Physics 396

Homework Set 10

1. A distant galaxy has a redshift of $z = 0.2$. According to Hubble's law, how far away was the galaxy, in light years, when the light was emitted if the Hubble parameter is 72 km/s/Mpc?

2. When the *flat* Robertson-Walker line element, given by Eq. (18.1) in your text, and the perfect fluid energy-momentum tensor are inserted into the Einstein equations of GR, one arrives at a set of non-linear, ordinary differential equations of the form

$$\begin{aligned} 8\pi\rho &= 3\frac{\dot{a}^2}{a^2} && (1) \\ 8\pi p &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) && (2) \\ 0 &= \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) && (3) \end{aligned}$$

where ρ and p are the density and pressure, respectively. It is noted that a dot represents a time derivative, a double-dot represents two time derivatives. Also notice that Eq. (3) is not a field equation, but rather a conservation statement. These differential equations represent the *Standard Model of Cosmology.*

a) Show that Eq. (3) is a redundant equation by inserting Eqs. (1) and (2) for ρ and p into Eq. (3). *Hint:* when performing derivatives, notice that $\frac{\dot{a}^2}{a^2} = \left(\frac{\dot{a}}{a}\right)^2$.

Having 2 equations and 3 unknowns (ρ, p, and a), we have an underdetermined system of equations. At this stage one usually employs an *equation of state* (EoS) of the form

$$p = w\rho \qquad (4)$$

where w is called the equation of state parameter.[1] In general, w can depend on time, however, here we will treat it as a constant. We now have 3 equations and 3 unknowns. At this point, Eq. (3) will yield an expression relating the density to the scale factor.

[1] Notice: $w = 0$ corresponds to *pressureless matter*, $w = 1/3$ corresponds to *radiation*, and $w = -1$ corresponds to *vacuum energy.*

This will allow us to decouple Eqs. (1) and (2), thus yielding two differential equations involving only the scale factor.

b) Using Eqs. (4) and (3), find $\rho = \rho(a)$. *Hint:* eliminate the pressure from Eq. (3) in favor of the density, separate variables, and integrate.

Answer:

$$\rho(t) = \rho_0 a(t)^{-3(1+w)} \tag{5}$$

c) Using Eqs.(1)-(3) and (4), obtain an expression for the acceleration of the Universe, $\ddot{a}/a$, in terms of the density of the Universe.

d) For what values of w does one obtain an *accelerated* expansion of the Universe?

3. Reconsider the *flat* Friedman-Robertson-Walker field equations of cosmology, given by Eqs. (1) - (3).

a) Show that $a(t) = a_0\, t^{2/3(1+w)}$ is a solution to Eq. (1).

b) Show that $a(t) = a_0\, t^{2/3(1+w)}$ is a solution to Eq. (2).

c) Solve Eqs. (1) and (2) for when $w = -1$.

Hint: Use the fact that

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \tag{6}$$

4. In class, we arrived at an equation of motion describing the scale factor of the form

$$\frac{1}{2H_0^2}\left(\frac{da}{dt}\right)^2 + U_{eff}(a) = 0, \tag{7}$$

where

$$U_{eff}(a) \equiv -\frac{1}{2}\left(\Omega_v a^2 + \frac{\Omega_m}{a} + \frac{\Omega_r}{a^2}\right) \qquad (a(t_0) = 1). \tag{8}$$

It is noted that Eq. (7) is of the form of a 1D conservation of energy expression for a particle with zero energy in Newtonian mechanics. Using your favorite plotting software (i.e. Maple, Matlab, Wolframalpha, etc.), plot $U_{eff}(a)$ vs a for

a) *Matter-dominated flat FRW spacetime*, $\Omega_m = 1,\ \Omega_r = 0,\ \Omega_v = 0,$

b) *Radiation-dominated flat FRW spacetime*, $\Omega_m = 0,\ \Omega_r = 1,\ \Omega_v = 0,$

c) *Vacuum-dominated flat FRW spacetime*, $\Omega_m = 0,\ \Omega_r = 0,\ \Omega_v = 1$,

d) *Our flat FRW spacetime*, $\Omega_m = 0.3,\ \Omega_r = 5 \times 10^{-5},\ \Omega_v = 0.7$.

Your a-axis should have a range of $0.5 < a < 3.5$ to properly show the curves. These four curves should be positioned on the same plot.

e) Using the 1D conservation of energy expression given by Eq. (7) and your four plots, discuss qualitatively the behavior of the scale factor $a(t)$ for all time for each of these cases. Does one get accelerated or decelerated expansion? Do we get expansion and then contraction? Do we get contraction and then expansion?