MAT 202
Larson – Section 6.2
Differential Equations: Growth & Decay

In section 6.1 we analyzed the solutions of differential equations graphically, using slope fields. We skipped numerical approximations using Euler's method. In this section, we will begin to learn how to solve certain types of differential equations. The strategy we will use is known as ***separation of variables***.

<u>Solving Differential Equations by Separation of Variables</u>:
Given a differential equation that can be written in the form: $y' = F(x, y)$ or $\frac{dy}{dx} = F(x, y)$. To solve by **separation of variables**:

1) ***Rewrite*** the differential equation as: $G(y)dy = H(x)dx$. You do this by manipulating the equation algebraically.
2) ***Integrate*** both sides: $\int G(y)dy = \int H(x)dx$ to get $g(y) = h(x) + C_1$. **Why do we only have one constant of integration?**
3) The ***solution*** may be written in multiple ways, due to algebraic manipulations; however, $g(y) = h(x) + C_1$ would suffice.

Ex: Solve the following differential equations:

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$$\frac{dy}{6-y} = \frac{(6-y)\,dx}{6-y}$$

$$\int \frac{-1}{6-y}\,dy = \int -1\,dx \qquad \int \frac{1}{u}\,du = \int -1\,dx$$

$6-y=u$

$-1\,dy = du$

$$\ln|u| = -x + C_1$$

$$e^{(\ln|6-y|)} = e^{(-x+C_1)}$$

$$|6-y| = e^{(-x+C_1)}$$

$$|6-y| = e^{-x} \cdot e^{C_1} = C$$

$$|6-y| = Ce^{-x}$$

$6-y = Ce^{-x}$ or $6-y = -Ce^{-x}$

$y = 6 - Ce^{-x}$ $\qquad$ $y = 6 + Ce^{-x}$

$$6-y = Ce^{-x}$$

$$\boxed{y = 6 - Ce^{-x}} \leftarrow \text{General Solution}$$

b) $(1 + x^2)y' - 2xy = 0$

Ex: Find the function $y = f(t)$ passing through the point (0, 7) given that $\frac{dy}{dt} = 5\sqrt{t}$.

Now we will discuss certain applications that translate to a rate of change of a variable y (such as the rate of population growth or decay) that is proportional to the value of y. When y is a function of time, the proportion can be written as follows:

$$\frac{dy}{dt} = ky$$

So what does the general solution of this type of differential equation look like?

b) $\left(1+x^2\right)y' - 2xy = 0$

$(1+x^2)\,y' = 2xy$

$$\frac{\cancel{(1+x^2)}\,dy}{y\cancel{(1+x^2)}} = \frac{2x\cancel{y}\,dx}{\cancel{y}(1+x^2)}$$

$$\int \frac{1}{y}\,dy = \int \frac{2x}{1+x^2}\,dx$$

$u = 1+x^2$
$du = 2x\,dx$

$$e^{\left(\ln|y|\right)} = e^{\left(\ln|1+x^2| + C_1\right)}$$

$$y = e^{\ln(1+x^2)} \cdot e^{C_1}$$

$$y = C(1+x^2)$$

Ex: Find the function $y = f(t)$ passing through the point (0, 7)

initial condition

given that $\frac{dy}{dt} = 5\sqrt{t}$.

$$\int dy = \int 5t^{1/2}\, dt$$

$$y = 5 \cdot \frac{2}{3} t^{3/2} + C$$

$$y = \frac{10}{3} t^{3/2} + C$$

$$7 = \frac{10}{3}(0)^{3/2} + C$$

$$7 = C$$

Particular Solution

$$y = \frac{10}{3} t^{3/2} + 7$$

<u>Exponential Growth & Decay Model</u>: If y is a differential function of t such that y > 0 and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

Where C is the initial value of y, and k is the ***proportionality constant***. ***Exponential growth*** occurs when $k > 0$, and ***exponential decay*** occurs when $k < 0$.

Ex: The population (in millions) of Egypt in 2011 was 82.1, and the expected continuous annual rate of change of the population, $k = 0.020$.

a) Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to the year 2010.
b) Use the module to predict the population of Egypt in 2020.

<u>Newton's Law of Cooling</u>: The rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium:

$$y' = k(y - T)$$

Ex: The population (in millions) of Egypt in 2011 was 82.1, and the expected continuous annual rate of change of the population, $k = 0.020$.

$(1, 82.1)$

a) Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to the year 2010.

b) Use the module to predict the population of Egypt in 2020.

a) $82.1 = Ce^{0.02 \cdot 1}$

$C = \frac{82.1}{e^{0.02}}$

$P = \frac{82.1}{e^{0.02}} e^{0.02t}$

b) In 2020, $t = 10$

$P = \frac{82.1}{e^{0.02}} e^{0.02(10)}$

$P \approx \boxed{98.29 \text{ million}}$

Where k is the proportionality constant, y is the object's temperature, and T is the temperature of the surrounding medium.

Ex: A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid's temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

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$$y' = k(y - T)$$

$(5, 60)$ (t, y)

$T = 20°F$

$(0, 160)$

I need a function y in terms of t.

$$y' = k(y-20)$$

$$\int \frac{1}{y-20}\,dy = \int k\,dt$$

$$e^{(\ln|y-20|)} = e^{(kt + C_1)}$$

$$y - 20 = Ce^{kt}$$

$$y = Ce^{kt} + 20 \qquad (\text{using } (0,160))$$

$$160 = Ce^{k\cdot 0} + 20$$

$$160 = C + 20$$

$$140 = C$$

$$y = 140e^{kt} + 20 \qquad (\text{using } (5,60))$$

$$60 = 140e^{k(5)} + 20$$

$$40 = 140e^{5k}$$

$$\ln\left(\frac{2}{7}\right) = \ln\left(e^{5k}\right)$$

$$\ln\left(\frac{2}{7}\right) = 5k$$

$$k = \frac{\ln(2/7)}{5}$$

$$y = 140e^{\left(\frac{\ln(2/7)}{5}t\right)} + 20$$

$$30 = 140e^{\left(\frac{\ln(2/7)}{5}t\right)} + 20$$

$$10 = 140e^{\left(\frac{\ln(2/7)}{5}t\right)}$$

$$\ln\left(\frac{1}{14}\right) = \ln\left(e^{\left(\frac{\ln(2/7)}{5}t\right)}\right)$$

$$\ln\left(\frac{1}{14}\right) = \frac{\ln(2/7)}{5}t$$

$$t = \frac{\ln\left(\frac{1}{14}\right)}{\frac{\ln(2/7)}{5}} = \frac{5\ln\left(\frac{1}{14}\right)}{\ln\left(\frac{2}{7}\right)}$$

$t \approx 10.53$ min.

$\approx$ 5.53 min. longer