Exercise 1

Problem 1 $|\psi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle$, $|\varphi_1\rangle = \frac{5i}{13}|+\hat{z}\rangle + \frac{12}{13}|-\hat{z}\rangle$, $|\varphi_2\rangle = \frac{1+2i}{\sqrt{10}}|+\hat{z}\rangle + \frac{1-2i}{\sqrt{10}}|-\hat{z}\rangle$

a.)

$|\varphi_1\rangle$ in terms of $\langle+\hat{z}|$, $\langle-\hat{z}|$

$$\boxed{\langle\varphi_1| = -\frac{5i}{13}\langle+\hat{z}| + \frac{12}{13}\langle-\hat{z}|}$$

$|\varphi_2\rangle$ in terms of $\langle+\hat{z}|$, $\langle-\hat{z}|$

$$\boxed{\langle\varphi_2| = \frac{1-2i}{\sqrt{10}}\langle+\hat{z} + \frac{1-2i}{\sqrt{10}}\langle-\hat{z}}$$

b.)

$$\langle\varphi_1|\psi\rangle = \left(-\frac{5i}{13}\langle+\hat{z}| + \frac{12}{13}\langle-\hat{z}|\right)\left(\frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle\right)$$

$$= -\frac{5i}{13}\cdot\frac{3}{5}\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} + \frac{12}{13}\cdot\frac{4i}{5}\overset{1}{\cancel{\langle-\hat{z}|-\hat{z}\rangle}} = -\frac{15i}{65} + \frac{48i}{65} = \frac{33i}{65}$$

$$\boxed{\langle\varphi_1|\psi\rangle = \frac{33i}{65}}$$

$$\langle\varphi_2|\psi\rangle = \left(\frac{1-2i}{\sqrt{10}}\langle+\hat{z} + \frac{1+2i}{\sqrt{10}}\langle-\hat{z}\right)\left(\frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle\right)$$

$$= \frac{3-6i}{5\sqrt{10}}\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} + \frac{4i+8i^2}{5\sqrt{10}}\overset{1}{\cancel{\langle-\hat{z}|-\hat{z}\rangle}} = \frac{3-6i}{5\sqrt{10}} + \frac{-8+4i}{5\sqrt{10}} = \frac{-5-2i}{5\sqrt{10}}$$

$$\boxed{\langle\varphi_2|\psi\rangle = \frac{-5-2i}{5\sqrt{10}}}$$

Problem 2

$$A = \begin{pmatrix} 2 & 2 \\ -i & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1+i & 2 \\ 1-i & 1 & 0 \end{pmatrix}$$

a.) Determine $A^\dagger$

$$\boxed{A^\dagger = \begin{pmatrix} 2 & i \\ 2 & 1 \end{pmatrix}}$$

b.) Determine B^t

$$\boxed{B^t = \begin{pmatrix} 0 & 1+i \\ 1-i & 1 \\ 2 & 0 \end{pmatrix}}$$

c.) Determine AB

$$AB = \begin{pmatrix} 2 & 2 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1+i & 2 \\ 1-i & 1 & 0 \end{pmatrix}$$

$$\boxed{AB = \begin{pmatrix} 2-2i & 4+2i & 4 \\ 1-i & 2-i & -2i \end{pmatrix}}$$

D.) Determine $(AB)^\dagger$

$$\boxed{(AB)^\dagger = \begin{pmatrix} 2+2i & 1+i \\ 4-2i & 2+i \\ 4 & 2i \end{pmatrix}}$$

$$B^t = \begin{pmatrix} 0 & 1+i \\ 1-i & 1 \\ 2 & 0 \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} 2 & i \\ 2 & 1 \end{pmatrix}$$

$$\boxed{B^t A^t = \begin{pmatrix} 2+2i & 1+i \\ 4-2i & 2+i \\ 4 & 2i \end{pmatrix}}$$

Problem 3

$$|\psi\rangle = \frac{1+i}{2}|+\hat{z}\rangle + \frac{1-i}{2}|-\hat{z}\rangle \quad , \quad \begin{aligned} |+\hat{n}\rangle &= \cos(\theta/2)|+\hat{z}\rangle + e^{i\varphi}\sin(\theta/2)|-\hat{z}\rangle \\ |-\hat{n}\rangle &= \sin(\theta/2)|+\hat{z}\rangle - e^{i\varphi}\cos(\theta/2)|-\hat{z}\rangle \end{aligned}$$

a.) $SG\hat{y}$: $|+\hat{y}\rangle \rightsquigarrow \theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2}$: $|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$

$$e^{i\pi/2} = \overset{0}{\cancel{\cos(\pi/2)}} + i\overset{1}{\cancel{\sin(\pi/2)}} = i$$

$$\boxed{|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle}$$

b.) $\langle +\hat{y}| = |+\hat{y}\rangle^{\dagger}$ $\therefore$

$$\boxed{\langle +\hat{y}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}|}$$

$$\text{Prob}(S_y = +\hbar/2) = |\langle +\hat{y}|\psi\rangle|^2$$

$$= \left(\frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}|\right)\left(\frac{1+i}{2}|+\hat{z}\rangle + \frac{1-i}{2}|-\hat{z}\rangle\right)$$

$$= \frac{1+i}{2\sqrt{2}}\overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} - \frac{i+i^2}{2\sqrt{2}}\overset{1}{\cancel{\langle -\hat{z}|-\hat{z}\rangle}} = \frac{1}{2\sqrt{2}}(1+i-i-1) = \frac{1}{2\sqrt{2}}(0) = 0$$

$$\boxed{\text{Prob}(S_y = +\hbar/2) = 0}$$

$$\text{Prob}(S_y = -\hbar/2) = |\langle -\hat{y}|\psi\rangle|^2 : \quad |-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{i}{\sqrt{2}}\langle -\hat{z}|\right)\left(\frac{1+i}{2}|+\hat{z}\rangle + \frac{1-i}{2}|-\hat{z}\rangle\right)$$

$$= \frac{1+i}{2\sqrt{2}}\overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} + \frac{i-i^2}{2\sqrt{2}}\overset{1}{\cancel{\langle -\hat{z}|-\hat{z}\rangle}} = \frac{1+i}{2\sqrt{2}} + \frac{1+i}{2\sqrt{2}} = \frac{2+2i}{2\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

$$\frac{(1+i)(1-i)}{2} = \frac{1-i+i-i^2}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$

$$\boxed{\text{Prob}(S_y = -\hbar/2) = 1}$$

Exercise 2

i) $|\varphi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4}{5}|-\hat{z}\rangle$: ii) $|\varphi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle$: iii) $|\varphi\rangle = \frac{12}{13}|+\hat{z}\rangle - \frac{5i}{13}|-\hat{z}\rangle$

a.) i.) $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\hat{z}\rangle$: $|\varphi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4}{5}|-\hat{z}\rangle$

$\varphi = 0$: $\cos(\theta/2) = \frac{3}{5}$: $\frac{\theta}{2} = \cos^{-1}(3/5)$: $\theta = 2\cdot\cos^{-1}(3/5) = 106°$

$$\boxed{\theta = 106°,\ \varphi = 0}$$

ii.) $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\hat{z}\rangle$: $|\varphi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle$

$\varphi = \pi/2$

$$\boxed{\theta = 106°,\ \varphi = 90°}$$

iii.) $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-\hat{z}\rangle$: $|\psi\rangle = \frac{12}{13}|+\hat{z}\rangle - \frac{5i}{13}|-\hat{z}\rangle$

$\cos(\theta/2) = \frac{12}{13}$: $\theta/2 = \cos^{-1}(12/13)$: $\theta = 2\cdot\cos^{-1}(12/13) = 45°$

$$\boxed{\theta = 45°,\ \varphi = 270°}$$

b.)

i.) Probability of SG$\hat{z}$: $|+\hat{z}\rangle = 1\cdot|+\hat{z}\rangle + 0\cdot|-\hat{z}\rangle$: $\langle+\hat{z}| = \langle+\hat{z}| + 0\cdot\langle-\hat{z}|$

$\langle+\hat{z}|\psi\rangle = \frac{3}{5}\langle+\hat{z}|+\hat{z}\rangle + 0\cdot\frac{4}{5}\langle-\hat{z}|-\hat{z}\rangle = \frac{3}{5}$: $|\langle+\hat{z}|\psi\rangle|^2 = \frac{9}{25}$

$$\boxed{|\langle+\hat{z}|\psi\rangle|^2 = 9/25}$$

ii.) Probability of SG$\hat{z}$: $|+\hat{z}\rangle = 1\cdot|+\hat{z}\rangle + 0\cdot|-\hat{z}\rangle$: $\langle+\hat{z}| = \langle+\hat{z}| + 0\cdot\langle-\hat{z}|$

$\langle+\hat{z}|\psi\rangle = \frac{3}{5}\langle+\hat{z}|+\hat{z}\rangle + 0\cdot\frac{4i}{5}\langle-\hat{z}|-\hat{z}\rangle = \frac{3}{5}$: $|\langle+\hat{z}|\psi\rangle|^2 = \frac{9}{25}$

$$\boxed{|\langle+\hat{z}|\psi\rangle|^2 = 9/25}$$

iii.) Probability of SG$\hat{z}$: $|+\hat{z}\rangle = 1\cdot|+\hat{z}\rangle + 0\cdot|-\hat{z}\rangle$: $\langle+\hat{z}| = \langle+\hat{z}| + 0\cdot\langle-\hat{z}|$

$\langle+\hat{z}|\psi\rangle = \frac{12}{13}\langle+\hat{z}|+\hat{z}\rangle - 0\cdot\frac{5i}{13}\langle-\hat{z}|-\hat{z}\rangle = \frac{12}{13}$: $|\langle+\hat{z}|\psi\rangle|^2 = 144/169$

$$\boxed{|\langle+\hat{z}|\psi\rangle|^2 = 144/169}$$

Exercise 2 Continued

c.) $\theta = \pi/2$, $\varphi = 0$

i.) Probability of SG $\hat{x}$: $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$: $\langle +\hat{x}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{1}{\sqrt{2}}\langle -\hat{z}|$

$$\langle +\hat{x}|\psi\rangle = \frac{1}{\sqrt{2}}\cdot\frac{3}{5}\langle +\hat{z}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}\cdot\frac{4}{5}\langle -\hat{z}|-\hat{z}\rangle = \frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} = \frac{7}{5\sqrt{2}}$$

$$\boxed{|\langle +\hat{x}|\psi\rangle|^2 = 49/50}$$

ii.) Probability of SG $\hat{x}$: $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$: $\langle +\hat{x}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{1}{\sqrt{2}}\langle -\hat{z}|$

$$\langle +\hat{x}|\psi\rangle = \frac{1}{\sqrt{2}}\cdot\frac{3}{5}\langle +\hat{z}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}\cdot\frac{4i}{5}\langle -\hat{z}|-\hat{z}\rangle = \left|\frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}}\right|^2 = \left(\frac{3+4i}{5\sqrt{2}}\right)\left(\frac{3-4i}{5\sqrt{2}}\right)$$

$$\boxed{|\langle +\hat{x}|\psi\rangle|^2 = 1/2}$$

iii.) Probability of SG $\hat{x}$: $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$: $\langle +\hat{x}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{1}{\sqrt{2}}\langle -\hat{z}|$

$$\langle +\hat{x}|\psi\rangle = \frac{1}{\sqrt{2}}\cdot\frac{12}{13}\langle +\hat{z}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}\cdot\frac{-5i}{13}\langle -\hat{z}|-\hat{z}\rangle = \left|\frac{12}{13\sqrt{2}} - \frac{5i}{13\sqrt{2}}\right|^2 = \left(\frac{12-5i}{13\sqrt{2}}\right)\left(\frac{12+5i}{13\sqrt{2}}\right)$$

$$\boxed{|\langle +\hat{x}|\psi\rangle|^2 = 1/2}$$

Exercise 3

Problem 1 $|\varphi\rangle = \cos(\theta/2)|+\hat{z}\rangle + \sin(\theta/2)|-\hat{z}\rangle$, $|-\hat{z}\rangle = 0\cdot|+\hat{z}\rangle + 1\cdot|-\hat{z}\rangle \longrightarrow \theta = \varphi = 0$
$|+\hat{z}\rangle = 1\cdot|+\hat{z}\rangle + 0\cdot|-\hat{z}\rangle \longrightarrow \theta = \varphi = 0$

$$\langle -\hat{z}|\varphi\rangle = 0\cdot\cos(\theta/2)\,\overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} + 1\cdot\sin(\theta/2)\,\overset{1}{\cancel{\langle -\hat{z}|-\hat{z}\rangle}} = \sin(\theta/2)$$

$\langle -\hat{z}|\hat{\varphi}\rangle = \sin(\theta/2) \longrightarrow$ we know θ cannot be $0°$, $\langle -\hat{z}|\hat{\varphi}\rangle$ would be $= 0$ if θ were $0°$.

$$\langle +\hat{z}|\varphi\rangle = 1\cdot\cos(\theta/2)\,\overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} + 0\cdot\sin(\theta/2)\,\overset{1}{\cancel{\langle -\hat{z}|-\hat{z}\rangle}} = \cos(\theta/2)$$

$\langle +\hat{z}|\hat{\varphi}\rangle = \cos(\theta/2) \longrightarrow$ we know θ cannot be $90°$, $\langle +\hat{z}|\hat{\varphi}\rangle$ would be $=0$ if θ were $90°$.

Problem 2 $|\varphi\rangle = \cos(\theta/2)|+\hat{z}\rangle + \sin(\theta/2)|-\hat{z}\rangle$, $|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$ $\theta = \pi/2$, $\varphi = 0$

$$\langle +\hat{x}|\varphi\rangle = \frac{1}{\sqrt{2}}\cos(\theta/2)\,\overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} + \frac{1}{\sqrt{2}}\sin(\theta/2)\,\overset{1}{\cancel{\langle -\hat{z}|-\hat{z}\rangle}} = \frac{1}{\sqrt{2}}\left(\cos(\theta/2) + \sin(\theta/2)\right)$$

$$|\langle +\hat{x}|\varphi\rangle|^2 = \frac{1}{2}\left(\cos(\theta/2) + \sin(\theta/2)\right)^2 = \frac{1}{2}\left(\cos^2(\theta/2) + \sin^2(\theta/2) + 2\cos(\theta/2)\sin(\theta/2)\right)$$
$$= \frac{1}{2}\left(1 + 2\cos(\theta/2)\sin(\theta/2)\right) = \frac{1}{2} + \cos(\theta/2)\sin(\theta/2) \;:$$

$$|\langle +\hat{x}|\varphi\rangle|^2 = \frac{1}{2} + \frac{1}{2}\sin(\theta) \;:\; \frac{1}{2}\left(1 + \sin(\theta)\right) = 0 \quad \therefore \sin(\theta) = -1 \quad \therefore \theta = \sin^{-1}(-1) = \frac{3\pi}{2}$$
$$\theta = \frac{\pi}{2}$$

We know that $\theta \neq \frac{3\pi}{2}$, because this value would yield a zero probability and that is not possible.

Problem 3 $|\varphi\rangle = \cos(\theta/2)|+\hat{z}\rangle + \sin(\theta/2)|-\hat{z}\rangle$, $|+\hat{z}\rangle = 0\cdot|+\hat{z}\rangle + 1\cdot|-\hat{z}\rangle \longrightarrow \theta = \varphi = 0$

$$\langle +\hat{z}|\varphi\rangle \longrightarrow \langle +\hat{z}|\varphi\rangle \longrightarrow \langle +\hat{z}|\varphi\rangle$$

This above sequence will not produce any more information than after the first measurement

Problem 4 $|\varphi\rangle = \cos(\theta/2)|+\hat{z}\rangle + \sin(\theta/2)|-\hat{z}\rangle$, $|+\hat{z}\rangle = 0\cdot|+\hat{z}\rangle + 1\cdot|-\hat{z}\rangle \longrightarrow \theta = \varphi = 0$
$|+\hat{x}\rangle = 1\cdot|+\hat{z}\rangle + 0\cdot|-\hat{z}\rangle \longrightarrow \theta = \pi/2$, $\varphi = 0$

$$\langle +\hat{z}|\varphi\rangle \xrightarrow{①} \underbrace{\langle +\hat{x}|\varphi\rangle \xrightarrow{②} \langle +\hat{x}|\varphi\rangle}$$

The sequence, ②, above will not depend on the outcome of ①, because $(\pm)$ is a possible result for ②.

Exercise 4

$$|\varphi\rangle = \frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle \; , \; |\psi\rangle = \frac{1}{12}|+\hat{z}\rangle - \frac{12}{13}|-\hat{z}\rangle$$

$$\langle\psi|\varphi\rangle = \left(\frac{1}{12}\langle+\hat{z}| - \frac{12}{13}\langle-\hat{z}|\right)\left(\frac{3}{5}|+\hat{z}\rangle + \frac{4i}{5}|-\hat{z}\rangle\right)$$

$$= \frac{1}{20}\cancel{\langle+\hat{z}|+\hat{z}\rangle}^{1} - \frac{48i}{65}\cancel{\langle-\hat{z}|-\hat{z}\rangle}^{1} = \frac{1}{20} - \frac{48i}{65}$$

$$\boxed{\langle\psi|\varphi\rangle = \frac{1}{20} - \frac{48i}{65}}$$

Exercise 5

$$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad |\psi\rangle = \frac{4}{5}|0\rangle + \frac{3i}{5}|1\rangle$$

$$\langle\varphi|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{4}{5}|0\rangle + \frac{3i}{5}|1\rangle\right)$$

$$= \frac{4}{5\sqrt{2}}\overbrace{\cancel{\langle 0|0\rangle}}^{1} + \frac{3i}{5\sqrt{2}}\overbrace{\cancel{\langle 1|1\rangle}}^{1} = \frac{4+3i}{5\sqrt{2}}$$

$$\boxed{\langle\varphi|\psi\rangle = \frac{4+3i}{5\sqrt{2}}}$$

Exercise 6

$$|-\hat{n}\rangle = \sin\theta/2\,|0\rangle - e^{i\varphi}\cos\theta/2\,|1\rangle \;,\; |+\hat{n}\rangle = \cos\theta/2\,|0\rangle + e^{i\varphi}\sin\theta/2\,|1\rangle$$

$SG\hat{z}$: $\theta = 0$, $\varphi = 0$

$$|-\hat{z}\rangle = \sin(0)\,|0\rangle - e^{0}\cos(0)\,|1\rangle$$
$$= 0\cdot|0\rangle - 1|1\rangle$$

$$\boxed{|-\hat{z}\rangle = -|1\rangle}$$

$\theta = 0$, $\varphi = 0$

$$|+\hat{z}\rangle = \cos(0)\,|0\rangle + e^{0}\sin(0)\,|1\rangle$$
$$= |0\rangle$$

$$\boxed{|+\hat{z}\rangle = |0\rangle}$$

$SG\hat{x}$: $\theta = \pi/2$, $\varphi = 0$

$$|-\hat{x}\rangle = \sin(\pi/4)|0\rangle - e^{0}\cdot\cos(\pi/4)|1\rangle$$
$$= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\boxed{|-\hat{x}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle}$$

$\theta = \pi/2$, $\varphi = 0$

$$|+\hat{x}\rangle = \cos(\pi/4)\,|0\rangle + e^{i\pi}\sin(\pi/4)$$

$$\boxed{|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$$

Exercise 7

1) $\left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \right\}$

$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\cdot\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{2}\cancelto{1}{\langle 0|0\rangle} + \frac{i}{2}\cancelto{1}{\langle 1|1\rangle} = \frac{1+i}{2} \neq 0$$

This is not a measurement basis

2) $\left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$

$$\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\cdot\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}\cancelto{1}{\langle 0|0\rangle} - \frac{1}{2}\cancelto{1}{\langle 1|1\rangle} = \frac{1}{2} - \frac{1}{2} = 0 \ \checkmark$$

This is a measurement basis

3) $\left\{ \frac{1}{5}(3|0\rangle + 4i|1\rangle),\ \frac{1}{5}(4|0\rangle - 3i|1\rangle) \right\}$

$$\frac{1}{5}(3\langle 0| - 4i\langle 1|)\cdot\frac{1}{5}(4|0\rangle - 3i|1\rangle) = \frac{1}{25}\left(12\cancelto{1}{\langle 0|0\rangle} + 12i^2\cancelto{1}{\langle 1|1\rangle}\right)$$

$$= \frac{1}{25}(12 - 12) = \frac{1}{25}(0) = 0 \ \checkmark$$

This is a measurement basis

4) $\left\{ \frac{1}{5}(3|0\rangle + 4i|1\rangle),\ \frac{1}{5}(4|0\rangle + 3i|1\rangle) \right\}$

$$\frac{1}{5}(3\langle 0| - 4i\langle 1|)\cdot\frac{1}{5}(4|0\rangle + 3i|1\rangle) = \frac{1}{25}\left(12\cancelto{1}{\langle 0|0\rangle} - 12i^2\cancelto{1}{\langle 1|1\rangle}\right)$$

$$= \frac{1}{25}(12 + 12) = \frac{24}{25} \neq 0$$

This is not a measurement basis