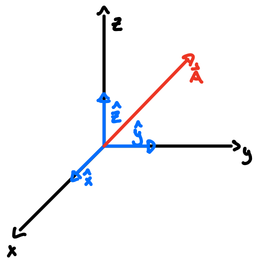
8-20-18

Ch. 1 Vector Analysis

Scalars
- Magnitude only
- Invariant under co-ordinate transformations

Vectors
- Direction and Magnitude
- Boldface in text, arrowed quantity in notes.
- Let $\hat{x}, \hat{y}, \hat{z}$ Be basis vector

Vector Space



Let $\vec{A}$ be expanded in terms of these basis vectors....

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$\hookrightarrow A_x, A_y, A_z$ are components of $\vec{A}$

Four vector operations : Addition : Three Kinds of Multiplication

i) Addition of two vectors

Geometrically : Tip to tail method

Algebraically : $\vec{A}+\vec{B} = (A_x\hat{x}+A_y\hat{y}+A_z\hat{z}) + (B_x\hat{x}+B_y\hat{y}+B_z\hat{z}) = (A_x+B_x)\hat{x} + (A_y+B_y)\hat{y} + (A_z+B_z)\hat{z}$

$= (B_x+A_x)\hat{x} + (B_y+A_y)\hat{y} + (B_z+A_z)\hat{z} = \vec{B}+\vec{A}$

Note: vector addition is commutative and associative

$\vec{A} + (\vec{B}+\vec{C}) = (\vec{A}+\vec{B}) + \vec{C}$

Notice:

$\vec{A} - \vec{A} = \vec{0} = \vec{A} + (-\vec{A})$

- To Subtract a vector, Add opposite :

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

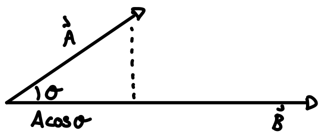
ii) Multiplication of a vector by a Scalar :

$a\vec{A} = a(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}) = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}$

$a(\vec{A}+\vec{B}) = a\vec{A} + a\vec{B}$

iii) Dot (Scalar) product of two vectors

$\vec{A}\cdot\vec{B} = AB\cos\theta$



So....

$\vec{A}\cdot\vec{B} = (A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\cdot(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$

$\hat{x}\cdot\hat{x}=1 \;:\; \hat{y}\cdot\hat{y}=1 \;:\; \hat{z}\cdot\hat{z}=1$

$\hat{x}\cdot\hat{y}=0 \;:\; \hat{x}\cdot\hat{z}=0 \; ....$

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ $\Rightarrow$ To calculate the scalar product, multiply like components and add

$\vec{B}\cdot\vec{A} = B_xA_x + B_yA_y + B_zA_z$ $\therefore$ Dot product is commutative

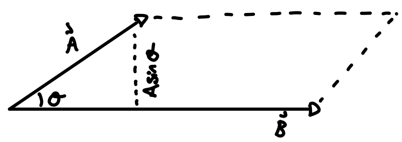
Notice:

$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2$

iv) Cross Product (Vector Product) of Two vectors

$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ $\quad\hat{n}$ - unit vector Normal to $\vec{A}$ & $\vec{B}$ plane

Direction given by RHR



Cross Product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x)$$

$$\vec{A}\times\vec{B} = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = -\vec{B}\times\vec{A}$$

$\therefore$ Cross Product is not commutative

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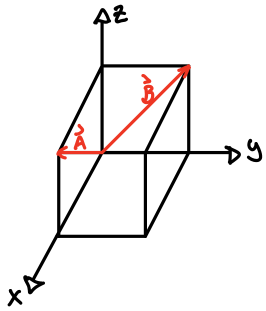
Geometrically: $|\vec{A}\times\vec{B}|$ is the area of the parallelogram generated by $\vec{A}$: $\vec{B}$

$\vec{A}\times\vec{A}=0$

ie. Find the angle between the face diagonals of a unit cube.



$\vec{A}=\hat{x}+\hat{z}$

$\vec{B}=\hat{y}+\hat{z}$

$\vec{A}\cdot\vec{B}=(\hat{x}+\hat{z})\cdot(\hat{y}+\hat{z})=1$

$1=AB\cos\theta$

$\vec{A}\cdot\vec{A}=A^2=2 \quad \therefore \quad A=\sqrt{2}$

$\vec{B}\cdot\vec{B}=B^2=2 \quad \therefore \quad B=\sqrt{2}$

$1=\sqrt{2}\sqrt{2}\cos\theta$

$1=2\cos\theta$

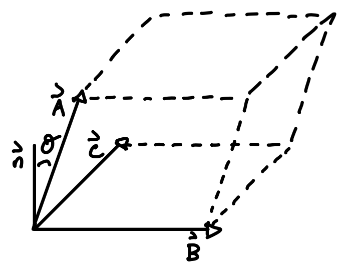
$\theta=\cos^{-1}(1/2)=60^\circ$

Scalar Triple Product

$$\vec{A}\cdot(\vec{B}\times\vec{C})=(A_x\hat{x}+A_y\hat{y}+A_z\hat{z})\cdot\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}=\begin{vmatrix}A_x&A_y&A_z\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}=A_x(B_yC_z-B_zC_y)+A_y(B_xC_z-B_zC_x)+A_z(B_xC_y-B_yC_x)$$

Geometrically: $|\vec{A}\cdot(\vec{B}\times\vec{C})|$ is the volume of the parallelpiped generated by $\vec{A},\vec{B},\vec{C}$

$|\vec{B}\times\vec{C}|$ is the area of the base.



Notice: $$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}A_x&A_y&A_z\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}=-\begin{vmatrix}A_x&A_y&A_z\\C_x&C_y&C_z\\B_x&B_y&B_z\end{vmatrix}=\begin{vmatrix}C_x&C_y&C_z\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

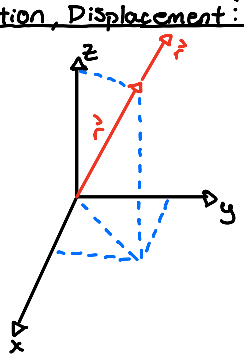
$=\vec{C}\cdot(\vec{A}\times\vec{B})=(\vec{A}\times\vec{B})\cdot\vec{C}$

$(\cdot,\times)\rightarrow(\times,\cdot)$

Vector Triple Product

$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

Position, Displacement : Seperation vectors



The position vector is ....

$[\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}]$ in 3D Cartesian coordinates

The magnitude is ....

$r=(\vec{r}\cdot\vec{r})^{1/2}=[(x\hat{x}+y\hat{y}+z\hat{z})\cdot(x\hat{x}+y\hat{y}+z\hat{z})]^{1/2}$

$[r=\sqrt{x^2+y^2+z^2}$

The radial unit vector.....

$$\hat{r}=\frac{\vec{r}}{|\vec{r}|}=\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

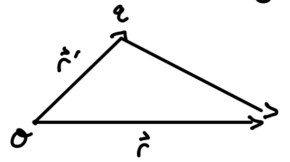
- The infinitesimal displacement vector is ...

$d\vec{l}=(\vec{r}+d\vec{r})-\vec{r}=[(x+dx)\hat{x}+(y+dy)\hat{y}+(z+dz)\hat{z}]$

$-[x\hat{x}+y\hat{y}+z\hat{z}]$

$d\vec{l}=dx\hat{x}+dy\hat{y}+dz\hat{z}$

Imagine an Electric charge located at $\vec{r}'$: we wish to Calculate the electric (or magnetic) Field at $\vec{r}$....



The Seperation vector, $\vec{\mathscr{r}}$, is...

$$[\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}']$$

$$\vec{\mathscr{r}} = (x\hat{x} + y\hat{y} + z\hat{z}) - (x'\hat{x} + y'\hat{y} + z'\hat{z})$$
$$= (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

It's magnitude is ...

$$\mathscr{r} = (\vec{\mathscr{r}} \cdot \vec{\mathscr{r}})^{1/2} = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

The unit vector in the direction of $\vec{\mathscr{r}}$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

8-24-18 Differential Calculus

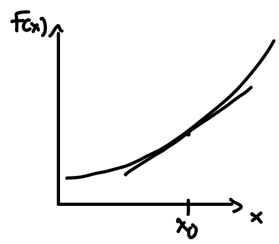
The derivative....

Consider $f(x)$....

Q: If we change x by a tiny amount, dx, how does $f(x)$ change.

A: $dF = \left(\frac{dF}{dx}\right) dx$

Geometrically: $\frac{dF}{dx}$ is the slope of the $f(x)$ vs. x curve



$$\text{Slope} = \left.\frac{dF}{dx}\right|_{x=x_0}$$

The gradient....

Consider $T(x,y,z)$....

Q: If we change x,y,z by a tiny amount, dx, dy, dz, how does $T(x,y,z)$ change?

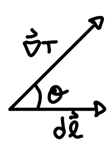
A: $$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$
$$= \left[\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right] \cdot \left[\hat{x}dx + \hat{y}dy + \hat{z}dz\right]$$
$$= \vec{\nabla} T \cdot d\vec{\ell}$$

The gradient of T

Note: T is a scalar, $\vec{\nabla} T$ is a vector.

Now....

$$dT = |\vec{\nabla} T||d\vec{\ell}|\cos\theta$$



Notice: For a fixed $|\vec{\ell}|$, dT is a maximum when $\theta = 0$

(since $\cos\theta = 1$)

$$(dT)_{max} = |\vec{\nabla} T| d\ell \quad \text{where } d\ell = |d\vec{\ell}|$$

$\therefore$ The gradient points in the direction of maximum increase of the function T.

$|\vec{\nabla}T|$ gives the rate of increase along this maximum direction.
i.e. Contour Map

$Z(x,y)$

now
$dz = \vec{\nabla}z \cdot d\vec{l} = |\vec{\nabla}z|\, dl \cos\theta$

$\vec{\nabla}z = 0\hat{x} + 0\hat{y} + 0\hat{z}$
- Stationary point

- Direction of steepest ascent is direction of $\vec{z}$
- $|\vec{\nabla}z|$ is the slope of the steepest ascent
- When $\theta = 90°$, $dz = |\vec{\nabla}z|\, dl \cos(90°) = 0$

$\Rightarrow$ Contour lines $\perp$ to $\vec{\nabla}z$, Steepest Slope along contour lines.

The $\vec{\nabla}$ operator

The gradient of T is ......

$$\vec{\nabla}T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

$\vec{\nabla}$ (DeL) operator in cartesian coordinates
$\Rightarrow$ ∇ is w/out meaning until it operates on a function
$\Rightarrow$ Vector Operator

Note:
$\vec{\nabla}$ Behaves like an ordinary vector, simply replace "multiply" with "Act upon"

3 ways ∇ can act:
1. $\vec{\nabla}T$ - The gradient of a scalar function
2. $\vec{\nabla} \cdot v$ - The divergence of a vector function
3. $\vec{\nabla} \times v$ - The curl of a vector function

The Divergence

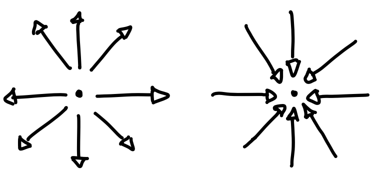
..... of a vector function $\vec{v}$ is of the form

$$\vec{\nabla} \cdot \vec{v} = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right] \cdot \left[\hat{x}v_x + \hat{y}v_y + \hat{z}v_z\right]$$

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Note:
$\vec{v}$ is a vector function

Geometrical Interpretation: $\vec{\nabla} \cdot \vec{v}$ is a measure of how much the vector $\vec{v}$ "spreads out" (Diverges) from the point in question



(+ Divergence) (-Divergence)

The curl

... of a vector function, $\vec{v}$, is of the form

$$\vec{\nabla}\times\vec{v}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix} = \hat{x}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)+\hat{y}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\hat{z}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)$$

Geometrical Interpretation : $\vec{\nabla}\times\vec{v}$ is a measure of how much the vector $\vec{v}$ "curls around" the point in Question

Consider the Scalar functions f : g

consider a couple of vector functions $\vec{A}$: $\vec{B}$

$$\vec{\nabla}(f+g) = \vec{\nabla}f+\vec{\nabla}g$$
$$\vec{\nabla}\cdot(A+B) = (\vec{\nabla}\cdot A)+(\vec{\nabla}\cdot B)$$
$$\vec{\nabla}\times(A+B) = (\vec{\nabla}\times A)+(\vec{\nabla}\times B)$$

Also consider the constant K

$$\vec{\nabla}(Kf) = K\vec{\nabla}f$$
$$\vec{\nabla}\cdot(KA) = K(\vec{\nabla}\cdot A)$$
$$\vec{\nabla}\times(K\vec{A}) = K(\vec{\nabla}\times\vec{A})$$

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Now, for the vector derivatives of a product. There are 6 product rules.

2 for Gradients

i) $\vec{\nabla}(fg) = f\vec{\nabla}g+g\vec{\nabla}f$

ii) $\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{A}\cdot\vec{\nabla})\vec{B}+(\vec{B}\cdot\vec{\nabla})\vec{A}$

2 for Divergences

iii) $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A})+\vec{A}\cdot\vec{\nabla}f$

iv) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$

2 for Curls .....

v) $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A})-\vec{A}\times(\vec{\nabla}f)$

vi) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$

Proof of iii)

$$\vec{\nabla}\cdot(f\vec{A}) = \left[\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right]\cdot\left[\hat{x}fA_x+\hat{y}fA_y+\hat{z}fA_z\right] = \frac{\partial}{\partial x}(fA_x)+\frac{\partial}{\partial y}(fA_y)+\frac{\partial}{\partial z}(fA_z)$$
$$= A_x\frac{\partial f}{\partial x}+f\frac{\partial A_x}{\partial x}+A_y\frac{\partial f}{\partial y}+f\frac{\partial A_y}{\partial y}+A_z\frac{\partial f}{\partial z}+f\frac{\partial A_z}{\partial z}$$
$$= \left(A_x\frac{\partial f}{\partial x}+A_y\frac{\partial f}{\partial y}+A_z\frac{\partial f}{\partial z}\right)+f\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) = \vec{A}\cdot\vec{\nabla}f+f(\vec{\nabla}\cdot\vec{A})$$

Second Derivatives

1. $\vec{\nabla}T$ is a vector, so we have ....
   (1) $\vec{\nabla}\cdot(\vec{\nabla}T)$ - The divergence of a gradient
   (2) $\vec{\nabla}\times(\vec{\nabla}T)$ - The curl of a gradient
2. $\vec{\nabla}\cdot\vec{v}$ is a Scalar, so we have.....
   (3) $\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$ - The gradient of the divergence.
3. $\vec{\nabla}\times\vec{v}$ is a vector, so we have ....
   (4) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$ - The divergence of the Curl
   (5) $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$ - The curl of the Curl

(1) The divergence of the gradient of T....

$$\vec{\nabla}\cdot(\vec{\nabla}T) = \left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right]\cdot\left[\hat{x}\frac{\partial}{\partial x}T + \hat{y}\frac{\partial}{\partial y}T + \hat{z}\frac{\partial}{\partial z}T\right]$$

$$= \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T \equiv \nabla^2 T$$ - The laplacian of T

Note: $\nabla^2 T$ is a scalar

(2) The Laplacian of a Vector ....

$$\nabla^2\vec{v} \equiv \hat{x}(\nabla^2 v_x) + \hat{y}(\nabla^2 v_y) + \hat{z}(\nabla^2 v_z)$$

The curl of the gradient of T....

one can show that..

$\vec{\nabla}\times(\vec{\nabla}T) = 0$ for any Scalar Function T

(3) The Gradient of the Divergence

$\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \neq \nabla^2\vec{v}$, is not equal to the laplacian of $\vec{v}$

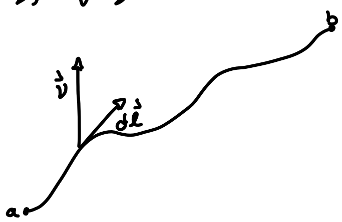
(4) The Divergence of the curl of $\vec{v}$.....

One can show that

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ For any Vector Function $\vec{v}$

(5) The curl of the curl of $\vec{v}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}$$



__Integral Calculus__

Line integrals of $\vec{v}$

$$\int_a^b \vec{v}\cdot d\vec{l} \implies$$

where

$$d\vec{l} = \hat{x}dx + \hat{y}dy + \hat{z}dz$$
$$\vec{v} = \hat{x}v_x + \hat{y}v_y + \hat{z}dz$$

$$\int_a^b \vec{v}\cdot d\vec{l} = \int_a^b (V_x dx + V_y dy + V_z dz)$$

Normally the value of the line integral depends on the particular path taken. There is a special class of vector functions for which the line integral is independent of the path.

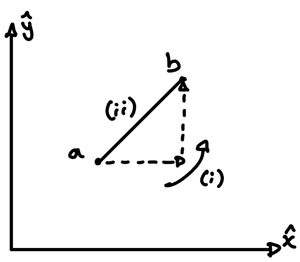
8-29-31

i.e #6

$\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$

$\vec{a} = (1,1,0)$; $\vec{b} = (2,2,0)$

$d\vec{l} = \hat{x}dx + \hat{y}dy$



(ii) $\int \vec{v}\cdot d\vec{l}$

$\vec{v}\cdot d\vec{l} = y^2dx + 2x(y+1)dy$

1^{st} along path (1) ....

(i) $dy=0$, $y=1$

$$\int_1^2 y^2 dx \Big|_{y=1} \int_1^2 dx = 1$$

(ii) $dx=0$ $x=2$

$$\int_1^2 2x(y+1)\,dy \Big|_{x=2} \int_1^2 4(y+1)\,dy = 4\left(\tfrac{y^2}{2}+y\right)\Big|_1^2 = 10$$

$$\int_a^b \vec{v}\cdot d\vec{l} = 11$$

$(i)+(ii) = 1 + 10 = 11$

2^{nd} along path 2

(2) $x=y$, $dx=dy$

$$\int_a^b \vec{v}\cdot d\vec{l} = \int \left(y^2dx + 2x(y+1)\,dy\right)\Big|_{\substack{x=y\\dx=dy}} = \int_1^2 x^2dx + 2x^2 + 2x\,dx = \int_1^2 3x^2 + 2x\,dx = x^3 + x^2\Big|_1^2$$

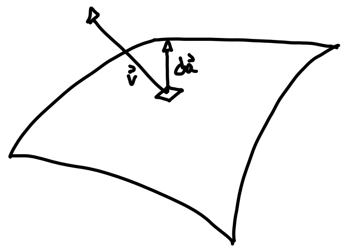
$12 - 2 = 10$

Surface Integral

$$\int_S \vec{v}\cdot d\vec{a}$$

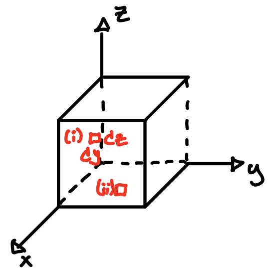
where $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$d\vec{a}$ = infinitesimal patch of area, direction $\perp$ to Surface



Note: The sign of $d\vec{a}$ is ambiguous. The convention is, if the surface is closed the outward normal is positive.

i.e.) $\vec{v} = 2xz\cdot\hat{x} + (x+2)\cdot\hat{y} + y(z^2-3)\hat{z}$



(i) $d\vec{a} = dydz\,\hat{x}$

$$\vec{v}\cdot d\vec{a} = 2xz\,dydz\Big|_{x=2} = 4z\,dydz$$

$$\int_{(i)} \vec{v}\cdot d\vec{a} = \int_0^2\int_0^2 4z\,dydz = 4y\Big|_0^2 \frac{z^2}{2}\Big|_0^2 = 4\cdot 2\cdot 2 = 16$$

(ii) $d\vec{a} = dxdy\cdot -\hat{z}$

$$\vec{v}\cdot d\vec{a} = \int_0^2\int_0^2 -y(z^2-3)\,dxdy\Big|_{z=0} = \int_0^2\int_0^2 -y(-3)\,dxdy = 3\int_0^2\int_0^2 y\,dxdy = 3\cdot 2\cdot 2 = 12$$

Volume Integral of T

$$\int_V T\,d\tau$$ where $T = T(x,y,z)$ is a Scalar Function

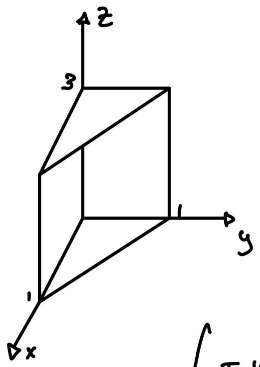
$d\tau = dxdydz$ is an infinitesimal volume element

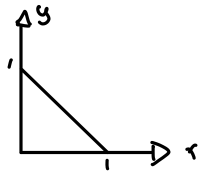
Volume Integral of $\vec{v}$

$$\int_V \vec{v}\,d\tau$$ where $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ is a vector function

$$= \int_V (v_x\hat{x}+v_y\hat{y}+v_z\hat{z})\,d\tau = \hat{x}\int v_x\,d\tau + \hat{y}\int v_y\,d\tau + \hat{z}\int v_z\,d\tau$$

ex 1.8 $T = xyz^2$, Volume integral of T over prism





$y = 1-x$
$x = 1-y$

$$\int_V T\,d\tau = \iiint xyz^2\,dx\,dy\,dz = \int_0^3 z^2\,dz \int_0^1 y\,dy \int_0^{1-y} x\,dx$$

$$= \int_0^3 z^2\,dz \int_0^1 y\,dy \cdot \frac{x^2}{2}\Big|_0^{1-y} = \frac{9}{2}\int_0^1 y\,dy\,(1-y^2)$$

$$= \frac{9}{2}\int_0^1 y(1-2y+y^2)\,dy = \frac{9}{2}\int_0^1 y^3-2y^2+y\,dy$$

$$= \frac{9}{2}\left(\frac{y^4}{4} - \frac{2}{3}y^3 + \frac{1}{2}y^2\right)\Big|_0^1 = \frac{9}{2}\left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right)$$

$$= \frac{9}{2}\left(\frac{3}{12} - \frac{8}{12} + \frac{6}{12}\right) = \frac{9}{2}\left(\frac{1}{12}\right) = \frac{9}{24} = \frac{3}{8}$$

Fundamental Theorem of Calculus

$$\int_a^b F(x)\,dx = f(b) = f(a)$$

$$F(x) = \frac{df}{dx}$$

so

$$\int_a^b \frac{df}{dx}\cdot dx = \int_a^b df = f\Big|_a^b = f(b) - f(a)$$

The integral of a derivative over an interval is given by the value of the function at the end points.

8-31-18

The Fundamental Theorem For Gradients

$$\int_a^b \vec{\nabla}T\cdot d\vec{l} = T(b) - T(a)$$

Proof: Recall from 1.2.2 gradients are

$$\vec{\nabla}T\cdot d\vec{l} = dT$$

So $\displaystyle\int_a^b \vec{\nabla}T\cdot d\vec{l} = \int_a^b dT = T\Big|_a^b = T(b) - T(a)$

Notice: RHS has no reference to the path, only to the end points

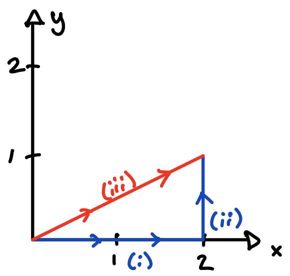
Corolary 1: $\displaystyle\int_a^b \vec{\nabla}T\cdot d\vec{l}$ is independent of the path from $\vec{a}$ to $\vec{b}$

Corolary 2: $\oint \vec{\nabla}T\cdot d\vec{l} = 0$

i.e 1.9 $T(x,y) = xy^2$ check Fundamental theorem for gradients.

$\vec{a} = [0,0,0]$

$\vec{b} = [2,1,0]$



$$\vec{\nabla}T = \hat{x}\frac{\partial}{\partial x}T + \hat{y}\frac{\partial}{\partial y}T$$

$$\vec{\nabla}T = y^2\hat{x} + 2xy\hat{y}$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$$

$\therefore$ $\vec{\nabla}T \cdot d\vec{l} = y^2 dx + 2xy\,dy$

Along path (i)....

$dy = 0$: $y = 0$

$$\int_0^2 y^2\,dx = 0$$

Along path (ii) ....

$dx = 0$: $x = 2$

$$\int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2(1)^2 - 2(0)^2 = 2$$

(i) + (ii) = 2

Along path (iii) .....

$y = \frac{1}{2}x$, $dy = \frac{1}{2}dx$

$$\vec{\nabla}T \cdot d\vec{l} = y^2 dx + 2xy\,dy \qquad y = \tfrac{1}{2}x,\quad dy = \tfrac{1}{2}dx$$

$$\vec{\nabla}T \cdot d\vec{l} = (\tfrac{1}{2}x)^2 dx + 2x(\tfrac{1}{2}x)(\tfrac{1}{2})\,dx = \tfrac{3}{4}x^2\,dx$$

$$\tfrac{3}{4}\int_0^2 x^2\,dx = \tfrac{3}{4}\cdot\frac{x^3}{3}\Big|_0^2 = 2$$

Now...

$T(b) - T(a)$: $T(2,1,0) - T(0,0,0)$

$2 \cdot 1^2 - 0 = 2$

The Divergence Theorem

$$\left[\int_V (\vec{\nabla}\cdot\vec{v})\,d\tau = \oint_S \vec{v}\cdot d\vec{a}\right]$$ - The integral of the divergence over a volume is equal to the value of the function at the surface.

Geometrical interpretation:

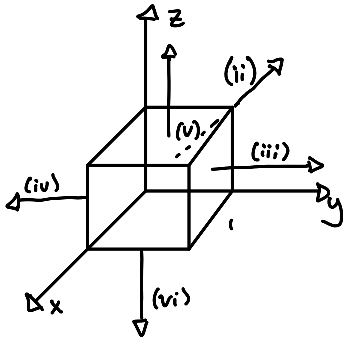
Let $\vec{v}$ be the flow of an incompressible fluid....

Then $\oint \vec{v}\cdot d\vec{a}$ is the flux (total amount of fluid passing through a surface, per unit time)

$\int(\vec{\nabla}\cdot\vec{v})\,d\tau$ measures the spreading out of the fluid through the volume

$\therefore$

$\int$ (Faucets within the volume) = $\oint$ (Flow out of the surface)



$\vec{v} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 2x + 2y = 2(x+y)$$

$$\int (\vec{\nabla}\cdot\vec{v})\, d\vec{l} \quad \int_0^1\int_0^1\int_0^1 2(x+y)\, dxdydz$$

$$2\int_0^1 dz \int_0^1 dy \int_0^1 (x+y)dx = 2\int_0^1 dy \left(\frac{x^2}{2} + yx\right)\Big|_0^1$$

$$= 2\int_0^1 dy\left(\frac{1}{2} + y\right) + 2\left[\frac{1}{2}y\Big|_0^1 + \frac{y^2}{2}\Big|_0^1\right] = 2\left[\frac{1}{2} + \frac{1}{2}\right] = 2$$

(i) : $d\vec{a} = dydz\,\hat{x}$

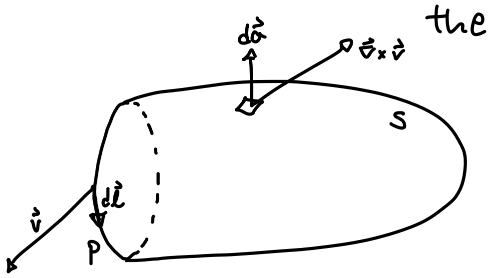
$$\vec{v}\cdot d\vec{a} = y^2 dydx \Big|_{x=1}$$

$$\int_{(i)} \vec{v}\cdot d\vec{a} = \int_0^1\int_0^1 y^2\, dydz = \frac{y^3}{3}\Big|_0^1 \cdot z\Big|_0^1 = \frac{1}{3}$$

Stokes Theorem

$$\left[\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}\right]$$

- The integral of the curl over a surface (Flux of curl) is equal to the value of the function at the perimeter. How much flow is following the boundary.



Geometrical interpretation :

$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$ - The total amount of swirl

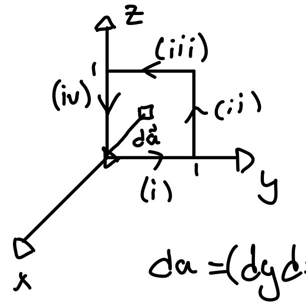
9-3-18

Corollary 1 :

$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a}$ - Does not depend on the particular surface used.

Corollary 2:

$\oint (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = 0$ - For any closed surface i.e. 1.11 $\vec{v} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$



$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xz+3y^2)\right) + \hat{y}\left(-\partial/\partial x(4yz^2)\right) + \hat{z}\left(\partial/\partial x(2xz+3y^2)\right)$$

$$= (4z^2 - 2x)\hat{x} + (2z)\hat{z}$$

$da = (dy\,dz)\,\hat{x}$

$$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_0^1\int_0^1 (4z^2 - 2x)\Big|_{x=0} dy\,dz = 4\int_0^1\int_0^1 z^2\,dy\,dz$$

$$4\cdot \frac{z^3}{3}\Big|_0^1 \cdot y\Big|_0^1 = 4/3 \qquad \boxed{4/3}$$

$\int_{(i)} v\cdot d\vec{l} \qquad d\vec{l} = dy\,\hat{y} \;:\; dx = dz = 0$

$$\int v\cdot d\vec{l} = \int_0^1 2xz + 3y^2\,dy\Big|_{x=0,\,z=0} = \int_0^1 3y^2\,dy = y^3\Big|_0^1 = 1$$

Integration By Parts.

The product rule is

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} \;\therefore\; f\frac{dg}{dx} = \frac{d}{dx}(fg) - (g)\frac{df}{dx}$$

Multiplying both sides by dx : integrating from a to b yields

$$\int_a^b f\frac{dg}{dx}\,dx = \int_a^b \frac{d}{dx}(fg)\,dx - \int_a^b g\frac{df}{dx}\,dx = fg\Big|_a^b - \int_a^b g\frac{df}{dx}\,dx$$

Now, taking the divergence of a scalar function times a vector function.....

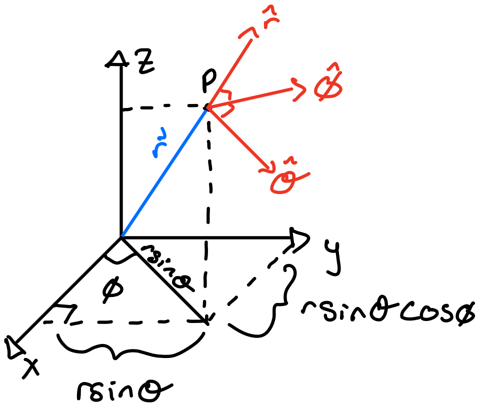
$$\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot(\vec{\nabla}f) \quad - \text{P.R (5)}$$

$$\therefore f(\vec{\nabla}\cdot\vec{A}) = \vec{\nabla}\cdot(f\vec{A}) - \vec{A}\cdot(\vec{\nabla}f)$$

multiplying both sides by $d\tau$ and integrating....

$$\int_V f(\vec{\nabla}\cdot\vec{A})\cdot d\tau = \int_V \vec{\nabla}\cdot(f\vec{A})\,d\tau - \int_V A\cdot(\vec{\nabla}f)\,d\tau = \int_S f\vec{A}\cdot d\vec{a} - \int_V \vec{A}\cdot(\vec{\nabla}f)\,d\tau$$

Spherical Polar Coordinates



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

The 3 spherical polar unit vectors can be written in terms of $\hat{x}, \hat{y}, \hat{z}$, we have...

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\hat{\theta} = \cos\theta\cos\phi + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

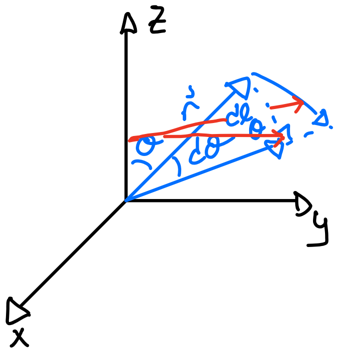
now, any vector can be written in terms of these unit vectors ....

$$\hat{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

=> Notice that $\hat{r}, \hat{\theta}, \hat{\phi}$ change directions as P moves so,

$$\hat{r} = \hat{r}(\theta, \phi)$$
$$\hat{\theta} = \hat{\theta}(\theta, \phi)$$
$$\hat{\phi} = \hat{\phi}(\theta, \phi)$$

now $d\vec{l} = dl_r\,\hat{r} + dl_\theta\,\hat{\theta} + dl_\phi\,\phi$



$dl_\theta = r\,d\theta$, $dl_r = dr$, $dl_\phi = r\sin\theta\,d\theta$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\theta\,\hat{\theta}$$

The infinitesimal volume element becomes

$$d\tau = dl_r\,dl_\theta\,dl_\phi = dr \cdot r\,d\theta \cdot r\sin\theta\,d\theta$$
$$= r^2\sin\theta \cdot dr\,d\theta\,d\phi$$

$0 \le r \le \infty$
$0 \le \theta \le \pi$
$0 \le \phi \le 2\pi$

9-5-18

Spherical Polar Coordinates

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,\phi$$
$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_\phi\hat{\phi}$$
$$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$$

i.e. 1.13

$$V = \int d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2\sin\theta\,dr\,d\theta\,d\phi$$
$$= \int_0^R r^2\,dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi$$
$$= \frac{r^3}{3}\Big|_0^R \cos\theta\Big|_{\pi}^{0}\ \phi\Big|_0^{2\pi} = \frac{R^3}{3}(2)(2\pi) = \frac{4}{3}\pi R^3$$

The Gradient of Some Scalar Function in cartesian Coordinates....

$$\vec{\nabla}T = \hat{x}\frac{\partial}{\partial x}T + \hat{y}\frac{\partial}{\partial y}T + \hat{z}\frac{\partial}{\partial z}T$$

The Gradient of Some Scalar Function in Spherical polar coordinates....

$$\vec{\nabla}T = \hat{r}\frac{\partial}{\partial r}T + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}T + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}T$$

The Divergence of $\vec{v}$ is ......

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

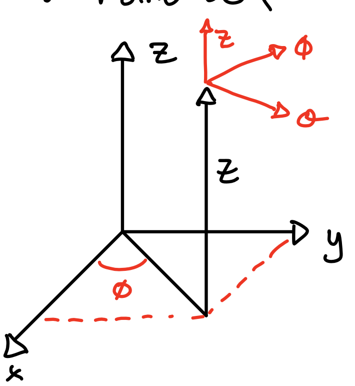
The Curl of $\vec{v}$ is ......

$$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

The Laplacian of T is ...

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}T\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}T$$

Cylindrical Coordinates



$x = s\cos\phi$
$y = s\sin\phi$
$z = z$

The three cylindrical unit vectors in terms of the cartesian coordinates are,

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

Now any vector can be written in terms of these unit vectors and these are of the form.....

$$\vec{A} = A_s\hat{s} + A_\phi\hat{\phi} + A_z\hat{z}$$

Notice that

$$\hat{s} = \hat{s}(\phi)$$
$$\hat{\phi} = \hat{\phi}(\phi)$$

Now

$$d\vec{l} = dl_s\hat{s} + dl_\phi\hat{\phi} + dl_z\hat{z}$$

where: $dl_s = ds$
$dl_\phi = s\,d\phi$
$dl_z = dz$

So

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

The infinitesimal volume element is....

$$d\tau = dl_s\,dl_\phi\,dl_z = ds \cdot s\,d\phi \cdot dz = s\,ds\,d\phi\,dz$$

notice $0 < s < \infty$
$0 < \phi < 2\pi$
$-\infty < z < \infty$

The Gradient of some Scalar Function is...

$$\vec{\nabla}T = \hat{s}\frac{\partial T}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial T}{\partial \phi} + \hat{z}\frac{\partial T}{\partial z}$$

The Divergence is.....

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

The Curl is.....

$$\vec{\nabla}\times\vec{v} = \left(\frac{1}{s}\frac{\partial}{\partial\phi}v_z - \frac{\partial}{\partial z}v_\phi\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right)\hat{z}$$

The Laplacian is.....

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\left(\frac{\partial^2}{\partial\phi^2}\right)T + \frac{\partial^2 T}{\partial z^2}$$

Potentials

Consider a vector Field $\vec{F}$ (Representing $\vec{E}$ & $\vec{B}$)

Say $\vec{\nabla} \times \vec{F} = 0$ then $\vec{F} = -\vec{\nabla} V$

$\vec{F}$ can be written as minus the gradient of some Scalar Function.

Proof: let $\vec{F} = -\vec{\nabla} V$

then $\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla} V = 0$

For any V - The curl of a gradient is always zero

Theorem 1: Curl-less Fields

$\vec{F}$ satisfies iff it satisfies all the others

a) $\vec{\nabla} \times \vec{F} = 0$
b) $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path
c) $\oint \vec{F} \cdot d\vec{l} = 0$
d) $\vec{F} = -\vec{\nabla} V$

9-7-18

If $\vec{\nabla} \times \vec{F} = 0$ then $\vec{F} = -\vec{\nabla} V$

notice

$V \to V + C$

then $\vec{F} = -\vec{\nabla} V \to -\vec{\nabla}(V + C) = -\vec{\nabla} V = \vec{F}$ - Any constant can be added to the potential V without changing $\vec{F}$

If $\vec{\nabla} \cdot \vec{F} = 0$, then $\vec{F} = \vec{\nabla} \times \vec{A}$ ($\vec{F}$ can be written as the curl of a vector potential)

Proof: Let $\vec{F} = \vec{\nabla} \times \vec{A}$, then $\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

Theorem 2: Divergence-less Fields

$\vec{F}$ satisfies one iff it satisfies all the others

a) $\vec{\nabla} \cdot \vec{F} = 0$
b) $\int_S \vec{F} \cdot d\vec{a}$ is independent of the surface
c) $\oint_S \vec{F} \cdot d\vec{a} = 0$
d) $\vec{F} \times \vec{A}$

Notice: Let $\vec{A} \to \vec{A} + \vec{\nabla} \phi$

then $\vec{F} = \vec{\nabla} \times \vec{A} \to \vec{\nabla} \times (\vec{A} \times \vec{\nabla} \phi)$

$= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} \phi = \vec{\nabla} \times \vec{A} = \vec{F}$

Any gradient of a scalar function can be added to $\vec{A}$ without changing $\vec{F}$