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MATH 360
CP Ch. 12.2

Figure 286

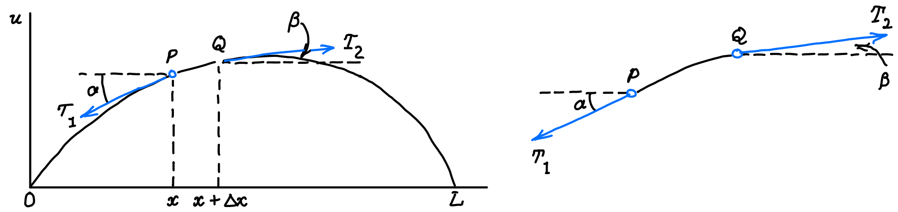

Fig. 286. Deflected string at fixed time t. Explanation on p. 544

Derivation of Wave Equation

The model of the vibrating string will consist of a PDE ("wave equation") and additional conditions. To obtain the PDE, we consider the forces acting on a small portion of the string (Fig. 286). This method is typical of modeling in mechanics and elsewhere.

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tension at the endpoints P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant. Using notation shown in Fig. 286, we thus obtain

(1)
$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{Const.}$$

In the vertical direction we have two forces, namely, the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2: here the minus sign appears because the component at P is directly downward. By Newton's second law (sec 2.4) the resultant of these two forces is equal to the mass $\rho\Delta x$ of the portion times the acceleration $\partial^2 u/\partial t^2$, evaluated at some point between x and $x+\Delta x$: here ρ is the mass of the undeflected string per unit length, and Δx is the length of the portion of the undeflected string. (Δ is generally used to denote small quantities: This has nothing to do with the Laplacian ∇^2, which is sometimes also denoted by Δ.) Hence,

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}$$

Using (1), we can divide this by $T_2 \cos\beta = T_1 \cos\alpha = T$, obtaining

(2)
$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

Now $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x+\Delta x$:

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Bigg|_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\Bigg|_{x+\Delta x}$$

Here we have to write partial derivatives because u also depends on time t. Dividing (2) by Δx, we thus have

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$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

If we let Δx approach zero, we obtain the linear PDE

(3) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

This is called the one-dimensional wave equation. We see that it is homogeneous and of the second order. The physical constant T/ρ is denoted by c^2 (instead of c) to indicate that this constant is positive, a fact that will be essential to the form of the solutions. "one dimensional" means that the equation involves only one space variable, x. In the next section we shall complete setting up the model and then show how to solve it by a general method that is probably the most important one for PDE'S in engineering mathematics.