MAT 202
Larson – Section 8.3
Trigonometric Integrals

In this section we will learn techniques for evaluating integrals of the form:

$$\int \sin^m x \cos^n x \, dx \ \ and \ \int \sec^m x \tan^n x \, dx$$

where either m or n is a positive integer. The process involves using trigonometric identities so that we can use a u-substitution with the Power Rule.

<u>Recall the following trigonometric identities</u>:

1. Pythagorean Identity:

$$\sin^2 x + \cos^2 x = 1 \quad or \quad \sin^2 x = 1 - \cos^2 x \quad or \quad \cos^2 x = 1 - \sin^2 x$$

2. Power Reducing Identities:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad or \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

<u>Tools for Integrals of the Form</u>: $\int \sin^m x \cos^n x \, dx$

1. If $m=2k+1$ (power of sine) is odd and positive, save one sine factor and convert the remaining factors to cosines. Expand and integrate:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx = \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx$$

2. If $n=2k+1$ (power of cosine) is odd and positive, save one cosine factor and convert the remaining factors to sines. Expand and integrate:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx$$

3. If m & n are both even and positive, make repeated use of the identities $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$ to convert the integrand to odd powers of the cosine. Then proceed to (2) above.

Ex: Integrate the following:

a) $\int \sin^3 x \cos^4 x \, dx$

b) $\int \sin^2 \frac{\pi x}{2} \, dx$

c) $\int \cos^3(\pi x - 1) \, dx$

Ex: Integrate the following:

a) $\int \sin^3 x \cos^4 x \, dx$

Recall: $\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\cos x] = -\sin x$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$= \int \sin^2 x \cdot \cos^4 x \cdot \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^4 x \, \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\int (1 - u^2) u^4 \, du$

$= -\int u^4 - u^6 \, du$

$= -\left[\frac{u^5}{5} - \frac{u^7}{7}\right] + C$

$= \boxed{-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C}$

b) $\int \sin^2 \frac{\pi x}{2} dx$

Power reducing formula:

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\int \frac{1-\cos\left(2\cdot\frac{\pi x}{2}\right)}{2} dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos(\pi x)\, dx$

$= \int \frac{1}{2} dx - \frac{1}{2}\int \cos \pi x\, dx$

$u = \pi x$

$du = \pi\, dx$

$\frac{1}{\pi} du = dx$

$= \frac{1}{2}x - \frac{1}{2\pi}\int \cos u\, du$

$= \frac{1}{2}x - \frac{1}{2\pi}\sin u + C$

$= \boxed{\frac{1}{2}x - \frac{\sin \pi x}{2\pi} + C}$

c) $\int \cos^3(\pi x - 1)\,dx$

$u = \sin(\pi x - 1)$

$du = \pi\cos(\pi x - 1)\,dx$

$\frac{1}{\pi}\,du = \cos(\pi x - 1)\,dx$

$$\int \cos^2(\pi x - 1) \cdot \cos(\pi x - 1)\,dx$$

Pythagorean Identity

$$\int \left(1 - \sin^2(\pi x - 1)\right)\cos(\pi x - 1)\,dx$$

$$\frac{1}{\pi}\int (1 - u^2)\,du$$

$$\frac{1}{\pi}\left[u - \frac{u^3}{3}\right] + C$$

$$\frac{1}{\pi}\left[\sin(\pi x - 1) - \frac{\sin^3(\pi x - 1)}{3}\right] + C$$

$$\boxed{\frac{1}{\pi}\sin(\pi x - 1) - \frac{\sin^3(\pi x - 1)}{3\pi} + C}$$

Recall the following trigonometric identities involving sec & tan:

1. Quotient Identity:

$$\tan x = \frac{\sin x}{\cos x}$$

2. Reciprocal Identity:

$$\sec x = \frac{1}{\cos x}$$

3. Pythagorean Identity:

$$1 + \tan^2 x = \sec^2 x \quad or \quad \tan^2 x = \sec^2 x - 1$$

Tools for Integrals of the Form: $\int \sec^m x \tan^n x \, dx$

1. If $m=2k$ (power of secant) is even and positive, save a secant squared factor and convert the remaining factors to tangents. Expand and integrate:

$$\int \sec^{2k} x \tan^n x \, dx = \int \left(\sec^2 x\right)^{k-1} \tan^n x \sec^2 x \, dx = \int \left(1 + \tan^2 x\right)^{k-1} \tan^n x \sec^2 x \, dx$$

2. If $n=2k+1$ (power of tangent) is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Expand and integrate:

$$\int \sec^m x \tan^{2k+1} x \, dx = \int \sec^{m-1} x \left(\tan^2 x\right)^k \sec x \tan x \, dx = \int \sec^{m-1} x \left(\sec^2 x - 1\right)^k \sec x \tan x dx$$

3. When there are no secant factors and the $n = 2k$ (power of tangent) is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, dx = \int \tan^{n-2} x \left(\tan^2 x\right) dx = \int \tan^{n-2} x \left(\sec^2 x - 1\right) dx$$

4. When the integral is of the form $\int \sec^m x \, dx$ where $m = 2k + 1$ (power of secant) is odd and positive, use ***integration by parts***.

5. When none of the first four guidelines applies, try converting to sines and cosines. Follow the guidelines for sines & cosines.

Ex: Integrate the following:

a) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

b) $\int \sec^4 2x\,dx$

c) $\int \tan^3 2x \sec^3 2x\,dx$

d) $\int \frac{\tan^2 x}{\sec^5 x}\,dx$

Note: In this section, we will skip Wallis's Formulas & integrals involving product to sum formulas

Ex: Integrate the following:

a) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$

Recall:

$\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$1 + \tan^2 x = \sec^2 x$

$\int \sec^{-1/2} x \cdot \tan^3 x \, dx$

$\int \sec^{-1/2} x \cdot \tan^2 x \cdot \tan x \, dx$

$\int \sec^{-1/2} x \, (\sec^2 x - 1) \tan x \, dx$

$\int \frac{\sec^{-1/2} x \, (\sec^2 x - 1) \sec x \tan x}{\sec x} \, dx$

$\int \sec^{-3/2} x \, (\sec^2 x - 1) \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$\int u^{-3/2} (u^2 - 1) \, du$

$\int u^{1/2} - u^{-3/2} \, du$

$\frac{2}{3} u^{3/2} + 2u^{-1/2} + C$

$\boxed{\frac{2}{3} \sec^{3/2} x + 2 \sec^{-1/2} x + C}$

b) $\int \sec^4 2x\, dx$

$u = \tan 2x$

$du = 2\sec^2 2x\, dx$

$\frac{1}{2} du = \sec^2 2x\, dx$

$\int \sec^2 2x \cdot \sec^2 2x\, dx$

$\int (1+\tan^2 2x) \sec^2 2x\, dx$

$\frac{1}{2} \int (1+u^2)\, du$

$\frac{1}{2} \left[u + \frac{u^3}{3} \right] + C$

$\boxed{\frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + C}$

c) $\int \tan^3 2x \sec^3 2x \, dx$

$$\int \tan^2 2x \sec^2 2x \sec 2x \tan 2x \, dx$$

$$\int (\sec^2 2x - 1) \sec^2 2x \sec 2x \tan 2x \, dx$$

$u = \sec 2x$

$du = 2 \sec 2x \tan 2x \, dx$

$\frac{1}{2} du = \sec 2x \tan 2x \, dx$

$$\frac{1}{2}\int (u^2 - 1) u^2 \, du$$

$$\frac{1}{2}\int u^4 - u^2 \, du$$

$$\frac{1}{2}\left[\frac{u^5}{5} - \frac{u^3}{3}\right] + C$$

$$\boxed{\frac{1}{10}\sec^5 2x - \frac{1}{6}\sec^3 2x + C}$$

d) $\displaystyle\int \frac{\tan^2 x}{\sec^5 x}\,dx$

$$\int \frac{\sin^2 x}{\cancel{\cos^3 x}} \cdot \cos^{\cancel{5}\,3} x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$\int \sin^2 x \cos^2 x \cos x \, dx$$

$$\int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$$\int u^2 (1 - u^2) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$