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MATH 360
HW Ch. 12.6 # 7

12.6 # 7, 21

Problem 12.6.7 $f(x) = x(10-x)$ $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$ $\lambda_n = \frac{cn\pi}{L}$

$c^2 = \frac{K}{\sigma\rho}$ $K = 1.04 \ cal/(cm\ °C)$, $\rho = 10.6 \ g/cm^3$, $\sigma = 0.056 \ cal/(g °C)$, $L = 10$

$u_t = c^2 u_{xx}$ $c^2 = \frac{1.04}{10.6(0.056)} = 1.752$ $\therefore$ $\lambda_n^2 = 0.01752\,(n\pi)^2$

$u(0,t) = u(10,t) = 0$

$u(x,0) = x(10-x)$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{10}x\right)dx = \frac{2}{10}\int_0^{10}(10x-x^2)\sin\left(\frac{n\pi}{10}x\right)dx$$

$$B_n = \frac{1}{5}\left[10\underbrace{\int_0^{10} x\sin\left(\frac{n\pi}{10}x\right)dx}_{①} - \underbrace{\int_0^{10} x^2\sin\left(\frac{n\pi}{10}x\right)dx}_{②}\right]$$

② $\int_0^{10} x^2 \sin\left(\frac{n\pi}{10}x\right)dx$ $u = x^2$, $du = 2x\,dx$ $uv - \int v\,du$

$v' = \sin\left(\frac{n\pi}{10}x\right)dx$, $v = -\frac{10}{n\pi}\cos\left(\frac{n\pi}{10}x\right)$

$$-\frac{10}{n\pi}x^2\cos\left(\frac{n\pi}{10}x\right)\Big|_0^{10} + \frac{20}{n\pi}\int_0^{10} x\cos\left(\frac{n\pi}{10}x\right)dx$$

$w = x$ $dw = dx$

$Q' = \cos\left(\frac{n\pi}{10}x\right)dx$ $Q = \frac{10}{n\pi}\sin\left(\frac{n\pi}{10}x\right)$

$$-\frac{10}{n\pi}(10)^2\cos(n\pi) + \frac{20}{n\pi}\left[\frac{10x}{n\pi}\sin\left(\frac{n\pi}{10}x\right)\Big|_0^{10} - \frac{10}{n\pi}\int_0^{10}\sin\left(\frac{n\pi}{10}x\right)dx\right]$$

$$\frac{(-1)^{n+1}(10)^3}{n\pi} - 2\left(\frac{10}{n\pi}\right)^2\int_0^{10}\sin\left(\frac{n\pi}{10}x\right)dx = \frac{(-1)^{n+1}(10)^3}{n\pi} - 2\left(\frac{10}{n\pi}\right)^2\left[-\frac{10}{n\pi}\cos\left(\frac{n\pi}{10}x\right)\right]_0^{10}$$

$$\frac{(-1)^{n+1}(10)^3}{n\pi} + 2\left(\frac{10}{n\pi}\right)^3\left[(-1)^n - 1\right] = \begin{cases} -\frac{(10)^3}{n\pi} & \text{iff } n = 2k \\ \frac{(10)^3}{n\pi} - 4\left(\frac{10}{n\pi}\right)^3 & \text{iff } n = 2k+1 \end{cases}$$

$$\int_0^{10} x^2\sin\left(\frac{n\pi}{10}x\right)dx = \frac{(-1)^{n+1}(1000)}{n\pi} + \frac{2000}{(n\pi)^3}\left((-1)^n - 1\right)$$

① $10\int_0^{10} x\sin\left(\frac{n\pi}{10}x\right)dx$ $u = x$, $du = dx$ $uv - \int v\,du$

$v' = \sin\left(\frac{n\pi}{10}x\right)dx$, $v = -\frac{10}{n\pi}\cos\left(\frac{n\pi}{10}x\right)$

$$10\left[-\frac{10x}{n\pi}\cos\left(\frac{n\pi}{10}x\right)\Big|_0^{10} + \frac{10}{n\pi}\int_0^{10}\cos\left(\frac{n\pi}{10}x\right)dx\right] = 10\left[\frac{-(10)^2(-1)^n}{n\pi} + \left(\frac{10}{n\pi}\right)^2\sin\left(\frac{n\pi}{10}x\right)\Big|_0^{10}\right]$$

$$= \frac{(-1)^{n+1}(10)^3}{n\pi} \quad \therefore \quad 10\int_0^{10} x\sin\left(\frac{n\pi}{10}x\right)dx = \frac{(-1)^{n+1}(1000)}{n\pi}$$

$$① + ② = \frac{(-1)^{n+1}(1000)}{n\pi} - \frac{(-1)^{n+1}(1000)}{n\pi} - \frac{2000}{(n\pi)^3}\left((-1)^n - 1\right) = \frac{-2000}{(n\pi)^3}\left((-1)^n - 1\right)$$

Problem 12.6.7 Continued

$$B_n = \frac{1}{5}\left[\frac{-2000\,((-1)^n - 1)}{(n\pi)^3} \right] = \frac{400\,(1-(-1)^n)}{(n\pi)^3} \quad \therefore \quad B_n = \frac{400\,(1-(-1)^n)}{(n\pi)^3}$$

$$B_n = \begin{cases} \frac{800}{(n\pi)^3} & \text{iff } n = 2k+1 \\ 0 & \text{iff } n = 2k \end{cases} \qquad \therefore \quad u(x,t) = \frac{800}{\pi^3} \sum_{n = \text{odd}}^{\infty} \frac{1}{n^3} e^{-0.01752\,(n\pi)^2 t} \sin\left(\frac{n\pi}{10} x\right)$$

$$\boxed{u(x,t) = \frac{800}{\pi^3} \sum_{n \text{ odd}}^{\infty} \frac{1}{n^3} e^{-0.01752(n\pi)^2 t} \sin\left(\frac{n\pi}{10} x\right)}$$

$$u(x,t) = \frac{800}{\pi^3}\left(e^{-0.01752(\pi)^2 t} \sin\left(\frac{\pi}{10} x\right) + \frac{1}{3^3} e^{-0.01752(3\pi)^2 t} \sin\left(\frac{3\pi}{10} x\right) + \frac{1}{5^3} e^{-0.01752(5\pi)^2 t} \sin\left(\frac{\pi}{2} y\right) \cdots \right)$$

$\boxed{\text{Problem } 12.6.7}$

$f(x) = x(10-x)$, $K = 1.04 \ cal/(cm\,°C)$, $\rho = 10.6 \ g/cm^3$, $\sigma = 0.056 \ cal/(g\,°C)$, $L = 10$, $c^2 = \frac{K}{\sigma\rho}$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{L}, \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$c^2 = \frac{K}{\sigma\rho} = \frac{1.04}{0.056 \cdot 10.6} \frac{cal/cm\,°C}{cal/g\,°C \cdot g/cm^3} = 1.75 \ : \ \lambda_n^2 = \frac{1.75\pi^2}{100} = 0.0175\cdot(n\pi)^2 \ : \ \lambda_n^2 = 0.0175\cdot(n\pi)^2$$

$$f(x) = 10x - x^2$$

$$B_n = \frac{1}{5}\left[10\int_0^{10} x\cdot\sin(0.1n\pi x)\,dx - \int_0^{10} x^2\cdot\sin(0.1n\pi x)\,dx\right]$$

$$\int_0^{10} x^2\cdot\sin(0.1n\pi x)\,dx \to u = x^2 \quad du = 2x\,dx : dx = \frac{du}{2x} : v' = \sin(0.1n\pi x)dx \quad v = \frac{-\cos(0.1n\pi x)}{0.1n\pi}$$

$$-\frac{x^2\cos(0.1n\pi x)}{0.1n\pi}\Bigg|_0^{10} + 2\int_0^{10}\frac{x\cos(0.1n\pi x)}{(0.1n\pi)}dx \ : \ u = x : du = dx \quad v' = \frac{\cos(0.1n\pi x)}{0.1n\pi}dx \quad v = \frac{\sin(0.1n\pi x)}{(0.1n\pi)^2}$$

$$-\frac{x^2\cos(0.1n\pi x)}{0.1n\pi}\Bigg|_0^{10} + 2\left[\frac{x\cdot\sin(0.1n\pi x)}{(0.1n\pi)^2}\Bigg|_0^{10} - \frac{1}{(0.1n\pi)^2}\int_0^{10}\sin(0.1n\pi x)\,dx\right]$$

$$-\frac{x^2\cos(0.1n\pi x)}{0.1n\pi}\Bigg|_0^{10} + 2\left[\frac{x\cdot\sin(0.1n\pi x)}{(0.1n\pi)^2}\Bigg|_0^{10} + \frac{1}{(0.1n\pi)^3}\cos(0.1n\pi x)\Bigg|_0^{10}\right]$$

$$\frac{-100\cdot\cos(n\pi)}{0.1n\pi} + 2\left[\cancelto{0}{\frac{10\cdot\sin(n\pi)}{(0.1n\pi)^2}} + \frac{\cos(n\pi)}{(0.1n\pi)^3} - \frac{1}{(0.1n\pi)^3}\right] \qquad 1000\cdot(-1)(-1)^n = 1000(-1)^{n+1}$$

$$\frac{-1000\cdot\cos(n\pi)}{(n\pi)} + \frac{2000}{(n\pi)^3}\left[(-1)^n - 1\right] \ : \ \left[\frac{2000}{(n\pi)^3}\left[(-1)^n - 1\right] + \frac{1000\cdot(-1)^{n+1}}{(n\pi)}\right] = \mathrm{I}$$

$$10\int_0^{10} x\cdot\sin(0.1n\pi x)\,dx \to u = x : du = dx : v' = \sin(0.1n\pi x)\,dx : v = \frac{-\cos(0.1n\pi x)}{0.1n\pi}$$

$$10\left[\frac{-x\cos(0.1n\pi x)}{0.1n\pi}\Bigg|_0^{10} + \frac{1}{0.1n\pi}\int_0^{10}\cos(0.1n\pi x)\,dx\right] = 10\left[\frac{-100\cos(n\pi)}{n\pi} + \frac{1}{(0.1n\pi)^2}\sin(0.1n\pi x)\Bigg|_0^{10}\right]$$

$$10\left[\frac{100(-1)(-1)^n}{n\pi} + \cancelto{0}{\frac{1}{(0.1n\pi)^2}\sin(n\pi)}\right] = \frac{1000\cdot(-1)^{n+1}}{(n\pi)} = \mathrm{II}$$

$$\mathrm{II} - \mathrm{I} = \cancel{\frac{1000\cdot(-1)^{n+1}}{(n\pi)}} - \cancel{\frac{1000\cdot(-1)^{n+1}}{(n\pi)}} - \frac{2000}{(\pi n)^3}\left[(-1)^n - 1\right] = \frac{2000}{(n\pi)^3}\left[1 - (-1)^n\right]$$

$$B_n = \frac{400}{(n\pi)^3}\left[1 - (-1)^n\right] = \begin{cases} \frac{800}{(n\pi)^3} & \text{iff } n = 2k+1 \\ 0 & \text{iff } n = 2k \end{cases} \qquad \boxed{u(x,t) = \frac{800}{\pi^3}\sum_{n=2k+1}^{\infty}\frac{1}{(n)^3}\sin(0.1n\pi x)\,e^{-0.0175(n\pi)^2}}$$

Example

$$c^2 = \frac{k}{\rho\sigma}, \quad k = 1.04 \text{ cal}/(\text{cm}\cdot\text{sec}^\circ\text{c}), \quad \rho = 10.6 \text{ g}/\text{cm}^3, \quad \sigma = 0.056 \text{ cal}/(\text{g}^\circ\text{c})$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t}, \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad \lambda_n = \frac{c^2(n\pi)^2}{L^2}$$

$$f(x) = 20x - x^2 : 0 \le x \le 20 \qquad \lambda_n^2 = 0.00438(n\pi)^2$$

$$B_n = \frac{2}{20}\left[20\int_0^{20} x\cdot\sin\left(\frac{n\pi}{20}x\right)dx - \int_0^{20} x^2\cdot\sin\left(\frac{n\pi}{20}x\right)dx\right]$$

$$20\int_0^{20} x\cdot\sin\left(\frac{n\pi}{20}x\right)dx \to u = x : du = dx : v' = \sin\left(\frac{n\pi}{20}x\right)dx : V = -\frac{20}{n\pi}\cos\left(\frac{n\pi}{20}x\right)$$

$$20\left[-\frac{20\cdot x}{(n\pi)}\cdot\cos\left(\frac{n\pi}{20}x\right)\Big|_0^{20} + \frac{20}{(n\pi)}\int_0^{20}\cos\left(\frac{n\pi}{20}x\right)dx\right]$$

$$20\left[\frac{-(20)^2}{(n\pi)}\cos(n\pi) + \left(\frac{20}{n\pi}\right)^2\cdot\sin\left(\frac{n\pi}{20}x\right)\Big|_0^{20}\right] = 20\left[\frac{-(20)^2}{(n\pi)}(-1)^n + \cancelto{0}{\left(\frac{20}{n\pi}\right)^2\sin(n\pi)} - 0\right]$$

$$\frac{-(20)^3(-1)^n}{(n\pi)} = \frac{(20)^3(-1)^{n+1}}{(n\pi)} \quad : \quad 20\int_0^{20} x\cdot\sin\left(\frac{n\pi}{20}x\right)dx = \frac{(20)^3(-1)^{n+1}}{(n\pi)}$$

$$① = \int_0^{20} x^2\cdot\sin\left(\frac{n\pi}{20}x\right)dx \to u = x^2 : du = 2x\,dx : v' = \sin\left(\frac{n\pi}{20}x\right)dx : V = -\frac{20}{(n\pi)}\cdot\cos\left(\frac{n\pi}{20}x\right)$$

$$\left[-\frac{20}{(n\pi)}\cdot x^2\cdot\cos\left(\frac{n\pi}{20}x\right)\Big|_0^{20} + \frac{40}{(n\pi)}\int_0^{20} x\cdot\cos\left(\frac{n\pi}{20}\right)dx\right] \qquad \begin{matrix} u = x & v' = \cos\left(\frac{n\pi}{20}x\right)dx \\ du = dx & V = \frac{20}{(n\pi)}\cdot\sin\left(\frac{n\pi}{20}x\right)\end{matrix}$$

$$\left[\frac{-20^3}{(n\pi)}\cos(n\pi) + \frac{40}{(n\pi)}\left(\cancelto{0}{\frac{20x}{(n\pi)}\sin\left(\frac{n\pi}{20}x\right)\Big|_0^{20}} - \frac{20}{(n\pi)}\int_0^{20}\sin\left(\frac{n\pi}{20}x\right)dx\right)\right]$$

$$\left[\frac{(20)^3(-1)^{n+1}}{(n\pi)} + \frac{40}{(n\pi)}\left(+\left(\frac{20}{n\pi}\right)^2\cos\left(\frac{n\pi}{20}x\right)\Big|_0^{20}\right)\right]$$

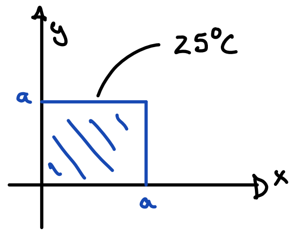
$$\left[\frac{20^3(-1)^{n+1}}{(n\pi)} + \frac{40}{(n\pi)}\left(+\left(\frac{20}{n\pi}\right)^2\left(\cos(n\pi) - \cos(0)\right)\right)\right] = \frac{20^3(-1)^{n+1}}{(n\pi)} + \frac{40}{(n\pi)}\left(\left(\frac{20}{n\pi}\right)^2\left((-1)^n - 1\right)\right)$$

$$② = \int_0^{20} x^2\cdot\sin\left(\frac{n\pi}{20}x\right)dx = \frac{20^3(-1)^{n+1}}{(n\pi)} + 2\cdot\frac{(20)^3}{(n\pi)^3}\left((-1)^n - 1\right)$$

$$① - ② = \frac{(20)^3(-1)^{n+1}}{(n\pi)} - \frac{(20)^3(-1)^{n+1}}{(n\pi)} - \frac{2\cdot(20)^3}{(n\pi)^3}\left((-1)^n - 1\right) = \frac{2\cdot(20)^3}{(n\pi)^3}\left(1 - (-1)^n\right)$$

$$B_n = \frac{(20)^3}{5\cdot(n\pi)^3}\left(1 - (-1)^n\right) = \begin{cases} \frac{2}{5}\cdot\frac{(20)^3}{(n\pi)^3} & \text{iff } n = 2k+1 \\ 0 & \text{iff } n = 2k\end{cases} \quad \therefore \quad \boxed{u(x,t) = \frac{3200}{(\pi)^3}\sum_{n=2k+1}^{\infty}\frac{1}{n^3}\cdot\sin\left(\frac{n\pi}{20}x\right)e^{-0.00438(n\pi)^2}}$$

Problem 12.6.21



$$u_t = c^2(u_{xx} + u_{yy})$$
$$0 = c^2(u_{xx} + u_{yy})$$
$$0 = u_{xx} + u_{yy}$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

$a = 24$, $b = 24$, $f(x) = 25°C$

$$A_n^* = \frac{2}{24\sinh(n\pi)}\int_0^{24} 25\cdot\sin\left(\frac{n\pi}{24}x\right)dx = \frac{25}{12\sinh(n\pi)}\left[-\frac{24}{n\pi}\cos\left(\frac{n\pi}{24}x\right)\right]_0^{24}$$

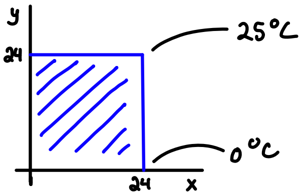
$$A_n^* = \frac{-50}{\sinh(n\pi)n\pi}\left[\cos(n\pi) - 1\right] = \frac{50}{\sinh(n\pi)n\pi}\left[1 - (-1)^n\right] \quad \therefore \quad A_n^* = \begin{cases} \dfrac{100}{\sinh(n\pi)n\pi} & \text{iff } n = 2k+1 \\ 0 & \text{iff } n = 2k \end{cases}$$

$$u(x,y) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2k+1)}\sin\left(\frac{(2k+1)\pi}{24}x\right)\frac{\sinh\left(\frac{(2k+1)\pi}{24}y\right)}{\sinh((2k+1)\pi)}$$

$$\boxed{u(x,y) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left(\frac{(2n-1)\pi}{24}x\right)\frac{\sinh\left(\frac{(2n-1)}{24}\pi y\right)}{\sinh((2n-1)\pi)}}$$

$$u(x,y) = \frac{100}{\pi}\left(\sin\left(\frac{\pi}{24}x\right)\frac{\sinh\left(\frac{\pi}{24}y\right)}{\sinh(\pi)} + \frac{1}{3}\sin\left(\frac{\pi}{8}x\right)\frac{\sinh\left(\frac{\pi}{8}y\right)}{\sinh(3\pi)} + \frac{1}{5}\sin\left(\frac{5\pi}{24}x\right)\frac{\sinh\left(\frac{5\pi}{24}y\right)}{\sinh(5\pi)} + \cdots\right)$$

Problem 12.6.21



$$u(x,y)=\sum_{n=1}^{\infty} A_n^{*}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

$$A_n^{*}=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$a \rightarrow x=24$ $\quad f(x)=25$

$b \rightarrow y=24$

$$A_n^{*}=\frac{2}{24\sinh(n\pi)}\int_0^{24} 25\cdot\sin\left(\frac{n\pi}{24}x\right)dx=\frac{25}{12\sinh(n\pi)}\int_0^{24}\sin\left(\frac{n\pi}{24}x\right)dx$$

$$=\frac{25}{12\sinh(n\pi)}\left[-\frac{24}{n\pi}\cos\left(\frac{n\pi}{24}x\right)\right]_0^{24}=\frac{25}{12\sinh(n\pi)}\left[\frac{-24}{n\pi}\cos(n\pi)+\frac{24}{n\pi}(1)\right]$$

$$=\frac{60}{\sinh(n\pi)}\cdot\frac{1}{(n\pi)}\left[1-\cos(n\pi)\right]=\frac{50}{\sinh(n\pi)}\cdot\frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$A_n^{*}=\begin{cases}\dfrac{100}{\sinh(n\pi)}\cdot\dfrac{1}{n\pi} & \text{iff } n=2k+1\\ 0 & \text{iff } n=0\end{cases}$$

$$\therefore \boxed{u(x,y)=\frac{100}{\pi}\cdot\sum_{n=2k+1}^{\infty}\frac{\sin(n\pi x/24)}{n}\cdot\frac{\sinh(n\pi y/24)}{\sinh(n\pi)}}$$

Example

$f(x)=1$, $L=\pi$, $c=1$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{L}, \quad A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n\pi}{L}x\right)dx$$

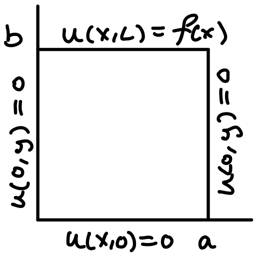
$$A_0 = \frac{1}{\pi}\int_0^{\pi} dx = \frac{1}{\pi}x\Big|_0^{\pi} = 1 \quad : \quad A_0 = 1$$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\,dx = \frac{2}{\pi}\left[\frac{\sin(n\pi)}{n}\right]_0^{\pi} = 0$$

$$\therefore \quad u(x,t) = A_0 + \sum_{n=1}^{\infty} 0\cdot\cos\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} = A_0$$

$$\boxed{u(x,t) = 1}$$

Example



$a=5$, $b=5$, $f(x) = 20°c$

$$u(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right), \quad A_n^* = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$A_n^* = \frac{2}{5\sinh(n\pi)}\int_0^5 20\cdot\sin\left(\frac{n\pi}{5}x\right)dx = \frac{8}{\sinh(n\pi)}\left[-\frac{5}{n\pi}\cos\left(\frac{n\pi}{8}x\right)\right]_0^5$$

$$= \frac{8}{\sinh(n\pi)}\left[-\frac{5}{n\pi}\left(\cos(n\pi)-\cos(0)\right)\right] = \frac{40}{\sinh(n\pi)(n\pi)}\left[1-(-1)^n\right]$$

$$A_n^* = \frac{40}{\sinh(n\pi)(n\pi)}\left[1-(-1)^n\right] \quad : \quad A_n^* = \begin{cases} \dfrac{80}{\sinh(n\pi)(n\pi)} & \text{iff } n=2k+1 \\ 0 & \text{iff } n=2k \end{cases}$$

$$\boxed{u(x,y) = \frac{80}{\pi}\sum_{n=2k+1}^{\infty} \frac{\sin\left(\frac{n\pi}{5}x\right)}{n}\cdot\frac{\sinh\left(\frac{n\pi}{5}y\right)}{\sinh(n\pi)}}$$