Exercise 1

Write $\hat{\rho}_{prep} = |0\rangle|0\rangle\langle 0|\langle 0| + |0\rangle|0\rangle\langle 0|\langle 1| + |0\rangle|0\rangle\langle 1|\langle 0| + |0\rangle|0\rangle\langle 1|\langle 1| \ldots\ldots\ldots$

$$\hat{\rho}_{prep} = \hat{u}_{prep}\,\hat{\rho}\,\hat{u}^{\dagger}_{prep} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir) \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir) & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\hat{\rho}_{prep} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| + (4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]}$$

Exercise 2

Determine $\hat{u}(\lambda)$ on $\hat{\rho}_{prep}$. keep in $\hat{\rho} = \ldots\, |0\rangle|0\rangle\langle 0|\langle 0| + |0\rangle|0\rangle\langle 0|\langle 1| + \ldots\ldots$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)e^{-i\lambda}|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| +$$
$$(4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)e^{i\lambda}|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]$$

$$\boxed{\begin{aligned}\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[&(4+4r^2)|0\rangle|0\rangle\langle 0|\langle 0| + (-8ir)e^{-i\lambda}|0\rangle|0\rangle\langle 1|\langle 1| + (4-4r^2)|0\rangle|1\rangle\langle 0|\langle 1| +\\ &(4-4r^2)|1\rangle|0\rangle\langle 1|\langle 0| + (8ir)e^{i\lambda}|1\rangle|1\rangle\langle 0|\langle 0| + (4+4r^2)|1\rangle|1\rangle\langle 1|\langle 1|\Big]\end{aligned}}$$

Exercise 3

Determine phase flip and get final density operator.

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (8ir)e^{i\lambda}|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

Phase flip acts on second part, i.e:

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes\underline{|0\rangle\langle 0|} + (-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes\underline{|0\rangle\langle 1|} + (4-4r^2)|0\rangle\langle 0|\otimes\underline{|1\rangle\langle 1|} +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes\underline{|0\rangle\langle 0|} + (8ir)e^{i\lambda}|1\rangle\langle 0|\otimes\underline{|1\rangle\langle 0|} + (4+4r^2)|1\rangle\langle 1|\otimes\underline{|1\rangle\langle 1|}\Big]$$

$\hat{\rho}_i = |0\rangle\langle 0| \longrightarrow \hat{\rho}_f = |0\rangle\langle 0|$: $\hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_f = |1\rangle\langle 1|$:

$\hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_f = (1-2\alpha)|0\rangle\langle 1|$: $\hat{\rho}_i = |1\rangle\langle 0| \longrightarrow \hat{\rho}_f = (1-2\alpha)|1\rangle\langle 0|$

These rules evolve $\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger}$ to

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (1-2\alpha)(-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (1-2\alpha)(8ir)e^{i\lambda}|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

$$\boxed{\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (1-2\alpha)(-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (1-2\alpha)(8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}}$$

Exercise 4

Calculate QFI. $\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (1-2\alpha)(-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (1-2\alpha)(8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}$

$$\hat{A}_1 = \frac{1}{16}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix}, \quad \hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4+4r^2) & (2\alpha-1)(8ir)e^{-i\lambda} \\ (1-2\alpha)(8ir)e^{i\lambda} & (4+4r^2) \end{pmatrix}$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial \lambda} - \frac{\partial \ln(\alpha)}{\partial \lambda}\hat{A}\right] + \frac{\partial}{\partial \lambda}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I}, \quad H_i = Tr\left[L_i \frac{\partial A_i}{\partial \lambda}\right]$$

i.) Calculate $\hat{L}_1$: $\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix}\begin{pmatrix} 4-4r^2 & 0 \\ 0 & 4-4r^2 \end{pmatrix} = \frac{1}{16^2}\begin{pmatrix} (4-4r^2)^2 & 0 \\ 0 & (4-4r^2)^2 \end{pmatrix}$$

$$Tr(\hat{A}_1^2) = \frac{1}{16^2}\left[(4-4r^2)^2 + (4-4r^2)^2\right] = \frac{2}{16^2}\left[16 - 32r^2 + 16r^4\right]$$

$$Tr(\hat{A}_1) = \frac{1}{16}\left[4-4r^2+4-4r^2\right] = \frac{1}{16}\left[8-8r^2\right] = \frac{1}{2}\left[1-r^2\right] : Tr(\hat{A}_1)^2 = \frac{1}{4}\left[1-2r^2+r^4\right]$$

$$\alpha = Tr(\hat{A}_1^2) - Tr(\hat{A}_1) = \frac{1}{128}\left[16-32r^2+16r^4\right] - \frac{32}{128}\left[1-2r^2+r^4\right]$$

$$= \frac{16}{128} - \frac{32r^2}{128} + \frac{16r^4}{128} - \frac{32}{128} + \frac{64r^2}{128} - \frac{32r^4}{128} = -\frac{16}{128} + \frac{32r^2}{128} - \frac{16r^4}{128} = -\frac{1}{8} + \frac{r^2}{4} - \frac{r^4}{8}$$

$\alpha = -\frac{1}{8}(1-2r^2+r^4)$, Since α does not depend upon λ, $\hat{L}$ changes to

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial \lambda} - \overset{0}{\cancel{\frac{\partial \ln(\alpha)}{\partial \lambda}}}\hat{A}\right] + \overset{0}{\cancel{\frac{\partial}{\partial \lambda}}}\left[\cancel{\ln(\alpha)} - \cancel{\ln(Tr(\hat{A}))}\right]\hat{I} = \frac{2}{Tr(\hat{A}_1)}\cdot\frac{\partial \hat{A}}{\partial \lambda}$$

$\hat{A}_1$ does not depend on λ $\therefore$ $\hat{L}_1 = 0$

$$\boxed{\hat{L}_1 = 0}$$

ii.) Calculate $\hat{L}_2$: $\alpha = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2) & (2\alpha-1)(8ir)e^{-i\lambda} \\ (1-2\alpha)(8ir)e^{i\lambda} & (4+4r^2) \end{pmatrix}\begin{pmatrix} (4+4r^2) & (2\alpha-1)(8ir)e^{-i\lambda} \\ (1-2\alpha)(8ir)e^{i\lambda} & (4+4r^2) \end{pmatrix}$$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)^2 + 64r^2(1-2\alpha)^2 & 2(4+4r^2)(1-2\alpha)(8ir)e^{-i\lambda} \\ 2(4+4r^2)(1-2\alpha)(8ir)e^{i\lambda} & 64r^2(1-2\alpha)^2 + (4+4r^2)^2 \end{pmatrix}$$

Exercise 4 Continued

$$Tr(\hat{A}_2^2) = \frac{1}{16^2}\left(2(4+4r^2)^2 + 128r^2(1-2\alpha)^2\right) \; : \; Tr(\hat{A}_2) = \frac{1}{16}\left(2(4+4r^2)\right) = \frac{1}{8}(4+4r^2)$$

$$\therefore \; \alpha = \frac{1}{16^2}\left(2(4+4r^2)^2 + 128r^2(1-2\alpha)^2\right) - \frac{1}{8}(4+4r^2)$$

Since α does not depend upon λ, $\hat{L}$ reduces to

$$\hat{L}_2 = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial \hat{A}}{\partial\lambda} - \overset{0}{\cancel{\frac{\partial \ln(\alpha)}{\partial\lambda}}}\hat{A}\right] + \overset{0}{\cancel{\frac{\partial}{\partial\lambda}}}\left[\ln(\alpha) - \ln(Tr(\hat{A}))\right]\hat{I} = \frac{2}{Tr(\hat{A}_2)}\cdot\frac{\partial \hat{A}_2}{\partial\lambda}$$

$$\hat{L}_2 = \frac{2}{Tr(\hat{A}_2)}\cdot\frac{\partial \hat{A}_2}{\partial\lambda} \; : \; \frac{\partial \hat{A}_2}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\hat{L}_2 = \frac{16}{(4+4r^2)}\cdot\frac{1}{16}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix} = \frac{1}{(4+4r^2)}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\boxed{\hat{L}_2 = \frac{1}{(4+4r^2)}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}}$$

iii.) Calculate QFI

$$H = H_1 + H_2 \; : \; H_1 = 0 \qquad H_i = Tr\left[L_i \frac{\partial A_i}{\partial\lambda}\right]$$

$$H_2 = Tr\left[\hat{L}_2\cdot\frac{\partial \hat{A}_2}{\partial\lambda}\right]$$

$$\hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4+4r^2) & (1-2\alpha)(-8ir)e^{-i\lambda} \\ (1-2\alpha)(8ir)e^{i\lambda} & (4+4r^2) \end{pmatrix} \; : \; \hat{L}_2 = \frac{1}{(4+4r^2)}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\frac{\partial \hat{A}_2}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & (1-2\alpha)(-8r)e^{-i\lambda} \\ (1-2\alpha)(-8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\hat{L}_2\cdot\frac{\partial \hat{A}_2}{\partial\lambda} = \frac{1}{16(4+4r^2)}\begin{pmatrix} 0 & (1-2\alpha)(-8r)e^{-i\lambda} \\ (1-2\alpha)(-8r)e^{i\lambda} & 0 \end{pmatrix}\begin{pmatrix} 0 & (1-2\alpha)(-8r)e^{-i\lambda} \\ (1-2\alpha)(-8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\hat{L}_2\cdot\frac{\partial \hat{A}_2}{\partial\lambda} = \frac{1}{16(4+4r^2)}\begin{pmatrix} (1-2\alpha)^2 64r^2 & 0 \\ 0 & (1-2\alpha)^2 64r^2 \end{pmatrix} \; : \; Tr\left[\hat{L}_2\cdot\frac{\partial \hat{A}_2}{\partial\lambda}\right] = \frac{(1-2\alpha)^2\, 128r^2}{64(1+r^2)}$$

$$\boxed{H = \frac{2r^2(1-2\alpha)^2}{(1+r^2)}}$$

Exercise 5

i.) Perform $\hat{\rho}_f = (1-\alpha)\hat{\rho}_i + \alpha\hat{\sigma}_x\hat{\rho}_i\hat{\sigma}_x$: $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\hat{\rho}_i = |0\rangle\langle 0|$: $\hat{\rho}_f = (1-\alpha)|0\rangle\langle 0| + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|0\rangle\langle 0|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= (1-\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

$\hat{\rho}_i = |0\rangle\langle 1|$: $\hat{\rho}_f = (1-\alpha)|0\rangle\langle 1| + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|0\rangle\langle 1|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= (1-\alpha)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \begin{pmatrix} 0 & 1-\alpha \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-\alpha \\ 0 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-\alpha \\ \alpha & 0 \end{pmatrix}$$

$\hat{\rho}_i = |1\rangle\langle 0|$: $\hat{\rho}_f = (1-\alpha)|1\rangle\langle 0| + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|1\rangle\langle 0|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= (1-\alpha)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = \begin{pmatrix} 0 & 0 \\ 1-\alpha & 0 \end{pmatrix} + \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1-\alpha & 0 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix}$$

$\hat{\rho}_i = |1\rangle\langle 1|$: $\hat{\rho}_f = (1-\alpha)|1\rangle\langle 1| + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}|1\rangle\langle 1|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$= (1-\alpha)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\rho}_f = (1-\alpha)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1-\alpha \end{pmatrix} + \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$$

$\hat{\rho}_i = |0\rangle\langle 0| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}$: $\hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & 1-\alpha \\ \alpha & 0 \end{pmatrix}$

$\hat{\rho}_i = |1\rangle\langle 0| \longrightarrow \hat{\rho}_f = \begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix}$: $\hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_f = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$

Exercise 5 Continued

ii.) Determine phase flip and get final density operator.

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2) & 0 & 0 & (-8ir)e^{-i\lambda} \\ 0 & (4-4r^2) & 0 & 0 \\ 0 & 0 & (4-4r^2) & 0 \\ (8ir)e^{i\lambda} & 0 & 0 & (4+4r^2) \end{pmatrix}$$

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes|0\rangle\langle 0| + (-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes|0\rangle\langle 1| + (4-4r^2)|0\rangle\langle 0|\otimes|1\rangle\langle 1| +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes|0\rangle\langle 0| + (8ir)e^{i\lambda}|1\rangle\langle 0|\otimes|1\rangle\langle 0| + (4+4r^2)|1\rangle\langle 1|\otimes|1\rangle\langle 1|\Big]$$

Phase Flip acts on second part, i.e:

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes\underline{\underline{|0\rangle\langle 0|}} + (-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes\underline{\underline{|0\rangle\langle 1|}} + (4-4r^2)|0\rangle\langle 0|\otimes\underline{\underline{|1\rangle\langle 1|}} +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes\underline{\underline{|0\rangle\langle 0|}} + (8ir)e^{i\lambda}|1\rangle\langle 0|\otimes\underline{\underline{|1\rangle\langle 0|}} + (4+4r^2)|1\rangle\langle 1|\otimes\underline{\underline{|1\rangle\langle 1|}}\Big]$$

$\hat{\rho}_i = |0\rangle\langle 0| \rightarrow \hat{\rho}_F = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}$: $\hat{\rho}_i = |0\rangle\langle 1| \longrightarrow \hat{\rho}_F = \begin{pmatrix} 0 & 1-\alpha \\ \alpha & 0 \end{pmatrix}$

$\hat{\rho}_i = |1\rangle\langle 0| \rightarrow \hat{\rho}_F = \begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix}$: $\hat{\rho}_i = |1\rangle\langle 1| \longrightarrow \hat{\rho}_F = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$

These rules evolve $\hat{u}\,\hat{\rho}_{prep}\hat{u}^{\dagger}$ to

$$\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\Big[(4+4r^2)|0\rangle\langle 0|\otimes\overset{1}{\begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}} + (-8ir)e^{-i\lambda}|0\rangle\langle 1|\otimes\overset{2}{\begin{pmatrix} 0 & 1-\alpha \\ \alpha & 0 \end{pmatrix}} + (4-4r^2)|0\rangle\langle 0|\otimes\overset{3}{\begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}} +$$
$$(4-4r^2)|1\rangle\langle 1|\otimes\overset{4}{\begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}} + (8ir)e^{i\lambda}|1\rangle\langle 0|\otimes\overset{5}{\begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix}} + (4+4r^2)|1\rangle\langle 1|\otimes\overset{6}{\begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}}\Big]$$

1: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 2: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & 1-\alpha \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1-\alpha \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 4: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$

5: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\otimes\begin{pmatrix} 0 & \alpha \\ 1-\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 1-\alpha & 0 & 0 & 0 \end{pmatrix}$ 6: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\otimes\begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix}$

Exercise 5 continued

$$① = \begin{pmatrix} (4+4r^2)(1-\alpha) & 0 & 0 & 0 \\ 0 & (4+4r^2)(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ② = \begin{pmatrix} 0 & 0 & 0 & (-8ir)e^{-i\lambda}(1-\alpha) \\ 0 & 0 & (-8ir)e^{-i\lambda}(\alpha) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$③ = \begin{pmatrix} (4-4r^2)(\alpha) & 0 & 0 & 0 \\ 0 & (4-4r^2)(1-\alpha) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ④ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (4-4r^2)(1-\alpha) & 0 \\ 0 & 0 & 0 & (4-4r^2)(\alpha) \end{pmatrix}$$

$$⑤ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & (8ir)e^{i\lambda}(\alpha) & 0 & 0 \\ (8ir)e^{i\lambda}(1-\alpha) & 0 & 0 & 0 \end{pmatrix} \quad ⑥ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (4+4r^2)(\alpha) & 0 \\ 0 & 0 & 0 & (4+4r^2)(1-\alpha) \end{pmatrix}$$

① + ② + ③ + ④ + ⑤ + ⑥

$$\boxed{\hat{u}\hat{\rho}_{prep}\hat{u}^{\dagger} = \frac{1}{16}\begin{pmatrix} (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) & 0 & 0 & (-8ir)e^{-i\lambda}(1-\alpha) \\ 0 & (4+4r^2)(\alpha)+(4-4r^2)(1-\alpha) & (-8ir)e^{-i\lambda}(\alpha) & 0 \\ 0 & (8ir)e^{i\lambda}(\alpha) & (4-4r^2)(1-\alpha)+(4+4r^2)(\alpha) & 0 \\ (8ir)e^{i\lambda}(1-\alpha) & 0 & 0 & (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) \end{pmatrix}}$$

$$\hat{A}_1 = \frac{1}{16}\begin{pmatrix} (4+4r^2)(\alpha)+(4-4r^2)(1-\alpha) & (-8ir)e^{-i\lambda}(\alpha) \\ (8ir)e^{i\lambda}(\alpha) & (4-4r^2)(1-\alpha)+(4+4r^2)(\alpha) \end{pmatrix}$$

$$\hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) & (-8ir)e^{-i\lambda}(1-\alpha) \\ (8ir)e^{i\lambda}(1-\alpha) & (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) \end{pmatrix}$$

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \frac{\partial \ln(\beta)}{\partial\lambda}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\ln(\beta) - \ln(Tr(\hat{A}))\right]\hat{I} \quad , \quad H_i = Tr\left[L_i \frac{\partial A_i}{\partial\lambda}\right]$$

iii.) Calculate $\hat{L}_1$: $\beta = Tr(\hat{A}_1^2) - Tr(\hat{A}_1)^2$

$$\hat{A}_1^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)(\alpha)+(4-4r^2)(1-\alpha) & (-8ir)e^{-i\lambda}(\alpha) \\ (8ir)e^{i\lambda}(\alpha) & (4-4r^2)(1-\alpha)+(4+4r^2)(\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (4+4r^2)(\alpha)+(4-4r^2)(1-\alpha) & (-8ir)e^{-i\lambda}(\alpha) \\ (8ir)e^{i\lambda}(\alpha) & (4-4r^2)(1-\alpha)+(4+4r^2)(\alpha) \end{pmatrix}$$

$$\hat{A}_1^2 = \frac{1}{16^2}\left(\begin{matrix} ((4+4r^2)(\alpha)+(4-4r^2)(1-\alpha))^2 + 64r^2\alpha^2 \\ (8ir)e^{i\lambda}(\alpha)((4+4r^2)(\alpha)+(4-4r^2)(1-\alpha)) + (8ir)e^{i\lambda}(\alpha)((4-4r^2)(1-\alpha)+(4+4r^2)(\alpha)) \end{matrix} \quad \vdots\right.$$

$$\left.\vdots \begin{matrix} (-8ir)e^{-i\lambda}(\alpha)((4+4r^2)(\alpha)+(4-4r^2)(1-\alpha)) + (-8ir)e^{-i\lambda}(\alpha)((4-4r^2)(1-\alpha)+(4+4r^2)(\alpha)) \\ ((4-4r^2)(1-\alpha)+(4+4r^2)(\alpha))^2 + 64r^2\alpha^2 \end{matrix}\right)$$

Exercise 5 continued

$$Tr(\hat{A}_1^2) = \frac{1}{16^2}\left(2((4+4r^2)(\alpha) + (4-4r^2)(1-\alpha))^2 + 128r^2\alpha^2\right)$$

$$Tr(\hat{A}_1)^2 = \frac{1}{16^2}\left(2((4+4r^2)(\alpha) + (4-4r^2)(1-\alpha))\right)^2 \qquad Tr(\hat{A}_1) = \frac{1}{2}((2\alpha-1)r^2+1)$$

$$\beta = \frac{1}{16^2}\left(2((4+4r^2)(\alpha) + (4-4r^2)(1-\alpha))^2 + 128r^2\alpha^2\right) - \frac{1}{16^2}\left(2((4+4r^2)(\alpha) + (4-4r^2)(1-\alpha))\right)^2$$

Since β does not depend upon λ, $\hat{L}$ simplifies to

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \cancel{\frac{\partial \ln(\beta)}{\partial\lambda}}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\cancel{\ln(\beta)} - \cancel{\ln(Tr(\hat{A}))}\right]\hat{I}, \quad H_i = Tr\left[\hat{L}_i\cdot\frac{\partial A_i}{\partial\lambda}\right]$$

$$\hat{L}_1 = \frac{1}{Tr(\hat{A}_1)}\cdot 2\frac{\partial\hat{A}_1}{\partial\lambda} \quad : 2\frac{\partial\hat{A}_1}{\partial\lambda} = \frac{1}{8}\begin{pmatrix} 0 & 8re^{-i\lambda}(\alpha) \\ -8re^{i\lambda}(\alpha) & 0 \end{pmatrix} = \begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_1 = \frac{2}{((2\alpha-1)r^2+1)}\begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix} = \frac{2}{((2\alpha-1)r^2+1)}\begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix}$$

$$\boxed{\hat{L}_1 = \frac{2}{((2\alpha-1)r^2+1)}\begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix}}$$

iv.) Calculate H_1 : $H_1 = Tr\left[\hat{L}_1\cdot\frac{\partial A_1}{\partial\lambda}\right]$

$$\hat{A}_1 = \frac{1}{16}\begin{pmatrix} (4+4r^2)(\alpha) + (4-4r^2)(1-\alpha) & (-8ir)e^{-i\lambda}(\alpha) \\ (8ir)e^{i\lambda}(\alpha) & (4-4r^2)(1-\alpha) + (4+4r^2)(\alpha) \end{pmatrix}$$

$$\frac{\partial\hat{A}_1}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & 8re^{-i\lambda}(\alpha) \\ -8re^{i\lambda}(\alpha) & 0 \end{pmatrix} \quad \hat{L}_1 = \frac{2}{((2\alpha-1)r^2+1)}\begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_1\cdot\frac{\partial\hat{A}_1}{\partial\lambda} = \frac{2}{((2\alpha-1)r^2+1)}\begin{pmatrix} 0 & re^{-i\lambda}(\alpha) \\ -re^{i\lambda}(\alpha) & 0 \end{pmatrix}\frac{1}{16}\begin{pmatrix} 0 & 8re^{-i\lambda}(\alpha) \\ -8re^{i\lambda}(\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_1\cdot\frac{\partial\hat{A}_1}{\partial\lambda} = \frac{1}{8((2\alpha-1)r^2+1)}\begin{pmatrix} -8r^2\alpha^2 & 0 \\ 0 & -8r^2\alpha^2 \end{pmatrix} \quad : Tr\left[\hat{L}_1\cdot\frac{\partial\hat{A}_1}{\partial\lambda}\right] = \frac{-16r^2\alpha^2}{8((2\alpha-1)r^2+1)} = \frac{-2r^2\alpha^2}{((2\alpha-1)r^2+1)}$$

$$\boxed{H_1 = \frac{-2r^2\alpha^2}{((2\alpha-1)r^2+1)}}$$

Exercise 5 continued

v.) Calculate $\hat{L}_2$: $\beta = Tr(\hat{A}_2^2) - Tr(\hat{A}_2)^2$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix} (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) & (-8ir)e^{-i\lambda}(1-\alpha) \\ (8ir)e^{i\lambda}(1-\alpha) & (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) \end{pmatrix}\begin{pmatrix} (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) & (-8ir)e^{-i\lambda}(1-\alpha) \\ (8ir)e^{i\lambda}(1-\alpha) & (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) \end{pmatrix}$$

$$\hat{A}_2^2 = \frac{1}{16^2}\begin{pmatrix} ((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))^2 + 64r^2(1-\alpha)^2 & (-8ir)e^{-i\lambda}(1-\alpha)((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha)) + (-8ir)e^{-i\lambda}(1-\alpha)((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha)) \\ (8ir)e^{i\lambda}(1-\alpha)((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha)) + (8ir)e^{i\lambda}(1-\alpha)((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha)) & ((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))^2 + 64r^2(1-\alpha)^2 \end{pmatrix}$$

$$Tr(\hat{A}_2^2) = \frac{1}{16^2}\left(2((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))^2) + 128r^2(1-\alpha)^2\right)$$

$$Tr(\hat{A}_2)^2 = \frac{1}{16^2}\left(2((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))\right) \qquad Tr(\hat{A}_2) = \frac{1-(2\alpha-1)r^2}{2}$$

$$\beta = \frac{1}{16^2}\left(2((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))^2) + 128r^2(1-\alpha)^2\right) - \frac{1}{16^2}\left(2((4+4r^2)(1-\alpha)+(4-4r^2)(\alpha))\right)^2$$

Since β does not depend upon λ, $\hat{L}$ Simplifies to

$$\hat{L} = \frac{1}{Tr(\hat{A})}\left[2\cdot\frac{\partial\hat{A}}{\partial\lambda} - \cancel{\frac{\partial \ln(\dot{\beta})}{\partial\lambda}}\hat{A}\right] + \frac{\partial}{\partial\lambda}\left[\cancel{\ln(\dot{\beta})} - \cancel{\ln(Tr(\dot{\hat{A}}))}\right]\hat{I}, \quad H_i = Tr\left[\hat{L}_i\cdot\frac{\partial A_i}{\partial\lambda}\right]$$

$$\hat{L}_2 = \frac{1}{Tr(\hat{A}_2)}\cdot 2\cdot\frac{\partial\hat{A}_2}{\partial\lambda} \quad : \quad \frac{\partial\hat{A}_2}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & 8re^{-i\lambda}(1-\alpha) \\ -8re^{i\lambda}(1-\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_2 = \frac{2}{(1-(2\alpha-1)r^2)}\begin{pmatrix} 0 & re^{-i\lambda}(1-\alpha) \\ -re^{i\lambda}(1-\alpha) & 0 \end{pmatrix}$$

$$\boxed{\hat{L}_2 = \frac{2}{(1-(2\alpha-1)r^2)}\begin{pmatrix} 0 & re^{-i\lambda}(1-\alpha) \\ -re^{i\lambda}(1-\alpha) & 0 \end{pmatrix}}$$

vi.) Calculate H_2 : $H_2 = Tr\left[\hat{L}_2\cdot\frac{\partial A_2}{\partial\lambda}\right]$

$$\hat{A}_2 = \frac{1}{16}\begin{pmatrix} (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) & (-8ir)e^{-i\lambda}(1-\alpha) \\ (8ir)e^{i\lambda}(1-\alpha) & (4+4r^2)(1-\alpha)+(4-4r^2)(\alpha) \end{pmatrix}$$

$$\frac{\partial\hat{A}_2}{\partial\lambda} = \frac{1}{16}\begin{pmatrix} 0 & 8re^{-i\lambda}(1-\alpha) \\ -8re^{i\lambda}(1-\alpha) & 0 \end{pmatrix} \quad \hat{L}_2 = \frac{2}{(1-(2\alpha-1)r^2)}\begin{pmatrix} 0 & re^{-i\lambda}(1-\alpha) \\ -re^{i\lambda}(1-\alpha) & 0 \end{pmatrix}$$

Exercise 5 continued

$$\hat{L}_2 \cdot \frac{\partial \hat{A}_2}{\partial \lambda} = \frac{2}{(1-(2\alpha-1)r^2)} \begin{pmatrix} 0 & re^{-i\lambda}(1-\alpha) \\ -re^{i\lambda}(1-\alpha) & 0 \end{pmatrix} \cdot \frac{1}{16} \begin{pmatrix} 0 & 8re^{-i\lambda}(1-\alpha) \\ -8re^{i\lambda}(1-\alpha) & 0 \end{pmatrix}$$

$$\hat{L}_2 = \frac{\partial \hat{A}_2}{\partial \lambda} = \frac{1}{8(1-(2\alpha-1)r^2)} \begin{pmatrix} -8r^2(1-\alpha)^2 & 0 \\ 0 & -8r^2(1-\alpha)^2 \end{pmatrix} : \quad Tr\left[\hat{L}_2 \cdot \frac{\partial \hat{A}_2}{\partial \lambda}\right] = \frac{-2r^2(1-\alpha)^2}{(1-(2\alpha-1)r^2)}$$

$$\boxed{H_2 = \frac{-2r^2(1-\alpha)^2}{(1-(2\alpha-1)r^2)}}$$

Vii.) Calculate H : $H = |H_1| + |H_2|$

$$\boxed{H = \frac{2r^2\alpha^2}{((2\alpha-1)r^2+1)} + \frac{2r^2(1-\alpha)^2}{(1-(2\alpha-1)r^2)}}$$

Viii.) Limiting cases:

$$\boxed{\alpha = 0 : \ H = \frac{2r^2}{(1+r^2)}, \quad \alpha = 1 : \ H = \frac{2r^2}{(1+r^2)}, \quad r \approx 0 : H = 2}$$

$$\hat{L}_2^2 = \frac{1}{(4+4r^2)^2}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}\begin{pmatrix} 0 & (2\alpha-1)(8r)e^{-i\lambda} \\ (2\alpha-1)(8r)e^{i\lambda} & 0 \end{pmatrix}$$

$$\hat{L}_2^2 = \frac{1}{(4+4r^2)^2}\begin{pmatrix} (2\alpha-1)^2(64r^2) & 0 \\ 0 & (2\alpha-1)^2(64r^2) \end{pmatrix}$$

$$\hat{P}_2 \cdot \hat{L}_2^2 = \frac{1}{16}\begin{pmatrix} 4+4r^2 & -8ir \\ 8ir & 4+4r^2 \end{pmatrix}\frac{1}{(4+4r^2)^2}\begin{pmatrix} (2\alpha-1)^2(64r^2) & 0 \\ 0 & (2\alpha-1)^2(64r^2) \end{pmatrix}$$

$$\hat{P}_2 \cdot \hat{L}_2^2 = \frac{1}{16(4+4r^2)^2}\begin{pmatrix} (4+4r^2)(2\alpha-1)^2(64r^2) & (-8ir)(2\alpha-1)^2(64r^2) \\ (8ir)(2\alpha-1)^2(64r^2) & (4+4r^2)(2\alpha-1)^2(64r^2) \end{pmatrix}$$

$$\mathrm{Tr}(\hat{P}_2 \cdot \hat{L}_2^2) = \frac{1}{16(4+4r^2)^2}\left(2(4+4r^2)(2\alpha-1)^2(64r^2)\right) = \frac{(\cancel{128}r^2)(\cancel{4+4r^2})(2\alpha-1)^2}{\cancel{16}(4+4r^2)^{\cancel{2}}} = \frac{8r^2(2\alpha-1)^2}{(4+4r^2)}$$

$$\boxed{H = \frac{2r^2(2\alpha-1)^2}{(1+r^2)}}$$