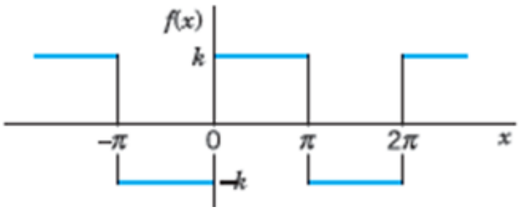
11.1 Fourier Series

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$



$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

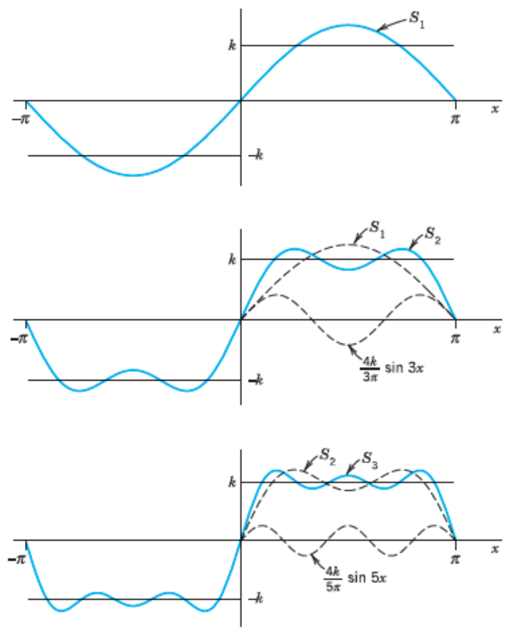


Fig. 261. First three partial sums of the corresponding Fourier series

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) *is orthogonal on the interval* $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

(a) $$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m)$$

(b) $$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m)$$

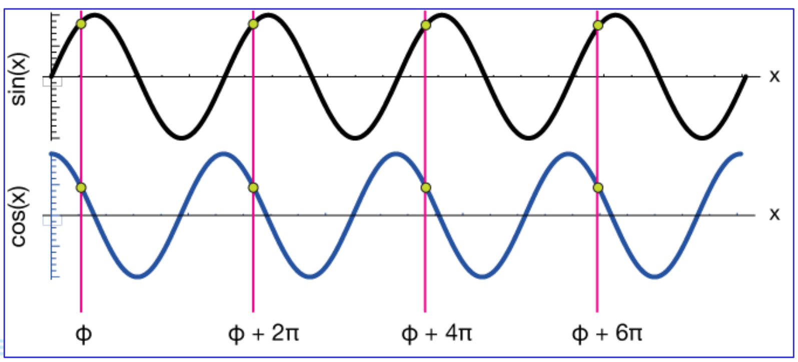
(c) $$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$



Periodic Rectangular Wave (Fig. 260)

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 260. The formula is

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x). \tag{7}$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

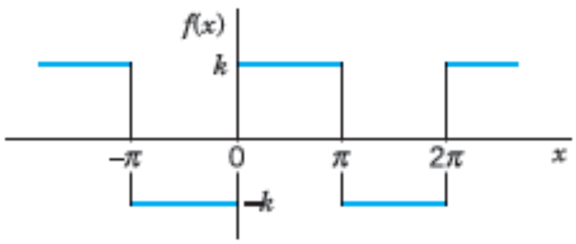
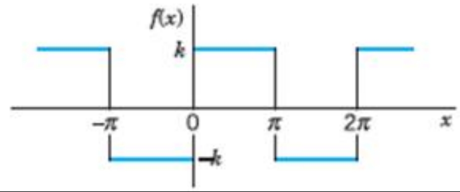


Fig. 260. Given function $f(x)$ (Periodic reactangular wave)

EXAMPLE 1

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\frac{\sin nx}{n}\Big|_{-\pi}^{0} + k\frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] = \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

$$= \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right).$$

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \quad \text{and thus} \quad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Since the a_n are zero, the Fourier series of $f(x)$ is

$$\frac{4k}{\pi}\left(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots\right). \tag{8}$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right). \qquad \text{etc.}$$

Their graphs in Fig. 261 seem to indicate that the series is convergent and has the sum $f(x)$, the given function. We notice that at $x = 0$ and $x = \pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the limits $-k$ and k of our function, at these points. This is typical.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

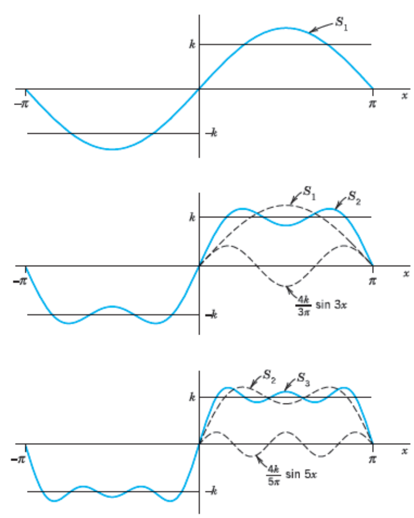


Fig. 261. First three partial sums of the corresponding Fourier series

Representation by a Fourier Series

Let $f(x)$ be periodic with period 2π and piecewise continuous (see Sec. 6.1) *in the interval $-\pi \leqq x \leqq \pi$. Furthermore, let $f(x)$ have a left-hand derivative and a right-hand derivative at each point of that interval. Then the Fourier series* (5) *of $f(x)$* [with coefficients (6)] *converges. Its sum is $f(x)$, except at points x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of the left- and right-hand limits*[2] *of $f(x)$ at x_0.*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

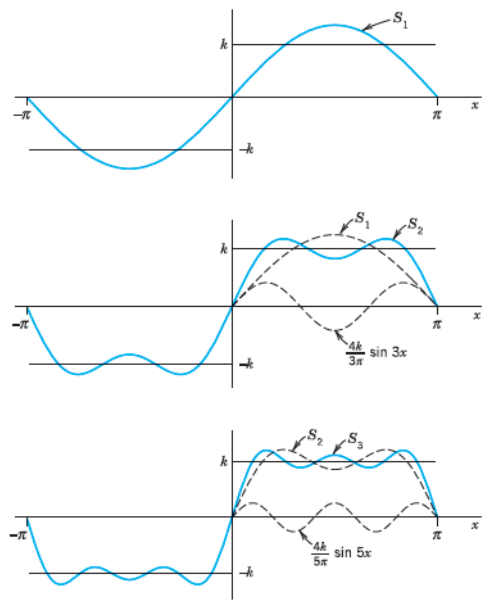


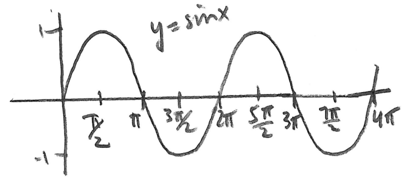
Fig. 261. First three partial sums of the corresponding Fourier series

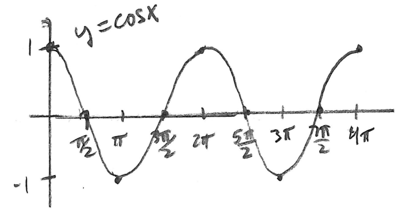
THEOREM 2

Ch11.1 Fourier Series: In Class Examples

Helpful formulas/results

1. $\cos(-x) = \cos(x)$
2. $\sin(-x) = -\sin(x)$
3. For $A\sin(Bx)$ & $A\cos(Bx)$, A = amplitude, period $= \frac{2\pi}{B}$, freq $= \frac{B}{2\pi}$
4.





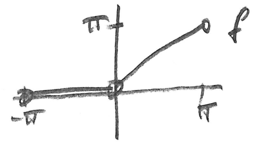
Examples

1. Observe from the graphs of $\sin x$ & $\cos x$ that they are periodic with period 2π. It can also be said that they are periodic with period 4π, since they repeat after 4π units. Thus the fundamental period is 2π, since it is the smallest period.

2. $\sin(3x)$ has fundamental period $\frac{2\pi}{3}$ (graph repeats after $[0, \frac{2\pi}{3}]$).
   $\cos(2\pi x)$ has fundamental period $\frac{2\pi}{2\pi} = 1$ (graph repeats after $[0,1]$).

(Note: Answer in back of book for #1 has errors. See if you can correct them.)

(3) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$



(a) Find a_0, a_n, and b_n.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{2\pi}\left(\frac{1}{2}x^2\right)\Big|_0^{\pi} = \frac{1}{4\pi}\left[\pi^2 - 0^2\right] = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{1}{\pi}\left[\overbrace{\frac{x}{n}\cancel{\sin(nx)}\Big|_0^{\pi}}^{0} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

$$\left[\begin{array}{ll} u = x & dv = \cos(nx)\,dx \\ du = dx & v = \frac{1}{n}\sin(nx) \end{array}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right]$$

$$= \frac{1}{n^2\pi}\left[\cos(n\pi) - 1\right]$$

$$= \begin{cases} 0, & \text{if } n = 2k \text{ (even)} \\ \frac{-2}{n^2\pi}, & \text{if } n = 2k+1 \text{ (odd)} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right]$$

$$\begin{array}{ll} u = x & dv = \sin(nx)\,dx \\ du = dx & v = -\frac{1}{n}\cos(nx) \end{array}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \overbrace{\frac{1}{n^2}\cancel{\sin}(nx)\Big|_0^{\pi}}^{0}\right]$$

$$= -\frac{1}{n}\cos(n\pi) = -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

(b) Find Fourier Series for $f(x)$.

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots\right)$$
$$+ \left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \cdots\right)$$

(c) Graph partial sums up to that including $\cos 5x$ and $\sin 5x$.

- Use Maple, Desmos.com or graphing calculator

Math 360 **Methods of Applied Math**

Ch11.1 Fourier Series

<u>In-Class Example</u>