3.6 Basis And Dimension

Span 3.6 # 10, 11, 14, 21, 22, 28, 34, 40, 64

The span of a set $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ of vectors in a vector space $\mathbb{V}$, denoted span $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$, is the set of all linear combinations of these vectors

$\mathbb{R}^3$

$$\text{Span} \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{is } \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\}$$

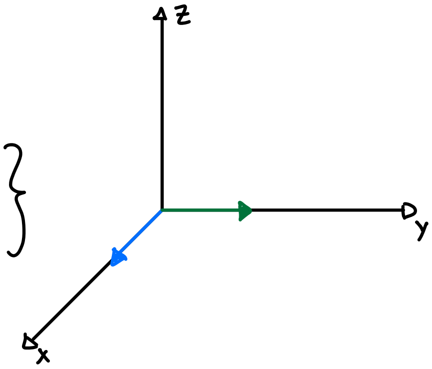
$$a \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 5 \end{bmatrix}$$

$$\begin{aligned} 2a + 0b &= 10 \\ a + 2b &= 8 \\ 5a + 5b &= 5 \end{aligned} \qquad \left[\begin{array}{cc|c} 2 & 0 & 10 \\ 1 & 2 & 8 \\ 5 & 5 & 5 \end{array}\right] = A \qquad \text{rref}(A) = I \text{ matrix } \therefore \text{ DNE}$$

Ex. 4

$\mathbb{R}^3$

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$



$$\text{Span} \left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\} \Rightarrow$$ can go anywhere but z direction

$\mathbb{R}^3$

$$\text{Span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right\} \qquad a \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} 3a &= x \\ 2a + 2b &= y \\ 5b &= z \end{aligned} \qquad \begin{aligned} a &= x/3 \\ \tfrac{2x}{3} + \tfrac{2z}{5} &= y \\ b &= z/5 \end{aligned}$$

$$\frac{2x}{3} - y + \frac{2z}{5} = 0$$

$$10x - 15y + 6z = 0$$

Linear Independence:

A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ of a vector space is linearly independent if no vector can be written as a linear combination of others.

Testing For Linear Independence

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{\vec{x}} = \vec{0}$$

Linearly independent if $\vec{x} = \vec{0}$ is the only solution

- A is invertible $\Rightarrow$ unique solution
- $|A| \neq 0$
- A has "n" pivot columns

Linear Indepence of Vector Functions

A set of vector functions $\{\vec{v}_1(t), \vec{v}_2(t) \dots \vec{v}_n(t)\}$ in a vector space $\mathbb{V}$ is linearly independent on an interval I if, for all t in I the only solution of

$$c_1 \vec{v}_1(t) + c_2 \vec{v}_2(t) + \dots c_n \vec{v}_n(t) = \vec{0}$$

for scalars $c_1 \dots c_n$ is $c_i = 0$

Ex. 9 $\vec{v}_1(t) = \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix}$, $\vec{v}_2(t) = \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix}$, and $\vec{v}_3(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$

$$c_1 \begin{bmatrix} e^t \\ 0 \\ 2e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ 3e^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^{t} \\ e^{2t} \\ e^{2t} \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \qquad |A| = -1 \neq 0 \therefore \text{ Not linearly independent}$$

The Wronskian of Functions

$$W(f_1, f_2, \dots, f_n)(t) = \begin{matrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{matrix}$$

If $W[f_1, f_2 \dots f_n](t) \neq 0$ for all t in Interval then $[f_1, f_2, \dots f_n]$ is linearly independent.

Ex 11. $\begin{bmatrix} t^2+1 \\ t^2-1 \\ 2t+5 \end{bmatrix} = \vec{v}$

$$W(\vec{v}) = \left| \begin{bmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{bmatrix} \right|$$

$$= 2\cdot(-1)^4 \left| \begin{bmatrix} t^2-1 & 2t+5 \\ 2t & 2 \end{bmatrix} \right|$$

$$+ \; 2\cdot(-1)^5 \left| \begin{bmatrix} t^2+1 & 2t+5 \\ 2t & 2 \end{bmatrix} \right|$$

$$2[(t^2-1)2-(2t+5)(2t)] = 2(\cancel{2t^2}-2-\cancel{2t^2}+10t)$$
$$-2[(t^2+1)2-(2t+5)(2t)] = -2(2t^2+2-4t^2+10t)$$
$$= -8 \neq 0 \;\therefore$$ linearly independent

Basis of A Vector Space

The set $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a basis for vector space $\mathbb{V}$ provided:

(i) $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is linearly independent

(ii) $\text{Span}[\vec{v}_1, \vec{v}_2, \dots \vec{v}_n] = \mathbb{V}$

Standard basis for $\mathbb{R}^n$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The dimension of the column space, called the rank, is the number of pivot columns of A.

$$\text{ran } A = \dim(\text{col}(A))$$

Basis For A Hyperplane

For $2x_1 + 3x_2 - 4x_3 - x_4 = 0$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

$$\text{rref} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$