Statistical and Thermal Physics: Homework 4

Due: 4 February 2020

1 Work done on a gas: general considerations

The work done on any gas can be computed via

$$W = -\int P \, dV.$$

a) There is only one type of process where it is legitimate to compute the work via

$$W = -P \int dV.$$

Describe what conditions this process must satisfy for this to give the correct expression for work. What curve would represent this process on a PV diagram?

b) For ideal gases $P = NkT/V$ and thus

$$W = -\int_{V_i}^{V_f} \frac{NkT}{V} \, dV.$$

It may appear that this gives

$$W = -NkT \int_{V_i}^{V_f} \frac{1}{V} \, dV = -NkT \ln\left(\frac{V_f}{V_i}\right).$$

But there is also only one type of process for which this is true. What condition must T satisfy during the process for this to be true? What curve would represent this process on a PV diagram?

c) During an isobaric process, is T independent of V? Would it be true for this process that

$$W = -NkT \int_{V_i}^{V_f} \frac{1}{V} \, dV = -NkT \ln\left(\frac{V_f}{V_i}\right) ?$$

Explain your answer.

In general one must be careful when calculating work. One must ask: Does P (or T) depend on V during the process? If it does, then the process of integration will be more complicated that extracting a constant and integrating with respect to V as done in the cases above.

2 Work done on an ideal gas: specific examples

An ideal gas which undergoes the following three-stage process, with initial volume V_i and pressure P_i. First it expands to double its initial volume while the pressure remains constant. Then the volume is held constant and the pressure increases to three times the original pressure. Then it returns to its initial state via a process which can be represented as a straight line on a PV diagram. Sketch the entire process on a PV diagram. Determine an expression for the work done on the gas in for the entire process.

3 Monoatomic ideal gas process

A monoatomic ideal gas undergoes a cyclical process, starting at an initial volume V_1 and initial pressure P_1 and eventually ending at the same pressure and volume. During the first stage of the process, the pressure and volume are related by

$$P = aV^2$$

where a is a constant with units of Pa/m^6. During this process the volume is reduced to $V_1/2$. During the second and third stages either the pressure or else the volume remains constant. At the end of the second stage the gas reaches a pressure of P_1.

a) Sketch the process on a PV diagram.

b) Determine the work done during each stage of the process.

c) Determine the change in internal energy during each stage of the process.

d) Determine the heat that leaves or enters the gas during each stage of the process.

e) Over the entire cycle, does the gas do work on its surroundings or is work done on the gas?

f) Consider the second stage. As the gas evolves during this stage is heat *constantly* supplied or *constantly* removed? Or is heat supplied during some portions of this stage and removed during other portions of this stage? Repeat this for the third stage.

g) Suppose that the only cost to running the gas through this cycle is the heat added (the heat removed is just lost). Determine what fraction of the heat added over the entire cycle is converted to work.

4 Energy for a van der Waals gas process.

A monoatomic van der Waals gas satisfies the equation of state,

$$\left(P + a\frac{N^2}{V^2}\right)(V - Nb) = NkT$$

and has thermal energy given by

$$E = \frac{3}{2}NkT - a\frac{N^2}{V}$$

where a and b are constants that depend on the particular gas. Suppose that the gas undergoes a constant volume process in which the pressure doubles. Determine expressions (in terms of the initial pressure and volume) for the work done, the change in internal energy and the heat supplied during this process.

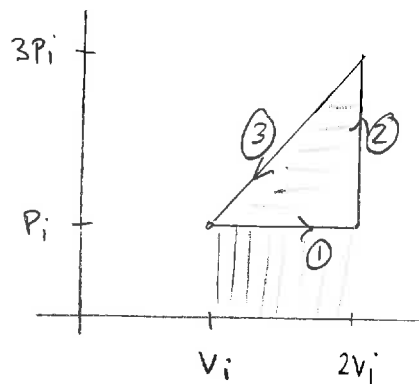a)   This requires that   P is constant.

$$P \quad \xrightarrow{\quad\quad}$$
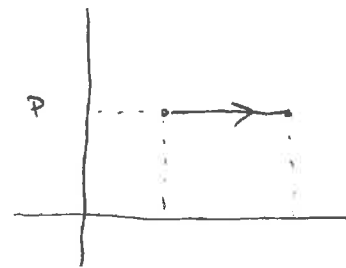
b) This requires that   T be constant

isothermal

$$PV = const$$
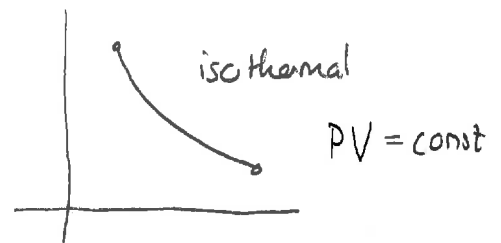
c) No $PV = NkT \Rightarrow$ $T = \dfrac{P}{Nk} V$

P, Nk constant

V varies.

So T varies and we cannot do the suggested calculation.

We need $W = -$ area under P vs V

Using ①, ②, ③

expansion

$W_① = -$ area rectangle $= -(2V_i - V_i)P_i = -P_iV_i$

$W_② = 0$

because compression

$W_③ = +[$area rectangle $+$ triangle$] = P_iV_i + \frac{1}{2}(2V_i - V_i)2P_i$

$= P_iV_i + P_iV_i = 2P_iV_i$

So $W = 2P_iV_i - P_iV_i = P_iV_i$ $\Rightarrow$ $\boxed{W = P_iV_i}$
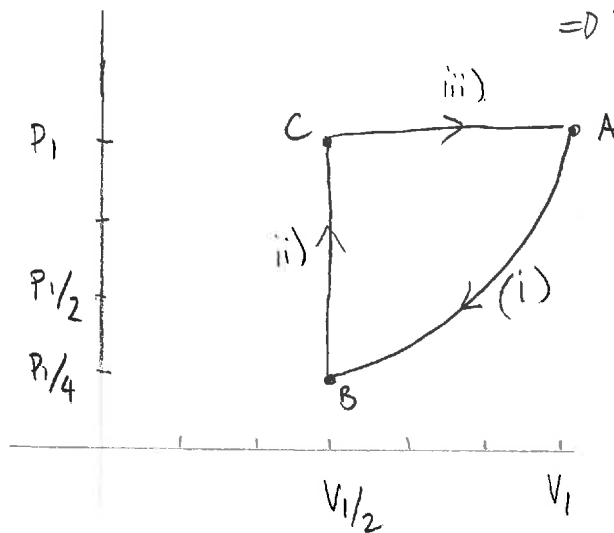
a) initially V_1, P_1 and on curve $P_1 = aV_1^2$

after first stage $V = V_2 = V_1/2$

$P = P_2$

$P_2 = aV_2^2$

$= aV_1^2/4 = P_1/4$

$\Rightarrow P_2 = P_1/4$

b) $W = -\int P\,dV$

(i) $P = aV^2 \Rightarrow W = -\int_{V_1}^{V_2} aV^2\,dV = -\frac{a}{3}V^3 \Big|_{V_1}^{V_2}$

$\Rightarrow W_{(i)} = \frac{a}{3}\left(V_1^3 - \frac{V_1^3}{8}\right) = \frac{7a}{24}V_1^3$

(ii) $dV = 0 \Rightarrow W_{(ii)} = 0$

(iii) $P = const \Rightarrow W = -\int_{V_1/2}^{V_1} P\,dV = -P_1\left(V_1 - V_1/2\right)$

$= -\frac{P_1 V_1}{2} = -\frac{aV_1^3}{2}$

Summary:
$$W_{(i)} = \frac{7}{24} a V_1^3$$

$$W_{(ii)} = 0$$

$$W_{(iii)} = -\frac{a V_1^3}{2}$$

c) $E = \frac{3}{2} NkT = \frac{3}{2} PV$

(i) $\Delta E = E_B - E_A = \frac{3}{2}(PV)_B - \frac{3}{2}(PV)_A$

$$= \frac{3}{2} \frac{P_1}{4} \frac{V_1}{2} - \frac{3}{2} P_1 V_1$$

$$= -\frac{3}{2} \cdot \frac{7}{8} P_1 V_1 = -\frac{21}{16} P_1 V_1$$

$$= -\frac{21}{16} a V_1^3$$

(ii) $\Delta E = E_C - E_B = \frac{3}{2}(PV)_C - \frac{3}{2}(PV)_B$

$$= \frac{3}{2}\left[P_1 V_1/2 - \frac{P_1 V_1}{4 \cdot 2} \right] = \frac{9}{16} P_1 V_1 = \frac{9}{16} a V_1^3$$

(iii) $\Delta E = E_A - E_C = \frac{3}{2}\left[P_1 V_1 - P_1 V_1/2 \right] = \frac{3}{4} a V_1^3 = \frac{12}{16} a V_1^3$

Summary
$$\Delta E_i = -\frac{21}{16} a V_1^3$$
$$\Delta E_{ii} = \frac{9}{16} a V_1^3$$
$$\Delta E_{iii} = \frac{12}{16} a V_1^3$$

d) $\quad Q = \Delta E - W$

$$Q_{(i)} = -\frac{21}{16} a V_1{}^3 - \frac{7}{24} a V_1{}^3 = -\frac{77}{48} a V_1{}^3$$

$$Q_{(ii)} = \frac{9}{16} a V_1{}^3 - 0 = \frac{9}{16} a V_1{}^3$$

$$Q_{(iii)} = \frac{12}{16} a V_1{}^3 - \left(-\frac{a V_1{}^3}{2}\right) = \frac{20}{16} a V_1{}^3$$

e) $\quad W \text{ total} = W_i + W_{ii} + W_{iii} = -\frac{5}{24} a V_1{}^3$

does work on surroundings.

f) any small section $\qquad dE = Q + \underbrace{PdV}_{=0}$

$$Q = dE = d(PV)$$
$$\underset{\uparrow \text{increases}}{} \Rightarrow Q > 0$$

g) heat added $= Q_{(ii)} + Q_{(iii)} = \frac{29}{16} a V_1{}^3$

$$\frac{W \text{ total}}{\text{heat added}} = \frac{\frac{5}{24} a V_1{}^3}{\frac{29}{16} a V_1{}^3} = \frac{5}{29} \cdot \frac{16}{24} = \frac{5 \cdot 2}{29 \cdot 3} = \frac{10}{87}$$

$$\left(P + \frac{aN^2}{V^2}\right)(V-Nb) = NkT$$

$$E = \frac{3}{2}NkT - a\frac{N^2}{V}$$

First we rewrite E in terms of P, V rather that T, V. So

$$E = \frac{3}{2}\left(P + \frac{aN^2}{V^2}\right)(V-Nb) - a\frac{N^2}{V}$$

Then

$$W = -\int P dV = 0 \qquad \text{since volume constant}$$

$$\Delta E = Ef - Ei$$

Let P_i be the initial temperature. So $Pf = 2P_i$

$$\Delta E = \frac{3}{2}\left(Pf + \frac{aN^2}{Vf^2}\right)(Vf-Nb) - a\frac{N^2}{Vf}$$

$$- \left[\frac{3}{2}\left(P_i + \frac{aN^2}{V_i^2}\right)(V_i-Nb) - a\frac{N^2}{V_i}\right]$$

Now $Vf = V_i$ and so

$$\Delta E = \frac{3}{2}\left(2P_i + \frac{aN^2}{V_i^2}\right)(V_i-Nb) - \cancel{a\frac{N^2}{V_i}}$$

$$- \frac{3}{2}\left(P_i + \frac{aN^2}{V_i^2}\right)(V_i-Nb) + \cancel{a\frac{N^2}{V_i}}$$

Thus

$$\Delta E = \frac{3}{2} P_i (V_i - Nb)$$

For heat, first law gives:

$$\Delta E = Q + W$$

$$\Rightarrow \quad Q = \Delta E - W$$

$$= \Delta E$$

So $\quad W = 0$

$$\Delta E = \frac{3}{2} P_i (V_i - Nb)$$

$$Q = \frac{3}{2} P_i (V_i - Nb)$$