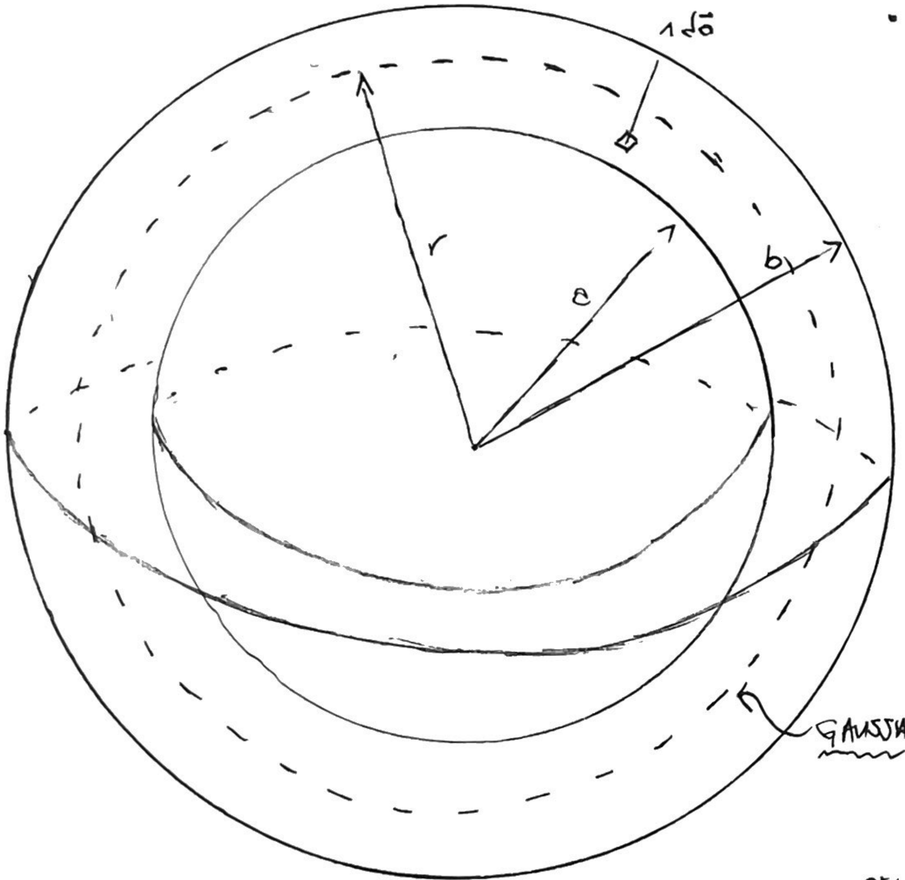
PHYS 311 HOMEWORK SET 12



- IF THEY ARE MAINTAINED A A POTENTIAL DIFFERENCE ΔV, THEN THESE SHELLS HAVE DIFFERENT NET CHARGES ON EACH.
- SUPPOSE THE INNER METAL SHELL HAS A NET CHARGE $+Q$ PLACED ON IT...
- SINCE THE SPHERICAL SHELL IS METAL & SPHERICALLY-SYMMETRIC, IT WILL HAVE A UNIFORM CHARGE DENSITY

GAUSSIAN SURFACE OF RADIUS $a<r<b$

SINCE THE MAGNITUDE OF E MUST BE THE SAME AT A GIVEN r

)NOW GAUSS' LAW..

$$\int \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{so} \quad \int \vec{E}\cdot d\vec{a} = \int E\,da = E\int da = E4\pi r^2$$

SINCE $\vec{E} = E\hat{r}$ & $d\vec{a} = da\,\hat{r}$

SO GAUSS' LAW YIELDS...

$$E4\pi r^2 = \frac{Q}{\varepsilon_0} \quad \therefore \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \quad \therefore \quad \left[\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\right]$$

NOW CALCULATE THE POTENTIAL DIFFERENCE BETWEEN THE SPHERICAL SHELLS...

$$\Delta V = -\int_b^a \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\varepsilon_0}Q\int_b^a \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\Big|_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

SINCE $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

NOW, SINCE THE SHELLS HAVE A POTENTIAL DIFFERENCE AND SINCE THE GAP CONTAINS A WEAKLY CONDUCTING MATERIAL OF CONDUCTIVITY σ, A CURRENT WILL FLOW BETWEEN THE SHELLS.

THE CURRENT BETWEEN THE SHELLS CAN BE CALCULATED VIA...

$$I = \int \vec{J} \cdot d\vec{a} = \sigma \int \vec{E} \cdot d\vec{a} = \sigma \int \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \cdot r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$$

↳ SINCE $\vec{J} = \sigma \vec{E}$ (OHM'S LAW)

$$= \sigma \cdot \frac{Q}{4\pi\varepsilon_0} \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\varphi = \frac{\sigma Q}{4\pi\varepsilon_0} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{\sigma Q}{\varepsilon_0}$$

SO

$$I = \frac{\sigma Q}{\varepsilon_0} \quad \therefore \quad Q = \frac{I \varepsilon_0}{\sigma}$$

AND

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{I\varepsilon_0}{\sigma}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right) I = \frac{1}{4\pi\sigma}\left(\frac{b-a}{ab}\right) I$$

SO

$$\boxed{I = 4\pi\sigma \Delta V \left(\frac{ab}{b-a}\right)}$$

b) NOW $I = \frac{\Delta V}{R}$ SO $\frac{1}{R} = 4\pi\sigma\left(\frac{ab}{b-a}\right)$

$$\therefore \boxed{R = \frac{1}{4\pi\sigma}\left(\frac{b-a}{ab}\right)}$$

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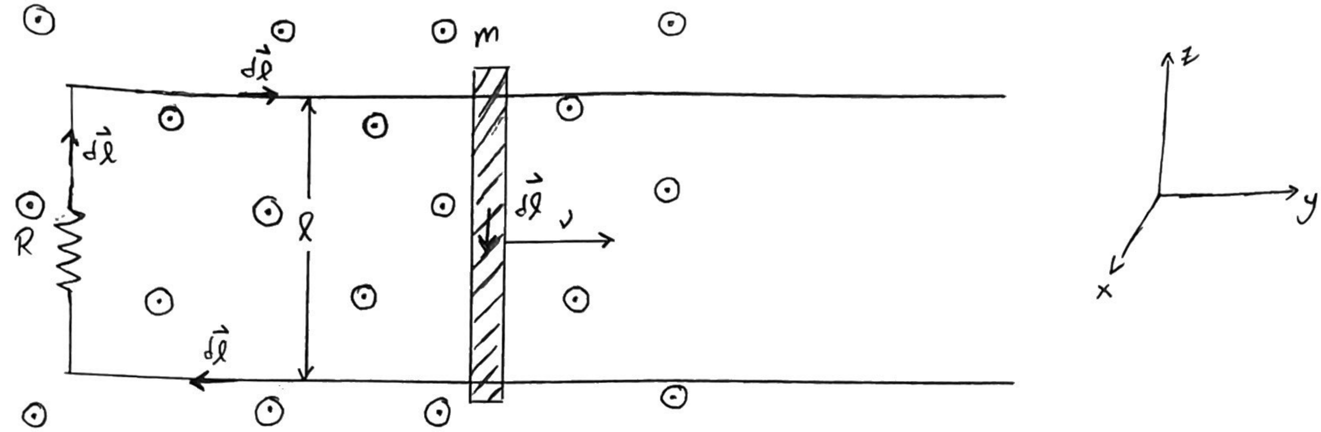


a) $\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{l} \quad =$

NOW
$$\vec{f}_{MAG} = \vec{v} \times \vec{B} \qquad \text{NOW } \vec{v} = v\hat{y} \qquad \text{SO } \vec{v} \times \vec{B} = -vB\hat{z}$$
$$\vec{B} = B\hat{x}$$

NOW AS ONLY THE CHARGES IN THE METAL BAR ARE ACTUALLY MOVING IN THE PRESENCE OF THE $\vec{B}$-FIELD THE CLOSED CONTOUR LOOP BECOMES ONE ONLY OVER THE BAR

SO
$$d\vec{l} = -dz\,\hat{z} \qquad \text{SO} \qquad \vec{f}_{MAG} \cdot d\vec{l} = vB\,dz$$

AND
$$\mathcal{E} = \int_0^l vB\,dz = vB \int_0^l dz = vBl$$

THIS $\mathcal{E}$ DRIVES A CURRENT GIVEN BY...
$$I = \frac{\mathcal{E}}{R}$$

SO
$$\left[I = \frac{vBl}{R} \right]$$ W/ THE CURRENT FLOWING IN THE CLOCKWISE DIRECTION.

b) NOW WE HAVE A CURRENT-CARRYING WIRE IN THE PRESENCE OF AN EXTERNAL $\vec{B}$-FIELD
$$\vec{F}_{MAG} = I \int d\vec{l} \times \vec{B} \quad = -IB\hat{y} \int_0^l dz \quad = -IBl\,\hat{y} \quad = -\frac{vBl}{R} \cdot Bl\,\hat{y}$$

NOW, IN THE METAL BAR...
$$d\vec{l} = -dz\,\hat{z} \qquad \text{SO} \qquad d\vec{l} \times \vec{B} = -B\,dz\,\hat{y}$$
$$\vec{B} = B\hat{x}$$

SO

$$\boxed{\vec{F}_{MAG} = -\frac{B^2 \ell^2}{R} v \, \hat{y}}$$

NOTICE:

- THIS FORCE ON THE BAR POINTS IN THE DIRECTION OPPOSING THE MOTION.
- IT'S PROPORTIONAL TO THE SPEED! LARGER SPEED, LARGER RESISTIVE FORCE.

c) NEWTONS 2^ND LAW TAKES THE FORM...

$$\vec{F}_{NET} = m\vec{a}$$

HERE, THE NET FORCE IS THE MAGNETIC FORCE SO..

$$F_{NET,y} = m\frac{dv_y}{dt} \quad \text{so} \quad \boxed{-\frac{B^2\ell^2}{R} v = m\frac{dv}{dt}}$$

NOW SEPARATING VARIABLES...

$$\int \frac{dv}{v} = -\frac{B^2 \ell^2}{mR}\int dt$$

SO

$$\ln v(t) = C - \frac{B^2\ell^2}{mR} t$$

EXPONENTIATING YIELDS..

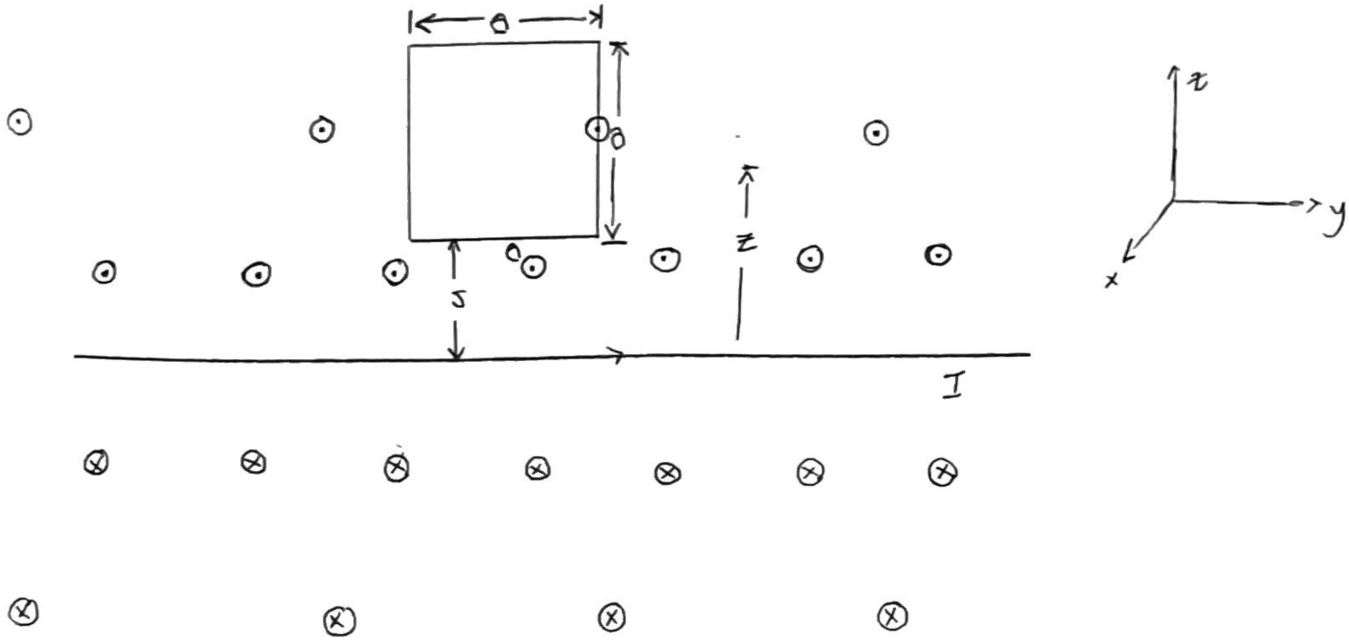
$$v(t) = e^C \exp\left[-\frac{B^2\ell^2}{mR} t\right]$$

NOW, APPLYING INITIAL CONDITIONS.

$$v(t=0) = e^C = v_0 \quad \text{so}$$

$$\boxed{v(t) = v_0 \exp\left[-\frac{B^2\ell^2}{mR} t\right]}$$

3.



NOTICE THAT THE $\vec{B}$-FIELD OF THE WIRE POINTS OUT OF THE PAGE FOR THE CURRENT-CARRYING WIRE AT THE LOCATION OF THE SQUARE LOOP.

a) NOW THE FLUX OF B THROUGH THE LOOP IS ...

$$\Phi = \int \vec{B} \cdot d\vec{a}$$

NOW $\vec{B} = \dfrac{\mu_0 I}{2\pi z}\,\hat{x}$; $d\vec{a} = dy\,dz\,\hat{x}$

SO $\vec{B} \cdot d\vec{a} = \dfrac{\mu_0 I}{2\pi z}\,dy\,dz$

SO

$$\Phi = \int_0^a\!\!\int_0^a \frac{\mu_0 I}{2\pi z}\,dy\,dz = \frac{\mu_0 I}{2\pi}\int_0^a dy \int_s^{s+a} \frac{dz}{z} = \frac{\mu_0 I}{2\pi}\,a \ln z \Big|_s^{s+a} = \frac{\mu_0 I}{2\pi}\,a\left[\ln(s+a) - \ln(s)\right]$$

$$\boxed{\Phi = \frac{\mu_0 I}{2\pi}\,a \ln\left(1 + \frac{a}{s}\right)}$$

b) IF SOMEONE PULLS THE LOOP DIRECTLY AWAY FROM THE WIRE AT SPEED v

THEN $v = \dfrac{ds}{dt}$ SO $s = s(t)$

NOW, THE UNIVERSAL FLUX RULE IS...

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

so

$$\mathcal{E} = -\frac{d}{dt}\left[\frac{\mu_0 I}{2\pi} a \ln\left(1+\frac{a}{s}\right)\right] = -\frac{\mu_0 I}{2\pi} a \frac{d}{dt} \ln\left(1+\frac{a}{s}\right)$$

NOW

$$\frac{d}{dt}\ln\left(1+\frac{a}{s}\right) = \frac{ds}{dt}\frac{d}{ds}\ln\left(1+\frac{a}{s}\right) = v \cdot \frac{1}{\left(1+\frac{a}{s}\right)} \cdot \frac{-a}{s^2} = \frac{-va}{s(s+a)}$$

so

$$\mathcal{E} = -\frac{\mu_0 I}{2\pi} a \cdot \frac{-va}{s(s+a)} = \frac{\mu_0 I a^2}{2\pi s(s+a)} v$$

$$\therefore \boxed{\mathcal{E} = \frac{\mu_0 I a^2}{2\pi s(s+a)} v}$$

DIRECTION?

SINCE THE MAGNETIC FLUX IS DECREASING, A CURRENT WILL BE GENERATED IN THE WIRE THAT CREATES ITS OWN $\vec{B}$ FIELD TO OPPOSE THE CHANGE (LENZ' LAW)

$\therefore$ THE CURRENT WILL FLOW COUNTER CLOCKWISE DIRECTION

(THIS WILL PRODUCE A $\vec{B}$-FIELD POINTING IN THE $\hat{x}$ DIRECTION)

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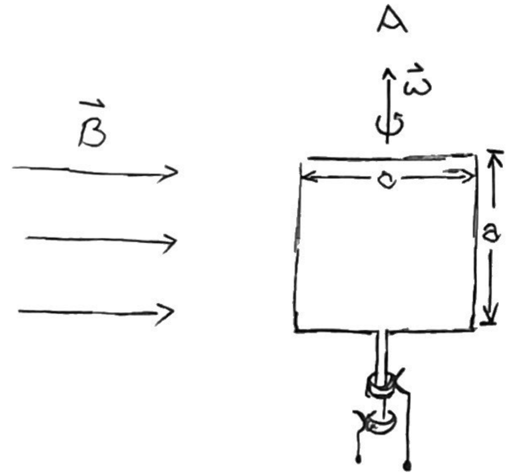
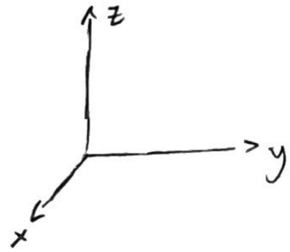
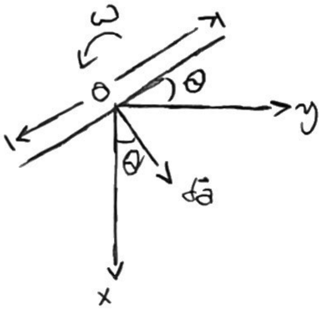
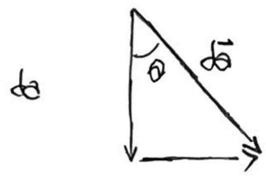




LOOKING DOWN THE Z-AXIS ...





NOW

$$d\vec{a} = da\cos\theta\,\hat{x} + da\sin\theta\,\hat{y}$$

NOW, IF THE SQUARE LOOP ROTATES AT A CONSTANT ANGULAR VELOCITY, THEN $\theta = \omega t$

AND

$$d\vec{a} = da\cos(\omega t)\,\hat{x} + da\sin(\omega t)\,\hat{y}$$

NOW $\vec{B} = B\hat{y}$

SO

$$\vec{B}\cdot d\vec{a} = B\,da\sin(\omega t)$$

SO

$$\Phi = \int \vec{B}\cdot d\vec{a} = B\sin(\omega t)\int da = Ba^2\sin(\omega t)$$

NOW, THE UNIVERSAL FLUX RULE IS ...

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}Ba^2\sin(\omega t) = -Ba^2\frac{d}{dt}\sin(\omega t) = -\omega Ba^2\cos(\omega t)$$

SO

$$\boxed{\mathcal{E}(t) = -\omega Ba^2\cos(\omega t)}$$

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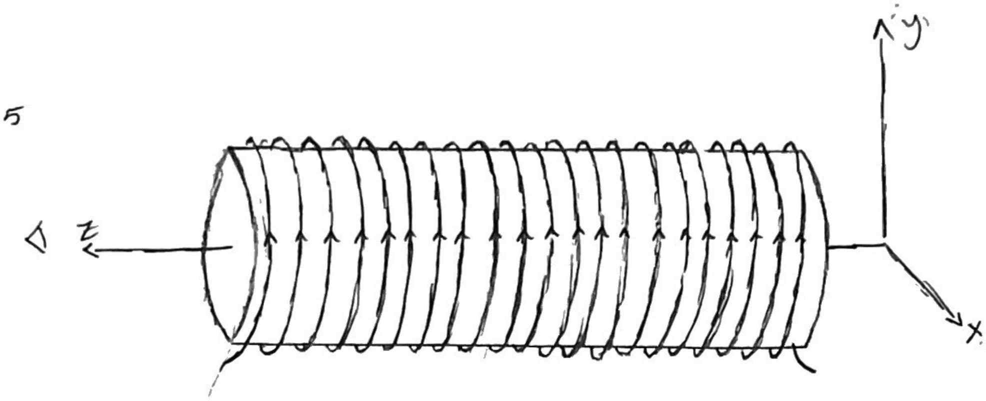
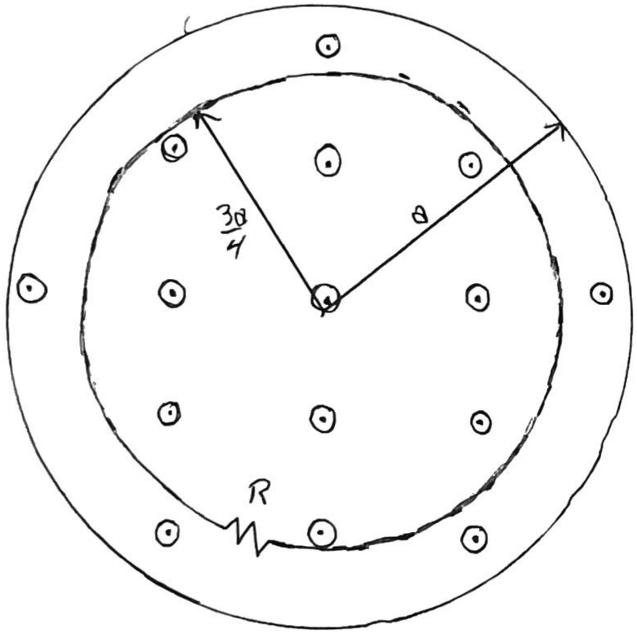


LOOKING DOWN THE Z-AXIS...



NOW

$$\vec{B}(t) = B_0 \sin(\omega t)\hat{z}$$

$$d\vec{a} = s\,d\varphi\,ds\,\hat{z}$$

SO THE MAGNETIC FLUX THROUGH THE WIRE IS...

$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{3a/4} B_0 \sin(\omega t)\, s\,ds\,d\varphi$$

$$= B_0 \sin(\omega t) \int_0^{2\pi} d\varphi \int_0^{3a/4} s\,ds$$

$$= B_0 \sin(\omega t)\cdot 2\pi \cdot \frac{s^2}{2}\Big|_0^{3a/4}$$

$$= B_0 \sin(\omega t)\,\pi\,\frac{9a^2}{16}$$

SO

$$\Phi = \frac{9\pi a^2}{16} B_0 \sin(\omega t)$$

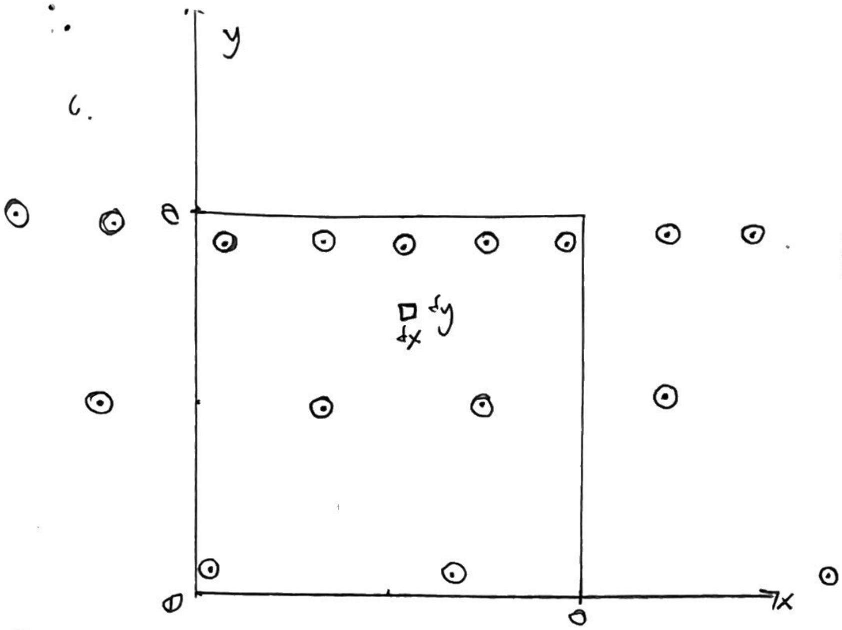
THE EMF IS FOUND VIA THE UNIVERSAL FLUX RULE...

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{9\pi}{16} a^2 B_0 \frac{d}{dt}\sin(\omega t) = -\frac{9\pi}{16} a^2 B_0 \omega \cos(\omega t)$$

AND THE CURRENT IS...

$$\boxed{I(t) = -\frac{9\pi}{16}\frac{a^2 B_0 \omega}{R}\cos(\omega t)}$$

6.



$\vec{B} = k y^2 t^3 \hat{z}$

FIRST CALCULATE THE FLUX THROUGH THE SQUARE LOOP OF WIRE...

$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_0^a\!\!\int_0^a k y^2 t^3 \, dx\, dy = k t^3 \int_0^a dx \int_0^a y^2 dy = k t^3 x \Big|_0^a \frac{y^3}{3}\Big|_0^a = \frac{1}{3} k t^3 a^4$$

NOW $d\vec{a} = dx\,dy\,\hat{z}$ SO $\vec{B}\cdot d\vec{a} = k y^2 t^3 dx\,dy$

SO

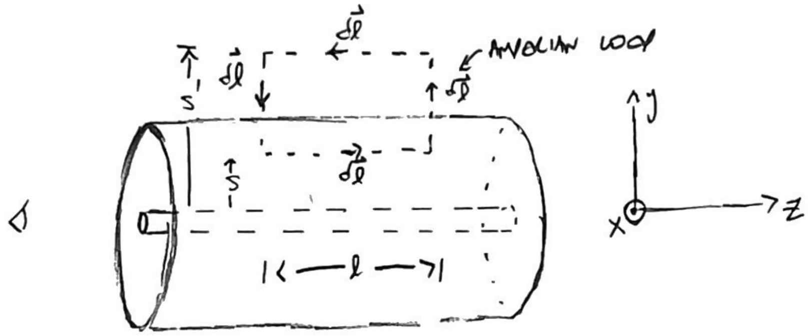
$$\Phi = \frac{1}{3} k t^3 a^4$$

SO THE EMF IS...

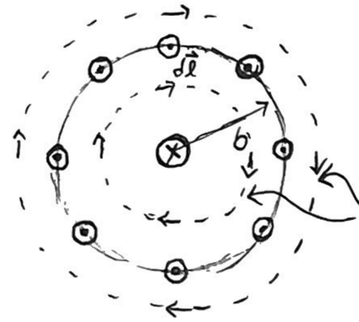
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{1}{3} k t^3 a^4\right) = -\frac{1}{3} k a^4 \frac{d}{dt} t^3 = -k a^4 t^2$$

$$\left[\mathcal{E} = -k a^4 t^2\right]$$

7.



LOOKING DOWN THE AXLE OF THE WIRE...



TWO AMPERIAN LOOPS OF RADII
- $s<b$
- $s>b$

AMPERE'S LAW IN INTEGRAL FORM IS...

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

NOW B MUST BE CIRCUMFERENTIAL SO

$$\vec{B} = B\hat{\varphi}$$

NOW...

$$\int \vec{B}\cdot d\vec{l} = \int_0^{2\pi} Bs\,d\varphi = B2\pi s$$

$$d\vec{l} = s\,d\varphi\,\hat{\varphi}$$

FOR $s<b$...

$I_{enc} = I(t)$ SO AMPERE'S LAW GIVES $B2\pi s = \mu_0 I(t)$

SO $$\vec{B}(s,t) = \frac{\mu_0 I(t)}{2\pi s}\hat{\varphi} \quad \text{FOR} \quad s<b$$

FOR $s>b$...

$I_{enc} = 0$ SO $$\vec{B}(s,t) = 0 \quad \text{FOR} \quad s>b$$

1) AS THE CHANGING $B(t)$ IS CIRCUMFERENTIAL, THE INDUCED $\vec{E}$-FIELD IS LONGITUDINAL

$$\vec{E} = E(s,t)\hat{z}$$

2) NOW CONSIDER THE AMPERIAN LOOP SHOWN IN THE FIGURE ABOVE...

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = \int_{\text{BOTTOM}} \vec{E}\cdot d\vec{l} + \underbrace{\int_{\text{RIGHT SIDE}} \vec{E}\cdot d\vec{l}}_{0} + \cancelto{0}{\int_{\text{TOP}} \vec{E}\cdot d\vec{l}} + \underbrace{\int_{\text{LEFT SIDE}} \vec{E}\cdot d\vec{l}}_{0} = \int_{\text{BOTTOM}} E\,dl = E\int_0^l dl = El$$

NOTICE:

- $\vec{E} \perp d\vec{l}$ ON LEFT & RIGHT SIDES $\therefore 0$
- $\vec{E} = 0$ AT TOP SIDE SINCE $B = 0$

∴ NOW, CALCULATE THE MAGNETIC FLUX THROUGH THE AMPERIAN LOOP ...

$$\Phi = \int \vec{B} \cdot d\vec{a}$$

NOW IN THE PLANE OF THE AMPERIAN LOOP...

$$\vec{B} = \frac{\mu_0 I(t)}{2\pi y} \hat{x} \quad \text{for } s<b \qquad ; \; d\vec{a} = dy\,dz\,\hat{x}$$

$$= 0 \qquad \text{for } s>b$$

so

$$\Phi = \int_0^l \int_s^b \frac{\mu_0 I(t)}{2\pi y}\, dy\,dz = \frac{\mu_0 I(t)}{2\pi} \int_0^l dz \int_s^b \frac{dy}{y} = \frac{\mu_0 I(t)}{2\pi} l \ln y \Big|_s^b = \frac{\mu_0 I(t)}{2\pi} l \left[\ln b - \ln s\right]$$

$$= \frac{\mu_0 I(t)}{2\pi} l \ln\left(b/s\right)$$

so USING THE FLUX LAW..

$$\mathcal{E} = -\frac{d}{dt}\Phi \quad \text{yields}$$

$$El = -\frac{d}{dt}\left[\frac{\mu_0 I(t)}{2\pi} l \ln\left(b/s\right)\right] = -\frac{\mu_0 l}{2\pi} \ln\left(b/s\right) \frac{d}{dt} I(t)$$

NOW IF $I(t) = I_0 \sin(\omega t)$

then

$$\frac{dI}{dt} = \omega I_0 \cos(\omega t)$$

so

$$E = -\frac{\mu_0}{2\pi} \ln(b/s)\, \omega I_0 \cos(\omega t)$$

so

$$\boxed{\vec{E}(s,t) = -\frac{\mu_0}{2\pi} \ln\left(b/s\right) \omega I_0 \cos(\omega t)\, \hat{z}}$$