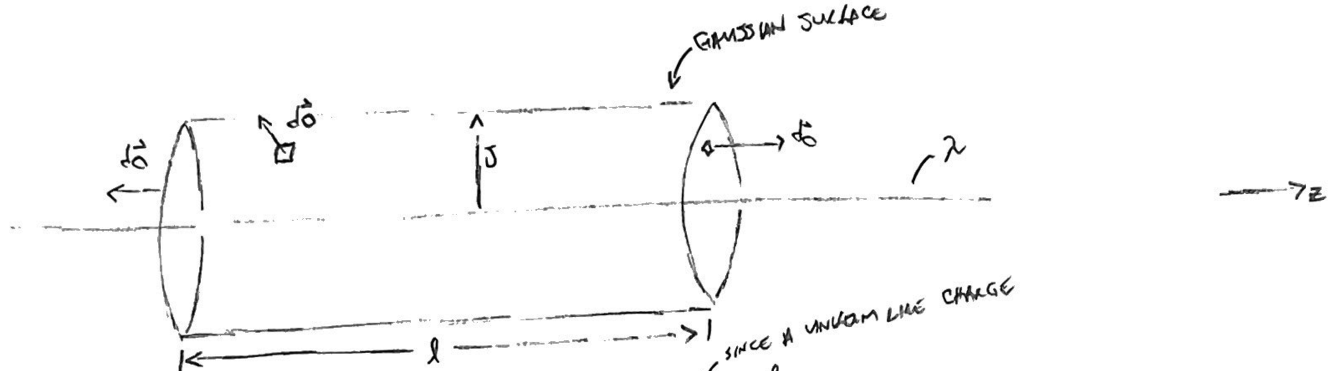
PHYS 311 HOMEWORK SET 5

1.



SINCE A UNIFORM LINE CHARGE

$$\lambda = \frac{dq}{dz} \quad \therefore \quad dq = \lambda dz \quad \text{so } q = \int dq = \int \lambda dz = \lambda \int_0^l dz = \lambda l$$

NOW GAUSS' LAW IS OF THE FORM...

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

FOR MY GAUSSIAN SURFACE (TUBE) : $d\vec{a} = s\, d\phi\, dz\, \hat{s}$ (DO YOU AGREE?)

FOR THE ENDCAPS : $d\vec{a} = s\, d\phi\, ds\, (\pm \hat{z})$

DUE TO THE CYLINDRICAL-SYMMETRY OF THE PROBLEM : $\vec{E} = E\hat{s}$

SINCE E IS A CONSTANT AT GIVEN s

SO

$$\oint \vec{E} \cdot d\vec{a} = \underbrace{\int_{\text{LEFT ENDCAP}} \cancel{\vec{E} \cdot d\vec{a}}^{\,0} + \int_{\text{RIGHT ENDCAP}} \cancel{\vec{E} \cdot d\vec{a}}^{\,0}}_{\text{THESE ARE ZERO SINCE } \vec{E} \perp d\vec{a} \text{ AT ENDCAPS}} + \int_{\text{TUBE}} \vec{E} \cdot d\vec{a} = \int_{\text{TUBE}} E\, da = E \int_{\text{TUBE}} da = E \int_0^l \int_0^{2\pi} s\, d\phi\, dz = E \cdot 2\pi s l$$

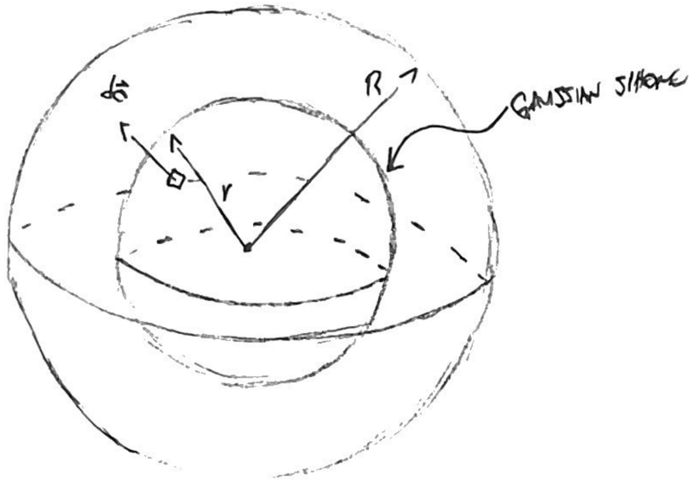
SINCE $E\hat{s} \cdot da\,\hat{s} = E\,da(\hat{s} \cdot \hat{s}) = E\,da$

SO GAUSS' LAW YIELDS

$$E \cdot 2\pi s l = \frac{\lambda l}{\varepsilon_0} \quad \text{so} \quad E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{s}$$

$$\text{so} \quad \boxed{\vec{E} = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{s} \hat{s}}$$

2.



NOW THE CHARGE ENCLOSED BY THE GAUSSIAN SPHERE IS...

$$Q_{enc} = \int \rho \, d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} k r^{\pi} \cdot r^2 \sin\theta \, dr\, d\theta\, d\varphi = k \int_0^{r} r^{(2+\pi)} dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= k \cdot \frac{r^{3+\pi}}{(3+\pi)} \Big|_0^{r} \cdot 2 \cdot 2\pi = \frac{4\pi k}{(3+\pi)} r^{3+\pi}$$

NOW

SINCE E IS A CONSTANT AT A GIVEN r

$$\oint \vec{E} \cdot d\vec{a} = \int E\, da = E \int da = E\, 4\pi r^2$$

SINCE $\vec{E} = E\hat{r}$; $d\vec{a} = da\,\hat{r}$

SO $\vec{E} \cdot d\vec{a} = E\, da$

NOW USING GAUSS' LAW...

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{YIELDS} \quad E\, 4\pi r^2 = \frac{4\pi k}{\varepsilon_0 (3+\pi)} r^{3+\pi}$$

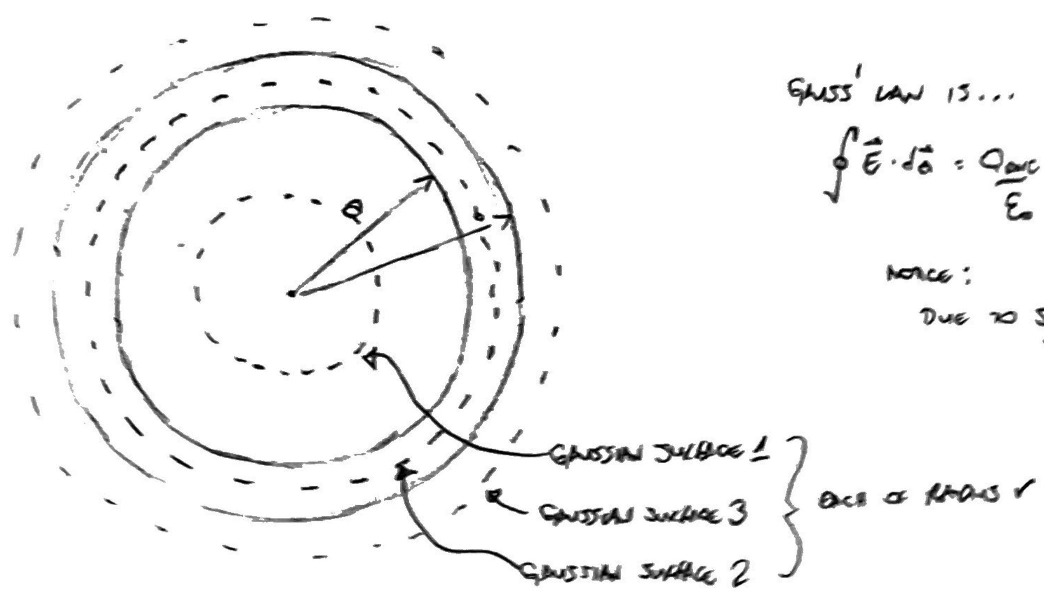
SO

$$E = \frac{k}{\varepsilon_0 (3+\pi)} r^{1+\pi}$$

SO

$$\boxed{\vec{E} = \frac{k}{\varepsilon_0 (3+\pi)} r^{1+\pi} \hat{r}}$$

3.



Gauss' law is...

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Notice:
Due to spherical symmetry, $\vec{E} = E\hat{r}$

a) For $r<a$, consider Gaussian surface 1...

$$Q_{enc} = 0 \quad \therefore \quad \vec{E} = 0$$

b) For $a<r<b$, consider Gaussian surface 2...

$$Q_{enc} = \int \rho \, d\tau = \int_0^{2\pi}\int_0^{\pi}\int_a^{r} k r^{\pi} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi = k \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_a^{r} r^{\pi+2} \, dr$$

$$= k \cdot 2\pi \cdot 2 \cdot \frac{r^{\pi+3}}{(\pi+3)}\Big|_a^r = \frac{4\pi k}{(\pi+3)}\left[r^{\pi+3} - a^{\pi+3}\right]$$

$$\oint \vec{E} \cdot d\vec{a} = \underbrace{\oint E \, da}_{\text{since } \vec{E} \parallel d\vec{a}} = \underbrace{E \oint da}_{\text{since the magnitude of } E \text{ must be a constant at a given } r} = E 4\pi r^2$$

Now Gauss' law reads...

$$E 4\pi r^2 = \frac{4\pi k}{\varepsilon_0(\pi+3)}\left[r^{\pi+3} - a^{\pi+3}\right] \quad \text{so} \quad E = \frac{k}{\varepsilon_0(\pi+3)}\left[r^{\pi+1} - \frac{a^{\pi+3}}{r^2}\right]$$

so

$$\vec{E} = \frac{k}{(\pi+3)\varepsilon_0}\left[r^{\pi+1} - \frac{a^{\pi+3}}{r^2}\right]\hat{r}$$

c) for $r > b$, CONSIDER GAUSSIAN SURFACE 3...

$$Q_{enc} = \int \rho \, d\tau = \int_0^{2\pi}\int_0^{\pi}\int_a^b k r^{\pi} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

– THIS IS THE SAME INTEGRATION AS THAT DONE IN b) W/ THE EXCEPTION OF THE UPPER LIMIT ON THE 'r INTEGRAL'

$$= 4\pi k \frac{r^{\pi+3}}{(\pi+3)}\Bigg|_a^b = \frac{4\pi k}{(\pi+3)}\left[b^{\pi+3} - a^{\pi+3}\right]$$

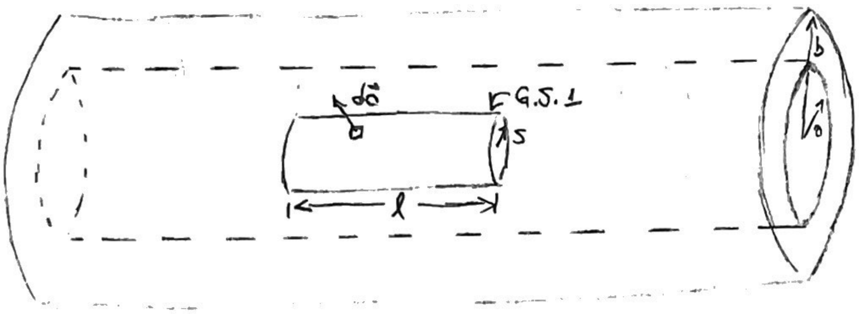
$$\oint \vec{E} \cdot d\vec{a} = E 4\pi r^2$$ – SAME RATIONALE AS THAT USED IN b)

so

$$E 4\pi r^2 = \frac{4\pi k}{(\pi+3)\varepsilon_0}\left[b^{\pi+3} - a^{\pi+3}\right]$$

so

$$\boxed{\vec{E} = \frac{k}{(\pi+3)\varepsilon_0}\left[b^{\pi+3} - a^{\pi+3}\right]\frac{\hat{r}}{r^2}}$$



NOTICE:

- DUE TO THE CYLINDRICAL-SYMMETRY THE $\vec{E}$ FIELD MUST BE OF THE FORM;

$$\vec{E} = E\hat{s}$$

NOW, WE HAVE A <u>VOLUME</u> CHARGE DENSITY...

$$\rho(s) = \rho_0 \frac{s}{a} \quad \text{for } s < a$$

AND A UNIFORM <u>SURFACE</u> CHARGE DENSITY..

$$\sigma = \sigma_0 \quad \text{for } s = b$$

a) FIRST CONSIDER $s < a$...

$$Q_{enc} = \int \rho \, d\tau = \int_0^l \int_0^{2\pi} \int_0^s \rho_0 \frac{s}{a} \cdot s \, ds \, d\phi \, dz = \frac{\rho_0}{a} \int_0^l dz \int_0^{2\pi} d\phi \int_0^s s^2 ds = \frac{\rho_0}{a} \cdot l \cdot 2\pi \cdot \frac{1}{3} s^3$$

$$= \frac{2\pi}{3} \rho_0 \frac{l s^3}{a}$$

NOW

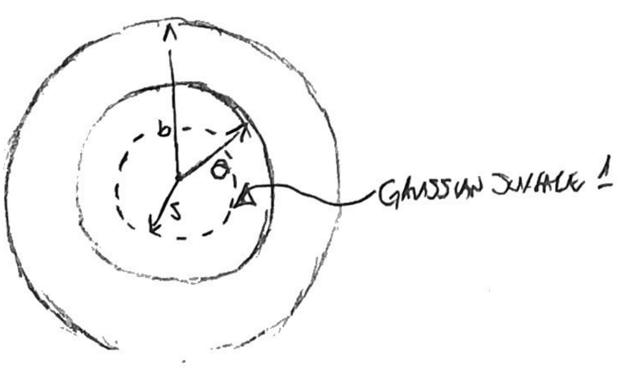
$$\oint \vec{E} \cdot d\vec{a} = \underbrace{\int_{\text{LEFT END CAP}} \cancel{\vec{E} \cdot d\vec{a}}^{\,0} + \int_{\text{RIGHT ENDCAP}} \cancel{\vec{E} \cdot d\vec{a}}^{\,0}}_{\text{SINCE } \vec{E} \perp d\vec{a} \text{ ON ENDCAPS}} + \int_{\text{TUBE}} \vec{E} \cdot d\vec{a} \overset{\text{SINCE } \vec{E} \parallel d\vec{a}}{=} \int_{\text{TUBE}} E \, da = E \int da = E 2\pi s l$$

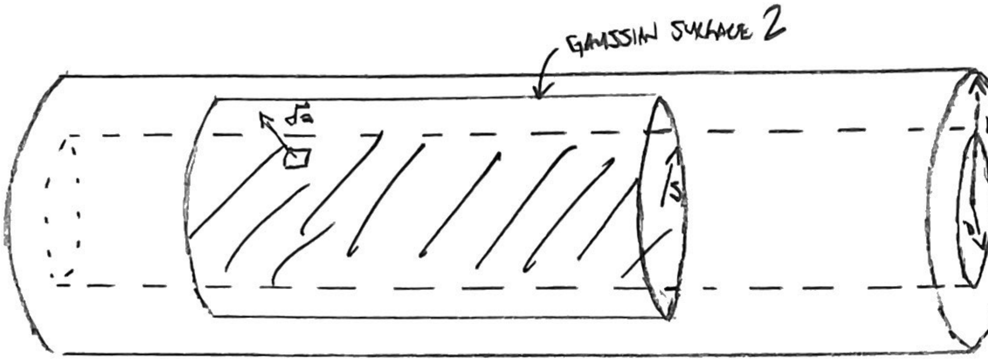
SINCE THE MAGNITUDE OF $\vec{E}$ MUST BE CONSTANT AT A GIVEN s

NOW, USING GAUSS' LAW...

$$E 2\pi s l = \frac{1}{\varepsilon_0} \frac{2\pi}{3} \rho_0 l \frac{s^3}{a} \quad \therefore \quad E = \frac{\rho_0}{3\varepsilon_0} \frac{s^2}{a} \quad \text{so} \quad \left[\vec{E}_1 = \frac{\rho_0}{3\varepsilon_0} \frac{s^2}{a} \hat{s} \right]$$

LOOKING DOWN THE Z-AXIS..





b) NOW CONSIDER $a<s<b$..

$$Q_{enc} = \int \rho \, d\tau = \int_0^l \int_0^{2\pi} \int_0^a \frac{\rho_0 s}{a} \cdot s \, ds \, d\phi \, dz = \frac{\rho_0}{a} \int_0^l dz \int_0^{2\pi} d\phi \int_0^a s^2 ds = \frac{\rho_0}{a} \cdot l \cdot 2\pi \cdot \frac{1}{3} s^3 \Big|_0^a$$

$$= \frac{2\pi}{3} \rho_0 l a^2$$

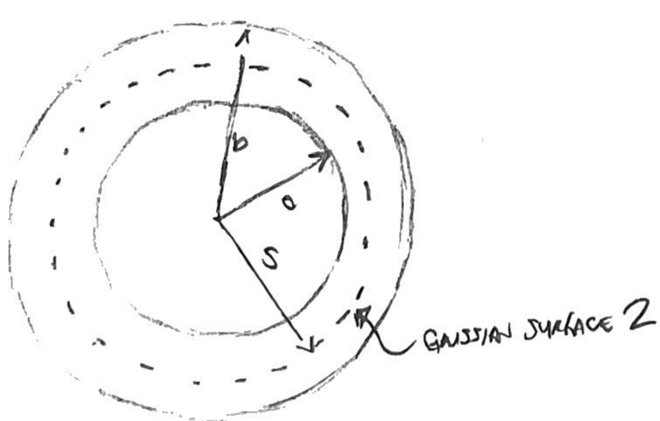
NOW

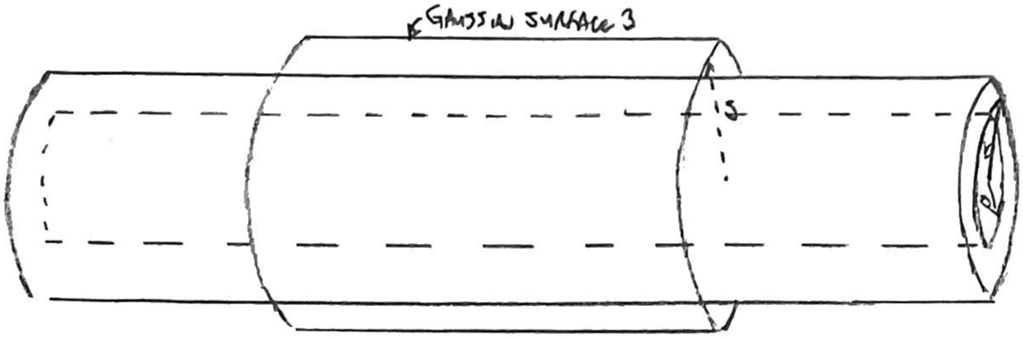
$$\oint \vec{E} \cdot d\vec{a} = E 2\pi s l \quad \text{(SAME RATIONAL AS PART a))}$$

NOW, USING GAUSS' LAW...

$$E 2\pi s l = \frac{1}{\varepsilon_0} \frac{2\pi}{3} \rho_0 l a^2 \quad \text{so} \quad E = \frac{\rho_0}{3\varepsilon_0} \frac{a^2}{s} \quad \text{so} \quad \left[\vec{E}_2 = \frac{\rho_0}{3\varepsilon_0} \frac{a^2}{s} \hat{s} \right]$$

LOOKING DOWN THE Z AXIS...





c) NOW CONSIDER $s > b$...

- THE CHARGE ENCLOSED BY THIS GAUSSIAN SURFACE OF THE VOLUME CHARGE DENSITY IS THE SAME AS THAT FOUND IN PART b). DO YOU SEE WHY?
- WE ALSO HAVE THE SURFACE CHARGE DENSITY, WHICH IS ENCLOSED BY G.S. 3

$$Q_{enc} = \int \sigma \, da = \underset{\uparrow}{\sigma} \int da = \sigma \, 2\pi b \ell$$

CONSTANT SURFACE CHARGE DENSITY

SO

$$Q_{enc,tot} = Q_{enc,volume} + Q_{enc,surface} = \frac{2\pi}{3} \rho_0 \ell a^2 + \sigma \, 2\pi b \ell$$

NOW

$$\oint \vec{E} \cdot d\vec{a} = E \, 2\pi s \ell \quad \text{(SAME RATIONAL AS PART a))}$$

NOW, USING GAUSS' LAW..

$$E \, 2\pi s \ell = \frac{1}{\varepsilon_0} \left[\frac{2\pi}{3} \rho_0 \ell a^2 + \sigma 2\pi b \ell \right] = \frac{2\pi \ell}{\varepsilon_0} \left[\frac{\rho_0}{3} a^2 + \sigma b \right]$$

SO

$$E = \frac{1}{\varepsilon_0} \left[\frac{1}{3} \rho_0 a^2 + \sigma b \right] \frac{1}{s} \qquad \text{SO} \qquad \boxed{\vec{E}_3 = \frac{1}{\varepsilon_0} \left[\frac{1}{3} \rho_0 a^2 + \sigma b \right] \frac{\hat{s}}{s}}$$

EVALUATE $\vec{E}_3$ AT $s = b + \varepsilon$, $\vec{E}_2$ AT $s = b - \varepsilon$, SUBTRACT, AND TAKE THE $\lim \varepsilon \to 0$..

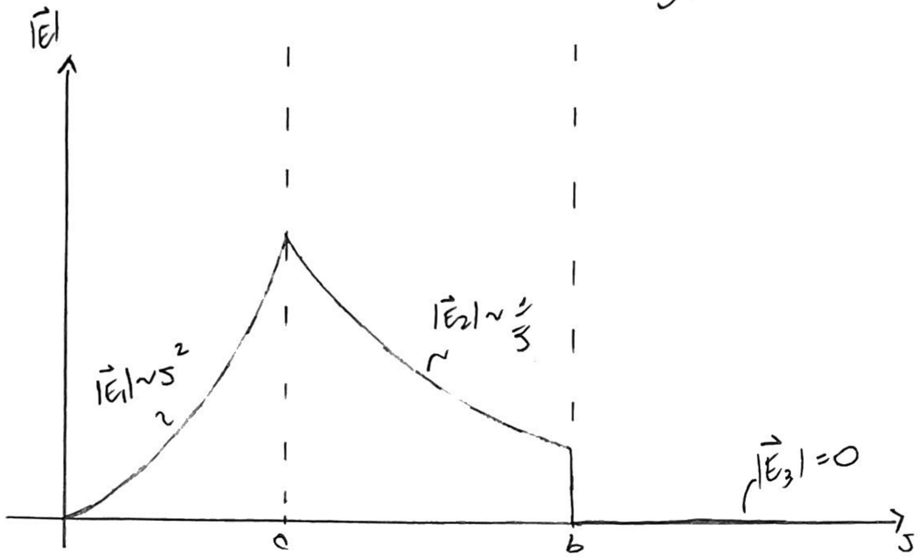
$$\vec{E}_3(s=b) - \vec{E}_2(s=b) = \frac{1}{\varepsilon_0} \left[\cancel{\frac{1}{3} \rho_0 \frac{a^2}{b}} + \sigma b \right] \frac{\hat{s}}{b} - \cancel{\frac{\rho_0}{3\varepsilon_0} \frac{a^2}{b}} \hat{s} = \frac{\sigma}{\varepsilon_0} \hat{s}$$

- THIS IS THE BOUNDARY CONDITION ON THE PERPENDICULAR COMPONENT OF THE $\vec{E}$-FIELD WHEN CROSSING A SURFACE CHARGE DENSITY!

NOW, FOR THIS PARTICULAR CHARGE ENSEMBLE, NOTICE THAT THE NET CHARGE ENCLOSED BY GAUSSIAN SURFACE #3 IS ZERO $\therefore$ $\vec{E}_3 = 0$ AS $\frac{1}{3}\rho_0 a^2 + \sigma b = 0$ SO

$$\sigma = -\frac{\rho_0 a^2}{3b}$$

d/



$\vec{E}_1 = K \frac{s^2}{a} \hat{s}$ WHERE $K \equiv \frac{\rho_0}{3\epsilon_0}$

$\vec{E}_2 = K \frac{a^2}{s} \hat{s}$

$\vec{E}_3 = \frac{1}{\epsilon_0}\left[\frac{1}{3}\rho_0 a^2 + \sigma b\right]\frac{\hat{s}}{s} = 0\,\hat{s}$ FOR THIS PARTICULAR SURFACE CHARGE DENSITY.

5.

AN ELECTROSTATIC FIELD MUST HAVE ZERO CURL, $\vec{\nabla} \times \vec{E} = 0$

a) $$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ kx^2y & 3kyz^2 & 2kxz^3 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}(2kxz^2) - \frac{\partial}{\partial z}(3kyz^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(kx^2y) - \frac{\partial}{\partial x}(2kxz^3)\right) + \hat{z}\left(\frac{\partial}{\partial x}(3kyz^2) - \frac{\partial}{\partial y}(kx^2y)\right)$$

$$= -(6kyz)\hat{x} - (2kz^3)\hat{y} - (kx^2)\hat{z}$$ $\therefore$ NOT AN ELECTROSTATIC FIELD

b) $$\vec{\nabla} \times \vec{E} = k\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y^2 & (6xy+3z^2) & 6yz \end{vmatrix}$$

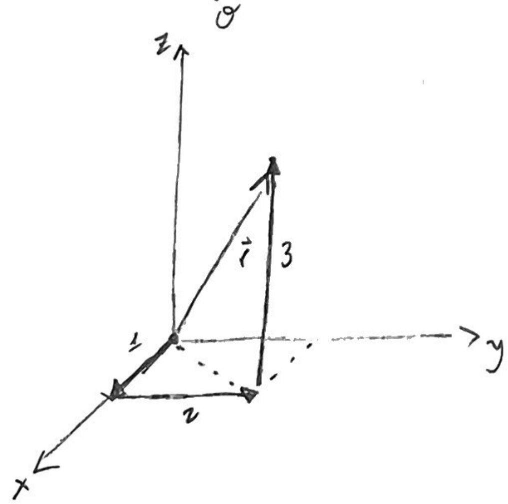
$$= \left[\hat{x}\left(\frac{\partial}{\partial y}(6yz) - \frac{\partial}{\partial z}(6xy+3z^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}(3y^2) - \frac{\partial}{\partial x}(6yz)\right) + \hat{z}\left(\frac{\partial}{\partial x}(6xy+3z^2) - \frac{\partial}{\partial y}(3y^2)\right)\right]k$$

$$= \left[\hat{x}(6z - 6z) + \hat{z}(6y - 6y)\right]k = \vec{0}$$ $\therefore$ COULD REPRESENT AN ELECTROSTATIC FIELD

NOW

$$V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

WE'LL INTEGRATE ALONG PATH 1, THEN PATH 2, THE PATH 3 TO GET TO $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$



FIRST ALONG PATH 1...

$$d\vec{l} = dx\,\hat{x}$$

SO $$\vec{E} \cdot d\vec{l} = 3ky^2 dx \Big|_{y=0=z} = 0$$

SO

$$\int_{\text{PATH 1}} \vec{E} \cdot d\vec{l} = 0$$

SECOND, ALONG PATH 2...

$$d\vec{l} = dy\,\hat{y} \quad \text{so} \quad \vec{E}\cdot d\vec{l} = k(6xy + 3z^2)\,dy\Big|_{z=0} = 6kxy\,dy$$

so

$$\int_{\text{PATH 2}} \vec{E}\cdot d\vec{l} = \int_0^y 6kxy\,dy = 6kx\int_0^y y\,dy = 6kx\,\frac{y^2}{2} = 3kxy^2$$

THIRD, ALONG PATH 3...

$$d\vec{l} = dz\,\hat{z} \quad \text{so} \quad \vec{E}\cdot d\vec{l} = 6kyz\,dz$$

so

$$\int_{\text{PATH 3}} \vec{E}\cdot d\vec{l} = \int_0^z 6kyz\,dz = 6ky\int z\,dz = 3kyz^2$$

NOW

$$V(\vec{r}) = -\int_{\text{PATH 1}} \vec{E}\cdot d\vec{l} - \int_{\text{PATH 2}} \vec{E}\cdot d\vec{l} - \int_{\text{PATH 3}} \vec{E}\cdot d\vec{l} = -3kxy^2 - 3kyz^2 = -3k(xy^2 + yz^2)$$

CHECK:

$$-\vec{\nabla}V = -\left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right]\left(-3k(xy^2 + yz^2)\right) = 3k\left[\hat{x}\frac{\partial}{\partial x}(xy^2 + yz^2) + \hat{y}\frac{\partial}{\partial y}(xy^2 + yz^2) + \hat{z}\frac{\partial}{\partial z}(xy^2 + yz^2)\right]$$

$$= 3k\left[\hat{x}\,y^2 + \hat{y}(2xy + z^2) + \hat{z}(2yz)\right]$$

$$= k\left[3y^2\hat{x} + (6xy + 3z^2)\hat{y} + (6yz)\hat{z}\right] \checkmark$$