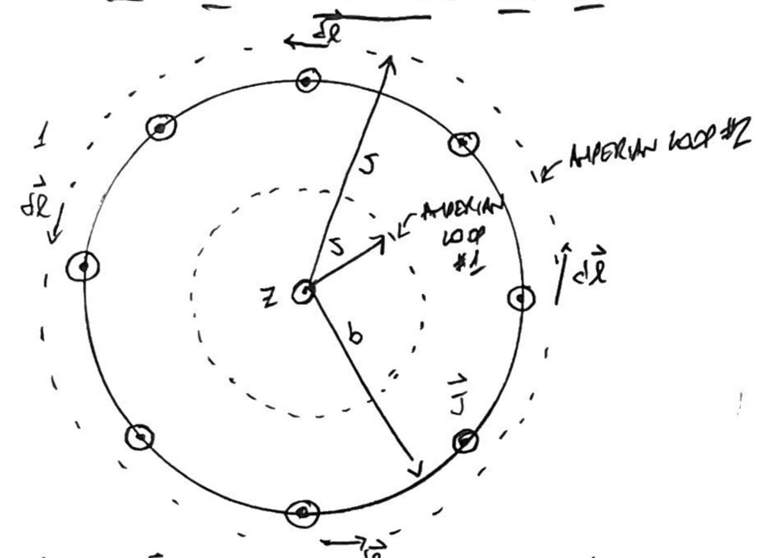
PHYS 311 HOMEWORK SET 11



a) FOR AMPERIAN LOOP #1...

$$I_{enc} = 0 \quad \therefore \quad \oint_C \vec{B} \cdot d\vec{l} = B 2\pi s = 0$$

$$\therefore B = 0$$

FOR AMPERIAN LOOP #2...

$$I_{enc} = I \quad \therefore \quad \oint \vec{B} \cdot d\vec{l} = B 2\pi s$$

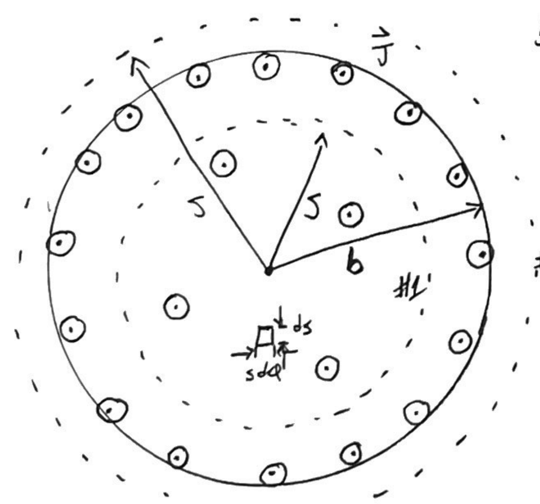
SO AMPERE'S LAW GIVES

$$B 2\pi s = \mu_0 I \quad \therefore \quad B = \frac{\mu_0 I}{2\pi s}$$

NOTE: $\vec{B}$ IS CIRCUMFERENTIAL $\therefore$ $\vec{B} = B\hat{\varphi}$

SO

$$\boxed{\begin{aligned} \vec{B} &= 0 \quad \text{WHEN } s < b \\ \vec{B} &= \frac{\mu_0 I}{2\pi s}\hat{\varphi} \quad \text{WHEN } s > b \end{aligned}}$$



b) NOW LET $\vec{J} = k s^2 \hat{z}$ WHERE k IS A CONSTANT OF PROPORTIONALITY

CONSIDER THE SAME TWO AMPERIAN LOOPS, MARKED BY THE DASHED LINES..

#2 THE TOTAL CURRENT FLOWING DOWN THE WIRE IS...

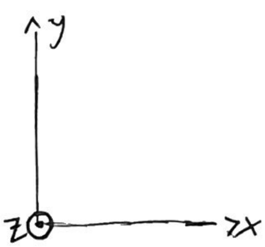
$$I = \int_{\text{WIRE}} \vec{J} \cdot d\vec{a} = \int_0^{2\pi}\int_0^{b} k s^3 \, ds \, d\varphi = \frac{k s^4}{4}\Big|_0^b 2\pi$$

WHERE $d\vec{a} = s\,ds\,d\varphi\,\hat{z}$

SO $\vec{J} \cdot d\vec{a} = k s^2 \hat{z} \cdot s\,ds\,d\varphi\,\hat{z} = k s^3 \, ds \, d\varphi$

SO

$$I = \frac{1}{2} k \pi b^4$$



NOW, FOR AMPERIAN LOOP #1..

$$I_{enc} = \int_0^{2\pi}\int_0^{s} k s^3 \, ds \, d\varphi = k \frac{1}{4} s^4 \Big|_0^s 2\pi = \frac{1}{2} k \pi s^4 = \frac{1}{2} k \pi b^4 \cdot \frac{s^4}{b^4} = I\frac{s^4}{b^4}$$

NOW

$$\oint \vec{B} \cdot d\vec{l} = B\, 2\pi s \quad \text{(AS BEFORE)}$$

SO AMPERE'S LAW YIELDS...

$$B\, 2\pi s = \mu_0 I \frac{s^4}{b^4} \qquad \text{so} \quad B = \frac{\mu_0}{2\pi} \frac{I s^3}{b^4}$$

For AMPERIAN LOOP #2 ...

$$I_{enc} = I \quad \text{so} \quad B = \frac{\mu_0 I}{2\pi s}$$

SO

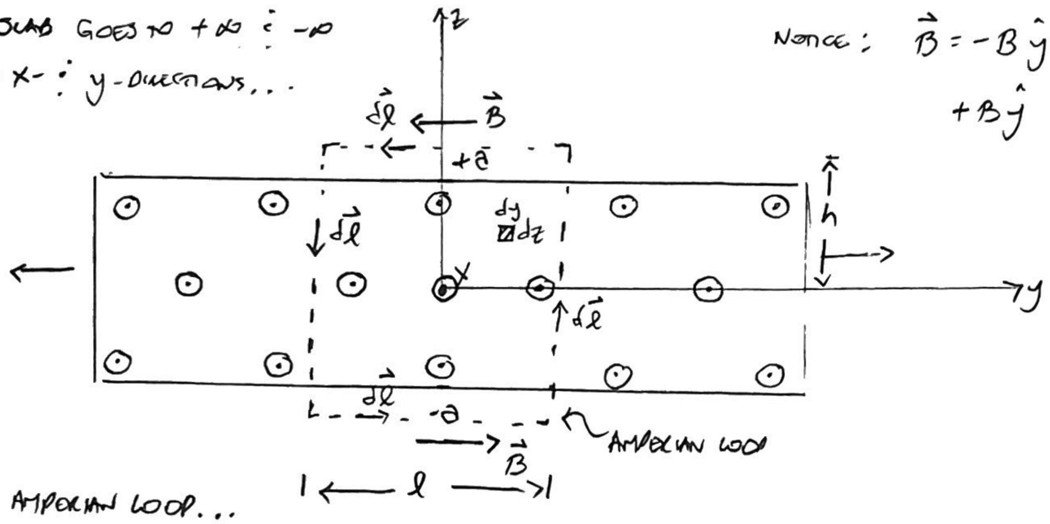
$$\boxed{\begin{aligned} \vec{B} &= \frac{\mu_0 I}{2\pi} \frac{s^3}{b^4} \hat{\varphi} \quad \text{for} \quad s<b \\ \vec{B} &= \frac{\mu_0 I}{2\pi s} \hat{\varphi} \quad \text{for} \quad s>b \end{aligned}}$$

2. LOOK DOWN THE X-AXIS, SO THAT THE VOLUME CURRENT DENSITY POINTS TOWARDS YOU.

NOTICE THAT THE SLAB GOES TO $+\infty$ & $-\infty$ IN BOTH THE X- & Y-DIRECTIONS...

NOTICE: $\vec{B} = -B\hat{y}$ FOR $z > +a$

$+B\hat{y}$ FOR $z < -a$



NOW FOR THE ABOVE AMPERIAN LOOP...

$$\oint \vec{B}\cdot d\vec{l} = \int_{TOP} \vec{B}\cdot d\vec{l} + \int_{BOTTOM} \vec{B}\cdot d\vec{l} + \underbrace{\int_{SIDES} \vec{B}\cdot d\vec{l}}_{0}$$

SINCE $\vec{B} = \pm B\hat{y}$ & $d\vec{l} = dz\,\hat{z}$ SO $\vec{B}\cdot d\vec{l} = 0$

$$= \int_{TOP} B_{TOP}\, dl + \int_{BOTTOM} B_{BOTTOM}\, dl = B_{TOP}\int_{TOP} dl + B_{BOTTOM}\int_{BOTTOM} dl$$

(SINCE B MUST BE A CONSTANT AND CONSTANT $z = h$)

SINCE $\vec{B} \parallel d\vec{l}$ ALONG THE TOP & BOTTOM CONTOUR

$$= (B_{TOP} + B_{BOTTOM})\, l$$

NOW

$$I_{ENC} = \int \vec{J}\cdot d\vec{a} = \int_{-a}^{+a}\int_{-l/2}^{+l/2} J\, dy\, dz = J(2a)\, l$$

WHERE $d\vec{a} = dy\,dz\,\hat{x}$

SO $\vec{J}\cdot d\vec{a} = J\hat{x}\cdot dy\,dz\,\hat{x} = J\,dy\,dz$

SO AMPERE'S LAW GIVES...

$$(B_{TOP} + B_{BOTTOM})\, l = \mu_0 J(2a)\, l \quad \text{SO} \quad \boxed{B_{TOP} + B_{BOTTOM} = \mu_0 J(2a)}$$

NOTICE:

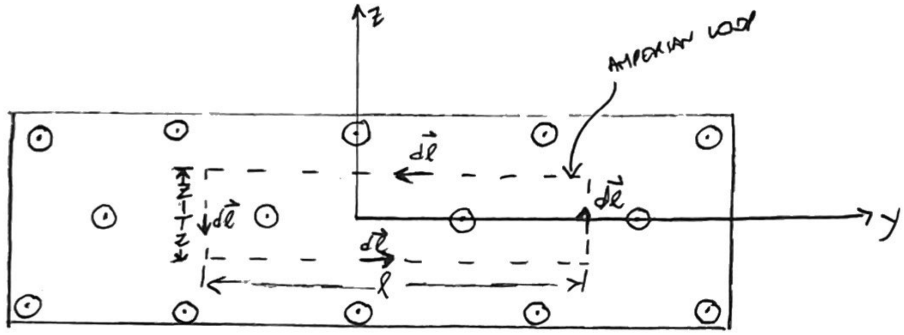
IF I TURN THIS PAPER UPSIDE-DOWN, THE PROBLEM LOOKS THE SAME $\therefore B_{TOP} = B_{BOTTOM}$

SO $B = \mu_0 J a$

SO

$$\left[\begin{aligned} \vec{B} &= -\mu_0 J a \,\hat{y} \quad \text{FOR } z > +a \\ \vec{B} &= +\mu_0 J a \,\hat{y} \quad \text{FOR } z < -a \end{aligned}\right]$$

TO FIND THE B-FIELD INSIDE, CONSIDER ANOTHER AMPERIAN LOOP...



BY THE EXACT SAME RATIONAL AS BEFORE,.

$$\oint \vec{B} \cdot d\vec{l} = (B_{BOTTOM} + B_{TOP})\, l = 2Bl$$

NOW

$$I_{ENC} = \int \vec{J} \cdot d\vec{a} = \int_{-z}^{z} \int_{-l/2}^{+l/2} J\, dy\, dz = J \int_{-z}^{z} dz \int_{-l/2}^{+l/2} dy = J(2z)\, l$$

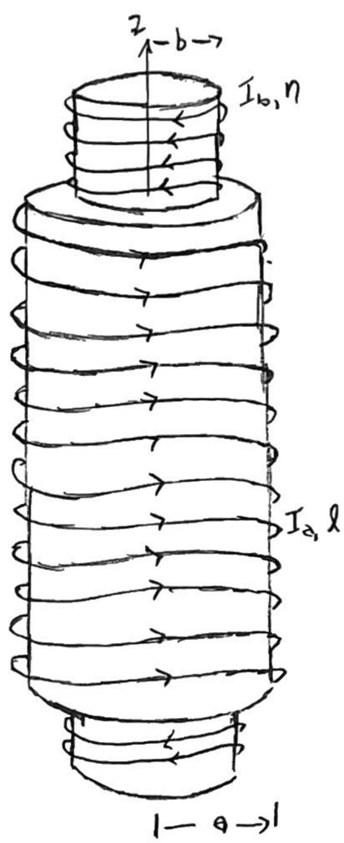
SO AMPERE'S LAW GIVES,.

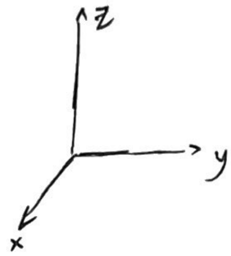
$$2Bl = \mu_0 J (2z)\, l \qquad \text{SO} \quad \left[\begin{aligned} \vec{B} &= -\mu_0 J |z| \,\hat{y} \quad \text{FOR } z > 0 \\ \vec{B} &= +\mu_0 J |z| \,\hat{y} \quad \text{FOR } z < 0 \end{aligned}\right]$$

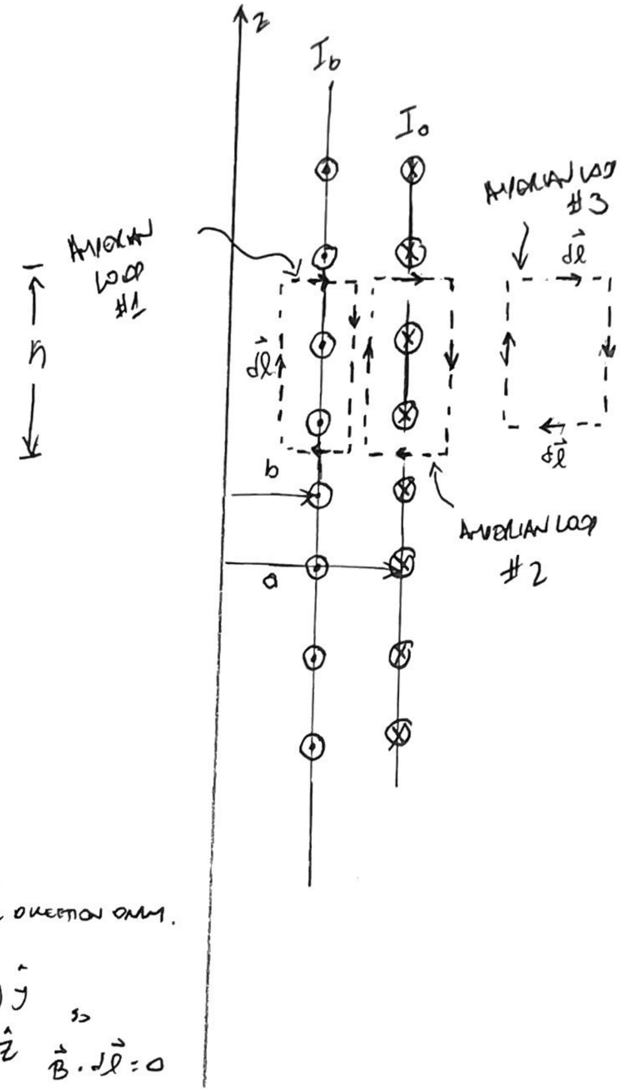
OR EQUIVALENTLY AS

$$\left[\vec{B}(z) = -\mu_0 J z \,\hat{y}\right]$$ AS THE SIGN OF z GIVES THE CORRECT DIRECTION FOR $\vec{B}$

3







NOTICE: BY THE RIGHT HAND RULE, THE $\vec{B}$-FIELD MUST POINT IN THE $\pm\hat{z}$ DIRECTION ONLY.

CONSIDER AMPERIAN LOOP #3...

$$\oint \vec{B}\cdot d\vec{l} = \underbrace{\int \vec{B}\cdot d\vec{l}}_{\text{INNERMOST PATH}} + \underbrace{\int \vec{B}\cdot d\vec{l}}_{\text{OUTERMOST PATH}} + \underbrace{\int \cancel{\vec{B}\cdot d\vec{l}}}_{\text{HORIZONTAL PATHS}} \rightarrow 0$$

SINCE $d\vec{l} = dy\,\hat{y}$
$\vec{B} = B\hat{z}$ SO $\vec{B}\cdot d\vec{l} = 0$

$$= +\underbrace{\int B\,dl}_{\text{INNERMOST PATH}} - \underbrace{\int B\,dl}_{\text{OUTERMOST PATH}} = (-B_{\text{OUTER RADIUS}} + B_{\text{INNER RADIUS}})\,h = I_{enc} = 0$$

SO $\boxed{B_{\text{OUTER RADIUS}} = B_{\text{INNER RADIUS}}}$

IF $\vec{B} = +B\hat{z}$

NOW, AS SHOWN IN CLASS, THE $\vec{B}$-FIELD MUST GO TO ZERO AS

$$\lim_{s\to\infty} B_{\text{OUTSIDE}} \to 0$$

$$\therefore \boxed{B_{\text{OUTER RADIUS}} = B_{\text{INNER RADIUS}} = 0}$$

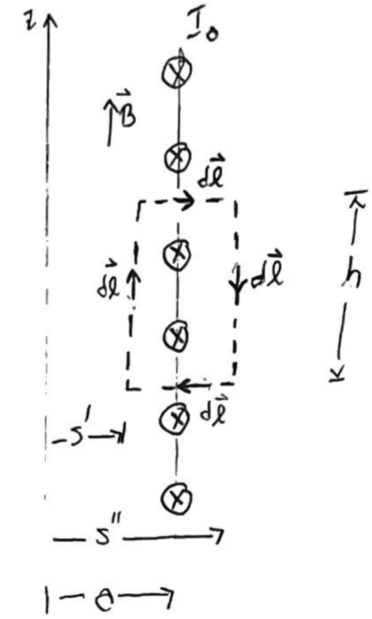
(4) SO

$$\boxed{\vec{B} = 0 \text{ FOR } s > a}$$

NOW LOOKING AT AMPERIAN LOOP #2...

let s' BE RADIAL DISTANCE TO INNERMOST PATH

s'' BE RADIAL DISTANCE TO OUTERMOST PATH...



$$\oint \vec{B}\cdot d\vec{l} = \int_{s=s'} \vec{B}\cdot d\vec{l} + \int_{s=s''} \vec{B}\cdot d\vec{l} + \underbrace{\int_{\text{HORIZONTAL PATHS}} \vec{B}\cdot d\vec{l}}_{0}$$ SINCE $\vec{B} \perp d\vec{l}$ HERE

$$= +\int_{s=s'} B\,dl + \int_{s=s''} B\,dl$$ SINCE $\vec{B} = B\hat{z}$; $d\vec{l} = \pm dz\,\hat{z}$

DEPENDING ON WHETHER $s=s'$ or $s=s''$

$$= +B(s')\int dl + \underbrace{B(s'')}_{0}\int dl$$ SINCE IT WAS ALREADY FOUND THAT $B=0$ FOR $s>a$

$$= +B(s')h$$

NOW

$$I_{enc} = +NI_a = +\frac{N}{h}hI_a = +lhI_a$$ WHERE $l = N$ TURNS PER h LENGTH

SO $+B(s')h = \mu_0 \cdot +lhI_a$ SO $B(s') = \mu_0 l I_a$

(**) SO $\left[\vec{B} = \mu_0 l I_a \hat{z} \text{ FOR } b<s<a\right]$

LASTLY, CONSIDER AMPERIAN LOOP #1...

let s' BE THE RADIAL DISTANCE TO THE INNERMOST PATH

s'' BE THE RADIAL DISTANCE TO THE OUTERMOST PATH

- WE CAN USE THE PICTURE ABOVE HERE IF WE REVERSE THE CURRENT DIRECTION AND LET $a \to b$
- OUR ANALYSIS WOULD YIELD..

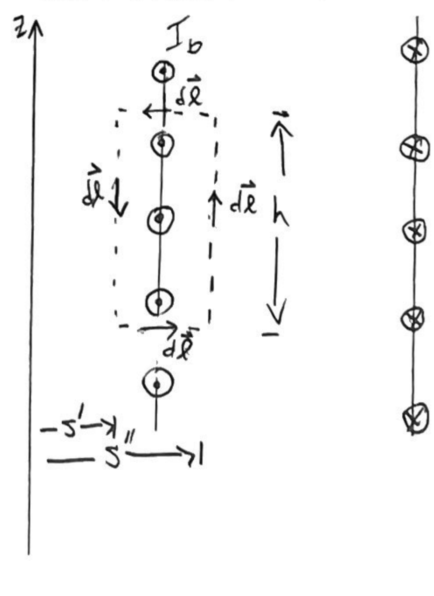
$$\oint \vec{B}\cdot d\vec{l} = +B(s')h + B(s'')h = \mu_0 n h I_b$$

WHERE $n = N'$ TURNS PER h LENGTH

SO

$$B(s') = \mu_0 n I_b - B(s') = \mu_0 n I_b - \mu_0 l I_a$$

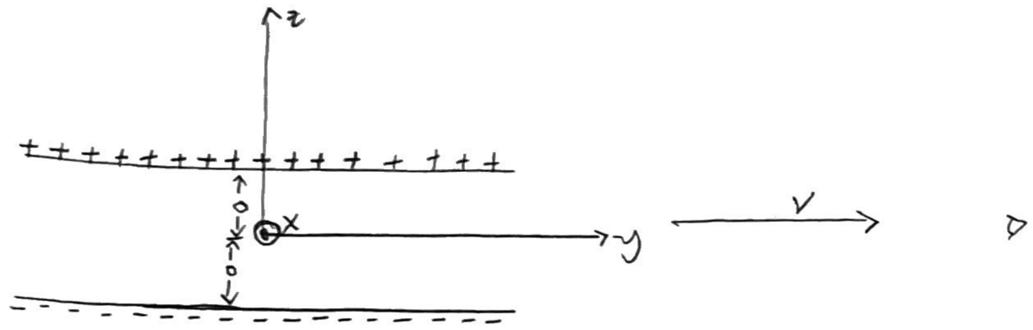
USING (**)



SO, USING THE RIGHT HAND RULE..

(***) $\left[\vec{B}(s) = \mu_0 (lI_a - nI_b)\hat{z} \text{ FOR } s<b\right]$

4. ..



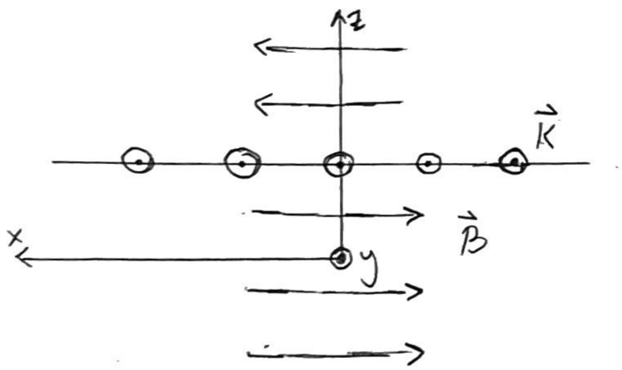
Allow the two infinite plates to lie in the xy plane and moving in the +y direction.

These moving surface charge densities are equivalent to surface current densities and are given by

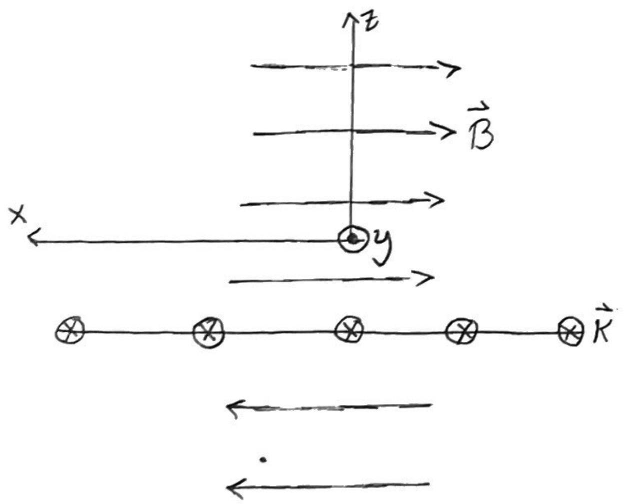
$$\vec{K}(z=a) = +\sigma v\, \hat{y}$$

$$\vec{K}(z=-a) = -\sigma v\, \hat{y}$$

Now look down the y axis and consider each plate separately..



For the bottom plate...



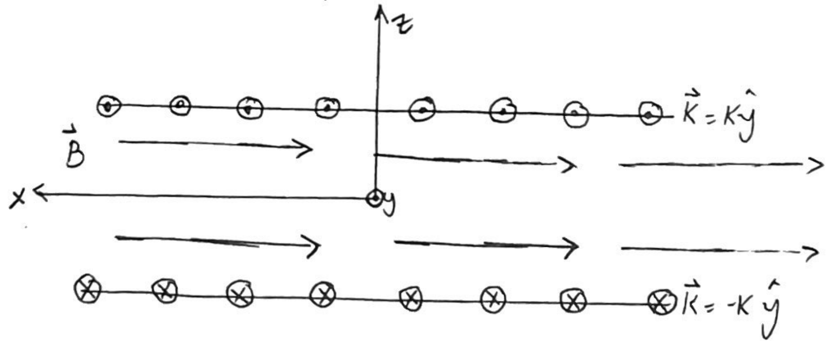
What does the field look like?

- The right hand gives the direction as either in the $\pm$ x-axis, depending on the direction of $\vec{K}$ and whether we are above or below.
- In class we found the $\vec{B}$-field of one sheet of surface current density to be..

$$B = \frac{1}{2}\mu_0 K$$

Independent of z!

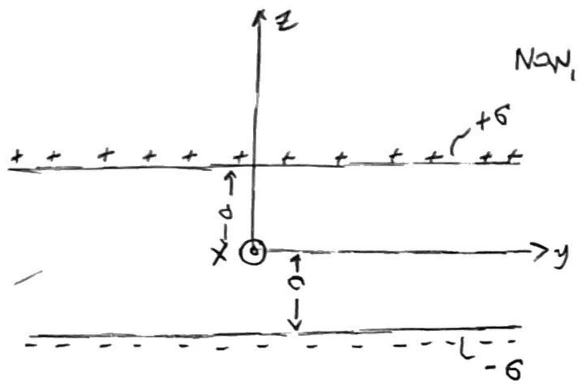
For both plates, what would we expect?



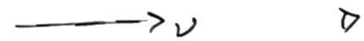
Notice:

- Above the top plate & below the bottom plates, the $\vec{B}$-fields are of equal magnitude but opposite direction, we should expect them to sum to zero!
- In between, we'd expect

$$\vec{B} = -\mu_0 K\, \hat{x} \quad \text{(twice one contribution)}$$



NOW, USING AMPERIAN LOOPS ...



ALLOW THE TWO ~~∞~~ PLATES TO LIE IN THE xy PLANE AND BE MOVING IN THE y DIRECTION,

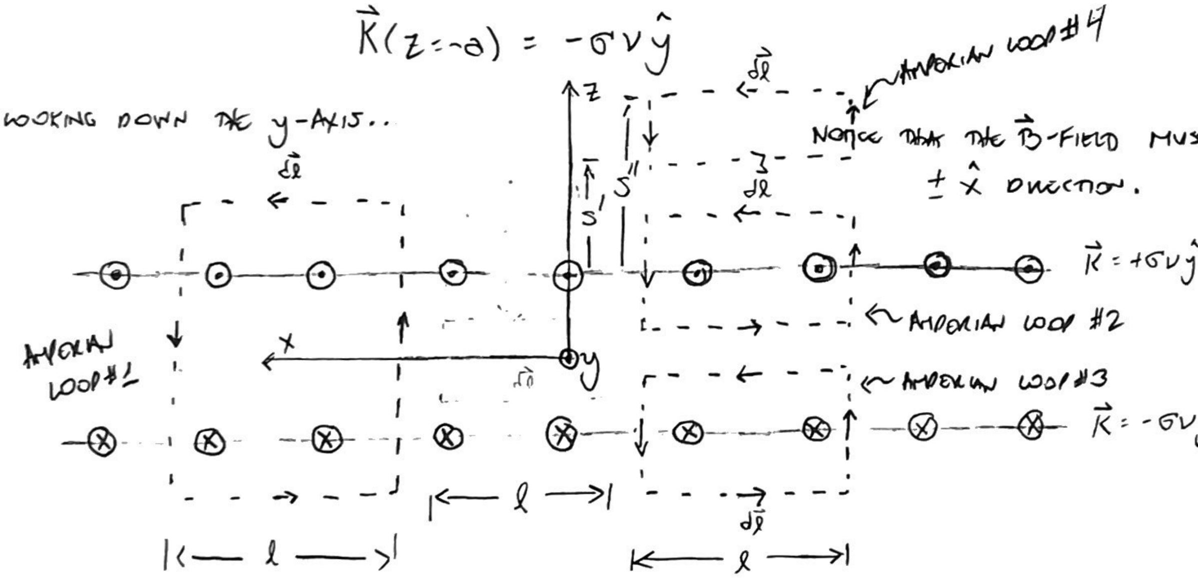
THESE MOVING SURFACE CHARGE DENSITIES ARE EQUIVALENT TO SURFACE CURRENT DENSITIES AND GIVEN BY

$$\vec{K}(z=a) = +\sigma v\,\hat{y}$$

$$\vec{K}(z=-a) = -\sigma v\,\hat{y}$$

NOW LOOKING DOWN THE y-AXIS..

NOTICE THAT THE $\vec{B}$-FIELD MUST POINT IN THE $\pm\hat{x}$ DIRECTION.



CONSIDER THE 4 AMPERIAN LOOPS SHOWN IN THE FIGURE ABOVE...

- ALL HAVE A HORIZONTAL LENGTH l

FOR ANY OF THE THREE...

$$\oint \vec{B}\cdot d\vec{l} = \int_{TOP} \vec{B}\cdot d\vec{l} + \int_{BOTTOM} \vec{B}\cdot d\vec{l} + \int_{SIDES} \cancel{\vec{B}\cdot d\vec{l}}^{\,0}$$

AS $\vec{B} = \pm B\,\hat{x}$ WHEREAS $d\vec{l} = dz\,\hat{z}$ SO $\vec{B}\cdot d\vec{l}\Big|_{SIDES} = 0$

$$= \int_{TOP} B\,dl - \int_{BOTTOM} B\,dl = B\int_{TOP} dl - B\int_{BOTTOM} dl = (B_{TOP} - B_{BOTTOM})\,l$$

↳ AS $\vec{B} \parallel d\vec{l}$

↳ AS B IS A CONSTANT AT CONSTANT z

ASSUMING $\vec{B}$ IS ANTI-$\parallel$ $d\vec{l}$

NOW FOR AMPERIAN LOOP #1...

$$I_{enc} = 0 \quad \text{so} \quad \left[B(z>a) \overset{(*)}{=} B(z<-a) \right] \quad \therefore$$

ABOVE THE TOP SHEET & BELOW THE BOTTOM SHEET, THE TOTAL $\vec{B}$-FIELD HAS THE SAME MAGNITUDES

FOR AMPERIAN LOOP #2...

AMPERE'S LAW YIELDS...

$$I_{enc} = K\ell \quad \text{so} \quad \left[B(z>a) - B(-a<z<+a) \right] \ell = \mu_0 K \ell$$

SO

$$\left[B(z>a) - B(-a<z<+a) = \mu_0 K \right]$$

FOR AMPERIAN LOOP # 3...

$$I_{enc} = -K\ell$$

SINCE CURRENT FLOWS INTO THE PAGE

SO AMPERE'S LAW YIELDS..

$$\left[B(-a<z<+a) - B(z<-a) \right] \ell = -\mu_0 K \ell$$

SO

$$\left[B(-a<z<+a) - B(z<-a) = -\mu_0 K \right]$$

NOTICE WE HAVE...

1) $B(z>a) = B(z<-a)$

2) $B(z>a) - B(-a<z<+a) = \mu_0 K$

3) $B(-a<z<+a) - B(z<-a) = -\mu_0 K$

ADDING 2) & 3) YIELDS...

$B(z>a) - B(z<-a) = 0$ WHICH IS EQUIVALENT TO 1)

SUBTRACTING 3) FROM 2) YIELDS...

$$B(z>a) + B(z<-a) - 2B(-a<z<a) = 2\mu_0 K$$

SO, USING (*)

$$(**)\ \left[2B(z>a) - 2B(-a<z<a) = 2\mu_0 K \right]$$

FOR AMPERIAN LOOP # 4...

$I_{enc} = 0$ SO $B(s') = B(s'')$

NOW

$\lim_{s'\to\infty} B(s') \to 0 \quad \therefore B(z>a) = B(z<-a)$

SO

(**) BECOMES...

$$B(-a<z<a) = -\mu_0 K$$

SO

$$\left[\begin{array}{l} \vec{B}(z) = 0 \quad \text{FOR } z>+a \text{ OR } z<-a \\ \vec{B}(z) = -\mu_0 K \hat{x} \quad \text{FOR } -a<z<+a \end{array} \right]$$

5. Let $\vec{A} = k\,\hat{\varphi}$ WHERE $\vec{B} = \vec{\nabla} \times \vec{A}$ NOTICE $A_\varphi = k$; $A_s = A_z = 0$

NOW THE CURL IN CYLINDRICAL COORDINATES IS OF THE FORM...

$$\vec{B} = \vec{\nabla} \times \vec{A} = \left(\cancel{-\frac{\partial}{\partial z} A_\varphi}\right)\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(s A_\varphi)\,\hat{z}$$ WHERE I SET $A_s = A_z = 0$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s k)\,\hat{z} = \frac{k}{s}\,\hat{z} = B_z\,\hat{z}$$ NOTICE: $B_\varphi = B_s = 0$

ALSO

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

SO

$$\vec{\nabla} \times \vec{B} = \cancel{\frac{1}{s}\frac{\partial B_z}{\partial \varphi}}\,\hat{s} - \frac{\partial B_z}{\partial s}\,\hat{\varphi} = -\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\hat{\varphi} = \frac{k}{s^2}\,\hat{\varphi} = \mu_0 \vec{J}$$

SO

$$\boxed{\vec{J} = \frac{k}{\mu_0}\frac{\hat{\varphi}}{s^2}}$$