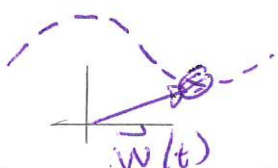
Tues: HW 5pm

Thurs: Read Ex 10.4, 11.1.1, 11.1.2.

Liénard Wiechert Potentials

The potential formalism of electromagnetism as applied to a moving point charge particle eventually gives the framework:

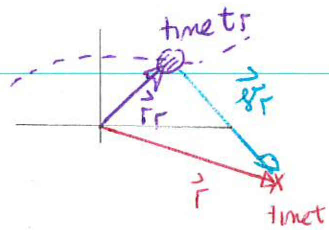
Particle with charge q and trajectory $\vec{w}(t)$



Want field at location $\vec{r}$ at time t. Determine retarded time via

$$c(t-t_r) = |\vec{r}-\vec{w}(t_r)|$$

Determine retarded position $\vec{r}_r = \vec{w}(t_r)$ and retarded separation vector $\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$



Scalar potential

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{c}{\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v}} \quad \text{with} \quad \vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r}$$

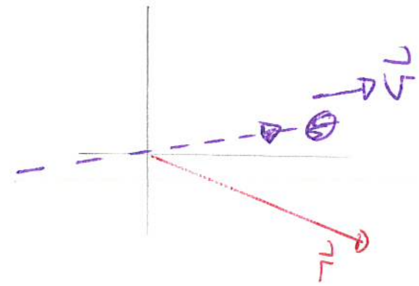
Vector potential

$$\vec{A}(\vec{r},t) = \frac{\vec{v}}{c^2}V(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\frac{\vec{v}}{\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v}}$$

Liénard-Weichert potentials.

Particle moving with constant velocity

Consider the potentials produced by a particle moving with constant velocity $\vec{v}$. To simplify the discussion assume that the particle passes through the origin at $t=0$. Then



$$\vec{w} = \vec{v}t$$

We first obtain the retarded time and show

$$t_r = \frac{c^2 t - \vec{r}\cdot\vec{v} - \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

Derivation: $\quad c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$

$$= |\vec{r} - \vec{v}t_r|$$

$$\Rightarrow \quad c^2 (t - t_r)^2 = |\vec{r} - \vec{v}t_r|^2 = (\vec{r} - \vec{v}t_r)\cdot(\vec{r} - \vec{v}t_r)$$

$$\Rightarrow \quad c^2 [t^2 - 2tt_r + t_r^2] = r^2 - v^2 t_r^2 - 2\vec{v}\cdot\vec{r}\, t_r$$

$$\Rightarrow \quad t_r^2 [c^2 - v^2] + t_r [-2c^2 t + 2\vec{v}\cdot\vec{r}] + c^2 t^2 - r^2 = 0$$

$$\Rightarrow \quad t_r = \frac{2c^2 t - 2\vec{v}\cdot\vec{r} \pm \sqrt{4(c^2 t - \vec{v}\cdot\vec{r})^2 - 4(c^2 t^2 - r^2)(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$= \frac{(c^2 t - \vec{v}\cdot\vec{r}) \pm \sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

By the previous argument only the $-$ sign gives $t_r < t$. ☒

We need this to calculate $\vec{\mathscr{r}}_r$ and $\mathscr{r}_r$. We get

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}\,t_r$$

and

$$\mathscr{r}_r = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

Then

$$\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r = \sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

Derivation:

$$\vec{v}\cdot\vec{\mathscr{r}}_r = \vec{v}\cdot\vec{r} - v^2 t_r$$

$$\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r = c^2(t - t_r) - \vec{v}\cdot\vec{r} + v^2 t_r$$

$$= c^2 t - \vec{v}\cdot\vec{r} + t_r(v^2 - c^2)$$

$$= \cancel{c^2 t^2 - \vec{v}\cdot\vec{r}} - (\cancel{c^2 t - \vec{v}\cdot\vec{r}}) + \sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

∎

Thus

$$\boxed{V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{\sqrt{(c^2 t - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}}$$

Point source
Constant $\vec{v}$

$$\vec{A}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 c}\,\frac{\vec{v}}{\sqrt{}}$$

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

$$\Rightarrow \epsilon_0 = \frac{1}{\mu_0 c^2}$$

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{qc\vec{v}}{\sqrt{(c^2 t^2 - \vec{v}\cdot\vec{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}}$$

Point source
Constant $\vec{v}$

1 Potentials for a particle moving with constant velocity.

A charged particle travels with constant velocity along the $+x$ axis, passing the origin at $t = 0$.

a) Determine the scalar and vector potentials at any point along the y axis.

b) Determine the potentials for the case where the velocity is zero. Are these consistent with those obtained by electrostatics and magnetostatics?

c) Determine the potentials for the case where $v \ll c$.

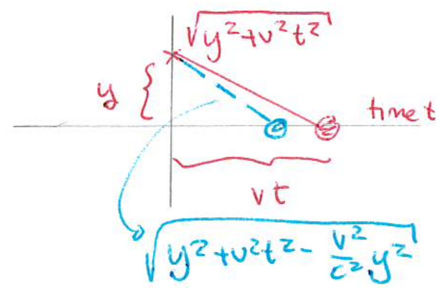
Answer: a)

$$\left.\begin{aligned}\vec{v} &= v\hat{x}\\ \vec{r} &= y\hat{y}\end{aligned}\right\} \Rightarrow \quad \vec{v}\cdot\vec{r} = 0$$

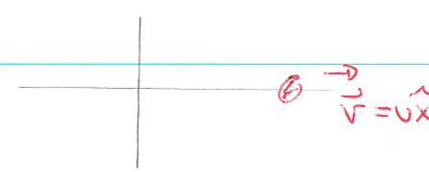
$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{\sqrt{c^4t^2 + (c^2-v^2)(r^2-c^2t^2)}}$$

$$= \frac{qc}{4\pi\epsilon_0} \frac{1}{\sqrt{(c^2-v^2)r^2 + v^2c^2t^2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(1-v^2/c^2)y^2 + v^2t^2}}$$



$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qv\,\hat{x}}{\sqrt{(1-v^2/c^2)y^2 + v^2t^2}}$$



b) $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0|y|}$ Yes

$\vec{A}(\vec{r},t) = 0$ Yes.

c) $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0} \dfrac{1}{\sqrt{y^2+v^2t^2}}$ $\}\rightarrow$ distance to current particle location

~ electrostatic

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qv\,\hat{x}}{\sqrt{y^2+v^2t^2}}$$

Fields produced by a moving point charge particle.

Given the potentials produced by a moving point charge we can calculate the fields by

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

where the spatial derivatives are w.r.t unprimed co-ordinates. We need to identify terms in which these appear in the potentials. These are:

Primary source for unprimed co-ordinates

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

explicit

→ Appears explicitly in

$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$

and $\mathscr{r}_r = |\vec{\mathscr{r}}_r|$

→ Appears implicitly in t_r since

$c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$

→ Appears implicitly in

$\vec{v} = \vec{v}(t_r)$

Each of these: $\vec{r}, \vec{\mathscr{r}}_r, \mathscr{r}_r, \vec{v}$ is therefore not necessarily constant when differentiation with respect to x, y, z arises.

The relevant derivatives appear as follows (Proofs Later)

Lemma 1

$$\boxed{\vec{\nabla} t_r = \frac{-\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r}}$$

Lemma 2

$$\vec{\nabla}(\vec{\mathscr{r}}_r\cdot\vec{v}) = \vec{v} + (\vec{\mathscr{r}}_r\cdot\vec{a} - v^2)\,\vec{\nabla} t_r$$

where the retarded acceleration is

$$\vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$$

Lemma 3

$$\boxed{\frac{\partial t_r}{\partial t} = \frac{\mathscr{r}_r c}{c\mathscr{r}_r - \vec{v}\cdot\vec{\mathscr{r}}_r}}$$

Lemma 4

$$\boxed{\frac{\partial}{\partial t}(\vec{v}\cdot\vec{\mathscr{r}}_r) = \frac{\partial t_r}{\partial t}(\vec{\mathscr{r}}_r\cdot\vec{a} - v^2)}$$

Lemma: 1: $\vec{\nabla} t_r = \dfrac{-\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v}}$

Proof: The retarded time depends implicitly on $\vec{r}$. Specifically

$$c(t-t_r) = \mathscr{r}_r$$

$$\Rightarrow c(t-t_r) = (\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r)^{1/2}$$

$$\Rightarrow \vec{\nabla}(c(t-t_r)) = \vec{\nabla}(\ldots)^{1/2}$$

$$\Rightarrow -c\vec{\nabla} t_r = \frac{1}{2}\frac{1}{[\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r]^{1/2}} \vec{\nabla}(\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r)$$

$$\Rightarrow -c\vec{\nabla} t_r = \frac{1}{2\mathscr{r}_r}\left\{ \vec{\mathscr{r}}_r \times (\vec{\nabla} \times \vec{\mathscr{r}}_r) + \vec{\mathscr{r}}_r \times (\vec{\nabla} \times \vec{\mathscr{r}}_r) + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})(\vec{\mathscr{r}}_r) \right\}$$

$$\Rightarrow -c\vec{\nabla} t_r = \frac{1}{\mathscr{r}_r}\left[\vec{\mathscr{r}}_r \times (\vec{\nabla} \times \vec{\mathscr{r}}_r) + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r \right]$$

We will show:

A: $\vec{\nabla} \times \vec{\mathscr{r}}_r = \vec{v} \times \vec{\nabla} t_r$

B: $(\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r - \vec{v}\,[\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r)]$

First to prove A:

$$\vec{\nabla} \times \vec{\mathscr{r}}_r = \vec{\nabla} \times (\vec{r} - \vec{w}(t_r))$$

$$= \cancel{\vec{\nabla} \times \vec{r}} - \vec{\nabla} \times \vec{w}(t_r)$$

$$= -\vec{\nabla} \times \vec{w}(t_r)$$

But $\vec{\nabla}\times\vec{W} = \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)\hat{z} + \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right)\hat{x} + \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}\right)\hat{y}$

and
$$\frac{\partial w_y}{\partial x} = \frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial x}$$
$$= \left.\frac{dw_y}{dt}\right|_{t_r}\frac{\partial t_r}{\partial x} \qquad \text{etc,} \ldots$$

gives:
$$\vec{\nabla}\times\vec{W} = \left(\frac{dw_y}{dt}\frac{\partial t_r}{\partial x} - \frac{dw_x}{dt}\frac{\partial t_r}{\partial y}\right)\hat{z} + \ldots$$
$$= \left[\left.\frac{dw_y}{dt}\right|_{t_r}(\vec{\nabla}t_r)_x - \left.\frac{dw_x}{dt}\right|_{t_r}(\vec{\nabla}t_r)_y\right]\hat{z} + \ldots$$
$$= \vec{\nabla}(t_r)\times\left.\frac{d\vec{w}}{dt}\right|_{t_r}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{\mathscr{r}}_r = -\vec{\nabla}(t_r)\times\vec{v} = \vec{v}\times\vec{\nabla}(t_r)$$

and this proves A

To prove (B): $\quad (\vec{\mathscr{r}}_r\cdot\vec{\nabla})\,\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r\cdot\vec{\nabla}\,(\vec{r} - \vec{w}(t_r))$.

Then for any vector $\vec{a}$ it is easily shown that $(\vec{a}\cdot\vec{\nabla})\vec{r} = \vec{a}$.

Thus
$$(\vec{\mathscr{r}}_r\cdot\vec{\nabla})\,\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r - \vec{\mathscr{r}}_r\cdot\vec{\nabla}(\vec{w}(t_r))$$

Now
$$\vec{b}\cdot\vec{\nabla}(\vec{w}(t_r)) = b_x\frac{\partial}{\partial x}\vec{w}(t_r) + b_y\frac{\partial}{\partial y}\vec{w}(t_r) + b_z\frac{\partial}{\partial z}\vec{w}(t_r)$$
$$= b_x\left.\frac{d\vec{w}}{dt}\right|_{t_r}\frac{\partial t_r}{\partial x} + b_y\left.\frac{d\vec{w}}{dt}\right|_{t_r}\frac{\partial t_r}{\partial y} + \ldots$$
$$= \frac{d\vec{w}}{dt}\left[\vec{b}\cdot\vec{\nabla}(t_r)\right].$$

Thus.

$$(\vec{\mathscr{r}} \cdot \vec{\nabla})\,\vec{\mathscr{r}} = \vec{\mathscr{r}} - \frac{d\vec{w}}{dt}\Big|_{t_r} \vec{\mathscr{r}}\cdot\vec{\nabla}(t_r)$$

$$= \vec{\mathscr{r}} - \vec{v}\,(\vec{\mathscr{r}}\cdot\vec{\nabla}(t_r))$$

which proves B. So

$$-c\,\vec{\nabla}(t_r) = \frac{1}{\mathscr{r}}\left\{\vec{\mathscr{r}}\times\left[\vec{v}\times\vec{\nabla}(t_r)\right] + (\vec{\mathscr{r}}\cdot\vec{\nabla})\,\vec{\mathscr{r}}\right\}.$$

$$= \frac{1}{\mathscr{r}}\left\{\vec{v}\,\cancel{(\vec{\mathscr{r}}\cdot\vec{\nabla}(t_r))} - \vec{\nabla}(t_r)(\vec{v}\cdot\vec{\mathscr{r}}) + \vec{\mathscr{r}} - \cancel{\vec{v}\,(\vec{\mathscr{r}}\cdot\vec{\nabla}(t_r))}\right\}$$

$$\Rightarrow c\,\vec{\nabla}(t_r) = \frac{1}{\mathscr{r}}\left\{\vec{\nabla}(t_r)(\vec{v}\cdot\vec{\mathscr{r}}) - \vec{\mathscr{r}}\right\}$$

$$\Rightarrow \vec{\nabla}(t_r)\left[\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}}\right] = -\vec{\mathscr{r}}$$

$$\Rightarrow \vec{\nabla}(t_r) = \frac{-\vec{\mathscr{r}}}{\mathscr{r}c - \vec{v}\cdot\vec{\mathscr{r}}} \qquad \blacksquare$$

Lemma 2 $\quad \vec{\nabla}(\vec{\imath}_r \cdot \vec{v}) = \vec{v} + (\vec{\imath}_r \cdot \vec{a} - v^2)\vec{\nabla}(t_r)$

where

$$\vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}.$$

Proof. $\quad \underset{\text{evaluated at } t_r}{\vec{\nabla}(\vec{\imath}_r \cdot \vec{v})} = \vec{\imath}_r \times (\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{\imath}_r) + (\vec{\imath}_r\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\cdot\vec{\imath}_r$

We saw in lemma 1 that.

$$\vec{\nabla}\times\vec{\imath}_r = \vec{v}\times\vec{\nabla}(t_r)$$

Then $\quad \vec{\nabla}\times\vec{v} = \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right)\hat{x} + \cdots$

$$= \left(\left.\frac{dv_y}{dt}\right|_{t_r}\frac{\partial t_r}{\partial z} - \left.\frac{dv_z}{dt}\right|_{t_r}\frac{\partial t_r}{\partial y}\right)\hat{x} + \cdots$$

$$= \left[a_y(\vec{\nabla}t_r)_z - a_z(\vec{\nabla}t_r)_y\right]\hat{x} + \cdots$$

$$= \vec{\nabla}(t_r)\times\vec{a}$$

So $\quad \vec{\nabla}(\vec{\imath}_r\cdot\vec{v}) = \vec{\imath}_r\times\left[\vec{\nabla}(t_r)\times\vec{a}\right] + \vec{v}\times\left[\vec{v}\times\vec{\nabla}(t_r)\right] + (\vec{\imath}_r\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\cdot\vec{\imath}_r$

Then $\quad (\vec{\imath}_r\cdot\vec{\nabla})\vec{v} = \imath_{rx}\frac{\partial}{\partial x}\vec{v} + \imath_{ry}\frac{\partial}{\partial y}\vec{v} + \cdots$

$$= \imath_{rx}\left.\frac{d\vec{v}}{dt}\right|_{t_r}\frac{dt_r}{dx} + \cdots$$

$$= \left[\vec{\imath}\cdot\vec{\nabla}(t_r)\right]\vec{a}$$

Also: $(\vec{v} \cdot \vec{\nabla}) \vec{\mathscr{r}}_r = v_x \frac{\partial}{\partial x} \vec{\mathscr{r}}_r + v_y \frac{\partial}{\partial y} \vec{\mathscr{r}}_r + \ldots$ and $\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$

$$= v_x \hat{x} - \left[\vec{v} \cdot \vec{\nabla}(t_r)\right] v_x \hat{x} + \ldots$$

$$= \vec{v} - \left[\vec{v} \cdot \vec{\nabla}(t_r)\right] \vec{v}$$

Thus: $\vec{\nabla}(\vec{\mathscr{r}}_r \cdot \vec{v}) = \left[\vec{\nabla}(t_r)\right] \vec{\mathscr{r}}_r \cdot \vec{a} - \vec{a}\cancel{\left[\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r)\right]}$

$$+ \cancel{\vec{v}\left[\vec{v} \cdot \vec{\nabla}(t_r)\right]} - \vec{\nabla}(t_r)\, v^2$$

$$+ \cancel{\vec{a}\left[\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r)\right]} + \vec{v} - \cancel{\vec{v}\left[\vec{v} \cdot \vec{\nabla}(t_r)\right]}$$

$$= \vec{v} + \vec{\nabla}(t_r)\left[\vec{\mathscr{r}}_r \cdot \vec{a} - v^2\right]$$

This proves the result. $\boxtimes$

Lemma 3 $\quad c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$

$$\Rightarrow \quad c^2(t-t_r)^2 = r^2 + w^2(t_r) - 2\vec{r}\cdot\vec{w}(t_r)$$

Differentiate w.r.t. $t \Rightarrow$

$$c^2 \cancel{2}(t-t_r)\left[1 - \frac{\partial t_r}{\partial t}\right] = \cancel{2} w \left.\frac{dw}{dt}\right|_{t_r} \frac{\partial t_r}{\partial t} - \cancel{2}\vec{r}\cdot \left.\frac{d\vec{w}}{dt}\right|_{t_r} \frac{\partial t_r}{\partial t}$$

$$\Rightarrow \quad c^2(t-t_r) = \frac{\partial t_r}{\partial t}\left[c^2(t-t_r) + \vec{w}\cdot\vec{v} - \vec{r}\cdot\vec{v}\right]$$

$$\Rightarrow \quad c\,\mathscr{r}_r = \frac{\partial t_r}{\partial t}\left[c^2(t-t_r) - \vec{v}\cdot\vec{\mathscr{r}}_r\right]$$

$$\Rightarrow \quad \frac{\partial t_r}{\partial t} = \frac{c\,\mathscr{r}_r}{c\,\mathscr{r}_r - \vec{v}\cdot\vec{\mathscr{r}}_r} \qquad \boxed{2}$$

Lemma 4 $\quad \frac{\partial}{\partial t}(\vec{v}\cdot\vec{\mathscr{r}}_r) = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\underbrace{\frac{\partial \vec{v}}{\partial t_r}}_{\vec{a}}\frac{\partial t_r}{\partial t}} \cdot \vec{\mathscr{r}}_r + \vec{v}\cdot\frac{\partial \vec{\mathscr{r}}_r}{\partial t}$

$$= \vec{a}\cdot\vec{\mathscr{r}}_r \frac{\partial t}{\partial t_r} + \vec{v}\left(-\frac{\partial \vec{w}}{\partial t}\right)$$

$$= \left(\vec{a}\cdot\vec{\mathscr{r}}_r - \vec{v}\cdot\vec{v}\right)\frac{\partial t}{\partial t_r}$$

$$= \left(\vec{\mathscr{r}}_r\cdot\vec{a} - v^2\right)\frac{\partial t}{\partial t_r} \qquad \boxed{3}$$

Electric field

Consider first

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \vec{\nabla}(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^{-1}$$

$$= \frac{-qc}{4\pi\epsilon_0}(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^{-2}\left[c\vec{\nabla}\mathscr{r}_r - \vec{\nabla}(\vec{v}\cdot\vec{\mathscr{r}}_r)\right]$$

Then $\vec{\nabla}\mathscr{r}_r = \vec{\nabla} c(t - t_r)$

$$= -c\vec{\nabla} t_r = \frac{c\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r}$$

$$\Rightarrow \vec{\nabla} V = -\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^2}\left[\frac{c^2\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r} - \vec{\nabla}(\vec{v}\cdot\vec{\mathscr{r}}_r)\right]$$

$$= \frac{-qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^2}\left[\frac{c^2\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r} - \vec{v} + \frac{\vec{\mathscr{r}}_r(\vec{\mathscr{r}}_r\cdot\vec{a} - v^2)}{(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)}\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^3}\left[\vec{v}(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r) - c^2\vec{\mathscr{r}}_r + v^2\vec{\mathscr{r}}_r - \vec{\mathscr{r}}_r(\vec{\mathscr{r}}_r\cdot\vec{a})\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)^3}\left[\vec{\mathscr{r}}_r(v^2 - c^2 - \vec{\mathscr{r}}_r\cdot\vec{a}) + \vec{v}(\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)\right]$$

Second

$$\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 qc}{4\pi}\frac{\partial}{\partial t}\frac{\vec{v}}{\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v}}$$

$$= \frac{\mu_0 qc}{4\pi}\frac{\frac{\partial\vec{v}}{\partial t}(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v}) - \vec{v}\frac{\partial}{\partial t}(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v})}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v})^2}$$

Now $$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial t_r}\frac{\partial t_r}{\partial t} = \vec{a}\,\frac{\partial t_r}{\partial t}$$

$$\frac{\partial}{\partial t}\mathscr{r}_r c = c\,\frac{\partial \mathscr{r}_r}{\partial t} = c\,\frac{\partial}{\partial t}\,c(t-t_r)$$

$$= c^2\left[1-\frac{\partial t_r}{\partial t}\right]$$

etc,...

Combining these gives:

$$\frac{\partial \vec{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v})^3}\left\{(\mathscr{r}_r c - \vec{\mathscr{r}}\cdot\vec{v})\left(\frac{\mathscr{r}_r\vec{a}}{c}-\vec{v}\right) + \frac{\mathscr{r}_r}{c}(c^2-v^2+\vec{\mathscr{r}}_r\cdot\vec{a})\,\vec{v}\right\}$$

Now

$$\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t} = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v})^3}\left\{(\mathscr{r}_r c - \vec{\mathscr{r}}\cdot\vec{v})\left(\frac{\mathscr{r}_r\vec{a}}{c}\right) + (c^2-v^2+\vec{\mathscr{r}}_r\cdot\vec{a})\left(\frac{\mathscr{r}_r}{c}\vec{v}-\vec{\mathscr{r}}_r\right)\right\}$$

$$\Rightarrow \quad \vec{E} = \frac{-q}{4\pi\epsilon_0}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec{v})^3}\left\{\mathscr{r}_r\vec{a}(\mathscr{r}_r c - \vec{\mathscr{r}}\cdot\vec{v}) + (c^2-v^2+\vec{\mathscr{r}}_r\cdot\vec{a})(\mathscr{r}_r\vec{v} - c\vec{\mathscr{r}}_r)\right\}$$

We let

$$\boxed{\vec{u} = c\hat{\mathscr{r}}_r - \vec{v}}$$

Thus

$$\vec{E} = \frac{-q}{4\pi\epsilon_0}\,\frac{1}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3}\left\{\mathscr{r}_r\vec{a}\,(\vec{\mathscr{r}}_r\cdot\vec{u}) - (c^2-v^2+\vec{\mathscr{r}}_r\cdot\vec{a})\,\mathscr{r}_r\vec{u}\right\}$$

$$= \frac{q}{4\pi\epsilon_0}\,\frac{1}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3}\left\{(c^2-v^2)\,\mathscr{r}_r\vec{u} - \underbrace{\left[\mathscr{r}_r\vec{a}\,(\vec{\mathscr{r}}_r\cdot\vec{u}) - (\vec{\mathscr{r}}_r\cdot\vec{a})\,\mathscr{r}_r\vec{u}\right]}_{\mathscr{r}_r\,\vec{\mathscr{r}}_r\times(\vec{a}\times\vec{u})}\right\}$$

This gives:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3} \left\{ (c^2 - v^2)\,\vec{u} + \vec{\mathscr{r}}_r \times (\vec{u} \times \vec{a}) \right\}.$$

Now using similar vector calculus for $\vec{B} = \vec{\nabla} \times \vec{A}$ we get

$$\vec{B} = \frac{1}{c}\left(\hat{\mathscr{r}}_r \times \vec{E}\right)$$

Thus the process is:

Given $\vec{w}(t)$ → Determine retarded time via $c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$

→ Determine retarded separation vector $\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$ → Determine $\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r}$ and $\vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$

↓

Determine $\vec{u} = c\,\hat{\mathscr{r}}_r - \vec{v}$

↓

Fields:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r \cdot \vec{u})^3} \left[(c^2 - v^2)\,\vec{u} + \vec{\mathscr{r}}_r \times (\vec{u} \times \vec{a}) \right]$$

$$\vec{B} = \frac{1}{c}\,\hat{\mathscr{r}}_r \times \vec{E}$$

2 Fields produced by a charge moving with constant velocity

Consider a charged particle moving with constant velocity. Thus

$$\mathbf{w}(t) = \mathbf{v}\, t$$

where $\mathbf{v}$ is the velocity.

a) Show that

$$\mathbf{u} = \frac{\mathbf{r} - \mathbf{v}\, t}{t - t_r}.$$

b) Determine an expression for the electric field.

c) Determine an expression for the magnetic field.

Answer: a) $\vec{u} = c\hat{\mathscr{r}}_r - \vec{v}$

$$= \frac{1}{\mathscr{r}_r} c\vec{\mathscr{r}}_r - \vec{v}$$

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}\, t_r$$

$$\Rightarrow \vec{u} = \frac{c}{\mathscr{r}_r}\vec{r} - \frac{c}{\mathscr{r}_r}\vec{v}\, t_r - \vec{v} \qquad \mathscr{r}_r = c\,(t - t_r)$$

$$= \frac{\vec{r} - \vec{v}\, t_r}{t - t_r} - \vec{v} = \frac{\vec{r} - \cancel{\vec{v} t_r} - vt + \cancel{\vec{v} t_r}}{t - t_r}$$

$$= \frac{\vec{r} - \vec{v}t}{t - t_r}$$

b) Here $\vec{a} = 0$. Thus:

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r \cdot \vec{u})^3}(c^2 - v^2)\, \vec{u}$$

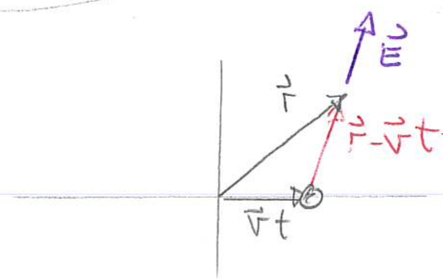
Then $\vec{\mathscr{r}}_r \cdot \vec{u} = (c\, \mathscr{r}_r - \vec{v} \cdot \vec{\mathscr{r}}_r)$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(c\,\mathscr{r}_r - \vec{\mathscr{r}}_r \cdot \vec{v})^3} (c^2 - v^2) \frac{\vec{r} - \vec{v}t}{t - t_r}$$

$$\Rightarrow \quad \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c\,\cancel{(t-t_r)}}{(c\mathscr{r}_r - \vec{\mathscr{r}}_r\cdot\vec{v})^3} \frac{(c^2-v^2)(\vec{r}-\vec{v}t)}{\cancel{t-t_r}}$$

$$\Rightarrow \quad \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c\,(c^2-v^2)\,(\vec{r}-\vec{v}t)\,\mathscr{r}_r}{c^3\,(\mathscr{r}_r - \vec{\mathscr{r}}_r\cdot\vec{v}/c)^3}$$

$$\Rightarrow \quad \boxed{\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r^2} \frac{1}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right)(\vec{r}-\vec{v}t)}$$



c) $\vec{B} = \frac{1}{c} \frac{\vec{\mathscr{r}}_r}{\mathscr{r}_r} \times \vec{E}$

Then $\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}t_r$

$$\vec{B} = \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r^3} \frac{1}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right) \underbrace{(\vec{r}-\vec{v}t_r)\times(\vec{r}-\vec{v}t)}_{\vec{r}\times\vec{v}\,\underbrace{(t_r-t)}_{-\mathscr{r}_r/c}}$$

$$\Rightarrow \vec{B} = \frac{1}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r^2} \frac{1}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right) \vec{v}\times\vec{r}$$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi} \frac{q}{\mathscr{r}_r^2} \frac{\vec{v}\times\vec{r}}{(1-\hat{\mathscr{r}}_r\cdot\vec{v}/c)^3} \left(1-\frac{v^2}{c^2}\right)$$