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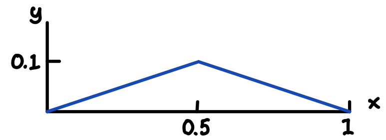
MATH 360
HW 12.3

12.3 #9

Problem 12.3.9



$x: 0 \le x \le 0.5 \quad f(x) = 0.2x$

$x: 0.5 \le x \le 1.0 \quad f(x) = -0.2x + 0.2$

$L=1,\ c^2=1,\ g(x)=0$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \ , \quad B_n^* = \frac{2}{Cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$L=1$

$$B_n^* \frac{2}{Cn\pi}\int_0^L 0\cdot\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

$$u(x,t) = \cos(n\pi t)\sin(n\pi x)$$

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\left[\underset{①}{\int_0^{0.5}} 0.2x\cdot\sin(n\pi x)\,dx + \underset{②}{\int_{0.5}^{1.0}} 0.2(1-x)\cdot\sin(n\pi x)\,dx\right]$$

$①^* \quad 0.2\int_0^{0.5} x\cdot\sin(n\pi x)\,dx$

$u = x \quad du = 1\,dx$

$v' = \sin(n\pi x)\,dx \quad v = -\frac{\cos(n\pi x)}{(n\pi)}$

$$0.2\left[-\frac{x\cdot\cos(n\pi x)}{(n\pi)} + \int\frac{\cos(n\pi x)}{(n\pi)}dx\right]$$

$$0.2\left[\frac{-x\cdot\cos(n\pi x)}{(n\pi)} + \frac{\sin(n\pi x)}{(n\pi)^2}\right]_0^{0.5} = 0.2\left[\frac{\sin(n\frac{\pi}{2})}{(n\pi)^2} - \frac{\cos(n\frac{\pi}{2})}{2n\pi}\right] \equiv c^*$$

$$①^* = 0.2\left[\frac{\sin(n\frac{\pi}{2})}{(n\pi)^2} - \frac{\cos(n\frac{\pi}{2})}{2(n\pi)}\right]$$

$②^* \quad 0.2\int_{0.5}^{1.0}(1-x)\cdot\sin(n\pi x)\,dx$

$u = (1-x),\ du = -dx,\ v' = \sin(n\pi x)\,dx,\ v = -\frac{\cos(n\pi x)}{n\pi}$

$$0.2\left[(x-1)\frac{\cos(n\pi x)}{n\pi}\Bigg|_{0.5}^{1.0} - \int\frac{\cos(n\pi x)}{n\pi}dx\right] = 0.2\left[\frac{\cos(n\frac{\pi}{2})}{2n\pi} - \frac{\sin(n\pi x)}{(n\pi)^2}\Bigg|_{0.5}^{1.0}\right]$$

$$0.2\left[\frac{\cos(n\frac{\pi}{2})}{2n\pi} - \left[\overset{0}{\frac{\sin(n\pi)}{(n\pi)^2}} - \frac{\sin(n\frac{\pi}{2})}{(n\pi)^2}\right]\right] = 0.2\left[\frac{\cos(n\frac{\pi}{2})}{2n\pi} + \frac{\sin(n\frac{\pi}{2})}{(n\pi)^2}\right]$$

$$②^* = 0.2\left[\frac{\cos(n\frac{\pi}{2})}{2n\pi} + \frac{\sin(n\frac{\pi}{2})}{(n\pi)^2}\right]$$

$$① + ② = 0.2\left[\frac{\sin(n\frac{\pi}{2})}{(n\pi)^2} - \cancel{\frac{\cos(n\frac{\pi}{2})}{2n\pi}} + \cancel{\frac{\cos(n\frac{\pi}{2})}{2n\pi}} + \frac{\sin(n\frac{\pi}{2})}{(n\pi)^2}\right] \qquad ① + ② \equiv ③$$

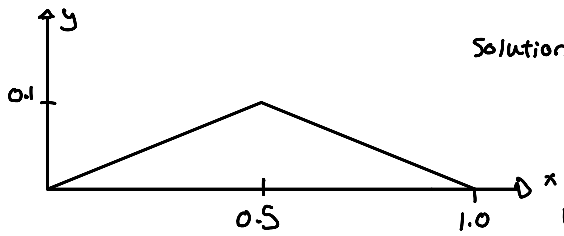
From original integral

$$③ = 0.2\cdot\left[\frac{2\sin(n\frac{\pi}{2})}{(n\pi)^2}\right] = \frac{0.4\cdot\sin(n\frac{\pi}{2})}{(n\pi)^2}\cdot 2 = \frac{4}{5(n\pi)^2}\cdot\sin(n\tfrac{\pi}{2}) = B_n$$

$$\boxed{u(x,t) = \frac{4}{5\pi^2}\left(\cos(\pi t)\sin(\pi x) - \frac{\cos(3\pi t)\sin(3\pi x)}{9} + \frac{\cos(5\pi t)\sin(5\pi x)}{25} - + \ldots\right)}$$

Problem 12.3.9

$L=1,\ c^2=1,\ V_0=0,\ k=0.01$



Solution to ODE : $u(x,t)=\sum_{n=1}^{\infty}\left[B_n\cos\lambda_n t + B_n^{*}\sin\lambda_n t\right]\sin\left(\frac{n\pi x}{L}\right)$

$B_n^{*}=0 \quad , \quad \lambda_n=\frac{cn\pi}{L}$

$u(x,t)=\sum_{n=1}^{\infty} B_n\cos\lambda_n t\cdot\sin\left(\frac{n\pi}{L}x\right), \quad B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

$B_n:\ 0\le x\le 0.5\ :\ f(x)=0.2x\ :\ \mathrm{I}$

$\quad\ \ 0.5\le x\le 1.0\ :\ f(x)=0.2(1-x)\ :\ \mathrm{II}$

$$\mathrm{I}:\quad B_n=2\left[\int_0^{0.5}0.2x\cdot\sin(n\pi x)\,dx = 0.2\int_0^{0.5}x\cdot\sin(n\pi x)\,dx\right]$$

$u=x\ ,\ du=dx$

$v'=\sin(n\pi x)\,dx,\ \ v=\frac{-\cos(n\pi x)}{n\pi}$ $\quad :\left[\frac{-x\cdot\cos(n\pi x)}{n\pi}+\frac{1}{n\pi}\int\cos(n\pi x)\,dx\right]\cdot 0.2$

$$0.2\left[\frac{-x\cdot\cos(n\pi x)}{n\pi}+\frac{1}{(n\pi)^2}\cdot\sin(n\pi x)\right]_0^{0.5}=0.2\left[\frac{-0.5\cos(\frac{n\pi}{2})}{n\pi}+\frac{\sin(\frac{n\pi}{2})}{(n\pi)^2}\right]$$

$$\mathrm{I}=0.2\left[\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}-\frac{0.5\cos(0.5\cdot n\pi)}{n\pi}\right]\cdot 2$$

$$\mathrm{II}:\ B_n=2\left[0.2\int_{0.5}^{1.0}(1-x)\sin(n\pi x)\,dx\right]$$

$u=(1-x)\ ,\ du=-dx$

$v'=\sin(n\pi x)\,dx,\ \ v=-\frac{\cos(n\pi x)}{n\pi}$ $\quad : 0.2\left[\frac{\cos(n\pi x)(x-1)}{n\pi}-\frac{1}{n\pi}\int\cos(n\pi x)\,dx\right]_{0.5}^{1.0}$

$$0.2\left[\frac{\cos(n\pi x)}{(n\pi)}\cdot(x-1)\Big|_{0.5}^{1.0}-\frac{\sin(n\pi x)}{(n\pi)^2}\Big|_{0.5}^{1.0}\right]=0.2\left[\overset{0}{\cancel{\frac{\cos(n\pi)}{(n\pi)}(1-1)}}-\frac{\cos(0.5\cdot n\pi)(-0.5)}{(n\pi)}\right. \longrightarrow$$

$$\left.-\cancel{\frac{\sin(n\pi)}{(n\pi)^2}}+\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}\right]=0.2\cdot\left[\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}+\frac{0.5\cdot\cos(0.5\cdot n\pi)}{(n\pi)}\right]$$

$$\mathrm{II}=0.2\cdot\left[\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}+\frac{0.5\cdot\cos(0.5\cdot n\pi)}{(n\pi)}\right]\cdot 2$$

$$\mathrm{I}+\mathrm{II}=0.4\left[\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}-\cancel{\frac{0.5\cos(0.5\cdot n\pi)}{n\pi}}+\frac{\sin(0.5\cdot n\pi)}{(n\pi)^2}+\cancel{\frac{0.5\cdot\cos(0.5\cdot n\pi)}{(n\pi)}}\right]$$

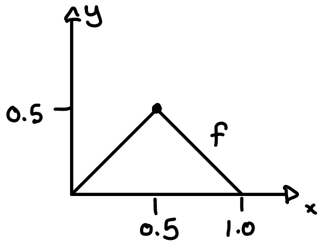
$$\mathrm{I}+\mathrm{II}=0.4\left[\frac{2\cdot\sin(\frac{n\pi}{2})}{(n\pi)^2}\right]=\frac{4}{5}\left[\frac{\sin(\frac{n\pi}{2})}{(n\pi)^2}\right]\ :\quad \sin\left(\frac{n\pi}{2}\right)=\begin{cases}(-1)^k, & n=2k+1\\ 0, & n=2k\end{cases}$$

$B_n=\frac{4}{5}\cdot\frac{1}{(n\pi)^2}\cdot(-1)^k$ $\quad\therefore$

$$\boxed{u(x,t)=\frac{4}{5}\left[\frac{\cos(\pi t)\sin(\pi x)}{(\pi^2)}-\frac{\cos(3\pi t)\sin(3\pi x)}{(9\pi^2)}+-\cdots\right]}$$

Example

$L=1\ ,\ c^2=1\ ,\ V_0=0$



$$u(x,t)=\sum_{n=1}^{\infty}\left[B_n\cos(\lambda_n t)+B_n^{*}\sin(\lambda_n t)\right]\sin\left(\frac{n\pi}{L}x\right)\ ,\quad \lambda_n=\frac{cn\pi}{L}$$

$V_0=0 \quad \therefore \quad B_n^{*}=0$

$$F(x):\begin{cases} x & : \ 0\le x\le 0.5 \\ 1-x & : \ 0.5\le x\le 1.0 \end{cases}$$

$$B_n:\quad B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$B_n=2\left[\int_0^{0.5} x\sin(n\pi x)\,dx+\int_{0.5}^{1.0}(1-x)\sin(n\pi x)\,dx\right]$$

$$① = \int_0^{0.5} x\sin(n\pi x)\,dx \longrightarrow u=x : du=dx : v'=\sin(n\pi x)\,dx : \ v=-\frac{\cos(n\pi x)}{(n\pi)}$$

$$\left.\frac{-x\cos(n\pi x)}{(n\pi)}\right|_0^{0.5}+\frac{1}{(n\pi)}\int_0^{0.5}\cos(n\pi x)\,dx=\frac{-0.5\cos(0.5n\pi)}{(n\pi)}+\frac{1}{(n\pi)}\left[\frac{\sin(n\pi x)}{(n\pi)}\right]_0^{0.5}$$

$$① = \frac{-0.5\cos\left(\frac{n\pi}{2}\right)}{(n\pi)}+\frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}$$

$$② \int_{0.5}^{1.0}(1-x)\sin(n\pi x)\,dx \longrightarrow u=(1-x) : du=-dx : v'=\sin(n\pi x)\,dx \quad v=-\frac{\cos(n\pi x)}{(n\pi)}$$

$$\left.\frac{(x-1)\cos(n\pi x)}{(n\pi)}\right|_{0.5}^{1.0}-\frac{1}{(n\pi)}\int_{0.5}^{1.0}\cos(n\pi x)\,dx=\frac{0.5\cos\left(\frac{n\pi}{2}\right)}{(n\pi)}-\frac{1}{(n\pi)^2}\Big[\sin(n\pi x)\Big]_{0.5}^{1.0}$$

$$② = \frac{0.5\cos\left(\frac{n\pi}{2}\right)}{(n\pi)}+\frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}$$

$$① + ② = \cancel{\frac{-0.5\cos\left(\frac{n\pi}{2}\right)}{(n\pi)}}+\frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}+\cancel{\frac{0.5\cos\left(\frac{n\pi}{2}\right)}{(n\pi)}}+\frac{\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}=\frac{2\cdot\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}$$

$$\therefore\quad B_n=\frac{4\cdot\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}\quad : \quad B_n=\begin{cases}\frac{4\cdot(-1)^k}{(n\pi)^2} & \text{iff } n=2k+1 \\ 0 & \text{iff } n=2k\end{cases}\qquad \lambda_n=(n\pi)$$

$$u(x,t)=\sum_{n=1}^{\infty}\left[B_n\cos(\lambda_n t)\right]\sin\left(\frac{n\pi}{L}x\right)\quad : \quad \boxed{u(x,t)=\frac{4}{\pi^2}\sum_{n=2k+1}^{\infty}\frac{(-1)^k}{n^2}\cos(n\pi t)\sin(n\pi x)}$$