Taylor Larrechea
Dr. Gustafson
MATH 366
CP 19.3

1) We are given the values of a function $f(x)$ at different points $x_0, x_1, \dots x_n$. We want to find approximate values of the function $f(x)$ for "new" x's that lie between these points for which the function values are given. This process is called interpolation.

2) A standard idea in interpolation now is to find a polynomial $p_n(x)$ of degree n (or less) that assumes the given values, thus

(1) $$p_n(x_0)=f_0, \quad p_n(x_1)=f_1, \quad \dots, \quad p_n(x)=f_n$$

We call this p_n an interpolation polynomial and $x_0, \dots, x_n$ the nodes. And if $f(x)$ is a mathematical function, we call p_n an approximation of f (or a polynomial approximation, because there are other kinds of approximations, as we shall see later). We use p_n to get (approximate) values of f for x's between x_0 and x_n ("interpolation") or sometimes outside this interval $x_0 \leqq x \leqq x_n$ ("extrapolation").

3) The two methods for finding p_n are the Linear Lagrange Interpolation and the Quadratic Lagrange Interpolation. The famous theorem that is mentioned in this part of the reading is Weierstrass Approximation Theorem.

4) $$L_0(x) = \frac{x-x_1}{x_0-x_1}, \quad L_1(x) = \frac{x-x_0}{x_1-x_0}$$

5) (2) $$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{x-x_1}{x_0-x_1}\cdot f_0 + \frac{x-x_0}{x_1-x_0}\cdot f_1$$

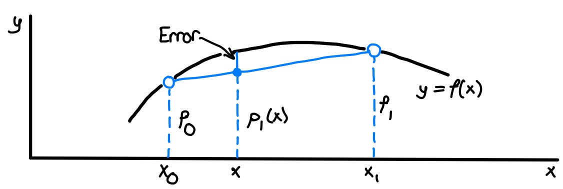

Fig. 431. Linear Interpolation

6) (3a) $$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2$$

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

(3b) $$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

7) Quadratic Lagrange Interpolation
Compute $\ln(9.2)$ by (3) from the data in Example 1 and the additional third value $\ln(11.0) = 2.3979$.

Solution. In (3),

$$L_0(x) = \frac{(x-9.5)(x-11.0)}{(9.0-9.5)(9.0-11.0)} = x^2 - 20.5x + 104.5, \quad L_0(9.2) = 0.5400,$$

$$L_1(x) = \frac{(x-9.0)(x-11.0)}{(9.5-9.0)(9.5-11.0)} = -\frac{1}{0.75}(x^2 - 20x + 99), \quad L_1(9.2) = 0.4800,$$

$$L_2(x) = \frac{(x-9.0)(x-9.5)}{(11.0-9.0)(11.0-9.5)} = \frac{1}{3}(x^2 - 18.5x + 85.5), \quad L_2(9.2) = -0.0200,$$

(see Fig. 433), so that (3a) gives, exact to 4D,

$$\ln(9.2) \approx p_2(9.2) = 0.5400 \cdot 2.1972 + 0.4800 \cdot 2.2513 - 0.0200 \cdot 2.3979 = 2.2192$$

8)

$$a_1 = f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

and in general

$$a_k = f[x_0, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0} \tag{8}$$

9)

$$p_1(x) = f_0 + (x - x_0) f[x_0, x_1]$$

$$p_2(x) = f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

10)

$$\binom{r}{0} = 1, \quad \binom{r}{s} = \frac{r(r-1)(r-2)\dots(r-s+1)}{s!} \qquad (s>0, \text{ integer})$$

11) **Newton's Forward and Backward Interpolations**

Compute a 7D-value of the Bessel function $J_0(x)$ for $x = 1.72$ from the four values in the following table, using (a) Newton's forward formula (14), (b) Newton's backward formula (18).

Solution. The computation of the differences is the same in both cases. Only their notation differs.
(a) **Forward.** In (14) we have $r = (1.72 - 1.70)/0.1 = 0.2$, and j goes from 0 to 3 (see first column). In each column we need the first given number, and (14) thus gives

$$J_0(1.72) \approx 0.3979849 + 0.2(-0.0579985) + \frac{0.2(-0.8)}{2}(-0.0001693) + \frac{0.2(-0.8)(-1.8)}{6} \cdot 0.0004093$$

$$= 0.3979849 - 0.0115997 + 0.0000196 = 0.3864183,$$

which is exact to 6D, the exact 7D-value being 0.3864185