MAT 202
Larson – Section 9.2
Series & Convergence

<u>Series</u>: The sum of the terms of a sequence is called a series. Note: A sequence is a list of numbers where order is important, whereas a series is a single number obtained by computing a sum.

<u>Infinite Series</u>: Let $\{a_n\}$ be an infinite sequence. The sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an ***infinite series*** (or simply ***series***). The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the terms of the series. For some series, it is convenient to begin the index at $n = 0$ (or some other integer). Sometimes, for convenience, it is common to represent an infinite series as $\sum a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

<u>Sequence of Partial Sums</u>: The sequence $S_1, S_2, S_3, \ldots, S_n$ is a sequence of partial sums.

$S_1 = a_1$

$S_2 = a_1 + a_2$ (*Sum of the first two terms*)

$S_3 = a_1 + a_2 + a_3$ (*Sum of the first three terms*)

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$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ (n^{th} *partial sum*)

If this sequence of partial sums converges, then the series is said to converge and has the sum indicated in the following definition.

<u>Definitions of Convergent and Divergent Series</u>: For the infinite series $\sum_{n=1}^{\infty} a_n$, the n^{th} partial sum is $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ ***converges***. The limit S is called the ***sum of the series***.

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

If $\{S_n\}$ diverges, then the series ***diverges***.

<u>Telescoping Sequence</u>: A series of the form

$$S_n = \sum_{n=1}^{\infty} b_n = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_{n-1} - b_n) + (b_n - b_{n+1})$$

is a ***telescoping series***. Notice that all the terms $b_2, b_3, b_4, \ldots, b_n$ cancel out leaving

$$S_n = b_1 - b_{n+1}$$

It follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Also, if the series converges, then its sum is

$$S = b_1 - \lim_{n \to \infty} b_{n+1}$$

Ex: Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Geometric Series: A series of the form

$$S_n = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \ a \neq 0$$

is a ***geometric series***, with ratio r, $r \neq 0$.

Convergence of a Geometric Series: A geometric series with ratio r diverges when $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$S_n = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \ 0 < |r| < 1.$$

Properties of Limits of Sequences: Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B, and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$

2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

Ex: Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$. $= \frac{2}{3} + \frac{2}{15} + \frac{2}{35} + \cdots$

$$\frac{2}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$$2 = A(2n-1) + B(2n+1)$$

$$2 = 2An - A + 2Bn + B$$

$2(B - A = 2)$

$2B + 2A = 0$

$4B = 4$

$B = 1 \quad A = -1$

Telescoping series

$$\sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right] = \underbrace{\left[1 - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{5}\right] + \left[\frac{1}{5} - \frac{1}{7}\right] + \cdots}_{\text{Telescoping series}}$$

$b_1 \quad b_2 \quad b_2 \quad b_3$

$$\text{Sum} = b_1 - \lim_{n \to \infty} b_{n+1}$$

$$= 1 - \left[\lim_{n \to \infty} \frac{1}{2n+1}\right]$$

$$= 1 - 0$$

$$= \boxed{1}$$

$$\lim_{n \to \infty} \left[1 - \frac{1}{2n+1}\right]$$

$$= 1$$

Ex: Find the sum of the series $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)$.

Ex: Given the repeating decimal $0.\overline{45}$.

a) Write the repeating decimal as a geometric series, and

b) Write its sum as the ratio of two integers.

<u>Limit of the *n*th Term of a Convergent Series</u>:

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

<u>*n*th Term Test for Divergence</u>:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Geometric Series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $0<|r|<1$

Not telescoping

Ex: Find the sum of the series $\sum_{n=0}^{\infty}\left(\frac{1}{2^n}-\frac{1}{3^n}\right)$.

$= \cancel{(1-1)} + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{4}-\frac{1}{9}\right)$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$$

$$= \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{2}\right)^n - \sum_{n=0}^{\infty} 1\cdot\left(\frac{1}{3}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}}$$

$$= 2 - \frac{3}{2}$$

$$= \frac{1}{2}$$

Ex: Given the repeating decimal $0.\overline{45}$. $= 0.45454545...$

a) Write the repeating decimal as a geometric series, and

b) Write its sum as the ratio of two integers.

a) $$\frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \cdots$$

$$\frac{45}{(100)^1} + \frac{45}{(100)^2} + \frac{45}{(100)^3} + \cdots$$

$$45 \cdot \left(\frac{1}{100}\right)^1 + 45 \cdot \left(\frac{1}{100}\right)^2 + 45\left(\frac{1}{100}\right)^3 + \cdots$$

$$\sum_{n=1}^{\infty} 45 \cdot \left(\frac{1}{100}\right)^n$$

Form: $\sum_{n=0}^{\infty} ar^n$

$$\sum_{n=0}^{\infty} 45 \cdot \left(\frac{1}{100}\right)^{n+1}$$

Geometric Series

$$\sum_{n=0}^{\infty} \left[45 \cdot \left(\frac{1}{100}\right)^1\right]\left(\frac{1}{100}\right)^n = \boxed{\sum_{n=0}^{\infty} 0.45 \left(\frac{1}{100}\right)^n}$$

$$= \frac{a}{1-r} = \frac{0.45}{1 - \frac{1}{100}}$$

$$= \frac{\frac{45}{100}}{\frac{99}{100}} = \frac{45}{99}$$

Ex: Determine the convergence or divergence of the following series:

a) $\sum_{n=0}^{\infty} (0.36)^n$

b) $\sum_{n=0}^{\infty} \frac{2n+1}{3n+2}$

Ex: Determine the convergence or divergence of the following series:

a) $\sum_{n=0}^{\infty} (0.36)^n$

Geometric Series

$0 < |r| < 1$

Converges

b) $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{3n+2} = \sum_{n=1}^{\infty} \frac{2(n-1)+1}{3(n-1)+2}$

$\displaystyle = \sum_{n=1}^{\infty} \underbrace{\frac{2n-1}{3n-1}}_{a_n}$

$\displaystyle\lim_{n\to\infty} a_n$

$\displaystyle\lim_{n\to\infty} \frac{2n-1}{3n-1} = \frac{2}{3} \neq 0$

by n^{th} term test $\displaystyle\sum_{n=0}^{\infty} \frac{2n+1}{3n+2}$ diverges