PHYS 396 HOMEWORK SET 5

1 $$ds^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$$

a) FOR THESE TWO EVENTS ...

$$x' = h \quad \therefore \quad dx' = dy' = dz' = 0$$

$$\therefore ds^2 = -\left(1 + \frac{gh}{c^2}\right)^2 c^2 dt'^2 = -c^2 d\tau_A^2 \quad \therefore \quad d\tau_A = \left(1 + \frac{gh}{c^2}\right) dt'$$

INTEGRATING YIELDS...

$$(*) \quad \left[\Delta\tau_A = \left(1 + \frac{gh}{c^2}\right)\Delta t'\right]$$

b) FOR THESE TWO EVENTS...

$$x' = 0 \quad \therefore \quad dx' = dy' = dz' = 0$$

$$\therefore \quad ds^2 = -c^2 dt'^2 = -c^2 d\tau_B^2 \quad \therefore \quad d\tau_B = dt'$$

INTEGRATING YIELDS

$$(**) \quad \left[\Delta\tau_B = \Delta t'\right]$$

c)

PLUGGING (**) INTO (*) TO ELIMINATE THE COORDINATE TIME INTERVAL, WE OBTAIN...

$$\left[\Delta\tau_A = \left(1 + \frac{gh}{c^2}\right)\Delta\tau_B\right]$$

NOTICE THAT

$\Delta\tau_A > \Delta\tau_B$ $\therefore$ ALICE MEASURES THE LONGER TIME INTERVAL

$$? \quad ct' = \frac{c^2}{g} \tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)$$

since $d \tanh^{-1}(x) = \frac{1}{1-x^2}\, dx$

so (*)

$$c\,dt' = \frac{c^2}{g}\, d\left[\tanh^{-1}\left(\frac{ct}{x+c^2/g}\right)\right] = \frac{c^2}{g} \cdot \frac{1}{1 - \frac{c^2t^2}{(x+c^2/g)^2}}\, d\left(\frac{ct}{x+c^2/g}\right)$$

NOW

$$d\left(\frac{ct}{x+c^2/g}\right) = \frac{c\,dt}{(x+c^2/g)} - \frac{ct}{(x+c^2/g)^2}\, dx$$

$$= \frac{c}{(x+c^2/g)^2}\left[(x+c^2/g)\,dt - t\,dx\right] \checkmark$$

PLUGGING THIS INTO (*) YIELDS..

$$c\,dt' = \frac{c^2}{g} \cdot \frac{1}{1 - \frac{c^2t^2}{(x+c^2/g)^2}} \cdot \frac{c}{(x+c^2/g)^2}\left[(x+c^2/g)\,dt - t\,dx\right]$$

$$= \frac{c^3}{g} \cdot \frac{1}{(x+c^2/g)^2 - c^2t^2}\left[(x+c^2/g)\,dt - t\,dx\right] \checkmark$$

NOW

$$x' = \left[\left(x + \frac{c^2}{g}\right)^2 - c^2t^2\right]^{1/2} - \frac{c^2}{g}$$

so

$$dx' = \frac{1}{2}\left[\left(x+\frac{c^2}{g}\right)^2 - c^2t^2\right]^{-1/2} \cdot \left[2\left(x+\frac{c^2}{g}\right)dx - 2c^2t\,dt\right]$$

$$= \left[\left(x+\frac{c^2}{g}\right)^2 - c^2t^2\right]^{-1/2}\left[\left(x+\frac{c^2}{g}\right)dx - c^2t\,dt\right] \checkmark$$

WHEREAS

$$dy' = dy$$

$$dz' = dz$$

NOW SQUARING ...

$$c^2 dt'^2 = \frac{c^6}{g^2} \cdot \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]^2}\left[(x+c^2/g)^2 dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\right] \checkmark$$

$$dx'^2 = \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]}\left[(x+c^2/g)^2 dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\right] \checkmark$$

$$dy'^2 = dy^2$$

$$dz'^2 = dz^2$$

NOW PLUGGING THIS INTO (1) YIELDS..

$$ds^2 = -\left[\cancel{1} + \frac{g}{c^2}\left[(x+c^2/g)^2 - c^2t^2\right]^{1/2} - \cancel{1}\right]^2 \frac{c^6}{g^2}\cdot\frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]^2}\left[(x+c^2/g)^2dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\right]$$

$$+ \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]}\left[(x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\right] + dy^2 + dz^2$$

$$= -\frac{\cancel{g^2}}{\cancel{c^4}}\left[(x+c^2/g)^2 - c^2t^2\right]\frac{c^{\cancel{6}\,2}}{\cancel{g^2}}\cdot\frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]^2}\left[(x+c^2/g)^2dt^2 + t^2dx^2 - 2(x+c^2/g)t\,dt\,dx\right]$$

$$+ \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]}\left[(x+c^2/g)^2dx^2 + c^4t^2dt^2 - 2(x+c^2/g)c^2t\,dt\,dx\right] + dy^2 + dz^2$$

$$= \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]}\left[\begin{array}{l} -c^2\left[(x+c^2/g)^2dt^2 + t^2dx^2 - \cancel{2(x+c^2/g)t\,dt\,dx}\right] \\ + (x+c^2/g)^2dx^2 + c^4t^2dt^2 - \cancel{2(x+c^2/g)c^2t\,dt\,dx}\end{array}\right] + dy^2 + dz^2$$

NOW COMBINING LIKE TERMS...

$$ds^2 = \frac{1}{\left[(x+c^2/g)^2 - c^2t^2\right]}\left[-c^2\left[(x+c^2/g)^2 - c^2t^2\right]dt^2 + \left[(x+c^2/g)^2 - c^2t^2\right]dx^2\right] + dy^2 + dz^2$$

so

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \checkmark$$

3

$$ct = \left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right)$$

so

$$c\,dt = \sinh\left(\frac{gt'}{c}\right)dx' + \frac{g}{c}\left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right)dt'$$

$$= \sinh\left(\frac{gt'}{c}\right)dx' + c\left(1 + \frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right)dt'$$

Whereas

$$x = \left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$$

$$dx = \cosh\left(\frac{gt'}{c}\right)dx' + \frac{g}{c}\left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right)dt'$$

$$= \cosh\left(\frac{gt'}{c}\right)dx' + c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)dt'$$

so squaring yields..

$$c^2dt^2 = \sinh^2\left(\frac{gt'}{c}\right)dx'^2 + c^2\left(1 + \frac{gx'}{c^2}\right)^2\cosh^2\left(\frac{gt'}{c}\right)dt'^2 + 2c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt'dx'$$

$$dx^2 = \cosh^2\left(\frac{gt'}{c}\right)dx'^2 + c^2\left(1 + \frac{gx'}{c^2}\right)^2\sinh^2\left(\frac{gt'}{c}\right)dt'^2 + 2c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)\cosh\left(\frac{gt'}{c}\right)dt'dx'$$

so

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$= \underbrace{\left[\cosh^2\left(\frac{gt'}{c}\right) - \sinh^2\left(\frac{gt'}{c}\right)\right]}_{1}dx'^2 - c^2\left(1 + \frac{gx'}{c^2}\right)^2\underbrace{\left[\cosh^2\left(\frac{gt'}{c}\right) - \sinh^2\left(\frac{gt'}{c}\right)\right]}_{1}dt'^2 + dy'^2 + dz'^2$$

$$= -\left(1 + \frac{gx'}{c^2}\right)c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \checkmark$$

4. SINCE

a) $$ct = \left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right)$$

$$x = \left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$$

THEN

1) $$\left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right) = ct$$

2) $$\left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right) = \frac{c^2}{g} + x$$

DIVIDING 1) BY 2)...

$$\tanh\left(\frac{gt'}{c}\right) = \frac{ct}{(c^2/g + x)} \quad \therefore \left[\frac{gt'}{c} = \tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)\right]\frac{c^2}{g}$$

SO

$$\left[ct' = \frac{c^2}{g}\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right)\right]$$

SQUARING 2) & 1) AND SUBTRACTING 1) FROM 2)...

$$\left(\frac{c^2}{g} + x'\right)^2\cosh^2\left(\frac{gt'}{c}\right) - \left(\frac{c^2}{g} + x'\right)^2\sinh^2\left(\frac{gt'}{c}\right) = \left(\frac{c^2}{g} + x\right)^2 - c^2t^2$$

SO

$$\left(\frac{c^2}{g} + x'\right)^2\left[\underbrace{\cosh^2\left(\frac{gt'}{c}\right) - \sinh^2\left(\frac{gt'}{c}\right)}_{1}\right] = \left(\frac{c^2}{g} + x\right)^2 - c^2t^2$$

SO

$$\left(\frac{c^2}{g} + x'\right)^2 = \left(\frac{c^2}{g} + x\right)^2 - c^2t^2$$

TAKING THE SQUARE ROOT & SOLVING FOR x'...

$$\left[x' = \left[\left(x + \frac{c^2}{g}\right)^2 - c^2t^2\right]^{1/2} - \frac{c^2}{g}\right]$$

NOW

$$\sinh \Theta \simeq \Theta + \frac{\Theta^3}{6} + \dots$$

$$\cosh \Theta \simeq 1 + \frac{\Theta^2}{2} + \frac{\Theta^4}{24} + \dots$$

SO APPLYING THESE SERIES EXPANSIONS TO (4) YIELDS..

$$ct = \left(\frac{c^2}{g} + x'\right)\left[\frac{gt'}{c} + \frac{1}{6}\frac{g^3 t'^3}{c^3} + \dots\right] \cong \left(\frac{c^2}{g} + x'\right)\frac{gt'}{c} \simeq ct' \text{ SINCE } x' \ll \frac{c^2}{g}$$

$$x = \left(\frac{c^2}{g} + x'\right)\left[1 + \frac{1}{2}\frac{g^2 t'^2}{c^2} + \dots\right] - \frac{c^2}{g} = \cancel{\frac{c^2}{g}} + \frac{c^2}{g}\cdot\frac{1}{2}\frac{g^2 t'^2}{c^2} + x' + \frac{1}{2}x'\frac{g^2 t'^2}{c^2} + \dots - \cancel{\frac{c^2}{g}}$$

$$= x' + \frac{1}{2}gt'^2 + \dots$$

c) $\frac{c^2}{g} = c \cdot \frac{3 \times 10^8 \, m/s}{9.8 \, m/s^2} = 3.1 \times 10^{7} \, s \cdot c \times \frac{1 \, hr}{3600 \, s} \times \frac{1 \, DAY}{24 \, hrs} \times \frac{1 \, yr}{365 \, DAYS}$

$= 0.97$ c yrs

5.

SINCE $ds^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2$

FOR 2 EVENTS THAT OCCUR AT THE SAME SPATIAL COORDINATES...

$$dx' = dy' = dz' = 0$$

AND THE LINE ELEMENT BECOMES...

$$ds^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 c^2 dt'^2 = -c^2 d\tau^2$$

SO

$$d\tau = \left(1 + \frac{gx'}{c^2}\right) dt' \quad \therefore \boxed{\frac{dt'}{d\tau} = \left(1 + \frac{gx'}{c^2}\right)^{-1}}$$

a) NOW

$$u^\alpha = \frac{dx^\alpha}{d\tau} \quad \text{SO} \quad u^0 = \frac{dt}{d\tau} = \frac{dt}{dt'}\frac{dt'}{d\tau}$$

NOW

$$\frac{dt}{dt'} = \left(\frac{c^2}{g} + x'\right)\frac{d}{dt'}\sinh\left(\frac{gt'}{c}\right) = \frac{g}{c}\left(\frac{c^2}{g} + x'\right)\cosh\left(\frac{gt'}{c}\right)$$

$$= c\left(1 + \frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right)$$

SO

$$u^0 = c\left(1 + \frac{gx'}{c^2}\right)\cosh\left(\frac{gt'}{c}\right)\left(1 + \frac{gx'}{c^2}\right)^{-1} = c\cosh\left(\frac{gt'}{c}\right)$$

$$u^1 = \frac{dx}{d\tau} = \frac{dx}{dt'}\frac{dt'}{d\tau} =$$

NOW

$$\frac{dx}{dt'} = \left(\frac{c^2}{g} + x'\right)\frac{d}{dt'}\cosh\left(\frac{gt'}{c}\right) = \frac{g}{c}\left(\frac{c^2}{g} + x'\right)\sinh\left(\frac{gt'}{c}\right) = c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right)$$

SO

$$u^1 = c\left(1 + \frac{gx'}{c^2}\right)\sinh\left(\frac{gt'}{c}\right) \cdot \left(1 + \frac{gx'}{c^2}\right)^{-1} = c\sinh\left(\frac{gt'}{c}\right)$$

$$u^2 = \frac{dy}{d\tau} = \frac{dy}{dt'}\frac{dt'}{d\tau} = 0 \quad \text{SINCE } \frac{dy}{dt'} = 0$$

$$u^3 = \frac{dz}{d\tau} = \frac{dz}{dt'}\frac{dt'}{d\tau} = 0 \quad \text{SINCE } \frac{dz}{dt'} = 0$$

$$\therefore \boxed{u^\alpha = \left(c\cosh\left(\frac{gt'}{c}\right), c\sinh\left(\frac{gt'}{c}\right), 0, 0\right)}$$

b) SINCE $a^\alpha = \dfrac{d^2x^\alpha}{d\tau^2} = \dfrac{d}{d\tau}u^\alpha$

SO

$$a^0 = \frac{d}{d\tau}u^0 = \frac{dt'}{d\tau}\frac{d}{dt'}\left(c\cosh\left(\frac{gt'}{c}\right)\right) = \left(1+\frac{gx'}{c^2}\right)^{-1}\cdot c\frac{g}{c}\sinh\left(\frac{gt'}{c}\right)$$

$$= g\left(1+\frac{gx'}{c^2}\right)^{-1}\sinh\left(\frac{gt'}{c}\right)$$

$$a^1 = \frac{d}{d\tau}u^1 = \frac{dt'}{d\tau}\frac{d}{dt'}\left(c\sinh\left(\frac{gt'}{c}\right)\right) = \left(1+\frac{gx'}{c^2}\right)^{-1}\cdot c\frac{g}{c}\cosh\left(\frac{gt'}{c}\right)$$

$$= g\left(1+\frac{gx'}{c^2}\right)^{-1}\cosh\left(\frac{gt'}{c}\right)$$

$$a^2 = a^3 = 0$$

SO

$$\boxed{a^\alpha = \left(g\left(1+\frac{gx'}{c^2}\right)^{-1}\sinh\left(\frac{gt'}{c}\right), g\left(1+\frac{gx'}{c^2}\right)^{-1}\cosh\left(\frac{gt'}{c}\right), 0, 0\right)}$$

c) NOW

$$\eta_{\alpha\beta}a^\alpha a^\beta = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$$

$$= -g^2\left(1+\frac{gx'}{c^2}\right)^{-2}\sinh^2\left(\frac{gt'}{c}\right) + g^2\left(1+\frac{gx'}{c^2}\right)^{-2}\cosh^2\left(\frac{gt'}{c}\right)$$

$$= g^2\left(1+\frac{gx'}{c^2}\right)^{-2}\left[\cosh^2\left(\frac{gt'}{c}\right) - \sinh^2\left(\frac{gt'}{c}\right)\right] = g^2(1+$$

SO, IF

$$a \equiv \left(\eta_{\alpha\beta}a^\alpha a^\beta\right)^{1/2} \quad \text{THEN} \quad \boxed{a = g\left(1+\frac{gx'}{c^2}\right)^{-1}}$$

SO

$$a(x'=h) = g\left(1+\frac{gh}{c^2}\right)^{-1}$$

$$a(x'=0) = g$$

SO $a(x'=0) > a(x'=h)$