PHYS 396 HOMEWORK SET 10

1. $z = 0.2$

let $H = 72 \frac{km}{s \cdot Mpc} \times \frac{1000 m}{1 km} \times \frac{1 Mpc}{1 \times 10^6 pc} \times \frac{1 pc}{3.09 \times 10^{16} m} = 2.33 \times 10^{-18} s^{-1}$

NOW $z = \frac{V}{C}$ $\therefore$ $V = 0.2C = 5.98 \times 10^7 m/s$

NOW HUBBLE'S LAW IS...

$V = H_o d$ $\therefore$ $d = \frac{V}{H_o} = \frac{5.98 \times 10^7 m/s}{2.33 \times 10^{-18} 1/s} = 2.6 \times 10^{25} m \times \frac{1 cy}{9.46 \times 10^{15} m} = 2.7 \times 10^9 cy$

NOW FROM EQN (1) ...

2a) $\rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2}$ so $\frac{d\rho}{dt} = \frac{3}{8\pi}\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)^2 = \frac{3}{4\pi}\frac{\dot{a}}{a}\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{3}{4\pi}\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)$

ALSO

$$\rho = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2} \qquad (1)$$

$$p = -\frac{1}{8\pi}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \qquad (2)$$

PLUGGING THESE INTO EQN (3)

$0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p)$ YIELDS $\frac{3}{4\pi}\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + 3\frac{\dot{a}}{a}\left[\frac{3}{8\pi}\frac{\dot{a}^2}{a^2} - \frac{1}{8\pi}\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\right]$

$$= \frac{3}{8\pi}\frac{\dot{a}}{a}\left[2\left(\cancel{\frac{\ddot{a}}{a}} - \cancel{\frac{\dot{a}^2}{a^2}}\right) + \left[3\cancel{\frac{\dot{a}^2}{a^2}} - \left(\cancel{\frac{2\ddot{a}}{a}} + \cancel{\frac{\dot{a}^2}{a^2}}\right)\right]\right] = 0 \quad \checkmark$$

b) NOW THE CONSERVATION EQN

$0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p)$ BECOMES $0 = \dot{\rho} + 3\frac{\dot{a}}{a}(1 + w)\rho$ USING $p = w\rho$

NOW SEPARATE & INTEGRATE..

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

a

$$\int\frac{d\rho}{\rho} = -3(1+w)\int\frac{da}{a}$$

$$\ln\rho(t) = \ln\rho_0 - 3(1+w)\ln a(t) = \ln\rho_0 + \ln a(t)^{-3(1+w)} = \ln\left(\rho_0 a(t)^{-3(1+w)}\right)$$

NOW EXPONENTIATING YIELDS...

$$\boxed{\rho(t) = \frac{\rho_0}{a(t)^{3(1+w)}}}$$

NOW

$$8\pi\rho = 3\frac{\dot{a}^2}{a^2} \qquad (1)$$

$$8\pi p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \qquad (2)$$

NOW, MULTIPLYING EQN (2) BY THREE & ADDING TO EQN (1) YIELDS..

$$8\pi(\rho+3p) = -\frac{6\ddot{a}}{a} = 8\pi(1+3w)\rho$$

$$\Rightarrow \left[\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(1+3w)\rho\right]$$

d) AS $\rho > 0$, NOTICE THAT $\frac{\ddot{a}}{a} > 0$ WHEN...

$$1+3w < 0$$

$$\Rightarrow$$

$$\left[w < -1/3\right]$$

3 a) let $a(t) = a_0 t^{2/3(1+w)}$

so

$$\dot{a} = \frac{da}{dt} = \frac{2}{3(1+w)} a_0 t^{2/3(1+w) - 1} = \frac{2}{3(1+w)} \cdot \frac{1}{t} a_0 t^{2/3(1+w)} = \frac{2}{3(1+w)} \cdot \frac{1}{t} a(t)$$

so

$$\frac{\dot{a}}{a} = \frac{2}{3(1+w)} \cdot \frac{1}{t} \qquad \therefore \frac{\dot{a}^2}{a^2} = \left[\frac{2}{3(1+w)}\right]^2 \cdot \frac{1}{t^2}$$

now

$$\rho(t) = \frac{\rho_0}{a^{3(1+w)}} = \frac{\rho_0}{a_0^{3(1+w)} t^2}$$

NOW, PLUGGING THESE INTO EQN. (1) YIELDS..

$$\frac{8\pi \rho_0}{a_0^{3(1+w)}} \cdot \frac{1}{t^2} = 3\left[\frac{2}{3(1+w)}\right]^2 \cdot \frac{1}{t^2}$$

SO, A VALID SOLUTION SO LONG AS ...

$$\left[\frac{8\pi \rho_0}{a_0^{3(1+w)}} = 3\left[\frac{2}{3(1+w)}\right]^2\right]$$

b) NOW, FROM BEFORE..

$$\dot{a} = \frac{2}{3(1+w)} a_0 t^{2/3(1+w) - 1} = \frac{2}{3(1+w)} a_0 t^{-(1+3w)/3(1+w)}$$

NOW

$$\frac{2}{3(1+w)} - 1 = \frac{1}{3(1+w)}\left[2 - 3(1+w)\right] = -\frac{(1+3w)}{3(1+w)}$$

so

$$\ddot{a} = \frac{2}{3(1+w)} \cdot -\frac{(1+3w)}{3(1+w)} a_0 t^{2/3(1+w) - 2} = \frac{-2(1+3w)}{[3(1+w)]^2} \cdot \frac{1}{t^2} a_0 t^{2/3(1+w)} = \frac{-2(1+3w)}{[3(1+w)]^2} \cdot \frac{1}{t^2} a(t)$$

$$\therefore \frac{\ddot{a}}{a} = \frac{-2(1+3w)}{[3(1+w)]^2} \cdot \frac{1}{t^2}$$

NOW PLUGGING THESE INTO EQN (2) YIELDS..

$$\frac{8\pi w \rho_0}{a_0^{3(1+w)}} \cdot \frac{1}{t^2} = -\left[\frac{-4(1+3w)}{[3(1+w)]^2} \cdot \frac{1}{t^2} + \left[\frac{2}{3(1+w)}\right]^2 \cdot \frac{1}{t^2}\right]$$

$$= -\frac{4}{[3(1+w)]^2} \cdot \frac{1}{t^2}\left[-(\cancel{1} + 3w) + \cancel{1}\right] = \frac{4 \cdot 3w}{[3(1+w)]^2} \cdot \frac{1}{t^2}$$

$$= \left[\frac{2}{3(1+w)}\right]^2 \cdot 3w \cdot \frac{1}{t^2}$$

So, A VALID SOLUTION SO LONG AS...

$$\frac{8\pi\rho_0}{a_0^{3(1+w)}} = 3\left[\frac{2}{3(1+w)}\right]^2$$

WHICH IS THE SAME RELATIONSHIP ON THE CONSTANTS THAT WAS FOUND IN PART a)!

c) WHEN $w=-1$, EQNS (1) & (2) BECOME...

$$8\pi\rho = 3\frac{\dot{a}^2}{a^2} \qquad (1)$$

$$-8\pi\rho = -\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right) \qquad (2)$$

NOW ADDING THESE TOGETHER YIELDS..

$$0 = 3\frac{\dot{a}^2}{a^2} - \left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right) = 2\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} = 2\left(\frac{\dot{a}^2}{a^2}-\frac{\ddot{a}}{a}\right) = -2\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)$$

so

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 0 \quad \therefore \quad \frac{\dot{a}}{a} = H_0 \text{ (A CONSTANT!)}$$

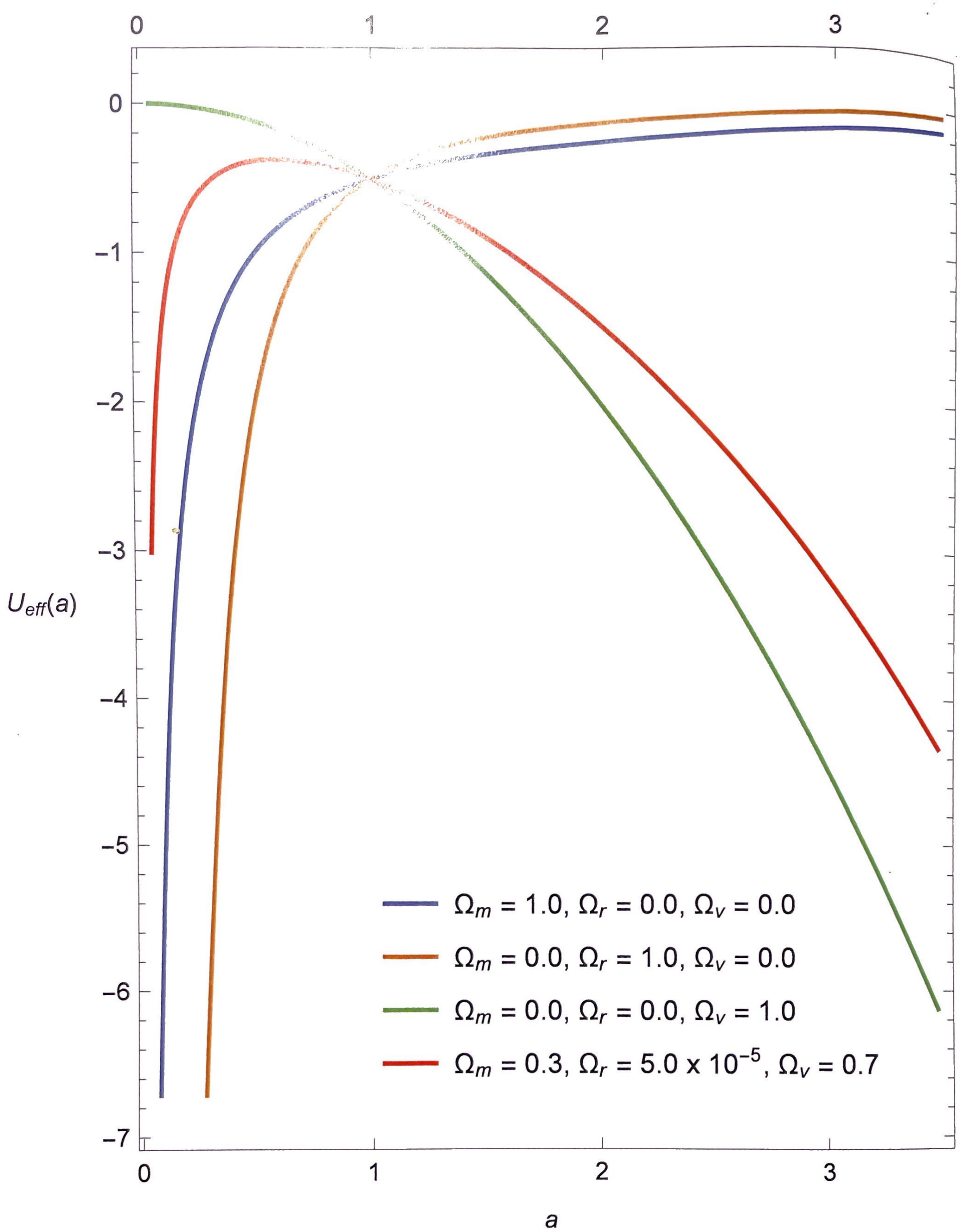
NOW SEPARATE & INTEGRATE...

$$\int \frac{da}{a} = H_0\int dt$$

$$\ln a(t) = H_0(t-t_0)$$

NOW EXPONENTIATING YIELDS..

$$\boxed{a(t) = e^{H_0(t-t_0)}}$$



0
1
2
3
0
-1
-2
-3
-4
-5
-6
-7
$U_{eff}(a)$
$\Omega_m = 1.0,\ \Omega_r = 0.0,\ \Omega_v = 0.0$
$\Omega_m = 0.0,\ \Omega_r = 1.0,\ \Omega_v = 0.0$
$\Omega_m = 0.0,\ \Omega_r = 0.0,\ \Omega_v = 1.0$
$\Omega_m = 0.3,\ \Omega_r = 5.0 \times 10^{-5},\ \Omega_v = 0.7$
0
1
2
3
a

4e) $\frac{1}{2H_0^2}\left(\frac{da}{dt}\right)^2 + U_{eff}(a) = 0$

THIS IS OF THE FORM OF A 1D CONSERVATION OF ENERGY EXPRESSION FOR ZERO TOTAL MECHANICAL ENERGY.

NOTICE

$$K = \frac{1}{2H_0^2}\left(\frac{da}{dt}\right)^2 \geq 0$$

WHEREAS

$$U_{eff}(a) = -\frac{1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right) < 0$$

SO NOTICE THAT AS THE 'POTENTIAL ENERGY' DECREASES THE 'KINETIC ENERGY' INCREASES $\therefore \frac{\dot{a}}{a}$ INCREASES AND VICE VERSA.

a) FOR MATTER-DOMINATED FLAT FRW SPACETIME...

$U_{eff}(a) = -\frac{1}{2a}$ SO AS a INCREASES, U INCREASES (SEE PLOT)

SO AS U INCREASES, K DECREASES $\therefore \frac{\dot{a}}{a}$ DECREASES $\therefore$ DECELERATED EXPANSION

b) FOR RADIATION-DOMINATED FLAT FRW SPACETIME...

$U_{eff}(a) = -\frac{1}{2a^2}$ SO AS a INCREASES, U INCREASES $\therefore K$ DECREASES $\therefore$ DECELERATED EXPANSION

c) FOR VACUUM-DOMINATED FLAT FRW SPACETIME...

$U_{eff}(a) = -\frac{1}{2}a^2$ SO AS a INCREASES, U DECREASES $\therefore K$ INCREASES $\therefore \frac{\dot{a}}{a}$ INCREASES

$\therefore$ ACCELERATED EXPANSION

d) FOR OUR FLAT FRW SPACETIME...

$$U_{eff} = -\frac{1}{2}\left(0.7a^2 + \frac{0.3}{a} + \frac{5.0\times10^{-5}}{a^2}\right)$$

SO AS a INCREASES, U INCREASES AND THEN DECREASES

$\therefore K$ DECREASES AND THEN INCREASES, AS DOES $\frac{\dot{a}}{a}$

$\therefore$ DECELERATED AND ACCELERATED EXPANSION