Ch2.2 Fixed Point Iteration

Let g be a function. The point p is a <u>fixed point</u> of g if $g(p) = p$. See graph on board.

Root-finding problems and fixed-point problems are equivalent:

(1) Suppose we want to solve $f(p)=0$. Define g by $g(x) = x - f(x)$. Then the fixed points of g are the roots of f.

(2) Suppose we want to solve $g(p)=p$. Define $f(x) = x - g(x)$. Then the zeros of f are the fixed points of g.

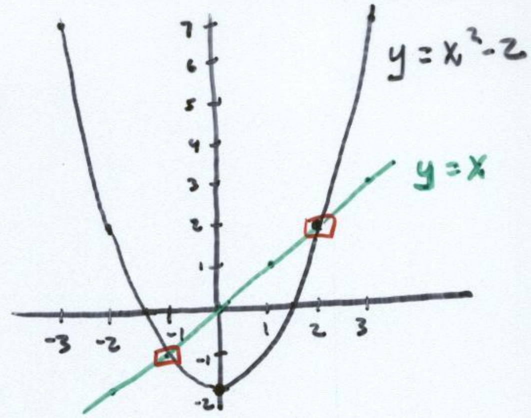
Typically we want to solve a root problem. However, the fixed-point point form is easier to analyze. Some fixed-point choices lead to good root finding methods.

<u>Example 1</u> $g(x) = x^2 - 2$ on $[-3, 3]$

The fixed points of g are $x = -1$ and $x = 2$:

$g(-1) = -1$

$g(2) = 2$



<u>Theorem 2.2</u>

(a) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$.

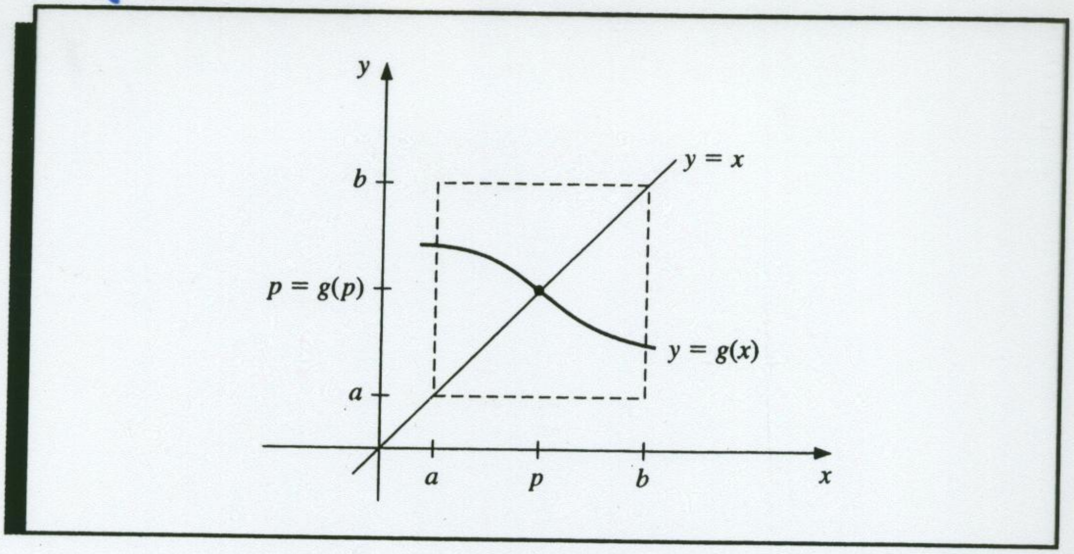
(b) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b),$$

then the fixed point in $[a,b]$ is unique.

<u>Proof</u> Uses Intermediate Value Theorem and Mean Value Theorem.

Figure 2.3



Proof (a) If $g(a)=a$ or $g(b)=b$, then g has a fixed point at an endpoint and we are done. If not, then $g(a)>a$ and $g(b)<b$, since $g(x)\in[a,b]$ by hypothesis. Let $h(x)=g(x)-x \in C[a,b]$ with $h(a)>0$ and $h(b)<0$.

By IVT, $\exists\, p\in(a,b)$ s.t. $h(p)=0$.
Then p is a fixed point of g.

(b) Suppose, in addition, that $|g'(x)|\le k<1$, and that both p and q are fixed points in $[a,b]$.

If $p\neq q$, then by MVT, $\exists\, c\in(p,q)$ s.t.

$$\frac{g(p)-g(q)}{p-q}=g'(c)$$

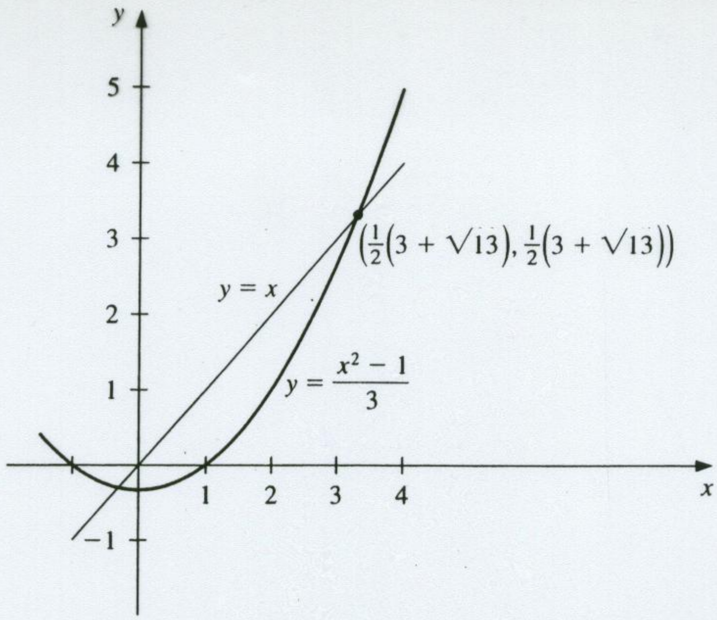
Thus, since $g(p)=p$ & $g(q)=q$,

$$|p-q|=|g(p)-g(q)|=|g'(c)|\cdot|p-q|<|p-q|$$

This is a contradiction, and hence $p=q$.

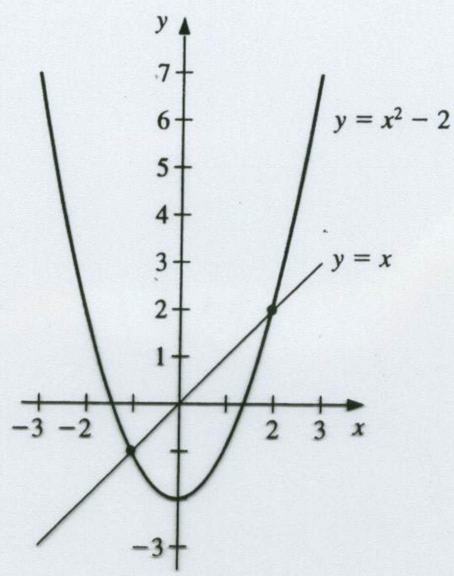
$\square$

Fig 2.4



y
5
4
3
2
1
−1
1
2
3
4
x
$y = x$
$y = \frac{x^2 - 1}{3}$
$\left(\frac{1}{2}(3 + \sqrt{13}), \frac{1}{2}(3 + \sqrt{13})\right)$

Fig 2.2



y
7
6
5
4
3
2
1
−3
−3 −2
2 3 x
$y = x^2 - 2$
$y = x$

Example 2

(a) $g(x) = \frac{1}{3}(x^2-1)$ on $[-1,1]$. (See graph)

Absolute min: $(0, -1/3)$

Absolute max: $(-1,0)$ and $(1,0)$

Note that $g \in C[-1,1]$ and

$$|g'(x)| = \left|\frac{2}{3}x\right| \le \frac{2}{3} \quad \text{on } (-1,1)$$

By Thm 2.2, g has a unique fixed pt. in $[-1,1]$.

Solving for p algebraically:

$$p = g(p) = \frac{p^2-1}{3} \Rightarrow p^2 - 3p - 1 = 0$$

Using quadratic formula, $p = \frac{1}{2}(3-\sqrt{13})$.

(b) $g(x) = 3^{-x} \in C[0,1]$ with

$$g'(x) = -3^{-x}\ln 3 < 0 \quad \text{on } [0,1].$$

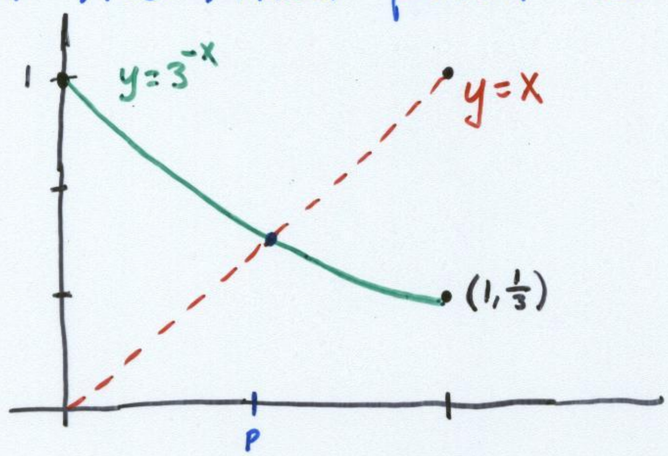
Thus g is decreasing on $[0,1]$ with

$$g(1) = \frac{1}{3} \le g(x) \le 1 = g(0), \quad x \in [0,1]$$

Thus $g(x) \in [0,1]$ for $x \in [0,1]$

By Thm 2.2a, g has a fixed pt. in $[0,1]$.

Now $g'(x) = -3^{-x} \ln 3$ with $|g'(0)| = |-\ln 3| > 1$,
and hence $|g'(x)| \not\leq 1$ on $(0,1)$.
Thus Thm 2.2 cannot be used to determine uniqueness.
However, g is always decreasing, and it is clear from the graph of g that the fixed point is unique.



Procedure for finding fixed points of g

Start by choosing an initial approx p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, $n \geq 1$.

If the sequence $\{p_n\}$ converges to p and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = \cancel{\lim_{n\to\infty}}$$

$$= g\left(\lim_{n\to\infty} p_{n-1}\right)$$

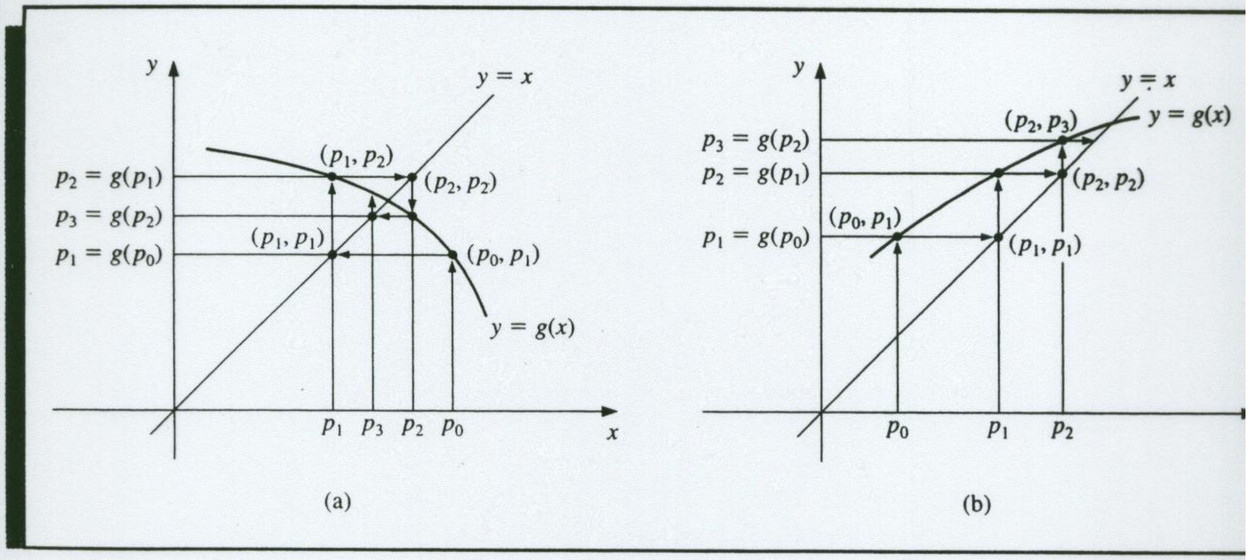
$$= g(p)$$

Thus a solution to $x = g(x)$ is obtained.

This technique is called <u>fixed-point iteration</u>, or <u>functional iteration</u>.

(This procedure is detailed in the Algorithm given next, and is illustrated in the following figure.)

Figure 2.6



y
y = x
(p₁, p₂)
p₂ = g(p₁)
(p₂, p₂)
p₃ = g(p₂)
(p₁, p₁)
p₁ = g(p₀)
(p₀, p₁)
y = g(x)
p₁ p₃ p₂ p₀
x
(a)
y
y = x
(p₂, p₃)
y = g(x)
p₃ = g(p₂)
p₂ = g(p₁)
(p₂, p₂)
(p₀, p₁)
p₁ = g(p₀)
(p₁, p₁)
p₀ p₁ p₂
(b)

Algorithm: Fixed-Point Iteration

To find a soln. to $p = g(p)$, given initial approx p_0:

INPUT initial approx. p_0, tolerance TOL, max iter. N_0.

OUTPUT approx soln p or failure message.

Step 1: Set $i = 1$.

Step 2 While $i \le N_0$ do steps 3-6.

Step 3 Set $p = g(p_0)$. (compute p_i.)

Step 4 If $|p - p_0| < TOL$ then

OUTPUT (p); (Procedure was succesful.)

STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (Update p_0.)

Step 7 OUTPUT ('The method ~~was suc~~ failed after N_0 iterations, N_0 =', N_0);

STOP

Example. Let $f(x) = x^3 + 4x^2 - 10$.
It turns out that f has a unique root in $[1,2]$. This root can be found using fixed-point iteration on $g(x) = x - (x^3 + 4x^2 - 10)$, hence solving $g(x) = x$.

There may be other representations of g that provide faster or more accurate results. For example,

$$x^3 + 4x^2 - 10 = 0 \iff 4x^2 = 10 - x^3$$

$$\iff x^2 = \frac{1}{4}(10 - x^3) \iff x = \underbrace{\pm \frac{1}{2}(10 - x^3)^{1/2}}_{g(x)}$$

Listed below are some possible forms for g.

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$ (e) $x = g_5(x) = \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} - x$

Table 2.2

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

Actual root: $p \cong 1.365230013$

Theorem 2.3 (Fixed-Point Theorem)

Let g [redacted] $\in C[a,b]$ be such that $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose, in addition, that g' exists on (a,b) and $\exists\, k$ constant s.t. $0 < k < 1$ and

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b).$$

Then, for any number $p_0 \in [a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1$$

converges to the unique fixed point p in $[a,b]$.

Proof By Thm 2.2, $\exists$ a unique fixed point $p \in [a,b]$. Also, $p_n \in [a,b]$ $\forall n \in \mathbb{N}$. Using the fact that $|g'(x)| \le k$ and the MVT, we have

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)| \cdot |p_{n-1} - p|$$
$$\le k\,|p_{n-1} - p|, \quad \xi_n \in (a,b).$$

It follows that $|p_n - p| \le k^n |p_0 - p|$

Thus $p_n \to p$ as $n \to \infty$. ▨

Corollary 2.4

If g satisfies the hypotheses of Theorem 2.3, then the bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0|, \quad n \geq 1$$

Proof (Partial):

For the first error bound, note that since $p \in [a,b]$, it follows that

$$|p_0 - p| \leq \max\{p_0 - a, b - p_0\}$$

From the inequality given at the end of Thm 2.3 proof, we have

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

For the proof of the second error bound, see text.

Note: Both error bounds depend on the bound k on first derivative $g'(x)$.

<u>Example</u> Recall from previous example that x^3+4x^2-10 has a unique root in $[1,2]$, and that the root may be approximated using fixed-point iteration for several different (but equivalent) $g(x)$:

(a) $g_1(x) = x - x^3 - 4x^2 + 10$

Note $g_1(1) = 6$ and $g_1(2) = -12$, so $g(x) \notin [1,2]$ for all $x \in [1,2]$. Further, $|g_1'(x)| > 1$ for all $x \in [1,2]$. Thus we cannot apply Thm 2.3, and hence we have no guarantee of convergence or divergence.

(b) $g_2(x) = \left[\left(\frac{10}{x} - 4x\right)\right]^{1/2}$

$g_2(1) = \sqrt{6} > 2$ and hence $g(x) \notin [1,2]$ for all $x \in [1,2]$. Also, recall that for $p_0 = 1.5$, $p_3 = (-8.65)^{1/2}$, which is undefined.

We cannot apply Thm 2.3 here.

(c) $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

Here, $g_3'(x) = \frac{1}{4}(10 - x^3)^{-1/2} \cdot (-3x^2)$

or

$$g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0 \text{ on } [1,2].$$

Thus g_3 is strictly decreasing on $[1,2]$.

However, $|g_3'(2)| \cong 2.12$. Recalling results of previous example, change interval to $[1, 1.5]$.

Note that $g_3''(x) = \frac{3}{8} \frac{x(x^3-40)}{(10-x^3)^{3/2}} < 0$ on $[1, 1.5]$

Thus $g_3'(x) < 0$ and is decreasing on $[1, 1.5]$

Hence $0 > g_3'(1) \geq g_3'(x) \geq g'(1.5)$ on $[1, 1.5]$

It follows that

$$|g_3'(x)| \leq |g_3'(1.5)| \cong 0.66 \quad \text{on } [1, 1.5]$$

Also $1.28 \cong g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5$

Thus $g(x) \in [1, 1.5]$, for all $x \in [1, 1.5]$.

Hence we can apply Thm 2.3 & Cor 2.4.

Ch2.2 (Example 4c)

$$g_3'(x) = -\frac{3}{4}x^2(10-x^3)^{-1/2} = -\frac{3}{4}\frac{x^2}{\sqrt{10-x^3}}$$

$$g''(x) = -\frac{3}{4}\left[\frac{2x\sqrt{10-x^3} - (x^2)(\frac{1}{2})(10-x^3)^{-1/2}(-3x^2)}{10-x^3}\right]$$

$$= -\frac{3}{4}\left[\frac{2x\sqrt{10-x^3} + \frac{3}{2}x^4(10-x^3)^{-1/2}}{10-x^3}\right] < 0 \text{ on } [1,1.5]$$

$\Rightarrow g_3'$ is decreasing on $[1,1.5]$, and $g_3' < 0$ on $[1,1.5]$.

$\Rightarrow 0 > g_3'(1) \geq g_3'(x) \geq g'(1.5)$ on $[1,1.5]$

$\Rightarrow |g_3'(x)| \leq |g_3'(1.5)| \cong 0.66$

$$-\frac{3}{4}\left[\frac{2x(10-x^3)+\frac{3}{2}x^4}{(10-x^3)\sqrt{10-x^3}}\right] = -\frac{3}{4}x\left[\frac{20-\frac{1}{2}x^3}{(10-x^3)\sqrt{10-x^3}}\right]$$

$$= -\frac{3}{8}x\left[\frac{40-x^3}{(10-x^3)^{3/2}}\right]$$

$$= \frac{3}{8}x\left[\frac{x^3-40}{(10-x^3)^{3/2}}\right] < 0 \text{ on } [1,1.5]$$

Alternatively, note that

$$|g_3'(x)| = \frac{3}{4}\frac{x^2}{\sqrt{10-x^3}} \leq \frac{3}{4}\frac{(1.5)^2}{\sqrt{10-(1.5)^3}} \cong 0.66$$

(d) $g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}\,(4+x)^{3/2}}\right| \le \frac{5}{\sqrt{10}\,(5)^{3/2}} < 0.15$$

on [1,2]. Also, note that

$$1 < g_4(2) \le g_4(x) \le g_4(1) < 2$$

Thus $g_4(x) \in [1,2]$ for all $x \in [1,2]$ $\therefore$ Can use Thm 2.4

Observe that the bound on $g_4'(x)$ is much smaller than the bound on $g_3'(x)$.

(e) The sequence defined by

$$g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$$

converges more rapidly than our other choices, and corresponds to Newton's Method (Ch 2.3)

Math 361 Extra Examples

Thm 2.3

(i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$. (See Figure 2.4.) ■

① $g(x) = 2 - 0.5x^2$ on $[0,2]$

(a) $g \in C[0,2]$

(b) $g(0) = 2$, $g(2) = 0$ (*)

$g'(x) = -x < 0$ on $[0,2] \Rightarrow g$ dec on $[0,2]$ (**)

By (*) and (**), it follows that $g(x) \in [0,2]$

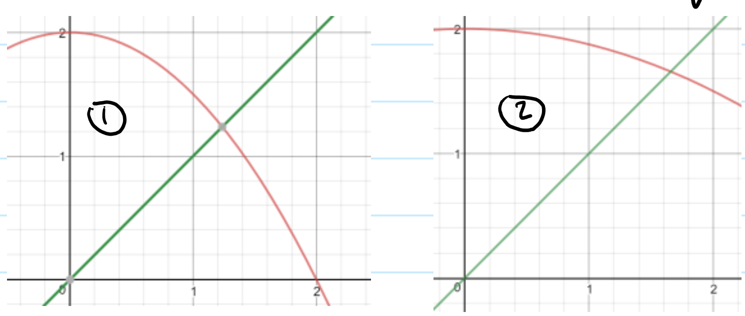
(c) Because $g \in C[0,2]$ and $g(x) \in [0,2]$, Thm 2.3(i) guarantees that g has a fixed point in $[0,2]$.

(d) Observe that

$$\max_{0 \le x \le 2} |g'(x)| = \max_{0 \le x \le 2} |-x| = 2 = k > 1.$$

Thus we cannot use Thm 2.3(ii) to guarantee a unique fixed point.

(e) Graph $g(x)$ & $y = x$



② $g(x) = 2 - \frac{1}{8}x^2$ on $[0,2]$

(a) $g \in C[0,2]$

(b) $g(0) = 2$, $g(2) = 1.5$; $g'(x) = -\frac{1}{4}x < 0$ on $[0,2]$

$\therefore g(x) \in [0,2]$

(c) Since $g \in C[0,2]$ and $g(x) \in [0,2]$, g has a fixed pt in $[0,2]$ by Thm 2.3(i).

(d) Observe that

$$\max_{0 \le x \le 2} |g'(x)| = \max_{0 \le x \le 2} \left|-\frac{1}{4}x\right| = \frac{1}{2} = k < 1$$

$\therefore$ Fixed pt is unique by Thm 2.3(ii).

(e) See graph above.

③ $g(x) = x^3 - 2.5x^2 + 0.75x + 1.675$

(a) $g \in C[0,2]$

(b) $g'(x) = 3x^2 - 5x + 0.75 \overset{\text{set}}{=} 0$

$$x = \frac{5 \pm \sqrt{25 - 4(3)(0.75)}}{2(3)} = \frac{5 \pm \sqrt{16}}{6} = \frac{3}{2}, \frac{1}{6}$$

$$\left.\begin{aligned} g(0) &= 1.675 \\ g(1/6) &\cong 1.685 \\ g(3/2) &= 1.5 \\ g(2) &= 1.125 \end{aligned}\right\}$$

By EVT (Ch1.1),

$1.125 \le g(x) \le 1.685$

$\therefore g(x) \in [0,2]$ for $x \in [0,2]$

(c) Since $g \in C[0,2]$ and $g(x) \in [0,2]$, g has a fixed point in $[0,2]$ by Thm 2.3(i)

(d) $g''(x) = 6x - 5 \overset{\text{set}}{=} 0 \Rightarrow x = 5/6$

$$\left.\begin{aligned} g'(0) &= 0.75 \\ g'(5/6) &= -1.333 \\ g'(1) &= -1.25 \end{aligned}\right\}$$

By EVT,

$\max\limits_{0 \le x \le 2} |g'(x)| = 1.333 = k > 1$

$\therefore$ Cannot use Thm 2.3(ii) to guarantee unique fixed pt. on $[0,2]$.

(e) Graph $g(x)$ & $y = x$.