MAT 202
Larson – Section 9.8
Power Series

In sections 9.2 & 9.3 we introduced series and determined whether they converge or diverge by using various different tests. In this section and section 9.10 we will focus our attention on three specific types of series: ***Power Series***, ***Taylor Series***, and ***MacLaurin Series***. We will begin with Power Series.

<u>Definition of Power Series</u>: If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a ***power series***. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a ***power series centered at c***, where c is a constant.

Ex: State where the power series is centered:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$

Ex: State where the power series is centered:

a) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \underbrace{\boxed{\frac{1}{n!}}}_{a_n}(x-\underbrace{\boxed{0}}_{c})^n$

Centered @ 0

b) $\sum_{n=1}^{\infty} \underbrace{\boxed{\frac{1}{n}}}_{a_n}(x-\underbrace{\boxed{2}}_{c})^n$

Centered @ 2

For What Values of "x" Does a Power Series Converge?

A power series in x can be viewed as a function of x: $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ where the **domain of f is the set of all x for which the power series converges**. Since $f(c) = \sum_{n=0}^{\infty} a_n (c-c)^n = a_0(1) + 0 + 0 + \cdots = a_0$. So c always lies in the domain of f. The domain of a power series can take three basic forms:

1) A single point.
2) An interval centered at c, or
3) The real number line

<u>Convergence of a Power Series</u>: Let c be a constant, R be the radius of convergence, and the set of all values of x for which the power series converges is the ***interval of convergence***. For a power series centered at c, precisely one of the following is true.

1. **The series converges only at c.** Means if $R = 0$, then the interval of convergence is the value $x = c$.
2. There exists a real number $R > 0$ such that the series
   a) **converges absolutely for** $|x-c| < R$. Means if radius of convergence $R > |x-c|$, then the interval of convergence ***MAY*** be $(c-R, c+R)$. We will discuss ***endpoint convergence*** soon.
   b) diverges for $|x-c| > R$.
3. **The series converges absolutely for all x.** Means if $R = \infty$, then the interval of convergence is $(-\infty, \infty)$.

To determine the radius of convergence we need the ***Ratio Test*** that was introduced in section 9.6.

Bothered with $(0)^0 = 1$

$$\ln y = \ln \lim_{x \to 0} x^x$$

$$\ln y = \lim_{x \to 0} \ln x^x$$

$$= \lim_{x \to 0} x \ln x$$

$$= \lim_{x \to 0} \frac{\ln x}{x^{-1}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \frac{-x^2}{1}$$

$$= \lim_{x \to 0} -x = 0$$

$$e^{\ln y} = e^0$$

$$y = 1$$

Ratio Test: Let $\sum a_n$ be a series with nonzero terms.

1. The series $\sum a_n$ converges absolutely when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

2. The series $\sum a_n$ diverges when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.

3. The Ratio Test is inconclusive when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} n! x^n$.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} \boxed{n!x^n} = a_n$.

centered @ 0

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\, x^{(n+1)}}{n!\, x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cdot \cancel{n!} \cdot \cancel{x^n} \cdot x}{\cancel{n!}\, \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| x(n+1) \right| = |x| \lim_{n\to\infty} |n+1|$$

$$= |x| \cdot \infty$$

converges if $|x| \cdot \infty < 1$

Diverges if $|x| \cdot \infty > 1$ or $|x| \cdot \infty = \infty$

Only converges if $x = 0$

$$\boxed{\text{Radius of convergence} = 0}$$

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$.

centered @ 2

Ratio Test: $\lim_{n\to\infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{\cancel{3}\cdot\cancel{(x-2)^n}\cdot(x-2)}{\cancel{3}\,\cancel{(x-2)^n}} \right| = \lim_{n\to\infty} |x-2|$

$= |x-2| < 1$

$= -1 < x-2 < 1$
$\;\; +2 \quad\; +2 \;\; +2$

$= 1 < x < 3$

$\boxed{\text{Radius of convergence} = 1}$

possible interval of convergence.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

<u>Interval of Convergence & Endpoint Convergence</u>: When finding the interval of convergence, it is necessary (at times) to **test the endpoints** to determine whether or not the series converges at those endpoints. For instance, if we get an interval of convergence $(c - R, c + R)$, depending on the type of series, we may need to test the values $x = c - R$, and $x = c + R$ in the series to determine whether or not we need to include those boundaries as points of convergence. We may get any one of the following intervals of convergence: $(c - R, c + R)$ or $(c - R, c + R]$ or $[c - R, c + R)$ or $[c - R, c + R]$. **Since we skipped much of chapter 9, our examples will be limited. Look at examples on page 650.**

Ex: Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

Ratio Test: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot x^{2(n+1)+1}}{(2(n+1)+1)!} \div \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \cdot (-1) \cdot \cancel{x^{2n}} \cdot x^{\cancel{3}\,2}}{(2n+3)!} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-1)^n}\, \cancel{x^{2n}} \cdot \cancel{x}} \right|$$

$$(2n+3)(2n+2)\cancel{(2n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{-x^2}{(2n+3)(2n+2)} \right| = 0 < 1$$

Means the series will always converge no matter what x is

$\therefore$ $\boxed{\text{Radius of Convergence} = \infty}$

Ex: Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$.

Radius of convergence:

$$\lim_{n\to\infty} \left| \frac{x^{5(n+1)}}{(n+1)!} \div \frac{x^{5n}}{n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{x^{5n}} \cdot x^{5}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^{5n}}} \right| = 0 < 1$$

Radius of convergence $= \infty$

$\therefore$ Interval of convergence: $\boxed{(-\infty, \infty)}$