Ch. 7 - The Description of Curved Spacetime

3-1-19

- Spacetime Geometry is summarized by a line element giving the spacetime distance between 2 nearby points.
- A line element specifies a geometry, but many different line elements describe the same Spacetime geometry.

i.e. : 4D Minkowski Spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Cartesian $\longrightarrow$ Spherical Polar

$$x = r\sin(\theta)\cos(\phi) \;,\quad y = r\sin(\theta)\sin(\phi) \;,\quad z = r\cos(\theta)$$

$$dx = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$

$$\left[ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$ - Same Geometry, different coordinates, different line element.

Question : $ds^2 = -dx^2 + dy^2 + y^2 dz^2 + y^2\sin^2(z)\, dt^2$

$x \to t ,\; y \to r ,\; z \to \theta ,\; t \to \phi$ $\longrightarrow$ t is an angle : Direction holding x is timeline

i.e. 2D Plane in Polar Coordinates

$$ds^2 = dr^2 + r^2 d\phi^2$$

Coordinate transformation :

$$r = \frac{a^2}{r'} \;:\; dr = \frac{-a^2}{r'^2}dr' \;\therefore\; dr^2 = \frac{a^4}{r'^4}dr'^2$$

$$d\sigma^2 = \frac{a^4}{r'^4}dr'^2 + \frac{a^4}{r'^2}d\phi^2 = \frac{a^4}{r'^4}\left[dr'^2 + r'^2 d\phi^2\right]$$

- What happens to the line element when $r' \to 0$? - 'Coordinate singularity'
- Is this a physical singularity ? - No, geometry of a 2D plane.

Metric

In 4D Minkowski Spacetime....

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta \quad w/ \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

is $\eta_{\alpha\beta} = \eta_{\alpha\beta}(x)$?
- No, geometry is independent of location

In a 4D curved Spacetime....

$$ds^2 = g_{\alpha\beta}(x)\,dx^\alpha dx^\beta \qquad w/ \qquad g_{\alpha\beta} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ & g_{11} & g_{12} & g_{13} \\ & & g_{22} & g_{23} \\ & & & g_{33} \end{bmatrix}$$

4x4 Symmetric matrix

$g_{\alpha\beta}$ is a symmetric, position independent matrix called the <u>Metric</u>

$$ds^2 = g_{\alpha\beta}\,dx^\beta dx^\alpha \quad {\scriptstyle \beta\to\alpha,\ \alpha\to\beta} \qquad ds^2 = g_{\beta\alpha}\,dx^\alpha dx^\beta = g_{\alpha\beta}\,dx^\alpha dx^\beta \quad \therefore \quad g_{\alpha\beta} = g_{\beta\alpha}$$

The Summation Convention

Rules for indices ....

1. Location of indices must be respected
   - upper indices : coordinates (x^α)
   - Lower indices : metric $g_{\alpha\beta}$

   For chain Rules

   $$dx^\alpha = \frac{\partial x^\alpha}{\partial x'^\beta}\,dx'^\beta$$

   Superscripts in denominators act as subscripts

2. Repeated indices always occur in Superscript - Subscript pairs

3. Free indices must balance on both sides of the equation

   i.e. $g_{\alpha\beta} = g_{\beta\alpha}$

<u>3-4-19</u>

Last time ....

$$\left[ds^2 = g_{\alpha\beta}(x)\,dx^\alpha dx^\beta\right]$$

Metric (Symmetric : Position Dependent)

i.e.

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

What is the metric?

$$g_{00} = -1,\ g_{11} = 1,\ g_{22} = r^2,\ g_{33} = r^2\sin^2\theta \qquad g_{\mu\nu}\Big|_{\mu\neq\nu} = 0$$

Local Inertial Frames

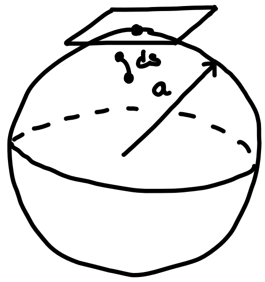
Given a metric $g_{\alpha\beta}(x)$ in 1 system of coordinates @ each point P of spacetime, it is possible to introduce new coordinates x'^α such that

$$\left[g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta}\right]$$

where x_p' have the coordinates locating the point p.

$$\therefore \left.\frac{\partial g_{\alpha\beta}'}{\partial x'}\right|_{x=x_p} = 0 \rightarrow \left[\begin{array}{c} g_{\alpha\beta}'(x_p') = \eta_{\alpha\beta} \\ \left.\frac{\partial g_{\alpha\beta}'}{\partial x'}\right|_{x=x_p} = 0 \end{array}\right] - \text{Local inertial Frame}$$

<u>i.e 7.2</u> The metric of a sphere @ North Pole



$$(*) \quad ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

@ the NP, $\theta = 0$, Can we find the metric of a flat plane?

$$\left[x = a\theta\cos\phi \quad , \quad y = a\theta\sin\phi \right]$$

Find θ & ϕ in terms of x & y : $\left[\tan\phi = \frac{y}{x} \quad : \quad \phi = \tan^{-1}(y/x) \right]$

$$x^2 + y^2 = a^2\theta^2$$

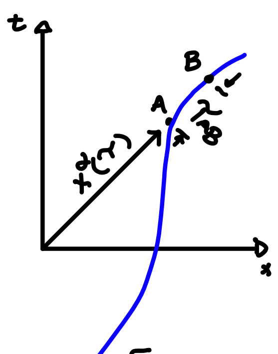
At the N.P, $\theta = 0$: $x = y$ are same ...
in the neighborhood (for same $x : y$)

$$g_{AB}(x,y) = \begin{pmatrix} 1 - 2y^2/3a^2 & 2xy/3a^2 \\ \frac{2xy}{3a^2} & 1 - \frac{2x^2}{3a^2} \end{pmatrix} + \ldots\ldots \quad g_{AB}\Big|_{\substack{x=0 \\ y=0}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; \therefore \; \left.\frac{\partial g_{AB}}{\partial x^c}\right|_{\substack{x=0 \\ y=0}} = 0$$

<u>Light Cones</u> : <u>Worldlines</u>

$$\left[\begin{array}{ll} ds^2 < 0 & \text{Timelike separation} \\ ds^2 = 0 & \text{Null separation} \\ ds^2 > 0 & \text{Spacelike separation} \end{array}\right]$$

Particles move on timelike worldlines



Parametrically specified by four functions $x^\alpha(\tau)$ of the distance τ along them

- In a curved spacetime, the distance between point A : point B along the worldline is ...

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -d\tau^2 \quad \therefore \quad d\tau = \left[-g_{\alpha\beta}(x)dx^\alpha dx^\beta\right]^{1/2}$$

$$\left[\tau_{AB} = \int_A^B \left[-g_{\alpha\beta}(x)dx^\alpha dx^\beta\right]^{1/2} \right]$$

i.e <u>7.3</u> <u>A</u> <u>worldline</u> : <u>Light</u> <u>cones</u> <u>in</u> <u>2D</u>

Consider ....

$$ds^2 = -x^2 dT^2 + dx^2 \quad \text{(2D metric)}$$

$\therefore$ worldline $\quad x(T) = A\cosh(T) \quad dx = A\sinh(T)\,dT$

Light cones are curves w/ $ds^2 = 0$

$$0 = -x^2 dT^2 + dx^2$$

$$\therefore \frac{dT}{dx} = \pm\frac{1}{x} \quad : \quad ds^2 = -A^2\cosh^2(T)dT^2 + A^2\sinh^2(T)dT^2 = -A^2(\cosh^2(T) - \sinh^2(T))dT^2$$

$$= -A^2 dT^2 = -d\tau^2 \quad \therefore \quad d\tau = A\,dT$$

$$\tau = AT + \cancel{c}^{\,0} \qquad \therefore \quad T = \frac{\tau}{A}$$

Worldline may be expressed parametrically as ....

$$\left[T = \tau/A \,, \quad x(\tau) = A\cosh(\tau/A) \right]$$

3-6-19

Length, Area, Volume : Four-Volume of Diagonal Metrics

Consider a general diagonal metric....

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

What is an element of area in x^1-x^2 surface w/ x^0 = constant
x^3 = constant

- dx^1 : dx^2 are the coordinate lengths
- dl^1 : dl^2 are the proper length

$d\theta$ is coordinate length
$r\,d\theta$ is proper length

$$dl^1 = \sqrt{g_{11}}\,dx^1 \,, \quad dl^2 = \sqrt{g_{22}}\,dx^2$$

The element of Area ?

$$dA = dl^1 dl^2 = \sqrt{g_{11}g_{22}}\,dx^1dx^2$$

The element of three-Volume ?

$$dv = dl^1dl^2dl^3 = \sqrt{g_{11}g_{22}g_{33}}\,dx^1dx^2dx^3$$

The element of four-Volume ?

$$dv = \sqrt{-g_{00}g_{11}g_{22}g_{33}}\,dx^0dx^1dx^2dx^3$$

Note:

$$\det g_{\alpha\beta} = g_{00}g_{11}g_{22}g_{33} = g \quad : \quad \left[dv = \sqrt{-g}\,d^4x \right]$$

- General expression for Four-Volume (Even if the metric is not orthogonal)

i.e 7.5

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

What is the element of area on the surface of the sphere ? $dA = \sqrt{g_{22}g_{33}}\,dx^2dx^3$

$dA = \sqrt{g_{22}g_{33}} \cdot dx^2 dx^3$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$, $dx^2 = d\theta$, $dx^3 = d\phi$

$r^2 \sin\theta \, d\theta d\phi \Big|_{r=R} = R^2 \sin\theta \, d\theta d\phi$: Three Volume? $dV = \sqrt{g_{11}g_{22}g_{33}} \, dx^1 dx^2 dx^3$
$dV = r^2 \sin\theta \, dr d\theta d\phi$

i.e. 7.6. Distance, Area : Volume of a homogeneous closed universe

$$ds^2 = \frac{dr^2}{1 - r^2/a^2} + r^2 (d\theta^2 + \sin^2 d\phi^2)$$

Consider a sphere of coordinate radius R centered on $r=0$ in this space....

What is the circumference of the equator on the sphere?
$\theta = \frac{\pi}{2}$, $r = R$

$$ds^2 = R^2 \sin^2(\pi/2) d\phi^2 = R^2 d\phi^2$$

$$C = \oint ds = \int_0^{2\pi} R \, d\phi = R \int_0^{2\pi} d\phi = 2\pi R \quad : \quad C = 2\pi R$$

What is the distance from the center to the surface? (of sphere)
θ = constant = ϕ
$d\theta = d\varphi = \varphi$

$$ds^2 = \frac{dr^2}{1 - r^2/a^2} \quad \therefore \quad ds = \frac{dr}{\sqrt{1 - r^2/a^2}} \quad : \quad S = \int_0^R ds = \int_0^R \frac{dr}{\sqrt{1 - r^2/a^2}}$$

$r = a\sin(\xi)$
$dr = a\cos(\xi) \, d\xi$
$r = 0$, $\xi = 0$
$r = R$, $\xi = \sin^{-1}\left(\frac{R}{a}\right)$

$$S = a \int_0^{\sin^{-1}\left(\frac{R}{a}\right)} d\xi = a\sin^{-1}(R/a) \quad : \quad \boxed{S = a\sin^{-1}(R/a)}$$

What is the area of a two-surface of a sphere?
$r = R$, $dr = 0$

$$dA = R^2 \sin\theta \, d\theta d\varphi \quad : \quad A = \int dA = \int_0^{2\pi} \int_0^{\pi} R^2 \sin\theta \, d\theta d\varphi = 4\pi R^2$$

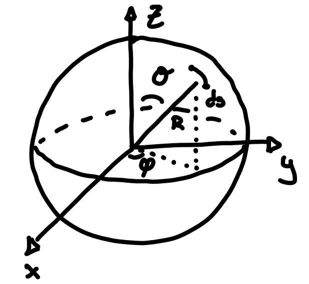
What is the volume inside the 3-sphere?

$$dV = \sqrt{g_{11}g_{22}g_{33}} \, dx^1 dx^2 dx^3 = \sqrt{\frac{1}{1 - r^2/a^2} \cdot r^2 \cdot r^2 \sin^2\theta} \; dr d\theta d\varphi = \frac{r^2 \sin\theta}{\sqrt{1 - r^2/a^2}} \, dr d\theta d\varphi$$

$$V = \int dV = \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{r^2 \sin\theta}{\sqrt{1 - r^2/a^2}} \, dr d\theta d\phi = 4\pi a^3 \left[\frac{1}{2} \sin^{-1}\left(\frac{R}{a}\right) - \frac{R}{2a}\left[1 - \frac{R^2}{a^2}\right]^{\frac{1}{2}} \right]$$

3-8-19

Embedding Diagrams : wormholes
Surface of 2-sphere, embedded in 3D flat space



$$ds^2 = dx^2 + dy^2 + dz^2$$
$$x = R\sin\theta\cos\varphi$$
$$y = R\sin\theta\sin\varphi$$
$$z = R\cos\theta$$
$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Considering the line element...

$$\left[ds^2 = -dt^2 + dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta\, d\varphi) \right]$$ - *Line element of wormhole*

What can you tell me about the geometry?
- 4D (1 Timelike direction & 3 Spacelike dimensions)
- Time-independent (Letting $t \to t + c$ Leaves ds^2 invariant)
- Spherically-Symmetric (const. t, const. r, yields $ds^2 = (R^2+b^2)(d\theta^2 + \sin^2 d\varphi^2)$)
- If $b \to 0$, 4D Minkowski
- $\lim\limits_{r \to \text{Huge}} ds^2 \to ds^2_{\text{Mink}}$

Embedding A Slice of A Wormhole Spacetime

$$ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

For a $t =$ const. Slice of the geometry

- Consider the "Equitorial Slice" $\theta = \pi/2$, $\sin(\pi/2) = 1$, $d\theta = 0$

(*) $$\left[d\Sigma^2 = dr^2 + (b^2+r^2)\, d\varphi^2 \right]$$ where $d\Sigma^2 = ds^2 \Big|_{\substack{t=\text{const.} \\ \theta = \pi/2}}$

Has rotational Symmetry
- $\varphi \to \varphi + $ Const., then $d\Sigma^2$ is invariant

$\Rightarrow$ Embed the slice as an axis Symmetric Surface in 3D flat space.

Flat space in polar coordinates...

$$\left[ds^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \right]^{(**)}$$

We need a function $z(r,\varphi)$ specifying a surface that has the same geometry as (*)

For our Axisymmetric Surface, we can have

$\psi = \varphi$: $z \neq z(\varphi)$, $\rho \neq \rho(\varphi)$

$\therefore z = z(r)$: $\rho = \rho(r)$

$$d\rho = \frac{d\rho}{dr} dr \longrightarrow \therefore d\rho^2 = \left(\frac{d\rho}{dr}\right)^2 dr^2$$

$$dz = \frac{dz}{dr}dr \longrightarrow \therefore \; dz^2 = \left(\frac{dz}{dr}\right)^2 dr^2$$

Plugging these into (**)...

$$ds^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2$$

1) $\rho^2 = b^2 + r^2 \quad \therefore \; \rho = \pm\sqrt{b^2+r^2} \qquad \frac{d\rho}{dr} = \frac{r}{\sqrt{b^2+r^2}}$

2) $\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = 1$

$$\left(\frac{dz}{dr}\right)^2 = 1 - \left(\frac{d\rho}{dr}\right)^2 = 1 - \frac{r^2}{b^2+r^2} = \frac{b^2}{b^2+r^2}$$

$$\frac{dz}{dr} = \pm\frac{b}{\sqrt{b^2+r^2}} \longrightarrow \int dz = \pm\int\frac{b}{\sqrt{b^2+r^2}}dr \rightarrow z = \pm b\int\frac{1}{\sqrt{b^2+r^2}}dr$$

$\sigma = \sinh^{-1}(r/b)$
$r = b\sinh(\sigma)$
$dr = b\cosh(\sigma)\,d\sigma$

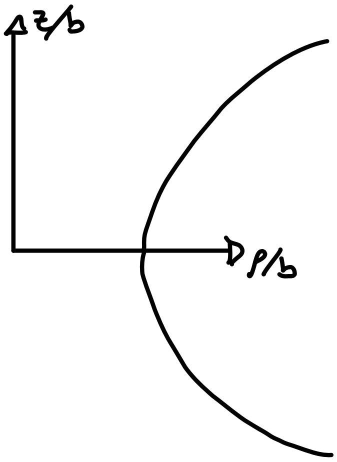
$$z = \pm b\int\frac{b\cosh(\sigma)}{\sqrt{b^2(1+\sinh^2(\sigma))}}d\sigma = \pm b\int\frac{b\cosh(\sigma)}{b\cosh(\sigma)}d\sigma = \pm b\int d\sigma = \pm b\sigma$$

$$z(r) = \pm b\,\mathrm{SINH}^{-1}(r/b) \quad : \quad r = b\,\mathrm{SINH}(z/b)$$

We can eliminate r in favor of ρ (so that $z = z(\rho)$)

$$\rho^2 = b^2 + r^2 = b^2 + b^2\sinh^2(z/b) = b^2\cosh^2(z/b)$$

$$\rho(z) = b\cosh(z/b) \quad \therefore \boxed{z(\rho) = b\cosh^{-1}(\rho/b)}$$



- $0 < r < \infty$ covers $z > 0$
- $z < 0$ covered by letting r range from $-\infty < r < \infty$

3-11-19

Vectors in Curved Spacetime

$$\boxed{\begin{aligned} \underset{\sim}{a} &= \underset{\sim}{a}(x) \\ \underset{\sim}{a}(x) &= a^\beta(x)\, e_\beta(x) \end{aligned}}$$

Components of the Vector $\underset{\sim}{a}$ (#'s)

What is the Scalar product of two vectors $\underset{\sim}{a} : \underset{\sim}{b}$ in curved spacetime

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a^\alpha \underset{\sim}{e}_\alpha) \cdot (b^\beta \underset{\sim}{e}_\beta) = (\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta) a^\alpha b^\beta$$

Orthonormal Bases

- Consists of Four mutually orthogonal vectors of unit length

$$\underset{\sim}{e}_{\hat\alpha} \text{ where } \alpha = 0, 1, 2, 3 \quad \text{so} \quad \underset{\sim}{a}(x) = a^{\hat\beta} \underset{\sim}{e}_{\hat\beta}$$

Where

$$\left[\underset{\sim}{e}_{\hat\alpha}(x) \cdot \underset{\sim}{e}_{\hat\beta}(x) = \eta_{\hat\alpha\hat\beta} = \text{diagonal}(-1, 1, 1, 1) \right]$$

What is the scalar product in an orthonormal basis?

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \eta_{\hat\alpha\hat\beta} a^{\hat\alpha} b^{\hat\beta}$$

Coordinate Bases

$$\left[\underset{\sim}{e}_\alpha(x) \cdot \underset{\sim}{e}_\beta(x) = g_{\alpha\beta}(x) \right]$$

What is the Scalar product?

$$\left[\underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\alpha\beta} a^\alpha b^\beta \right]$$

i.e 7.8 Polar coordinates in the plane

$$\left[ds^2 = dr^2 + r^2 d\varphi^2 \right]$$

A coordinate basis consists of two vectors $\underset{\sim}{e}_r$: $\underset{\sim}{e}_\varphi$ are orthogonal : $g_{r\varphi} = g_{\varphi r} = 0$

What about the magnitude $\underset{\sim}{e}_r$: $\underset{\sim}{e}_\varphi$? $(\underset{\sim}{e}_r \cdot \underset{\sim}{e}_\varphi = 0 : g_{r\varphi} = 0)$

$|e_r|^2 = \underset{\sim}{e}_r \cdot \underset{\sim}{e}_r = g_{rr} = 1 \quad \therefore \quad |e_r| = 1 \quad : \quad |e_\varphi|^2 = \underset{\sim}{e}_\varphi \cdot \underset{\sim}{e}_\varphi = g_{\varphi\varphi} = r^2 \quad \therefore \quad |e_\varphi| = 1$