Taylor Larrechea
Dr. Gustafson
MATH 360
CP 10.2

The Three Theorems

(Theorem 1) $\vec{F} = \text{grad}(f)$, where grad(f) is the gradient of f as explained in Sec. 9.7.
(Theorem 2) Integration around closed curves C in D always gives 0.
(Theorem 3) Curl$(\vec{F}) = 0$, provided D is simply connected, as defined below.

Examples 1 & 2

Ex 1 Path Independence

Show that the integral $\int_C \vec{F} \cdot d\vec{r} = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$ is path independent in any domain in space and find its value in the integration from $A: (0,0,0)$ to $B: (2,2,2)$.

Solution. $\vec{F} = [2x, 2y, 4z] = \text{grad}(f)$, where $f = x^2 + y^2 + 2z^2$ because $\partial f/\partial x = 2x = F_1$, $\partial f/\partial y = 2y = F_2$, $\partial f/\partial z = 4z = F_3$. Hence the integral is independent of path according to Theorem 1, and (3) gives $f(B) - f(A) = f(2,2,2) - f(0,0,0) = 4 + 4 + 8 = 16$.

If you want to check this, use the most convenient path $C: \vec{r}(t) = [t, t, t]$, $0 \leq t \leq 2$ on which $\vec{F}(\vec{r}(t)) = [2t, 2t, 4t]$, so that $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2t + 2t + 4t = 8t$, and integration from 0 to 2 gives $8 \cdot 2^2/2 = 16$. If you did not see the potential by inspection, use the method in the next example.

Ex 2 Path Independence. Determination of a Potential

Evaluate the integral $I = \int_C (3x^2\,dx + 2yz\,dy + y^2\,dz)$ from $A: (0,1,2)$ to $B: (1,-1,7)$ by showing that $\vec{F}$ has a potential and applying (3).

Solution. If $\vec{F}$ has a potential f, we should have

$$f_x = F_1 = 3x^2, \quad f_y = F_2 = 2yz, \quad f_z = F_3 = y^2$$

We show that we can satisfy these conditions. By integration of f_x and differentiation,

$$f = x^3 + g(y,z), \quad f_y = g_y = 2yz, \quad g = y^2 z + h(z), \quad f = x^3 + y^2 z + h(z)$$
$$f_z = y^2 + h' = y^2 \qquad h' = 0 \qquad h = 0, \text{ say.}$$

This gives $f(x,y,z) = x^3 + y^3 z$ and by (3),

$$I = f(1,-1,7) - f(0,1,2) = 1 + 7 - (0 + 2) = 6$$

Equations (6) & (6*)

$$\text{Curl}(\vec{F}) = 0$$

in components (see sec. 9.9)

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$