Ch. 8 Geodesics

3-11-19

The Geodesic Equation

Free particles move along paths of "Extremal Proper Time"

— Geodesics

Geodesics obey the Geodesic equation

4D Minkowski....

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = 0}$$

In arbitrary curved spacetime....

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$ — Geodesic equation ⟵ Equations of motion for free particles in curved spacetime

Notice : Let $\beta \to \gamma$: $\gamma \to \beta$

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\gamma\beta}\frac{dx^\gamma}{d\tau}\frac{dx^\beta}{d\tau} = -\Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \quad \therefore \quad \boxed{\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}}$$

• Γ's are related to the metric : Derivatives of the metric
∴ obey.

$$\boxed{g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)}$$

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i.e Calculate the Γ's for the wormhole geometry

$$ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{tt} = -1 \;,\quad g_{rr} = 1 \;,\quad g_{\theta\theta} = b^2+r^2 \;,\quad g_{\phi\phi} = (b^2+r^2)\sin^2(\theta)$$

$g_{\mu\nu}$ is diagonal

3 Free indices α, β, γ

1. $\alpha = \beta = \gamma = t$:
$$g_{t\delta}\Gamma^\delta_{tt} = \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial t} + \cancel{\frac{\partial g_{tt}}{\partial t}} - \cancel{\frac{\partial g_{tt}}{\partial t}}\right) = \frac{1}{2}(0) = 0$$

$$= g_{tt}\Gamma^t{}_{tt} + \cancel{g_{tr}\Gamma^r{}_{tt}} + \cancel{g_{t\theta}\Gamma^\theta{}_{tt}} + \cancel{g_{t\varphi}\Gamma^\varphi{}_{tt}}$$

$$\therefore \ \Gamma^t{}_{tt} = 0$$

2. $\alpha = \beta = t, \ \gamma = r$: $g_{t\rho}\Gamma^\rho{}_{tr} = \frac{1}{2}\left(\frac{\partial g_{tt}}{\partial r} + \cancel{\frac{\partial g_{tr}}{\partial t}} - \cancel{\frac{\partial g_{tr}}{\partial t}}\right) = \frac{1}{2}(0) = 0$

$$= g_{tt}\Gamma^t{}_{tr} = 0 \quad \therefore \quad \Gamma^t{}_{tr} = 0 = \Gamma^t{}_{rt}$$

3. $\alpha = \beta = \gamma = r$: $g_{r\rho}\Gamma^\rho{}_{rr} = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial r} + \cancel{\frac{\partial g_{rr}}{\partial r}} - \cancel{\frac{\partial g_{rr}}{\partial r}}\right) = \frac{1}{2}(0) = 0$

$$= g_{rr}\Gamma^r{}_{rr} = 0 \quad \therefore \quad \boxed{\Gamma^r{}_{rr} = 0}$$

4. $\alpha = r, \ \beta = \gamma = \theta$: $g_{r\rho}\Gamma^\rho{}_{\theta\theta} = \frac{1}{2}\left(\cancel{\frac{\partial g_{r\theta}}{\partial \theta}} + \cancel{\frac{\partial g_{r\theta}}{\partial \theta}} - \frac{\partial g_{\theta\theta}}{\partial r}\right) = -\frac{1}{2}\frac{\partial}{\partial r}(b^2+r^2)$

Only nonvanishing Christophel Symbols for the wormhole metric are...

$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2}, \quad \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{r}{b^2+r^2}$$

$$\Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta, \quad \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = \Gamma^\varphi{}_{\theta\varphi} = \cot\theta$$

What is the geodesic equation for the wormhole metric?

$\alpha = r$

$$\frac{d^2r}{d\tau^2} = -\Gamma^r{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^r{}_{\theta\theta}\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} - \Gamma^r{}_{\varphi\varphi}\frac{d\varphi}{d\tau}\frac{d\varphi}{d\tau} = r\left(\frac{d\theta}{d\tau}\right)^2 + r\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2$$

$\alpha = \theta \ldots.$

$$\frac{d^2\theta}{d\tau^2} = -\Gamma^\theta{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^\theta{}_{r\theta}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \Gamma^\theta{}_{\theta r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} - \Gamma^\theta{}_{\varphi\varphi}\frac{d\varphi}{d\tau}\frac{d\varphi}{d\tau}$$

$$= \frac{-2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

In Summary :

$$\left[\begin{array}{c} \dfrac{d^2t}{d\tau^2} = 0, \quad \dfrac{d^2\theta}{d\tau^2} = \dfrac{-2r}{b^2+r^2}\dfrac{dr}{d\tau}\dfrac{d\theta}{d\tau} + \sin\theta\cos\theta\left(\dfrac{d\varphi}{d\tau}\right)^2 \\ \dfrac{d^2\varphi}{d\tau^2} = \dfrac{-2r}{b^2+r^2}\dfrac{dr}{d\tau}\dfrac{d\varphi}{d\tau} - \dfrac{2\cos\theta}{\sin\theta}\dfrac{d\theta}{d\tau}\dfrac{d\varphi}{d\tau}, \quad \dfrac{d^2r}{d\tau^2} = r\left(\dfrac{d\theta}{d\tau}\right)^2 + r\sin^2\theta\left(\dfrac{d\varphi}{d\tau}\right)^2 \end{array}\right]$$

Notice that for radial geodesic, $\varphi \neq \varphi(\tau)$, $\theta \neq \theta(\tau)$

$$\frac{d\theta}{d\tau} = \frac{d\varphi}{d\tau} = 0 \quad : \left[\frac{d^2 t}{d\tau^2} = 0 \ , \ \frac{d^2 r}{d\tau^2} = 0 \right]$$

$$u^r = \frac{dr}{d\tau} \equiv v$$

Since

$$-1 = \underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\alpha\beta} u^\alpha u^\beta = \underbrace{g_{tt}}_{-1} u^t u^t + \underbrace{g_{rr}}_{1} u^r u^r + \underbrace{g_{\theta\theta}}_{r^2+b^2} \cancel{u^\theta u^\theta} + \underbrace{g_{\varphi\varphi}}_{(r^2+b^2)\sin^2\theta} \cancel{u^\varphi u^\varphi}$$

$$-1 = -(u^t)^2 + (u^r)^2 \ : \ (u^t)^2 = 1 + (u^r)^2 = 1 + v^2 \ : \ u^t = \sqrt{1+v^2}$$

$$\therefore \left[u^\alpha = (\sqrt{1+v^2}, v, 0, 0 \right] \qquad \therefore \ u^r = v \neq u^r(\tau)$$

Now,

$$u^r = \frac{dr}{d\tau} = v \ : \ \int dr = \int v \, d\tau \rightarrow r(\tau) = v\tau + c$$

$$r(\tau = 0) = 0 \ : \ c = 0 \rightarrow r(\tau) = vt$$

The elapsed proper time $\Delta\tau$ between $r = R$: $r = -R$ is $\Delta\tau = \frac{2R}{v}$

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i.e. Calculate Γ's for the 2D plane in Polar Coordinates

$$ds^2 = dr^2 + r^2 d\varphi^2 \ , \ g_{rr} = 1 \ , \ g_{\varphi\varphi} = r^2 \qquad g_{\alpha\delta} \Gamma^\delta_{\beta\gamma}$$

$$g_{r\delta}\Gamma^\delta_{rr} \ , \ g_{r\delta}\Gamma^\delta_{r\varphi} \ , \ g_{r\delta}\Gamma^\delta_{\varphi\varphi} \ , \ g_{\varphi\delta}\Gamma^\delta_{rr} \ , \ g_{\varphi\delta}\Gamma^\delta_{r\varphi} \ , \ g_{\varphi\delta}\Gamma^\delta_{\varphi\varphi}$$

$$g_{r\delta}\Gamma^\delta_{rr} = g_{rr}\Gamma^r_{rr} = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial r} + \cancel{\frac{\partial g_{rr}}{\partial r}} - \cancel{\frac{\partial g_{rr}}{\partial r}} \right) = 0 \ : \ g_{rr} = 1 \ \therefore \ \frac{\partial g_{rr}}{\partial r} = 0 \ : \left[\Gamma^r_{rr} = 0 \right]$$

$$g_{r\delta}\Gamma^\delta_{r\varphi} = g_{rr}\Gamma^r_{r\varphi} = \frac{1}{2}\left(\frac{\partial g_{rr}}{\partial \varphi} + \cancel{\frac{\partial g_{r\varphi}}{\partial r}} - \cancel{\frac{\partial g_{r\varphi}}{\partial r}} \right) = 0 \ : \ g_{rr} = 1 \ \therefore \ \frac{\partial g_{rr}}{\partial \varphi} = 0 \ : \left[\Gamma^r_{r\varphi} = 0 = \Gamma^r_{\varphi r} \right]$$

$$g_{\varphi\delta}\Gamma^\delta_{r\varphi} = g_{\varphi\varphi}\Gamma^\varphi_{r\varphi} = \frac{1}{2}\left(\frac{\partial g_{\varphi r}}{\partial \varphi} + \frac{\partial g_{\varphi\varphi}}{\partial r} - \cancel{\frac{\partial g_{r\varphi}}{\partial \varphi}} \right) = \frac{1}{2}\frac{\partial r^2}{\partial r} = r \quad \therefore \left[\Gamma^\varphi_{r\varphi} = \frac{1}{r} = \Gamma^\varphi_{\varphi r} \right]$$

Only non vanishing Christophel symbols are....

$$\Gamma^r_{\varphi\varphi} = -r, \ \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \quad : \quad g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right)$$

$$\frac{d^2 x^\alpha}{ds^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \ : \ ds^2 = -d\tau^2$$

For $\alpha = r$.....

$$\frac{d^2 r}{ds^2} = -\Gamma^r_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = -\Gamma^r_{r\gamma}\frac{dr}{ds}\frac{dx^\gamma}{ds} - \Gamma^r_{\varphi\gamma}\frac{d\varphi}{ds}\frac{dx^\gamma}{ds} = -\Gamma^r_{rr}\frac{dr}{ds}\frac{dr}{ds} - \Gamma^r_{r\varphi}\frac{dr}{ds}\frac{d\varphi}{ds} - \Gamma^r_{\varphi r}\frac{d\varphi}{ds}\frac{dr}{ds} - \Gamma^r_{\varphi\varphi}\frac{d\varphi}{ds}\frac{d\varphi}{ds}$$

$$= -\Gamma^{r}_{\varphi\varphi}\left(\frac{d\varphi}{ds}\right)^2 = +r\left(\frac{d\varphi}{ds}\right)^2 \quad : \quad \left[\frac{d^2r}{ds^2} = +r\left(\frac{d\varphi}{ds}\right)^2\right]$$

For $\alpha = \varphi$

$$\frac{d^2\varphi}{ds^2} = -\Gamma^{\varphi}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = -2\Gamma^{\varphi}_{r\varphi}\frac{dr}{ds}\frac{d\varphi}{ds} = -\frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds} \quad : \quad r^2\left[\frac{d^2\varphi}{ds^2} = -\frac{2}{r}\frac{dr}{ds}\frac{d\varphi}{ds}\right]r^2$$

$$r^2\frac{d^2\varphi}{ds^2} + 2r\frac{dr}{ds}\frac{d\varphi}{ds} = 0 = \frac{d}{ds}\left(r^2\frac{d\varphi}{ds}\right)$$

In Newtonian Mechanics Conservation laws are connected to Symmetries!

Conservation Law		Symmetry Under
Energy	$\Leftrightarrow$	Displacements in time
Linear Momentum	$\Leftrightarrow$	Displacements in Space
Angular Momentum	$\Leftrightarrow$	Rotations in Space

How does one tell if a spacetime geometry has a symmetry?
- The metric is independent of a coordinate.

i.e. Let $x' \longrightarrow x' + \text{Const.} \Rightarrow ds^2$ remains unchanged

The wormhole metric ....

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Let $t \to t + \text{const.}$ $\quad ds^2 = ds^2$

For example...

If $x' \to x' + c$ Leave ds^2 unchanged, then vector $\underset{\sim}{\xi}$ w/ Components

- $\xi^\alpha = (0, 1, 0, 0)$ - Lies along a direction in which metric doesn't change
- $\underset{\sim}{\xi}$ - "Killing Vector" Associated with the symmetry

i.e **8.6 The Killing Vectors of Flat Space**

$$ds^2 = dx^2 + dy^2 + dz^2$$

Translational Symmetries		Associated Killing Vectors
$x \to x + c$		$(1,0,0)$
$y \to y + c$	$\Leftrightarrow$	$(0,1,0)$
$z \to z + c$		$(0,0,1)$

In Spherical Polar Coordinates ....

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Rotational Symmetries	Associated Killing Vectors
$\varphi \to \varphi + c$	$(0,0,1)$

$(-y, x, 0)$ in cartesian coordinates

A symmetry implies a conserved quantity along a geodesic

In an arbitrary coordinate system, a conserved quantity along a geodesic is....

$$\left[\underset{\sim}{\xi} \cdot \underset{\sim}{u} = \text{constant}\right] \quad \text{where } \xi \text{ is a killing vector}$$

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$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 \quad : \quad ds^2 = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = -\frac{d\tau^2}{d\tau^2}$$

i.e 8.7 Geodesics in the plane using polar coordinates

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad : \quad g_{rr} = 1 \,, \; g_{\varphi\varphi} = r^2$$

Geodesic Equation from before....

$$\frac{d^2r}{d\tau^2} = r\left(\frac{d\varphi}{d\tau}\right)^2 \quad : \quad \frac{d^2r}{ds^2} = r\left(\frac{d\varphi}{ds}\right)^2$$

$$\Rightarrow$$

$$\frac{d}{d\tau}\left(r^2 \frac{d\varphi}{d\tau}\right) = 0 \quad : \quad \frac{d}{ds}\left(r^2 \frac{d\varphi}{ds}\right) = 0$$

Dividing (*) by ds^2

$$\left[1 = \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\varphi}{ds}\right)^2 \longrightarrow \left[\vec{u} \cdot \vec{u} = -1\right] \right]^{(**)}$$

$\varphi \to \varphi + \text{constant}$. Leaves ds^2 invariant $\Rightarrow$ Associated Killing Vector $\vec{\xi} = (0,1)$

w/ $\xi^r = 0$, $\xi^\varphi = 1$

A conserved quantity....

$$\ell = \vec{\xi} \cdot \vec{u} = g_{AB} \xi^A u^B = g_{\varphi\varphi} \xi^\varphi u^\varphi = r^2 \frac{d\varphi}{ds} \quad \therefore \; \ell = r^2 \frac{d\varphi}{ds} \quad \therefore \; \frac{d\varphi}{ds} = \frac{\ell}{r^2}$$

Plugging into (**)

$$1 = \left(\frac{dr}{ds}\right)^2 + r^2 \frac{\ell^2}{r^4} \quad \therefore \; \frac{dr}{ds} = \sqrt{1 - \frac{\ell^2}{r^2}}$$

integrating yields $r(s)$, However we want $r = r(\varphi)$ or $\varphi = \varphi(r)$

$$\frac{d\varphi}{dr} = \frac{d\varphi/ds}{dr/ds} = \frac{\ell/r^2}{\sqrt{1 - \ell^2/r^2}} = \frac{\ell}{r^2\sqrt{1 - \frac{\ell^2}{r^2}}} = \frac{\ell}{r\sqrt{r^2 - \ell^2}}$$

$$\int d\varphi = \ell\int \frac{dr}{r\sqrt{r^2-\ell^2}} = \ell\cdot\frac{\ell}{\ell}\int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{\ell^2(\sec^2\theta-1)}} = \int d\theta$$

$r = \ell\sec\theta \quad \therefore \quad \theta = \cos^{-1}(\ell/r)$
$dr = \ell\sec\theta\tan\theta\, d\theta$

Integrating yields ....

$$\varphi(r) = \varphi_0 + \cos^{-1}(\ell/r) \quad : \quad \ell = r\cos(\varphi-\varphi_0) = r\left[\cos\varphi\cos\varphi_0 + \sin\varphi\sin\varphi_0\right]$$

$x = r\cos\varphi \quad , \quad y = r\sin\varphi$

$\left[\ell = \cos\varphi_0 x + \sin\varphi_0 y\right]$ - General equation of a strait line

Null Geodesics

- Light rays move along Null worldlines $(ds^2 = 0 = g_{\alpha\beta}dx^\alpha dx^\beta)$
  - $x^\alpha(\lambda)$ is the path of a light ray through spacetime parameterized by λ
  - $u^\alpha = \frac{dx^\alpha}{d\lambda}$ then $\left[\underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0\right]$

we need the analog of the Geodesic equation....

$$\left[\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}\right]$$ - Geodesic EQN. for Null Geodesics

Local Inertial Frames : Freely Falling Frames

Local Inertial Frame

Coordinates centered on a point P in spacetime which...

- $g_{\alpha\beta}(x_P) = \eta_{\alpha\beta}$
- The Christophel Symbols vanish at that point

The geodesic equation in these coordinates, @ this point is ....

$$\left.\frac{d^2x^\alpha}{d\tau^2} = 0\right|_{x=x_P}$$

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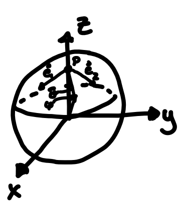
i.e 8.8 Rieman Normal Coordinates @ the North Pole of a sphere $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$



1. Point P is at north Pole
2. $\vec{e}_1 : \vec{e}_2$ Pointing in the $\varphi = 0 : \varphi = \pi/2$ directions
3. $\vec{n}$ in φ Direction has components
$n^A = (\cos\varphi, \sin\varphi) \quad S = a\theta$
$x^A = Sn^A = (a\theta\cos\varphi, a\theta\sin\varphi)$

See 7.2 ....

For small θ

$$g_{AB} = \mathrm{diag}(1,1) \therefore \frac{\partial g_{AB}}{\partial x^C} = 0$$