Phys 392 2018

Lecture 10

Fri: Picnic

Tues: HW

More than two qubits

In general systems can have more than two qubits. Suppose that a system has n qubits. If $\{|0\rangle, |1\rangle\}$ form a basis for single qubits then the following constitute a basis for n qubits:

$$|0\rangle|0\rangle|0\rangle \ldots |0\rangle \equiv |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \equiv |000\ldots00\rangle$$

$$|0\rangle|0\rangle|0\rangle \ldots |1\rangle \equiv \quad \text{- - - -} \quad \equiv |1000\ldots01\rangle$$

$$\vdots$$

$$|1\rangle|1\rangle|1\rangle \ldots |1\rangle \equiv \qquad \equiv |111\ldots11\rangle$$

qubit 1 qubit 2 qubit n

There are then 2^n basis elements. Thus

$$\boxed{\text{With } n \text{ qubits the dimension of the space of all kets is } 2^n\text{. This grows exponentially with } n.}$$

The general state of such an n-qubit system is

$$|\Psi\rangle = \alpha_0|00\ldots00\rangle + \alpha_1|00\ldots01\rangle + \alpha_2|0\ldots010\rangle + \cdots + \alpha_{N-1}|11\ldots1\rangle$$

where $N = 2^n$. The usual rules for measurement apply.

Measurement operators

We will now present an alternative formalism for measurements. Consider a single qubit. Suppose this is in state $|\Psi\rangle$ and is subjected to a measurement in the basis $\{|0\rangle, |1\rangle\}$. Then recall

Outcome:	prob	state after
0	$\|\langle 0\|\Psi\rangle\|^2$	$\|0\rangle$
1	$\|\langle 1\|\Psi\rangle\|^2$	$\|1\rangle$

We will show that we can describe the last two columns by rephrasing the measurement in terms of operators, or matrices. Consider

$$\text{Prob}(0) = |\langle 0|\Psi\rangle|^2 = (\langle 0|\Psi\rangle)^* \langle 0|\Psi\rangle$$

$$= (\langle 0|\Psi\rangle)^\dagger \langle 0|\Psi\rangle$$

But $\quad (\langle 0|\Psi\rangle)^\dagger = |\Psi\rangle^\dagger \langle 0|^\dagger = \langle\Psi|0\rangle \quad$ gives.

$$\text{Prob}(0) = \langle\Psi|0\rangle\langle 0|\Psi\rangle$$

and if we extract the middle quantity $|0\rangle\langle 0|$ it appears we can act on this with the bra and ket $\langle\Psi|$ and $|\Psi\rangle$ to obtain the probability. But what is $|0\rangle\langle 0|$? We can view this as an operator that maps kets onto kets:

Ket in $\qquad\qquad\qquad\qquad$ Ket out

$$|\Psi\rangle \longrightarrow \boxed{\begin{array}{c}\text{Operator}\\ |0\rangle\langle 0|\end{array}} \qquad |0\rangle\langle 0|\Psi\rangle = \underbrace{(\langle 0|\Psi\rangle)}_{\text{number}} \underbrace{|0\rangle}_{\text{ket}}$$

A more concrete version of this involves matrix representations of the bra and ket.

$$|0\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$$\langle 0| \leftrightarrow \begin{pmatrix}1 & 0\end{pmatrix}$$

so

$$|0\rangle\langle 0| \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

This matrix is a linear operation that maps a column vector to a column vector. We can denote this operator as:

"operator" $\rightarrow$ $$\hat{P}_0 := |0\rangle\langle 0| \leftrightarrow \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

and likewise we can construct

$$\hat{P}_1 := |1\rangle\langle 1| \leftrightarrow \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

Then it appears that

$$\text{Prob}(0) = \langle\Psi|\hat{P}_0|\Psi\rangle$$
$$\text{Prob}(1) = \langle\Psi|\hat{P}_1|\Psi\rangle$$

Example: Show that for $|\Psi\rangle = a_0|0\rangle + a_1|1\rangle$, the above formalism in terms of vectors yields correct probabilities.

Answer: $|\Psi\rangle \leftrightarrow \begin{pmatrix}\alpha_0\\\alpha_1\end{pmatrix}$

so $$\langle\Psi|\hat{P}_0|\Psi\rangle = \begin{pmatrix}\alpha_0^* & \alpha_1^*\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\alpha_0\\\alpha_1\end{pmatrix} = \begin{pmatrix}\alpha_0^* & \alpha_1^*\end{pmatrix}\begin{pmatrix}\alpha_0\\0\end{pmatrix} = |\alpha_0|^2$$

This is correct. Similarly for $\langle\Psi|\hat{P}_1|\Psi\rangle$

These operators also yield the state after measurement. Note that for $|\Psi\rangle = a_0|0\rangle + a_1|1\rangle$,

$$\hat{P}_0|\Psi\rangle = a_0|0\rangle\overbrace{\langle 0|0\rangle}^{1} + a_1|0\rangle\overbrace{\langle 0|1\rangle}^{0}$$

$$= a_0|0\rangle$$

This is not normalized. So after measurement the state is:

$$\frac{\hat{P}_0|\Psi\rangle}{\sqrt{\underbrace{\langle\Psi|\hat{P}_0|\Psi\rangle}_{a_0}}}$$

Clearly we can describe the measurement using the operators $\hat{P}_0, \hat{P}_1$. These measurement operators have the following properties:

1) Each operator is a projector, meaning $\hat{P}_i^2 = \hat{P}_i$
(repeated measurements give same results)
2) Operators corresponding to distinct outcomes are orthogonal, specifically $\hat{P}_0\hat{P}_1 = \hat{P}_1\hat{P}_0 = 0$.
(distinct measurements correspond to orthogonal states)
3) These operators are positive, meaning that for any state $|\Psi\rangle$
$$\langle\Psi|P_i|\Psi\rangle \geq 0$$
(measurement operators give positive or zero probabilities)
4) $\sum \hat{P}_i = 1$ (~~at lea~~ all probs add to 1)

In general this describes a class of measurements, regardless of the number of qubits.

A measurement is described by a set of projectors $\hat{P}_1, \hat{P}_1, \dots \hat{P}_n$ which satisfy

1) $\hat{P}_i^2 = \hat{P}_i$
2) $\hat{P}_i\hat{P}_j = 0 \quad i \neq j$
3) each $\hat{P}_i$ is positive.
4) $\sum \hat{P}_i = \hat{I}$ (identity)

The view of measurements becomes

state prior to measurement $|\Psi\rangle$ $\rightsquigarrow$ **measurement operators** $\hat{P}_1, \hat{P}_2, \dots \hat{P}_n$ and associated outcomes $a_1, a_2, \dots, a_n$ $\rightsquigarrow$ **Outcome a_i**: State out $\dfrac{\hat{P}_i|\Psi\rangle}{\langle\Psi|\hat{P}_i|\Psi\rangle}$, $\text{Prob} = \langle\Psi|\hat{P}_i|\Psi\rangle$

Note that given states associated with the outcomes

$$a_i \rightsquigarrow |\varphi_i\rangle$$

we can construct projectors by $\hat{P}_i = |\varphi_i\rangle\langle\varphi_i|$

Exercise: Consider a measurement in the basis

$$\left\{ \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle \,,\; \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle \right\}$$

a) Construct the two projectors and show that they satisfy the requirements for a set of projectors.

b) For the state $|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ determine the probability of the two outcomes, and the states after measurement for each outcome.

Answer: a) Two outcomes: $+, -$

$$\hat{P}_+ = \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right)\left(\tfrac{1}{\sqrt{2}}\langle 0| + \tfrac{1}{\sqrt{2}}\langle 1|\right)$$

$$= \tfrac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$$

Alternatively

$$\hat{P}_+ = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

and

$$\hat{P}_- = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$$

Then $\hat{P}_+\hat{P}_+ = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \hat{P}_+$

$$\hat{P}_-\hat{P}_- = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \hat{P}_-$$

$$\hat{P}_+\hat{P}_- = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = 0$$

$$\hat{P}_+ + \hat{P}_- = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \hat{I}$$

b) $\text{Prob}(+) = \langle\Psi|\hat{P}_+|\Psi\rangle$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ +\frac{i}{\sqrt{2}}\end{pmatrix} = \frac{1}{4}(1\ \ -i)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ +i\end{pmatrix}$$

$$= \frac{1}{4}(1\ \ -i)\begin{pmatrix}1+i\\ 1+i\end{pmatrix}$$

$$= \frac{1}{4}\left[(1+i) - i(1+i)\right]$$

$$= \frac{1}{2}$$

$$\text{Prob}(-) = \langle\Psi|\hat{P}_-|\Psi\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$= \frac{1}{4}(1\ \ -i)\begin{pmatrix}1-i\\ -1+i\end{pmatrix} = \frac{1}{4}\left[(1-i) + i + 1\right] = \frac{1}{2}$$

If outcome $+$ occurs, state is

$$\frac{\hat{P}_+ |\Psi\rangle}{\sqrt{\langle\Psi|P_+|\Psi\rangle}}$$

and $$\hat{P}_+|\Psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}$$

State is $$\frac{\frac{1+i}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{1/2}} = \frac{1+i}{\sqrt{2}\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{e^{i\pi/4}}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Ignoring the global phase this is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ ∎

Now consider a general construction of such bra ket products.

Suppose

$$|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$$\langle\Phi| = \beta_0\langle 0| + \beta_1\langle 1|$$

Then $$|\Psi\rangle\langle\Phi| = \alpha_0\beta_0|0\rangle\langle 0| + \alpha_0\beta_1|0\rangle\langle 1| + \alpha_1\beta_0|1\rangle\langle 0| + \alpha_1\beta_1|1\rangle\langle 1|$$

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}\begin{pmatrix} \beta_0 & \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 & \alpha_0\beta_1 \\ \alpha_1\beta_0 & \alpha_1\beta_1 \end{pmatrix}$$

We see that

$$|0\rangle\langle 0| \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|1\rangle\langle 1| \rightsquigarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple qubit measurements

The same formalism carries over to multiple qubit measurements. Suppose that we measure in basis

$$|00\rangle \rightsquigarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle \rightsquigarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle \rightsquigarrow \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle \rightsquigarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Then the associated projectors are

$$\hat{P}_{00} = |00\rangle\langle 00| \rightsquigarrow \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\hat{P}_{01} = |01\rangle\langle 01| \rightsquigarrow \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \text{etc...}$$

and we can do calculations as before. But now suppose that we want to measure just the left qubit in basis $\{|0\rangle, |1\rangle\}$. Then

$$\text{Prob}(0 \text{ regardless of right}) = \langle\Psi|\hat{P}_{00}|\Psi\rangle + \langle\Psi|\hat{P}_{01}|\Psi\rangle$$

$$= \langle\Psi|(\hat{P}_{00} + \hat{P}_{01})|\Psi\rangle$$

Then we can define a projector just for the left qubit:

$$\hat{P}_0 := \hat{P}_{00} + \hat{P}_{01}$$

$$\hat{P}_1 := \hat{P}_{10} + \hat{P}_{11}$$

One can easily show that these are projectors and that they satisfy measurement requirements for the left qubit.

Exercise: Two qubits are in the state

$$|\Psi\rangle = \frac{1}{2}\left\{|00\rangle - |01\rangle - |10\rangle - |11\rangle\right\}.$$

Suppose that the left qubit is measured in the basis $\{|0\rangle, |1\rangle\}$. List outcomes, probabilities and the state after each outcome.

Answer:

$$\hat{P}_0 = \hat{P}_{00} + \hat{P}_{01} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and } |\Psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\hat{P}_1 = \hat{P}_{10} + \hat{P}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Prob}(0) = \langle\Psi|\hat{P}_0|\Psi\rangle$$

$$= \frac{1}{4}(1\ -1\ -1\ -1)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{4}(1\ -1\ -1\ -1)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}$$

Similarly $\text{Prob}(1) = 1/2$

If outcome 0 is attained, state is:

$$\frac{\hat{P}_0|\Psi\rangle}{\sqrt{\langle\Psi|P_0|\Psi\rangle}} = \sqrt{2}\ \hat{P}_0|\Psi\rangle$$

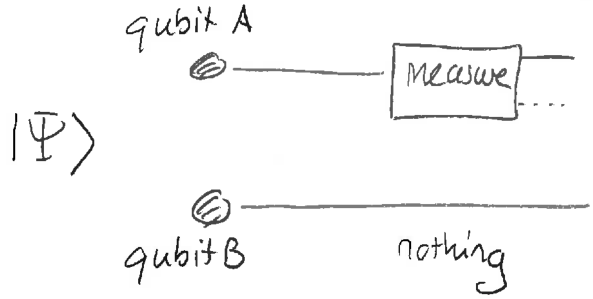
But $\hat{P}_0|\Psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ state is $\frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$

$$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Similarly if outcome 1 state is: $|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

We can see that in many cases:

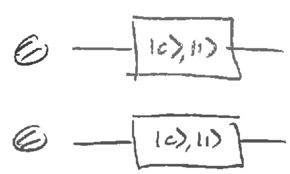


obtain this gives definite state for A

also gives definite state for B.

Tensor products of operators

We should be able to construct the projectors for the following measurement from projectors for individual measurements.



For example, how can one construct

$$|01\rangle\langle 01|$$

from $|0\rangle\langle 0|$ (for A) and $|1\rangle\langle 1|$ (for B)?

The tensor product of two operators accomplishes this:

let $\hat{A} = \begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}$ be an operator on the left qubit ket

$\hat{B} = \begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix}$ be " " " " right " "

Then the tensor product of A with B is a 4x4 matrix

$$\hat{A} \otimes \hat{B} = \begin{pmatrix} a_0 b_0 & a_0 b_1 & a_1 b_0 & a_1 b_1 \\ a_0 b_2 & a_0 b_3 & a_1 b_2 & a_1 b_3 \\ a_2 b_0 & a_2 b_1 & a_3 b_0 & a_3 b_1 \\ a_2 b_2 & a_2 b_3 & a_3 b_2 & a_3 b_3 \end{pmatrix}$$

We can show that

1) $$\underset{\text{left}}{\hat{A}} \otimes \overset{\text{right}}{\hat{B}} \; \underset{\text{left}}{|\Psi\rangle} \overset{\text{right}}{|\Phi\rangle} = \left(\hat{A}|\Psi\rangle\right) \otimes \left(\hat{B}|\Psi\rangle\right)$$

2) $$\hat{A} \otimes (\hat{B}_1 + \hat{B}_2) = \hat{A} \otimes \hat{B}_1 + \hat{A} \otimes \hat{B}_2$$

3) $$(\hat{A}_1 + \hat{A}_2) \otimes \hat{B} = \hat{A}_1 \otimes \hat{B} + \hat{A}_2 \otimes \hat{B}$$

Then it follows that for the $\{|0\rangle, |1\rangle\}$ measurements

$$\hat{P}_{00} = \hat{P}_0 \otimes \hat{P}_0$$

$$\hat{P}_{01} = \hat{P}_0 \otimes \hat{P}_1$$

$$\vdots$$

Reversing this $$\hat{P}_{00} + \hat{P}_{01} = \hat{P}_0 \otimes \hat{P}_0 + \hat{P}_0 \otimes \hat{P}_1 = \hat{P}_0 \otimes (\hat{P}_0 + \hat{P}_1) = \hat{P}_0 \otimes \hat{I}$$

Thus the single qubit measurement operators are $\hat{P}_0 \otimes \hat{I}$, $\hat{P}_1 \otimes \hat{I}$

<u>Exercise:</u> Suppose the right qubit is measured in $\{\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$ and left in $\{|0\rangle, |1\rangle\}$. Here

$$\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

a) Determine the four operators for the pairs of outcomes

b) " " operators for measurements on left alone

c) " " " " " on right alone

Answer: a) Outcome $0+$ $\rightsquigarrow$ $\hat{P}_0 \otimes \hat{P}_+ \equiv \hat{P}_{0+}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$0-$ $\rightsquigarrow$ $\hat{P}_0 \otimes \hat{P}_- \equiv \hat{P}_{0-}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$1+$ $\rightsquigarrow$ $\hat{P}_1 \otimes \hat{P}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$\equiv \hat{P}_{1+}$

$1-$ $\rightsquigarrow$ $\hat{P}_1 \otimes \hat{P}_-$ ..... $= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

$\equiv \hat{P}_{1-}$

b) $$\hat{P}_0 = \hat{P}_{0+} + \hat{P}_{0-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{P}_1 = \hat{P}_{1+} + \hat{P}_{1-} = \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \end{pmatrix}$$

c) $$\hat{P}_+ = \hat{P}_{0+} + \hat{P}_{1+} = \frac{1}{2}\begin{pmatrix} \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \end{pmatrix}$$

$$\hat{P}_- = \hat{P}_{0-} + \hat{P}_{1-} = \frac{1}{2}\begin{pmatrix} \begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix} \end{pmatrix}$$