9.3 Epidemic dynamics

The dynamics of infection and the spread of an epidemic can be modeled using a linear dynamical system. (More sophisticated nonlinear epidemic dynamic models are also used.) In this section we describe a simple example.

A disease is introduced into a population. In each period (say, days) we count the fraction of the population that is in four different infection states:

- *Susceptible.* These individuals can acquire the disease the next day.
- *Infected.* These individuals have the disease.
- *Recovered* (and immune). These individuals had the disease and survived, and now have immunity.
- *Deceased.* These individuals had the disease, and unfortunately died from it.

We denote the fractions of each of these as a 4-vector x_t, so, for example, $x_t = (0.75, 0.10, 0.10, 0.05)$ means that in day t, 75% of the population is susceptible, 10% is infected, 10% is recovered and immune, and 5% has died from the disease.

There are many mathematical models that predict how the disease state fractions x_t evolve over time. One simple model can be expressed as a linear dynamical system. The model assumes the following happens over each day.

- 5% of the susceptible population will acquire the disease. (The other 95% will remain susceptible.)
- 1% of the infected population will die from the disease, 10% will recover and acquire immunity, and 4% will recover and not acquire immunity (and therefore, become susceptible). The remaining 85% will remain infected.

(Those who have have recovered with immunity and those who have died remain in those states.)

We first determine $(x_{t+1})_1$, the fraction of susceptible individuals in the next day. These include the susceptible individuals from today, who did not become infected, which is $0.95(x_t)_1$, plus the infected individuals today who recovered without immunity, which is $0.04(x_t)_2$. All together we have $(x_{t+1})_1 = 0.95(x_t)_1 + 0.04(x_t)_2$. We have $(x_{t+1})_2 = 0.85(x_t)_2 + 0.05(x_t)_1$; the first term counts those who are infected and remain infected, and the second term counts those who are susceptible and acquire the disease. Similar arguments give $(x_{t+1})_3 = (x_t)_3 + 0.10(x_t)_2$, and $(x_{t+1})_4 = (x_t)_4 + 0.01(x_t)_2$. We put these together to get

$$x_{t+1} = \begin{bmatrix} 0.95 & 0.04 & 0 & 0 \\ 0.05 & 0.85 & 0 & 0 \\ 0 & 0.10 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{bmatrix} x_t,$$

which is a time-invariant linear dynamical system of the form (9.1).

Figure 9.5 shows the evolution of the four groups from the initial condition $x_0 = (1, 0, 0, 0)$. The simulation shows that after around 100 days, the state converges to one with a little under 10% of the population deceased, and the remaining population immune.

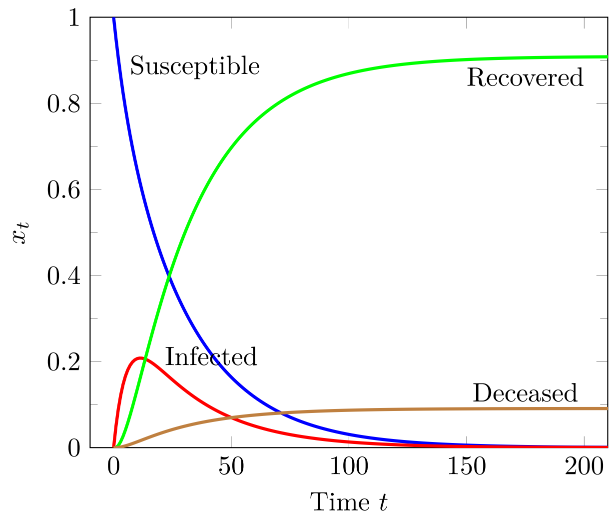

Figure 9.5 Simulation of epidemic dynamics.

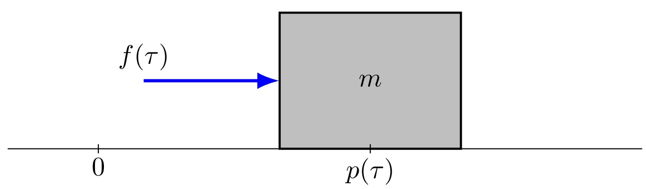

Figure 9.6 Mass moving along a line.

9.4 Motion of a mass

Linear dynamical systems can be used to (approximately) describe the motion of many mechanical systems, for example, an airplane (that is not undergoing extreme maneuvers), or the (hopefully not too large) movement of a building during an earthquake. Here we describe the simplest example: A single mass moving in 1-D (*i.e.*, a straight line), with an external force and a drag force acting on it. This is illustrated in figure 9.6. The (scalar) position of the mass at time τ is given by $p(\tau)$. (Here τ is continuous, *i.e.*, a real number.) The position satisfies Newton's law of motion, the differential equation

$$m\frac{d^2p}{d\tau^2}(\tau) = -\eta\frac{dp}{d\tau}(\tau) + f(\tau),$$

where $m > 0$ is the mass, $f(\tau)$ is the external force acting on the mass at time τ, and $\eta > 0$ is the drag coefficient. The right-hand side is the total force acting on the mass; the first term is the drag force, which is proportional to the velocity and in the opposite direction.