MAT 202
Larson – Section 9.1
Sequences

In mathematics, recognizing patterns and behavior of numbers that are written sequentially helps us to build intuition and write equations that describe the pattern. Once we understand the pattern and have an algebraic representation, we are able to determine whether the sequence converges, diverges, is monotonic and/or bounded. In this section we will be working with sequences.

Sequence: A function whose domain is a set of successive (consecutive) positive integers. A sequence can be described as a list of objects in which the order is important. An ***infinite sequence*** has an infinite amount of terms, where a ***finite sequence*** has a finite (or countable) number of terms.

Term: When we list a sequence in order, the range values are called the terms of the sequence.

Subscript Notation: Since a sequence is a function we can use $f(n)$ to describe the n^{th} term of the sequence. Subscript notation is used in lieu of function notation, that is, a_n will now describe the n^{th} term of the sequence and the entire sequence is denoted $\{a_n\}$. Occasionally, it is convenient to begin a sequence with a_0, so that the terms of the sequence become $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ and the domain is the set of nonnegative integers.

Ex: Write the first five terms of the sequence $a_n = \left(-\frac{2}{5}\right)^n$.

Ex: Write the first five terms of the sequence $a_n = \left(-\frac{2}{5}\right)^n$

.

$a_1 = \frac{-2}{5}$

$a_2 = \frac{4}{25}$

$a_3 = \frac{-8}{125}$

$a_4 = \frac{16}{625}$

$a_5 = -\frac{32}{3125}$

$a_n = (-1)^n \left(\frac{2}{5}\right)^n$

Recursively Defined Sequences: A sequence is defined recursively if each term of the sequence is defined in terms of its predecessors.

Ex: Write the first five terms of the recursively defined sequence $a_1 = 6,\ a_{n+1} = 2a_n - 4$.

n Factorial (n!): If n is a positive integer, n factorial is defined as: $n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots 3\cdot 2\cdot 1$. Note: $0! = 1$

Ex: Evaluate the following:

a) $\dfrac{13!}{11!2!}$

b) $\dfrac{n!}{(n-2)!}$

The primary focus of this chapter concerns sequences whose terms approach limiting values. Such sequences are said to ***converge.***

Definition of the Limit of a Sequence: Let L be a real number. The limit of a sequence $\{a_n\}$ is L, written as

$$\lim_{n \to \infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. (Recall ***delta-epsilon*** definition of a limit in section 1.2). If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of the sequence does not exist, then the sequence **diverges**.

Ex: Write the first five terms of the recursively defined sequence $\underline{\text{1st}}\, a_1 = 6,\ a_{n+1} = 2a_n - 4$

$a_n = 2a_{n-1} - 4$

Next term = 2 · preceding term − 4

$a_2 = a_{1+1} = 2a_1 - 4 = 2(6) - 4 = 8$

$\underline{\text{2nd}}\ a_2 = 8$

$a_3 = 12$

$a_4 = 20$

$a_5 = 36$

6, 8, 12, 20, 36

Ex: Evaluate the following:

a) $\dfrac{13!}{11!2!} = \dfrac{13 \cdot \cancel{12}^{6} \cdot \cancel{11!}}{\cancel{11!} \cdot \cancel{2}_{1} \cdot 1}$

$= \boxed{78}$

b) $\dfrac{n!}{(n-2)!} = \dfrac{n \cdot (n-1) \cdot \cancel{(n-2)!}}{\cancel{(n-2)}!}$

$= n(n-1)$

Limit of a Sequence: Let L be a real number. Let f be a function of a real variable such that $\lim_{x \to \infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n \to \infty} a_n = L$.

Properties of Limits of Sequences: Let $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} b_n = K$.

1. $\lim_{n \to \infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n \to \infty} ca_n = cL$, c is any real number.
3. $\lim_{n \to \infty} (a_n b_n) = LK$
4. $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ *and* $K \neq 0$

Notice that these rules are synonymous with properties of limits in section 1.3.

Ex: For the following sequences, find the limit (if possible) and determine if the sequence converges or diverges.

a) $a_n = 2 - \frac{1}{4^n}$

b) $a_n = \sin \frac{n\pi}{2}$

c) $a_n = \frac{n^2}{2^n - 1}$

Ex: For the following sequences, find the limit (if possible) and determine if the sequence converges or diverges.

a) $a_n = 2 - \dfrac{1}{4^n}$

$$\lim_{n \to \infty} \left(2 - \frac{1}{4^n}\right) = 2 \text{ ; converges}$$

b) $a_n = \sin\frac{n\pi}{2}$

$\lim_{n \to \infty} \sin\frac{n\pi}{2} \longrightarrow$ This oscillates between -1 & 1

$\therefore$ no clear limit.

Diverges

c) $a_n = \dfrac{n^2}{2^n - 1}$

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = \frac{\infty}{\infty}$$ ← Ind. form

L'Hopital's (twice)

$$\frac{d}{dx}\left[b^x\right] = (\ln b)\, b^x$$

$$\lim_{n\to\infty} \frac{2n}{(\ln 2)\, 2^n} = \frac{\infty}{\infty}$$

$$\lim_{n\to\infty} \frac{2}{(\ln 2)(\ln 2)\, 2^n}$$

$$\lim_{n\to\infty} \frac{2}{(\ln 2)^2\, 2^n} = 0$$

converges

Tools that may help you find the limit:

1. Theorem 8.4 – ***L'Hôpital's Rule*** (pg. 558)
2. Theorem 9.3 – ***Squeeze Theorem for Sequences*** (pg. 587)
3. Theorem 9.4 – ***Absolute Value Theorem*** (pg. 588)

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the n^{th} term of the sequence. In such cases, we may be required to discover a pattern in the sequence and to describe the n^{th} term. Once the n^{th} term is determined, we can investigate the convergence or divergence of the sequence.

Ex: Write an expression for the nth term of the sequence. (There is more than one correct answer.)

a) -5, -2, 3, 10, 19, …

b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{25}, \frac{1}{121}, \ldots$

Up to this point we need the limit to determine whether a sequence converges. We will now provide a test for convergence of sequences without determining the limit.

Definition of Monotonic Sequence:

A sequence $\{a_n\}$ is ***monotonic*** when its terms are nondecreasing or when its terms are nonincreasing.

Ex: Write an expression for the nth term of the sequence.

(There is more than one correct answer.)

a) -5, -2, 3, 10, 19, …

$\underbrace{-5, -2}_{3}\ \underbrace{-2, 3}_{5}\ \underbrace{3, 10}_{7}\ \underbrace{10, 19}_{9}$

$\boxed{a_n = n^2 - 6}$

b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{25}, \frac{1}{121}, \ldots$

$$\boxed{a_n = \frac{1}{n! + 1}}$$

Ex: Determine whether each sequence having the n^{th} term is monotonic:

a) $a_n = 3 + (-1)^n$

b) $b_n = \dfrac{2n}{1+n}$

c) $c_n = \dfrac{n^2}{2^n - 1}$

<u>**Definition of Bounded Sequence**</u>**:**

1. A sequence $\{a_n\}$ is ***bounded above*** when there is a real number M such that $a_n \le M$ for all n. The number M is called an ***upper bound*** of the sequence.
2. A sequence $\{a_n\}$ is ***bounded below*** when there is a real number N such that $N \le a_n$ for all n. The number N is called a ***lower bound*** of the sequence.
3. A sequence $\{a_n\}$ is ***bounded*** when it is bounded above and bounded below.

GRAND FINALE!!!

<u>**Bounded Monotonic Sequences**</u>**:**

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges…YAY!

Ex: Determine whether each sequence having the n^{th} term is monotonic:

a) $a_n = 3 + \left(-1\right)^n$

$a_1 = 2$

$a_2 = 4$

$a_3 = 2$

Not monotonic

b) $b_n = \dfrac{2n}{1+n}$

$b_1 = 1$

$b_2 = \dfrac{4}{3}$

$b_3 = \dfrac{3}{2}$

$b_4 = \dfrac{8}{5}$

monotonic

c) $c_n = \dfrac{n^2}{2^n - 1}$

$c_1 = 1$

$c_2 = \dfrac{4}{3}$

$c_3 = \dfrac{9}{7}$

Not monotonic

Ex: Determine whether the sequence with the given n^{th} term is monotonic and bounded. What can you conclude? Use a graphing calculator to confirm your results.

$$a_n = 5 + \frac{1}{n^2}$$

Ex: Determine whether the sequence with the given n^{th} term is monotonic and bounded. What can you conclude? Use a graphing calculator to confirm your results.

$$a_n = 5 + \frac{1}{n^2}$$

$a_1 = 6$

$a_2 = 5\frac{1}{4}$

$a_3 = 5\frac{1}{9}$

$a_4 = 5\frac{1}{16}$

Nonincreasing

$\therefore$ monotonic

bounded Above

Upper bound = 6

bounded below

lower bound = 5

$$\lim_{n\to\infty}\left(5+\frac{1}{n^2}\right)=5$$

$\therefore$ bounded

Converges