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PHYS 396
HW 6

Problem 1

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dv^2 + 2\,dv\,dr + r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

a.) Light Cone $ds^2=0$, $d\theta = d\phi = 0$

$$0 = -\left(1-\frac{R_s}{r}\right)dv^2 + 2\,dv\,dr \rightarrow 0 = -\left(1-\frac{R_s}{r}\right)dv + 2\,dr \rightarrow \left(1-\frac{R_s}{r}\right)dv = 2\,dr$$

$$\boxed{\frac{dv}{dr} = \frac{2}{\left(1-\frac{R_s}{r}\right)},\ dv=0}$$

b.)

$$(0, R_s/2) = \frac{2}{1-\frac{R_s}{R_s/2}} = \frac{2}{1-2} = -1 \qquad V = 2\cdot(\ln(|r-R_s|)\cdot R_s + r)$$

$$(0, R_s) = \frac{2}{1-\frac{R_s}{R_s}} = \frac{2}{1-1} = \frac{2}{0} \rightarrow \text{undefined}$$

$$(0, 2R_s) = \frac{2}{1-\frac{R_s}{2R_s}} = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} \rightarrow 4$$

$$(0, 3R_s) = \frac{2}{1-\frac{R_s}{3R_s}} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} \rightarrow 3$$

$$(0, 4R_s) = \frac{2}{1-\frac{R_s}{4R_s}} = \frac{2}{1-\frac{1}{4}} = \frac{2}{\frac{3}{4}} \rightarrow \frac{8}{3}$$

$$\boxed{\begin{aligned} &(0, R_s/2) = -1\\ &(0, R_s) = \text{undefined}\\ &(0, 2R_s) = 4\\ &(0, 3R_s) = 3\\ &(0, 4R_s) = 8/3 \end{aligned}}$$

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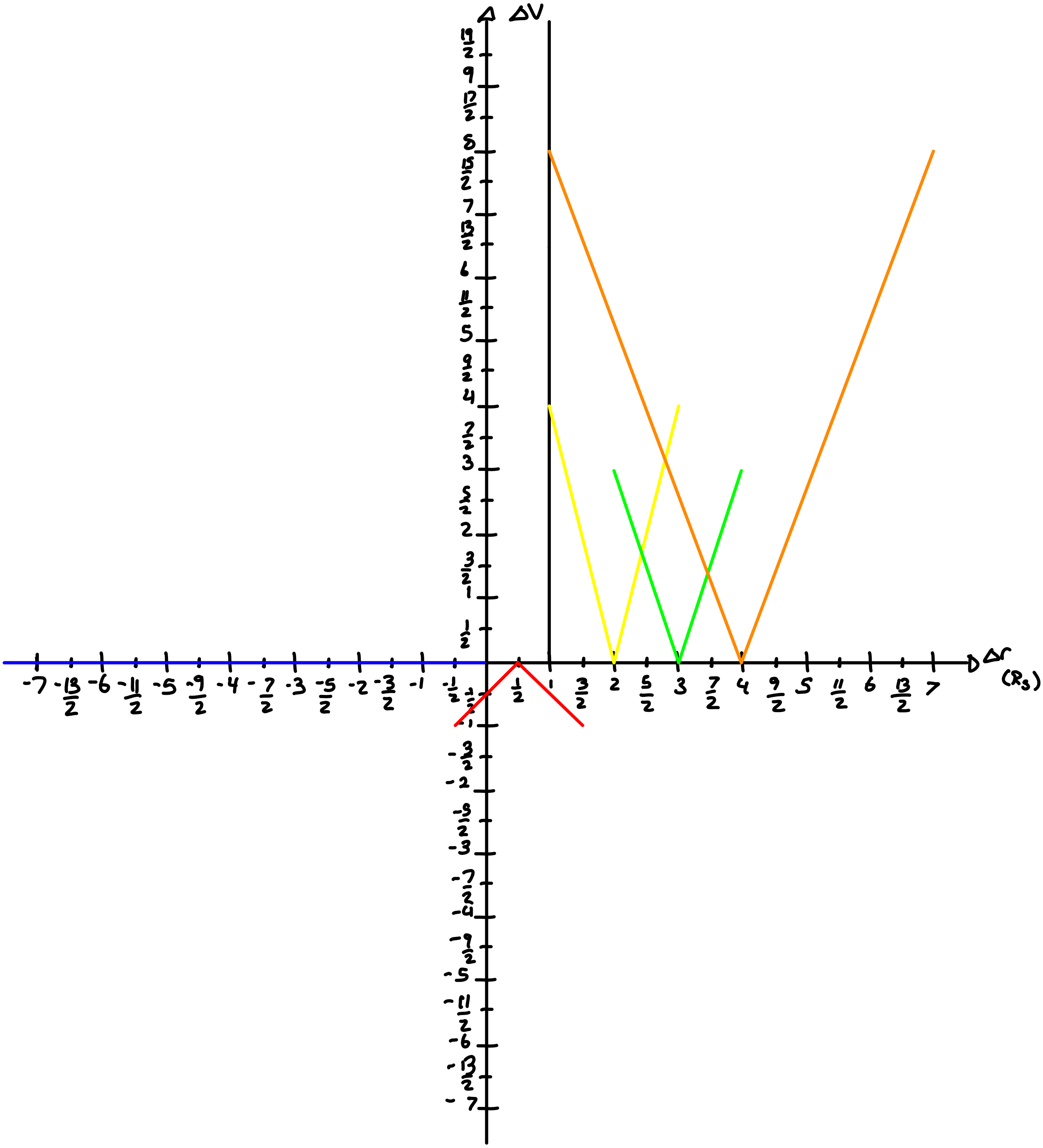
Problem 1 Continued



c.)

$\boxed{\text{As } r \text{ increases, the slopes of the light cones decrease.}}$

Problem 2

$$ds^2 = -c^2dt^2 + dr^2 + (b^2+r^2)(d\sigma^2 + \sin^2(\sigma)\,d\varphi^2)$$

a.) $ds^2 = -c^2(0) + 0^2 + (b^2+R^2)(d\sigma^2 + \sin^2(\sigma)\,d\varphi^2) \qquad \sigma = \frac{\pi}{2}\ ,\ d\sigma = 0$

$$ds^2 = (b^2+R^2)(0^2 + (1)^2 \cdot d\varphi^2) = (b^2+R^2)\,d\varphi^2$$

$$ds = \sqrt{b^2+R^2}\,d\varphi \quad : \quad S = \int_0^{2\pi} \sqrt{b^2+R^2}\,d\varphi = 2\pi\sqrt{b^2+R^2}$$

$$\boxed{S = 2\pi\sqrt{b^2+R^2}}$$

b.) $\sigma = \text{const.}$, $d\sigma = 0$, $\varphi = \text{const.}$, $d\varphi = 0$, $t = \text{const}$, $dt = 0$

$$ds^2 = -c^2(0) + dr^2 + (b^2+R^2)(0+0) = dr^2$$

$$ds^2 = dr^2 \ ; \ ds = dr$$

$$S = \int_{-R}^{R} dr = 2R$$

$$\boxed{S = 2R}$$

c.) $r = R \qquad dA = \sqrt{g_{22}g_{33}}\,dx^2dx^3 \rightarrow dA = \sqrt{(b^2+r^2)^2}\,\sin\sigma\,d\sigma\,d\varphi$

$$A = (b^2+R^2)\int_0^{2\pi}\int_0^{\pi} \sin\sigma\,d\sigma\,d\varphi$$

$$= (b^2+R^2)\int_0^{2\pi} (2)\,d\phi = (b^2+R^2)\,4\pi$$

$$\boxed{A = (b^2+R^2)\,4\pi}$$

d.) $dv = dl^1dl^2dl^3 = \sqrt{g_{11}g_{22}g_{33}}\,dx^1dx^2dx^3 \rightarrow dv = \sqrt{(b^2+r^2)^2}\,\sin\sigma\,dr\,d\sigma\,d\varphi$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (b^2+r^2)\,\sin\sigma\,dr\,d\sigma\,d\varphi$$

$$= \int_0^R (b^2+r^2)\cdot 4\pi\,dr \rightarrow 4\pi\left[b^2r + \frac{r^3}{3}\right]_0^R$$

$$\boxed{V = 4\pi\left[b^2R + \frac{R^3}{3}\right]}$$

Problem 3

$$x = R\sin(\chi)\sin(\theta)\cos(\varphi)$$
$$y = R\sin(\chi)\sin(\theta)\sin(\varphi)$$
$$z = R\sin(\chi)\cos(\theta)$$
$$w = R\cos(\chi)$$

a.) $x^2 = R^2\sin^2(\chi)\sin^2(\theta)\cos^2(\varphi)$, $y^2 = R^2\sin^2(\chi)\sin^2(\theta)\sin^2(\varphi)$, $z^2 = R^2\sin^2(\chi)\cos^2(\theta)$, $w^2 = R^2\cos^2(\chi)$

$$x^2+y^2+z^2+w^2 = R^2\sin^2(\chi)\sin^2(\theta)\cos^2(\varphi) + R^2\sin^2(\chi)\sin^2(\theta)\sin^2(\varphi) + R^2\sin^2(\chi)\cos^2(\theta) + R^2\cos^2(\chi)$$

$$= R^2\sin^2(\chi)\left(\sin^2(\theta)\cos^2(\varphi) + \sin^2(\theta)\sin^2(\varphi) + \cos^2(\theta)\right) + R^2\cos^2(\chi)$$
$$= R^2\sin^2(\chi)\left(\sin^2(\theta)\left(\cos^2(\varphi)+\sin^2(\varphi)\right) + \cos^2(\theta)\right) + R^2\cos^2(\chi)$$
$$= R^2\sin^2(\chi)\left(\sin^2\theta + \cos^2\theta\right) R^2\cos^2(\chi)$$
$$= R^2\sin^2(\chi) + R^2\cos^2(\chi) \rightarrow R^2\left(\sin^2(\chi)+\cos^2(\chi)\right) = R^2 \checkmark$$

$$\boxed{x^2+y^2+z^2+w^2 = R^2}$$

b.) dx, dy, dz, dw

(i) : $dx = \frac{dx}{d\chi} + \frac{dx}{d\theta} + \frac{dx}{d\varphi}$

$\frac{dx}{d\chi} = R\cos(\chi)\sin(\theta)\cos(\varphi)$, $\frac{dx}{d\theta} = R\sin(\chi)\cos(\theta)\cos(\varphi)$, $\frac{dx}{d\varphi} = -R\sin(\chi)\sin(\theta)\sin(\varphi)$

$$\boxed{dx = R\cos(\chi)\sin(\theta)\cos(\varphi)\,d\chi + R\sin(\chi)\cos(\theta)\cos(\varphi)\,d\theta - R\sin(\chi)\sin(\theta)\sin(\varphi)\,d\varphi}$$

(ii) : $dy = \frac{dy}{d\chi} + \frac{dy}{d\theta} + \frac{dy}{d\varphi}$

$\frac{dy}{d\chi} = R\cos(\chi)\sin(\theta)\sin(\varphi)$, $\frac{dy}{d\theta} = R\sin(\chi)\cos(\theta)\sin(\varphi)$, $\frac{dy}{d\varphi} = R\sin(\chi)\sin(\theta)\cos(\varphi)$

$$\boxed{dy = R\cos(\chi)\sin(\theta)\sin(\varphi)\,d\chi + R\sin(\chi)\cos(\theta)\sin(\varphi)\,d\theta + R\sin(\chi)\sin(\theta)\cos(\varphi)\,d\varphi}$$

(iii) : $dz = \frac{dz}{dR} + \frac{dz}{d\chi} + \frac{dz}{d\theta} + \frac{dz}{d\varphi}$

$\frac{dz}{d\chi} = R\cos(\chi)\cos(\theta)$, $\frac{dz}{d\theta} = -R\sin(\chi)\sin(\theta)$, $\frac{dz}{d\varphi} = 0$

$$\boxed{dz = R\cos(\chi)\cos(\theta)\,d\chi - R\sin(\chi)\sin(\theta)\,d\theta}$$

(iv) : $dw = \frac{dw}{d\chi} + \frac{dw}{d\theta} + \frac{dw}{d\varphi}$, $\frac{dw}{d\chi} = -R\sin(\chi)$, $\frac{dw}{d\theta} = 0$, $\frac{dw}{d\varphi} = 0$

$$\boxed{dw = -R\sin(\chi)\,d\chi}$$

Problem 3) continued

c.)

$$dx = \overbrace{R\cos(x)\sin(\sigma)\cos(\varphi)dx}^{A} + \overbrace{R\sin(x)\cos(\sigma)\cos(\varphi)\,d\sigma}^{B} - \overbrace{R\sin(x)\sin(\sigma)\sin(\varphi)\,d\varphi}^{C}$$

$$dx^2 = (A+B+C)(A+B+C)$$

$$A^2 + AB + AC$$
$$+ BA + B^2 + BC$$
$$+ CA + CB + C^2$$

$$dx^2 = 2R^2\cos(x)\sin(\sigma)\cos(\varphi)\sin(x)\cos(\sigma)\cos(\varphi)\,dx\,d\sigma - 2R^2\cos(x)\sin(\sigma)\cos(\varphi)\sin(x)\sin(\sigma)\sin(\varphi)\,dx\,d\varphi$$

$$- 2R^2\sin(x)\cos(\sigma)\cos(\varphi)\sin(x)\sin(\sigma)\sin(\varphi)\,d\sigma\,d\varphi + R^2\cos^2(x)\sin^2(\sigma)\cos^2(\varphi)\,dx^2 +$$

$$R^2\sin^2(x)\cos(\sigma)^2\cos(\varphi)^2\,d\sigma^2 + R^2\sin^2(x)\sin^2(\sigma)\sin^2(\varphi)\,d\varphi^2$$

$$dy = \overbrace{R\cos(x)\sin(\sigma)\sin(\varphi)dx}^{A} + \overbrace{R\sin(x)\cos(\sigma)\sin(\varphi)\,d\sigma}^{B} + \overbrace{R\sin(x)\sin(\sigma)\cos(\varphi)\,d\varphi}^{C}$$

$$dy^2 = (A+B+C)(A+B+C)$$

$$A^2 + AB + AC$$
$$+ BA + B^2 + BC$$
$$+ CA + CB + C^2$$

$$dy^2 = 2R^2\cos(x)\sin(\sigma)\sin(\varphi)\sin(x)\cos(\sigma)\sin(\varphi)\,dx\,d\sigma + 2R^2\cos(x)\sin(\sigma)\sin(\varphi)\sin(x)\sin(\sigma)\cos(\varphi)\,dx\,d\varphi$$

$$+ 2R^2\sin(x)\cos(\sigma)\sin(\varphi)\sin(x)\sin(\sigma)\cos(\varphi)\,d\sigma\,d\varphi + R^2\cos^2(x)\sin^2(\sigma)\sin^2(\varphi)\,dx^2 +$$

$$R^2\sin^2(x)\cos^2(\sigma)\sin^2(\varphi)\,d\sigma^2 + R^2\sin^2(x)\sin^2(\sigma)\cos^2(\varphi)\,d\varphi^2$$

$$dx^2 + dy^2 : 2R^2\cos(x)\sin(\sigma)\sin(x)\cos(\sigma)\big(\overset{1}{\cancel{\cos^2(\varphi) + \sin^2(\varphi)}}\big)\,dx\,d\sigma$$

$$+$$

$$2R^2\cos(x)\sin(\sigma)\sin(x)\sin(\sigma)\big(\overset{0}{\cancel{\sin(\varphi)\cos(\varphi) - \sin(\varphi)\cos(\varphi)}}\big)\,dx\,d\varphi$$

$$+$$

$$2R^2\sin(x)\cos(\sigma)\sin(x)\sin(\sigma)\big(\overset{0}{\cancel{\sin(\varphi)\cos(\varphi) - \sin(\varphi)\cos(\varphi)}}\big)\,d\sigma\,d\varphi$$

$$+$$

$$R^2\cos^2(x)\sin^2(\sigma)\,dx^2\big(\overset{1}{\cancel{\sin^2(\varphi) + \cos^2(\varphi)}}\big) + R^2\sin^2(x)\cos(\sigma)^2\,d\sigma^2\big(\overset{1}{\cancel{\sin^2(\varphi) + \cos^2(\varphi)}}\big)$$

$$+$$

$$R^2\sin^2(x)\sin^2(\sigma)\,d\varphi^2\big(\overset{1}{\cancel{\cos^2(\varphi) + \sin^2(\varphi)}}\big)$$

$$\boxed{dx^2 + dy^2 = 2R^2\cos(x)\sin(\sigma)\sin(x)\cos(\sigma)\,dx\,d\sigma + R^2\cos^2(x)\sin^2(\sigma)\,dx^2 + R^2\sin^2(x)\cos(\sigma)^2\,d\sigma^2 + R^2\sin^2(x)\sin^2(\sigma)\,d\varphi^2}$$

d.) $dz^2 = R^2\cos^2(x)\cos^2(\sigma)\,dx^2 - 2R^2\cos(x)\sin(\sigma)\sin(x)\cos(\sigma)\,dx\,d\sigma + R^2\sin^2(x)\sin^2(\sigma)\,d\sigma^2$

$$dx^2 + dy^2 + dz^2 = 2R^2\cos(x)\sin(\sigma)\cancel{\sin(x)}\cos(\sigma)\,dx\,d\sigma + R^2\cos^2(x)\sin^2(\sigma)\,dx^2 + R^2\sin^2(x)\cos(\sigma)^2\,d\sigma^2$$

$$+ R^2\sin^2(x)\sin^2(\sigma)\,d\varphi^2 + R^2\cos^2(x)\cos^2(\sigma)\,dx^2 - 2R^2\cos(x)\cancel{\sin(\sigma)}\sin(x)\cos(\sigma)\,dx\,d\sigma +$$

$$R^2\sin^2(x)\sin^2(\sigma)\,d\sigma^2 = R^2\sin^2(x)\,d\sigma^2\big(\cos^2(\sigma) + \sin^2(\sigma)\big) + R^2\cos^2(x)\,dx^2\big(\sin^2(\sigma) + \cos^2(\sigma)\big) +$$

$$+ R^2\sin^2(x)\sin^2(\sigma)\,d\varphi^2 \quad : \quad dx^2 + dy^2 + dz^2 = R^2\left[\sin^2(x)\,d\sigma^2 + \cos^2(x)\,dx^2 + \sin^2(x)\sin^2(\sigma)\,d\varphi^2\right]$$

Problem 3] Continued

$$\boxed{dx^2+dy^2+dz^2=R^2\left[\cos^2(\chi)\,d\chi^2+\sin^2(\chi)\left(d\theta^2+\sin^2(\theta)\,d\phi^2\right)\right]}$$

e.) $dw^2 = R^2\sin^2(\chi)\,d\chi^2$

$$dx^2+dy^2+dz^2+dw^2=R^2\left[d\chi^2\left(\sin^2(\chi)+\cos^2(\chi)\right)+\sin^2(\chi)\left(d\theta^2+\sin^2(\theta)\,d\phi^2\right)\right]$$

$$\boxed{dx^2+dy^2+dz^2+dw^2=R^2\left[d\chi^2+\sin^2(\chi)\left(d\theta^2+\sin^2(\theta)\,d\phi^2\right)\right]}$$

f.) $r=\sin(\chi)$ $\therefore$ $\chi=\sin^{-1}(r)$: $\dfrac{d\chi}{dr}=\dfrac{1}{\sqrt{1-r^2}}$: $d\chi=\dfrac{dr}{\sqrt{1-r^2}}$: $d\chi^2=\dfrac{dr^2}{1-r^2}$

ds^2 thus becomes : $\boxed{ds^2=R^2\left[\dfrac{dr^2}{1-r^2}+r^2\left(d\theta^2+\sin^2(\theta)\,d\phi^2\right)\right]}$

g.)

$$\boxed{-1\le r\le 1}$$

Problem 4

a.) $$ds^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \qquad \theta = \pi/2 \ \therefore\ d\theta = 0$$

$$= R^2\left[\frac{dr^2}{1-r^2} + r^2(0^2 + d\varphi^2)\right] = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right]$$

$$\boxed{d\Sigma^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right]}$$

b.) $$ds^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right] \longrightarrow (1), \quad ds^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \longrightarrow (2)$$

$$z = z(r,\varphi), \quad \rho = \rho(r,\varphi), \quad \psi = \psi(r,\varphi)$$

$$z = z(r), \quad \rho = \rho(r), \quad \psi = \varphi$$

$$\left[dz = \frac{dz}{dr}dr, \quad d\rho = \frac{d\rho}{dr}dr, \quad d\psi = d\varphi\right] \longrightarrow (3)$$

putting (3) into (2)

$$ds^2 = \left(\frac{d\rho}{dr}\right)^2 dr^2 + \rho^2 d\varphi^2 + \left(\frac{dz}{dr}\right)^2 dr^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2$$

$$ds^2 = \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2 \longrightarrow (4)$$

Setting (4) = (1),

$$\left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\varphi^2 = R^2\left[\frac{dr^2}{1-r^2} + r^2 d\varphi^2\right] \longrightarrow (5)$$

$$\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2 = \frac{R^2}{1-r^2}, \qquad \rho^2 = R^2 r^2, \quad \rho = \pm Rr, \quad \frac{d\rho}{dr} = \pm R$$

$$(\pm R)^2 + \left(\frac{dz}{dr}\right)^2 = \frac{R^2}{1-r^2} \longrightarrow \left(\frac{dz}{dr}\right)^2 = \frac{R^2}{1-r^2} - R^2 \longrightarrow \left(\frac{dz}{dr}\right)^2 = \frac{R^2}{1-r^2} - \frac{R^2(1-r^2)}{1-r^2}$$

$$\left(\frac{dz}{dr}\right)^2 = \frac{R^2 - R^2 + R^2 r^2}{1-r^2} \longrightarrow \left(\frac{dz}{dr}\right)^2 = \frac{R^2 r^2}{1-r^2} \longrightarrow \frac{dz}{dr} = \pm R\frac{r}{\sqrt{1-r^2}} \longrightarrow (6)$$

(6) is now our answer $$\boxed{\frac{dz}{dr} = \pm R\frac{r}{\sqrt{1-r^2}}}$$

Problem 4) Continued

c.) $\frac{dz}{dr} = \pm R \frac{r}{\sqrt{1-r^2}} \longrightarrow \int \frac{dz}{\pm R} = \int \frac{r}{\sqrt{1-r^2}} dr \longrightarrow \frac{z}{\pm R} = \int \frac{r}{\sqrt{1-r^2}} dr \longrightarrow (7)$

(7) : $u = 1-r^2$, $du = -2r\,dr \rightarrow dr = -\frac{du}{2r}$

(7) : $\frac{z}{\pm R} = -\frac{1}{2}\int \frac{1}{\sqrt{u}} du \longrightarrow \frac{z}{\pm R} = -\frac{1}{2}\int u^{-1/2} du \longrightarrow \frac{z}{\pm R} = -u^{1/2} \longrightarrow (8)$

putting $u = 1-r^2$ into (8) : $\frac{z}{\pm R} = -\sqrt{1-r^2} \rightarrow \frac{z^2}{R^2} = 1-r^2 \rightarrow \frac{z^2}{R^2} + r^2 = 1$

$$\boxed{\frac{z^2}{R^2} + r^2 = 1}$$

d.) $\frac{z^2}{R^2} = 1 - r^2 \rightarrow \frac{z}{R} = \sqrt{1-r^2}$