}{2}\right)^2} \\ w_0^2 \gg \frac{\gamma^2}{4} \end{array}\right]$$

General solution is of the form,

- $x(t) = A_0 e^{-\gamma t/2}\cos(wt+\varnothing)$, $w = \sqrt{w_0^2 - \left(\frac{\gamma}{2}\right)^2}$, $\varnothing \equiv$ A constant

Notice
- γ & w are determined by the properties of the oscillator
- A_0 & $\varnothing$ are determined by the initial conditions

Graph of $x(t) = A_0 e^{-\gamma t/2}\cos(wt)$



Notice that when $\gamma \to \varnothing$
- $x(t) = A_0\cos(wt+\varnothing)$ when $w = w_0$ < Solution for SHM >
- $x(t') = 0$ when $wt' = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$
- period $T = \frac{2\pi}{w}$

Succesive maxima occur when $\varnothing = 0$
- A_n occurs at $t = t_0$, so
  - $x(t) = A_0 e^{-\gamma t/2}\cos(wt_0) = A_0 e^{-\gamma t/2} = A_n$
- A_{n+1} occurs at $t = t_0 + T$
  - $x(t_0+T) = A_0 e^{-\frac{\gamma(t_0+T)}{2}}\cos(w(t_0+T)) = A_{n+1}$

Since $\cos(wt_0) = \cos w(t_0+T)$
- $\cos(wt_0 + wT) = \cos(wt_0 + w2\pi) = \cos(wt_0) = 1$
- $A_{n+1} = A_0 e^{-\frac{\gamma(t_0+T)}{2}} = A_0 e^{-\gamma t/2} = A_n^{-\gamma T/2}$

- $$\left[\frac{A_{n+1}}{A_n} = e^{-\gamma T/2} = \frac{A_n}{A_{n+1}} = e^{\gamma T/2}\right]$$

Succesive maxima decrease by the same fractional amount

The natural logarithm of A_n/A_{n+1}

- $\ln\left(\frac{A_n}{A_{n+1}}\right) = \ln e^{\gamma T/2} = \frac{\gamma T}{2}$ $\Rightarrow$ This is called the Logarithmic Decrement

- Heavy Damping:

This occurs when $\frac{\gamma^2}{4} > w_0^2$
- $$\left[\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + w_0^2 x = 0\right] \qquad \gamma = \frac{b}{m} \quad w_0^2 = \frac{k}{m}$$
  $m\ddot{x} + b\dot{x} + kx = 0$

Choose a solution of the form
- $x(t) = e^{-\beta t}f(t)$
- $\dot{x} = -\beta e^{-\beta t} + e^{-\beta t}\dot{f} = e^{-\beta t}(-\beta f + \dot{f})$
- $\ddot{x} = -\beta e^{-\beta t}(-\beta f + \dot{f}) + e^{-\beta t}(-\beta\dot{f} + \ddot{f}) = e^{-\beta t}(\beta^2 f - 2\beta\dot{f} + \ddot{f})$

with $\beta = \gamma/2$
- $e^{-\beta t}(\ddot{f} + 2\beta\dot{f} + \beta^2 f) + \gamma e^{-\beta t}(\dot{f} - \beta f) + w_0^2 e^{-\beta t}f = 0$
  - $e^{-\beta t}[\ddot{f} + (\gamma - 2\beta)\dot{f} + (\beta^2 - \beta\gamma + w_0^2)f] = 0 \quad \beta = \gamma/2$
- $\beta^2 - \beta\gamma + w_0^2 = \frac{\gamma^2}{4} - \frac{\gamma}{2}\cdot\gamma + w_0^2 = w_0^2 - \frac{\gamma^2}{4}$

This becomes
- $\ddot{f} + (w_0^2 - \gamma^2/4)f = 0 \qquad \ddot{f} = \underbrace{(\gamma^2/4 - w_0^2)}_{\to\,\alpha^2} f \qquad \left[\ddot{f} = \alpha^2 f\right]$

This solution takes the form of $\qquad \alpha = \sqrt{\left(\frac{\gamma}{2}\right)^2 - w_0^2}$
- $f(t) = Ae^{\alpha t} + Be^{-\alpha t} \Rightarrow x(t) = e^{-\beta t}f(t) \rightarrow \left[x(t) = e^{-\gamma t/2}\left[Ae^{\alpha t} + Be^{-\alpha t}\right]\right]$

Critical Damping

- This occurs when $\gamma^2/4 = \omega_0^2$

The trial solution would be,

- $\frac{d^2 f}{dt^2} = 0$ with a solution of the form $f(t) = A + Bt$

This becomes,

- $x(t) = Ae^{-\gamma t/2} + Bte^{-\gamma t/2} = e^{-\gamma t/2}(A + Bt) \Rightarrow \boxed{x(t) = e^{-\gamma t/2}(A + Bt)}$

When critical damping occurs, the mass returns to the equilibrium position in the shortest possible time without oscillating.

Ch 2)

$m = 2.5\,kg$
$K = 600\,N/m$
$V_0 = 1.5\,m/s$

a.) Determine the value of the damping constant "b" required to produce critical damping

b.) Determine max value of displacement at time in which impulse occurs.

a.) For critical damping

$$\gamma^2/4 = b^2/4m^2 = \omega_0^2 = K/m$$

$$\frac{b^2}{4m^2} = \frac{K}{m} \; : \; b^2 = 4mk \; : \; b = \sqrt{4mk} \; : \; b = \sqrt{4(2.5)kg(600)N/m} = 77.5\,kg/s$$

$$\boxed{b = 77.5\,kg/s}$$

b.) General solution for critical damping is ...

$$x(t) = e^{-\gamma t/2}(A + Bt) \quad \therefore \quad V = \frac{dx}{dt} = Bte^{-\gamma t/2} \cdot -\frac{\gamma}{2}(A + Bt)e^{-\gamma t/2}$$

$$V = \left(B - \frac{\gamma}{2}(A + Bt)\right)e^{-\gamma t/2}$$

Initial conditions of..

$$v(t) = Be^{-\gamma t/2}\left(1 - \frac{\gamma}{2}t\right)$$

- $x = 0$
- $V = V_i$
- $t = 0$

- $x(t=0) = 0 = A$
- $\dot{x}(t=0) = V_0 = B = 1.5\,m/s$
- $x(t) = V_0 te^{-\gamma t/2}$

$\therefore$ $x(t) = Bte^{-\gamma t/2}$

$v(t) = Be^{-\gamma t/2}\left(1 - \frac{\gamma}{2}t\right)$

Max displacement occurs when $dx/dt = 0$

$$V_i e^{-\gamma t/2}(1 - \gamma t/2) = 0$$

$$t = \frac{2}{\gamma} = \frac{2m}{b} = \frac{2(2.5\,kg)}{(77.5\,kg/s)} = 6.5\times10^{-2}\,s$$

$$\boxed{x = 3.6\times10^{-2}\,m}$$

$$x = \frac{2V_i}{e\gamma} = \frac{2mV_i}{eb} = \frac{2(2.5kg)(1.5\,m/s)}{e(77.5\,kg/s)} = 3.6\times10^{-2}\,m$$

Rate of Energy Loss in a Damped Harmonic oscillator

The mechanical enegy of a damped oscillator

- $E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ $\qquad E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ $\qquad x \equiv x(t)$, $\dot{x} \equiv \frac{dx}{dt}$

For the case of a very lightly damped oscillator i.e $(\phi = 0)$: $\frac{\gamma^2}{4} \ll \omega_0^2$

- $x = A_0 e^{-\gamma t/2} \cos(\omega t)$ $\qquad \omega = \sqrt{\omega_0^2 - (\frac{\gamma}{2})^2} \approx \omega_0$
- $x(t) = A_0 e^{-\gamma t/2} \cos(\omega_0 t)$

very lightly damped

Now...

- $\dot{x}(t) = A_0\left[\frac{-\gamma}{2} e^{\frac{-\gamma t}{2}} \cos(\omega_0 t) - \omega_0 e^{\frac{-\gamma t}{2}} \sin(\omega_0 t)\right]$

$$= -A_0 e^{\frac{-\gamma t}{2}}\left[\frac{\gamma}{2}\cos(\omega_0 t) + \omega_0 \sin(\omega_0 t)\right]$$

Since $\frac{\gamma}{2} \ll \omega_0$, we can neglect the first term in square brackets giving us

- $\dot{x}(t) = -A_0\omega_0 e^{\frac{-\gamma t}{2}} \sin(\omega_0 t)$ $\qquad$ "$\frac{\gamma}{2}\cos(\omega_0 t)$" is practically zero because $\omega_0 \gg \frac{\gamma}{2}$

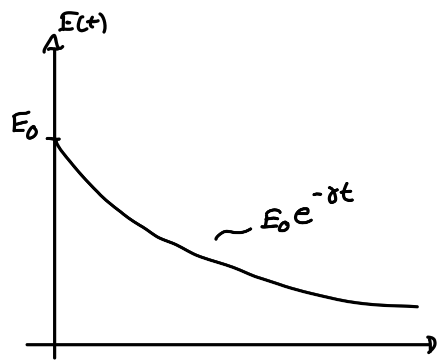
Plugging into Energy equation yields

- $E(t) = \frac{1}{2} m\omega_0^2 A_0^2 e^{-\gamma t} \sin^2(\omega_0 t) + \frac{1}{2} k A_0^2 e^{-\gamma t} \cos^2(\omega_0 t)$

$$\boxed{E(t) = \frac{1}{2} A_0^2 e^{-\gamma t}\left[m\omega_0^2 \sin^2(\omega_0 t) + k\cos^2(\omega_0 t)\right]}$$

Substituting for $\omega_0^2 = \frac{k}{m}$, we obtain...

- $E(t) = \frac{1}{2} k A_0^2 e^{-\gamma t}$, giving ... $\boxed{E(t) = E_0 e^{-\gamma t}}$
- where $E_0 \equiv$ the total energy of the oscillator at $t = 0$



γ has a physical meaning, the reciprocal of γ: $1/\gamma$ is known as the decay time for the energy of the oscillator.

- $\tau = 1/\gamma$, we obtain ... * τ has dimensions of time and is called decay time.
- $E(t) = E_0 e^{\frac{-t}{\tau}}$

Calculate the time rate of change of Energy

- $\frac{dE(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right] = mv\frac{dv}{dt} + kx\frac{dx}{dt} = (ma + kx)v$ $\qquad m\dot{x}\ddot{x} + kx\dot{x}$, $\dot{x}(m\ddot{x} + kx)$
- Remember that .. $ma = -kx$, so $ma + kx = -bv$

This then gives

- $\boxed{\frac{dE(t)}{dt} = -bv^2 = F_D v \qquad bv \equiv F_D}$

* The energy of an oscillator is dissipated because it does work against the damping force at the rate (damping force x velocity) <Fluid is heated>

The Quality Factor Q of A Damped Harmonic Oscillator

Define the quality factor, Q:

- $Q = \frac{\omega_0}{\gamma}$ ⟨Dimensionless⟩

- Describes how good an oscillator is
- Smaller degree of damping, the higher the quality of the oscillator

Consider the energy of a very lightly damped oscillator one period apart,

- $E_1 = E(t_1) = E_0 e^{-\gamma t}$
- $E_2 = E(t_2) = E_0^{-\gamma(t_1+T)}$ where $t_2 = t_1 + T$

Taking the ratio yields

- $\frac{E_2}{E_1} = e^{-\gamma T} = 1 - \gamma T + \ldots\ldots$ ← (since γ is small) · $\gamma T \ll 1$

Thus

- $\gamma T = 1 - \frac{E_2}{E_1} = \frac{E_1 - E_2}{E_1} = \frac{\omega_0 T}{Q} = \frac{2\pi}{Q}$ - This is the fractional change in energy per cycle.

Notice

- $\frac{1}{Q} = \frac{1}{2\pi} \cdot \frac{E_1 - E_2}{E_1}$ - This is the fractional change in energy per radian

We Define Q by,

$$\left[Q = \frac{\text{Energy Stored in the oscillator}}{\text{Energy dissipated per radian}} \right]$$

Ex

i) $E(t) = E_0 e^{-t/\tau}$

$$\ln\left(\frac{E(t)}{E_0}\right) = -\frac{t}{\tau}$$

$$\frac{t}{\tau} = \ln\left(\frac{E_0}{E(t)}\right)$$

$$\tau = \frac{t}{\ln(E_0/E(t))}$$

When $t = 4s$ $E(t=4s) = \frac{1}{2}E_0$

$$\tau = \frac{4}{\ln(2)} = 5.8s$$

$$\gamma = \frac{1}{\tau} = \frac{1}{5.8s} = 0.17s^{-1}$$

ii) $Q = \frac{\omega_0}{\gamma} = \frac{2\pi\nu}{\gamma} = \frac{2\pi(330\ Hz)}{0.17s^{-1}} = 1.2 \times 10^4$

$\nu = 330s^{-1}$

iii) $\frac{\Delta E}{E_1} = \frac{2\pi}{Q} = \frac{2\pi}{1.2\times10^{-4}} = 5.2\times10^{-4}$