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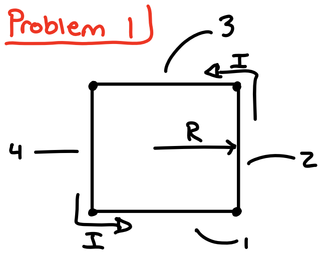
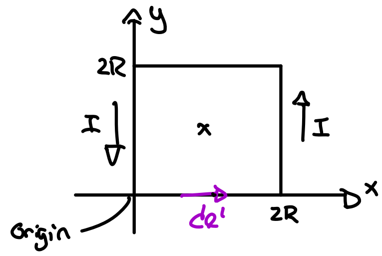
PHYS 311
HW 10

## Problem 1




$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$




①: $\vec{\mathscr{r}}_1 = \vec{r}_1 - \vec{r}_1'$ : $\vec{r}_1 = R\hat{x} + R\hat{y}$ , $\vec{r}_1' = x'\hat{x}$ , $d\vec{l}' = dx'\hat{x}$

$$\vec{\mathscr{r}}_1 = (R,R) - (x',0) = (R-x')\hat{x} + (R)\hat{y} \,, \quad \mathscr{r}^2 = (R-x')^2 + (R)^2 \qquad \hat{\mathscr{r}}_1 = \frac{(R-x')\hat{x} + (R)\hat{y}}{\sqrt{(R-x')^2 + (R)^2}}$$

$$d\vec{l}' \times \hat{\mathscr{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{(R-x')}{\sqrt{(R-x')^2+(R)^2}} & \frac{R}{\sqrt{(R-x')^2+(R)^2}} & 0 \end{vmatrix} = \hat{z}\left(\frac{R}{\sqrt{(R-x')^2+(R)^2}^{\,2}}\, dx'\right)$$

$(R-x') = u$

$$\vec{B}(\vec{r}_1) = \frac{\mu_0}{4\pi} I \cdot R \cdot \hat{z} \int_0^{2R} \frac{1}{((R-x')^2+(R)^2)^{3/2}}\, dx' = \frac{\mu_0 I}{4\pi} \cdot R \cdot \hat{z} \left[ \frac{x'-R}{\sqrt{x'^2 - 2x'R + 2R^2} \cdot R^2} \right]_0^{2R}$$

$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi} \cdot R \cdot \hat{z} \left[ \frac{R}{\sqrt{4R^2 - 4R^2 + 2R^2} \cdot R^2} - \left( \frac{-R}{\sqrt{2R^2} \cdot R^2} \right) \right] = \frac{\mu_0 I}{4\pi} R \left[ \frac{1}{\sqrt{2}R^2} + \frac{1}{\sqrt{2}R^2} \right] \hat{z}$$

$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi} \cdot R \left[ \frac{\sqrt{2}}{R^2} \right] \hat{z} = \frac{\mu_0 I}{4\pi} \cdot \frac{\sqrt{2}}{R} \hat{z} \quad : \quad \vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi} \cdot \frac{\sqrt{2}}{R} \hat{z} \longleftarrow (1^*)$$

The result of (1*) can be used for the other three sides of the square since the current flows in the same direction such that the magnetic field will be in the same direction for all four sides. The sides of the square are also all symmetric such that all the magnitudes will be the same. Thus we can say,

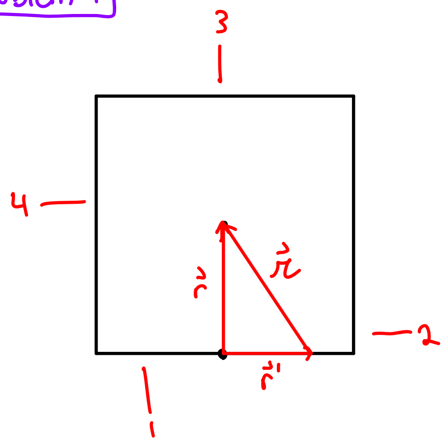
$$\vec{B}(\vec{r}_1) = \vec{B}(\vec{r}_2) = \vec{B}(\vec{r}_3) = \vec{B}(\vec{r}_4)$$

with

$$\vec{B}_{net}(\vec{r}') = 4\vec{B}(\vec{r}_1) = \cancel{4} \cdot \frac{\mu_0 I}{\cancel{4}\pi} \cdot \frac{\sqrt{2}}{R} \hat{z} = \frac{\mu_0 I}{\pi} \cdot \frac{\sqrt{2}}{R} \hat{z}$$

$$\boxed{\vec{B}(\vec{r}') = \frac{\mu_0 I}{\pi} \cdot \frac{\sqrt{2}}{R} \hat{z}}$$

## Problem 1




$\vec r = R\hat y$ , $\vec r' = x'\hat x$ , $\vec{\mathscr{r}} = \vec r - \vec r' = R\hat y - x'\hat x$

$|\vec{\mathscr{r}}| = \sqrt{R^2+x'^2}$ , $\mathscr{r}^2 = R^2+x'^2$ , $\hat{\mathscr{r}} = \dfrac{-x'\hat x + R\hat y}{\sqrt{R^2+x'^2}}$

$$\vec B(\vec r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec l' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

loop 1: $\vec B(\vec r) = \dfrac{\mu_0 I}{4\pi}\displaystyle\int_{-\frac{R}{2}}^{\frac{R}{2}}\left(\frac{dx' \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right)$ : $dl' \times \hat{\mathscr{r}} = \begin{vmatrix} \hat x & \hat y & \hat z \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{R^2+x'^2}} & \frac{R}{\sqrt{R^2+x'^2}} & 0 \end{vmatrix}$

$$dl' \times \hat{\mathscr{r}}' = \frac{R}{\sqrt{R^2+x'^2}}\, dx' \ (\hat z)$$

$\vec B_1(\vec r_1') = \dfrac{\mu_0 I}{4\pi}\displaystyle\int_{-R}^{R}\frac{R}{(R^2+x'^2)^{3/2}}\, dx' \ (\hat z)$ $\quad x' = R\cdot\tan\theta$ , $dx' = R\cdot\sec^2\theta\, d\theta$

$x'^2 = R^2\cdot\tan^2\theta$

$$= \frac{\mu_0 I}{4\pi}\int_{-R}^{R}\frac{R^2\cdot\sec^2\theta}{(R^2+R^2\tan^2\theta)^{3/2}}\, d\theta = \frac{\mu_0 I}{4\pi}\int_{-R}^{R}\frac{R^2\cdot\sec^2\theta}{(R^2\cdot\sec^2\theta)^{3/2}}\, d\theta = \frac{\mu_0 I}{4\pi}\int_{-R}^{R}\frac{R^2\cdot\sec^2\theta}{R^3\cdot\sec^3\theta}\, d\theta$$

$$= \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\int_{-R}^{R}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\sin\theta \Big|_{\theta = \tan^{-1}\left(\frac{x'}{R}\right)}\right]_{-R}^{R}$$

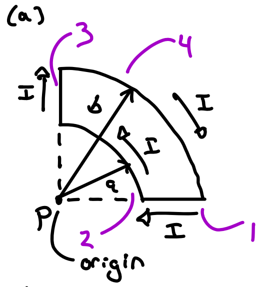
$$= \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{x'}{\sqrt{R^2+x'^2}}\right]_{-R}^{R} = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{\cancel{R}}{\sqrt{2}\cdot\cancel{R}} + \frac{\cancel{R}}{\sqrt{2}\cdot\cancel{R}}\right] = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{2}{\sqrt{2}}\right]$$

$= \dfrac{\mu_0 I}{4\pi}\cdot\dfrac{\sqrt{2}}{R}$ : $\vec B_1(\vec r') = \dfrac{\mu_0 I}{4\pi}\cdot\dfrac{\sqrt{2}}{R}\ (\hat z)$

4 sides, so $\boxed{\vec B_{net}(\vec r') = \dfrac{\mu_0 I}{\pi}\cdot\dfrac{\sqrt{2}}{R}\ (\hat z)}$

Problem 2)

(a)




$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$\vec{B}(\vec{r}_1'):\ \vec{\mathscr{r}}_1 = \vec{r}_1 - \vec{r}_1' :\ \vec{r}_1 = 0\hat{s} + 0\hat{\phi} \quad \vec{r}_1' = (b-s')\hat{s} - (0)\phi\ ,\ d\vec{l}' = -ds'\,\hat{s}$

$\vec{\mathscr{r}}_1 = (0-(b-s'))\hat{s} + (0-0)\hat{\phi} = (s'-b)\hat{s} \quad \therefore \quad \hat{\mathscr{r}}_1 = \dfrac{(s'-b)\hat{s}}{(s'-b)} = \hat{s}$

$|\vec{\mathscr{r}}_1| = (s'-b)$

$$d\vec{l}' \times \hat{\mathscr{r}}_1 = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ -ds' & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0\hat{s} + 0\hat{\phi} + 0\hat{z} \quad \therefore \quad \boxed{\vec{B}(\vec{r}_1') = 0}$$

$\vec{B}(\vec{r}_2'):\ \vec{\mathscr{r}}_2 = \vec{r}_2 - \vec{r}_2' :\ \vec{r}_2 = 0\hat{s} + 0\hat{\phi},\ \vec{r}_2' = (a)\hat{s},\ d\vec{l}' = a\,d\phi\,\hat{\phi}$

$\vec{\mathscr{r}}_2 = (0-a)\hat{s} = (-a)\hat{s} \quad \therefore \quad \hat{\mathscr{r}}_2 = \dfrac{-a}{a}\hat{s} = -\hat{s}$

$|\vec{\mathscr{r}}_2| = a\ ,\ \mathscr{r}_2^2 = a^2$

$$d\vec{l}' \times \hat{\mathscr{r}}_2 = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ 0 & a\,d\phi & 0 \\ -1 & 0 & 0 \end{vmatrix} = a\,d\phi\ \hat{z}$$

$$\vec{B}(\vec{r}_2') = \frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}} \frac{a}{a^2}\, d\phi\, \hat{z} = \frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}} \frac{1}{a}\, d\phi\, \hat{z} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{a}\phi\right]_0^{\frac{\pi}{2}} \hat{z} = \frac{\mu_0 I}{8}\cdot\frac{1}{a}\,\hat{z}$$

$$\boxed{\vec{B}(\vec{r}_2') = \frac{\mu_0 I}{8}\cdot\frac{1}{a}\,\hat{z}}$$

$\vec{B}(\vec{r}_3'):\ \vec{\mathscr{r}}_3 = \vec{r}_3 - \vec{r}_3' :\ \vec{r}_3 = 0\hat{s} + 0\hat{\phi},\ \vec{r}_3' = (a+s')\hat{s},\ d\vec{l} = ds'\,\hat{s}$

$\vec{\mathscr{r}}_3 = (0-(a+s'))\hat{s} + (0-0)\hat{\phi} = -(a+s')\hat{s} \quad \therefore \quad \hat{\mathscr{r}}_3 = \dfrac{(a+s')}{-(a+s')}\hat{s} = -\hat{s}$

$$d\vec{l}' \times \hat{\mathscr{r}}_3 = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ ds' & 0 & 0 \\ -(a+s') & 0 & 0 \end{vmatrix} = 0\hat{s} + 0\hat{\phi} + 0\hat{z} \quad \therefore \quad \boxed{\vec{B}(\vec{r}_3') = 0}$$

$\vec{B}(\vec{r}_4'):\ \vec{\mathscr{r}}_4 = \vec{r}_4 - \vec{r}_4' :\ \vec{r}_4 = 0\hat{s} + 0\hat{\phi}\ ,\ \vec{r}_4' = (b)\hat{s}\ ,\ d\vec{l}' = b\,d\phi\,\hat{\phi}$

$\vec{\mathscr{r}}_4 = (0-b)\hat{s} = -b\hat{s} \quad \therefore \quad \hat{\mathscr{r}}_4 = \dfrac{-b}{b}\hat{s} = -\hat{s}$

$|\vec{\mathscr{r}}_4| = b\ ,\ \mathscr{r}_4^2 = b^2$

$$d\vec{l}' \times \hat{\mathscr{r}}_4 = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ 0 & b\,d\phi & 0 \\ -1 & 0 & 0 \end{vmatrix} = b\,d\phi\ \hat{z}$$

$$\vec{B}(\vec{r}_4') = \frac{\mu_0 I}{4\pi}\int_{\frac{\pi}{2}}^{0} \frac{b}{b^2}\, d\phi\, \hat{z} = \frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}} \frac{1}{b}\, d\phi \cdot -\hat{z} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{b}\phi\right]_0^{\frac{\pi}{2}} \cdot -\hat{z} = \frac{\mu_0 I}{8}\,\frac{1}{b}\cdot -\hat{z}$$

Problem 2) Continued

$$\therefore \vec{B}(\vec{r}\,'_4) = \frac{\mu_0 I}{8b}\cdot\frac{1}{b}\cdot\hat{z}$$

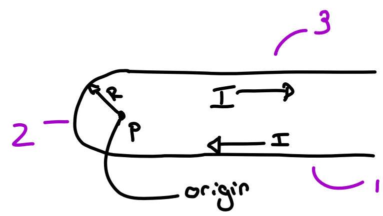
$$\vec{B}(\vec{r}\,')_{net} = \sum_{n=1}^{4} \vec{B}_n(\vec{r}\,') = \cancelto{0}{\vec{B}(\vec{r}\,'_1)} + \vec{B}(\vec{r}\,'_2) + \cancelto{0}{\vec{B}(\vec{r}\,'_3)} + \vec{B}(\vec{r}\,'_4)$$

$$= \frac{\mu_0 I}{8}\cdot\frac{1}{a}\hat{z} + \frac{\mu_0 I}{8}\cdot\frac{1}{b}\cdot\hat{z}$$

$$= \frac{\mu_0 I}{8}\left[\frac{1}{a} - \frac{1}{b}\right]\hat{z}$$

$$\boxed{\vec{B}(\vec{r}\,') = \frac{\mu_0 I}{8}\left[\frac{1}{a}-\frac{1}{b}\right]\hat{z}}$$

(b)

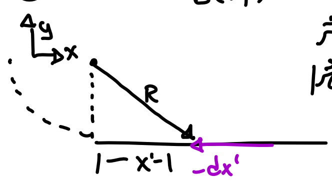



$$\vec{B}(\vec{r}\,') = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

① $\vec{B}(\vec{r}_1)$ : $\vec{\mathscr{r}}_1 = \vec{r}_1 - \vec{r}_1\,'$ : $\vec{r}_1 = 0\hat{x} + 0\hat{y}$ : $\vec{r}_1\,' = x'\hat{x} - R\hat{y}$ : $d\vec{l}\,' = -dx'\,\hat{x}$

$\vec{\mathscr{r}}_1 = (0-x')\hat{x} + (0-(-R))\hat{y} = -x'\hat{x} + R\hat{y}$ $\therefore$ $\hat{\mathscr{r}}_1 = \dfrac{R\hat{y} - x'\hat{x}}{\sqrt{x^2+R^2}}$

$|\vec{\mathscr{r}}_1| = \sqrt{x'^2+R^2}$ , $\mathscr{r}_1^2 = x'^2 + R^2$




$$d\vec{l}\,' \times \hat{\mathscr{r}}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -dx' & 0 & 0 \\ \frac{-x'}{\sqrt{x'^2+R^2}} & \frac{R}{\sqrt{x'^2+R^2}} & 0 \end{vmatrix} = \frac{R}{\sqrt{x'^2+R^2}}\,dx'\,(-\hat{z})$$

- Negative $\hat{z}$-direction

$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi}\cdot -\int_{\infty}^{0} \frac{R}{(x'^2+R^2)^{3/2}}\,dx'\,\hat{z} = \frac{\mu_0 I}{4\pi}\cdot R\cdot\int_0^{\infty}\frac{1}{(x'^2+R^2)^{3/2}}\,dx'\hat{z}$$

Definition

$$\alpha = \frac{\mu_0 I}{4\pi}R$$

$$\vec{B}(\vec{r}_1) = \alpha\int_0^{\infty}\frac{R\cdot\sec^2\theta}{(R^2\tan^2\theta + R^2)^{3/2}}\,d\theta\,\hat{z} = \alpha\int_0^{\infty}\frac{R\cdot\sec^2\theta}{(R^2\sec^2\theta)^{3/2}}\,d\theta\,\hat{z} = \alpha\cdot\int_0^{\infty}\frac{R\cdot\sec^2\theta}{R^3\cdot\sec^3\theta}\,d\theta\,\hat{z}$$

$x' = R\tan\theta$

$dx' = R\cdot\sec^2\theta\,d\theta$

$$\theta = \tan^{-1}\left(\frac{x'}{R}\right) \qquad \sin\left(\tan^{-1}\left(\frac{x'}{R}\right)\right) = \frac{x'}{\sqrt{R^2+x'^2}}$$

$$\vec{B}(\vec{r}_1) = \frac{\alpha}{R^2}\int_0^{\infty}\frac{1}{\sec\theta}\,d\theta\,\hat{z} = \frac{\alpha}{R^2}\Big[\sin\theta\Big]_0^{\infty}\Big|_{\theta=\tan^{-1}\left(\frac{x'}{R}\right)} = \frac{\alpha}{R^2}\left[\sin\left(\tan^{-1}\left(\frac{x'}{R}\right)\right)\right]_0^{\infty}\hat{z}$$

$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi}\cdot\frac{R}{R^2}\cdot\frac{x'}{\sqrt{R^2+x'^2}}\Big|_0^{\infty}\hat{z} = \frac{\mu_0 I}{4\pi}\cdot\frac{x'}{\sqrt{R^2+x'^2}\cdot R}\Big|_0^{\infty}\hat{z} = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\hat{z}$$

$$\boxed{\vec{B}(\vec{r}_1) = \frac{\mu_0 I}{4\pi}\,\frac{1}{R}(-\hat{z})}$$

Problem 2) Continued

② 

$\vec{B}(\vec{r}_2)$: $\vec{\mathscr{r}}_2 = \vec{r}_2 - \vec{r}_2'$ : $\vec{r}_2 = 0\hat{s} + 0\hat{\phi}$ : $\vec{r}_2' = -R\hat{s}$ : $d\vec{l}' = R\,d\phi\,\hat{\phi}$

$\vec{\mathscr{r}}_2 = (0-(-R))\hat{s} = R\hat{s}$ $\quad \therefore \hat{\mathscr{r}}_2 = \frac{R}{R}\hat{s} = \hat{s}$

$|\vec{\mathscr{r}}_2| = R,\ \mathscr{r}_2^2 = R^2$

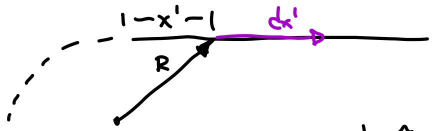
$$d\vec{l}' \times \hat{\mathscr{r}}_2 = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ 0 & R\,d\phi & 0 \\ 1 & 0 & 0 \end{vmatrix} = -R\,d\phi\,\hat{z}$$

- Negative $z$-direction

$$\vec{B}(\vec{r}_2') = \frac{\mu_0 I}{4\pi}\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{-R}{R^2}\,d\phi\,\hat{z} = \frac{\mu_0 I}{4\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{R}\,d\phi\,\hat{z} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{R}\phi\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\hat{z}$$

$$\vec{B}(\vec{r}_2') = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{3\pi}{2} - \frac{\pi}{2}\right]\hat{z} = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{\cancel{2}\pi}{\cancel{2}}\right]\hat{z} = \frac{\mu_0 I}{4}\cdot\frac{1}{R}\,\hat{z}$$

$$\boxed{\vec{B}(\vec{r}_2') = \frac{\mu_0 I}{4}\cdot\frac{1}{R}(-\hat{z})}$$

③ 


$\vec{B}(\vec{r}_3')$: $\vec{\mathscr{r}}_3 = \vec{r}_3 - \vec{r}_3'$ : $\vec{r}_3 = 0\hat{x} + 0\hat{y}$ ; $\vec{r}_3' = x'\hat{x} + R\hat{y}$ : $d\vec{l}' = dx'\,\hat{x}$

$\vec{\mathscr{r}}_3 = (0-x')\hat{x} + (0-R)\hat{y} = -x\hat{x} - R\hat{y}$

$|\vec{\mathscr{r}}_3| = \sqrt{x'^2+R^2},\ \mathscr{r}_3^2 = x'^2+R^2$ $\quad \therefore \hat{\mathscr{r}}_3 = \frac{-x'\hat{x} - R\hat{y}}{\sqrt{x'^2+R^2}}$

$$d\vec{l}' \times \hat{\mathscr{r}}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{x'^2+R^2}} & \frac{-R}{\sqrt{x'^2+R^2}} & 0 \end{vmatrix} = \frac{R}{\sqrt{x'^2+R^2}}\,dx'\,(-\hat{z})$$

- Negative $z$-direction

$$\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}\int_0^\infty \frac{R}{(x'^2+R^2)^{3/2}}\,dx'\,(-\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot R\int_0^\infty \frac{1}{(x'^2+R^2)^{3/2}}\,dx'\,(-\hat{z})$$

$x' = R\cdot\tan\theta$ : $\theta = \tan^{-1}\left(\frac{x'}{R}\right)$

$dx' = R\cdot\sec^2\theta\,d\theta$

$$\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}\cdot R\int_0^\infty \frac{R\cdot\sec^2\theta}{(R^2\tan^2\theta + R^2)^{3/2}}\,d\theta\,(-\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot R^2\int_0^\infty \frac{\sec^2\theta}{(R^2\sec^2\theta)^{3/2}}\,d\theta\,(-\hat{z})$$

$$\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}R^2\int_0^\infty \frac{\sec^2\theta}{R^3\sec^3\theta}\,d\theta\,(-\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\int_0^\infty \frac{1}{\sec\theta}\,d\theta\,(-\hat{z})$$

$\sin\left(\tan^{-1}\left(\frac{x'}{R}\right)\right) = \frac{x'}{\sqrt{R^2+x'^2}}$

$$\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\sin\theta\,\Big|_{\theta=\tan^{-1}\left(\frac{x'}{R}\right)}\right]_0^\infty(-\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\sin\left(\tan^{-1}\left(\frac{x'}{R}\right)\right)\right]_0^\infty(-\hat{z})$$

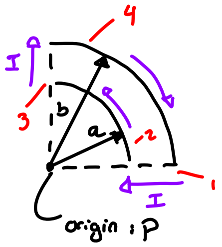
$$\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{x'}{\sqrt{R^2+x'^2}}\right]_0^\infty(-\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}(-\hat{z}) \quad \vdots \quad \boxed{\vec{B}(\vec{r}_3') = \frac{\mu_0 I}{4\pi}\frac{1}{R}(-\hat{z})}$$

$$\vec{B}_{net}(\vec{r}') = \vec{B}(\vec{r}_1') + \vec{B}(\vec{r}_2') + \vec{B}(\vec{r}_3') = \left[\frac{\mu_0 I}{4}\cdot\frac{1}{R} + \frac{\mu_0 I}{4\pi}\cdot\frac{2}{R}\right](-\hat{z})$$

$$\boxed{\vec{B}(\vec{r}') = \frac{\mu_0 I}{4}\cdot\frac{1}{R}\left[1 + \frac{2}{\pi}\right](-\hat{z})}$$

Problem 2

(a)




$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \qquad \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

loop 1 : $d\vec{l}' = dx\,\hat{x}$ : $\vec{r} = (0)\hat{r} + (0)\hat{\phi}$ : $\vec{r}' = -x'(\hat{r})$ : $\vec{\mathscr{r}} = x'(\hat{r})$ : $|\vec{\mathscr{r}}| = x'$ $\therefore$ $\hat{\mathscr{r}} = (\hat{r})$

$d\vec{l}' \times \hat{\mathscr{r}} = dx'(\hat{r}) \times (\hat{r}) = 0$ : loop $1 = 0$

loop 3: Same as loop 1 but in the y-direction. $\therefore$ loop $2 = 0$

loop 2: $\vec{B}_2(\vec{r}') = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$

$\vec{r} = 0(\hat{r}) + 0(\hat{\phi})$ : $\vec{\mathscr{r}} = -a(\hat{r})$ : $|\vec{\mathscr{r}}| = a$ : $\mathscr{r}^2 = a^2$ : $\hat{\mathscr{r}} = -\frac{a}{a}(\hat{r}) = -(\hat{r})$
$\vec{r}' = a(\hat{r})$

$d\vec{l}' = a\,d\phi\,(\hat{\phi})$ : $\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$$\hat{\phi} \times \hat{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = (-\sin^2\phi - \cos^2\phi)(\hat{z}) = (-\hat{z})$$

$d\vec{l}' \times \hat{\mathscr{r}} = -a\,d\phi\,(-\hat{z}) = a\,d\phi\,(\hat{z})$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}} \frac{a}{a^2}\,d\phi\,(\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{a}\Big[\phi\Big]_0^{\frac{\pi}{2}}(\hat{z}) = \frac{\mu_0 I}{8}\cdot\frac{1}{a}(\hat{z})$$

$$\boxed{\vec{B}_2(\vec{r}) = \frac{\mu_0 I}{8}\cdot\frac{1}{a}(\hat{z})}$$

loop 4: $\vec{B}_4(\vec{r}_4') = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$ : $\vec{r} = 0(\hat{r}) + 0(\hat{\phi})$ : $\vec{r}' = b(\hat{r})$ : $\vec{\mathscr{r}} = -b\,\hat{r}$
$|\vec{\mathscr{r}}| = b$ : $\hat{\mathscr{r}} = -\hat{r}$ : $\mathscr{r}^2 = b^2$

$d\vec{l}' = b\,d\phi\,\hat{\phi}$ : $d\vec{l}' \times \hat{\mathscr{r}} = b\,d\phi(\hat{\phi}) \times -(\hat{r})$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$
$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$$-b\,d\phi\,(\hat{\phi} \times \hat{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = -b\,d\phi(-\sin^2\phi - \cos^2\phi)(\hat{z})$$

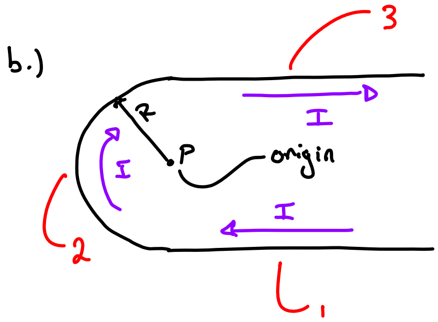
$d\vec{l}' \times \hat{\mathscr{r}} = b\,d\phi\,(\hat{z})$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_{\frac{\pi}{2}}^{0} \frac{b}{b^2}\,d\phi\,(\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{b}\Big[\phi\Big]_{\frac{\pi}{2}}^{0}(\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{b}\Big[-\frac{\pi}{2}\Big](\hat{z})$$

$$\boxed{\vec{B}_4(\vec{r}') = -\frac{\mu_0 I}{8}\cdot\frac{1}{b}(\hat{z})} \quad \therefore \quad \boxed{B_{net} = \frac{\mu_0 I}{8}\Big[\frac{1}{a} - \frac{1}{b}\Big](\hat{z})}$$

Problem 2| Continued

b.)




$$\vec{B}(\vec{r}\,') = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\,' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \quad : \vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$$

loop 1: $\vec{r} = 0(\hat{x}) + 0(\hat{y})$ : $\vec{r}\,' = x'(\hat{x}) - R(\hat{y})$ : $\vec{\mathscr{r}} = -x'(\hat{x}) + R(\hat{y})$ : $|\vec{\mathscr{r}}| = \sqrt{x'^2 + R^2}$ : $\mathscr{r}^2 = x'^2 + R^2$

$$\hat{\mathscr{r}} = \frac{-x'(\hat{x}) + R(\hat{y})}{\sqrt{x'^2+R^2}} \quad :$$

$$d\vec{l}\,' = dx'(\hat{x})$$

$$d\vec{l}\,' \times \hat{\mathscr{r}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ \frac{-x'}{\sqrt{x'^2+R^2}} & \frac{R}{\sqrt{x'^2+R^2}} & 0 \end{vmatrix} = \frac{R}{\sqrt{x'^2+R^2}}\, dx'\ (\hat{z})$$

$$\vec{B}(\vec{r}\,') = \frac{\mu_0 I}{4\pi}\int_{\infty}^{0} \frac{R}{(x'^2+R^2)^{3/2}}\, dx' = \frac{\mu_0 I}{4\pi}\int_{\infty}^{0} \frac{R^2 \cdot \sec^2\theta}{R^3 \cdot \sec^3\theta}\, d\theta = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\int_{\infty}^{0} \cos\theta\, d\theta$$

$x' = R\cdot\tan\theta$ , $x'^2 = R^2\cdot\tan^2\theta$

$dx' = R\cdot\sec^2\theta\, d\theta$

$$= \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[ \sin\theta \Big|_{\theta = \tan^{-1}\left(\frac{x'}{R}\right)} \right]_{\infty}^{0} = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\left[\frac{x'}{\sqrt{R^2+x'^2}}\right]_{\infty}^{0} = -\frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\,(\hat{z})$$

$$\boxed{\vec{B}_1(\vec{r}\,') = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\,(-\hat{z})}$$

loop 2: $\vec{r} = 0(\hat{s}) + 0(\hat{\phi})$ : $\vec{r}\,' = R(\hat{s}) + 0(\hat{\phi})$ : $\vec{\mathscr{r}} = -R(\hat{s})$ : $|\vec{\mathscr{r}}| = R$ : $\mathscr{r}^2 = R^2$

$\hat{\mathscr{r}} = \frac{-R}{R}(\hat{s}) = (-\hat{s})$ : $\hat{\mathscr{r}} = (-\hat{s})$ : $d\vec{l}\,' = R\, d\phi\ (\hat{\phi})$

$$d\vec{l}\,' \times \hat{\mathscr{r}} = R\,d\phi\ (\hat{\phi} \times -\hat{s}) = -R\,d\phi\ (\hat{\phi}\times\hat{s}) = -R\,d\phi \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = -R\,d\phi\,(-\sin^2\phi - \cos^2\phi)(\hat{z}) = R\,d\phi\ (\hat{z})$$

$$\vec{B}(\vec{r}\,') = \frac{\mu_0 I}{4\pi}\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \frac{R}{R^2}\, d\phi\ (\hat{z}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\cdot\Big[\phi\Big]_{\frac{3\pi}{2}}^{\frac{\pi}{2}}(\hat{z}) = \frac{\mu_0}{4\pi}\,\frac{1}{R}\left[\frac{\pi}{2} - \frac{3\pi}{2}\right](\hat{z})$$

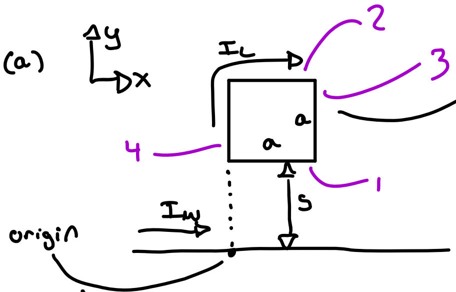
$$\boxed{\vec{B}_2(\vec{r}\,') = \frac{\mu_0 I}{4}\cdot\frac{1}{R}\,(-\hat{z})}$$

we can say $\vec{B}_1(\vec{r}\,') = \vec{B}_3(\vec{r}\,')$ :

$$\boxed{\vec{B}_3(\vec{r}\,') = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{R}\,(-\hat{z})} \qquad \boxed{\vec{B}_{net}(\vec{r}\,') = \frac{\mu_0 I}{4R}\left[1 + \frac{2}{\pi}\right](-\hat{z})}$$

Problem 3

(a)




Current on object

$$\vec{F}_{mag} = \int I_l (d\vec{l}' \times \vec{B}_w)$$

current from source

Magnetic Field due to a long straight wire : $\vec{B} = \frac{\mu_0 I}{2\pi d}(-\hat{z})$

①: $\vec{B}_1 = \frac{\mu_0 I}{2\pi s}(-\hat{z})$ $\quad d\vec{l}' = dx'\,\hat{x}$ $\quad d\vec{l}' \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ 0 & 0 & -\frac{\mu_0 I_w}{2\pi s} \end{vmatrix} = \frac{\mu_0 I}{2\pi s} dx'\,\hat{y}$

$$\vec{F} = I_l \int_a^0 \frac{\mu_0 I_w dx'}{2\pi s}\hat{y} = I_l \left[\frac{\mu_0 I_w x'}{2\pi s}\right]_a^0 \hat{y} = I_l \cdot -\frac{\mu_0 I_w a}{2\pi s}\hat{y}$$

$$\boxed{\vec{F}_1 = I_l \frac{\mu_0 I_w}{2\pi s} a\,(-\hat{y})}$$

② $\vec{B}_2 = \frac{\mu_0 I}{2\pi(s+a)}(-\hat{z})$, $d\vec{l}' = dx'\,\hat{x}$ $\quad d\vec{l} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ 0 & 0 & -\frac{\mu_0 I}{2\pi(s+a)} \end{vmatrix} = \frac{\mu_0 I}{2\pi(s+a)} dx'\,(\hat{y})$

$$\vec{F}_2 = I_l \int_0^a \frac{\mu_0 I_w}{2\pi(s+a)} dx'\,(\hat{y}) = I_l \cdot \frac{\mu_0 I_w}{2\pi(s+a)} x' \Big|_0^a (\hat{y}) = I_l \cdot \frac{\mu_0 I_w}{2\pi(s+a)} a\,(\hat{y})$$

$$\boxed{\vec{F}_2 = I_l \frac{\mu_0 I_w}{2\pi(s+a)} a\,(\hat{y})}$$

Due to sides 3 & 4 being equal distance from the wire, the force on these sections will be equal and opposite thus canceling out.

$$\boxed{\vec{F}_3 = -\vec{F}_4}$$

$$F_{net} = \sum_{n=1}^{4} \vec{F}_n = \vec{F}_1 + \vec{F}_2 + \cancel{\vec{F}_3} + \cancel{\vec{F}_4} = \frac{I_l \mu_0 I_w a}{2\pi s}(-\hat{y}) + \frac{I_l \mu_0 I_w}{2\pi s + 2\pi a} a\,(\hat{y})$$

$$= \frac{I_l \mu_0 I_w a(2\pi s + 2\pi a)}{2\pi s(2\pi s + 2\pi a)}(-\hat{y}) + \frac{I_l \mu_0 I_w a\, 2\pi s}{2\pi s(2\pi s + 2\pi a)}(\hat{y})$$

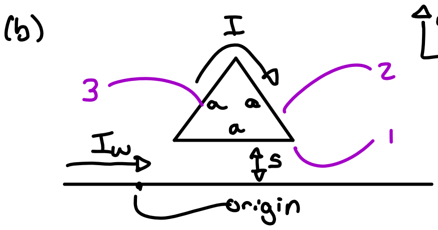
$$= \frac{I_l I_w \mu_0 a \cdot 2\pi s + I_l I_w \mu_0 a\, 2\pi a}{2\pi s(2\pi s + 2\pi a)}(-\hat{y}) + \frac{I_l I_w \mu_0 a \cdot 2\pi s}{2\pi s(2\pi s + 2\pi a)}(\hat{y})$$

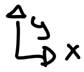
$$= \frac{I_l I_w \mu_0 \cdot \cancel{2\pi} a^2}{\cancel{2\pi} s(2\pi s + 2\pi a)}(-\hat{y}) = \frac{I_l I_w \mu_0 \cdot a^2}{2\pi s(s+a)}(-\hat{y})$$

$$\boxed{\vec{F} = \frac{I_l I_w \mu_0 a^2}{2\pi s(s+a)}(-\hat{y})}$$

Problem 3 | Continued

(b)

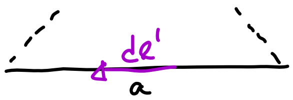



$\vec{F}_{mag} = \int I_\ell (d\vec{\ell}' \times \vec{B}_w)$ $\qquad$ $\vec{B}$ due to $\infty$ wire: $\vec{B} = \frac{\mu_0 I_w}{2\pi d}(-\hat{z})$

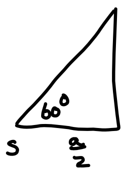
$d\vec{\ell}' = dx'\,\hat{x}$

①




$$\vec{B}_1 = \frac{\mu_0 I_w}{2\pi s}(-\hat{z}) \qquad d\vec{\ell}' \times \vec{B}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & 0 & 0 \\ 0 & 0 & -\frac{\mu_0 I_w}{2\pi s} \end{vmatrix} = \frac{\mu_0 I_w}{2\pi s}\,dx'\,(\hat{y})$$

$$\vec{F}_1 = I_\ell \int_a^0 \frac{\mu_0 I_w}{2\pi s}\,dx'\,(\hat{y}) = I_\ell\left[\frac{\mu_0 I_w}{2\pi s}x'\right]_a^0(\hat{y}) = \frac{I_\ell I_w \mu_0 a}{2\pi s}(-\hat{y})$$

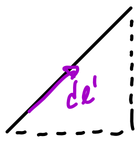



$\tan\theta = \frac{y}{x}$

$y = \frac{a}{2}\tan 60 = \frac{\sqrt{3}a}{2}$

$$\boxed{\vec{F}_1 = \frac{I_\ell I_w \mu_0 a}{2\pi s}(-\hat{y})}$$

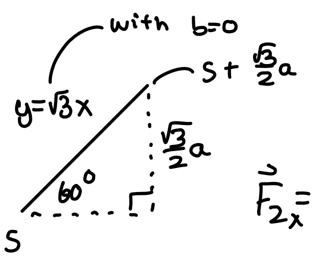
②




$$\vec{B}_2 = \frac{\mu_0 I_w}{2\pi y}(-\hat{z})$$

$$d\vec{\ell}' = dx'\,\hat{x} + dy'\,\hat{y}$$

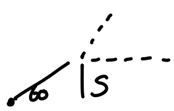
$$d\vec{\ell}' \times \vec{B}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & dy' & 0 \\ 0 & 0 & -\frac{\mu_0 I_w}{2\pi y'} \end{vmatrix} = \frac{\mu_0 I_w}{2\pi y'}\,dy'\,(-\hat{x}) + \frac{\mu_0 I_w}{2\pi y'}\,dx'\,(\hat{y})$$




$$\vec{F}_{2x} = I_\ell \int_s^{s+\frac{\sqrt{3}}{2}a} \frac{\mu_0 I_w}{2\pi y'}\,dy'\,(-\hat{x}) = \frac{\mu_0 I_\ell I_w}{2\pi}\ln(y)\Big|_s^{s+\frac{\sqrt{3}}{2}a}(-\hat{x})$$

$$\vec{F}_{2x} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(s+\frac{\sqrt{3}}{2}a\right) - \ln(s)\right](-\hat{x}) = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(\frac{s+\frac{\sqrt{3}}{2}a}{s}\right)\right](-\hat{x})$$

$$\vec{F}_{2x} = \frac{\mu_0 I_\ell I_w}{2\pi}\ln\left(1+\frac{\sqrt{3}}{2}a\right)(-\hat{x})$$




$\tan 60 = \frac{s}{x} \quad \therefore \quad x = \frac{s}{\tan 60} = \frac{s}{\sqrt{3}}$

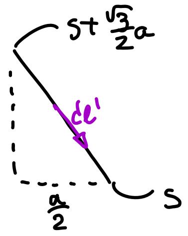
$$\vec{F}_{2y} = I_\ell \int_0^{\frac{a}{2}} \frac{\mu_0 I_\ell I_w}{2\pi y}\,dx'\,(\hat{y})\Big|_{y=\sqrt{3}x'} = \frac{\mu_0 I_\ell I_w}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}} \frac{1}{\sqrt{3}x'}\,dx'\,(\hat{y}) = \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\ln(x')\Big|_{\frac{s}{\sqrt{3}}}^{\frac{s}{\sqrt{3}}+\frac{a}{2}}(\hat{y})$$

$$\vec{F}_{2y} = \frac{\mu_0 I_\ell I_w}{2\pi}\cdot\frac{1}{\sqrt{3}}\left[\ln\left(\frac{s}{\sqrt{3}}+\frac{a}{2}\right) - \ln\left(\frac{s}{\sqrt{3}}\right)\right]\hat{y} = \frac{\mu_0 I_\ell I_w}{2\pi}\cdot\frac{1}{\sqrt{3}}\left[\ln\left(\frac{\frac{s}{\sqrt{3}}+\frac{a}{2}}{s/\sqrt{3}}\right)\right]\hat{y}$$

$$\vec{F}_{2y} = \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\left[\ln\left(1+\frac{\sqrt{3}}{2}\frac{a}{s}\right)\right]\hat{y}$$

$$\boxed{\vec{F}_2 = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(1+\frac{\sqrt{3}}{2}a\right)(-\hat{x}) + \frac{1}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}}{2}\frac{a}{s}\right)(\hat{y})\right]}$$

Problem 3| Continued




$$\vec{B}_3 = \frac{\mu_0 I_w}{2\pi y}(-\hat{z})$$

$$d\vec{\ell}' = dx'\,\hat{x} - dy'\,\hat{y}$$

$$d\ell' \times \vec{B}_3 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dx' & -dy' & 0 \\ 0 & 0 & -\frac{\mu_0 I_w}{2\pi y} \end{vmatrix}$$

$$d\ell' \times \vec{B}_3 = \frac{\mu_0 I_w}{2\pi y'}\,dy'\,(\hat{x}) + \frac{\mu_0 I_w}{2\pi y'}\,dx'\,(\hat{y})$$

$$\vec{F}_{3x} = I_\ell \int_{s+\frac{\sqrt{3}}{2}a}^{s} \frac{\mu_0 I_w}{2\pi y'}\,dy'\,(\hat{x}) = \frac{\mu_0 I_\ell I_w}{2\pi} \ln(y')\Big|_{s+\frac{\sqrt{3}}{2}a}^{s} (\hat{x})$$

$$\vec{F}_{3x} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln(s) - \ln\left(s+\tfrac{\sqrt{3}}{2}a\right)\right](\hat{x}) = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(\frac{s}{s+\frac{\sqrt{3}}{2}a}\right)\right](\hat{x})$$

$$\vec{F}_{3x} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(\frac{s}{s+\frac{\sqrt{3}}{2}a}\right)\right](\hat{x}) \;:\; \vec{F}_{3x} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(1+\tfrac{\sqrt{3}}{2}a\right)\right](\hat{x})$$

$$\vec{F}_{3y} = I_\ell \int_{\frac{a}{2}+\frac{\sqrt{3}}{s}}^{\frac{\sqrt{3}}{s}+a} \frac{\mu_0 I_w}{2\pi y}\,dx'\,(\hat{y})\Big|_{y=-\sqrt{3}x'} = \frac{\mu_0 I_\ell I_w}{2\pi}\int_{\frac{a}{2}+\frac{\sqrt{3}}{s}}^{\frac{\sqrt{3}}{s}+a} \frac{1}{\sqrt{3}}\frac{1}{x'}\,dx'\,(-\hat{y})$$

$$\vec{F}_{3y} = \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\Big[\ln(x')\Big]_{\frac{a}{2}+\frac{\sqrt{3}}{s}}^{\frac{\sqrt{3}}{s}+a}(-\hat{y}) = \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\left[\ln\left(\tfrac{\sqrt{3}}{s}+a\right) - \ln\left(\tfrac{a}{2}+\tfrac{\sqrt{3}}{s}\right)\right](-\hat{y})$$

$$\vec{F}_{3y} = \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\left[\ln\left(1+\tfrac{\sqrt{3}}{2}\tfrac{a}{s}\right)\right](\hat{y})$$

$$\boxed{\vec{F}_3 = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\ln\left(1+\tfrac{\sqrt{3}}{2}a\right)\hat{x} + \frac{1}{\sqrt{3}}\ln\left(1+\tfrac{\sqrt{3}}{2}\tfrac{a}{s}\right)\hat{y}\right]}$$

$$\vec{F}_{net} = \sum_{n=1}^{3}\vec{F}_n = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \qquad \vec{F}_{3x} = -\vec{F}_{2x}$$

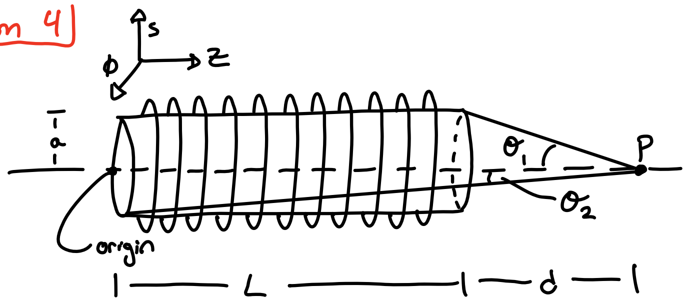
$$= \frac{I_\ell I_w \mu_0 a}{2\pi s}(-\hat{y}) + \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\ln\left(1+\tfrac{\sqrt{3}}{2}\tfrac{a}{s}\right)(\hat{y}) + \frac{\mu_0 I_\ell I_w}{2\pi}\frac{1}{\sqrt{3}}\ln\left(1+\tfrac{\sqrt{3}}{2}\tfrac{a}{s}\right)(\hat{y})$$

$$= \frac{\mu_0 I_\ell I_w}{2\pi}\left[-\frac{a}{s} + \frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)\right](\hat{y})$$

$$\boxed{\vec{F} = \frac{\mu_0 I_\ell I_w}{2\pi}\left[\frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right) - \frac{a}{s}\right](\hat{y})}$$

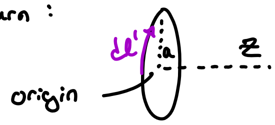
# Problem 4

(a)




$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

One turn:




$\vec{\mathscr{r}} = \vec{r} - \vec{r}' : \vec{r} = 0\hat{s} + z\hat{z} : \vec{r}' = a\hat{s} + 0\hat{z} : d\vec{l}' = a\,d\phi\,\hat{\phi}$

$\vec{\mathscr{r}} = (0-a)\hat{s} + (z-0)\hat{z} = -a\hat{s} + z\hat{z} : |\vec{\mathscr{r}}| = \sqrt{a^2+z^2} : \mathscr{r}^2 = a^2 + z^2$

$$\hat{\mathscr{r}} = \frac{z\hat{z} - a\hat{s}}{\sqrt{a^2+z^2}}$$

$$d\vec{l}' \times \hat{\mathscr{r}} = a\,d\phi\,\hat{\phi} \times \left(\frac{z\hat{z} - a\hat{s}}{\sqrt{a^2+z^2}}\right) = \frac{az}{\sqrt{a^2+z^2}} d\phi\,(\hat{\phi}\times\hat{z}) - \frac{a^2 d\phi}{\sqrt{a^2+z^2}}(\hat{\phi}\times\hat{s})$$

$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\phi = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

$$\hat{\phi}\times\hat{s} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = \hat{z}(-\sin^2\phi - \cos^2\phi) = -\hat{z}$$

$$\hat{\phi}\times\hat{z} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\phi\,(\hat{x}) + \sin\phi\,(\hat{y})$$

$$d\vec{l}' \times \hat{\mathscr{r}} = \frac{az}{\sqrt{a^2+z^2}} d\phi\,(\cos\phi(\hat{x}) + \sin\phi(\hat{y})) + \frac{a^2 d\phi}{\sqrt{a^2+z^2}}\hat{z}$$

$$\vec{B}(\vec{r}') = \frac{\mu_0 I}{4\pi}\left[\overbrace{\cancel{\int_0^{2\pi} \frac{az\cos\phi}{(a^2+z^2)^{3/2}} d\phi\,(\hat{x})}}^{0} + \overbrace{\cancel{\int_0^{2\pi} \frac{az\sin\phi}{(a^2+z^2)^{3/2}} d\phi\,(\hat{y})}}^{0} + \int_0^{2\pi} \frac{a^2}{(a^2+z^2)^{3/2}} d\phi\,(\hat{z})\right]$$

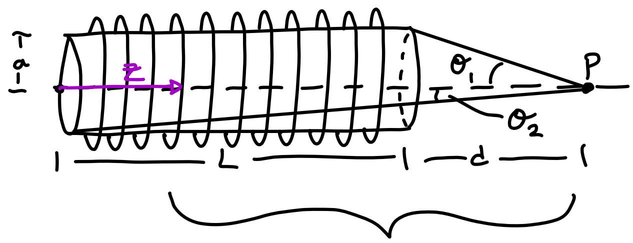
$$\vec{B}(\vec{r}') = \frac{\mu_0 I}{4\pi}\left[\frac{a^2}{(a^2+z^2)^{3/2}}\phi\,(\hat{z})\right]_0^{2\pi} = \frac{\mu_0 I}{2}\left[\frac{a^2}{(a^2+z^2)^{3/2}}\right](\hat{z})$$

$\vec{B}$-Field of one turn: $\vec{B} = \frac{\mu_0 I}{2}\left(\frac{a^2}{(a^2+z^2)^{3/2}}\right)(\hat{z})$

$a$ - radius of coil

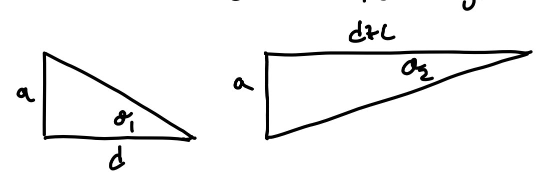
$z$ - distance along $z$-axis from coil

$(d+L-z)$ - distance to P from point in solenoid




$\underbrace{\phantom{xxxxxxxx}}_{d+L-z}$

$\lambda = \frac{NI}{L}$ — current per length · $n$ turns




For entire solenoid at P

$$\vec{B} = \frac{\mu_0\lambda}{2}\int_0^L \frac{a^2}{(a^2+(d+L-z)^2)^{3/2}} dz\,(\hat{z})$$

$u = d+L-z$

$du = -dz$

$d = \frac{a}{\tan\theta_1}$ $\quad a = (d+L)\tan\theta_2$

$L = \frac{a}{\tan\theta_2} - d = \frac{a}{\tan\theta_2} - \frac{a}{\tan\theta_1}$

Problem 4) continued

$$\vec{B} = \frac{\mu_0 \lambda}{2} a^2 \int_{d+L}^{d} \frac{-1}{(a^2+u^2)^{3/2}} du = \frac{\mu_0 \lambda}{2} a^2 \int_{d}^{d+L} \frac{1}{(a^2+u^2)^{3/2}} du \qquad \begin{aligned} u &= a\tan\theta \\ du &= a\cdot\sec^2\theta\, d\theta \end{aligned}$$

$$\vec{B} = \frac{\mu_0 \lambda}{2} a^2 \int_{\tan^{-1}\left(\frac{u(d)}{a}\right)}^{\tan^{-1}\left(\frac{u(d)+L}{a}\right)} \frac{a\cdot\sec^2\theta}{(a^2\cdot\sec^2\theta)^{3/2}} d\theta = \frac{\mu_0 \lambda}{2} a^2 \int_{\tan^{-1}\left(\frac{u(d)}{a}\right)}^{\tan^{-1}\left(\frac{u(d)+L}{a}\right)} \frac{a\cdot\sec^2\theta}{a^3\sec^3\theta} d\theta \qquad \theta = \tan^{-1}\left(\frac{u}{a}\right)$$

$$\vec{B} = \frac{\mu_0 \lambda}{2} \int_{\tan^{-1}\left(\frac{u(d)}{a}\right)}^{\tan^{-1}\left(\frac{u(d)+L}{a}\right)} \frac{1}{\sec\theta} d\theta = \frac{\mu_0 \lambda}{2} \left[ \sin\theta \Big|_{\theta=\tan^{-1}\left(\frac{u}{a}\right)} \Big|_{u=d+L-z} \right]_0^L$$

$$\vec{B} = \frac{\mu_0 \lambda}{2} \left[ \frac{u}{\sqrt{u^2+a^2}} \Big|_{u=d+L-z} \right]_0^L = \frac{\mu_0 \lambda}{2} \left[ \frac{d+L-z}{\sqrt{(d+L-z)^2+a^2}} \right]_0^L$$

$$\vec{B} = \frac{\mu_0 \lambda}{2} \left[ \frac{d}{\sqrt{d^2+a^2}} - \frac{d+L}{\sqrt{(d+L)^2+a^2}} \right] (\hat{z})$$

In terms of $a, \theta_1, \theta_2$

$$\vec{B} = \frac{\mu_0 \lambda}{2} \left[ \frac{a}{\tan\theta_1 \sqrt{a^2\cdot\sec^2\theta_1}} - \frac{a}{\tan\theta_2 \sqrt{a^2\cdot\sec^2\theta_2}} \right] (\hat{z}) = \frac{\mu_0 \lambda}{2} \left[ \frac{a}{\tan\theta_1\, a\sec\theta_1} - \frac{a}{\tan\theta_2\cdot a\cdot\sec\theta_2} \right] (\hat{z})$$

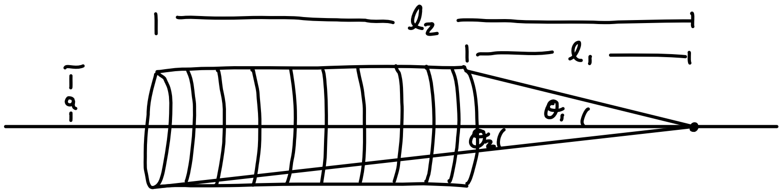
$$\boxed{\vec{B} = \frac{\mu_0 N I}{2L} \left[ \frac{\cos^2\theta_1}{\sin\theta_1} - \frac{\cos^2\theta_2}{\sin\theta_2} \right] (\hat{z})}$$

(b) $$\lim_{L\to\infty} \frac{\mu_0 N I}{2L} \left[ \frac{\cos^2\theta_1}{\sin\theta_1} - \frac{\cos^2\theta_2}{\sin\theta_2} \right] (\hat{z}) = 0$$

$$\boxed{\vec{B} = 0}$$

Problem 4

a.)




$n = \frac{N}{L}$

$$\vec{B}_{Loop}(z) = \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{\frac{3}{2}}} (\hat{z}) = \frac{\mu_0 I}{2} \cdot \frac{a^2}{(a^2+z^2)^{\frac{3}{2}}} (\hat{z})$$

$$B_{Tot} = \int_{l_1}^{l_2} B(z) n\, dz = \frac{\mu_0 I}{2} . na^2 \int_{l_1}^{l_2} \frac{1}{(a^2+z^2)^{\frac{3}{2}}} dz\, (\hat{z})$$

$z = a \cdot \tan\theta$, $z^2 = a^2 \cdot \tan^2\theta$

$dz = a \cdot \sec^2\theta\, d\theta$

$$\vec{B}_{net} = \frac{\mu_0 I}{2} . na^2 \int_{l_1}^{l_2} \frac{a \cdot \sec^2\theta}{(a^2 + a^2 \cdot \tan^2\theta)^{\frac{3}{2}}} d\theta\, (\hat{z}) = \frac{\mu_0 I}{2} . na^2 \int_{l_1}^{l_2} \frac{a \cdot \sec^2\theta}{(a^2 \cdot \sec^2\theta)^{\frac{3}{2}}} d\theta\, (\hat{z})$$

$$= \frac{\mu_0 I}{2} . na^2 \int_{l_1}^{l_2} \frac{a \cdot \sec^2\theta}{a^3 \cdot \sec^3\theta} d\theta\, (\hat{z}) = \frac{\mu_0 I n}{2} \int_{l_1}^{l_2} \cos\theta\, d\theta\, (\hat{z}) = \frac{\mu_0 I n}{2} . \sin\theta \Big|_{\theta = \tan^{-1}\left(\frac{z}{a}\right)} \Big|_{l_1}^{l_2}$$

$$= \frac{\mu_0 I n}{2} \left[\frac{z}{\sqrt{z^2+a^2}}\right]_{l_1}^{l_2} (\hat{z}) = \frac{\mu_0 I n}{2} \left[\frac{l_2}{\sqrt{l_2^2+a^2}} - \frac{l_1}{\sqrt{l_1^2+a^2}}\right] (\hat{z})$$

$$\boxed{\vec{B} = \frac{\mu_0 I n}{2} \left[\frac{l_2}{\sqrt{l_2^2+a^2}} - \frac{l_1}{\sqrt{l_1^2+a^2}}\right] (\hat{z})}$$

b.)

$\therefore$ $\boxed{\vec{B}_{Tot} = \mu_0 n I}$