Exercise 1

$$|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\varphi}\frac{1}{\sqrt{2}}|1\rangle$$

a.) For $|\varphi\rangle$, $\theta = \pi/2$

$\boxed{\text{SG}\hat{n} \text{ must be oriented in the x-y plane since } \varphi \text{ is unknown}}$

b.) SG$\hat{z}$ has a φ value of 0

$|+\hat{z}\rangle = |0\rangle$, $|-\hat{z}\rangle = -|1\rangle$

$$\langle +\hat{z}|\varphi\rangle = \langle 0|\cdot\left(\tfrac{1}{\sqrt{2}}|0\rangle + e^{i\varphi}\cdot\tfrac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}\overset{1}{\cancel{\langle 0|0\rangle}} + \frac{e^{i\varphi}}{\sqrt{2}}\overset{0}{\cancel{\langle 0|1\rangle}} = \frac{1}{\sqrt{2}}$$

$$|\langle +\hat{z}|\varphi\rangle|^2 = 1/2$$

$$\langle -\hat{z}|\varphi\rangle = \langle 1|\cdot\left(\tfrac{1}{\sqrt{2}}|0\rangle - e^{i\varphi}\tfrac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}\overset{0}{\cancel{\langle 1|0\rangle}} - \frac{e^{i\varphi}}{\sqrt{2}}\langle 1|1\rangle = \frac{-e^{i\varphi}}{\sqrt{2}}$$

$$|\langle -\hat{z}|\varphi\rangle|^2 = \left(\frac{-e^{i\varphi}}{\sqrt{2}}\cdot\frac{-e^{-i\varphi}}{\sqrt{2}}\right) = \frac{1}{2}\left(e^0\right) = 1/2 \quad \therefore$$

$$|\langle -\hat{z}|\varphi\rangle|^2 = 1/2$$

$\boxed{\text{We can not learn anything about } \varphi \text{ by measuring SG}\hat{z}}$

c.) SG$\hat{x}$ has a $\theta = \pi/2$ and $\varphi = 0$ value

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad |-\hat{x}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\langle +\hat{x}|\varphi\rangle = \frac{1}{2}\overset{1}{\cancel{\langle 0|0\rangle}} + \frac{1}{2}e^{i\varphi}\overset{1}{\cancel{\langle 1|1\rangle}} = \left(\frac{1}{2} + \frac{1}{2}e^{i\varphi}\right) = \frac{1}{2}(1+e^{i\varphi})$$

$$|\langle +\hat{x}|\varphi\rangle|^2 = \frac{1}{4}(1+e^{i\varphi})(1+e^{-i\varphi}) = \frac{1}{4}(1+e^{-i\varphi}+e^{i\varphi}+1) = \frac{1}{4}(2+e^{i\varphi}+e^{-i\varphi})$$

$$\frac{1}{4}(2+\cos(\varphi)+\cancel{i\sin(\varphi)}+\cos(\varphi)-\cancel{i\sin(\varphi)}) = \frac{1}{4}(2+2\cos(\varphi)) = \frac{1}{2}(1+\cos(\varphi))$$

$$|\langle +\hat{x}|\varphi\rangle|^2 = \frac{1}{2}(1+\cos(\varphi)) = n_+/N$$

$$\langle -\hat{x}|\varphi\rangle = \frac{1}{2}\overset{1}{\cancel{\langle 0|0\rangle}} - \frac{1}{2}e^{i\varphi}\overset{1}{\cancel{\langle 1|1\rangle}} = \left(\frac{1}{2} - \frac{1}{2}e^{i\varphi}\right) = \frac{1}{2}(1-e^{i\varphi})$$

$$|\langle -\hat{x}|\varphi\rangle|^2 = \frac{1}{4}(1-e^{i\varphi})(1-e^{-i\varphi}) = \frac{1}{4}(1-e^{-i\varphi}-e^{i\varphi}+1) = \frac{1}{4}(2-e^{-i\varphi}-e^{i\varphi})$$

$$\frac{1}{4}(2-\cos(\varphi)+\cancel{i\sin(\varphi)}-\cos(\varphi)-\cancel{i\sin(\varphi)}) = \frac{1}{4}(2-2\cos(\varphi)) = \frac{1}{2}(1-\cos(\varphi))$$

$$|\langle -\hat{x}|\varphi\rangle|^2 = \frac{1}{2}(1-\cos(\varphi)) = n_-/N$$

Exercise 1 Continued

we can find the value of φ by the following relationships

$$\boxed{\frac{1}{2}(1+\cos(\varphi)) = n_+/N \quad \text{or} \quad \frac{1}{2}(1-\cos(\varphi)) = n_-/N}$$

d.)

The SG$\hat{z}$ direction cannot determine φ since the $|\varphi\rangle$ particle is in the x-y plane and SG$\hat{z}$ is orthogonal to this measurement and thus we cannot determine φ because of that.

Exercise 2

a.) Do $\hat{P}_0|\varphi\rangle$, $\hat{P}_1|\varphi\rangle$ b.) Probabilities of outcomes 0, 1

$\hat{P}_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\hat{P}_1 = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\text{Prob}(0) = \langle\varphi|\hat{P}_0|\varphi\rangle$, $\text{Prob}(1) = \langle\varphi|\hat{P}_1|\varphi\rangle$

1.) $|\varphi\rangle = |0\rangle$

a.) $\hat{P}_0|\varphi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\boxed{\hat{P}_0|\varphi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\hat{P}_1|\varphi\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\boxed{\hat{P}_1|\varphi\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

b.) $\langle\varphi|\hat{P}_0|\varphi\rangle = (1\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$: $\boxed{\langle\varphi|\hat{P}_0|\varphi\rangle = 1}$

$\langle\varphi|\hat{P}_1|\varphi\rangle = (1\ 0)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$: $\boxed{\langle\varphi|\hat{P}_1|\varphi\rangle = 0}$

2.) $|\varphi\rangle = 1/\sqrt{2}\,|0\rangle + 1/\sqrt{2}\,|1\rangle$

a.) $\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\boxed{\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\boxed{\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

b.) $\langle\varphi|\hat{P}_0|\varphi\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}(1\ 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_0|\varphi\rangle = 1/2}$

$\langle\varphi|\hat{P}_1|\varphi\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}(1\ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_1|\varphi\rangle = 1/2}$

3.) $|\varphi\rangle = 1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle$

a.) $\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\boxed{\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$: $\boxed{\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \end{pmatrix}}$

Exercise 2 Continued

b.) $\langle\varphi|\hat{P}_0|\varphi\rangle = \frac{1}{2}(1\ \ {-1})\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}(1\ \ {-1})\begin{pmatrix}1\\0\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_0|\varphi\rangle = 1/2}$

$\langle\varphi|\hat{P}_1|\varphi\rangle = \frac{1}{2}(1\ \ {-1})\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}(1\ \ {-1})\begin{pmatrix}0\\-1\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_1|\varphi\rangle = 1/2}$

4.) $|\varphi\rangle = 1/\sqrt{2}\,|0\rangle + i/\sqrt{2}\,|1\rangle$

a.) $\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}$: $\boxed{\hat{P}_0|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}}$

$\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\end{pmatrix}$: $\boxed{\hat{P}_1|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\i\end{pmatrix}}$

b.) $\langle\varphi|\hat{P}_0|\varphi\rangle = \frac{1}{2}(1\ \ {-i})\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}(1\ \ {-i})\begin{pmatrix}1\\0\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_0|\varphi\rangle = 1/2}$

$\langle\varphi|\hat{P}_1|\varphi\rangle = \frac{1}{2}(1\ \ {-i})\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{2}(1\ \ {-i})\begin{pmatrix}0\\i\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_1|\varphi\rangle = 1/2}$

5.) $|\varphi\rangle = 3/5\,|0\rangle + 4i/5\,|1\rangle$

a.) $\hat{P}_0|\varphi\rangle = \frac{1}{5}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}3\\4i\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3\\0\end{pmatrix}$: $\boxed{\hat{P}_0|\varphi\rangle = \frac{1}{5}\begin{pmatrix}3\\0\end{pmatrix}}$

$\hat{P}_1|\varphi\rangle = \frac{1}{5}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}3\\4i\end{pmatrix} = \frac{1}{5}\begin{pmatrix}0\\4i\end{pmatrix}$: $\boxed{\hat{P}_1|\varphi\rangle = \frac{1}{5}\begin{pmatrix}0\\4i\end{pmatrix}}$

b.) $\langle\varphi|\hat{P}_0|\varphi\rangle = \frac{1}{25}(3\ \ {-4i})\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}3\\4i\end{pmatrix} = \frac{1}{25}(3\ \ {-4i})\begin{pmatrix}3\\0\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_0|\varphi\rangle = 9/25}$

$\langle\varphi|\hat{P}_1|\varphi\rangle = \frac{1}{25}(3\ \ {-4i})\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}3\\4i\end{pmatrix} = \frac{1}{25}(3\ \ {-4i})\begin{pmatrix}0\\4i\end{pmatrix}$: $\boxed{\langle\varphi|\hat{P}_1|\varphi\rangle = 16/25}$

Exercise 3

a.) Construct measurement operators for the measurement $SG\hat{y}$.

$$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \quad , \quad |-\hat{y}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

$$\langle +\hat{y}| = \frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1| \quad , \quad \langle -\hat{y}| = \frac{1}{\sqrt{2}}\langle 0| + \frac{i}{\sqrt{2}}\langle 1|$$

$$\hat{P}_+ = |\hat{y}\rangle\langle\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} \quad :$$

$$\hat{P}_- = |-\hat{y}\rangle\langle-\hat{y}| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix} \quad :$$

$$\boxed{\hat{P}_+ = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} \qquad \hat{P}_- = \frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}}$$

b.) Calculate probabilities of pre-measurement states

i.) $|\varphi\rangle = |0\rangle$

$$\langle\varphi|\hat{P}_+|\varphi\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{2}$$

$$\boxed{\langle\varphi|\hat{P}_+|\varphi\rangle = 1/2}$$

$$\langle\varphi|\hat{P}_-|\varphi\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{2}$$

$$\boxed{\langle\varphi|\hat{P}_-|\varphi\rangle = 1/2}$$

ii.) $|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$\langle\varphi|\hat{P}_+|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1\end{pmatrix}\cdot\begin{pmatrix}1-i\\ 1+i\end{pmatrix} = \frac{1}{4}(1-i+1+i) = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{\langle\varphi|\hat{P}_+|\varphi\rangle = 1/2}$$

$$\langle\varphi|\hat{P}_-|\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1+i\\ 1-i\end{pmatrix} = \frac{1}{4}(1+i+1-i) = \frac{2}{4} = \frac{1}{2}$$

$$\boxed{\langle\varphi|\hat{P}_-|\varphi\rangle = 1/2}$$

Exercise 3 Continued

iii.) $|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

$$\langle\varphi|\hat{P}_+|\varphi\rangle = \frac{1}{\sqrt{2}}(1\ -i)\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{4}(1\ -i)\begin{pmatrix}2\\ 2i\end{pmatrix} = \frac{1}{4}(2+2) = \frac{4}{4}$$

$$\boxed{\langle\varphi|\hat{P}_+|\varphi\rangle = 1}$$

$$\langle\varphi|\hat{P}_-|\varphi\rangle = \frac{1}{\sqrt{2}}(1\ -i)\cdot\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{4}(1\ -i)\begin{pmatrix}0\\ 0\end{pmatrix} = \frac{1}{4}(0+0) = 0$$

$$\boxed{\langle\varphi|\hat{P}_-|\varphi\rangle = 0}$$

iv.) $|\varphi\rangle = \frac{4}{5}|0\rangle + \frac{3i}{5}|1\rangle$

$$\langle\varphi|\hat{P}_+|\varphi\rangle = \frac{1}{5}(4\ -3i)\cdot\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\cdot\frac{1}{5}\begin{pmatrix}4\\ 3i\end{pmatrix} = \frac{1}{50}(4\ -3i)\begin{pmatrix}7\\ 7i\end{pmatrix} = \frac{1}{50}(28+21) = \frac{49}{50}$$

$$\boxed{\langle\varphi|\hat{P}_+|\varphi\rangle = 49/50}$$

$$\langle\varphi|\hat{P}_-|\varphi\rangle = \frac{1}{5}(4\ -3i)\cdot\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}\cdot\frac{1}{5}\begin{pmatrix}4\\ 3i\end{pmatrix} = \frac{1}{50}(4\ -3i)\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{50}(4-3) = \frac{1}{50}$$

$$\boxed{\langle\varphi|\hat{P}_-|\varphi\rangle = 1/50}$$

Exercise 4

1) $\hat{P}_i \hat{P}_i = \hat{P}_i$

a.) Operators for SG$\hat{z}$ $\{|0\rangle, |1\rangle\}$, $|+\hat{z}\rangle = |0\rangle$, $|-\hat{z}\rangle = -|1\rangle$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \therefore \quad \hat{P}_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_{+z} \cdot \hat{P}_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\hat{P}_{+z} \cdot \hat{P}_{+z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \therefore \quad \hat{P}_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{P}_{-z} \cdot \hat{P}_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\hat{P}_{-z} \cdot \hat{P}_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

b.) Operators for SG$\hat{y}$ $\{|0\rangle, |1\rangle\}$, $|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$, $|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \quad \therefore \quad \hat{P}_{+y} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\hat{P}_{+y} \cdot \hat{P}_{+y} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$\boxed{\hat{P}_{+y} \cdot \hat{P}_{+y} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = \hat{P}_{+y}}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \therefore \quad \hat{P}_{-y} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\hat{P}_{-y} \cdot \hat{P}_{-y} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\boxed{\hat{P}_{-y} \cdot \hat{P}_{-y} = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \hat{P}_{-y}}$$

Exercise 4 Continued

2) $\hat{P}_i \cdot \hat{P}_j = 0$

a.) $\hat{P}_{+z} \cdot \hat{P}_{-z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\boxed{\hat{P}_{+z} \cdot \hat{P}_{-z} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0}$$

b.) $\hat{P}_{+y} \cdot \hat{P}_{-y} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1+i^2 & i-i \\ i-i & i^2+1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\boxed{\hat{P}_{+y} \cdot \hat{P}_{-y} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0}$$

3) $\sum P_i = \hat{I}$

a.)

$\sum \hat{P}_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\sum \hat{P}_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

b.)

$\sum \hat{P}_y = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\sum \hat{P}_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

4)

$\boxed{\text{Both measurement operators are positive}}$

Exercise 5

$$\bar{P}_{EST} = \frac{1}{N}\bar{n}_H \ , \ \bar{P}^2_{EST} = \frac{1}{N^2}\bar{n}^2_H \ : \ \bar{n}_H = \sum_{n_H=0}^{N} n_H \cdot \text{prob}(n_H) \ , \ \bar{n}^2_H = \sum_{n_H=0}^{N} n_H^2 \cdot \text{prob}(n_H)$$

$$\binom{N}{n_H} = \frac{N!}{n_H!(N-n_H)!} \longrightarrow \text{prob}(n_H) = \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H}$$

$$\bar{n}_H = \sum_{n_H=0}^{N} n_H \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H} \ , \ (a+b)^N = \sum_{K=0}^{N}\binom{N}{K} a^K b^{N-K}$$

$$\sum_{K=0}^{N}\binom{N}{K} a^K (1-a)^{N-K} K = Na \ , \ \sum_{K=0}^{N}\binom{N}{K} a^K (1-a)^{N-K} K^2 = Na\left[1+a(N-1)\right]$$

Problem 1 | Calculate $\bar{P}_{EST}$

$$\bar{P}_{EST} = \frac{1}{N}\bar{n}_H = \frac{1}{N}\cdot\sum_{n_H=0}^{N} n_H \cdot \text{prob}(n_H) = \frac{1}{N}\cdot\sum_{n_H=0}^{N} n_H \cdot \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H}$$

$$\bar{P}_{EST} = \frac{1}{N}\cdot\sum_{n_H=0}^{N} \frac{n_H \cdot N!}{n_H!(N-n_H)!} P^{n_H}(1-P)^{N-n_H} = \frac{1}{N}\cdot\sum_{n_H=0}^{N} n_H \cdot \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H}$$

Lets allow $n_H \to K$: $\bar{P}_{EST} = \frac{1}{N}\cdot\sum_{K=0}^{N} K\cdot\binom{N}{K} P^K (1-P)^{N-K} \longrightarrow (1)$

Using, $\sum_{K=0}^{N}\binom{N}{K} a^K (1-a)^{N-K}$, (1) can be re-written as, $\frac{1}{N}\cdot N\cdot P$, and thus

$$\boxed{\bar{P}_{EST} = P}$$

Problem 2 | Calculate Variance

Variance $\equiv \nu = \bar{P}^2_{EST} - (\bar{P}_{EST})^2$, First we need to calculate $\bar{P}^2_{EST}$.

$$\bar{P}^2_{EST} = \frac{1}{N^2}\bar{n}^2_H = \frac{1}{N^2}\sum_{n_H=0}^{N} n_H^2 \cdot \text{prob}(n_H) = \frac{1}{N^2}\sum_{n_H=0}^{N} n_H^2 \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H}$$

$$\bar{P}^2_{EST} = \frac{1}{N^2}\sum_{n_H=0}^{N} n_H^2 \binom{N}{n_H} P^{n_H}(1-P)^{N-n_H}$$

Lets allow $n_H \to K$: $\bar{P}^2_{EST} = \frac{1}{N^2}\sum_{K=0}^{N} K^2 \binom{N}{K} P^K(1-P)^{N-K} \longrightarrow (2)$

Using, $\sum_{K=0}^{N}\binom{N}{K} a^K (1-a)^{N-K} K^2 = Na\left[1+a(N-1)\right]$, we can re-write (2) as

$$\frac{1}{N^2}\left[N\cdot P(1+P(N-1))\right] = \frac{1}{N}\left(P(1+PN-P)\right) = \frac{1}{N}(P+P^2N-P^2) = \frac{1}{N}(P+P^2(N-1))$$

$$\therefore \quad \bar{P}^2_{EST} = \frac{1}{N}(P+P^2(N-1))$$

Exercise 5 Continued

$$\mathcal{V} = \frac{(P + P^2 N - P^2)}{N} - P^2 = \frac{(P + P^2 N - P^2)}{N} - \frac{P^2 N}{N} = \frac{P + P^2 N - P^2 - P^2 N}{N}$$

$$= \frac{P - P^2}{N}$$

$$\therefore \quad \boxed{\mathcal{V} = \frac{P}{N}(1-P)}$$

Problem 3 | Calculate $F_{N\,toss}$

$$\sum_{\substack{\text{All possible} \\ \text{outcomes}}}^{\infty} \frac{1}{\text{Prob(outcome)}} \left(\frac{\partial \text{Prob(outcome)}}{\partial P} \right)^2 = F$$

$$\frac{1}{P} + \frac{1}{(1-P)} = \frac{(1-P)}{P(1-P)} + \frac{P}{P(1-P)} = \frac{1}{P(1-P)}$$

$$F_{N\,toss} = N \cdot F_{\text{single coin toss}} = N \cdot \frac{1}{P(1-P)} = \frac{N}{P(1-P)}$$

$F_{N\,toss} = \frac{N}{P(1-P)}$, this result tells us that our variance is the smallest it can possibly be. $\mathcal{V}(P_{est}) \geq \frac{1}{F_{N\,toss}}$