MAT 202
Larson – Section 5.7 (Part I)
Inverse Trigonometric Functions: Integration

Recall the derivatives of the six inverse trigonometric functions. Again, notice that (2) is the negative of (1), (4) is the negative of (3), and (6) is the negative of (5).

<u>Derivatives of Inverse Trigonometric Functions</u>: Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1-u^2}}$
2. $\dfrac{d}{dx}[\arccos u] = -\dfrac{u'}{\sqrt{1-u^2}}$
3. $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1+u^2}$
4. $\dfrac{d}{dx}[arc\cot u] = -\dfrac{u'}{1+u^2}$
5. $\dfrac{d}{dx}[arc\sec u] = \dfrac{u'}{|u|\sqrt{u^2-1}}$
6. $\dfrac{d}{dx}[arc\csc u] = -\dfrac{u'}{|u|\sqrt{u^2-1}}$

When listing the ***antiderivative*** that corresponds to each of the inverse trigonometric functions, we only need to use one member of each pair. Since it is easier to manage the "positive" member, we will use only (1), (3), & (5) to define the following integration rules.

<u>Integrals Involving Inverse Trigonometric Functions</u>: Let u be a differentiable function of x and let $a > 0$.

1. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$
2. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$
3. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}arc\sec\frac{|u|}{a} + C$

For a list of all of the basic integration rules, go to page 378.

Ex: Find the indefinite integral of the following:

a) $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

b) $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

c) $\int \frac{x-3}{x^2+1}\,dx$

Ex: Find the indefinite integral of the following:

a) $\int \frac{1}{\sqrt{1-(x+1)^2}}dx = \int \frac{1}{\sqrt{(1)^2-(x+1)^2}}dx$

$u = x+1$

$du = dx$

$= \int \frac{1}{\sqrt{1^2-u^2}}du$

a

$= \arcsin\frac{u}{1} + C$

$= \boxed{\arcsin(x+1) + C}$

b) $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \int \frac{1}{\sqrt{1^2-u^2}}\,du$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \arcsin u + C$

$= \boxed{\arcsin(\ln x) + C}$

c) $\displaystyle\int \frac{x-3}{x^2+1}\,dx = \underbrace{\int \frac{x}{x^2+1}\,dx}_{} - \int \frac{3}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$\displaystyle - 3\int \frac{1}{x^2+1}\,dx$

$\displaystyle \frac{1}{2}\int \frac{1}{u}\,du \qquad - 3\arctan x + C$

$= \frac{1}{2}\ln|u| - 3\arctan x + C$

$= \boxed{\frac{1}{2}\ln(x^2+1) - 3\arctan x + C}$

Ex: Evaluate the following definite integrals:

a) $\displaystyle\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx$

b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$

Ex: Evaluate the following definite integrals:

a) $\int_0^{\ln 5} \frac{e^x}{1+e^{2x}}\,dx = \int_0^{\ln 5} \frac{e^x}{1+(e^x)^2}\,dx$

$u = e^x$

$du = e^x\,dx$

$= \int \frac{1}{1+u^2}\,du$

$= \arctan u$

$= \arctan(e^x)\Big]_0^{\ln 5}$

$= \arctan(e^{\ln 5}) - \arctan(e^0)$

$= \arctan(5) - \arctan(1)$

$= \arctan(5) - \frac{\pi}{4}$

$\approx \boxed{0.588}$

b) $$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{1}{1+u^2}\,du$$

$u = \sin x$

$du = \cos x \, dx$

$$= \arctan(\sin x)\Big]_0^{\pi/2}$$

$$= \arctan(\sin \pi/2) - \arctan(\sin 0)$$

$$= \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4} - 0$$

$$\pi/4$$

$$\int \frac{\cos x}{3+\sin^2 x}\,dx = \int \frac{\cos x}{(\sqrt{3})^2+(\sin x)^2}\,dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int \frac{1}{(\underset{a}{\sqrt{3}})^2 + u^2}\,du$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{\sin x}{\sqrt{3}}\right) + C$$