which looks more intimidating. This inequality is attributed to the mathematician Augustin-Louis Cauchy; Hermann Schwarz gave the derivation given below.

The Cauchy–Schwarz inequality can be shown as follows. The inequality clearly holds if $a = 0$ or $b = 0$ (in this case, both sides of the inequality are zero). So we suppose now that $a \neq 0$, $b \neq 0$, and define $\alpha = \|a\|$, $\beta = \|b\|$. We observe that

$0 = 2\|a\|^2\|b\|^2 - 2\|a\|\|b\|a^Tb$

$\cancel{2}\|a\|^{\cancel{2}}\|b\|^{\cancel{2}} = \cancel{2\|a\|\|b\|}a^Tb$

$\|a\|\|b\| \geq a^Tb$, $a^Tb \leq \|a\|\|b\|$

$$\begin{aligned} 0 &\leq \|\beta a - \alpha b\|^2 \\ &= \|\beta a\|^2 - 2(\beta a)^T(\alpha b) + \|\alpha b\|^2 \\ &= \beta^2\|a\|^2 - 2\beta\alpha(a^Tb) + \alpha^2\|b\|^2 \\ &= \|b\|^2\|a\|^2 - 2\|b\|\|a\|(a^Tb) + \|a\|^2\|b\|^2 \\ &= 2\|a\|^2\|b\|^2 - 2\|a\|\,\|b\|(a^Tb). \end{aligned}$$

Dividing by $2\|a\|\,\|b\|$ yields $a^Tb \leq \|a\|\,\|b\|$. Applying this inequality to $-a$ and b we obtain $-a^Tb \leq \|a\|\,\|b\|$. Putting these two inequalities together we get the Cauchy–Schwarz inequality, $|a^Tb| \leq \|a\|\,\|b\|$.

This argument also reveals the conditions on a and b under which they satisfy the Cauchy–Schwarz inequality with equality. This occurs only if $\|\beta a - \alpha b\| = 0$, *i.e.*, $\beta a = \alpha b$. This means that each vector is a scalar multiple of the other (in the case when they are nonzero). This statement remains true when either a or b is zero. So the Cauchy–Schwarz inequality holds with equality when one of the vectors is a multiple of the other; in all other cases, it holds with strict inequality.

Verification of triangle inequality. We can use the Cauchy–Schwarz inequality to verify the triangle inequality. Let a and b be any vectors. Then

When $\theta = \pi/2 \rightarrow$ Pythag. Thm.

$$\begin{aligned} \|a+b\|^2 &= \|a\|^2 + 2a^Tb + \|b\|^2 \\ a^Tb = \|a\|\|b\| \longrightarrow\ &\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 \\ &= (\|a\| + \|b\|)^2, \longleftarrow x^2 + 2xy + y^2 = (x+y)^2 \end{aligned}$$

where we used the Cauchy–Schwarz inequality in the second line. Taking the squareroot we get the triangle inequality, $\|a+b\| \leq \|a\| + \|b\|$.

Angle between vectors. The *angle* between two nonzero vectors a, b is defined as

$$\cos^{-1}(\theta) \equiv \mathrm{arcos}(\theta) \qquad \theta = \arccos\left(\frac{a^Tb}{\|a\|\,\|b\|}\right)$$

where arccos denotes the inverse cosine, normalized to lie in the interval $[0, \pi]$. In other words, we define θ as the unique number between 0 and π that satisfies

$$a^Tb = \|a\|\,\|b\| \cos\theta.$$

$0 < \theta < \pi/2$
↳ Acute
$\frac{\pi}{2} < \theta < \pi$
↳ obtuse

The angle between a and b is written as $\angle(a, b)$, and is sometimes expressed in degrees. (The default angle unit is *radians*; 360° is 2π radians.) For example, $\angle(a, b) = 60°$ means $\angle(a, b) = \pi/3$, *i.e.*, $a^Tb = (1/2)\|a\|\|b\|$.

The angle coincides with the usual notion of angle between vectors, when they have dimension two or three, and they are thought of as displacements from a

$$a^Tb = \|a\|\|b\|\cos(\theta) \rightarrow \frac{a^Tb}{\|a\|\|b\|} = \cos(\theta) \rightarrow \theta = \cos^{-1}\left(\frac{a^Tb}{\|a\|\|b\|}\right)$$

$\theta = \frac{n\pi}{2}$ w/ $n = 2k+1 \Rightarrow \perp$ or perpendicular, orthogonal

$\theta = n\pi$ w/ $n = 2k \Rightarrow \parallel$ or parallel, aligned