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PHYS 396
HW 1

Problem 1

a.) $G \to \frac{m^3}{kg \cdot s^2}$, $M \to kg$, $R \to m$, $c \to m/s$

$$c^2 = m^2/s^2 \quad : \quad \frac{GM}{Rc^2} = \frac{m^3/kg \cdot s^2 \cdot kg}{m \cdot m^2/s^2} = \frac{\cancel{m^3 \cdot kg}/\cancel{kg \cdot s^2}}{\cancel{m^3/s^2}} = \text{Dimensionless}$$

$\boxed{\text{This is a dimensionless quantity}}$

b.)

i.) $R = 6{,}378{,}000\ m$ $\quad c = 2.998 \times 10^8\ m/s$
$M = 5.972 \times 10^{24}\ kg$ $\quad G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

$$\frac{GM}{Rc^2} = \frac{6.67 \times 10^{-11}\ m^3/kg \cdot s^2\ (5.972 \times 10^{24}\ kg)}{(6.378 \times 10^6\ m)(2.998 \times 10^8\ m/s)^2} = 6.948 \times 10^{-10}$$

$$\boxed{\frac{GM}{Rc^2} = 6.95 \times 10^{-10}}$$

ii.) $R = 6.597 \times 10^8\ m$ $\quad c = 2.998 \times 10^8\ m/s$
$M = 1.989 \times 10^{30}\ kg$ $\quad G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

$$\frac{GM}{Rc^2} = \frac{(6.67 \times 10^{-11}\ m^3/kg \cdot s^2)(1.989 \times 10^{30}\ kg)}{(6.597 \times 10^8\ m)(2.998 \times 10^8\ m/s)^2} = 2.214 \times 10^{-6}$$

$$\boxed{\frac{GM}{Rc^2} = 2.214 \times 10^{-6}}$$

c.)

i) $G = 6.67 \times 10^{-11}\ m^3/kg \cdot s^2$ $\quad c = 2.998 \times 10^8\ m/s$
$M = 5.972 \times 10^{24}\ kg$

$$R_s = \frac{2GM}{c^2} = \frac{2 \cdot (6.67 \times 10^{-11}\ m^3/kg \cdot s^2)(5.972 \times 10^{24}\ kg)}{(2.998 \times 10^8\ m/s)^2} = 8.863 \times 10^{-3}\ m$$

$$\boxed{R_s = 8.863 \times 10^{-3}\ m}$$

Problem 1] Continued

ii) $G = 6.67\times10^{-11}\ m^3/kg\cdot s^2$ $c = 2.998\times10^{8}\ m/s$
$M = 1.989\times10^{30}\ kg$

$$R_s = \frac{2GM}{c^2} = \frac{2\cdot(6.67\times10^{-11}\ m^3/kg\cdot s^2)(1.989\times10^{30}\ kg)}{(2.998\times10^{8}\ m/s)^2} = 2{,}952.07\ m$$

$$\boxed{R_s = 2{,}952\ m}$$

d.)

i) $R_{s\odot}/R_\odot$

$R_{s\odot} = 2{,}952\ m$
$R_\odot = 6.597\times10^{8}\ m$

$$\frac{R_{s\odot}}{R_\odot} = \frac{2{,}952\ (m)}{6.597\times10^{8}\ (m)} = 4.4747\times10^{-6}$$

$$\boxed{\frac{R_{s\odot}}{R_\odot} = 4.475\times10^{-6}}$$

ii) $R_{s\oplus}/R_\oplus$

$R_{s\oplus} = 8.863\times10^{-3}\ m$
$R_\oplus = 6.378\times10^{6}\ m$

$$\frac{R_{s\oplus}}{R_\oplus} = \frac{8.863\times10^{-3}\ m}{6.378\times10^{6}\ m} = 1.390\times10^{-9}$$

$$\boxed{\frac{R_{s\oplus}}{R_\oplus} = 1.390\times10^{-9}}$$

The Schwarzschild radius is smaller than the radius of the respective object.

Problem 2

a.) $x = r\cos(\phi)$, $y = r\sin(\phi)$

$$\frac{dx}{dr} = \cos(\phi) , \quad \frac{dx}{d\phi} = -r\sin(\phi) , \quad \frac{dy}{dr} = \sin(\phi) , \quad \frac{dy}{d\phi} = r\cos(\phi)$$

$$dx = \cos(\phi)\cdot dr , \quad dx = -r\sin(\phi)\,d\phi , \quad dy = \sin(\phi)\,dr , \quad dy = r\cos(\phi)\,d\phi$$

$$\boxed{dx = \cos(\phi)\,dr - r\sin(\phi)\,d\phi , \quad dy = \sin(\phi)\,dr + r\cos(\phi)\,d\phi}$$

b.) $dx^2 = \cos^2(\phi)\,dr^2 - 2r\cos(\phi)\sin(\phi)\,dr\,d\phi + r^2\sin^2(\phi)\,d\phi^2$

$dy^2 = \sin^2(\phi)\,dr^2 + 2r\cos(\phi)\sin(\phi)\,dr\,d\phi + r^2\cos^2(\phi)\,d\phi^2$

$dx^2 + dy^2 = \cos^2(\phi)\,dr^2 + r^2\sin^2(\phi)\,d\phi^2 + \sin^2(\phi)\,dr^2 + r^2\cos^2(\phi)\,d\phi^2$

$= dr^2\left(\cos^2(\phi) + \sin^2(\phi)\right) + r^2 d\phi^2\left(\cos^2(\phi) + \sin^2(\phi)\right)$

$= (dr^2 + r^2 d\phi^2)(\cos^2(\phi) + \sin^2(\phi)) = (dr^2 + r^2 d\phi^2)$ ✓

$$\boxed{ds^2 = dr^2 + r^2 d\phi^2}$$

Problem 3

a.) $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$

$\frac{dx}{d\theta} = a\cos\theta\cos\phi$, $\frac{dx}{d\phi} = -a\sin\theta\sin\phi$, $\frac{dy}{d\theta} = a\cos\theta\sin\phi$, $\frac{dy}{d\phi} = a\sin\theta\cos\phi$, $\frac{dz}{d\theta} = -a\sin\theta$, $\frac{dz}{d\phi} = 0$

$$\boxed{dx = a\cos\theta\cos\phi\, d\theta - a\sin\theta\sin\phi\, d\phi \ , \quad dy = a\cos\theta\sin\phi\, d\theta + a\sin\theta\cos\phi\, d\phi \ , \quad dz = -a\sin\theta\, d\theta}$$

b.)

$$dx^2 = a^2\cos^2\theta\cos^2\phi\, d\theta^2 - 2a^2\sin\theta\cos\theta\sin\phi\cos\phi\, d\theta d\phi + a^2\sin^2\theta\sin^2\phi\, d\phi^2$$

$$dy^2 = a^2\cos^2\theta\sin^2\phi\, d\theta^2 + 2a^2\sin\theta\cos\theta\sin\phi\cos\phi\, d\theta d\phi + a^2\sin^2\theta\cos^2\phi\, d\phi^2$$

$$dz^2 = a^2\sin^2\theta\, d\theta^2$$

$$dx^2 + dy^2 + dz^2 = a^2\cos^2\theta\cos^2\phi\, d\theta^2 + a^2\sin^2\theta\sin^2\phi\, d\phi^2 + a^2\cos^2\theta\sin^2\phi\, d\theta^2 + a^2\sin^2\theta\cos^2\phi\, d\phi^2 + a^2\sin^2\theta\, d\theta^2$$

$$= a^2\cos^2\theta\, d\theta^2(\underbrace{\cos^2\phi + \sin^2\phi}_{=1}) + a^2\sin^2\theta\, d\phi^2(\underbrace{\sin^2\phi + \cos^2\phi}_{=1}) + a^2\sin^2\theta\, d\theta^2$$

$$= a^2\cos^2\theta\, d\theta^2 + a^2\sin^2\theta\, d\phi^2 + a^2\sin^2\theta\, d\theta^2$$

$$= a^2 d\theta^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) + a^2\sin^2\theta\, d\phi^2$$

$$= a^2(d\theta^2 + \sin^2\theta\, d\phi^2) \checkmark$$

$$\boxed{ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

Problem 4

$$ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$$

a.) constant θ means $d\theta = 0$: $ds^2 = a^2(0 + \sin^2\theta\cos^4\theta\, d\phi^2)$

$ds^2 = a^2 \cdot \sin^2\theta\cos^4\theta\, d\phi^2 \rightarrow ds = a \cdot \sin\theta\cos^2\theta\, d\phi$

Let $\theta \rightarrow \Theta$ to represent constant θ

$\therefore$ $\boxed{ds = a \cdot \sin\Theta\cos^2\Theta\, d\phi}$

b.) $C = \oint ds$

$$C = \int_0^{2\pi} a \cdot \sin\Theta\cos^2\Theta\, d\phi = 2\pi a\sin\Theta\cos^2\Theta$$

$$\boxed{C = 2\pi a\sin\Theta\cos^2\Theta}$$

c.) $C = 2\pi a\sin\Theta\cos^2\Theta$

$\frac{dc}{d\Theta} = 2\pi a\cos^3\Theta - 4\pi a\sin^2\Theta\cos\Theta$: $0 = 2\pi a\cos^3\Theta - 4\pi a\sin^2\Theta\cos\Theta$

$0 = 2\pi a\cos\Theta(\cos^2\Theta - 2\sin^2\Theta)$

$0 = 2\pi a\cos\Theta(1 - 3\sin^2\Theta)$: $0 = 2\pi a \cdot \cos\Theta$: $0 = 1 - 3\sin^2\Theta$

$0 = 2\pi a \cdot \cos\Theta \rightarrow \cos\Theta = 0$ $\Theta = \frac{\pi}{2}, \frac{3\pi}{2}$: $\Theta = \sum_{K=1}^{\infty} \frac{(2k+1)\pi}{2}$

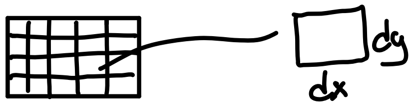
$0 = 1 - 3\sin^2\Theta \rightarrow \sin^2\Theta = \frac{1}{3} \rightarrow \sin\Theta = \sqrt{\frac{1}{3}}$: $\Theta = 35.26°, 144.74°$

Zeros occur @ $\Theta = 35.26°, 90°, 144.74°, 270°$

$$\boxed{\begin{array}{l} \Theta = 90°, 270° \text{ — A minimum} \\ \Theta = 35.26°, 144.74° \text{ — A maximum} \end{array}}$$

Problem 5

a.)



$dA_{map} = dxdy$

$$\boxed{dA = dxdy}$$

b.) $ds^2 = a^2\left(\left(\frac{d\lambda}{dy}\right)^2 dy^2 + \left(\frac{2\pi}{L}\right)^2 \cos^2[\lambda(y)]\, dx^2\right)$

$ds = a\left(\frac{d\lambda}{dy} dy + \left(\frac{2\pi}{L}\right)\cos[\lambda(y)]\, dx\right)$

y component   x componen

$$\boxed{dA = a\cdot\left(\frac{2\pi}{L}\right)\cos(\lambda(y))\cdot a\,\frac{d\lambda}{dy}\, dxdy}$$

c.) $dA = dA$:

$dx\cdot dy = a^2\cdot\left(\frac{2\pi}{L}\right)\cos(\lambda(y))\cdot\left(\frac{d\lambda}{dy}\right) dxdy$

$1 = a^2\cdot\left(\frac{2\pi}{L}\right)\cos(\lambda(y))\cdot\left(\frac{d\lambda}{dy}\right)$

$\frac{L}{a^2\cdot 2\pi}\cdot\frac{1}{\cos(\lambda(y))} = \frac{d\lambda}{dy}$

$$\boxed{\frac{d\lambda}{dy} = \frac{L}{a^2\cdot 2\pi}\cdot\sec(\lambda(y))}$$

d.)

$\frac{d\lambda}{dy} = \frac{L}{a^2\cdot 2\pi}\cdot\frac{1}{\cos(\lambda(y))} \to \cos(\lambda(y))\, d\lambda = \frac{L}{a^2\cdot 2\pi}\cdot dy$

$\int_0^\lambda \cos(\lambda(y))\, d\lambda = \frac{L}{a^2\cdot 2\pi}\int_0^y dy \to \sin(\lambda(y))\Big|_0^\lambda = \frac{L}{a^2\cdot 2\pi}\cdot y(\lambda)$

$\sin(\lambda) = \frac{L}{a^2\cdot 2\pi}\cdot y \to y = \frac{a^2\cdot 2\pi}{L}\sin(\lambda)$, $\lambda = \sin^{-1}\left(\frac{L}{a^2\cdot 2\pi}y\right)$

$$\boxed{y(\lambda) = \frac{a^2\cdot 2\pi}{L}\sin(\lambda)\ ,\quad \lambda(y) = \sin^{-1}\left(\frac{L}{a^2\cdot 2\pi}y\right)}$$

e.) $ds^2 = a^2\left(\left(\frac{d\lambda}{dy}\right)^2 dy^2 + \left(\frac{2\pi}{L}\right)^2\cos^2[\lambda(y)]\, dx^2\right)$

$\frac{d\lambda}{dy} = \frac{L}{\sqrt{(-Ly+2a^2\pi)(Ly+2a^2\pi)}} = \frac{L}{\sqrt{-L^2y^2+4\pi^2a^4}} = \frac{1}{\sqrt{\frac{4\pi^2a^4}{L^2} - y^2}}$: $\left(\frac{d\lambda}{dy}\right)^2 = \frac{1}{(4\pi^2a^4/L^2 - y^2)}$

$(-Ly+2a^2\pi)(Ly+2a^2\pi) = -L^2y^2 + 4\pi^2a^4 = 4\pi^2a^4 - L^2y^2 = \frac{4\pi^2a^4}{L^2} - y^2$

dy part

$\cos^2(\lambda(y)) = \cos^2\left(\sin^{-1}\left(\frac{L}{a^2\cdot 2\pi}y\right)\right) = \frac{(-Ly+2a^2\pi)(Ly+2a^2\cdot\pi)}{4a^4\pi^2} = \frac{-L^2y^2+4a^4\pi^2}{4a^4\pi^2}$

Problem 5 | continued

$$ds^2 = a^2\left[\frac{1}{\left(\frac{4\pi^2 a^4}{L^2} - y^2\right)} dy^2 + \left(\frac{4\pi^2}{L^2}\right)\left(\frac{-L^2y^2 + 4a^4\pi^2}{4a^4\pi^2}\right)dx^2\right] \qquad f(y) = \frac{a}{\sqrt{4\pi^2a^4/L^2 - y^2}}$$

$$\frac{4\pi^2}{L^2}\left(\frac{-L^2y^2 + 4a^4\pi^2}{4a^4\pi^2}\right) = \frac{1}{L^2}\left(\frac{-L^2y^2 + 4a^4\pi^2}{a^4}\right) = 1\left(\frac{-y^2 + 4a^4\pi^2/L^2}{a^4}\right) = \frac{-y^2 + 4a^4\pi^2/L^2}{a^4}$$

$$a^2 \cdot \left(\frac{4a^4\pi^2/L^2 - y^2}{a^4}\right) = \frac{\frac{4a^4\pi^2}{L^2} - y^2}{a^2}$$ dx part

If we make the assumption that $f(y) = \frac{a}{\sqrt{4\pi^2a^4/L^2 - y^2}}$ the line element becomes:

$$\boxed{ds^2 = \frac{1}{f^2(y)} dx^2 + f^2(y) dy^2}$$