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MATH 360
CP Ch. 5.4

One of the most important ODEs in applied mathematics are Bessel's equation,

(1)
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Where the parameter ν (nu) is a given real number which is positive or zero. Bessel's equation often appears if a problem shows cylindrical symmetry, for example, as the membranes in sec 12.9. The equation satisfies the assumptions of Theorem 1. To see this, divide (1) by x^2 to get the standard form $y'' + y'/x + (1 - \nu^2/x^2)y = 0$. Hence, according to the Frobenius theory, it has a solution of the form

(2)
$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0)$$

Substituting (2) and its first and second derivatives into Bessel's equation, we obtain

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

We equate the sum of the coefficients of x^{s+r} to zero. Note that this power x^{s+r} corresponds to $m=s$ in the first, second, and fourth series, and to $m=s-2$ in the third series. Hence for $s=0$ and $s=1$, the third series does not contribute since $m \geqq 0$. For $s = 2, 3, \ldots$ all four series contribute, so that we get a general formula for all these s. We find

(3)
(a) $$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \qquad (s=0)$$
(b) $$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \qquad (s=1)$$
(c) $$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0 \qquad (s = 2, 3, \ldots).$$

From (3a) we obtain the indical equation by dropping a_0,

(4)
$$(r+\nu)(r-\nu) = 0.$$

The roots are $r_1 = \nu (\geqq 0)$ and $r_2 = -\nu$.

Coefficient Recursion for $r = r_1 = \nu$. For $r = \nu$, Eq (3b) reduces to $(2\nu+1)a_1 = 0$. Hence $a_1 = 0$ since $\nu \geqq 0$. Substituting $r = \nu$ in (3c) and combining the three terms containing a_s gives simply

(5)
$$(s+2\nu)sa_s + a_{s-2} = 0.$$

Since $a_1 = 0$ and $\nu \geqq 0$, it follows from (5) that $a_3 = 0$, $a_5 = 0$, ... Hence we have to deal only with even-numbered coefficients a_s with $s = 2m$. For $s = 2m$, Eq (5) becomes

$$(2m + 2\nu)2ma_{2m} + a_{2m-2} = 0.$$

Solving for a_{2m} gives the recursion formula

(6) $$a_{2m} = -\frac{1}{2^2 m(\nu+m)} a_{2m-2}, \qquad m = 1, 2, \ldots\ldots$$

From (6) we can now determine $a_2, a_4, \ldots\ldots$ successively. This gives

$$a_2 = -\frac{a_0}{2^2(\nu+1)}$$

$$a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$$

and so on, and in general

(7) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\ldots\ldots(\nu+m)}, \qquad m = 1, 2, \ldots\ldots$$

Bessel Functions $J_n(x)$ for Integer $\nu = n$

Integer values of ν are denoted by n. This is standard. For $\nu = n$ the relation (7) becomes

(8) $$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} \qquad m = 1, 2, \ldots\ldots$$

a_0 is still arbitrary, so that the series (2) with these coefficients would contain this arbitrary factor a_0. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly, we have to make a choice. The choice $a_0 = 1$ would be possible. A simpler series (2) could be obtained if we could absorb the growing product $(n+1)(n+2)\ldots\ldots(n+m)$ into a factorial function $(n+m)!$ What should be our choice? Our choice should be

(9) $$a_0 = \frac{1}{2^n n!}$$

because then $n!(n+1)\ldots\ldots(n+m) = (n+m)!$ in (8), so that (8) simply becomes

(10) $$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}, \qquad m = 1, 2, \ldots\ldots$$

By inserting these coefficients into (2) and remembering that $c_1 = 0$, $c_3 = 0$, $\ldots\ldots$ we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

(11) $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \qquad (n \geqq 0).$$

$J_n(x)$ is called the Bessel function of the first kind of order n. The series (11) converges for all x, as the ratio test shows. Hence $J_n(x)$ is defined for all x. The series converges very rapidly because of the factorials in the denominator.

Example 1 Bessel Functions $J_0(x)$ and $J_1(x)$

For $n=0$ we obtain from (11) the Bessel function of order 0

(12) $$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

which looks similar to a cosine (Fig. 110). For $n=1$ we obtain the Bessel function of order 1

(13) $$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + - \cdots,$$

which looks similar to a sine (Fig. 110). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the "waves" decreases with increasing x. Heuristically, n^2/x^2 in (1) in standard form [(1) divided by x^2] is zero (if $n=0$) or small in absolute value for large x, and so is y'/x, so that then Bessel's equation comes close to $y''+y=0$, the equation of $\cos(x)$ and $\sin(x)$; also y'/x acts as a "damping term," in part responsible for the decrease in height. One can show that for large x,

(14) $$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

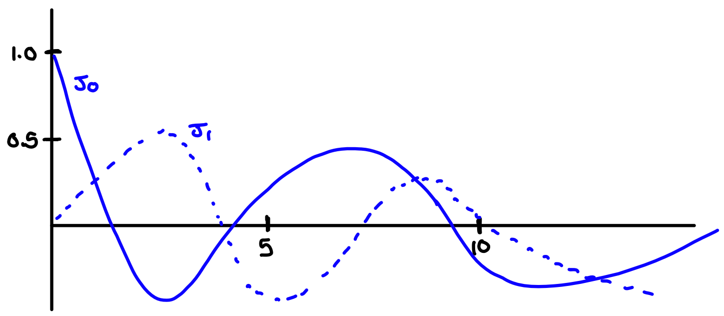

Fig. 110 Bessel Functions of The First kind

Formula (14) is surprisingly accurate even for smaller $x(>0)$. For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of J_0 you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.

Bessel Functions $J_\nu(x)$ for any $\nu \geqq 0$. Gamma Function

We now proceed from integer $\nu = n$ to any $\nu \geqq 0$. We had $a_0 = 1/(2^n n!)$ in (9). So we have to extend the factorial function $n!$ to any $\nu \geqq 0$. For this we choose

(15) $$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

with the gamma function $\Gamma(\nu+1)$ defined by

(16) $$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt \qquad (\nu > -1)$$

(Caution! Note the convention $\nu+1$ on the left but ν in the integral.) Integration by parts gives

$$\Gamma(\nu+1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} dt = 0 + \nu\Gamma(\nu).$$

This is the basic functional relation of the gamma function

(17) $$\Gamma(\nu+1) = \nu\Gamma(\nu).$$

Now from (16) with $\nu=0$ and then by (17) we obtain

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 0-(-1) = 1$$

and then $\Gamma(2) = 1\cdot\Gamma(1) = 1!$, $\Gamma(3) = 2\Gamma(1) = 2!$ and in general

(18) $$\Gamma(n+1) = n! \qquad (n=0,1,\dots)$$

Hence the gamma function generalizes the factorial function to arbitrary positive ν. Thus (15) with $\nu=n$ agrees with (9).

Furthermore, from (7) with a_0 given by (15) we first have

$$a_{2m} = \frac{(-1)^m}{2^{2m}m!(\nu+1)(\nu+2)\dots(\nu+m)2^\nu\Gamma(\nu+1)}.$$

Now (17) gives $(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2)$, $(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3)$ and so on, so that

$$(\nu+1)(\nu+2)\dots(\nu+m)\Gamma(\nu+1) = \Gamma(\nu+m+1).$$

Hence because of our (standard!) choice (15) of a_0 the coefficients (7) are simply

(19) $$a_{2m} = \frac{(-1)^m}{2^{2m+\nu}m!\Gamma(\nu+m+1)}.$$

With these coefficients and $r=r_1=\nu$ we get from (2) a particular solution of (1), denoted by $J_\nu(x)$ and given by

(20) $$J_\nu(x) = x^\nu\sum_{m=0}^\infty \frac{(-1)^m x^{2m}}{2^{2m+\nu}m!\Gamma(\nu+m+1)}.$$

$J_\nu(x)$ is called the Bessel function of the first kind of order ν. The series (20) converges for all x, as one can verify by the ratio test.