Example. As a simple example, we consider the case with $m = 1$ (kilogram), $\eta = 1$ (Newtons per meter per second), and sampling period $h = 0.01$ (seconds). The external force is

$$f(\tau) = \begin{cases} 0.0 & 0.0 \leq \tau < 0.5 \\ 1.0 & 0.5 \leq \tau < 1.0 \\ -1.3 & 1.0 \leq \tau < 1.4 \\ 0.0 & 1.4 \leq \tau. \end{cases}$$

We simulate this system for a period of 2.5 seconds, starting from initial state $x_1 = (0, 0)$, which corresponds to the mass starting at rest (zero velocity) at position 0. The simulation involves iterating the dynamics equation from $k = 1$ to $k = 250$. Figure 9.7 shows the force, position, and velocity of the mass, with the axes labeled using continuous time τ.

9.5 Supply chain dynamics

The dynamics of a supply chain can often be modeled using a linear dynamical system. (This simple model does not include some important aspects of a real supply chain, for example limits on storage at the warehouses, or the fact that demand fluctuates.) We give a simple example here.

We consider a supply chain for a single divisible commodity (say, oil or gravel, or discrete quantities so small that their quantities can be considered real numbers). The commodity is stored at n warehouses or storage locations. Each of these locations has a target (desired) level or amount of the commodity, and we let the n-vector x_t denote the *deviations* of the levels of the commodities from their target levels. For example, $(x_5)_3$ is the actual commodity level at location 3, in period 5, minus the target level for location 3. If this is positive it means we have more than the target level at the location; if it is negative, we have less than the target level at the location.

The commodity is moved or transported in each period over a set of m transportation links between the storage locations, and also enters and exits the nodes through purchases (from suppliers) and sales (to end-users). The purchases and sales are given by the n-vectors p_t and s_t, respectively. We expect these to be positive; but they can be negative if we include returns. The net effect of the purchases and sales is that we add $(p_t - s_t)_i$ of the commodity at location i. (This number is negative if we sell more than we purchase at the location.)

We describe the links by the $n \times m$ incidence matrix A^{sc} (see §7.3). The direction of each link does not indicate the direction of commodity flow; it only sets the *reference direction* for the flow: Commodity flow in the direction of the link is considered positive and commodity flow in the opposite direction is considered negative. We describe the commodity flow in period t by the m-vector f_t. For example, $(f_6)_2 = -1.4$ means that in time period 6, 1.4 units of the commodity are moved along link 2 in the direction opposite the link direction (since the flow is negative). The n-vector $A^{\mathrm{sc}} f_t$ gives the net flow of the commodity into the n locations, due to the transport across the links.

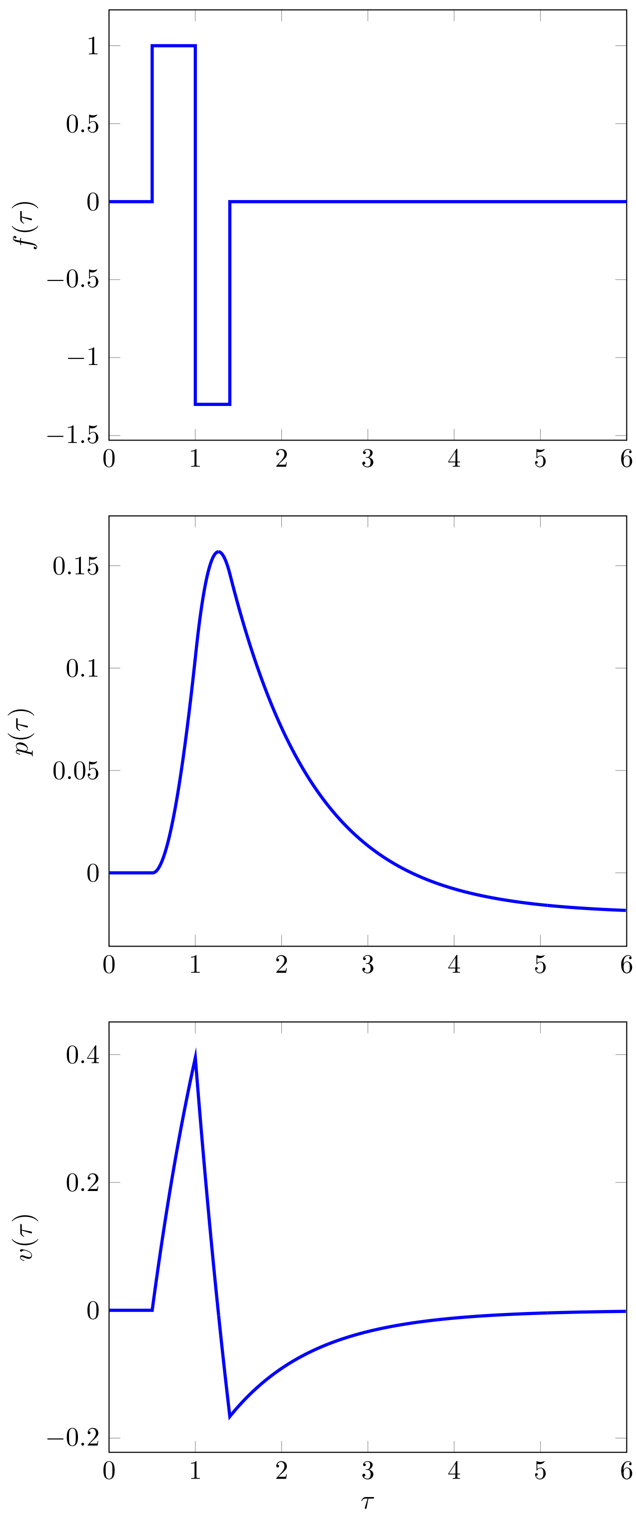

Figure 9.7 Simulation of mass moving along a line. Applied force (top), position (middle), and velocity (bottom).

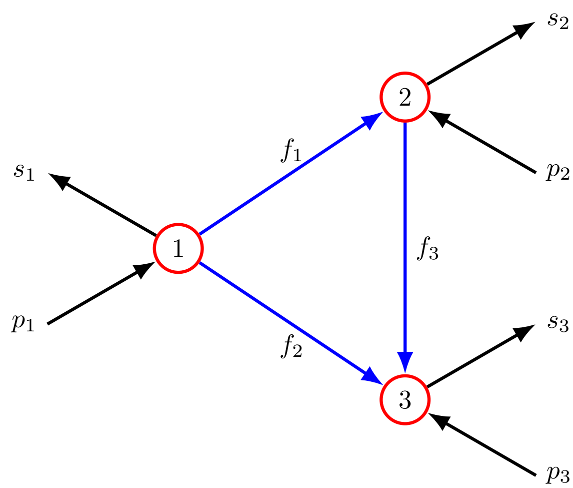

Figure 9.8 A simple supply chain with $n = 3$ storage locations and $m = 3$ transportation links.

Taking into account the movement of the commodity across the network, and the purchase and sale of the commodity, we get the dynamics

$$x_{t+1} = x_t + A^{\mathrm{sc}} f_t + p_t - s_t, \quad t = 1, 2, \ldots.$$

In applications where we control or run a supply chain, s_t is beyond our control, but we can manipulate f_t (the flow of goods between storage locations) and p_t (purchases at locations). This suggests treating s_t as the offset, and $u_t = (f_t, p_t)$ as the input in a linear dynamical system with input (9.2). We can write the dynamics equations above in this form, with dynamics and input matrices

$$A = I, \qquad B = \left[\begin{array}{cc} A^{\mathrm{sc}} & I \end{array} \right].$$

(Note that A^{sc} refers to the supply chain graph incidence matrix, while A is the dynamics matrix in (9.2).) This gives

$$x_{t+1} = Ax_t + B(f_t, p_t) - s_t, \quad t = 1, 2, \ldots, .$$

A simple example is shown in figure 9.8. The supply chain dynamics equation is

$$x_{t+1} = x_t + \left[\begin{array}{rrrrrr} -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} f_t \\ p_t \end{array} \right] - s_t, \quad t = 1, 2, \ldots.$$

It is a good exercise to check that the matrix-vector product (the middle term of the right-hand side) gives the amount of commodity added at each location, as a result of shipment and purchasing.