Ch. 12 Gravitational Collapse and Black Holes

4-8-19

$r = 2M$ is a coordinate singularity
$r = 0$ is a physical singularity

Peform a coordinate transformation .....

$$t = v - r - 2M \ln\left|r/2M - 1\right|$$

4-10-19

The Schwarzschild Black Hole

(*)
$$ds^2 = -(1 - 2M/r)\,dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Perform a coordinate transformation ...

$$t = v - r - 2M \ln\left|r/2M - 1\right|$$

For $r > 2M$.....

(**)
$$t = v - r - 2M \ln\left|r/2M - 1\right|$$

$$dt = dv - dr - \frac{2M}{r/2M - 1} \cdot \frac{1}{2M}\, dr = dv - dr\left(1 + \frac{1}{r/2M - 1}\right) = dv - dr\, \frac{r/2M \cdot \frac{2M}{r}}{r/2M - 1 \cdot \frac{2M}{r}}$$

$$= dv - \frac{dr}{(1 - 2M/r)}$$

For $r < 2M$....

$$t = v - r - 2M \ln\left(1 - r/2M\right)$$

$$dt = dv - dr - 2M \frac{1}{1 - r/2M} \cdot -\frac{1}{2M}\, dr = dv - dr\left(1 - 1/(1 - r/2M)\right) = dv - dr\left(1 + 1/(r/2M - 1)\right)$$

$$dt = dv - \frac{dr}{(1 - 2M/r)} \quad \therefore \quad dt^2 = dv^2 - \frac{2\,dv\,dr}{(1 - 2M/r)} + \frac{dr^2}{(1 - 2M/r)^2}$$

$$ds^2 = -(1 - 2M/r)\left[dv^2 - \frac{2\,dv\,dr}{(1 - 2M/r)} + \frac{dr^2}{(1 - 2M/r)^2}\right] + \frac{dr^2}{(1 - 2M/r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\left[ds^2 = -(1 - 2M/r)\,dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$ – Schwarzschild geometry in Eddington Finkelstein coordinates

Light Cones of the Schwarzchild Geometry

Consider radial light rays ($d\theta = d\varphi = 0$: $ds^2 = 0$)

$$0 = -(1-2m/r)\,dv^2 + 2\,dv\,dr = \frac{dv}{dr}dr^2\left(2-(1-2m/r)\,dv/dr\right) = 0$$

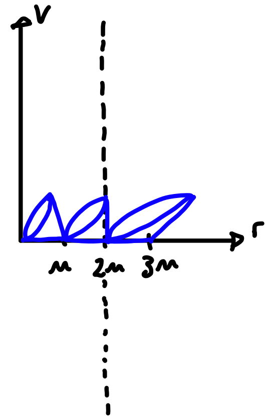
$$\frac{dv}{dr} = 0 \quad , \quad \frac{dv}{dr} = \frac{2}{1-2m/r}$$

$V = \text{constant}$

$\updownarrow$

An inward going radial light ray

How do you know? A: Ref (**) As $t\uparrow$ $r\downarrow$



$$v(r) - 2(r + 2m\ln|r/2m - 1|) = \text{Const}$$

Or, there exists a special case :

$[r = 2m] \rightarrow$ The light rays are neither inward or outward but stationary!

$$\left.\frac{dv}{dr}\right|_{r=m} = -2 \qquad \left.\frac{dv}{dr}\right|_{r=2m} = \infty \qquad \left.\frac{dv}{dr}\right|_{r\gg 2m} = 2 \qquad \left.\frac{dv}{dr}\right|_{r=3m} = 6$$

@ $r = 2m$, the light cones tip over! All Future directed paths in the direction of decreasing r.

Geometry of the Horizon : Singularity

Horizon ($r = 2m$) is a 3D null surface in Spacetime

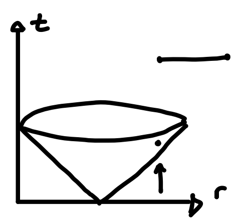
$\Rightarrow$ One-way property - Can cross only once

4D Minkowski.....

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Radial Light Ray ($d\theta = d\varphi = 0$, $ds^2 = 0$)

$$0 = -dt^2 + dr^2 \quad : \quad \frac{dt}{dr} = \pm 1$$



$V = \text{constant}$ Slice of the horizon

$$ds^2 = 4m^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad : \quad \boxed{A = 16\pi m^2} \rightarrow \text{Area of the Horizon}$$

Inside $r = 2m$ surfaces of constant r are spacelike $g_{rr} > 0$ for $r < 2m$

- $r = 0$ is a spacelike surface

· $r=0$ Singularity is not a place in space but rather a moment in time

4-12-19

Kruski - Szekeres Coordinates

Replace Schwarzschild coordinates t : r by V : U

For $r > 2M$ ....

$$U = (r/2M - 1)^{1/2} e^{r/4M} \cosh(t/4M)$$

$$V = (r/2M - 1)^{1/2} e^{r/4M} \sinh(t/4M)$$

For $r < 2M$ ....

(**)

$$U = (1 - r/2M)^{1/2} e^{r/4M} \sinh(t/4M)$$

$$V = (1 - r/2M)^{1/2} e^{r/4M} \cosh(t/4M)$$

The line element becomes .....

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad \text{when } r = r(U,V)$$

$r > 2M$

$$U = f(r)\cosh(t/4M) \qquad dU = \frac{df}{dr}\cosh(t/4M)dr + \frac{1}{4M} f(r)\sinh(t/4M)dt$$

$$V = f(r)\sinh(t/4M) \qquad dV = \frac{df}{dr}\sinh(t/4M)dr + \frac{1}{4M} f(r)\cosh(t/4M)dt$$

where $f(r) = (r/2M - 1)^{1/2} e^{r/4M}$

$$dU^2 = \left(\frac{df}{dr}\right)^2 \cosh^2(t/4M)dr^2 + \frac{1}{2M} f(r)\left(\frac{df}{dr}\right)\sinh(t/4M)\cosh(t/4M)drdt + \frac{1}{(4M)^2} f^2(r)\sinh^2(t/4M)dt^2$$

$$dV^2 = \left(\frac{df}{dr}\right)^2 \sinh^2(t/4M)dr^2 + \frac{1}{2M} f(r)\left(\frac{df}{dr}\right)\sinh(t/4M)\cosh(t/4M)drdt + \frac{1}{(4M)^2} f^2(r)\cosh^2(t/4M)dt^2$$

now,

$$-dV^2 + dU^2 = \left(\frac{df}{dr}\right)^2 \underbrace{(\cosh^2(t/4M) - \sinh^2(t/4M))}_{1} - \frac{1}{(4M)^2} f^2(r)\underbrace{(\cosh^2(t/4M) - \sinh^2(t/4M))}_{1} dt^2$$

$$\left[-dV^2 + dU^2 = \left(\frac{df}{dr}\right)^2 dr^2 - \frac{1}{16M^2} f^2(r)dt^2\right] \text{(*)} :$$

$$\frac{df}{dr} = \frac{1}{2}\left(\frac{r}{2M} - 1\right)^{-1/2} \cdot \frac{1}{2M} e^{r/4M} + \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M}$$

$$= \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{r/4M}\left[1 + r/2M - 1\right]$$

$$\frac{df}{dr} = \frac{1}{4M}\cdot\left(\frac{r}{2M}-1\right)^{-1/2}\frac{r}{2M}\,e^{r/4M}$$

$$\left[\left(\frac{df}{dr}\right)^2 = \frac{1}{16M^2}\cdot\frac{r^2}{4M^2}\cdot\frac{1}{(r/2M-1)}\,e^{r/2M} = \frac{r^2}{64M^4}\,\frac{e^{r/2M}}{(r/2M-1)}\right]^{(**)}$$

Plugging $(**)$ into $(*)$ yields.....

$$-dv^2 + du^2 = \frac{r^2}{64M^4}\,\frac{e^{r/2M}}{(r/2M-1)}\,dr^2 - \frac{1}{16M^2}\,(r/2M-1)\,e^{r/2M}\,dt^2$$

$$= \frac{1}{16M^2}\left[-\frac{r}{2M}\left(1-\frac{2M}{r}\right)dt^2 + \frac{r^2}{4M^2}\cdot\frac{1}{r/2M\,(1-2M/r)}\,dr^2\right]e^{r/2M}$$

$$= \frac{r}{32M^3}\left[-\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1-2M/r)}\right]e^{r/2M}$$

Solving bracketed term.....

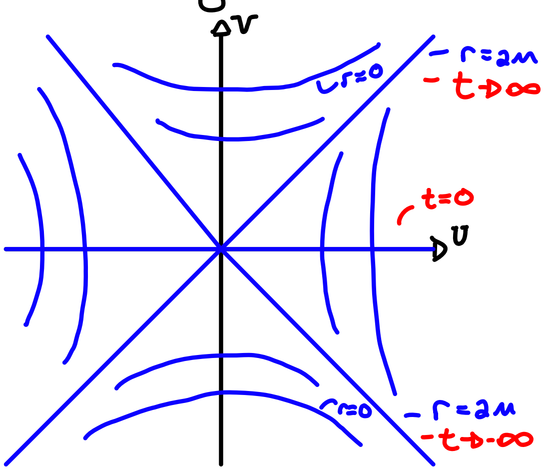
$$-(1-2M/r)\,dt^2 + \frac{dr^2}{(1-2M/r)} = \frac{32M^3}{r}\left[-dv^2+du^2\right]e^{-r/2M}$$

$$\left[ds^2 = \frac{32M^3}{r}\left(-dv^2+du^2\right)e^{-r/2M} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

$$u^2 - v^2 = (r/2M-1)\,e^{r/2M}\left(\cosh^2(t/4M) - \sinh^2(t/4M)\right) \quad \therefore \quad \left[(r/2M-1)\,e^{r/2M} = u^2 - v^2\right]$$

4-15-19

Kruskal Diagram



Lines of constant r...

$$\left[U^2 - V^2 = \text{constant.}\right]$$ − Hyperbolas in UV plane

For $r=2M$:

$U^2 - V^2 = 0 \quad \therefore \left[V = \pm U\right]$ − Straight lines of 45° slopes

− Two event horizons?

For $r=0$:

$$-1 = U^2 - V^2 \quad \therefore \quad V^2 = U^2 + 1 \longrightarrow \left[V = \pm\sqrt{U^2+1}\right]^{(*)}$$

Two physical singularities

$(*)$ using $(**)$

$$U^2 + 1 = (1 - r/2M)\,e^{r/2M}\sinh^2\left(\frac{t}{4M}\right) + 1\,\Big|_{V=0} = \sinh^2\left(\frac{t}{4M}\right) + 1 = \cosh^2\left(\frac{t}{4M}\right) = V\,\Big|_{r=0}$$

How about t?

$$\frac{V}{U} = \tanh\left(\frac{t}{4m}\right) \quad r > 2m$$

$$\frac{U}{V} = \tanh\left(\frac{t}{4m}\right) \quad r < 2m$$

Lines of constant t ......

$$\frac{V}{U} = \text{constant} \quad \therefore \quad \left[V = (\text{constant})\,U\right]$$

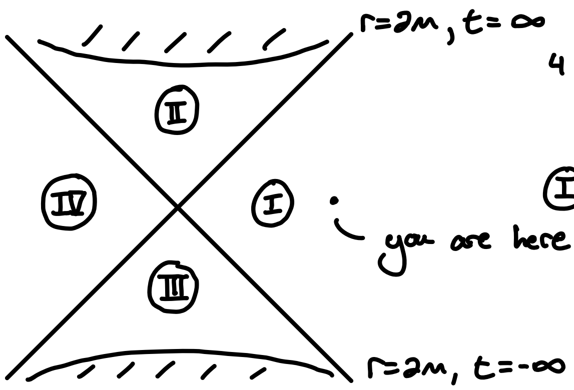
$$\lim_{t\to\infty} \frac{u}{V} \longrightarrow 1 \quad : \quad \tanh(\sigma) = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}$$

$$\lim_{t\to-\infty} \frac{u}{V} \longrightarrow -1$$

$$\lim_{t\to 0} \tanh\left(\frac{t}{4m}\right) \longrightarrow 0 \quad \therefore \quad \frac{V}{u} = 0 \quad \therefore \quad V = 0 \quad \text{for } r > 2m$$

$$\frac{u}{V} = 0 \quad \therefore \quad u = 0 \quad \text{for } r < 2m$$

Kruskal Extension of the Schwarzschild Geometry



4 Regions

I : Covered by Schwarzschild coordinates
I : II : Covered by Eddington-Finklestein coordinates
all regions covered by Kruskal coordinates

Notice :

- g_{vv} is always negative
- $g_{uu}, g_{\theta\theta}, g_{\varphi\varphi}$ are always positive
- $r=0$ is clearly a spacelike surface
- The horizon at $r=2m$ is the $V=u$ line null surface governed by radial light rays that remain stationary @ $r=2m$
- For $r<2m$, decreasing r is simply the time-like direction
- Two asymptotically flat regions, 1 where $\lim u \to \infty$, 1 wher $v \to -\infty$

III is the time-reverse of II $\Rightarrow$ Part of spacetime from which things can escape to us called a White Hole

Radial light rays ....

$$d\theta = d\varphi = 0 \ \vdots \ ds^2 = 0$$

$$① = -dv^2 + du^2 \quad \therefore \quad \frac{dv}{du} = \pm 1 \longrightarrow \text{As was the case in 4D Minkowski Spacetime}$$

Notice

$\left|\frac{dv}{du}\right| > 1$ — For a particle world line