Physics 396

Homework Set 9

1. In geometrized units, the Schwarzschild line element takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where M is the total mass of the central object. Consider the coordinate transformation

$$t = v - r - 2M\ln\left|\frac{r}{2M} - 1\right|, \tag{2}$$

which brings the Schwarzschild coordinates (t, r, θ, ϕ) to the Eddington-Finkelstein coordinates (v, r, θ, ϕ). In class, we performed this coordinate transformation in the regime where $r > 2M$. *Explicitly* perform this coordinate transformation in the regime where $r < 2M$.

2. Consider the Schwarzschild geometry exterior to a spherically symmetric massive object. In Eddington-Finkelstein coordinates, in geometrized units, the line element takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{3}$$

where M is the total mass of the central object. It is noted that the above line element has *off-diagonal* terms in the metric.

   a) The line element of Eq. (3) contains two symmetries associated with the v and ϕ coordinates. Construct the two *first integrals* associated with these symmetries of the form

$$e \equiv -\underset{\sim}{\xi} \cdot \underset{\sim}{u} \tag{4}$$

$$\ell \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u}, \tag{5}$$

   where e and ℓ are constants, $\underset{\sim}{u}$ is the four-velocity of the massive particle, and ξ, η are the Killing vectors associated with the v, ϕ coordinates, respectively.

   b) Another *first integral* can be constructed from the fact that freely-falling massive particles follow *timelike* four-velocities, namely,

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1. \tag{6}$$

Using Eq. (3), construct this first integral.

c) As the angular momentum of this freely-falling massive particle is conserved, one can choose the motion to occur in the $\theta = \pi/2$ plane. Using this fact and Eqs. (4) and (5), show that Eq. (6) can be written in the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r), \tag{7}$$

where $\mathcal{E} \equiv (e^2 - 1)/2$ and

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3} \tag{8}$$

3. Consider the Schwarzschild geometry exterior to a spherically symmetric massive object, where the Schwarzschild line element takes the form of Eq. (1). This line element contains two symmetries associated with the t and ϕ coordinates. The two first integrals associated with these symmetries are of the form

$$e \equiv -\underset{\sim}{\xi} \cdot \underset{\sim}{u} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \tag{9}$$

$$\ell \equiv \underset{\sim}{\eta} \cdot \underset{\sim}{u} = r^2 \sin^2\theta\, \frac{d\phi}{d\tau}, \tag{10}$$

and are stated in Eqs. (9.21) and (9.22) of your text.

Consider the coordinate transformation given by Eq. (2), which brings the Schwarzschild line element to that of the Eddington-Finkelstein coordinates. Show that the first integrals of Eqs. (9) and (10) take the form of those that were found through Eqs. (4) and (5) when the coordinate transformation of Eq. (2) is employed.

4. In class, we arrived at an equation describing the world lines of *radial light rays* in the Schwarzschild geometry in Eddington-Finkelstein coordinates, which took the form

$$v - 2\left(r + 2M\ln\left|\frac{r}{2M} - 1\right|\right) = \text{const.}, \tag{11}$$

where this solution corresponds to *outgoing* radial light rays when $r > 2M$ and *ingoing* radial light rays when $r < 2M$.

a) Starting with Eq. (3), derive this result.

b) Setting $r \equiv 2Mx$, show that Eq. (11) can be written in the form

$$\frac{v(x)}{4M} = \kappa + x + \ln|x-1|, \tag{12}$$

where κ is a constant of integration.

c) Using your favorite plotting software (i.e. Maple, Matlab, Wolframalpha, etc.), plot $v(x)/4M$ vs x for $\kappa = 0$, $\kappa = 1$, $\kappa = 2$, $\kappa = 3$, and $\kappa = 4$, for both $0 < x < 1$ and $x > 1$. Your x-axis should have a range of $0.1 < x < 0.99$ for the $x < 1$ curve and $1.01 < x < 3$ for the $x > 1$ curve.

5. In class, we arrived at an equation of motion describing the radial coordinate of a freely-falling particle about a spherically-symmetric massive object of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r), \tag{13}$$

where M is the mass of the central object, $\mathcal{E} \equiv (e^2-1)/2$, and

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}. \tag{14}$$

a) Now consider an observer who falls *radially* into a spherical black hole of mass M. Is $\mathcal{E}$ positive, negative, or zero for ...

i. the observer who starts from *rest* relative to a stationary observer at a Schwarzschild coordinate radius R.

ii. the observer who starts at infinity with an *initial speed* relative to a stationary observer.

iii. the observer who starts at infinity from *rest* relative to a stationary observer.

b) Starting with Eq. (13), derive $\tau(r)$ for the observer who starts at infinity from rest.

c) Starting with Eq. (13) and using Eq. (9), derive $t(r)$ for the observer who starts at infinity from rest.

6. Reconsider the radially in-falling observer who starts from rest at infinity.

   a) Calculate the time it takes for the observer, who starts from rest at infinity, as measured by this observer's watch, to pass from the event horizon to the $r = 0$ singularity.

   b) What initial radial coordinate r yields the same time interval found in a), as measured by this observer's watch, to pass from this r coordinate to the event horizon?

   c) Although the final fate of our sun will not be to collapse to a black hole, imagine a black hole of one solar mass. How much time passes on the observer's watch, in seconds, for the scenario outlined in part a).