Physics 396

Homework Set 11

1. In Homework Set 10, we showed that

$$a(t) = a_0\, t^{2/3(1+w)} \tag{1}$$

is a solution to the *flat Friedman equation*, for a constant equation of state parameter w, given by

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = 0. \tag{2}$$

It is again noted that $w = 0$ for pressureless matter, $w = 1/3$ for radiation, and $w = -1$ for a constant vacuum energy density. It is further noted that the Hubble constant H_0 is defined by the expression

$$H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}, \tag{3}$$

where a dot indicates a time-derivative. The Hubble time, t_H, is defined by

$$t_H \equiv \frac{1}{H_0}, \tag{4}$$

which gives an upper bound and a rough estimate for the age of a *decelerating* Universe.

a) Given a Hubble constant of $H_0 = 72.0$ (km/s)/Mpc, calculate the Hubble time in years to three significant figures.

b) Inserting Eq. (1) into Eq. (3), find an expression for t_0, the current age of the Universe in terms of the Hubble time.

c) For a Universe containing only radiation, calculate the current age of this Universe.

d) For a Universe containing only pressureless-matter, calculate the current age of this Universe.

2. In Homework Set 6, we arrived at a 3D *non-Euclidean* line element of a three-sphere of radius R from a fictitious *flat 4D Euclidean space* via the coordinate transformation

$$\begin{aligned} x &= R\sin\chi\sin\theta\cos\phi \\ y &= R\sin\chi\sin\theta\sin\phi \\ z &= R\sin\chi\cos\theta \\ w &= R\cos\chi, \end{aligned} \tag{5}$$

where $0 \leq \chi < \pi$, $0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. Here we wish to arrive at a three-surface in *flat 4D spacetime* that is the analog of the three-surface of a sphere in *flat 4D Euclidean space*. This geometry describes that of a *Lorentz hyperboloid*.

a) Consider the coordinate transformation

$$
\begin{aligned}
x &= R\sinh\chi\sin\theta\cos\phi \\
y &= R\sinh\chi\sin\theta\sin\phi \\
z &= R\sinh\chi\cos\theta \\
t &= R\cosh\chi, \qquad (6)
\end{aligned}
$$

where $0 \leq \chi < \infty$, $0 \leq \theta < \pi$, and $0 \leq \phi < 2\pi$. Setting $R = 1$, show that the above transformation obeys the equation of constraint

$$-t^2 + x^2 + y^2 + z^2 = -1. \qquad (7)$$

b) Calculate $dx,\ dy,\ dz,\ dt$.

c) Calculate $dx^2 + dy^2$.

d) Calculate $dx^2 + dy^2 + dz^2$.

e) Show that the 4D Minkowski spacetime line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (8)$$

becomes that of a 3D *non-Euclidean* line element of a three-surface known as the *Lorentz hyperboloid* of the form

$$dS^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \qquad (9)$$

under the above coordinate transformation.

f) By defining $r \equiv \sinh\chi$, show that the above line element takes the form

$$dS^2 = \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (10)$$

Notice that the line element of Eqs. (9) and (10) equate to a $t =$ const. slice of a *FRW homogeneous open universe*

3. In Homework Set 6, we found the *embedding diagram* for a 2D equatorial slice ($t = \text{const.}$, $\theta = \pi/2$) of a *FRW homogeneous closed universe.* This equated to finding a curved 2D surface in 3D Euclidean space with the *same* intrinsic geometry as a 2D equatorial slice ($t = \text{const.}$, $\theta = \pi/2$) of the 4D homogeneous closed universe.

   a) Consider the $t = \text{const.}$ slice of a *FRW homogeneous open universe* given by Eq. (9). Construct the corresponding 2D equatorial slice ($\theta = \pi/2$) for this homogenous opene universe, analogous to Eq. (7.41) of your text.

   b) Following a procedure similar to that of Section 7.7 of your text, show that the curved 2D surface obeys the differential equation

$$\frac{dz}{d\chi} = \pm R\sqrt{1 - \cosh^2(\chi)}. \tag{11}$$

   c) Show that this whole 2D equatorial slice of the *FRW homogeneous open universe* can't be embedded as an axisymmetric surface in flat 3D Euclidean space.

Extra Credit: Create your own Universe

4. In class, we arrived at an equation of motion describing the scale factor of the form

$$\frac{1}{2}\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 + U_{eff}(\tilde{a}) = \frac{\Omega_c}{2}, \tag{12}$$

where

$$U_{eff}(\tilde{a}) \equiv -\frac{1}{2}\left(\Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} + \frac{\Omega_r}{\tilde{a}^2}\right). \tag{13}$$

Eq. (12) is the Friedman equation for a flat, open, or closed Universe and is of the form of a 1D conservation of energy expression for a particle in Newtonian mechanics with either zero, negative, or positive total energy, respectively. This expression connects the time evolution of the Universe to its spatial geometry and energy density.

It is noted that the cosmological parameters obey the constraint equation

$$\Omega_v + \Omega_m + \Omega_r + \Omega_c = 1, \tag{14}$$

where $0 \leq \Omega_m, \Omega_r \leq 1$, however, Ω_c, Ω_v can be positive, negative, or zero. In this problem, your task is to find values for the cosmological parameters $\Omega_m, \Omega_v, \Omega_r$ and Ω_c, subjected to the constraint given by Eq. (14), that will generate a Universe that either *i. expands* until the scale factor reaches a *maximum* value and then contracts to a "Big Crunch" or

ii. contracts until the scale factor reaches a *minimum* value before 'bouncing' to an expanding Universe.

Using your favorite plotting software (i.e. Maple, Matlab, Wolframalpha, etc.)

a) plot $U_{eff}(\tilde{a})$ vs $\tilde{a}$ for your choice of cosmological parameters with $-1 \leq \Omega_c \leq 1$.

b) plot $\Omega_c/2$ with $-1 \leq \Omega_c \leq 1$.

c) Explicitly state your choice of parameters that satisfy *i.* and *ii.* These two curves should be positioned on the same plot.