I). Why Probability?

In the last lesson we explored some basic Descriptive Statistics (mean, median, IQR).

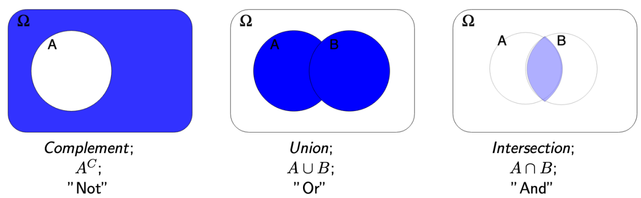
Descriptive statistics help us understand the collective properties of the elements of a data sample and form the basis for testing hypotheses and making predictions using **Inferential Statistics**.

***Inferential statistics* is built on the foundation of probability theory, so we turn our attention to probability!**

II). Defining Probability

A). Review of Set Notation:

Basic Set Operations



B). Probability Terminology

Experiment: A procedure that can be repeated, involves and element of chance, and has well defined outcomes.
Outcome: Result of an experiment
Sample Space Ω: The set of all possible outcomes
Cardinality of Ω (denoted |Ω|): The number of elements in the sample space set.
Event: Subset of Ω

	Example 1	Example 2
Experiment	Flipping a coin	Flipping a coin twice
Sample Space Ω		
|Ω|		
Example of an event	Getting tails:	Getting at least one tail:

DEFINITION:

A **probability function, *P*,** assigns a value in [0, 1] to each outcome or event in the sample space Ω, such that:

- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Comment: It's a great idea to think of probability as a *measure* like length, area, or volume. Indeed, advanced classes in probability are based on an area of mathematics called "measure theory".

C). The Philosophy of Probability:

You can think of probability as a numerical measure of uncertainty. Exactly what this means is the subject of considerable philosophical debate. For example:

What do we mean by "the probability that this coin lands on heads is 0.5"?

Does it mean that the coin has a certain *physical property* that causes it to land on heads 50% of the time?
Does it mean that the coin exhibits certain *behaviors in the long run*?
Or does probability refer to something about *my subjective degree of belief* that the coin will land on heads?

To further complicate things, what does it mean to say that "the probability that democrats win the US presidency in 2024 is 0.4"? (What physical property could "probability" refer to here? There is no "long run" for a single event like an election…)

The fact that there doesn't seem to be a single clear correct answer to the question *what kind of thing is probability*? is philosophically interesting. But it is also statistically and scientifically interesting, because so many issues in statistics and science arise because there are different, plausible interpretations of probability theory.

Some Interpretations of Probability

Classical Interpretation of Probability: : Symmetrical outcomes:

Probability of equally likely outcomes:
If Ω is a finite nonempty sample space of ***equally likely*** **outcomes**, and E is an event, that is, a subset of Ω, then the probability of E is

$$P(E) = \frac{|E|}{|\Omega|}$$

Pros:
- Applies easily to fair coins/die/card examples

Cons:
- Doesn't handle situations when outcomes are not equally likely
- Doesn't handle situations when the # of possible outcomes is infinite.

Objective (Frequentist view) Interpretation:

Defines probabilities as relative frequencies. So, what occurs in the long run is the probability.

Ex: Saying that "there's a 10% probability of it raining today" means if we keep track of all days on which it is forecast to rain with probability 10%, then in the long run the number of days it actually does rain gets closer and closer to 10%.

Subjective (sometimes called Bayesian) Interpretation:

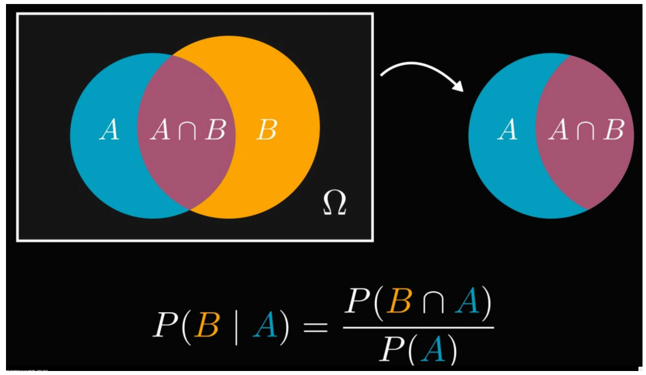
Defines probabilities as subjective degree of belief.

Ex: Saying that "there's a 10% probability of it raining today" means that someone or group (e.g., the people who constructed the model) believe with 10% confidence that it will rain today. Subjectivists often justify their views based on what people would be willing to bet (and avoiding so-called Dutch Book Arguments).

We've just scratched the surface! If you're interested in exploring this more check out **STAT4700: Philosophy of Statistics**

D). Rules of Probability

Definition: Conditional Probability



Ex 1). Suppose 48% of all teenagers own a skateboard and 39% of all teenagers own a skateboard and roller blades. What is the probability that a teenager owns roller blades given that the teenager owns a skateboard?

Ex 2). I have 3 cards: Ace of Hearts, King of Diamonds and Queen of Spades.

I shuffle them and draw two cards at random without replacement.

a). What is the chance that I get the Queen followed by the King?

Multiplication Rule: $P(A \cap B) = P(A)P(B|A)$
$=$

The answer is *less than or equal to* each of the two chances being multiplied.
The more conditions you have to satisfy, the less likely you are to satisfy them all

Ex 2). I have 3 cards: Ace of Hearts, King of Diamonds and Queen of Spades. I shuffle them and draw two cards at random without replacement.

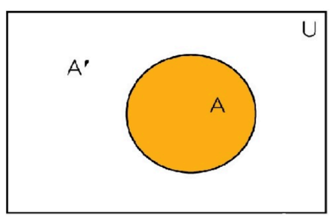
b). What is the chance that one of the cards I draw is a King and the other is a Queen?

Addition Rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

The answer is *greater than or equal to* the chance of each individual way.

Ex 3). What is the probability of getting at least one head in 3 coin tosses?

Complement Rule: $P(A') = 1 - P(A)$



Probability Problem Solving Techniques:

Ask yourself what event must happen on the first trial.

- If there's <u>a clear answer</u> (e.g. "not a six") whose probability you know, you can most likely use the **multiplication rule**.

1. Suppose you have a **biased** coin, where the **probability of heads** is p.

   (a) (3 pts) You flip the coin 8 times. What is the probability that you get the sequence HTTHTTTT (in that exact order)?

- If there's <u>no clear answer</u> (e.g. "could be K or Q, but then the next one would have to be Q or K …"), list all the **distinct ways** your event could occur and **add up their chances**.

   (b) (3 pts) You flip the coin 8 times. What is the probability that you get an equal number of heads and tails?

- If the <u>list above is long</u> and complicated, look at the **complement**. If the complement is simpler (e.g. the complement of "at least one" is "none"), you can find its chance and subtract that from 1.

Ex 4). Suppose you're in a classroom with a total of ____ students.

What is the probability that there are no students who share a birthday?

To simplify the problem assume the following 2 things:

- Nobody was born on Feb 29th
- Birthdays are uniformly distributed throughout the year, i.e. each day out of the 365 is equally likely to be the birthday of a randomly selected person

Let's generalize this!

What is the probability that in a group of **n** students here are no students who share a birthday?

Define the following events:

- B_i : Event that the ith person chosen has a unique birthday (different than all i-1 people before them)

$P(no\ shared\ bdays\ in\ group\ of\ n\ people) = P(B_1 \cap B_2 \cap B_3 \ldots \cap B_n) =$

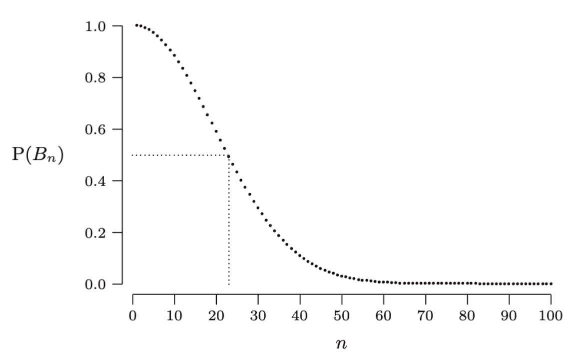


Fig. 3.1. The probability $P(B_n)$ of no coincident birthdays for $n = 1, \ldots, 100$.