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MATH 360
CP 11.6

Example 1 Fourier-Legendre Series

A Fourier-Legendre series is an eigenfunction expansion

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \ldots = a_0 + a_1 x + a_2\left(\tfrac{3}{2}x^2 - \tfrac{1}{2}\right) + \ldots$$

in terms of Legendre polynomials (sec 5.3). The latter are the eigenfunctions of the Sturm-Liouville problem in Example 4 of Sec 11.5 on the interval $-1 \leq x \leq 1$. We have $r(x) = 1$ for Legendre's equation, and (2) gives

(3) $$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\, dx \qquad (m = 0, 1, \ldots)$$

because the norm is

(4) $$\|P_m\| = \sqrt{\int_{-1}^{1} P_m(x)^2\, dx} = \sqrt{\frac{2}{2m+1}} \qquad (m = 0, 1, \ldots)$$

as we state without proof. The proof of (4) is tricky; it uses Rodrigues's Formula in Problem Set. 5.2 and a reduction of the resulting integral to a quotient of gamma functions.

For instance, let $f(x) = \sin(\pi x)$. Then we obtain the coefficients

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} (\sin(\pi x)) P_m(x)\, dx, \quad \text{Thus} \quad a_1 = \frac{3}{2}\int_{-1}^{1} x \sin(\pi x)\, dx = \frac{3}{\pi} = 0.95493, \text{ etc}$$

Hence the Fourier-Legendre series of $\sin(\pi x)$ is

$$\sin(\pi x) = 0.95493 \cdot P_1(x) - 1.15824 \cdot P_3(x) + 0.21929 \cdot P_5(x) - 0.001664 \cdot P_7(x) + 0.00068 \cdot P_9(x) - 0.00002 \cdot P_{11}(x) + \ldots$$

The coefficient of P_{13} is about $3 \cdot 10^{-7}$. The sum of the first three non-zero terms gives a curve that practically coincides with the sine curve. Can you see why the even-numbered coefficients are zero? Why a_3 is the absolutely biggest coefficient.