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MATH 360
CP Ch. 12.6

One-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Boundary Conditions

$$u(0,t)=0, \quad u(L,t)=0 \qquad \text{For all } t \geq 0 \qquad (2)$$

Initial Condition

$$u(x,0) = f(x) \qquad [f(x) \text{ given}] \qquad (3)$$

Step 1. Two ODE'S from the heat equation (1). Substitution of a product $u(x,t)=F(x)G(t)$ into (1) gives $F\dot{G} = c^2 F''G$ with $\dot{G} = dG/dt$ an $F'' = d^2F/dx^2$. To separate the variables, we divide by c^2FG, obtaining

(4) $$\frac{\dot{G}}{c^2 G} = \frac{F''}{F}$$

The left side depends only on t and the right side only on x, so that both sides must equal a constant k (as in Sec. 12.3). You may show that for $k=0$ or $k>0$ the only solution $u=FG$ satisfying (2) is $u \equiv 0$. For negative $k=-p^2$ we have the form (4)

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2$$

Multiplication by the denominators immediately gives the two ODE'S

(5) $$F'' + p^2 F = 0$$

and

(6) $$\dot{G} + c^2 p^2 G = 0$$

Step 2. Satisfying the boundary conditions (2). We first solve (5). A general solution is

(7) $$F(x) = A\cos(px) + B\sin(px)$$

From the boundary conditions (2) it follows that

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(L,t) = F(L)G(t) = 0.$$

Since $G \equiv 0$ would give $u \equiv 0$, we require $F(0)=0$, $F(L)=0$ and get $F(0)=A=0$ by (7) and then $F(L)=B\sin(pL)=0$, with $B \neq 0$ (to avoid $F \equiv 0$); thus,

$$\sin(pL)=0, \quad \text{hence} \quad p=\frac{n\pi}{L}, \qquad n=1,2,\ldots\ldots$$

Setting $B=1$, we thus obtain the following solutions of (5) satisfying (2):

$$F_n(x)=\sin\left(\frac{n\pi}{L}x\right), \quad n=1,2,\ldots$$

(As in sec. 12.3, we need not consider negative integer values or n.)

All this was literally the same as in sec. 12.3. From now on it differs since (6) differs from (6) in sec. 12.3. We now solve (6). For $p=n\pi/L$, as just obtained, (6) becomes

$$\dot{G}+\lambda_n^2 G=0 \quad \text{where} \quad \lambda_n=\frac{cn\pi}{L}.$$

It has the general solution

$$G_n(t)=B_n e^{-\lambda_n^2 t} \qquad n=1,2,\ldots$$

where B_n is a constant. Hence the functions

(8)
$$u_n(x,t)=F_n(x)G_n(t)=B_n\sin\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t} \qquad (n=1,2,\ldots)$$

are solutions of the heat equation (1), satisfying (2). These are the eigenfunctions of the problem, corresponding to the eigenvalues $\lambda_n=cn\pi/L$.

Step 3. Solution of the entire problem. Fourier series. So far we have solutions (8) satisfying the boundary conditions (2). To obtain a solution that also satisfies the initial condition (3), we consider a series of these eigenfunctions,

(9)
$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} \qquad \left(\lambda_n=\frac{cn\pi}{L}\right)$$

From this and (3) we have

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)=f(x)$$

Hence for (9) to satisfy (3), the B_n's must be the coefficients of the Fourier sine series, as given by (4) in Sec. 11.3; thus

(10)
$$B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) \qquad (n=1,2,\ldots)$$