MAT 202
Larson – Section 5.1
The Natural Logarithmic Function: Differentiation

In calculus I, we focused mainly on differentiating and integrating algebraic functions (with the exception of some trigonometric functions). In calculus II, we will begin by placing our focus on differentiation and integration of ***transcendental functions*** (non-algebraic functions). Logarithmic, exponential, trigonometric, inverse trigonometric functions & hyperbolic functions are all examples of the types of transcendental functions that we will see in chapter 5. In this section we will learn how to differentiate ***natural logarithmic functions***. But first, we will review basic properties of logarithms.

Logarithmic Function: For x > 0, $b > 0$ and $b \neq 1$, $y = \log_b x$ iff $x = b^y$. The function $f(x) = \log_b x$ is called the logarithmic function with base *b*.

What should we think when we see the expression $\log_b x$? Think: "What power do I raise b to get x?"

The Natural Logarithmic Function:

$$f(x) = \log_e x = \ln x, \quad where\, x > 0$$

Note: "LOG" on your calculator refers to the common logarithm which is a base 10 logarithm ($y = \log_{10} x$). "LN" on your calculator refers to the natural logarithm in which the base is *e*.

The Natural Base *e*:

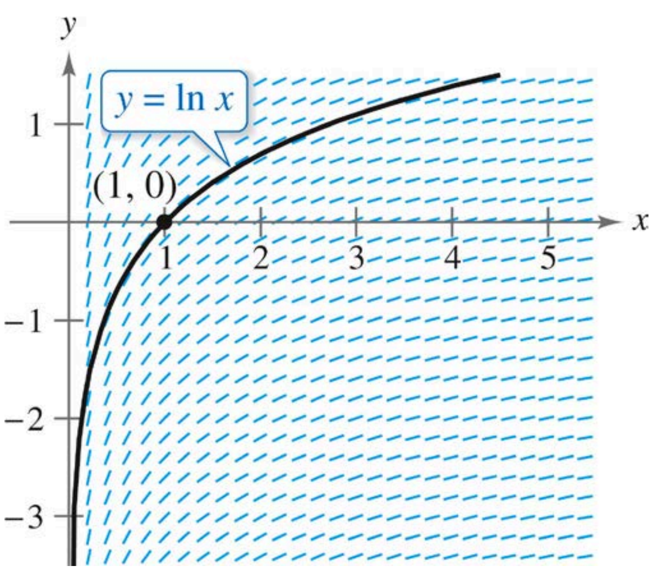
$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828282846$$

Note: *e* is an irrational number. That is, it is a decimal number that does not repeat and it does not terminate.

Properties of Logarithms: Let $b > 0$, $b \neq 1$, and let n be a real number. If u and v are positive real numbers, the following properties are true:

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $\ln(uv) = \ln u + \ln v$
5. $\ln\left(\frac{u}{v}\right) = \ln u - \ln v$
6. $\ln u^n = n \cdot \ln u$

Graph of a Natural Logarithmic Function:
Domain? Range? Continuous? Increasing or Decreasing? One-to-one? Concavity? Strictly Monotonic (Increasing or Decreasing on the functions entire domain)?



Recall the General Power Rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

Notice that this rule does not apply when $n = -1$. This means that we have not yet found the antiderivative for $f(x) = \frac{1}{x}$.

Graph of a Natural Logarithmic Function:

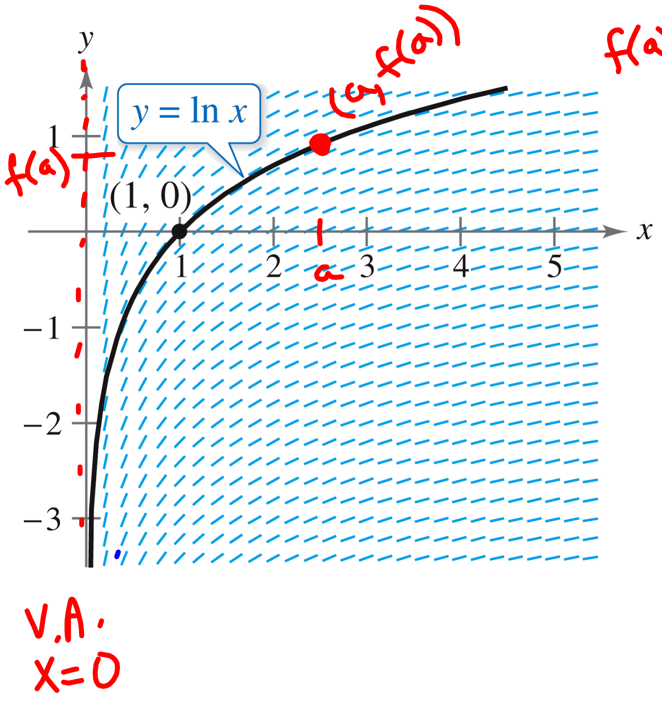
Domain? Range? Continuous? Increasing or Decreasing?

One-to-one? Concavity? Strictly Monotonic (Increasing or Decreasing on the functions entire domain)?

$y = \ln x = \log_e x$

$e^y = x$

$f(a)=f(b) \Rightarrow a=b$



Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Increasing

One-to-one

Concave down

Strictly monotonic

Calculus Definition of Natural Logarithmic Function:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

This definition gives us the means for which we can find $\int \frac{1}{x}\,dx$ as well as the derivative of $\ln x$.

Derivative of the Natural Logarithmic Function:
Let u be a differentiable function of x. u > 0 and x > 0.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}$, that is, if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$.
2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}$, that is, if $f(x) = \ln[g(x)]$, then $f'(x) = \frac{1}{g(x)} \cdot g'(x)$.

Derivative Involving Absolute Value:
If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

Note: Remember to use all of the tools gained in calculus I for calculating the derivative (i.e., product rule, quotient rule, chain rule, etc.). Also, with logarithms, it is sometimes helpful to rewrite the function using logarithmic properties.

Ex: Find the derivative of the following functions:

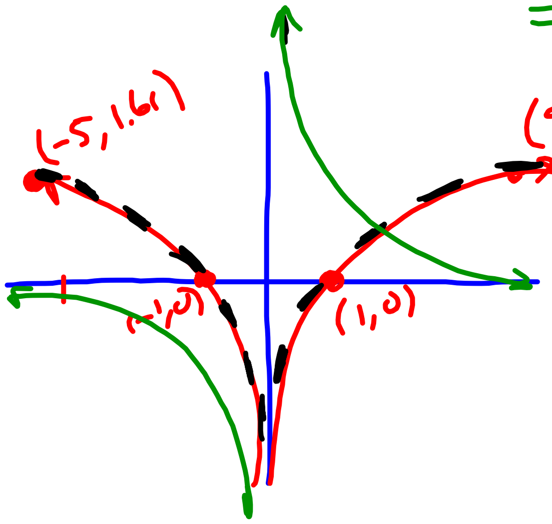
a) $y = \ln\sqrt{x^2 - 4}$

Derivative Involving Absolute Value:

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$$

$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$



Ex: Find the derivative of the following functions:

a) $y = \ln\sqrt{x^2 - 4}$

$\ln x^{1/2} \neq (\ln x)^{1/2}$

$y = \ln (x^2 - 4)^{1/2}$

u

$y = \frac{1}{2} \ln (x^2 - 4)$ $\frac{1}{u}$ u'

$y' = \frac{1}{\cancel{2}} \cdot \frac{1}{x^2 - 4} \cdot \cancel{2}x$

$$\boxed{y' = \frac{x}{x^2 - 4}}$$

b) $f(x) = \dfrac{\ln x}{x^3}$

c) $f(x) = \ln\left(\dfrac{\sqrt{4 + x^2}}{x}\right)$

d) $y = \ln\left|\sec x + \tan x\right|$

Quotient Rule

b) $f(x)=\dfrac{\ln x}{x^3}$

$$f'(x) = \frac{\frac{1}{x}\cdot x^3 - 3x^2\cdot \ln x}{(x^3)^2}$$

$$f'(x) = \frac{x^2 - 3x^2 \ln x}{x^6}$$

$$f'(x) = \frac{x^2(1-3\ln x)}{x^6}$$

$$\boxed{f'(x) = \frac{1-3\ln x}{x^4}}$$

c) $$f(x) = \ln\left(\frac{\sqrt{4+x^2}}{x}\right)$$

Rewrite!

$$f(x) = \ln\sqrt{4+x^2} - \ln x$$

$$f(x) = \frac{1}{2}\ln(4+x^2) - \ln x$$

$$f'(x) = \frac{1}{\cancel{2}} \cdot \frac{1}{4+x^2} \cdot \cancel{2}x - \frac{1}{x}$$

$$f'(x) = \frac{x}{4+x^2} - \frac{1}{x} \quad \text{or} \quad \frac{-4}{x(4+x^2)}$$

d) $y = \ln|\sec x + \tan x|$

$y' = \dfrac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$

$y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$y' = \dfrac{\sec x(\cancel{\tan x + \sec x})}{\cancel{\sec x + \tan x}}$

$\boxed{y' = \sec x}$

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\csc x] = -\csc x \cot x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

Ex: Find an equation of the tangent line to the graph of $y = \ln x^4$ at the point (1, 0).

Ex: Use ***implicit differentiation*** to find $\dfrac{dy}{dx}$: $4x^3 + \ln y^2 + 2y = 2x$

Ex: Use ***logarithmic differentiation*** to find $\dfrac{dy}{dx}$: $y = \dfrac{x^2\sqrt{3x-2}}{(x+1)^2}, \quad x > \dfrac{2}{3}$

Ex: Find an equation of the tangent line to the graph of

$y = \ln x^4$ at the point $(\overset{x}{1}, \overset{y}{0})$.

$$y = \underset{?}{m}x + \underset{?}{b}$$

$$y = \ln x^4$$

$$y = 4 \ln x$$

$$y' = \frac{4}{x}$$

$$m = \frac{4}{1} = 4 \checkmark$$

$$0 = 4(1) + b$$

$$-4 = b \checkmark$$

$$\boxed{y = 4x - 4}$$

Ex: Use ***implicit differentiation*** to find $\dfrac{dy}{dx}$:

$$4x^3 + \ln y^2 + 2y = 2x$$

$$\frac{d}{dx}\left[4x^3 + 2\ln y + 2y = 2x\right]$$

$$12x^2 + 2\cdot\frac{1}{y}\frac{dy}{dx} + 2\frac{dy}{dx} = 2$$

$$12x^2 + \frac{2}{y}\frac{dy}{dx} + 2\frac{dy}{dx} = 2$$
$$-12x^2 \qquad\qquad\qquad -12x^2$$

$$\frac{2}{y}\frac{dy}{dx} + 2\frac{dy}{dx} = 2-12x^2$$

$$\frac{\left(\frac{2}{y}+2\right)\frac{dy}{dx}}{\frac{2}{y}+2} = \frac{2-12x^2}{\frac{2}{y}+2}$$

$$\boxed{\frac{dy}{dx} = \frac{2-12x^2}{\frac{2}{y}+2}} \quad \text{or} \quad \frac{y(1-6x^2)}{1+y}$$

Ex: Use ***logarithmic differentiation*** to find $\frac{dy}{dx}$:

$$\ln y = \ln\left(\frac{x^2\sqrt{3x-2}}{(x+1)^2}\right), \quad x > \frac{2}{3}$$

$$\ln y = \ln\left(\frac{x^2\sqrt{3x-2}}{(x+1)^2}\right)$$

$$\ln y = \ln\left(x^2(3x-2)^{1/2}\right) - \ln(x+1)^2$$

$$\ln y = \ln x^2 + \ln(3x-2)^{1/2} - \ln(x+1)^2$$

$$\frac{d}{dx}\left[\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2}\cdot\frac{1}{3x-2}\cdot 3 - 2\cdot\frac{1}{x+1}$$

$$y\left(\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right)$$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{x^2\sqrt{3x-2}}{(x+1)^2}\left(\frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}\right)}$$

OR

$$\boxed{y' = \frac{3x^3+15x^2-8x}{2(x+1)^3\sqrt{3x-2}}}$$

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$