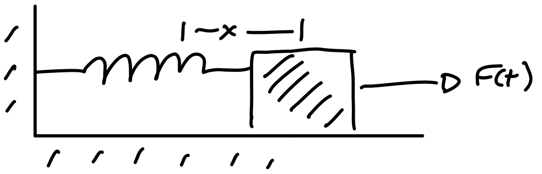
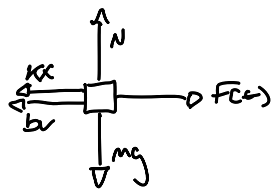
Ch. 3 Forced Oscillations

Consider a damped, driven harmonic oscillator





Consider the driving force to be harmonic and of the form

- $F(t) = F_0 \cos(\omega t)$

$\gamma = \frac{b}{m} \qquad \omega_0 = \sqrt{\frac{k}{m}}$

Newtons 2nd Law....

- $\vec{F}_{net} = m\vec{a}_x$
- $F_0 \cos(\omega t) - kx - bv = max$
- $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$

$$* \left[\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)\right]$$

For *

- Try a solution of the form

$$\left.\begin{aligned} x(t) &= A(\omega)\cos(\omega t - \delta) \\ \frac{dx}{dt} &= -\omega A(\omega)\sin(\omega t - \delta) \\ \frac{d^2x}{dt^2} &= -\omega^2 A(\omega)\cos(\omega t - \delta) \end{aligned}\right\} \text{Plugging into } (*)$$

Notice:

- The amplitude is a function of the frequency
- δ is the phase angle between the driving force and the resultant displacement

$$-\omega^2 A(\omega)\cos(\omega t - \delta) - \omega\gamma A(\omega)\sin(\omega t - \delta) + \omega_0^2 A(\omega)\cos(\omega t - \delta) = \frac{F_0}{m}\cos(\omega t)$$

$$\left[(\omega_0^2 - \omega^2)\cos(\omega t - \delta) - \omega\gamma\sin(\omega t - \delta)\right]A(\omega) = \frac{F_0}{m}\cos(\omega t)$$

Remember...

$\cos(\omega t - \delta) = \cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)$

$\sin(\omega t - \delta) = \sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)$

Plugging into *

$$\left[(\omega_0^2 - \omega^2)\left[\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)\right]\right] - \omega\gamma\left[\sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)\right]A(\omega) = \frac{F_0}{m}\cos(\omega t)$$

$$\left[\left[(\omega_0^2 - \omega^2)\cos(\delta) + \omega\gamma\sin(\delta)\right]\cos(\omega t) + \left[(\omega_0^2 - \omega^2)\sin(\delta) - \omega\gamma\cos(\delta)\right]\sin(\omega t)\right]A(\omega) = \frac{F_0}{m}\cos(\omega t)$$

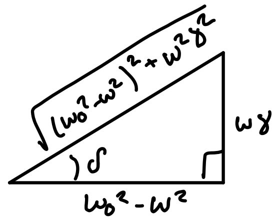
This equation must hold for all time

1.) $\left[(\omega_0^2-\omega^2)\cos(\delta)+\omega\gamma\sin(\delta)\right]A(\omega)=F_0/m$

2.) $\left[(\omega_0^2-\omega^2)\sin(\delta)-\omega\gamma\cos(\delta)\right]A(\omega)=0$

For eqn 2) . . . .

$$\tan(\delta)=\frac{\omega\gamma}{(\omega_0^2-\omega^2)} \quad \therefore \quad \delta=\tan^{-1}\left(\frac{\omega\gamma}{(\omega_0^2-\omega^2)}\right)$$



(**)

$$\sin(\delta)=\frac{\omega\gamma}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

$$\cos(\delta)=\frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$$

For (**)

$$\left[\frac{(\omega_0^2-\omega^2)(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)+\omega^2\gamma^2}}+\frac{\omega\gamma\cdot\omega\gamma}{\sqrt{(\omega_0^2-\omega^2)+\omega^2\gamma^2}}\right]A(\omega)=\frac{F_0}{m}$$

$$\left[\frac{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}\right]A(\omega)=F_0/m$$

$$\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\,A(\omega)=F_0/m$$

The solution is then . . .

$$\left[x(t)=A(\omega)\cos(\omega t-\delta)\right]$$

where

$$\tan(\delta)=\frac{\omega\gamma}{(\omega_0^2-\omega^2)} \qquad \delta \equiv \text{Phase angle}$$

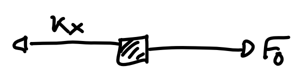
$$\left[A(\omega)=\frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}\right]$$

Explore limiting cases

CI: $\lim\limits_{\omega\to 0}\tan(\delta)\to 0 \qquad \delta\to 0$ (or π)

$$\lim_{\omega\to 0}A(\omega)\to\frac{F_0/m}{\omega_0^2}=\frac{F_0}{m\omega_0^2}=\frac{F_0}{k}$$

$$x(t)=F_0/k$$



$$\bar{F}_x=m\bar{a}_x$$
$$F_0-kx=0$$
$$x=\frac{F_0}{k}$$

Notice

- For small frequencies, mass moves with the driving force.

CII: $\lim\limits_{\omega\to\infty}\tan(\delta)\to\frac{\gamma}{-\omega}\to 0 \quad \therefore \quad \delta=\pi$

$$\lim_{\omega\to\infty}A(\omega)\to 0$$

Notice

- As ω increases, amplitude decreases, and The mass is in the opposite direction of the driving force
- As $\omega\to\infty$, mass doesn't move due to inertia

CT III: $\lim_{\omega=\omega_0} \tan(\delta) \to \infty \therefore \delta \to \pi/2$

$$\lim_{\omega=\omega_0} A(\omega) = \frac{F_0/m}{\omega\gamma}$$

Notice

- When driving frequency is the same as the natural frequency, the displacement lags the driving force by $\pi/2$
- Velocity leads the displacement by $\pi/2$. The mass is moving in the same direction as the driving force

What frequency ω yields $A_{max} = A(\omega_{max})$?

$$0 = \frac{dA(\omega)}{d\omega} = \frac{F_0}{m}\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]^{-1/2} = -\frac{F_0}{2m}\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]^{-3/2}\left[2(\omega_0^2-\omega^2)\cdot -2\omega + 2\omega\gamma^2\right]$$

$$0 = \frac{F_0\omega}{m}\left[(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right]^{-3/2}\underbrace{\left[2(\omega_0^2-\omega^2)-\gamma\right]}_{0} \text{ - must be } = 0$$

$$2(\omega_0^2-\omega^2)-\gamma^2 = 0$$

$$\omega_0^2 - \frac{\gamma^2}{2} = \omega^2$$

$$\omega_0^2\left(1-\frac{\gamma^2}{2\omega_0^2}\right) = \omega^2$$

$$\omega = \omega_0\sqrt{1-\gamma^2/2\omega_0^2}$$

$$\left[\omega_{max} = \omega_0\sqrt{1-\gamma^2/2\omega_0^2}\right]$$

← maximizes amplitude

$$A_{max} = A(\omega_{max}) = \frac{F_0/m}{\sqrt{\frac{\gamma^4}{4}+\left(\omega_0^2-\frac{\gamma^2}{2}\right)\gamma^2}} = \frac{F_0/m}{\sqrt{\omega_0^2\gamma^2-\frac{\gamma^4}{4}}}$$

$$\left[\begin{array}{l} A_{max} = \dfrac{F_0/m}{\gamma\omega_0\sqrt{1-\gamma^2/4\omega_0^2}} \\ \omega_{max} = \omega_0\sqrt{1-\gamma^2/2\omega_0^2} \end{array}\right] \qquad Q = \frac{\omega_0}{\gamma}$$

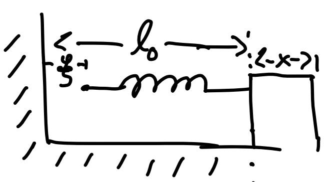
$$\frac{F_0 Q}{k}\left(1-1/4Q^2\right)^{1/2} = \frac{F_0}{k}\frac{Q}{\sqrt{1-1/4Q^2}}$$

For very light damping, $\omega_0 \gg \gamma$ & $Q \gg 1$

$$\lim_{Q\gg 1} A_{max} \sim Q$$

Harmonic driving force acts as an amplifier @ $\omega = \omega_{max}$

Ex



$m = 1.5$ kg

$k = 150$ N/m

$\xi = a\cos(\omega t)$

$a = 5.0\times10^{-3}$ m

$\omega = 6\pi$ rad/s

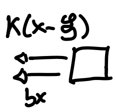
$b = 3.0\ \frac{Ns}{m}$

Determine $A(\omega)$: δ

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{150\text{ N/m}}{1.5\text{ kg}}} \qquad \gamma = \frac{b}{m} = \frac{3.0\text{ kg/s}}{1.5\text{ kg}} = 2\text{ s}^{-1}$$

$\omega_0 = 10$ rad/s $\quad \gamma = 2\text{ s}^{-1}$

FBD



$$\vec{F}_{net} = m\vec{a}_x$$

$$-k(x-\xi) - b\dot{x} = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = k\xi$$

$$= ka\cos(\omega t)$$

$F_0 = ka = (150\text{ N/m})(5.0\times10^{-3}\text{ m}) = 0.750$ N

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} = \frac{0.750\text{ N}/1.5\text{ kg}}{\sqrt{\left((10\tfrac{rad}{s})^2-(6\pi\tfrac{rad}{s})^2\right)^2+(6\pi\tfrac{rad}{s})^2(2.0\text{ s}^{-1})^2}} = 1.9\times10^{-3}\text{ m}$$

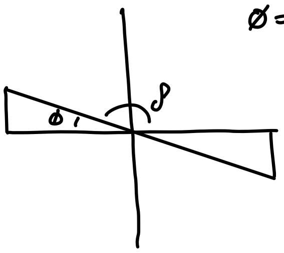
$$\delta = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^2-\omega^2}\right)$$

$$= \tan^{-1}\left(\frac{(6\pi\ \text{rad/s})(2.0\,s^{-1})}{(10\ \text{rad/s})^2-(6\pi\ \text{rad/s})^2}\right) = \tan^{-1}(-0.15)$$

$$\omega_{max} = \omega_0\sqrt{1-\frac{\gamma^2}{2\omega_0^2}} = (10\ \text{Rad/s})\sqrt{1-\frac{(2.0s^{-1})^2}{2(10s^{-1})^2}} = 9.9 \text{ radians}$$

$$A_{max} = \frac{F_0/m}{\gamma\omega_0}\frac{1}{\sqrt{1-\gamma^2/4\omega_0^2}} = \frac{0.75N/1.5kg}{(2.0s^{-1})(10s^{-1})}\cdot\frac{1}{\sqrt{1-\frac{(2.0s^{-1})^2}{4(10s^{-1})^2}}}$$

$$A_{max} = 2.5\times10^{-2}\ m$$



$$\phi = \tan^{-1}(0.15) = 8.5°$$

$$\delta = 171.5° \cdot \frac{\pi\ \text{rad}}{180°} = 3.0 \text{ rad}$$

$$\delta = 3.0 \text{ rad}$$

Transient Phenomena

Claim : Let x_1 be a solution to the damped driven harmonic oscillator

$$\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2x_1 = \frac{F_0}{m}(\cos(\omega t))$$

Let x_2 be a solution to the damped harmonic oscillator

$$\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2x_2 = 0$$

Claim : $x = x_1 + x_2$ is a solution to the damped driven harmonic oscillator

Proof

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2x = \frac{F_0}{m}\cos(\omega t)$$

$$\frac{d^2(x_1+x_2)}{dt^2} + \gamma\frac{d(x_1+x_2)}{dt} + \omega_0^2(x_1+x_2) = \frac{F_0}{m}\cos(\omega t)$$

$$\underbrace{\left(\frac{d^2x_1}{dt^2} + \gamma\frac{dx_1}{dt} + \omega_0^2x_1\right)}_{\frac{F_0}{m}\cos(\omega t)} + \underbrace{\left(\frac{d^2x_2}{dt^2} + \gamma\frac{dx_2}{dt} + \omega_0^2x_2\right)}_{0} = \frac{F_0}{m}\cos(\omega t)$$

$$\left[\begin{array}{l} x_1(t) = A(\omega)\cos(\omega t-\delta) \text{ - steady state} \\ x_2(t) = Be^{-\gamma t/2}\cos\left[\sqrt{\omega_0^2-(\gamma/2)^2}\,t+\phi\right] \text{ - transient Solution}\end{array}\right]$$

The sum of the two is the solution

$x(t) = x_1(t) + x_2(t)$

- $A(\omega)$ & δ are functions of ω
- B & ϕ are initial conditions