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Equation 1

Let $w=f(x,y,z)$ be continuous and have continuous first partial derivatives in a domain D in xyz-space. Let $x=x(u,v)$, $y=y(u,v)$, $z=z(u,v)$ be functions that are continuous and first partial derivatives in a domain B in the uv-plane, where B is such that for every point (u,v) in B, the corresponding point $[x(u,v), y(u,v), z(u,v)]$ lies in D.

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \qquad (1)$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

Equation 2

If, $w=f(x,y)$ & $x=x(u,v)$, $y=y(u,v)$ as before, then (1) becomes

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \qquad (2)$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

Example 1

If $w=x^2-y^2$ and we define polar coordinates r,θ by $x=r\cos\theta$, $y=r\sin\theta$, then (2) gives

$$\frac{\partial w}{\partial r}=2x\cos\theta-2y\sin\theta=2r\cos^2\theta-2r\sin^2\theta=2r\cos 2\theta$$

$$\frac{\partial w}{\partial \theta}=2x(-r\sin\theta)-2y(r\cos\theta)=-2r^2\cos\theta\sin\theta-2r^2\sin\theta\cos\theta=-2r^2\sin 2\theta$$

Partial Derivatives on a Surface $z=g(x,y)$

Let $w=f(x,y,z)$ and let $z=g(x,y)$ represent a surface S in space. Then on S the function becomes

$$\tilde{w}(x,y)=f(x,y,g(x,y))$$

Hence, by (1) the partial derivatives are

$$\frac{\partial \tilde{w}}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}, \quad \frac{\partial \tilde{w}}{\partial y}=\frac{\partial f}{\partial y}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} \qquad [z=g(x,y)]$$