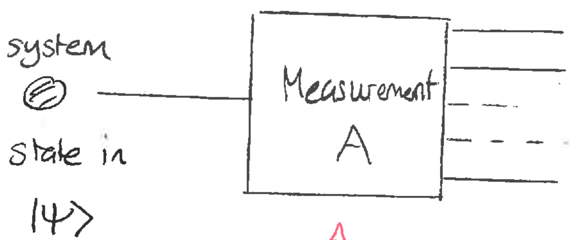
Thurs: Seminar

Tues: HW

General quantum measurements + states.

In general the relationship between states + measurements for quantum systems is described by:



	measurement outcomes	associated states
	a_1	$\|\varphi_1\rangle$
	a_2	$\|\varphi_2\rangle$
	$\vdots$	$\vdots$
	a_n	$\|\varphi_n\rangle$

specify state: e.g $|+\hat{x}\rangle$ or $|\nearrow\rangle$

specify choice e.g. $SG\hat{y}$, $SG\hat{n}$, PBS, ...

number of outcomes and values depend on physical situation

e.g. $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$

The crucial rules are:

1) the meaning of associated state is:

> If the system is in state $|\varphi_k\rangle$ prior to the measurement then the measurement will yield a_k with certainty

e.g. $|+\hat{x}\rangle$ yields $+\hbar/2$ with certainty if $SG\hat{x}$ meas. is done

$|L\rangle$ yields left circular " " if L/R pol meas is done

2) after the measurement the following is true:

> If the measurement yielded outcome a_k then the state of the system after measurement is $|\varphi_k\rangle$

If SG$\hat{y}$ yields $-\hbar/2$ then state is $|-\hat{y}\rangle$ after

If PBS yields reflection " " after is $|\leftrightarrow\rangle$.

3) For a given system there are many possible choices of measurement. But

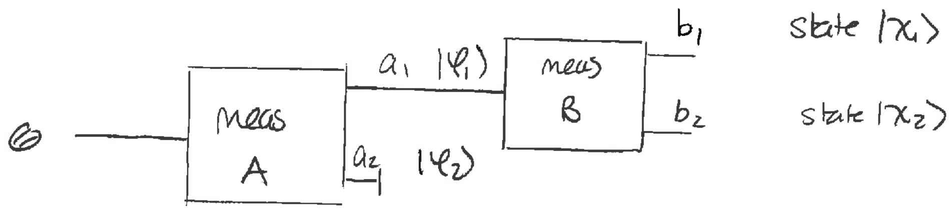
> For any given measurement the associated states are orthonormal and form a basis for all possible states. So
>
> $$\langle \varphi_i | \varphi_j \rangle = \delta_{ij} = \begin{cases} 0 & i=j \\ 1 & i \neq j \end{cases}$$

For SG$\hat{x}$ the states are $\{|+\hat{x}\rangle, |-\hat{x}\rangle\}$. These are orthonormal and constitute a basis.

4) The key computational rule regards measurement outcome probabilities.

> If the state prior to measurement is $|\psi\rangle$ and the associated states and outcomes are as listed then
>
> $$\text{Prob}(\text{outcome } a_k) = |\langle \varphi_k | \psi \rangle|^2$$

The remaining task is to relate states pertaining to different measurements. This will enable us to address questions like



Then if A yields a_1 the probability that B yields b_1 is

$$\text{Prob}(b_1) = |\langle \chi_1 | \varphi_1 \rangle|^2$$

given a_1

We need to be able to express $|\varphi_1\rangle$ and $|\chi_1\rangle$ in terms of a common basis to do the calculation. Examples are:

a) spin-½

$$|+\hat{n}\rangle = \cos\theta/2 \; |+\hat{z}\rangle + e^{i\phi} \sin\frac{\theta}{2} \; |-\hat{z}\rangle$$

↳ choice on $\hat{n}$ gives θ, ϕ

b) polarization for conventional polarization state given by

$$\vec{E}_0 = E_{0x}\,\hat{x} + e^{i\phi} E_{0y}\,\hat{y}$$

state is

$$|\psi\rangle = \left[E_{0x} \,|\rightarrow\rangle + e^{i\phi} E_{0y} \,|\uparrow\rangle \right] \Big/ \sqrt{E_{0x}^2 + E_{0y}^2}$$

These rules are driven by physical observations. Their justifications are that, eventually they yield correct probabilities attained in experiments

Describing measurements

Although we usually describe measurements in terms of physical quantities, we can also resort to the following mathematical description.

With every measurement there is an associated orthonormal basis. Thus we can refer to the measurement in terms of the basis. So:

SG$\hat{z}$	measurement	$\equiv$	measurement	in	basis	$\{\|+\hat{z}\rangle, \|-\hat{z}\rangle\}$
SG$\hat{x}$	"	$\equiv$	"	"	"	$\{\|+\hat{x}\rangle, \|-\hat{x}\rangle\}$
PBS	"	$\equiv$	"	"	"	$\{\|\rightarrow\rangle, \|\uparrow\rangle\}$

We then just need to associate outcomes with each measurement basis element. So generically we can say:

"Measure in basis $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots |\varphi_n\rangle\}$"

We leave the actual physical construction of the measurement for later but in the meanwhile we can compute probabilities.

Generic description of two-state systems.

A two state quantum system is one for which the most precise choice / choices of measurement yield one of two possible outcomes.

A spin-$1/2$ system is such a system because any SG$\hat{n}$ measurement yields one of two possible outcomes.

The photon polarization is another such system. Any polarization measurement yields one of two outcomes

In quantum information such a two state system is called a <u>qubit</u>. There are many physical examples (including systems for which there are measurements that yield more than two outcomes but are managed so that they only ever involve two states and their superpositions).

We will use a generic notation for qubits as follows

	Example spin 1/2	Polarization PBS
Fix a particular measurement choice / measurement basis.	$\{\lvert +\hat{z}\rangle, \lvert -\hat{z}\rangle\}$	
↓ Denote the two associated states by $\lvert 0\rangle, \lvert 1\rangle$ "computational basis"	$\lvert +z\rangle \equiv \lvert 0\rangle$ $\lvert -z\rangle \equiv \lvert 1\rangle$	$\lvert \rightarrow\rangle \equiv \lvert 0\rangle$ $\lvert \uparrow\rangle \equiv \lvert 1\rangle$
↓ Denote the measurement outcomes by "0", "1"	$S_z = +\hbar/2 \equiv 0$ $S_z = -\hbar/2 \equiv 1$	$Pol = H \equiv 0$ $Pol = V \equiv 1$
↓ A generic state is $\lvert\psi\rangle = a_0\lvert 0\rangle + a_1\lvert 1\rangle$		

Calculations are done with the bra/ket scheme and the orthonormality rules:

$$\langle 0|0\rangle = 1$$

$$\langle 0|1\rangle = \langle 1|0\rangle = 0$$

$$\langle 1|1\rangle = 1$$

In terms of column and row vectors, the following is standard

$$|0\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\langle 0| \leftrightarrow \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\langle 1| \leftrightarrow \begin{pmatrix} 0 & 1 \end{pmatrix}$$

and thus

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \leftrightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

1 Generic qubit measurements

A qubit is subjected to two measurements. First the qubit is measured in the basis

$$\left\{\frac{1}{\sqrt{2}}|0\rangle+\frac{i}{\sqrt{2}}|1\rangle\,,\frac{1}{\sqrt{2}}|0\rangle-\frac{i}{\sqrt{2}}|1\rangle\right\}$$

and the result is recorded. The qubit is then measured in the basis

$$\left\{\frac{3i}{5}|0\rangle+\frac{4}{5}|1\rangle\,,-\frac{4i}{5}|0\rangle+\frac{3}{5}|1\rangle\right\}.$$

and the result is recorded. Prior to the first measurement the state of the qubit is $|1\rangle$.

a) Determine the probability with which the first measurement yields either outcome.

b) Each measurement yields one of two possible outcomes. How many combinations of outcomes does the pair of measurements yield? Determine the probabilities for all possible outcomes of this entire situation.

Answer: a) Call the first measurement A and denote the outcomes and states:

outcome	state
a_1	$\lvert\varphi_1\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle + i\lvert 1\rangle)$
a_2	$\lvert\varphi_2\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle - i\lvert 1\rangle)$

$$\text{Prob}(a_1) = |\langle\varphi_1|\psi\rangle|^2 \qquad |\psi\rangle = |0\rangle$$

$$\langle\varphi_1|1\rangle = \frac{i}{\sqrt{2}} \quad \Rightarrow \quad \text{Prob}(a_1) = 1/2$$

$$\text{Prob}(a_2) = |\langle\varphi_2|\psi\rangle|^2$$

$$\langle\varphi_2|1\rangle = -\frac{i}{\sqrt{2}} \quad \Rightarrow \quad \text{Prob}(a_2) = 1/2$$

b) Label measurement via

outcome	state
b_1	$\lvert\chi_1\rangle = \frac{3i}{5}\lvert 0\rangle + \frac{4}{5}\lvert 1\rangle$
b_2	$\lvert\chi_2\rangle = -\frac{4i}{5}\lvert 0\rangle + \frac{3}{5}\lvert 1\rangle$

There are four possible combinations a_1b_1, a_1b_2, a_2b_1, a_2b_2

Suppose a_1 occured. Then state after A is $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ so

$$\text{Prob}(b_1|a_1) = |\langle \chi_1|\varphi_1\rangle|^2$$

"b_1 given a_1"

and $$\langle \chi_1|\varphi_1\rangle = -\frac{3i}{5}\frac{1}{\sqrt{2}} + \frac{4i}{5}\frac{1}{\sqrt{2}} = \frac{i}{5\sqrt{2}}$$

$$|\langle \chi_1|\varphi_1\rangle|^2 = \frac{1}{50} \quad \Rightarrow \quad \text{Prob}(b_1|a_1) = \frac{1}{50}$$

$$\text{Prob}(b_2|a_1) = |\langle \chi_2|\varphi_1\rangle|^2$$

and $$\langle \chi_2|\varphi_1\rangle = +\frac{4i}{5}\frac{1}{\sqrt{2}} + \frac{3i}{5\sqrt{2}} = \frac{7i}{5\sqrt{2}}$$

$$|\langle \chi_2|\varphi_1\rangle|^2 = \frac{49}{50} \quad \Rightarrow \quad \text{Prob}(b_2|a_1) = \frac{49}{50}$$

Suppose a_2 occured. State after is $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$$\text{Prob}(b_1|a_2) = |\langle \chi_1|\varphi_2\rangle|^2$$

$$= \left|-\frac{3i}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}}\right|^2 = \frac{49}{50}$$

$$\text{Prob}(b_2|a_2) = |\langle \chi_2|\varphi_2\rangle|$$

$$= \left|\frac{4i}{5}\frac{1}{\sqrt{2}} - \frac{3i}{5\sqrt{2}}\right|^2 = \frac{1}{50}$$

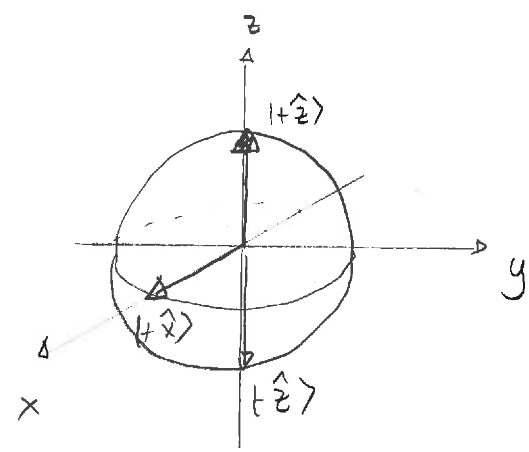
So

A	B	Total prob
a_1	b_1	1/100
a_1	b_2	49/100
a_2	b_1	49/100
a_2	b_2	1/100

Bloch sphere

A convenient representation of qubit states refers to spin-½ particles. Any spin-½ particle state can be represented as $|+\hat{n}\rangle$ for some vector $\hat{n}$. All such vectors can be represented on a unit sphere. We attach spin-½ kets to this.

But we have



$$|+\hat{z}\rangle \equiv |0\rangle$$

$$|-\hat{z}\rangle \equiv |1\rangle$$

and

$$|+\hat{x}\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$$

$$|-\hat{x}\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)$$

etc. So we can represent qubits using

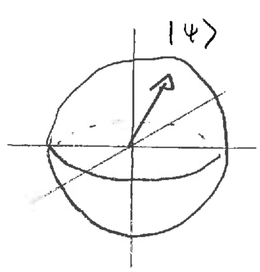
$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$ ⟶ consider $|\psi\rangle = a_0|+\hat{z}\rangle + a_1|-\hat{z}\rangle$

find vector $\hat{n}$ s.t

$$|+\hat{n}\rangle = a_0|+\hat{z}\rangle + a_1|-\hat{z}\rangle$$

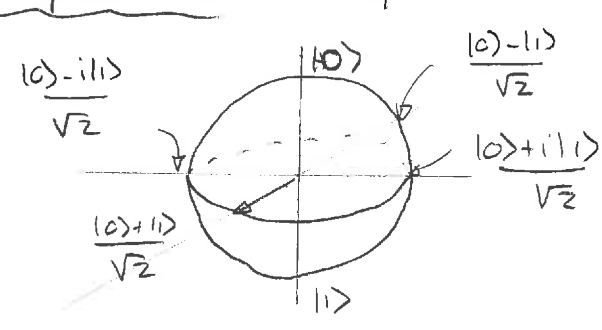
- ignore global phase

then



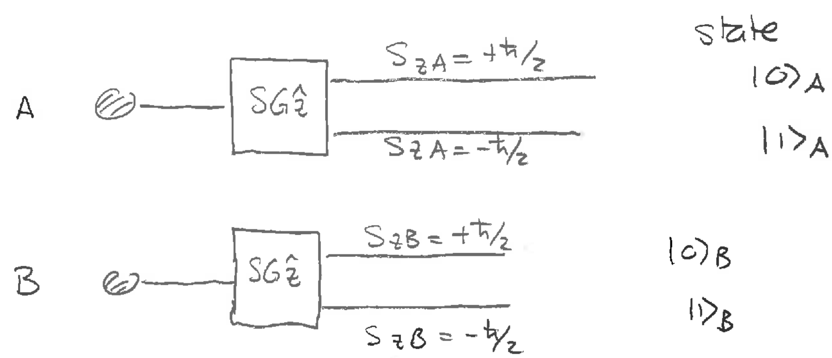
This is the <u>Bloch-sphere representation</u> of qubit states

This will eventually be useful for single qubit dynamics



Multiple quantum systems / qubits

In general we will need to consider multiple qubits. For example, suppose we have two distinct qubits. We could label these A and B. We can subject each of these independently to measurements. For spin-½ systems an example would be



There are clearly four combinations of measurement outcomes and there must be four states that correspond to each pair of outcomes:

S_{zA}	S_{zB}	Associated state
$+\hbar/2$	$+\hbar/2$	$\|0\rangle_A \|0\rangle_B$
$+\hbar/2$	$-\hbar/2$	$\|0\rangle_A \|1\rangle_B$
$-\hbar/2$	$+\hbar/2$	$\|1\rangle_B \|0\rangle_A$
$-\hbar/2$	$-\hbar/2$	$\|1\rangle_B \|1\rangle_A$

We can start by writing the states adjacent to each other. However, we will need to give these a vector structure. This involves the "tensor product of two vector spaces."

The scheme is:

V is a vector space with dimension m and basis $|\alpha_1\rangle, |\alpha_2\rangle, \dots |\alpha_m\rangle$

W is a vector space with dimension n and basis $|\beta_1\rangle, |\beta_2\rangle, \dots, |\beta_m\rangle$

The tensor product $V \otimes W$ is a vector space with dimension mn and a set of basis vectors:

$$|\alpha_i\rangle|\beta_j\rangle$$

for $i = 1, \dots, m$ and $j = 1, \dots n$

Notes:

1) the order of the products matters, for $V \otimes W$ we always have

$$\underbrace{|\ \rangle}_{\text{from } V}\underbrace{|\ \rangle}_{\text{from } W}$$

2) the tensor product of two basis vectors is

$$|\alpha_i\rangle \otimes |\beta_j\rangle \equiv |\alpha_i\rangle|\beta_j\rangle$$

This establishes a vector space with dimension mn.
Additionally the tensor product is related to the underlying vector spaces via:

If $|v_1\rangle, |v_2\rangle \in V$ and $|w_1\rangle, |w_2\rangle \in W$ then for all numbers a, b,

1) $(a|v_1\rangle + b|v_2\rangle) \otimes |w_1\rangle = a|v_1\rangle \otimes |w_1\rangle + b|v_2\rangle \otimes |w_1\rangle$

2) $|v_1\rangle \otimes (a|w_1\rangle + b|w_2\rangle) = a|v_1\rangle \otimes |w_1\rangle + b|v_1\rangle \otimes |w_2\rangle$

Note that the order of multiplication cannot be reversed because this would reverse the original vector spaces. So

$$\underset{\text{from } V}{|v\rangle} \otimes \underset{\text{from } W}{|w\rangle} \neq \underset{\text{from } W}{|w\rangle} \otimes \underset{\text{from } V}{|v\rangle}$$

Thus we insist that the states that represent two qubits are elements of the tensor product of the two ket spaces. Generically any ket in this space can be represented as a superposition of $|0\rangle_A|0\rangle_B, |0\rangle_A|1\rangle_B, \ldots$

$$|0\rangle_A|0\rangle_B \equiv |0\rangle_A \otimes |0\rangle_B$$

2 Tensor products of qubit states

Consider the following states for a single qubit:

$$
\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \\
|\phi_3\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \\
|\phi_4\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle .
\end{aligned}
$$

Express the following tensor products in terms of $|0\rangle|0\rangle, \ldots$.

a) $|\phi_1\rangle \otimes |\phi_1\rangle$
b) $|\phi_1\rangle \otimes |\phi_2\rangle$
c) $|\phi_2\rangle \otimes |\phi_2\rangle$
d) $|\phi_1\rangle \otimes |\phi_3\rangle$
e) $|\phi_3\rangle \otimes |\phi_3\rangle$

Answer: a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$= \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$$

b) Similarly

$$\frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right)$$

c) $\frac{1}{2}\left(|0\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle + |1\rangle|1\rangle\right)$

d) $\frac{1}{2}\left(|0\rangle|0\rangle + i|0\rangle|1\rangle + |1\rangle|0\rangle + i|1\rangle|1\rangle\right)$

e) $\frac{1}{2}\left(|0\rangle|0\rangle - i|0\rangle|1\rangle + |1\rangle|0\rangle - i|1\rangle|1\rangle\right)$