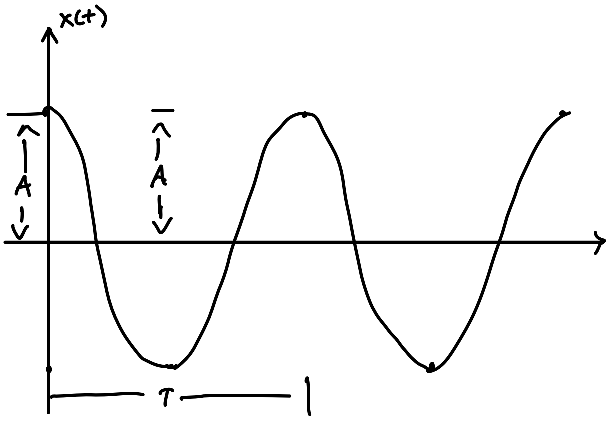
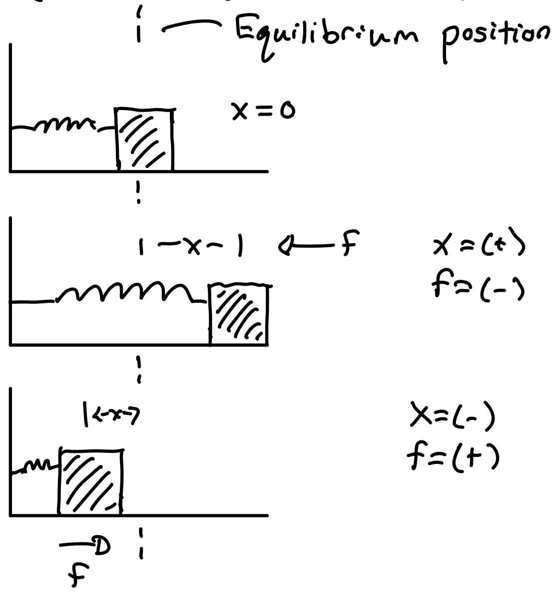
Simple Harmonic Motion

Consider a mass attached to vertical string...
- Call downward (+) x direction
- Displace & Release from rest...
  Plot $x(t)$ vs. t



- A - Amplitude of motion
  $x(t=0) = A$ w/ $[A] = m$
- T is the period of the motion
  $x(t=0) = x(t=T) = A$ w/ $[T] = s$
- ν is the frequency (# of oscillations per unit time)
  w/ $[\nu] = s^{-1}$

Now consider a mass on a frictionless surface attached to a spring....

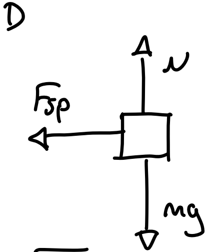


$x = (+)$
$f = (-)$

$x = (-)$
$f = (+)$

In General
- $F = -kx$
  - Force acts in opposite direction as the displacement

FBD



$a_x = \frac{dx}{dt} = \frac{d^2x}{dt^2}$

ΣF_y : $N = mg$

ΣF_x : $-kx = ma_x$

$-kx = m\frac{d^2x}{dt^2}$

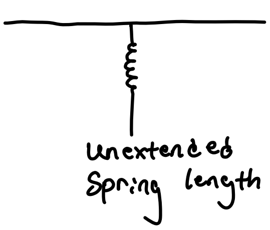
$\frac{d^2x}{dt^2} = -w^2 x$

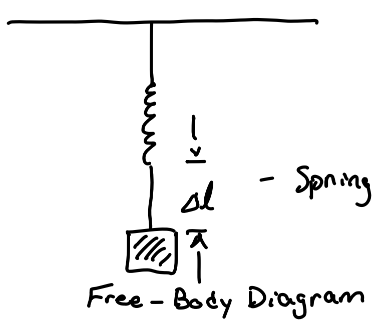
$w = \sqrt{k/m}$

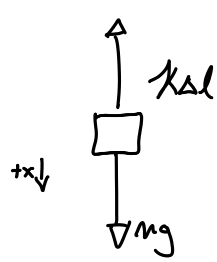
$$\boxed{\frac{d^2x}{dt^2} = -w^2 x} \quad \text{where } w = \sqrt{\frac{k}{m}}$$

- $x(t) = A\cos(wt)$ — show this is a solution
- $\dot{x}(t) = -Aw\sin(wt)$
- $\ddot{x}(t) = -Aw^2\cos(wt) = -w^2[A\cos(wt)] = -w^2 x$

Return to vertical Spring

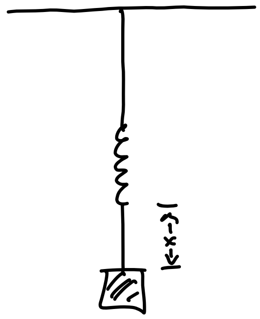






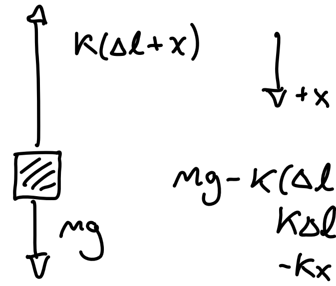
$F_{net} = ma_x$ $a_x = 0$

$mg - k\Delta l = 0$
$mg = k\Delta l$



Now displaced by an additional amount x

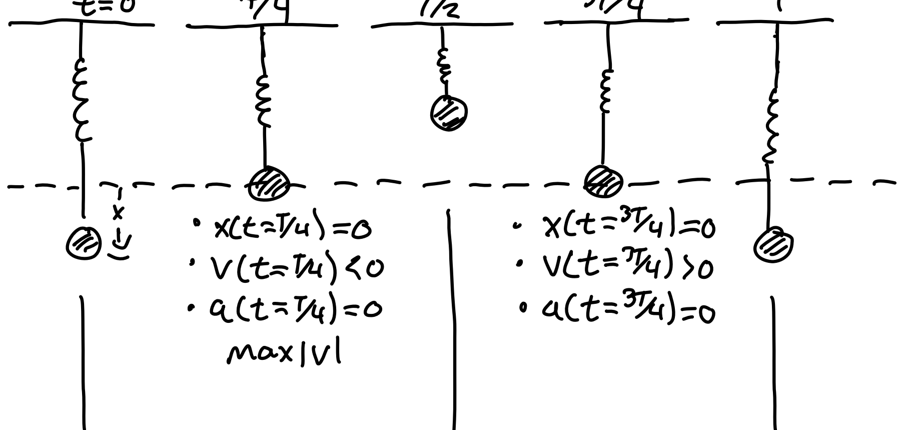
Free Body Diagram....



$$mg - K(\Delta l + x) = ma$$
$$K\Delta l = mg$$
$$-Kx = ma_x = m\frac{d^2x}{dt^2}$$

• Same Diffy Q ← $\left[\frac{d^2x}{dt^2} = -\omega x\right]$

Displacement, velocity, Acceleration in SMH



• $x(t=T/4)=0$
• $V(t=T/4)<0$
• $a(t=T/4)=0$
max $|V|$

• $x(t=3T/4)=0$
• $V(t=3T/4)>0$
• $a(t=3T/4)=0$

• $X(t=0) = A > 0$
• $V(t=0) = 0$
• $a(t=0) = \frac{F(t=0)}{m} < 0$

max $|a|$ since

• $x(t=T/2) = -A < 0$
• $V(t=T/2) = 0$
• $a(t=T/2) > 0$

• $X(t=0) = A > 0$
• $V(t=0) = 0$
• $a(t=0) = \frac{F(t=0)}{m} < 0$

$-Kx = m\frac{d^2x}{dt^2}$
• $x(t) = A\cos(\omega t)$ $\omega = \sqrt{\frac{K}{m}}$ For Simple Harmonic Oscillator
• $V(t) = -A\omega\sin(\omega t)$
• $a(t) = -A\omega^2\cos(\omega t) = -\omega^2 x$
• $\omega = \frac{2\pi}{T}$ — Angular frequency of oscillation

Turning Pts. $|x| = A = x_{max}$, $x(t') = 0$

General Solutions for SHM: Phase Angle

What if instead of letting go from rest, I input an initial velocity

The general solution to...

• $\frac{d^2x}{dt^2} = -\omega^2 x$ $x(t) = a\cos(\omega t) + b\sin(\omega t)$

Another Equivalent Solution is...

$x(t) = A\cos(\omega t + \phi)$ $\phi \equiv$ Phase Angle

- $A\left[\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)\right]$
- $\underbrace{(A\cos(\phi))}_{a}\cos(\omega t) - \underbrace{(A\sin(\phi))}_{b}\sin(\omega t)$

$a\cos(\omega t) + b\sin(\omega t)$

Proof: To General Solution

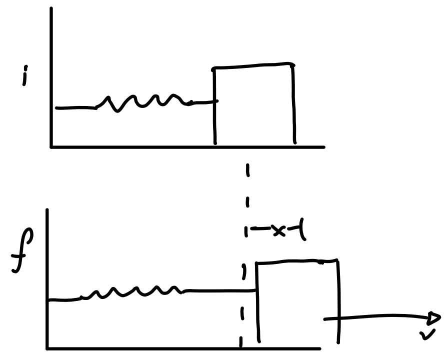
- $\frac{dx}{dt} = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$
- $\frac{d^2x}{dt^2} = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)$

$= -\omega^2(a\cos(\omega t) + b\sin(\omega t)) = -\omega^2 x$

Notice:

- a & b or A & ϕ are constants determined by initial conditions
- ω is a constant determined by properties of springs & mass

Ex



$x_0 = 0.04\,m$

$V_0 = 0.50\,m/s$

$m = 0.80\,kg$

$K = 180\,N/m$

$V(t) = -\omega A\sin(\omega t + \phi)$

$\omega = \sqrt{K/m} = 15\,s^{-1}$

$x(t) = A\cos(\omega t + \phi)$

applying initial conditions

1. $x(t=0) = A\cos(\phi) = 0.04\,m$
2. $V(t=0) = -\omega A\sin(\phi) = 0.50\,m/s$

$2/1$: $\frac{-\omega\sin(\phi)}{\cos(\phi)} = \frac{0.50\,m/s}{0.04\,m}$

$-\omega\tan(\phi) = 12.5\,s^{-1}$

$\phi = \tan^{-1}\left(\frac{12.5}{-\omega}\right)$

$\phi = -39.8° = 320°$

1: $A\cos(320) = 0.04$

$A = \frac{0.04\,m}{\cos(320)} = 0.052\,m$

$A = 0.052\,m$

$\phi = 320°$

The energy of a SHO

- $F = -Kx$ - Hookes law
- $\Delta U = -W = -\int_i^f F\,dx = -\int_0^x -Kx'\,dx' = K\int_0^x x\,dx$

$= \frac{1}{2}Kx^2\Big|_0^x = \frac{1}{2}Kx^2$

- $\Delta U = \frac{1}{2}Kx^2$

Now Newtons Second law....

- $\vec{F}_{net} = m\vec{a}_x$
- $-Kx = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$

multiplying by $dx = v\,dt$

$-Kx\,dx = m\frac{dv}{dt}\cdot v\,dt$

$-Kx\,dx = mv\,dv$

$d(x^2) = 2x\,dx$

$x\,dx = \frac{1}{2}d(x^2)$

$v\,dv = \frac{1}{2}d(v^2)$

$\int \frac{1}{2}d(x^2) = \int \frac{1}{2}m\,d(v^2)$

- $E - \frac{1}{2}Kx^2 = \frac{1}{2}mv^2$

$\left[E = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2\right]$

conservation of energy

The energy of a SHO

$$\left[E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2\right]$$ - conservation of mechanical energy

The position of the mass of the spring...

- $x(t) = A\cos(\omega t + \phi)$
- $v(t) = -\omega A\sin(\omega t + \phi)$

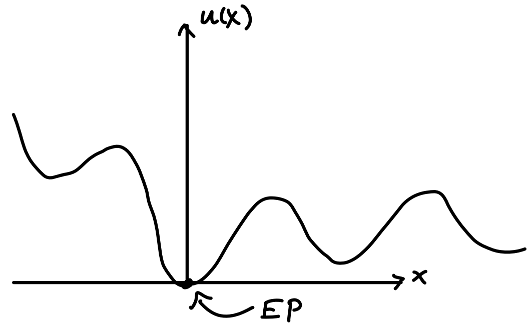
- $E = \frac{1}{2}kA^2\cos^2(\omega t+\phi) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t+\phi)$

$$m\omega^2 = m\frac{k}{m} = k$$

$$= \frac{1}{2}kA^2\left[\cos^2(\omega t+\phi) + \sin^2(\omega t+\phi)\right]$$

- $E = \frac{1}{2}kA^2$

The Physics of Small Vibrations



- Consider the oscillatory motion of a particle in one dimension....

EQ points exist : one is placed at origin $x=0$

For the generic 1D potential, expand the equilibrium point...

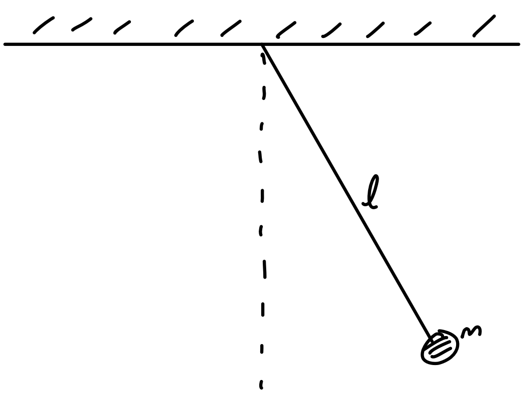
- $$U(x) = U(x=0) + \frac{1}{1!}\frac{du}{dx}(x-0) + \frac{1}{2!}\left.\frac{d^2u}{dx^2}\right|_{x=0}(x-0)^2 + \dots$$

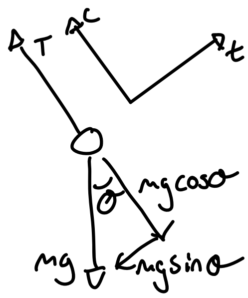
$=0$ since its a constant

- $$F\Big|_{x=0} = -\left.\frac{du}{dx}\right|_{x=0} = 0$$

- $$U(x) = \underbrace{\frac{1}{2}\left.\frac{d^2u}{dx^2}\right|_{x=0}}_{+k} x^2$$
- $$k \equiv \left.\frac{d^2u}{dx^2}\right|_{x=0} > 0$$
- $$U(x) = \frac{1}{2}kx^2$$

The Simple Pendulum





Newton's 2nd Law says....

- $\vec{F}_{net} = m\vec{a}_x$
- $a_t = l\ddot{\theta} = l\frac{d^2\theta}{dt^2}$

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

- $$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta$$

The Taylor Series Expansion

- $$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{6!}\theta^5 + \dots$$

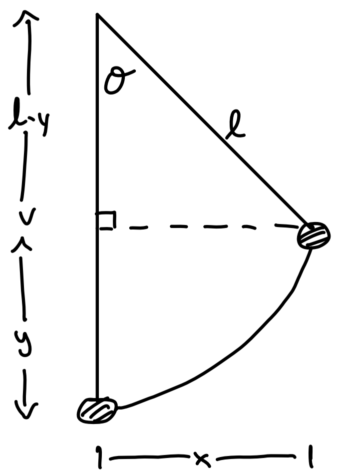
Small θ...

$$\sin\theta \simeq \theta$$

- $$\frac{d^2\theta}{dt^2} = \frac{-g}{l}\sin\theta \qquad \omega_0 = \sqrt{g/l}$$
- $$\frac{d^2\theta}{dt^2} = -\omega_0^2\theta$$ ← A simple harmonic oscillator
  - $\theta_0 \equiv$ Angular amplitude of oscillation

- $$\theta(t) = \theta_0\cos(\omega_0 t + \phi) \qquad \omega_0 = \sqrt{g/l} = \frac{2\pi}{T} \qquad T = 2\pi\sqrt{g/l}$$

The energy of a Simple pendulum



- $l^2 = (l-y)^2 + x^2 = l^2 - 2ly + y^2 + x^2$
- $0 = -2ly + y^2 + x^2$
- $y = \dfrac{y^2+x^2}{2l}$

$y \ll x$ for small θ $\therefore$ $y^2 \ll x^2$

- $y = \dfrac{x^2}{2l}$

The mechanical energy is .....

- $E = \frac{1}{2}mv^2 + mgy$
  $= \frac{1}{2}mv^2 + \dfrac{mgx^2}{2l}$

$x = A$ $\quad v = 0$

- $E = \dfrac{mgA^2}{2l} = \dfrac{1}{2}mv^2 + \dfrac{mgx^2}{2l}$

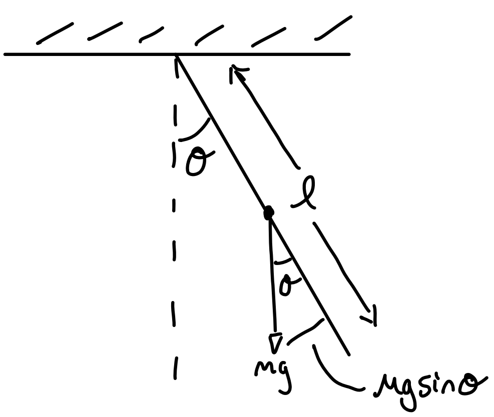
$$V = \sqrt{\frac{g}{l}(A^2 - x^2)} = \frac{dx}{dt}$$

So $\displaystyle\int \sqrt{\frac{g}{l}}\, dt = \int \frac{dx}{\sqrt{A^2 - x^2}}$ $\qquad x = A\sin(\alpha)$, $\quad dx = A\cos(\alpha)\,d\alpha$

- $\displaystyle w_0(t) + \phi = \int \frac{A\cos(\alpha)\,d\alpha}{\sqrt{A^2(1-\sin^2 x)}} = \int dx$
  $= \alpha = \sin^{-1}\left(\dfrac{x}{A}\right)$
- $w_0(t) + \phi = \sin^{-1}\left(\dfrac{x}{A}\right)$

$x = A\sin(w_0 t + \phi)$

The Physical Pendulum



- $\tau_{net} = I\alpha$
- $\tau = \frac{1}{2}l \cdot mg\sin\theta$ $\qquad \alpha = \dfrac{d^2\theta}{dt}$
  $I = \frac{1}{3}ml^2$
- $-\frac{1}{2}mgl\sin\theta = \frac{1}{3}ml^2 \cdot \dfrac{d^2\theta^2}{dt^2}$
- $\dfrac{d^2\theta^2}{dt^2} = \dfrac{-3g}{2l}\sin\theta$

In the small angle approximation ...

- $\sin\theta \simeq \theta$ $\qquad w = \sqrt{\dfrac{3g}{2l}}$
- $\dfrac{d^2\theta^2}{dt^2} = -w^2\theta$