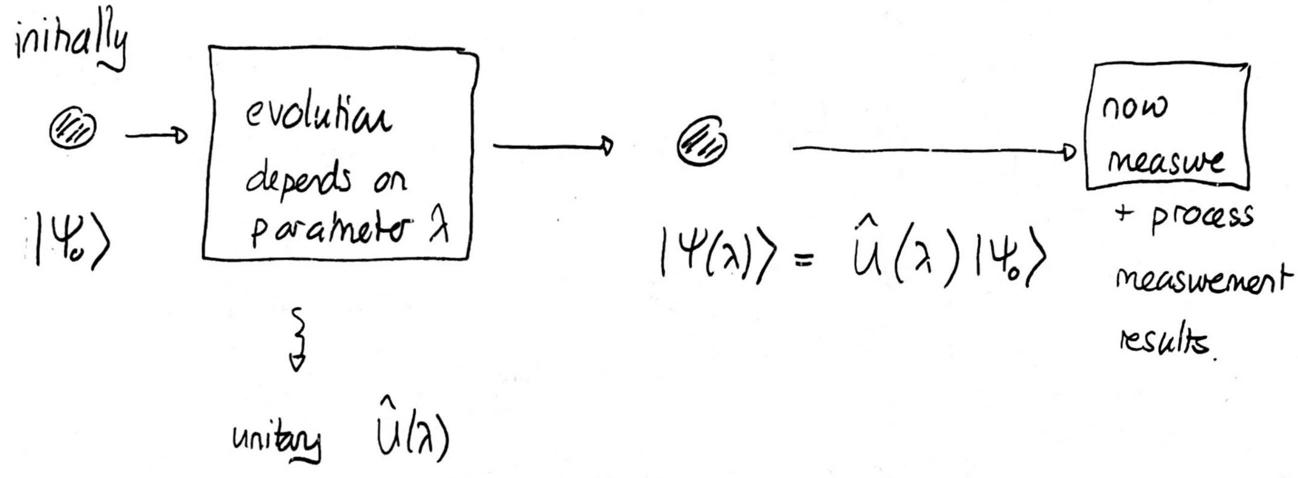
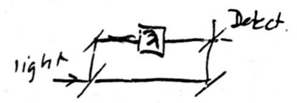
Estimating a parameter that arises from evolution of a state.

One way that a parameter can arise is via parameter independent evolution:



Exercise: Suppose it is known



"Phase shift"

$$\hat{U}(\lambda) = \begin{pmatrix} e^{-i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}$$

but value of λ is unknown. Want to estimate λ.

First get an idea of what the operation does:

a) Consider $|\psi_0\rangle = |0\rangle \rightarrow$ determine $|\psi(\lambda)\rangle$

Determine vector/directions $\hat{n}_0$ and $\hat{n}$ for each.

b) Consider $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \rightarrow$ determine $|\psi(\lambda)\rangle$

Determine vector/directions $\hat{n}_0$ and $\hat{n}$ for each

c) Consider $|\psi_0\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle$ $\rightarrow$ determine $|\psi(\lambda)\rangle$

Determine vector / directions for each.

Now consider feeding various states followed by various measurements

d) $|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ $\rightarrow$ measure $SG\hat{z}$ on $|\psi(\lambda)\rangle$

$\rightarrow$ measure $SG\hat{x}$ on $|\psi(\lambda)\rangle$

get probabilities for each and classical FI.

e) $|\psi_0\rangle = |0\rangle$ $\rightarrow$ $|\psi(\lambda)\rangle$ $\rightarrow$ measure $SG\hat{z}$

$\rightarrow$ " $SG\hat{x}$

get probs + classical FI for each.

f) $|\psi_0\rangle = \frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle \rightarrow |\psi(\lambda)\rangle$ $\rightarrow$ measure $SG\hat{z}$

$\rightarrow$ " $SG\hat{x}$

get probs + classical FI ....

Does the classical FI depend on choice of measurement AND/OR choice of initial state?

Evolution operators

It is possible to construct evolution operators by exponentiation, such as.

$$U(\lambda) = e^{-i\lambda\hat{\sigma}_z}$$

To compute this

$$e^{\hat{A}} = I + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^3 + \frac{1}{4!}\hat{A}^4 + \dots\dots$$

Note that this means

$$e^{-i\lambda\hat{\sigma}_z} = I + (-i\lambda\hat{\sigma}_z) + \frac{1}{2!}(-i\lambda\hat{\sigma}_z)^2 + \frac{1}{3!}(-i\lambda\hat{\sigma}_z)^3$$

$$= I - i\lambda\hat{\sigma}_z + \frac{1}{2!}(-\lambda^2)\hat{\sigma}_z^2 + \frac{1}{3!}i\lambda^3\hat{\sigma}_z^3 + \dots$$

Then $\hat{\sigma}_z{}^2 = \hat{I}$

$$e^{-i\lambda\sigma_z} = I - i\lambda\hat{\sigma}_z - \frac{1}{2!}\lambda^2 I + \frac{1}{3!}i\lambda^3\hat{\sigma}_z + \frac{1}{4!}\lambda^4\hat{I} + \dots$$

$$= I\left(1 - \frac{1}{2!}\lambda^2 + \frac{1}{4!}\lambda^4 + \dots\right) - i\sigma_z\left(\lambda - \frac{1}{3!}\lambda^3 + \frac{1}{5!}\lambda^5 + \dots\right.$$

$$= I\cos\lambda - i\sigma_z\sin\lambda$$

Get

$$e^{-i\lambda\sigma_z} = \cos\lambda\,\hat{I} - i\sin\lambda\,\hat{\sigma}_z$$

In general if $\hat{\sigma}^2 = \hat{I}$ then

$$\boxed{e^{-i\lambda\hat{\sigma}} = \cos\lambda\,\hat{I} - i\sin\lambda\,\hat{\sigma}}$$

Exercises: Show

a) $\hat{\sigma}_x^2 = \hat{I}$

$\hat{\sigma}_y^2 = \hat{I}$

$\hat{\sigma}_z^2 = \hat{I}$

b) $\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z$

$\hat{\sigma}_y\hat{\sigma}_z = i\hat{\sigma}_x$

$\hat{\sigma}_z\hat{\sigma}_x = i\hat{\sigma}_y$

c) $\hat{\sigma}_x\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_x$

$\hat{\sigma}_y\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_y$

d) If $\hat{\sigma} = n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z$

then $\sigma^2 = (n_x^2 + n_y^2 + n_z^2)\,\hat{I}$ (without matrices)

Exercises: Finding matrices for exponentials. Find matrix for

a) $e^{-i\frac{\lambda}{2}\hat{\sigma}_z}$

b) $e^{-i\frac{\lambda}{2}\hat{\sigma}_x}$

c) $e^{-i\frac{\lambda}{2}\hat{\sigma}_y}$

d) $e^{-i\frac{\lambda}{2}\left(\frac{1}{\sqrt{2}}\hat{\sigma}_x + \frac{1}{\sqrt{2}}\hat{\sigma}_y\right)}$

* Can show that for qubits, 2x2 unitaries, any unitary has form

$$\hat{U} = \underbrace{\left[e^{-i(n_x \hat{\sigma}_x + n_y \hat{\sigma}_y + n_z \hat{\sigma}_z)\alpha/2} \right]}_{\text{rotation through angle } \alpha \text{ about axis } n_x \hat{x} + n_y \hat{y} + n_z \hat{z}} \cancel{e^{i\beta}}$$

where (n_x, n_y, n_z) = unit vector

* Spin half particle in unif mag. field $\vec{B}$, that evolution is

$$e^{-i(B_x \hat{\sigma}_x + B_y \hat{\sigma}_y + B_z \hat{\sigma}_z) t \gamma/2}$$