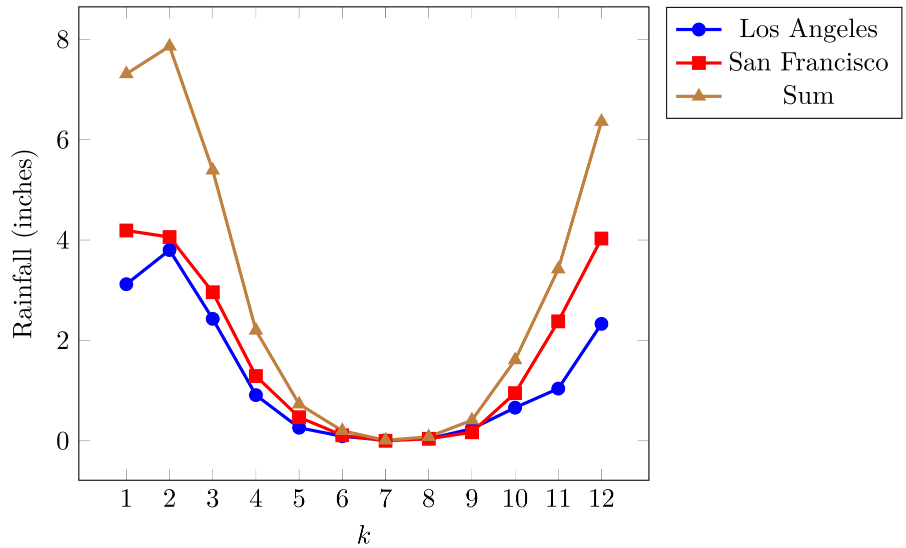

Figure 1.9 Average monthly rainfall in inches measured in downtown Los Angeles and San Francisco International Airport, and their sum. Averages are 30-year averages (1981–2010).

1.3 Scalar-vector multiplication

Another operation is *scalar multiplication* or *scalar-vector multiplication*, in which a vector is multiplied by a scalar (*i.e.*, number), which is done by multiplying every element of the vector by the scalar. Scalar multiplication is denoted by juxtaposition, typically with the scalar on the left, as in

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}.$$

Scalar-vector multiplication can also be written with the scalar on the right, as in

$$\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix}(1.5) = \begin{bmatrix} 1.5 \\ 13.5 \\ 9 \end{bmatrix}.$$

The meaning is the same: It is the vector obtained by multiplying each element by the scalar. A similar notation is $a/2$, where a is a vector, meaning $(1/2)a$. The scalar-vector product $(-1)a$ is written simply as $-a$. Note that $0\,a = 0$ (where the left-hand zero is the scalar zero, and the right-hand zero is a vector zero of the same size as a).

Properties. By definition, we have $\alpha a = a\alpha$, for any scalar α and any vector a. This is called the *commutative property* of scalar-vector multiplication; it means that scalar-vector multiplication can be written in either order.

Scalar multiplication obeys several other laws that are easy to figure out from the definition. For example, it satisfies the associative property: If a is a vector and β and γ are scalars, we have

$$(\beta\gamma)a = \beta(\gamma a).$$

On the left-hand side we see scalar-scalar multiplication ($\beta\gamma$) and scalar-vector multiplication; on the right-hand side we see two scalar-vector products. As a consequence, we can write the vector above as $\beta\gamma a$, since it does not matter whether we interpret this as $\beta(\gamma a)$ or $(\beta\gamma)a$.

The associative property holds also when we denote scalar-vector multiplication with the scalar on the right. For example, we have $\beta(\gamma a) = (\beta a)\gamma$, and consequently we can write both as $\beta a\gamma$. As a convention, however, this vector is normally written as $\beta\gamma a$ or as $(\beta\gamma)a$.

If a is a vector and β, γ are scalars, then

$$(\beta + \gamma)a = \beta a + \gamma a.$$

(This is the left-distributive property of scalar-vector multiplication.) Scalar multiplication, like ordinary multiplication, has higher precedence in equations than vector addition, so the right-hand side here, $\beta a+\gamma a$, means $(\beta a)+(\gamma a)$. It is useful to identify the symbols appearing in this formula above. The $+$ symbol on the left is addition of scalars, while the $+$ symbol on the right denotes vector addition. When scalar multiplication is written with the scalar on the right, we have the right-distributive property:

$$a(\beta + \gamma) = a\beta + a\gamma.$$

Scalar-vector multiplication also satisfies another version of the right-distributive property:

$$\beta(a + b) = \beta a + \beta b$$

for any scalar β and any n-vectors a and b. In this equation, both of the $+$ symbols refer to the addition of n-vectors.

Examples.

- *Displacements.* When a vector a represents a displacement, and $\beta > 0$, βa is a displacement in the same direction of a, with its magnitude scaled by β. When $\beta < 0$, βa represents a displacement in the opposite direction of a, with magnitude scaled by $|\beta|$. This is illustrated in figure 1.10.

- *Materials requirements.* Suppose the n-vector q is the bill of materials for producing one unit of some product, *i.e.*, q_i is the amount of raw material required to produce one unit of product. To produce α units of the product will then require raw materials given by αq. (Here we assume that $\alpha \geq 0$.)

- *Audio scaling.* If a is a vector representing an audio signal, the scalar-vector product βa is perceived as the same audio signal, but changed in volume (loudness) by the factor $|\beta|$. For example, when $\beta = 1/2$ (or $\beta = -1/2$), βa is perceived as the same audio signal, but quieter.

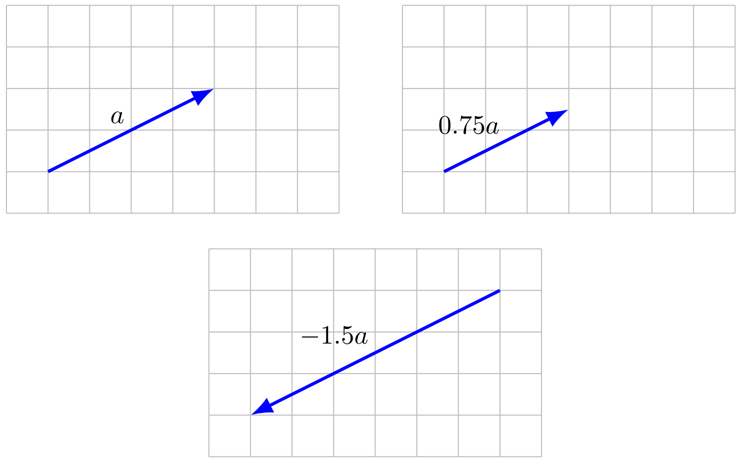

Figure 1.10 The vector $0.75a$ represents the displacement in the direction of the displacement a, with magnitude scaled by 0.75; $(-1.5)a$ represents the displacement in the opposite direction, with magnitude scaled by 1.5.

Linear combinations. If $a_1, \ldots, a_m$ are n-vectors, and $\beta_1, \ldots, \beta_m$ are scalars, the n-vector

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

is called a *linear combination* of the vectors $a_1, \ldots, a_n$. The scalars $\beta_1, \ldots, \beta_m$ are called the *coefficients* of the linear combination.

Linear combination of unit vectors. We can write any n-vector b as a linear combination of the standard unit vectors, as

$$b = b_1 e_1 + \cdots + b_n e_n. \tag{1.1}$$

In this equation b_i is the ith entry of b (*i.e.*, a scalar), and e_i is the ith unit vector. In the linear combination of $e_1, \ldots, e_n$ given in (1.1), the coefficients are the entries of the vector b. A specific example is

$$\begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Special linear combinations. Some linear combinations of the vectors $a_1, \ldots, a_m$ have special names. For example, the linear combination with $\beta_1 = \cdots = \beta_m = 1$, given by $a_1 + \cdots + a_m$, is the *sum* of the vectors, and the linear combination with $\beta_1 = \cdots = \beta_m = 1/m$, given by $(1/m)(a_1 + \cdots + a_m)$, is the *average* of the vectors. When the coefficients sum to one, *i.e.*, $\beta_1 + \cdots + \beta_m = 1$, the linear combination is called an *affine combination*. When the coefficients in an affine combination are nonnegative, it is called a *convex combination*, a *mixture*, or a *weighted average*. The coefficients in an affine or convex combination are sometimes given as percentages, which add up to 100%.

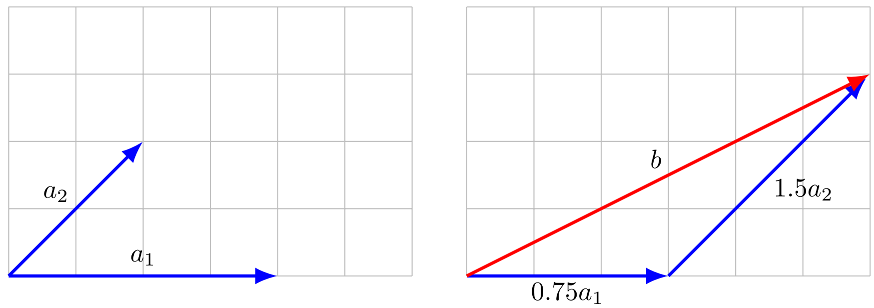

Figure 1.11 *Left.* Two 2-vectors a_1 and a_2. *Right.* The linear combination $b = 0.75a_1 + 1.5a_2$.

Examples.

- *Displacements.* When the vectors represent displacements, a linear combination is the sum of the scaled displacements. This is illustrated in figure 1.11.
- *Audio mixing.* When $a_1, \ldots, a_m$ are vectors representing audio signals (over the same period of time, for example, simultaneously recorded), they are called *tracks*. The linear combination $\beta_1 a_1 + \cdots + \beta_m a_m$ is perceived as a mixture (also called a *mix*) of the audio tracks, with relative loudness given by $|\beta_1|, \ldots, |\beta_m|$. A producer in a studio, or a sound engineer at a live show, chooses values of $\beta_1, \ldots, \beta_m$ to give a good balance between the different instruments, vocals, and drums.
- *Cash flow replication.* Suppose that $c_1, \ldots, c_m$ are vectors that represent cash flows, such as particular types of loans or investments. The linear combination $f = \beta_1 c_1 + \cdots + \beta_m c_m$ represents another cash flow. We say that the cash flow f has been *replicated* by the (linear combination of the) original cash flows $c_1, \ldots, c_m$. As an example, $c_1 = (1, -1.1, 0)$ represents a \$1 loan from period 1 to period 2 with 10% interest, and $c_2 = (0, 1, -1.1)$ represents a \$1 loan from period 2 to period 3 with 10% interest. The linear combination

  $$d = c_1 + 1.1c_2 = (1, 0, -1.21)$$

  represents a two period loan of \$1 in period 1, with compounded 10% interest. Here we have replicated a two period loan from two one period loans.
- *Line and segment.* When a and b are different n-vectors, the affine combination $c = (1 - \theta)a + \theta b$, where θ is a scalar, describes a point on the *line* passing through a and b. When $0 \leq \theta \leq 1$, c is a convex combination of a and b, and is said to lie on the *segment* between a and b. For $n = 2$ and $n = 3$, with the vectors representing coordinates of 2-D or 3-D points, this agrees with the usual geometric notion of line and segment. But we can also talk about the line passing through two vectors of dimension 100. This is illustrated in figure 1.12.

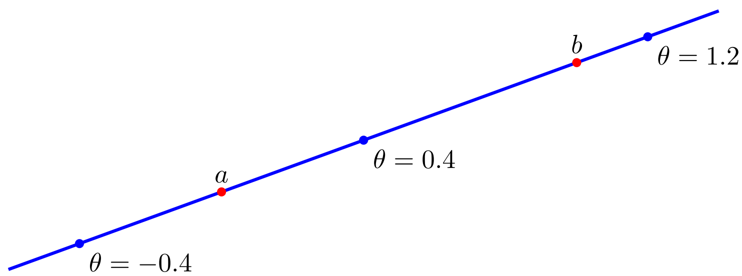

Figure 1.12 The affine combination $(1-\theta)a + \theta b$ for different values of θ. These points are on the line passing through a and b; for θ between 0 and 1, the points are on the line segment between a and b.

1.4 Inner product

The (standard) *inner product* (also called *dot product*) of two n-vectors is defined as the scalar

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

the sum of the products of corresponding entries. (The origin of the superscript 'T' in the inner product notation $a^T b$ will be explained in chapter 6.) Some other notations for the inner product (that we will not use in this book) are $\langle a, b\rangle$, $\langle a|b\rangle$, (a, b), and $a \cdot b$. (In the notation used in this book, (a, b) denotes a stacked vector of length $2n$.) As you might guess, there is also a vector *outer product*, which we will encounter later, in §10.1. As a specific example of the inner product, we have

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = (-1)(1) + (2)(0) + (2)(-3) = -7.$$

When $n = 1$, the inner product reduces to the usual product of two numbers.

Properties. The inner product satisfies some simple properties that are easily verified from the definition. If a, b, and c are vectors of the same size, and γ is a scalar, we have the following.

- *Commutativity.* $a^T b = b^T a$. The order of the two vector arguments in the inner product does not matter.
- *Associativity with scalar multiplication.* $(\gamma a)^T b = \gamma(a^T b)$, so we can write both as $\gamma a^T b$.
- *Distributivity with vector addition.* $(a+b)^T c = a^T c + b^T c$. The inner product can be distributed across vector addition.

These can be combined to obtain other identities, such as $a^T(\gamma b) = \gamma(a^T b)$, or $a^T(b + \gamma c) = a^T b + \gamma a^T c$. As another useful example, we have, for any vectors a, b, c, d of the same size,

$$(a+b)^T(c+d) = a^T c + a^T d + b^T c + b^T d.$$