3.4 Determinants and Cramers Rule

3.4 # 4, 19, 29, 43

Determinant of a 2x2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}) - a_{21}(a_{12})$$

Determinant of an nxn Matrix

i^{th} row $\quad |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} (-1)^{ij} |M_{ij}|$

j^{th} column $\quad |A| = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} |M_{ij}|$

$$C_{ij} = (-1)^{i+j} |M_{ij}|$$

Ex: $\left| \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix} \right| = 35 - 2 = 33$

det(a)

Ex: $\left| \begin{bmatrix} 2 & 1 & 5 \\ 7 & 2 & 1 \\ 0 & 3 & 5 \end{bmatrix} \right| = \sum_{j=1}^{3} a_{ij} (-1)^{i+j} |M_{ij}|$

$$\left.\begin{aligned} &= 0 + 3(-1)^{3+2} \left| \begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix} \right| \; : \; 0 - 3(-33) = 99 \\ &= 5(-1)^{3+3} \left| \begin{bmatrix} 2 & 1 \\ 7 & 2 \end{bmatrix} \right| \; : \; 5(-3) = -15 \end{aligned}\right\} 99 - 15 = 84$$

Ex: $\begin{bmatrix} 2 & 2 & 3 & 1 \\ 1 & 5 & 7 & 2 \\ 5 & 1 & 1 & 3 \\ 7 & 8 & 4 & 4 \end{bmatrix} = 2 \cdot (-1)^{1+1} \left| \begin{bmatrix} 5 & 7 & 2 \\ 1 & 1 & 3 \\ 8 & 4 & 4 \end{bmatrix} \right| + 2(-1)^{1+2} \left| \begin{bmatrix} 1 & 7 & 2 \\ 5 & 1 & 3 \\ 7 & 4 & 2 \end{bmatrix} \right| + 3(-1)^{1+3} \left| \begin{bmatrix} 1 & 5 & 2 \\ 5 & 1 & 3 \\ 7 & 8 & 4 \end{bmatrix} \right| + 1(-1)^{1+4} \left| \begin{bmatrix} 1 & 5 & 7 \\ 5 & 1 & 1 \\ 7 & 8 & 4 \end{bmatrix} \right|$

Any matrix with a row full of all 0's, then the determinant of that matrix is zero

$A = \begin{bmatrix} 3 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 5 & 7 \\ 2 & 1 & 8 & 7 \end{bmatrix}$

iff $\det |A| = 0$ A is not invertible

nearly 0 = Singular matrix

Let A be a square matrix

- If B is produced by interchanging two rows of A. Then
  $|B| = -|A|$
- If B is produced by multiplying a row of A by a constant K, and add it to another row
  $|B| = |A|$
- If B is produced by multiplying a row of A by a constant K, then
  $|B| = K|A|$
- $|AB| = |A||B|$
- $|A^{-1}| = \dfrac{1}{|A|}$ if $|A| \neq 0$

- $|A| = |A^T|$
- If A is triangular, then $|A| = \prod_{i=1}^{n} a_{ii}$ $\quad \Delta - \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & 2 \end{bmatrix}$
- If two rows or Columns of A are equal then $|A| = 0$

Cramers Rule

For an nxn matrix A, with $|A| \neq 0$, denote A_i the matrix obtained by replacing the i-th column of A with $\vec{b}$.
The i-th component of the solution of

$A\vec{x} = \vec{b}$ is given by $\quad x_i = \dfrac{|A_i|}{|A|}$

$$x + 2y = 5 \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$
$$2x + 3y = 8$$

$$A_1 = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}$$

$A_1 = X = A^{-1}$

$A_2 = Y = A^{-1}$