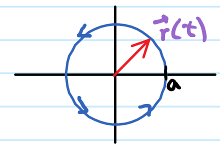
Math 360 Ch 9.5: Curves & Arc Length

Class Prep: Write up Examples 1-4 & Eqn 10 for Arclength

Example 1: Parametric representation of circle (vector function)

$$\vec{r}(t) = [a\cos t, a\sin t], \quad 0 \le t \le 2\pi$$

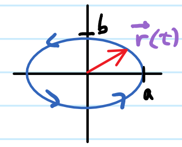
$$\Leftrightarrow \; x^2 + y^2 = a^2$$



Example 2: Parametric representation of ellipse

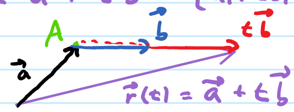
$$\vec{r}(t) = [a\cos t, b\sin t], \quad 0 \le t \le 2\pi$$

$$\Leftrightarrow \; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



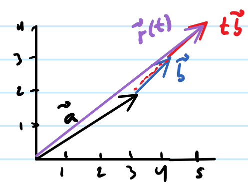
Example 3 Parametric representation of line through point A (with position vector $\vec{a}$) in direction $\vec{b}$

$$\vec{r}(t) = \vec{a} + t\vec{b} = [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$$



Suppose $A = (3,2)$, so $\vec{a} = [3,2]$, and $\vec{b} = [1,1]$. Then

$$\vec{r}(t) = [3,2] + t[1,1] = [3+t, 2+t]$$



Example 4 Parametric representation of a circular helix

$$\vec{r}(t) = [a\cos t, a\sin t, ct] = a\cos t\,\vec{i} + a\sin t\,\vec{j} + ct\,\vec{k}, \quad t > 0, \; c \ne 0$$

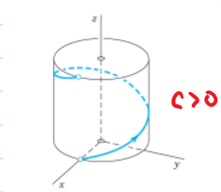


Fig. 204. Right-handed circular helix

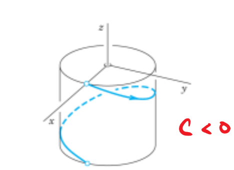
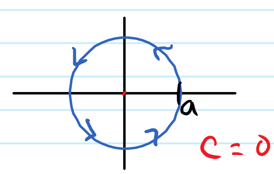


Fig. 205. Left-handed circular helix



Tangent to Curve

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)],$$

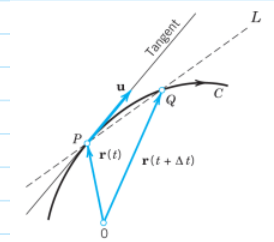
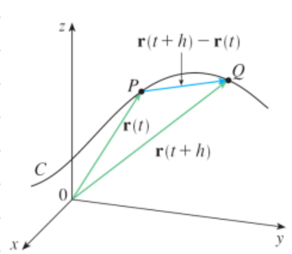


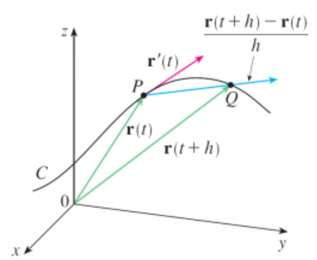
Fig. 207. Tangent to a curve

Unit tangent vector: $\mathbf{u} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'.$

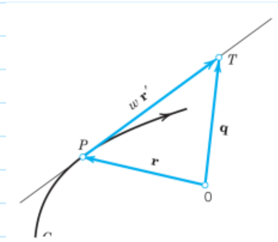
Tangent line: $\boxed{\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}'}$



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector $\mathbf{r}'(t)$



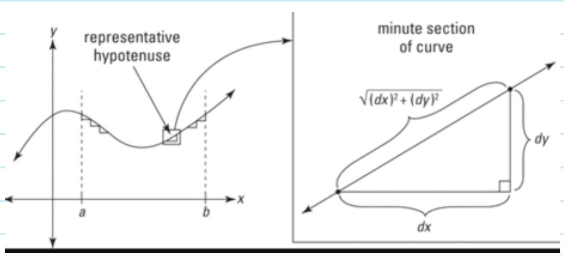
$\mathbf{q}(w) = \mathbf{r} + w\mathbf{r}'$

Example See handout

Arc Length of a Curve

1) Length of curve C, given by $\vec{r}(t) = [x(t), y(t), z(t)]$

$$\ell = \int_a^b \sqrt{\vec{r}'(t) \cdot \vec{r}'(t)}\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$



2) Arc length s of a Curve C

$$s(w) = \int_a^w \sqrt{\vec{r}'(t) \cdot \vec{r}'(t)}\, dt$$

Example See handout

Ch 9.5 Curves. Arc Length. Curvature.

Ex 1| What curve is represented by $[\cos t, 2+\sin t, 3]$. Sketch.

Soln Let $[x(t), y(t), z(t)] = [\cos t, 2+\sin t, 3]$. Then a point (x,y,z) on C is given by

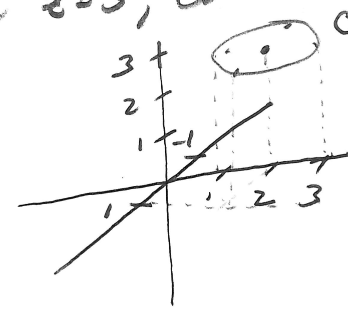
$$x(t) = \cos t, \quad y(t) = 2+\sin t, \quad z(t) = 3$$

Then, ignoring $z=3$ for the moment,

$$x^2 + (y-2)^2 = \cos^2 t + \sin^2 t = 1$$

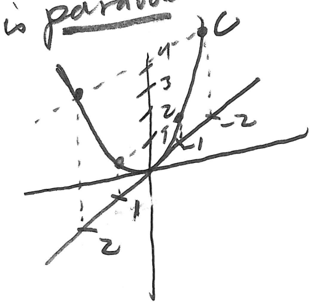
In xy plane, C is a circle centered at $(0,2)$ with $r=1$.

Thus C is a circle in the plane $z=3$, centered at $(0,2,3)$ with radius $r=1$.



Ex 2| C: $[t, 0, t^2]$

Soln $x=t, \; y=0, \; z = t = x^2 \Rightarrow$ C is parabola $z=x^2$ in xz plane. $(y=0)$



Ex 3| C: $[2\cos t, 2\sin t, t]$

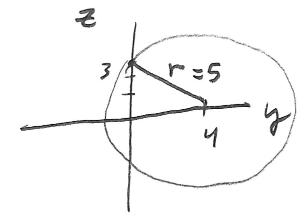
Helix with radius $r=2$ about z-axis

(Compare with Ex 1 above, and p. 383 from reading)

$$\left.\begin{aligned} x &= 2\cos t \\ y &= 2\sin t \end{aligned}\right\} x^2+y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

$$\Rightarrow x^2+y^2 = 4$$

Ex 4 Circle in yz plane with center (4,0) and passing through (0,3). Sketch.



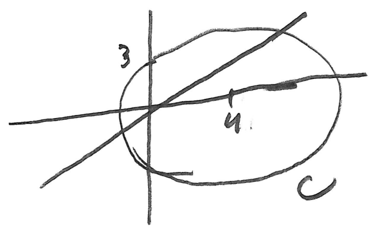
Soln Since C is a circle in yz plane,

$$x = 0, \quad \underbrace{y = 4 + 5\cos t, \quad z = 0 + 5\sin t}$$

$\hookrightarrow (y-4)^2 + (z-0)^2 = 25\cos^2 t + 25\sin^2 t = 25$

$\Rightarrow (y-4)^2 + z^2 = 25$

$\Rightarrow$ circle centered at (4,0) in yz-plane with radius $r = 5$.



Ex 5 Straight line through (1,1,0) and (2,0,1).

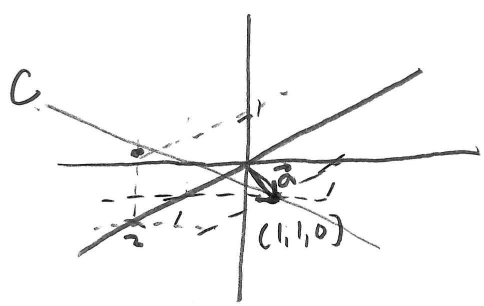
Soln C is a line, so use line formula from reading:

$$\vec{r} = \vec{a} + t\vec{b}$$

$\vec{a}$: position vector; $\vec{b}$: direction vector

$$\left.\begin{aligned} \vec{a} &= [1,1,0] \\ \vec{b} &= [2-1, 0-1, 1-0] = [1,-1,1] \end{aligned}\right\}$$

$$\Rightarrow \vec{r} = [1,1,0] + t[1,-1,1] = [1+t, 1-t, t]$$



From $\vec{a} = [1,1,0]$ position, curve C (a line) goes in the direction of $\vec{b} = [1,-1,1]$, i.e., out 1 in x-direction, back 1 in y-direction (negative), and up 1 in z-direction.

- Can graph by hand or Maple.

Ex 6 $\vec{r}(t) = [4\cos t, 3, 4\sin t]$; $P: (4, 3, 0)$

Find $\vec{r}\,'(t)$, $\vec{u}\,'(t)$, and tangent of C at P. (See reading)

Soln $\vec{r}\,'(t) = [-4\sin t, 0, 4\cos t]$

$$|\vec{r}\,'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$$

Note: Dependence on t disappears. Not all problems have t disappear here.

$$\vec{u}\,'(t) = \frac{1}{4}[-4\sin t, 0, 4\cos t]$$

$$= [-\sin t, 0, \cos t]$$

Tangent of C at P:

$$\vec{q}(w) = \vec{r} + w\vec{r}\,' \quad \text{at point } P$$

position direction

$$\vec{q}(w) = [4\cos t, 3, 4\sin t] + w[-4\sin t, 0, 4\cos t]$$

Now plug in point P information:

At what t does $\vec{r}(t)$ give $(4,3,0)$?

Soln:

$4\cos t = 4 \Rightarrow \cos t = 1$

$4\sin t = 0 \Rightarrow \sin t = 0$

$\Big\} \Rightarrow t = 0, 2\pi, \ldots$

$3 = 3$

Then

$$\vec{q}(w) = [4, 3, 0] + w[0, 0, 4]$$

$$= [4, 3, 4w]$$

Can graph by hand or Maple, etc.

Ex 7 $\vec{r}(t) = [\cosh t, t]$, from $t=0$ to $t=1$. Find length of C.

Soln $L = \int_0^1 \sqrt{\vec{r}' \cdot \vec{r}'}\, dt$; $\vec{r}'(t) = [\sinh t, 1]$

$= \int_0^1 \sqrt{\sinh^2 t + 1}\, dt$

(identity) $= \int_0^1 \sqrt{\cosh^2 t}\, dt$

$= \int_0^1 \cosh t\, dt$

$= \sinh t \Big|_0^1$

$= \sinh(1) - \sinh(0)$

$= \sinh(1) - 0$

$= \sinh(1)$

$\sinh(t) = \dfrac{e^t - e^{-t}}{2}$

$\cosh(t) = \dfrac{e^t + e^{-t}}{2}$

$\cosh^2 t - \sinh^2 t = 1$