Ch 4.2 Richardson's Extrapolation

This method is used to generate high-accuracy results while using low order formulas.

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form that depends on a parameter such as step size h.

For example,

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6}f'''(\xi_1)$$

$$= N(h) + O(h^2)$$

where

$$N(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

Suppose that for each $h \neq 0$ we have a formula $N(h)$ that approximates an unkown value M and that the truncation error involved with the approximation has the form

$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \cdots$$

for constants $k_1, k_2, k_3, \ldots$

Thus the truncation error is $O(h)$. The object of extrapolation is to find a way to combine $O(h)$ approximations in an appropriate way to obtain higher order truncation error, such as $O(h^2)$.

For example, we could combine the $N(h)$ formulas to produce an $O(h^2)$ formula $\hat{N}(h)$ for M, with

$$M - \hat{N}(h) = \hat{k}_2 h^2 + \hat{k}_3 h^3 + \cdots$$

These $\hat{N}(h)$ may then be combined to obtain an $O(h^3)$ formula, etc.

We have $M = N_3(h) + \frac{K_3}{8} h^3 + \cdots$

where

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3}$$

This process is continued by constructing an $O(h^4)$ approximation

$$N_4(h) = N_3\left(\frac{h}{2}\right) + \frac{N_3(h/2) - N_3(h)}{7}$$

and $O(h^5)$ approximation

$$N_5(h) = N_4\left(\frac{h}{2}\right) + \frac{N_4(h/2) - N_4(h)}{15}$$

In general,

$$\boxed{N_k(h) = \overset{\text{Previous term}}{N_{k-1}\left(\frac{h}{2}\right)} + \overset{\text{Update}}{\frac{N_{k-1}(h/2) - N_{k-1}(h)}{2^{k-1} - 1}}}$$

In Table form:

$O(h)$	$O(h^2)$	$O(h^3)$	$O(h^4)$
1: $N_1(h) = N(h)$			
2: $N_1(h/2) = N(h/2)$	3: $N_2(h)$		
4: $N_1\left(\frac{h}{4}\right) = N\left(\frac{h}{4}\right)$	5: $N_2(h/2)$	6: $N_3(h)$	
7: $N_1\left(\frac{h}{8}\right) = N\left(\frac{h}{8}\right)$	8: $N_2(h/4)$	9: $N_3(h/2)$	10: $N_4(h)$

Example The centered difference formula for $f'(x_0)$ can be expressed with an error formula:

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

Since this error formula contains only even powers of h, extrapolation is more effective than outlined above.

Here, we immediately have the $O(h^2)$ approx

$$(1) \quad f'(x_0) = N_1(h) - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots$$

where $N_1(h) = N(\cancel{h}) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$

Replacing h by h/2, we obtain

$$f'(x_0) = N_1(h/2) - h^2/24\, f'''(x_0) - h^4/1920\, f^{(5)}(x_0) - \cdots$$

Subtracting 4 times this equation from (1), then dividing by 3, we obtain

$$f'(x_0) = N_2(h) + \frac{h^4}{480}f^{(5)}(x_0) + \cdots, \quad N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N\left(\frac{h}{2}\right) - N_1(h)}{3}$$

We have

$$f'(x_0) = N_2(h) + \overbrace{\frac{h^4}{480} f^{(5)}(x_0)}^{O(h^4)} + \cdots$$

where $N_2(h) = N_1\left(\frac{h}{2}\right) + \dfrac{N_1(h/2) - N_1(h)}{3}$

Replacing h by $h/2$ again and doing the algebra, we obtain

$$f'(x_0) = N_3(h) + O(h^6)$$

where $$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{15}$$

In general, for the centered difference approximation for $\underline{f'(x_0)}$, we have

$$\boxed{N_k(h) = N_{k-1}\left(\frac{h}{2}\right) + \frac{N_{k-1}(h/2) - N_{k-1}(h)}{4^{k-1} - 1}},$$

which provides an $\underline{O(h^{2k})}$ approximation because of the presence of only even powers of h in the original error expression for $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right] - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(x_0) - \cdots$$

Example, continued

Suppose $f(x) = xe^x$, $x_0 = 2.0$, $h = 0.2$.

Then

$$N_1(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$N_1(0.2) = \frac{1}{2(0.2)}\left[2.2e^{2.2} - 1.8e^{1.8}\right]$$

$$= 22.414161$$

$$N_1(h/2) = N_1(0.1) = \frac{1}{2(0.1)}\left[2.1e^{2.1} - 1.9e^{1.9}\right]$$

$$= 22.228787$$

$$N_1(h/4) = N_1(0.05) = \frac{1}{2(0.05)}\left[2.05e^{2.05} - 1.95e^{1.95}\right]$$

$$= 22.182565$$

Recall our extrapolation table:

$N_1(0.2) = 22.41\ldots$

$N_1(0.1) = 22.22\ldots$ $\quad N_2(0.2) = N_1(0.1) + \dfrac{N_1(0.1) - N_1(0.2)}{3}$

$= 22.166996$

$N_1(0.05) = 22.18\ldots$ $\quad N_2(0.1) = N_1(0.05) + \dfrac{N_1(0.05) - N_1(0.1)}{3}$ $\quad \boxed{N_3(.2)}$

$= 22.167158$

$$N_3(0.2) = N_2(0.1) + \frac{1}{15}\left[N_2(0.1) - N_2(0.2)\right] = 22.167169$$

On the previous slide, we had

$O(h^2)$	$O(h^4)$	$O(h^6)$
$N_1(0.2) = 22.414161$		
$N_1(0.1) = 22.\cancel{182565}\,228767$	$N_2(0.2) = 22.166996$	
$N_1(0.05) = 22.182567$	$N_2(0.1) = 22.167158$	$N_3(.2) = \underline{22.167169}$

Our original function & data were

$$f(x) = xe^x, \quad x_0 = 2.0, \quad h = 0.2.$$

We are approximating $f'(2.0)$ using extrapolation on the centered difference formula for $f'(x_0)$.

To compare our extrapolation approximations with the actual value of $f'(2)$, we have

$$f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x$$

$$\Rightarrow f'(2) = e^2 + 2e^2 = \underline{22.167168}$$

Thus $N_3(0.2)$ has five accurate decimal digits, while our original first approximation $N_1(0.2)$ has no decimal digits accurate. At best, our original approx formula with $h = 0.05$, i.e., $N_1(0.05)$, has 1 accurate decimal digit.

Since each column beyond the first in the extrapolation table is obtained by a simple averaging proces (roughly speaking), the extrapolation technique can produce higher-order approximations with minimal computationalcost.

However, as k increases, the roundoff error in $N_1\left(\frac{h}{2^k}\right)$ will generally increase because the instability of numerical differentiation is related to the step size $h/2^k$. (Recall discusion at end of Ch4.1) (Three point formulas for $f'(x_0)$ are inherently unstable, and extrapolation methods are based on these formulas.)

The method of extrapolation can be applied to all formulas $N(h)$ approximating M, with form

$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \cdots,$$

and hence is used throughout remainder of text (Not just for numerical differentiation only).