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PHYS 231 HW 1

Problem 1

$$\sum_{n=0}^{\infty} \frac{F^n(a)(x-a)^n}{n!} \quad \text{Taylor Series Expansions} \quad \lambda = 10^5$$

i.) $F(x) = \ln(\lambda + 100x), x=0$

$$F(0) = \ln(\lambda + 100(0))$$
$$= \ln(10^5)$$
$$= 5\ln(10)$$

$$F'(x) = \frac{100}{\lambda + 100x} = \frac{100}{100(x+10^3)} = \frac{1}{(x+10^3)} : F'(0) = \frac{1}{10^3}$$

$$F'(x) = \frac{1}{(x+10^3)} : F''(x) = \frac{(x+10^3)(0) - 1(1)}{(x+10^3)^2} = \frac{-1}{(x+10^3)^2} : F''(0) = \frac{-1}{10^6}$$

$$F''(x) = \frac{-1}{(x+10^3)^2} : F'''(x) = \frac{(x+10^3)^2(0) + 1(2(x+10^3)(1))}{(x+10^3)^4} = \frac{2}{(x+10^3)^3} : F'''(0) = \frac{2}{(10)^9} = \frac{1}{5.0\times10^8}$$

$$F'''(x) = \frac{2}{(x+10^3)^3} : F^4(x) = \frac{(x+10^3)^3(0) - 2(3(x+10^3)^2)}{(x+10^3)^6} = \frac{-6}{(x+10^3)^4} : F^4(0) = \frac{-6}{(10^3)^4} = \frac{-3}{5.0\times10^{11}}$$

$$\sum_{n=0}^{\infty} \frac{F^n(a)(x-a)^n}{n!} : 5\ln(10) + \frac{1/10^3\,(x-0)}{1!} + \frac{-1/10^6(x-0)^2}{2!} + \frac{1/5.0\times10^8\,(x-0)^3}{3!} + \frac{-3/5.0\times10^{11}\,(x-0)^4}{4!}$$

$$\boxed{\sum \frac{F^n(a)(x-a)^n}{n!} = 5\ln(10) + \frac{1}{1000}x + \frac{1}{2\cdot10^6}x^2 + \frac{1}{3\cdot10^9}x^3 - \frac{1}{4\cdot10^{12}}x^4}$$

ii.) $F(x) = \sqrt{(\lambda^2 + \lambda x^2)}, \quad x=0$

$$F(0) = \sqrt{(\lambda^2)} = \lambda$$

$$F'(x) : (\lambda^2+\lambda x^2)^{1/2} : \tfrac{1}{2}(\lambda^2+\lambda x^2)^{-1/2}\cdot 2\lambda x = \frac{\lambda x}{\sqrt{\lambda^2+\lambda x^2}} : F'(0) = 0$$

$$F''(x) : f'(x) = \frac{\lambda x}{\sqrt{\lambda^2+\lambda x^2}} = \frac{\lambda\sqrt{\lambda^2+\lambda x^2} - \lambda x \cdot \frac{\lambda x}{\sqrt{\lambda^2+\lambda x^2}}}{(\lambda^2+\lambda x^2)} = \frac{\lambda\sqrt{\lambda^2+\lambda x^2} - \frac{\lambda^2 x^2}{\sqrt{\lambda^2+\lambda x^2}}}{(\lambda^2+\lambda x^2)}$$

$$\frac{\lambda(\lambda^2+\lambda x^2) - \lambda^2 x^2}{\sqrt{\lambda^2+\lambda x^2}\,(\lambda^2+\lambda x^2)} = \frac{\lambda(\lambda^2+\lambda x^2) - \lambda^2 x^2}{(\lambda^2+\lambda x^2)^{3/2}} : F''(0) = \frac{\lambda^3}{(\lambda^2)^{3/2}} = \frac{\lambda^3}{\lambda^3} = 1$$

$$F'''(x) : F''(x) = \frac{\lambda(\lambda^2+\lambda x^2) - \lambda^2 x^2}{(\lambda^2+\lambda x^2)^{3/2}} = \frac{\lambda^3 + \lambda^2 x^2 - \lambda^2 x^2}{(\lambda^2+\lambda x^2)^{3/2}} = \frac{\lambda^3}{(\lambda^2+\lambda x^2)^{3/2}} = \lambda^3(\lambda^2+\lambda x^2)^{-3/2}$$

$$F'''(x) = 0(\lambda^2+\lambda x^2)^{-3/2} + \lambda^3\left(-\tfrac{3}{2}(\lambda^2+\lambda x^2)^{-5/2}(2\lambda x)\right) = \lambda^3\left(-3\lambda x(\lambda^2+\lambda x^2)^{-5/2}\right)$$

$$F'''(x) = \frac{-3\lambda^4 x}{(\lambda^2+\lambda x^2)^{5/2}} \quad : \quad F'''(0) = 0$$

$$F^4(x): \; F'''(x) = -3\lambda^4 x\,(\lambda^2+\lambda x^2)^{-5/2} : F^4(x) = -3\lambda^4(\lambda^2+\lambda x^2)^{-5/2} - 3\lambda^4 x \cdot -\tfrac{5}{2}(\lambda^2+\lambda x^2)^{-7/2} = \frac{1.2\times10^8\sqrt{10}\,(x^2-25000)}{(x^2+1.0\times10^5)^{7/2}}$$

$$F^4(0) = -3\lambda^4(\lambda^2)^{-5/2} = \frac{-3}{100{,}000}$$

$$\lambda + \frac{x^2}{2!} - \frac{\frac{3}{100{,}000}}{4!}x^4 = \lambda + \frac{1}{2}x^2 - \frac{1}{800{,}000}x^4$$

$$1.2\times10^8\sqrt{10}\,x^2 - 1.2\times10^8\sqrt{10}\,(2500)$$

$$f^5(x): \; f^4(x) = 1.2\times10^8\sqrt{10}\,(x^2-2500)(x^2+1.0\times10^5)^{-7/2}$$

$$= (1.2\times10^8\sqrt{10}\,x^2 - 1.423\times10^{12})(x^2+1.0\times10^5)^{-7/2}$$

$$= (2.4\times10^8\sqrt{10}\,x)(x^2+1.0\times10^5)^{-7/2} + (1.2\times10^8\sqrt{10}\,x^2 - 1.423\times10^{12})\left(-\tfrac{7}{2}(x^2+1.0\times10^5)^{-9/2}(2x)\right)$$

$$= \frac{100{,}000^3 \cdot 150000000000\,x(-400000x^2+30000000000)}{(100{,}000^2+100{,}000x^2)^{9/2}}$$

$F^5(0): \; f^5(0) = 0$

I Looked up derivatives for these remaining few

$$F^6(x): \; f^6(x) = \frac{100{,}000^3 \cdot 150000000000(2400000000000x^4 - 2.4\times10^{16}x^2 - 100000^2\cdot1200000x^2 + 100000^2\cdot30000000000)}{(100000^2+100000x^2)^{11/2}}$$

$$F^6(0): \; f^6(0) = \frac{4.5\times10^{21}}{2^{30}\cdot5^{30}}$$

$$f^7(x): \; f^7(x) = \frac{1.5\times10^2(-1.68\times10^{17}x^5 + 4.0\times10^{22}x^3 - 1.08\times10^{27}x)}{(10000000000+100000x^2)^{13/2}}$$

$$f^7(0) = 0$$

$$F^8(x): \; f^8(x) = \frac{1.5\times10^{26}(1.344\times10^{23}x^6 - 5.04\times10^{28}x^4 + 2.52\times10^{33}x^2 - 1.05\times10^{37})}{(10000000000+100000x^2)^{15/2}}$$

$$F^8(0) = -1.575\times10^{-12}$$

$$\boxed{\sum_{n=0}^{\infty}\frac{F^n(a)(x-a)^n}{n!} = \lambda + \frac{1}{2}x^2 - \frac{1}{800000}x^4 - 6.25\times10^{-12}x^6 - 17x^8}$$

Problem 2

i.) $\cosh x = \frac{1}{2}(e^{x}+e^{-x})$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^n}{n!}$$

$f(x) = \cosh(x)$ $\quad f'''(x) = \sinh(x)$ $\quad$ odd = 0
$f(0) = 1$ $\quad f'''(0) = 0$ $\quad$ even = 1

$f'(x) = \sinh(x)$ $\quad f^{4}(x) = \cosh(x)$
$f'(0) = 0$ $\quad f^{4}(0) = 1$

$f''(x) = \cosh(x)$
$f''(0) = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^n}{n!} : 1+0+\frac{x^2}{2}+0+\frac{x^4}{4!}+0+\frac{x^6}{6!}+0+\frac{x^8}{8!} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}$$

Taylor: $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}$ $\quad$: Cosine is an even function $\therefore$ there are no odd's in the expansion

$\cosh(0) = 1$ ✓ $\qquad \cosh(1) = 1.543$ ✓

$\frac{1}{2}(e^{0}+e^{0}) = \frac{1}{2}(1+1) = \frac{1}{2}(2) = 1$ ✓ $\qquad 1+\frac{1^2}{2!}+\frac{1^4}{4!}+\frac{1^6}{6!}+\frac{1^8}{8!} = 1.543$ ✓

$1+\frac{0^2}{2!}+\frac{0^4}{4!}+\frac{0^6}{6!}+\frac{0^8}{8!} = 1$ ✓ $\qquad \frac{1}{2}(e^{1}+e^{-1}) = 1.543$ ✓

From the above calculations, and the Taylor Series expansion about zero, we can say...

$$\boxed{\cosh(x) = \frac{1}{2}(e^{x}+e^{-x})\ ✓}$$

ii.) $\sinh x = \frac{1}{2}(e^{x}-e^{-x})$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^n}{n!}$$

$f(x) = \sinh(x)$ $\quad f''(x) = \sinh(x)$ $\quad f^{4}(x) = \sinh(x)$
$f(0) = 0$ $\quad f''(0) = 0$ $\quad f^{4}(0) = 0$

$f'(x) = \cosh(x)$ $\quad f'''(x) = \cosh(x)$ $\quad$ odd = 1
$f'(0) = 1$ $\quad f'''(0) = 1$ $\quad$ even = 0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^n}{n!} = 0+\frac{x}{1!}+0+\frac{x^3}{3!}+0+\frac{x^5}{5!}+0+\frac{x^7}{7!}+\frac{x^9}{9!} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}$$

Taylor: $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}$ $\quad$: Sine is an odd function, there are no even terms in the expansion

$\sinh(0) = 0$ ✓ $\qquad \sinh(1) = 1.175$ ✓

$\frac{1}{2}(e^{0}-e^{0}) = \frac{1}{2}(0) = 0$ ✓ $\qquad \frac{1}{2}(e^{1}-e^{-1}) = \frac{1}{2}(2.350) = 1.175$ ✓

$0+\frac{0^3}{3!}+\frac{0^5}{5!}+\frac{0^7}{7!}+\frac{0^9}{9!} = 0$ ✓ $\qquad 1+\frac{1^3}{3!}+\frac{1^5}{5!}+\frac{1^7}{7!}+\frac{1^9}{9!} = 1.175$ ✓

$$\boxed{\sinh(x) = \frac{1}{2}(e^{x}-e^{-x})}$$

From the above calculations, and the Taylor Series expansion about zero, we can say...

Problem 3

i.) Prove $1 = \sin^2\theta + \cos^2\theta$ from $e^{i\theta} = \cos\theta + i\sin\theta$

Proof:

$e^{i\theta} = \cos\theta + i\sin\theta \quad (1)$

$e^{-i\theta} = \cos\theta - i\sin\theta \quad (2)$

$(1)\cdot(2)$

$e^{i\theta}\cdot e^{-i\theta} \quad (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$ $\qquad i^2 = -1 \quad \therefore -i^2 = 1$

$e^{0} = \cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta$

$1 = \cos^2\theta + \sin^2\theta$

$$\boxed{1 = \cos^2\theta + \sin^2\theta \quad \checkmark}$$

ii.) Prove $1 = \sin^2\theta + \cos^2\theta$ from Taylor of $\sin^2\theta$ & $\cos^2\theta$ about 0

$\sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$ $\qquad 1 = \sin^2\theta + \cos^2\theta \;:\; \sin^2\theta = 1 - \cos^2\theta$ $\qquad 2\sin(\theta)\cos(\theta) = \sin(2\theta)$

$\sin^2\theta$: $f(\theta) = \sin^2\theta$, $f(0) = 0$ $\quad f'(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta)$, $f'(0) = 0$ $\quad f''(\theta) = 2\cos(2\theta)$, $f''(0) = 2$ $\quad f'''(\theta) = -4\sin(2\theta)$, $f'''(0) = 0$

$f^4(\theta) = -8\cos(2\theta)$, $f^4(0) = -8$ $\quad f^5(\theta) = 16\sin(2\theta)$, $f^5(0) = 0$ $\quad f^6(\theta) = 32\cos(2\theta)$, $f^6(0) = 32$ $\quad f^7(\theta) = -64\sin(2\theta)$, $f^7(0) = 0$

$f^8(\theta) = -128\cos(2\theta)$, $f^8(0) = -128$ $\qquad$ odd - 0, Even - ?

$$\sin^2\theta : \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!} = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32x^6}{6!} - \frac{128x^8}{8!}$$

$\cos^2\theta$: $f(\theta) = \cos^2\theta$, $f(0) = 1$ $\quad f'(\theta) = -\sin(2\theta)$, $f(0) = 0$ $\quad f''(\theta) = -2\cos(2\theta)$, $f''(0) = -2$ $\quad f'''(\theta) = 4\sin(2\theta)$, $f'''(0) = 0$ $\quad f^4(\theta) = 8\cos(2\theta)$, $f^4(0) = 8$

$f^5(\theta) = -16\sin(2\theta)$, $f^5(0) = 0$ $\quad f^6(\theta) = -32\cos(2\theta)$, $f^6(0) = -32$ $\quad f^7(\theta) = +64\sin(2\theta)$, $f^7(\theta) = 0$ $\quad f^8(\theta) = +128\cos(2\theta)$, $f^8(\theta) = \cdot 128$

$$\cos^2\theta \quad \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!} = 1 - \frac{2x^2}{2!} + \frac{8x^4}{4!} - \frac{32x^6}{6!} + \frac{128x^8}{8!}$$

Taking the expansions of $\sin^2\theta + \cos^2\theta$ and summing them we get,

$$\cancel{\frac{2x^2}{2!}} - \cancel{\frac{8x^4}{4!}} + \cancel{\frac{32x^6}{6!}} - \cancel{\frac{128x^8}{8!}} + 1 - \cancel{\frac{2x^2}{2!}} + \cancel{\frac{8x^4}{4!}} - \cancel{\frac{32x^6}{6!}} + \cancel{\frac{128x^8}{8!}} = 1$$

$$\therefore \boxed{\sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}(\cos^2\theta) + \sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}(\sin^2\theta) = 1 \quad \checkmark}$$

Problem 4) $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ $\quad \gamma = 2 \quad \omega_0 = 0.1, 0.5, 1, 2, 4$ $\quad x = 1$ @ $t = 0$, $\dot{x} = 0$ @ $t = 0$

i.) $y'' + 2y' + (0.1)^2 y = 0 \qquad \gamma = 2, \ \omega_0 = 0.1$

Calculator Solution:

$$y = Ae^{\frac{3\sqrt{11}-10}{10}t} + Be^{-\frac{3\sqrt{11}+10}{10}t}$$

$A = 1.00252 \qquad B = -0.002519$

$$\boxed{y = (1.00252)e^{\frac{3\sqrt{11}-10}{10}t} - (0.002519)e^{\frac{-3\sqrt{11}+10}{10}t}}$$

ii.) $y'' + 2y' + (0.5)^2 y = 0 \qquad \gamma = 2, \ \omega_0 = 0.5$

Calculator Solution

$$y = Ae^{\frac{\sqrt{3}-2}{2}t} + Be^{\frac{-2+\sqrt{3}}{2}t}$$

$A = 1.07735 \qquad B = -0.07735$

$$\boxed{y = (1.07735)e^{\frac{\sqrt{3}-2}{2}t} - (0.07735)e^{\frac{-2+\sqrt{3}}{2}t}}$$

iii.) $y'' + 2y' + 1y = 0 \qquad \gamma = 2, \ \omega_0 = 1$

$(r+1)(r+1) \qquad r = -1$

$y = Ae^{-t} + Bte^{-t}$

$1 = A$

$y' = -Ae^{-t} + Be^{-t} - Bte^{-t}$

$0 = -1 + B$

$B = 1$

$$\boxed{y = e^{-t} + te^{-t}}$$

iv.) $y'' + 2y' + 4y = 0 \qquad \gamma = 2, \ \omega_0 = 2$

$\alpha = \frac{-2}{2} = -1 \qquad \beta = \frac{\sqrt{16-4}}{2(1)} = \frac{\sqrt{12}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$y_h = e^{-t}\left(A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)\right)$

$t = 0$: $\quad 1 = A$

$y' = -e^{-t}(A\cos\sqrt{3}t + B\sin\sqrt{3}t) + e^{-t}(-\sqrt{3}A\sin(\sqrt{3}t) + \sqrt{3}B\cos(\sqrt{3}t))$

$0 = -(A) + (B\sqrt{3})$

$1 = B\sqrt{3}$

$B = \frac{1}{\sqrt{3}}$

$$\boxed{y = e^{-t}\left(\cos(\sqrt{3}t) + \frac{\sin(\sqrt{3}t)}{\sqrt{3}}\right)}$$

v.) $y''+2y'+16y=0 \qquad \gamma=2,\ \omega_0=4$

$\alpha=-1 \qquad \beta=\frac{\sqrt{60}}{2}=\frac{2\sqrt{15}}{2}=\sqrt{15}$

$\beta=\sqrt{15}$

$y=e^{-t}\left(A\cos(\sqrt{15}t)+B\sin(\sqrt{15}t)\right)$

$t=0 \qquad 1=(A)$

$y'=-e^{-t}\left(A\cos(\sqrt{15}t)+B\sin(\sqrt{15}t)\right)+e^{-t}\left(-\sqrt{15}A\sin(\sqrt{15}t)+\sqrt{15}B\cos(\sqrt{15}t)\right)$

$t=0 \qquad 0=-(A)+(\sqrt{15}B)$

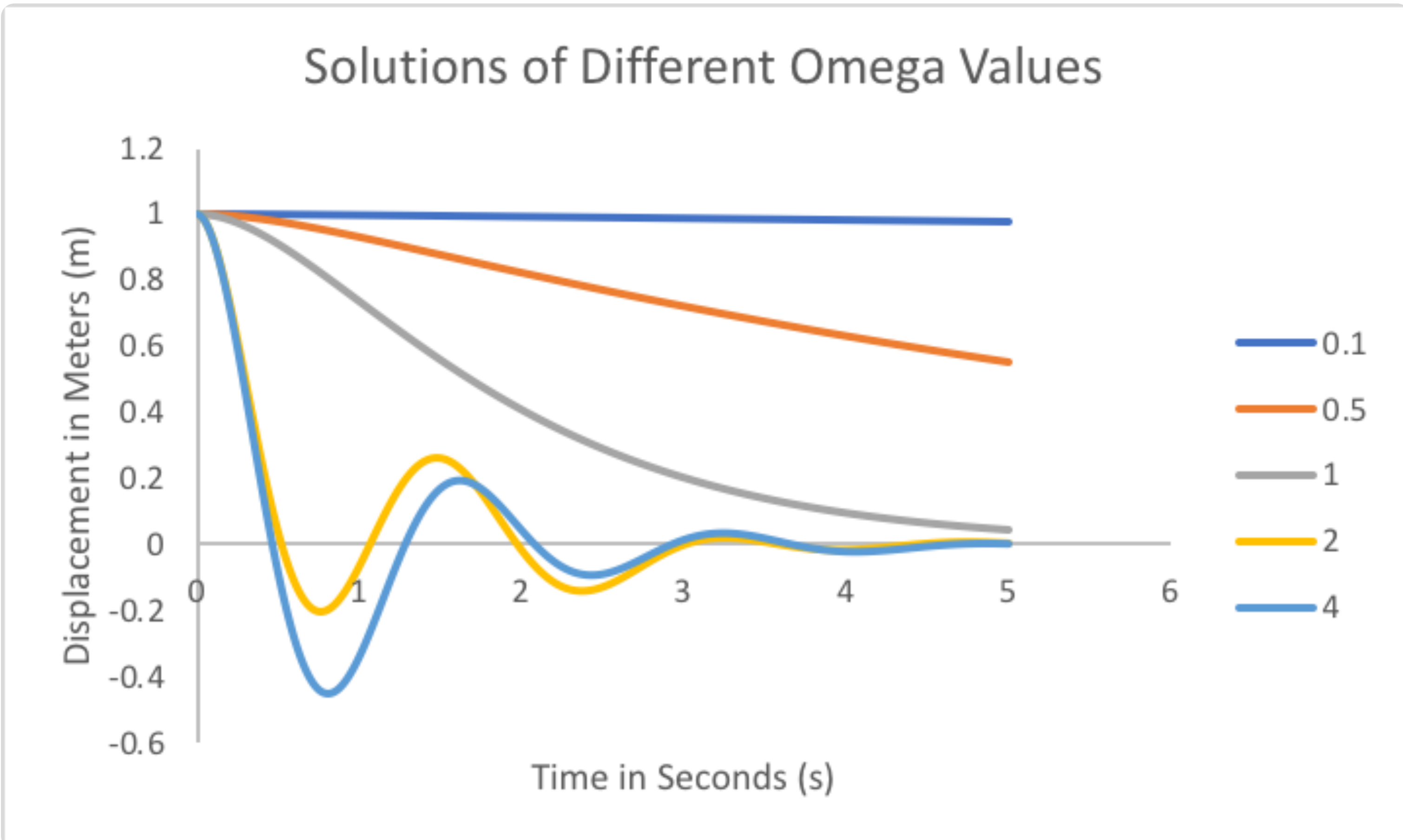
$1=\sqrt{15}B$

$B=1/\sqrt{15}$

$$\boxed{y=e^{-t}\left(\cos(\sqrt{15}t)+\frac{\sin(\sqrt{15}t)}{\sqrt{15}}\right)}$$

Graph of all Equations

Note: The different numerical values found in the legend are the seperate omega values for each subpart.



Problem 5 $\quad U(\bar{r},2) = \epsilon_{well}\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right] \qquad r_0 \equiv y \quad \delta r \equiv x$

$r = r_0 + \delta r$

i.) $U(\bar{r},2) = E\left(\frac{r_0}{r_0+\delta r}\right)^{12} - 2E\left(\frac{r_0}{r_0+\delta r}\right)^{6} \qquad \delta r=0: \; E(1) - 2E = -E$

$\frac{du}{d\delta r}: (r_0)^{12}\cdot -12(r_0+\delta r)^{-13} = \frac{-12 r_0^{12}}{(r_0+\delta r)^{13}}E \qquad (r_0)^6 (r_0+\delta r)^{-6}: \; (r_0)^6 \cdot -6(r_0+\delta r)^{-7} = \frac{-6(r_0)^6}{(r_0+\delta r)^7} = \frac{12E(r_0)^6}{(r_0+\delta r)^7}$

$\frac{du}{d\delta r} = \frac{-12 r_0^{12}}{(r_0+\delta r)^{13}}E + 12E\frac{r_0^6}{(r_0+\delta r)^7} = 12E\left(\frac{r_0^6}{(r_0+\delta r)^7} - \frac{r_0^{12}}{(r_0+\delta r)^{13}}\right) \qquad \delta r=0: \; 12E\left(\frac{1}{r} - \frac{1}{r}\right) = 0$

$\frac{du^2}{d\delta r^2}: -12 r_0^{12} E (r_0+\delta r)^{-13} = -12E r_0^{12}(-13(r_0+\delta r)^{-14}) = \frac{156 r_0^{12} E}{(r_0+\delta r)^{14}} \qquad 12E(r_0)^6(r_0+\delta r)^{-7} = 12E(r_0)^6(-7(r_0+\delta r)^{-8})$

$\frac{-84E(r_0)^6}{(r_0+\delta r)^8}$

$\frac{du^2}{d\delta r^2} = \frac{156E(r_0)^{12}}{(r_0+\delta r)^{14}} - \frac{84E(r_0)^6}{(r_0+\delta r)^8} = 4E\left(\frac{39(r_0)^{12}}{(r_0+\delta r)^{14}} - \frac{21(r_0)^6}{(r_0+\delta r)^8}\right) \qquad \delta r=0: \left(\frac{156E}{r_0^2} - \frac{84E}{r_0^2}\right) = \frac{72E}{r_0^2}$

$\frac{du^3}{d\delta r^3}: 156E(r_0)^{12}(r_0+\delta r)^{-14} = 156E(r_0)^{12}(-14(r_0+\delta r)^{-15}) \qquad -84E(r_0)^6(r_0+\delta r)^{-8} = -84E(r_0)^6(-8(r_0+\delta r)^{-9})$

$\frac{-2184E(r_0)^{12}}{(r_0+\delta r)^{15}} \qquad \frac{672(r_0)^6}{(r_0+\delta r)^9}$

$\frac{du^3}{d\delta r^3} = E\left(\frac{-2184(r_0)^{12}}{(r_0+\delta r)^{15}} + \frac{672(r_0)^6}{(r_0+\delta r)^9}\right) \qquad \delta r=0: \left(\frac{-2184E}{r_0^3} + \frac{672E}{r_0^3}\right) = \frac{-1512E}{r_0^3}$

$\frac{du^4}{d\delta r^4}: 2184E(r_0)^{12}(r_0+\delta r)^{-15} = -2184E(r_0)^{12}(-15(r_0+\delta r)^{-16}) \qquad 672E(r_0)^6(r_0+\delta r)^{-9} = 672E(r_0)^6(-9(r_0+\delta r)^{-10})$

$\frac{32760(r_0)^{12}\cdot E}{(r_0+\delta r)^{-16}} \qquad \frac{-6048E(r_0)^6}{(r_0+\delta r)^{-10}}$

$\frac{du^4}{d\delta r} = E\left(\frac{32760(r_0)^{12}}{(r_0+\delta r)^{16}} - \frac{6048(r_0)^6}{(r_0+\delta r)^{10}}\right) \qquad \delta r=0: \left(\frac{32760E}{r_0^4} - \frac{6048E}{r_0^4}\right) = \frac{26{,}658}{r_0^4}E$

After the second derivative vanishes, the other terms only make very small contributions, but it makes it minutely more accurate. This is the base of perturbation theory.

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-a)^n}{n!} = -1.312\times10^{-21} + \frac{\left(\frac{72E}{r_0^2}\right)x^2}{2!} - \frac{\left(\frac{1512E}{r_0^3}\right)x^3}{3!} + \frac{\left(\frac{26{,}658E}{r_0^4}\right)x^4}{4!}$$

Since the 0^{th} term of this expansion is a constant (-1.312×10^{-21}), and the 3^{rd}, 4^{th}, and suceeding terms are only small contributions, the series expansion can be approximated as follows;

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(x-a)^n}{n!} = U + \frac{\left(\frac{72E}{r_0^2}\right)x^2}{2!} \Rightarrow \boxed{\sum_{n=0}^{\infty}\frac{f^{(n)}(x-a)^n}{n!} = U + \frac{1}{2}\left(\frac{72E}{r_0^2}\right)x^2 = \frac{1}{2}K\delta r^2 \quad K \equiv \left(\frac{72E}{r_0^2}\right) \quad x \equiv \delta r \quad U = -1.312\times10^{-21}J}$$

$(-1.312\times10^{-21} \equiv U)$

ii.) K in terms of L·J parameters

$$u(\vec{r}_o + \delta\vec{r}) \sim \frac{1}{2}K\delta r^2 \Rightarrow \frac{\left(\frac{72E}{r_o^2}\right)x^2}{2!} = \frac{1}{2}\left(\frac{72E}{r_o^2}\right)x^2 \Rightarrow \frac{1}{2}K\delta r^2$$

$$K \equiv \left(\frac{72E}{r_o^2}\right) \quad \delta r \equiv x$$

$$\boxed{K = \left(\frac{72E}{r_o^2}\right)}$$

iii.) Natural frequency,

$$\omega_o = \sqrt{\frac{K}{m}}$$

$$m = 1.16 \times 10^{-26}\,kg$$

$$r_o = 0.415\,nm$$

$$E = 1.312 \times 10^{-21}\,J$$

$$K = \left(\frac{72(1.312 \times 10^{-21})J}{(0.415 \times 10^{-9}m)^2}\right) = 0.5485\,\frac{N}{m} \qquad K = 0.5485\,\frac{N}{m}$$

$$\omega_o = \sqrt{\frac{(0.5485)\frac{N}{m}}{(1.16 \times 10^{-26})kg}} = 6.876 \times 10^{12}\,s^{-1}$$

$$\frac{\frac{kg \cdot m/s^2}{m}}{\frac{kg}{1}} = \frac{1}{s^2} \checkmark$$

$$\boxed{\omega_o = 6.876 \times 10^{12}\,s^{-1}}$$