MAT 202
Larson – Section 9.3
The Integral Test and *p*-Series

In section 9.2 we introduced series and determined whether they converge or diverge. In this section we will study a couple of tests for convergence that apply to ***series with positive terms***.

The Integral Test: If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad and \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Ex: Confirm that the integral test can be applied to the series. Then use the integral test to determine the convergence or divergence of the series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Ex: Confirm that the integral test can be applied to the series. $\int_1^\infty \frac{x}{x^2+1}\,dx$

Then use the integral test to determine the convergence or divergence of the series:

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \frac{1}{2} + \frac{2}{5} + \dots$

$f(x) = \frac{x}{x^2+1}$

positive? ✓

continuous? ✓

decreasing? ✓

$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2}$

$f'(x) = \frac{-x^2+1}{(x^2+1)^2}$

$0 = -x^2+1 \qquad x = \pm 1$

$(-\infty,-1) \quad (-1,1) \quad (1,\infty)$

Only concerned w/ this interval

$f'(2) < 0$

decreasing

$\int_1^\infty \frac{x}{x^2+1}\,dx$

$\lim_{b\to\infty}\left[\int_1^b \frac{x}{x^2+1}\,dx\right]$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\lim_{b\to\infty}\left[\frac{1}{2}\ln|x^2+1| \Big|_1^b\right]$

$\lim_{b\to\infty}\left[\frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln|2|\right] = \infty$

$\therefore \sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

positive? ✓

continuous? ✓

decreasing? ✓

$f(x) = \dfrac{1}{x^2+1}$ $\quad f'(x) = \dfrac{-2x}{(x^2+1)^2}$

$0 = -2x$

$x = 0$

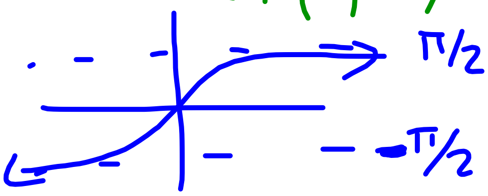
$(-\infty, 0)$ $(0, \infty)$

$f'(3) < 0$

decreasing on $(1, \infty)$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx$$

$$\lim_{b\to\infty}\left[\int_1^{b} \frac{1}{x^2+1}\,dx\right]$$



$$\lim_{b\to\infty}\left[\arctan x \Big|_1^b\right] = \lim_{b\to\infty}\left[\arctan b - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

∴ Converges

Since the integral converges $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges

Note: In example (b) above, the fact that the improper integral converges to $\frac{\pi}{4}$ does not imply that the infinite series converges to $\frac{\pi}{4}$. We would have to use a different process to approximate the sum of the series (see page 606 in your text).

<u>***p*-Series**</u>**:** A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is a ***p*-series**, where p is a positive constant.

<u>**Harmonic Series**</u>**:** In a p-series, if $p = 1$, we get $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which is the ***harmonic series***.

<u>**Convergence of *p*-Series**</u>**:** The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ **converges** for $p > 1$, and **diverges** for $0 < p \le 1$.

Ex: Determine the convergence or divergence of the following p-series:

a) $\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$

Ex: Determine the convergence or divergence of the following *p*-series:

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.04}}$

p-series where $p = 1.04 > 1$

Converges

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$

p-series where $p = \frac{3}{4} < 1$

$0 < \frac{3}{4} \leq 1$ Diverges

Ex: Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

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$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \cdots$$

Positive for $n \geq 2$? ✓

Continuous? ✓

decreasing? ✓

$$f(x) = \frac{1}{x \ln x}$$

$$f'(x) = -\frac{(\ln x + 1)}{(x \ln x)^2}$$

$$0 = -\ln x - 1$$

$$\ln x = -1$$

$$x = e^{-1}$$

$$\left(\frac{1}{e}, \infty\right)$$

$$f'(3) < 0$$

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx$$

$$\lim_{b \to \infty} \left[\int_2^b \frac{1}{x \ln x}\,dx \right]$$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$$\lim_{b \to \infty} \left[\ln|\ln x| \Big|_2^b \right]$$

$$\lim_{b \to \infty} \left[\ln|\ln b| - \ln|\ln 2| \right] = \infty$$

Series diverges

The Integral Test: If f is positive, continuous, and

$x \geq N > 1$

decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=N}^{\infty} a_n \quad and \quad \int_{N}^{\infty} f(x)\,dx$$

either both converge or both diverge.