MAT 202
Larson – Section 6.4
First-Order Linear Differential Equations

Recall in section 6.1, the ***order*** of a differential equation is determined by the highest order derivative in the equation. Hence, a ***first-order differential equation*** is a differential equation that only involves the first derivative and nothing higher. A ***first-order linear differential equation*** is a special type of first-order differential equation that can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where P and Q are continuous functions of x. This first-order linear differential equation is said to be in ***standard form***.

<u>Solving First-Order Linear Differential Equations</u>:

1. Given a first-order linear differential equation in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$.
2. Multiply this equation by $u(x) = e^{\int P(x)dx}$, which is called the ***integrating factor***. We get: $y'e^{\int P(x)dx} + P(x)ye^{\int P(x)dx} = Q(x)e^{\int P(x)dx}$. **Why did we choose $e^{\int P(x)dx}$ to multiply through the differential equation? Ingenious!!!**
3. This equation turns into: $\frac{d}{dx}\left[ye^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx}$.
4. Integrate both sides of this equation to: $ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + C$
5. Isolate y and get the ***General Solution***: $y = \frac{1}{e^{\int P(x)dx}}\int Q(x)e^{\int P(x)dx}dx + C$, some books write: $y = \frac{1}{u(x)}\int Q(x)u(x)dx + C$, where $u(x) = e^{\int P(x)dx}$.

$$\frac{dy}{dx} + P(x)y = Q(x) \leftarrow$$ First order Linear D.E.

$$\frac{P(x)y}{P(x)} = \frac{-\frac{dy}{dx}}{P(x)} + \frac{Q(x)}{P(x)}$$

$$y = \frac{dy}{dx} \cdot \frac{-1}{P(x)} + \frac{Q(x)}{P(x)}$$

$$y = m\,x + b$$

Ex: Solve the following first-order linear differential equations:

a) $\frac{dy}{dx} - y = 10$ **(I know this can be done with separation of variables, but I want you to use an integrating factor.)**

b) $e^x y' + 4e^x y = 1$

c) $(x+3)y' + 2y = 2(x+3)^2$

Ex: Solve the following first-order linear differential equations:

$\frac{dy}{dx} + P(x)\,y = Q(x)$ $\quad y = \frac{1}{e^{\int P(x)dx}}\left[\int Q(x)e^{\int P(x)dx}dx + C\right]$

a) $\frac{dy}{dx} - y = 10$ **(I know this can be done with separation of variables, but I want you to use an integrating factor.)**

Separation of variables: $\int \frac{1}{y+10}\frac{dy}{} = \int 1\,dx$ $\quad e^{\ln|y+10|} = e^{x+C_1}$

$y + 10 = Ce^x$ $\quad \boxed{y = -10 + Ce^x}$

$\frac{dy}{dx} + -1 \cdot y = 10$

$P(x) = -1$ $\quad Q(x) = 10$

$$y = \frac{1}{e^{\int -1\,dx}}\left[\int 10e^{\int -1\,dx}dx + C\right]$$

$$y = \frac{1}{e^{-x+C_1}}\left[\int 10e^{-x+C_1}dx + C\right]$$

$$y = \frac{1}{C_2e^{-x}}\left[-10e^{-x+C_1} + C\right]$$

$$y = \frac{1}{C_2e^{-x}}\left[-10 \cdot C_2e^{-x} + C\right]$$

$$y = -10 + \frac{C}{C_2e^{-x}}$$

$$y = 10 + \frac{C}{C_2}e^{x}$$

$$y = -10 + Ke^{x}$$

b) $\frac{e^x y'}{e^x} + \frac{4e^x y}{e^x} = \frac{1}{e^x}$

$y' + P(x)\,y = Q(x)$

$y' + 4y = e^{-x}$

$P(x) = 4$

$Q(x) = e^{-x}$

$$y = \frac{1}{e^{\int 4\,dx}}\left[\int e^{-x}\cdot e^{\int 4\,dx}\,dx + C\right]$$

$$y = \frac{1}{e^{4x}}\left[\int e^{-x}\cdot e^{4x}\,dx + C\right]$$

$$y = \frac{1}{e^{4x}}\left[\int e^{3x}\,dx + C\right]$$

$$y = \frac{1}{e^{4x}}\left[\frac{1}{3}e^{3x} + C\right]$$

$$\boxed{y = \frac{1}{3e^{x}} + \frac{C}{e^{4x}}}$$

OR
=
$$\boxed{y = \frac{1}{3}e^{-x} + Ce^{-4x}}$$

c) $\dfrac{(x+3)y'}{x+3} + \dfrac{2y}{x+3} = \dfrac{2(x+3)^2}{x+3}$

$$y' + \underbrace{\left(\frac{2}{x+3}\right)}_{P(x)} y = \underbrace{2(x+3)}_{Q(x)}$$

$$\int \frac{2}{x+3}\,dx = 2\ln|x+3| = \ln(x+3)^2$$

$$y = \frac{1}{e^{\int \frac{2}{x+3}dx}} \left[\int (2x+6)\, e^{\int \frac{2}{x+3}dx}\, dx + C \right]$$

$$y = \frac{1}{e^{\ln(x+3)^2}} \left[\int (2x+6)\, e^{\ln(x+3)^2}\, dx + C \right]$$

$$y = \frac{1}{(x+3)^2} \left[\int 2(x+3)(x+3)^2\, dx + C \right]$$

$$y = \frac{1}{(x+3)^2} \left[2\int (x+3)^3\, dx + C \right]$$

$$y = \frac{1}{(x+3)^2} \left[\frac{2(x+3)^4}{4} + C \right]$$

$$\boxed{y = \frac{(x+3)^2}{2} + \frac{C}{(x+3)^2}}$$

<u>Mixture Applications</u>:
Consider a tank that at time $t = 0$ contains v_0 gallons of a solution of which, by weight, q_0 pounds is soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute. Given the amount of concentrate, Q, in the solution at any time, t, the change of Q with respect to t is given in the first-order linear differential equation:

$$\frac{dQ}{dt} + \frac{r_2}{v_0 + (r_1 - r_2)t} Q = q_1 r_1$$

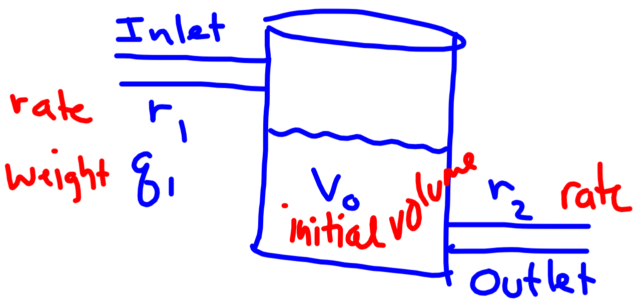
Ex: A 200-gallon tank is half full of distilled water. At time t = 0, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at a the rate of 3 gallons per minute.

a) At what time will the tank be full?
b) At the time the tank is full, how many pounds of concentrate will it contain?

Mixture Applications:

Consider a tank that at time $t = 0$ contains v_0 gallons of a solution of which, by weight, q_0 pounds is soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute. Given the amount of concentrate, Q, in the solution at any time, t, the change of Q with respect to t is given in the first-order linear differential equation:

$$\underbrace{\frac{dQ}{dt}}_{\text{Change in amount of concentrate. w.r.t. time}} + \overbrace{\left[\frac{r_2}{v_0 + (r_1 - r_2)t}\right]}^{p(t)} Q = \overbrace{\left[q_1 r_1\right]}^{q(t)}$$



Ex: A 200-gallon tank is half full of $\boxed{\text{distilled water}}$. At time $\boxed{t = 0}$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at a the rate of 3 gallons per minute.

$V_0 = 100$ $q_1 = 0.5$ $r_2 = 3$ $r_1 = 5$ $\boxed{t = 50 \text{ min.}}$

a) At what time will the tank be full? $r \cdot t = d$ $(5-3)t = 100$

b) At the time the tank is full, how many pounds of concentrate will it contain?

@ 50 min we want Q.

at time $t=0$, distilled water means $Q=0$
(i.e. 0 lbs. of concentrate) I.C.: $(0,0)$

$$\frac{dQ}{dt} + \left[\frac{r_2}{V_0 + (r_1 - r_2)t}\right] Q = g_1 r_1$$

$$\frac{dQ}{dt} + \left[\frac{3}{100 + 2t}\right] Q = \underbrace{0.5 \cdot 5}_{g(t)}$$

$$\underbrace{\frac{\frac{3}{2}}{50+t}}_{p(t)}$$

$$Q = \frac{1}{e^{\int \frac{3}{100+2t} dt}} \left[\int 2.5\, e^{\int \frac{3}{100+2t} dt} dt + C \right]$$

$u = 100 + 2t$
$du = 2\,dt$
$\frac{1}{2} du = dt$

$$Q = \frac{1}{e^{\frac{3}{2} \ln(100+2t)}} \left[\int 2.5\, e^{\frac{3}{2}\ln(100+2t)} dt + C \right]$$

$$Q = \frac{1}{(100+2t)^{3/2}} \left[2.5 \int (100+2t)^{3/2} dt + C \right]$$

$$Q = \frac{1}{(100+2t)^{3/2}} \left[2.5 \cdot \frac{1}{\cancel{2}} \frac{\cancel{2}}{5} (100+2t)^{5/2} + C \right]$$

$u = 100 + 2t$
$du = 2\,dt$
$\frac{1}{2} du = dt$

$$Q = \frac{1}{(100+2t)^{3/2}} \left[\frac{1}{2} (100+2t)^{5/2} + C \right]$$

$\boxed{@ (0,0)}$

$$0 = \frac{1}{100^{3/2}} \left[\frac{1}{2} (100)^{5/2} + C \right]$$

$$0 = \frac{1}{2}(100)^{5/2} + C$$

$$C = -\frac{1}{2}(100)^{5/2}$$

$$Q = \frac{1}{(100+2t)^{3/2}} \left[\frac{1}{2}(100+2t)^{5/2} - \frac{1}{2}(100)^{5/2} \right]$$

$$Q(50) = \frac{1}{(100+2(50))^{3/2}} \left[\frac{1}{2}(100+2(50))^{5/2} - \frac{1}{2}(100)^{5/2} \right]$$

$$\approx \boxed{82.32 \text{ lb}}$$