Improving Quantum Parameter Estimation

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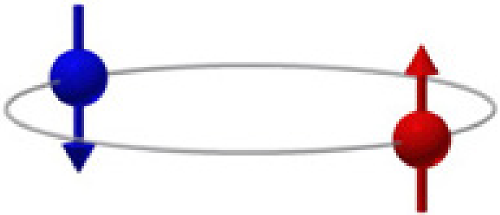
Spin-1/2 Particles

Quantum estimation is the use of quantum mechanical systems as measuring devices for physical parameters.



In the context of this presentation, these quantum mechanical systems are spin-1/2 particles.

Spin-1/2 Particles in Magnetic Field

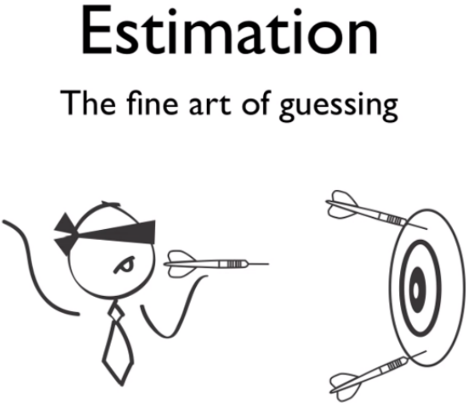
These spin-1/2 particles can be subjected to something like a magnetic field.

Can measure the change in the spin vector.

Example of Quantum Estimation

An example of spin-1/2 particles and parameters is in MRI machines.

Estimation



Coin Toss

The simplest example of estimation is a coin toss.

The result of this coin flip is either heads or tails.

Probability

Wish to estimate the probability of obtaining heads.

N is the number of times flipped, n_H is how many times heads appears.

Probability estimate of obtaining heads is

$$P_{est} = \frac{n_H}{N}. \quad (1)$$

Fluctuating Estimates

Multiple runs of N flips will give different estimates.

Fluctuation of estimates is calculated via

$$\nu = \bar{P}^2_{est} - (\bar{P}_{est})^2. \tag{2}$$

$\bar{P^2}_{est}$ and $\bar{P}_{est}$ are calculated via

$$\bar{P}_{est} = \frac{1}{N}\bar{n}_H \qquad \text{and} \qquad \bar{P}^2_{est} = \frac{1}{N^2}\bar{n}^2_H. \tag{3}$$

We aim to have the smallest variance possible.

Variance and Fisher Information

The variance binomial outcomes is

$$\nu = \frac{P_{est}}{N}(1 - P_{est}). \tag{4}$$

The Fisher Information is calculated via

$$F = \sum_{\text{All Possible Outcomes}}^{\infty} \frac{1}{\text{Prob(Outcome)}} \left(\frac{\partial \text{Prob(Outcome)}}{\partial \text{P}} \right)^2. \tag{5}$$

The Cramer-Rao bound tells us that

$$\nu \geq \frac{1}{F}. \tag{6}$$

where F is calculated in equation (5). The Fisher Information of our coin flip experiment via

$$F_{\text{N Toss}} = N \cdot F_{\text{Single Toss}} = N \cdot \frac{1}{P(1-P)} = \frac{N}{P(1-P)}. \tag{7}$$

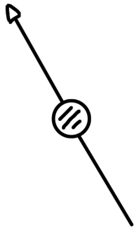
Single Particles

The state of a single particle mathematically is represented by

$$|+\hat{n}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \tag{8}$$

and

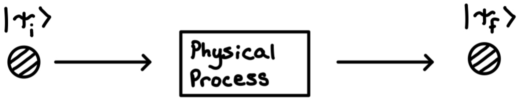
$$|-\hat{n}\rangle = \sin(\theta/2)|0\rangle - e^{i\phi}\cos(\theta/2)|1\rangle . \tag{9}$$



Unitary Evolution

States can evolve via physical processes, mathematically this is calculated by

$$|\psi_f\rangle = \hat{U}\,|\psi_i\rangle\,. \tag{10}$$



It should be noted that $\hat{U}$ are $2^n \times 2^n$ matrices where n is the number of particles.

Unitary Parameter Evolution

This evolution is calculated by

$$|\psi_f\rangle = \hat{U}(\lambda)\,|\psi_i\rangle\,. \tag{11}$$

An example of this unitary operator is

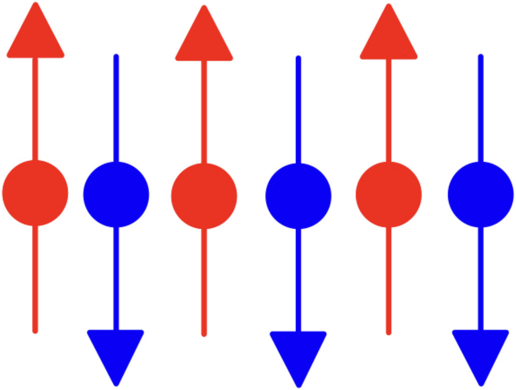
$$\hat{U}(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}. \tag{12}$$

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow |\psi_f\rangle = \frac{1}{\sqrt{2}}(e^{-i\lambda/2}\,|0\rangle + e^{i\lambda/2}\,|1\rangle) \tag{13}$$

λ is the parameter that we wish to estimate.

Multiple Particles

We can now discuss quantum systems with multiple particles.



Product States

A generic state is

$$|\psi\rangle = c_0 |0\rangle |0\rangle + c_1 |0\rangle |1\rangle + c_2 |1\rangle |0\rangle + c_3 |1\rangle |1\rangle . \tag{14}$$

A product state can be written as

$$|\psi\rangle = |\psi_A\rangle |\psi_B\rangle . \tag{15}$$

An example is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{16}$$

where

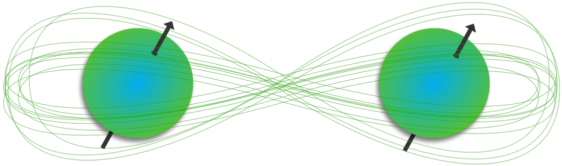
$$|\psi\rangle = \frac{1}{2}(|0\rangle |0\rangle + |0\rangle |1\rangle - |1\rangle |0\rangle - |1\rangle |1\rangle). \tag{17}$$

Entangled States

An example of an entangled state is

$$|\psi\rangle = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle) \tag{18}$$

Entangled states do not exist in classical physics.



Density Operators

Exact state unknown. Represented by state $|\psi_1\rangle$ with probability a_1 and $|\psi_2\rangle$ with probability a_2

$$\hat{\rho} = a_1 |\psi_1\rangle \langle\psi_1| + a_2 |\psi_2\rangle \langle\psi_2| . \tag{19}$$

For example

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \text{ with } a_1 = \frac{1+r}{2} \tag{20}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \text{ with } a_2 = \frac{1-r}{2} \tag{21}$$

the density operator is then

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} . \tag{22}$$

Equation (22) is the density operator that will be used without the rest of the talk.

Density Operators With Parameters

Parameter dependent density operators evolve via

$$\hat{\rho}(\lambda) = \hat{U}(\lambda)\hat{\rho}_0\hat{U}^\dagger(\lambda) \tag{23}$$

An example

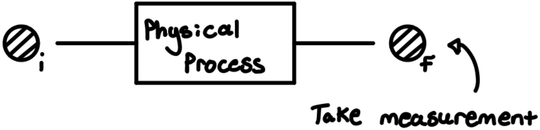
$$\hat{U}(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix} \quad \text{and} \quad \hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \tag{24}$$

then evolves to

$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 & re^{-i\lambda} \\ re^{i\lambda} & 1 \end{pmatrix}. \tag{25}$$

Quantum Estimation

A key component of quantum estimation is called the Quantum Fisher Information.



Relationship between F and H is

$$F \leq H. \tag{26}$$

H is not dependent upon measurement choice, only initial state.

QFI Calculation

The way this Quantum Fisher Information is calculated is by

$$H = \mathrm{Tr}[\hat{\rho}\hat{L}^2] \tag{27}$$

where

$$\frac{\partial \hat{\rho}}{\partial \lambda} = \frac{1}{2}\left[\hat{\rho}\hat{L} + \hat{L}\hat{\rho}\right]. \tag{28}$$

Want to maximize Quantum Fisher Information.

Phase Shifts

We wish to estimate parameter λ with $\hat{\rho}_f = \hat{U}(\lambda)\hat{\rho}_i\hat{U}(\lambda)^\dagger$.

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 + r_{zi} & r_{xi} - ir_{yi} \\ r_{xi} + ir_{yi} & 1 - r_{zi} \end{pmatrix} \tag{29}$$

Where $\hat{\rho}_f$ is

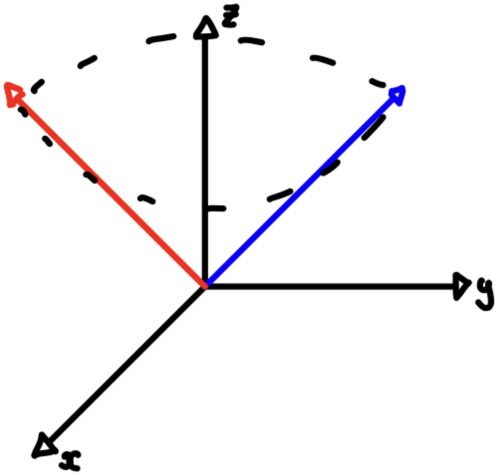
$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 + r_{zf} & r_{xf} - r_{yf} \\ r_{xf} + r_{yf} & 1 - r_{zf} \end{pmatrix} \tag{30}$$

Where

$$\begin{aligned} r_{xf} &= r_{xi}\cos(\lambda) - r_{yi}\sin(\lambda) \\ r_{yf} &= r_{xi}\sin(\lambda) + r_{yi}\cos(\lambda) \\ r_{zf} &= r_{zi}. \end{aligned} \tag{31}$$

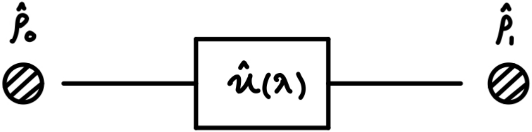
Visualizing of Phase Shifts

We can picture this Phase Shift in the figure.



One Particle Phase Shift

We now examine single particle phase shifts.

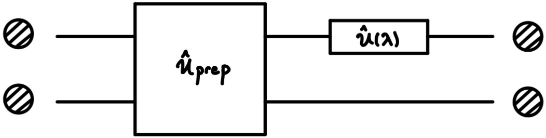


The Quantum Fisher Information is

$$H = r^2. \quad (32)$$

Two Particle one Channel Phase Shift

We now examine a two particle phase shift.



The $\hat{U}_{prep}$ matrix involves single qubit operations, one interaction is between two particles.

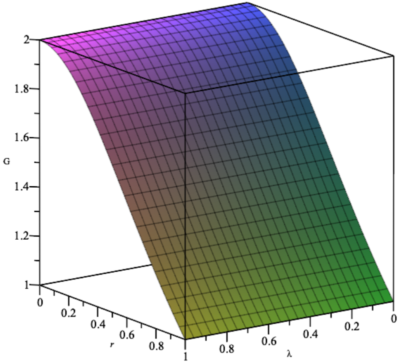
The Quantum Fisher Information is

$$H = \frac{2r^2}{1 + r^2}. \tag{33}$$

Gain of Phase Shifts

One can calculate what is called the gain of two scenarios by $G = \frac{H_{both}}{H_{single}}$. The gain of the last two scenarios is

$$G = \frac{2}{1 + r^2}. \tag{34}$$



When $G > 1$ the correlated scheme is estimating better.

Phase Flips

We wish to estimate parameter λ with
$\hat{\rho}_f = (1-\lambda)\hat{\rho}_i + \lambda\hat{\sigma}_z\hat{\rho}_i\hat{\sigma}_z$.

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1+r_{zi} & r_{xi}-ir_{yi} \\ r_{xi}+ir_{yi} & 1-r_{zi} \end{pmatrix} \tag{35}$$

Where $\hat{\rho}_f$ is

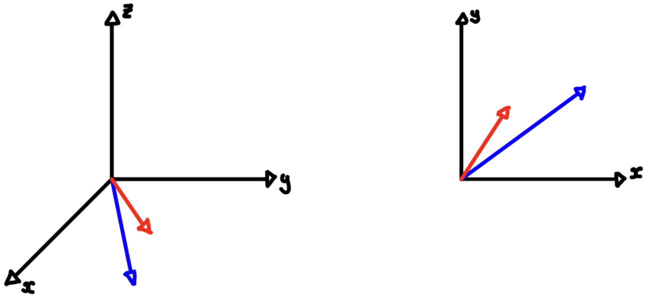
$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1+r_{zi} & (1-2\lambda)(r_{xi}-ir_{yi}) \\ (1-2\lambda)(r_{xi}+ir_{yi}) & 1-r_{zi} \end{pmatrix}. \tag{36}$$

The evolution of each component of these particles is

$$\begin{aligned} r_{xf} &= r_{xi}(1-2\lambda) \\ r_{yf} &= r_{yi}(1-2\lambda) \\ r_{zf} &= r_{zi}. \end{aligned} \tag{37}$$

Visualizing of Phase Flips

We can visualize how this Phase Flip evolves in the figure.



QFI of Phase Flips

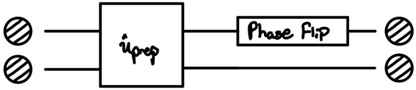
We examine the simplest scenario for a phase flip.

The Quantum Fisher Information is

$$H = \frac{4r^2}{1 - r^2(1 - 2\lambda)^2}. \tag{38}$$

QFI of Phase Flips Cont.

The next scenario is with two particles, one channel phase flip.



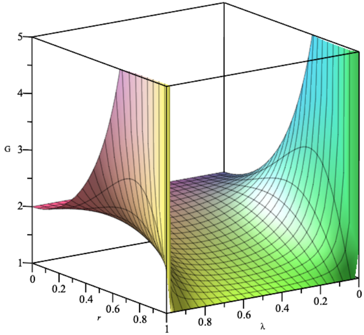
The Quantum Fisher Information is

$$H = \frac{8r^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}. \tag{39}$$

QFI of Phase Flips Cont.

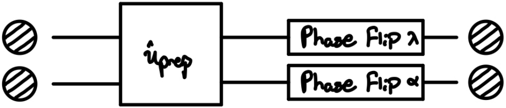
The gain of the last two scenarios is

$$G = \frac{2(1+r^2)(1-r^2(1-2\lambda)^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2}. \tag{40}$$



QFI of Phase Flips Cont.

We now examine calculating the Quantum Fisher Information of a measurement with noise in a spectator.



The Quantum Fisher Information is

$$H = \frac{8r^2(1-2\alpha)^2(1+r^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}. \tag{41}$$

QFI of Phase Flips Cont.

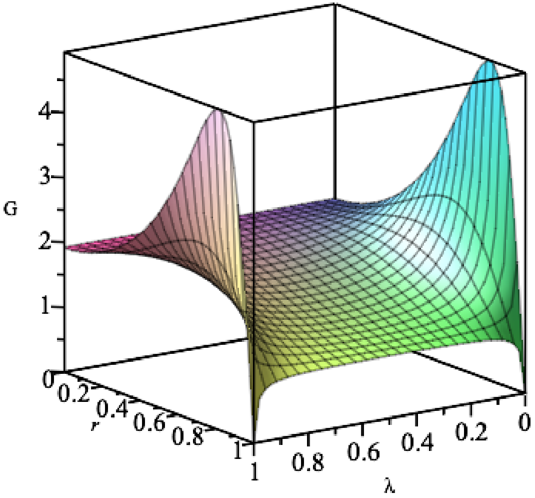
The gain of the last scenario is

$$G = \frac{2(1-2\alpha)^2(1+r^2)(1-r^2(1-2\lambda)^2)}{(1+r^2)^2 - 4r^2(1-2\lambda)^2(1-2\alpha)^2}. \tag{42}$$

We will examine gains of when $\alpha = 0.01, 0.1, 0.3, 0.5$.

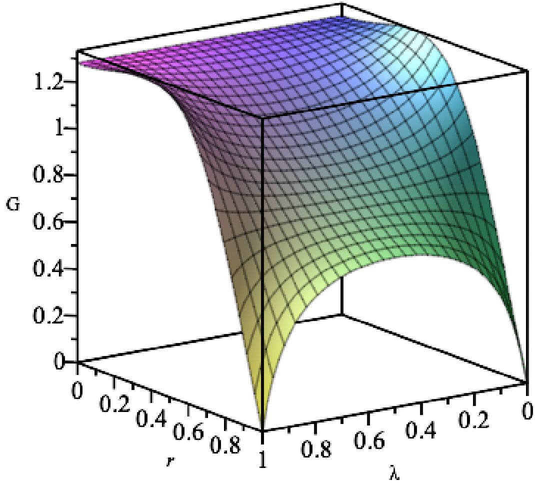
QFI of Phase Flips Cont.

Gain of when $\alpha = 0.01$.



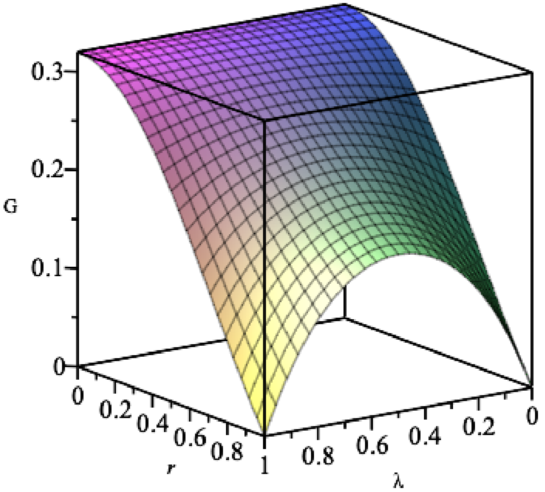
QFI of Phase Flips Cont.

Gain of when $\alpha = 0.1$.



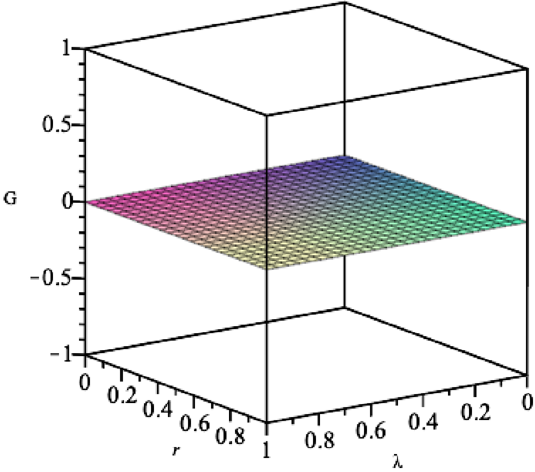
QFI of Phase Flips Cont.

Gain of when $\alpha = 0.3$.



QFI of Phase Flips Cont.

Gain of when $\alpha = 0.5$.



Phase Flip Conclusions

When the noise in the spectator is small or large enough (i.e α is small or large) there is still advantageous gain.

As α approaches 0.5 the gain disappears.

Depolarizing Channels

We now wish to estimate parameter λ with $\hat{\rho}_f = \frac{(1-\lambda)}{2}\mathrm{Tr}[\hat{\rho}_i]\hat{I} + \lambda\hat{\rho}_i$.

$$\hat{\rho}_i = \frac{1}{2}\begin{pmatrix} 1 + r_{zi} & r_{xi} - ir_{yi} \\ r_{xi} + ir_{yi} & 1 - r_{zi} \end{pmatrix}. \tag{43}$$

Where $\hat{\rho}_f$ is

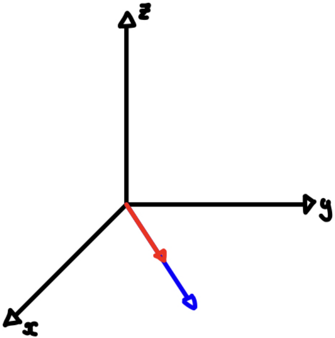
$$\hat{\rho}_f = \frac{1}{2}\begin{pmatrix} 1 + \lambda r_{zi} & \lambda(r_{xi} - ir_{yi}) \\ \lambda(r_{xi} + ir_{yi}) & 1 - \lambda r_{zi} \end{pmatrix} \tag{44}$$

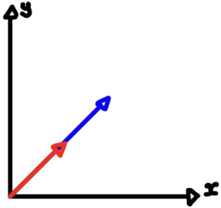
The way the components of these vectors change can be described mathematically as

$$\begin{aligned} r_{xf} &= \lambda r_{xi} \\ r_{yf} &= \lambda r_{yi} \\ r_{zf} &= \lambda r_{zi}. \end{aligned} \tag{45}$$

Visualizing of Depolarizing Channel

We can visualize how this Depolarizing evolution.





QFI of Depolarizing Channels

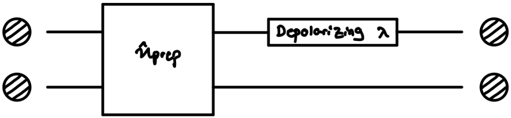
We now examine the simplest depolarizing channel evolution.

The Quantum Fisher Information is

$$H = \frac{r^2}{1 - r^2\lambda^2}. \tag{46}$$

QFI of Depolarizing Channels

We now examine the next scenario with two particles and one channel depolarizing evolution.



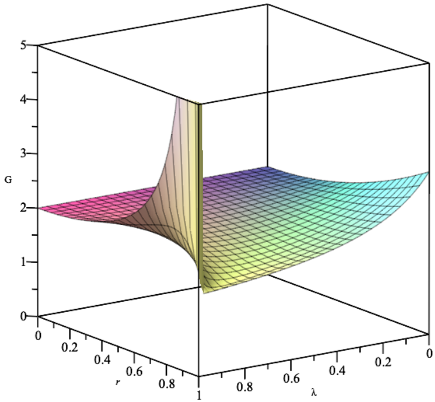
The Quantum Fisher Information is

$$H = \frac{r^2(r^2(1-4\lambda)+r^4\lambda+2)}{(1+2r\lambda+r^2\lambda)(1-2r\lambda+r^2\lambda)(1-r^2\lambda)}. \qquad (47)$$

QFI of Depolarizing Channels

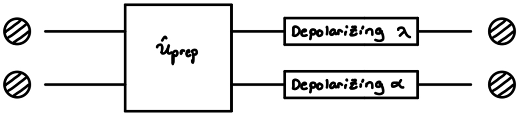
The gain of the last two scenarios is

$$G = \frac{(r^2\lambda^2 - 1)(r^4\lambda - 4r^2\lambda + r^2 + 2)}{(1 + 2r\lambda + r^2\lambda)(1 - 2r\lambda + r^2\lambda)(r^2\lambda - 1)}. \qquad (48)$$



QFI of Depolarizing Channels

We now examine a two particle two channel depolarizing evolution.



The Quantum Fisher Information is

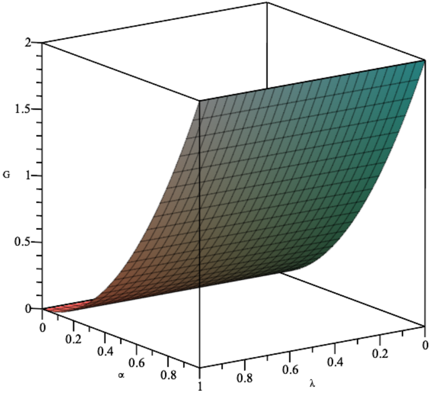
$$H = \frac{r^2\alpha^2(\alpha\lambda r^4 - 4r^2\lambda\alpha + r^2 + 2)}{(1 - r^2\lambda\alpha)(1 + r\alpha(r+2)\lambda)(1 + r\alpha(r-2)\lambda)}. \quad (49)$$

QFI of Depolarizing Channels

The gain of the last scenario is

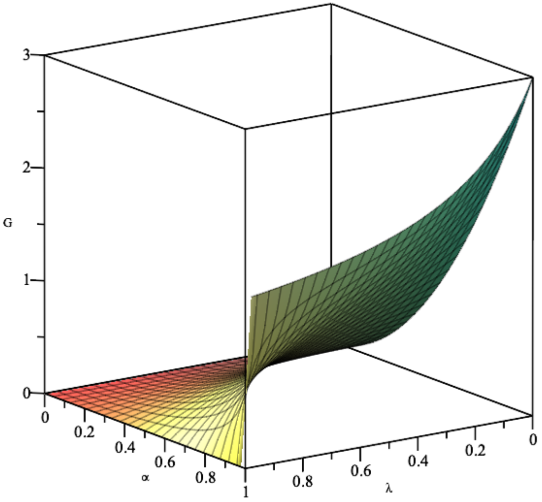
$$G = \frac{\alpha(\alpha\lambda r^4 - 4\lambda r^2\alpha + r^2 + 2)^2(r^2\lambda^2 - 1)}{(\lambda r^2\alpha - 1)(1 + r\alpha(r+2)\lambda)(1 + r\alpha(r-2)\lambda)}. \quad (50)$$

With $r = 0.01$.



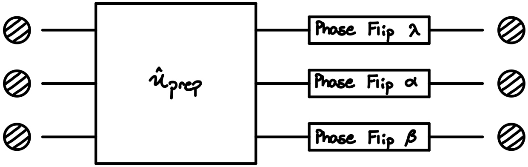
QFI of Depolarizing Channels

With $r = 1.0$.



QFI of 3 Phase Flip Channels

We now examine a three particle measurement.



QFI of 3 Phase Flip Channels

The Quantum Fisher Information of this measurement comes out to be

$$H = r^2(1-2\alpha)^2(1-2\beta)^2 \cdot \left(\frac{(r+3)^2(3r^2+1)(\eta) + 3(1-r^2)^3(\xi)}{(\eta)(\xi)} \right) \tag{51}$$

where $\eta = (1-r^2)^2 - \gamma^2 r^2 (1-r^2)^2$,
$\xi = (3r^2+1)^2 - r^2(r+3)^2\gamma^2$, and
$\gamma = (1-2\lambda)(1-2\alpha)(1-2\beta)$.

QFI of 3 Phase Flip Channels

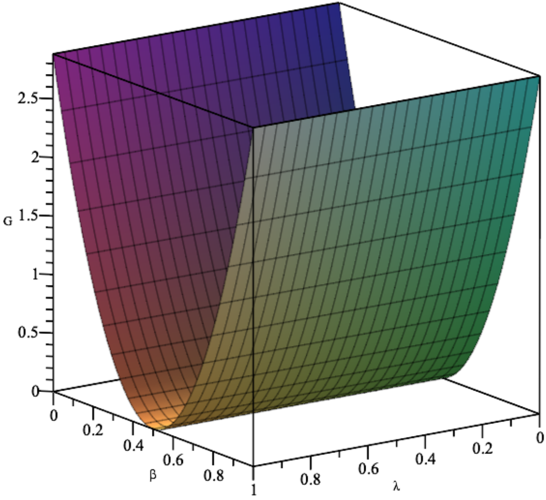
The gain of the three particle scenario comes out to be

$$G = \frac{1 - r^2(1-2\lambda)^2(1-2\alpha)^2(1-2\beta)^2}{4} \cdot \left(\frac{(r+3)^2(3r^2+1)(\eta) + 3(1-r^2)^3(\xi)}{(\eta)(\xi)} \right). \tag{52}$$

With the next gain scenarios $r = 0.01$.

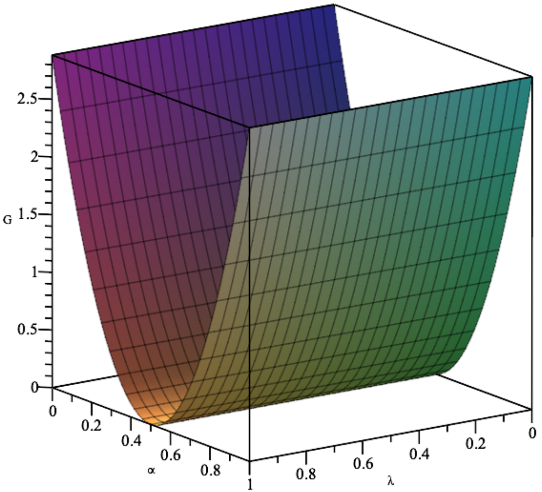
QFI of 3 Phase Flip Channels

With $\alpha = 0.01$.



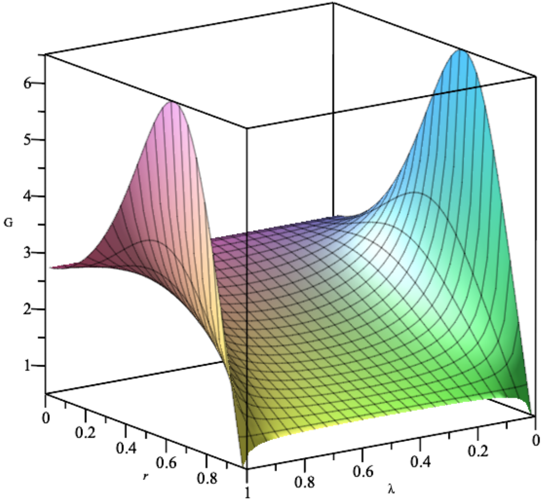
QFI of 3 Phase Flip Channels

With $\beta = 0.01$.



QFI of 3 Phase Flip Channels

With this last scenario, $\alpha = 0.01$ and $\beta = 0.01$.



Conclusion

In the presence of Phase Shifts, Phase Flips, and Depolarizing Channels the Quantum Fisher Information will vary depending on initial conditions.

Pure states are difficult to obtain, so in practicality noisy states are commonly used for estimation.

Typically the use of multiple particles in a measurement yield better outcomes with some exceptions.

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Questions?

Any Questions?