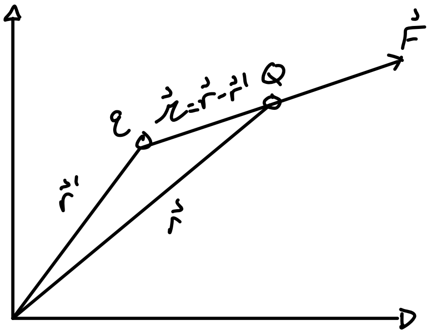
Ch.2 : Electrostatics

9-7-18



What is the force on a test charge Q due to a single charge q, at best a distance $\mathscr{r}$ away?

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}}$$ - Coulomb's law

$\varepsilon_0 = 8.85\times 10^{-12}\ \frac{C^2}{Nm^2}$ - Permativity of Free Space

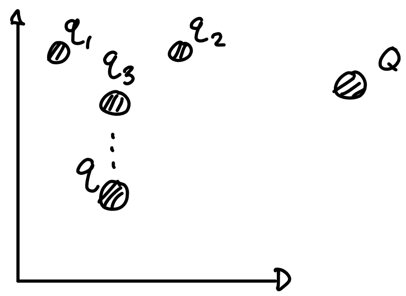
$\vec{r}$ is the position vector of the test charge
$\vec{r}'$ is the position vector of the Source charge
$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ - Is the seperation vector
$\mathscr{r}$ - The magnitude of the seperation vector
$\hat{\mathscr{r}}$ - Unit vector in direction of the Seperation vector

The Electric Field

If we have several point charges $q_1, q_2, \ldots\ldots, q_n$ at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots \mathscr{r}_n$ from our test charge Q,

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots\ldots + \vec{F}_N$$ - Principle of Superposition

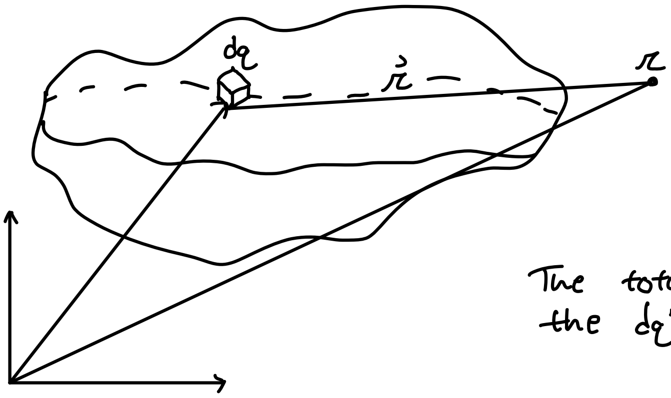
$$= \frac{1}{4\pi\varepsilon_0} Q\left[\frac{q_1\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \ldots + \frac{q_n\hat{\mathscr{r}}_n}{\mathscr{r}_n^2}\right]$$



$$\vec{E} \equiv \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1\hat{\mathscr{r}}_1}{\mathscr{r}_1^2} + \frac{q_2\hat{\mathscr{r}}_2}{\mathscr{r}_2^2} + \ldots \frac{q_n\hat{\mathscr{r}}_n}{\mathscr{r}_n^2}\right]$$ - Electric Field of the Source Charges

$$\therefore \boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}}$$ - The Force per unit charge

Continuous Charge Distribution



Consider a charge distributed continuously over some region . . . . . .

The $\vec{E}$-Field due to the infinitesimal charge is

$$d\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{dq\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The total electric field is found by "summing" over the dq's

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}dq$$

Now — charge of length

$dq = \lambda dl'$ — Infinitesimal length element

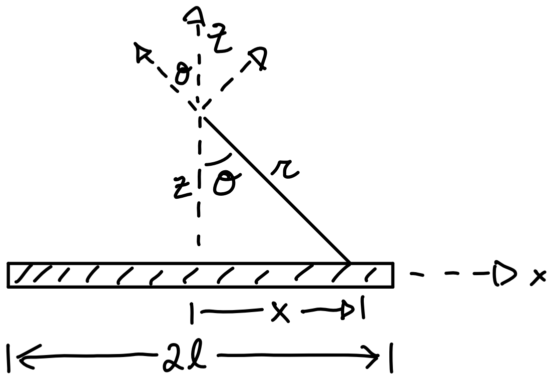
$dq = \sigma da'$ — Infinitesimal area element — Change/Area

$dq = \rho dv'$ — Initesimal volume element — Charge/volume

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')dv'}{\mathscr{r}^2}\hat{\mathscr{r}}}$$ - Continuous charge distribution

9-10-18

ie 2.2



$\vec{E} = E_z\hat{z}$ since $\hat{x}$'s cancel

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{dq}{\mathscr{r}^2}\cos\theta$$

$$\mathscr{r}^2 = z^2 + x^2$$

$$\cos\theta = \frac{z}{\mathscr{r}} = \frac{z^2}{\sqrt{z^2+x^2}}$$

$$\tan\theta_m = L/z$$

$$\sin\theta_m = \frac{L}{\sqrt{z^2+L^2}}$$

$$\therefore dE_z = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda dx}{z^2+x^2}\cdot\frac{z}{\sqrt{z^2+x^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda z dx}{(z^2+x^2)^{3/2}} \therefore E_z = 2\frac{\lambda}{4\pi\epsilon_0}z\int_0^L \frac{dx}{(z^2+x^2)^{3/2}} = \frac{\lambda}{2\pi\epsilon_0}z\int_0^{\theta_m}\frac{z\sec^2\theta d\theta}{(z^2+z^2\tan^2\theta)^{3/2}} = \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\int_0^{\theta_m}\frac{\sec^2\theta d\theta}{\sec\theta}$$

Let $x = z\tan\theta$

$dx = z\sec^2\theta d\theta$

$$= \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\int_0^{\theta_m}\cos\theta d\theta$$

$$= \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\sin\theta_m\Big|_0^{\theta_m}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\sin\theta_m$$

$$= \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\cdot\frac{L}{\sqrt{z^2+L^2}}$$

$$\therefore \boxed{\vec{E}(z) = \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\cdot\frac{L}{\sqrt{z^2+L^2}}\hat{z}}$$

Notice:

$$\lim_{z\gg L}\vec{E}(z) = \frac{\lambda}{2\pi}\cdot\frac{L}{z^2}\cdot\left(1+L^2/z^2\right)^{-1/2}\hat{z} \simeq \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{L}{z^2}\cdot\left(1-\frac{L^2}{2z^2}+z\ldots\right)\hat{z} \therefore \lim_{z\gg L} \simeq \frac{1}{4\pi\epsilon_0}\cdot\frac{2L\lambda}{z^2}\hat{z}$$

$$\lim_{z\gg L} \simeq \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\hat{z}$$ ← $\vec{E}$ field of a point charge

$$\lim_{L\to\infty}\vec{E}(z) = \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{L}\cdot\left(1+z^2/L^2\right)^{-1/2}\hat{z} \simeq \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{z}\hat{z}$$ ← $\vec{E}$ Field of infinitely long strait wire!

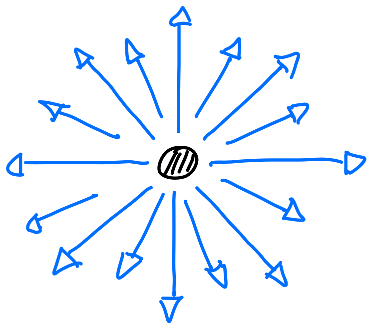
Field Lines, Flux, and Gauss' Law

Place a single point charge q @ the origin....

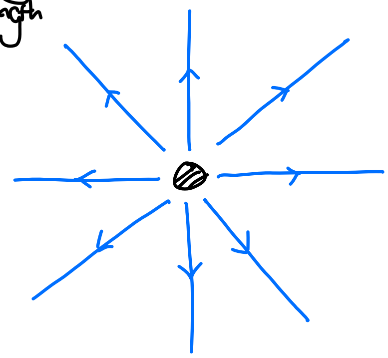
Then $\vec{r}' = 0 \therefore \vec{\mathscr{r}} = \vec{r}$

$$\boxed{\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$$

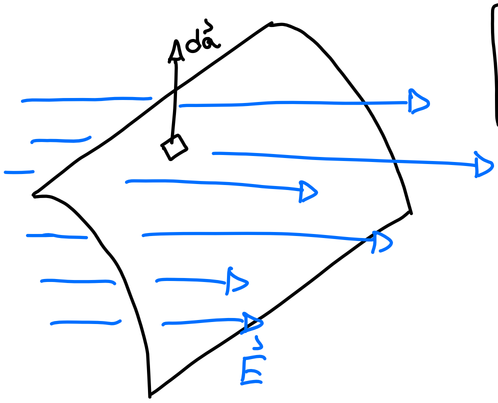
Sketch out this vector-function . . . . .



Alternatively, we can draw field lines
The density of the field lines determines the strength

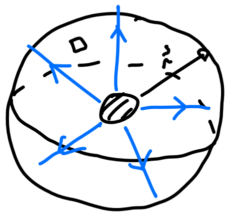


The flux of $\vec{E}$ through a surface S is . . . .



$$\boxed{\Phi_E = \int_S \vec{E} \cdot d\vec{a}}$$ —— Electric flux

Consider a point charge at the origin of the coordinate system . . . . . .
Calculate the flux of $\vec{E}$ through a sphere of radius r.



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \text{so} \quad \vec{E}\cdot d\vec{a} = E\,da = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\,r^2\sin\theta\,d\theta\,d\phi$$

$$d\vec{a} = r^2\sin\theta\,d\theta\,d\phi \qquad\qquad = \frac{1}{4\pi\varepsilon_0}\,q\sin\theta\,d\theta\,d\phi$$

$$\Phi_E = \int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\varepsilon_0}q\sin\theta\,d\theta\,d\phi = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta = \frac{q}{4\pi\varepsilon_0}\cdot 2\pi\cos\theta\Big|_{\pi}^{0} = \frac{q}{\varepsilon_0}$$

The flux through the sphere is independent of the size of the sphere.

9-14-18

Last time . . . . .

The flux of $\vec{E}$-field through surface, S is . . .

$$\Phi_E = \int \vec{E}\cdot d\vec{a}$$

For a charge at origin, flux through sphere of radius r

$$\oint \vec{E}\cdot d\vec{a} = q/\varepsilon_0$$

Now consider a bunch of charges scattered about....

The total E-field is....

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots + \vec{E}_n = \sum_{i=1}^{n} \vec{E}_i$$

The Flux through a closed surface is.....

$$\oint \vec{E}\cdot d\vec{a} = \oint \sum_{i=1}^{n} \vec{E}_i \cdot d\vec{a} = \sum_{i=1}^{n} \oint \vec{E}_i \cdot d\vec{a} = \sum_{i=1}^{n} \frac{q_i}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\sum_{i=1}^{n} q_i\right) = \frac{1}{\varepsilon_0} Q_{Enc}$$

Total charge enclosed by the surface

$$\left[\oint \vec{E}\cdot d\vec{a} = Q_{ENC}/\varepsilon_0\right]$$ - Gauss' Law in Integral Form

Gauss' Law contains no new info not already contained in Coulomb's law.

Now use the Divergence Theorem....

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{E})\, d\tau$$

Also,

$$Q_{ENC} = \int \rho\, d\tau$$

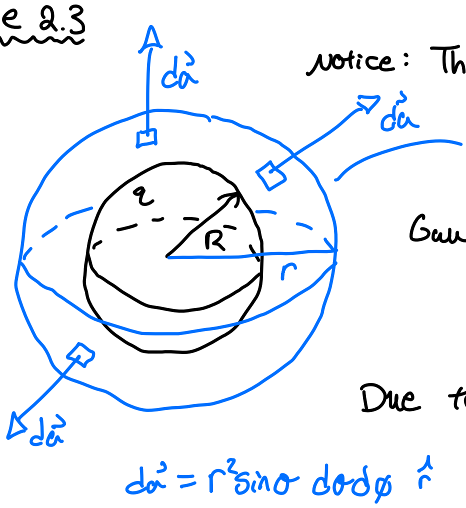
So Gauss' Law Becomes....

$$\int_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$ for any volume V

Integrands are equal $\therefore$

$$\left[\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0\right]$$ - Gauss' Law in Differential Law

i.e 2.3



$d\vec{a} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

Notice: The problem has spherical symmetry

Gaussian Sphere

Gauss' Law in Integral form is....

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{Enc}}{\varepsilon_0} \quad : \quad Q_{Enc} = q$$

Due to spherical symmetry, $\vec{E} = E\hat{r}$

$\vec{E}\cdot d\vec{a} = \int_S E\,da$

$\oint_S \vec{E}\cdot d\vec{a} = \oint_S E\,da$ —— Magnitude of $\vec{E}$ is constant at a given r

$E\int_0^{\pi}\int_0^{2\pi} r^2\sin\theta\,d\phi\,d\theta = 4\pi r^2$

Using Gauss' Law

$$E\cdot 4\pi r^2 = \frac{q}{\varepsilon_0} \quad : \quad E = \frac{q}{4\pi r^2\varepsilon_0}$$

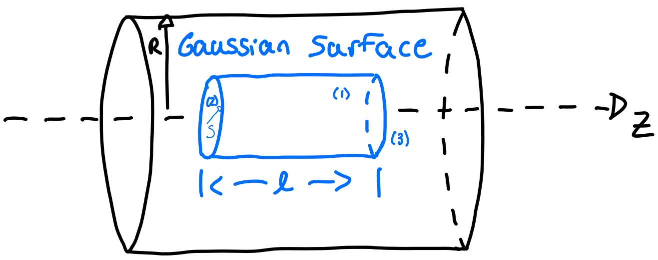
$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}\hat{r}}$$

3 Kinds of Symmetry, 3 types of Gaussian Sphere

1. Spherical Symmetry ⇒ use concentric Sphere
2. Cylindrical Symmetry ⇒ use coaxial Cylinder
3. Plane Symmetry ⇒ use pillbox

i.e. 2.4

$\rho(s) = ks$



Draw a "Gaussian Surface" inside the cylinder?

$$Q_{ENC} = \int \rho\,d\tau' = \iiint ks'\,s'ds'd\phi'dz' = \int_0^{\ell}\int_0^{2\pi}\int_0^{s} ks'\cdot s'\,ds'd\phi'dz'$$

$$= k\int_0^s s'^2ds'\int_0^{2\pi}d\phi'\int_0^{\ell}dz' = \frac{ks'^3}{3}\Big|_0^s\cdot 2\pi\cdot\ell = \frac{2\pi k\ell s^3}{3}$$

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$$\oint \vec{E}\cdot d\vec{a} = \int_1 \vec{E}\cdot d\vec{a} + \int_2 \vec{E}\cdot d\vec{a} + \int_3 \vec{E}\cdot d\vec{a}$$

Due to cylindrical Symmetry, $\vec{E} = E\hat{s}$

⇒ Magnitude of E must be constant at a given S

$$\oint \vec{E}\cdot d\vec{a} = \int_{①} + \int_{②} + \int_{③} \quad \therefore \oint \vec{E}\cdot d\vec{a} = \int_{①} \vec{E}\cdot d\vec{a} = \int_{①} E\,da = E\int_{①} da = E\cdot 2\pi\cdot s\cdot L$$

$d\vec{a} = da\,\hat{s}$

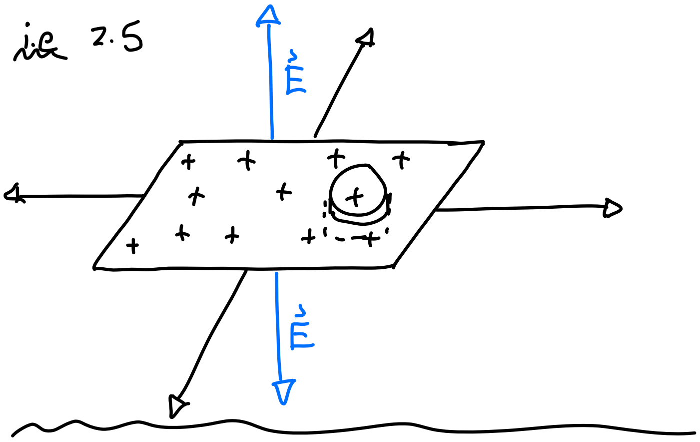
2 & 3 = 0 no Flux through 2 & 3

$d\vec{a} = s\,d\phi\,dz\,\hat{s}$

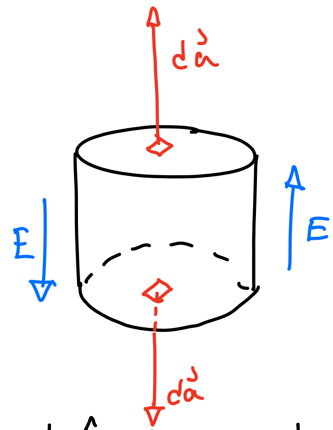
$$E\,2\pi s L = \frac{2\pi}{3\varepsilon_0}\pi l s^2$$

$$E = \frac{K}{3\varepsilon_0} s^2 \longrightarrow \left[\vec{E} = \frac{K}{3\varepsilon_0} s^2\,\hat{s}\right]$$

ie 2.5



$\sigma = \frac{\text{Charge}}{\text{Area}}$



Due to the plane Symmetry

$\vec{E} = E\hat{z}$ - Above

$\vec{E} = -E\hat{z}$ - Below

$$\oint \vec{E}\cdot d\vec{a} = \int_{\text{Top}} \vec{E}\cdot d\vec{a} + \int_{\text{Bottom}} \vec{E}\cdot d\vec{a} + \int_{\text{side}} \vec{E}\cdot d\vec{a}$$

Same

$d\vec{a} = da\,\hat{z}$ above , $da = s\,ds\,d\phi\ \hat{z}$

$d\vec{a} = -da\,\hat{z}$ below , $da = -s\,ds\,d\phi\ \hat{z}$

$\vec{E}\cdot d\vec{a} = E\cdot da$

Area of pillbox

$$2\int \vec{E}\cdot d\vec{a} = 2E\int da = 2E\cdot A$$

$$Q_{Enc} = \sigma A$$

$da = s\,ds\,d\phi$

→ The magnitude of the electric field must be constant

Using Gauss' Law:

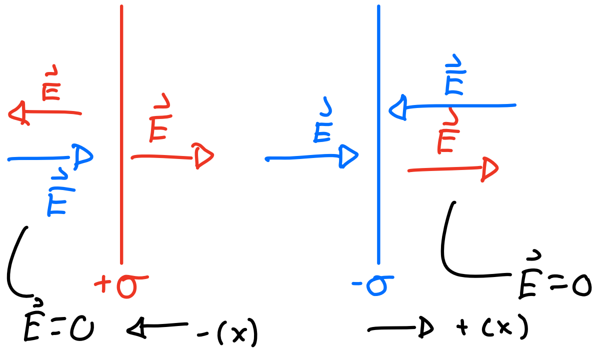
$$2EA = \frac{\sigma A}{\varepsilon_0} \ : \ E = \frac{\sigma}{2\varepsilon_0} \ : \ \left[\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}\right]$$

~ $\hat{n}$ is a unit vector pointing away from the Surface.

The $\vec{E}$. Field of .....

- A sphere falls off like $\frac{1}{r^2}$
- An infinite line falls of like $\frac{1}{r}$
- An infinite Sheet is constant, doesn't fall off

ie 2.6

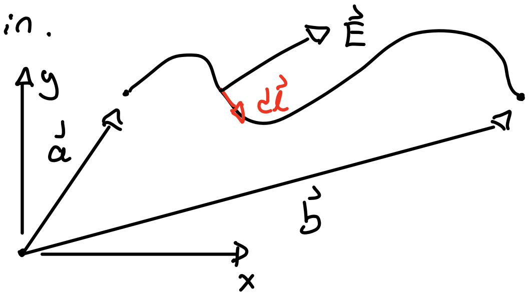


Net $\vec{E}$ field

$$\boxed{\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}}$$ Parallel plate capacitor

The Curl of The Electric Field

Consider a point charge at the origin.



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

calculate the line integral of this E-field

$r_a = |\vec{a}|$, $r_b = |\vec{b}|$

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} \quad : \quad \text{Where } d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r^2}dr \quad : \quad \int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} = \frac{-q}{4\pi\varepsilon_0}r^{-1}\Big|_{r_a}^{r_b}$$

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

now if $r_a = r_b$, (Integration around a closed path) yields....

$$\oint_C \vec{E}\cdot d\vec{l} = 0 = \int_A (\vec{\nabla}\times\vec{E})\cdot d\vec{a} \quad \therefore \text{ Curl of } \vec{E} \text{ is } 0$$

Consider N charges $q_1, q_2, \ldots q_n$

then $\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n)$

$$0 = (\vec{\nabla}\times\vec{E}_1) + (\vec{\nabla}\times\vec{E}_2) + (\vec{\nabla}\times\vec{E}_n)$$

Electrostatic's are curless

9-19-18

Electric potential

The $\vec{E}$ field of an ensemble of static charges is curlless : $\vec{\nabla}\times\vec{E} = 0$

Now, $\vec{E}$ can be written as the gradient of a scalar

$$\vec{E} = -\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}V = 0 \qquad \left[\vec{E} = -\vec{\nabla}V\right] \leftarrow \text{Electric potential}$$

Likewise ....

$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$ is independent of the path

Define a function ...

$$V(\vec{r}) \equiv -\int_{O}^{\vec{r}} \vec{E}\cdot d\vec{l} \qquad O \text{ – is a reference point}$$

So potential difference between two $\vec{a}$ & $\vec{b}$ is ....

$$V(\vec{b}) - V(\vec{a}) = -\int_{O}^{\vec{b}} \vec{E}\cdot d\vec{l} + \int_{O}^{\vec{a}} \vec{E}\cdot d\vec{l} \rightarrow -\int_{O}^{\vec{b}} \vec{E}\cdot d\vec{l} - \int_{\vec{a}}^{O} \vec{E}\cdot d\vec{l}$$

$$\rightarrow = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l}$$

Comments on Potential

(i) Potential & Potential Energy are completely different

(ii) $\vec{E}$ is a vector, V is a scalar

$$\vec{\nabla}\times\vec{E} = 0 \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x}\underbrace{\left(\frac{\partial}{\partial y}(E_z) - \frac{\partial}{\partial z}(E_y)\right)}_{0} + \hat{y}\underbrace{\left(\frac{\partial}{\partial z}(E_x) - \frac{\partial}{\partial x}(E_z)\right)}_{0} + \hat{z}\underbrace{\left(\frac{\partial}{\partial x}(E_y) - \frac{\partial}{\partial y}(E_x)\right)}_{0}$$

(iii) There is an ambiguity in the definition of the potential

Change Refrence point from $O \rightarrow O'$ Then $V(\vec{r}) \rightarrow V'(\vec{r})$

so

$$V'(\vec{r}) = -\int_{O'}^{\vec{r}} \vec{E}\cdot d\vec{l} = \underbrace{-\int_{O'}^{O} \vec{E}\cdot d\vec{l}}_{\equiv K} \underbrace{-\int_{O}^{\vec{r}} \vec{E}\cdot d\vec{l}}_{\equiv V(\vec{r})} = K + V(\vec{r})$$

Notice: Potential difference does not change

$$V'(\vec{b}) - V'(\vec{a}) = (K + V(\vec{b})) - (K + V(\vec{a})) = V(\vec{b}) - V(\vec{a})$$

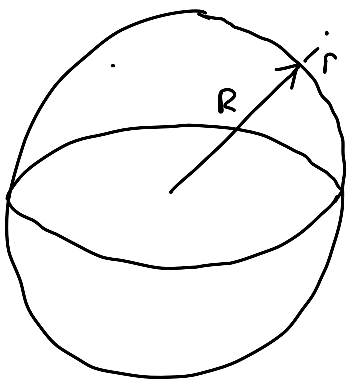
(iv) Potential obey's the superposition principle....

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots \vec{E}_n \qquad \vec{E}_1 = -\vec{\nabla} V_1 ,\ \vec{E}_2 = -\vec{\nabla} V_2 ,\ \vec{E}_n = -\vec{\nabla} V_n = -\vec{\nabla}(\underbrace{V_1 + V_2 + \dots + V_n}_{V})$$

Where $V = V_1 + V_2 + \dots + V_n$

(v) $[F] = N,\ [Q] = C,\ [E] = \frac{F}{Q} = \frac{N}{C},\ [V] = [E][l] = \frac{N}{C}\cdot m = \frac{J}{C} = V$

i.e



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \text{when } r > R$$

$$\vec{E} = 0 \quad \text{when } r < R$$

For points outside the sphere $(r > R)$....

$$V(r) = -\int_{O}^{r} \vec{E}\cdot d\vec{l}$$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin(\theta)\,d\phi\,\hat{\phi}$$

$$V(r) = -\int_{O}^{r} \frac{1}{4\pi\varepsilon_0}\frac{q}{r'^2}\,dr' = -\frac{q}{4\pi\varepsilon_0}\int_{O}^{r} r^{-2}\,dr$$

$$V(r) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r}\Big|_{\infty}^{r} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r} - \frac{q}{4\pi\varepsilon_0}\frac{1}{\infty} = \frac{q}{4\pi\varepsilon_0 r}$$

$O \equiv \infty$

For points inside the sphere $(r < R)$....

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} \Rightarrow -\int_{\infty}^{R} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr - \cancel{\int_{R}^{r} 0\,dr} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R} \Longleftrightarrow \text{not zero}$$

Check:

Outside: $$\vec{E}(\vec{r}) = -\vec{\nabla} V = -\left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]\frac{q}{4\pi\varepsilon_0} r^{-1}$$

$$-\hat{r}\frac{q}{4\pi\varepsilon_0} r^{-1} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

Inside: $\vec{E}(\vec{r}) = -\vec{\nabla} V = 0$

Poisson's EQN: Laplace's EQN

The fundamental EQN's for $\vec{E}$, in differential form are

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{E} = 0$$

now $\vec{E} = -\vec{\nabla} V$ so, $-\vec{\nabla}\cdot(\vec{\nabla} V) = \frac{\rho}{\varepsilon_0}$

$$\left[\nabla^2 V = -\frac{\rho}{\varepsilon_0}\right] \leftarrow \text{Poisson's equation} \Rightarrow \text{Know } V\text{, find } \rho$$

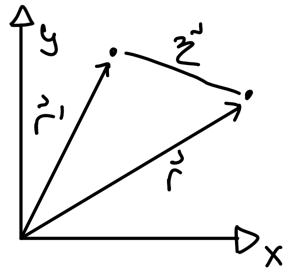
9-21-18

In regions where there is no charge, $\rho = 0$

$$\boxed{\nabla^2 V = 0} \text{ - Laplace's equation}$$

<u>Potential of a localized charge distribution</u>

The potential of a point charge q is . . . .



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$$

For a collection of charges . . . . .

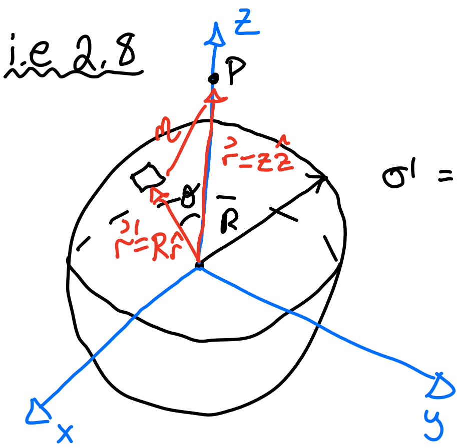
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=0}^{N}\frac{q_i}{\mathscr{r}_i}$$ where we used the principle of super position

For a continuous distribution of charge

$$dq = \rho(\vec{r}\,')\,d\tau'$$

$$\left[V(\vec{r}) = \int dv = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}\right]$$

i.e 2.8



$$\sigma' = \frac{\text{Charge}}{\text{Area}} = \frac{q}{4\pi R^2} \qquad dq = \sigma(r')\,da$$

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\vec{r}\,')\,da}{\mathscr{r}} \qquad da = R^2\sin\theta\,d\theta\,d\phi$$

$$= \vec{r} - \vec{r}\,' = z\hat{z} - R\hat{r}$$

$$\mathscr{r}\mathscr{r} = (z\hat{z} - R\hat{r})\cdot(z\hat{z} - R\hat{r}) = z^2 + R^2 - 2zR(\underbrace{\hat{z}\cdot\hat{r}}_{\cos\theta})$$

$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$$

$$V(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{R^2\sin\theta}{\sqrt{z^2+R^2-2zR\cos\theta}}\,d\theta\,d\phi$$

$$V(z) = \frac{\sigma R^2}{4\pi\varepsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{z^2+R^2-2zR\cos\theta}}$$

$$u = z^2 + R^2 - 2zR\cos\theta$$
$$du = 2Rz\sin\theta\,d\theta \;:\; d\theta = \frac{du}{2Rz\sin\theta}$$

$$V(z) = \frac{\sigma R^2}{4\pi\varepsilon_0} \cdot 2\pi \cdot \frac{1}{2Rz} \int_{(z-R)^2}^{(z+R)^2} \frac{du}{u^{1/2}} \Rightarrow u\Big|_{\theta=\pi} = z^2 + R^2 + 2zR = (z+R)^2$$

$$u\Big|_{\theta=0} = z^2 + R^2 - 2zR = (z-R)^2$$

$$= \frac{\sigma R}{2\varepsilon_0 z} \frac{u^{1/2}}{1/2}\Bigg|_{(z-R)^2}^{(z+R)^2}$$

$$= \frac{\sigma R}{2\varepsilon_0 z}\left[\sqrt{(z+R)^2} - \sqrt{(z-R)^2}\right]$$

Notice: $\sqrt{(z-R)^2} = \pm(z-R)$
$= +(z-R)$ when $z > R$
$= +(R-z)$ when $R > z$

$$V(z) = \frac{\sigma R}{2\varepsilon_0 z}\left[z + R - (z-R)\right] = \frac{\sigma R^2}{\varepsilon_0 z} = \frac{q}{4\pi R^2} \cdot \frac{R^2}{\varepsilon_0 z} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$

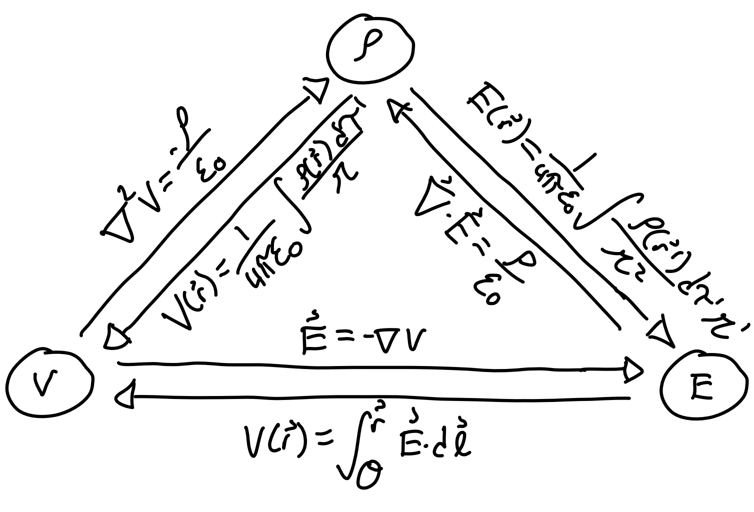
when inside $(z < R)$

$$V(z) = \frac{\sigma R}{2\varepsilon_0 z}\left[z + R - (R - z)\right] = \frac{\sigma R}{\varepsilon_0} = \frac{q}{4\pi R^2} \cdot \frac{R}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

Summary

Two Experimental observations:
(1) Principle of Superposition
(2) Coulomb's law

yields 6 Formulas relating $\rho, \vec{E}, V$

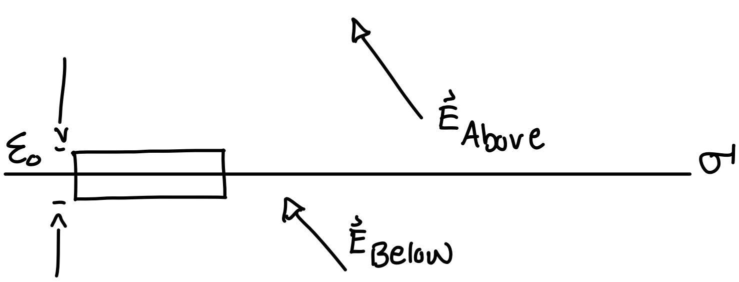


9-24-18

Electrostatic Boundary Conditions

Consider a sheet of surface charge σ

How does $\vec{E}$ change over such a boundary?



Consider a wafer - thin pillbox : Gauss' Law

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{ENC}}{\varepsilon_0}$$

- $\vec{E}$ Flux through side surface is $\emptyset$ in $\lim_{\varepsilon \to 0}$

$$E^{\perp}_{Above} A - E^{\perp}_{Below} A = \frac{1}{\varepsilon_0} \sigma A \quad \text{or} \quad \left[E^{\perp}_{Above} - E^{\perp}_{Below} = \frac{\sigma}{\varepsilon_0} \right]$$

Component of E $\perp$ to surface

The normal component of $\vec{E}$ is discontinuous at the boundary by an amount,

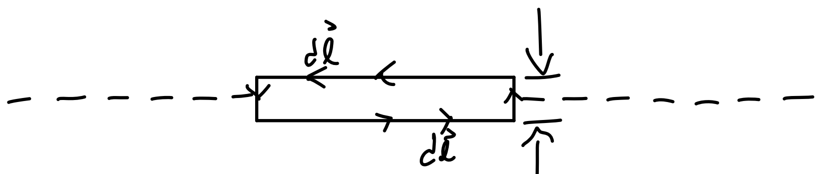
$$\frac{\sigma}{\varepsilon_0}$$

Components of $\vec{E}$ $\parallel$ to Surface

For the magnetic component of $\vec{E}$ . . .

consider a thin rectangular loop

$$\oint \vec{E} \cdot d\vec{l} = 0 \;\therefore\; E^{\parallel}_{Above}\, l - E^{\parallel}_{Below}\, l = 0$$



$$\left[E^{\parallel}_{Above} = E^{\parallel}_{Below} \right]$$

The ends yield Zero as $\lim_{\varepsilon \to 0}$

$\therefore$ The tangential component of $\vec{E}$ is Continuous @ the boundary

These two can be combined into a Singular Formula . . . .

$$\left[\vec{E}_{Above} - \vec{E}_{Below} = \frac{\sigma}{\varepsilon_0} \hat{n} \right]$$

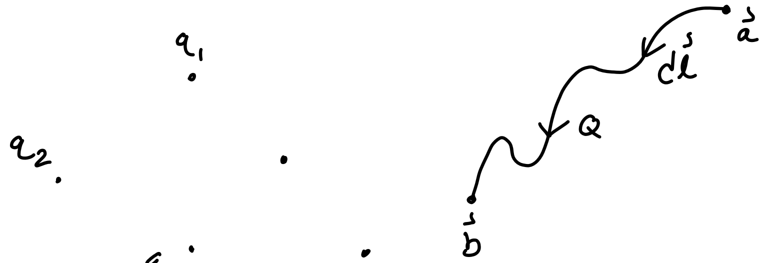
where $\hat{n}$ is a unit vector pointing normal to the surface.

The potential is continuous across any Boundary.

$$V_{above} = V_{below}$$

The work done to move a charge

A static configuration



How much work will you have to do?

$$W = + \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l}$$

where $\vec{F} = -Q\vec{E}$

The Force you must exert in opposition to the E Force

$$W = -Q\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = +Q\left[V(\vec{b}) - V(\vec{a})\right] \text{ since } V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

So,

$$V(\vec{b}) - V(\vec{a}) = \frac{W}{Q}$$

To bring a charge Q in from far away ($r \to \infty$) to point $\vec{r}$.

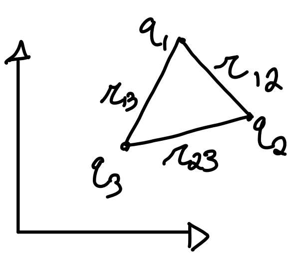
$$\left[W = Q\,V(\vec{r})\right]$$

The Energy of a Point Ditribution

How much work would it take to assemble a collection of point particles

1^{st}, bring in q_1 from ∞ to $\vec{r}_1$ ....

work? $= 0$ Since there is no field to fight against



2^{nd}, bring in q_2 from ∞ to $\vec{r}_2$ ....

$$W_2 = q_2 V_1(\vec{r}_2)$$

The work done on q_2 — Potential of q_1 at $\vec{r}_2$

$$W_2 = q_2 \cdot \frac{1}{4\pi\varepsilon_0}\cdot\frac{q_1}{r_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

The work done on q_3 from ∞ to $\vec{r}_3$ ....

$$W_3 = q_3\left(V_1(\vec{r}_3) + V_2(\vec{r}_3)\right) = q_3\cdot\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_{23}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{23}}$$

The total work neccessary to assemble the first 3 charges is ....

$$W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{q_iq_j}{r_{ij}}$$

$$W = \frac{1}{4\pi\varepsilon_0}\Bigg[\underbrace{\sum_{\substack{i=1\\ j>i}}^{3}\frac{q_1q_j}{r_{ij}}}_{\frac{q_1q_2}{r_{12}}+\frac{q_1q_3}{r_{13}}} + \underbrace{\sum_{\substack{i=1\\ j>i}}^{3}\frac{q_2q_j}{r_{2j}}}_{\frac{q_2q_3}{r_{23}}} + \underbrace{\sum_{\substack{j=1\\ j>i}}^{3}\frac{q_3q_j}{r_{3j}}}_{0}\Bigg]$$

9-26-18

$W = QV(\vec{r})$

In general, the total work necessary to assemble N charges is...

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{r_{ij}} \quad : \; j > i, \text{ so we don't count the same pair twice}$$

Alternatively, count the same pair twice : divide by two.

$$W = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i\left(\sum_{j=1}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right) \Leftrightarrow$$

Potential @ point $\vec{r}$: Due to all the other charges $V(\vec{r})$

$$* \left[W = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)\right]$$

- Amount of work it takes to assemble a configuration of point charges

The Energy of A continuous charge distribution

Let $\sum_{i=1}^{N} q_i \to \int dq = \int \rho \, d\tau$

(*) Becomes . . . .

$$(**) \left[W = \frac{1}{2}\int \rho V \, d\tau\right]$$

- Amount of work it takes to assemble a continuous distribution of charge (In terms $\rho : V$)

Now write this in terms of the $\vec{E}$-field

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

(**) Becomes

$$W = \frac{\varepsilon_0}{2}\int V(\vec{\nabla}\cdot\vec{E})\, d\tau$$

Now, using integration by parts in 3D....

$$\int_{\mathcal{V}} f(\vec{\nabla}\cdot\vec{A})\, d\tau = -\int_{\mathcal{V}} \vec{A}\cdot(\vec{\nabla} f)\, d\tau + \oint_S f\vec{A}\cdot d\vec{a} \qquad (1.59)$$

$$W = -\frac{\varepsilon_0}{2}\int \vec{E}\cdot(\vec{\nabla} V)\, d\tau + \frac{\varepsilon_0}{2}\oint_S V\vec{E}\cdot d\vec{a}$$

$$\left[W = \frac{\varepsilon_0}{2}\int E^2 \, d\tau + \frac{\varepsilon_0}{2}\oint V\vec{E}\cdot d\vec{a}\right]$$

— Notice this volume integral is over the region where the charge distribution is.

- One can integrate over a larger volume where the extra volume will contribute nothing as $\rho = 0$ there.

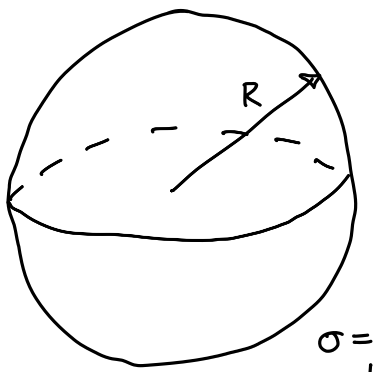
Now,

$V \sim 1/r$, $E \sim 1/r^2$, $da \sim r^2$ So $V\vec{E}\cdot d\vec{a} \sim 1/r$

As the volume goes up, $\oint \vec{v}\cdot d\vec{a}$ goes down

(***) $$\boxed{W = \frac{\varepsilon_0}{2}\int_{\text{All space}} E^2 d\tau}$$

i.e. Find the energy of a uniformly charged spherical shell of total charge q with radius R



$\sigma = \frac{q}{4\pi R^2}$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \quad \text{when } r \le R$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad \text{when } r \ge R$$

using (**).....

$$W = \frac{1}{2}\int \sigma V\, da = \frac{1}{2}\cdot\frac{q}{4\pi R^2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}\int da = \frac{1}{2}\frac{q}{4\pi R^2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R}\cdot 4\pi R^2$$

$$\boxed{W = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{r}}$$

using (***).....

$$W = \frac{\varepsilon_0}{2}\int_{\text{All space}} E^2 d\tau \qquad \text{Now } \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \quad \text{—} \quad r > R$$
$$= 0 \quad \text{—} \quad r < R$$

$$W = \frac{\varepsilon_0}{2}\int_{\text{All space}} E^2 d\tau = \cancel{\frac{\varepsilon_0}{2}\iiint_0^R 0\, d\tau} + \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(\frac{1}{4\pi\varepsilon_0}\right)^2\frac{q^2}{4}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\varepsilon_0}{2}\cdot\frac{1}{16\pi^2\varepsilon_0^2}\, q^2\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\, d\theta\int_R^{\infty}\frac{dr}{r^2}$$

$$W = \frac{1}{2}\cdot\frac{1}{16\pi^2\varepsilon_0}\cdot q^2\cdot 2\pi\cdot 2\cdot\frac{1}{r}\Big|_{\infty}^{R} = \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\cdot\frac{q^2}{R}$$

<u>Comments</u> on <u>electrostatic</u> <u>energy</u>

(i) An inconsistency?

The total energy for a point charge

$$W = \frac{1}{2}\varepsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{q^2}{r^4} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{2}\varepsilon_0 \cdot \frac{q^2}{16\pi^2\varepsilon_0{}^2} \cdot 2\pi \cdot 2 \int_0^{\infty} \frac{dr}{r^2} = \frac{1}{2}\frac{\varepsilon^2}{4\pi\varepsilon_0}\frac{1}{r}\Big|_{\infty}^{0} \rightarrow \infty$$

9-28-18

(ii) where is the energy stored?

$w = \frac{1}{2}\int \rho V \, d\tau$ Suggests energy is stored in charges

$w = \frac{1}{2}\varepsilon_0 \int_{\text{All space}} E^2 d\tau$ Suggests energy is stored in fields

$\Big\}$ unanswerable

$$\frac{\text{Energy}}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2$$

(iii) The superposition principle.....

Electrostatic energy does not obey the superposition principle. Consider a system of two charges...

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{All space}} E^2 d\tau = \frac{1}{2}\varepsilon_0 \int_{\text{All space}} (\vec{E}_1 + \vec{E}_2)\cdot(\vec{E}_1 + \vec{E}_2)\, d\tau = \frac{1}{2}\varepsilon_0 \int_{\text{All space}} (E_1^2 + E_2^2 + 2E_1 \cdot E_2)\, d\tau$$

$$W = W_1 + W_2 + \varepsilon_0 \int \underbrace{\vec{E}_1 \cdot \vec{E}_2}_{\text{Cross terms}} d\tau$$

Conductors

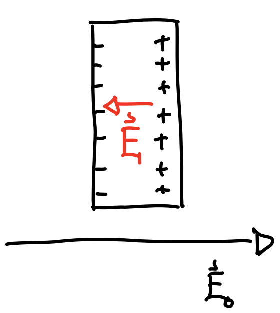
Insulators : electrons are attached to a particular atom (i.e. Glass, Rubber, ETC.)
Conductors : 1 or more electrons/atom are free to move through material.

$\rightarrow$ Perfect Conductor : unlimited supply of completely free electrons.

Properties of Conductors

(i) $\vec{E} = 0$ inside a conductor

Consider a conductor subjected to an External $\vec{E}$-Field



- Free charges produce induced $\vec{E}$-Field, $\vec{E}_0$
- Charges Continue to flow until $\vec{E} = \vec{E}_0 + \vec{E}_1 = 0$

(ii) $\rho = 0$ Inside a conductor

$\rho = \varepsilon_0 (\vec{\nabla} \cdot \vec{E})$ - If $\vec{E}$ Field is zero, $\vec{\nabla} \cdot \vec{E}$ inside is zero
- ρ inside a zero

(iii) Net charge resides on Surface

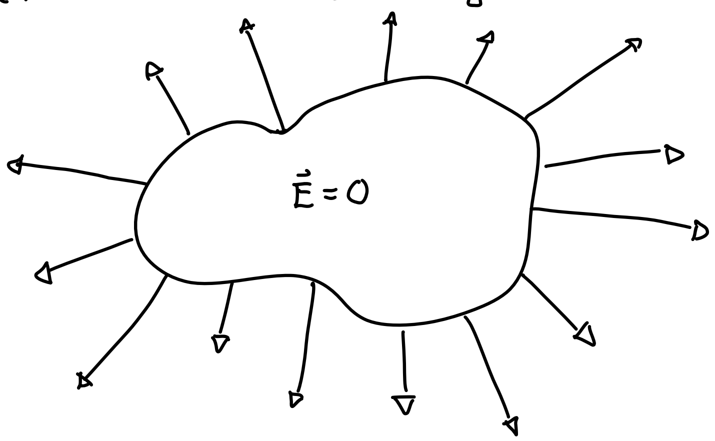
(iv) A conductor is an equipotential

Consider 2 points w/in conductor

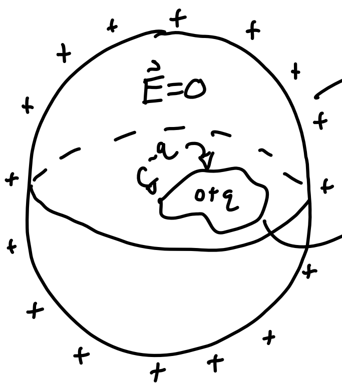
$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \underbrace{\vec{E}}_{0} \cdot d\vec{l} = 0$$

0 since $\vec{E}_{inside} = 0$ $\therefore$ $V(b) = V(a)$

(v) $\vec{E}$ is $\perp$ to the surface just outside the conductor



i.e 2.10

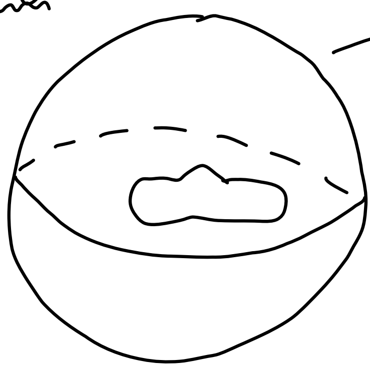


Distributed uniformly

Cavity

$$\left[\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} \right] \text{- outside}$$

i.e



Empty cavity (Free of charge) . . . .

$\vec{E} = 0$ inside the cavity

Proof by contradiction

Let $\vec{E} \neq 0$ inside cavity

$$\oint \vec{E} \cdot d\vec{l} = \underbrace{\int \vec{E} \cdot d\vec{l}}_{\text{Inside cavity}} + \underbrace{\int \cancel{\vec{E} \cdot d\vec{l}}^{\,0}}_{\text{Inside conductor}}$$

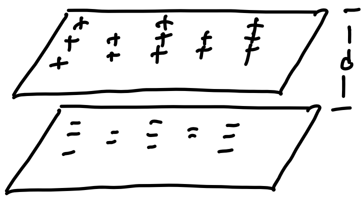
$\oint \vec{E} \cdot d\vec{l} > 0$ if $\vec{E} \neq 0$

But $\oint \vec{E} \cdot d\vec{l} = 0$ $\therefore$ $\vec{E}_{inside} = 0$

10-1-18

Capacitors

consider two conductors, one with charge $+Q$, one with charge $-Q$

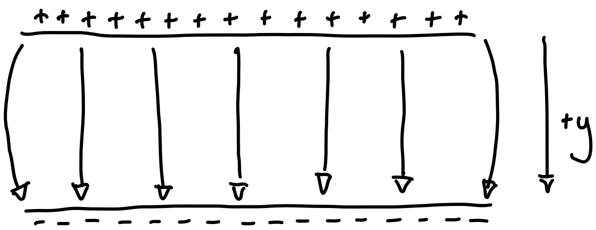


So, $E \sim Q$

Since $V = -\int \vec{E} \cdot d\vec{l}$, $V \sim Q$ Let $Q = CV$ — constant of proportionality; capacitance

$$[C] = \frac{[Q]}{[V]} = \frac{C}{V} = F \text{ - Farad}$$

i.e 2.11

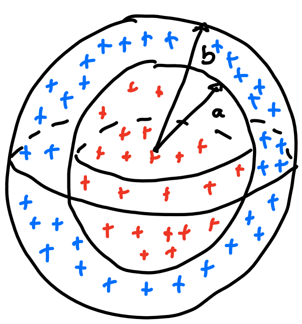


$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{y}$$

$$\Delta V = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}, \quad d\vec{l} = -dy\,\hat{y}, \quad \vec{E} \cdot d\vec{l} = -\frac{\sigma}{\varepsilon_0} dy$$

$$\Delta V = \frac{\sigma}{\varepsilon_0} \int_0^d dy = \frac{\sigma}{\varepsilon_0} \cdot d = \left(\frac{d}{\varepsilon_0 A}\right) \cdot q \qquad \therefore \boxed{C = \frac{\varepsilon_0 A}{d}}$$

i.e 2.12



$+Q$ on inner
$-Q$ on outer

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2}\hat{r} \quad \text{for } a < r < b$$

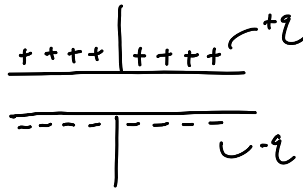
So

$$V(a) - V(b) = -\int_a^b \vec{E} \cdot d\vec{l}, \quad d\vec{l} = dr\,\hat{r}, \quad \vec{E} \cdot d\vec{l} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{dr}{r^2}$$

$$-\int_a^b \frac{Q}{4\pi\varepsilon_0} \cdot \frac{dr}{r^2} = \frac{-Q}{4\pi\varepsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_a^b = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right] = \frac{1}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right) Q \qquad \therefore \boxed{C = 4\pi\varepsilon_0 \left(\frac{ab}{a-b}\right)}$$

How much work does it take to charge the capacitor to a final amount Q?



At some intermediate stage...: $\Delta V = \frac{q}{C}$

The work needed to transport dq is ....

$$dW = dq\,\Delta V = \frac{q}{C}\,dq$$

The total work is ...

$$W = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\cdot\frac{q^2}{2}\Big|_0^Q = \frac{Q^2}{2C}$$

, or in terms of the potential

$$W = \frac{Q^2}{2C} = \frac{C^2V^2}{2C} = \frac{CV^2}{2} \quad : \quad W = \frac{1}{2}\cdot CV^2$$

$$\left[\, W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \,\right]$$ — where V is the potential across the capacitor

Chapter 2 Review

$\vec{E} = \frac{\vec{F}}{Q}$ ← Definition of electric field

$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i$ ← one point charge

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ $\quad$ $\vec{r}$ – point looking @

$\vec{r}'$ – point @ charge

$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')\hat{\mathscr{r}}\,d\tau'}{\mathscr{r}^2}$ ← Continuous charge distribution

$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{r}')\hat{\mathscr{r}}\,da'}{\mathscr{r}^2}$ ← Area

$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\vec{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl'$ ← line

$$\Phi_E \equiv \oint \vec{E}\cdot d\vec{a} = \frac{Q_{ENC}}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \text{or} \quad \vec{\nabla}\times\vec{E} = 0$$

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$\vec{E} = -\vec{\nabla}V$$

$$\nabla^2 V = \frac{-\rho}{\varepsilon_0} \quad : \quad \nabla^2 V = 0 \text{ when } \rho = 0$$

$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$ — n source charges

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau' \longleftarrow \text{Volume}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\vec{r}\,')}{\mathscr{r}}\,da' \longleftarrow \text{Area}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\vec{r}\,')}{\mathscr{r}}\,dl' \longleftarrow \text{Line}$$

$$\vec{E}_{Above} - E_{Below} = \frac{\sigma}{\varepsilon_0}\hat{n} \longleftarrow \text{Electrostatic B.C's}$$

$$W = Q\,V(\vec{r}) \longleftarrow \text{work done}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

$$W = \frac{1}{2}\int \rho V\,d\tau$$

$$W = \frac{1}{2}\varepsilon_0\int_{\text{All space}} E^2\,d\tau$$