5.4 Bessel's Equation & Bessel Functions

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\,(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!\,2!} + \frac{x^5}{2^5 2!\,3!} - \frac{x^7}{2^7 3!\,4!} + - \cdots$$

Friedrich Wilhelm Bessel (1784 – 1846) studied disturbances in planetary motion, which led him in 1824 to make the first systematic analysis of solutions of this equation. The solutions became known as Bessel functions.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$x^2 y'' + xy' + x^2 y = 0 \qquad (\upsilon = 0)$

(1) $y(x) = \sum_{m=0}^{\infty} a_m x^{r+m}, \ y'(x) = \sum_{m=0}^{\infty} a_n (r+m) x^{r+m-1}, \ y''(x) = \sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2}$

(2) $\sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m} + \sum_{m=0}^{\infty} a_m (r+m) x^{r+m} + \sum_{m=0}^{\infty} a_m x^{r+m+2} = 0$

(3) $a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{m=2}^{\infty} \left\{ a_m (r+m)^2 + a_{m-2} \right\} x^{r+m} = 0$

(4) $r^2 = 0$ (indicial eqn), $a_m = -\dfrac{a_{m-2}}{(r+m)^2} = -\dfrac{a_{m-2}}{m^2}$, $m = 2, 3, \dots$ (recursion relation)

(5) $a_0 = arb \Rightarrow a_{2k} = -\dfrac{a_{2k-2}}{(2k)^2} = -\dfrac{a_{2k-2}}{2^2 k^2} = \dfrac{a_{2k-4}}{2^4 k^2 (k-1)^2} = \cdots = (-1)^k \dfrac{a_0}{2^{2k} (k!)^2}$

(6) $a_1 = 0 \Rightarrow a_{2k+1} = 0$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m$$

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln \frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right]$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

indicial equation

$$(r + \nu)(r - \nu) = 0.$$

$J_\nu(x)$ is called the **Bessel function of the first kind of order** ν.

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\, \Gamma(\nu + m + 1)}. \quad \nu \geqq 0.$$

For ν not an integer

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m!\, \Gamma(m - \nu + 1)}.$$

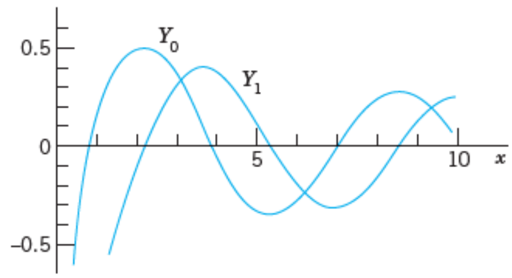

Fig. 112. Bessel functions of the second kind Y_0 and Y_1.

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}[J_\nu(x) \cos \nu\pi - J_{-\nu}(x)]$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x).$$

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln \frac{x}{2} + \gamma\right) + \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n} m!\, (m+n)!} x^{2m}$$

$$- \frac{x^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n - m - 1)!}{2^{2m-n} m!} x^{2m}$$

Hankel functions

Bessel functions of the third kind *of order* ν

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

$J_\nu(x)$ is called the **Bessel function of the first kind of order ν.**

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\, \Gamma(\nu + m + 1)}. \qquad \nu \geqq 0.$$

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\, (n+m)!}$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!\, (m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!\, 2!} + \frac{x^5}{2^5 2!\, 3!} - \frac{x^7}{2^7 3!\, 4!} + - \cdots$$

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

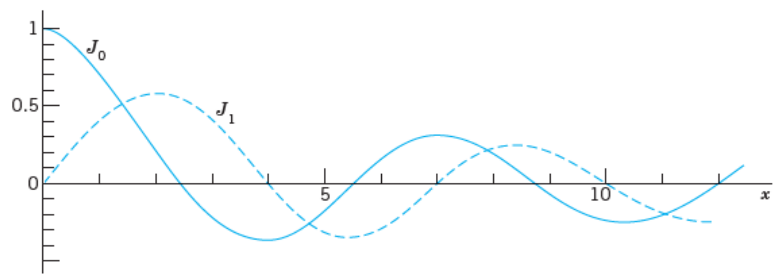

Fig. 110. Bessel functions of the first kind J_0 and J_1

Derivatives, Recursions

$$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$$

$$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x).$$

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x).$$

the **gamma function** $\Gamma(\nu + 1)$ defined by

$$\boxed{\Gamma(\nu + 1) = \int_0^\infty e^{-t} t^\nu \, dt} \qquad (\nu > -1).$$

(**CAUTION!** Note the convention $\nu + 1$ on the left but ν in the integral.) Integration by parts gives

$$\Gamma(\nu + 1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu - 1} \, dt = 0 + \nu\Gamma(\nu).$$

This is the basic functional relation of the gamma function

$$\boxed{\Gamma(\nu + 1) = \nu\Gamma(\nu).} \qquad \textbf{(17)}$$

Now from (16) with $\nu = 0$ and then by (17) we obtain

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t} \Big|_0^\infty = 0 - (-1) = 1$$

and then $\Gamma(2) = 1 \cdot \Gamma(1) = 1!$, $\Gamma(3) = 2\Gamma(1) = 2!$ and in general

$$\boxed{\Gamma(n + 1) = n!} \qquad (n = 0, 1, \cdots). \qquad \textbf{(18)}$$

Hence ***the gamma function generalizes the factorial function* to arbitrary positive ν.**