University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

Chris Ketelsen

Sets and Functions

Sets and Set Operations

Sets

Def: A **set** is an unordered collection of objects, called elements or members of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that A is an element of set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Notation: For sets with a small number of elements, we write the set with it's members inside braces: $\{a, b, c, d\}$

Example: The set V of all vowels: $V =$ ___________

Example: The set of all primes less than 10: $P =$ _________

Example: Thet set of all positive integers between 1 and 200:

$$A = \underline{\hspace{5cm}}$$

Sets

Listing all elements is called the **roster method**. Equally popular is the **builder method**

Example: The set of all positive even integer less than 20

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even and } x < 20\}$$

Note that this looks a lot like a quantified logic statement, but inside braces because it's a set

Sets

Listing all elements is called the **roster method**. Equally popular is the **builder method**

Example: The set of all positive even integer less than 20

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even and } x < 20\}$$

Note that this looks a lot like a quantified logic statement, but inside braces because it's a set

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ - Natural numbers
- $\mathbb{Z} = \{\ldots, -2 - 1, 0, 1, 2, \ldots\}$ - Integers
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ - Positive integers
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R} =$ the set of **real** numbers

Sets

Sets can have pretty much anything in them, even other sets

Example: $A = \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

Note: Sets have no ordering. So $\{1, 2, 3, 4\} = \{4, 2, 3, 1\}$

Note: Repeated elements don't matter: $\{a, b, b, a\} = \{a, b\}$

In [1]:
```
mylist= [4,2,3,1,3,2,1]
myset = set(mylist)
print(myset)
```

```
set([1, 2, 3, 4])
```

Def: Two sets are equal if and only if they have the same elements.
So if A and B are sets, we say A and B are equal if and only if

$$\forall x \; (x \in A \leftrightarrow x \in B)$$

Subsets

Def: The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

Question: Use quantifiers to express the definition of $A \subseteq B$

Finish the Sentence:

Example: The set of all integers, $\mathbb{Z}$, is a subset of ...

Example: The set of all rational numbers, $\mathbb{Q}$, is a subset of ...

Example: The set of all 2824 students is a subset of ...

Subsets

Strategy: To show that $A \subseteq B$ you have to show that every element of A is also in B

Strategy: To show that $A \nsubseteq B$ you have to find just one element in A that is not in B

Question: How do you feel about the following?

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

When we wish to emphasize that $A \subseteq B$ but $A \neq B$ we write $A \subset B$. This says that A is a **proper** subset of B

Example: The set of all even integers is a proper subset of $\mathbb{Z}$

$$\{x \in \mathbb{Z} \mid x \text{ is even}\} \subset \mathbb{Z}$$

Subsets

Venn Diagrams

When thinking about subsets, drawing a picture can help

Example: Draw a Venn Diagram relating vowels to the letters of the alphabet

Example: Draw a Venn Diagram relating prime numbers and odd numbers

The Empty Set

Special Sets

The Empty Set: The set that has no elements, written $\emptyset$ or $\{\,\}$

Every nonempty set has at least two subsets

Theorem: For every set S, $\emptyset \subseteq S$, and $S \subseteq S$

The Power Set

Special Sets

The Power Set: $\mathcal{P}(S)$ is set of all subsets of the set S

Example: $\mathcal{P}(\{0,1,2\}) =$

$$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Question: How many elements does $\mathcal{P}(S)$ contain?

The Power Set

Question: How many elements does $\mathcal{P}(S)$ contain?

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Answer: If S has n elements then $\mathcal{P}(S)$ has 2^n elements

The number of elements in a set is called the sets **cardinality**. If a sets cardinality is a finite number we say the set is **finite**:

Example: The cardinality of the english alphabet is ______

Example: The cardinality of the set of vowels is ______

Example: The cardinality of the set of natural numbers is ______

Cartesian Products

Sometimes we want to talk about things that come from sets, but have a defined order

Example: Think about ordered pairs of integers representing points in the $xy-$plane. $(1, 2)$ is distinctly different from $(2, 1)$.

We can generate all ordered pairs from two sets using the **Cartesian Product**;

Def: Let A and B be sets. The Cartesian product of A and B, written $A \times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

Cartesian Products

Example: What is the Cartesian product of $A = \{Chris, Aly\}$ and $B = \{1, 2, 3\}$?

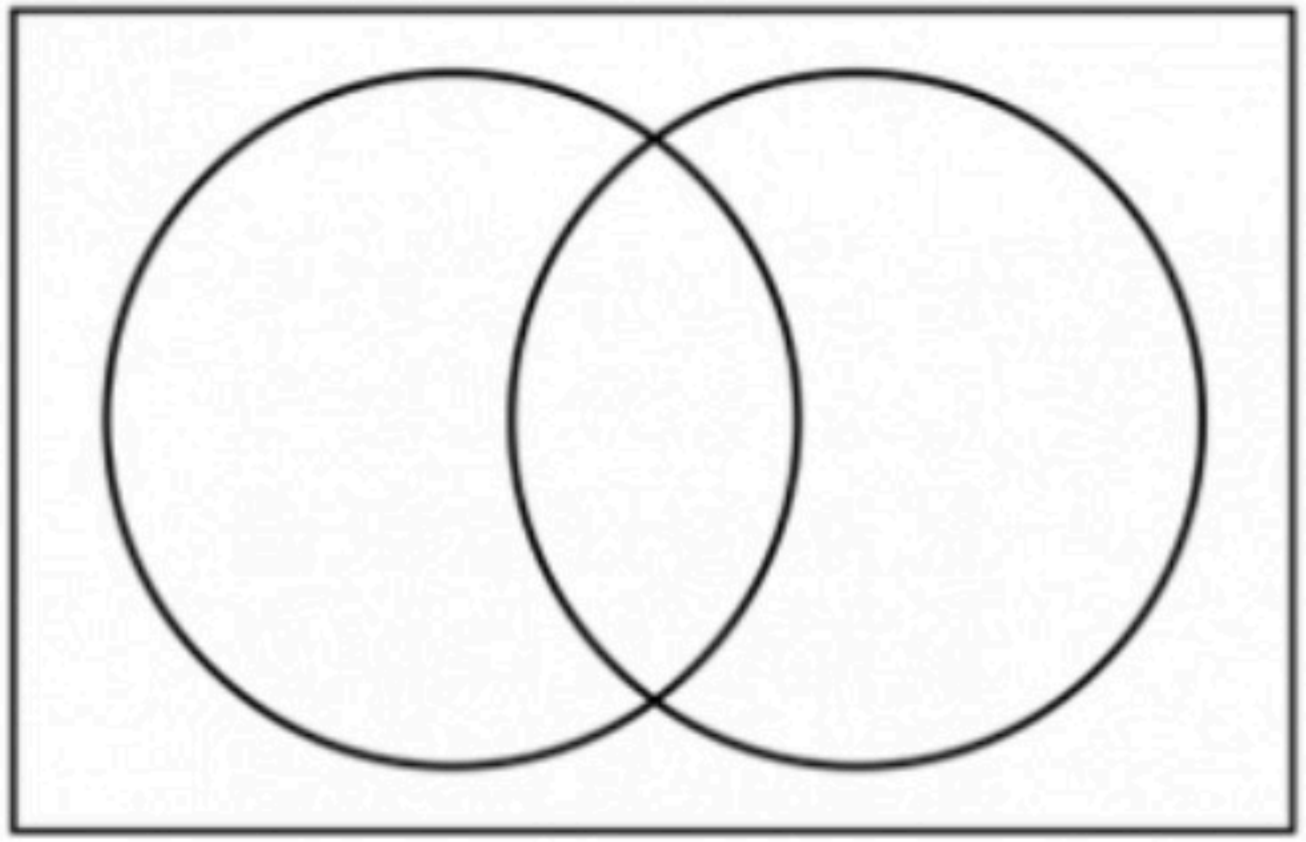
Note: $A \times B \neq B \times A$. Why?

The Union of Sets

Two or more sets can be combined in different ways.

Def: Let A and B be sets. The union of the sets A and B, denoted $A \cup B$, is the set that contains those elements that are either in A or in B, or in both

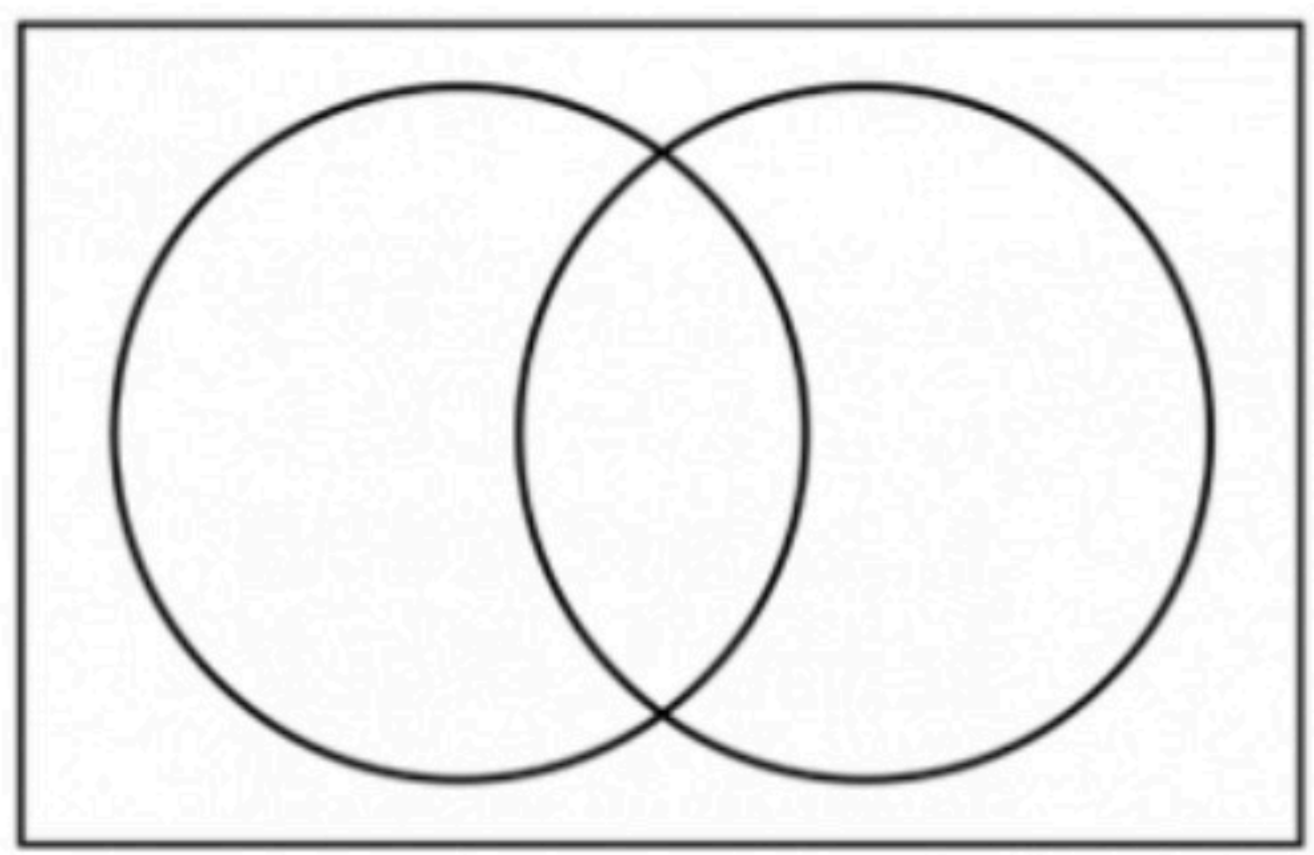
$$A \cup B = \{x \mid x \in A \lor x \in B\}$$



The Union of Sets

Consider the sets of all CS majors and Applied Math majors at CU (sets C and A, respectively)

Example: The union of CS majors and Applied Math majors, $C \cup A$ is the set of all students that are either majoring in CS or in Applied Math

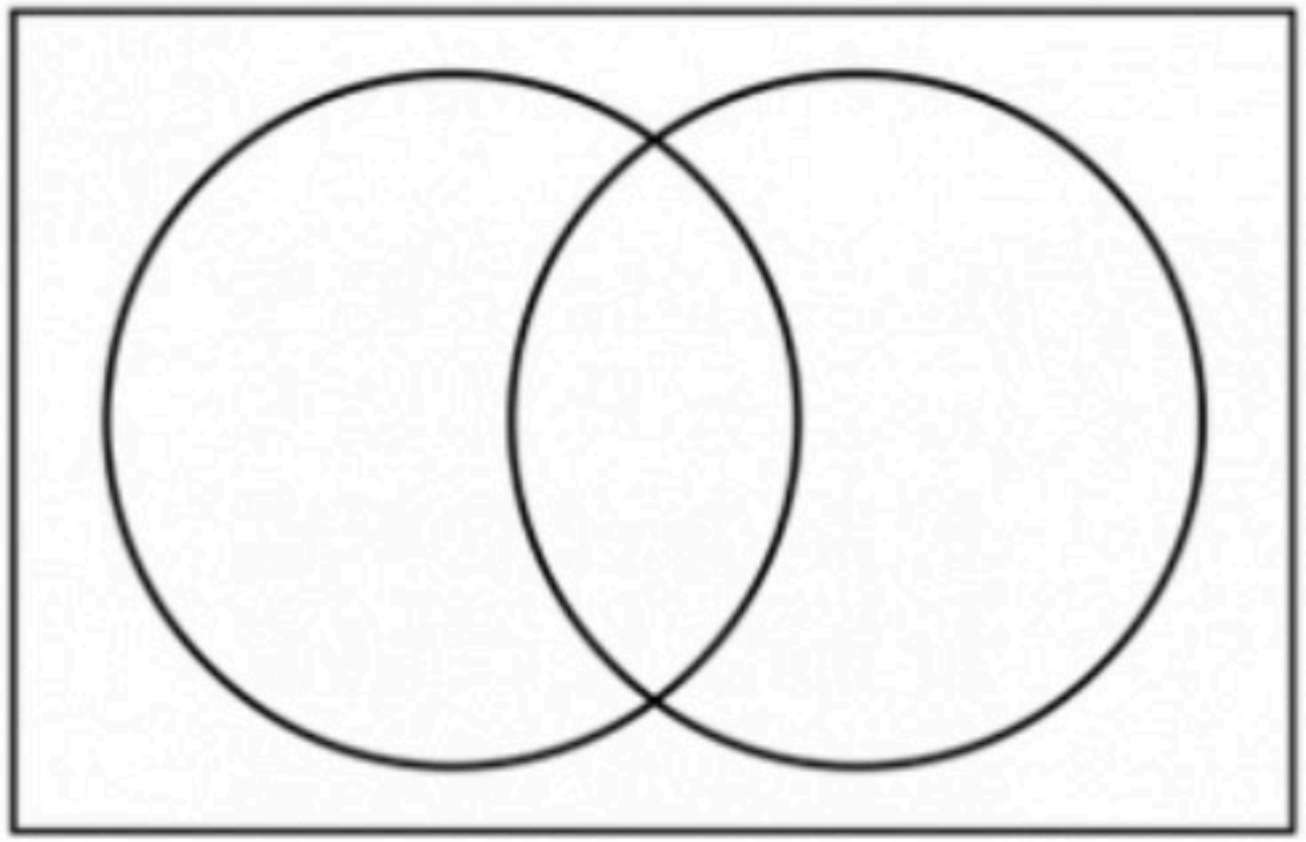


The Intersection of Sets

Def: Let A and B be sets. The **intersection** of the sets A and B, denoted $A \cap B$, is the set containing those elements in both A and B

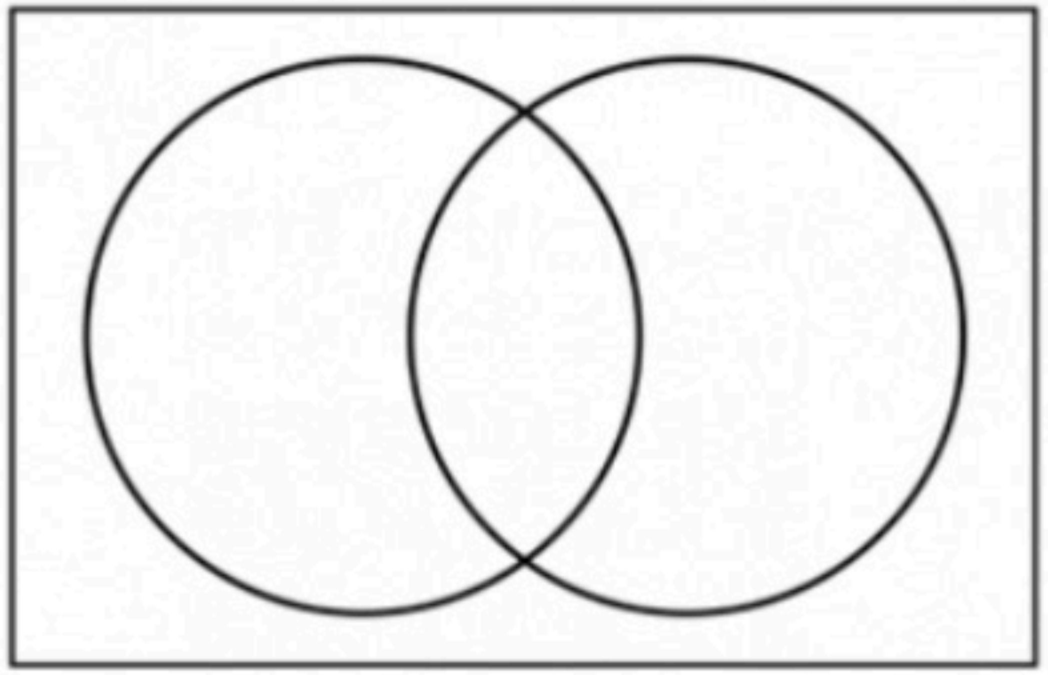
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

Example: Describe the set $A \cap C$ from the previous example



Cardinality

Question: How many elements are there in $A \cup B$?



The Difference of Sets

Def: Let A and B be sets. The difference of A and B, denoted $A - B$ or $A \backslash B$, is the set containing those elements that are in A but not in B.

Question: How could you represent $A - B$ is set builder logic?

Question: What is the difference of the set of positive integers less than 10 and the set of prime numbers?

The Complement of a Set

Last One!

Lots of times we want to talk about the set of all elements under consideration, which we call the universal set, U.

Think of the universal set U as your domain of discourse

Def: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set $U - A$

An element belongs to $\bar{A}$ if and only if $x \notin A$, so

$$\bar{A} = \{x \in U \mid x \notin A\} \text{ or just } \{x \mid x \notin A\}$$

Set Identities

Exercise: Show that $A - B = A \cap \bar{B}$

Strategy: To show that two sets R and S are equal, we show that $R \subseteq S$ and $S \subseteq R$

Proof:

($\Rightarrow$) Let x be an arbitrary element in $A - B$.

Set Identities

Exercise: Show that $A - B = A \cap \bar{B}$

Strategy: To show that two sets R and S are equal, we show that $R \subseteq S$ and $S \subseteq R$

Proof:

($\Rightarrow$) Let x be an arbitrary element in $A - B$.

This means that $x \in A$ and $x \notin B$

But $x \notin B$ implies that x is in the complement of B ($x \in \bar{B}$)

Since $x \in A$ and $x \in \bar{B}$ we know $x \in A \cap \bar{B}$

Since x was any element in $A - B$ we've shown $A - B \subseteq A \cap \bar{B}$

Set Identities

To complete our proof that $A - B = A \cap \bar{B}$ we need to show

$$A \cap \bar{B} \subseteq A - B$$

($\Leftarrow$) Let x be an arbitrary element in $A \cap \bar{B}$

Set Identities

To complete our proof that $A - B = A \cap \bar{B}$ we need to show

$$A \cap \bar{B} \subseteq A - B$$

($\Leftarrow$) Let x be an arbitrary element in $A \cap \bar{B}$

This means that $x \in A$ and $x \in \bar{B}$

But $x \in \bar{B}$ implies that x not in B ($x \notin B$)

Since $x \in A$ and $x \notin B$ we know $x \in A - B$

Since x was any element in $A \cap \bar{B}$ we've shown $A \cap \bar{B} \subseteq A - B$

Since we've shown both $A - B \subseteq A \cap \bar{B}$ and $A \cap \bar{B} \subseteq A - B$

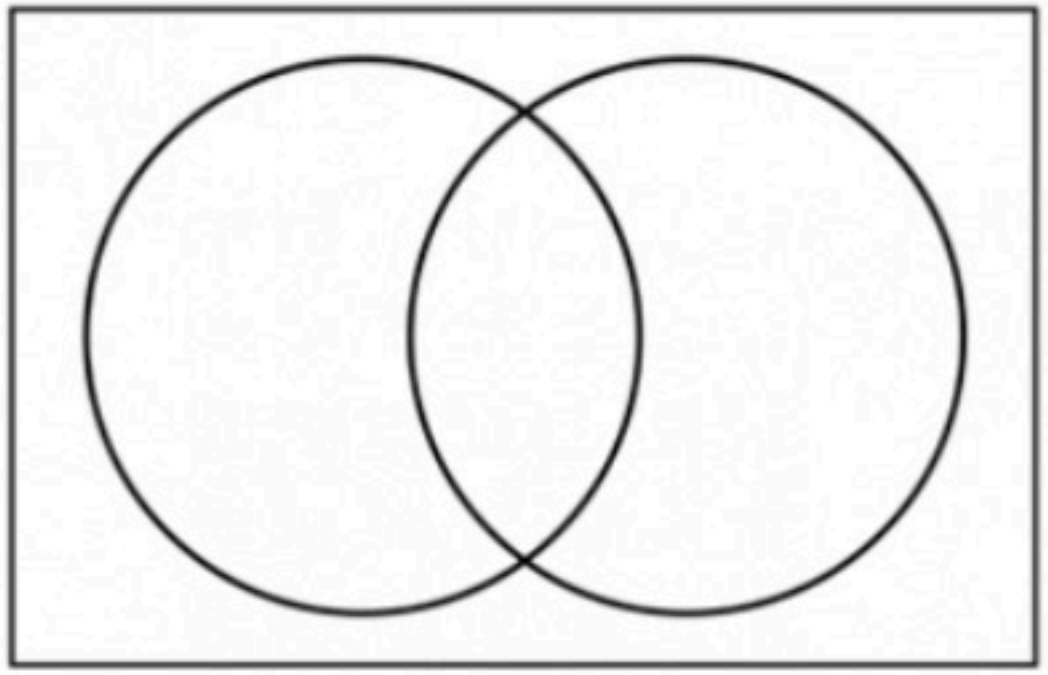
which completes the proof that $A - B = A \cap \bar{B}$

Set Identities

It turns out that when sets are combined using only ∪, ∩, and complements, there is a nice set of Set Identities that completely mirrors the logical equivalences we saw previously in the course.

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Venn Diagram "Proof":



Set Identities

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Set Builder Proof: Using only logical equivalences

$$\overline{A \cap B} \quad = \quad \{x \mid x \notin A \cap B\} \quad \text{(def. complement)}$$

Set Identities

Example: DeMorgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Set Builder Proof: Using only logical equivalences

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cap B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Set Identities

We of course have the other DeMorgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

EFY: Prove this identity using set builder notation

There are loads more Set Identities that mirror logical equivalences. They're summarized in the Table 1 from Section 2.2 of Rosen.

Set Identities

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set Wrap-Up