CHAPTER SIX

6.3 Permutations and Combinations

Often when we count things, we are concerned with their arrangement, or lack thereof. The wording of the question changes the actual count

Five letters = $\{A, B, C, D, E\}$

How many sets of three distinct letters are there when choosing from five letters?

$\{CBA, BAE, EBA, \dots\}$

How many sets of three letters are there, in alphabetical order, when choosing from five letters.

$\{ABC, ABD, ABE, ACD, \dots\}$

How many sets of three letters, with repetition allowed, are there when choosing from five letters.

$\{AAA, ABB, ACA, \dots\}$

In today's counting problems we will employ two ideas:

1] The permutation

2] The combination

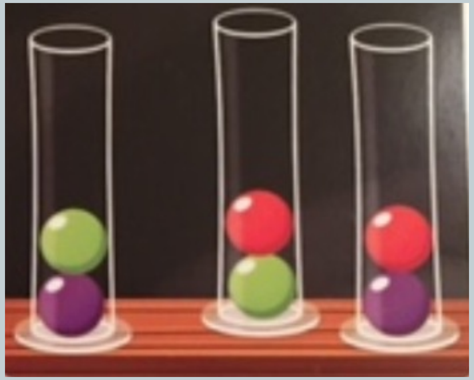
Given a set of objects to choose from, a permutation of the objects refers to choosing some (or all) of the objects and regarding the order to be important.

Given a set of objects to choose from, a combination of the objects refers to choosing some (or all) of the objects and order is not important.

Choosing subsets of three from $\{A, B, C, D, E\}$:

Permutations: $\{A, B, C\}$, $\{A, C, B\}$, $\{B, A, C\}$, $\{B, C, A\}$, $\{C, A, B\}$, $\{C, B, A\}$, $\{A, B, D\}$, ...

Combinations: $\{A, B, C\}$, $\{A, B, D\}$, $\{A, B, E\}$, $\{A, C, D\}$, $\{A, C, E\}$, $\{A, D, E\}$, $\{B, C, D\}$ ...



Order IS important

Order is NOT important

Rather than attempting to count every instance in a permutation or in a combination, there exist formulas that will calculate the answer.

When choosing r items from a set consisting of n items, the number of…

<u>permutations</u>: $P(n,r) = \frac{n!}{(n-r)!}$

<u>combinations</u>: $C(n,r) = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

What remains is figuring out which occasions call for which counting method

How many ways can 4 students be chosen from a group of 7 to be representatives in a meeting of other groups?

ANS:

Combination or permutation?

How many ways can 4 students be chosen from a group of 7 to be representatives in a meeting of other groups?

ANS:

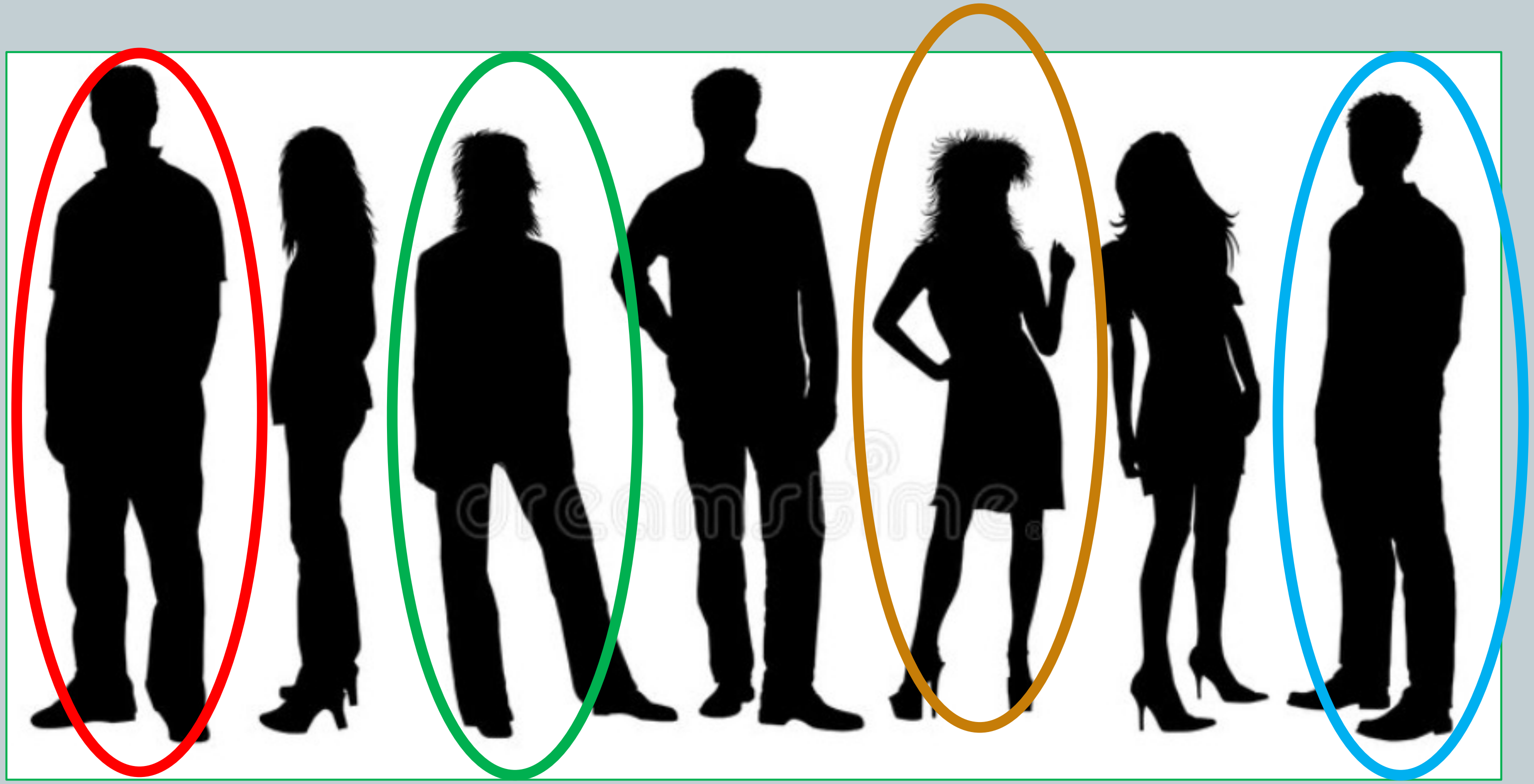
Combination!

$$C(7,4) = \frac{7!}{3!\,4!} = 35$$

How many ways can 4 students be chosen from a group of 7 to be a representatives in a meeting?

The representatives need a president, vice president, captain, and a princess.

ANS:



How many ways can 4 students be chosen from a group of 7 to be a representatives in a meeting?
The representatives need a president, vice president, captain, and a princess.

ANS:
This is a permutation. Technically this is called a 4-permutation.

$$P(7,4) = \frac{7!}{3!} = 840$$

Consider the set A = {Hippo, dinosaur, bunny}

How many 2-permutations of set A are there?

How many 2-combinations of set A are there?

Consider the set A = {Hippo, dinosaur, bunny}

How many 2-permutations of set A are there?

$$P(3,2) = \frac{3!}{(3-2)!} = 6$$

(Hippo, Dino), (Dino, Hippo), (Hippo, Bunny), (Bunny, Hippo), (Dino, Bunny), (Bunny, Dino)

How many 2-combinations of set A are there?

$$C(3,2) = \frac{3!}{(3-2)!2!} = 3$$

(Hippo, Dino), (Hippo, Bunny), (Dino, Bunny)

Proving the permutation formula:

Suppose n is a positive integer and r is an integer with $1 \leq r \leq n$.

According to the multiplication rule, to create a permutation,
the first element can be chosen in n ways.
Then there are $n - 1$ ways to choose the second element,
and there are $n - 2$ ways to choose the third element, and so on,
until there are $n - (r - 1)$ ways to choose the r^{th} element.

So, there are $P(n, r) = n(n-1)(n-2)\ldots(n-(r-1))$ r-permutations of a set with n distinct elements.

But $P(n, r) = n(n-1)(n-2)\ldots\big(n-(r-1)\big) = \frac{n!}{(n-r)!}$.

Q.E.D.

$P(n, 0) = ?$

How many ways are there to permutate n items taken 0 at a time?

$P(n, n) = ?$

How many ways are there to permutate n items taken n at a time?

$P(n, 0) = ?$

How many ways are there to permutate n items taken 0 at a time?

$$P(n, 0) = \frac{n!}{(n-0)!} = 1$$

Take none of them.

$P(n, n) = ?$

How many ways are there to permutate n items taken n at a time?

$$P(n, n) = \frac{n!}{(n-n)!} = n!$$

Multiplication Principle.

$C(n, 0) = ?$

How many combinations of n items taken 0 at a time?

$C(n, n) = ?$

How many combinations of n items taken n at a time?

$C(n, 0) = ?$

How many combinations of n items taken 0 at a time?

$$C(n, 0) = \frac{n!}{0!\,(n-0)!} = 1$$

Take none of them.

$C(n, n) = ?$

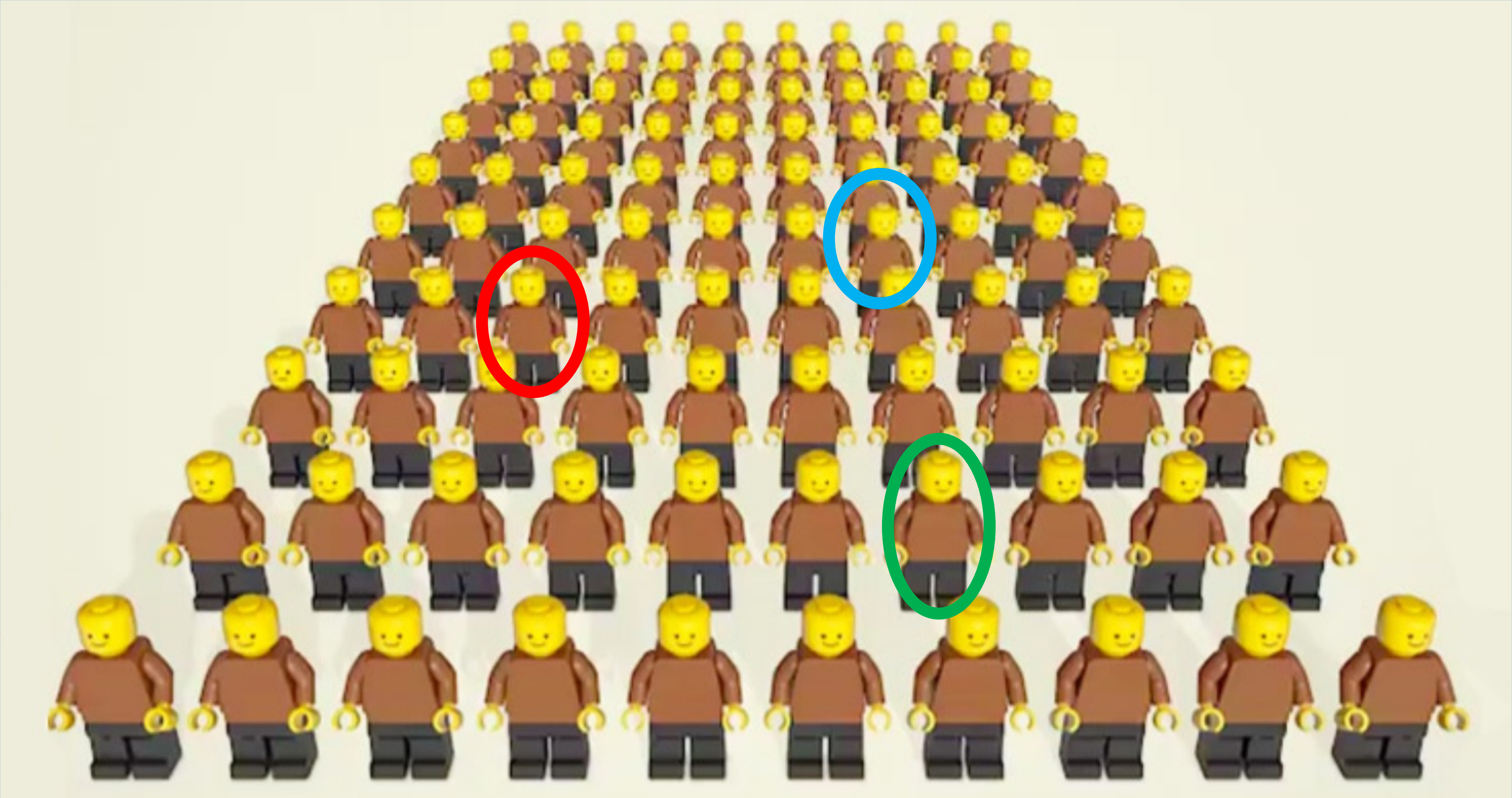
How many combinations of n items taken n at a time?

$$C(n, n) = \frac{n!}{n!\,(n-n)!} = 1$$

Take them all.

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people in a contest?

ANS:



How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people in a contest?

ANS:

The order is relevant, therefore, this is a Permutation

$$P(100,3) = \frac{100!}{(100-3)!} = 98 \cdot 99 \cdot 100 = 970{,}200$$

How many ways are there to award a gold, silver, and bronze medal to eight racers. No ties occur.

ANS:

How many ways are there to award a gold, silver, and bronze medal to eight racers. No ties occur.

ANS:

The order is relevant, therefore, $P(8,3) = \frac{8!}{(8-3)!} = 6 \cdot 7 \cdot 8 = 336$

A traveler is going to visit 8 cities. The first city is nearby and considered the starting point.

How man possible orders can the rest of the cities be visited?

ANS:

A traveler is going to visit 8 cities. The first city is nearby and considered the starting point.

How man possible orders can the rest of the cities be visited?

ANS:

There are actually only 7 cities to choose from (order wise) and you are going to them all, and the order is relevant; therefore, this is a permutation of 7 items taken 7 at a time, or just 7!:

$$P(7,7) = \frac{7!}{(7-7)!} = 7! = 5040$$

How many 8-letter strings of ABCDEFGH contain the string ABC ?

ANS:

The order in a string makes a difference, therefore, this is a permutation.

There are 8 letters total to choose from.

But this isn't quite 8 permutate 8; $P(8,8)$.

We need to consider "ABC" to be one item that is a choice.

How many 8-letter strings of ABCDEFGH contain the string ABC ?

ANS:

The order in a string makes a difference, therefore, this is a permutation.

Furthermore, consider "ABC" to be one item that is a choice.

$$P(6,6) = \frac{6!}{0!} = 6! = 720$$

$$\underbrace{\boxed{\text{ABC}}\ \boxed{\text{D}}\ \boxed{\text{E}}\ \boxed{\text{F}}\ \boxed{\text{G}}\ \boxed{\text{H}}}_{\text{6 "things" to permutate, 6 at a time}}$$

What is $C(4,2)$?

What does this mean relative to the set $\{a, b, c, d\}$?

ANS:

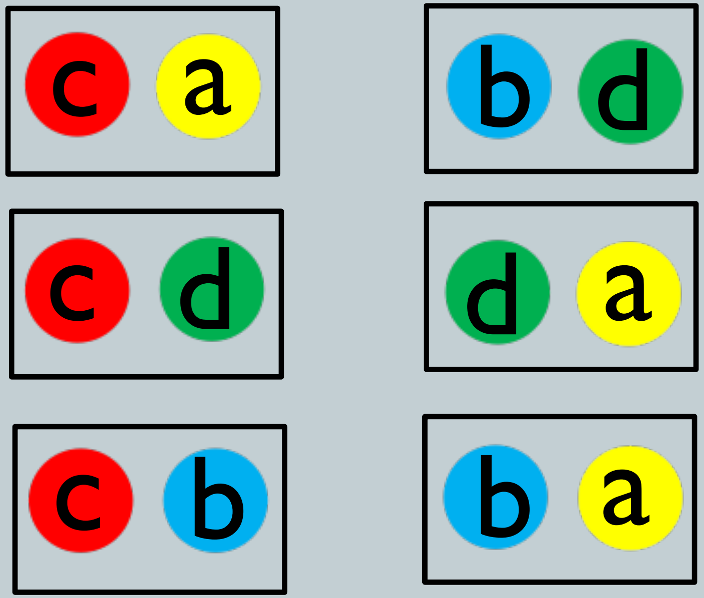
What is $C(4,2)$?

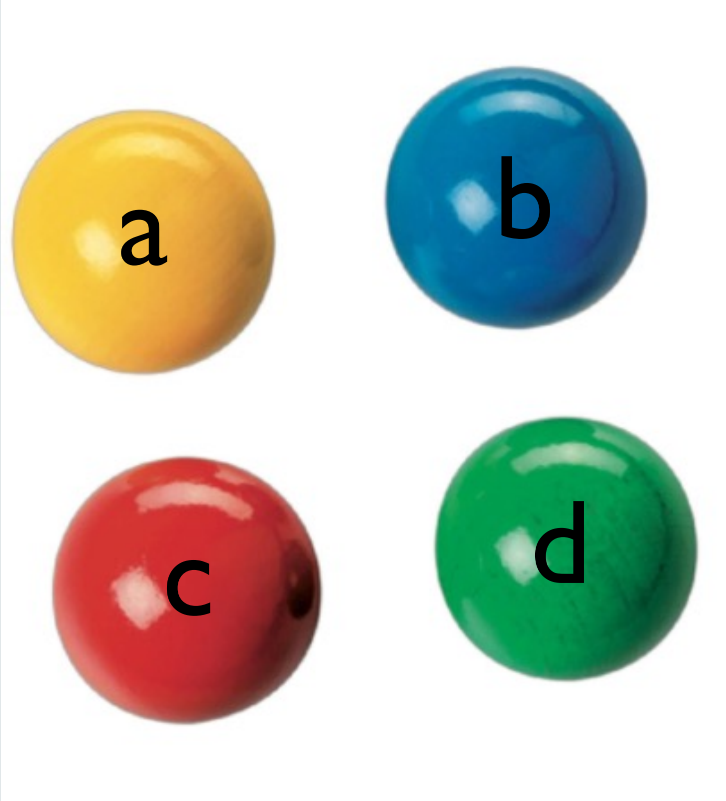
What does this mean relative to the set $\{a, b, c, d\}$?

ANS:

$$C(4,2) = \frac{4!}{2!2!} = 6$$

There are 6 collections (order irrelevant) of 4 items taken 2 at a time.





Notice the pattern: $(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$$C(4,0) = \binom{4}{0} = 1$$

$$C(4,1) = \binom{4}{1} = 4$$

$$\left.\begin{array}{l}\text{aaab}\\ \text{aaba}\\ \text{abaa}\\ \text{baaa}\end{array}\right\} C(4,1)$$

$$C(4,2) = \binom{4}{2} = 6$$

$$C(4,3) = \binom{4}{3} = 4$$

$$C(4,4) = \binom{4}{4} = 1$$

This pattern will be used extensively in the next section. It is a good idea to see it now and figure out how it works.

Notice that a set with {A, B, C} has $P(3, 2) = 6$ **permutations**:
{A, B}, {B, A}, {A, C}, {C, A}, {B, C}, {C, B}

Given a permutation with 2 elements, there are $P(2, 2) = 2$ ways to arrange these elements.

Therefore, there are $C(3,2) = 3$ **combinations**, because $\frac{6}{2} = 3$.
{A, B}, {A, C}, {B, C}

in general, $P(n, r) = C(n, r) \cdot P(r, r)$ which implies:

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\,(n-r)!}$$

Calculations for a combination can get out of hand fast.
How can they be simplified?

$$C(16,2) = \frac{16!}{14!2!} = \frac{15 \cdot 16}{2} = 120$$

$$C(27,24) = \frac{27!}{3!24!} = \frac{25 \cdot 26 \cdot 27}{1 \cdot 2 \cdot 3} = 2925$$

$$C(100,3) = \frac{100!}{97!3!} = \frac{98 \cdot 99 \cdot 100}{1 \cdot 2 \cdot 3} = 161{,}700$$

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

ANS:

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

ANS:

You are merely taking a set of 5 from 52, order is irrelevant (combination):

$$C(52,5) = \frac{52!}{47!\,5!} = 2{,}598{,}960$$

How many ways can you select 47 cards from a standard deck of 52 cards?

Recall the previous slide question:

How many ways can you select 5 cards from a standard deck of 52 cards?

ANS:

Pile A

Pile B

How many ways can you select 47 cards from a standard deck of 52 cards?

Recall the previous slide question:

How many ways can you select 5 cards from a standard deck of 52 cards?

ANS:

You are taking a set of 5 from 52, order is irrelevant.

This is the same as taking 47 from 52, order being irrelevant.

$$C(52,5) = \frac{52!}{47!\,5!} = 2{,}598{,}960$$

$$C(52,47) = \frac{52!}{5!\,47!} = 2{,}598{,}960$$

Are we choosing 5 cards to stay, or 5 cards to go? There is no difference.

How many ways are there to select five players from a 9-member tennis team to make a trip to a match at another school?

ANS:

How many ways are there to select five players from a 9-member tennis team to make a trip to a match at another school?

ANS:
This is a combination since the order is not important:

$C(9,5) = \frac{9!}{4!5!} = \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 126$ different selections are possible.

A group of 30 people from the CU aerospace program have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission? (Everyone has equivalent jobs)

ANS: ?

How many ways are there to select a crew of six people to go on this mission? (Everyone has equivalent jobs)

ANS:

This is a combination since all job positions are considered the same:

$$C(30,6) = \frac{30!}{24!6!} = \frac{25 \cdot 26 \cdot 27 \cdot 28 \cdot 29 \cdot 30}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 593{,}775$$

- This only becomes a permutation if the six people chosen are going with job descriptions: Captain, pilot, co-pilot, engineer, medic, alien hunter.
- $P(30, 6) = \frac{30!}{24!} = 427{,}518{,}000$

How many bit strings of length n contain exactly r 1's ?

ANS:

$$\underbrace{\underline{\ 0\ }\ \ \underline{\ 0\ }\ \ \underline{\ 1\ }\ \ \underline{\ 0\ }\ \ \underline{\ 1\ }\ \ \underline{\ 0\ }\ \ \ldots\ \ \underline{\ 1\ }}$$

n number of slots,
r of which contain a 1.

How many bit strings of length n contain exactly r 1's ?

ANS:

Our only concern is finding out how many strings of length n contain exactly r 1's, we are not concerned with where the 1's are located.

Therefore, this is a combination: $C(n, r) = \frac{n!}{(n-r)!r!}$

A committee of 7 is to be chosen from a group of 20 people.
The committee must have 3 mathematics faculty and 4 CS faculty.

Suppose there are 9 faculty members from the mathematics department,
and there are 11 faculty members from the CS department.

How many different ways could this committee of 7 be created?

ANS:

There are two choices to make in succession.

This calls for the multiplication principle.

So multiply (what?) · (what?)

There are 9 faculty members from the mathematics department.
There are 11 faculty members from the CS department.

The committee must have 3 mathematics faculty and 4 CS faculty.

How many different ways could this committee be created?

ANS:

$$C(9,3) \cdot C(11,4) =$$

$$84 \cdot 330 = \mathbf{27,720}$$