Vector Calculus

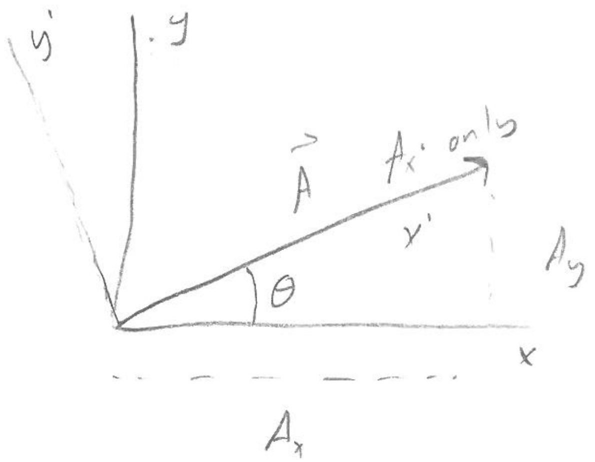
1. Scalar $\rightarrow$ magnitude only

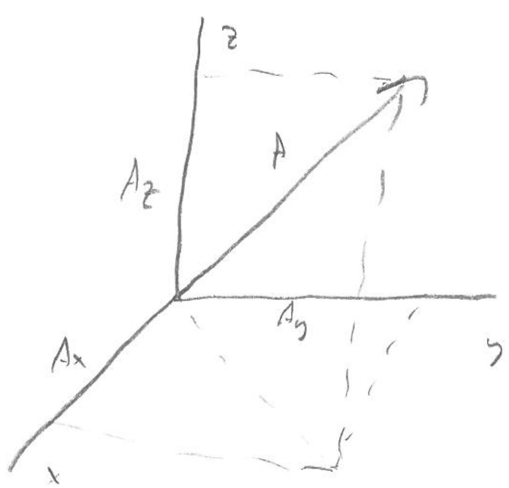
   Invariant under coordinate transforms

2. Vector $\vec{A} = \sum_i x_i \hat{e}_i \rightarrow$ Ex (A_x, A_y, A_z)

   $\vec{r}, \vec{v}, \vec{a}, \vec{F}, \vec{p}$

   depends on choice of coordinate system





Vector Algebra

#1. if $\vec{A} = \vec{B}$ then $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

#2. $\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$

#3. $c\vec{A} = (cA_x, cA_y, cA_z)$

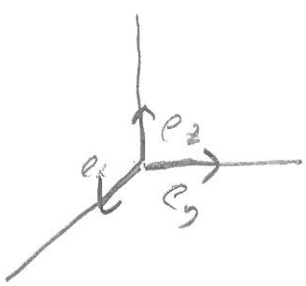
#4. $\vec{0} = (0,0,0)$

#5. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

#6. $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$

Unit Vectors

$\hat{e}_x = (1,0,0)$ $\hat{e}_y = (0,1,0)$ $\hat{e}_z = (0,0,1)$



Scalar Product

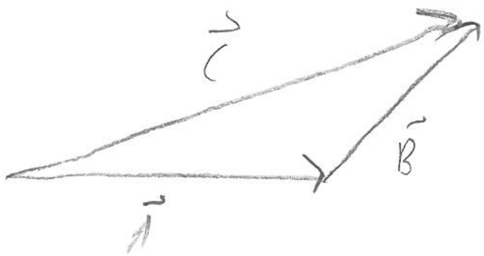
$$\vec{A} \cdot \vec{B} = |A||B| \cos\theta = A_x B_x + A_y B_y + A_z B_z = (A_x, A_y, A_z) \cdot (B_x, B_y, B_z) = \sum_i A_i B_i$$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \underline{\underline{\sqrt{\vec{A} \cdot \vec{A}}}}$$

$$\rightarrow \frac{\vec{A} \cdot \vec{B}}{|A||B|} = \cos\theta$$

USEFUL

$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = |C|^2$$



$$|\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}\cdot\vec{B} = |C|^2$$

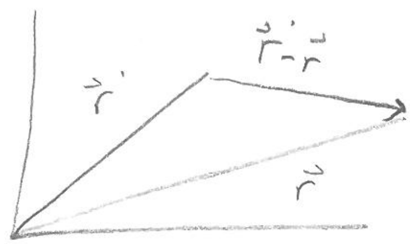
$$|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta = |C|^2$$

Pythagorean Theorem?

→ Work $= \int \vec{F}\cdot d\vec{r}$ → Component of Force along direction of Travel

Ex.

→



$$|\vec{r}'-\vec{r}| = \sqrt{(\vec{r}'-\vec{r})\cdot(\vec{r}'-\vec{r})}$$

Ex.

SCALAR PRODUCT

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \rightarrow \underline{\underline{\text{Show}}}$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

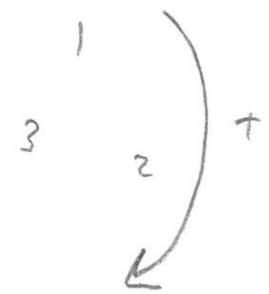
$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

USEFUL

If $\vec{C} = \vec{A} \times \vec{B}$ $\quad C_i = \sum_{j,k} \epsilon_{ijk} A_j B_k$

$$\begin{aligned} \epsilon_{ijk} &= 0 \quad i=j,\ i=k,\ j=k \\ &= 1 \quad \epsilon_{123},\ \epsilon_{231},\ \epsilon_{312} \\ &= -1 \quad \epsilon_{132},\ \epsilon_{213},\ \epsilon_{321} \end{aligned}$$

Hint



$$C_1 = \underset{+1}{\epsilon_{123}} A_2 B_3 + \underset{-1}{\epsilon_{132}} A_3 B_2 = A_2 B_3 - A_3 B_2$$

$$|\vec{A} \times \vec{B}|^2 = (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_x B_x + A_y B_y + A_z B_z)^2$$

$$= \vec{A}^2 \vec{B}^2 - \vec{A}^2 \vec{B}^2 \cos^2\theta$$

$$= \vec{A}^2 \vec{B}^2 \sin^2\theta$$

So $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta\, \hat{n}$

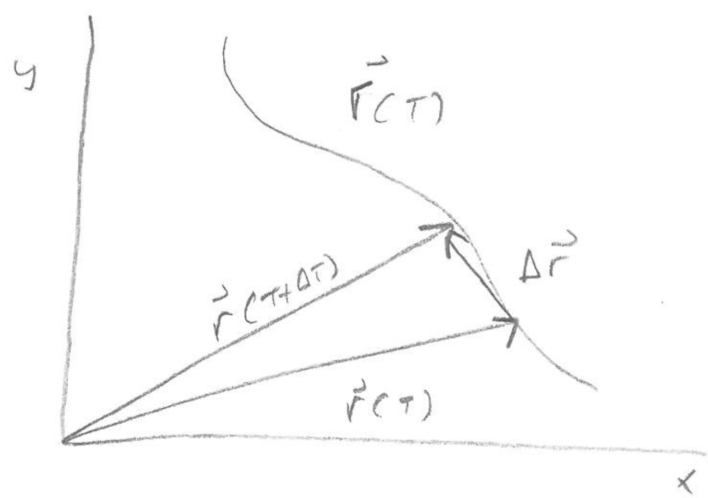
Finally

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

and

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Vector differentiation with respect to a scalar



$$\vec{r}(T) + \Delta\vec{r} = \vec{r}(T+\Delta T)$$

$$\frac{d\vec{r}}{dT} = \lim_{\Delta T \to 0} \frac{\vec{r}(T+\Delta T) - \vec{r}(T)}{\Delta T} \Rightarrow \underline{\frac{dr_x(T)}{dT}\hat{e}_x + \frac{dr_y(T)}{dT}\hat{e}_y}$$

$$\frac{d}{dT}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dT} + \frac{d\vec{B}}{dT}$$

$$\frac{d}{dT}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dT}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dT}$$

$$\frac{d}{d\tau}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{d\tau}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{d\tau}$$

$$\frac{d\,\phi\vec{A}}{d\tau} = \frac{d\phi}{d\tau}\vec{A} + \phi\frac{d\vec{A}}{d\tau} \rightarrow e_x \quad \tau^3(3\tau^2, 2\tau, 1)$$

Mechanics

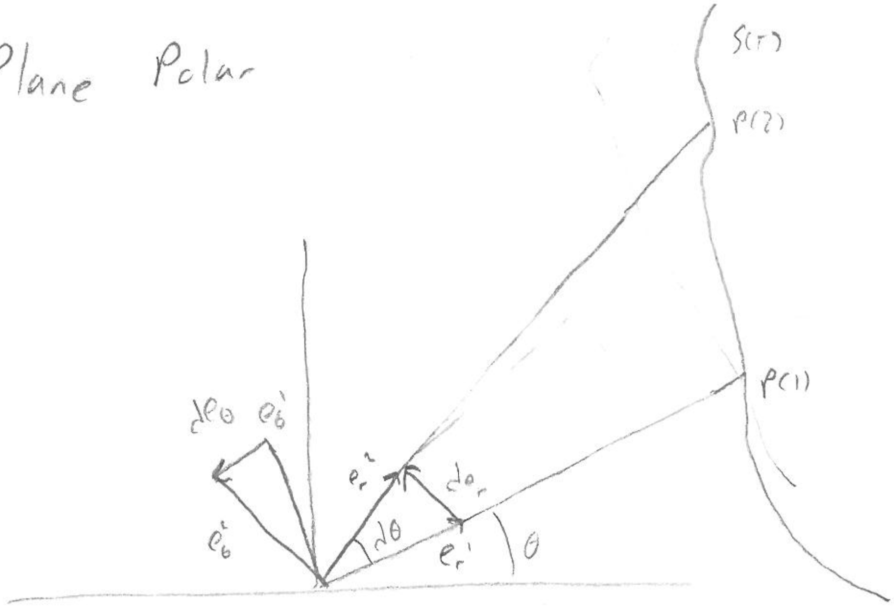
$$\vec{r} = \vec{r}(\tau) \qquad \vec{v} = \frac{d\vec{r}}{d\tau} = \dot{r} \quad \vec{a} = \frac{d^2\vec{r}}{d\tau^2} = \frac{d\vec{v}}{d\tau} = \ddot{r} = \dot{v}$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

$$\vec{a} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y + \ddot{z}\hat{e}_z$$

Plane Polar

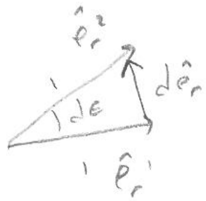


$\hat{\imath}, \hat{\jmath}, \hat{k} \rightarrow$ Static

$\hat{e}_r, \hat{e}_\theta$ NOT!!

$\hat{e}_r \perp \hat{e}_\theta$

$d\hat{e}_r = \hat{e}_r^2 - \hat{e}_r^1 \rightarrow \parallel \hat{e}_\theta$ magnitude $|d\theta|$



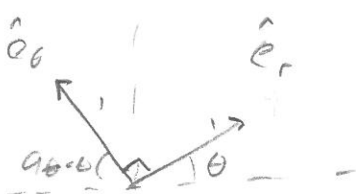
So $\underline{\underline{d\hat{e}_r = d\theta \hat{e}_\theta}}$

$d\hat{e}_\theta = \hat{e}_\theta^2 - \hat{e}_\theta^1 \parallel -\hat{e}_r$

$|d\hat{e}_r| = |d\theta|$

So $d\hat{e}_\theta = -d\theta \hat{e}_r$

Or



$\hat{e}_r = \cos\theta \, \hat{\imath} + \sin\theta \, \hat{\jmath}$

$\hat{e}_\theta = -\cos(90-\theta)\hat{\imath} + \sin(90-\theta)\hat{\jmath}$

$\hat{e}_\theta = -\sin\theta \, \hat{\imath} + \cos\theta \, \hat{\jmath}$

$\sin(A \pm B) = SACB \pm CASB$

$\cos(A \pm B) = CACB \mp SASB$

$d\hat{e}_r = -\sin\theta\, d\theta\, \hat{i} + \cos\theta\, d\theta\, \hat{j} = d\theta\, \hat{e}_\theta$

$d\hat{e}_\theta = -\cos\theta\, d\theta\, \hat{i} - \sin\theta\, d\theta\, \hat{j} = -d\theta\, \hat{e}_r$

$$\boxed{\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \dot{\theta}\, \hat{e}_\theta \,, \quad \dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\, \hat{e}_r}$$

Now $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \underline{\underline{\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta}}$

$\dot{r}\hat{e}_r = \vec{v}_r$, $r\dot{\theta}\hat{e}_\theta \rightarrow r\omega\hat{e}_\theta \rightarrow$ circular motion spin velocity

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta]$

$$\vec{a} = \underset{1}{\ddot{r}\hat{e}_r} + \underset{2}{\dot{r}\dot{\hat{e}}_r} + \underset{3}{\dot{r}\dot{\theta}\hat{e}_\theta} + \underset{4}{r\ddot{\theta}\hat{e}_\theta} + \underset{5}{r\dot{\theta}\dot{\hat{e}}_\theta}$$

$$\vec{a} = (\underset{1}{\ddot{r}} - \underset{5}{r\dot{\theta}^2})\hat{e}_r + (\underset{2+3}{2\dot{r}\dot{\theta}} + \underset{4}{r\ddot{\theta}})\hat{e}_\theta$$

$\frac{v^2}{r}, r\omega^2$ centripetal

Coriolis

$r\alpha \rightarrow$ spin up spin down

Cartesian

$d\vec{s} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z \longleftarrow$ Line element

Important

$d\vec{s} \cdot d\vec{s} = |d\vec{s}|^2 = dx^2 + dy^2 + dz^2$

$V^2 = \frac{|ds|^2}{|d\tau|^2} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$

Cylindrical

$d\vec{s} = dr\hat{e}_r + rd\theta\hat{e}_\theta + dz\hat{e}_z$

$|d\vec{s}|^2 = dr^2 + r^2 d\theta^2 + dz^2$

$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$

$\vec{V} = \dot{r}e_r + r\dot{\theta}\hat{e}_\theta + \dot{z}e_z$

Spherical

$d\vec{s} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$

$|d\vec{s}|^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$

$V^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$

$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi$

Important Later

Gradient Operator

$\phi = \phi(x,y,z) \rightarrow$ Scalar Function

$$\nabla = \sum_i \hat{e}_i \frac{\partial}{\partial x_i}$$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{e}_x + \frac{\partial \phi}{\partial y}\hat{e}_y + \frac{\partial \phi}{\partial z}\hat{e}_z$$

$\rightarrow$ if that was a hill $\nabla\phi$ would pt uphill

Normal to $\phi =$ constant at point where change is maximum

$$\vec{\nabla}\cdot\vec{A} = \sum \hat{e}_i \frac{\partial}{\partial x_i} \cdot \vec{A}_i$$

$$\vec{\nabla}\cdot\vec{A} = \sum \frac{\partial \vec{A}_i}{\partial x_i} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$\rightarrow$ How much "Flow" into or out of Area

$$\vec{\nabla}\times\vec{A} = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial A_k}{\partial x_j}\hat{e}_i \quad \text{or} \quad \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Measure of Rotation in Field

$$\nabla^2 \phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Ex: $d\phi \rightarrow$ Perfect differential

$$d\phi = \sum \frac{\partial \phi}{\partial x_i} dx_i = \sum_i (\nabla \phi)_i \, dx_i$$

or $d\phi = (\vec{\nabla}\phi) \cdot d\vec{s}$ $\quad d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

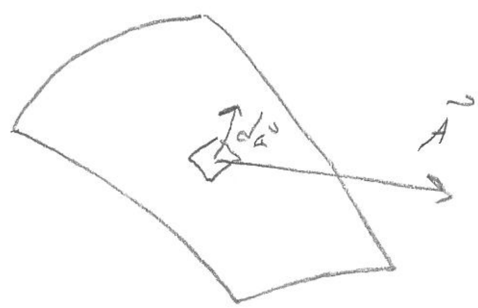
Vector Integration

Let $\vec{A} = A_1(x,y,z)\hat{e}_x + A_2(x,y,z)\hat{e}_y + A_3(x,y,z)\hat{e}_z$

Then $\int \vec{A}\, d^3r = \int A_1 d^3r + \int A_2 d^3r + \int A_3 d^3r$

Surface Integral

$$\oiint_S \vec{A} \cdot d\vec{a} = \oiint_S \vec{A} \cdot \hat{n}\, da$$



$$= \oiint [A_1 da_1 + A_2 da_2 + A_3 da_3]$$

$\hat{n}$ is normal to Surface (unit normal)

da_1 projection of vector component on Surface normal

$da_1 = dx_2 dx_3$

$\uparrow d\vec{a} \; \xrightarrow{\vec{A}}$ $\vec{A} \cdot d\vec{a} = 0$ $\quad \uparrow d\vec{a} \uparrow \vec{A}$ $\vec{A} \cdot d\vec{a} = 1$

$\uparrow d\vec{a} \nearrow \vec{A}$ $\vec{A} \cdot d\vec{a} = |A| \cos\theta$ $\quad \uparrow d\vec{a} \downarrow \vec{A}$ $\vec{A} \cdot d\vec{a} = -1$

Ex Let $r\hat{r} = \vec{A}$

FIND $\oiint \vec{A}\cdot d\vec{a}$ over Sphere

$$d\vec{a} = r^2 \sin\theta \, d\phi \, d\theta \, \hat{r}$$

$$\int_0^{2\pi}\int_0^{\pi} r^3 \sin\theta \, d\phi \, d\theta = R^3 \cdot \frac{4\pi}{3}$$

Tricky → generally need to form $\hat{N}$ via gradients or cross products. Then need to parameterize the surface, We'll use $\oiint$ primarily in cases with high degrees of Symmetry

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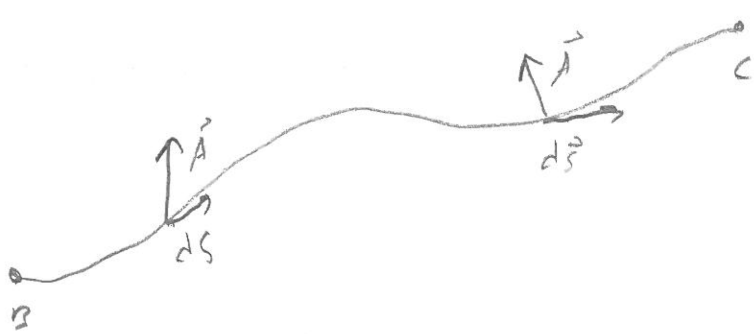
Line integrals

$$\oint_B^C \vec{A}\cdot d\vec{s} = \int_B^C (A_x dx + A_y dy + A_z dz)$$



Let $\vec{F} = -\nabla\phi \rightarrow$ W done is independent of Path

$$-\int_{P_1}^{P_2}\left[\frac{\partial\phi}{\partial x}\hat{e}_x + \frac{\partial\phi}{\partial y}\hat{e}_y + \frac{\partial\phi}{\partial z}\hat{e}_z\right]\cdot\left[dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z\right]$$

$$= -\int_{P_1}^{P_2}\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -\int_{P_1}^{P_2}d\phi = -\left[\phi(P_2) - \phi(P_1)\right]$$

and $= 0$ if $P_1 = P_2 \rightarrow$ closed Path

Gauss Theorem

$$\underbrace{\iiint_V \vec{\nabla}\cdot\vec{A}}_{\text{divergence}}\,dv = \oiint_S \vec{A}\cdot d\vec{a}$$

$\rightarrow$ essentially what is made or lost in a volume is how much "Flows" in or out

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Stokes Theorem

$\oiint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} \rightarrow$ "Swirl on Surface"

$= \oint \vec{A}\cdot d\vec{s}$ Flowing along boundary

Examples

Find the normal to the Surface $z = x^2 + y^2$

$\rightarrow$ if $F(x,y,z) = 0$ Then $\frac{\nabla F}{|\nabla F|} = \hat{n}$

$F(x,y,z) = x^2 + y^2 - z \quad \nabla F = 2x\hat{i} + 2y\hat{j} - 1\hat{k}$

$|\nabla F| = \sqrt{\nabla F \cdot \nabla F} = \sqrt{4x^2 + 4y^2 + 1}$

$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} - \hat{k}}{\sqrt{4x^2 + 4y^2 + 1}}$ at $1, -1, 2$ $\quad \frac{2\hat{i} - 2\hat{j} + 1k}{\sqrt{33}}$

Or let $r = x\hat{i} + y\hat{j} + (x^2 + y^2)\hat{k}$

Then $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$ yields Same

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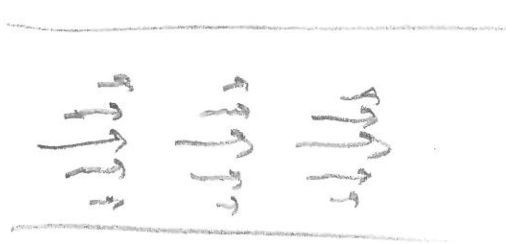
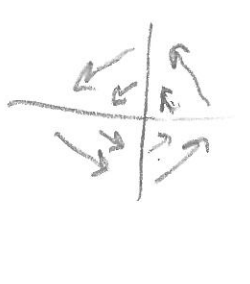
Divergences and Curls

$\vec{F}_1 = (1 - y^2, 0)$



$\vec{F}_2 = (-y, x)$



google rotation vector field

radial Field $\vec{F_3} = (x, y, z)$

$\nabla \cdot \vec{F_3} = \underline{\underline{3}}$ $\quad \nabla \times \vec{F_1}$? $\frac{\partial(1-y^2)}{\partial z}\hat{e_y} - \frac{\partial(1-y^2)}{\partial y}\hat{e_z} = 2y\hat{e_z}$

Note $\vec{\nabla} \times \vec{\nabla}\Phi = 0$ $\underline{\text{Always}}$

an $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ Always

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Ex. Let $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

evaluate $\oint \vec{A} \cdot d\vec{r}$ on the line joining 0,0,0 and 1,1,1

$d\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ let $x = y = z = t$ Then $t \in 0,1$

$\vec{A} \cdot d\vec{r} = (3t^2 + 6t) - 14t^2 + 20t^3 = 6t - 11t^2 + 20t^3$

$\int \vec{A} \cdot d\vec{r} = \int_0^1 6t - 11t^2 + 20t^3 \, dt$

Ex. Find the work done moving

a particle around the origin on a circle with R=3

and $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$

In the $z=0$ plane

$$\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\vec{F}\cdot d\vec{r} = (2x-y)dx + (x+y)dy$$

Parametarize $x = 3\cos(T)$, $y = 3\sin T$

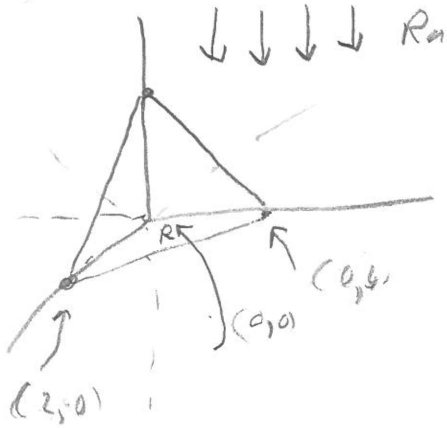
$$dx = -3\sin(T) \quad dy = 3\cos(T)$$

$$\oint_0^{2\pi} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left[6\cos(T) - 3\sin(T)\right](-3\sin(T)) + \left[3\cos(T) + 3\sin(T)\right]3\cos(T)\,dT$$

$$= \int_0^{2\pi} 9 - 9\sin(T)\cos(T)\,dT = 18\pi$$

I'll leave 2 examples of Stokes and Gauss's Theorems for HW

Surface integral → rain on the roof



$\vec{Rain} = 0\hat{i} + 0\hat{j} - A\hat{k}$

$z = 4-2x-y$

$\vec{r} = u\hat{i} + v\hat{k} + (4-2u-v)\hat{k}$

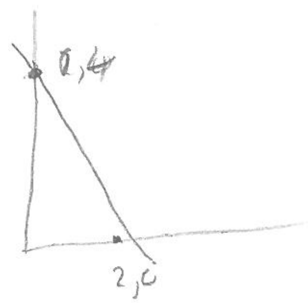
$d\vec{r} = du\,\hat{i} + dv\,\hat{j} + (-2du - dv\,\hat{k})$

$\underset{\vec{T}_u}{(\hat{i} - 2\hat{k})}\,du + \underset{\vec{T}_v}{(\hat{j} - \hat{k})}\,dv$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix}$ only $\hat{k}$ matters → $1\hat{k}$ but $\hat{n} = 2\hat{i} + \hat{j} + \hat{k}$

$\downarrow \hat{n} = -2\hat{i} - \hat{j} - \hat{k}$

$\iint \vec{R} \cdot d\vec{a} = \iint +A\,du\,dy$



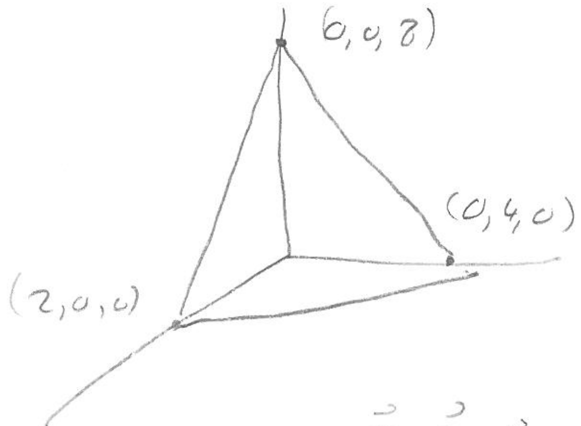
$y = -2x+4$

$\int_0^2 \int_0^{y=4-2x} A\,dy\,dx \qquad \int_0^2 [-2x+4\,dy]\,A \to -x^2+4x \Big]_0^2 A$

$\underline{-4+8 = 4A}$

Same as no roof

What is $\oint \vec{F}\cdot d\vec{r}$ for $\vec{F} = z\hat{i} - z\hat{j} + (x^2-y^2)\hat{k}$ for boundary of $z = 8 - 4x - 2y$



$z - 8 + 4x + 2y = 0$

$+4\hat{i} + 2\hat{j} + \hat{k} = \vec{n}$

$\vec{\nabla}\times\vec{F} = (1-2y)\hat{i} + (1-2x)\hat{j}$

$$\vec{\nabla}\times\vec{F}\cdot\vec{n} = 4(1-2y) + 2(1-2x) = 4+2-4x-8y$$

$$= 6-4x-8y$$

$$\iint \vec{\nabla}\times\vec{F}\cdot\hat{n}\,da = \int_0^2\int_0^{4-2x} 6-4x-8y \; dx\,dy$$

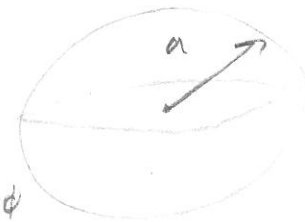
$= -\frac{88}{3}$ or could do 3 separate line integrals

$$\vec{F} = \frac{\vec{r}}{(\vec{r}\cdot\vec{r})^{3/2}} \; ? \qquad \frac{1}{r^2} = \frac{1}{\vec{r}\cdot\vec{r}} \; , \; \hat{r} = \frac{\vec{r}}{(\vec{r}\cdot\vec{r})^{1/2}}$$

$$\frac{1}{r^2}\hat{r} = \frac{\vec{r}}{(\vec{r}\cdot\vec{r})^{1/2}\cdot(\vec{r}\cdot\vec{r})^{2/2}} = \frac{\vec{r}}{(\vec{r}\cdot\vec{r})^{3/2}} = \frac{\vec{r}}{|r|^3}$$

$\oiint \vec{F}\cdot\hat{n}\,|da|$?

$a^2 \sin\theta\, d\theta\, d\phi$



$$\iint \underbrace{\frac{\vec{r}}{|r|^3}\cdot\frac{\vec{r}}{|r|}}\,|da| = \underbrace{\frac{\vec{r}\cdot\vec{r}}{|r|^4}} = \underbrace{\frac{\vec{r}\cdot\vec{r}}{(\vec{r}\cdot\vec{r})^2}}$$

$$= \underbrace{\frac{1}{\vec{r}\cdot\vec{r}}} = \frac{1}{a^2}$$

$$= \frac{4\pi a^2}{a^2} = 4\pi$$

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{|r|^3} \qquad \oiint \vec{E}\cdot\hat{n}\,|d\vec{a}| = \frac{Q}{\epsilon_0}$$

$$\iiint \nabla\cdot\vec{E}\, dV = \frac{Q}{\epsilon_0}$$

$$Q = \iiint q(x,y,z)\,dx\,dy\,dz$$