PHYS 396 Homework Set 7 Solutions

For $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$g_{\theta\theta} = a^2$$
$$g_{\varphi\varphi} = a^2\sin^2\theta$$

a) All possible combinations are...

$$\left[g_{\theta\delta}\Gamma^{\delta}_{\theta\theta},\ g_{\theta\delta}\Gamma^{\delta}_{\theta\varphi},\ g_{\varphi\delta}\Gamma^{\delta}_{\theta\theta},\ g_{\theta\delta}\Gamma^{\delta}_{\varphi\varphi},\ g_{\varphi\delta}\Gamma^{\delta}_{\varphi\theta},\ g_{\varphi\delta}\Gamma^{\delta}_{\varphi\varphi} \right]$$

b) Since $g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)\ldots$

1) $$g_{\theta\delta}\Gamma^{\delta}_{\theta\theta} = \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial\theta} + \cancel{\frac{\partial g_{\theta\theta}}{\partial\theta}} - \cancel{\frac{\partial g_{\theta\theta}}{\partial\theta}}\right) = \frac{1}{2}\frac{\partial}{\partial\theta}a^2 = 0$$
$$= g_{\theta\theta}\Gamma^{\theta}_{\theta\theta} \quad \therefore \boxed{\Gamma^{\theta}_{\theta\theta} = 0}$$

2) $$g_{\theta\delta}\Gamma^{\delta}_{\theta\varphi} = \frac{1}{2}\left(\frac{\partial g_{\theta\theta}}{\partial\varphi} + \cancelto{0}{\frac{\partial g_{\theta\varphi}}{\partial\theta}} - \cancelto{0}{\frac{\partial g_{\theta\varphi}}{\partial\theta}}\right) = \frac{1}{2}\frac{\partial}{\partial\varphi}a^2 = 0$$
$$= g_{\theta\theta}\Gamma^{\theta}_{\theta\varphi} \quad \therefore \boxed{\Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = 0}$$

3) $$g_{\varphi\delta}\Gamma^{\delta}_{\theta\theta} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial g_{\varphi\theta}}{\partial\theta}} + \cancelto{0}{\frac{\partial g_{\varphi\theta}}{\partial\theta}} - \frac{\partial}{\partial\varphi}g_{\theta\theta}\right) = -\frac{1}{2}\frac{\partial}{\partial\varphi}a^2 = 0$$
$$= g_{\varphi\varphi}\Gamma^{\varphi}_{\theta\theta} \quad \therefore \boxed{\Gamma^{\varphi}_{\theta\theta} = 0}$$

4) $$g_{\theta\delta}\Gamma^{\delta}_{\varphi\varphi} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial g_{\theta\varphi}}{\partial\varphi}} + \cancelto{0}{\frac{\partial g_{\theta\varphi}}{\partial\varphi}} - \frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = -\frac{1}{2}\frac{\partial}{\partial\theta}a^2\sin^2\theta = -\frac{1}{2}\cdot 2a^2\sin\theta\cos\theta = -a^2\sin\theta\cos\theta$$
$$= g_{\theta\theta}\Gamma^{\theta}_{\varphi\varphi} = a^2\Gamma^{\theta}_{\varphi\varphi} \quad \therefore \boxed{\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta}$$

5) $$g_{\varphi\delta}\Gamma^{\delta}_{\varphi\theta} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial\theta} + \cancel{\frac{\partial g_{\varphi\theta}}{\partial\varphi}} - \cancel{\frac{\partial g_{\varphi\theta}}{\partial\varphi}}\right) = \frac{1}{2}\frac{\partial}{\partial\theta}a^2\sin^2\theta = a^2\sin\theta\cos\theta$$
$$= g_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} = a^2\sin^2\theta\,\Gamma^{\varphi}_{\varphi\theta} \quad \therefore \boxed{\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}}$$

6) $$g_{\varphi\delta}\Gamma^{\delta}_{\varphi\varphi} = \frac{1}{2}\left(\frac{\partial g_{\varphi\varphi}}{\partial\varphi} + \cancel{\frac{\partial g_{\varphi\varphi}}{\partial\varphi}} - \cancel{\frac{\partial g_{\varphi\varphi}}{\partial\varphi}}\right) = \frac{1}{2}\frac{\partial}{\partial\varphi}a^2\sin^2\theta = 0$$
$$= g_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\varphi} \quad \therefore \boxed{\Gamma^{\varphi}_{\varphi\varphi} = 0}$$

SO, IN SUMMARY, THE NON-ZERO CHRISTOFFEL SYMBOLS ARE...

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta$$

$$\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}$$

2. THE GEODESIC EQUATION IS OF THE FORM...

a) $$\frac{d^2x^\alpha}{dS^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dS}\frac{dx^\gamma}{dS}.$$

FOR A LINE ELEMENT OF THE FORM

$$dS^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

THE GEODESIC EQUATION HAS COMPONENTS OF THE FORM...

$$\frac{d^2\theta}{dS^2} = -\Gamma^\theta_{\beta\gamma}\frac{dx^\beta}{dS}\frac{dx^\gamma}{dS} = -\Gamma^\theta_{\phi\phi}\frac{d\phi}{dS}\frac{d\phi}{dS} = \sin\theta\cos\theta\left(\frac{d\phi}{dS}\right)^2$$

AND

$$\frac{d^2\phi}{dS^2} = -\Gamma^\phi_{\beta\gamma}\frac{dx^\beta}{dS}\frac{dx^\gamma}{dS} = -\Gamma^\phi_{\phi\theta}\frac{d\phi}{dS}\frac{d\theta}{dS} - \Gamma^\phi_{\theta\phi}\frac{d\theta}{dS}\frac{d\phi}{dS} = -2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{dS}\frac{d\phi}{dS}$$

SO IN SUMMARY...

$$\left[\begin{aligned} \frac{d^2\theta}{dS^2} &= \sin\theta\cos\theta\left(\frac{d\phi}{dS}\right)^2 \\ \frac{d^2\phi}{dS^2} &= -2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{dS}\frac{d\phi}{dS} \end{aligned}\right]$$

b) NOTICE THAT $[\phi \to \phi + \text{CONSTANT}]$ LEAVES THE LINE ELEMENT UNCHANGED AS $d\phi \to d\phi \therefore ds^2 \to ds^2$

c) THE ASSOCIATED KILLING VECTOR IS

$$\left[\vec{\xi} = (0, 1)\right] \text{ SINCE } \xi^\theta = 0,\ \xi^\phi = \underline{1}$$

d) $$l \equiv \vec{\xi}\cdot\vec{u} = g_{AB}\xi^A u^B = g_{\phi\phi}\xi^\phi u^\phi = a^2\sin^2\theta \cdot \underline{1} \cdot \frac{d\phi}{dS}$$

SO

$$\left[l = a^2\sin^2\theta\frac{d\phi}{dS}\right]$$

e) EQUATION (5) CAN BE WRITTEN AS

$$0 = \frac{d^2\varphi}{ds^2} + \frac{2\cos\theta}{\sin\theta}\frac{d\theta}{ds}\frac{d\varphi}{ds} = \frac{1}{\sin^2\theta}\left[\sin^2\theta\frac{d^2\varphi}{ds^2} + 2\sin\theta\cos\theta\frac{d\theta}{ds}\frac{d\varphi}{ds}\right]$$

BUT $\frac{d}{ds}\left(\sin^2\theta\frac{d\varphi}{ds}\right) = \sin^2\theta\frac{d^2\varphi}{ds^2} + 2\sin\theta\cos\theta\frac{d\theta}{ds}\frac{d\varphi}{ds}$

SO

$$0 = \frac{1}{\sin^2\theta}\frac{d}{ds}\left(\sin^2\theta\frac{d\varphi}{ds}\right) \quad \therefore \sin^2\theta\frac{d\varphi}{ds} = \text{CONSTANT} \equiv \frac{\ell}{a^2}$$

$$\therefore \left[\ell = a^2\sin^2\theta\frac{d\varphi}{ds}\right] \checkmark$$

3. $\vec{u}\cdot\vec{u} = 1$

a) so $\vec{u}\cdot\vec{u} = g_{AB}u^A u^B = g_{\theta\theta}(u^\theta)^2 + g_{\varphi\varphi}(u^\varphi)^2$

$$= a^2\left(\frac{d\theta}{ds}\right)^2 + a^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 = 1$$

so

$$(*)\quad \boxed{1 = a^2\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right]}$$

b) NOW $\frac{d\varphi}{ds} = \frac{\ell}{a^2\sin^2\theta}$. PLUGGING THIS INTO (*) YIELDS..

$$1 = a^2\left(\frac{d\theta}{ds}\right)^2 + a^2\sin^2\theta\cdot\frac{\ell^2}{a^4\sin^4\theta} = a^2\left(\frac{d\theta}{ds}\right)^2 + \frac{\ell^2}{a^2\sin^2\theta}$$

NOW SOLVING FOR $\frac{d\theta}{ds}$...

$$a^2\left(\frac{d\theta}{ds}\right)^2 = 1 - \frac{\ell^2}{a^2\sin^2\theta} = 1 - \frac{\ell^2}{a^2}\csc^2\theta$$

NOW TAKING A SQUARE ROOT AND SOLVING FOR $\frac{d\theta}{ds}$...

$$\frac{d\theta}{ds} = \frac{1}{a}\left[1 - \frac{\ell^2}{a^2}\csc^2\theta\right]^{1/2}$$

c) NOW

$$(**)\quad \frac{d\theta}{d\varphi} = \frac{d\theta/ds}{d\varphi/ds}$$

AND

$$\frac{d\varphi}{ds} = \frac{\ell}{a^2}\csc^2\theta$$

$$\frac{d\theta}{ds} = \frac{1}{a}\left[1 - \frac{\ell^2}{a^2}\csc^2\theta\right]^{1/2}$$

PLUGGING THESE INTO (**) YIELDS...

$$\frac{d\theta}{d\varphi} = \frac{1}{a}\left[1 - \frac{\ell^2}{a^2}\csc^2\theta\right]^{1/2}\cdot\frac{a^2}{\ell}\cdot\frac{1}{\csc^2\theta} = \frac{a}{\ell}\cdot\frac{1}{\csc^2\theta}\left[1 - \frac{\ell^2}{a^2}\csc^2\theta\right]^{1/2}$$

NOW, SEPARATING VARIABLES...

$$\int\frac{\ell}{a}\frac{\csc^2\theta\, d\theta}{\sqrt{1 - \frac{\ell^2}{a^2}\csc^2\theta}} = \int d\varphi$$

so

$$\varphi(\theta) = \frac{l}{a}\int \frac{\csc^2\theta\, d\theta}{\sqrt{1 - \frac{l^2}{a^2}\csc^2\theta}}$$ ✓

$\sin^2\theta + \cos^2\theta = 1$

so

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore\ 1 + \cot^2\theta = \csc^2\theta$$

d)
$$= \frac{l}{a}\int \frac{\csc^2\theta\, d\theta}{\sqrt{\left(1 - \frac{l^2}{a^2}\right) - \frac{l^2}{a^2}\left(\csc^2\theta - 1\right)}}$$

$$= \frac{l}{a}\int \frac{\csc^2\theta\, d\theta}{\sqrt{\left(1 - l^2/a^2\right) - l^2/a^2 \cot^2\theta}} = -\frac{l}{a}\int \frac{dx}{\sqrt{\left(1 - l^2/a^2\right) - \frac{l^2}{a^2}x^2}} = -\int \frac{dx}{\sqrt{\left(\frac{a^2}{l^2} - 1\right) - x^2}}$$

let $x \equiv \cot\theta$

$dx = -\csc^2\theta\, d\theta$

let $K^2 = \frac{a^2}{l^2} - 1$

$$= -\int \frac{dx}{\sqrt{K^2 - x^2}} = -K\int \frac{\cos\alpha\, d\alpha}{\sqrt{K^2(1 - \sin^2\alpha)}} = -\int d\alpha = -\alpha + \varphi_0 = -\sin^{-1}\left(\frac{x}{K}\right) + \varphi_0$$

let $x = K\sin\alpha \quad \therefore\ \alpha = \sin^{-1}\left(\frac{x}{K}\right)$

$dx = K\cos\alpha\, d\alpha$

SINCE
$\cos^2\alpha + \sin^2\alpha = 1$

so

$$-(\varphi - \varphi_0) = \sin^{-1}\left(\frac{x}{K}\right) \quad \text{or} \quad x = K\sin(-(\varphi - \varphi_0))$$

or

$$\frac{\cos\theta}{\sin\theta} = \left[\cot\theta = K\sin(\varphi_0 - \varphi) = \sqrt{a^2/l^2 - 1}\,\sin(\varphi_0 - \varphi)\right]$$

NOW MULTIPLYING BOTH SIDES BY $a\sin\theta$..

(*)
$$a\cos\theta = a\sqrt{a^2/l^2 - 1}\,\sin(\varphi_0 - \varphi)\sin\theta$$

$$= a\sqrt{a^2/l^2 - 1}\left[\sin\varphi_0\cos\varphi - \sin\varphi\cos\varphi_0\right]\sin\theta$$

$$= a\sqrt{a^2/l^2 - 1}\,\sin\varphi_0(\sin\theta\cos\varphi) - a\sqrt{a^2/l^2 - 1}\,\cos\varphi_0(\sin\varphi\sin\theta)$$

NOW

$z = a\cos\theta$

$y = a\sin\theta\sin\varphi$

$x = a\sin\theta\cos\varphi$

let $\sqrt{a^2/l^2 - 1}\,\sin\varphi_0 \equiv A$

$\sqrt{a^2/l^2 - 1}\,\cos\varphi_0 \equiv B$

$\therefore$ (*) BECOMES...

$$\left[\ z = Ax - By\ \right]$$

THIS IS THE EQUATION DESCRIBING A 2D PLANE SLICING THROUGH THE CENTER OF A 3D AKA A GREAT CIRCLE!

4. THE LINE ELEMENT OF 3D FLAT SPACETIME IN POLAR COORDINATES IS..

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\varphi^2$$

$$\therefore \; g_{tt} = -c^2, \; g_{rr} = 1, \; g_{\varphi\varphi} = r^2$$

a) ALL POSSIBLE COMBINATIONS OF t ARE..

$$g_{t\delta}\Gamma^{\delta}_{tt}, \; g_{t\delta}\Gamma^{\delta}_{t\gamma}, \; g_{\alpha\delta}\Gamma^{\delta}_{tt}, \; g_{t\delta}\Gamma^{\delta}_{\beta\gamma}, \; g_{\alpha\delta}\Gamma^{\delta}_{t\gamma}, \; g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma}$$

NOW ALL POSSIBLE COMBINATIONS OF r : φ ARE...

$$g_{t\delta}\Gamma^{\delta}_{tt} : g_{t\delta}\Gamma^{\delta}_{tt}$$

$$g_{t\delta}\Gamma^{\delta}_{t\gamma} : g_{t\delta}\Gamma^{\delta}_{tr}, \; g_{t\delta}\Gamma^{\delta}_{t\varphi}$$

$$g_{\alpha\delta}\Gamma^{\delta}_{tt} : g_{r\delta}\Gamma^{\delta}_{tt}, \; g_{\varphi\delta}\Gamma^{\delta}_{tt}$$

$$g_{t\delta}\Gamma^{\delta}_{\beta\gamma} : g_{t\delta}\Gamma^{\delta}_{rr}, \; g_{t\delta}\Gamma^{\delta}_{r\varphi}, \; g_{t\delta}\Gamma^{\delta}_{\varphi\varphi}$$

$$g_{\alpha\delta}\Gamma^{\delta}_{t\gamma} : g_{r\delta}\Gamma^{\delta}_{tr}, \; g_{r\delta}\Gamma^{\delta}_{t\varphi}, \; g_{\varphi\delta}\Gamma^{\delta}_{tr}, \; g_{\varphi\delta}\Gamma^{\delta}_{t\varphi}$$

$$g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} : g_{r\delta}\Gamma^{\delta}_{rr}, \; g_{r\delta}\Gamma^{\delta}_{r\varphi}, \; g_{\varphi\delta}\Gamma^{\delta}_{rr}, \; \left[g_{r\delta}\Gamma^{\delta}_{\varphi\varphi}\right], \; \left[g_{\varphi\delta}\Gamma^{\delta}_{r\varphi}\right], \; g_{\varphi\delta}\Gamma^{\delta}_{\varphi\varphi}$$

$\therefore$ 18 POSSIBLE COMBINATIONS

b) THE <u>ONLY</u> NONVANISHING DERIVATIVES OF THE METRIC TENSOR ARE...

$\dfrac{\partial g_{\varphi\varphi}}{\partial r}$ WHICH COME FROM TWO φ'S AND ONE r SO...

SO

$$g_{r\delta}\Gamma^{\delta}_{\varphi\varphi} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial g_{r\varphi}}{\partial\varphi}} + \cancelto{0}{\frac{\partial g_{r\varphi}}{\partial\varphi}} - \frac{\partial g_{\varphi\varphi}}{\partial r}\right) = -\frac{1}{2}\frac{\partial}{\partial r}r^2 = -r = g_{rr}\Gamma^{r}_{\varphi\varphi} \quad \therefore \left[\Gamma^{r}_{\varphi\varphi} = -r\right]$$

$$g_{\varphi\delta}\Gamma^{\delta}_{r\varphi} = \frac{1}{2}\left(\cancelto{0}{\frac{\partial g_{\varphi r}}{\partial\varphi}} + \frac{\partial g_{\varphi\varphi}}{\partial r} - \cancelto{0}{\frac{\partial g_{r\varphi}}{\partial\varphi}}\right) = \frac{1}{2}\frac{\partial}{\partial r}r^2 = r = g_{\varphi\varphi}\Gamma^{\varphi}_{r\varphi} = r^2\Gamma^{\varphi}_{r\varphi}$$

$$\therefore \left[\Gamma^{\varphi}_{r\varphi} = \frac{1}{r}\right]$$

c) THE GEODESIC EQN

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$$ ... COMPONENTS...

for $\alpha = t$...

$$\frac{d^2t}{d\lambda^2} = -\Gamma^t_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$ AS ALL $\Gamma^t_{\beta\gamma} = 0$ FOR ANY β & γ

FOR $\alpha = r$...

$$\frac{d^2r}{d\lambda^2} = -\Gamma^r_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = -\Gamma^r_{\varphi\varphi}\frac{d\varphi}{d\lambda}\frac{d\varphi}{d\lambda} = +r\left(\frac{d\varphi}{d\lambda}\right)^2$$

for $\alpha = \varphi$...

$$\frac{d^2\varphi}{d\lambda^2} = -\Gamma^\varphi_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = -\Gamma^\varphi_{r\varphi}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} - \Gamma^\varphi_{\varphi r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = -2\Gamma^\varphi_{r\varphi}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = -\frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$$

SO IN SUMMARY...

$$\frac{d^2t}{d\lambda^2} = 0$$

$$\frac{d^2r}{d\lambda^2} = r\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$\frac{d^2\varphi}{d\lambda^2} = -\frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}$$ ✓

d) SINCE $\underset{\sim}{u}\cdot\underset{\sim}{u} = 0$ FOR A NULL GEODESIC

$$0 = \underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\alpha\beta}u^\alpha u^\beta = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\varphi\varphi}(u^\varphi)^2$$
$$= -c^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2$$

e) SINCE $t \to t + \text{CONST.}$ LEAVES ds^2 UNCHANGED, THERE IS AN ASSOCIATED KILLING VECTOR: $\underset{\sim}{\xi} \equiv (1, 0, 0)$

SINCE $\varphi \to \varphi + \text{CONST.}$ LEAVES ds^2 UNCHANGED, THERE IS AN ASSOCIATED KILLING VECTOR: $\underset{\sim}{\eta} = (0, 0, 1)$

NOW

$$-e \equiv \underset{\sim}{\xi}\cdot\underset{\sim}{u} = g_{\alpha\beta}\xi^\alpha u^\beta = g_{tt}\xi^t u^t = -c^2\frac{dt}{d\lambda} \quad \therefore \left[\frac{dt}{d\lambda} = \frac{e}{c^2}\right]$$ WHICH IS EQUIVALENT TO $\frac{d^2t}{d\lambda^2} = 0$

$$\ell \equiv \underset{\sim}{\eta}\cdot\underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta = g_{\varphi\varphi}\eta^\varphi u^\varphi = r^2\frac{d\varphi}{d\lambda} \quad \therefore \left[r^2\frac{d\varphi}{d\lambda} = \ell\right]$$

THIS IS EQUIVALENT TO (16)! TAKE A DERIVATIVE TO ... (16)

5 B) FROM THE PREVIOUS PROBLEM, $-e = \underset{\sim}{\xi} \cdot \underset{\sim}{u}$ YIELDED

$$\frac{dt}{d\lambda} = \frac{e}{c^2}.$$

NOW, TAKING A λ DERIVATIVE..

$$\frac{d^2t}{d\lambda^2} = 0 \quad \checkmark$$

ALSO, $\ell = \underset{\sim}{\eta} \cdot \underset{\sim}{u}$ YIELDED $r^2 \frac{d\varphi}{d\lambda} = \ell$

TAKING A λ DERIVATIVE...

$$\frac{d}{d\lambda}\left(r^2 \frac{d\varphi}{d\lambda}\right) = r^2 \frac{d^2\varphi}{d\lambda^2} + 2r \frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = r^2\left[\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}\right] = 0 \quad \checkmark$$

EQN (17) YIELDED: $0 = -c^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2$

EQN (18) YIELDED: $\frac{dt}{d\lambda} = \frac{e}{c^2}$

EQN (19) YIELDED: $\frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$

PLUGGING (18) & (19) INTO (17),..

$$0 = -c^2 \cdot \frac{e^2}{c^4} + \left(\frac{dr}{d\lambda}\right)^2 + r^2 \cdot \frac{\ell^2}{r^4}$$

$$= \left(\frac{dr}{d\lambda}\right)^2 - \frac{e^2}{c^2} + \frac{\ell^2}{r^2}$$

NOW SOLVING FOR $\frac{dr}{d\lambda}$ YIELDS..

$$\frac{dr}{d\lambda} = \left[\frac{e^2}{c^2} - \frac{\ell^2}{r^2}\right]^{1/2} \quad \checkmark$$

c) SEPARATING VARIABLES YIELDS...

$$\int \frac{dr}{\left[\frac{e^2}{c^2} - \frac{\ell^2}{r^2}\right]^{1/2}} = \int d\lambda \quad \text{or} \quad \lambda(r) = \lambda_0 + \int \frac{r\,dr}{\left[\frac{e^2}{c^2}r^2 - \ell^2\right]^{1/2}}$$

NOW LET $x = \frac{e^2}{c^2}r^2 - \ell^2$

$$dx = 2\frac{e^2}{c^2} r\,dr \quad \therefore \quad r\,dr = \frac{c^2}{2e^2}\,dx$$

SO

$$\lambda(r) = \lambda_0 + \frac{c^2}{2e^2}\int \frac{dx}{x^{1/2}} = \lambda_0 + \frac{c^2}{e^2}x^{1/2} = \lambda_0 + \frac{c^2}{e^2}\left[\frac{e^2}{c^2}r^2 - \ell^2\right]^{1/2}$$

NOW SOLVING FOR $r = r(\lambda)$...

$$\left[\frac{e^2}{c^2}r^2 - \ell^2\right]^{1/2} = \frac{e^2}{c^2}(\lambda - \lambda_0)$$

SQUARING

$$\frac{e^2}{c^2}r^2 - \ell^2 = \frac{e^4}{c^4}(\lambda - \lambda_0)^2$$

OR

$$r^2 = \frac{e^2}{c^2}(\lambda - \lambda_0)^2 + \frac{c^2\ell^2}{e^2} = \frac{e^2}{c^2}\left[(\lambda - \lambda_0)^2 + \frac{c^4\ell^2}{e^4}\right]$$

SO

(*)

$$r(\lambda) = \frac{e}{c}\left[(\lambda - \lambda_0)^2 + \frac{c^4\ell^2}{e^4}\right]^{1/2}$$

d) EQN (19) YIELDED : $\frac{d\varphi}{d\lambda} = \frac{\ell}{r^2}$. NOW USING (*)...

$$r^2 = \frac{e^2}{c^2}\left[(\lambda - \lambda_0)^2 + \frac{c^4\ell^2}{e^4}\right]$$

SO

$$\frac{d\varphi}{d\lambda} = \frac{\ell}{\frac{e^2}{c^2}\left[(\lambda - \lambda_0)^2 + \frac{c^4\ell^2}{e^4}\right]}$$

SO SEPARATING VARIABLES YIELDS ..

$$\int d\varphi = \frac{\ell c^2}{e^2}\int \frac{d\lambda}{\left[(\lambda - \lambda_0)^2 + \frac{c^4\ell^2}{e^4}\right]} = K\int \frac{dx}{[x^2 + K^2]} = K\int \frac{K\sec^2\theta\, d\theta}{K^2(1 + \tan^2\theta)} = \int d\theta$$

let $x = \lambda - \lambda_0$

$dx = d\lambda$

$K \equiv \frac{\ell c^2}{e^2}$

let $x = K\tan\theta$

$dx = K\sec^2\theta\, d\theta$

SO

$$\theta = \tan^{-1}\left(\frac{x}{K}\right) = \tan^{-1}\left(\frac{\lambda - \lambda_0}{K}\right)$$

NOW $\sin^2\theta + \cos^2\theta = 1$

SO $\tan^2\theta + 1 = \sec^2\theta$

SO

$$\varphi(\lambda) = \varphi_0 + \tan^{-1}\left(\frac{\lambda - \lambda_0}{K}\right) = \varphi_0 + \tan^{-1}\left(\frac{e^2}{\ell c^2}(\lambda - \lambda_0)\right)$$