Exercise 1

$$\hat{\rho}_f = (1-\alpha)\hat{\rho}_i + \alpha\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z$$

a.)

i.) Initial State is $|0\rangle$

$$\hat{\rho}_i = |\psi\rangle\langle\psi| = |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \; : \; \hat{\rho}_i = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \; \hat{\sigma}_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\hat{\rho}_f = (1-\alpha)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \alpha\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}1-\alpha & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}\alpha & 0\\0 & -\alpha\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1-\alpha & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}\alpha & 0\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}}$$

ii.) Initial State is $|1\rangle$

$$\hat{\rho}_i = |\psi\rangle\langle\psi| = |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \; : \; \hat{\rho}_i = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \; \hat{\sigma}_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\hat{\rho}_f = (1-\alpha)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \alpha\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0\\0 & 1-\alpha\end{pmatrix} + \begin{pmatrix}\alpha & 0\\0 & -\alpha\end{pmatrix}\begin{pmatrix}0 & 0\\0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1-\alpha\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & \alpha\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}}$$

b.) Initial State is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\hat{\rho}_i = |\psi\rangle\langle\psi| = |+\hat{x}\rangle\langle+\hat{x}| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \; : \; \hat{\rho}_i = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}, \; \hat{\sigma}_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$\hat{\rho}_f = (1-\alpha)\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} + \alpha\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1-\alpha & 1-\alpha\\1-\alpha & 1-\alpha\end{pmatrix} + \begin{pmatrix}\alpha & 0\\0 & -\alpha\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & -1\\1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1-\alpha & 1-\alpha\\1-\alpha & 1-\alpha\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\alpha & -\alpha\\-\alpha & \alpha\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1-2\alpha\\1-2\alpha & 1\end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\begin{pmatrix}1 & 1-2\alpha\\1-2\alpha & 1\end{pmatrix}}$$

Exercise 1 Continued

c.) Initial state is $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$$\hat{\rho}_i = |\psi\rangle\langle\psi| = |{-}\hat{\varphi}\rangle\langle{-}\hat{\varphi}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & +i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} : \hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\rho}_f = (1-\alpha)\cdot\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \alpha\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (1-\alpha) & i(1-\alpha) \\ i(\alpha-1) & (1-\alpha) \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1-\alpha) & i(1-\alpha) \\ i(\alpha-1) & (1-\alpha) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha & -i\alpha \\ i\alpha & \alpha \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & i(1-2\alpha) \\ i(2\alpha-1) & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & i(1-2\alpha) \\ i(2\alpha-1) & 1 \end{pmatrix}}$$

Exercise 2

$$\frac{1}{2}\left(\hat{I} + r_x\hat{\sigma}_x + r_y\hat{\sigma}_y + r_z\hat{\sigma}_z\right) \quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a.)

i.) $\hat{\rho}_f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad r_x = r_y = 0, \ r_z = 1$

$$\frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\left(\hat{I} + \hat{\sigma}_z\right)}$$

ii.) $\hat{\rho}_f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad r_x = r_y = 0, \ r_z = -1$

$$\frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\left(\hat{I} - \hat{\sigma}_z\right)}$$

b.)

$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & 1-2\alpha \\ 1-2\alpha & 1 \end{pmatrix} \quad r_y = r_z = 0 \ : \ r_x = 1-2\alpha$

$$\frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1-2\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1-2\alpha \\ 1-2\alpha & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1-2\alpha \\ 1-2\alpha & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\left(\hat{I} + (1-2\alpha)\hat{\sigma}_x\right)}$$

c.) $\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & i(1-2\alpha) \\ i(2\alpha-1) & 1 \end{pmatrix} \quad r_x = r_z = 0 \ : \ r_y = (2\alpha-1)$

$$\frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (2\alpha-1)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i(1-2\alpha) \\ i(2\alpha-1) & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & i(1-2\alpha) \\ i(2\alpha-1) & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = \frac{1}{2}\left(\hat{I} + (2\alpha-1)\hat{\sigma}_y\right)}$$

Exercise 3

$$\hat{\rho}_f = (1-\alpha)\hat{\rho}_i + \alpha\,\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a.) $\hat{\rho}_i = |0\rangle\langle 0|$

$$\hat{\rho}_f = (1-\alpha)|0\rangle\langle 0| + \alpha\hat{\sigma}_z|0\rangle\langle 0|\hat{\sigma}_z$$

$$\hat{\sigma}_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; : \; \langle 0|\hat{\sigma}_z = (1\;0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (1\;0)$$

$$\hat{\sigma}_z|0\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\;0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; : \; \alpha\cdot\hat{\sigma}_z|0\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\;0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; : \; (1-\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \quad (1-\alpha)|0\rangle\langle 0| + \alpha\hat{\sigma}_z|0\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = (1-\alpha)|0\rangle\langle 0| + \alpha\hat{\sigma}_z|0\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

b.) $\hat{\rho}_i = |0\rangle\langle 1|$

$$\hat{\rho}_f = (1-\alpha)|0\rangle\langle 1| + \alpha\hat{\sigma}_z|0\rangle\langle 1|\hat{\sigma}_z$$

$$\hat{\sigma}_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; : \; \langle 1|\hat{\sigma}_z = (0\;1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (0\;-1)$$

$$\hat{\sigma}_z|0\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0\;-1) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \; : \; \alpha\hat{\sigma}_z|0\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & -\alpha \\ 0 & 0 \end{pmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0\;1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; : \; (1-\alpha)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-\alpha \\ 0 & 0 \end{pmatrix}$$

$$\therefore \quad (1-\alpha)|0\rangle\langle 1| + \alpha\hat{\sigma}_z|0\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & 1-\alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1-2\alpha \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\hat{\rho}_f = (1-\alpha)|0\rangle\langle 1| + \alpha\hat{\sigma}_z|0\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & 1-2\alpha \\ 0 & 0 \end{pmatrix}}$$

Exercise 3 continued

c.) $\hat{\rho}_i = |1\rangle\langle 0|$

$$\hat{\rho}_F = (1-\alpha)|1\rangle\langle 0| + \alpha\hat{\sigma}_z|1\rangle\langle 0|\hat{\sigma}_z$$

$$\hat{\sigma}_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \; : \quad \langle 0|\hat{\sigma}_z = (1\,0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (1\,0)$$

$$\hat{\sigma}_z|1\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 0 \\ -1 \end{pmatrix}(1\,0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \; : \quad \alpha\hat{\sigma}_z|1\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1\,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \; : \quad (1-\alpha)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1-\alpha & 0 \end{pmatrix}$$

$$\therefore \quad (1-\alpha)|1\rangle\langle 0| + \alpha\hat{\sigma}_z|1\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ 1-\alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1-2\alpha & 0 \end{pmatrix}$$

$$\boxed{\hat{\rho}_F = (1-\alpha)|1\rangle\langle 0| + \alpha\hat{\sigma}_z|1\rangle\langle 0|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ 1-2\alpha & 0 \end{pmatrix}}$$

d.) $\hat{\rho}_i = |1\rangle\langle 1|$

$$\hat{\rho}_F = (1-\alpha)|1\rangle\langle 1| + \alpha\hat{\sigma}_z|1\rangle\langle 1|\hat{\sigma}_z$$

$$\hat{\sigma}_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \; : \quad \langle 1|\hat{\sigma}_z = (0\,1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (0\,-1)$$

$$\hat{\sigma}_z|1\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 \\ -1 \end{pmatrix}(0\,-1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; : \quad \alpha\hat{\sigma}_z|1\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\,1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; : \quad (1-\alpha)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1-\alpha \end{pmatrix}$$

$$\therefore \quad (1-\alpha)|1\rangle\langle 1| + \alpha\hat{\sigma}_z|1\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ 0 & 1-\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho}_F = (1-\alpha)|1\rangle\langle 1| + \alpha\hat{\sigma}_z|1\rangle\langle 1|\hat{\sigma}_z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$