6.7 Nonhomogeneous Linear Systems

#9-13

Ex 3) $\bar{x}' = A\bar{x} + \bar{f} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\bar{x} + \begin{bmatrix} \sin 2t \\ 3\cos 2t \end{bmatrix}$

$$\bar{x}_p = \begin{bmatrix} A\cos 2t + B\sin 2t \\ C\cos 2t + D\sin 2t \end{bmatrix} \qquad \bar{x}' = \begin{bmatrix} -2A\sin(2t) + 2B\cos(2t) \\ -2C\sin(2t) + 2D\cos(2t) \end{bmatrix}$$

$$\begin{bmatrix} -2A\sin(2t) + 2B\cos(2t) \\ -2C\sin(2t) + 2D\cos(2t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} A\cos(2t) + B\sin(2t) \\ C\cos(2t) + D\sin(2t) \end{bmatrix} + \begin{bmatrix} \sin 2t \\ 3\cos 3t \end{bmatrix}$$

$$\cos(2t)\begin{bmatrix} 2B \\ 2D \end{bmatrix} + \sin(2t)\begin{bmatrix} -2A \\ -2C \end{bmatrix}$$

$$= -2A\cos(2t) - 2B\sin(2t) + C\cos(2t) + D\sin(2t) + \sin 2t$$

$$= A\cos(2t) + B\sin(2t) - 2C\cos(2t) - 2D\sin(2t) + 3\cos 3t$$

$$\cos(2t)\begin{bmatrix} 2B \\ 2D \end{bmatrix} + \sin(2t)\begin{bmatrix} -2A \\ -2C \end{bmatrix} = \cos(2t)\begin{bmatrix} -2A + C \\ A - 2C + 3 \end{bmatrix} + \sin(2t)\begin{bmatrix} -2B + D + 1 \\ B - 2C - 2D \end{bmatrix}$$

$$\cos(2t)\begin{bmatrix} 2B \\ 2D \end{bmatrix} \qquad \sin(2t)\begin{bmatrix} -2A \\ -2C \end{bmatrix}$$

$-2A + C = 2B$
$A - 2C + 3 = 2D$

$-2B + D + 1 = -2A$
$B - 2D = -2C$

$-2A - 2B + C = 0$
$A - 2C - 2D = -3$
$2A - 2B + D = -1$
$B + 2C - 2D = 0$

$$\left[\begin{array}{cccc|c} -2 & -2 & 1 & 0 & 0 \\ 1 & 0 & -2 & -2 & -3 \\ 2 & -2 & 0 & 1 & -1 \\ 0 & 1 & 2 & -2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -21/65 \\ 0 & 1 & 0 & 0 & 38/65 \\ 0 & 0 & 1 & 0 & 34/65 \\ 0 & 0 & 0 & 1 & 53/65 \end{array}\right]$$

$A = -21/65$
$B = 38/65$
$C = 34/65$
$D = 53/65$

$$x_p = \begin{bmatrix} -\frac{21}{65}\cos(2t) + \frac{38}{65}\sin(2t) \\ \frac{34}{65}\cos(2t) + \frac{53}{65}\sin(2t) \end{bmatrix}$$

x_h: $|A - \lambda I| = 0$

$$x_h = c_1 e^{\lambda_1}\bar{v}_1 + c_2 e^{\lambda_2}\bar{v}_2$$

$$\left|\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix}\right| = 0$$

$(-2-\lambda)(-2-\lambda) - 1(1)$
$\lambda^2 + 4\lambda + 4 - 1$
$\lambda^2 + 4\lambda + 3$
$(\lambda+3)(\lambda+1)$
$\lambda = -1$
$\lambda = -3$

$(A - \lambda I)\bar{v}_1$
$\lambda = -1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x = y \quad \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$(A - \lambda I)\bar{v}_2$
$\lambda = -3$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x = -y \quad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{x(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{65}\begin{bmatrix} -21\cos(2t) + 38\sin(2t) \\ 34\cos(2t) + 53\sin(2t) \end{bmatrix}}$$

6.7.9

$\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

x_p: $x_p = \begin{bmatrix} A \\ B \end{bmatrix}$ $x'_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A + B \\ 4A + B \end{bmatrix} + \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

$A + B = 3$
$4A + B = 9$

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 1 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$ $A = 2$, $B = 1$

$x_p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda) - 1(4)$

$1 - 2\lambda + \lambda^2 - 4$

$\lambda^2 - 2\lambda - 3$

$(\lambda - 3)(\lambda + 1)$

$\lambda = -1, 3$

$(A - \lambda I)\bar{v}_i$

$\lambda = -1$ $x = -\frac{1}{2}y$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = 3$ $x = \frac{1}{2}y$

$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

6.7.10

$\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} e^t \\ -4e^t \end{bmatrix}$

$x_p = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$ $x_p' = \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix}$

$\begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} Ae^t \\ Be^t \end{bmatrix} + \begin{bmatrix} e^t \\ -4e^t \end{bmatrix}$

$\begin{bmatrix} Ae^t + Be^t + e^t \\ 4Ae^t + Be^t - 4e^t \end{bmatrix}$

$e^t \begin{bmatrix} A + B + 1 \\ 4A + B - 4 \end{bmatrix} = e^t \begin{bmatrix} A \\ B \end{bmatrix}$

$A + B + 1 = A$: $B + 1 = 0$: $B = -1$
$4A + B - 4 = B$: $4A - 4 = 0$: $A = 1$

$x_p = e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

x_h: $|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda + 1)(\lambda - 3)$

$\lambda = -1, 3$

$(A - \lambda I)\bar{v}_1 = 0$

$\lambda = -1$ $x = -\frac{1}{2}y$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$(A - \lambda I)\bar{v}_2 = 0$

$\lambda = 3$ $x = \frac{1}{2}y$

$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6.7.11)

$$\bar{x}' = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}\bar{x} + \begin{bmatrix} 3t \\ 9 \end{bmatrix}$$

$\bar{x}_p$: $\begin{bmatrix} At+B \\ Ct+D \end{bmatrix}$ $\bar{x}'_p = \begin{bmatrix} A \\ C \end{bmatrix}$

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} At+B \\ Ct+D \end{bmatrix} + \begin{bmatrix} 3t \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} -Ct-D+3t \\ 3At+3B+4Ct+4D+9 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix} = t\begin{bmatrix} -C+3 \\ 3A+4C \end{bmatrix} + \begin{bmatrix} -D \\ 3B+4D+9 \end{bmatrix}$$

$-C+3=0$ $\quad -D=A$

$3A+4C=0$ $\quad C=3B+4D+9$

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 3 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 4 & -9 \end{array}\right] \begin{bmatrix} 1 & 0 & 4/3 & 0 & 0 \\ 0 & 1 & -1/3 & 4/3 & -3 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$a = -\frac{4}{3}c$: $a=-4$

$b = \frac{1}{3}c - \frac{4}{3}d - 3$: $b = -\frac{22}{3}$

$c = \frac{3}{4}d$: $c=3$

$d=4$: $d=4$

$-2 - \frac{16}{3}$

$-\frac{6}{3} - \frac{16}{3} = -\frac{22}{3}$

$$x_p = \begin{bmatrix} At+B \\ Ct+D \end{bmatrix}$$

$$x_p = t\begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -22/3 \\ 4 \end{bmatrix}$$

x_h: $|A-\lambda I|=0$ $\quad \lambda=1,3$

$$\begin{bmatrix} -\lambda & -1 \\ 3 & 4-\lambda \end{bmatrix}=0$$

$-4\lambda+\lambda^2+3$

$\lambda^2-4\lambda+3$

$(\lambda-3)(\lambda-1)$

$(A-\lambda I)\bar{v}_1=0$

$\lambda=1$ $\quad x=-y$

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$(A-\lambda I)\bar{v}_2=0$

$\lambda=3$ $\quad x=-\frac{1}{3}y$

$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$x(t) = c_1e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + t\begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -22/3 \\ 4 \end{bmatrix}$$

6.7.13

$\bar{x}' = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ -t \end{bmatrix}$

x_p: $\bar{x}_p = \begin{bmatrix} At + B \\ Ct + D \end{bmatrix}$

$\bar{x}'_p = \begin{bmatrix} A \\ C \end{bmatrix}$

$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} At+B \\ Ct+D \end{bmatrix} + \begin{bmatrix} 1 \\ -t \end{bmatrix}$

$\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 2At + 2B + 2Ct + 2D + 1 \\ At + B + 3Ct + 3D - t \end{bmatrix}$

$\begin{bmatrix} A \\ C \end{bmatrix} = t \begin{bmatrix} 2A + 2C \\ A + 3C - 1 \end{bmatrix} + \begin{bmatrix} 2B + 2D + 1 \\ B + 3D \end{bmatrix}$

$2A + 2C = 0$ $\quad -A + 2B + 2D = -1$

$A + 3C = 1$ $\quad B - C + 3D = 0$

$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 \\ -1 & 2 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & -1.375 \\ 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 6.25 \end{array}\right]$

$\bar{x}_p = t \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -11/8 \\ 5/8 \end{bmatrix}$

x_h: $|A - \lambda I| = 0$

$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$

$(2-\lambda)(3-\lambda) - 2(1)$

$6 - 5\lambda + \lambda^2 - 2$

$\lambda^2 - 5\lambda + 4$

$(\lambda - 4)(\lambda - 1)$

$\lambda = 1, 4$

$\lambda = 1$ $\quad (A - \lambda I)\bar{v}_1 = 0$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$x = -2y$ $\quad v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda = 4$ $\quad (A - \lambda I)\bar{v}_2 = 0$

$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x = y$ $\quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x(t) = c_1 e^{t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{t}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} -11 \\ 5 \end{bmatrix}$