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PHYS 231 HW 6

Ch. 6 # 12, 15, 23, 25, 26, 34, 40, 63, 64, # 10

Problem 12)

a.)

$$E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2} = \frac{h^2 n^2}{2mL^2} = \frac{h^2 n^2}{8mL^2}$$

$$\hbar = \frac{h}{2\pi} \qquad \hbar^2 = \frac{h^2}{4\pi^2}$$

$$E_{n+1} - E_n : \frac{h^2 (n+1)^2}{8mL^2} - \frac{h^2 (n^2)}{8mL^2} = \frac{h^2}{8mL^2}\left(n^2 + 2n + 1 - n^2\right) = (2n+1)\frac{h^2}{8mL^2}$$

$$\boxed{E_{n+1} - E_n = (2n+1)\frac{h^2}{8mL^2}}$$

b.) $E_{n=1} : (2(1)+1)\frac{h^2}{8mL^2} = \frac{3h^2}{8mL^2}$

$$\boxed{E_1 = \frac{3h^2}{8mL^2}}$$

c.) $E_{n=8} : (2(8)+1)\frac{h^2}{8mL^2} = \frac{17h^2}{8mL^2}$

$$\boxed{E_8 = \frac{17h^2}{8mL^2}}$$

d.) $E_{n=800} : (2(800)+1)\frac{h^2}{8mL^2} = \frac{1601h^2}{8mL^2}$

$$\boxed{E_{800} = \frac{1601 h^2}{8mL^2}}$$

Problem 15

$L = 2.0\times10^{-6}\,m \quad T = 13\,K \quad KE = \frac{3KT}{2}$

a.)

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$$

$m = 9.1094\times10^{-31}\,kg$
$h = 6.626\times10^{-34}\,J\cdot s$
$L = 2.0\times10^{-6}\,m$
$n = 1$

$$E_{n=1} = \frac{(1)^2(6.626\times10^{-34}\,J\cdot s)^2}{8(9.1\times10^{-31}\,kg)(2.0\times10^{-6}\,m)^2} = 1.508\times10^{-26}\,J$$

$$E_1 = 1.508\times10^{-26}\,J\;\frac{eV}{1.602\times10^{-19}\,J} = 9.411\times10^{-8}\,eV$$

$\boxed{E_1 = 9.40\times10^{-8}\,eV}$

$m = 9.1094\times10^{-31}\,kg$
$h = 6.626\times10^{-34}\,J\cdot s$
$L = 2.0\times10^{-6}\,m$
$n = 2$

$$E_{n=2} = \frac{(2)^2(6.626\times10^{-34}\,J\cdot s)^2}{8(9.1094\times10^{-31}\,kg)(2.0\times10^{-6}\,m)^2} = 6.031\times10^{-26}\,J$$

$$E_2 = 6.031\times10^{-26}\,J\;\frac{1eV}{1.602\times10^{-19}\,J} = 3.765\times10^{-7}\,eV$$

$\boxed{E_2 = 3.77\times10^{-7}\,eV}$

$m = 9.1094\times10^{-31}\,kg$
$h = 6.626\times10^{-34}\,J\cdot s$
$L = 2.0\times10^{-6}\,m$
$n = 3$

$$E_{n=3} = \frac{(3)^2(6.626\times10^{-34}\,J\cdot s)^2}{8(9.1094\times10^{-31}\,kg)(2.0\times10^{-6}\,m)^2} = 8.46\times10^{-7}\,eV$$

$\boxed{E_3 = 8.46\times10^{-7}\,eV}$

$E_1 = 9.40\times10^{-8}\,eV$
$E_2 = 3.77\times10^{-7}\,eV$
$E_3 = 8.46\times10^{-7}\,eV$

b.) $E = \frac{3}{2}kT$

$E = \frac{n^2h^2}{8mL^2}$

$$\frac{3}{2}kT = \frac{n^2h^2}{8mL^2}$$

$$n^2 = \frac{12mL^2kT}{h^2} = \frac{12(9.1094\times10^{-31}\,kg)(2.0\times10^{-6}\,m)^2(1.3807\times10^{-23}\,J/K)(13K)}{(6.626\times10^{-34}\,J\cdot s)^2} = 17,876$$

$m = 9.1094\times10^{-31}\,kg$
$L = 2.0\times10^{-6}\,m$
$k = 1.3807\times10^{-23}\,J/K$
$T = 13\,K$
$h = 6.626\times10^{-34}\,J\cdot s$

$n^2 = 17,876$
$n = 133.7 \approx 134$

$\boxed{n = 134}$

Problem 23

a.)

$$K_f = \sqrt{\frac{2m(V-E_n)}{\hbar^2}}$$

$$K_i = \sqrt{\frac{2mE_n}{\hbar^2}}$$

$K \propto \frac{1}{\lambda}$ $\therefore$ since $K_f < K_i$, $\lambda_f > \lambda_i$

The wavelength for a finite well is larger.

b.) $K_f = \sqrt{\frac{2m(V-E_n)}{\hbar^2}}$ $\quad$ $K_i = \sqrt{\frac{2m(E_n)}{\hbar^2}}$

$K_f^2 = \frac{2m(V-E_n)}{\hbar^2}$ $\quad$ $K_i^2 = \frac{2m(E_n)}{\hbar^2}$

$\frac{K_f^2\hbar^2}{2m} = V - E_n$ $\quad$ $\frac{K_i^2\hbar^2}{2m} = E_n$

$E_n = V - \frac{K_f^2\hbar^2}{2m}$

larger λ, smaller P, smaller E

Finite: $E_n = V - \frac{K_f^2\hbar^2}{2m}$

Infinite: $E_n = \frac{K_i^2\hbar^2}{2m}$

Since the values for the finite well are going to be less than that of the infinite, we can say $E_f < E_i$. In terms of a physical sense, a larger λ implies a smaller E.

c.) $K_f = \sqrt{\frac{2m(V-E_n)}{\hbar^2}}$

By looking at the wave number equation, there can only be a finite number of E_n because once $E_n > V$, the wave number equation yields an imaginary result which is not possible. No bound states when $E_n > V$.

Problem 25 $E<0$ for bound states

Region 1: $\frac{-\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} = E$

$$\frac{d^2\psi}{dx^2} = \frac{E\psi_I}{} \cdot \frac{2m}{-\hbar^2}$$

$$\frac{d^2\psi_I}{dx^2} = -\frac{2mE\psi_I}{\hbar^2} \qquad \alpha = \sqrt{\frac{-2mE}{\hbar^2}} \quad \text{for } E<0$$

The solution to equation $\frac{d^2\psi_I}{dx^2}$ is $\psi_I(x) = Ae^{\alpha x}$ for region I

Region 2: $\frac{-\hbar^2}{2m}\frac{1}{\psi_{II}}\frac{d^2\psi_{II}}{dx^2} = E + V_0$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} = \psi_{II}(E+V_0)$$

$$\frac{d^2\psi_{II}}{dx^2} = \frac{-2m}{\hbar^2}(E+V_0)\psi_{II} \qquad K = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\frac{d^2\psi_{II}}{dx^2} = -K^2\psi_{II} \longrightarrow \psi_{II}(x) = Ce^{ikx} + De^{-ikx} \quad \text{for region II}$$

Region 3: $\frac{-\hbar^2}{2m}\frac{1}{\psi_{III}}\frac{d^2\psi_{III}}{dx^2} = E$

$$\frac{d^2\psi_{III}}{dx^2} = \frac{-2m}{\hbar^2}E\psi_{III} \qquad \alpha = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\frac{d^2\psi_{III}}{dx^2} = -\alpha^2\psi_{III} \longrightarrow \psi_{III}(x) = Ge^{-\alpha x}$$

At the boundary condition where $x=L$,

$$\psi_{II}(x) = \psi_{III}(x)$$
$$Ce^{ikx} + De^{-ikx} = Ge^{-\alpha x}$$

$$\frac{d\psi_{II}(x)}{dx} = ik(Ce^{ikx} - De^{-ikx})$$

$$\frac{d\psi_{III}(x)}{dx} = -\alpha Ge^{-\alpha x}$$

$$ik(Ce^{ikx} - De^{-ikx}) = -\alpha Ge^{-\alpha x}$$
$$Ce^{ikx} - De^{-ikx} = \frac{i\alpha}{k}Ge^{-\alpha x}$$

$$\frac{k}{i\alpha}(Ce^{ikx} - De^{-ikx}) = Ge^{-\alpha x}$$

$$\frac{k}{i\alpha}(Ce^{ikx} - De^{-ikx}) = Ce^{ikx} + De^{-ikx}$$

$$\frac{k}{i\alpha}Ce^{ikx} - \frac{k}{i\alpha}De^{-ikx} = Ce^{ikx} + De^{-ikx}$$

$$\frac{k}{i\alpha}(Ce^{ikx} - Ce^{ikx}) = \frac{k}{i\alpha}De^{-ikx} + De^{-ikx}$$

$$Ce^{ikx}\left(\frac{k}{i\alpha} - 1\right) = De^{-ikx}\left(\frac{k}{i\alpha} + 1\right)$$

$$\frac{C}{D} = \frac{e^{-ikx}\left(\frac{k}{i\alpha}+1\right)}{e^{ikx}\left(\frac{k}{i\alpha}-1\right)}$$

$$\frac{C}{D} = \frac{e^{-ikx}\left(\frac{k+i\alpha}{i\alpha}\right)}{e^{ikx}\left(\frac{k-i\alpha}{i\alpha}\right)}$$

$$\frac{C}{D} = e^{-2ikx}\left(\frac{k+i\alpha}{k-i\alpha}\right)\frac{i}{i}$$

$$\frac{C}{D} = e^{-2ikx}\left(\frac{ik-\alpha}{ik+\alpha}\right) \qquad x=L$$

$$\boxed{\begin{gathered} Ce^{ikL} + De^{-ikL} = Ge^{-\alpha L} \\ \frac{C}{D} = e^{-2ikL}\left(\frac{ik-\alpha}{ik+\alpha}\right) \end{gathered}}$$

Problem 26

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) + V\psi = E\psi \qquad V = \begin{cases} 0 & x<L,\ y<L,\ z<L \\ \infty \end{cases} \qquad \therefore\ V=0 \qquad \psi = A\sin\left(\frac{n\pi}{L}x\right)$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z) = E\psi(x,y,z) \longrightarrow \psi_x:\ -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x) \longrightarrow \psi(x) = \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L_x}x\right)$$

$$\psi_y:\ -\frac{\hbar^2}{2m}\frac{\partial^2\psi(y)}{\partial y^2} = E\psi(y) \longrightarrow \psi(y) = \sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi}{L_y}y\right)$$

$$\psi_z:\ -\frac{\hbar^2}{2m}\frac{\partial^2\psi(z)}{\partial z^2} = E\psi(z) \longrightarrow \psi(z) = \sqrt{\frac{2}{L_z}}\sin\left(\frac{n_z\pi}{L_z}z\right)$$

Energy Eigen Values

$$E_x = \frac{\pi^2\hbar^2}{2mL_x^2}n_x^2\ ,\quad E_y = \frac{\pi^2\hbar^2}{2mL_y^2}n_y^2\ ,\quad E_z = \frac{\pi^2\hbar^2}{2mL_z^2}n_z^2$$

$$\sum E:\quad \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right) = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)$$

$$E = \frac{\pi h^2}{8m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}+\frac{n_z^2}{L_z^2}\right)$$

	E_n	(n_x, n_y, n_z)
$6E_0$	2	(211), (121), (112)
$9E_0$	3	(221), (212), (122)
$11E_0$	4	(311), (131), (113)
$12E_0$	5	(222), (222), (222)

$2^{nd}, 3^{rd}, 5^{th}$ levels are degenerate

Problem 34

$$\psi_1 = Axe^{-\alpha x^2/2}$$

a.) $\int_0^\infty \psi_1^* \psi_1 \, dx = 1$

$\psi^* = Axe^{-\alpha x^2/2}$

$\psi = Axe^{-\alpha x^2/2}$

$$\int_0^\infty Axe^{-\alpha x^2/2} \cdot Axe^{-\alpha x^2/2} \, dx = 1$$

$$A^2 \int_0^\infty x^2 e^{-\left(\frac{\alpha x^2}{2} + \frac{\alpha x^2}{2}\right)} dx = 1$$

$$A^2 \int_0^\infty x^2 e^{-\alpha x^2} dx = 1$$

$\gamma = \alpha x^2 \quad \therefore \quad x^2 = \frac{\gamma}{\alpha}$

$d\gamma = 2\alpha x \, dx$

$dx = \frac{d\gamma}{2\alpha x} = \frac{d\gamma}{2\alpha\sqrt{\frac{\gamma}{\alpha}}}$

$$2A^2 \int_0^\infty \frac{\gamma}{\alpha} e^{-\gamma} \cdot \frac{1}{2\alpha\sqrt{\gamma/\alpha}} \, d\gamma = 1$$

$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1)$

$$\frac{A^2}{\alpha^2} \int_0^\infty \frac{\gamma e^{-\gamma}}{(\gamma/\alpha)^{1/2}} \, d\gamma = 1$$

$$\frac{A^2}{\alpha^2} \int_0^\infty \frac{\gamma e^{-\gamma}}{(\gamma)^{1/2}(\alpha)^{1/2}} \, d\gamma = 1$$

$$\frac{A^2}{\alpha^{3/2}} \int_0^\infty \gamma^{1/2} e^{-\gamma} \, d\gamma = 1$$

$$\frac{A^2}{\alpha^{3/2}} \left(\Gamma\left(\tfrac{1}{2}+1\right) \right) = 1$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\frac{A^2}{\alpha^{3/2}} \left(\Gamma\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \right) = 1$$

$$\boxed{\psi = \sqrt{\frac{2\alpha^{3/2}}{\sqrt{\pi}}} \, x e^{-\frac{\alpha x^2}{2}}}$$

$$\frac{A^2}{\alpha^{3/2}} \left(\sqrt{\pi}\left(\tfrac{1}{2}\right) \right) = 1$$

$$\frac{A^2}{\alpha^{3/2}} \frac{\sqrt{\pi}}{2} = 1 \implies A = \sqrt{\frac{2\alpha^{3/2}}{\sqrt{\pi}}}$$

b.) $\langle x \rangle = \int_{-\infty}^\infty x \, \psi^* \cdot \psi \, dx$

$$\langle x \rangle = A^2 \int_{-\infty}^\infty x^3 e^{-\alpha x^2} dx$$

Since $x^3 e^{-\alpha x^2}$ is an odd function of x, we can say that $\langle x \rangle = 0$

$\langle x^2 \rangle = \int_{-\infty}^\infty x^2 \cdot \psi^* \cdot \psi \, dx$

$$\langle x^2 \rangle = A^2 \int_{-\infty}^\infty x^2 \cdot x e^{-\alpha x^2/2} \cdot x e^{-\alpha x^2/2} \, dx$$

$\gamma = \alpha x^2 \quad \therefore \quad x^2 = \gamma/\alpha$

$d\gamma = 2\alpha x \, dx \qquad x = \sqrt{\gamma/\alpha}$

$dx = \frac{d\gamma}{2\alpha x}$

$$= 2A^2 \int_{-\infty}^\infty x^4 e^{-\alpha x^2} dx$$

$$= 2A^2 \int_{-\infty}^\infty \frac{\frac{\gamma^2}{\alpha^2} e^{-\gamma}}{2\alpha x} \, d\gamma$$

$$= 2A^2 \int_0^\infty \frac{\gamma^2 e^{-\gamma}}{2\alpha^3 \cdot \frac{1}{\sqrt{\alpha}} \sqrt{\gamma}} \, d\gamma$$

$\Big|_0^\infty$

$$= \frac{A^2}{\alpha^{5/2}} \int_0^\infty \gamma^{3/2} e^{-\gamma} \, d\gamma$$

$n = 3/2$

$$= \frac{A^2}{\alpha^{5/2}} \left(\Gamma(n+1) \right)$$

$$= \frac{A^2}{\alpha^{5/2}} \left(\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) \right)$$

$$\langle x^2 \rangle = \frac{A^2}{\alpha^{5/2}} \frac{3\sqrt{\pi}}{4} \implies \frac{\frac{2\alpha^{3/2}}{\sqrt{\pi}}}{\frac{\alpha^{5/2}}{1}} \cdot \frac{\frac{3\sqrt{\pi}}{4}}{\frac{1}{1}} 2 = \frac{3}{2a}$$

$\langle x^2 \rangle = \frac{3}{2a} \qquad \langle x \rangle = 0$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{2a} - 0} = \sqrt{\frac{3}{2a}}$$

$$\boxed{\langle x \rangle = 0, \ \langle x^2 \rangle = \frac{3}{2a} \quad \Delta x = \sqrt{3/2a}}$$

Problem 40

$T = 12{,}000\ K$ $\quad {}^{12}C$ $\qquad T = \left[1 + \dfrac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)}\right]^{-1}$

$\kappa = \dfrac{\sqrt{2m(V_0 - E)}}{\hbar}$ $\quad \hbar = 1.054 \times 10^{-34}\ J\cdot s$

$E = \frac{3}{2}kT$, $\quad k = 8.617\times10^{-5}\ eV/K$ $\quad m = 1.67\times10^{-27}\ kg$

$V = \dfrac{kqq}{r}$ $\quad r = r_0 A^{1/3}$ $\quad r_0 = 1.2\times10^{-15}\ m$ $\quad q = 6$

$L = 2r$ $\quad A = 12$ $\quad q = 1$

$V_0 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r} = \dfrac{1}{4\pi(8.85\times10^{-12}\ F/m)}\dfrac{6(1.6\times10^{-19}\ C)^2}{12(1.2\times10^{-15}\ m)} = 5.03\times10^{-13}\ J$

$V_0 = 3.14\ MeV$

$E = \frac{3}{2}(8.617\times10^{-5}\ eV/K)(12{,}000\ K) = 1.55\ eV \qquad E = 2.48\times10^{-19}\ J$

$\kappa = \dfrac{\sqrt{2(1.67\times10^{-27}\ kg)(5.03\times10^{-13}\ J - 2.48\times10^{-19}\ J)}}{1.054\times10^{-34}\ J\cdot s} = 3.89\times10^{14}\ m^{-1}$

$L = 2(1.2\times10^{-15}(12)^{1/3}) = 5.49\times10^{-15}\ m$

$$T = \left[1 + \frac{(5.03\times10^{-13}\ J)^2 \sinh^2(3.89\times10^{14}\ m^{-1}(5.49\times10^{-15}\ m))}{4(2.48\times10^{-19}\ J)(5.03\times10^{-13}\ J - 2.48\times10^{-19}\ J)}\right]^{-1} = 1.14\times10^{-7}$$

$$\boxed{T = 1.14\times10^{-7}}$$

Problem 63 $V(x) = D(1-e^{-\alpha(r-r_0)})^2$

$x = \alpha(r-r_0)$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^{(n)}}{n!}$; e^{-x}

$F(0) = 1$

$f'(x) = -e^{-x}$ $f'(0) = -1$ even $= +$

$f''(x) = e^{-x}$ $f''(0) = 1$ odd $= -$

$\sum_{n=0}^{\infty} 1-x \longrightarrow$ Taylor of $(e^{-x}) = 1-x$

$V(x) = D(1-(1-\alpha)(r-r_0))^2$

$= D(1-1+\alpha(r-r_0))^2$

$= D(\alpha(r-r_0))^2$

$V(x) = D\alpha^2(r-r_0)^2$

$\boxed{V(x) = D\alpha^2(r-r_0)^2}$

Problem 64

$$E_v = \hbar\omega\left(n+\tfrac{1}{2}\right) - \frac{\hbar^2\omega^2}{4D}\left(n+\tfrac{1}{2}\right)^2 \qquad \omega = \alpha\sqrt{\frac{2D}{m_1}}$$

$$\hbar\omega = \hbar\alpha\sqrt{\frac{2D}{m_1}} = \hbar\alpha\sqrt{\frac{2D}{\frac{m_1+m_2}{m_1 m_2}}} = \hbar\alpha\sqrt{\frac{2Dm_1m_2}{m_1+m_2}}$$

$$\hbar\omega = \hbar\alpha\sqrt{\frac{2Dm_1m_2}{m_1+m_2}} = \left(6.582\times10^{-16}\,ev\cdot s\right)\left(7.8\times10^{-9}\,m\right)^{-1}\sqrt{\frac{2(4.42\,ev)(39.10\,u)(35.45\,u)\ (3.0\times10^{8}\,m/s)^2}{(39.10\,u)+(35.45\,u)\ (931.5\times10^{6}\,ev/c^2)}}$$

$\alpha = 7.8\times10^{-9}\,m^{-1}$

$m_1 = 39.10\ u$

$m_2 = 35.45\ u$

$\hbar = 6.582\times10^{-16}\ ev\cdot s$

$D = 4.42\ ev$

$\hbar\omega = 0.0106\ ev \qquad \dfrac{\hbar^2\omega^2}{4D} = 6.36\times10^{-6}\ ev$

$$E_0 = 0.0106\,ev\left(\tfrac{1}{2}\right) - 6.36\times10^{-6}\,ev\left(\tfrac{1}{2}\right)$$
$$E_1 = 0.0106\,ev\left(1+\tfrac{1}{2}\right) - 6.36\times10^{-6}\,ev\left(1+\tfrac{1}{2}\right)$$
$$E_2 = 0.0106\,ev\left(2+\tfrac{1}{2}\right) - 6.36\times10^{-6}\,ev\left(2+\tfrac{1}{2}\right)$$
$$E_3 = 0.0106\,ev\left(3+\tfrac{1}{2}\right) - 6.36\times10^{-6}\,ev\left(3+\tfrac{1}{2}\right)$$

Problem 10

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - \lambda)\psi \quad , \quad \psi_n(\xi) = H_n(\xi)e^{-\xi^2/2}$$

$$-2\xi[2a_2\xi + \ldots . n a_n \xi^{n-1}] + (\lambda - 1)[a_0 + \ldots . + a_n \xi^n] = 0$$ only need first two terms

$$-2n a_n \xi^n + (\lambda-1)a_n \xi^n = 0$$
$$(\lambda-1)a_n \xi^n = 2n a_n \xi^n$$
$$(\lambda-1) = 2n$$
$$\lambda = 2n+1$$

$\lambda = 2n+1$

$\frac{\lambda}{2} = n + \frac{1}{2}$

$$\boxed{\hbar\omega_0 \lambda_n/2 = \hbar\omega_0(n+\tfrac{1}{2})} \checkmark$$