29.2 Formulating problems as linear programs

Although we shall focus on the simplex algorithm in this chapter, it is also important to be able to recognize when we can formulate a problem as a linear program. Once we cast a problem as a polynomial-sized linear program, we can solve it in polynomial time by the ellipsoid algorithm or interior-point methods. Several linear-programming software packages can solve problems efficiently, so that once the problem is in the form of a linear program, such a package can solve it.

We shall look at several concrete examples of linear-programming problems. We start with two problems that we have already studied: the single-source shortest-paths problem (see Chapter 24) and the maximum-flow problem (see Chapter 26). We then describe the minimum-cost-flow problem. Although the minimum-cost-flow problem has a polynomial-time algorithm that is not based on linear programming, we won't describe the algorithm. Finally, we describe the multicommodity-flow problem, for which the only known polynomial-time algorithm is based on linear programming.

When we solved graph problems in Part VI, we used attribute notation, such as $v.d$ and $(u, v).f$. Linear programs typically use subscripted variables rather than objects with attached attributes, however. Therefore, when we express variables in linear programs, we shall indicate vertices and edges through subscripts. For example, we denote the shortest-path weight for vertex v not by $v.d$ but by d_v. Similarly, we denote the flow from vertex u to vertex v not by $(u, v).f$ but by f_{uv}. For quantities that are given as inputs to problems, such as edge weights or capacities, we shall continue to use notations such as $w(u, v)$ and $c(u.v)$.

Shortest paths

We can formulate the single-source shortest-paths problem as a linear program. In this section, we shall focus on how to formulate the single-pair shortest-path problem, leaving the extension to the more general single-source shortest-paths problem as Exercise 29.2-3.

In the single-pair shortest-path problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbb{R}$ mapping edges to real-valued weights, a source vertex s, and destination vertex t. We wish to compute the value d_t, which is the weight of a shortest path from s to t. To express this problem as a linear program, we need to determine a set of variables and constraints that define when we have a shortest path from s to t. Fortunately, the Bellman-Ford algorithm does exactly this. When the Bellman-Ford algorithm terminates, it has computed, for each vertex v, a value d_v (using subscript notation here rather than attribute notation) such that for each edge $(u, v) \in E$, we have $d_v \leq d_u + w(u, v)$.

The source vertex initially receives a value $d_s = 0$, which never changes. Thus we obtain the following linear program to compute the shortest-path weight from s to t:

$$\begin{aligned}
&\text{maximize} && d_t && && && (29.44)\\
&\text{subject to} \\
& && d_v &\le\ & d_u + w(u,v) && \text{for each edge } (u,v) \in E\ , && (29.45)\\
& && d_s &=\ & 0\ . && && (29.46)
\end{aligned}$$

You might be surprised that this linear program maximizes an objective function when it is supposed to compute shortest paths. We do not want to minimize the objective function, since then setting $\bar{d}_v = 0$ for all $v \in V$ would yield an optimal solution to the linear program without solving the shortest-paths problem. We maximize because an optimal solution to the shortest-paths problem sets each $\bar{d}_v$ to $\min_{u:(u,v)\in E}\left\{\bar{d}_u + w(u,v)\right\}$, so that $\bar{d}_v$ is the largest value that is less than or equal to all of the values in the set $\left\{\bar{d}_u + w(u,v)\right\}$. We want to maximize d_v for all vertices v on a shortest path from s to t subject to these constraints on all vertices v, and maximizing d_t achieves this goal.

This linear program has $|V|$ variables d_v, one for each vertex $v \in V$. It also has $|E| + 1$ constraints: one for each edge, plus the additional constraint that the source vertex's shortest-path weight always has the value 0.

Maximum flow

Next, we express the maximum-flow problem as a linear program. Recall that we are given a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \ge 0$, and two distinguished vertices: a source s and a sink t. As defined in Section 26.1, a flow is a nonnegative real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the capacity constraint and flow conservation. A maximum flow is a flow that satisfies these constraints and maximizes the flow value, which is the total flow coming out of the source minus the total flow into the source. A flow, therefore, satisfies linear constraints, and the value of a flow is a linear function. Recalling also that we assume that $c(u, v) = 0$ if $(u, v) \notin E$ and that there are no antiparallel edges, we can express the maximum-flow problem as a linear program:

$$\begin{aligned}
&\text{maximize} && \sum_{v\in V} f_{sv} && - && \sum_{v\in V} f_{vs} && && (29.47)\\
&\text{subject to} \\
& && f_{uv} &\le\ & c(u,v) && \text{for each } u, v \in V\ , && (29.48)\\
& && \sum_{v\in V} f_{vu} &=\ & \sum_{v\in V} f_{uv} && \text{for each } u \in V - \{s,t\}\ , && (29.49)\\
& && f_{uv} &\ge\ & 0 && \text{for each } u, v \in V\ . && (29.50)
\end{aligned}$$

This linear program has $|V|^2$ variables, corresponding to the flow between each pair of vertices, and it has $2|V|^2 + |V| - 2$ constraints.

It is usually more efficient to solve a smaller-sized linear program. The linear program in (29.47)–(29.50) has, for ease of notation, a flow and capacity of 0 for each pair of vertices u, v with $(u, v) \notin E$. It would be more efficient to rewrite the linear program so that it has $O(V + E)$ constraints. Exercise 29.2-5 asks you to do so.

Minimum-cost flow

In this section, we have used linear programming to solve problems for which we already knew efficient algorithms. In fact, an efficient algorithm designed specifically for a problem, such as Dijkstra's algorithm for the single-source shortest-paths problem, or the push-relabel method for maximum flow, will often be more efficient than linear programming, both in theory and in practice.

The real power of linear programming comes from the ability to solve new problems. Recall the problem faced by the politician in the beginning of this chapter. The problem of obtaining a sufficient number of votes, while not spending too much money, is not solved by any of the algorithms that we have studied in this book, yet we can solve it by linear programming. Books abound with such real-world problems that linear programming can solve. Linear programming is also particularly useful for solving variants of problems for which we may not already know of an efficient algorithm.

Consider, for example, the following generalization of the maximum-flow problem. Suppose that, in addition to a capacity $c(u, v)$ for each edge (u, v), we are given a real-valued cost $a(u, v)$. As in the maximum-flow problem, we assume that $c(u, v) = 0$ if $(u, v) \notin E$, and that there are no antiparallel edges. If we send f_{uv} units of flow over edge (u, v), we incur a cost of $a(u, v) f_{uv}$. We are also given a flow demand d. We wish to send d units of flow from s to t while minimizing the total cost $\sum_{(u,v)\in E} a(u, v) f_{uv}$ incurred by the flow. This problem is known as the ***minimum-cost-flow problem***.

Figure 29.3(a) shows an example of the minimum-cost-flow problem. We wish to send 4 units of flow from s to t while incurring the minimum total cost. Any particular legal flow, that is, a function f satisfying constraints (29.48)–(29.49), incurs a total cost of $\sum_{(u,v)\in E} a(u, v) f_{uv}$. We wish to find the particular 4-unit flow that minimizes this cost. Figure 29.3(b) shows an optimal solution, with total cost $\sum_{(u,v)\in E} a(u, v) f_{uv} = (2 \cdot 2) + (5 \cdot 2) + (3 \cdot 1) + (7 \cdot 1) + (1 \cdot 3) = 27$.

There are polynomial-time algorithms specifically designed for the minimum-cost-flow problem, but they are beyond the scope of this book. We can, however, express the minimum-cost-flow problem as a linear program. The linear program looks similar to the one for the maximum-flow problem with the additional con-

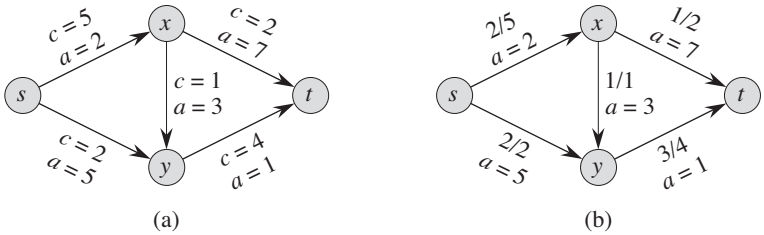

Figure 29.3 **(a)** An example of a minimum-cost-flow problem. We denote the capacities by c and the costs by a. Vertex s is the source and vertex t is the sink, and we wish to send 4 units of flow from s to t. **(b)** A solution to the minimum-cost flow problem in which 4 units of flow are sent from s to t. For each edge, the flow and capacity are written as flow/capacity.

straint that the value of the flow be exactly d units, and with the new objective function of minimizing the cost:

$$\text{minimize} \quad \sum_{(u,v)\in E} a(u,v)\,f_{uv} \tag{29.51}$$

subject to

$$\begin{aligned}
f_{uv} &\le c(u,v) && \text{for each } u, v \in V\ , \\
\sum_{v\in V} f_{vu} - \sum_{v\in V} f_{uv} &= 0 && \text{for each } u \in V - \{s,t\}\ , \\
\sum_{v\in V} f_{sv} - \sum_{v\in V} f_{vs} &= d\ , \\
f_{uv} &\ge 0 && \text{for each } u, v \in V\ .
\end{aligned} \tag{29.52}$$

Multicommodity flow

As a final example, we consider another flow problem. Suppose that the Lucky Puck company from Section 26.1 decides to diversify its product line and ship not only hockey pucks, but also hockey sticks and hockey helmets. Each piece of equipment is manufactured in its own factory, has its own warehouse, and must be shipped, each day, from factory to warehouse. The sticks are manufactured in Vancouver and must be shipped to Saskatoon, and the helmets are manufactured in Edmonton and must be shipped to Regina. The capacity of the shipping network does not change, however, and the different items, or ***commodities***, must share the same network.

This example is an instance of a ***multicommodity-flow problem***. In this problem, we are again given a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \ge 0$. As in the maximum-flow problem, we implicitly assume that $c(u, v) = 0$ for $(u, v) \notin E$, and that the graph has no antipar-

allel edges. In addition, we are given k different commodities, $K_1, K_2, \ldots, K_k$, where we specify commodity i by the triple $K_i = (s_i, t_i, d_i)$. Here, vertex s_i is the source of commodity i, vertex t_i is the sink of commodity i, and d_i is the demand for commodity i, which is the desired flow value for the commodity from s_i to t_i. We define a flow for commodity i, denoted by f_i, (so that f_{iuv} is the flow of commodity i from vertex u to vertex v) to be a real-valued function that satisfies the flow-conservation and capacity constraints. We now define f_{uv}, the ***aggregate flow***, to be the sum of the various commodity flows, so that $f_{uv} = \sum_{i=1}^{k} f_{iuv}$. The aggregate flow on edge (u, v) must be no more than the capacity of edge (u, v). We are not trying to minimize any objective function in this problem; we need only determine whether such a flow exists. Thus, we write a linear program with a "null" objective function:

$$
\begin{aligned}
&\text{minimize} && 0 && && \\
&\text{subject to} && && && \\
& && \sum_{i=1}^{k} f_{iuv} && \le && c(u,v) && \text{for each } u, v \in V \ , \\
& \sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} && && = && 0 && \text{for each } i = 1, 2, \ldots, k \text{ and} \\
& && && && && \text{for each } u \in V - \{s_i, t_i\} \ , \\
& \sum_{v \in V} f_{i,s_i,v} - \sum_{v \in V} f_{i,v,s_i} && && = && d_i && \text{for each } i = 1, 2, \ldots, k \ , \\
& && f_{iuv} && \ge && 0 && \text{for each } u, v \in V \text{ and} \\
& && && && && \text{for each } i = 1, 2, \ldots, k \ .
\end{aligned}
$$

The only known polynomial-time algorithm for this problem expresses it as a linear program and then solves it with a polynomial-time linear-programming algorithm.

Exercises

29.2-1

Put the single-pair shortest-path linear program from (29.44)–(29.46) into standard form.

29.2-2

Write out explicitly the linear program corresponding to finding the shortest path from node s to node y in Figure 24.2(a).

29.2-3

In the single-source shortest-paths problem, we want to find the shortest-path weights from a source vertex s to all vertices $v \in V$. Given a graph G, write a