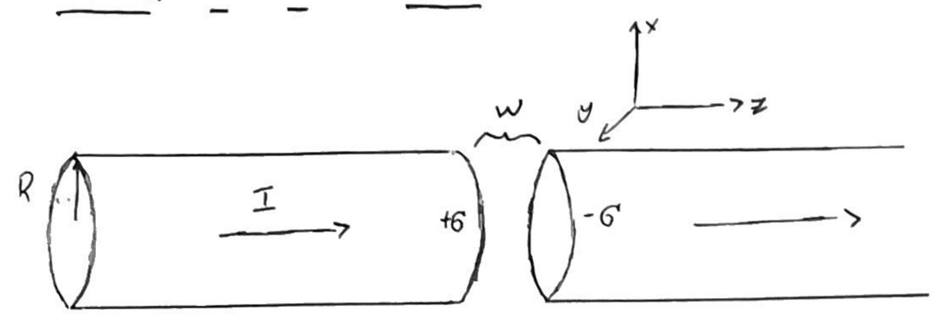
PHYS 311 HOMEWORK SET 13 SOLUTIONS



NOTICE THAT AS THE GAP FORMS A PARALLEL-PLATE CAPACITOR, THE $\hat{E}$-FIELD IS OF THE FORM...

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z} \qquad \text{WHERE} \qquad \sigma = \frac{Q}{\pi R^2}$$

NOW

$$I = \frac{dq}{dt} \qquad \text{SO} \qquad Q(t) = It \quad \text{FOR A CONSTANT CURRENT}$$

SO

$$\vec{E}(t) = \frac{1}{\varepsilon_0}\frac{It}{\pi R^2}\hat{z}$$

NOW THE DISPLACEMENT CURRENT DENSITY IS

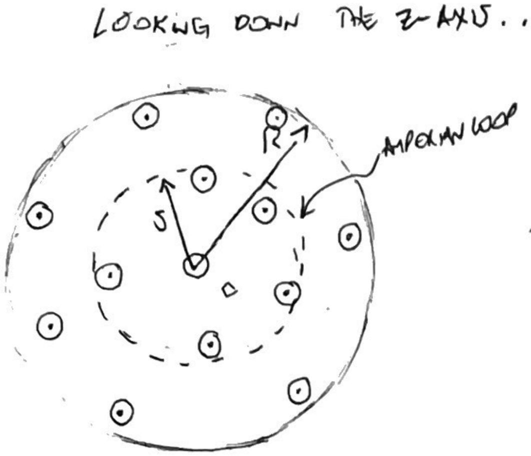
$$\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 \cdot \frac{\partial}{\partial t}\left(\frac{I}{\varepsilon_0 \pi R^2} t\,\hat{z}\right) = \frac{I}{\pi R^2}\hat{z}$$

NOW, AMPERE'S LAW IN INTEGRAL FORM IS...

$$\oint \vec{B}\cdot d\vec{l} = \underbrace{\mu_0 I_{enc}}_{0} + \mu_0\varepsilon_0\int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

ALSO CONSIDER AN AMPERIAN LOOP COAXIAL W/ THE WIRE OF RADIUS $s < R$.

LOOKING DOWN THE z-AXIS..



NOW

$$\mu_0\varepsilon_0\iint \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a} = \mu_0 \int_0^{2\pi}\int_0^{s} \frac{I}{\pi R^2}\hat{z}\cdot s\,ds\,d\varphi\,\hat{z}$$

$$= \frac{\mu_0 I}{\pi R^2}\int_0^{2\pi} d\varphi \int_0^{s} s\,ds = \frac{\mu_0 I}{\pi R^2}\cdot 2\pi \cdot \frac{1}{2}s^2 = \mu_0 I \frac{s^2}{R^2}$$

WHEREAS

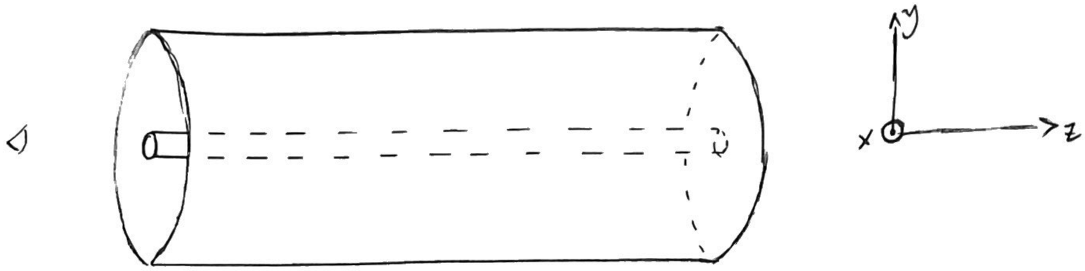
$$\int \vec{B} \cdot d\vec{l} = \int_0^{2\pi} B s \, d\varphi = B 2\pi s$$

$\hookrightarrow$ AS $\vec{B} = B\hat{\varphi}$ & $d\vec{l} = s \, d\varphi \, \hat{\varphi}$

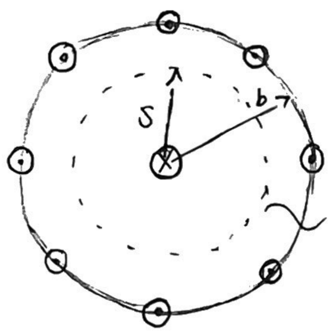
SO AMPERE'S LAW BECOMES...

$$B 2\pi s = \mu_0 I \frac{s^2}{R^2} \quad \text{so} \quad B = \frac{\mu_0 I}{2\pi} \frac{s}{R^2} \quad \text{so} \quad \left[\vec{B} = \frac{\mu_0 I}{2\pi} \frac{s}{R^2} \hat{\varphi} \right]$$

2.



LOOKING DOWN THE AXIS OF THE WIRE...



NOW, FOR $s<b$ NOTICE THAT THE CHANGING $\vec{E}$-FIELD POINTS IN THE $\hat{z}$ DIRECTION.

THE INDUCED $\vec{B}$-FIELD WILL BE CIRCUMFERENTIAL.

a) THE DISPLACEMENT CURRENT IS FOUND VIA...

$$\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{WHERE} \quad \vec{E} = -\frac{\mu_0 I_0 \omega}{2\pi} \ln\left(b/s\right) \cos(\omega t)$$

SO

$$\vec{J}_D = \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\mu_0 I_0 \omega}{2\pi} \ln\left(b/s\right) \cos(\omega t)\hat{z}\right) = -\frac{\mu_0 \varepsilon_0 I_0 \omega}{2\pi} \ln(b/s) \frac{\partial}{\partial t} \cos(\omega t)\hat{z}$$

$$= \frac{\mu_0 \varepsilon_0 I_0 \omega^2}{2\pi} \ln(b/s) \sin(\omega t)\hat{z}$$

b) TO GET THE TOTAL DISPLACEMENT CURRENT...

$$I_D = \int \vec{J}_D \cdot d\vec{a} \qquad = \frac{\mu_0 \varepsilon_0 I_0 \omega^2}{2\pi} \sin(\omega t) \int_0^{2\pi}\int_0^{b} \ln(b/s)\, s\, ds\, d\varphi$$

NOW

$$\vec{J}_D = \frac{\mu_0 \varepsilon_0 I_0 \omega^2}{2\pi} \ln\left(b/s\right) \sin(\omega t)\hat{z}$$

$$d\vec{a} = s\, d\varphi\, ds\, \hat{z}$$

so

$$I_D = \frac{\mu_0 \varepsilon_0 I_0 \omega^2}{2\pi} \sin(\omega t) \int_0^{2\pi} d\varphi \int_0^b \ln(b/s)\, s\, ds \;=\; \mu_0 \varepsilon_0 I_0 \omega^2 \sin(\omega t) \int_0^b \ln(b/s)\, s\, ds$$

now

$$\ln(b/s) = \ln b - \ln s$$

so

$$\int_0^b \ln(b/s)\, s\, ds = \ln b \int_0^b s\, ds - \int_0^b \ln s\; s\, ds = \cancel{\ln b \frac{s^2}{2}\Big|_0^b} - \left(\cancel{\frac{s^2}{2} \ln s}\Big|_0^b - \int \frac{1}{s} ds \cdot \frac{s^2}{2} \right)$$

let $u = \ln s \quad dv = s\,ds$

$du = \frac{1}{s} ds \quad v = \frac{s^2}{2}$

$$= \frac{1}{2} \int_0^b s\, ds = \frac{s^2}{4}\Big|_0^b = \frac{1}{4} b^2$$

so

$$I_D = \frac{1}{4} \mu_0 \varepsilon_0 I_0 \omega^2 b^2 \sin(\omega t)$$

so

$$\boxed{\frac{I_D}{I} = \frac{1}{4} \mu_0 \varepsilon_0 \omega^2 b^2} \qquad \text{since } I = I_0 \sin(\omega t)$$

for $I_D = 0.01\, I$, we need . . .

$$\frac{1}{100} = \frac{1}{4} \mu_0 \varepsilon_0 \omega^2 b^2 \quad \text{so} \quad \omega = \sqrt{\frac{1}{25 \mu_0 \varepsilon_0 b^2}} = \frac{1}{5b} \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{c}{5b}$$

$$= \frac{3.0 \times 10^8 \text{ m/s}}{5(5.0 \times 10^{-3} \text{ m})}$$

so

$$\boxed{\omega = 1.2 \times 10^{10} \text{ s}^{-1}}$$