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PHYS 396
HW 2

Problem 1

a.) $x(t) = x_0 + V_x t$
$y(t) = y_0$
$z(t) = z_0 + V_z t$

$$\frac{dx}{dt} = V_x \ , \ \frac{d^2x}{dt^2} = 0 \ : \ \frac{dy}{dt} = 0 \ , \ \frac{d^2y}{dt^2} = 0 \ : \ \frac{dz}{dt} = V_z \ , \ \frac{d^2z}{dt^2} = 0$$

$$\therefore \boxed{\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0} \quad \checkmark$$

b.) $x'(t) = \cos(\omega t)x + \sin(\omega t)y \rightarrow X = \cos(\omega t)(x_0 + V_x(t)) + \sin(\omega t)(y_0)$
$y'(t) = -\sin(\omega t)x + \cos(\omega t)y \rightarrow Y = -\sin(\omega t)(x_0 + V_x(t)) + \cos(\omega t)(y_0)$
$z'(t) = z \rightarrow Z_1 = z_0 + V_z(t)$

$X(t) = \cos(\omega t)x_0 + V_x(t)\cos(\omega t) + \sin(\omega t)y_0$
$Y(t) = -\sin(\omega t)x_0 - V_x(t)\sin(\omega t) + \cos(\omega t)y_0$
$Z_1(t) = z_0 + V_z(t)$

$$\frac{dX}{dt} = -\omega\sin(\omega t)x_0 + V_x(\cos(\omega t) - \omega t\sin(\omega t)) + \omega\cos(\omega t)y_0$$

$$\frac{dY}{dt} = -\omega\cos(\omega t)x_0 - V_x(\sin(\omega t) + \omega t\cos(\omega t)) - \omega\sin(\omega t)y_0$$

$$\frac{dZ_1}{dt} = V_z$$

$$\frac{d^2X}{dt^2} = -\omega^2\cos(\omega t)x_0 + V_x(-\omega\sin(\omega t) - \omega\sin(\omega t) - \omega^2 t\cos(\omega t)) - \omega^2\sin(\omega t)y_0$$

$$\frac{d^2Y}{dt^2} = \omega^2\sin(\omega t)x_0 - V_x(\omega\cos(\omega t) + \omega\cos(\omega t) - \omega^2 t\sin(\omega t)) - \omega^2\cos(\omega t)y_0$$

$$\boxed{\begin{aligned} \frac{d^2X}{dt^2} &= -\omega^2(\cos(\omega t)x_0 + \sin(\omega t)y_0) + V_x(-2\omega\sin(\omega t) - \omega^2 t\cos(\omega t)) \\ \frac{d^2Y}{dt^2} &= \omega^2(\sin(\omega t)x_0 - \cos(\omega t)y_0) - V_x(2\omega\cos(\omega t) - \omega^2 t\sin(\omega t)) \\ \frac{d^2Z_1}{dt^2} &= 0 \end{aligned}}$$

Problem 1

a.) $\frac{\partial x}{\partial t} = V_x$, $\frac{\partial^2 x}{\partial t^2} = 0$: $\frac{\partial y}{\partial t} = 0$, $\frac{\partial^2 y}{\partial t^2} = 0$: $\frac{\partial z}{\partial t} = 0$, $\frac{\partial^2 z}{\partial t^2} = 0$

$$\boxed{\frac{\partial^2 x}{\partial t^2} = \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 z}{\partial t^2} = 0}$$

b.)

$$x'(t) = \cos(\omega t)\, x_0 + V_x \cos(\omega t)\, t + \sin(\omega t)\, y_0$$
$$y'(t) = -\sin(\omega t)\, x_0 - V_x \sin(\omega t)\, t + \cos(\omega t)\, y_0$$
$$z'(t) = z_0 + V_z t$$

$$\frac{dx'}{dt} = -\omega \sin(\omega t)\, x_0 + V_x \cos(\omega t) - \omega V_x \sin(\omega t)\, t + \omega \cos(\omega t)\, y_0$$
$$\frac{dy'}{dt} = -\omega \cos(\omega t)\, x_0 - V_x \sin(\omega t) - \omega V_x \cos(\omega t)\, t - \omega \sin(\omega t)\, y_0$$
$$\frac{dz'}{dt} = V_z$$

$$\frac{d^2x'}{dt^2} = -\omega^2 \cos(\omega t)\, x_0 - \omega V_x \sin(\omega t) - \omega V_x \sin(\omega t) - \omega^2 V_x \cos(\omega t)\, t - \omega^2 \sin(\omega t)\, y_0$$
$$\frac{d^2y'}{dt^2} = \omega^2 \sin(\omega t)\, x_0 - \omega V_x \cos(\omega t) - \omega V_x \cos(\omega t) + \omega^2 V_x \sin(\omega t)\, t - \omega^2 \cos(\omega t)\, y_0$$
$$\frac{d^2z'}{dt^2} = 0$$

$$\boxed{\begin{aligned} \frac{d^2x'}{dt^2} &= -\omega^2(\cos(\omega t)x_0 + \sin(\omega t)y_0) - V_x(2\omega \sin(\omega t) + \omega^2 \cos(\omega t)t) \\ \frac{d^2y'}{dt^2} &= \omega^2(\sin(\omega t)x_0 - \cos(\omega t)y_0) + V_x(\omega^2 \sin(\omega t)t - 2\omega \cos(\omega t)) \\ \frac{d^2z'}{dt^2} &= 0 \end{aligned}}$$

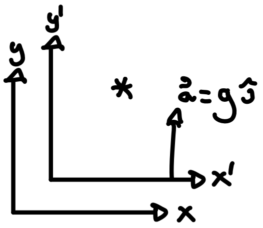
Problem 2

a.) $\vec{a} = g\hat{j}$ $\qquad y(0) = 0 \ , \ y'(0) = 0$

$$\frac{d^2y}{dt^2} = g \ , \quad \frac{dy}{dt} = gt + V_0 \ , \quad y(t) = \frac{gt^2}{2} + V_0 t + y_0$$

$$y'(0) = 0 \ : \quad 0 = 0 \cdot g + V_0 \quad \therefore V_0 = 0 \ , \quad y(0) = 0 \ : \quad 0 = \frac{g(0)^2}{2} + 0(0) + y_0$$

$$0 = g(0) + y_0 \quad \therefore y_0 = 0$$



$$\boxed{\begin{aligned} x'(t) &= x \\ y'(t) &= y - \frac{gt^2}{2} \\ z'(t) &= z \end{aligned}}$$

b.) $x(t) = x_0 + V_x t \ \rightarrow \ x(t) = x' + V_x(t)$

$y(t) = y_0 \ \rightarrow \ y(t) = gt^2/2 + y'$

$z(t) = z_0 + V_z t \ \rightarrow \ z(t) = z'$

$$\frac{dx}{dt} = V_x \ , \quad \frac{d^2x}{dt^2} = 0$$

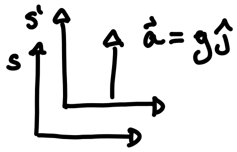
$$\frac{dy}{dt} = gt \ , \quad \frac{d^2y}{dt^2} = g$$

$$\frac{dz}{dt} = 0 \ , \quad \frac{d^2z}{dt^2} = 0$$

$$\boxed{\frac{d^2x}{dt^2} = 0 \ , \quad \frac{d^2y}{dt^2} = -g \ , \quad \frac{d^2z}{dt^2} = 0}$$

Problem 2)

a.)



$x(t) = z(t) = 0 \leftarrow$ no movement in x or z

$y(t)$: $\frac{d^2y}{dt^2} = g$, $\frac{dy}{dt} = gt + V_{oy}$, $y(t) = \cancelto{0}{y_0} + V_{oy}'\Delta t + g\frac{\Delta t^2}{2}$

$x'(t) = z'(t) = 0 \leftarrow$ no x or z movement

$$\boxed{x'(t) = 0,\ y'(t) = V_{oy}\Delta t - g\frac{\Delta t^2}{2},\ z'(t) = 0}$$

b.)

$\frac{dx'}{dt} = 0$, $\frac{d^2x'}{dt^2} = 0$: $\frac{dy'}{dt} = V_{oy} - g\Delta t$, $\frac{d^2y'}{dt^2} = -g$: $\frac{dz'}{dt} = 0$, $\frac{d^2z'}{dt^2} = 0$

$$\boxed{\frac{d^2x'}{dt^2} = 0,\ \frac{d^2y'}{dt^2} = -g,\ \frac{d^2z'}{dt^2} = 0}$$

Problem 3

a.)

$\vec{F}_y = -m_G \cdot g\,(\hat{j})$

$x(t) = 0$ since no motion in x
$z(t) = 0$ since no motion in z
$y(t) = ?$

$$x(t) = x_0 + v_x t \,,\quad y(t) = y_0 + v_y t - g\frac{t^2}{2} \,,\quad z(t) = z_0 + v_z t$$

$$\frac{dx}{dt} = v_x \,,\ \frac{d^2x}{dt^2} = 0 \ : \ \frac{dy}{dt} = v_y - gt \,,\ \frac{d^2y}{dt^2} = -g \ : \ \frac{dz}{dt} = v_z \,,\ \frac{d^2z}{dt^2} = 0$$

$$\boxed{\frac{d^2x'}{dt^2} = 0 \,,\quad \frac{d^2y'}{dt^2} = -g \,,\quad \frac{d^2z'}{dt^2} = 0}$$

b.) $m_I = m_G$

$$F_y = -mg \ : \ a_y = -g \ : \ x(t) = x_0 + v_x t \,,\ y(t) = y_0 + v_y t - g\frac{t^2}{2} \,,\ z(t) = z_0 + v_z t$$

$$\frac{dx}{dt} = v_x \,,\ \frac{d^2x}{dt^2} = 0 \ : \ \frac{dy}{dt} = v_y - gt \,,\ \frac{d^2y}{dt^2} = -g \ : \ \frac{dz}{dt} = v_z \,,\ \frac{d^2z}{dt^2} = 0$$

$$\frac{d^2x'}{dt^2} = 0 \,,\quad \frac{d^2y'}{dt^2} = -g \,,\quad \frac{d^2z'}{dt^2} = 0$$

This result shows that inertial and gravitational math are the same.

Problem 3

a.) No x-movement : $\frac{d^2x}{dt^2} = 0$

No z-movement : $\frac{d^2z}{dt^2} = 0$

Movement in y: $y(t) = y_0 + V_{0y}\Delta t - \frac{1}{2}g\Delta t^2$, $\frac{dy}{dt} = V_{0y} - g\Delta t$, $\frac{d^2y}{dt^2} = -g$

$$\boxed{\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g, \quad \frac{d^2z}{dt^2} = 0}$$

b.) No x-movement : $\frac{d^2x}{dt^2} = 0$

No z-movement : $\frac{d^2z}{dt^2} = 0$

Movement in y: $y(t) = y_0 + V_{0y}\Delta t - \frac{1}{2}g\Delta t^2$, $\frac{dy}{dt} = V_{0y} - g\Delta t$, $\frac{d^2y}{dt^2} = -g$

$$\boxed{\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g, \quad \frac{d^2z}{dt^2} = 0}$$

Problem 4

$$\mu(r) = kr^n \quad \text{for} \quad r<R$$

a.) $M = \int_V \mu(r)\, d\tau \quad : \quad d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$M = k \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^{2+n} \sin\theta\, dr\, d\theta\, d\phi$$

$$= k\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi \cdot \frac{r^{3+n}}{3+n}\Big|_0^R = k\int_0^{2\pi} d\phi \cdot -\cos\theta\Big|_0^{\pi} \cdot \left[\frac{R^{3+n}}{3+n} - \frac{0^{3+n}}{3+n}\right]$$

$$= k\cdot \phi\Big|_0^{2\pi} \cdot 2 \cdot \left(\frac{R^{3+n}}{3+n} - 0\right)$$

$$M = k \cdot 4\pi \cdot \frac{R^{3+n}}{3+n} \longrightarrow \frac{(3+n)M}{4\pi R^{3+n}} = k$$

$$\boxed{k = \frac{(3+n)}{4\pi \cdot R^{3+n}} \cdot M}$$

b.) n can take on values greater than -3, since at this point the mass would be undefined.

c.) $\oint \vec{g}\cdot d\vec{A} = -4\pi G M$

Outside : $r > R$ .....

$\oint \vec{g}\cdot d\vec{A}$: $\vec{g} = -g(\hat{r})$, $d\vec{A} = r^2 \sin\theta\, d\theta\, d\phi\, (-\hat{r})$

$$\oint \vec{g}\cdot d\vec{A} = g r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = -g \cdot r^2 \cdot 2\pi \cdot 2 = -4\pi g r^2$$

$$g \cdot 4\pi r^2 = -4\pi G M \quad : \quad g = -\frac{GM}{r^2} \quad : \quad \vec{g} = -\frac{GM}{r^2}(\hat{r})$$

Inside : $r < R$

$d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$M = \int_V k r^n \cdot d\tau$$

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} k r^n \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= k\int_0^{2\pi}\int_0^{\pi} \frac{r^{3+n}}{(3+n)}\Big|_0^r \sin\theta\, d\theta\, d\phi$$

$$= \frac{k r^{3+n}}{(3+n)} \int_0^{2\pi} -\cos\theta\Big|_0^{\pi} d\phi$$

$$= \frac{2\cdot k r^{3+n}}{(3+n)} \cdot \phi\Big|_0^{2\pi} = \frac{4\pi k r^{3+n}}{(3+n)} = \frac{M r^{n+3}}{R^{n+3}}$$

$$-g 4\pi r^2 = 4\pi G M \frac{r^{n+3}}{R^{n+3}} \quad : \quad g = -\frac{GM r^{n+1}}{R^{n+3}} \quad : \quad \vec{g} = -\frac{GM r^{n+1}}{R^{n+3}}(\hat{r})$$

$$\boxed{\begin{aligned} r<R &: \vec{g} = \frac{-GM}{r^2}(\hat{r}) \\ r>R &: \vec{g} = \frac{-GM r^{n+1}}{R^{n+3}}(\hat{r}) \end{aligned}}$$

Problem 4 Continued

d.) $\vec{g} = -\vec{\nabla}\Phi = -\left[\frac{\partial\Phi}{\partial r}\hat{e}_r + \frac{1}{r}\cancel{\frac{\partial\Phi}{\partial\theta}}^{0}\hat{\theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}}^{0}\hat{\phi}\right]$

$g = \frac{-d\Phi}{dr} \quad \therefore \quad \Phi(r) = -\int g\, dr$

For $r < R$ ...

$$\Phi(r) = -\int -\frac{GMr^{n+1}}{R^{n+3}}\, dr = \frac{GM}{R^{n+3}} \cdot \frac{r^{n+2}}{(n+2)}$$

For $r > R$ ...

$$\Phi(r) = -\int -\frac{GM}{r^2}\, dr = -\frac{GM}{r}$$

$$\boxed{\begin{array}{ll} r<R: & \Phi(r) = \frac{GM}{R^{n+3}} \cdot \frac{r^{n+2}}{(n+2)} \\ r>R: & \Phi(r) = -\frac{GM}{r} \end{array}}$$

Problem 4

$$\mu(r) = kr^n$$

a.) $M = \int_V \mu(r)\, d\tau \; : \; d\tau = r'^2 \sin\theta\, dr' d\theta d\phi \; : \; 0 \le \phi \le 2\pi,\ 0 \le \theta \le \pi,\ 0 \le r' \le r$

$$= k\int_0^{2\pi}\int_0^{\pi}\int_0^{r} r'^{2+n} \sin\theta\, dr' d\theta d\phi$$

$$M = k \cdot 4\pi \cdot \frac{r'^{3+n}}{3+n}\Big|_0^r = 4\pi \cdot k \cdot \frac{r^{3+n}}{(3+n)} \; : \; M = 4\pi k \cdot \frac{r^{3+n}}{(3+n)}$$

$$\boxed{k = \frac{(3+n)}{r^{3+n}} \cdot \frac{M}{4\pi}}$$

b.)

$\boxed{\text{n must be greater than } -3}$

c.) $\oint \vec{g} \cdot d\vec{a} = -4\pi G M$

Outside : $r > R$

$$\oint \vec{g} \cdot d\vec{a} = -g\int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta\, d\theta d\phi = -4\pi r^2 \cdot g$$

$\vec{g} = -g(\hat{r})$

$d\vec{A} = r^2 \sin\theta\, d\theta d\phi\, (\hat{r})$

$$-g \cdot 4\pi r^2 = -4\pi G M$$

$$g = \frac{GM}{r^2} \; : \; \vec{g} = -\frac{GM}{r^2}(\hat{r})$$

Inside : $r < R$

$$M = k\int_0^{2\pi}\int_0^{\pi}\int_0^{r} r'^{2+n} \sin\theta\, dr' d\theta d\phi = k \cdot 4\pi \cdot \frac{r^{3+n}}{(3+n)} = M\frac{r^{n+3}}{R^{n+3}}$$

$$-g \cdot 4\pi r^2 = -4\pi G M \frac{r^{n+3}}{R^{n+3}}$$

$$\boxed{r<R: \vec{g} = -GM\frac{r^{n+1}}{R^{n+3}}(\hat{r}) \quad r>R: \vec{g} = -\frac{GM}{r^2}(\hat{r})}$$

d.) $\phi(r) = -\int V\, dr$

$$r>R : \; \phi(r) = -\int_0^r -\frac{GM}{r^2}\, dr = -\frac{GM}{r}$$

$$r>R : \; \phi(r) = \int_0^r GM\frac{r^{n+1}}{R^{n+3}} = \frac{GMr^{n+2}}{R^{n+3}(n+2)}$$

$$\boxed{\begin{gathered}\phi(r) = -\frac{GM}{r} \; : \; r<R \\ \phi(r) = \frac{GM}{(n+2)}\frac{r^{n+2}}{R^{n+3}} \; : \; r>R\end{gathered}}$$