# PHYS 230 HOMEWORK SET 13 SOLUTIONS

$m = 0.550\,kg$

$F_0 = 2.50N$

$\gamma = \frac{1}{8}\gamma_{cr}$

$k = 46.0\,N/m$

FOR A DAMPED, DRIVEN HARMONIC OSCILLATOR, THE TIME DEPENDENCE OF THE DISPLACEMENT IS GIVEN BY...

$$x(t) = A(\omega)\cos(\omega t - \delta) \quad \text{WHERE}$$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \quad ; \quad \tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$$

THE NATURAL FREQUENCY OF THE SPRING WAS FOUND TO BE...

$$\omega_0 = \sqrt{\frac{k}{m}} = 9.15\,RAD/s$$

THE CRITICAL VALUE OF THE DAMPING PARAMETER WAS FOUND TO BE..

$$\gamma_{cR} = 18.3\,s^{-1}$$

SO OUR DAMPING PARAMETER HERE IS

$$\gamma = \frac{1}{8}(18.3\,s^{-1}) = 2.29\,s^{-1}$$

a) IN CLASS, WE FOUND A FORMULA FOR THE FREQUENCY THAT GIVES RISE TO A MAXIMUM AMPLITUDE, $\omega_{MAX}$, OF THE FORM

$$\omega_{MAX} = \omega_0\sqrt{1 - \gamma^2/2\omega_0^2} = (9.15\,RAD/s)\sqrt{1 - (2.29\,s^{-1})^2/2(9.15\,RAD/s)^2} = 9.00\,s^{-1}$$

b) THE MAXIMUM AMPLITUDE IS GIVEN BY...

$$A(\omega_{MAX}) \equiv A_{MAX} = \frac{F_0\omega_0}{k\gamma}\left(1 - \gamma^2/4\omega_0^2\right)^{-1/2} = \frac{(2.50N)(9.15\,RAD/s)}{(46.0\,N/m)(2.29\,s^{-1})}\left(1 - (2.29\,s^{-1})^2/4(9.15\,RAD/s)^2\right)^{-1/2}$$

$$\boxed{A_{MAX} = 0.219m}$$

c) $$Q = \frac{\omega_0}{\gamma} = \frac{9.15\,RAD/s}{2.29\,s^{-1}} = 4.00$$

d) $$A(\omega) = \frac{2.50N/0.550\,kg}{\sqrt{\left((9.15\,RAD/s)^2 - (2\pi\,RAD/s)^2\right)^2 + (2\pi\,RAD/s)^2(2.29\,s^{-1})^2}} = \frac{4.545\,m/s^2}{\sqrt{1958\,s^{-4} + 207.0\,s^{-4}}}$$

$$\boxed{A(\omega) = 0.0977\,m}$$

e) $\delta = \tan^{-1}\left(\dfrac{(2\pi \text{ rad/s})(2.29 s^{-1})}{(9.15 \text{ rad/s})^2 - (2\pi \text{ rad/s})^2}\right) = \tan^{-1}(0.325) = 18.0°$

so, in summary...

$x(t) = A(\omega)\cos(\omega t - \delta)$ where $A(\omega) = 0.0977 m^2$

$\delta = 18.0°$

2. $x(t) = A(\omega)\cos(\omega t - \delta) + Be^{-\gamma t/2}\cos(\omega_1 t + \phi)$

NOW

$\omega = 2\pi$ RAD/s

$A(\omega) = 0.0977m$

$\delta = 18.0°$

$\gamma = 2.29s^{-1}$

NOW

$\omega_1 = \sqrt{\omega_0^2 - (\gamma/2)^2}$

$= \sqrt{(9.15 \text{RAD/s})^2 - (2.29s^{-1}/2)^2} = 9.08$ RAD/s

$\omega_1 = 9.08$ RAD/s

WITH INITIAL CONDITIONS...

$x(t=0) = +0.0550m$

$\dot{x}(t=0) = +0.450m/s$

NOW

a) $\frac{dx}{dt} = \frac{d}{dt}\left[A(\omega)\cos(\omega t - \delta) + Be^{-\gamma t/2}\cos(\omega_1 t + \phi)\right]$

$= -\omega A(\omega)\sin(\omega t - \delta) + B\left[-\frac{\gamma}{2}e^{-\gamma t/2}\cos(\omega_1 t + \phi) + e^{-\gamma t/2} \cdot -\omega_1 \sin(\omega_1 t + \phi)\right]$

$= -\omega A(\omega)\sin(\omega t - \delta) - Be^{-\gamma t/2}\left[\frac{\gamma}{2}\cos(\omega_1 t + \phi) + \omega_1 \sin(\omega_1 t + \phi)\right]$

b) APPLYING THE FIRST INITIAL CONDITION...

(*) $x(t=0) = A(\omega)\cos(-\delta) + B\cos(\phi) = +0.0550m$

$= (0.0977m)\cos(-18.0°) + B\cos(\phi) = 0.0929m + B\cos(\phi) = +0.0550m$

SO $\left[B\cos(\phi) = -0.0379m\right]$

ALSO

(*) $\dot{x}(t=0s) = -\omega A(\omega)\sin(-\delta) - B\left[\frac{\gamma}{2}\cos(\phi) + \omega_1 \sin(\phi)\right] = +0.450m/s$

$= -(2\pi \text{RAD/s})(0.0977m)\sin(-18.0°) - B\left[\frac{(2.29s^{-1})}{2}\cos(\phi) + (9.08 \text{RAD/s})\sin(\phi)\right]$

$= 0.190m/s - B\left[(1.15s^{-1})\cos\phi + (9.08s^{-1})\sin\phi\right] = +0.450m/s$

SO $B\left[(1.15s^{-1})\cos(\phi) + (9.08s^{-1})\sin(\phi)\right] = -0.260\ m/s$

SO IN SUMMARY...

1) $B\cos\phi = -0.0379\,m$

2) $B\left[(1.15\,s^{-1})\cos\phi + (9.08\,s^{-1})\sin\phi\right] = -0.260\,m/s$

) NOW

$B = -\dfrac{0.0379\,m}{\cos\phi}$ . PLUGGING THIS INTO THE $2^{nd}$ EXPRESSION YIELDS..

$$-\frac{0.0379\,m}{\cos\phi}\left[(1.15\,s^{-1})\cos\phi + (9.08\,s^{-1})\sin\phi\right] = -0.260\,m/s$$

or

$$1.15\,s^{-1} + (9.08\,s^{-1})\tan\phi = 6.86\,s^{-1}$$

or

$$\tan\phi = 0.629 \quad \therefore \quad \boxed{\phi = 32.2°}$$

THIS YIELDS..

$$B = -\frac{0.0379\,m}{\cos(32.2°)} = -0.0448\,m$$

SO IN SUMMARY...

$$\boxed{\begin{aligned} B &= -0.0448\,m \\ \phi &= 32.2° \end{aligned}}$$

CHECK: PLUGGING OUR RESULTS INTO THE LEFT HAND SIDE OF (*) YIELDS..

$$x(t=0s) = (0.0977\,m)\cos(-18.0°) + (-0.0448\,m)\cos(32.2°)$$

$$= 0.0929\,m - 0.0379\,m = 0.0550\,m \checkmark$$

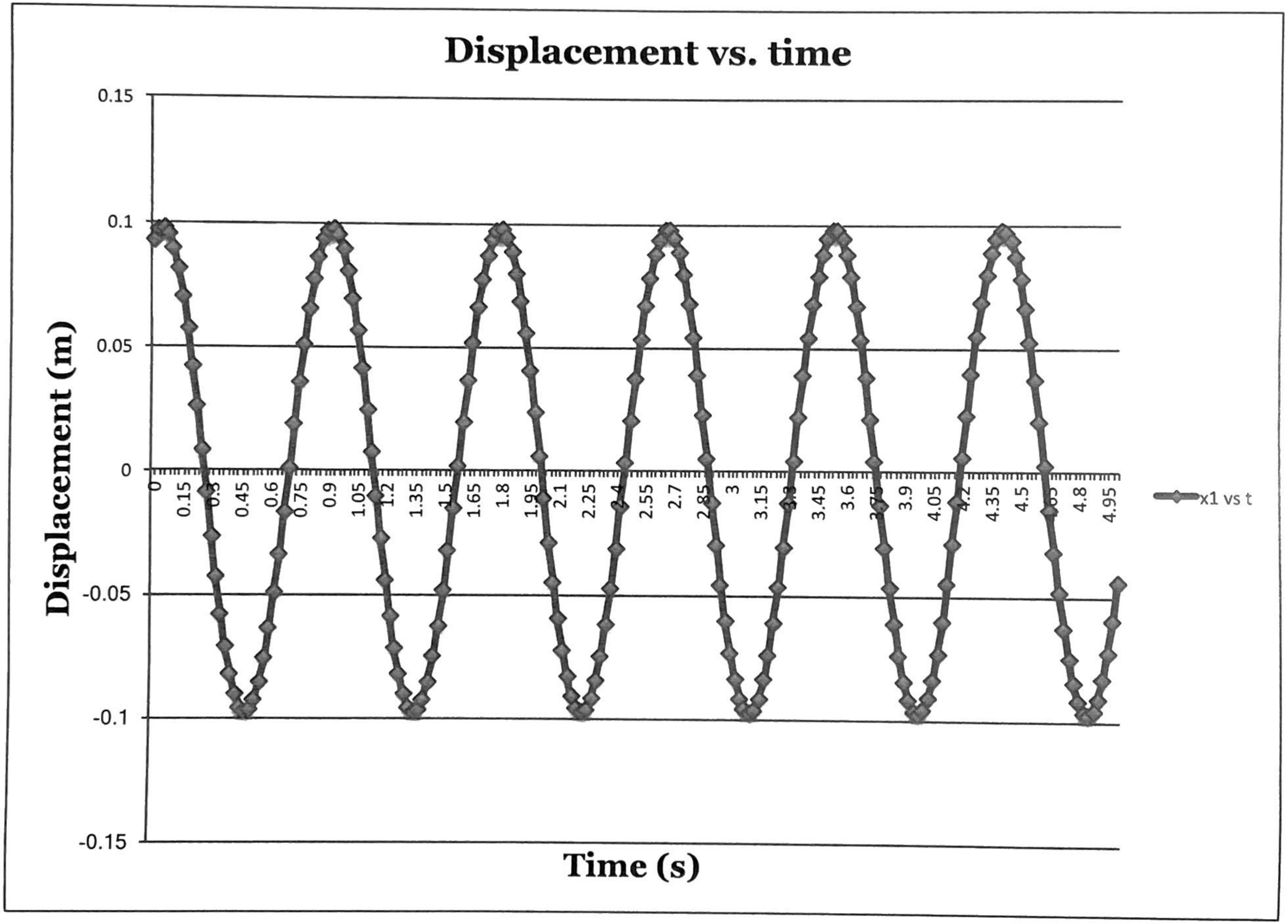
NOW PLUGGING OUR RESULTS INTO THE LHS OF (**) YIELDS...

$$\dot{x}(t=0s) = -(2\pi\,rad/s)(0.0977\,m)\sin(-18.0°) - (-0.0448\,m)\left[\left(\frac{2.29\,s^{-1}}{2}\right)\cos(32.2°) + \left(9.08\,\frac{rad}{s}\right)\sin(32.2°)\right]$$

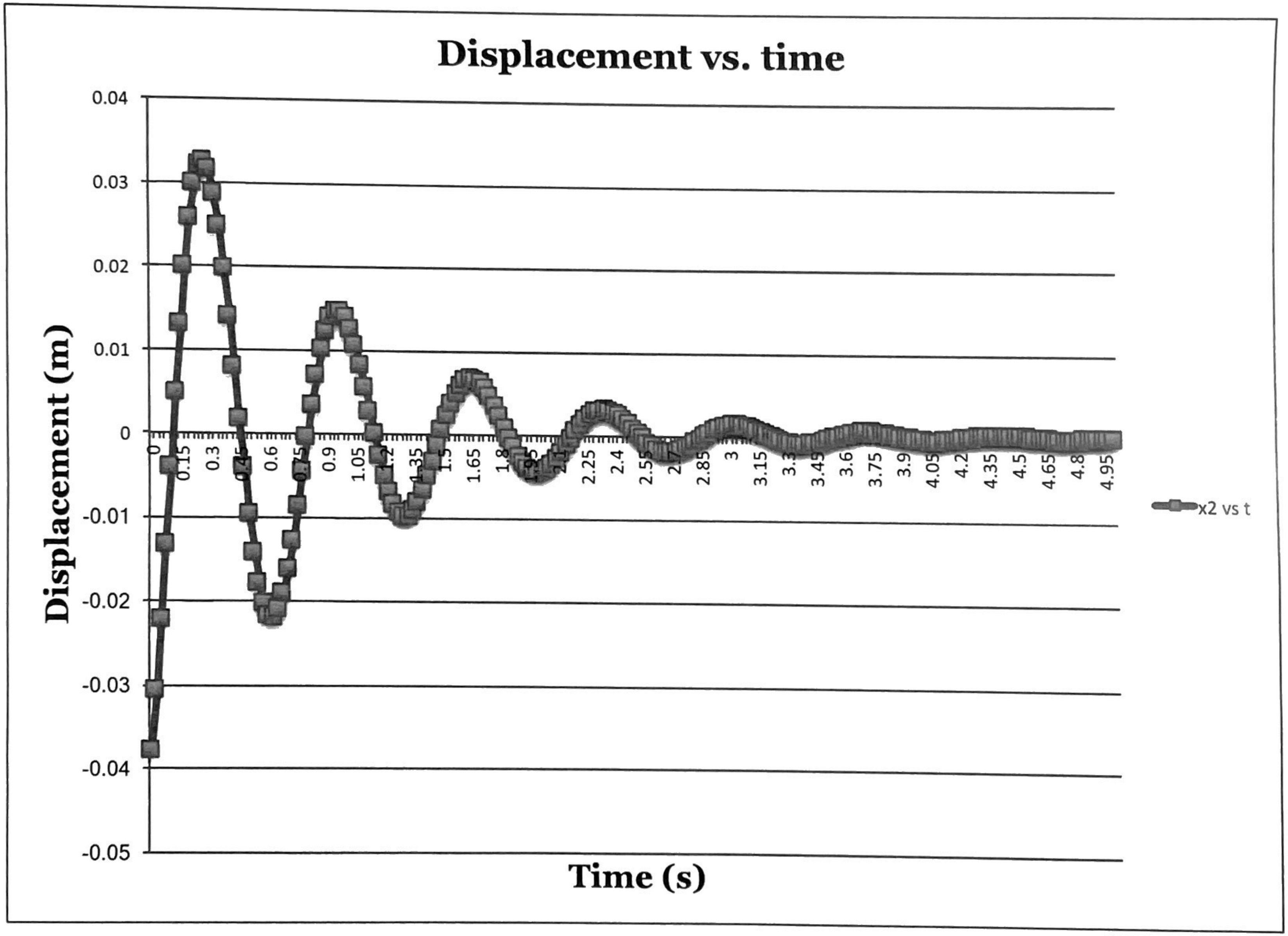
$$= (0.190\,m/s) + (0.0448\,m)\left[(0.969\,s^{-1}) + (4.84\,s^{-1})\right] = +0.450\,m/s \checkmark$$

| | | |
|---|---|---|
| $A =$ | 0.0977 | m |
| $\omega_0 =$ | 9.15 | 1/s |
| $\gamma =$ | 2.29 | 1/s |
| $\omega =$ | 6.2832 | 1/s |
| $\delta =$ | 0.31416 | |
| $B =$ | -0.0448 | m |
| $\varphi =$ | 0.562 | rad |
| $\omega_1 =$ | 9.08 | 1/s |

| t(s) | $\gamma t/2$ | $x_1=A*\cos(\omega t-\delta)$ | $x_2=B*\exp(-\gamma t/2)*\cos(\omega_1 t+\varphi)$ | $x_1+x_2$ |
|---|---|---|---|---|
| 0 | 0 | 0.09291823 | -0.037909359 | 0.055008871 |
| 0.025 | 0.02863 | 0.09682021 | -0.030673366 | 0.066146844 |
| 0.05 | 0.05725 | 0.097598645 | -0.022286191 | 0.075312454 |
| 0.075 | 0.08588 | 0.095228422 | -0.013236754 | 0.081991668 |
| 0.1 | 0.1145 | 0.089786007 | -0.004020318 | 0.08576569 |
| 0.125 | 0.14313 | 0.08144698 | 0.004886953 | 0.086333933 |
| 0.15 | 0.17175 | 0.070480369 | 0.013051048 | 0.083531417 |
| 0.175 | 0.20038 | 0.05723997 | 0.020099755 | 0.077339725 |
| 0.2 | 0.229 | 0.042152936 | 0.025738073 | 0.067891009 |
| 0.225 | 0.25763 | 0.025705993 | 0.029758862 | 0.055464855 |
| 0.25 | 0.28625 | 0.008429743 | 0.032048443 | 0.040478186 |
| 0.275 | 0.31488 | -0.009118462 | 0.032587164 | 0.023468702 |
| 0.3 | 0.3435 | -0.026372494 | 0.031445156 | 0.005072662 |
| 0.325 | 0.37213 | -0.042775715 | 0.028773788 | -0.014001927 |
| 0.35 | 0.40075 | -0.057798937 | 0.02479348 | -0.033005457 |
| 0.375 | 0.42938 | -0.07095749 | 0.019778691 | -0.051178799 |
| 0.4 | 0.458 | -0.081826863 | 0.014041015 | -0.067785848 |
| 0.425 | 0.48663 | -0.090056396 | 0.007911326 | -0.08214507 |
| 0.45 | 0.51525 | -0.095380594 | 0.001721945 | -0.093658649 |



Displacement vs. time
Displacement (m)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
0.15
0.3
0.45
0.6
0.75
0.9
1.05
1.2
1.35
1.5
1.65
1.8
1.95
2.1
2.25
2.4
2.55
2.7
2.85
3
3.15
3.3
3.45
3.6
3.75
3.9
4.05
4.2
4.35
4.5
4.8
4.95
x1 vs t
Time (s)



Displacement vs. time
Displacement (m)
Time (s)
x2 vs t
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.15
0.3
0.45
0.6
0.75
0.9
1.05
1.2
1.35
1.5
1.65
1.8
1.95
2.1
2.25
2.4
2.55
2.7
2.85
3
3.15
3.3
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3.6
3.75
3.9
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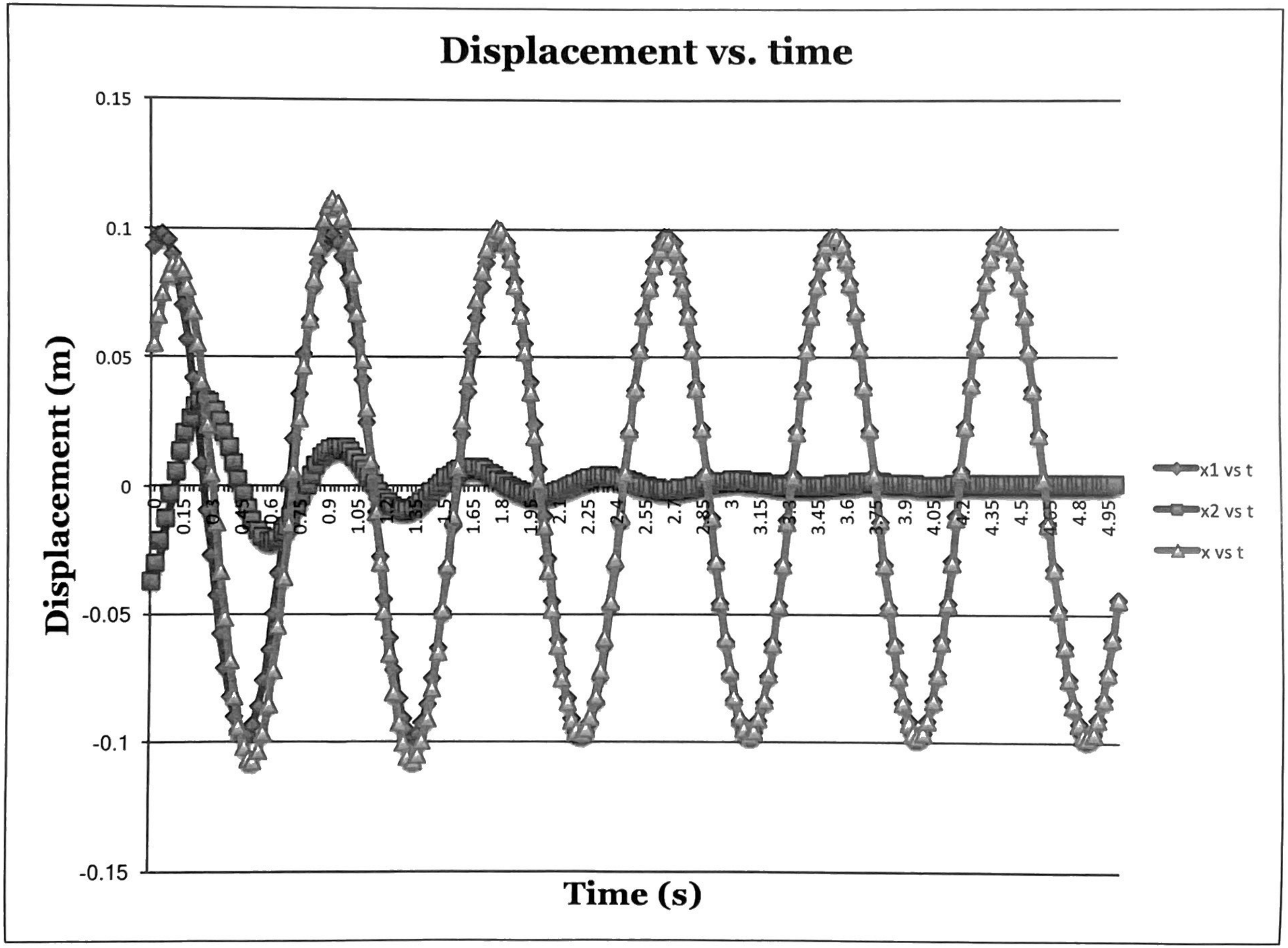
4.35
4.5
4.65
4.8
4.95



Displacement vs. time
Displacement (m)
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
0
0.15
0.3
0.45
0.6
0.75
0.9
1.05
1.2
1.35
1.5
1.65
1.8
1.95
2.1
2.25
2.4
2.55
2.7
2.85
3
3.15
3.3
3.45
3.6
3.75
3.9
4.05
4.2
4.35
4.5
4.65
4.8
4.95
Time (s)
x1 vs t
x2 vs t
x vs t

4. $z(x,t) = (0.21m)\sin\left((0.36m^{-1})x + (0.52s^{-1})t\right)$

$= A\sin(kx + \omega t)$

So

a) $A = 0.21m$ - THE AMPLITUDE

$k = (0.36m^{-1}) = 2\pi/\lambda$ so $\lambda = \frac{2\pi}{(0.36m^{-1})} = 17m$ - THE WAVELENGTH

$v = \frac{\omega}{k} = \frac{(0.52s^{-1})}{(0.36m^{-1})} = 1.4m/s$ - THE VELOCITY OF THE WAVE

b) THE WAVE IS TRAVELING IN THE $-x$ DIRECTION.

c) NOW

$\frac{\partial z}{\partial x} = kA\cos(kx+\omega t)$ $\quad$ $\frac{\partial z}{\partial t} = \omega A\cos(kx+\omega t)$

$\frac{\partial^2 z}{\partial x^2} = -k^2 A\sin(kx+\omega t)$ $\quad$ $\frac{\partial^2 z}{\partial t^2} = -\omega^2 A\sin(kx+\omega t)$

THE WAVE EQN IS OF THE FORM..

$\frac{\partial^2 z}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$ so

$-k^2 A\sin(kx+\omega t) = \frac{1}{v^2}\cdot -\omega^2 A\sin(kx+\omega t)$

or

$k^2 = \frac{\omega^2}{v^2}$ $\therefore$ $v = \frac{\omega}{k}$ ✓

d) THE SPEED OF A POINT ON THE STRING IS GIVEN BY..

$\frac{\partial z}{\partial t} = \omega A\cos(kx+\omega t)$ WHICH HAS A MAXIMUM VALUE WHEN $\cos(kx+\omega t) = 1$

so

$\left(\frac{\partial z}{\partial t}\right)_{max} = \omega A = (0.52s^{-1})(0.21m) = 0.11m/s$

THE ACCELERATION OF A POINT ON THE STRING IS GIVEN BY...

$\frac{\partial^2 z}{\partial t^2} = -\omega^2 A\sin(kx+\omega t)$ WHICH HAS A MAXIMUM VALUE OF $\left(\frac{\partial^2 z}{\partial t^2}\right)_{max} = \omega^2 A = (0.52s^{-1})^2(0.21m)$

$= 0.057m/s^2$