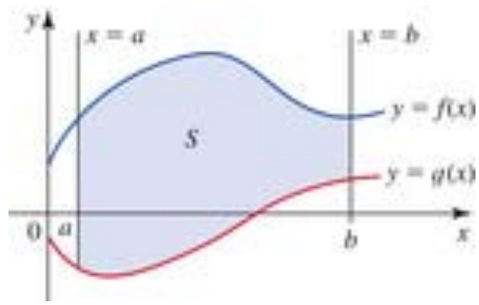
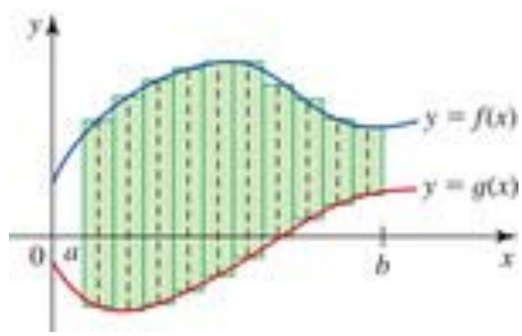
MAT 201
Larson/Edwards – Section 7.1
Area of a Region Between Two Curves

In Chapter 4 we discussed ways to find the area under a curve by using a definite integral. In this chapter we will be discussing how to find the area of a region between two curves.

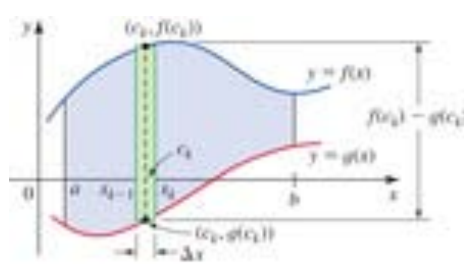
For example, suppose we would like to find the area of the shaded region *S* in the diagram below:



Will summing rectangles work with a problem like this?



How do we use a definite integral to express this area?



Area of a Region Between Two Curves:

If f and g are continuous on [a, b] and $f(x) \geq g(x)$ for all x in [a, b], then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is:

$$A = \int_a^b \left[f(x) - g(x) \right] dx.$$

Does it matter if one or both of the functions are below the x-axis on the interval [a, b]? Explain.

No; Subtracting a "−" becomes "+".

Since every rectangle that is being added has a height of $f(c_k) - g(c_k)$, we can use the definite integral $\int_a^b [f(x) - g(x)]dx$ to express the area of the shaded region.

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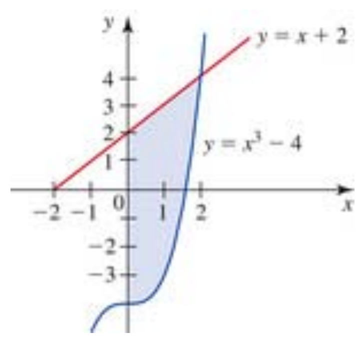
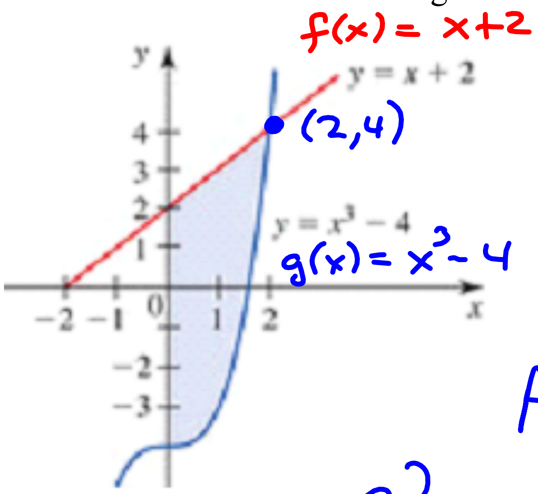
Ex: Find the area of the shaded region below:



Ex: Find the area of the shaded region below:



$[0, 2]$

$$A = \int_a^b [f(x) - g(x)]\, dx$$

$$A = \int_0^2 [(x+2) - (x^3 - 4)]\, dx$$

$$A = \int_0^2 x + 2 - x^3 + 4 \; dx$$

$$= \int_0^2 x + 6 - x^3 \; dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^4}{4} \Big|_0^2$$

$$= \left(\frac{2^2}{2} + 6(2) - \frac{2^4}{4}\right) - (0)$$

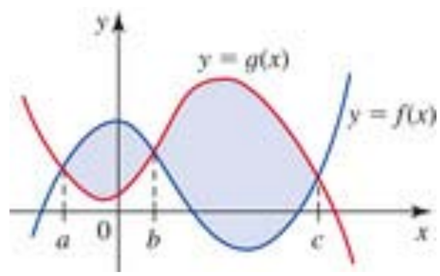
$$= 2 + 12 - 4$$

$$= 10 \text{ sq. units}$$

Suppose now that you were asked to find the area of the region bounded between intersecting curves (without a graph or limits of integration). How would we proceed?

Ex: Find the area of the region bounded by the graphs of $f(x) = x^2 + 2x$ and $g(x) = x + 2$

What if the curves intersect in more than two points? How would we proceed?

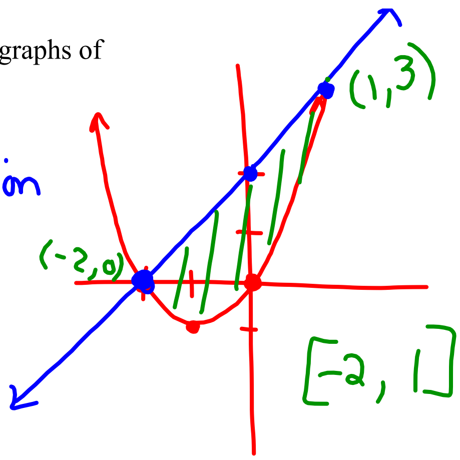


Notice that $f(x) \geq g(x)$ on the interval $[a, b]$, and $g(x) \geq f(x)$ on the interval $[b, c]$.

Ex: Find the area of the region bounded by the graphs of

$f(x) = x^2 + 2x$ and $g(x) = x + 2$

$f(x) = x(x+2)$



Find limits of integration

$x^2 + 2x = x + 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2, 1$

$$A = \int_{-2}^{1} \left[(x+2) - (x^2+2x)\right] dx$$

$$= \int_{-2}^{1} \left[x + 2 - x^2 - 2x\right] dx$$

$$= \int_{-2}^{1} -x + 2 - x^2 \, dx$$

$$= -\frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_{-2}^{1}$$

$$= \left(-\frac{(1)^2}{2} + 2(1) - \frac{(1)^3}{3}\right) - \left(-\frac{(-2)^2}{2} + 2(-2) - \frac{(-2)^3}{3}\right)$$

$$= -\frac{1}{2} + 2 - \frac{1}{3} + 2 + 4 - \frac{8}{3}$$

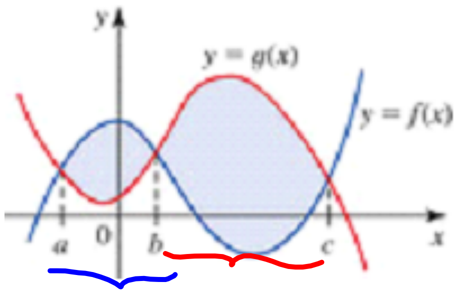
$$= 8 - 3 - \frac{1}{2}$$

$$= 4.5 \text{ units}^2$$

What if the curves intersect in more than two points? How would we proceed?

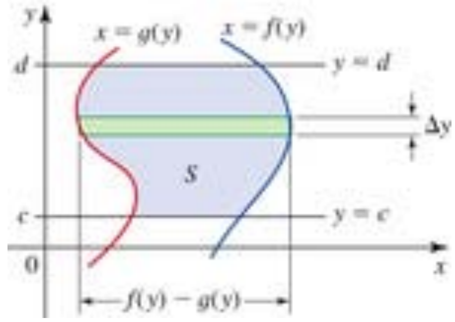
$f(x) \geq g(x)$

Split it up!



$$\int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx$$

Can we find the area between two curves by using horizontal rectangles?



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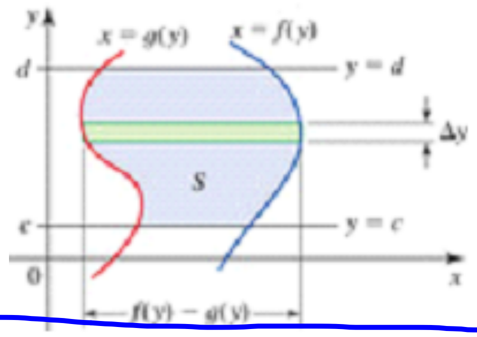

$$\boxed{\int_c^d [f(y) - g(y)]\, dy}$$