MAT 201
Larson/Edwards – Section 3.3
Increasing and Decreasing Functions & First Derivative Test

In this section we will learn how the derivative can be used to classify relative extrema as either relative maxima or relative minima. In order for us to do this we must first understand how to determine where a function ins increasing or decreasing.

Increasing & Decreasing Functions (As I see it):

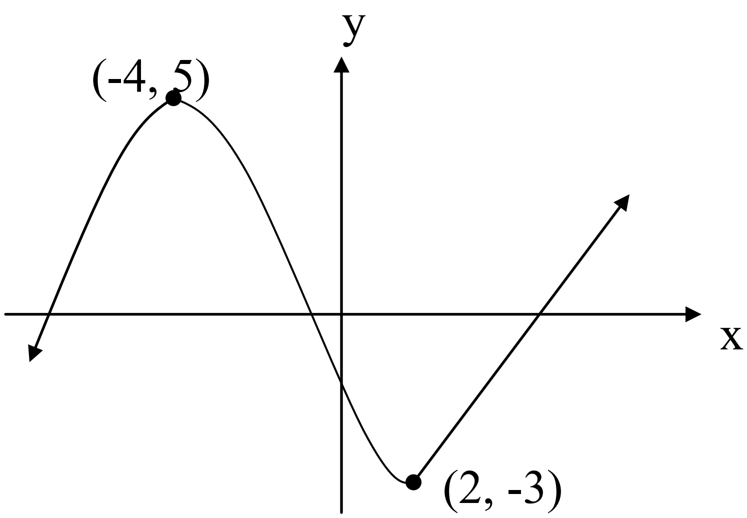
- A function *f* is ***increasing*** on an open interval if, as *x* increases, *y* (or *f(x)*) increases with it.
- A function *f* is ***decreasing*** on an open interval if, as *x* increases, *y* (or *f(x)*) decreases with it.
- A function *f* is ***constant*** on an open interval if, as *x* increase, *y* (or *f(x)*) stays the same.

The "Official" definition for Increasing/Decreasing:

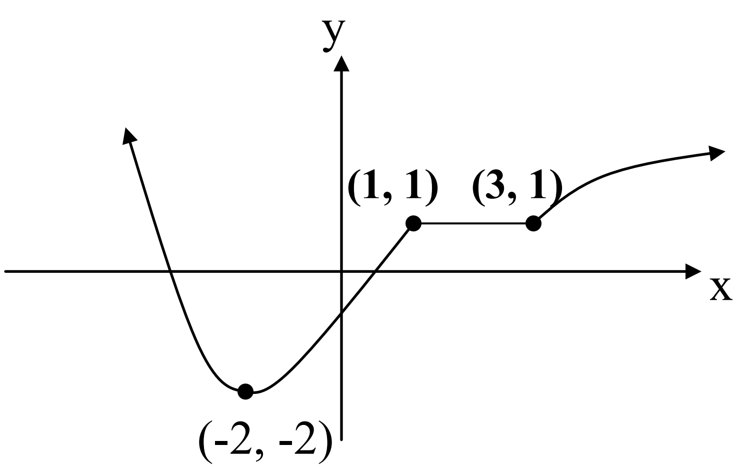
- A function *f* is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function *f* is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A function *f* is constant on an interval if for any two numbers x_1 and x_2 in the interval, $x_2 > x_1$ implies $f(x_2) = f(x_1)$.

Ex: Determine, from the graph, the intervals where f is increasing, decreasing, or constant.

a)



b)



Ex: Determine, from the graph, the intervals where f is increasing, decreasing, or constant.

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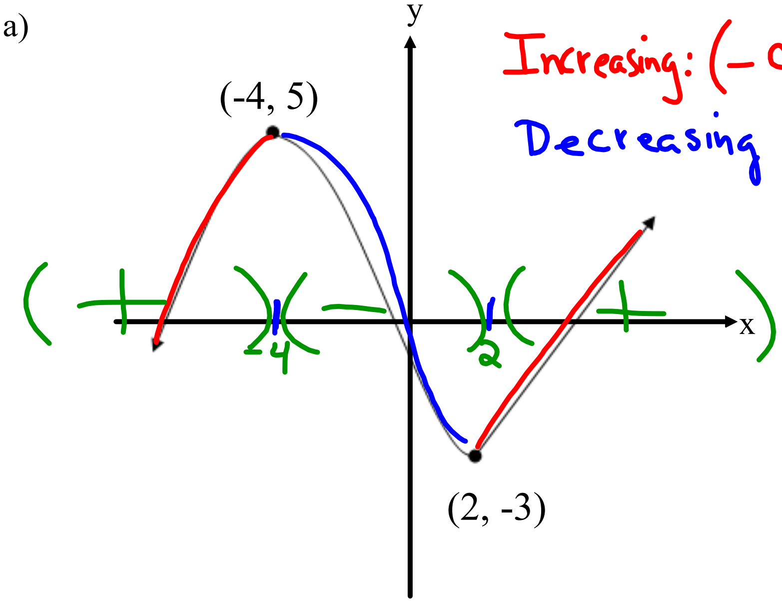
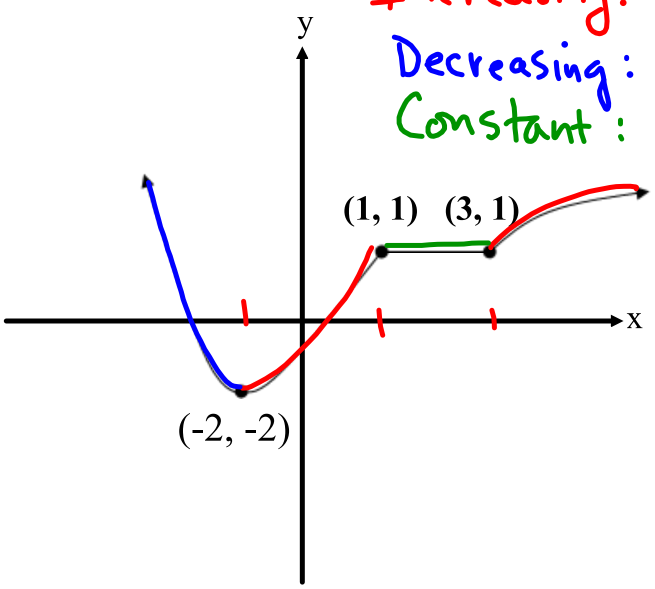


Increasing: $(-\infty,-4)\cup(2,\infty)$

Decreasing: $(-4,2)$

b)

Increasing: $(-2,1) \cup (3,\infty)$
Decreasing: $(-\infty,-2)$
Constant: $(1,3)$



Ex: Without graphing the function, find the interval(s) where the function $f(x) = \frac{2}{3}x^3 - x^2 - 12x + 3$ is increasing and decreasing.

Tools for finding increasing/decreasing intervals of *f*:

1. Find the derivative of the function *f*.
2. Find the critical numbers of the function:
   a. Set the derivative equal to 0 ($f'(x) = 0$).
   b. Solve for *x*. These values AND those values that are not differentiable are your critical numbers.
3. Use your critical numbers to partition the function into different intervals
4. Test an arbitrary value in each interval to determine whether the function is increasing or decreasing on that interval
   a. If $f'(x) > 0$ for all *x* in (a, b), then *f* is increasing on (a, b).
   b. If $f'(x) < 0$ for all *x* in (a, b), then *f* is decreasing on (a, b).
   c. If $f'(x) = 0$ for all *x* in (a, b), then *f* is constant on (a, b).

Ex: Without graphing the function, find the interval(s) where the function $f(x) = \frac{2}{3}x^3 - x^2 - 12x + 3$ is increasing and decreasing.

$$f'(x) = 2x^2 - 2x - 12$$

$$0 = 2x^2 - 2x - 12$$

$$0 = 2(x^2 - x - 6)$$

$$0 = 2(x-3)(x+2)$$

Critical #'s: $x = 3, -2$

Intervals: $(-\infty, -2)$ $(-2, 3)$ $(3, \infty)$

choose -3 choose 0 choose 4

$f'(-3) = 12 > 0$ $f'(0) = -12 < 0$ $f'(4) = 12 > 0$

Increasing on $(-\infty, -2)$ Dec. on $(-2, 3)$ Inc. on $(3, \infty)$

Increasing on $(-\infty, -2) \cup (3, \infty)$
Decreasing on $(-2, 3)$

How do we find the Relative Extremum?

First-Derivative Test:
Let f be continuous on the interval (a, b) in which c is the only critical number. If f is differentiable on the interval (except possibly at c), then $f(c)$ can be classified as a relative minimum, a relative maximum, or neither, as shown.

1. On the interval (a, b), if $f'(x)$ is negative to the left of $x = c$ and positive to the right of $x = c$, then $f(c)$ is a ***relative minimum***.
2. On the interval (a, b), if $f'(x)$ is positive to the left of $x = c$ and negative to the right of $x = c$, then $f(c)$ is a ***relative maximum***.
3. On the interval (a, b), if $f'(x)$ is positive on both sides of $x = c$ OR negative on both sides of $x = c$, then $f(c)$ is not a relative extremum of f.

Why do (1), (2) and (3) above make sense?

Ex: Find the relative extrema of $f(x) = \dfrac{x^2}{1 - x^2}$.

How do we find the Relative Extremum?

Test intervals for increase & decrease.

If a function transitions from increase to decrease, at that point of transition, you have a relative max.

If a function transitions from decrease to increase, at that point of transition, you have a relative min.

Ex: Find the relative extrema of $f(x) = \dfrac{x^2}{1-x^2}$.

$$f'(x) = \frac{2x(1-x^2) - (-2x)(x^2)}{(1-x^2)^2}$$

$$f'(x) = \frac{2x - \cancel{2x^3} \cancel{+2x^3}}{(1-x^2)^2}$$

$$f'(x) = \frac{2x}{(1-x^2)^2} \quad \begin{matrix} + \\ + \end{matrix}$$

Critical #'s:

$1-x^2=0 \Rightarrow x=\pm 1$

$2x=0 \Rightarrow x=0$

$$0 = \frac{2x}{(1-x^2)^2}$$

$0 = 2x \quad x=0$

Decreasing	Decreasing $\leftrightarrow$	Increasing	Increasing
$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(-2)<0$	$f(-0.5)<0$	$f(0.5)>0$	$f(2)>0$

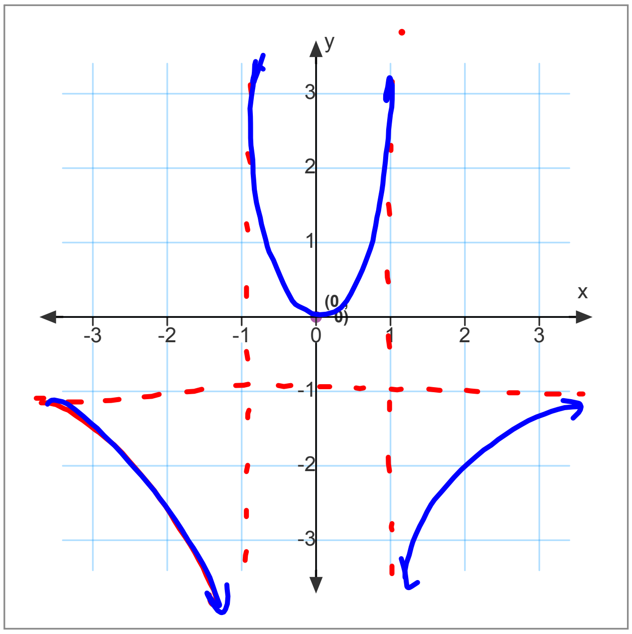
$$\boxed{\begin{matrix}\text{Relative Minimum} \\ @ \\ (0,0)\end{matrix}}$$

Ex: Find the relative extrema of $f(x) = \dfrac{x^2}{1-x^2}$.

V.A.: $x = 1$
$x = -1$

H.A.: $y = -1$

x-int: $(0,0)$ ← rel. min.



Tools for Finding Relative Extrema:

1. Determine the critical numbers, c, of the function f.
2. Determine the sign of $f'(x)$ to the left and right of each critical point.
   a. If the sign changes from positive to negative then $(c, f(c))$ is a ***relative maximum***.
   b. If the sign changes from negative to positive, then $(c, f(c))$ is a ***relative minimum***.
   c. If the sign does not change, then $(c, f(c))$ is not a relative extrema.

Ex: The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by the function: $C(t) = \dfrac{3t}{27 + t^3} \quad (t \geq 0)$.

At what time is the concentration the greatest?

Ex: The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by the function:

$$C(t) = \frac{3t}{27 + t^3} \quad (t \geq 0)$$

At what time is the concentration the greatest?

$$C'(t) = \frac{3(27+t^3) - 3t(3t^2)}{(27+t^3)^2}$$

$$C'(t) = \frac{81 + 3t^3 - 9t^3}{(27+t^3)^2} = \frac{81 - 6t^3}{(27+t^3)^2} \quad \frac{-}{+}$$

Crit. #'s: $t^3 + 27 = 0 \quad t^3 = -27 \quad t = -3$

$81 - 6t^3 = 0$

$81 = 6t^3$

$\sqrt[3]{\frac{27}{2}} = \sqrt[3]{t^3}$

$2.38 = t$

Increasing $(0, 2.38)$ Decreasing $(2.38, \infty)$

$C'(1) > 0 \qquad C'(3) < 0$

$\approx$ 2.38 hours

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