University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

Chris Ketelsen

# Graphs
# Connectedness and Eulerian Tours

University of Colorado **Boulder**

Department of Computer Science
CSCI 2824: Discrete Structures
Chris Ketelsen

# Graphs
# Connectedness and Eulerian Tours

# Walks, Paths, and Eulerian Tours

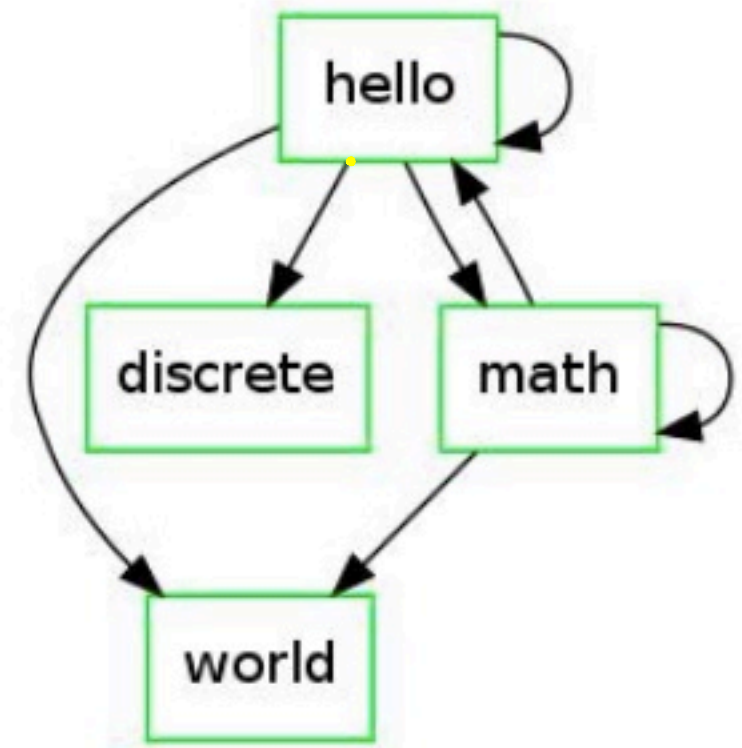
**Def**: Let $G = (V, E)$ be a graph (directed, or undirected). A *walk* of a graph is a sequence of alternating vertices and edges such that

- we start on any vertex and end on any vertex
- a single step in the walk proceeds upon an outgoing edge from the current vertex

**Note**: A walk has to respect edge direction in a directed graph. In an undirected graph it doesn't matter.

# Walks, Paths, and Eulerian Tours

**Example:** Consider the directed graph below




**Valid Walk:** $hello \to hello \to math \to hello \to discrete$

**Valid Walk:** $hello \to hello \to math \to math$

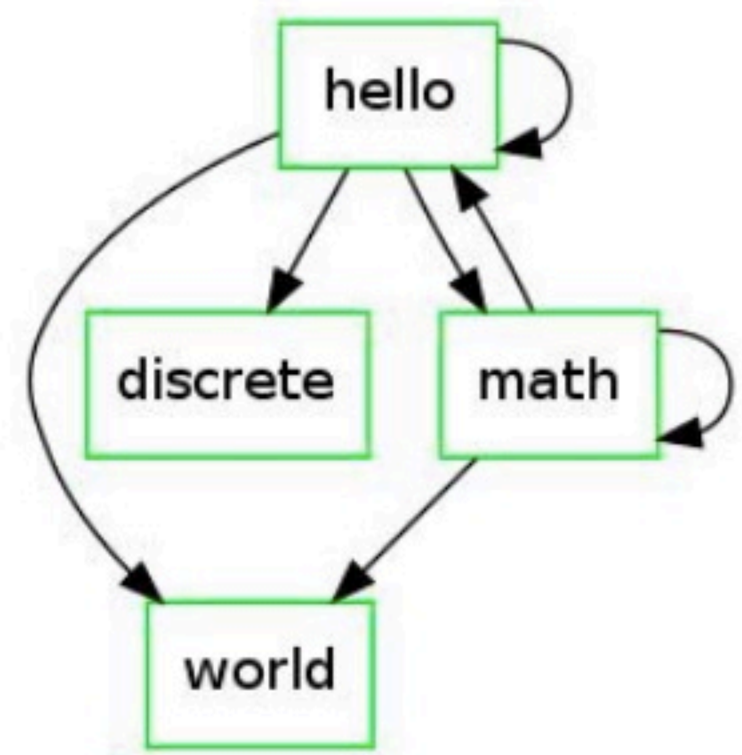
**Valid Walk:** $hello$

**Invalid Walk:** $hello \to discrete \to world \to world$

## Walks, Paths, and Eulerian Tours

**Def**: Let $G = (V, E)$ be a graph (directed, or undirected). A *path* of a graph is a walk in which no vertex is repeated. The *length* of a path is the number of edges that it traverses.

**Example**: Consider the directed graph below



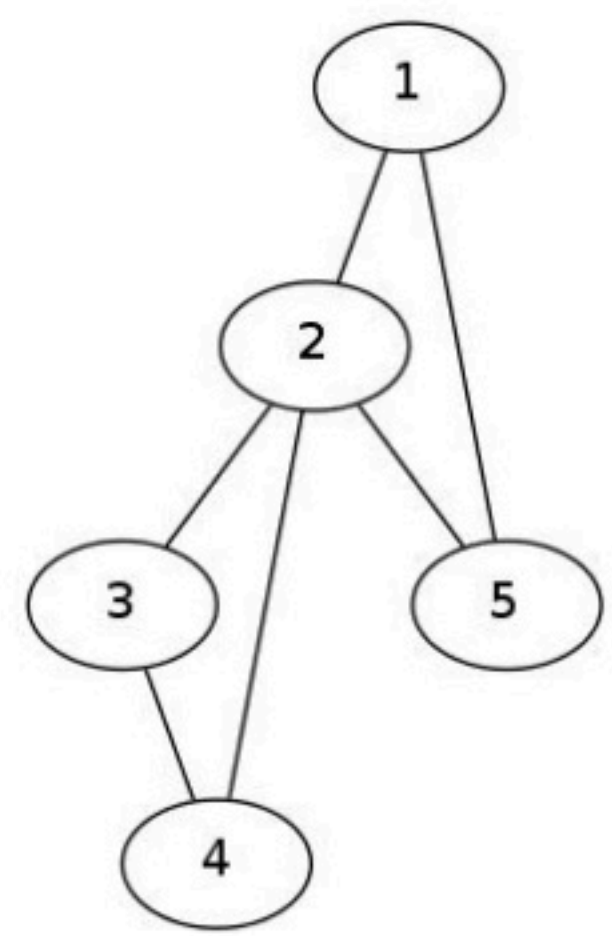

**Valid Path**: $math \to hello \to discrete$ (path of length 2)

**Invalid Path**: $math \to hello \to hello \to math$

# Connectedness

**Def**: An undirected graph $G$ is called **connected** if there is a path between every pair of distinct vertices in the graph. An undirected graph is called **disconnected** if it is not connected.

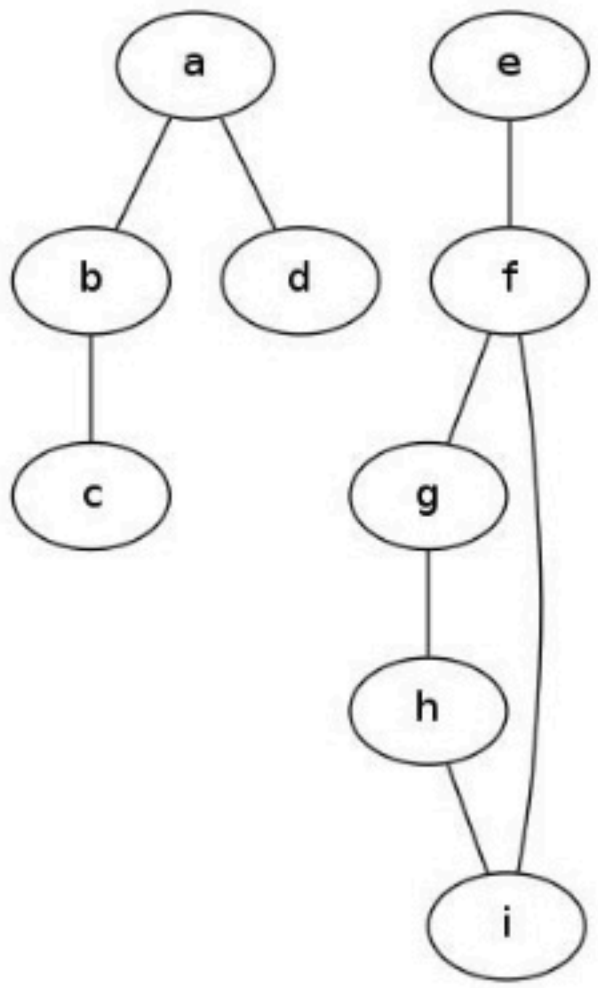
**Example**: The following graph is connected.



# Connectedness

**Def**: An undirected graph $G$ is called **connected** if there is a path between every pair of distinct vertices in the graph. An undirected graph is called **disconnected** if it is not connected.

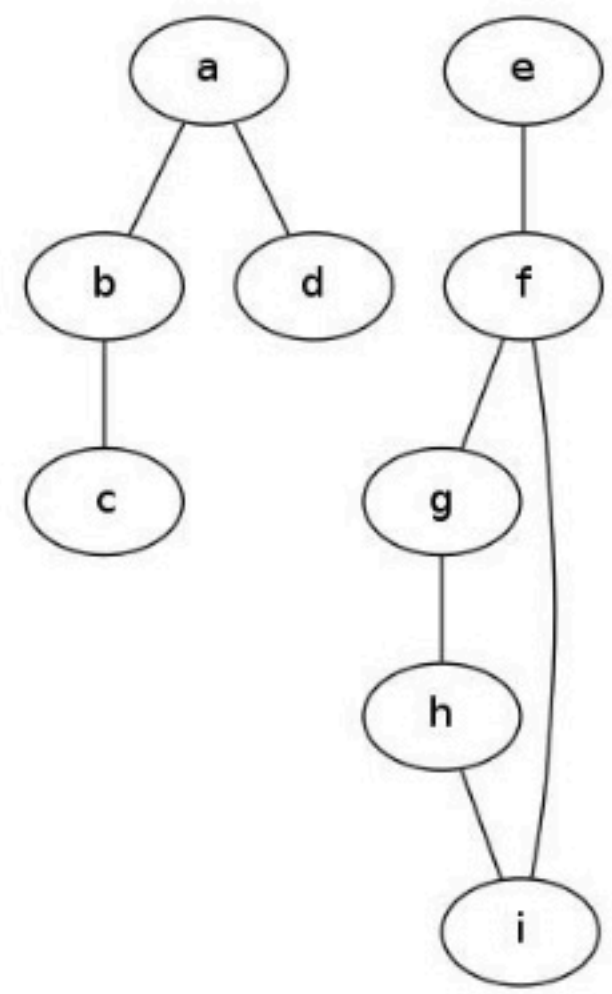
**Example**: The following graph is disconnected



# Connectedness

**Def**: A subgraph of a disconnected graph $G$ is called a **connected component** of $G$ if it is a maximal connected subgraph of $G$.

**Example**: $\{a, b, c, d\}$ and $\{e, f, g, h, i\}$ are connected components

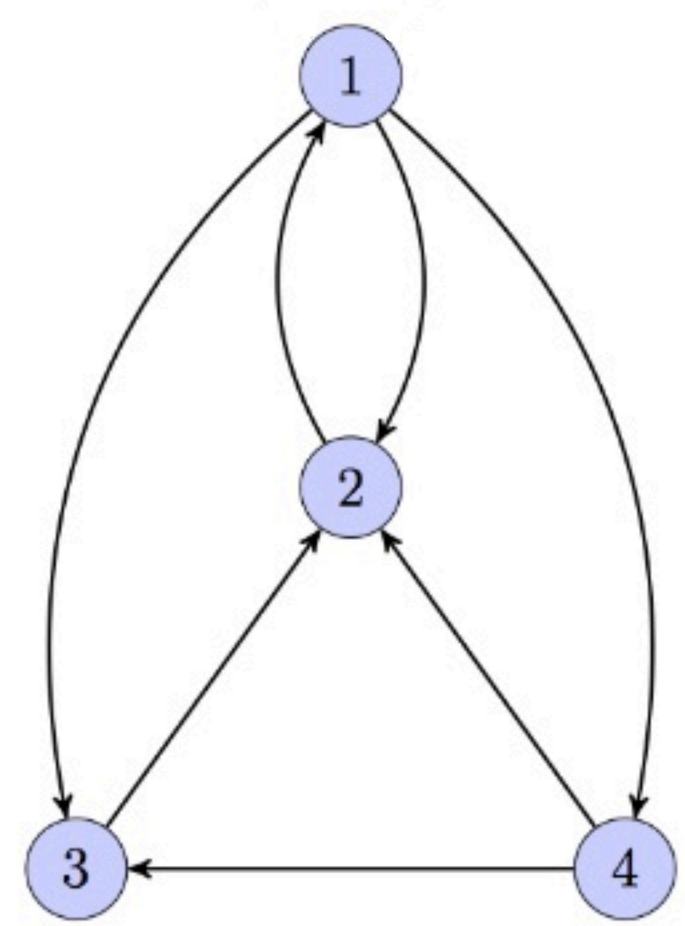



**Example**: But $\{a, b, c\}$ isn't because it's not maximal

# Connectedness in Digraphs

Connectedness is directed graphs is a little trickier. We define two notions of connectedness.

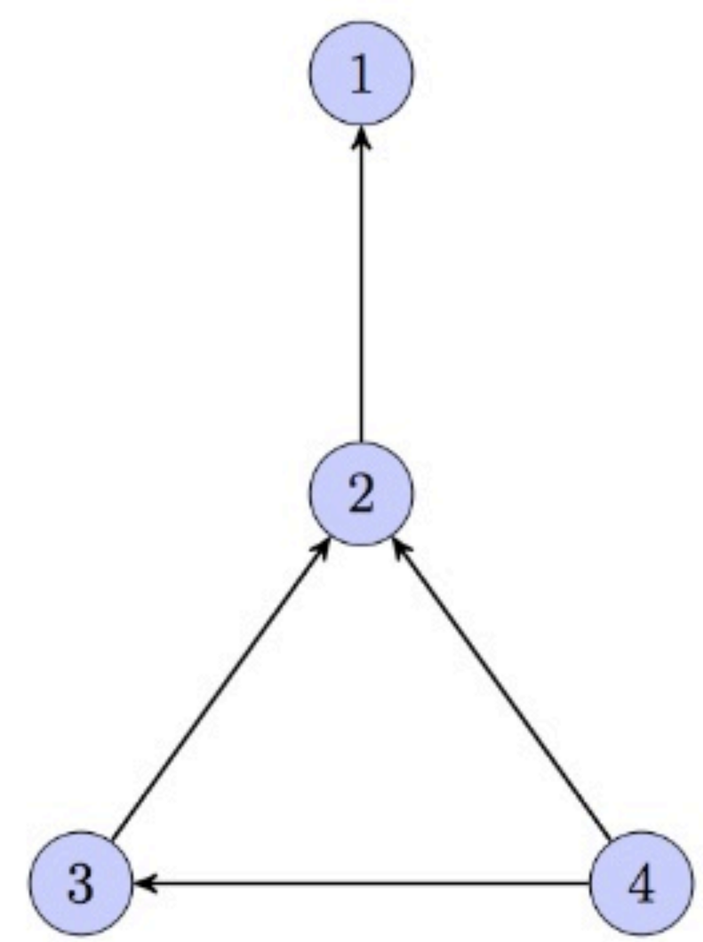
**Def**: A digraph $G$ is called **strongly connected** if for each pair of distinct vertices $a$ and $b$ there is a path from $a$ to $b$ **AND** a path from $b$ to $a$



# Connectedness in Digraphs

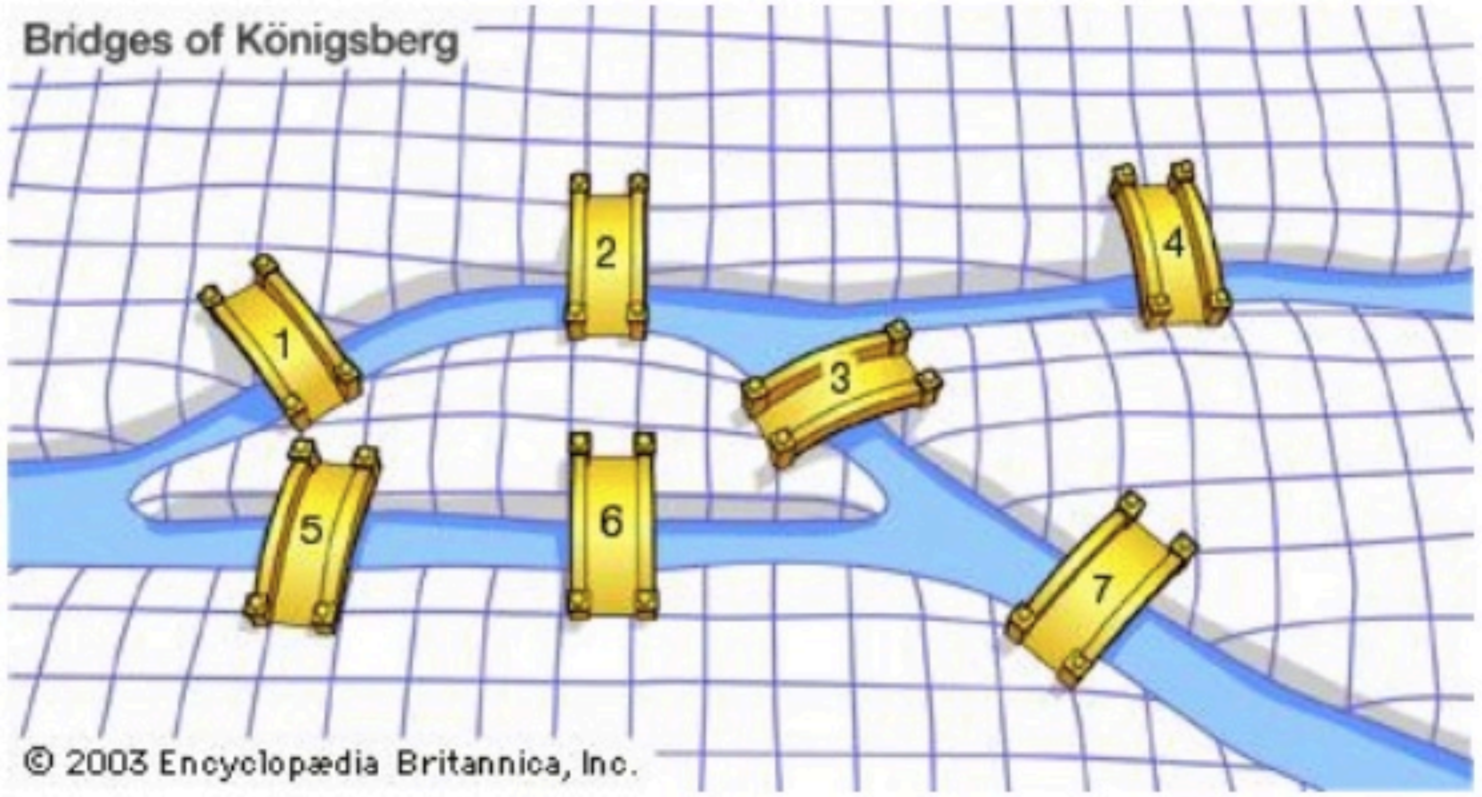
Connectedness is directed graphs is a little trickier. We define two notions of connectedness.

**Def**: A digraph $G$ is called **weakly connected** if for each pair of distinct vertices $a$ and $b$ there is a path from $a$ to $b$ **OR** a path from $b$ to $a$



# Walks, Paths, and Eulerian Tours

**Puzzle** - **The Seven Bridges of Konigsberg:** In the city of Konigsberg there are two islands formed by a river with seven bridges connecting the islands to the city mainland. Can you devise a walk that traverse each bridge exactly once and starts and ends in the same place?



# Walks, Paths, and Eulerian Tours

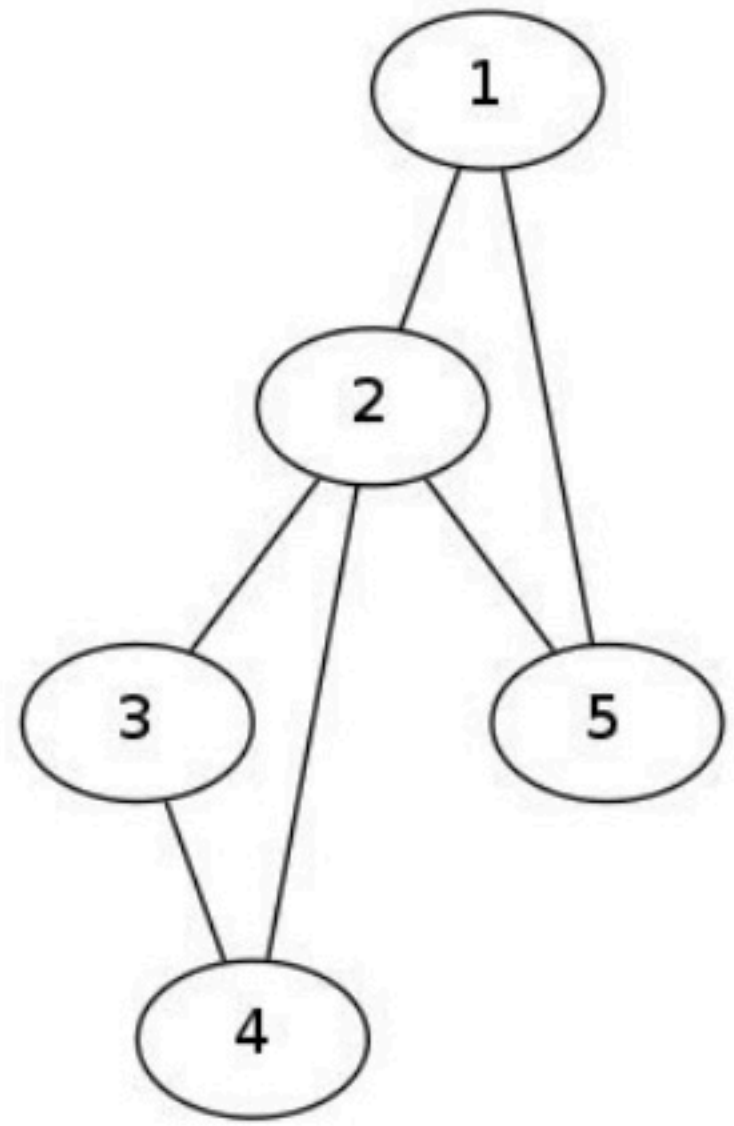
**Def**: Let $G = (V, E)$ be a graph (directed, or undirected). An *Eulerian Tour* of a graph is a special walk which starts and ends at the same vertex and traverses each edge exactly one time.

**Let's Summarize the Terminology**:

- A **walk** is literally anything
- A **path** does not repeated vertices ($\Rightarrow$ or edges [why?])
- An **Eulerian Tour** does not repeat edges (and starts and ends at same vertex)

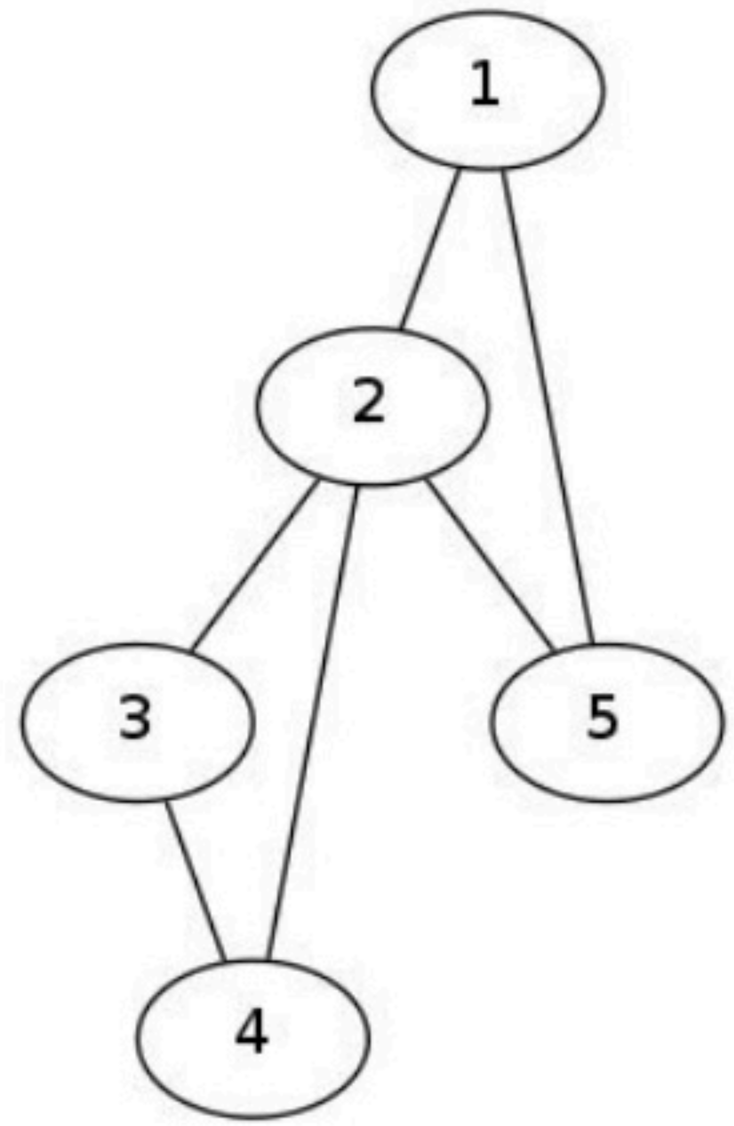
# Walks, Paths, and Eulerian Tours

**Example:** Is there an Eulerian Tour in the following graph?



# Walks, Paths, and Eulerian Tours

**Example:** Is there an Eulerian Tour in the following graph?

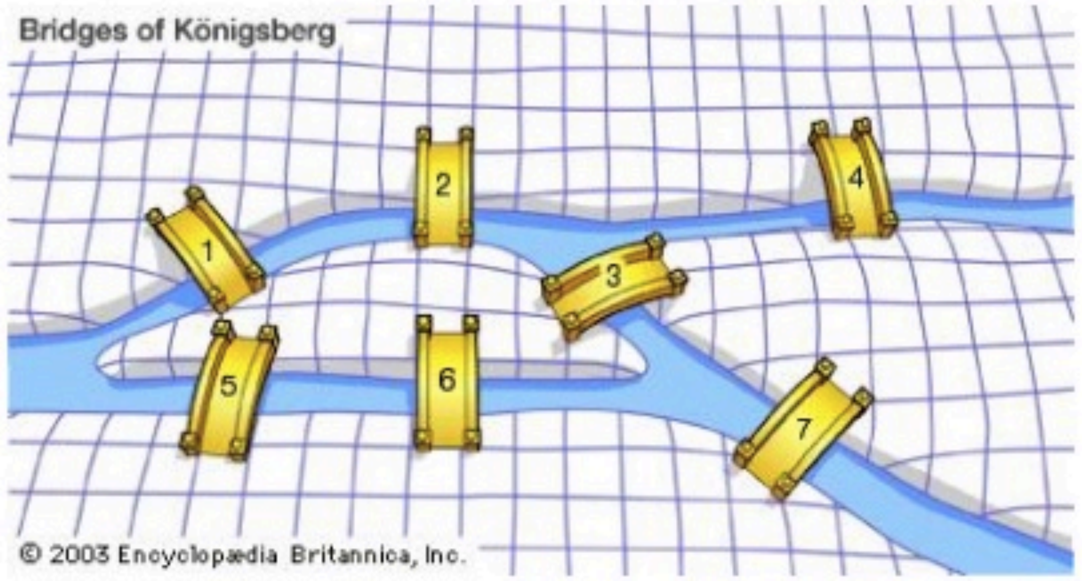



$$2 \rightarrow 1 \rightarrow 5 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 2$$

**Yes!**

# Walks, Paths, and Eulerian Tours

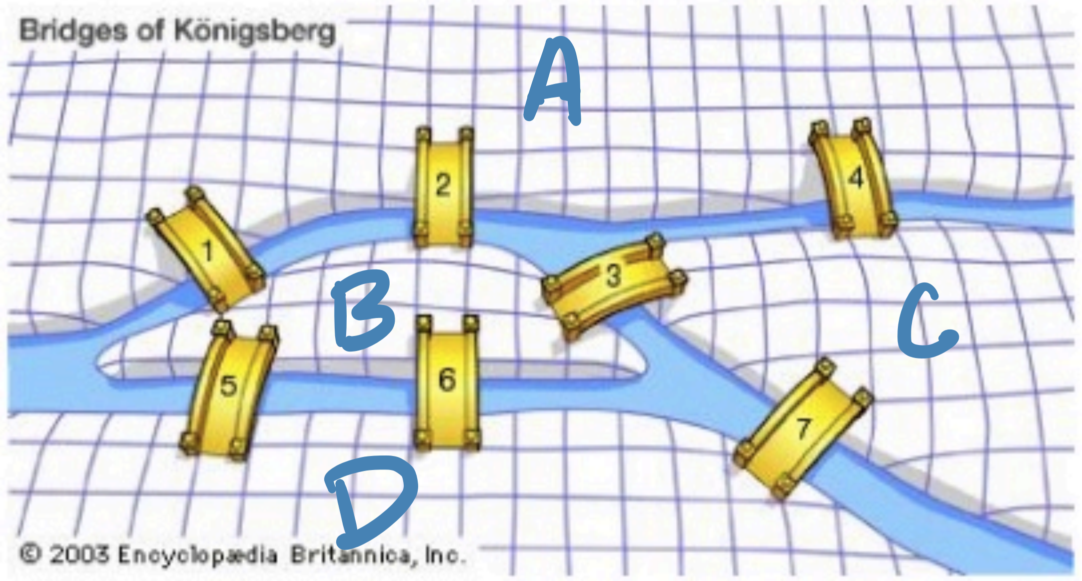
## Seven Bridges of Koenigsberg



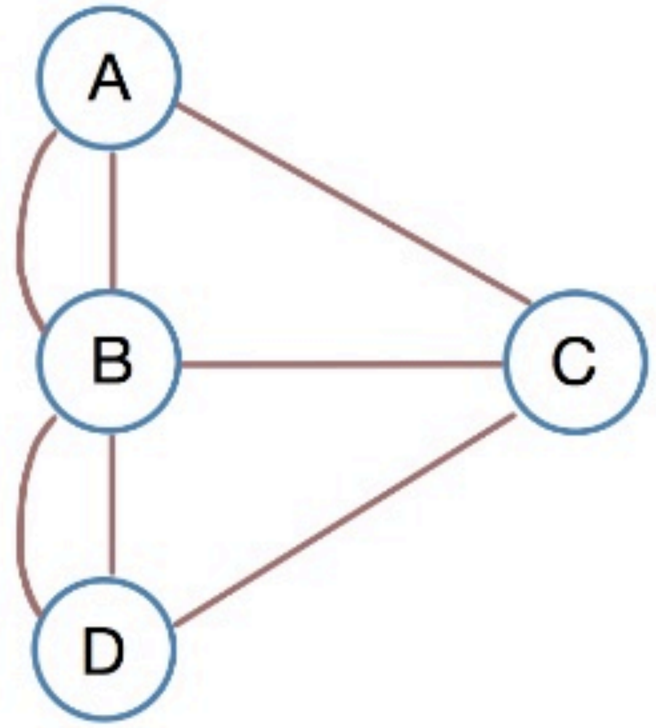

Let's represent this as a graph and see if it has an Eulerian Tour

# Walks, Paths, and Eulerian Tours

## Seven Bridges of Koenigsberg

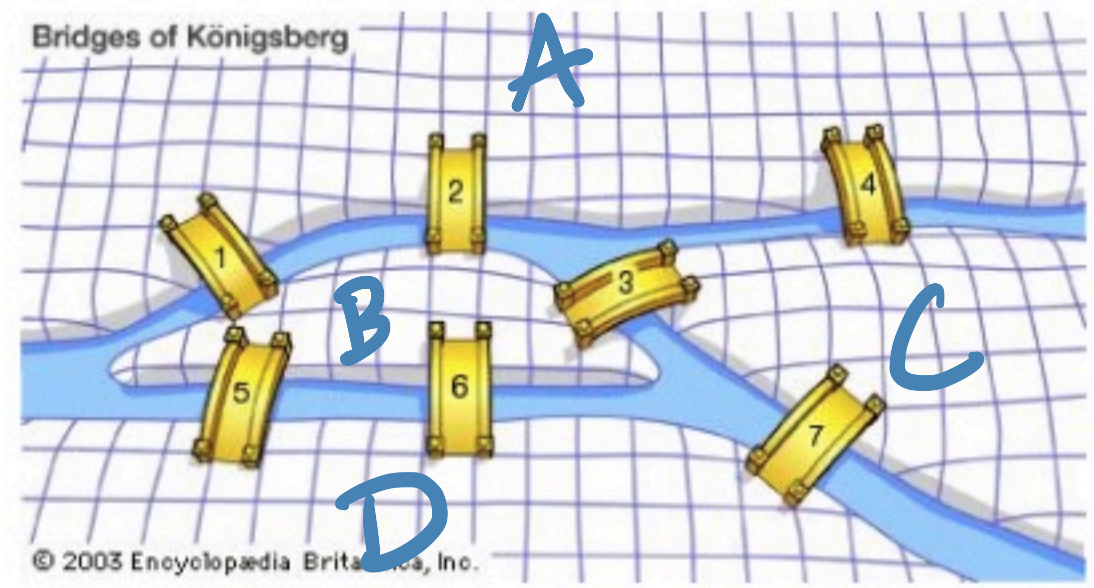



Here's one way. This is called a *multigraph* pairs of vertices have multiple edges connecting them. Let's make it into a simple graph

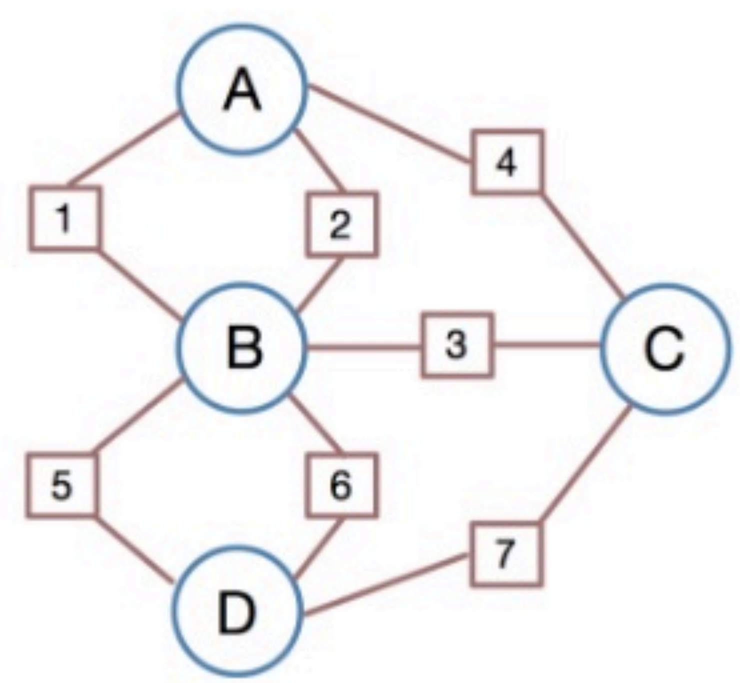


# Walks, Paths, and Eulerian Tours

## Seven Bridges of Koenigsberg

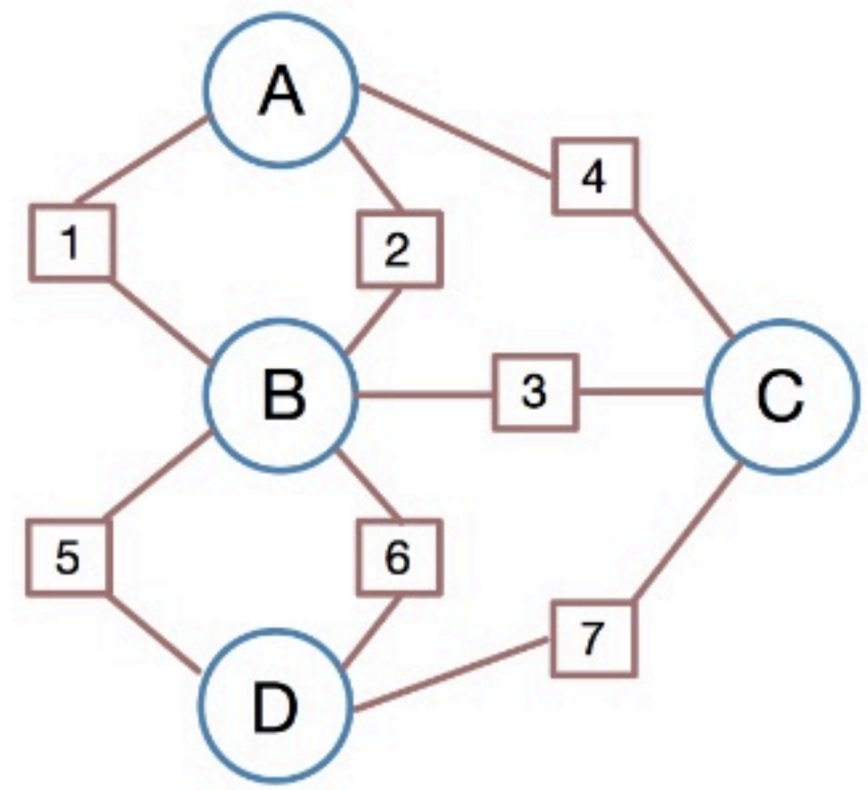



Here's one way. This is called a *multigraph* pairs of vertices have multiple edges connecting them. Let's make it into a simple graph



# Walks, Paths, and Eulerian Tours

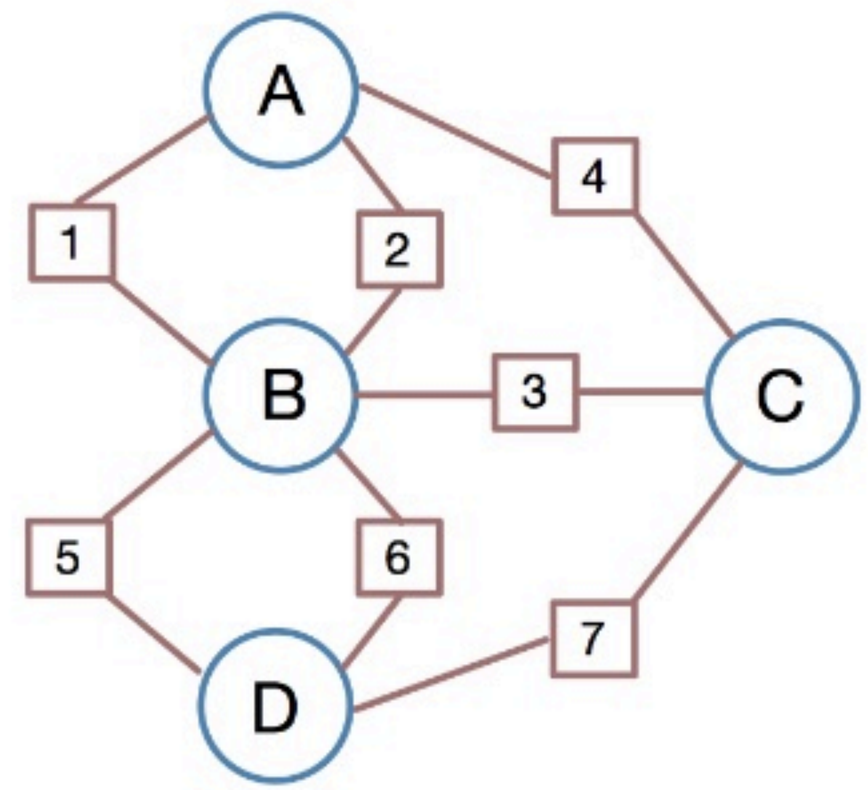
## Seven Bridges of Koenigsberg




OK, does *this* graph have an Eulerian Tour?

# Walks, Paths, and Eulerian Tours

## Seven Bridges of Koenigsberg

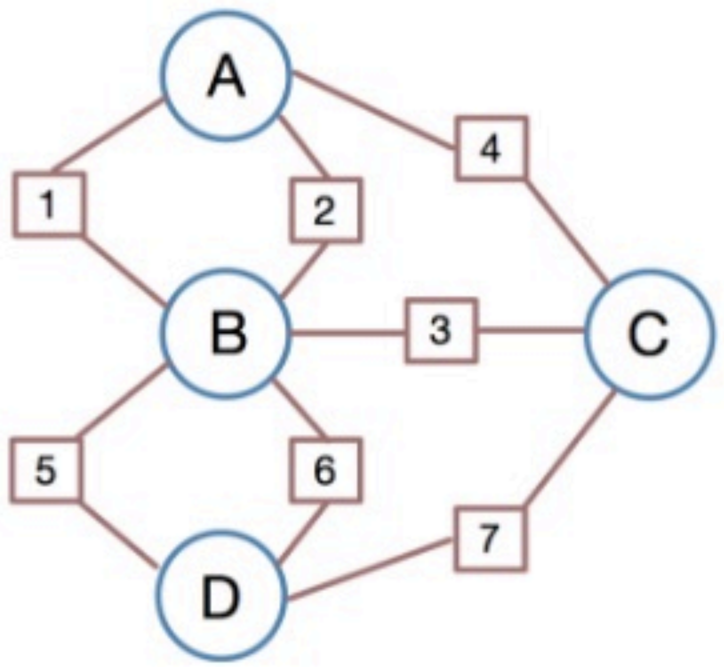



OK, does *this* graph have an Eulerian Tour?

**Nope!** And here's why ...

# Walks, Paths, and Eulerian Tours

## Seven Bridges of Koenigsberg

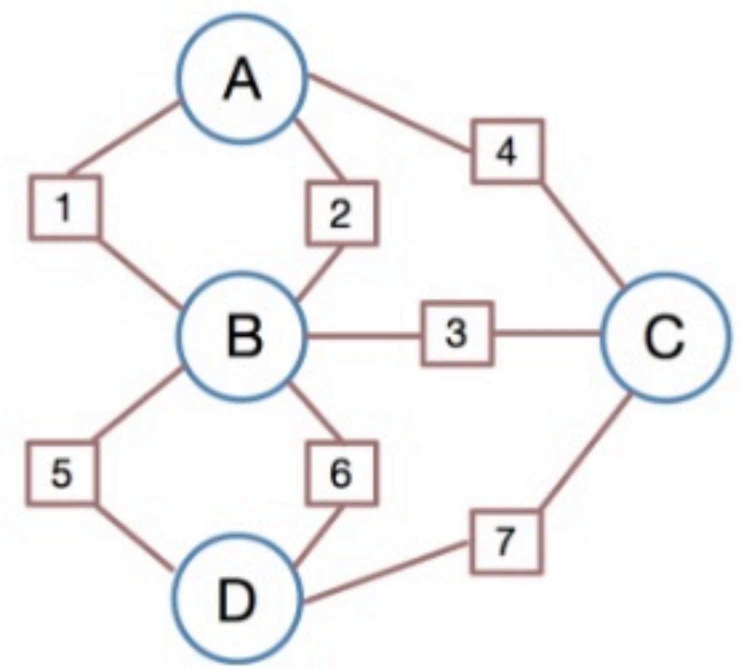



Suppose we start at vertex $C$

- Suppose we leave on Bridge $4$
- Then eventually come back on Bridge $3$
- We then have to leave on Bridge $7$
- But then there's no way to come back to end on $C$ !

# Walks, Paths, and Eulerian Tours

## Seven Bridges of Koenigsberg



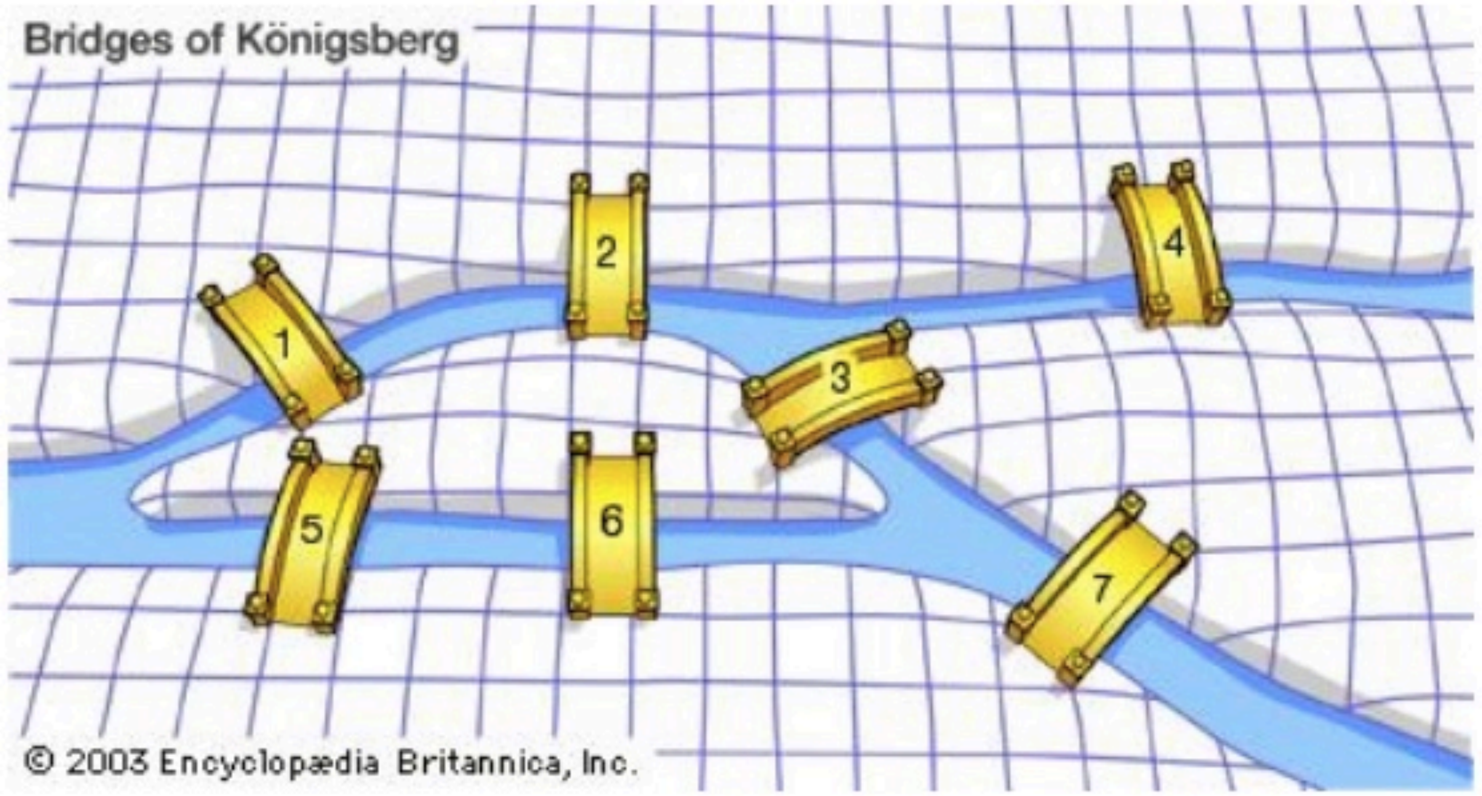

Similarly, if we started somewhere other than vertex $C$

- Suppose we enter $C$ on Bridge $4$
- Then we leave on Bridge $3$
- We then have to come back on Bridge $7$
- But then we're stuck! No more unused edges out of $C$ !

# Walks, Paths, and Eulerian Tours

**Theorem**: A *connected* undirected graph $G$ (with no self-loops) has an Eulerian Tour if and only if each vertex has even degree.

**Seven Bridges of Koenigsberg**: Since the graph has at least one vertex with odd degree (if you check, all of the vertices representing land have odd degree) there cannot be an Eulerian Tour.



# Walks, Paths, and Eulerian Tours

**Theorem**: A *connected* undirected graph $G$ (with no self-loops) has an Eulerian Tour if and only if each vertex has even degree.

**Seven Bridges of Koenigsberg**: Since the graph has at least one vertex with odd degree (if you check, all of the vertices representing land have odd degree) there cannot be an Eulerian Tour.