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CSCI 2824: Discrete Structures

Section 7.2: Probability Theory and its Applications

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Warm-Up:

Ex). What is the probability of drawing a Full House in a 5-card poker hand? (A Full House is 3 cards of one value and 2 cards of another value).

Full House

A standard deck of cards contains 52 cards of 4 different suits.

There are 13 cards per suit. ♥ ♣ ♦ ♠

Each of the 13 cards has a different value {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}.

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Card games are a substantial source of probability type questions.

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There are 13 cards per suit. ♥ ♣ ♦ ♠

Each of the 13 cards has a different value $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$.

Ex). What is the probability of drawing a Full House in a 5-card poker hand?

A Full House is 3 cards of one value and 2 cards of another value.

Full House

What is the probability of drawing a Full House in a 5-card poker hand?
A Full House is 3 cards of one value and 2 cards of another value.

ANS:

$$|E| = \underbrace{P(13,2)}_{\text{values}} \cdot \underbrace{C(4,3)}_{\text{suits}} \cdot \underbrace{C(4,2)}_{\text{suits}} = 3744$$

or equivalently,

$$|E| = \underbrace{\underbrace{C(13,1)}_{\text{value}} \cdot \underbrace{C(4,3)}_{\text{suit}}}_{\text{Three of a kind}} \cdot \underbrace{\underbrace{C(12,1)}_{\text{value}} \cdot \underbrace{C(4,2)}_{\text{suit}}}_{\text{Two of a kind}} = 3744$$

$|S| = C(52,5) = 2,598,960$

Full House

Therefore, $p(E) = \frac{|E|}{|S|} = \frac{3744}{2,598,960} \approx 0.0014$.

Discrete Probability

Example: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is a 1?

A sequence of 10 bits is randomly generated. What is the probability that **at least one** of these bits is 0?

ANS:

Let E_1 be the event that one of the 10 bits is a 0.
Let E_2 be the event that two of the 10 bits are 0.
Let E_3 be the event that three of the 10 bits are 0.
Let E_4 be the event that four of the 10 bits are 0.
Let E_5 be the event that five of the 10 bits are 0.
Let E_6 be the event that six of the 10 bits are 0.
Let E_7 be the event that seven of the 10 bits are 0.
Let E_8 be the event that eight of the 10 bits are 0.
Let E_9 be the event that nine of the 10 bits are 0.
Let E_{10} be the event that ten of the 10 bits are 0.

Option 1:
Find the probability of all of these events,
and then add them together

$$\frac{\binom{10}{1}}{2^{10}} + \frac{\binom{10}{2}}{2^{10}} + \cdots + \frac{\binom{10}{10}}{2^{10}}$$

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

ANS:

Let E be the event that **at least one** of the 10 bits is a 0.

Then $\bar{E}$ is the event that **none** of the 10 bits are 0, i.e. $\bar{E}$ is the event that all the bits are 1.

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{|\bar{E}|}{|S|} = 1 - \frac{\binom{10}{0}}{2^{10}} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

The only string left out is the string of all 1's.

Discrete Probability

<u>**Theorem**</u>: Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complement of E, is given by

$$p(\bar{E}) = 1 - p(E)$$

Discrete Probability

Theorem: Let E_1 and E_2 be two events in the sample space S

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example: Suppose you draw one card from a standard deck. What is the probability that is is an Ace or a Diamond?

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

ANS:

Let E_1 be the event that the random integer selected is divisible by 2:

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, \ldots\}$$

Let E_2 be the event that the random integer selected is divisible by 5:

$$\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, \ldots\}$$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}$$

Conditional Probability

How likely is it that it will be 32°F today?

How likely is it that it will be 32°F today, given that you live on a Caribbean island?

How likely is it that it will be 32°F today, given that you live in Antarctica?

Knowing additional information changes the probability.

The answer is dependent upon the additional conditions.

Probability Theory – Conditional Probability

You flip a coin three times.

a). What is the probability of getting exactly 2 heads?

b). ***Given that the first flip is tails***, what is the probability of getting exactly 2 heads?

Probability Theory – Conditional Probability

You flip a coin three times.

a). What is the probability of getting exactly 2 heads?

By the multiplication principle, $2 \cdot 2 \cdot 2 = 8$ different outcomes:

S = $\{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$

E = $\{\ HHT,\ HTH,\ THH\}$

P(two heads) =

b). ***Given that the first flip is tails***, what is the probability of getting exactly 2 heads?

New S = $\{THH,\ THT,\ TTH,\ TTT\}$

Probability Theory – Conditional Probability

The probability that E occurs given that F occurred is called the ***conditional probability*** of E given F and denoted $\boldsymbol{p(E|F)}$.

Let E and F be events with $p(F) > 0$. The conditional probability of E given F is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

- In general, to compute the conditional probability of E given F we use F as the sample space. For E to occur it must be the case that E belongs to $E \cap F$.

Probability Theory – Conditional Probability

Let E and F be events with $p(F) > 0$. The conditional probability of E given F is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

<u>Example</u>: Suppose that we flip a fair coin three times.
Now suppose that the first coin flip is Tails.
Given this information, what is the probability that an odd number of coins comes up Tails?

Probability Theory – Conditional Probability

Let E and F be events with $p(F) > 0$. The conditional probability of E given F is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Example: Suppose that we flip a fair coin three times.
Now suppose that the first coin flip is Tails.
Given this information, what is the probability that an odd number of coins comes up Tails?

$F = \{THH, THT, TTH, TTT\}$

$E = \{HHT, HTH, THH, TTT, \}$

$E \cap F = \{THH, TTT\}$

Recalling that the original sample space S has size 8, we have

$$P(E \cap F) = \frac{2}{8} = \frac{1}{4} \quad \text{and} \quad p(F) = \frac{4}{8} = \frac{1}{2}$$

Thus, by the definition of conditional probability

$$p(E \mid F) = \frac{1/4}{1/2} = \frac{1}{2} \quad ✓$$

Notice the definition of conditional probability,

In general, the conditional probability of A given C is:

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, \quad \text{as long as } P(C) > 0.$$

leads us to the multiplication rule:

The multiplication rule:

$$P(A \cap C) = P(A|C)P(C)$$

- Useful when the conditional probability is easy to compute, but the probability of the intersections is not.

Probability Theory – Conditional Probability

Example: A bit string of length four is generated at random. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Probability Theory – Conditional Probability

Example: A bit string of length four is generated at random. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

There are $2^4 = 16$ possible outcomes.
Eight of these start with a 0.

$\underline{\;0\;}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$

Let E be the event that a bit string of length four contains at least two consecutive 0's.
Let F be the event that the first bit of a bit string of length four is a 0.

Then $E \cap F = \{0000,\ 0001,\ 0010,\ 0011,\ 0100\}$

$p(E \cap F) = {}^{5}\!/_{16}$

$p(F) = {}^{8}\!/_{16}$

Therefore, $p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{5/16}{8/16} = \frac{5}{16} \cdot \frac{16}{8} = \frac{5}{8}$

Terminology

We say that E and F are *independent* events when the occurrence of one event has no effect on the outcome of the second event. Just because you are given extra info…

Consider two games

1] roll a 6-sided die and win \$2 if the outcome is odd, win \$4 if the outcome is a five, win \$0 for any other outcome.

What is the probability that you roll a 5, given that you rolled an odd?

2] Roll a 6-sided die and flip a coin. Win \$5 if you roll a 6 and flip T.

What is the probability that you roll a 6, given that you flipped T?

Does the extra information change the probability of the outcome?

If not, then the events are independent.

Probability – Independence

An event A is said to be **independent** of event B if $p(A \mid B) = p(A)$

Probability Theory

When two events are independent, something nice happens

Assume that E and F are independent, then

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = p(E)$$

If we solve the second equality for $p(E \cap F)$ we find that

$$p(E \cap F) = p(E)p(F)$$

So if two events are independent, we can compute the probability of their intersection (i.e. that they both occur) by simply multiplying their individual probabilities!

Probability – Independence

An event A is said to be **independent** of event B if $p(A \mid B) = p(A)$

This definition, combined with the multiplication rule and the definition of conditional probability, give us a few equivalent tests for independence of two events:

1) $p(A \mid B) = p(A)$
2) $p(B \mid A) = p(B)$
3) $p(A \cap B) = p(A)P(B)$

Events $A_1, A_2, \ldots, A_m$ are **independent** if $P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1)P(A_2) \ldots P(A_m)$

Probability Theory

Example: Suppose you draw two cards from a deck without replacement.

a). Are the events that the first card is Red and that the second card is Red independent?

b). What is the probability that the first card is red and the second card is red?

Probability Theory

<u>**Example**</u>: Suppose you roll a fair die and flip a coin.
What is the probability that you roll a 4 and that the coin lands Tails?

Probability Theory

Example: Suppose you draw two cards from a deck.
What is the probability that the 1st card is a club and the 2nd card is a heart?

Probability Theory

Example: Suppose you draw two cards from a deck. What is the probability that the 1st card is a club and the 2nd card is a heart?

Note: This is NOT independent because you are more likely to draw a heart after one of the clubs is removed.

Probability – Review

Conditional Probability: The probability that A occurs given that C has occurred is

$$P(A\,|C) = \frac{P(A\ \cap C)}{P(C)}$$

Multiplication Rule: $P(A\ \cap C) = P(A\mid C)\,P(C)$

Independence: Events A and B are independent if and only if

1) $P(A\mid B) = P(A)$
2) $P(B\mid A) = P(B)$
3) $P(A\ \cap B) = P(A)P(B)$

The definition of probability that we have used so far: $p(E) = \frac{|E|}{|S|}$

assumes that all outcomes are equally likely.

This is like flipping a coin or rolling a die.

But what if the outcomes are not equally likely?

Like flipping an unbalanced coin or rolling a 6-sided die with four sides having a 6 and the other two sides having a 1.

Remark: We will not discuss probabilities of events when the set of outcomes is not finite or countable, such as when the outcome of an experiment can be any real number.
In such cases, integral calculus is usually required for the study of the probabilities of events.

Frequentist (Classical Statistics) view of Probability:

To model an experiment, the probability $p(s)$ assigned to an outcome s should equal ***the limit*** of the number of times s occurs divided by the number of times the ***experiment is performed as this number grows without bound.***

For example:
What is the probability of a 32°F temperature on November 24 each year?

This probability can be assigned by looking at past years' data.

(This is assuming that successive trials do not depend on past results).

Probability Theory

To solve this issue, ***we will assign probabilities to events*** rather than simply divide by the number of outcomes.

Let S be the sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome $s \in S$. We require two conditions be met:

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

- Since E is a set of possible outcomes, we compute the probability of E by summing the probabilities of the outcomes in E.

The probability of an event E is the sum of the probabilities of the outcomes in E.

$$p(E) = \sum_{s \in E} p(s)$$

- Before we thought of $p(E)$ as a number, but now we think of p as a function that maps outcomes to a value between 0 and 1.

Probability Theory

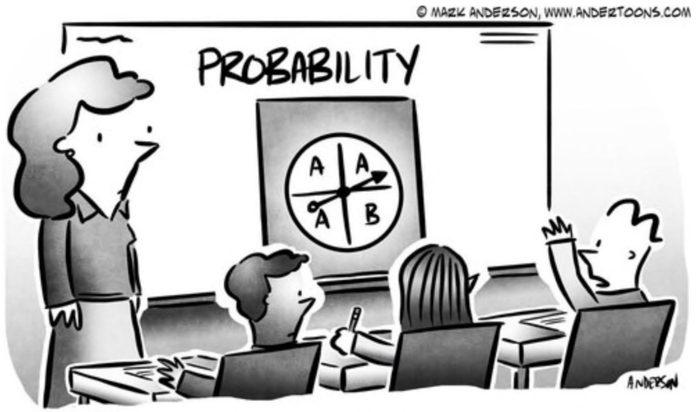
The function p from the set of all outcomes of the sample space S (of an experiment with a finite or countable number of outcomes) is called a ***probability distribution***.

Example Part A: Give the probability distribution that models the outcomes H and T when a fair coin is flipped.

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

1. $p(H) =$, $p(T) =$
2. $p(H) + p(T) =$

Two common ways to write probability distributions:



"I know mathematically that A is more likely, but I gotta say, I feel like B wants it more."

Probability Theory

The function p from the set of all outcomes of the sample space S is called a ***probability distribution***.

Example Part A: Specify the probability distribution that models the outcomes H and T when a fair coin is flipped.

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

1. $p(H) = \frac{1}{2}, \quad p(T) = \frac{1}{2}$
2. $p(H) + p(T) = \frac{1}{2} + \frac{1}{2} = 1$

Example Part B: Suppose the coin is now unbalanced and biased to show H twice as often as T. What probabilities do you assign?

Suppose the coin is biased show H twice as often as tails.

What probabilities do you assign?

$p(H) = 2\ p(T)$

$p(H) + p(T) = 1$

$\therefore\ \ 2\,p(T) + p(T) = 3p(T) = 1$

So, $p(T) = {}^{1}/_{3}$ and $p(H) = {}^{2}/_{3}$ Both (i) and (ii) conditions are met.

Probability Theory

Example: Suppose that a die is biased so that 3 appears twice as often as each other number but the other five outcomes are equally likely.

a). What is the probability distribution?

b). What is the probability that an odd number is rolled?

c). What is the probability that an even number is rolled?

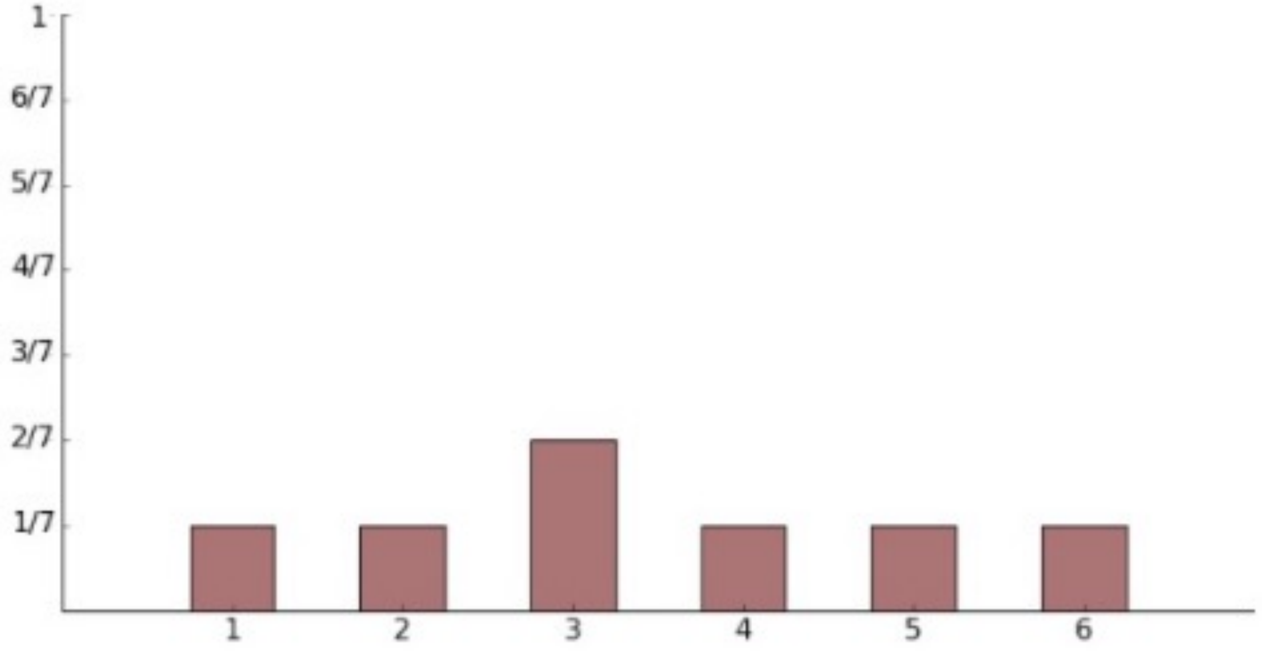
Probability Theory

Example: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but the other five outcomes are equally likely.

a). What is the probability distribution?

Two common ways to write down distribution p

s	1	2	3	4	5	6
$p(s)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$



Probability Theory

Example: Suppose that a die is biased so that 3 appears twice as often as each other number but the other five outcomes are equally likely.

b). What is the probability that an odd number is rolled?

What do we know?

$p(1) = p(2) = p(4) = p(5) = p(6),$ $\qquad p(3) = 2p(k)$ where $k = 1, 2, 4, 5, 6$

$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$ ← Recall $\sum_{s \in S} p(s) = 1$

Therefore,

$$p(E) = \sum_{s \in E} p(s)$$

$$p(1) + p(1) + 2p(1) + p(1) + p(1) + p(1) = 1$$

$$\Longrightarrow \ \mathrm{p}(1) = \frac{1}{7} \ [= p(2) = p(4) = p(5) = p(6)] \Longrightarrow \ p(3) = \frac{2}{7}$$

$$\boldsymbol{p(odd) = p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}}$$

Probability Theory

Example:

Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but the other five outcomes are equally likely.

c). What is the probability that an even number is rolled?

Probability Theory

Example: For the loaded die in the previous example, what is the probability that the die comes up even?

Recall:

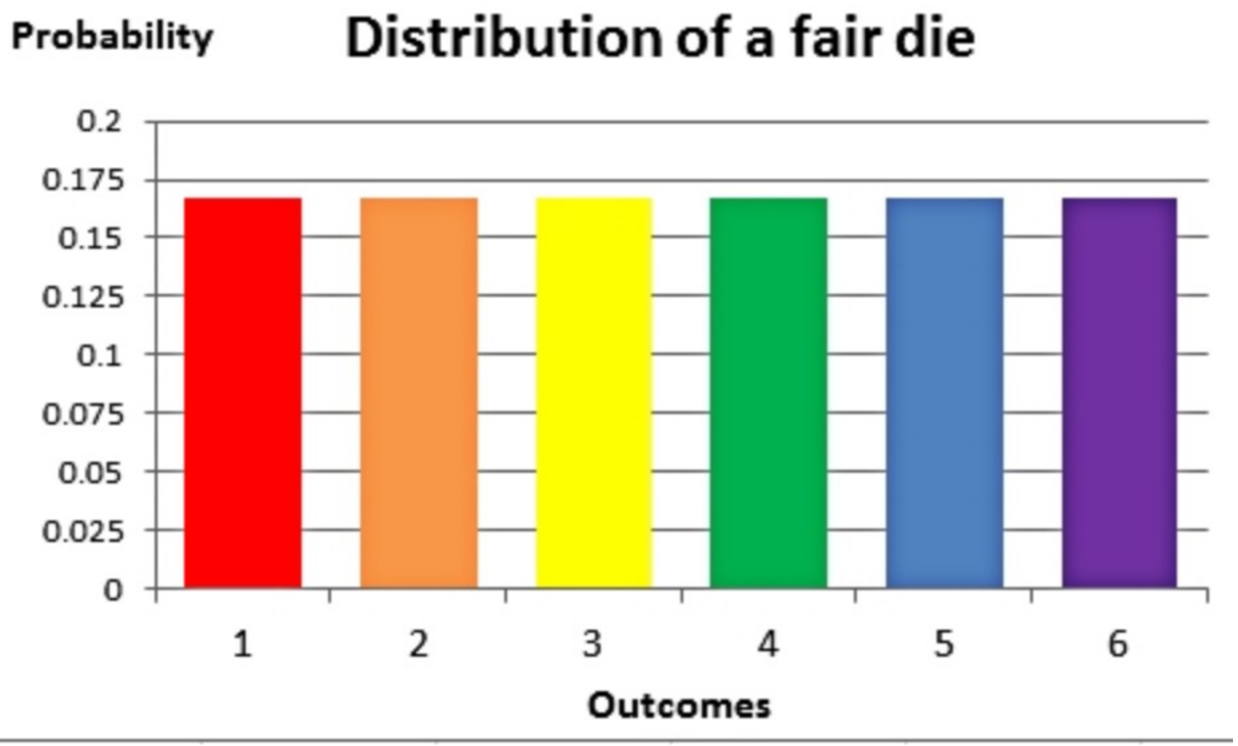
Complement: $p(\bar{E}) = 1 - p(E)$

Union: $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Uniform Distribution

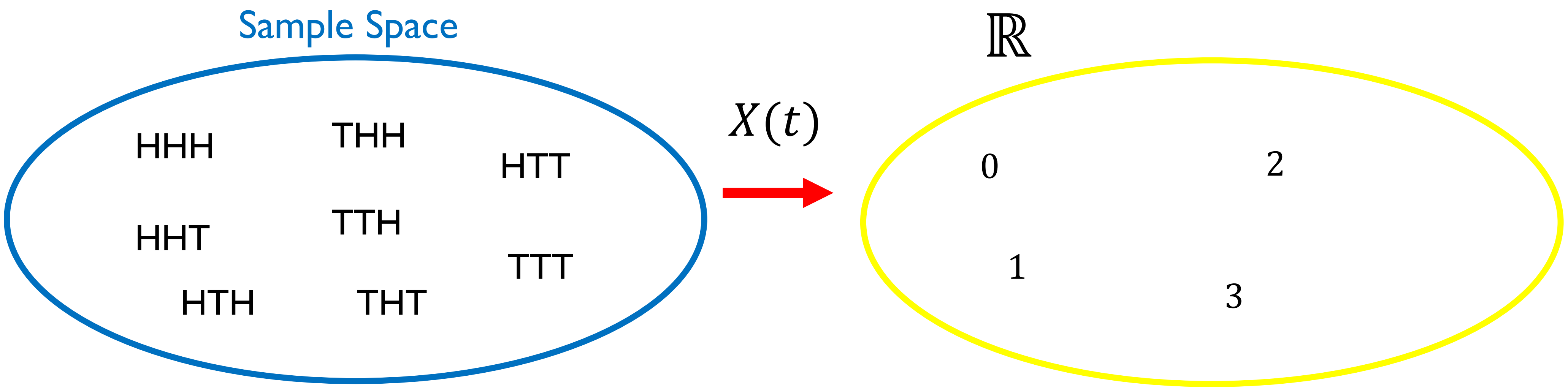
Suppose that S is a set with n elements. The ***uniform distribution*** assigns the probability $1/n$ to each element of S.

e.g. The results of rolling a single fair die is a uniform distribution with $n = 6$.



Random Variables

Many problems are concerned with a numerical value associated with the outcome of an experiment. For instance, we may be interested in the total number of one bits in a randomly generated string of 10 bits; or in the number of times tails come up when a coin is flipped 20 times. To study problems of this type we introduce the concept of a random variable.



Random Variables

This is the unfortunate name of a function.

Random variables are NOT random.

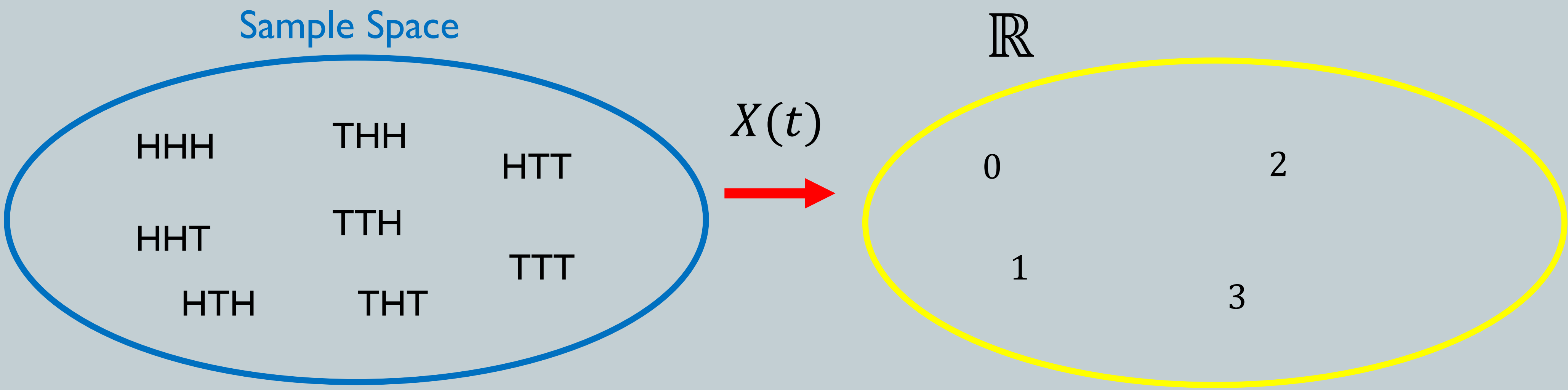
Random variables are NOT variables.

Random variables ARE functions.

Random Variable:

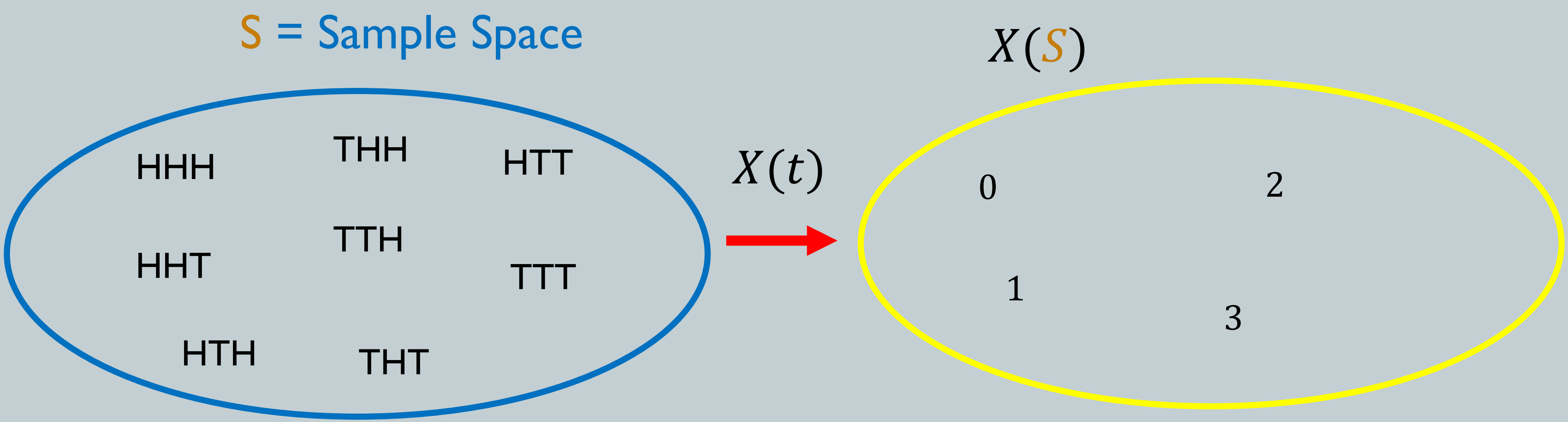
A function from the sample space of an experiment to the set of real numbers.

i.e. A random variable (function) assigns a real number to each possible outcome.

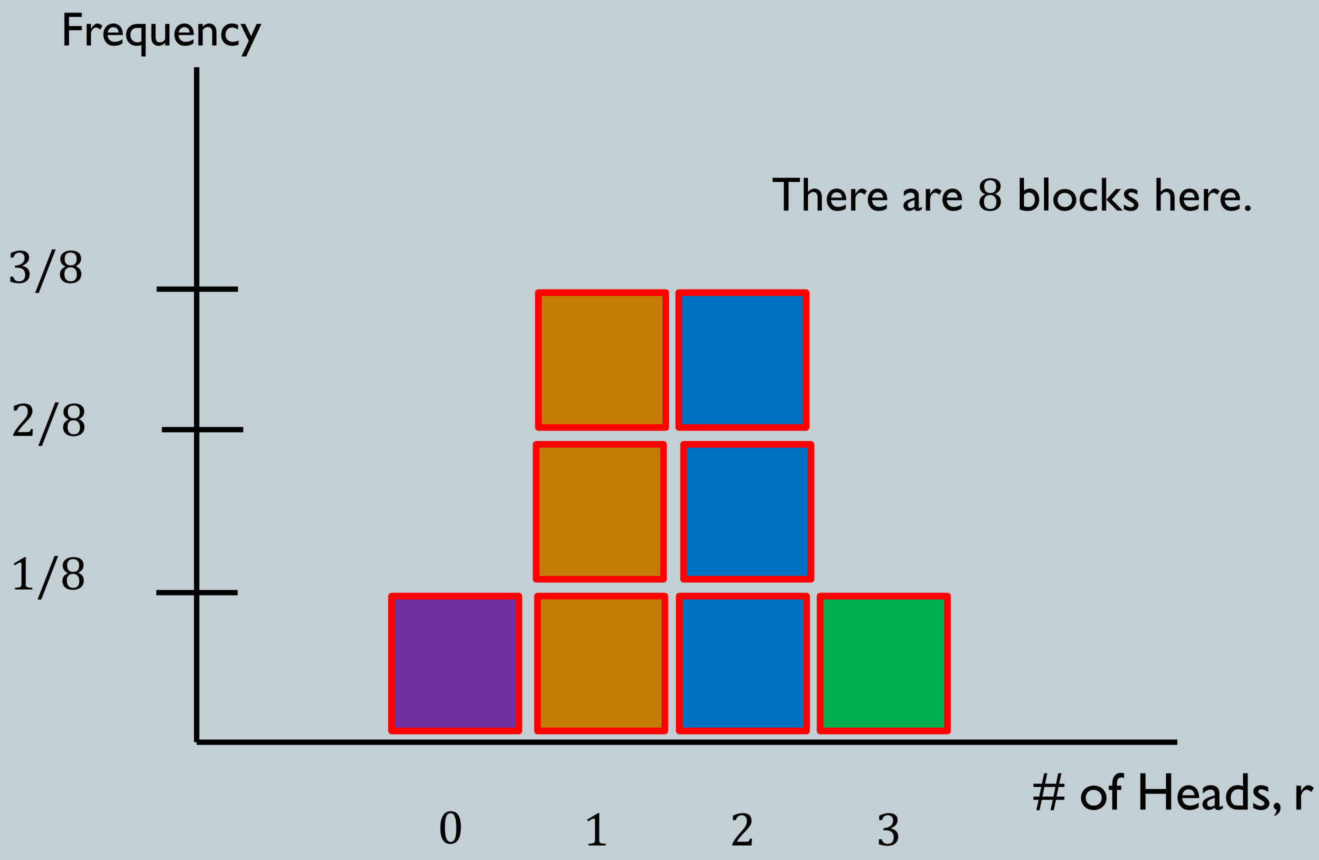


Let $X(t)$ be the function (random variable) that equals the number of heads that appear when t is the outcome of the experiment of tossing a coin three times.

$$X(TTT) = 0$$
$$X(TTH) = X(THT) = X(HTT) = 1$$
$$X(HHT) = X(HTH) = X(THH) = 2$$
$$X(HHH) = 3$$



The *distribution* of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(s)$,
where $p(X = r)$ is the probability that X takes the value r.
The set of pairs in this distribution is determined by the probabilities $p(X = r)$ for $r \in X(S)$.



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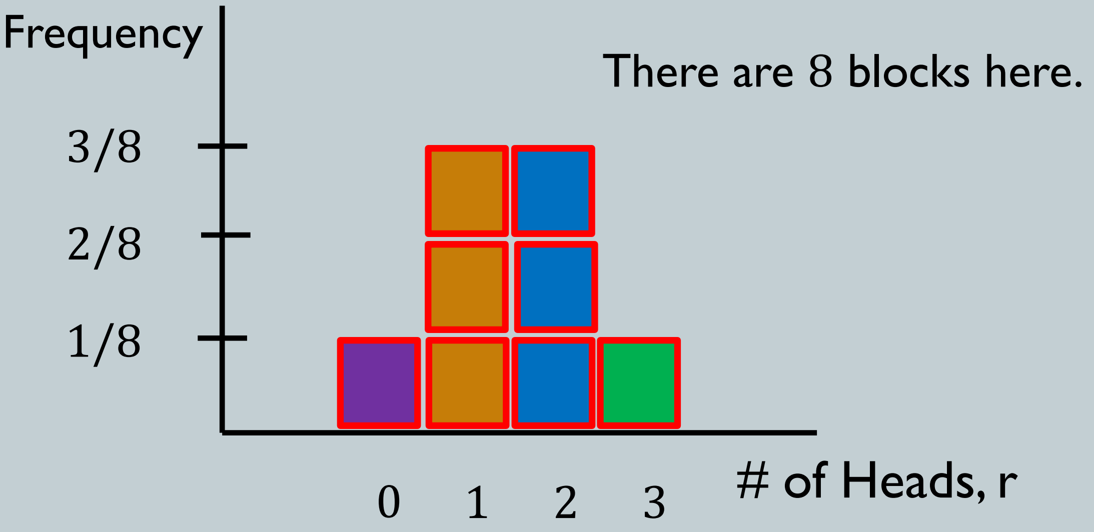
The set of pairs in this distribution is determined by the probabilities $p(X = r)$ for $r \in X(S)$.

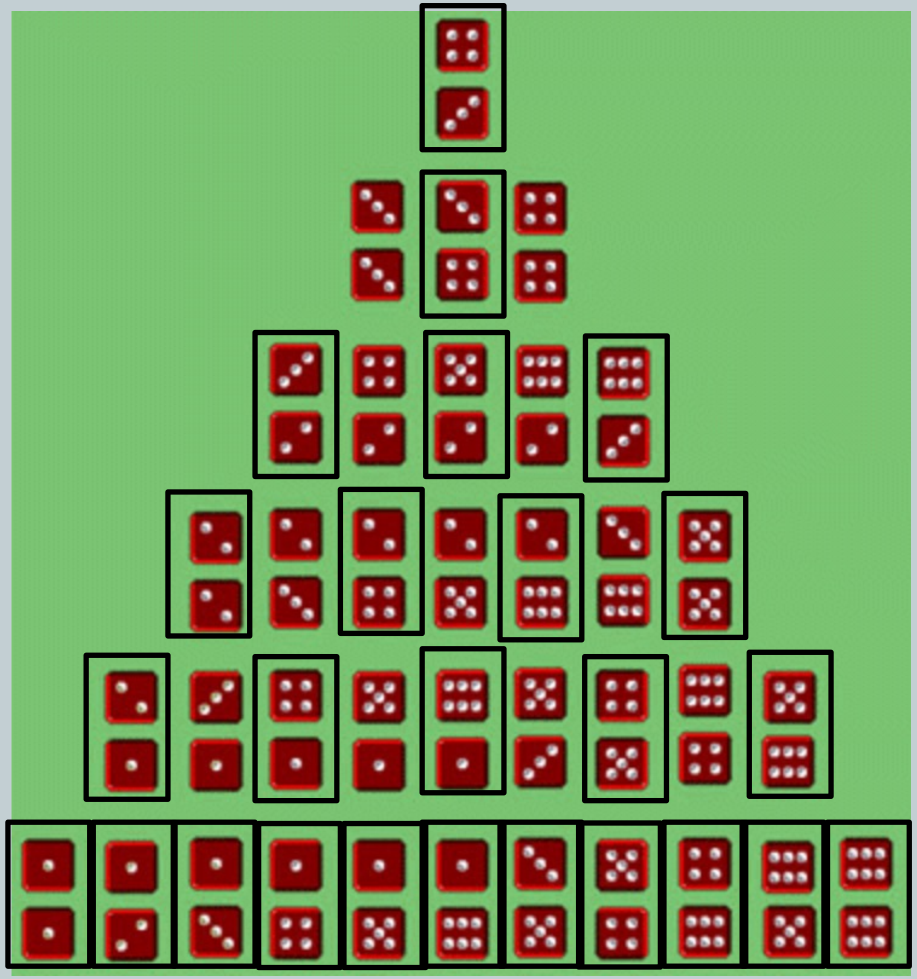
$X(TTT) = 0$
$X(TTH) = X(THT) = X(HTT) = 1$
$X(HHT) = X(HTH) = X(THH) = 2$
$X(HHH) = 3$



Each of eight possible outcomes when a fair coin is flipped three times has probability $^1/_8$.
So, the distribution of the random variable $X(t)$ is determined by the probabilities:
$P(X = 0) = {^1/_8}, P(X = 1) = {^3/_8}, P(X = 2) = {^3/_8}, P(X = 3) = {^1/_8}$.

The *distribution* is therefore, $(0, {^1/_8}), (1, {^3/_8}), (2, {^3/_8}), (3, {^1/_8})$.

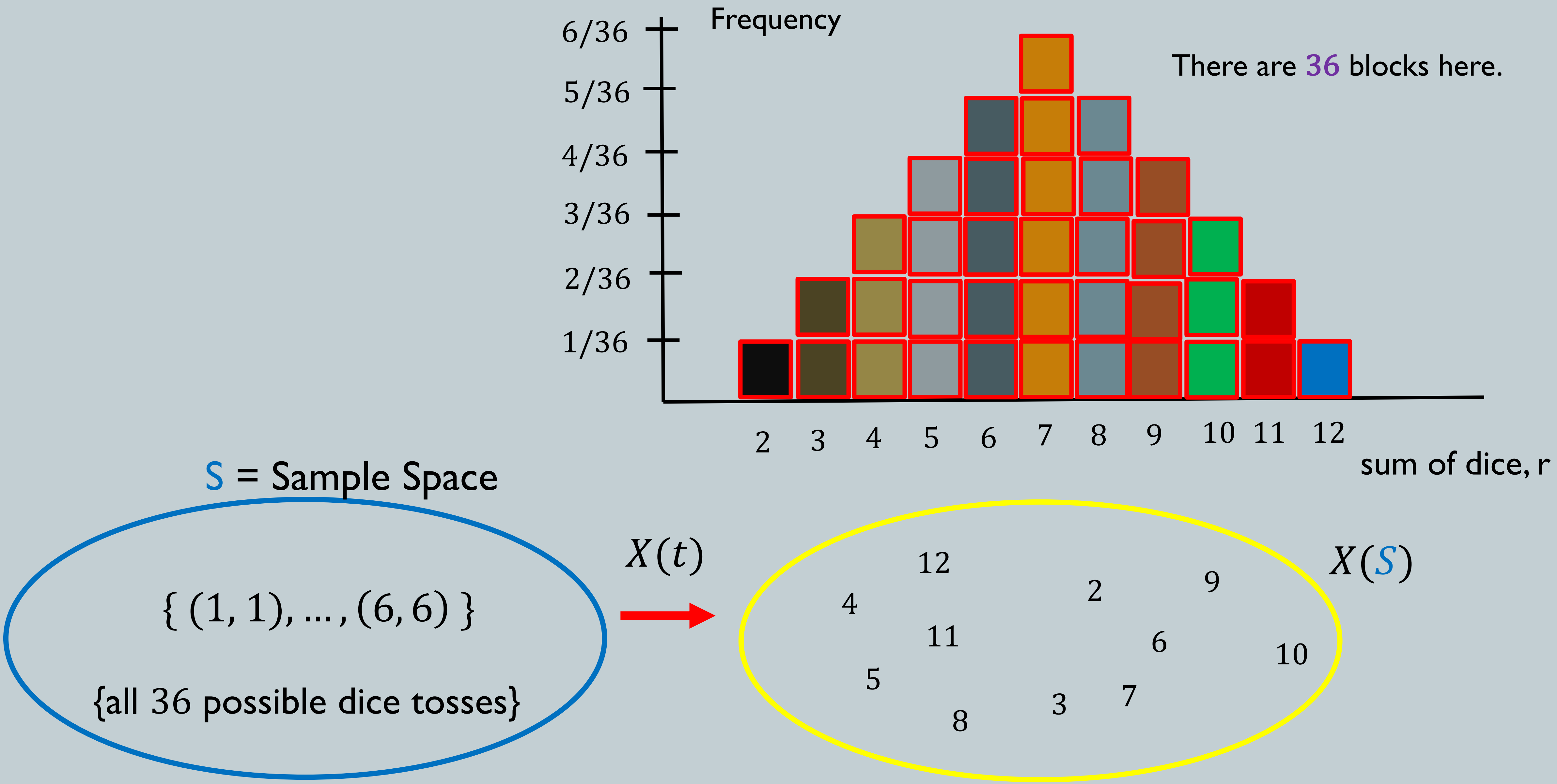
Suppose X is the sum of the numbers that appear when a pair of dice is rolled.
What are the values of this random variable for the outcomes (i, j) ?
(i and j are the numbers that appear on the first & second die when rolled)



Suppose X is the sum of the numbers that appear when a pair of dice is rolled. What are the values of this random variable for the outcomes (i, j) ?

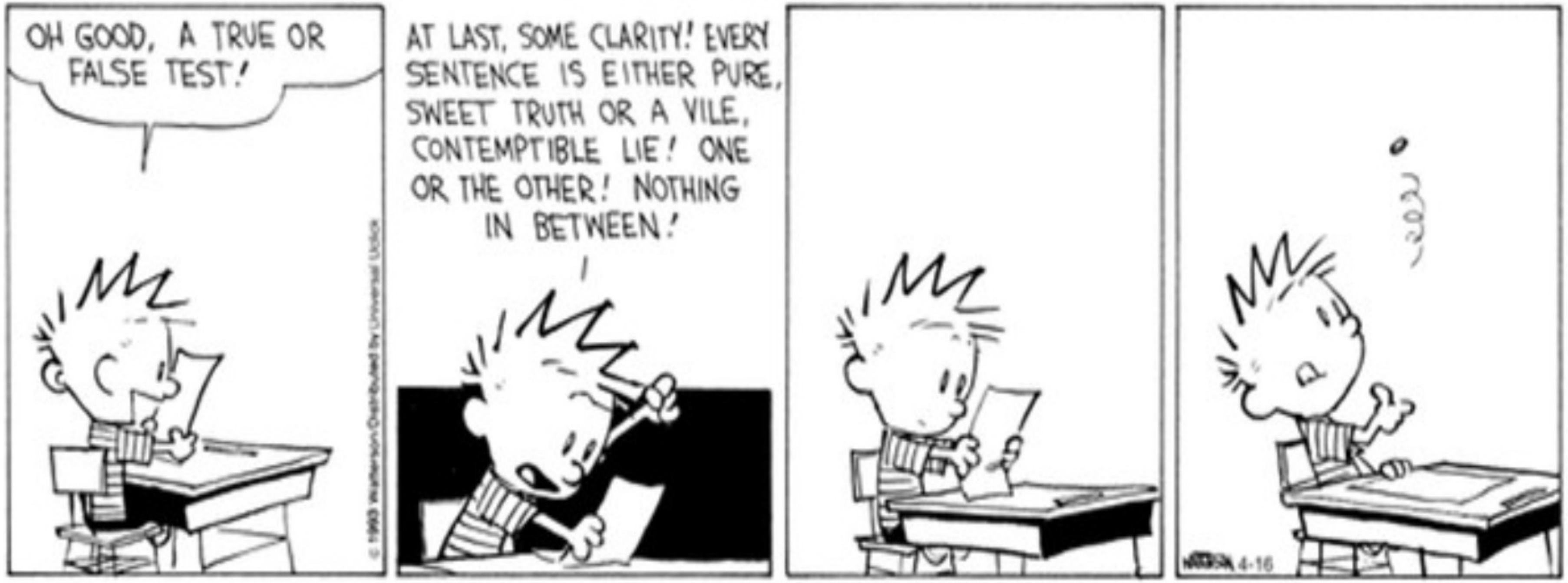
$$
\begin{aligned}
&X((1,1)) = 2,\\
&X((1,2)) = X((2,1)) = 3,\\
&X((1,3)) = X((2,2)) = X((3,1)) = 4,\\
&X((1,4)) = X((2,3)) = X((3,2)) = X((4,1)) = 5,\\
&X((1,5)) = X((2,4)) = X((3,3)) = X((4,2)) = X((5,1)) = 6,\\
&X((1,6)) = X((2,5)) = X((3,4)) = X((4,3)) = X((5,2)) = X((6,1)) = 7,\\
&X((2,6)) = X((3,5)) = X((4,4)) = X((5,3)) = X((6,2)) = 8,\\
&X((3,6)) = X((4,5)) = X((5,4)) = X((6,3)) = 9,\\
&X((4,6)) = X((5,5)) = X((6,4)) = 10,\\
&X((5,6)) = X((6,5)) = 11,\\
&X((6,6)) = 12.
\end{aligned}
$$

The *distribution* of random variable X on a sample space S. X is the sum of the dice.



The Bernoulli Distribution

- The Bernoulli distribution is used to model experiments with only two possible outcomes; often referred to as “success” and “failure”, and encoded as 1 and 0, respectively.



Bernoulli Trials

Suppose we are running an experiment that can have only two possible outcomes.

Perhaps a bit is to be created: it is either 0 or 1. (cosmic rays, particles)

Perhaps a coin is flipped: it is either H or T.

Each performance of an experiment with two possible outcomes is called a Bernoulli trial

In a Bernoulli trial we will label the two outcomes as "success" and "failure"

$\{p, q\}$ {Heads, Tails} {0, 1} {Green, Red} {Small, Large} etc.

If p is the probability of success, and $1\text{-}p$ is the probability of failure, then $p + (1 - p) = 1$.

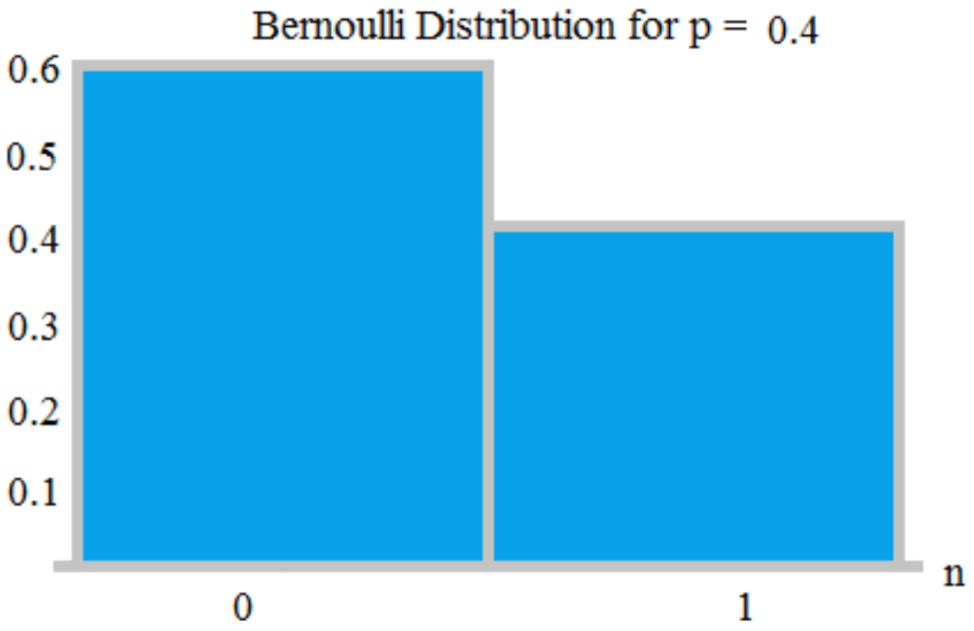
The Bernoulli Distribution

A discrete random variable X has a **Bernoulli distribution** with parameter p, where $0 \leq p \leq 1$, if its probability function p_X is given by

$$p_X(1) = P(X = 1) = p \quad \text{and} \quad p_X(0) = P(X = 0) = 1 - p$$

We denote this distribution by Ber(p).

Example:



Independent Bernoulli Trials

A coin is biased so that the probability of heads is $^2/_3$.

What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

Independent Bernoulli Trials

A coin is biased so that the probability of heads is $^{2}/_{3}$.

What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

This is a Bernoulli trial repeated seven times.

How many outcomes are possible?

How many of these outcomes have exactly four heads?

How likely is it that a 'Head' appears on any given flip?

A coin is biased so that the probability of heads is $^2/_3$. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

ANS:

How many outcomes are possible? $2^7 = 128$

How many of these outcomes have exactly four heads?

$$C(7,4) = \frac{7!}{4!\,3!} = \frac{5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3} = 5 \cdot 7 = 35$$

$$\{HHHHTTT, \quad HHHTHTT, \quad HHHTTHT, \quad etc.\}$$

How likely is it that a 'Head' appears on any given flip?

$p = \frac{2}{3}$ which also implies that $1 - \mathrm{p} = \frac{1}{3}$.

So, for instance $HHHHTTT$ has a probability of $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$, and there are 35 of these expressions in various orders. Commutative.

A coin is biased so that the probability of heads is $^2/_3$. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

ANS:

$$C(7,4) \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^3 = \frac{35 \cdot 16}{3^7} = \frac{560}{2187}$$

Theorem:

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is:

$$C(n, k) \cdot p^k \cdot q^{n-k}$$

Proof:

When n Bernoulli trials are carried out, the outcome is an n-tuple $(t_1, t_2, \ldots, t_n)$, where $t_i = S$ or $t_i = F$ for $i = 1, \ldots, n$.

Because the n trials are independent, the probability of each outcome of n trials consisting of k successes and $n - k$ failures (in any order) is $p^k q^{n-k}$.

Because there are $C(n, k)$ n-tuples of S's and F's that contain exactly k amount of S's, the probability of exactly k successes is $C(n, k) \cdot p^k \cdot q^{n-k}$. Q.E.D.

In repeated Bernoulli trials we are interested in:

n amount of trials,

k amount of successes,

p being the probability of a success,

q is therefore the probability of failure: $q = 1 - p$.

So, we could encapsulate all this information as a function of 3 inputs,

This function is called the binomial distribution.

$$b(k; n, p) = C(n, k) \cdot p^k \cdot (1 - p)^{n-k}$$

Binomial Distribution

A discrete random variable X has a **Binomial Distribution** with parameters *n* and *p*, where $n = 1, 2, \ldots$ and $0 \le p \le 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n$$

We denote this distribution by Bin(n, p).

Suppose that the probability that a 0 bit is generated is 0.9,
that the probability that a 1 bit is generated is 0.1,
and that bits are generated independently.

What is the probability that exactly eight 0 bits are generated when 10 total bits are generated?

Suppose that the probability that a 0 bit is generated is 0.9, that the probability that a 1 bit is generated is 0.1, and that bits are generated independently.

What is the probability that exactly eight 0 bits are generated when 10 total bits are generated?

ANS:

$$b(8; 10, 0.9) = C(10, 8) \cdot (0.9)^8 \cdot (0.1)^2 = 0.1937102445$$

Independent Bernoulli Trials

Example: Suppose that you show up to a quiz completely unprepared. The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way.

Let X be the discrete random variable that gives the number of questions you guess correctly.

Sum of Bernoulli Random Variables

Example: Suppose that you show up to a quiz completely unprepared. The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. Let X be the discrete random variable that gives the number of questions you guess correctly.

What is the probability that you get 0 questions correct?

What is the probability that you get 1 problem correct?

Sum of Bernoulli Random Variables

<u>Example:</u> Suppose that you show up to a quiz completely unprepared. The quiz has 5 questions, each with 4 multiple choice options. You decide to guess each answer in a completely random way. Let X be the discrete random variable that gives the number of questions you guess correctly.
What is the probability that you get 3 questions correct?

Sum of Bernoulli Random Variables → A Binomial Distribution!

Example : What is the probability that you get *k* questions correct out of *n* problems total?