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MATH 360
CP Ch. 10.7

<u>Equations 1 & 2</u>
The divergence of a vector function $F=[F_1,F_2,F_3]=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$

$$\operatorname{div}\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} \qquad (1)$$

The divergence theorem is precisely,

$$\iiint_T \operatorname{div}\vec{F}\,dv=\iint_S \vec{F}\cdot\vec{n}\,dA$$

<u>Example 1</u> Evaluation of a Surface Integral by the Divergence Theorem
Before we prove the theorem, let us show a typical application. Evaluate

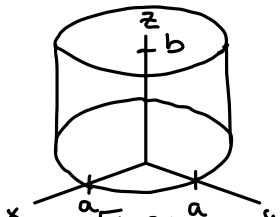

Fig. 252

$$I=\iint_S x^3\,dy\,dz+x^2y\,dz\,dx+x^2z\,dx\,dy$$

where S is the closed surface in Fig. 252 consisting of the cylinder $x^2+y^2=a^2$ $(0\leqq z\leqq b)$ and the circular disks $z=0$ and $z=b\,(x^2+y^2\leqq a^2)$.

Solution. $F_1=x^3$, $F_2=x^2y$, $F_3=x^2z$. Hence $\operatorname{div}\vec{F}=3x^2+x^2+x^2=5x^2$. The form of the surface suggests that we introduce polar coordinates r,θ defined by $x=r\cos\theta$, $y=r\sin\theta$ (thus cylindrical coordinates r,θ,z). Then the volume element is $dx\,dy\,dz=r\,dr\,d\theta\,dz$, and we obtain

$$I=\iiint_T 5x^2\,dx\,dy\,dz=\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}(5r^2\cos^2\theta)\,r\,dr\,d\theta\,dz$$

$$=5\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\frac{a^4}{4}\cos^2\theta\,d\theta\,dz=5\int_{z=0}^{b}\frac{a^4\pi}{4}\,dz=\frac{5\pi}{4}a^4b$$

Example 2 Verification of the Divergence Theorem

Evaluate $\iint_S (7x\hat{i} - z\hat{k}) \cdot \vec{n}\, dA$ over the sphere $S: x^2+y^2+z^2=4$ (a) by 2, (b) directly

Solution. (a) $\operatorname{div} \vec{F} = \operatorname{div}[7x, 0, -z] = \operatorname{div}[7x\hat{i} - zk\hat{j}] = 7-1=6$. Answer: $6 \cdot (4/3)\pi \cdot 2^3 = 64\pi$.

(b) We can represent S by (3), sec. 10.5 (with a=2), and we shall use $\vec{n}\, dA = \vec{N}\, du\, dv$ [see (3*), sec. 10.6]

Accordingly,

$$S: \vec{r} = [2\cos(v)\cos(u),\ 2\cos(v)\sin(u),\ 2\sin(v)]$$
$$\vec{r}_u = [-2\cos(v)\sin(u),\ 2\cos(v)\cos(u),\ 0]$$
$$\vec{r}_v = [-2\sin(v)\cos(u),\ -2\sin(v)\sin(u),\ 2\cos(v)]$$
$$\vec{N} = \vec{r}_u \times \vec{r}_v = [4\cos^2(v)\cos(u),\ 4\cos^2(v)\sin(u),\ 4\cos(v)\sin(v)]$$

Now on S we have $x = 2\cos(v)\cos(u)$, $z = 2\sin(v)$, so that $\vec{F} = [7x, 0, -z]$ becomes on S

$$\vec{F}(S) = [14\cos(v)\cos(u),\ 0,\ -2\sin(v)]$$

and

$$\vec{F}(S)\cdot\vec{N} = (14\cos(v)\cos(u))\cdot 4\cos^2(v)\cos(u) + (-2\sin(v))\cdot 4\cos(v)\sin(v)$$

$$= 56\cos^3(v)\cos^2(u) - 8\cos(v)\sin^2(v)$$

on S we have to integrate over u from 0 to 2π. This gives

$$\pi \cdot 56\cos^3(v) - 2\pi \cdot 8\cos(v)\sin^2(v)$$

The integral of $\cos(v)\sin^2(v)$ equals $(\sin^3(v))/3$, and that of $\cos^3(v) = \cos(v)(1-\sin^2(v))$ equals $\sin(v) - (\sin^3(v))/3$. On S we have $-\pi/2 \leqq v \leqq \pi/2$, so that by substituting these limits we get

$$56\pi(2-\tfrac{2}{3}) - 16\pi \cdot \tfrac{2}{3} = 64\pi$$

as hoped for. To see the point of Gauss's theorem, compare the amounts of work.