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PHYS 396 HW 10

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1, 2, 3, 4

Problem 1

$$H_0 = 72 \frac{\cancel{km}}{Sec \cdot \cancel{MPC}} \times \frac{1000 \cancel{m}}{1 \cancel{km}} \times \frac{1 \cancel{MPC}}{1 \times 10^6 \cancel{PC}} \times \frac{1 \cancel{PC}}{3.09 \times 10^{16} \cancel{m}} = 2.33 \times 10^{-18} s^{-1}$$

$$H_0 = 2.33 \times 10^{-18} s^{-1} \quad : \quad z = \frac{\Delta \lambda}{\lambda} = \left[\frac{\dot{a}(t_0)}{a(t_0)} \right] d \longrightarrow (1)$$

$$H_0 = H(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} \longrightarrow (2)$$

Putting (2) into (1) we get: $z^* = H_0 d$ $\qquad z = 0.2 : H_0 = 2.33 \times 10^{-18} s^{-1}$

Galaxies velocity $= 0.2 \times 3.0 \times 10^8 m/s = 6.0 \times 10^7 m/s \longrightarrow z^*$

$$\therefore \quad d = \frac{z^*}{H_0} = \frac{6.0 \times 10^7 m/s}{2.33 \times 10^{-18} 1/s} = 2.58 \times 10^{25} m$$

$$d = 2.58 \times 10^{25} m \longrightarrow (3)$$

1 Year in Seconds: $\frac{1 \cancel{min}}{60 s} \cdot \frac{1 \cancel{hr}}{60 \cancel{min}} \cdot \frac{1 \cancel{day}}{24 \cancel{hr}} \cdot \frac{1 \text{ year}}{365 \cancel{days}} = \frac{1 \text{ year}}{3600 \cdot 24 \cdot 365 \cdot (sec)}$

$$1 \text{ year} = 3.1536 \times 10^7 (sec)$$

Speed of light: $c = 3.0 \times 10^8 \frac{m}{s}$: Light year $= c \cdot yr = 3.0 \times 10^8 \frac{m}{s} \cdot 3.1536 \times 10^7 (s)$

$$c yr = 9.4608 \times 10^{15} m \quad \therefore \quad \frac{1 \cdot c yr}{9.4608 \times 10^{15} m}$$

Therefore (3) in terms (4) is

$$d = 2.58 \times 10^{25} m \cdot \frac{1 \cdot c yr}{9.4608 \times 10^{15} m} = 2.73 \times 10^9 c \cdot yr$$

$$\boxed{d = 2.73 \times 10^9 c \cdot yr}$$

Problem 2|

$$8\pi\rho = 3\frac{\dot{a}^2}{a^2} \longrightarrow (4) : 8\pi p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \longrightarrow (5) : 0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) \longrightarrow (6)$$

a.) putting (4) into (5) we have : $\rho = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2$: $p = -\frac{1}{8\pi}\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\left(\frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{8\pi}\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right)$$

$$= \dot{\rho} + 3\frac{\dot{a}}{a}\left(\frac{1}{4\pi}\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{4\pi}\left(\frac{\ddot{a}}{a}\right)\right) = \dot{\rho} + \frac{3}{4\pi}\left(\left(\frac{\dot{a}}{a}\right)^3 - \frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}\right)\right) \longrightarrow (7)$$

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)^2 = 2\left(\frac{\dot{a}}{a}\right)\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 2\left(\frac{\dot{a}}{a}\right)\left(-\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) = -\frac{2\dot{a}^3}{a^3} + 2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}\right)$$

$\dot{\rho}$: $\rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2}$: $\frac{d}{dt}(\rho) = \frac{3}{8\pi}\left(-2a^{-3}\dot{a}^2\cdot\dot{a} + 2\cdot\frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)\right)$

$= \frac{3}{8\pi}\dot{a}^2 a^{-2}$

$$\dot{\rho} = \frac{3}{4\pi}\left(-\frac{\dot{a}^3}{a^3} + \frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}\right)\right) \longrightarrow (8)$$

Putting (8) into (7) we see:

$$0 = -\frac{3}{4\pi}\left(\left(\frac{\dot{a}}{a}\right)^3 - \frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}\right)\right) + \frac{3}{4\pi}\left(\left(\frac{\dot{a}}{a}\right)^3 - \frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}\right)\right) = 0$$

$\therefore$ $\boxed{0=0, \text{ redundant}}$

b.) $p = w\rho \longrightarrow (9)$: $0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) \longrightarrow (6)^*$

Putting (9) into $(6)^*$ we have :

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + w\rho) = \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\rho(1+w)$$

$$\dot{\rho} = -3\left(\frac{\dot{a}}{a}\right)\rho(1+w) \quad : \quad \frac{\dot{\rho}}{\rho} = -3(1+w)\left(\frac{\dot{a}}{a}\right) \longrightarrow (10)$$

(10) becomes : $\int \frac{1}{\rho}\frac{d\rho}{dt}dt = -3(1+w)\int \frac{1}{a}\frac{da}{dt}dt$

$$\ln(\rho(t)) = -3(1+w)\cdot\ln(a(t)) + k$$

$$e^{(\ln(\rho(t)))} = e^{(-3(1+w)\cdot\ln(a(t)) + k)}$$

$e^{-3\ln(x)} = x^{-3}$ $\qquad \rho(t) = e^{(k)}e^{(-3(1+w)\ln(a(t)))}$ $\qquad e^{(k)} \longrightarrow \rho_0$

$$\boxed{\rho(t) = \rho_0\, a(t)^{-3(1+w)}}$$

Problem 2 | Continued

c.) $8\pi p = 3\frac{\dot{a}^2}{a^2} \longrightarrow$ (4) : $8\pi p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \longrightarrow$ (5) $\quad p = w\rho \longrightarrow$ (9)

$((4) \longrightarrow (5)) \longleftarrow (9)$

$(4) \rightarrow (5)$: $\quad \frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho \quad : \quad 8\pi p = -\left(2\frac{\ddot{a}}{a} + \frac{8\pi}{3}\rho\right) \longrightarrow$ (11)

Putting (9) into (11) we get $8\pi w\rho = -\frac{2\ddot{a}}{a} - \frac{8\pi}{3}\rho$

$-2\frac{\ddot{a}}{a} = \frac{8\pi\rho}{3} + 8\pi w\rho \quad : \quad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho - 4\pi w\rho \quad : \quad \frac{\ddot{a}}{a} = -4\pi\rho\left(\frac{1}{3} + w\right)$

$$\boxed{\frac{\ddot{a}}{a} = -4\pi\rho\left(w + \frac{1}{3}\right)}$$

d.) There is an accelerated expansion of the universe when $\frac{\ddot{a}}{a} > 0$, in this case $\frac{\ddot{a}}{a} > 0$ when the $(w + \frac{1}{3})$ term is negative. $\therefore$

$$\boxed{-1 \leq w < -\frac{1}{3}}$$

Problem 3

a.) $a(t) = a_0 t^{\frac{2}{3}(1+w)}$: $\rho(t) = \rho_0 a(t)^{-3(1+w)}$: $8\pi\rho = 3\frac{\dot{a}^2}{a^2}$

$a = a_0 t^n$: $\dot{a} = a_0 \cdot n t^n \cdot t^{-1}$: $\frac{\dot{a}}{a} = \frac{\cancel{a_0} \cdot n \cancel{t^n} t^{-1}}{\cancel{a_0 t^n}} = \frac{n}{t}$ $\therefore$ $\frac{\dot{a}}{a} = \frac{n}{t}$: $\frac{\dot{a}^2}{a^2} = \frac{n^2}{t^2}$

$n = 2/3(1+w)$: $n^2 = 4/9(1+w)^2$

$$\rho = \frac{3}{8\pi} \cdot \frac{\dot{a}^2}{a^2} = \frac{3}{8\pi} \cdot \frac{4/9}{t^2(1+w)^2} = \frac{1}{6\pi}\frac{1}{(1+w)^2 t^2} \quad : \quad \rho(t) = \frac{1}{6\pi}\frac{1}{(1+w)^2 t^2}$$

$$\rho(t) = \rho_0 \cdot a(t)^{-3(1+w)} = \rho_0 a_0 t^{2/3(1+w)^{-3(1+w)}} = \rho_0 a_0^{-3(1+w)} t^{-2}$$

$$\therefore \quad \rho_0 a_0^{-3(1+w)} \cdot \frac{1}{\cancel{t^2}} = \frac{1}{6\pi} \cdot \frac{1}{\cancel{t^2}(1+w)^2} \quad \therefore \quad \rho_0 = \frac{1}{6\pi}\frac{a_0^{3(1+w)}}{(1+w)^2}$$

$$\boxed{\rho_0 = \frac{1}{6\pi}\frac{a_0^{3(1+w)}}{(1+w)^2}}$$

b.) $8\pi p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$: $p = w\rho$: $p = w\rho(t) = w\rho_0 a(t)^{-3(1+w)}$

$a = a_0 t^n$: $\dot{a} = a_0 n t^{n-1}$: $\ddot{a} = a_0 n(n-1) t^{n-2}$: $\frac{\ddot{a}}{a} = \frac{\cancel{a_0} n(n-1) \cancel{t^n} t^{-2}}{\cancel{a_0 t^n}}$

$\frac{\ddot{a}}{a} = \frac{n(n-1)}{t^2}$: $\frac{\dot{a}^2}{a^2} = \frac{n^2}{t^2}$: $n = \frac{2}{3(1+w)}$: $n^2 = \frac{4}{9(1+w)^2}$

$$8\pi p = -\left(2 \cdot \frac{n(n-1)}{t^2} + \frac{n^2}{t^2}\right) = -\left(2 \cdot \frac{2/3(1+w)(2/3(1+w)-1)}{t^2} + \frac{4}{9}\frac{1}{(1+w)^2 t^2}\right)$$

$$8\pi w\rho_0 a(t)^{-3(1+w)} = 8\pi w \rho_0 a_0 t^{2/3(1+w)^{-3(1+w)}} = 8\pi w \rho_0 a_0^{-3(1+w)} t^{-2}$$

$$\frac{8\pi w \rho_0 a_0^{-3(1+w)}}{\cancel{t^2}} = -\frac{4}{3}\frac{(2/3(1+w)-1)}{(1+w)\cancel{t^2}} - \frac{4}{9}\frac{1}{(1+w)^2 \cancel{t^2}}$$

$$8\pi w \rho_0 a_0^{-3(1+w)} = -\frac{4}{3}\left(\frac{\frac{2}{3(1+w)} - 1}{(1+w)} + \frac{1}{3(1+w)^2}\right)$$

$$\rho_0 = -\frac{1}{6\pi w} a_0^{3(1+w)} \left(\frac{\frac{2}{3(1+w)} - 1}{1+w} + \frac{1}{3(1+w)^2}\right) = \frac{-1}{6\pi w} a_0^{3(1+w)} \left(\frac{-3w-1}{3(1+w)^2} + \frac{1}{3(1+w^2)}\right)$$

$$\rho_0 = \frac{-1}{6\pi \cancel{w}} a_0^{3(1+w)} \left(\frac{-\cancel{3w}}{\cancel{3}(1+w)^2}\right) = \frac{1}{6\pi} a_0^{3(1+w)} \cdot \frac{1}{(1+w)^2}$$

$$\boxed{\rho_0 = \frac{1}{6\pi(1+w)^2} a_0^{3(1+w)}}$$

Problem 3 | Continued

c.) $8\pi\rho = 3\dfrac{\dot{a}^2}{a^2}$: $8\pi w\rho = -\left(2\dfrac{\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2}\right)$

w/ $w=-1$: $$8\pi\rho = 3\frac{\dot{a}^2}{a^2}$$
$$-8\pi\rho = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$

$$0 = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}^2}{a^2} = -2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} = 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) : \frac{d}{dt}\left(\frac{\dot{a}}{a}\right)$$

$$0 = 2\cdot\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) : H_0 = \left(\frac{\dot{a}}{a}\right) : 0 = 2\cdot\frac{d}{dt}H_0$$

$$2\cdot\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 2\cdot\frac{d}{dt}H_0 : \cancel{2}\cdot\frac{\cancel{d}}{\cancel{d}t}\cdot\frac{da}{dt}\cdot\frac{1}{a} = \cancel{2}\cdot\frac{\cancel{d}}{\cancel{d}t}\cdot H_0$$

$$\int\frac{da}{a} = \int H_0\,dt \longrightarrow (12)$$

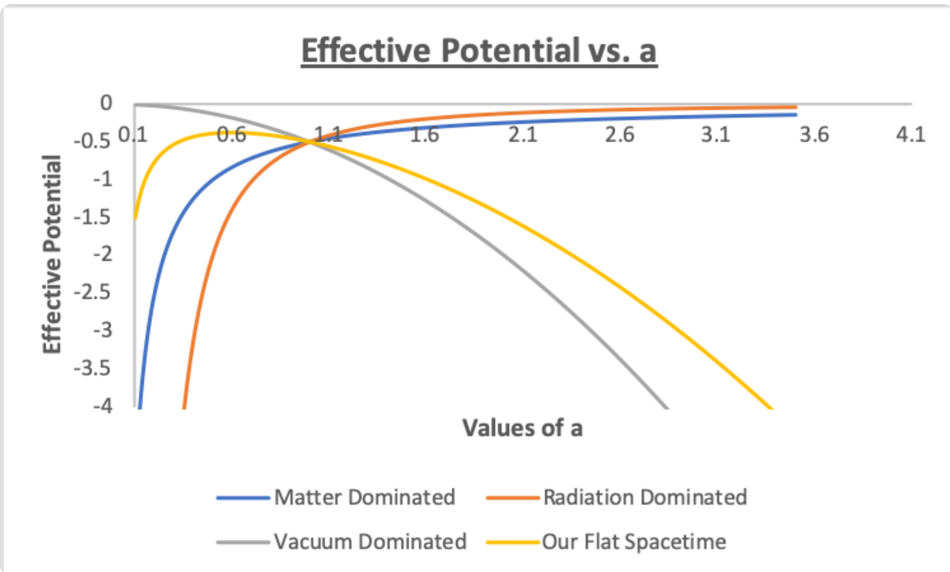
Solving equation (12) gives : $\ln(a) = H_0 t - H_0 t_0$

$$\boxed{a(t) = e^{H_0(t-t_0)}}$$

Problem 4

$$U_{eff}(a) = -\frac{1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right) \quad (a(t_0)=1)$$

a.) – d.)



e.)

$$\frac{1}{2H_0^2}\left(\frac{da}{dt}\right)^2 + U_{eff}(a) = 0$$

i: Matter-dominated spacetime

Decelerated expansion

ii: Radiation-dominated Spacetime

Decelerated expansion

iii: Vacuum-dominated spacetime

Accelerated expansion

iv: Our-Flat Spacetime

Decelerated expansion, then accelerated expansion