Taylor Carrechea
Dr. Gustafson
MATH 360
HW 11 Ch.5.3

Ch.5.3 # 3,5,8

Problem 5.3.3 $xy''+2y'+xy=0$

Assume $y(x)=\sum_{m=0}^{\infty} a_m x^{m+r}$

$$y''=\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}, \quad y'=\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}, \quad y=\sum_{m=0}^{\infty}a_m x^{m+r}$$

$$x\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}+2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1}+x\sum_{m=0}^{\infty}a_m x^{m+r}=0$$

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1}+\sum_{m=0}^{\infty}2(m+r)a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r+1}=0$$

$$\sum_{m=0}^{\infty}\Big[(m+r)(m+r-1)+2(m+r)\Big]a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r+1}=0$$

$$\sum_{m=0}^{\infty}(m+r)\Big[(m+r-1)+2\Big]a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r+1}=0$$

$$\sum_{m=0}^{\infty}(m+r)\Big[(m+r+1)\Big]a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r+1}=0$$

$$(r)(r+1)a_0x^{r-1}+(1+r)(r+2)a_1x^{r}+\sum_{m=2}^{\infty}(m+r)(m+r+1)a_m x^{m+r-1}+\sum_{m=0}^{\infty}a_m x^{m+r+1}$$

odd's are 0! when $r=0$

$$r(r+1)a_0x^{r-1}+\sum_{m=0}^{\infty}\Big[(m+r+2)(m+r+3)a_{m+2}+a_m\Big]x^{m+r+1}=0$$

$r(r+1)a_0=0 \longrightarrow$ Indicial Equation $\quad r=0,-1$

So, $y_1(x)=x^{r_1}(a_0+a_1x+a_2x^2+\dots)$
$y_2(x)=Ky_1(x)\ln(x)+x^{r_2}(A_0+A_1x+A_2x^2+\dots)$: $y_2(x)=y_1(x)\ln(x)+x^r\sum A_m x^m$

$$(m+r+2)(m+r+3)a_{m+2}+a_m=0$$

$$a_{m+2}=\frac{-a_m}{(m+r+2)(m+r+3)}$$

$r=0$:

$$a_{m+2}=\frac{-a_m}{(m+2)(m+3)}$$

$r=-1$

$$a_{m+2}=\frac{-a_m}{(m+1)(m+2)}$$

Problem 5.3.3) Continued

$r=0$: $a_0 = arb$, $a_1 = 1$ $r=-1$:

$$a_2 = \frac{-a_0}{(2)(3)} \quad m=0$$

$$a_3 = \frac{-a_1}{(2)(3)} \quad m=1$$

$$a_4 = \frac{-a_2}{(5)(4)} = \frac{a_0}{(5)(4)(3)(2)} \quad m=2$$

$$a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}$$

$$a_6 = \frac{-a_4}{(7)(6)} = \frac{-a_0}{(7)(6)(5)(4)(3)(2)} \quad m=4$$

$$a_7 = \frac{-a_5}{(8)(7)} = \frac{-a_1}{(8)(7)(6)(5)(4)(3)(2)}$$

$r=0$ $$y_1(x) = \left[a_0 + x - \frac{a_0}{3!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0 x^4}{5!} + \frac{a_1}{5!}x^5 - + \cdots\right]$$

$$y_2(x) = K\cdot y_1(x)\cdot \ln(x) + x^{-1}\left(\sum_{m=0}^{\infty} A_m x^m\right)$$: Original ODE: $xy'' + 2y' + xy = 0$

$$y_2(x) = K\cdot y_1\cdot \ln(x) + \sum_{m=0}^{\infty} A_m x^{m-1}$$: $$y_2(x)' = K\cdot y_1'\cdot \ln(x) + K\cdot y_1\cdot\frac{1}{x} + \sum_{m=1}^{\infty}(m-1)A_m x^{m-2}$$

$$y_2(x)'' = K\cdot y_1''\cdot \ln(x) + K\cdot y_1'\cdot\frac{1}{x} + K\cdot y_1'\cdot\frac{1}{x} - K\cdot y_1\cdot\frac{1}{x^2} + \sum_{m=2}^{\infty}(m-2)(m-1)A_m x^{m-3}$$

$$xy_2'' = \left[K\cdot x\cdot y_1''\cdot \ln(x) + 2\cdot K\cdot y_1' - K\cdot y_1'\cdot\frac{1}{x} + \sum_{m=2}^{\infty}(m-2)(m-1)A_m x^{m-2}\right]$$

$$2y_2' = \left[2\cdot K\cdot y_1'\cdot \ln(x) + 2\cdot K\cdot y_1\cdot\frac{1}{x} + 2\sum_{m=1}^{\infty}(m-1)A_m x^{m-2}\right]$$

$$xy_2 = \left[K\cdot x\cdot y_1\cdot \ln(x) + \sum_{m=0}^{\infty} A_m x^m\right]$$

Putting into ODE: $xy'' + 2y' + xy = 0$

$$\left[K\cdot x\cdot y_1''\cdot \ln(x) + 2\cdot K\cdot y_1' - K\cdot y_1'\cdot\frac{1}{x} + \sum_{m=2}^{\infty}(m-2)(m-1)A_m x^{m-2}\right] +$$
$$\left[2\cdot K\cdot y_1'\cdot \ln(x) + 2\cdot K\cdot y_1\cdot\frac{1}{x} + 2\sum_{m=1}^{\infty}(m-1)A_m x^{m-2}\right] + \left[K\cdot x\cdot y_1\cdot \ln(x) + \sum_{m=0}^{\infty} A_m x^m\right] \rightarrow *$$

$$\sum_{m=2}^{\infty}(m-2)(m-1)A_m x^{m-2} \rightarrow \sum_{m=0}^{\infty}(m)(m+1)A_{m+2}x^m$$

$$2\sum_{m=1}^{\infty}(m-1)A_m x^{m-2} \rightarrow 2\sum_{m=-1}^{\infty}(m+1)A_{m+2}x^m \rightarrow 2\cancelto{0}{(-1+1)}A_1 x^{-1} + 2\sum_{m=0}^{\infty}(m+1)A_{m+2}x^m$$

$$\sum_{m=0}^{\infty}A_m x^m \rightarrow \sum_{m=0}^{\infty}A_m x^m$$

Problem 5.3.3 Continued

$$k\cdot x\cdot y_1''\cancel{\ln(x)} + 2\cdot\cancel{k}\cdot y_1' - k\cdot\cancel{y_1'}\cdot\frac{1}{x} + 2\cdot k\cdot\cancel{y_1'}\cdot\ln(x) + 2\cdot k\cdot\cancel{y_1}\cdot\frac{1}{x} + k\cdot x\cdot\cancel{y_1}\cdot\ln(x) + \sum_{m=0}^{\infty}\Big[(m)(m+1)A_{m+2} + \ldots$$

$$\ldots\ 2(m+1)A_{m+2} + A_m\Big]x^m = 0 \qquad \text{original ODE: } xy''+2y'+xy=0$$

$$k\cdot x\cdot y_1''\cdot\ln(x) + 2\cdot k\cdot y_1'\cdot\ln(x) + k\cdot x\cdot y_1\cdot\ln(x) + 2\cdot k\cdot y_1' - k\cdot y_1'\cdot\frac{1}{x} + 2\cdot k\cdot y_1\cdot\frac{1}{x}$$

$$\underbrace{(x\cdot y_1'' + 2\cdot y_1' + x\cdot y_1)\cdot k\cdot\ln(x)}_{=0\,!} + k\cdot y_1'\left(2-\frac{1}{x}\right) + 2\cdot k\cdot y_1\cdot\frac{1}{x} + \sum_{m=0}^{\infty}\Big[(m)(m+1)A_{m+2} + 2(m+1)A_{m+2} + A_m\Big]x^m = 0$$

$$y_1(x) = \left[a_0 + x - \frac{a_0}{3!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0x^4}{5!} + \frac{a_1}{5!}x^5 - + \ldots\ldots\right]$$

$$y_1'(x) = \left[1 - \frac{2a_0}{3!}x - \frac{3a_1}{3!}x^2 + \frac{4a_0}{5!}x^3 + \frac{5a_1}{5!}x^4\right]$$

$$\sum_{m=0}^{\infty}\Big[(m)(m+1)A_{m+2} + 2(m+1)A_{m+2} + A_m\Big]x^m = k\left[-y_1'(2-1/x) - 2y_1\cdot\frac{1}{x}\right]$$

$$\sum_{m=0}^{\infty}\Big[(m)(m+1)A_{m+2} + 2(m+1)A_{m+2} + A_m\Big]x^{m+1} = k\Big[-y_1'(2x-1) - 2y_1\Big]$$

$$\sum_{m=0}^{\infty}\Big[A_{m+2}(m+1)(m+2) + A_m\Big]x^{m+1} = k\Big[-2x\cdot y_1' + y_1' - 2y_1\Big] \qquad \text{let, } k=1$$

$$-2x\cdot y_1' = \left[-2x + \frac{4a_0}{3!}x^2 + \frac{6a_1}{3!}x^3 - \frac{8a_0}{5!}x^4 - \frac{10a_1}{5!}x^5\right],\quad +y_1' = \left[1 - \frac{2a_0}{3!}x - \frac{3a_1}{3!}x^2 + \frac{4a_0}{5!}x^3 + \frac{5a_1}{5!}x^4\right]$$

$$-2y_1 = \left[-2a_0 - 2x + \frac{2a_0}{3!}x^2 + \frac{2a_1}{3!}x^3 - \frac{2a_0x^4}{5!} - \frac{2a_1x^5}{5!}\right]$$

$$(1-2a_0) - \left(2 + \frac{2a_0}{3!} + 2\right)x + \left(\frac{4a_0}{3!} - \frac{3a_1}{3!} + \frac{2a_0}{3!}\right)x^2 + \left(\frac{6a_1}{3!} + \frac{4a_0}{5!} + \frac{2a_1}{3!}\right)x^3 + \left(-\frac{10a_1}{5!} + \frac{5a_1}{5!} - \frac{2a_0}{5!}\right)x^4$$

$$-\left(\frac{10a_1}{5!} + \frac{2a_1}{5!}\right)x^5$$

$$= (1-2a_0) - \left(4 + \frac{2a_0}{3!}\right)x + \left(\frac{6a_0}{3!} - \frac{3a_1}{3!}\right)x^2 + \left(\frac{4a_0}{5!} + \frac{8a_1}{3!}\right)x^3 + \left(-\frac{2a_0}{5!} - \frac{5a_1}{5!}\right)x^4 - \left(\frac{12a_1}{5!}\right)x^5$$

① ② ③ ④ ⑤ ⑥

$$m=0: (2A_2 + A_0)x = \left(4 + \frac{2a_0}{3!}\right)(-x) \rightarrow 2A_2 + A_0 = -4 - \frac{2a_0}{3!} \ :\ A_2 = -2,\ A_0 = -\frac{2a_0}{3}$$

$$m=1: (6A_3 + A_1)x^2 = \left(\frac{6a_0}{3!} - \frac{3a_1}{3!}\right)x^2 \rightarrow 6A_3 + A_1 = \frac{6a_0}{3!} - \frac{3a_1}{3!} \ :\ A_3 = \frac{a_0}{3!},\ A_1 = -\frac{a_1}{2}$$

$$y_2(x) = k\cdot y_1(x)\cdot\ln(x) + \sum_{m=0}^{\infty}A_m x^{m-1} \quad \rightarrow \quad y_2(x) = k\cdot y_1(x)\cdot\ln(x) + \left(-\tfrac{2}{3}a_0\cdot\tfrac{1}{x} - a_1/2 - 2x + \tfrac{a_0}{3!}x^2\right)$$

$$y_2(x) = kx\cdot y_1(x)\ln(x) + \left(-\tfrac{2}{3}a_0 - a_1/2\,x - 2x^2 + a_0/3!\,x^3 + \ldots\right)$$

$$\boxed{\begin{aligned} y_1(x) &= a_0 + x - \frac{a_0}{3!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{5!}x^4 + \ldots\ldots \\ y_2(x) &= kx\cdot y_1(x)\ln(x) + \left(-\tfrac{2}{3}a_0 - a_1/2\,x - 2x^2 + a_0/3!\,x^3 + \ldots\right)\end{aligned}}$$

Problem 5.3.5

$$xy'' + (2x+1)y' + (x+1)y = 0$$

Assume: $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}, \quad y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}, \quad y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$x\sum_{m=0}^{\infty} (m+r)\cancel{(m+r-1)}a_m x^{m+r-2} + 2x\sum_{m=0}^{\infty} \cancel{(m+r)}a_m x^{m+r-1} + \sum_{m=0}^{\infty} \cancel{(m+r)}a_m x^{m+r-1} + x\cancel{\sum_{m=0}^{\infty} a_m x^{m+r}} + \cancel{\sum_{m=0}^{\infty} a_m x^{m+r}}$$

$$\sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}\big((m+r-1)+1\big) + 2\sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1}$$

$$\sum_{m=0}^{\infty} (m+r)^2 a_m x^{m+r-1} + 2\sum_{m=0}^{\infty} (m+r+1)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1}$$

$$r^2 a_0 x^{r-1} + (1+r)^2 a_1 x^r + \sum_{m=2}^{\infty} (m+r)^2 a_m x^{m+r-1} + 2(r+1)a_0 x^r + \sum_{m=1}^{\infty} 2(m+r+1)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+1} = 0$$

$$r^2 a_0 x^{r-1} + (1+r)^2 a_1 x^r + 2(r+1)a_0 x^r + \sum_{m=0}^{\infty}\Big[(m+r+2)^2 a_{m+2} + 2(m+r+2)a_{m+1} + a_m\Big]x^{m+r+1} = 0$$

Indicial EQ: $r^2 a_0 = 0 \quad \therefore \quad r = 0$: $y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \ldots)$

$y_2(x) = y_1(x)\ln(x) + x^r(A_0 + A_1 x + A_2 x^2)$

$r=0$: $(1+r)^2 a_1 x^r + 2(r+1)a_0 x^r$

$a_1 + 2a_0 = 0 \quad \therefore \quad a_1 = -2a_0$

$$(m+r+2)^2 a_{m+2} + 2(m+r+2)a_{m+1} + a_m = 0$$

$$a_{m+2} = \frac{-2(m+2)a_{m+1} - a_m}{(m+2)^2}$$

$m=0$: $a_2 = \dfrac{-4a_1 - a_0}{4} = \dfrac{8a_0 - a_0}{4} = \dfrac{7a_0}{4}$ $\qquad -\dfrac{42}{36} = -\dfrac{14}{12} = -\dfrac{7}{6}$

$m=1$: $a_3 = \dfrac{-6a_2 - a_1}{9} = \dfrac{-6a_2}{9} - \dfrac{a_1}{9} = -\dfrac{7}{6}a_0 + \dfrac{4a_0}{9} = \dfrac{-13}{18}a_0$

$m=2$: $a_4 = \dfrac{-8a_3 - a_2}{16} = -\dfrac{a_3}{2} - \dfrac{a_2}{16} = \dfrac{13}{36}a_0 - \dfrac{7}{64}a_0 = \dfrac{45}{64}a_0$

$$\boxed{y_1(x) = \left(a_0 - 2a_0 x + \tfrac{7}{4}a_0 x^2 - \tfrac{13}{18}a_0 x^3 + \tfrac{45}{64}a_0 x^4 - + \ldots\right.}$$

Problem 5.3.5 Continued

$$y_2(x) = y_1(x)\ln(x) + \sum_{m=0}^{\infty} A_m x^m \quad , \quad y_2(x)' = y_1(x)'\cdot\ln(x) + y_1(x)\cdot\frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2(x)'' = y_1(x)''\cdot\ln(x) + y_1'(x)\cdot\frac{1}{x} + y_1'(x)\cdot\frac{1}{x} - y_1(x)\cdot\frac{1}{x^2} + \sum_{m=2}^{\infty} m(m-1) A_m x^{m-2}$$

ODE : $xy'' + (2x+1)y' + (x+1)y = 0$

$$x\cdot\left[y_1''\ln(x) + 2\cdot y_1'\cdot\frac{1}{x} - y_1\cdot\frac{1}{x^2} + \sum_{m=2}^{\infty} m(m-1)A_m x^{m-2}\right] + 2x\cdot\left[y_1'\cdot\ln(x) + y_1\cdot\frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}\right] + \ldots\ldots$$

$$\left[y_1'\cdot\ln(x) + y_1\cdot\frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}\right] + x\left[y_1\cdot\ln(x) + \sum_{m=0}^{\infty} A_m x^m\right] + \left[y_1\cdot\ln(x) + \sum_{m=0}^{\infty} A_m x^m\right] = 0$$

$$\left[x\cdot y_1''\cdot\ln(x) + 2\cdot y_1' - y_1\cdot\frac{1}{x} + \sum_{m=2}^{\infty} m(m-1)A_m x^{m-1}\right] + \left[2x\cdot y_1'\cdot\ln(x) + 2\cdot y_1 + 2\sum_{m=1}^{\infty} m A_m x^m\right]$$

$$+ \left[y_1'\cdot\ln(x) + y_1\cdot\tfrac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}\right] + \left[xy_1\cdot\ln(x) + \sum_{m=0}^{\infty} A_m x^{m+1}\right] + \left[y_1\cdot\ln(x) + \sum_{m=0}^{\infty} A_m x^m\right] = 0$$

$$\left[x\cdot y_1''\cdot\ln(x) + 2\cdot y_1' - y_1\cdot\frac{1}{x} + 2x\cdot y_1'\cdot\ln(x) + 2\cdot y_1 + y_1'\cdot\ln(x) + y_1\cdot\tfrac{1}{x} + xy_1\cdot\ln(x) + y_1\cdot\ln(x)\right] + \text{———} \; 1^*$$

$$\left[\underbrace{\sum_{m=2}^{\infty} m(m-1)A_m x^{m-1}}_{a} + \underbrace{2\sum_{m=1}^{\infty} m A_m x^m}_{b} + \underbrace{\sum_{m=1}^{\infty} m A_m x^{m-1}}_{c} + \underbrace{\sum_{m=0}^{\infty} A_m x^{m+1}}_{d} + \underbrace{\sum_{m=0}^{\infty} A_m x^m}_{e}\right] \text{———} \; 2^*$$

2^*: a: $\displaystyle\sum_{m=2}^{\infty} m(m-1)A_m x^{m-1} \rightarrow \sum_{m=1}^{\infty} (m+1)(m)A_{m+1} x^m$

b: $\displaystyle 2\sum_{m=1}^{\infty} m A_m x^m \rightarrow \sum_{m=1}^{\infty} 2(m)A_m x^m$

c: $\displaystyle\sum_{m=1}^{\infty} m A_m x^{m-1} \rightarrow \sum_{m=0}^{\infty} (m+1)A_{m+1} x^m \rightarrow A_1 + \sum_{m=1}^{\infty} (m+1)A_{m+1} x^m$

d: $\displaystyle\sum_{m=0}^{\infty} A_m x^{m+1} \rightarrow \sum_{m=1}^{\infty} A_{m-1} x^m$

e: $\displaystyle\sum_{m=0}^{\infty} A_m x^m \rightarrow A_0 + \sum_{m=1}^{\infty} A_m x^m$ $\qquad$ 2^* Becomes : a+b+c+d+e

$$2^* = \left[A_0 + A_1 + \sum_{m=1}^{\infty}\left[(m+1)(m) + 2(m)A_m + (m+1)A_{m+1} + A_{m-1} + A_m\right]x^m\right]$$

Regrouping 1^*

$$(x\cdot y_1'' + 2x\cdot y_1' + y_1' + x\cdot y_1 + y_1)\ln(x) + 4\cdot y_1'$$

$$1^* = \left[\left(xy'' + (2x+1)y_1' + (x+1)y_1\right)\cdot\ln(x) + 4\cdot y_1'\right]$$

Problem 5.3.5) Continued

$$1^* + 2^* = \left(xy'' + \cancel{(2x+1)y_1'} + (x+1)y_1 \right)^{0} \cdot \ln(x) + 4 \cdot y_1' +$$

$$A_0 + A_1 + \sum_{m=1}^{\infty} \left[A_{m+1}(m+1)(m) + 2(m)A_m + (m+1)A_{m+1} + A_{m-1} + A_m \right] x^m = 0$$

$$4y_1' + A_0 + A_1 + \sum_{m=1}^{\infty} \left[A_{m+1}(m+1)(m) + 2(m)A_m + (m+1)A_{m+1} + A_{m-1} + A_m \right] x^m = 0$$

$$A_0 + A_1 + \sum_{m=1}^{\infty} \left[A_{m+1}(m+1)(m) + 2(m)A_m + (m+1)A_{m+1} + A_{m-1} + A_m \right] x^m = -4 \cdot y_1'$$

$$y_1 = \left(a_0 - 2a_0 x + \tfrac{7}{4} a_0 x^2 - \tfrac{13}{18} a_0 x^3 + \tfrac{45}{64} a_0 x^4 - + \ldots \right)$$

$$y_1' = \left(-2a_0 + \tfrac{7}{2} a_0 x - \tfrac{13}{6} a_0 x^2 + \tfrac{45}{16} \cdot x^3 - + \ldots \right)$$

$$A_0 + A_1 + \sum_{m=1}^{\infty} \left[A_{m+1}(m+1)(m) + 2(m)A_m + (m+1)A_{m+1} + A_{m-1} + A_m \right] x^m = 8a_0 - 14a_0 x + \frac{26}{3} a_0 x^2 - \frac{45}{4} a_0 x^3$$

$$A_0 = 8a_0 \quad , \quad A_1 = -14a_0$$

$$m=1 : \left(2(A_2) + 2A_1 + 2A_2 + A_0 + A_1 \right) x^2 = \frac{26}{3} a_0 x^2$$

$$4A_2 + 3A_1 + A_0 = \frac{26}{3} a_0$$

$$4A_2 - 42a_0 + 8a_0 = \frac{26}{3} a_0 \rightarrow 4A_2 - 34a_0 = \frac{26}{3} a_0 \rightarrow A_2 - \frac{17a_0}{2} = \frac{13}{6} a_0$$

$$A_2 = \frac{13}{6} a_0 + \frac{17}{2} a_0 = \frac{13}{6} a_0 + \frac{81}{6} a_0 = \frac{64}{6} a_0 = \frac{32}{3} a_0$$

$$A_2 = \frac{32}{3} a_0$$

$$m=2 : \left(6A_3 + 4A_2 + 3A_3 + A_1 + A_2 \right) x^3 = \left(-\frac{45}{4} a_0 \right) x^3$$

$$9A_3 + 5A_2 + A_1 = -\frac{45}{4} a_0 \rightarrow 9 \cdot A_3 + \frac{160}{3} a_0 - \frac{42}{3} a_0 = -\frac{45}{4} a_0$$

$$9A_3 - \frac{118}{3} a_0 = -\frac{45}{4} a_0 \rightarrow 9A_3 = \frac{91}{3} \rightarrow A_3 = \frac{91}{27} a_0$$

$$A_3 = \frac{91}{27} a_0$$

$$\boxed{\begin{aligned} y_1(x) &= \left(a_0 - 2a_0 x + \tfrac{7}{4} a_0 x^2 - \tfrac{13}{18} a_0 x^3 + \tfrac{45}{64} a_0 x^4 - + \ldots \right. \\ y_2(x) &= y_1(x) \cdot \ln(x) + \left(8a_0 - 14a_0 x + \tfrac{32}{3} a_0 x^2 + \tfrac{91}{27} a_0 x^3 \right) \end{aligned}}$$

Problem 5.3.8

$$xy'' + y' - xy = 0$$

Assume: $y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y''(x) = \sum_{m=0}^{\infty} (m+r-1)(m+r)a_m x^{m+r-2}, \quad y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}, \quad y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$x \cdot \sum_{m=0}^{\infty} (m+r-1)(m+r)a_m x^{m+r-2} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} + x \cdot \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$\sum_{m=0}^{\infty} \left[(m+r-1) + 1\right](m+r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = \sum_{m=0}^{\infty} (m+r)^2 a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$m=0: \quad r^2 a_0 x^{r-1} + \sum_{m=1}^{\infty} (m+r)^2 a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0 \rightarrow r^2 a_0 x^{r-1} + \sum_{m=0}^{\infty} (m+r+1)^2 a_{m+1} x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r}$$

Indicial EQ. : $r^2 a_0 = 0 \quad \therefore r = 0$: $y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \ldots\ldots)$, $y_2(x) = y_1(x)\ln(x) + x^r(A_1 x + A_2 x^2 + \ldots)$

Recursion: $\sum_{m=0}^{\infty} \left[(m+r+1)^2 a_{m+1} + a_m\right] x^{m+r} = 0$: $(m+r+1)^2 a_{m+1} + a_m = 0$: $a_{m+1} = \dfrac{-a_m}{(m+r+1)^2}$

$$a_{m+1} = \left.\frac{-a_m}{(m+r+1)^2}\right|_{r=0} \rightarrow a_{m+1} = \frac{-a_m}{(m+1)^2}$$

$m=0$: $a_1 = \dfrac{-a_0}{(1)^2} = -a_0$ $\qquad$ $m=2$: $a_3 = \dfrac{-a_2}{(3)^2} = \dfrac{a_0}{36}$

$m=1$: $a_2 = \dfrac{a_1}{(2)^2} = \dfrac{-a_0}{4}$ $\qquad$ $m=3$: $a_4 = \dfrac{-a_3}{(4)^2} = \dfrac{-a_0}{576}$

So, $\boxed{y_1(x) = a_0 - a_0 x - \frac{a_0}{4}x^2 + \frac{a_0}{36}x^3 - \frac{a_0}{576}x^4 + - \ldots\ldots}$

$$y_2(x) = y_1(x)\ln(x) + \sum_{m=0}^{\infty} A_m x^m, \quad y_2(x)' = y_1(x)'\ln(x) + y_1(x)\cdot\frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2(x)'' = y_1(x)''\ln(x) + y_1(x)'\cdot\frac{1}{x} + y_1(x)'\cdot\frac{1}{x} - y_1(x)\cdot\frac{1}{x^2} + \sum_{m=2}^{\infty} (m-1)m A_m x^{m-2}$$

original ODE : $xy'' + y' - xy = 0$

$$x \cdot y_2(x)'' = x \cdot y_1''\ln(x) + y_1' + y_1' - y_1 \cdot \frac{1}{x} + \sum_{m=2}^{\infty} (m-1)m A_m x^{m-1} \quad : ①$$

$$y_2' = y_1'\ln(x) + y_1 \cdot \frac{1}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1} \quad : ②$$

$$-xy_2' = -xy_1\ln(x) - \sum_{m=0}^{\infty} A_m x^{m+1} \quad : ③$$

① + ② + ③

$$x \cdot y_1''\cancel{\ln(x)} + 2y_1' - \cancel{\frac{y_1}{x}} + y_1'\cancel{\ln(x)} + \cancel{\frac{y_1}{x}} - xy_1\cancel{\ln(x)} + \sum_{m=2}^{\infty} (m-1)m A_m x^{m-1} + \sum_{m=1}^{\infty} m A_m x^{m-1} - \sum_{m=0}^{\infty} A_m x^{m+1} = 0$$

Problem 5.3.8) Continued

$$\overbrace{(x y_1'' + \cancel{y_1'} - x y_1)}^{0}\ln(x) + 2y_1' + \sum_{m=0}^{\infty}(m+1)(m+2)A_{m+2}x^{m+1} + \sum_{m=0}^{\infty}(m+1)A_{m+1}x^{m} - \sum_{m=0}^{\infty}A_m x^{m+1} = 0$$

$$2y_1' = \sum_{m=0}^{\infty}\Big[A_m - (m+1)(m+2)A_{m+2}\Big]x^{m+1} + A_1 + \sum_{m=1}^{\infty}(m+1)A_{m+1}x^{m}$$

$$2y_1' = \sum_{m=0}^{\infty}\Big[A_m - (m+1)(m+2)A_{m+2} + (m+2)A_{m+2}\Big]x^{m+1} + A_1$$

$$2y_1' = \sum_{m=0}^{\infty}\Big[A_m - A_{m+2}(m+2)\big[(m+1)+1\big]\Big]x^{m+1} + A_1$$

$$2y_1' = \sum_{m=0}^{\infty}\Big[A_m - A_{m+2}(m+2)^2\Big]x^{m+1} + A_1$$

$$y_1' = -a_0 - \frac{a_0x}{2} + \frac{a_0}{12}x^2 - \frac{a_0}{144}x^3 \quad : \quad 2\cdot y_1' = -2a_0 - a_0x + \frac{a_0x^2}{6} - \frac{a_0}{72}x^3$$

$$-2a_0 - a_0x + \frac{a_0x^2}{6} - \frac{a_0x^3}{72} = \sum_{m=0}^{\infty}\Big[A_m - A_{m+2}(m+2)^2\Big]x^{m+1} + A_1$$

$A_1 = -2a_0$

$m=0$: $(A_0 - 4A_2)x = -(a_0)x$: $A_0 - 4A_2 = -1$ $\therefore$ $4A_2 = A_0 + 1$: $A_2 = \frac{A_0+1}{4}$

$m=1$: $(A_1 - 9A_3)x^2 = \frac{(a_0)}{6}x^2$: $A_1 - 9A_3 = \frac{a_0}{6}$: $-2a_0 - 9A_3 = \frac{a_0}{6}$: $-9A_3 = \frac{11a_0}{6}$

$A_3 = \frac{-11}{54}a_0$

$m=2$: $(A_2 - 16A_4)x^3 = \frac{-1}{72}a_0x^3$: $16A_4 = A_2 + \frac{1}{72}a_0$: $A_4 = \frac{A_2}{16} + \frac{1}{72\cdot16}a_0 = \frac{A_0+1}{4\cdot16} + \frac{1}{72\cdot16}.a_0$

$A_4 = \frac{A_0+1}{64} + \frac{a_0}{72\cdot16}$

so,

$$y_1(x) = a_0 - a_0x - \frac{a_0}{4}x^2 + \frac{a_0}{36}x^3 - \frac{a_0}{576}x^4 + - \ldots\ldots$$

$$y_2(x) = y_1(x)\cdot\ln(x) + A_0 - 2a_0x + \frac{(A_0+1)}{4}x^2 - \frac{11}{54}a_0x^3 + \left[\frac{A_0+1}{64} + \frac{a_0}{72\cdot16}\right]x^4$$