University of Colorado **Boulder**

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CSCI 2824: Discrete Structures

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Relations

Closures of Relations

Closures of Relations

Example: Let A be the set of airports in the United States and $R \subseteq A \times A$ be the relation such that $(a, b) \in R$ if there is a direct flight from airport a to airport b.

Question: Can we determine, by inspecting the elements of R, if it's possible to travel by plane (possibly using multiple connections) starting from airport c and ending at airport d?

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Question: Can we determine, by inspecting the elements of R, if it's possible to travel by plane (possibly using multiple connections) starting from airport c and ending at airport d?

Answer: Not easily (though, algorithmically we could). It would be easier if we could define a relation S on the set A such that $(c, d) \in S$ if it is possible to start at airport c and fly through multiple airports and eventually land at airport d.

Such a relation is called the **transitive closure** of R.

Closures of Relations

In general, let R be a relation on a set A.

R might not have some property $\mathbf{P}$ such as reflexivity or transitivity

Can we find a larger relation S that contains R that has property $\mathbf{P}$?

Def: If there is a relation S with property $\mathbf{P}$ containing R such that S is a subset of every every relation with property $\mathbf{P}$ that contains R, then S is called the **closure** of R with respect to $\mathbf{P}$.

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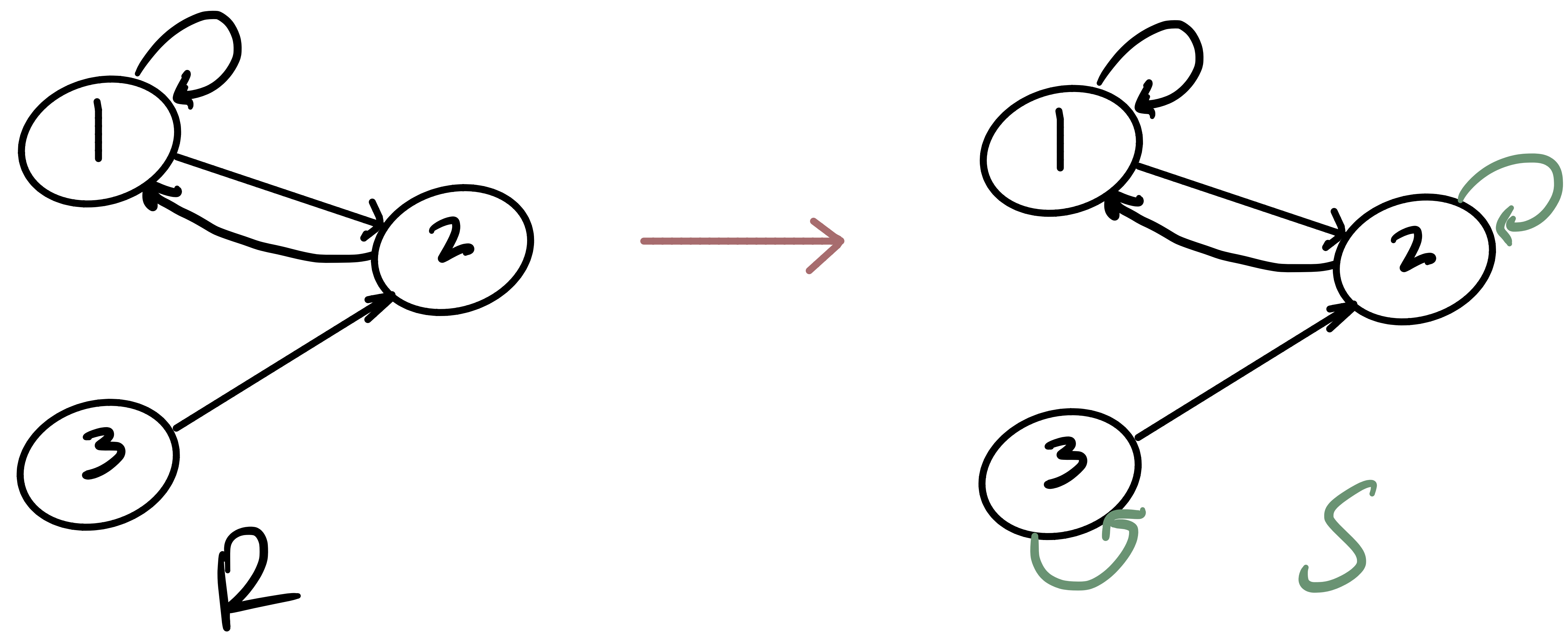
Translation: If there is a relation S that contains R that has property $\mathbf{P}$ and is **AS SMALL AS POSSIBLE**, we say S is the closure of R with respect to $\mathbf{P}$.

Closure WRT Reflexivity

Example: Let $R = \{(1,1),(1,2),(2,1),(3,2)\}$ be a relation on the set $A = \{1,2,3\}$. Note that R is not reflexive.

Question: How can we find a relation S, that is as small as possible, that contains R and is reflexive?



Closure WRT Reflexivity

Note: The closure of a relation is often somewhat to very different from the original relation.

Example: What is the reflexive closure of $R = \{(a, b) \mid a < b\}$ on the set of integers?

$$S = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in \mathbb{Z}\}$$

$$= \{(a,b) \mid a \leq b\}$$

Closure WRT Transitivity

As described in the airport example, the closure S of a relation R wrt transitivity is the relation such that $(a, b) \in S$ if there is a sequence of elements in R of the form $(c, x_1), (x_1, x_2), \dots, (x_k, d)$.

This is much easier to think about in terms of directed graphs.

But first we need some new graph terminology

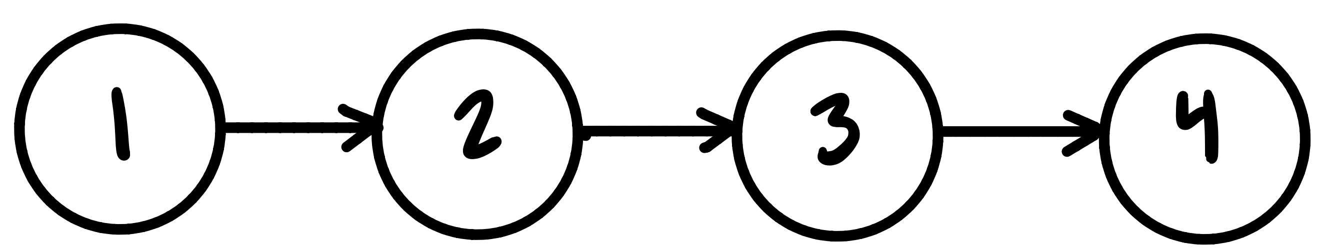
Def: A *path* from a to b in a directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2) \dots, (x_{n-1}, x_n)$ in G, where $x_0 = a$ and $x_n = b$. We say such a path has length n.

Closure WRT Transitivity

Def: R^n is the relation on a set A such that $(a, b) \in R^n$ if there is a path of length n that starts at a and ends at b.

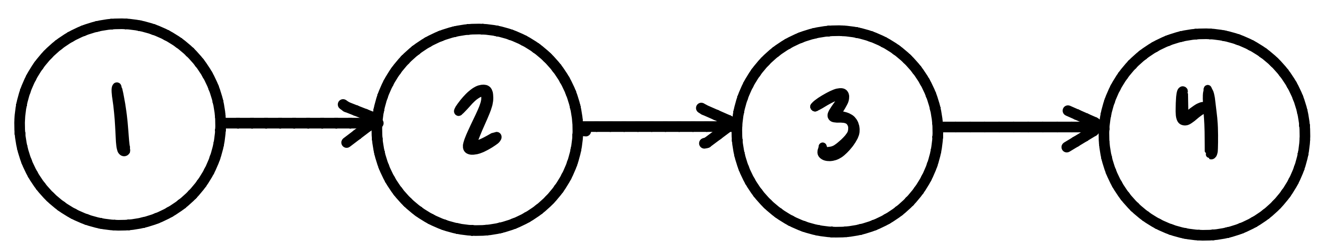
Theorem: The transitive closure of R on a set A with n elements is the relation $S = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$

Example: Find the transitive closure of R represented by

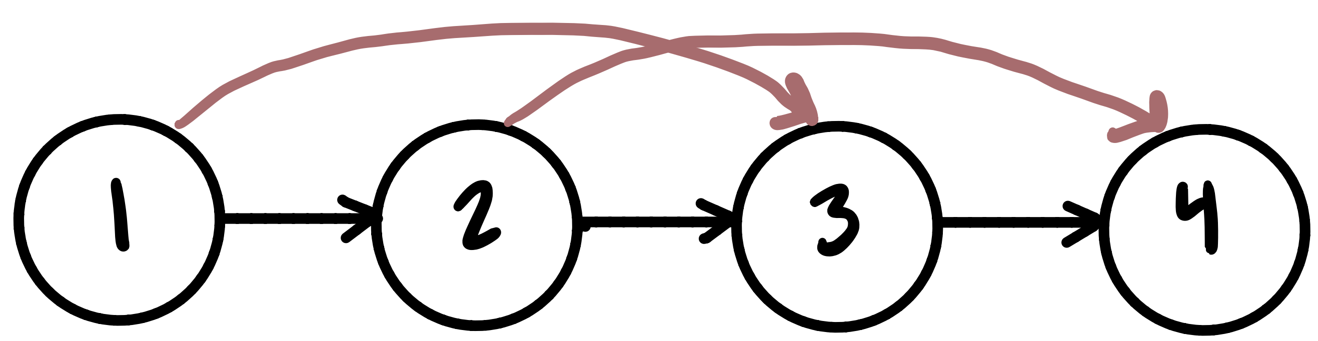


Closure WRT Transitivity

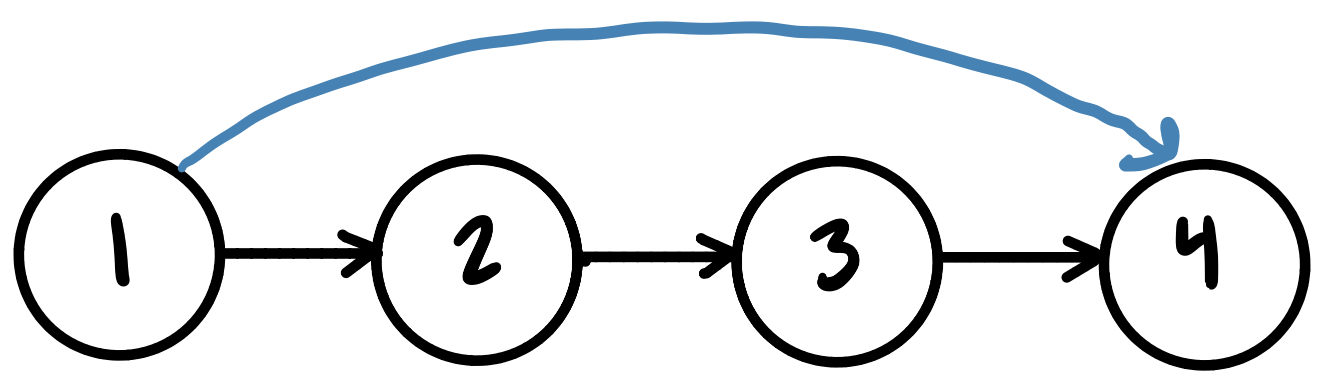
Example: Find the transitive closure of R represented by



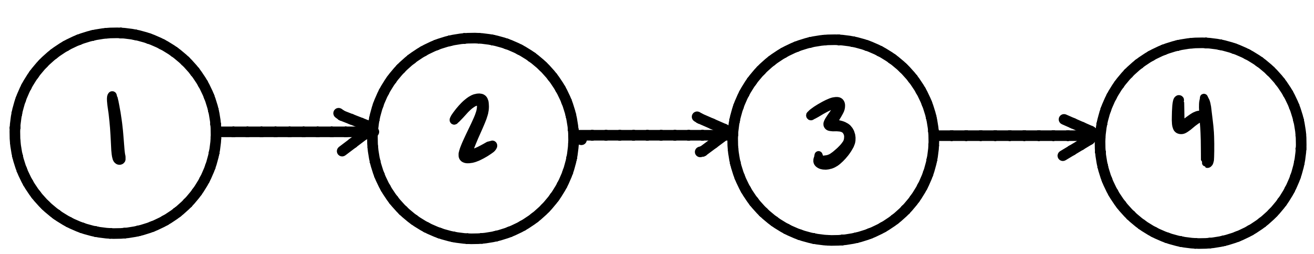
$R = \{(1,2), (2,3), (3,4)\}$



$R^2 = \{(1,3), (2,4)\}$



$R^3 = \{(1,4)$



$R^4 = \{\}$

Closure WRT Transitivity

Example: Find the transitive closure of R represented by

$R = \{(1,2), (2,3), (3,4)\}$

$R^2 = \{(1,3), (2,4)\}$

$R^3 = \{(1,4)\}$

$R^4 = \{\}$

$S = R \cup R^2 \cup R^3 \cup R^4 = \{(1,2), (2,3), (3,4), (1,3), (2,4), (1,4)\}$