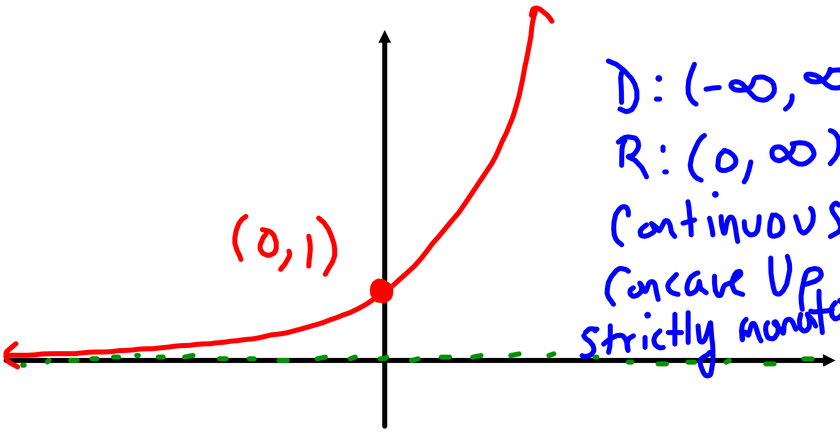
How would we graph the function $f(x) = e^x$? What is the relationship between $f(x) = e^x$ and $g(x) = \ln x$? Inverses of each other

Domain? Range? Continuous? Concavity? Strictly Monotonic?



D: $(-\infty, \infty)$

R: $(0, \infty)$

Continuous

Concave Up

Strictly monotonic

H.A.

$y = 0$

Recall from Intermediate Algebra:

Laws of Exponents:

1. $a^1 = a$
2. $a^0 = 1$
3. $0^0 = undefined$
4. $a^m \cdot a^n = a^{m+n}$
5. $\dfrac{a^m}{a^n} = a^{m-n}$
6. $(a^m)^n = a^{mn}$
7. $(ab)^n = a^n b^n$
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Laws of Negative Exponents:

1. $a^{-n} = \dfrac{1}{a^n}$
2. $\dfrac{1}{a^{-n}} = a^n$
3. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Derivative of the Natural Exponential Function:

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}\left[e^x\right] = e^x$, that is, if $f(x) = e^x$, then $f'(x) = e^x$.
2. $\dfrac{d}{dx}\left[e^u\right] = e^u \dfrac{du}{dx}$, that is, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$.

Proof of 1:

Derivative of the Natural Exponential Function:

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}\left[e^x\right] = e^x$, that is, if $f(x) = e^x$, then $f'(x) = e^x$.
2. $\dfrac{d}{dx}\left[e^u\right] = e^u \dfrac{du}{dx}$, that is, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$.

Proof of 1:

$$\ln y = \ln e^x$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[x\right]$$

$$y\left(\frac{1}{y}\frac{dy}{dx} = 1\right)$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = e^x \quad \boxed{MW}$$

Ex: Find the derivative of the following functions:

a) $y = 5e^{x^2 + 5}$

b) $g(t) = \left(e^{-t} + e^{t}\right)^3$

c) $y = \dfrac{2}{\left(e^{x} + e^{-x}\right)}$

Ex: Find the equation of the tangent line to the graph of $f(x) = e^{-x}\ln x$ at the point (1, 0).

Ex: Find the derivative of the following functions:

a) $y = 5e^{(x^2 + 5)}$

$y' = 5e^{x^2+5}(2x)$

$\boxed{y' = 10xe^{x^2+5}}$

b) $g(t)=\left(e^{-t}+e^{t}\right)^{3}$

$$\boxed{g'(t)=3\left(e^{-t}+e^{t}\right)^{2}\left(-e^{-t}+e^{t}\right)}$$

$f(x)=e^{x}$ $\quad$ $f^{100}(x)=e^{x}$

$f(x)=e^{-x}$ $\quad$ $f^{100}(x)=e^{-x}$

c) $y = \dfrac{2}{\left(e^{x} + e^{-x}\right)}$

$e^{-x} \cdot -1 = -e^{-x}$

$$y' = \frac{0 \cdot (e^{x} + e^{-x}) - 2(e^{x} - e^{-x})}{(e^{x} + e^{-x})^{2}}$$

$$\boxed{y' = \frac{-2(e^{x} - e^{-x})}{(e^{x} + e^{-x})^{2}}}$$

Ex: Find the equation of the tangent line to the graph of
$f(x) = e^{-x}\ln x$ at the point (1, 0).

$y = mx + b$

$m = f'(1) = \frac{1}{e}$

$f'(x) = -e^{-x}\cdot \ln x + \frac{1}{x}\cdot e^{-x}$

$f'(1) = \cancel{-e^{-1}\ln(1)} + \frac{1}{1}e^{-1} = \frac{1}{e}$

$0 = \frac{1}{e}(1) + b$

$0 = \frac{1}{e} + b$

$-\frac{1}{e} = b$

$\boxed{y = \frac{1}{e}x - \frac{1}{e}}$

Ex: Use ***implicit differentiation*** to find $\frac{dy}{dx}$: $\cos x^2 = xe^y$

Ex: Find the extrema and the points of inflection (if any exist) of the function $f(x) = x^2 e^{-x}$

Integrals Involving Exponential and Logarithmic Functions: Let u be a differentiable function of x.

1. $\int e^x \, dx = e^x + C$
2. $\int e^u \frac{du}{dx} dx = \int e^u du = e^u + C$

Note: When integrating the above functions, it often helps to use the method of u-substitution (or change of variables), or pattern recognition. Also, it helps to rewrite the function so that you can use the rules of integration that we've established earlier.

Ex: Find the following indefinite integrals:

a) $\int e^{1-3x} \, dx$

Ex: Use ***implicit differentiation*** to find $\dfrac{dy}{dx}$:

$\cos(x^2) = xe^y$

$$\frac{d}{dx}\left[\cos x^2\right] = \frac{d}{dx}\left[xe^y\right]$$

$$-2x\sin x^2 = 1\cdot e^y + e^y\cdot\frac{dy}{dx}\cdot x$$

$$-2x\sin x^2 = e^y + xe^y\frac{dy}{dx}$$

$$-2x\sin x^2 - e^y = xe^y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{-2x\sin x^2 - e^y}{xe^y}}$$

Ex: Find the extrema and the points of inflection (if any exist) of the function $f(x) = x^2 e^{-x}$

$f'(x) = -x^2 e^{-x} + 2xe^{-x}$

Crit. #'s: $0 = -xe^{-x}(x-2)$

$x = 0, 2$

$f''(x) = -2xe^{-x} + x^2 e^{-x} + 2e^{-x} - 2xe^{-x}$

$f''(x) = -4xe^{-x} + x^2 e^{-x} + 2e^{-x}$

$f''(0) = 2$ $\boxed{\text{Relative Min @ } (0,0)}$

$f''(2) = -8e^{-2} + 4e^{-2} + 2e^{-2} = -2e^{-2} < 0$

$\boxed{\text{Relative Max @ } \left(2, \frac{4}{e^2}\right)}$

$0 = e^{-x}(x^2 - 4x + 2)$

$x \approx 0.586, 3.414$

$(-\infty, 0.586)$ $(0.586, 3.414)$ $(3.414, \infty)$

@ $x=0$ >0 @ $x=1$ <0 @ $x=4$ >0

$\boxed{\begin{array}{l} \text{P.O.I: } (0.586, 0.191) \\ \qquad\quad\; (3.414, 0.384) \end{array}}$

Ex: Find the following indefinite integrals:

a) $\int e^{1-3x}\,dx$

$u = 1-3x$

$du = -3\,dx$

$-\frac{1}{3}du = dx$

$= -\frac{1}{3}\int e^{u}\,du$

$= -\frac{1}{3}e^{u} + C$

$= \boxed{-\frac{1}{3}e^{1-3x} + C}$

b) $\int \frac{e^{-x}}{1+e^{-x}}\,dx$

c) $\int e^{-x}\tan\left(e^{-x}\right)dx$

Ex: Evaluate the definite integral: $\int_0^1 xe^{-x^2}\,dx$

Ex: Find the area of the region bounded by the graphs of the equations:

$y = xe^{-x^2/4}, \quad y = 0, \quad x = 0, \quad x = \sqrt{6}$

b) $\displaystyle\int \frac{e^{-x}}{1+e^{-x}}\,dx$

$u = 1+e^{-x}$

$du = -e^{-x}\,dx$

$-du = \boxed{e^{-x}\,dx}$

$-\displaystyle\int \frac{1}{u}\,du$

$= -\ln|u| + C$

$= \boxed{-\ln|1+e^{-x}| + C}$

c) $\int e^{-x}\tan\left(e^{-x}\right)dx$

$u = e^{-x}$

$du = -e^{-x}\,dx$

$-du = e^{-x}\,dx$

$-\int \tan u \, du$

$-\left(-\ln|\cos u| + C\right)$

$= \boxed{\ln\left|\cos e^{-x}\right| + C}$

Ex: Evaluate the definite integral: $\int_0^1 xe^{-x^2}\,dx$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$-\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^u$

$= -\frac{1}{2}e^{-x^2}\Big]_0^1$

$= -\frac{1}{2}e^{-1} - -\frac{1}{2}$

$= \boxed{-\frac{1}{2e} + \frac{1}{2}}$

Ex: Find the area of the region bounded by the graphs of the equations:

$$y = xe^{-x^2/4}, \quad y = 0, \quad x = 0, \quad x = \sqrt{6}$$

$$A = \int_0^{\sqrt{6}} xe^{-x^2/4}\,dx \approx \boxed{1.554 \ u^2}$$