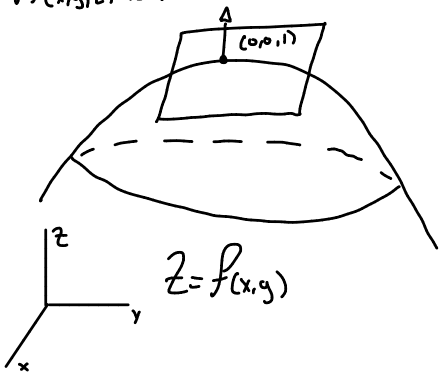
$f(x,y)$

- $\nabla f(x,y)$ is $\perp$ to level curves of f in the xy-plane

$F(x,y,z)$

- $\nabla f(x,y,z)$ is $\perp$ to level surfaces of F in R^3



2^{nd} Derivatives Test

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

$D>0$ & $f_{xx}(a,b) > 0$ $\therefore$ $f(a,b)$ is a local min

$D>0$ & $f_{xx}(a,b) < 0$ $\therefore$ $f(a,b)$ is a local max

$D<0$, inconclusive

$$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

$z = 4 + x^3 + y^3 - 3xy - z$

$\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x, -1 \rangle$

$f_{xx} = 6x$ $\quad f_{yy} = 6y$ $\quad f_{xy} = -3$

$0 = 6x$, $x = 0$ $\quad 0 = 6y$, $y = 0$

$(0,0)$

$f_{xx}(0,0)\, f_{yy}(0,0) - [f_{xy}(a,b)]^2$

$0(0) - [-3]^2$

$0 - (9) = -9$

$D \le 0$ $\therefore$ inconclusive (Saddle point)

$f_{xx}(1,1)\, f_{yy}(1,1) - (f_{xy}(1,1))^2$

$6(6) - (-3)^2$

$36 - (9)$

27 $\quad D > 0$ $\therefore$ minimum

Ex) $f(x,y) = x^2 + y^2 + x^{-2}y^{-2} + 8$

$f_x = 2x - 2y^{-2}x^{-3}$ $\qquad 2x - \frac{2}{y^2x^3} = 0$ (1)

$f_y = 2y - 2x^{-2}y^{-3}$ $\qquad 2y - \frac{2}{x^2y^3} = 0$ (2)

$x - \frac{1}{y^2x^3} = 0$ $\qquad y - \frac{1}{x^2y^3} = 0$

$\frac{-1}{y^2x^3} = -x$ $\qquad -y = \frac{-1}{x^2y^3}$

$-1 = -y^2x^4$ $\qquad -x^2y^4 = -1$

$1 = y^2x^4$ $\qquad 1 = x^2y^4$

$y^2 = \frac{1}{x^4}$

$y = \pm\frac{1}{x^2}$

$\frac{1}{x^2} - \frac{1}{x^2(\frac{1}{x^2})^3} = 0$

$\frac{1}{x^2} - x^4 = 0$

$\frac{1}{x^2} = x^4$

$1 = x^6$

$x = \pm 1$

$1 - \frac{1}{y^2} = 0$

$-\frac{1}{y^2} = -1$

$-1 = -y^2$

$y^2 = 1$

$y = \pm 1$

Critical points

$(1,1), (-1,1)$

$(1,-1), (-1,-1)$

2^{nd} derivative

$f_x = 2x - 2y^{-2}x^{-3}$ $\quad f_y = 2y - 2x^{-2}y^{-3}$ $\quad f_x = 2x - 2y^{-2}x^{-3}$

$f_{xx} = 2 + 6y^{-2}x^{-4}$ $\quad f_{yy} = 2 + 6x^{-2}y^{-4}$ $\quad f_{xy} = 4y^{-3}x^{-3}$

f_{xx}	f_{yy}	f_{xy}
$(1,1) = 8$	$(1,1) = 8$	$(1,1) = 4$
$(1,-1) = 8$	$(1,-1) = 8$	$(1,-1) = -4$
$(-1,1) = 8$	$(-1,1) = 8$	$(-1,1) = -4$
$(-1,-1) = 8$	$(-1,-1) = 8$	$(-1,-1) = 4$

$8(8) - (4)^2$ $\qquad 8(8) - (-4)^2$

$64 - 16$ $\qquad 64 - (16)$

48 $\qquad 48$

$8(8) - (-4)^2$ $\qquad 8(8) - (4)^2$

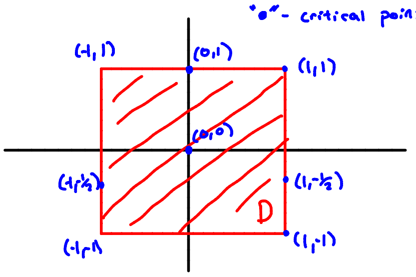
$64 - (16)$ $\qquad 64 - (16)$

48 $\qquad 48$

$D > 0$ $\therefore$ minimum

14.7.3 $D = \{(x,y) \mid |x| \le 1 \text{ \& } |y| \le 1\}$

"•" - critical point



$f(x,y) = x^2 + y^2 + x^2y + 8$

$-1 \le x \le 1$
$-1 \le y \le 1$

$f_x = 2x + 2xy$ $\quad$ $f_y = 2y + x^2$
$f_{xx} = 2 + 2y$ $\quad$ $f_{yy} = 2y$

$\vec{r}(t) = \langle -1+2t, -1 \rangle$

$2x + 2xy = 0$ $\quad$ $2y + x^2 = 0$
$2x(1+y) = 0$
$x = 0$ $\quad$ $X = 0$ $\quad x = \pm\sqrt{2}$
$y = -1$ $\quad$ $Y = 0$ $\quad y = -1$

$(0,0)$
$(\sqrt{2}, -1)$
$(-\sqrt{2}, -1)$

Critical points
$(0,0)\ (\sqrt{2},-1)\ (-\sqrt{2},-1)$
$-1 \le x \le 1$
$-1 \le y \le 1$ $\quad$ omitted

$\vec{r}_1(t) = \langle 1,-1 \rangle + T\langle 0,2 \rangle$
$0 \le t \le 1$

$f(t) = f(1, -1+2t) = 1^2 + (-1+2t)^2 + 1^2(-1+2t) + 8$ $\quad (0,0)$

$f(t) = 1 + 1 - 4t + 4t^2 - 1 + 2t + 8$
$f(t) = 4t^2 - 2t + 9 \quad 0 \le t \le 1$
$f'(t) = 8t - 2$
$f' = 0 : \ 0 = 8t - 2$
$2 = 8t$
$t = 1/4$

$\vec{r}_1(1/4) = \langle 1,-1 \rangle + t\langle 0,2 \rangle$
$\langle 1, -1/2 \rangle$

$\vec{r}_3(t) = \langle -1, 1-2t \rangle \quad 0 \le t \le 1$ $\quad$ $\langle -1,1 \rangle + t\langle 0,-2 \rangle$
$(-1)^2 + (1-2t)^2 + (-1)^2(1-2t) + 8$ $\quad$ $P = (-1, -1/2)$
$1 + 1 - 4t + 4t^2 + 1 - 2t + 8$
$f_{③} = 4t^2 - 6t + 11$
$0 = 8t - 6$
$6 = 8t$
$t = 3/4$

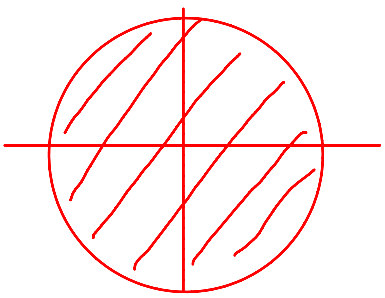
$\vec{r}_4(t) = \langle -1+2t, -1 \rangle \quad 0 \le t \le 1$
$(-1+2t)^2 + (-1)^2 + (-1+2t)^2(-1) + 8$
$1 - 4t + 4t^2 + 1 + (1 - 4t + 4t^2)(-1) + 8$
$2 - \cancel{4t} + \cancel{4t^2} + (-1 + \cancel{4t} - \cancel{4t^2}) + 8$
9

$f(0,0) = 8$
$f(1,-1) = 9$

$\vec{r}_2(t) = \langle 1-2t, 1 \rangle \quad 0 \le t \le 1$
$f|_2 = (1-2t)^2 + 1^2 + (1-2t)^2(1) + 8$
$1 - 4t + 4t^2 + 1 + 1 - 4t + 4t^2 + 8$
$f_{②} = 8t^2 - 8t + 11$
$0 = 16t - 8$
$8 = 16t$
$t = 1/2$

$t(1/2) = P(1 - 2(1/2), 1)$
$P(1-1, 1)$
$P(0,1)$

$f(x,y) = 2x^3 + y^4 + 4$



$\nabla f = \langle 6x^2, 4y^3 \rangle = \langle 0,0 \rangle$
$(x,y) = (0,0)$

Boundary of D : $\vec{r}(t) = \langle \cos t, \sin t \rangle$
$0 \le t \le 2\pi$

$f(t) = 2 \cdot \cos^3 t + \sin^4 t + 4$
$f'(t) = 6\cos^2 t \cdot -\sin t + 4\sin^3 t \cdot \cos t$
$f'(t) = -6\cos^2 t \sin t + 4\sin^3 t \cos t$
$2\cos t \sin t(-3\cos t + 2\sin^2 t) = 0$

$2\cos t \sin t = 0$ $\quad$ $-3\cos t + 2\sin^2 t$
$t = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $-3\cos t + 2(1-\cos^2 t) = 0$
$t = 0, \pi, 2\pi$ $\quad$ $-3\cos t - 2\cos^2 t + 2 = 0$
$-2\cos^2 t - 3\cos t + 2 = 0$
$t = 0.5$
$t = \frac{\pi}{3}, \frac{5\pi}{3}$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$t = 0 : (1, 0)$
$t = \frac{\pi}{3} : (\frac{1}{2}, \frac{\sqrt{3}}{2})$
$t = \frac{\pi}{2} : (0, 1)$
$t = \pi : (-1, 0)$
$t = \frac{5\pi}{3} : (\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$f(x,y) = x^2+y^2+4x-2y \qquad D = \{(x,y) \mid x^2+y^2 \le 16\}$

$x = 4\cos t$
$y = 4\sin t$

$f_x = 2x+4 \qquad 2x+4=0 \qquad 2x=-4 \qquad x=-2$

$f_y = 2y-2 \qquad 2y-2=0 \qquad 2y=2 \qquad y=1$

$(-2,1)$

$x^2 \le 16$
$0 \le x \le 4$
$y^2 \le 16$
$0 \le y \le 4$

$x^2+y^2 \le 16$
$y^2 \le 16-x^2$
$y = \sqrt{16-x^2}$

$f(x(t),y(t)) = (4\cos t)^2 + (4\sin t)^2 + 4(4\cos t) - 2(4\sin t)$
$= 16\cos^2 t + 16\sin^2 t + 16\cos t - 8\sin t$
$= 16(\cos^2 t + \sin^2 t)$
$= 16 + 16\cos t - 8\sin t$
$= 0 - 16\sin t - 8\cos t$
$= -8(2\sin t + \cos t)$

$1+15-4-2\sqrt{15}$
$1+15-4+2\sqrt{15}$
$4+1-8-2$

$f(x,x) = x^2 + (16-x^2) + 4x - 2(16-x^2)$
$f(x,x) = x^2+16-x^2+4x-32+2x^2$
$f(x,x) = 2x^2+4x-16$
$f'(x,x) = 4x+4$
$0 = 4x+4$
$-4 = 4x$
$x = -1$
$(-1,\sqrt{15})\ (-1,-\sqrt{15})$

$y^2 = 16-x^2$
$y^2 = 16-(-1)^2$
$y^2 = 16-(1)$
$y^2 = 15$
$y = \pm\sqrt{15}$

$f(-1,\sqrt{15}) = 12-2\sqrt{15} = -5.75$
$f(-1,-\sqrt{15}) = 12+2\sqrt{15} = 19.746$
$f(-2,1) = -5$
$f(0,4) = -8$
$f(4,0) = 32$

$x = 4\cos(t)$
$y = 4\sin(t)$

$x^2+y^2 = 16$

$f(x,y) = x^2+y^2+4x-2y$

$\vec{r}(4\cos(t), 4\sin(t))$

$f(t) = (4\cos(t))^2 + (4\sin(t))^2 + 4(4\cos(t)) - 2(4\sin(t))$

$0 \le t \le 2\pi$

$16\cos^2(t) + 16\sin^2(t) + 16\cos(t) - 8\sin(t)$
$16(\cos^2(t)+\sin^2(t)) + 16\cos(t) - 8\sin(t)$
$f(t) = 16 + 16\cos(t) - 8\sin(t)$
$f'(t) = -16\sin(t) - 8\cos(t)$
$0 = -8(2\sin(t) - \cos(t))$

$2\sin(t) - \cos(t) = 0$
$-\cos(t) = -2\sin(t)$
$\cot(t) = 2$