MAT 202
Larson – Section 8.8
Improper Integrals

In section 8.7 we used ***L'Hopital's rule*** to evaluate limits. We will apply this rule to ***improper integrals***.

Recall that an ***indefinite integral*** has no limits of integration, and a ***definite integral*** has an upper and lower limit of integration in which these limits are finite.

An ***improper integral*** is an integral of the form $\int_a^b f(x)\,dx$ in which one or both of the limits of integration (a or b) are infinite **OR** if the function f has a finite number of infinite discontinuities in the interval $[a, b]$.

For example, suppose we would like to find the area under the curve $f(x) = \dfrac{1}{x^2}$;

a) bounded by the lines $y = 0$, $x = 1$, and infinity as our upper limit. This translates to the improper integral: $\int_1^\infty \dfrac{1}{x^2}\,dx$

b) bounded by the lines $y = 0$, $x = -1$, and $x = 1$. This translates to the improper integral: $\int_{-1}^1 \dfrac{1}{x^2}\,dx$ where there is an ***infinite discontinuity*** in the interval [-1, 1] at $c = 0$.

In order for us to evaluate improper integrals we will need to use limits.

<u>Definition of Improper Integrals with Infinite Integration Limits</u>:

1. If f is continuous on the interval $[a, \infty)$, then
$\int_a^{\infty} f(x)\, dx = \lim\limits_{b \to \infty} \int_a^b f(x)\, dx$.
2. If f is continuous on the interval $(-\infty, b]$, then
$\int_{-\infty}^{b} f(x)\, dx = \lim\limits_{a \to -\infty} \int_a^b f(x)\, dx$.
3. If f is continuous on the interval $(-\infty, \infty)$, then
$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx$, where c is any real number.

<u>Convergence or Divergence</u>?

1. An improper integral ***converges*** when the limit exists.
2. An improper integral ***diverges*** when the limit does not exist.

For case number (3) above, the improper integral on the left diverges when EITHER of the improper integrals on the right diverges.

Ex: Evaluate the following and determine whether it converges or diverges.

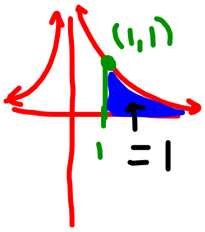
a) $\int_1^{\infty} \frac{1}{x^2}\, dx$

b) $\int_1^{\infty} \frac{1}{x}\, dx$

c) $\int_1^{\infty} (1 - x)e^{-x}\, dx$

Ex: Evaluate the following and determine whether it converges or diverges.

a) $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x^2}\,dx\right]$



$= \lim_{b\to\infty}\left[-\frac{1}{x}\Big|_1^b\right]$

$= \lim_{b\to\infty}\left[\cancel{-\frac{1}{b}}^{\,0} - (-1)\right] = \boxed{1}$

$\boxed{\text{Converges}}$

b) $\int_1^{\infty} \frac{1}{x}\,dx$

$$\lim_{b\to\infty}\left[\int_1^b \frac{1}{x}\,dx\right] = \lim_{b\to\infty}\left[\ln|x|\Big|_1^b\right]$$

$$\lim_{b\to\infty}\left[\ln|b| - \cancel{\ln|1|}^{0}\right]$$

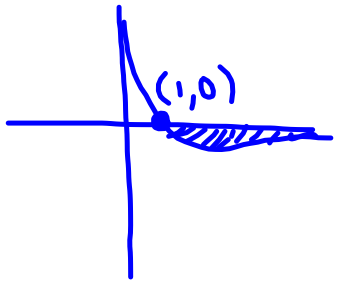
$$= \boxed{\infty \quad \text{Diverges}}$$

c) $\int_1^\infty (1-x)e^{-x}\,dx$

$u = 1-x \qquad dv = e^{-x}\,dx$

$du = -dx \qquad v = -e^{-x}$

$$\lim_{b\to\infty}\left[\underbrace{\int_1^b (1-x)e^{-x}\,dx}_{uv-\int v\,du}\right]$$



$$-(1-x)e^{-x} - \int +e^{-x}\,dx$$

$$-(1-x)e^{-x} + e^{-x} = xe^{-x}$$

$$\cancel{-e^{-x}} + xe^{-x}\ \cancel{+e^{-x}}$$

$$\lim_{b\to\infty}\left[xe^{-x}\Big|_1^b\right] = \lim_{b\to\infty}\left[be^{-b} - e^{-1}\right]$$

$$\lim_{b\to\infty}\left[\frac{b}{e^b}\right] - \lim_{b\to\infty}\left[\frac{1}{e}\right]$$

L'Hopital's

$$\lim_{b\to\infty}\left[\frac{1}{e^b}\right] \to 0 \quad - \lim_{b\to\infty}\left[\frac{1}{e}\right] = -\frac{1}{e}$$

converges

Notice that the previous examples (a) & (b) were of the form: $\int_1^{\infty} \frac{1}{x^p}\,dx$.

This is a special type of improper integral in which the following theorem (Theorem 8.5) can be applied:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \textit{diverges}, & p \le 1 \end{cases}$$

Ex: Use Theorem 8.5 to evaluate the following and determine whether the improper integral converges or diverges:

a) $\int_1^{\infty} \frac{8}{x^5}\,dx$

b) $\int_1^{\infty} \frac{7}{\sqrt[6]{x}}\,dx$

The second basic type of improper integral is one that has an infinite discontinuity at or between the limits of integration.

Notice that the previous examples (a) & (b) were of the form: $\int_1^\infty \frac{1}{x^p}\,dx$. This is a special type of improper integral in which the following theorem (Theorem 8.5) can be applied:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p>1 \\ diverges, & p \le 1 \end{cases}$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$$

$$\lim_{b\to\infty}\left[\int_1^b x^{-p}\,dx\right] = \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\Big|_1^b\right]$$

$$\lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

If $p>1$

$$\lim_{b\to\infty}\left[\frac{1}{b^{|1-p|}(1-p)} - \frac{1}{1-p}\right] = 0 - \frac{1}{1-p} = \boxed{\frac{1}{p-1}}$$

If $p<1$

$$\lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1}{1-p}\right] = \infty$$

Ex: Use Theorem 8.5 to evaluate the following and determine whether the improper integral converges or diverges:

a) $\int_1^\infty \frac{8}{x^5}\,dx = 8\int_1^\infty \frac{1}{x^5}\,dx$ $\quad P=5>1$

$= 8 \cdot \frac{1}{5-1} = 8 \cdot \frac{1}{4} = \boxed{2}$

Converges

b) $\int_1^{\infty} \frac{7}{\sqrt[6]{x}}\,dx = 7\int_1^{\infty} \frac{1}{x^{1/6}}\,dx$

$P = \frac{1}{6} < 1 \quad \therefore$ D.N.E.

Diverges

Definition of Improper Integrals with Infinite Discontinuities:

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$.
2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$.
3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Note: Same rules apply for convergence and divergence.

Ex: Evaluate the following and determine whether it converges or diverges.

a) $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$

b) $\int_{-1}^1 \frac{1}{x^2}\,dx$

c) $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$

Ex: Evaluate the following and determine whether it converges or diverges.

case #2

a) $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \int_0^1 x^{-1/3}\,dx$

discontinuity @ $x=0$

$= \lim_{c\to 0^+}\left[\int_c^1 x^{-1/3}\,dx\right]$

$= \lim_{c\to 0^+}\left[\frac{3}{2}x^{2/3}\Big|_c^1\right]$

$= \lim_{c\to 0^+}\left[\frac{3}{2} - \cancel{\frac{3}{2}c^{2/3}}^{\,0}\right]$

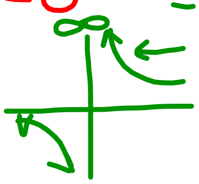
$= \frac{3}{2}$; converges

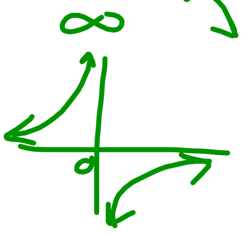
Case #3

b) $$\int_{-1}^{1}\frac{1}{x^2}\,dx = \int_{-1}^{0}\frac{1}{x^2}\,dx + \int_{0}^{1}\frac{1}{x^2}\,dx$$

Discontinuity @ $x=0$

$$= \lim_{c\to 0^-}\left[\int_{-1}^{c}\frac{1}{x^2}\,dx\right] + \lim_{c\to 0^+}\left[\int_{c}^{1}\frac{1}{x^2}\,dx\right]$$





$$= \lim_{c\to 0^-}\left[-\frac{1}{x}\Big|_{-1}^{c}\right] + \lim_{c\to 0^+}\left[-\frac{1}{x}\Big|_{c}^{1}\right]$$

$$= \lim_{c\to 0^-}\left[-\frac{1}{c}+1\right] + \lim_{c\to 0^+}\left[-1+\frac{1}{c}\right]$$

$$\infty + \infty$$

diverges

case #1

c) $$\int_0^\infty \frac{1}{\sqrt{x}\,(x+1)}\,dx =$$

disc.

$$\lim_{c\to 0^+}\left[\int_c^1 \frac{1}{\sqrt{x}(x+1)}\,dx\right] + \lim_{b\to\infty}\left[\int_1^b \frac{1}{\sqrt{x}(x+1)}\,dx\right]$$

$$\int \frac{1}{\sqrt{x}(x+1)}\,dx \qquad u=\sqrt{x} \qquad x=u^2$$

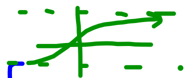
$$du=\frac{1}{2}x^{-1/2}\,dx$$

$$=2\int \frac{1}{u^2+1}\,du \qquad du=\frac{1}{2\sqrt{x}}\,dx$$

$$2\,du=\frac{1}{\sqrt{x}}\,dx$$

$$=2\arctan u \to 2\arctan\sqrt{x}$$

$$\lim_{c\to 0^+}\left[2\arctan\sqrt{x}\,\Big|_c^1\right] + \lim_{b\to\infty}\left[2\arctan\sqrt{x}\,\Big|_1^b\right]$$



$$\lim_{c\to 0^+}\left[2\arctan\sqrt{1}-2\arctan\sqrt{c}\right] + \lim_{b\to\infty}\left[2\arctan\sqrt{b}-2\arctan\sqrt{1}\right]$$

$$=\left[2\cdot\frac{\pi}{4}-0\right]+\left[2\cdot\frac{\pi}{2}-2\cdot\frac{\pi}{4}\right]$$

$$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$ converges