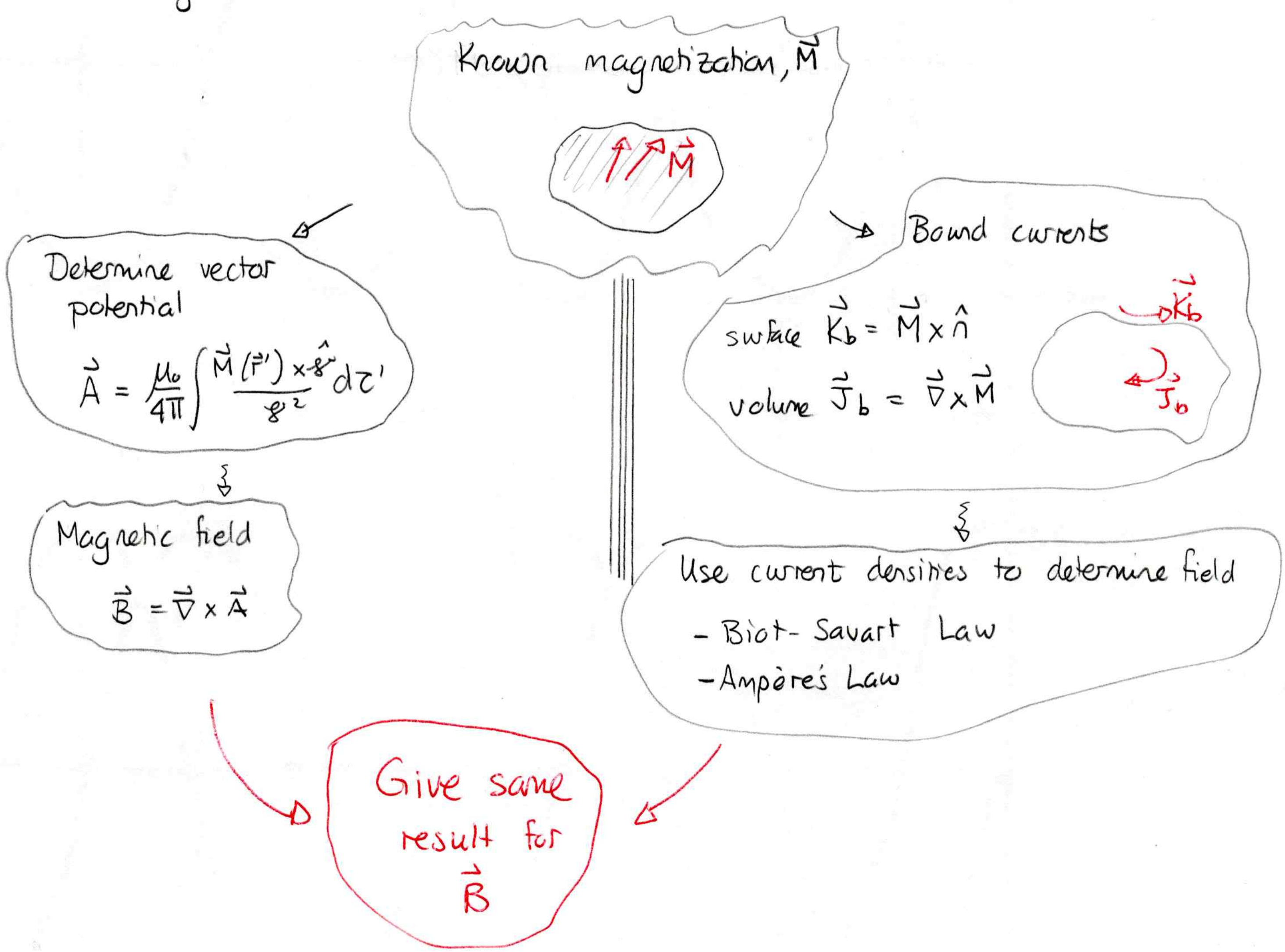
Fri: HW by 5pm

Tues: Read 7.3.5, 7.3.6, 8.1

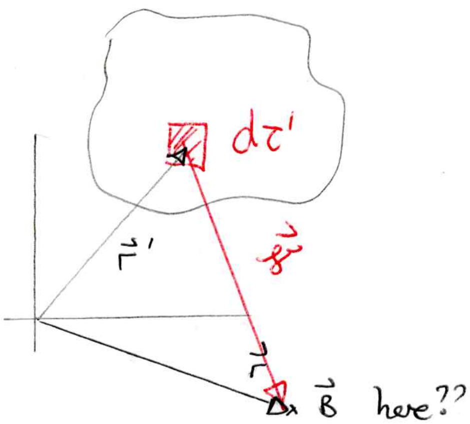
Magnetization and bound currents

The magnetic properties of a material can be described in terms of a distribution of point dipoles. The distribution is quantified by the magnetization $\vec{M}(\vec{r}')$, which is the dipole moment per unit volume. This then gives the scheme:



Recall that the computational tools, adapted for bound currents are as follows. The Biot-Savart Law

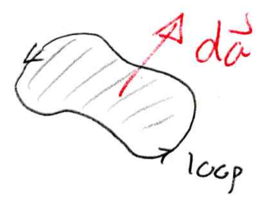
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\text{volume}} \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau' + \frac{\mu_0}{4\pi}\int_{\text{surface}} \frac{\vec{K}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da'$$



The more convenient technique involves Ampère's Law.

For any closed loop

$$\oint_{\text{loop}} \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$



where the enclosed current is

$$I_{enc} = \int_{\text{loop surface}} \vec{J}_b\cdot d\vec{a}$$

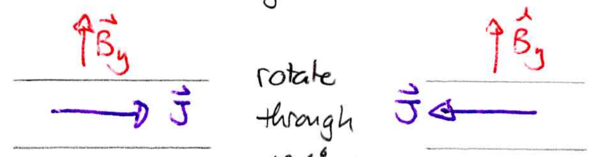
b) Recall that the Biot-Savart law implies that the magnetic field cannot have a component along the current direction.

Thus $B_x = 0 \Rightarrow \vec{B} = B_y \hat{y} + B_z \hat{z}$

The "current-reversal" argument addresses the $\hat{y}$ - component

Then the current reverses but the $\hat{y}$-component does not.

This is impossible. So $B_y = 0$.



Thus

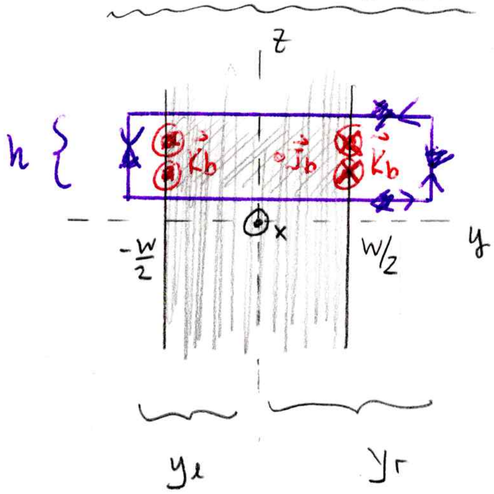
$$\vec{B} = B_z(y)\,\hat{z}$$

c) We need to argue in parts

- field outside the slab is uniform
- value of field outside the slab
- field inside the slab.

field outside the slab

Construct a rectangular loop in the y - z plane.



Around this loop

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

But for bound currents the enclosed current across this section is:

$$I_{enc} = I_{enc\ \text{right edge}} + I_{enc\ \text{interior}} + I_{enc\ \text{left edge}}$$

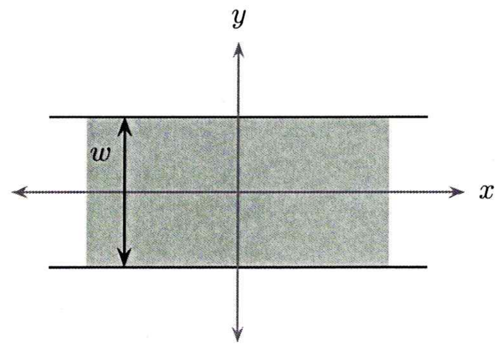
$$= M\left(-\tfrac{w}{2}\right)h + \int \vec{J}_b \cdot d\vec{a} - M\left(\tfrac{w}{2}\right)h$$

$$= M\left(-\tfrac{w}{2}\right)h + \iint \frac{dM}{dy}\, dy\, dz - M\left(\tfrac{w}{2}\right)h$$

$$= h\left\{ M\left(-\tfrac{w}{2}\right) + M\left(\tfrac{w}{2}\right) - M\left(-\tfrac{w}{2}\right) - M\left(\tfrac{w}{2}\right)\right\} = 0$$

1 Magnetized ~~sheet~~ slab

A sheet of material extends infinitely along the z and x axes and has width w along the y axis. The origin of the axes is centered as illustrated. The material is magnetized in such a way that $\mathbf{M} = M(y)\hat{\mathbf{z}}$.



a) Determine expressions for the bound current densities and sketch these qualitatively.

b) In order to determine the magnetic field, we need to determine its direction. In general

$$\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}.$$

Use the Biot-Savart law to eliminate one component and the "current-reversal" argument to eliminate another.

c) Choose an~~d~~ appropriate Ampèrian loop and apply Ampère's law to determine the magnetic field at all points.

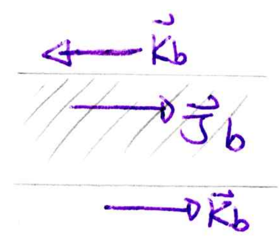
Answer

a) Surface current $\quad \vec{K}_b = \vec{M} \times \hat{n} = M(y)\, \hat{z} \times \hat{n}$

At surface $y = +w/2 \quad \hat{n} = \hat{y} \Rightarrow \vec{K}_b = -M(\frac{w}{2})\hat{x}$

At surface $y = -w/2 \quad \hat{n} = -\hat{y} \Rightarrow \vec{K}_b = M(-\frac{w}{2})\hat{x}$

e.g. $M > 0$



Volume current.

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & M(y) \end{vmatrix}$$

$$= \hat{x}\frac{\partial M}{\partial y} + \hat{y}\left(-\cancel{\frac{\partial M(y)}{\partial x}}\right) + \hat{z}\, 0$$

$$\vec{J}_b = \frac{dM}{dy}\hat{x}$$

Thus

$$\oint \vec{B}\cdot d\vec{l} = 0$$

$$\Rightarrow \int_{\text{right}} \vec{B}\cdot d\vec{l} + \cancel{\int_{\text{top}} \vec{B}\cdot d\vec{l}}^{\,0} + \int_{\text{left}} \vec{B}\cdot d\vec{l} + \cancel{\int_{\text{bottom}} \vec{B}\cdot d\vec{l}}^{\,0} = 0$$

$\vec{B}\cdot d\vec{l} = 0$ $\quad\quad$ $\vec{B}\cdot d\vec{l} = 0$

$$\int_0^h \vec{B}\cdot\hat{z}\,dz_{\text{right}} - \int_0^h \vec{B}\cdot\hat{z}\,dz_{\text{left}}$$

$$\Rightarrow \int_0^h B_z(y_l)\,dz - \int_0^h B_z(y_r)\,dz = 0$$

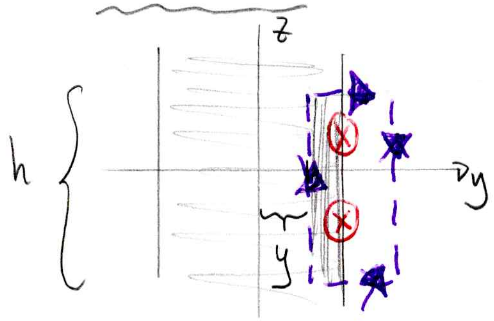
$$\Rightarrow h\left[B_z(y_l) - B_z(y_r)\right] = 0 \Rightarrow B_z(y_l) = B_z(y_r)$$

Thus everywhere outside the slab B_z is uniform. We can determine the value by noting that as $y_r \to \infty$ the current density will appear to approach zero. Then $\vec{B} \to 0 \Rightarrow B_z = 0$ outside.

So

$$\vec{B} = \begin{cases} B_z(y)\,\hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

Inside the slab



Use the illustrated loop. Then

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$ Only left edge contributes

$$\Rightarrow B_z(y)\,h = \mu_0 I_{enc}$$

$$\Rightarrow B_z(y) = \frac{\mu_0 I_{enc}}{h}$$

Now

$$I_{enc} = \underbrace{I_{enc} \text{ in shaded volume}} + \underbrace{I_{enc} \text{ surface at } y = w/2}$$

$$= \iint_{\text{shaded volume}} \vec{J}_b \cdot d\vec{a} \quad - K_{b \text{ at } w/2}\, h$$

↳ Because $\vec{K}_b$ into, loop sense out.

For the volume $\left.\begin{matrix} y \le y' \le w/2 \\ 0 \le z' \le h \end{matrix}\right\}$ $d\vec{a} = -dy'dz'\hat{x}$

$$\vec{J}_b \cdot d\vec{a} = -\frac{dM}{dy'}dy'dz'$$

$$I_{enc} = -\int_0^h dz' \int_y^{w/2} \frac{dM}{dy'}dy' - \left(-M\left(\tfrac{w}{2}\right)h\right)$$

$$= -h\left[M\left(\tfrac{w}{2}\right) - M(y)\right] - hM\left(\tfrac{w}{2}\right)$$

$$= +hM(y).$$

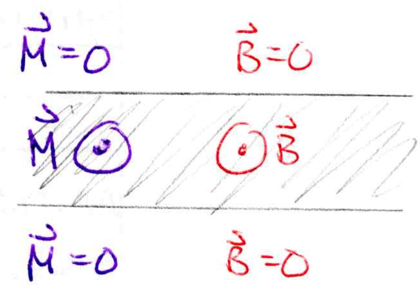
So $\quad B_z = \dfrac{+\mu_0 hM(y)}{h} = \mu_0 M(y)$

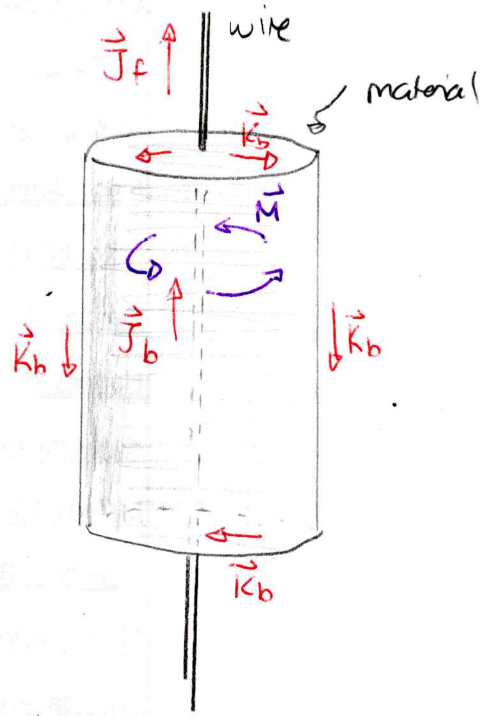
Thus

$$\vec{B} = \begin{cases} \mu_0 M(y)\hat{z} & \text{inside slab} \\ 0 & \text{outside slab} \end{cases}$$

Auxiliary magnetic fields

In the previous example, the magnetic field ended up being proportional to the magnetization. Is there an easier way to reach this result?





In general the magnetic field satisfies

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

and the current density consists of free and bound current densities

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

Then

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_f + \mu_0\vec{\nabla}\times\vec{M}$$

$$\Rightarrow \vec{\nabla}\times(\vec{B}-\mu_0\vec{M}) = \mu_0\vec{J}_f$$

$$\Rightarrow \vec{\nabla}\times\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right) = \vec{J}_f$$

Thus we define the auxiliary (magnetic) field as:

$$\boxed{\vec{H} := \frac{1}{\mu_0}\vec{B}-\vec{M}}$$

Then:

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}_f}$$

Separately

$$\vec{\nabla}\cdot\vec{H} = \frac{1}{\mu_0}\underbrace{\vec{\nabla}\cdot\vec{B}}_{0} - \vec{\nabla}\cdot\vec{M} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

Thus

Given free current density the auxiliary field satisfies

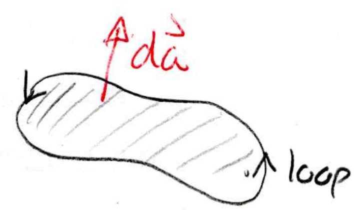
$$\vec{\nabla}\times\vec{H} = \vec{J}_f$$
$$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

We can adopt some strategies from magnetostatics to determine $\vec{H}$.

For example Stoke's theorem will give:

For any closed loop

$$\oint \vec{H}\cdot d\vec{l} = I_{\text{free enc}}$$



where the enclosed free current is

$$I_{\text{free enc}} = \int_{\text{loop surface}} \vec{J}_{\text{free}}\cdot d\vec{a}$$

loop surface → any surface bounded by the loop.

So we can determine fields via:

Known free current density $\vec{J}_f$ → Determine $\vec{H}$ → Magnetic field is $\vec{B} = \mu_0(\vec{H}+\vec{M})$

Known magnetization $\vec{M}$ → Magnetic field is $\vec{B} = \mu_0(\vec{H}+\vec{M})$

Note that unless $\vec{\nabla}\cdot\vec{M} = 0$ we cannot use a variant of the Biot-Savart law since that requires $\vec{\nabla}\cdot\vec{H} = 0$. Also in the absence of free current, i.e. $\vec{J}_f = 0$, it is not necessarily true that $\vec{H} = 0$.

2 Magnetic field in the presence of a cylindrical material

A cylindrical material with radius R carries magnetization $\mathbf{M} = M\hat{\phi}$ where $M > 0$ is constant.

a) Suppose that the material has a finite length with ends that are perpendicular to the axis. Determine the direction of the bound volume and surface currents.

b) Determine the auxiliary field if the cylinder is infinite in length and use this to determine the magnetic field.

Answer: a) Interior:

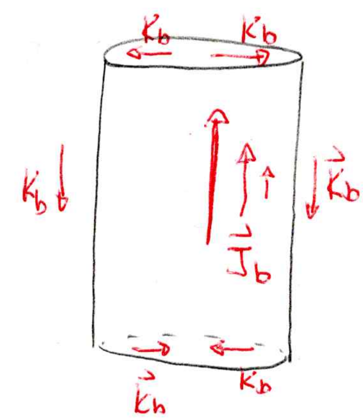
$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

$$= \left[\frac{1}{s}\frac{\partial M_z}{\partial \phi}^{0} - \frac{\partial M_\phi}{\partial z}^{0}\right]\hat{s}$$

$$+ \left[\frac{\partial M_s}{\partial z}^{0} - \frac{\partial M_z}{\partial s}^{0}\right]\hat{\phi}$$

$$+ \frac{1}{s}\left[\frac{\partial}{\partial s}(sM) - \frac{\partial M_s}{\partial \phi}^{0}\right]\hat{z}$$

$$\Rightarrow \vec{J}_b = \frac{1}{s} M \hat{z}$$



Curved surface: $\vec{K}_b = \vec{M} \times \hat{n}$ and $\hat{n} = \hat{s}$

$$= M\hat{\phi} \times \hat{s} = -M\hat{z}$$

Top surface: $\vec{K}_b = \vec{M} \times \hat{n}$ and $\hat{n} = \hat{z}$

$$= M\hat{\phi} \times \hat{z}$$

$$= M\hat{s}$$

Bottom surface: $\vec{K}_b = \vec{M} \times \hat{n}$ and $\hat{n} = -\hat{z}$

$$= -M\hat{\phi} \times \hat{z}$$

$$= -M\hat{s}$$

b) The current only flows along $+\hat{z}$. Thus $B_z = 0$ and

$$\vec{B} = B_s\,\hat{s} + B_\phi\,\hat{\phi}$$

A rotation through 180° about x establishes that $B_s = 0$

$$\Rightarrow \quad \vec{B} = B_\phi(s)\,\hat{\phi} \quad \Rightarrow \quad \vec{H} = H_\phi(s)\,\hat{\phi}$$

Using an Ampèrian loop that is a circle in the x-y plane with radius s, We get

$$\oint \vec{H}\cdot d\vec{l} = I_{\text{free enc}} = 0$$

$$\Rightarrow \quad 2\pi s\, H_\phi = 0 \quad \Rightarrow \quad H_\phi = 0 \quad \Rightarrow \quad \vec{H} = 0.$$

Thus: $\vec{H} = 0$ everywhere. Then

$$\frac{1}{\mu_0}\vec{B} - \vec{M} = 0 \quad \Rightarrow \quad \vec{B} = \mu_0 \vec{M}$$

Thus

$$\vec{B} = \begin{cases} \mu_0 M\,\hat{\phi} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

Magnetic Media

In order to use the previous scheme we need to describe how magnetization is produced. We have seen that external magnetic fields could realign dipoles or change the magnitude of dipole moments.

Exactly how this happens will depend on the material. For linear magnetic materials

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the magnetic susceptibility (unitless). Then

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

$$= \mu_0 (1 + \chi_m) \vec{H}$$

and we define

$$\boxed{\mu = \mu_0 (1 + \chi_m)}$$

as the permeability of the material. Thus

$$\boxed{\vec{B} = \mu \vec{H}}$$

3 Magnetic field in the presence of a cylindrical material surrounding a current-carrying wire

A cylindrical material with radius R surrounds a wire that carries current I. The material is linear with permittivity μ and is surrounded by free space.

~~perme~~ permeability

a) Determine the auxiliary field at all locations.

b) Determine the magnetic field at all locations.

c) Determine the magnetization at all locations and determine the bound current densities.

Answer: a) The auxiliary field satisfies

$$\oint_{\text{loop}} \vec{H}\cdot d\vec{l} = I_{\text{free enc}}.$$

We need to establish the direction of $\vec{H}$. We know that

$$\vec{\nabla}\times\vec{H} = \vec{J}_{\text{free}}.$$

Then inside either material

$$\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\frac{1}{\mu}\vec{B} = \frac{1}{\mu}\vec{\nabla}\cdot\vec{B} = 0$$ (delta fn across boundary

So $\vec{H}$ satisfies (except at the boundary)

$$\vec{\nabla}\times\vec{H} = \vec{J}_{\text{free}}$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

We can use usual magnetostatics. By symmetry

$$\vec{H} = H_\phi(s)\,\hat{\phi}$$

Use a loop of radius s as an Amperian loop. So

$$\left.\begin{array}{l} 0\leqslant \phi' \leqslant 2\pi \\ s' = s \end{array}\right\} \quad d\vec{l} = s\,d\phi'\,\hat{\phi}$$

and

$$\vec{H}\cdot d\vec{l} = H_\phi(s)\,s\,d\phi'$$

Thus

$$\oint \vec{H}\cdot d\vec{l} = I_{\text{free enc}} = I$$

$$\Rightarrow \quad 2\pi s\, H_\phi(s) = I \qquad \Rightarrow \quad H_\phi(s) = \frac{I}{2\pi s}$$

$$\Rightarrow \quad \vec{H} = \frac{I}{2\pi s}\hat{\phi}$$

b) $\vec{B} = \mu \vec{H}$ inside material

$\vec{B} = \mu_0 \vec{H}$ outside "

$$\Rightarrow \quad \vec{B} = \begin{cases} \frac{\mu}{2\pi}\frac{I}{s}\hat{\phi} & \text{inside} \quad s<R \\ \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi} & \text{outside} \quad s>R \end{cases}$$

c) $\vec{M} = \chi_m \vec{H}$ inside the material But $\mu = \mu_0(1+\chi_m)$

$$\Rightarrow \quad \frac{\mu}{\mu_0} - 1 = \chi_m$$

$$\Rightarrow \quad \vec{M} = \begin{cases} \left(\frac{\mu}{\mu_0} - 1\right)\frac{I}{2\pi s}\hat{\phi} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

The bound surface current is (at $s=R$)

$$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{s} = \left(\frac{\mu}{\mu_0}-1\right)\frac{I}{2\pi R}(-\hat{z})$$

$$\Rightarrow \quad \vec{K}_b = -\left(\frac{\mu}{\mu_0}-1\right)\frac{I}{2\pi R}\hat{z}$$

Then $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0 \quad \Rightarrow \quad \vec{J}_b = 0$