CHAPTER

7 Discrete Probability

Combinatorics and probability theory share common origins. The theory of probability was first developed more than 300 years ago, when certain gambling games were analyzed. Although probability theory was originally invented to study gambling, it now plays an essential role in a wide variety of disciplines. For example, probability theory is extensively applied in the study of genetics, where it can be used to help understand the inheritance of traits. Of course, probability still remains an extremely popular part of mathematics because of its applicability to gambling, which continues to be an extremely popular human endeavor.

In computer science, probability theory plays an important role in the study of the complexity of algorithms. In particular, ideas and techniques from probability theory are used to determine the average-case complexity of algorithms. Probabilistic algorithms can be used to solve many problems that cannot be easily or practically solved by deterministic algorithms. In a probabilistic algorithm, instead of always following the same steps when given the same input, as a deterministic algorithm does, the algorithm makes one or more random choices, which may lead to different output. In combinatorics, probability theory can even be used to show that objects with certain properties exist. The probabilistic method, a technique in combinatorics introduced by Paul Erdős and Alfréd Rényi, shows that an object with a specified property exists by showing that there is a positive probability that a randomly constructed object has this property. Probability theory can help us answer questions that involve uncertainty, such as determining whether we should reject an incoming mail message as spam based on the words that appear in the message.

7.1 An Introduction to Discrete Probability

Introduction

Probability theory dates back to 1526 when the Italian mathematician, physician, and gambler Girolamo Cardano wrote the first known systematic treatment of the subject in his book *Liber de Ludo Aleae* (*Book on Games of Chance*). (This book was not published until 1663, which may have held back the development of probability theory.) In the seventeenth century the French mathematician Blaise Pascal determined the odds of winning some popular bets based on the outcome when a pair of dice is repeatedly rolled. In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined the probability of an event as the number of successful outcomes divided by the number of possible outcomes. For instance, the probability that a die comes up an odd number when it is rolled is the number of successful outcomes—namely, the number of ways it can come up odd—divided by the number of possible outcomes—namely, the number of different ways the die can come up. There are a total of six possible outcomes—namely, 1, 2, 3, 4, 5, and 6—and exactly three of these are successful outcomes—namely, 1, 3, and 5. Hence, the probability that the die comes up an odd number is $3/6 = 1/2$. (Note that it has been assumed that all possible outcomes are equally likely, or, in other words, that the die is fair.)

In this section we will restrict ourselves to experiments that have finitely many, equally likely, outcomes. This permits us to use Laplace's definition of the probability of an event. We will continue our study of probability in Section 7.2, where we will study experiments with finitely many outcomes that are not necessarily equally likely. In Section 7.2 we will also introduce some key concepts in probability theory, including conditional probability, independence of events, and random variables. In Section 7.4 we will introduce the concepts of the expectation and variance of a random variable.

Finite Probability

An **experiment** is a procedure that yields one of a given set of possible outcomes. The **sample space** of the experiment is the set of possible outcomes. An **event** is a subset of the sample space. Laplace's definition of the probability of an event with finitely many possible outcomes will now be stated.

DEFINITION 1 If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is $p(E) = \frac{|E|}{|S|}$.

The probability of an event can never be negative or more than one!

According to Laplace's definition, the probability of an event is between 0 and 1. To see this, note that if E is an event from a finite sample space S, then $0 \leq |E| \leq |S|$, because $E \subseteq S$. Thus, $0 \leq p(E) = |E|/|S| \leq 1$.

Examples 1–7 illustrate how the probability of an event is found.

EXAMPLE 1 An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue?

Extra Examples

Solution: To calculate the probability, note that there are nine possible outcomes, and four of these possible outcomes produce a blue ball. Hence, the probability that a blue ball is chosen is $4/9$. ◀

EXAMPLE 2 What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution: There are a total of 36 equally likely possible outcomes when two dice are rolled. (The product rule can be used to see this; because each die has six possible outcomes, the total

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GIROLAMO CARDANO (1501–1576) Cardano, born in Pavia, Italy, was the illegitimate child of Fazio Cardano, a lawyer, mathematician, and friend of Leonardo da Vinci, and Chiara Micheria, a young widow. In spite of illness and poverty, Cardano was able to study at the universities of Pavia and Padua, from where he received his medical degree. Cardano was not accepted into Milan's College of Physicians because of his illegitimate birth, as well as his eccentricity and confrontational style. Nevertheless, his medical skills were highly regarded. One of his main accomplishments as a physician is the first description of typhoid fever.

Cardano published more than 100 books on a diverse range of subjects, including medicine, the natural sciences, mathematics, gambling, physical inventions and experiments, and astrology. He also wrote a fascinating autobiography. In mathematics, Cardano's book *Ars Magna*, published in 1545, established the foundations of abstract algebra. This was the most comprehensive book on abstract algebra for more than a century; it presents many novel ideas of Cardano and of others, including methods for solving cubic and quartic equations from their coefficients. Cardano also made several important contributions to cryptography. Cardano was an advocate of education for the deaf, believing, unlike his contemporaries, that deaf people could learn to read and write before learning to speak, and could use their minds just as well as hearing people.

Cardano was often short of money. However, he kept himself solvent through gambling and winning money by beating others at chess. His book about games of chance, *Liber de Ludo Aleae*, written in 1526 (but published in 1663), offers the first systematic treatment of probability; it also describes effective ways to cheat. Cardano was considered to be a man of dubious moral character; he was often described as a liar, gambler, lecher, and heretic.

number of outcomes when two dice are rolled is $6^2 = 36$.) There are six successful outcomes, namely, (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), and (6, 1), where the values of the first and second dice are represented by an ordered pair. Hence, the probability that a seven comes up when two fair dice are rolled is $6/36 = 1/6$. ◀

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Lotteries are extremely popular throughout the world. We can easily compute the odds of winning different types of lotteries, as illustrated in Examples 3 and 4. (The odd of winning the popular Mega Millions and Powerball lotteries are studied in the supplementary exercises.)

EXAMPLE 3 In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

Solution: There is only one way to choose all four digits correctly. By the product rule, there are $10^4 = 10{,}000$ ways to choose four digits. Hence, the probability that a player wins the large prize is $1/10{,}000 = 0.0001$.

Players win the smaller prize when they correctly choose exactly three of the four digits. Exactly one digit must be wrong to get three digits correct, but not all four correct. By the sum rule, to find the number of ways to choose exactly three digits correctly, we add the number of ways to choose four digits matching the digits picked in all but the ith position, for $i = 1, 2, 3, 4$.

To count the number of successes with the first digit incorrect, note that there are nine possible choices for the first digit (all but the one correct digit), and one choice for each of the other digits, namely, the correct digits for these slots. Hence, there are nine ways to choose four digits where the first digit is incorrect, but the last three are correct. Similarly, there are nine ways to choose four digits where the second digit is incorrect, nine with the third digit incorrect, and nine with the fourth digit incorrect. Hence, there is a total of 36 ways to choose four digits with exactly three of the four digits correct. Thus, the probability that a player wins the smaller prize is $36/10{,}000 = 9/2500 = 0.0036$. ◀

EXAMPLE 4 There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40?

Solution: There is only one winning combination. The total number of ways to choose six numbers out of 40 is

$$C(40, 6) = \frac{40!}{34!\,6!} = 3{,}838{,}380.$$

Consequently, the probability of picking a winning combination is $1/3{,}838{,}380 \approx 0.00000026$. (Here the symbol $\approx$ means approximately equal to.) ◀

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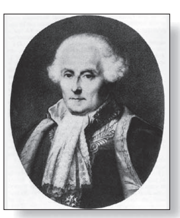

PIERRE-SIMON LAPLACE (1749–1827) Pierre-Simon Laplace came from humble origins in Normandy. In his childhood he was educated in a school run by the Benedictines. At 16 he entered the University of Caen intending to study theology. However, he soon realized his true interests were in mathematics. After completing his studies, he was named a provisional professor at Caen, and in 1769 he became professor of mathematics at the Paris Military School.

Laplace is best known for his contributions to celestial mechanics, the study of the motions of heavenly bodies. His *Traité de Mécanique Céleste* is considered one of the greatest scientific works of the early nineteenth century. Laplace was one of the founders of probability theory and made many contributions to mathematical statistics. His work in this area is documented in his book *Théorie Analytique des Probabilités*, in which he defined the probability of an event as the ratio of the number of favorable outcomes to the total number of outcomes of an experiment.

Laplace was famous for his political flexibility. He was loyal, in succession, to the French Republic, Napoleon, and King Louis XVIII. This flexibility permitted him to be productive before, during, and after the French Revolution.

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Poker, and other card games, are growing in popularity. To win at these games it helps to know the probability of different hands. We can find the probability of specific hands that arise in card games using the techniques developed so far. A deck of cards contains 52 cards. There are 13 different kinds of cards, with four cards of each kind. (Among the terms commonly used instead of "kind" are "rank," "face value," "denomination," and "value.") These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces. There are also four suits: spades, clubs, hearts, and diamonds, each containing 13 cards, with one card of each kind in a suit. In many poker games, a hand consists of five cards.

EXAMPLE 5 Find the probability that a hand of five cards in poker contains four cards of one kind.

Solution: By the product rule, the number of hands of five cards with four cards of one kind is the product of the number of ways to pick one kind, the number of ways to pick the four of this kind out of the four in the deck of this kind, and the number of ways to pick the fifth card. This is

$$C(13, 1)C(4, 4)C(48, 1).$$

By Example 11 in Section 6.3 there are $C(52, 5)$ different hands of five cards. Hence, the probability that a hand contains four cards of one kind is

$$\frac{C(13, 1)C(4, 4)C(48, 1)}{C(52, 5)} = \frac{13 \cdot 1 \cdot 48}{2{,}598{,}960} \approx 0.00024.$$ ◀

EXAMPLE 6 What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Solution: By the product rule, the number of hands containing a full house is the product of the number of ways to pick two kinds in order, the number of ways to pick three out of four for the first kind, and the number of ways to pick two out of four for the second kind. (Note that the order of the two kinds matters, because, for instance, three queens and two aces is different from three aces and two queens.) We see that the number of hands containing a full house is

$$P(13, 2)C(4, 3)C(4, 2) = 13 \cdot 12 \cdot 4 \cdot 6 = 3744.$$

Because there are $C(52, 5) = 2{,}598{,}960$ poker hands, the probability of a full house is

$$\frac{3744}{2{,}598{,}960} \approx 0.0014.$$ ◀

EXAMPLE 7 What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers $1, 2, \ldots, 50$ if (a) the ball selected is not returned to the bin before the next ball is selected and (b) the ball selected is returned to the bin before the next ball is selected?

Solution: (a) By the product rule, there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254{,}251{,}200$ ways to select the balls because each time a ball is drawn there is one fewer ball to choose from. Consequently, the probability that 11, 4, 17, 39, and 23 are drawn in that order is $1/254{,}251{,}200$. This is an example of **sampling without replacement**.

(b) By the product rule, there are $50^5 = 312{,}500{,}000$ ways to select the balls because there are 50 possible balls to choose from each time a ball is drawn. Consequently, the probability that 11, 4, 17, 39, and 23 are drawn in that order is $1/312{,}500{,}000$. This is an example of **sampling with replacement**. ◀

Probabilities of Complements and Unions of Events

We can use counting techniques to find the probability of events derived from other events.

THEOREM 1 Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

Proof: To find the probability of the event $\overline{E} = S - E$, note that $|\overline{E}| = |S| - |E|$. Hence,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$

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There is an alternative strategy for finding the probability of an event when a direct approach does not work well. Instead of determining the probability of the event, the probability of its complement can be found. This is often easier to do, as Example 8 shows.

EXAMPLE 8 A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Solution: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all the bits are 1s. Because the sample space S is the set of all bit strings of length 10, it follows that

$$\begin{aligned} p(E) = 1 - p(\overline{E}) &= 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} \\ &= 1 - \frac{1}{1024} = \frac{1023}{1024}. \end{aligned}$$

Hence, the probability that the bit string will contain at least one 0 bit is 1023/1024. It is quite difficult to find this probability directly without using Theorem 1. ◀

We can also find the probability of the union of two events.

THEOREM 2 Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Proof: Using the formula given in Section 2.2 for the number of elements in the union of two sets, it follows that

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|.$$

Hence,

$$
\begin{aligned}
p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} \\
&= \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\
&= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\
&= p(E_1) + p(E_2) - p(E_1 \cap E_2).
\end{aligned}
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EXAMPLE 9 What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Extra Examples

Solution: Let E_1 be the event that the integer selected at random is divisible by 2, and let E_2 be the event that it is divisible by 5. Then $E_1 \cup E_2$ is the event that it is divisible by either 2 or 5. Also, $E_1 \cap E_2$ is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10. Because $|E_1| = 50$, $|E_2| = 20$, and $|E_1 \cap E_2| = 10$, it follows that

$$
\begin{aligned}
p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \\
&= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.
\end{aligned}
$$

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Probabilistic Reasoning

A common problem is determining which of two events is more likely. Analyzing the probabilities of such events can be tricky. Example 10 describes a problem of this type. It discusses a famous problem originating with the television game show *Let's Make a Deal* and named after the host of the show, Monty Hall.

EXAMPLE 10 **The Monty Hall Three-Door Puzzle** Suppose you are a game show contestant. You have a chance to win a large prize. You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is behind each door, does the following. First, whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors. Which strategy should you use? Should you change doors or keep your original selection, or does it not matter?

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Solution: The probability you select the correct door (before the host opens a door and asks you whether you want to change) is $1/3$, because the three doors are equally likely to be the correct door. The probability this is the correct door does not change once the game show host opens one of the other doors, because he will always open a door that the prize is not behind.

The probability that you selected incorrectly is the probability the prize is behind one of the two doors you did not select. Consequently, the probability that you selected incorrectly is $2/3$. If you selected incorrectly, when the game show host opens a door to show you that the prize is not behind it, the prize is behind the other door. You will always win if your initial choice was incorrect and you change doors. So, by changing doors, the probability you win is $2/3$. In other words, you should always change doors when given the chance to do so by the game show host. This doubles the probability that you will win. (A more rigorous treatment of this puzzle can be found in Exercise 15 of Section 7.3. For much more on this famous puzzle and its variations, see [Ro09].)

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