5. Linear independence

5.1. Linear independence

5.2. Basis

Cash flow replication. Let's consider cash flows over 3 periods, given by 3-vectors. We know from VMLS page 93 that the vectors

$$e_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad l_1 = \begin{bmatrix} 1 \\ -(1+r) \\ 0 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ -(1+r) \end{bmatrix}$$

form a basis, where r is the (positive) per-period interest rate. The first vector e_1 is a single payment of 1 in period (time) $t = 1$. The second vector l_1 is loan of \$ 1 in period $t = 1$, paid back in period $t = 1$ with interest r. The third vector l_2 is loan of \$ 1 in period $t = 2$, paid back in period $t = 3$ with interest r. Let's use this basis to replicate the cash flow $c = (1, 2, -3)$ as

$$c = \alpha_1 e_1 + \alpha_2 l_1 + \alpha_3 l_2 = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -(1+r) \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ -(1+r) \end{bmatrix}.$$

From the third component we have $c_3 = \alpha_3(-(1+r))$, so $\alpha_3 = -c_3/(1+r)$. From the second component we have

$$c_2 = \alpha_2(-(1+r)) + \alpha_3 = \alpha_2(-(1+r)) - c_3/(1+r),$$

so $\alpha_2 = -c_2/(1+r) - c_3/(1+r)^2$. Finally, from $c_1 = \alpha_1 + \alpha_2$, we have

$$\alpha_1 = c_1 + c_2/(1+r) + c_3/(1+r)^2,$$