CSCI 2824: Discrete Structures

Section 7.1: Intro to Discrete Probability

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Warm-Up:

How many distinct ways can you rearrange these objects?

Review: Counting

Warm-Up:

How many distinct ways can you rearrange these objects?

Discrete Probability

Previously, we've always started with a set of n distinct items

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

But what if you start with n objects and some of them are repeats?

The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Course Logistics

- ❑ Quiz 8 TONIGHT
- ❑ HW 8 (Written and Coding) Due Friday 6pm
- ❑ Read Section 7.2 before Fri’s class
- ❑ *Friday class live on Zoom (using my office hour Zoom link)*
- ❑ *Exam 3 NEXT Wednesday (Nov. 9th) IN CLASS, 11:15am-12:05pm*
  - ❑ *Covers* 5.1-5.4, & 6.1-6.5

Binomial Theorem

Example:

a). Give a closed form expression for: $\sum_{j=0}^{n} \binom{n}{j} \mathrm{x}^{\mathrm{n}-j} \mathrm{y}^{j}$

Binomial Theorem

The Binomial Theorem: Let x and y be variables, and let n be a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} \mathrm{x}^{\mathrm{n}-j} \mathrm{y}^{j}$$

But why?

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

1 way to choose $\boldsymbol{x^3}$ term: $x \cdot x \cdot x$

3 ways to choose $\boldsymbol{x^2 y}$ term: $x \cdot x \cdot y,\ \ x \cdot y \cdot x,\ \ y \cdot x \cdot x$

3 ways to choose $\boldsymbol{x y^2}$ term: $x \cdot y \cdot y,\ \ y \cdot x \cdot y, y \cdot y \cdot x$

1 way to choose $\boldsymbol{y^3}$ term: $y \cdot y \cdot y$

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Notice the pattern of exponents and coefficients:

For instance,→

$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

$$\binom{7}{0} x^7y^0 + \binom{7}{1} x^{7-1}y^1 + \binom{7}{2} x^{7-2}y^2 + \binom{7}{3} x^{7-3}y^3 + \binom{7}{4} x^{7-4}y^4 + \ldots + \binom{7}{7} x^{7-7}y^7$$

Binomial Theorem

Example:

b). Give a closed form expression for: $\sum_{j=0}^{n} \binom{n}{j} 2^j 3^{\mathrm{n}-j}$

c). What is the value of the following sum? $\sum_{j=0}^{5} \binom{5}{j} 4^j (-2)^{\mathrm{n}-j}$

Binomial Theorem

Example: Suppose we were to expand $(2x - 3y)^{25}$

a). How many terms would the expansion have?

b). What is the coefficient of the term with $x^{12}y^{13}$?

Binomial Theorem

Example: Suppose we were to expand $(2x-3y)^{25}$

a). How many terms would the expansion have? **26**

b). What is the coefficient of the term with $x^{12}y^{13}$?

$$\binom{25}{13}(2x)^{25-13}(-3y)^{13} = \frac{25!}{13!\,12!}\cdot(2x)^{12}(-3y)^{13} = -5{,}200{,}300\cdot 2^{12}\cdot 3^{13}\cdot x^{12}y^{13}$$

Thus the coefficient is $-5{,}200{,}300\cdot 2^{12}\cdot 3^{13}$

Discrete Probability

Discrete Probability: concerned with the probability of a finite event occurring

Probability of an Event: $$\frac{\text{number of ways the event can occur}}{\text{number of possible outcomes}}$$

Example: Suppose the experiment is to roll a die and the event is that an even number is rolled. Find the probability of this event.

Discrete Probability

Experiment: A procedure that yields one of a given set of possible outcomes.

Sample Space: The set of possible outcomes of an experiment.

Event: A subset of the sample space.

Example: Suppose the experiment is flipping 3 coins. We want to know the probability that all three coins show the same side of the coin.

Sample Space: The set of all possible combos of the three flips:

$$\mathrm{S} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Event: The set of all combos where all three coins are the same: $E = \{HHH, TTT\}$

Discrete Probability

If S is a finite nonempty sample space of *equally likely* outcomes, and E is an event, that is, a subset of S, then the **probability of E**, written $p(E)$, is

$$p(E) = \frac{|E|}{|S|}$$

Note that: $0 \leq p(E) \leq 1$

- So from the coin flip example on the previous slide: $p(E) = \frac{|E|}{|S|} = \frac{2}{8} = \frac{1}{4}$

Experiment: Roll a 6-sided die.
Event: A multiple of 3 is revealed.

a). How many outcomes are possible?

b). How many of the the possible outcomes are considered "success"?

c). What is $P(E) = ?$
(i.e what is the ratio of successes to outcomes)?

What is the probability that when a 6-sided die is rolled, the result will be a multiple of 3?

ANS:

Possible outcomes: $S = \{1, 2, 3, 4, 5, 6\}$

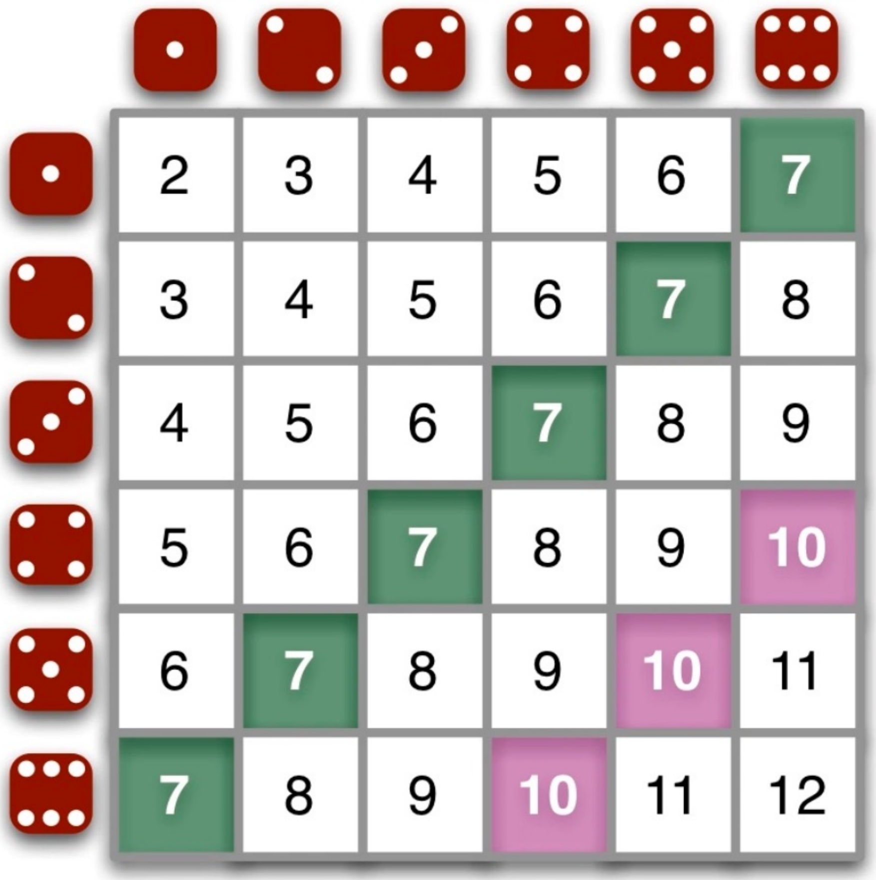
Events considered "Success": $E = \{3, 6\}$

$$p(E) = \frac{|E|}{|S|} = {}^{2}/_{6} = \frac{1}{3}, \quad \text{or } 33\tfrac{1}{3}\% \text{ chance.}$$

Discrete Probability

Example: What is the probability that when two dice are rolled, the numbers on the two dice sum to 7?

1] What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?



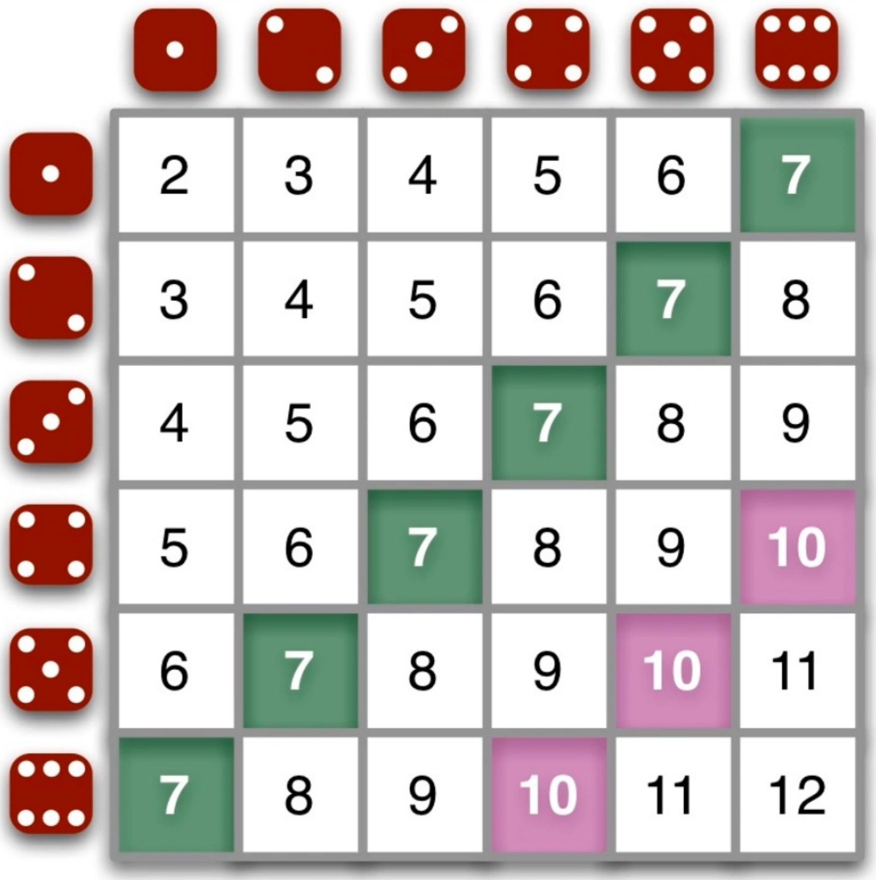
1] What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

ANS:

1] There are 36 possible outcomes.

6 of the outcomes are considered "Success".

$p(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6}$



1] What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

2] What is the probability that the sum shown is 10?

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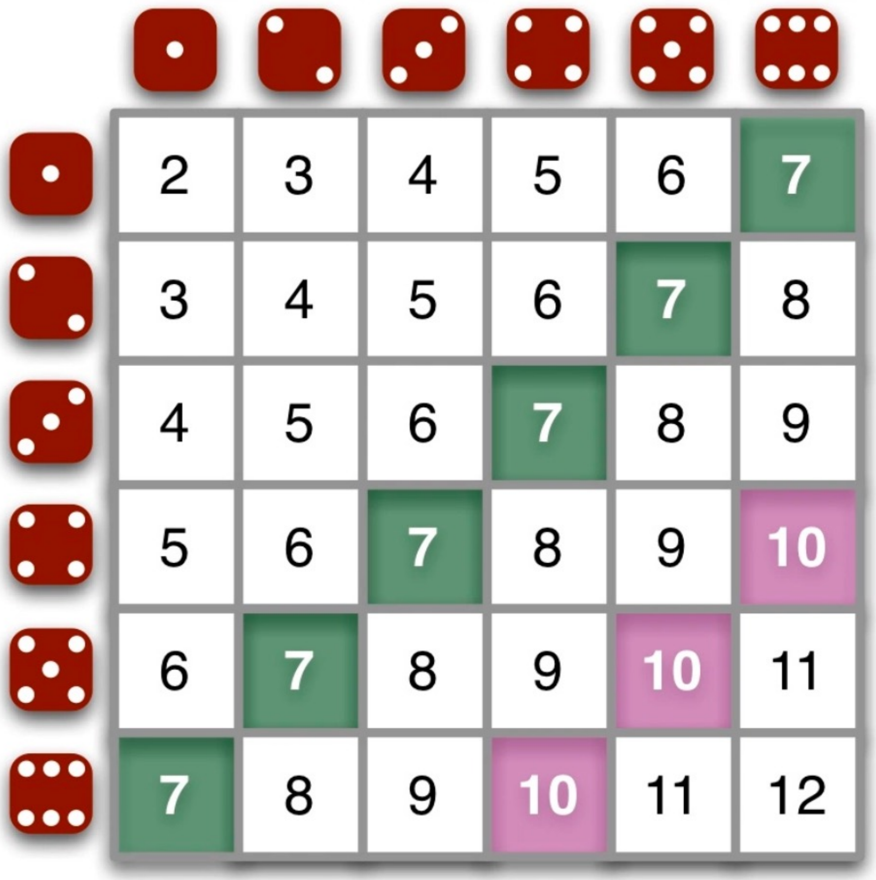

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ANS:

1] There are 36 possible outcomes.

6 of the outcomes are considered "Success".

$$p(\text{sum is } 7) = \frac{6}{36} = \frac{1}{6}$$

2] There are 36 possible outcomes.

3 of the outcomes are considered "Success".

$$p(\text{sum is } 10) = \frac{3}{36} = \frac{1}{12}$$

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

There are 38 slots on a roulette wheel.
If you watch the game for 20 minutes and notice that the last 7 ”wins” landed on black, should you bet on red or black for the eighth spin?

There are 38 slots on a roulette wheel.
If you watch the game for 20 minutes and notice that the last 7 ”wins” landed on black, should you bet on red or black for the eighth spin?

Don’t fall for the “gambler's fallacy. “

The wheel is equally likely to fall on any number REGARDLESS of previous performance.

Suppose there is a lottery in which players win a big prize if they pick 4 digits, in order (repetition allowed), that are selected by a random mechanical process.

What is the probability that a player will win the big prize?

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What is the probability that a player will win the big prize?

ANS:
There is only one way to choose all four digits correctly: $|E| = 1$

By the product rule, there exist $10^4 = 10{,}000$ different possibilities: $|S| = 10{,}000$

A permutation with repetition allowed

Therefore, $p(E) = {}^{1}/_{10{,}000} = 0.0001$

Suppose in this same lottery a player will win a small prize if they pick 3 of 4 digits, in order, that are selected by a random mechanical process.

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Suppose in this same lottery a player will win a small prize if they pick 3 of 4 digits, in order, that are selected by a random mechanical process.

What is the probability that a player will win the small prize?

ANS:

To win, exactly one digit must be wrong to get three digits (but not all four) correct.

Your choices → ___ ___ X ___

The winning numbers → 3 1 4 1

$X \in \underbrace{\{0, 1, 2, 3, 5, 6, 7, 8, 9\}}$

Any of these 9 digits, note there is no 4

ANS:
Exactly one digit must be wrong to get three digits correct, but not all four correct.

To count the number of successes with the first digit incorrect, note that there are 9 possible choices for the first digit, and only 1 choice for each of the remaining 3 choices.

This means there are 9 ways to choose four digits where the first digit is incorrect, but the last three are correct.

Similar reasoning goes for the next 3 slots/choices. Hence, in total there are $4 \cdot 9 = 36$ ways to choose four digits with exactly three of the four digits correct.

$$|S| = 10{,}000$$
$$|E| = 36$$

$$p(E) = {}^{36}/_{10{,}000} = 0.0036$$

Discrete Probability

Example: Consider a lottery that awards a large prize to anyone that correctly chooses a set of 6 distinct numbers out of a possible n numbers. What is the probability of winning a lottery that selects 6 numbers out of 40?

Consider a lotto/powerball type ticket where six numbers are chosen from a set of 40 numbers. How likely is it that a person picks the correct six numbers?

ANS:

There is only one winning combination.

The total number of ways to choose six numbers out of 40 is:

$$C(40,6) = \frac{40!}{34!6!} = 3{,}838{,}380$$

Consequently, the probability of picking a winning combination is:

$$1/_{3,838,380} \approx 0.00000026.$$

Card games are a substantial source of probability type questions.

A standard deck of cards contains 52 cards of 4 different suits.

There are 13 cards per suit. ♥ ♣ ♦ ♠

Each of the 13 cards has a different value {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}.

Ex). What is the probability of drawing a Full House in a 5-card poker hand?

A Full House is 3 cards of one value and 2 cards of another value.

Full House

What is the probability of drawing a Full House in a 5-card poker hand?
A Full House is 3 cards of one value and 2 cards of another value.

ANS:

Three of a kind | Two of a kind

$$|E| = C(13,1)\cdot C(4,3)\cdot C(12,1)\cdot C(4,2) = 3744$$

value suit value suit

or equivalently,

$$|E| = P(13,2)\cdot C(4,3)\cdot C(4,2) = 3744$$

values suits suits

Therefore, $p(E) = \dfrac{3744}{2{,}598{,}960} \approx 0.0014.$

Full House

How likely is it that the numbers $11, 4, 17, 39,$ and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, 3, 4, \ldots, 50$ if the ball selected is not returned to the bin before the next ball is selected ?

.

How likely is it that the numbers $11, 4, 17, 39,$ and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, 3, 4, \ldots, 50$ if the ball selected is not returned to the bin before the next ball is selected ?

ANS:

By the product rule, there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254{,}251{,}200$ ways to select the balls because each time a ball is drawn there is one fewer ball to choose from.

There is only 1 way to draw $11, 4, 17, 39, 23$.

Therefore, the probability is $\frac{1}{254{,}251{,}200}$

Sampling without replacement.

How likely is it that the numbers $11, 4, 17, 39,$ and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, 3, 4, \ldots, 50$ if the ball selected is returned to the bin before the next ball is selected ?

How likely is it that the numbers $11, 4, 17, 39,$ and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, 3, 4, \ldots, 50$ if the ball selected is returned to the bin before the next ball is selected ?

ANS:

By the product rule, there are $50 \cdot 50 \cdot 50 \cdot 50 \cdot 50 = 312{,}500{,}000$ ways to select the balls because there are 50 possible balls to choose from each time a ball is drawn.

Consequently, the probability of drawing $11, 4, 17, 39, 23$ is $\frac{1}{312{,}500{,}000}$

Sampling with replacement

Discrete Probability

Theorem: Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complement of E, is given by

$$p(\bar{E}) = 1 - p(E)$$

Example: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is a 1?

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

ANS:

Let E be the event that **at least one** of the 10 bits is a 0.

Then $\bar{E}$ is the event that **none** of the 10 bits are 0, i.e. $\bar{E}$ is the event that all the bits are 1.

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{|\bar{E}|}{|S|} = 1 - \frac{\binom{10}{0}}{2^{10}} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

The only string left out is the string of all 1's.

A sequence of 10 bits is randomly generated. What is the probability that **at least one** of these bits is 0?

ANS:

Let E_1 be the event that one of the 10 bits is a 0.
Let E_2 be the event that two of the 10 bits are 0.
Let E_3 be the event that three of the 10 bits are 0.
Let E_4 be the event that four of the 10 bits are 0.
Let E_5 be the event that five of the 10 bits are 0.
Let E_6 be the event that six of the 10 bits are 0.
Let E_7 be the event that seven of the 10 bits are 0.
Let E_8 be the event that eight of the 10 bits are 0.
Let E_9 be the event that nine of the 10 bits are 0.
Let E_{10} be the event that ten of the 10 bits are 0.

Find the probability of all of these events,

and then add them together?

$$\frac{\binom{10}{1}}{2^{10}} + \frac{\binom{10}{2}}{2^{10}} + \cdots + \frac{\binom{10}{10}}{2^{10}}$$

Discrete Probability

Theorem: Let E_1 and E_2 be two events in the sample space S

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example: Suppose you draw one card from a standard deck. What is the probability that is is an Ace or a Diamond?

Theorem:

Let E_1 and E_2 be events in the sample space S.

Then $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

Proof:

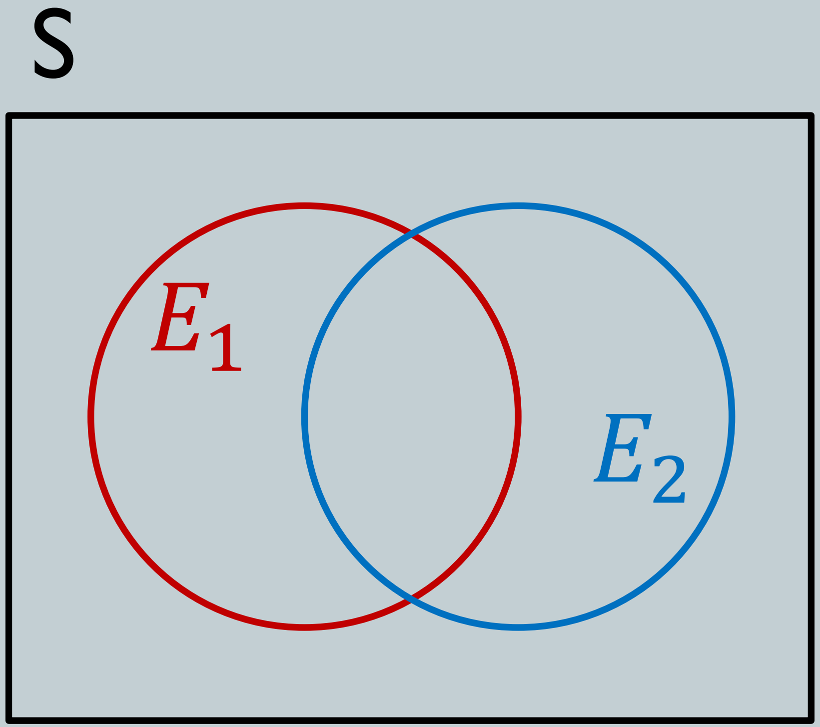
$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} =$$

$$\frac{|E_1|+|E_2|-|E_1 \cap E_2|}{|S|} =$$

$$\frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} =$$

$$p(E_1) + p(E_2) - p(E_1 \cap E_2)$$



What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

ANS:

Let E_1 be the event that the random integer selected is divisible by 2:

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, \dots\}$$

Let E_2 be the event that the random integer selected is divisible by 5:

$$\{5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, \dots\}$$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}$$

Extra Practice

EX. 1 What is the probability that when a coin is flipped 6 times it comes up heads all 6 times?

EX. 2 What is the probability that a five-card poker hand contains at least one ace?

EX. 3 What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive values?

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

EX. 5 What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

Solutions

EX. 1 What is the probability that when a coin is flipped 6 times it comes up heads all 6 times?

There are $2^6 = 64$ ways to flip 6 coins

In only 1 of those ways do the coins come up all heads

So $p(E) = \dfrac{1}{64} \approx 0.016$

EX. 2 What is the probability that a five-card poker hand contains at least one ace?

Easier to compute the probability that no aces appear and then use the complement rule. There are 48 cards in the deck that *aren't* aces, so the number of ways to choose 5 non-aces is $C(48, 5)$

Then the probability of picking no aces is

$$p(E) = \frac{C(48, 5)}{C(52, 5)}$$

The probability of picking at least one ace is

$$p(\bar{E}) = 1 - p(E) = 1 - \frac{C(48, 5)}{C(52, 5)} \approx 0.34$$

EX. 3 What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive values?

We can specify the hand by first choosing the lowest card in the straight from the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (Note, there is a straight that starts with 10 because 10-J-K-Q-A is a straight)

There are $C(10, 1)$ ways to do this

Next we need to choose the suits of each card. Since there are 5 cards and 4 suits there are $C(4, 1)^5 = 4^5$ ways to do this

So by the product rule this gives $10 \cdot 4^5 = 10,240$ hands

The probability is then $p(\mathrm{E}) = \frac{10{,}240}{2{,}598{,}960} \approx 0.00394$

Note: Excluding straight flushes yields a slightly smaller probability

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

First we need to compute the number of hands that contain cards of 5 different values, then we'll subtract off the flushes and straights

There are $C(13, 5)$ ways to choose 5 values, and for each card there are $C(4, 1)$ ways to choose the suit. Thus there are

$$C(13, 5) \cdot C(4, 1)^5 = 1317888 \text{ hands with } 5 \text{ different values}$$

Of those, 10240 of them are straights and 5148 of those are flushes. But of those, 40 of them are straight flushes. So the number of hands of the desired type is

$$1317888 - (10240 + 5148 - 40) = 1302540$$

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

The probability of drawing a 5-card hand with 5 different values and does not contain a flush or a straight is then

$$p(\mathrm{E}) = \frac{1,302,540}{2,598,960} \approx 0.501$$

EX. 5 What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

There are 20 numbers between 1 and 100 that are divisible by 5

There are 14 numbers between 1 and 100 that are divisible by 7

But 2 of those numbers (35 and 70) are divisible by both 5 and 7

Thus the probability is

$$p(E) = \frac{20 + 14 - 2}{100} = \frac{32}{100} = 0.32$$