Statistical and Thermal Physics: Final Exam

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Instructions

- There are 8 questions on 12 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

$$R = 8.31\,\mathrm{J/mol\ K} \qquad N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1} \qquad 1\,\mathrm{atm} = 1.01 \times 10^5\,\mathrm{Pa}$$

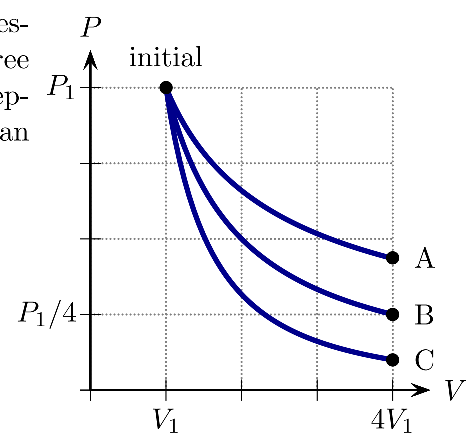
$$k = 1.38 \times 10^{-23}\,\mathrm{J/K} = 8.61 \times 10^{-5}\,\mathrm{eV/K} \qquad e = 1.60 \times 10^{-19}\,\mathrm{C} \qquad \hbar = 1.05 \times 10^{-34}\,\mathrm{Js}$$

$$N! \approx N^N e^{-N}\sqrt{2\pi N} \qquad \ln N! \approx N \ln N - N \qquad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad e^x \approx 1 + x \quad \text{if } x \ll 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}} \qquad \int_0^{\infty} x e^{-ax^2}\,\mathrm{d}x = \frac{1}{2a} \qquad \int_0^{\infty} x^2 e^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{4a^3}} \qquad \int_0^{\infty} x^3 e^{-ax^2}\,\mathrm{d}x = \frac{1}{2a^2}$$

Question 1

A monoatomic ideal gas initially has volume V_1 and pressure P_1. It can subsequently undergo any of the three indicated expansions. Describe which of these best represents an isothermal expansion and which represents an adiabatic expansion. Explain your choices.



The isothermal expansion is curve B.

The adiabatic expansion is curve C.

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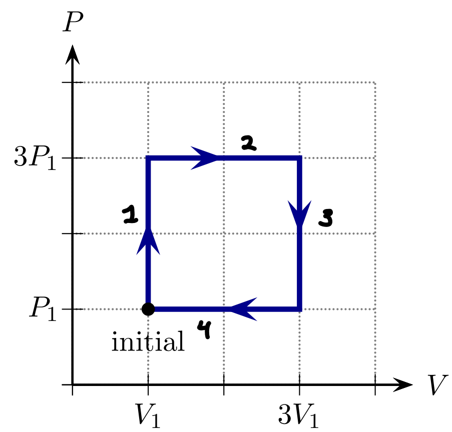
Question 2

A heat engine constructed from a monoatomic ideal gas undergoes the cycle illustrated in the PV diagram.



a) Determine the efficiency of the engine in terms of P_i and V_i. $E = Q + W \,, \quad Q = E - W$

$W_1 = W_3 = 0$

$W_2 = -P\Delta V = -3P_1(2V_1) = -6P_1V_1$

$W_4 = -P\Delta V = -P_1(-2V_1) = 2P_1V_1$

$\boxed{W_{net} = -4P_1V_1}$

$Q_1 = E_1 - \cancel{W_1}^{0} \; : \; Q_1 = \frac{3}{2}\Delta(PV) = \frac{3}{2}(3P_1V_1 - P_1V_1) = \frac{3}{2}(2P_1V_1) = 3P_1V_1$

$Q_2 = E_2 - W_2 = \frac{3}{2}\Delta(PV) + 6P_1V_1 = \frac{3}{2}(9P_1V_1 - 3P_1V_1) + 6P_1V_1 = \frac{3}{2}(6P_1V_1) + 6P_1V_1 = 15P_1V_1$

$Q_3 = E_3 - \cancel{W_3}^{0} = \frac{3}{2}\Delta(PV) = \frac{3}{2}(3P_1V_1 - 9P_1V_1) = \frac{3}{2}(-6P_1V_1) = -9P_1V_1$

$Q_4 = E_4 - W_4 = \frac{3}{2}\Delta(PV) - 2P_1V_1 = \frac{3}{2}(P_1V_1 - 3P_1V_1) - 2P_1V_1 = \frac{3}{2}(-2P_1V_1) - 2P_1V_1 = -5P_1V_1$

$\boxed{Q_+ = 18P_1V_1}$

$$\eta = \left|\frac{W_{net}}{Q_+}\right| = \left|\frac{-4P_1V_1}{18P_1V_1}\right| = \frac{2}{9}$$

$$\boxed{\eta = \frac{2}{9}}$$

Question 2 continued . . .

b) Determine the optimal efficiency of the engine that operates between two heat reservoirs, such that the low temperature heat reservoir matches the lowest temperature attained by the gas during the cycle and the high temperature reservoir matches the highest temperature attained by the gas during the cycle.

$\eta \leq 1 - \frac{T_L}{T_H}$, $PV = NkT$, $T = \frac{PV}{Nk}$

$T_L = \frac{P_1 V_1}{Nk}$, $T_H = \frac{9 P_1 V_1}{Nk}$

$\frac{T_L}{T_H} = \frac{\cancel{P_1 V_1}}{\cancel{Nk}} \cdot \frac{\cancel{Nk}}{9 \cancel{P_1 V_1}} = \frac{1}{9}$ $\eta \leq 1 - \frac{1}{9} \leq 8/9$

$\boxed{\eta \leq 8/9}$

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Question 3

Answer either part a) or part b) for full credit for this problem.

a) Show that, for any gas,

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P.$$

$$dE = TdS - PdV \ : \ dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

$$dE = T\cdot\left(\frac{\partial S}{\partial V}\right)_V dT + \left(T\left(\frac{\partial S}{\partial V}\right)_T - P\right)dV \ : \ dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV$$

$$\left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV = T\cdot\left(\frac{\partial S}{\partial V}\right)_V dT + \left(T\left(\frac{\partial S}{\partial V}\right)_T - P\right)dV$$

$$\therefore \quad \boxed{\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P}$$

b) A monoatomic ideal gas undergoes a free expansion in which its volume increases to 10 times the original volume. Determine the change in entropy of the gas.

Free expansion $\Rightarrow \Delta T = 0$: $\Delta S = fNk\ln(\overset{0}{\cancel{T_f/T_i}}) + Nk\ln(V_f/V_i)$

$V_f = 10\,V_i$

$\Delta S = Nk\ln(10V_i/V_i) = Nk\ln(10)$

$$\boxed{\Delta S = Nk\ln(10)}$$

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Question 4

The entropy of a system of spin-1/2 particles, each with magnetic dipole moment μ and in magnetic field with magnitude B is

$$S = kN\ln(N) - \frac{1}{2}\left(N - \frac{E}{\mu B}\right)\ln\left[\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{E}{\mu B}\right)\ln\left[\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right].$$

Determine an expression for the temperature of this system.

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N$$

$$\frac{\partial S}{\partial E} = \frac{1}{2\mu B}\ln\left[\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right] - \cancel{\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\cdot\frac{\frac{1}{2\mu B}}{\frac{1}{2}\left(N - \frac{E}{\mu B}\right)}}$$

$$- \frac{1}{2\mu B}\ln\left[\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right] - \cancel{\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\frac{\frac{1}{2\mu B}}{\frac{1}{2}\left(N + \frac{E}{\mu B}\right)}}$$

$$\frac{1}{T} = \frac{K}{2\mu B}\left[\ln\left(\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right) - \ln\left(\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right)\right]$$

$$\frac{1}{T} = \frac{k}{2\mu B}\left[\ln\left(\frac{N - E/\mu B}{N + E/\mu B}\right)\right]$$

$$\boxed{T = \frac{2\mu B}{K}\left[\ln\left(\frac{N - E/\mu B}{N + E/\mu B}\right)\right]^{-1}}$$

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Question 5

Consider two Einstein solids, labeled A and B, which contain identical oscillators. System A has four oscillators and system B has two oscillators. Initially each system is isolated and contains three energy units. The systems are then allowed to interact.

a) List the macrostates available to the interacting pair of Einstein solids. Provide the multiplicity for each macrostate.

q_A	ε_B	Ω_A	Ω_B	Ω
6	0	84	1	84
5	1	56	2	112
4	2	35	3	105
3	3	20	4	80
2	4	10	5	50
1	5	4	6	24
0	6	1	7	7

Question 5 continued . . .

b) Determine the equilibrium macrostate. Explain your answer.

Equilibrium macrostat = most probable macrosta = highest Ω

$$\boxed{q_A = 5 \,,\; q_B = 1 \,,\; \Omega = 112}$$

c) In which direction does the energy flow as the system approaches equilibrium? Explain your answer.

Energy flows from B to A since @ equilibrium there are more energy units for A

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Question 6

Answer either part a) or part b) for full credit for this problem.

a) An ensemble of 8 distingushable spin-1/2 particles is in a region in which there is no magnetic field. Let p_4 be the probability that the system is in the *macrostate* where exactly 4 particles have spin up and p_1 is the *macrostate* where exactly 1 particle has spin up. Someone claims that $p_4 = 4p_1$. Is this claim true or false? Explain your answer.

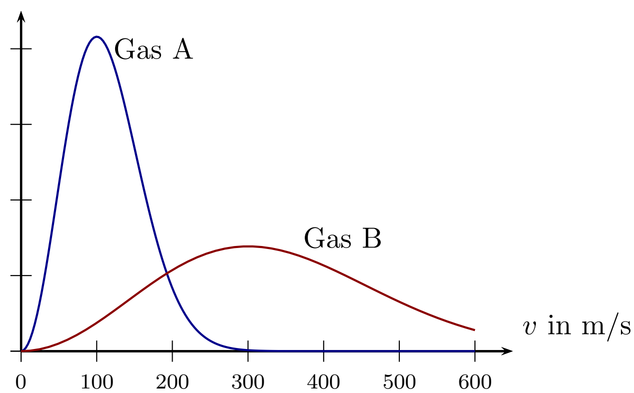
$$\binom{8}{4} = 70 \quad , \quad \binom{8}{1} = 8$$

$$P_4 = \frac{70}{2^8} \quad , \quad P_1 = \frac{8}{2^8}$$

$$\frac{P_4}{P_1} = \frac{70}{8} = \frac{35}{4} \neq 4$$

False

b) The distributions of speed for two gases, whose particles are the same, are provided. Describe as accurately as possible how the temperature of gas A is related to the temperature of gas B. Explain your answer.



$$V_{mp} = \sqrt{\frac{2kT}{m}}$$

$$\frac{V_{mp}^2 \cdot m}{2k} = T_A \quad , \quad \frac{V_{mp}^2 \cdot m}{2k} = T_B$$

$$T_A = \frac{10{,}000\,(m/s)^2 m}{2k} \quad , \quad T_B = \frac{90{,}000\,(m/s)^2 m}{2k}$$

$\frac{T_B}{T_A} = 9$, Gas B has Temp 9 times higher!

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Question 7

Consider an ensemble of N distinguishable harmonic oscillators, each with the same energy,

$$E = \hbar\omega\left(n + \frac{1}{2}\right).$$

The ensemble is at equilibrium with a bath at temperature T.

a) Use the canonical ensemble formalism to show that the mean energy of the system is

$$\overline{E} = N\hbar\omega\left(\frac{1}{e^{\hbar\omega\beta} - 1} + \frac{1}{2}\right)$$

where $\beta = 1/kT$.

$$Z = \sum_i e^{-E_s\beta} = e^{-\hbar\omega(n+\frac{1}{2})\beta} \; : \; Z_{single} = e^{-\hbar\omega(n+\frac{1}{2})\beta}, \; Z_N = Z_s^N$$

$$Z_s = e^{-\hbar\omega\beta/2}\sum_i e^{-\hbar\omega n\beta} = e^{-\hbar\omega\beta/2}\cdot\frac{1}{1 - e^{-\hbar\omega\beta}}$$

$$\bar{E} = -N\cdot\frac{\partial}{\partial\beta}\ln(Z) = -N\cdot\frac{\partial}{\partial\beta}\ln\left(\frac{e^{-\hbar\omega\beta/2}}{1-e^{-\hbar\omega\beta}}\right)$$

$$\bar{E} = -N\cdot\frac{\partial}{\partial\beta}\left[\ln(e^{-\hbar\omega\beta/2}) - \ln(1-e^{-\hbar\omega\beta})\right]$$

$$= -N\cdot\left[-\frac{\hbar\omega}{2} - \frac{\hbar\omega e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}}\right] = N\hbar\omega\left[\frac{e^{-\hbar\omega\beta}}{1-e^{-\hbar\omega\beta}} + \frac{1}{2}\right]$$

$$= N\hbar\omega\left[\frac{1}{e^{\hbar\omega\beta} - 1} + \frac{1}{2}\right]$$

$$\boxed{\bar{E} = N\hbar\omega\left[\frac{1}{e^{\hbar\omega\beta} - 1} + \frac{1}{2}\right]}$$

Question 7 continued . . .

b) Determine an expression for the mean energy in the high temperature approximation, $kT \gg \hbar\omega$ and determine the heat capacity of the system in this limit.

$$\bar{E} = N\hbar\omega\left[\frac{1}{e^{\hbar\omega\beta}-1} + \frac{1}{2}\right] \qquad kT >> \hbar\omega \qquad 1 >> \frac{\hbar\omega}{kT} \qquad 1 >> \hbar\omega\beta$$

$e^{x} \approx 1 + x$ when $1 >> x$

$$\bar{E} = N\hbar\omega\left[\frac{1}{1+\hbar\omega\beta-1} + \frac{1}{2}\right] = N\hbar\omega\left[\frac{1}{\hbar\omega\beta} + \frac{1}{2}\right]$$

$$\bar{E} = N\hbar\omega\left[\frac{2+\hbar\omega\beta}{2\hbar\omega\beta}\right] : \hbar\omega\beta \approx 0 \;\therefore\; \bar{E} = N\cancel{\hbar\omega}\left[\frac{\cancel{2}}{\cancel{2}\cancel{\hbar\omega}\beta}\right]$$

$$\boxed{\bar{E} = NkT}$$

$$C = \frac{\partial \bar{E}}{\partial T} = Nk \qquad \boxed{C = Nk}$$

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Question 8

Answer either part a) or part b) for full credit for this problem.

a) Carbon monoxide, a diatomic molecule vibrates as a harmonic oscillator with angular frequency 4.04×10^{14} Hz. For the quantum harmonic oscillator, the energy levels are $\hbar\omega(n + 1/2)$ where $n = 0, 1, 2, \ldots$. Let P_n be the probability with which the molecule is in state n.

i) Show that the ratio of particles in the first excited state to those in the ground state is P_1/P_0 and use this to determine an expression for this ratio in terms of frequency, temperature and constants.

$$P = \frac{e^{-E_s\beta}}{Z} \quad \therefore \quad P_0 = \frac{e^{-E_0\beta}}{Z}, \quad P_1 = \frac{e^{-E_1\beta}}{Z}$$

$$E_0 = \frac{\hbar\omega}{2}, \quad E_1 = \frac{3\hbar\omega}{2}$$

$$\frac{P_1}{P_0} = \frac{e^{-E_1\beta}}{e^{-E_0\beta}} = e^{-(E_1-E_0)\beta} = e^{-\hbar\omega\beta}$$

$$\boxed{\frac{P_1}{P_0} = e^{-\hbar\omega\beta}}$$

ii) Determine this ratio if the temperature is 600 K and describe what will happen to this ratio as the temperature increases.

$$\frac{P_1}{P_0} = e^{\left(-\frac{1.05\times10^{-34}\,Js\cdot(4.04\times10^{14}\,s^{-1})}{1.38\times10^{-23}\,J/K\,(600K)}\right)} = 0.0059 \approx 0.006$$

$$\boxed{\frac{P_1}{P_0} = 0.006}$$

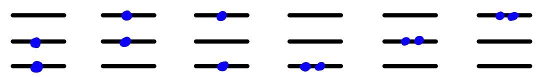
The ratio will increase as T increases

Question 8 continued ...

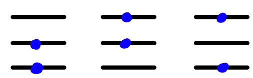
b) i) A “toy” system has three states with energies ϵ_1, ϵ_2 and ϵ_3. Suppose that the system contains two particles. The particles could either be Bosons or Fermions. List the possible states available to the set of particles for each case.

$e^{-\epsilon_1\beta}, e^{-\epsilon_2\beta}, e^{-\epsilon_3\beta}$

Boson



Fermions



ii) Consider a system of either Bosons or Fermions. Describe how the mean occupancy number of any state of the Boson system compares (e.g always equal, always larger, always smaller, sometimes larger, ect, ...) to that of the Fermion system. Explain your answer.

$\bar{n}_k = \frac{1}{e^{(\epsilon-\mu)\beta}-1} \rightarrow$ Bosons , $\bar{n}_k = \frac{1}{e^{(\epsilon-\mu)\beta}+1} \rightarrow$ Fermions

$\bar{n}_{k\,\text{Bosons}} > \bar{n}_{k\,\text{Fermions}}$ ALWAYS !

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