the order in which the breadth-first search of G' colors vertices in V black is the same as the order in which Dijkstra's algorithm extracts the vertices of V from the priority queue when it runs on G.

24.3-8

Let $G = (V, E)$ be a weighted, directed graph with nonnegative weight function $w : E \to \{0, 1, \ldots, W\}$ for some nonnegative integer W. Modify Dijkstra's algorithm to compute the shortest paths from a given source vertex s in $O(WV + E)$ time.

24.3-9

Modify your algorithm from Exercise 24.3-8 to run in $O((V + E) \lg W)$ time. (*Hint:* How many distinct shortest-path estimates can there be in $V - S$ at any point in time?)

24.3-10

Suppose that we are given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, all other edge weights are nonnegative, and there are no negative-weight cycles. Argue that Dijkstra's algorithm correctly finds shortest paths from s in this graph.

24.4 Difference constraints and shortest paths

Chapter 29 studies the general linear-programming problem, in which we wish to optimize a linear function subject to a set of linear inequalities. In this section, we investigate a special case of linear programming that we reduce to finding shortest paths from a single source. We can then solve the single-source shortest-paths problem that results by running the Bellman-Ford algorithm, thereby also solving the linear-programming problem.

Linear programming

In the general ***linear-programming problem***, we are given an $m \times n$ matrix A, an m-vector b, and an n-vector c. We wish to find a vector x of n elements that maximizes the ***objective function*** $\sum_{i=1}^{n} c_i x_i$ subject to the m constraints given by $Ax \le b$.

Although the simplex algorithm, which is the focus of Chapter 29, does not always run in time polynomial in the size of its input, there are other linear-programming algorithms that do run in polynomial time. We offer here two reasons to understand the setup of linear-programming problems. First, if we know that we can cast a given problem as a polynomial-sized linear-programming problem, then we immediately have a polynomial-time algorithm to solve the problem. Second, faster algorithms exist for many special cases of linear programming. For example, the single-pair shortest-path problem (Exercise 24.4-4) and the maximum-flow problem (Exercise 26.1-5) are special cases of linear programming.

Sometimes we don't really care about the objective function; we just wish to find any ***feasible solution***, that is, any vector x that satisfies $Ax \le b$, or to determine that no feasible solution exists. We shall focus on one such ***feasibility problem***.

Systems of difference constraints

In a ***system of difference constraints***, each row of the linear-programming matrix A contains one 1 and one -1, and all other entries of A are 0. Thus, the constraints given by $Ax \le b$ are a set of m ***difference constraints*** involving n unknowns, in which each constraint is a simple linear inequality of the form

$$x_j - x_i \le b_k ,$$

where $1 \le i, j \le n$, $i \ne j$, and $1 \le k \le m$.

For example, consider the problem of finding a 5-vector $x = (x_i)$ that satisfies

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \le \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix} .$$

This problem is equivalent to finding values for the unknowns x_1, x_2, x_3, x_4, x_5, satisfying the following 8 difference constraints:

$$x_1 - x_2 \le 0 , \tag{24.3}$$
$$x_1 - x_5 \le -1 , \tag{24.4}$$
$$x_2 - x_5 \le 1 , \tag{24.5}$$
$$x_3 - x_1 \le 5 , \tag{24.6}$$
$$x_4 - x_1 \le 4 , \tag{24.7}$$
$$x_4 - x_3 \le -1 , \tag{24.8}$$
$$x_5 - x_3 \le -3 , \tag{24.9}$$
$$x_5 - x_4 \le -3 . \tag{24.10}$$

One solution to this problem is $x = (-5, -3, 0, -1, -4)$, which you can verify directly by checking each inequality. In fact, this problem has more than one solution. Another is $x' = (0, 2, 5, 4, 1)$. These two solutions are related: each component of x' is 5 larger than the corresponding component of x. This fact is not mere coincidence.

Lemma 24.8
Let $x = (x_1, x_2, \ldots, x_n)$ be a solution to a system $Ax \le b$ of difference constraints, and let d be any constant. Then $x + d = (x_1 + d, x_2 + d, \ldots, x_n + d)$ is a solution to $Ax \le b$ as well.

Proof For each x_i and x_j, we have $(x_j + d) - (x_i + d) = x_j - x_i$. Thus, if x satisfies $Ax \le b$, so does $x + d$. ■

Systems of difference constraints occur in many different applications. For example, the unknowns x_i may be times at which events are to occur. Each constraint states that at least a certain amount of time, or at most a certain amount of time, must elapse between two events. Perhaps the events are jobs to be performed during the assembly of a product. If we apply an adhesive that takes 2 hours to set at time x_1 and we have to wait until it sets to install a part at time x_2, then we have the constraint that $x_2 \ge x_1 + 2$ or, equivalently, that $x_1 - x_2 \le -2$. Alternatively, we might require that the part be installed after the adhesive has been applied but no later than the time that the adhesive has set halfway. In this case, we get the pair of constraints $x_2 \ge x_1$ and $x_2 \le x_1 + 1$ or, equivalently, $x_1 - x_2 \le 0$ and $x_2 - x_1 \le 1$.

Constraint graphs

We can interpret systems of difference constraints from a graph-theoretic point of view. In a system $Ax \le b$ of difference constraints, we view the $m \times n$ linear-programming matrix A as the transpose of an incidence matrix (see Exercise 22.1-7) for a graph with n vertices and m edges. Each vertex v_i in the graph, for $i = 1, 2, \ldots, n$, corresponds to one of the n unknown variables x_i. Each directed edge in the graph corresponds to one of the m inequalities involving two unknowns.

More formally, given a system $Ax \le b$ of difference constraints, the corresponding ***constraint graph*** is a weighted, directed graph $G = (V, E)$, where

$$V = \{v_0, v_1, \ldots, v_n\}$$

and

$$\begin{aligned} E = \{&(v_i, v_j) : x_j - x_i \le b_k \text{ is a constraint}\} \\ &\cup \{(v_0, v_1), (v_0, v_2), (v_0, v_3), \ldots, (v_0, v_n)\} \ . \end{aligned}$$

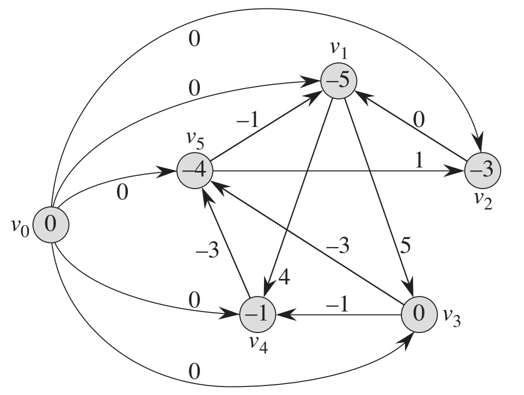

Figure 24.8 The constraint graph corresponding to the system (24.3)–(24.10) of difference constraints. The value of $\delta(v_0, v_i)$ appears in each vertex v_i. One feasible solution to the system is $x = (-5, -3, 0, -1, -4)$.

The constraint graph contains the additional vertex v_0, as we shall see shortly, to guarantee that the graph has some vertex which can reach all other vertices. Thus, the vertex set V consists of a vertex v_i for each unknown x_i, plus an additional vertex v_0. The edge set E contains an edge for each difference constraint, plus an edge (v_0, v_i) for each unknown x_i. If $x_j - x_i \le b_k$ is a difference constraint, then the weight of edge (v_i, v_j) is $w(v_i, v_j) = b_k$. The weight of each edge leaving v_0 is 0. Figure 24.8 shows the constraint graph for the system (24.3)–(24.10) of difference constraints.

The following theorem shows that we can find a solution to a system of difference constraints by finding shortest-path weights in the corresponding constraint graph.

Theorem 24.9
Given a system $Ax \le b$ of difference constraints, let $G = (V, E)$ be the corresponding constraint graph. If G contains no negative-weight cycles, then

$$x = (\delta(v_0, v_1), \delta(v_0, v_2), \delta(v_0, v_3), \ldots, \delta(v_0, v_n)) \tag{24.11}$$

is a feasible solution for the system. If G contains a negative-weight cycle, then there is no feasible solution for the system.

Proof We first show that if the constraint graph contains no negative-weight cycles, then equation (24.11) gives a feasible solution. Consider any edge $(v_i, v_j) \in E$. By the triangle inequality, $\delta(v_0, v_j) \le \delta(v_0, v_i) + w(v_i, v_j)$ or, equivalently, $\delta(v_0, v_j) - \delta(v_0, v_i) \le w(v_i, v_j)$. Thus, letting $x_i = \delta(v_0, v_i)$ and

$x_j = \delta(v_0, v_j)$ satisfies the difference constraint $x_j - x_i \le w(v_i, v_j)$ that corresponds to edge (v_i, v_j).

Now we show that if the constraint graph contains a negative-weight cycle, then the system of difference constraints has no feasible solution. Without loss of generality, let the negative-weight cycle be $c = \langle v_1, v_2, \ldots, v_k \rangle$, where $v_1 = v_k$. (The vertex v_0 cannot be on cycle c, because it has no entering edges.) Cycle c corresponds to the following difference constraints:

$$\begin{aligned} x_2 - x_1 &\le w(v_1, v_2) , \\ x_3 - x_2 &\le w(v_2, v_3) , \\ &\vdots \\ x_{k-1} - x_{k-2} &\le w(v_{k-2}, v_{k-1}) , \\ x_k - x_{k-1} &\le w(v_{k-1}, v_k) . \end{aligned}$$

We will assume that x has a solution satisfying each of these k inequalities and then derive a contradiction. The solution must also satisfy the inequality that results when we sum the k inequalities together. If we sum the left-hand sides, each unknown x_i is added in once and subtracted out once (remember that $v_1 = v_k$ implies $x_1 = x_k$), so that the left-hand side of the sum is 0. The right-hand side sums to $w(c)$, and thus we obtain $0 \le w(c)$. But since c is a negative-weight cycle, $w(c) < 0$, and we obtain the contradiction that $0 \le w(c) < 0$. ■

Solving systems of difference constraints

Theorem 24.9 tells us that we can use the Bellman-Ford algorithm to solve a system of difference constraints. Because the constraint graph contains edges from the source vertex v_0 to all other vertices, any negative-weight cycle in the constraint graph is reachable from v_0. If the Bellman-Ford algorithm returns TRUE, then the shortest-path weights give a feasible solution to the system. In Figure 24.8, for example, the shortest-path weights provide the feasible solution $x = (-5, -3, 0, -1, -4)$, and by Lemma 24.8, $x = (d-5, d-3, d, d-1, d-4)$ is also a feasible solution for any constant d. If the Bellman-Ford algorithm returns FALSE, there is no feasible solution to the system of difference constraints.

A system of difference constraints with m constraints on n unknowns produces a graph with $n + 1$ vertices and $n + m$ edges. Thus, using the Bellman-Ford algorithm, we can solve the system in $O((n+1)(n+m)) = O(n^2 + nm)$ time. Exercise 24.4-5 asks you to modify the algorithm to run in $O(nm)$ time, even if m is much less than n.