Physics 396

Homework Set 5

1. Consider the line element of the form

$$ds^2 = -\left(1 + \frac{gx'}{c^2}\right)^2 c^2 dt'^2 + dx'^2 + dy'^2 + dz'^2, \tag{1}$$

where g is a constant with units of acceleration. This line element describes an observer in an *accelerating* frame of reference with coordinates (t', x', y', z').

a) Consider an observer in this frame of reference, Alyce, who remains at a fixed location $x' = h$. Alyce *emits* two light signals and therefore measures the *proper time* interval between these two events. Obtain an expression for this proper time, $\Delta\tau_A$, in terms of the coordinate time $\Delta t'$.

b) Consider a second observer in this same frame of reference, Bob, who remains at a fixed location $x' = 0$. Bob *receives* the two light signals at his location and therefore measures the *proper time* interval between these two events. Obtain an expression for this proper time interval, $\Delta\tau_B$, in terms of the coordinate time $\Delta t'$.

c) As the coordinate times in a) and b) are the same, obtain an expression connecting $\Delta\tau_A$ and $\Delta\tau_B$. Who measures the longer time interval?

2. Consider the line element in Eq. (1) and the coordinate transformations

$$\begin{aligned} ct' &= \frac{c^2}{g}\tanh^{-1}\left(\frac{ct}{x + c^2/g}\right), \\ x' &= \left[\left(x + \frac{c^2}{g}\right)^2 - c^2t^2\right]^{1/2} - \frac{c^2}{g}, \\ y' &= y \quad , \quad z' = z. \end{aligned} \tag{2}$$

By inserting the coordinate transformations of Eqs. (2) into the line element of Eq. (1), show that Eq. (1) takes the form of that of flat Minkowski spacetime. *Hint*: Keep the combination $(x + c^2/g)$ as one when performing the calculation.

Since this coordinate transformation brings the line element to the form of Minkowski spacetime, and since the line element is an *invariant* quantity, this implies that Eq. (1) is also that of Minkowski spacetime, but in the coordinates of an *accelerating* frame.

3. Consider the Minkowski line element of the form

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{3}$$

in (t, x, y, z) coordinates. Also consider the coordinate transformations

$$\begin{aligned}
ct &= \left(\frac{c^2}{g} + x'\right) \sinh\left(\frac{gt'}{c}\right), \\
x &= \left(\frac{c^2}{g} + x'\right) \cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}, \\
y &= y' \ , \quad z = z',
\end{aligned} \tag{4}$$

where g is a constant with units of acceleration. By inserting the coordinate transformations of Eqs. (4) into the line element of Eq. (3), show that Eq. (3) takes the form of Eq. (1).

4. a) Show that the coordinate transformations of Eqs. (4) equate to the *inverse* coordinate transformations of Eqs. (2).

   b) Show that in the limit when $(ct', x') \ll c^2/g$, the coordinate transformations of Eqs. (4) reduce to the approximate form

$$\begin{aligned}
ct &= ct' \left(1 + o(gx'/c^2)\right) \simeq ct' \\
x &= x' + \frac{1}{2} g t'^2 \left(1 + o(gx'/c^2)\right) \simeq x' + \frac{1}{2} g t'^2.
\end{aligned} \tag{5}$$

   Notice that this corresponds to the coordinate transformations of a *uniformly accelerating frame* relative to an inertial frame in Newtonian mechanics.

   c) Notice that the ratio c^2/g defines a length scale (and therefore a time scale) where Eqs. (5) are valid. Consider an accelerating frame where g is set equal to the value of acceleration for a freely-falling object. Calculate the length scale (in light years) and time scale (in years) of this ratio of constants.

5. The coordinate transformations given in Eq. (4) of Problem 3 equate to going from the inertial coordinates of Minkowski spacetime (t, x, y, z) to the coordinates of an *accelerating* observer (t', x', y', z') relative to this Minkowski spacetime.

   a) Using Eqs. (1) and (4) and the chain rule, compute the *velocity four-vector* of a point in this accelerating frame, namely,

$$u^\alpha = \frac{dx^\alpha}{d\tau}. \tag{6}$$

   b) Compute the *acceleration four-vector* of a point in this accelerating frame, namely,

$$a^\alpha = \frac{d^2x^\alpha}{d\tau^2}. \tag{7}$$

   c) Compute the *invariant* magnitude of the acceleration four vector, namely,

$$a \equiv \left(\eta_{\alpha\beta} a^\alpha a^\beta\right)^{1/2}. \tag{8}$$

   d) How does this invariant quantity differ for an observer at $x' = h$ compared to one at $x' = 0$?