Chapter 9

Linear dynamical systems

In this chapter we consider a useful application of matrix-vector multiplication, which is used to describe many systems or phenomena that change or evolve over time.

9.1 Linear dynamical systems

Suppose $x_1, x_2, \ldots$ is a sequence of n-vectors. The index (subscript) denotes time or period, and is written as t; x_t, the value of the sequence at time (or period) t, is called the *state* at time t. We can think of x_t as a vector that changes over time, *i.e.*, one that changes dynamically. In this context, the sequence $x_1, x_2, \ldots$ is sometimes called a *trajectory* or *state trajectory*. We sometimes refer to x_t as the *current state* of the system (implicitly assuming the current time is t), and x_{t+1} as the *next state*, x_{t-1} as the *previous state*, and so on.

The state x_t can represent a portfolio that changes daily, or the positions and velocities of the parts of a mechanical system, or the quarterly activity of an economy. If x_t represents a portfolio that changes daily, $(x_5)_3$ is the amount of asset 3 held in the portfolio on (trading) day 5.

A *linear dynamical system* is a simple model for the sequence, in which each x_{t+1} is a linear function of x_t:

$$x_{t+1} = A_t x_t, \quad t = 1, 2, \ldots. \tag{9.1}$$

Here the $n \times n$ matrices A_t are called the *dynamics matrices*. The equation above is called the *dynamics* or *update* equation, since it gives us the next value of x, *i.e.*, x_{t+1}, as a function of the current value x_t. Often the dynamics matrix does not depend on t, in which case the linear dynamical system is called *time-invariant*.

If we know x_t (and $A_t, A_{t+1}, \ldots$) we can determine $x_{t+1}, x_{t+2}, \ldots$ simply by iterating the dynamics equation (9.1). In other words: If we know the *current* value of x, we can find all *future* values. In particular, we do not need to know the *past* states. This is why x_t is called the *state* of the system. It contains all the information needed at time t to determine the future evolution of the system.

Linear dynamical system with input. There are many variations on and extensions of the basic linear dynamical system model (9.1), some of which we will encounter later. As an example, we can add additional terms to the update equation:

$$x_{t+1} = A_t x_t + B_t u_t + c_t, \quad t = 1, 2, \ldots. \tag{9.2}$$

Here u_t is an m-vector called the *input*, B_t is the $n \times m$ *input matrix*, and the n-vector c_t is called the *offset*, all at time t. The input and offset are used to model other factors that affect the time evolution of the state. Another name for the input u_t is *exogenous variable*, since, roughly speaking, it comes from outside the system.

Markov model. The linear dynamical system (9.1) is sometimes called a *Markov* model (after the mathematician Andrey Markov). Markov studied systems in which the next state value depends on the current one, and not on the previous state values $x_{t-1}, x_{t-2}, \ldots$. The linear dynamical system (9.1) is the special case of a Markov system where the next state is a linear function of the current state.

In a variation on the Markov model, called a (linear) K-Markov model, the next state x_{t+1} depends on the current state and $K-1$ previous states. Such a system has the form

$$x_{t+1} = A_1 x_t + \cdots + A_K x_{t-K+1}, \quad t = K, K+1, \ldots. \tag{9.3}$$

Models of this form are used in time series analysis and econometrics, where they are called (vector) *auto-regressive models*. When $K = 1$, the Markov model (9.3) is the same as a linear dynamical system (9.1). When $K > 1$, the Markov model (9.3) can be reduced to a standard linear dynamical system (9.1), with an appropriately chosen state; see exercise 9.4.

Simulation. If we know the dynamics (and input) matrices, and the state at time t, we can find the future state trajectory $x_{t+1}, x_{t+2}, \ldots$ by iterating the equation (9.1) (or (9.2), provided we also know the input sequence $u_t, u_{t+1}, \ldots$). This is called *simulating* the linear dynamical system. Simulation makes predictions about the future state of a system. (To the extent that (9.1) is only an approximation or model of some real system, we must be careful when interpreting the results.) We can carry out what-if simulations, to see what would happen if the system changes in some way, or if a particular set of inputs occurs.

9.2 Population dynamics

Linear dynamical systems can be used to describe the evolution of the age distribution in some population over time. Suppose x_t is a 100-vector, with $(x_t)_i$ denoting the number of people in some population (say, a country) with age $i-1$ (say, on January 1) in year t, where t is measured starting from some base year, for $i = 1, \ldots, 100$. While $(x_t)_i$ is an integer, it is large enough that we simply consider