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PHYS 362 HW 16
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Problem 1

a.) $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N$: $S = k\ln[\Omega_n]$

$E = -\mu B \cdot M$: $M = (n_- - n_+)$: $E = \mu B(n_- - n_+) = \mu B(N - n_+ - n_+) = \mu B(N - 2n_+)$: $E = \mu B(N - 2n_+)$

$N = n_- + n_+$

$E = \mu B(N - 2n_+)$: $\frac{E}{\mu B} = N - 2N_+$: $2N_+ = N - \frac{E}{\mu B}$: $N_+ = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)$

$S = k\ln[\Omega(n_+)]$: $\Omega(n_+) = \frac{N!}{n_+!(N-n_+)!}$: $S = k\ln\left(\frac{N!}{n_+!(N-n_+)!}\right)$

$S = k[\ln(N!) - \ln(n_+!(N-n_+)!)] = k[\ln(N!) - \ln(n_+!) - \ln((N-n_+)!)]$

$S = k[\ln(N!) - \ln(n_+!) - \ln((N-n_+)!)]$ $\longrightarrow$ using sterlings approximation

$S = k[N\ln(N) - n_+\ln(n_+) - (N-n_+)\ln((N-n_+))]$

$\frac{\partial S}{\partial E} = \frac{\partial S}{\partial n_+}\cdot\frac{\partial n_+}{\partial E}$: $\frac{\partial n_+}{\partial E} = -\frac{1}{2\mu B}$, $\frac{\partial S}{\partial n_+} = k\ln\left(\frac{N-n_+}{n_+}\right)$

$\frac{\partial S}{\partial n_+} = k\left[-\ln(n_+) - \cancel{\frac{n_+}{n_+}} + \ln((N-n_+)) + \cancel{\frac{(N-n_+)}{(N-n_+)}}\right] = k[\ln((N-n_+)) - \ln(n_+)] = k\ln\left(\frac{N-n_+}{n_+}\right)$

$\frac{1}{T} = -\frac{1}{2\mu B}k\ln\left(\frac{N-n_+}{n_+}\right) = -\frac{k}{2\mu B}\ln\left(\frac{N-n_+}{n_+}\right)$: $\frac{1}{T} = -\frac{k}{2\mu B}\ln\left(\frac{N-n_+}{n_+}\right)$

$N - n_+ = N - \frac{1}{2}\left(N - \frac{E}{\mu B}\right) = N - \frac{N}{2} + \frac{E}{2\mu B} = \frac{N}{2} + \frac{E}{2\mu B} = \frac{1}{2}\left(N + \frac{E}{\mu B}\right)$

$\frac{1}{T} = -\frac{k}{2\mu B}\ln\left(\frac{\cancel{1/2}(N+E/\mu B)}{\cancel{1/2}(N-E/\mu B)}\right) = \frac{k}{2\mu B}\ln\left(\frac{N-E/\mu B}{N+E/\mu B}\right)$

$$\boxed{\frac{1}{T} = \frac{k}{2\mu B}\ln\left(\frac{N-E/\mu B}{N+E/\mu B}\right)}$$

$\frac{2\mu B}{Tk} = \ln\left(\frac{N-E/\mu B}{N+E/\mu B}\right)$: $e^{2\mu B/Tk} = \frac{N-E/\mu B}{N+E/\mu B}$: $Ne^{2\mu B/Tk} + \frac{E}{\mu B}e^{2\mu B/Tk} = N - \frac{E}{\mu B}$

$\frac{E}{\mu B}e^{2\mu B/Tk} + \frac{E}{\mu B} = N - Ne^{2\mu B/Tk}$: $\frac{E}{\mu B}(e^{2\mu B/Tk}+1) = N(1-e^{2\mu B/Tk})$: $E = N\mu B\left(\frac{1-e^{2\mu B/Tk}}{1+e^{2\mu B/Tk}}\right)$

$$\boxed{E = N\mu B\left(\frac{1-e^{2\mu B/Tk}}{1+e^{2\mu B/Tk}}\right)}$$

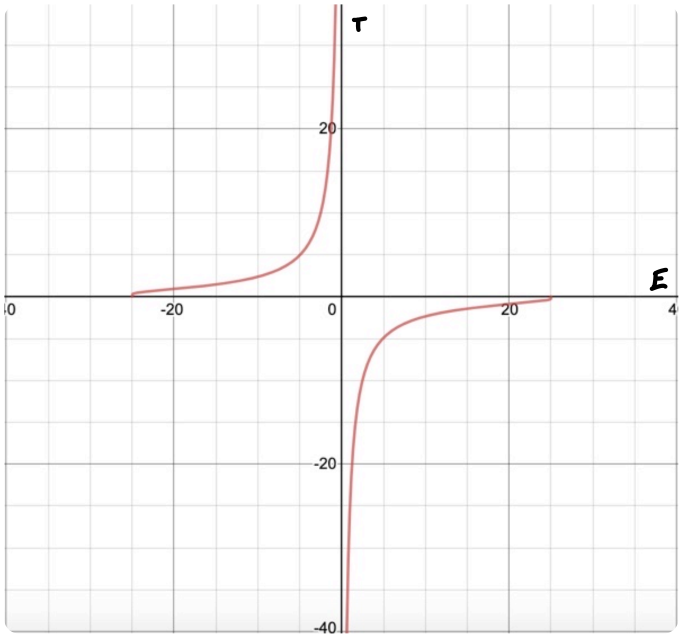
Problem 1 Continued

b.) $E = N\mu B = \mu B(N - 2n_+)$: $E = \mu B(N - 2n_+)$, $n_+ = 0 \to n_+ = N$

$E = N\mu B$, $E = -N\mu B$

$$\boxed{-N\mu B \le E \le N\mu B}$$

$k = \mu B = 1$: $\frac{1}{T} = \frac{1}{2}\ln\left(\frac{N-E}{N+E}\right) \rightsquigarrow T = \frac{2}{\ln(N-E/N+E)}$: $T(E) = \frac{2}{\ln(N-E/N+E)}$



$$\frac{1}{T} = \frac{k}{2\mu B}\ln\left(\frac{N - E/\mu B}{N + E/\mu B}\right) \qquad \frac{2\mu B}{k\ln(N - E/\mu B / N + E/\mu B)} = T$$

$$\frac{2\mu B}{k\ln(N - E/\mu B / N + E/\mu B)} > 1 \quad \text{when} \quad N - E/\mu B > 0 \quad N > \frac{E}{\mu B}$$

$$\frac{2\mu B}{k\ln(N - E/\mu B / N + E/\mu B)} < 1 \quad \text{when} \quad N - E/\mu B < 0 \quad N < \frac{E}{\mu B}$$

$$\boxed{\begin{array}{l} T > 0 \text{ when } N > \frac{E}{\mu B} \\ T < 0 \text{ when } N < \frac{E}{\mu B} \end{array}}$$

c.) $\bar{E} = \sum E_i \cdot P_i$: $\frac{1}{T} = \frac{k}{2\mu B}\ln\left(\frac{n_+}{N - n_+}\right)$: $P_+ = \frac{n_+}{N}$, $P_- = 1 - \frac{n_+}{N}$

$$\frac{2\mu B}{kT} = \ln\left(\frac{n_+}{N - n_+}\right) : e^{2\mu B/Tk} = \frac{n_+}{N - n_+} : Ne^{2\mu B/Tk} - n_+ e^{2\mu B/Tk} = n_+$$

$$Ne^{2\mu B/Tk} = n_+(1 + e^{2\mu B/Tk}) : n_+ = \frac{Ne^{2\mu B/Tk}}{(1 + e^{2\mu B/Tk})}$$

$$P_+ = \frac{e^{2\mu B/Tk}}{1 + e^{2\mu B/Tk}} , \quad P_- = 1 - \frac{e^{2\mu B/Tk}}{1 + e^{2\mu B/Tk}} = \frac{1 + \cancel{e^{2\mu B/Tk}} - \cancel{e^{2\mu B/Tk}}}{1 + e^{2\mu B/Tk}} = \frac{1}{1 + e^{2\mu B/Tk}}$$

Problem 1 continued

$$\bar{E} = -\mu B \cdot \frac{e^{2\mu B/Tk}}{1 + e^{2\mu B/Tk}} + \mu B \cdot \frac{1}{1+e^{2\mu B/Tk}} = \mu B \left(\frac{1 - e^{2\mu B/Tk}}{1 + e^{2\mu B/Tk}} \right)$$

$$\boxed{\bar{E} = \mu B \left(\frac{1 - e^{2\mu B/Tk}}{1 + e^{2\mu B/Tk}} \right)}$$

The energy of the entire system is the same as the average energy $\bar{E}$.

Problem 2

a.) $T = \dfrac{2\mu B}{k \ln(N - E/\mu B / N + E/\mu B)}$: $\cancel{N} - \dfrac{E}{\mu B} > \cancel{N} + \dfrac{E}{\mu B}$ $-\dfrac{2E}{2\mu B} > 0$ $\dfrac{E}{2\mu B} < 0$

$E < 0$: $(N - 2n_+) < 0 \rightsquigarrow \dfrac{N}{2} < n_+$

$\boxed{n_+ > \dfrac{N}{2}}$, This shows that the number of particles in spin up is greater than spin down

b.) $T = \dfrac{2\mu B}{k \ln(n_+/n_-)}$: As T increases, $\ln\left(\dfrac{n_+}{n_-}\right)$ must decrease, this only happens as n_+ approaches n_-. Therefore the $\dfrac{n_+}{n_-}$ must decrease

$\boxed{\dfrac{n_+}{n_-} \text{ must decrease}}$

Problem 3

a.) $T = \dfrac{2\mu B}{k \ln(n_+/n_-)} = \dfrac{2\mu B}{k \ln\left(\frac{n_+}{N-n_+}\right)}$

For $n_+ > n_-$ with certainty, $P_{n_+} = 1$: $P_{n_+} = \dfrac{n_+}{N}$

$$T = \frac{2\mu B}{k \ln\left(\frac{n_+}{N-n_+}\right)} \rightsquigarrow \frac{kT}{2\mu B} = \frac{1}{\ln\left(\frac{n_+}{N-n_+}\right)} \rightsquigarrow \ln\left(\frac{n_+}{N-n_+}\right) = \frac{2\mu B}{kT}$$

$$\ln\left(\frac{N-n_+}{n_+}\right) = -\frac{2\mu B}{kT} \rightsquigarrow \frac{N-n_+}{n_+} = e^{-\frac{2\mu B}{kT}} \rightsquigarrow N = n_+ e^{-\frac{2\mu B}{kT}} + n_+$$

$$N = n_+\left(1 + e^{-\frac{2\mu B}{kT}}\right) \rightsquigarrow \frac{N}{n_+} = 1 + e^{-\frac{2\mu B}{kT}} \rightsquigarrow \frac{n_+}{N} = \frac{1}{1+e^{-\frac{2\mu B}{kT}}}$$

$$1 = \frac{1}{1+e^{-2\mu B/kT}} \rightsquigarrow \cancel{1} + e^{-2\mu B/kT} = \cancel{1} \rightsquigarrow e^{-2\mu B/kT} = 0$$

For $e^{-\frac{2\mu B}{kT}}$ to be equal to zero, the value in the exponent must be infinite: i.e $e^{-\infty} = 0$

This only happens when $T=0$ $\therefore$ $\boxed{T=0 \text{ guarantees } n_+ > n_- \text{ with certainty}}$

b.) For $n_+ = n_-$, $P_{n_+} = \frac{1}{2}$: $P_{n_+} = \dfrac{1}{1+e^{-2\mu B/kT}}$

$$\frac{1}{2} = \frac{1}{1+e^{-2\mu B/kT}} \rightsquigarrow 1 + e^{-2\mu B/kT} = 2 \rightsquigarrow e^{-2\mu B/kT} = 1$$

For $e^{-2\mu B/kT} = 1$, the value in the exponent must be zero. This only happens when T goes to infinity

$$\boxed{T=\infty \text{ guarantees that } n_+ = n_-}$$

c.) It would be better to decrease the temperature so that $n_+ \gg n_-$. This way the signal from the spin up particles will cancel those of spin down. This can only happen if there is a significant more spin up particles in comparison to spin down. This will only happen if the temperature approaches 0.

Problem 4

a.) $\Omega(N,q) \approx \left(\frac{eq}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$, $E = \hbar\omega\left(q + \frac{N}{2}\right)$, $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N$, $S = k\ln(\Omega)$

$\Omega(N,q) \approx e^N\left(\frac{q}{N}\right)^N \frac{1}{\sqrt{2\pi N}}$: $\frac{E}{\hbar\omega} = q + \frac{N}{2}$, $\frac{E}{N\hbar\omega} = \frac{q}{N} + \frac{1}{2}$ $\therefore$ $\frac{q}{N} = \frac{E}{N\hbar\omega} - \frac{1}{2}$

$\Omega(E,q) \approx e^N\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^N \frac{1}{\sqrt{2\pi N}}$

$S = k\ln\left(e^N\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^N \frac{1}{\sqrt{2\pi N}}\right) = k\ln\left(\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^N \frac{e^N}{\sqrt{2\pi N}}\right)$

$S = k\left[\ln\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^N + \ln\left(\frac{e^N}{\sqrt{2\pi N}}\right)\right]$: $\frac{\partial S}{\partial E} = k\frac{1}{\cancel{\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^N}} \cdot \cancel{N}\left(\frac{E}{N\hbar\omega} - \frac{1}{2}\right)^{N-1} \cdot \frac{1}{\cancel{N}\hbar\omega}$

$\frac{\partial S}{\partial E} = \frac{k}{\hbar\omega} \cdot \frac{1}{\frac{E}{N\hbar\omega} - \frac{1}{2}} = \frac{k}{\hbar\omega}\left(\frac{1}{\frac{2E - N\hbar\omega}{2N\hbar\omega}}\right) = \frac{k}{\cancel{\hbar\omega}}\left(\frac{2N\cancel{\hbar\omega}}{2E - N\hbar\omega}\right) = \frac{2kN}{2E - N\hbar\omega}$

$\frac{\partial S}{\partial E} = \frac{kN}{E - \frac{N\hbar\omega}{2}} \rightsquigarrow \frac{1}{T} = \frac{kN}{E - \frac{N\hbar\omega}{2}}$ $\therefore$ $T = \frac{E - \frac{N\hbar\omega}{2}}{kN}$

$$\boxed{T = \frac{E - \frac{N\hbar\omega}{2}}{kN}}$$

b.) $T = \frac{E - \frac{N\hbar\omega}{2}}{kN} \rightsquigarrow NkT = E - \frac{N\hbar\omega}{2} \rightsquigarrow E = NkT + \frac{N\hbar\omega}{2} \rightsquigarrow E = N\left(kT + \frac{\hbar\omega}{2}\right)$

$$\boxed{E = N\left(kT + \frac{\hbar\omega}{2}\right)}$$

$E = \hbar\omega\left(\frac{NkT}{\hbar\omega} + \frac{N}{2}\right)$: $E = \hbar\omega\left(q + \frac{N}{2}\right) \rightsquigarrow q = \frac{NkT}{\hbar\omega}$

This energy has the expected behavior in terms of the number of particles

c.) $E = \hbar\omega\left(q + \frac{N}{2}\right)$, $q \gg N \rightsquigarrow E = q\hbar\omega = NkT$

$E = q\hbar\omega = NkT$ is only valid if $kT \gg \hbar\omega$

$E = NkT = (6.02\times10^{23}\,\text{mol}^{-1})(1.38\times10^{-23}\,\text{J K}\cdot\text{mol}^{-1})(300\,\text{K}) = 2{,}492.28\ \text{J}$

$$\boxed{E = 2{,}492\ \text{J}}$$

Problem 5

a.) Show that $\Omega(N,q) \approx \left(\frac{eN}{q}\right)^q \frac{1}{\sqrt{2\pi q}}$ and $S = kq\left[\ln(N/q)+1\right]$ when $1 \ll q \ll N$

$$\Omega(N,q) \approx \left(1+\frac{q}{N}\right)^N\left(1+\frac{N}{q}\right)^q\sqrt{\frac{N+q}{2\pi Nq}}$$

$$\lim_{N\gg q}\left(1+\frac{N}{q}\right)^q \approx \left(\frac{N}{q}\right)^q, \quad \lim_{N\to\infty}\left(1+\frac{q}{N}\right)^N \approx e^q, \quad \lim_{N\gg q}\sqrt{\frac{N+q}{2\pi Nq}} \approx \sqrt{\frac{N}{2\pi Nq}} \approx \sqrt{\frac{1}{2\pi q}}$$

$$\therefore \quad \Omega(N,q) \approx \left(\frac{N}{q}\right)^q e^q \frac{1}{\sqrt{2\pi q}} \approx \left(\frac{Ne}{q}\right)^q \frac{1}{\sqrt{2\pi q}}$$

$$\boxed{\Omega(N,q) \approx \left(\frac{eN}{q}\right)^q \frac{1}{\sqrt{2\pi q}}}$$

$$S = k\ln(\Omega) = k\ln\left[\left(\frac{eN}{q}\right)^q\frac{1}{\sqrt{2\pi q}}\right] = k\left(\ln\left(\frac{eN}{q}\right)^q + \ln\left(\frac{1}{\sqrt{2\pi q}}\right)\right)$$

$$= k\left(\ln((eN)^q) - \ln(q^q) + \overset{0}{\cancel{\ln(1)}} - \ln(\sqrt{2\pi q})\right) \quad \text{as } N \gg q: \ln(\sqrt{2\pi q}) \approx 0$$

$$= k\left(\ln(e^q) + \ln(N^q) - \ln(q^q)\right) = k\left(q\overset{1}{\cancel{\ln(e)}} + q\ln(N) - q\ln(q)\right)$$

$$= k\left(q + q\left(\ln(N/q)\right)\right) = kq\left(1+\ln(N/q)\right)$$

$$\boxed{S = kq\left(1+\ln\left(\frac{N}{q}\right)\right)}$$

b.) $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N$ $\quad E = \hbar\omega(q+N/2) \rightsquigarrow q = \frac{E}{\hbar\omega} - \frac{N}{2}$: $\frac{N}{2} = \frac{E}{\hbar\omega} - q$ $\quad \therefore N = 2\left(\frac{E}{\hbar\omega} - q\right)$

$$N = 2\left(\frac{E - q\hbar\omega}{\hbar\omega}\right)$$

$$\left(\frac{\partial S}{\partial E}\right)_N = \left(\frac{\partial S}{\partial q}\right)_N\left(\frac{\partial q}{\partial E}\right)_N \; : \; \left(\frac{\partial q}{\partial E}\right)_N = \frac{1}{\hbar\omega}, \quad S = kq + kq\ln\left(\frac{N}{q}\right)$$

$$\left(\frac{\partial S}{\partial q}\right)_N = k + k\ln\left(\frac{N}{q}\right) + kq\left(\frac{-1}{q}\right) = \cancel{k} + k\ln\left(\frac{N}{q}\right) - \cancel{k} = k\ln\left(\frac{N}{q}\right)$$

$$\left(\frac{\partial S}{\partial q}\right)_N = k\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right) \quad \therefore \quad \left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{\hbar\omega}\cdot k\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right)$$

$$\frac{1}{T} = \frac{k}{\hbar\omega}\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right) \quad \therefore \quad \boxed{T = \frac{\hbar\omega}{k}\cdot\frac{1}{\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right)}}$$

Problem 5 Continued

$$T = \frac{\hbar\omega}{k} \cdot \frac{1}{\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right)} \rightsquigarrow \frac{kT}{\hbar\omega} = \frac{1}{\ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right)} \rightsquigarrow \ln\left(\frac{2(E-q\hbar\omega)}{q\hbar\omega}\right) = \frac{\hbar\omega}{kT}$$

$$\frac{2(E-q\hbar\omega)}{q\hbar\omega} = e^{\frac{\hbar\omega}{kT}} \rightsquigarrow (E-q\hbar\omega) = \frac{q\hbar\omega}{2} e^{\frac{\hbar\omega}{kT}} \quad \therefore\ E = q\hbar\omega + \frac{q\hbar\omega}{2} e^{\frac{\hbar\omega}{kT}} = q\hbar\omega\left(1 + \frac{e^{\frac{\hbar\omega}{kT}}}{2}\right)$$

$$\boxed{E = q\hbar\omega\left(1 + \frac{e^{\frac{\hbar\omega}{kT}}}{2}\right)}$$

Problem 6

Relative abundance : $R = \frac{P_1}{P_2} = \frac{\cancel{Z^{-1}} e^{-\beta E_{s_1}}}{\cancel{Z^{-1}} e^{-\beta E_{s_2}}} = \frac{e^{-\beta_1 E_{s_1}}}{e^{-\beta_1 E_{s_2}}}$: $\frac{\Delta E}{k} = 4180\ k$

$\beta = \frac{1}{kT}$

$$R = e^{-\frac{1}{kT}(E_{s_1} - E_{s_2})} = e^{-\frac{\Delta E}{kT}} = e^{\frac{-4180k}{T}}$$

When $T = 300\ k$: $R = e^{\frac{-4180\ k}{300\ k}} = 8.89 \times 10^{-7}$

$$\boxed{R = 8.89 \times 10^{-7}}$$

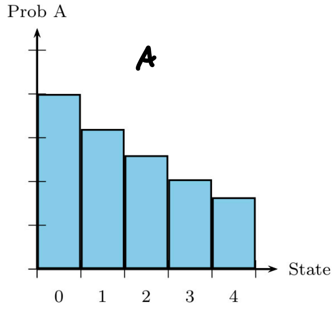
Wheen $T = 1000k$: $R = e^{\frac{-4180k}{1000\ k}} = 0.0153$

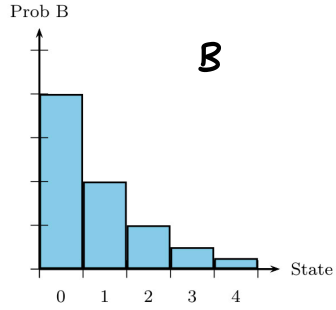
$$\boxed{R = 0.0153}$$

Problem 7

$$P_s = \frac{e^{-E_s/kT}}{Z}$$

$$Z = \sum_i e^{-E_s/kT}$$





$$P_s Z = e^{-E_s/kT} \rightsquigarrow \ln(P_s Z) = \frac{-E_s}{kT} \quad \therefore \quad T = \frac{-E_s}{k\ln(P_s Z)} \quad ; \quad E_s < 0$$

From $T = \frac{-E_s}{k\ln(P_s Z)}$, we can see that smaller differences in probabilities will yield a higher temperature. Therefore plot A will have higher temperatures.

$$\boxed{\text{Plot A}}$$

Problem 8

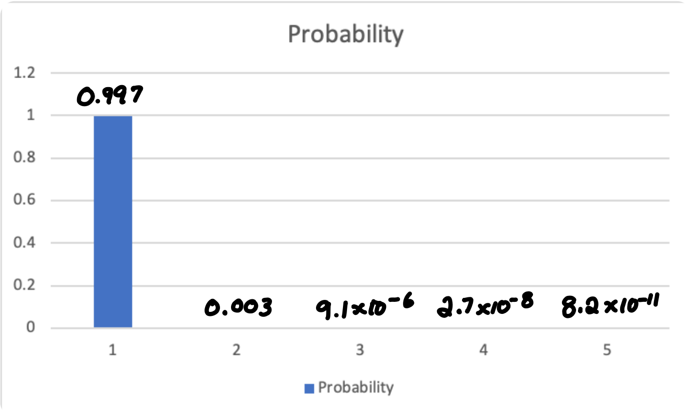
a.) $\epsilon = 0.050\ eV$, $T = 100\ K$

$\epsilon = 8.01 \times 10^{-21}\ J$

$K = 1.88 \times 10^{-23}\ \frac{J}{mol\ K}$

State	Energy	$e^{-E_s/KT}$	Probability $e^{-E_s/KT}/Z$
1	0	1	0.997
2	ϵ	0.003	0.003
3	2ϵ	9.0×10^{-6}	9.1×10^{-6}
4	3ϵ	2.7×10^{-8}	2.7×10^{-8}
5	4ϵ	8.3×10^{-11}	8.2×10^{-11}

$Z = 1.00301$



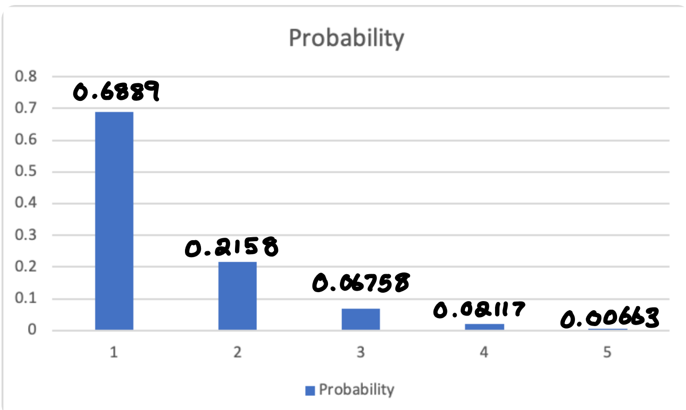
b.) $\epsilon = 0.050\ eV$, $T = 500\ K$

$\epsilon = 8.01 \times 10^{-21}\ J$

$K = 1.88 \times 10^{-23}\ \frac{J}{mol\ K}$

State	Energy	$e^{-E_s/KT}$	Probability $e^{-E_s/KT}/Z$
1	0	1	0.6889
2	ϵ	0.313	0.2158
3	2ϵ	0.098	0.06758
4	3ϵ	0.0307	0.02117
5	4ϵ	0.00962	0.00663

$Z = 1.45332$



Problem 8 continued

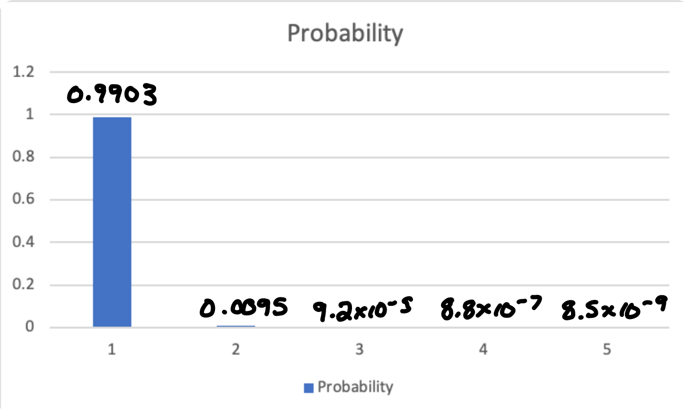
c.) $\epsilon = 0.20\,eV$, $T = 500\,K$

$\epsilon = 3.204 \times 10^{-20}\,J$

$K = 1.38 \times 10^{-23}\,\frac{J}{mol\,K}$

State	Energy	$e^{-E_s/KT}$	Probability $e^{-E_s/KT}/Z$
1	0	1	0.9903
2	ϵ	0.00962	0.0095
3	2ϵ	9.3×10^{-5}	9.2×10^{-5}
4	3ϵ	8.9×10^{-7}	8.8×10^{-7}
5	4ϵ	8.6×10^{-9}	8.5×10^{-9}

$Z = 1.00971$



d.) From the graphs in part A and B we observe an increase in probabilities amongst states 2 through 5 as temperature increases. State 1's probability decreases. Energy is more evenly distributed.

e.) From the graphs in part A, B, and C we obsere a slight increase in probabilities for states 2 through 5 as the energy gap increases. State 1's probability decreases. Energy is less evenly distributed.