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PHYS 342
HW I

Problem 1

$A = \hat{\imath} + 2\hat{\jmath} - k \quad , \quad B = -2\hat{\imath} + 3\hat{\jmath} + \hat{k}$

a.) $A-B$ & $|A-B|$

$A-B: (1+2)\hat{\imath} + (2-3)\hat{\jmath} + (-1-1)\hat{k}$

$A-B = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$|A-B| : \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{9+1+4} = \sqrt{14}$

$\boxed{\begin{aligned} A-B &= 3\hat{\imath} - \hat{\jmath} - 2\hat{k} \\ |A-B| &= \sqrt{14} \end{aligned}}$

b.) $Comp_A B = \frac{B \cdot A}{|A|}$

$\vec{B} = -2\hat{\imath} + 3\hat{\jmath} + \hat{k} \qquad |A| = \sqrt{6}$

$\vec{A} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$B \cdot A = -2(1) + 3(2) + 1(-1) = -2 + 6 - 1 = 3$

$\boxed{Comp_A B = \frac{3}{\sqrt{6}}}$

c.) $\vec{A} \cdot \vec{B} = AB\cos\theta$

$\vec{A} \cdot \vec{B} = 1(-2) + 2(3) - 1(1) = -2 + 6 - 1 = 3$

$|A| = \sqrt{6}$

$|B| = \sqrt{14}$

$3 = \sqrt{6} \cdot \sqrt{14} \cdot \cos\theta$

$3 = \sqrt{84} \cos\theta$

$3 = 2\sqrt{21} \cos\theta$

$\cos\theta = \frac{3}{2\sqrt{21}}$

$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) = 70.9°$

$\boxed{\theta = 70.9°}$

d.) $\vec{A} \times \vec{B}$

$\vec{A} = \langle 1, 2, -1 \rangle$

$\vec{B} = \langle -2, 3, 1 \rangle$

$\vec{A} \times \vec{B} : \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{vmatrix} = \langle 2(1) + 1(3), -1(-2) - 1(1), 1(3) - 2(-2) \rangle = \langle 5, 1, 7 \rangle$

$\boxed{\begin{aligned} \vec{A} \times \vec{B} &= \langle 5, 1, 7 \rangle \\ \vec{A} \times \vec{B} &= 5\hat{\imath} + \hat{\jmath} + 7\hat{k} \end{aligned}}$

e.) $(A-B) \times (A+B)$

$(A-B) = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$(A+B) = -\hat{\imath} + 5\hat{\jmath} + 0\hat{k}$

$(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) : \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & -1 & -2 \\ -1 & 5 & 0 \end{vmatrix} = \langle -1(0) + 2(5), -2(-1) - 3(0), 3(5) + 1(-1) \rangle = \langle 10, 2, 7 \rangle$

$\boxed{\begin{aligned} (\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) &= \langle 10, 2, 7 \rangle \\ (\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) &= 10\hat{\imath} + 2\hat{\jmath} + 7\hat{k} \end{aligned}}$

Problem 2

$r = 2b\sin(\omega t)\hat{i} + b\cos(\omega t)\hat{j}$

a.)

$$\frac{dr}{dt} = 2\omega b\cos(\omega t)\hat{i} - \omega b\sin(\omega t)$$

$$\frac{d^2r}{dt^2} = -2\omega^2 b\sin(\omega t)\hat{i} - \omega^2 b\cos(\omega t)$$

Speed is magnitude of velocity

$\cos^2(x) + \sin^2(x) = 1 \quad \therefore \quad \sin^2(x) = 1 - \cos^2(x)$

$$|\vec{v}| = \sqrt{4\omega^2 b^2\cos^2(\omega t) + \omega^2 b^2 \sin^2(\omega t)}$$
$$= \sqrt{\omega^2 b^2(4\cos^2(\omega t) + \sin^2(\omega t))}$$
$$= \sqrt{\omega^2 b^2(4\cos^2(\omega t) + 1 - \cos^2(\omega t))}$$
$$= \sqrt{\omega^2 b^2(3\cos^2(\omega t) + 1)}$$
$$|\vec{v}| = \omega b\sqrt{3\cos^2(\omega t) + 1}$$

$$\boxed{\begin{aligned} &\vec{v} = 2\omega b\cos(\omega t)\hat{i} - \omega b\sin(\omega t) \\ &\vec{a} = -2\omega^2 b\sin(\omega t)\hat{i} - \omega^2 b\cos(\omega t) \quad \text{or} \quad -\omega^2\hat{r} \\ &|\vec{v}| = \omega b\sqrt{3\cos^2(\omega t) + 1} \end{aligned}}$$

b.)

$\vec{v} = 2\omega b\cos(\omega t)\hat{i} - \omega b\sin(\omega t)\hat{j}$ $\qquad t = \frac{\pi}{2\omega}$

$\vec{a} = -2\omega^2 b\sin(\omega t)\hat{i} - \omega^2 b\cos(\omega t)\hat{j}$

$$\vec{v}\left(\tfrac{\pi}{2\omega}\right) = 2\omega b\cos\left(\tfrac{\pi}{2}\right)\hat{i} - \omega b\sin\left(\tfrac{\pi}{2}\right)\hat{j} = 0\hat{i} - \omega b\hat{j}$$
$$\vec{a}\left(\tfrac{\pi}{2\omega}\right) = -2\omega^2 b\sin\left(\tfrac{\pi}{2}\right)\hat{i} - \omega^2 b\cos\left(\tfrac{\pi}{2}\right)\hat{j} = -2\omega^2 b\hat{i} + 0\hat{j}$$

$$\vec{v}\cdot\vec{a} = 0\hat{i}\cdot(-2\omega^2 b)\hat{i} - \omega b\hat{j}\cdot(0)\hat{j} = 0 - 0 = 0$$
$$\vec{v}\cdot\vec{a} = 0$$

$\vec{v}\cdot\vec{a} = |v|\cdot|a|\cdot\cos(\theta)$ $\qquad 0 = (\omega b)(2\omega^2 b)\cos(\theta)$

$|v| = \omega b$ $\qquad 0 = \cos(\theta)$

$|a| = 2\omega^2 b$ $\qquad \theta = \cos^{-1}(0) = 90° \text{ \& } 270°$

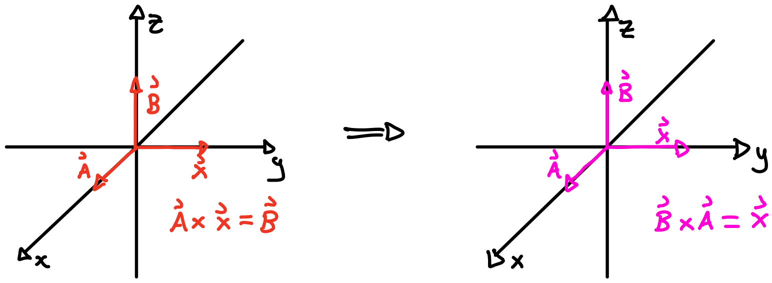
$$\boxed{\theta = 90° \ @\ t = \tfrac{\pi}{2\omega}}$$

Problem 3

$A \times X = B \qquad A \cdot X = \emptyset$

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$



$\vec{A} \times \vec{x} = \vec{B} \rightarrow \vec{x} = \vec{B} \times \vec{A}$

$$A \times B = \underset{A}{A} \times \underset{B}{A} \times \underset{C}{X}$$

$$(A \cdot X)A - (A \cdot A)X = A \times B$$

$$\emptyset A - (A \cdot A)X = A \times B$$

$$\emptyset A - A \times B = (A \cdot A)X$$

$$x = \frac{A\emptyset - A \times B}{|A|^2}$$

$$\boxed{X = \frac{\emptyset A - A \times B}{|A|^2}}$$

Problem 4

(a) $\nabla r^n = n r^{(n-2)} \vec{r}$

$$r = \sqrt{x^2+y^2+z^2}$$
$$r = (x^2+y^2+z^2)^{1/2} \Rightarrow r_n = (x^2+y^2+z^2)^{n/2}$$

$$\frac{\partial r}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2x = xn \cdot (x^2+y^2+z^2)^{n/2} \cdot (x^2+y^2+z^2)^{-1} = xn \cdot r^{n/2} \cdot r^{-1}$$

$$\frac{\partial r}{\partial y} = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2y = yn \cdot (x^2+y^2+z^2)^{n/2} \cdot (x^2+y^2+z^2)^{-1} = yn \cdot r^{n/2} \cdot r^{-1}$$

$$\frac{\partial r}{\partial z} = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \cdot 2z = zn \cdot (x^2+y^2+z^2)^{n/2} \cdot (x^2+y^2+z^2)^{-1} = zn \cdot r^{n/2} \cdot r^{-1}$$

$$\left.\begin{aligned}
\frac{\partial r}{\partial x} &= n \cdot x r^{\frac{n}{2}-1} = n x r^{n-2} \\
\frac{\partial r}{\partial y} &= n \cdot y r^{\frac{n}{2}-1} = n y r^{n-2} \\
\frac{\partial r}{\partial z} &= n \cdot z r^{\frac{n}{2}-1} = n z r^{n-2}
\end{aligned}\right\} n r^{n-2} \cdot \vec{r}$$

$$\vec{r} = [x, y, z]$$

$\therefore$

$$\boxed{\nabla r^n = n r^{n-2} \vec{r}}$$

(b) $\nabla f(r) = \frac{\vec{r}}{r}\frac{df}{dr}$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$$\nabla f(r) = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} \qquad \vec{r} = [\hat{x}, \hat{y}, \hat{z}]$$

$$\frac{\partial r}{\partial x} = 2x \cdot 1/2 \cdot (x^2+y^2+z^2)^{-1/2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}_x}{r}$$

$$\frac{\partial r}{\partial y} = 2y \cdot 1/2 \cdot (x^2+y^2+z^2)^{-1/2} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}_y}{r}$$

$$\frac{\partial r}{\partial z} = 2z \cdot 1/2 \cdot (x^2 \cdot y^2+z^2) = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}_z}{r}$$

$$\nabla f(r) = \frac{\partial f}{\partial r} \cdot \frac{\hat{x}}{r} + \frac{\partial f}{\partial r} \cdot \frac{\hat{y}}{r} + \frac{\partial f}{\partial r} \cdot \frac{\hat{z}}{r} = \frac{\vec{r}}{r}\frac{\partial f}{\partial r} \Rightarrow \frac{\vec{r}}{r}\frac{df}{dr}$$

$\therefore$ $\boxed{\nabla f(r) = \frac{\vec{r}}{r}\frac{df}{dr}}$

(c) $\nabla^2(\ln(r)) = \frac{1}{r^2}$: $r=\sqrt{x^2+y^2+z^2}$, $\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 z}{\partial z^2}$

$\frac{\partial^2 f}{\partial x^2}$: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}$

$r=(x^2+y^2+z^2)^{-1/2}$ $\quad 2$
$r^{-2} = (x^2+y^2+z^2)^{-1/2} = (\;)^{-1}$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\ln(r)\right] = \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial x}\frac{\partial \ln(r)}{\partial r}\right] = \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial x}\frac{1}{r}\right]$$

$$\frac{\partial}{\partial x}\left[\frac{\partial r}{\partial x}\frac{1}{r}\right] = \frac{\partial}{\partial x}\left[\frac{x}{r}\cdot\frac{1}{r}\right] = \frac{\partial}{\partial x}\left[\frac{x}{r^2}\right] = \frac{1}{r^2}$$

$$\frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-1}\right] = \left[\frac{1}{r^2} - x\cdot 2x\cdot(x^2+y^2+z^2)^{-2}\right] = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

$\frac{\partial^2 f}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}$: $\frac{\partial^2 f}{\partial y^2} = \frac{1}{r^2} - \frac{2y^2}{r^4}$: $\frac{\partial^2 f}{\partial z^2} = \frac{1}{r^2} - \frac{2z^2}{r^4}$ Same procedure for y & z as x.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + \left(\frac{1}{r^2} - \frac{2y^2}{r^4}\right) + \left(\frac{1}{r^2} - \frac{2z^2}{r^4}\right)$$

$$= \frac{3}{r^2} - \frac{2(x^2+y^2+z^2)}{r^4}$$

$(x^2+y^2+z^2) = r^2$

$$= \frac{3}{r^2} - \frac{2r^2}{r^4} = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$$\boxed{\nabla^2(\ln(r)) = \frac{1}{r^2}}$$

Problem 5

a.) $\vec{F} = m\ddot{x} = -kx \qquad k=4, \; m=1$

$\ddot{x} = -4x$

$\ddot{x} + 4x = 0 \quad : \quad r^2+4$

$\Delta = b^2 - 4ac = 0^2 - 4(1)(4) = -16$

$y_h = e^{\alpha t}(C_1\cos(\beta t) + C_2\sin(\beta t))$

$\alpha = \frac{-b}{2a} \qquad \beta = \frac{\sqrt{4ac-b^2}}{2a}$

$a=1 \quad b=0 \quad c=4$

$\alpha = 0 \qquad \beta = \frac{\sqrt{4(1)(4)-0}}{2} = \frac{4}{2} = 2$

$\beta = 2$

$x(t) = C_1\cos(2t) + C_2\sin(2t)$

$x(t=0) = C_1\cos(0) + C_2\sin(0)$

$1 = C_1$

$\dot{x}(t) = -2\cdot C_1\cdot\sin(2t) + 2\cdot C_2\cdot\cos(2t)$

$\dot{x}(t=0) = -2\cdot C_1\cdot\sin(0) + 2\cdot C_2\cdot\cos(0)$

$0 = 2C_2$

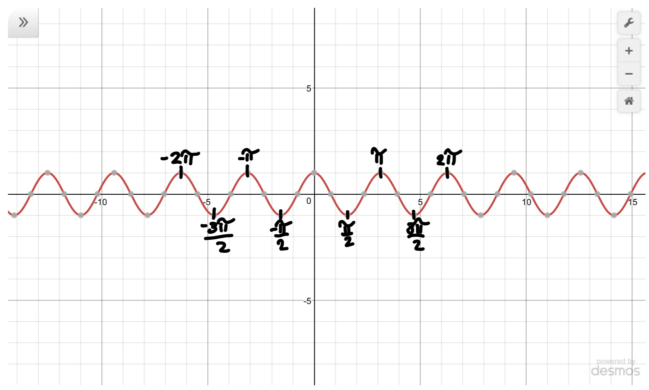
$C_2 = 0$

$x(t) = C_1\cos(2t) + C_2\sin(2t)$

$C_1 = 1 \qquad C_2 = 0$

$\boxed{x(t) = \cos(2t)}$

The motion of mass is undamped so the oscillation doesn't change as time goes on. If Friction were involved, the oscillation would slowly die down and eventually come to rest.



b.) $m\ddot{x} = -kx - b\dot{x}$

$m\ddot{x} + b\dot{x} + kx = 0$

$\underset{a}{m} \quad \underset{b}{b} \quad \underset{c}{k}$

$b^2 > 4mk$: Real Roots

$\Delta = b^2 - 4ac$

$= b^2 - 4mk > 0 \quad \therefore \quad x(t) = C_1e^{r_1t} + C_2e^{r_2t}$

$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4mk}}{2m}$

$r_1 = \frac{-b+\sqrt{b^2-4mk}}{2m}, \quad r_2 = \frac{-b-\sqrt{b^2-4mk}}{2m}$

(i) $\boxed{x(t) = C_1e^{\frac{-b+\sqrt{b^2-4mk}}{2m}t} + C_2e^{\frac{-b-\sqrt{b^2-4mk}}{2m}t}}$

$b^2 = 4mk$: Repeated Roots

$\Delta = b^2 - 4ac$

$\Delta = b^2 - 4mk = 0 \quad \therefore \quad x(t) = C_1e^{rt} + C_2te^{rt}$

$r = \frac{-b}{2a} = \frac{-b}{2m}$

(ii) $\boxed{x(t) = e^{\frac{-b}{2m}t}(C_1 + C_2t)}$

$b^2 < 4mk$: Imaginary roots

$\Delta = b^2 - 4ac$

$\Delta = b^2 - 4mk < 0 \quad \therefore \quad x(t) = e^{\alpha t}(C_1\cos(\beta t) + C_1\sin(\beta t))$

$\alpha = \frac{-b}{2a} = \frac{-b}{2m}$

$\beta = \frac{\sqrt{4ac-b^2}}{2a} = \frac{\sqrt{4mk-b^2}}{2m}$

(iii) $\boxed{x(t) = e^{\frac{-b}{2m}t}\left(C_1\cos\left(\frac{\sqrt{4mk-b^2}}{2m}t\right) + C_2\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)\right)}$

See next page for $C_1 = C_2$

if $C_1 = C_2$:

$$\boxed{\begin{aligned}
&(i) \rightarrow \quad x(t) = C\left(e^{\frac{-b+\sqrt{b^2-4mK}}{2m}t} + e^{\frac{-b-\sqrt{b^2-4mK}}{2m}t}\right) \\
&(ii) \rightarrow \quad x(t) = C e^{\frac{-b}{2m}t}(1+t) \\
&(iii) \rightarrow \quad x(t) = C e^{\frac{-b}{2m}t}\left(\cos\left(\frac{\sqrt{4mK-b^2}}{2m}t\right) + \sin\left(\frac{\sqrt{4mK-b^2}}{2m}t\right)\right)
\end{aligned}}$$

Problem 6

(a)

$$\int_0^1 \frac{1-e^x}{\sqrt{x}}\,dx$$

$$\sum_i^n \frac{f^{(n)}(x-a)^{(n)}}{n!} \quad : \quad f(x)=e^x \ @ \ a=0$$

$$f(0)=1$$
$$f'(x)=e^x \ : \ f'(0)=e^0=1$$
$$f''(x)=e^x \ : \ f''(0)=e^0=1$$
$$f'''(x)=e^x \ : \ f'''(0)=e^0=1$$
$$f''''(x)=e^x \ : \ f''''(0)=e^0=1$$

$$\sum_i = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

$$\int_0^1 \frac{1-(1+x+x^2/2+x^3/6+x^4/24)}{\sqrt{x}}\,dx$$

$$\int_0^1 \frac{-x-x^2/2-x^3/6-x^4/24}{\sqrt{x}}\,dx$$

$$\int_0^1 \frac{-x}{\sqrt{x}}-\frac{x^2}{2\sqrt{x}}-\frac{x^3}{6\sqrt{x}}-\frac{x^4}{24\sqrt{x}}\,dx = \int_0^1 -(x)^{\frac{1}{2}}-\frac{1}{2}\cdot(x)^{3/2}-\frac{1}{6}(x)^{5/2}-\frac{1}{24}(x)^{7/2}\,dx$$

$$\left[-\tfrac{2}{3}x^{3/2}-\tfrac{1}{5}x^{5/2}-\tfrac{1}{21}x^{7/2}-\tfrac{1}{108}(x)^{9/2}\right]\Big|_0^1$$

$$\left[-\tfrac{2}{3}(1)-\tfrac{1}{5}(1)-\tfrac{1}{21}(1)-\tfrac{1}{108}(1)\right] = -0.9235$$

$$\boxed{\text{Analytical result} = -0.9235}$$

The numerical results from MATLAB are on the next page. When the expansion of e^x is @ 0, the results yield a very accurate integral. It agrees very well with the numerical results in MATLAB.

Problem 7

$f(x) = \ln(x+1) \qquad f(x) = \ln(x)$

$\underline{f(x) = \ln(x+1)}$

$$\sum_{i=0}^{n} \frac{f^{(n)}(x-a)^{(n)}}{n!} \quad @\ a=0 \qquad f^{(0)} = \ln(1) \qquad\qquad \frac{d}{dx}(x+1)^2 = 2(x+1)'$$

$$f'(x) = \frac{1}{x+1} \qquad f'(0) = 1$$

$$f''(x) = \frac{-1}{(x+1)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2(x+1)}{(x+1)^4} = \frac{2}{(x+1)^3} \qquad f'''(0) = 2$$

$$\boxed{f(x) = \ln(x+1) = \sum_{i}^{n} \frac{f^{(n)}(x-a)^{(n)}}{n!} = \ln(1) + x - \frac{x^2}{2} + \frac{x^3}{3}}$$

$\underline{f(x) = \ln(x)}$

$$\sum_{i=0}^{n} \frac{f^{(n)}(x-a)^{(n)}}{n!} \quad @\ a=0$$

$$f^{(0)} = \ln(0) \equiv \text{undefined}$$

$$f_x^{(1)} = \frac{1}{x} \ : \ f_x^{(0)} = \frac{1}{0} \equiv \text{undefined}$$

> The function $\ln(x)$ cannot be represented by a series evaluated @ $x=0$ because all of the terms of the expansion are undefined.