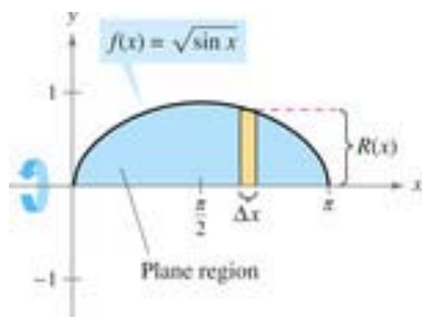
MAT 201
Larson/Edwards – Section 7.2 (Part 1)
Volume: The Disk Method

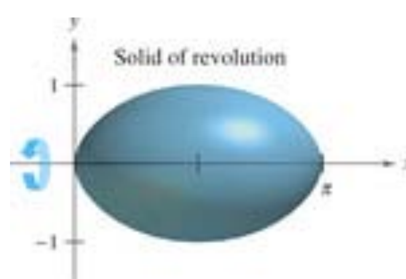
In section 7.1 we discussed ways to find the area between two curves by using definite integrals. We will now investigate how the definite integral can be used to determine the volume of ***solids of revolution***.

What is a solid of revolution?

Suppose we have described an area under a curve as in the picture below:



If we were to rotate this area about the x-axis we would obtain the following ***solid of revolution***:



By using formulas in geometry we can only estimate the volume of this solid. Using calculus we can find the exact volume!

Recall that in order to find the area under a curve we added rectangles. If these rectangles are rotated around the x-axis they become disks. We will be finding the volume by adding cylindrical disks.

What is the volume of a cylindrical disk with radius R and height w?

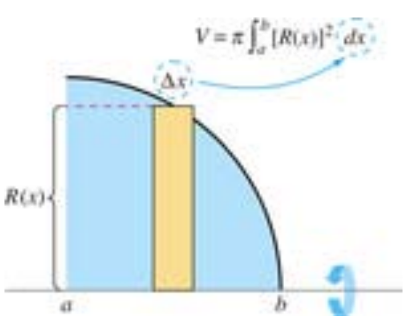
The radius of each disk being added depends on the x-value in the interval, and the height of the disk, w, is the change in x (Δx).

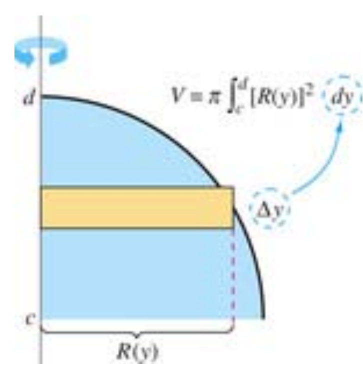
Our volume equation will then take on the following transformation:

$$V = \pi R^2 w \quad \Rightarrow \quad \Delta V = \pi \left[R(x_i)\right]^2 \Delta x \quad \Rightarrow \quad V = \pi \int_a^b \left[R(x)\right]^2 dx$$

<u>**The Disk Method**</u>**:**

- Horizontal Axis of Revolution (x-axis):
  $Volume = V = \pi \int_a^b \left[R(x)\right]^2 dx$
- Vertical Axis of Revolution (y-axis):
  $Volume = V = \pi \int_c^d \left[R(y)\right]^2 dy$





Ex: Find the solid generated by revolving the plane region bounded by the equations $y = x$, $y = 0$, $x = 3$, about the x-axis.

$$V = \pi \int_a^b [R(x)]^2 \, dx$$

$a = 0$
$b = 3$

$R(x) = x$

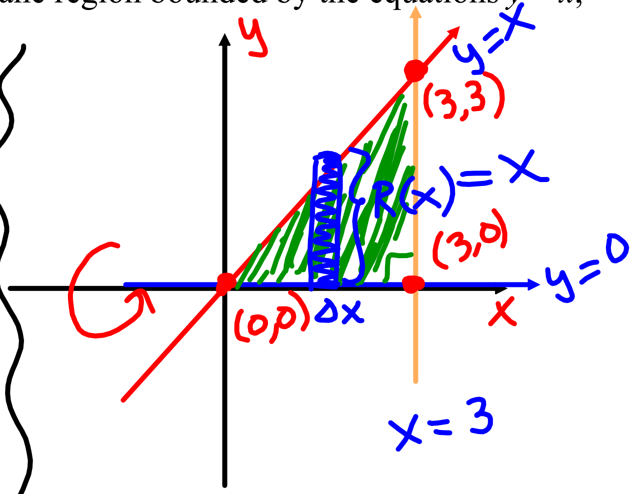
$$V = \pi \int_0^3 (x)^2 \, dx$$

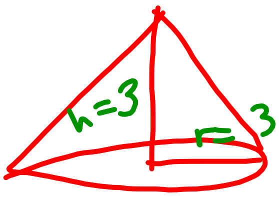
$$V = \pi \left[\frac{x^3}{3} \Big|_0^3 \right]$$

$$= \pi \left[\frac{27}{3} - 0 \right]$$

$$= \pi \cdot 9$$

$$= 9\pi \text{ units}^3$$





$$V = \frac{\pi r^2 h}{3}$$

$$V = \frac{\pi (3)^2 \cancel{(3)}}{\cancel{3}}$$

$$V = 9\pi \text{ units}^3$$

The Washer Method (If The Solid Has A "Hole"):
Horizontal Axis of Revolution (x-axis):

$$Volume = V = \pi\int_a^b \left(\left[R(x)\right]^2 - \left[r(x)\right]^2\right)dx$$

Vertical Axis of Revolution (y-axis):

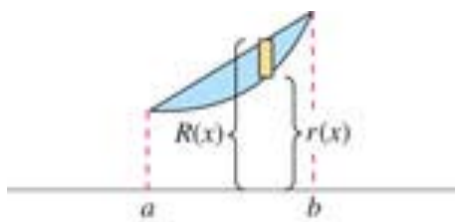
$$Volume = V = \pi\int_c^d \left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right)dy$$

Note: $R(x)$ (or $R(y)$) is the ***outer radius*** and $r(x)$ (or $r(y)$) is the ***inner radius***.

Would this method work even if there isn't a hole??? Yes; $r(x)=0$.

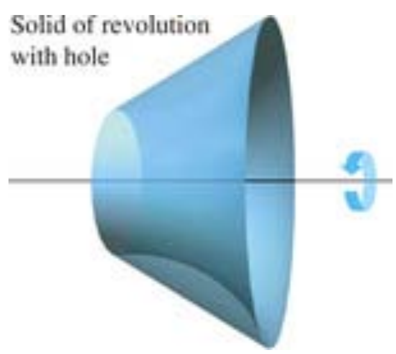
Ex: Find the solid generated by revolving the plane region bounded by the equations $y = x$, $y = 0$, $x = 3$, about the x-axis.

Suppose now that we would like to find the volume of solid created by rotating the following plane region about the x-axis.



Plane region

We would obtain the following solid of revolution with a hole:



Ex: Find the solid generated by revolving the plane region bounded by the equations $y = x$, $y = 0$, $x = 3$, about the y-axis.

$$V = \pi \int_c^d \left[(R(y))^2 - (r(y))^2 \right] dy$$

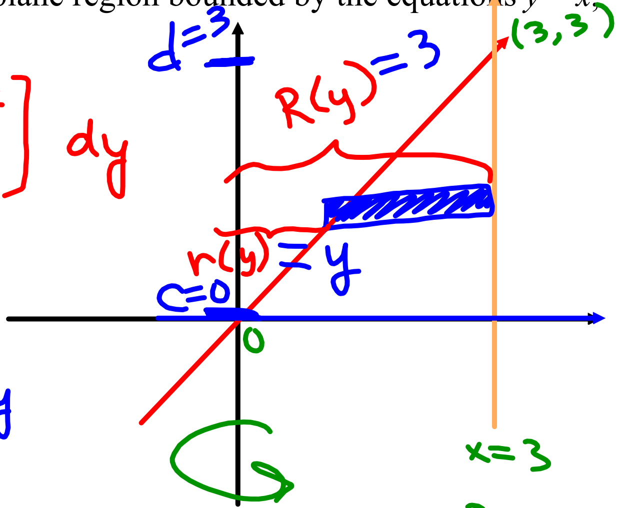
$$V = \pi \int_0^3 \left[(3)^2 - (y)^2 \right] dy$$

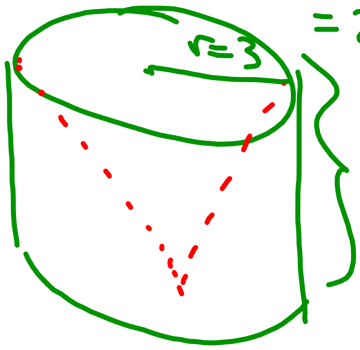
$$= \pi \left[9y - \frac{y^3}{3} \Big|_0^3 \right]$$

$$= \pi \left[(27 - 9) - (0) \right]$$

$$= \pi (18)$$

$$= 18\pi \text{ units}^3$$



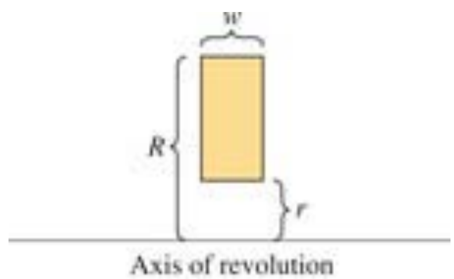


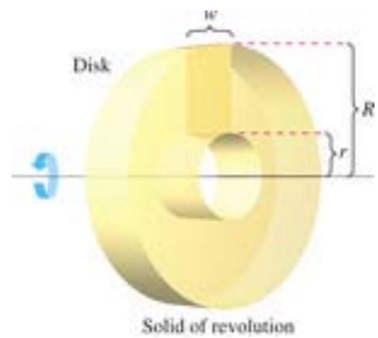
$\pi (3)^2 (3)$

$= 27\pi - 9\pi$

$= 18\pi$

With this problem the disk method can be extended by replacing the representative disk with a representative ***washer***.





<u>The Washer Method (If The Solid Has A "Hole")</u>:

- Horizontal Axis of Revolution (x-axis):

  $$Volume = V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$$

- Vertical Axis of Revolution (y-axis):

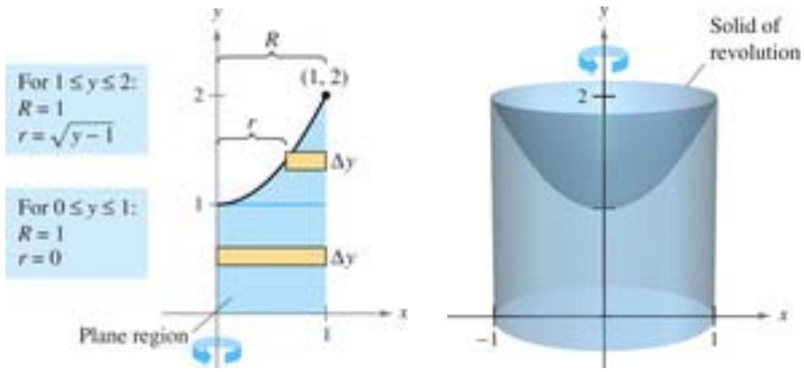
  $$Volume = V = \pi\int_c^d \left([R(y)]^2 - [r(y)]^2\right)dy$$

Note: *R(x)* (or *R(y)*) is the ***outer radius*** and *r(x)* (or *r(y)*) is the ***inner radius***.

Would this method work even if there isn't a hole???

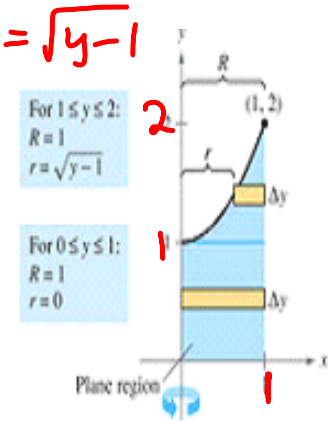
Ex: Find the solid generated by revolving the plane region bounded by the equations $y = x$, $y = 0$, $x = 3$, about the y-axis.

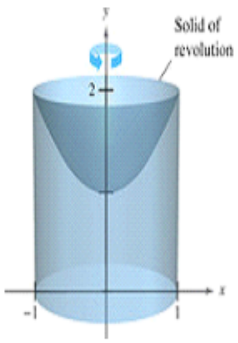
Note: If the solid of revolution involves BOTH disks and washers, separate each volume over the appropriate interval and add them in the end!



Note: If the solid of revolution involves BOTH disks and washers, separate each volume over the appropriate interval and add them in the end!

$r = \sqrt{y-1}$





$$V = \pi \int_0^1 [1]^2 \, dy + \pi \int_1^2 \left[(1)^2 - \left(\sqrt{y-1}\right)^2\right] dy$$