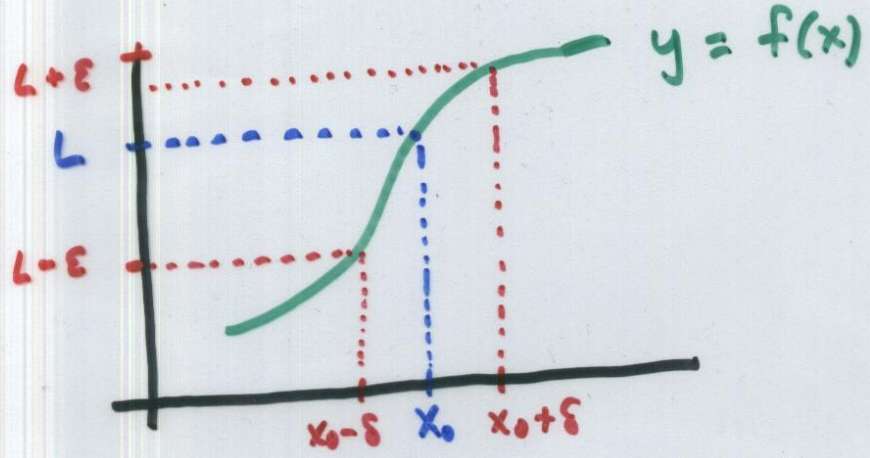
Ch1.1 Review of Calculus

Limit A function f defined on a set X of real numbers has the limit L at x_0,

$$\lim_{x \to x_0} f(x) = L$$

if, given any $\varepsilon > 0$, there is a $\delta > 0$ s.t.

$|f(x) - L| < \varepsilon$ whenever $x \in X$ & $0 < |x - x_0| < \delta$



Continuous

A function is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Notation for continuous functions:

$$C(X) = \{f : f \text{ is continuous on } X\}$$

$$C[a,b] = \{f : f \text{ is continuous on } [a,b]\}$$

Limit of a sequence

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ has limit x (converges to x) if, for any $\varepsilon > 0$, there is an $N(\varepsilon) > 0$ such that

$$|x_n - x| < \varepsilon \quad \text{whenever } n > N(\varepsilon)$$

Notation

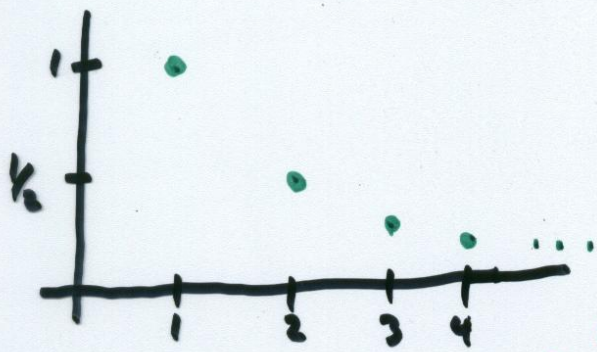
$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \text{ as } n \to \infty$$

Theorem If f is a function defined on a set X of real numbers, and $x_0 \in X$, then f is continuous at x_0 iff for all $\{x_n\}_{n=1}^{\infty}$ s.t. $x_n \to x_0$, $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Ex Let $x_n = \frac{1}{n}$ for $n = 1, 2, \ldots$

Then $\{x_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

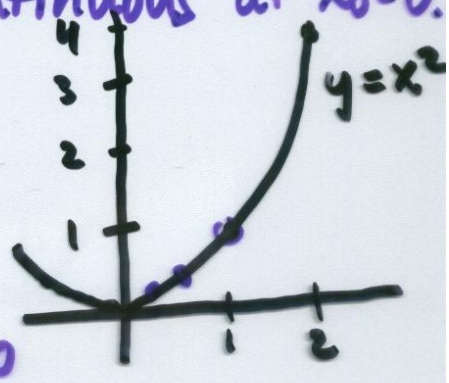
and
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$



Ex Let $f(x) = x^2$ and $\{x_n\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. We know from Calculus that f is continuous at $x_0 = 0$.

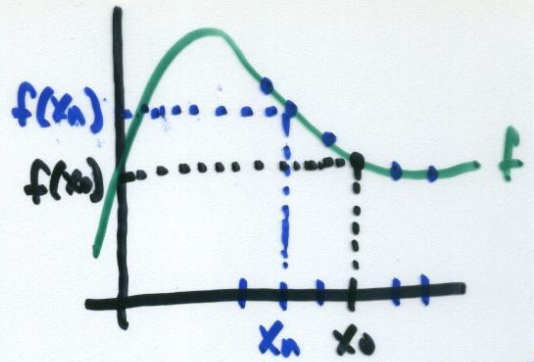
Note that

$$f(x_n) = (x_n)^2 = \left(\frac{1}{n}\right)^2 = \frac{1}{n^2} \to 0 \text{ as } n \to \infty$$



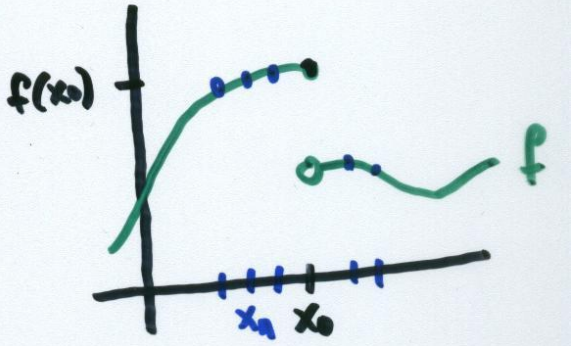
and $f(0) = 0$. Thus $\lim_{n\to\infty} f(x_n) = f(x_0)$.

All the functions that we will consider when discussing numerical methods will be assumed to be continuous, since this is the minimal requirement for predictable behavior.



$[x_n \to x_0] \Rightarrow [f(x_n) \to f(x_0)]$

Functions that are not continuous can skip over points of interest, which can cause difficulties when attempting to approximate a solution to a problem.



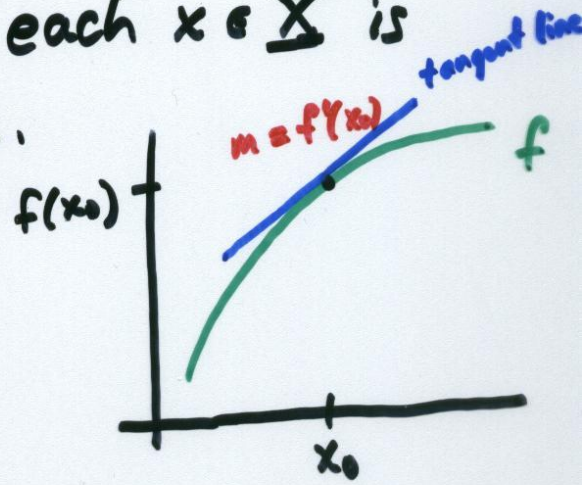
$[x_n \to x_0] \not\Rightarrow [f(x_n) \to f(x_0)]$

More sophisticated assumptions about a function generally lead to better approximation results. A <u>smooth</u> function is more predictable than a jagged one, for example. Smoothness relies on the concept of a derivative.

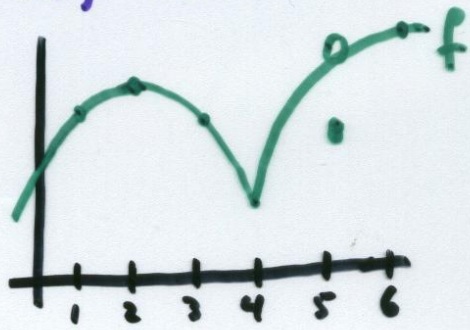
Derivative Let f be a function defined in an open interval containing x_0. Then f is <u>differentiable at x_0</u> if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. A function f that has a derivative $f'(x)$ for each $x \in X$ is <u>differentiable on X</u>.



Theorem If the function f is differentiable at x_0, then f is continuous at x_0.



Notation

$C^n(\mathbb{X}) = \{f : f$ has n continuous derivatives$\}$

$C^\infty(\mathbb{X}) = \{f : f$ has continuous derivs. of all orders$\}$

Ex $\{\text{Polynomials}\} \subseteq C^\infty(\mathbb{R})$

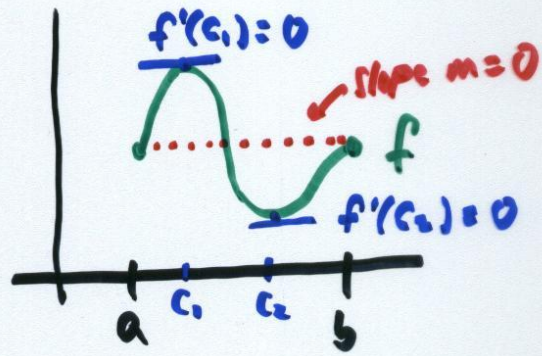
$\{f : f(x) = a x^{1/2}\} \subseteq C^\infty(0,\infty)$

$\{f : f(x) = \frac{1}{x-c}\} \subseteq C^\infty[(-\infty,c)\cup(c,\infty)]$

$\{f : f(x) = a \ln(x)\} \subseteq C^\infty(0,\infty)$

Rolle's Theorem Let $f \in C[a,b]$ and suppose f is differentiable on (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.

Note: Slope m of line connecting endpoints of graph of f is zero.

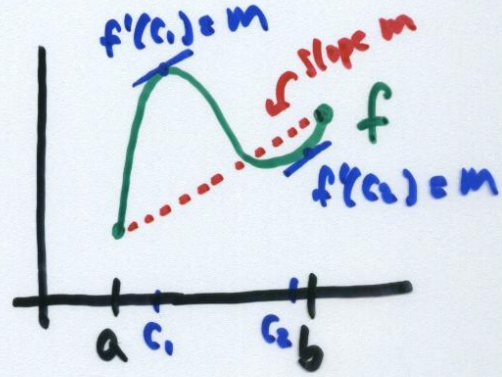


Mean Value Theorem

If $f \in C[a,b]$ and f is differentiable on (a,b), then $\exists c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Note: Slope m of line connecting endpoints of graph of f is

$$m = \frac{f(b)-f(a)}{b-a}$$

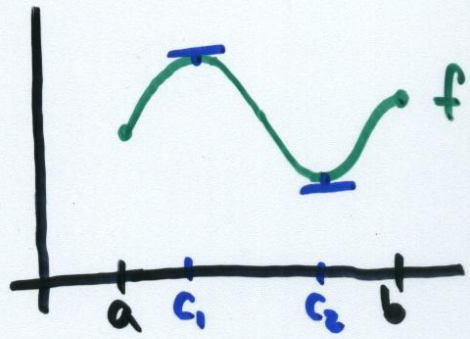


Extreme Value Theorem

If $f \in C[a,b]$, then $\exists c_1, c_2 \in [a,b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$, $\forall x \in [a,b]$

In addition, if f is differentiable on (a,b), then c_1, c_2 are the endpoints of $[a,b]$ or $f'(c_1) = 0$ & $f'(c_2) = 0$. (endpts. or critical pts.)

Ex See graph:
max at $x = c_1$
min at $x = c_2$



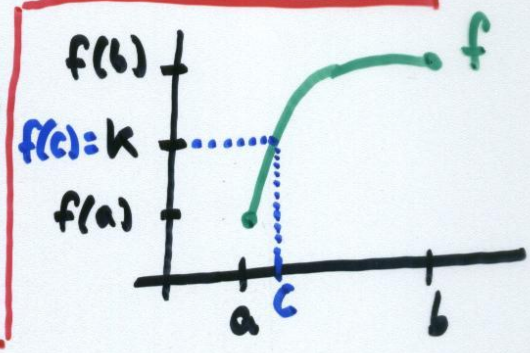
Intermediate Value Theorem

If $f \in C[a,b]$ and $K \in \mathbb{R}$ s.t.

$$f(a) \le K \le f(b),$$

then $\exists c \in [a,b]$ s.t.

$$f(c) = K$$



<u>Note</u> This c-value is not necessarily unique, especially in the case where f is not 1-1.

<u>Example</u> Suppose we needed to solve $f(x) = 0$ (find root of f), where

$$f(x) = x^5 - 2x^3 + 3x^2 - 1, \quad x \in [0,1].$$

We might first want to know if such a root of f even exists, before trying to find it.

Use IVT: Note first that $f \in C[0,1]$. Next,

$$f(0) = -1 < 0 \quad \text{and} \quad f(1) = 1 > 0.$$

Thus IVT guarantees that a root exists in $[0,1]$. We will learn how to find such a root later in this course.

```
plot(x^5-2*x^3+3*x^2-1, x = 0..1, color = blue,
title = "f(x) = x^5 - 2x^3 + 3x^2 - 1");
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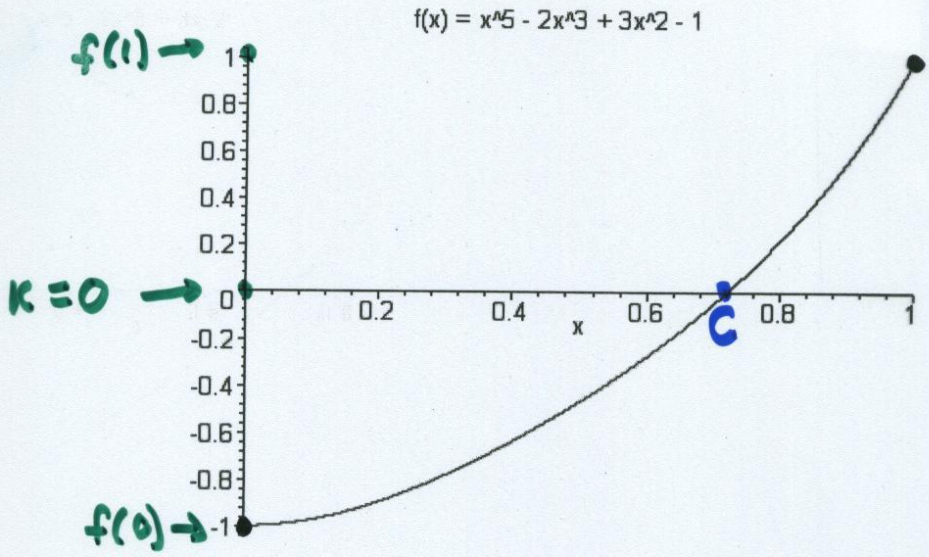


Ch1.1 Problems

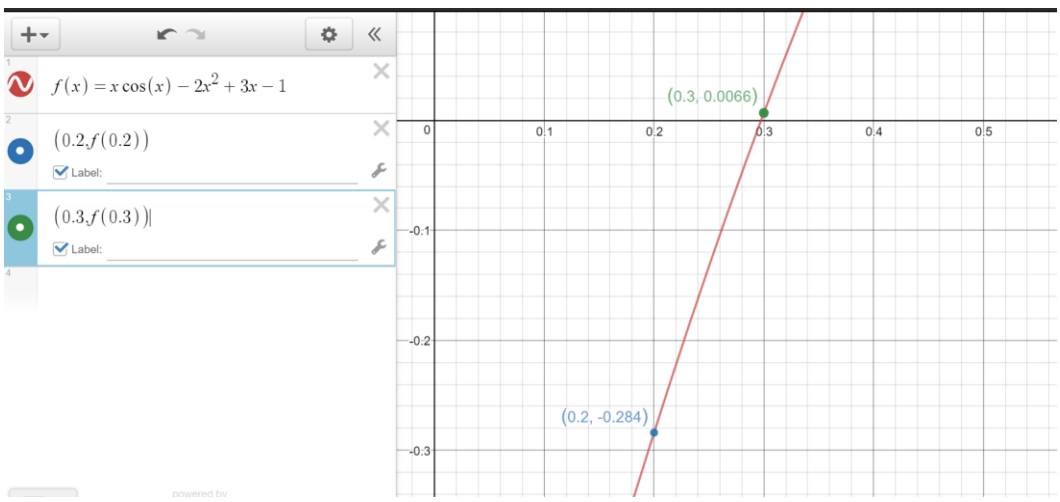
1. Show that the following function has at least one solution in the given intervals.

$f(x) = x \cdot cos(x) - 2x^2 + 3x - 1$, [0.2,0.3] and [1.2,1.3]

(a) [0.2,0.3]

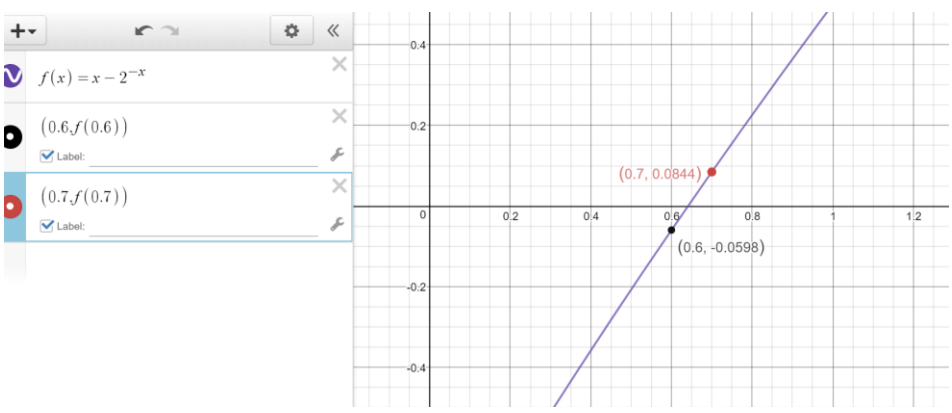
$$f \in C[0.2,0.3]$$
$$f(0.2) = -0.284 \leq 0 \leq f(0.3) = 0.0066$$

Therefore by the IVT, $\exists c \in (0.2,0.3)\ s.t. f(c) = 0$. See graph below.



2. $f(x) = x - 2^{-x}$

From the graph below, we see that $x = 0.6$ and $x = 0.7$ bracket the root. Since $f \in C[0.6,0.7]$ and $f(0.6) < 0$ and $f(0.7) > 0$, we conclude that f has a root in (0.6,0.7) by the IVT.



Taylor's Theorem

Suppose $f \in C^n[a,b]$, that $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. Then for all $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 and x such that

$$f(x) = P_n(x) + R_n(x)$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

and

$$R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}\cdot(x-x_0)^{n+1}$$

Here, $P_n(x)$ is called the n-th Taylor polynomial for f about x_0, and $R_n(x)$ is called the remainder term associated with $P_n(x)$.

Recall that for each $x \in [a,b]$,

$$f(x) = \underline{P_n(x)} + \underline{R_n(x)}$$

where

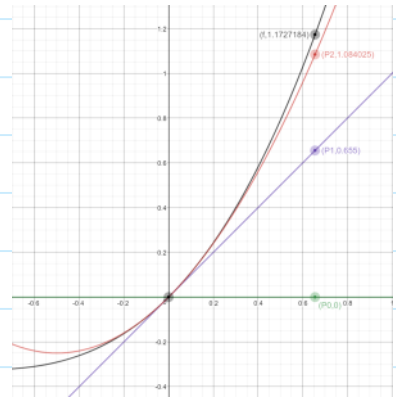
$$\underline{R_n(x)} = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \cdot (x-x_0)^{n+1}$$

and $\underline{\xi(x)}$ is some number between x & x_0.

We usually don't know the value of $\xi(x)$; we just know an interval that it lies in. That is, $\xi(x) \in (x, x_0)$ or $\xi(x) \in (x_0, x)$, depending on whether $x < x_0$ or $x_0 < x$.

Thus we typically cannot find the exact value of the remainder term. Instead, we will often try to find a bound on $|R_n(x)|$, which gives us an error bound when approximating $f(x)$ with $P_n(x)$.

Ch 1.1 Taylor Polynomials



Example (≈11) $f(x) = e^x \sin x$, $x_0 = 0$

a) Find $P_2(x)$

$$f(x) = e^x \sin x$$

$$f'(x) \overset{\text{product rule}}{=} e^x \sin x + e^x \cos x$$

$$f''(x) \overset{\text{product rule}}{=} e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$\overset{\text{algebra}}{=} 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x = 2e^x(\cos x - \sin x)$$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 2$

$f'''(\xi(x)) = 2e^{\xi(x)}\left[\cos(\xi(x)) - \sin(\xi(x))\right]$, $\xi(x)$ between 0 & x.

$\therefore P_2(x) = 0 + 1(x-0) + \frac{2}{2!}(x-0)^2 = x + x^2$ ← Plot in Desmos with $f(x)$

b) Find $R_2(x)$

$$R_2(x) = \frac{f^{(3)}(\xi(x))}{3!}(x-0)^3 = \frac{2e^{\xi(x)}\left[\cos(\xi(x)) - \sin(\xi(x))\right]}{6} x^3,$$

where $\xi(x)$ between 0 & x.

c) Find $f(0.5)$, $P_2(0.5)$, and $R_2(0.5)$

$f(0.5) \cong 0.79044$, $P_2(0.5) = 0.75$, $R_2(0.5) = \frac{0.125}{3} e^{\xi(0.5)}\left[\cos(\xi(0.5)) - \sin(\xi(0.5))\right]$,

where $\xi(0.5) \in (0, 0.5)$.

d) Find actual error and error bound for $|R_2(0.5)|$.

Actual error $= |f(0.5) - P_2(0.5)| \cong \underline{0.04044}$ (Desmos)

← Answer

$$|R_3(0.5)| = \frac{0.125}{3} e^{\xi(0.5)} \left|\cos(\xi(0.5)) - \sin(\xi(0.5))\right| \quad ①$$

$$\overset{①}{\leq} \frac{0.125}{3} e^{0.5} \left|\cos(\xi(0.5)) - \sin(\xi(0.5))\right| \quad ②$$

$$\overset{②}{\leq} \frac{0.125}{3} e^{0.5} \left|\cos(0) - \sin(0)\right|$$

$$\cong 0.0686967196\ldots \text{ (Desmos)}$$

$$\leq \underline{0.07} \text{ (Answer)}$$

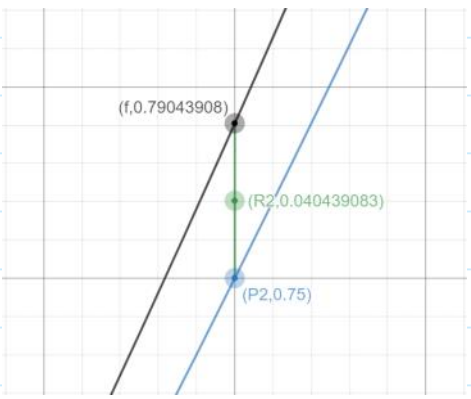
↳ Compare to actual error above

① Since $\xi(0.5) \in (0, 0.5)$, and e^x is positive and increasing on $(0, 0.5)$

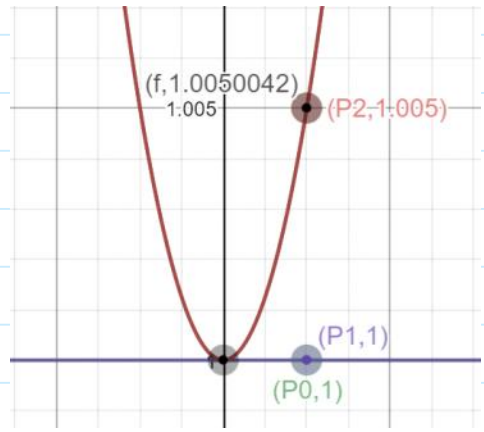
② Since $\xi(0.5) \in (0, 0.5)$ and $y = \cos x - \sin x$ is positive and decreasing on $(0, 0.5)$:

(a) $\cos x > \sin x > 0$ on $(0, 0.5)$

(b) $y' = -\sin x - \cos x$
$= -(\sin x + \cos x) < 0$ on $(0, 0.5)$



Example $f(x) = x^2 + \cos x, \quad x_0 = 0$



(a) Find $P_2(x)$

$f(x) = x^2 + \cos x \Rightarrow f(0) = 1$

$f'(x) = 2x - \sin x \Rightarrow f'(0) = 0$

$f''(x) = 2 - \cos x \Rightarrow f''(0) = 1$

$f'''(x) = \sin x \Rightarrow f'''(\xi(x)) = \sin(\xi(x))$, ↳ $\xi(x)$ between 0 and x

$\therefore P_2(x) = 1 + 0(x-0) + \frac{1}{2!}(x-0)^2 = 1 + \frac{1}{2}x^2$

(b) Find $R_2(x)$

$R_2(x) = \frac{f'''(\xi(x))}{3!}(x-0)^3 = \frac{\sin(\xi(x))}{6}x^3$, $\xi(x)$ between 0 & x.

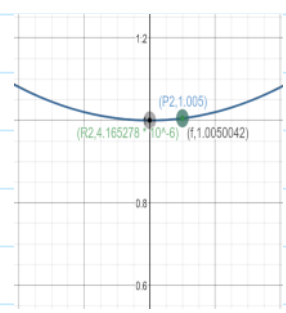
(c) Find $f(0.1)$, $P_2(0.1)$, $R_2(0.1)$

$f(0.1) \cong 1.005004$, $P_2(0.1) = 1 + \frac{1}{2}(0.1)^2 = 1.005$, $R_2(\xi(0.1)) = \frac{\sin(\xi(0.1))}{6}(0.1)^3 = \frac{0.001}{6}\sin(\xi(0.1))$,

↳ $\xi(0.1) \in (0, 0.1)$

(d) Find actual error and error bound for $|R_2(0.1)|$.

Actual error $= |f(0.1) - P_2(0.1)| \cong 0.000004 = 4 \times 10^{-6}$



$|R_3(0.1)| = \frac{0.001}{6}|\sin(\xi(0.1))| \overset{①}{\leq} \frac{0.001}{6}|\xi(0.1)| \overset{②}{\leq} \frac{0.001}{6}(0.1)$

$= 0.000016\overline{6} \leq 2 \times 10^{-5}$

① From Ch1.1 Notes: $|\sin x| \leq |x|$ ② Since $\xi(0.1) \in (0, 0.1)$

Example $f(x) = \sqrt{x}, \quad x_0 = 1$

(a) Find $P_3(x)$

$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-3/2} = \frac{-1}{4\sqrt{x^3}} \Rightarrow f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}} \Rightarrow f'''(1) = 3/8$

$f^{(4)}(x) = \frac{-15}{16}x^{-7/2} = \frac{-15}{16\sqrt{x^5}} \Rightarrow f^{(4)}(\xi(x)) = \frac{-15}{16} \cdot \frac{1}{\sqrt{\xi(x)^5}}$

↳ $\xi(x)$ between 1 & x

$\therefore P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$

$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$

(b) Find $R_3(x)$

$R_3(x) = \frac{-15/16\,(\xi(x))^{-7/2}}{4!}(x-1)^4$, $\xi(x)$ between 1 & x

(c) Find $f(2)$, $P_3(2)$, and $|f(2) - P_3(2)|$

$f(2) = \sqrt{2} \cong 1.414214$, $P_3(2) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} \cong 1.4375$

$|f(2) - P_3(2)| \cong 0.023286$

(d) Find error bound for $|R_3(2)|$.

Example Let $f(x) = \cos x$.

(a) Find $P_2(x)$ about $x_0 = 0$.

Recall $P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

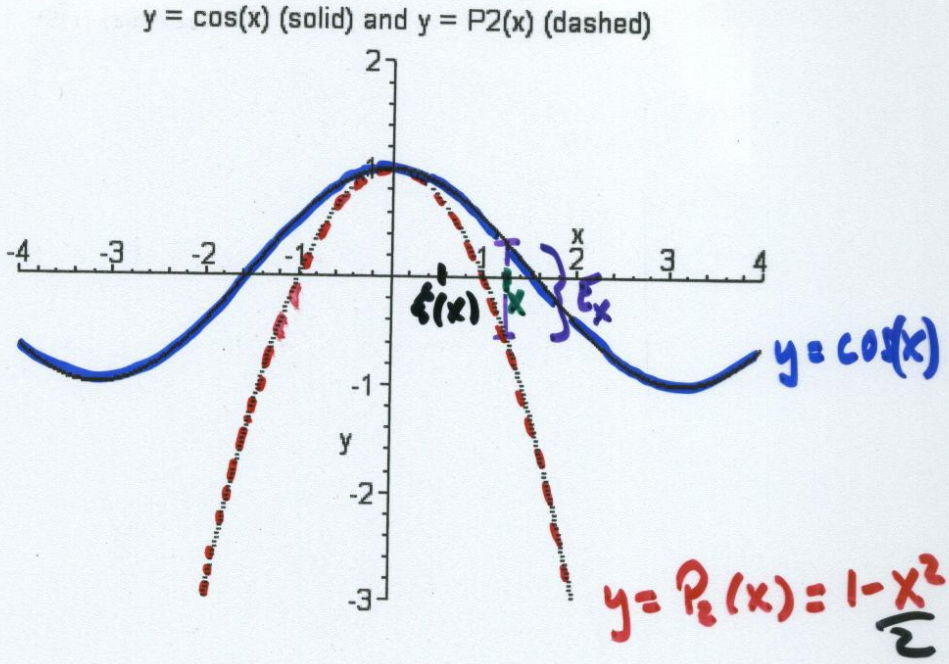
Thus
$$P_2(x) = f(0) + f'(0)\cdot(x-0) + \frac{f''(0)}{2!}(x-0)^2$$
$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2}x^2$$

Now
$$f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$$
$$f'(x) = -\sin x \Rightarrow f'(0) = -\sin(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -\cos(0) = -1$$

Thus $P_2(x) = 1 - \frac{1}{2}x^2$ (see graph on next slide)

To find the remainder term $R_2(x)$:
$$f'''(x) = \sin x \Rightarrow f'''(\xi(x)) = \sin(\xi(x))$$
where $\xi(x)$ is between x and 0.

Thus $R_2(x) = \frac{\sin(\xi(x))}{3!}x^3$, $\xi(x)$ between 0 & x



<u>Error</u> at $x = R_2(x) = \frac{\sin(\xi(x))}{3!} x^3 = E_x$

where $\xi(x)$ is somewhere between 0 and x.

<u>Error Bound</u>:

$$|E_x| = \left| \frac{\sin(\xi(x))}{6} x^3 \right| \le \frac{|\sin(\xi(x))|}{6} x^3 \le \frac{x^3}{6}$$

(b) Use $P_2(x)$ to approximate $f(0.01)$. Find $R_2(0.01)$ and also the actual error. Compare the error to an error bound on $|R_2(0.01)|$.

<u>Soln</u> Recall that $f(x) = \cos(x)$ and $P_2(x) = 1 - \frac{1}{2}x^2$. Then

$$P_2(0.01) = 1 - \frac{1}{2}(0.01)^2 = 1 - 0.00005$$
$$= 0.99995$$

This is the approx to $\cos(0.01)$.

The actual value of $\cos(0.01)$ is

$$\cos(0.01) = 0.999950000417,$$

to twelve decimal places. Thus the actual error is

$$|\cos(0.01) - P_2(0.01)| \cong 0.000000000417$$
$$\cong 4.1667 \times 10^{-10} \quad \text{(TI-86)}$$

Next, we find $R_2(0.01)$:

$$R_2(0.01) = \frac{\sin(\xi(0.01))}{6}(0.01)^3$$

$$= (1.\bar{6} \times 10^{-7}) \sin(\xi(0.01))$$

where $\xi(0.01)$ is between 0 and 0.01.

To find an error bound using $R_2(0.01)$, we examine $|R_2(0.01)|$:

$$|R_2(0.01)| = 1.\bar{6} \times 10^{-7} \cdot |\sin(\xi(0.01))|$$

$$\leq 1.\bar{6} \times 10^{-7} \quad (\text{since } |\sin(\xi(0.01))| \leq 1)$$

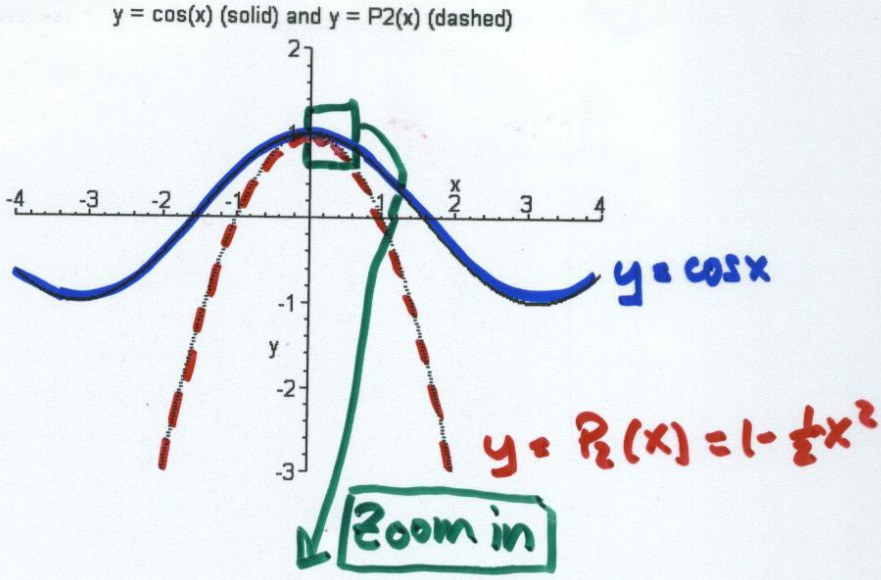
where $\xi(0.01) \in (0, 0.01)$.

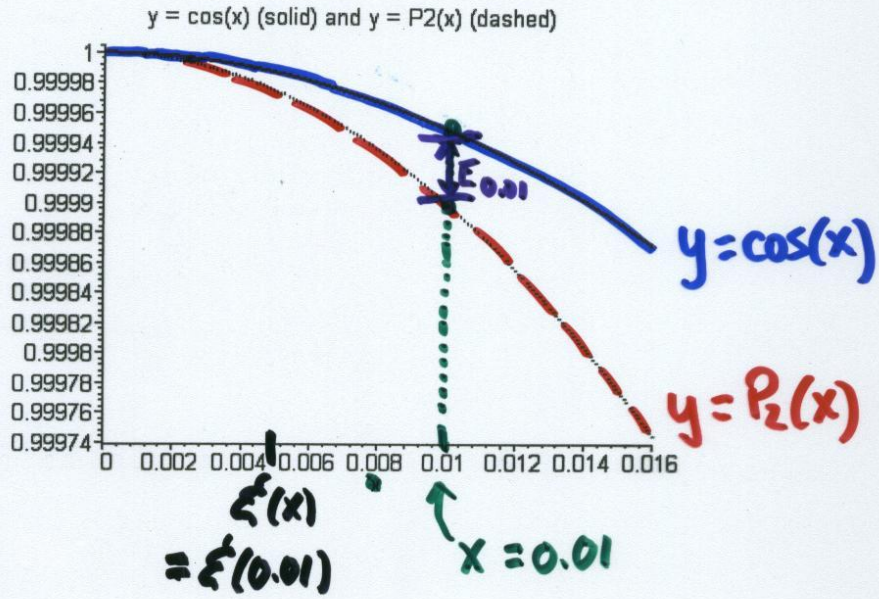
This error bound tells us that

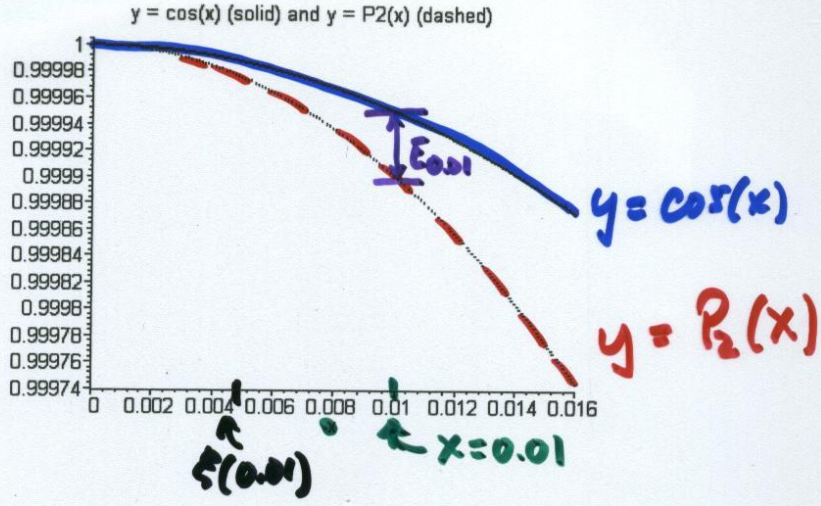
$$|\cos(0.01) - P_2(0.01)| \leq 0.0000001\bar{6}, = 1.\bar{6} \times 10^{-7}$$

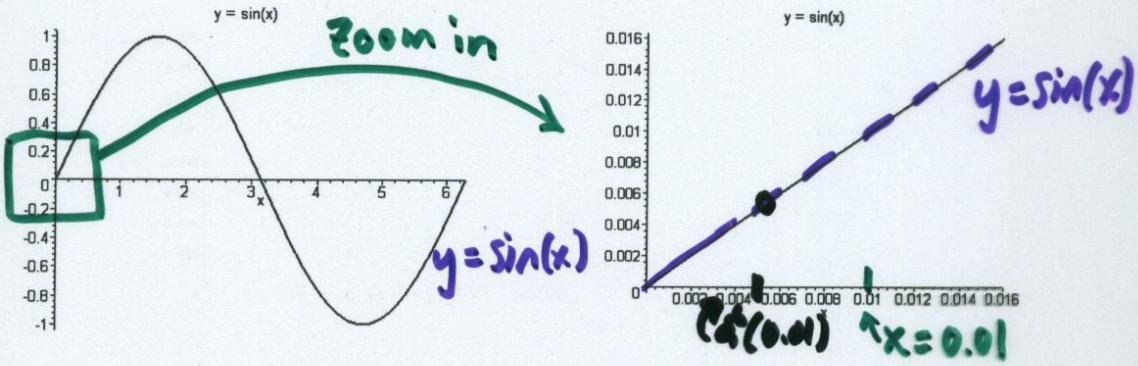
which is about 40[redacted] times the actual error, which we found earlier to be approx

$$E_{0.01} = 4.1667 \times 10^{-10}$$

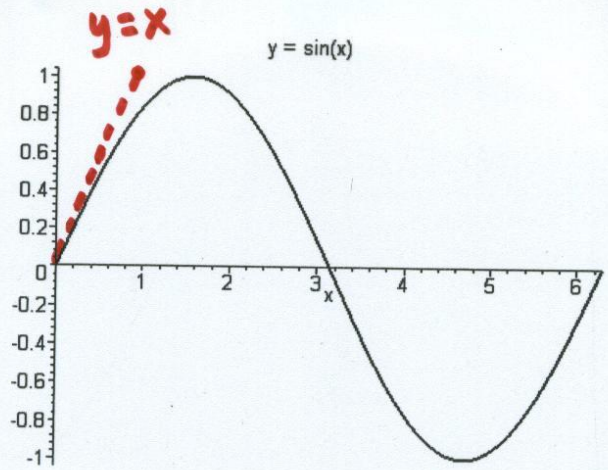

y = cos(x) (solid) and y = P2(x) (dashed)
y = cos x
y = P2(x) = 1 - ½x²
Zoom in


y = cos(x) (solid) and y = P2(x) (dashed)
E0.01
y = cos(x)
y = P2(x)
ε(x)
= ε(0.01)
x = 0.01

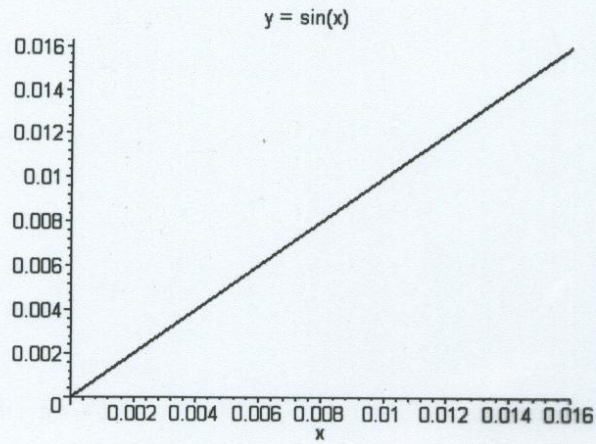




$$E_{0.01} = R_2(0.01) = \left(1.\bar{6} \times 10^{-7}\right) \sin\left(\xi(0.01)\right)$$



For small x, $\sin(x) \cong x$



It can be shown that $|\sin(x)| \leq |x|$ for all x.

Since $|\sin(x)| \le |x|$ for all x, we can improve our error bound:

$$
\begin{aligned}
|R_2(0.01)| &= 1.\bar{6} \times 10^{-7} \cdot |\sin(\xi(0.01))| \\
&\le 1.\bar{6} \times 10^{-7} \cdot |\xi(0.01)| \\
&\le 1.\bar{6} \times 10^{-7} \cdot (0.01) \\
&= 1.\bar{6} \times 10^{-9}
\end{aligned}
$$

(Recall that $\xi(0.01) \in (0, 0.01)$.)

This error bound tells us that

$$|\cos(0.01)| - P_2(0.01)| \le 1.\bar{6} \times 10^{-9}$$

which is 4 [redacted] times the actual error, which we found earlier to be

$$E_{0.01} = 4.1667 \times 10^{-10}$$

Similarly, for any x we have

$$|\cos(x) - P_2(x)| \le \frac{|\sin(\xi(x))|}{3!} x^3 \le \frac{x^3}{6}$$

since $|\xi(x)| \le |x|$ & $|\sin(\xi(x))| \le |\xi(x)|$

This bound works best for $|x| \ll 1$.

(c) Approximate $\int_0^{0.1} \cos(x)\,dx$ using P_2.

<u>Soln</u> Recall that $\cos(x) = P_2(x) + R_2(x)$.

Thus

$$\int_0^{0.1} \cos(x)\,dx = \int_0^{0.1} [P_2(x) + R_2(x)]\,dx$$

$$\cong \int_0^{0.1} P_2(x)\,dx$$

$$= \int_0^{0.1} \left[1 - \tfrac{1}{2}x^2\right] dx$$

$$= \left[x - \tfrac{1}{6}x^3\right]\Big|_0^{0.1}$$

$$\cong 0.09983333333 \text{ (Maple)}$$

A bound for the error in this approx can be found as follows.

$$\left| \int_0^{0.1} R_2(x)\,dx \right| \le \int_0^{0.1} \left| \frac{\sin(\xi(x))}{3!} x^3 \right| dx$$

$$\le \int_0^{0.1} \frac{1}{6} x^3\,dx = \frac{1}{24} x^4 \Big|_0^{0.1}$$

$$= 0.41\bar{6} \times 10^{-5}$$

On the previous slide, we found the error bound E_B to be

$$\left| \int_0^{0.1} R_2(x)\,dx \right| \le 0.41\bar{6} \times 10^{-5} = E_B$$

Thus

$$\left| \int_0^{0.1} \cos(x)\,dx - \int_0^{0.1} P_2(x)\,dx \right| \le 0.41\bar{6} \times 10^{-5}$$

Now

$$\int_0^{0.1} \cos(x)\,dx = \sin(x)\Big|_0^{0.1} = \sin(0.1) - \sin(0)$$

$$\cong 0.\underline{09983341}665 \quad \text{(Maple)}$$

Recall

$$\int_0^{0.1} P_2(x)\,dx \cong 0.\underline{09983333}333$$

Thus the actual error is about

$$E_A \cong 0.8332 \times 10^{-7}$$

Comparing the actual error to the error bound, we find that $E_B \cong 50 \cdot E_A$

<u>Note</u>: If we use P_3 instead, then $E_B \cong E_A$ (See text)

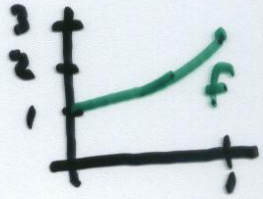
In the previous example, we were able to find bounds on functions by using identities, inequalities, and/or max values. That is, we used

$$|\sin(x)| \le |x| \quad \text{and} \quad |\sin(x)| \le 1$$

In general, and in HW, you will likely need to determine maxs of functions in order to bound them. From Calculus, this typically involves finding critical points, finding where f is increasing and/or decreasing, and checking endpoints.

<u>Ex</u> $f(x) = e^x$ on $[0,1]$.

Since $f'(x) = e^x > 0$ on $[0,1]$, there are no critical points & f increasing on $[0,1]$. Thus



$$|f(x)| \le |e^1| = e \le 2.72 \le 3$$

on $[0,1]$.

Ex Let $f(x) = 4x^3 - x^4$ on $[0,4]$.

Find $\max\limits_{0 \le x \le 4} |f(x)|$.

Soln $f'(x) = 12x^2 - 4x^3$

$= 4x^2(3-x)$

The critical points are $x=0$ & $x=3$.

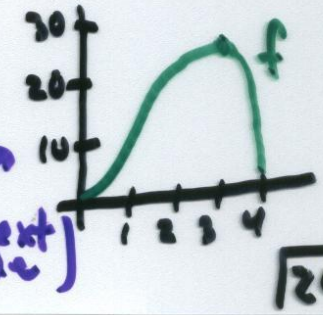
Use a sign chart to determine where f is increasing & decreasing:

		0		1		2		3		4
$f'(x) = 4x^2(3-x)$	+	0	+	+	+	+	+	0	−	

Thus f is increasing on $(0,3)$ and decreasing on $(3,4)$, and hence there is a max at $x=3$.

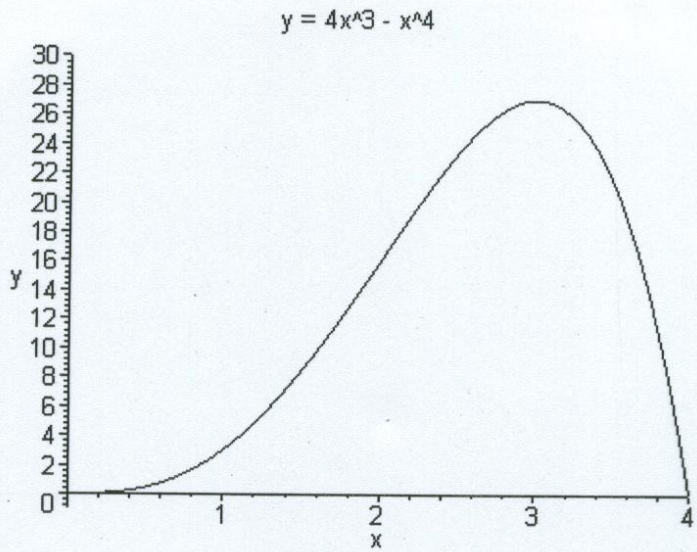
Next, check endpoints & c.p.'s for abs max:

$$\left.\begin{array}{l} f(0) = 0 \\ f(3) = 27 \\ f(4) = 0 \end{array}\right\} \max\limits_{0 \le x \le 4} |f(x)| = 27$$



(see next slide)

```
plot(4*x^3-x^4,x=0..4,y = -1..30,color=blue,title = " y = 4x^3 - x^4 ");
```



Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Ex $\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$

$u = e^x \qquad dv = \sin x\,dx$

$du = e^x dx \qquad v = -\cos x$

$u = e^x \qquad dv = \cos x\,dx$

$du = e^x dx \qquad v = \sin x$

$$\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$