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MATH 360 HW 6

11.5 # 7, 10

Problem 11.5.7

$y'' + \lambda y = 0 \,,\ y(0) = 0 \,,\ y(10) = 0$ $\qquad [p(x)y']' + [q(x) + \lambda r(x)]y = 0$

$P(x) = 1 \,,\ q(x) = 0 \,,\ r(x) = 1$ $\qquad [y']' + [0 + \lambda]y = 0$

$k_1 y(0) + k_2 y'(0) = 0$ at $x = 0$ $\qquad k_1 = 1 \,,\ k_2 = 0$

$l_1 y(10) + l_2 y'(10) = 0$ $\qquad l_1 = 1 \,,\ l_2 = 0$

$$\int_0^{10} y_m(x) y_n(x) \cdot 10 \, dx = K_{mn} \delta_{mn}$$

Case 1 ($\lambda = 0$) $\qquad y'' + \lambda y = 0 \Rightarrow y'' = 0 \Rightarrow y = c_1 x + c_2$

$y(0) = 0 : \quad 0 = c_1(0) + c_2 : \quad c_2 = 0$
$y(10) = 0 : \quad 0 = c_1(10) + \cancel{c_2}^{0} : \quad c_1 = 0$ $\qquad \Rightarrow \underline{y(x) = 0}$

Case 2 ($\lambda < 0$) & $\lambda = -\nu^2$ $\quad \nu \neq 0$ $\qquad y'' - \nu^2 y = 0 \Rightarrow r^2 - \nu^2 = 0 : \ r = \pm \nu$

$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \Leftrightarrow y(x) = c_1 \cosh(\nu x) + c_2 \sinh(\nu x)$

$y(0) = 0 : \quad 0 = c_1 \cosh(0) + \cancel{c_2 \sinh(0)}^{0}$ $\quad : \quad y(10) = 0 : \quad 0 = \cancel{c_1 \cosh(10\nu)}^{0} + c_2 \sinh(10\nu)$
$0 = c_1$ $\qquad 0 = c_2 \sinh(10\nu)$

$\sinh(10\nu) = 0$ when $\nu_m = \frac{\pi}{10} x \quad \forall_x = 1, 2, 3 \ldots.$

So Eigenvalue is $\quad \lambda_m = \left(\frac{\pi}{10} \cdot x\right)^2 \quad m = 1, 2, 3, \ldots.$

Eigenfunction is $\quad y_m = \sin\left(\frac{\pi}{10} \cdot mx\right)$

Case 3 ($\lambda > 0$) & $\lambda = \nu^2$ $\quad \nu \neq 0$ $\qquad y'' + \nu^2 y = 0 \Rightarrow r^2 + \nu^2 = 0 : \ r = \pm \nu$

$\alpha = \frac{-b}{2a} = \frac{0}{2} = 0$ $\qquad y(x) = c_1 \cos(\nu x) + c_2 \sin(\nu x)$

$\beta = \frac{\sqrt{4ac - b^2}}{2a} = \nu$

$y(0) = 0 : \quad 0 = c_1 \cos(0) + c_2 \sin(0)$ $\qquad y(10) = 0 : \quad 0 = \cancel{c_1 \cos(10\nu)}^{0} + c_2 \sin(10\nu)$
$0 = c_1$ $\qquad 0 = c_2 \sin(10\nu)$
$\hookrightarrow$ Same result as case 2!

$$\boxed{\begin{aligned} \lambda_m &= \left(\frac{\pi}{10} \cdot x\right)^2 \quad x = 1, 2, 3, \ldots. \\ y_m &= \sin\left(\frac{\pi}{10} \cdot mx\right) \end{aligned}}$$

Problem 11.5.10

$y'' + \lambda y = 0$, $y(0) = y(1)$, $y'(0) = y'(1)$

$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$

$p(x) = 1$ $\quad q(x) = 0$ $\quad r(x) = 1$

$[1 \cdot y']' + [0 + \lambda \cdot 1]y = 0$

Case 1 $\lambda = 0$: $y'' = 0 \Rightarrow y = c_1 x + c_2$ $\quad y(0) = y(1)$: $y(1) = c_1(0) + c_2$

$c_2 = y(1)$

$y' = c_1$: $y'(0) = y'(1) = c_1$ $\quad$: $c_1 = 0$

$c_2 = y(1)$ $\quad y(1) = 0$

Case 2 $\lambda < 0$: $\lambda = -\nu^2$: $y'' - \nu^2 y = 0$: $r^2 - \nu^2 = 0$: $r = \pm \nu$

$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \Leftrightarrow$ $y(x) = c_1 \cosh(\nu x) + c_2 \sinh(\nu x)$

$y'(x) = \nu \cdot c_1 \sinh(\nu x) + \nu \cdot c_2 \cosh(\nu x)$

$y(0) = y(1) \Rightarrow c_1 = c_1 \cosh(\nu x) + c_2 \sinh(\nu x)$

$y'(0) = y'(1) \Rightarrow \nu c_2 = \nu c_1 \sinh(\nu x) + \nu c_2 \cosh(\nu x)$

$0 = c_1(\cosh(\nu x) - 1) + c_2 \sinh(\nu x)$

$0 = c_1 \cdot \nu \sin(\nu x) + c_2 \nu(\cosh(\nu x) - 1)$

$A = \begin{bmatrix} \cosh(\nu x) - 1 & \sinh(\nu x) \\ \nu \sinh(\nu x) & \nu(\cosh(\nu x) - 1) \end{bmatrix}$

$\det(A) = \nu(\cosh(\nu x) - 1)^2 - \nu(\sinh^2(\nu x)) = \nu([\cosh(\nu x) - 1]^2 - [\sinh^2(\nu x)])$

$\nu \neq 0$, cosh & sinh never $= 0$ $\therefore$ $y = 0$, trivial Solution

Case 3 $\lambda > 0$: $\lambda = \nu^2$: $y'' + \nu^2 y = 0 \Rightarrow r^2 + \nu^2 = 0$: $r = \pm i\nu$

$\alpha = \frac{-b}{2a} = \frac{0}{2a} = 0$

$\beta = \frac{\sqrt{4ac - b^2}}{2a} = \nu$

$y(x) = c_1 \cos(\nu x) + c_2 \sin(\nu x)$

$y'(x) = -\nu c_1 \cdot \sin(\nu x) + \nu c_2 \cos(\nu x)$

$y(0) = y(1) \Rightarrow$ $c_1 = c_1 \cos(\nu) + c_2 \sin(\nu) \Rightarrow 0 = c_1(\cos(\nu) - 1) + c_2 \sin(\nu)$

$y'(0) = y'(1) \Rightarrow \nu \cdot c_2 = -\nu c_1 \sin(\nu) + \nu c_2 \cos(\nu) \Rightarrow 0 = -c_1 \cdot \nu \sin(\nu) + c_2 \cdot \nu(\cos(\nu) - 1)$

$A = \begin{bmatrix} \cos(\nu) - 1 & \sin(\nu) \\ -\nu \sin(\nu) & \nu(\cos(\nu) - 1) \end{bmatrix}$ $\quad \det(A) = \nu\left[(\cos(\nu) - 1)^2\right] + \nu \sin^2(\nu)$

$(\cos(\nu) - 1)(\cos(\nu) - 1) = \nu[\cos^2(\nu) - 2\cos(\nu) + 1]$

$\det(A) = \nu \cos^2(\nu\pi) - 2\nu \cos(\nu\pi) + \nu + \nu \sin^2(\nu\pi)$

$= \nu - 2\nu\cos(\nu\pi) + \nu = 2\nu - 2\nu\cos(\nu\pi) = 2\nu[1 - \cos(\nu\pi)]$

$\nu \neq 0$: $1 - \cos(\nu\pi) = 0$ $\therefore$ $\cos(\nu\pi) = 1$ $\therefore$ $\nu = 2m$

$m = 1, 2, 3 \ldots .$

$\lambda_m = 2\pi m$

$\boxed{\begin{array}{l} \text{Eigenvalues: } \lambda_m = (2m\pi)^2 \\ \text{Eigenfunctions: } y(x) = c_m \cos(2\pi m x) + K_m \cos(2\pi m x) \end{array}}$

7.) $y'' + \lambda y = 0 \quad y(0) = 0 \quad y(10) = 0$

Case 1 $\lambda = 0$

$y'' = 0 \quad \therefore \quad y = c_2 x + c_1$

$0 = c_2(0) + c_1$

$\therefore \; c_1 = 0$

$10 = c_2(0)$ Trivial Solution

$\therefore \; c_2 = 0$

Case 2 $\lambda < 0 \quad \lambda = -\nu^2$

$y'' - \nu^2 y = 0 \; \vdots \; r^2 - \nu^2 = r^2 = \nu^2 \; \therefore \; r = \pm \nu$

$b^2 - 4ac = 0 - 4(1)(-\nu^2) = 4\nu^2 > 0 \qquad y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$y = c_1 e^{\nu t} + c_2 e^{-\nu t} = c_1 \cosh(\nu t) + c_2 \sinh(\nu t)$

$0 = \underbrace{c_1 \cosh(0) + c_2 \sinh(0)}$ never zero!

$y = 0$, trivial

Case 3 $\lambda > 0 \quad \lambda = \nu^2$

$y'' + \nu^2 y = 0 \; \vdots \; r^2 + \nu^2 = 0 \; \vdots \; r^2 = \nu^2 \; \vdots \; r = \pm i\nu$

$\Delta = 0 - 4\nu^2(1) < 0 \quad \therefore \quad y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$

$\alpha = \frac{-b}{2a} = 0$

$\beta = \frac{\sqrt{4ac - b^2}}{2a} = \frac{\sqrt{4(1)\nu^2 - 0}}{2} = \frac{\sqrt{4\nu^2}}{2} = \frac{2\nu}{2} = \nu$

$y = c_1 \cos(\nu t) + c_2 \sin(\nu t)$

$0 = c_1 \qquad 0$

$0 = c_1 \cancel{\cos}(10\nu) + c_2 \sin(\nu t)$

$0 = c_2 \sin(\nu t)$

$\sin(10\nu) = 0$ when $\nu_m = \frac{\pi}{10} m$, $m = 1, 2, 3 \ldots$

Eigen Value $= \nu_m = \left(\frac{\pi}{10} m\right)^2$ $\qquad [P(x)y']' + [Q(x) + \lambda r(x)] = 0$

Eigen Function $= y_m = c_2 \sin\left(\frac{\pi}{10} m x\right)$