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MATH 360
CP 10.9

Equation 2 Stokes's Theorem

$$\iint_S \text{Curl}(\vec{F})\cdot\vec{n}\,dA = \oint_C \vec{F}\cdot\vec{r}\,'(s)\,ds$$

Example 1 Verification of Stokes's Theorem
Before we prove stokes's theorem, let us first get used to it by verifying for it for $\vec{F} = [y, z, x]$ and S the paraboloid (Fig 255)

$$z = f(x,y) = 1-(x^2+y^2), \quad z \geq 0$$

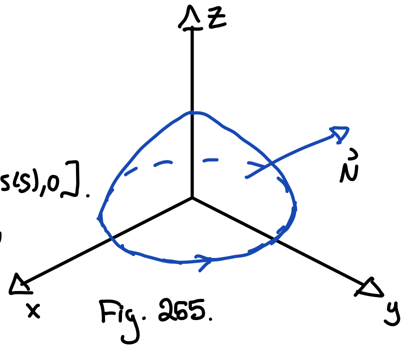

Fig. 265.

Solution. The curve C, oriented as in Fig. 255, is the circle $\vec{r}(s) = [\cos(s), \sin(s), 0]$. It's unit tangent vector is $r'(s) = [-\sin(s), \cos(s), 0]$. The function $\vec{F} = [y, z, x]$ on C is $\vec{F}(\vec{r}(s)) = [\sin(s), 0, \cos(s)]$. Hence,

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(s))\cdot\vec{r}\,'(s)\,ds = \int_0^{2\pi}[(\sin(s))(-\sin(s))+0+0]\,ds = -\pi$$

we now consider the surface integral. We have $F_1 = y$, $F_2 = z$, $F_3 = x$, so that in (2^*) we obtain,

$$\text{Curl}(\vec{F}) = \text{Curl}[F_1, F_2, F_3] = \text{Curl}[y, z, x] = [-1, -1, -1]$$

A normal vector of S is $\vec{N} = \text{grad}(z - f(x,y)) = [2x, 2y, 1]$. Hence $(\text{curl}(\vec{F}))\cdot\vec{N} = -2x-2y-1$. Now $\vec{n}\,dA = \vec{N}\,dx\,dy$ (see (3^*) in sec. 10.6 with x, y instead of u, v). Using polar coordinates r, θ defined by $x = r\cos(\theta)$, $y = r\sin(\theta)$ and denoting the projection of S into the xy-plane by R, we thus obtain

$$\iint_S (\text{Curl}(\vec{F}))\cdot\vec{n}\,dA = \iint_R (\text{curl}(F))\cdot\vec{N}\,dx\,dy = \iint_R (-2x-2y-1)\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (-2r(\cos(\theta)+\sin(\theta))-1)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(-\frac{2}{3}(\cos(\theta)+\sin(\theta))-\frac{1}{2}\right)d\theta = 0+0-\frac{1}{2}(2\pi) = -\pi$$

$$\boxed{-\pi}$$

Example 3 Evaluation of a Line Integral by Stokes's Theorem

Evaluate $\int_C \vec{F}\cdot\vec{r}'\,ds$, where C is the circle $x^2+y^2=4$, $z=-3$, oriented counterclockwise as seen by a person standing at the origin, and, with respect to right-handed cartesian coordinates,

$$\vec{F}=[y,\,xz^3,\,-zy^3]=y\hat{\imath}+xz^3\hat{\jmath}-zy^3\hat{k}$$

Solution. As a surface S bounded by C we can take the plane circular disk $x^2+y^2\le 4$ in the plane $z=-3$. Then $\vec{n}$ in Stokes's theorem points in the positive z-direction: thus $\vec{n}=\vec{k}$. Hence $(\text{curl}(\vec{F}))\cdot\vec{n}$ is simply the component of $\text{curl}(\vec{F})$ in the positive z-direction. Since $\vec{F}$ with $z=-3$ has the components $F_1=y$, $F_2=-27x$, $F_3=3y^3$, we thus obtain

$$\text{curl}(\vec{F})\cdot\vec{n}=\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=-27-1=-28 \qquad \boxed{-28}$$

Hence the integral over S in Stokes's theorem equals -28 times the area 4π of the disk S. This yields the answer $-28\cdot 4\pi=-112\pi\approx-352$. Confirm this by direct calculation, which involves somewhat more work.