MAT 201
Larson/Edwards – Section 2.6
Related Rates

In this section we will apply what we've learned about implicit differentiation and the chain rule to help solve problems that involve related rates with respect to time.

What is a related rate?

In order for us to understand what is meant by ***related rate***, we need to understand what is meant by ***related variables***. For instance, the perimeter of a square can be modeled with the following formula: $p = 4s$, where p is perimeter and s is the length of the side.

With this formula, $p = 4s$ we say that p and s are related. And since p and s are related, the rates of change of p and s are related with respect to time, that is:

$$\frac{dp}{dt} = 4\frac{ds}{dt}$$

We see that p always has four times the value of s, so it follows that the change in perimeter with respect to time is four times the change in the side with respect to time.

Ex: Suppose the side of a square is increasing at a rate of 15 ft/min. Find the rate of change of the perimeter of the square. Does your answer make sense?

Ex: If the radius of a spherical balloon increases at a rate of 1.5 inches per minute, find the rate at which the surface area changes when the radius is 6 inches.

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$$P = 4s$$

$$\frac{ds}{dt} = \frac{15\text{ ft.}}{\text{min}} \qquad \frac{dP}{dt}$$

$$\frac{d}{dt}[P] = \frac{d}{dt}[4s]$$

$$\frac{dP}{dt} = 4\frac{ds}{dt}$$

$$\frac{dP}{dt} = 4 \cdot 15 \text{ ft./min.}$$

$$\boxed{\frac{dP}{dt} = 60 \text{ ft./min.}}$$

Ex: If the radius of a spherical balloon increases at a rate of 1.5 inches per minute, find the rate at which the surface area changes when the radius is 6 inches.

Given: $\frac{dr}{dt} = 1.5$ in/min.

$r = 6$ in.

Want: $\frac{dS}{dt}$

Formula that relates the variables:

Surface area of a sphere:

$$S = 4\pi r^2$$

$$\frac{d}{dt}[S] = \frac{d}{dt}[4\pi r^2]$$

$$\frac{dS}{dt} = 8\pi \underset{6}{r} \underset{1.5}{\frac{dr}{dt}}$$

$$\frac{dS}{dt} = 8\pi(6\text{ in.})\cdot(1.5)\text{ in/min.}$$

$$\frac{dS}{dt} = 72\pi \text{ in/min.} \approx 226.2 \text{ in}^2\text{/min.}$$

<u>Solving Related Rate Problems</u>:

1. **Identify all given quantities and all quantities to be determined.**
   a. Given: change in radius with respect to time = 1.5 in./min.
      This translates to: $\frac{dr}{dt} = 1.5\ in./\min$
   b. Determine: change in surface area with respect to time when the radius is 6 inches. This translates to: Find $\frac{dS}{dt}$ when $r = 6\ inches$.
2. **Write an equation that relates all variables whose rates of change are either given or to be determined.**
   a. Equation: $S = 4\pi r^2$
3. **Use the chain rule to differentiate both sides of the equation with respect to time.**
   a. Differentiate: $\frac{d}{dt}[S] = \frac{d}{dt}\left[4\pi r^2\right]$ (Remember how to find the derivative implicitly?)
   b. We get: $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$
4. **Substitute into the resulting equation all known values of the variables and their rates of change. Then solve for the required rate of change.**
   a. Substitute: $\frac{dr}{dt} = 1.5\ in./\min$ and $r = 6\ inches$ into the equation $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ to get $\frac{dS}{dt} = 8\pi(6)(1.5)$.
   b. Solving/Simplifying: $\frac{dS}{dt} = 72\pi \approx 226.2\ in.^2/\min$

Our answer is $\approx 226.2\ in.^2/\min$

Ex: All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is 2 centimeters?

Ex: An airplane is flying in still air with an airspeed of 275 miles per hour. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.

Ex: All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is 2 centimeters?

Given: $\frac{de}{dt} = 6 \text{ cm/s}$

$e = 2 \text{ cm}$

Want: $\frac{dV}{dt}$ when

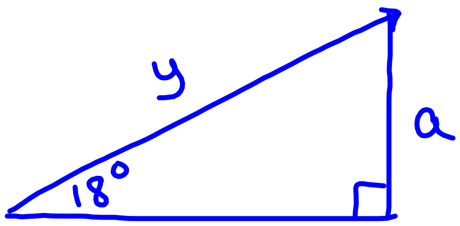
Formula: $V = e^3$

$$\frac{dV}{dt} = 3e^2 \frac{de}{dt}$$

$$\frac{dV}{dt} = 3 \cdot (2 \text{ cm})^2 \cdot 6 \text{ cm/s}$$

$$\boxed{\frac{dV}{dt} = 72 \text{ cm}^3/\text{s}}$$

Ex: An airplane is flying in still air with an airspeed of 275 miles per hour. If it is climbing at an angle of 18°, find the rate at which it is gaining altitude.



Given: $\frac{dy}{dt} = 275$ mph

$$\sin 18^\circ = \frac{a}{y}$$

$$y \sin 18^\circ = a$$

$$\frac{d}{dt}\left[(\sin 18^\circ) y\right] = \frac{d}{dt}\left[a\right]$$

$$\sin 18^\circ \cdot \frac{dy}{dt} = \frac{da}{dt}$$

$$\frac{da}{dt} = 275 \sin 18^\circ$$

$$\frac{da}{dt} \approx \boxed{84.98 \text{ mph.}}$$