University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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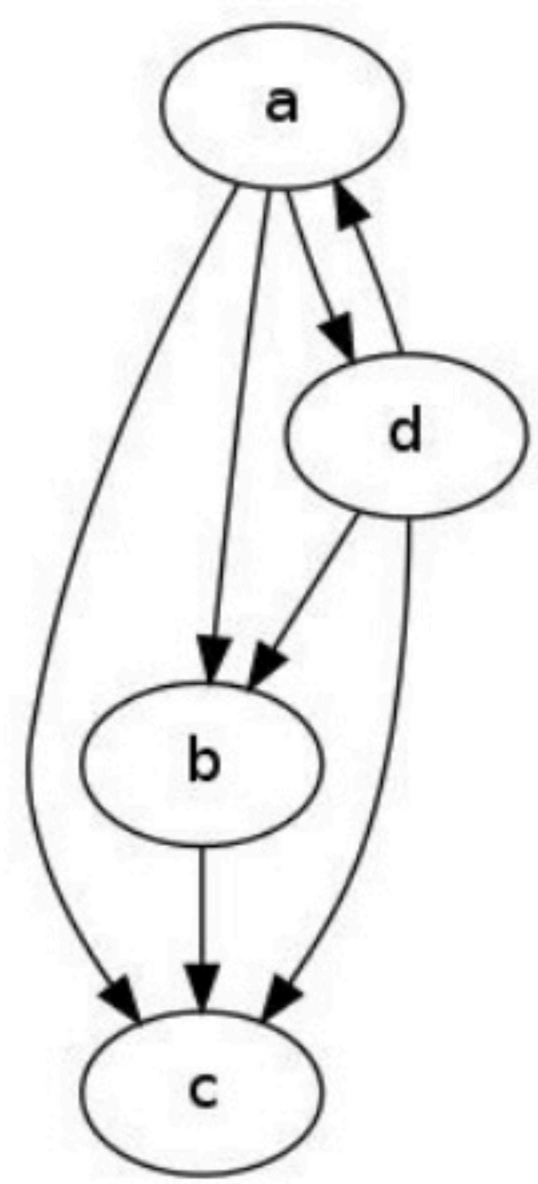
Graphs

Essential Graph Theory

Directed Graphs

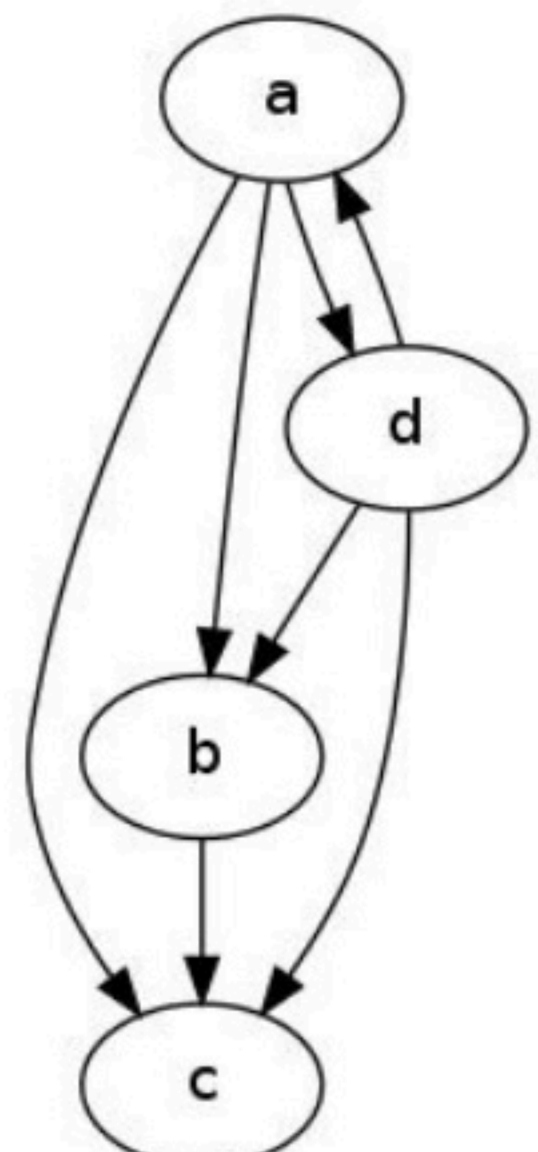
We've encountered graphs before when we studied relations.

We viewed graphs as a way to visual relations over sets



Directed Graphs

Def: A *directed graph*, or *digraph*, G consists of a finite set of vertices V and a set of edges $E \subseteq V \times V$ where each edge is represented by a tuple $(u, v) \in E$, where $u, v \in V$

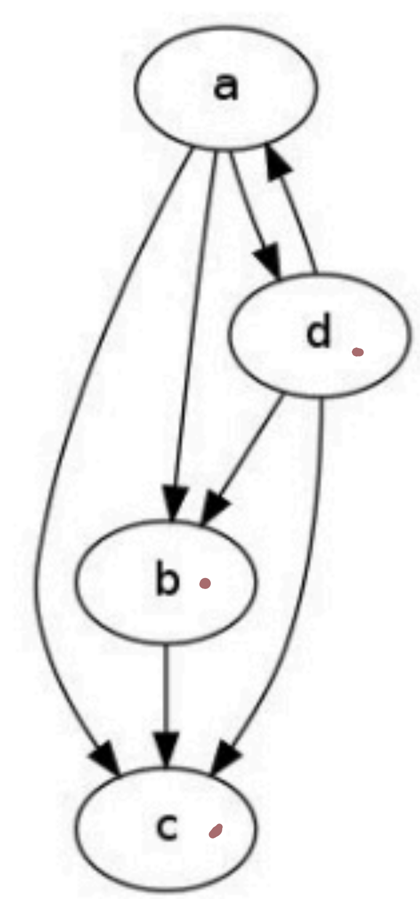


Directed Graphs

Example: Let $V = \{a, b, c, d\}$ and

$$E = \{(a, b), (b, c), (a, c), (a, d), (d, a), (d, b), (d, c)\}$$

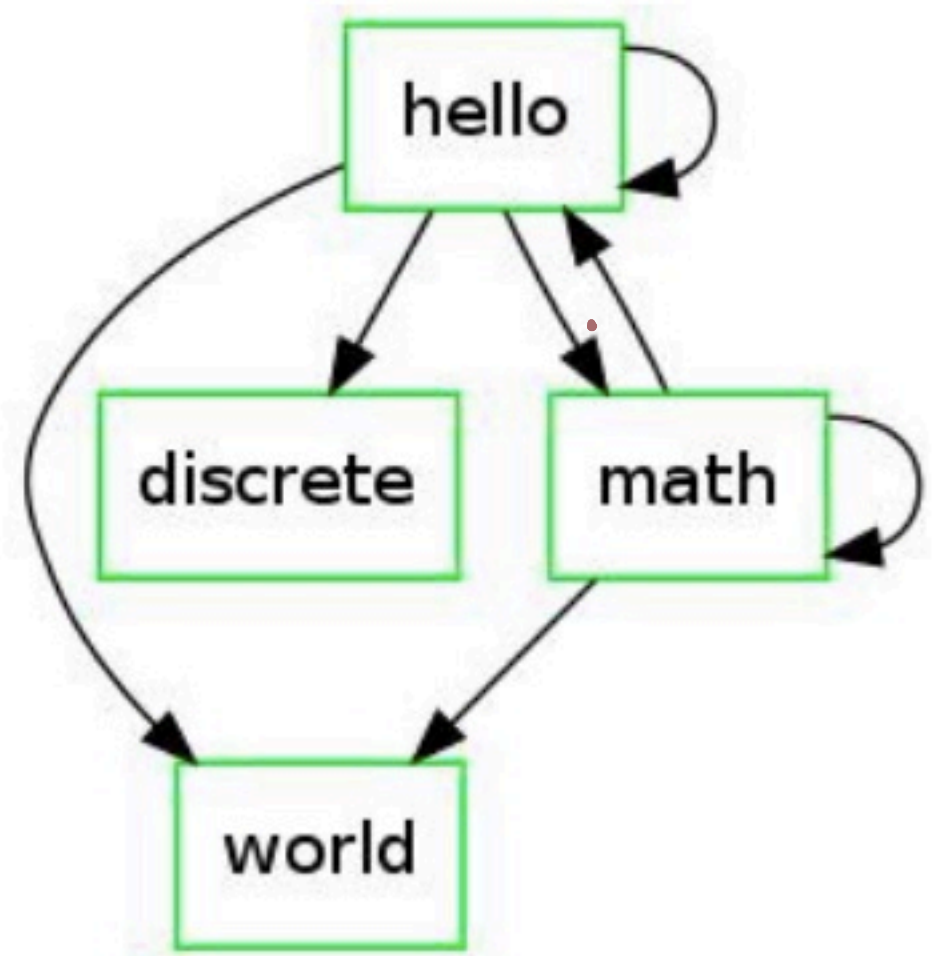
We draw the graph by drawing a circle for each vertex, and an arrow going from vertex u to vertex v if $(u, v) \in E$



Directed Graphs

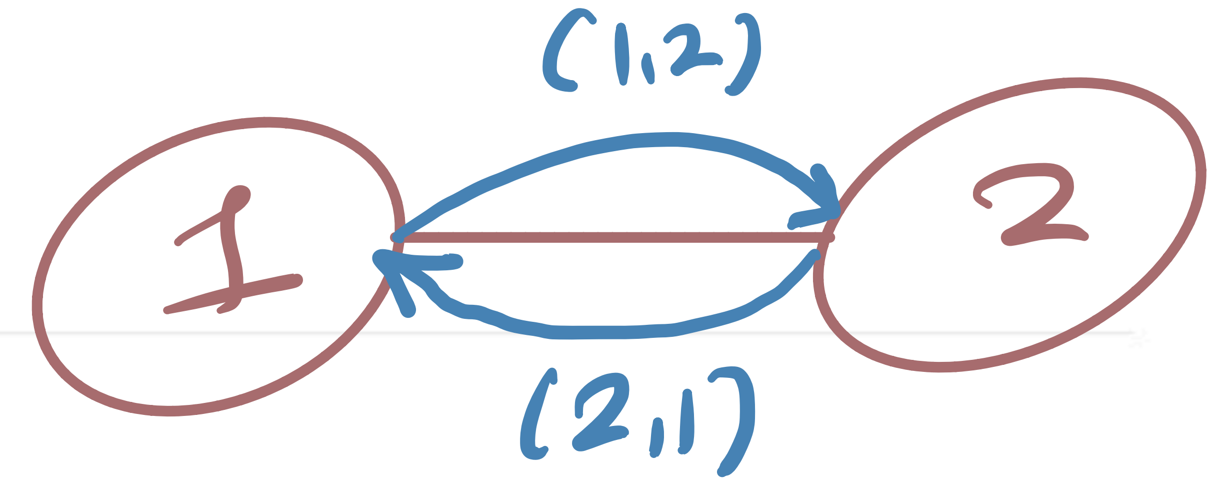
Example: Let $V = \{hello, world, math, discrete\}$ and

$E = \{(hello, hello), (hello, world), (hello, math), (hello, discrete), \ldots$



The edges $(hello, hello)$ and $(math, math)$ are called *self-loops*

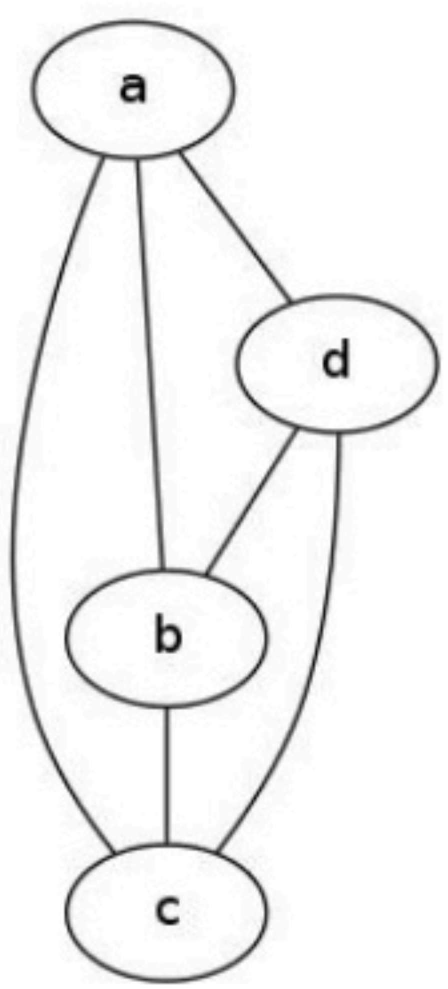
Undirected Graphs



So far we've studied directed graphs.

An *undirected graph* is a special type of directed graph where the represented edge relation is symmetric

Def: A *undirected graph* G consists of a finite set of vertices V and a set of edges $E \subseteq V \times V$ such that the relation E is symmetric

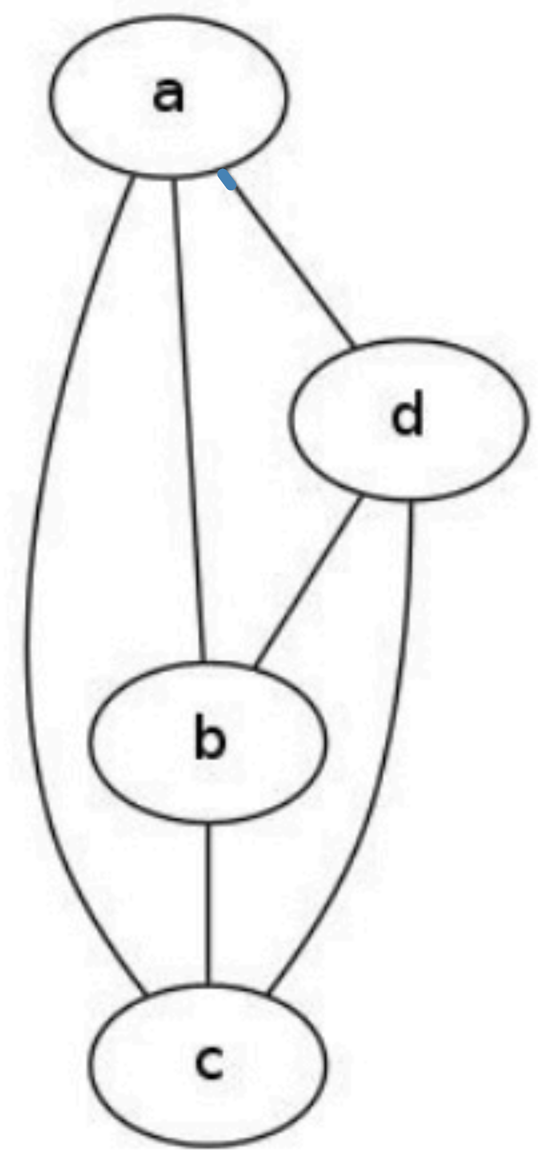


Undirected Graphs

Example: Let $V = \{a, b, c, d\}$ and

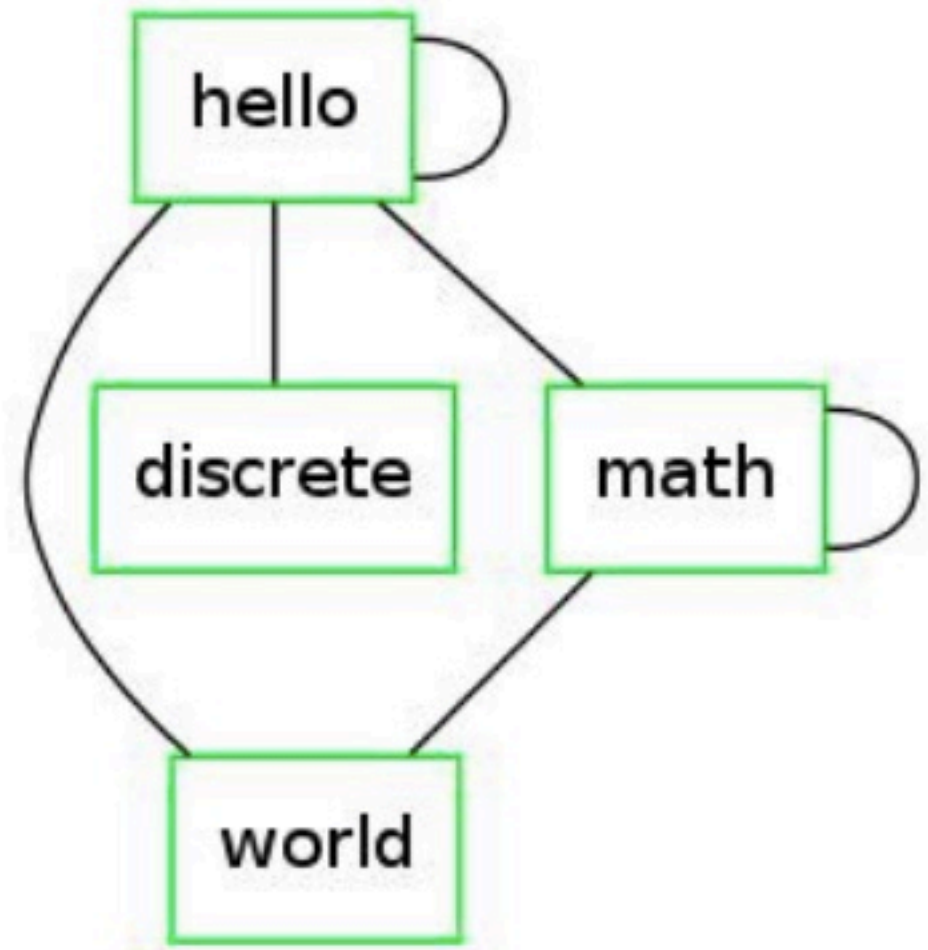
$$E = \{(a,b),(b,a),(b,c),(c,b),(a,c),(c,a),(a,d),(d,a),(d,b),(b,d)$$

$$(c,d),(d,c)\}$$



Undirected Graphs

Example: Let $V = \{hello, world, math, discrete\}$



Once again, this graph has self-loops.

Shortly we will ban self-loops because it makes the math cleaner

Vertex Degree

Let $G = (V, E)$ be a graph with vertex set V and edge set E

The set of incoming edges to a vertex v is the set

$$\text{incoming}(v) = \{(u, v) \mid (u, v) \in E\}$$

Def: The *in-degree* of a vertex v is the number of incoming edges into v. In other words:

$$\text{in-degree}(v) = |\text{incoming}(v)|$$

Directed Graphs

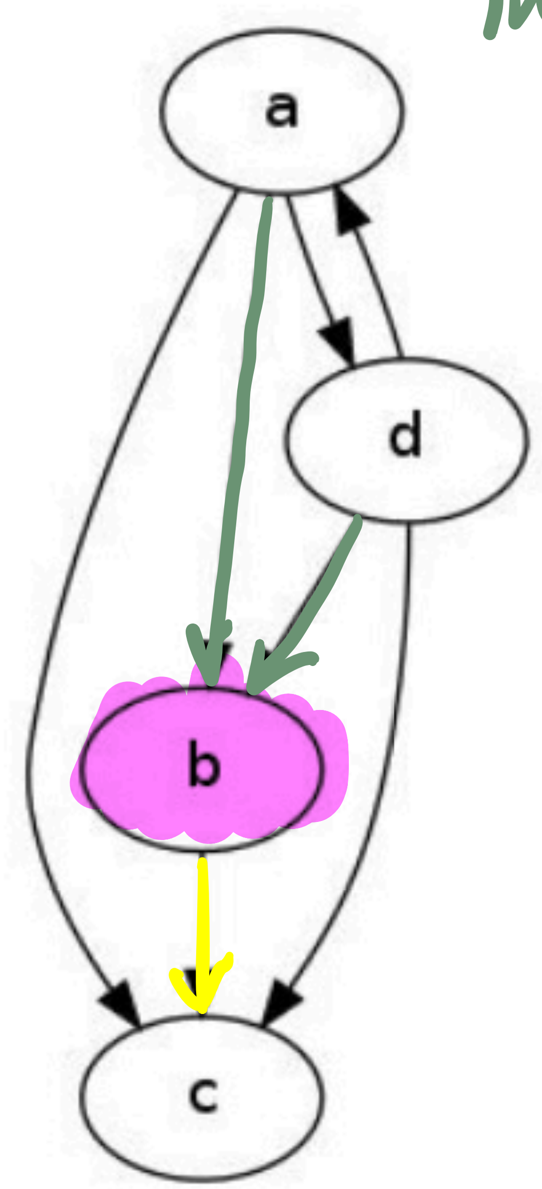
incoming(b) =

$\{(a,b), (d,b)\}$

in-degree(b) =

$|\{(a,b), (d,b)\}|$

$= 2$



Vertex Degree

Let $G = (V, E)$ be a graph with vertex set V and edge set E

The set of outgoing edges to a vertex v is the set

$$\text{outgoing}(v) = \{(v, u) \mid (v, u) \in E\}$$

Def: The *out-degree* of a vertex v is the number of outgoing edges leaving v. In other words:

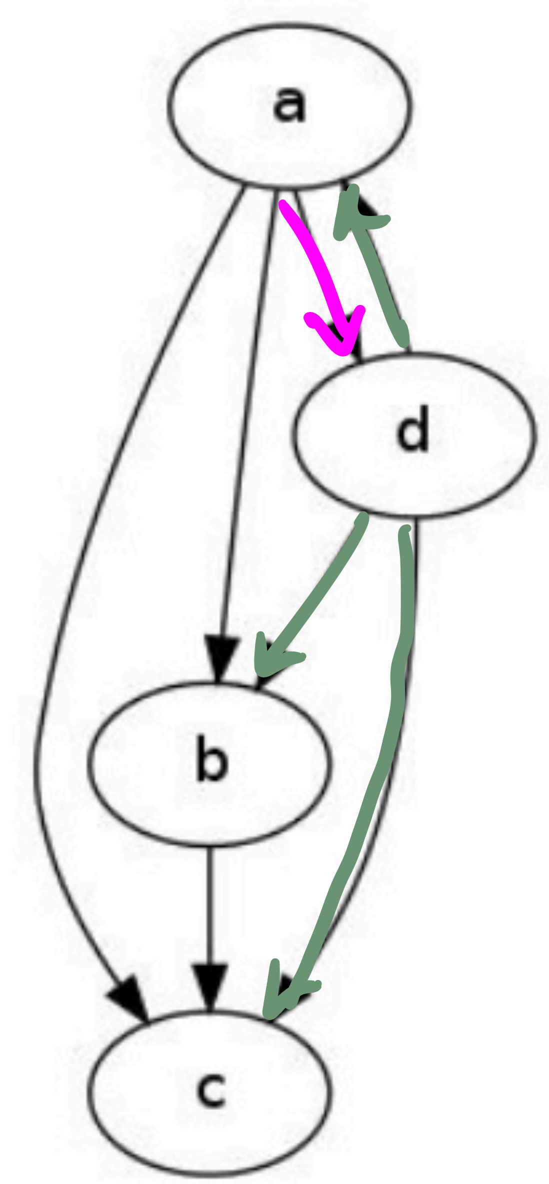
$$\text{out-degree}(v) = |\text{outgoing}(v)|$$

Directed Graphs

outgoing(d) =

$\{(d,a), (d,b), (d,c)\}$

out-degree(d) = 3



Vertex Degree

Let $G = (V, E)$ be a graph with vertex set V and edge set E

The set of outgoing edges to a vertex v is the set

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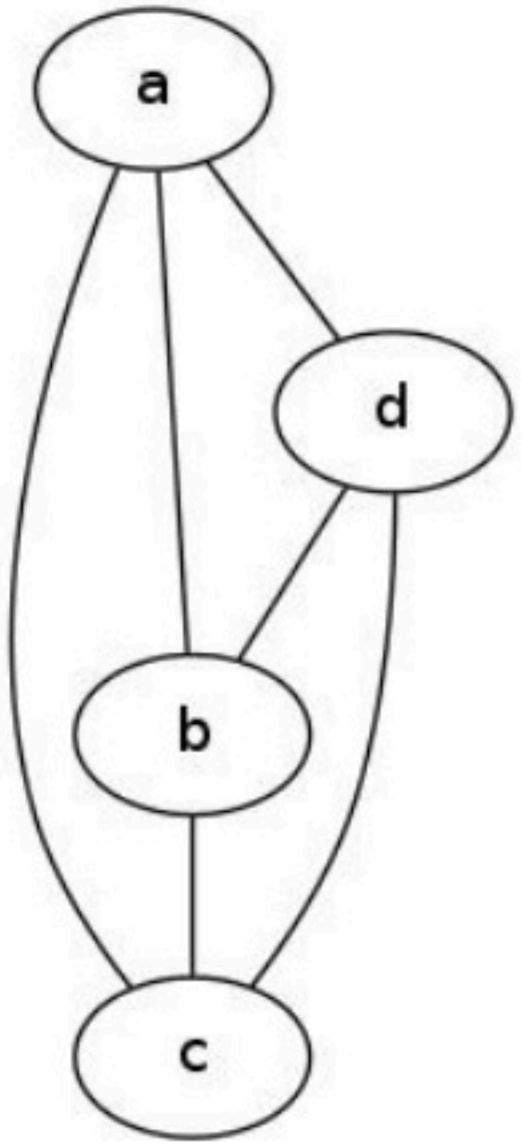
Def: The *out-degree* of a vertex v is the number of outgoing edges leaving v. In other words:

$$\text{out-degree}(v) = |\text{outgoing}(v)|$$

In an undirected graph the set of incoming edges and outgoing edges are equivalent, because the edge relation is symmetric

$$\text{degree}(v) = \text{in-degree}(v) = \text{out-degree}(v)$$

Undirected Graphs

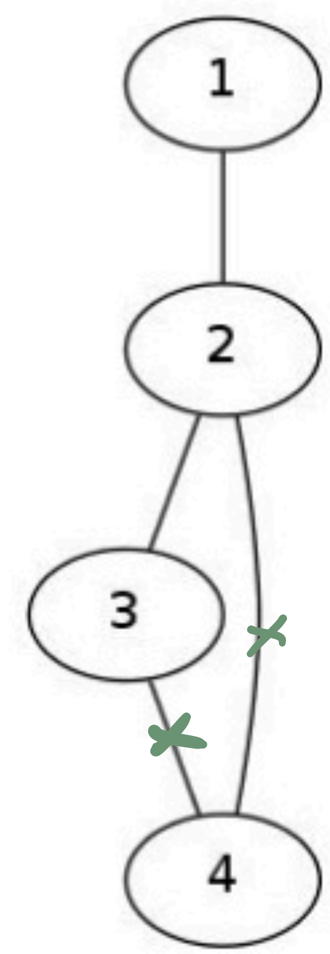


deg(a) = 3
deg(d) = 3
deg(b) = 3
deg(c) = 3

Degree Sequences

Let $G = (V, E)$ be an undirected graph **without any self-loops**

The **degree sequence** of the graph is simply a list (usually ordered by vertex name) of the degree of each vertex

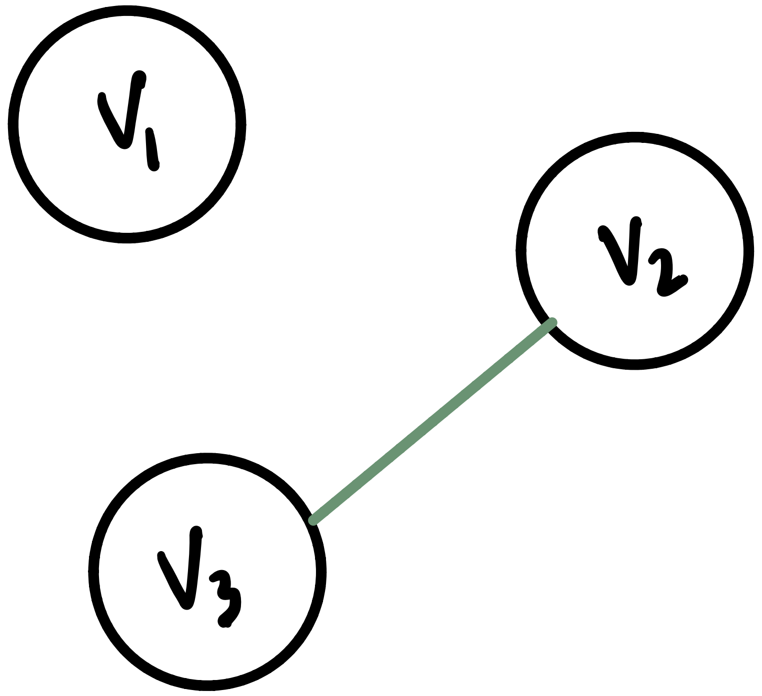


deg(1) = 1
deg(2) = 3
deg(3) = 2
deg(4) = 2

G's degree sequence is $\underline{1, 3, 2, 2}$

Degree Sequences

Example: Can you construct a graph G (with no self-loops) with degree sequence $0, 1, 1$?



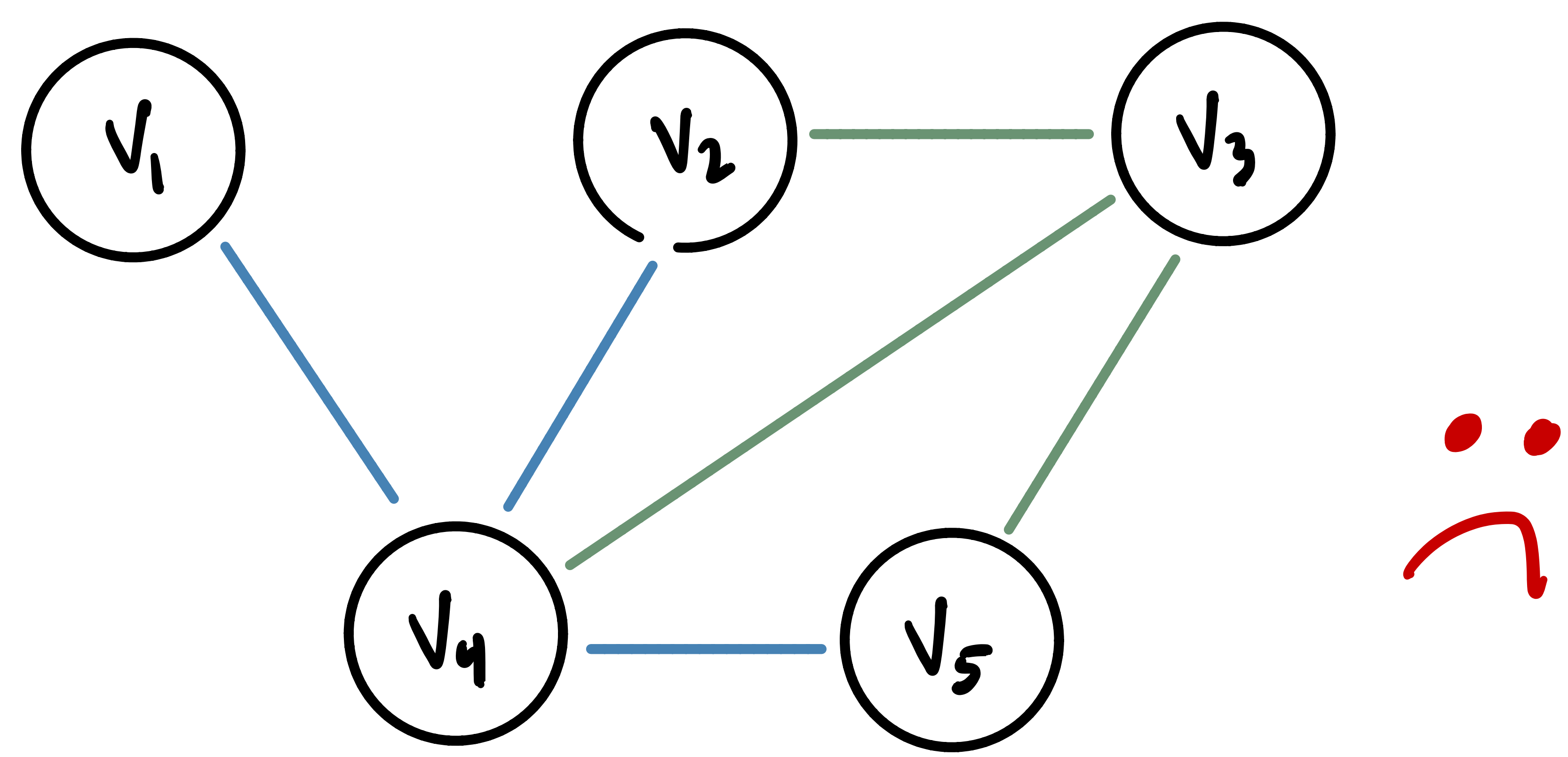
Degree Sequences

Example: Can you construct a graph G (with no self-loops) with degree sequence $1, 2, 3, 4, 5$?

Degree Sequences

Example: Can you construct a graph G (with no self-loops) with degree sequence $1, 2, 3, 4, 5$?

Answer: No! But let's try anyway

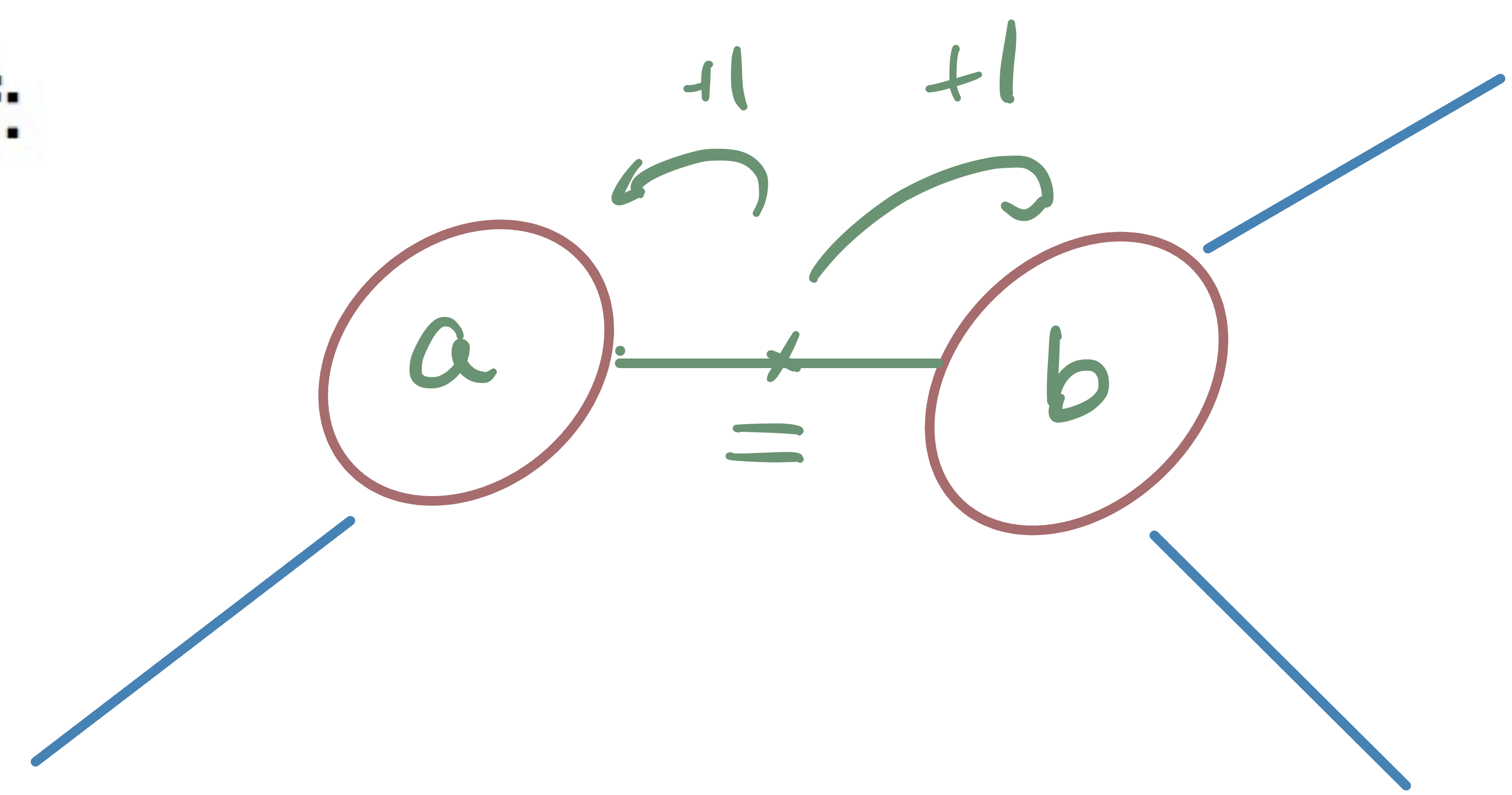


Degree Sequences

The Handshake Theorem: Let $G = (V, E)$ be an undirected graph with no self-loops and m be the number of *undirected edges* in G. Then the sum of each degree in its degree sequence is twice the number of undirected edges. In other words,

$$\sum_{v \in V} \text{degree}(v) = 2m$$

Informal Proof:



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Corollary 1: The sum of the degree sequence of an undirected graph (with no self-loops) must be even

Corollary 2: If any vertices have odd degree then there must be an *even* number of vertices with odd degree.

EVEN EVEN

$$\underline{\underline{2m}} = \sum_{v \in V} \deg(v) = \sum_{v \in V_{\underline{EVEN}}} \deg(v) + \sum_{v \in V_{ODD}} \deg(v)$$

ODD

Degree Sequences

There is another reason there can't be a graph with degree sequence $1, 2, 3, 4, 5$

Theorem: Let $G = (V, E)$ be an undirected graph with no self-loops such that each vertex has degree at least 1 (i.e. no isolated vertices). Then at least two vertices must have the same degree.

Degree Sequences

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Theorem: Let $G = (V, E)$ be an undirected graph with no self-loops such that each vertex has degree at least 1 (i.e. no isolated vertices). Then at least two vertices must have the same degree.

Proof: We'll use the Pigeonhole Principle to prove this. Let $n = |V|$.

- Each vertex must have degree between $1, \ldots, (n - 1)$
- This means that there are $(n - 1)$ possible degrees (holes)
- But there are n vertices (pigeons)
- Therefore two vertices must have the same degree

Degree Sequences

In/Out Degrees for Directed Graphs

For a directed graph $G = (V, E)$ we have

$$\sum_{v \in V} \text{in-degree}(v) = \sum_{v \in V} \text{out-degree}(v) = |E|$$

This makes sense because each directed edge contributes to the in-degree of one vertex and the out-degree of another vertex