College of Engineering & Applied Sciences

CSPB 2820

Linear Algebra With Computer Science Applications

Exam Notes

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Exam 1 Notes

Inner Product: Calculates the overlap between two vectors

$$a^T b = \|a\|\|b\|\cos(\theta).$$

When $\theta = \pi/2$ the vectors are perpendicular and when $\theta = 0$ the vectors are parallel.

Taylor Series: Approximates a differentiable function to a polynomial at a given point of the original function

$$\mathbb{T}(x,a) = \sum_{n=0}^{\infty} \frac{f^n(x-a)^n}{n!}$$

$$\hat{f}(x) = f(x,z) + \nabla f(z)^T(x-z)$$

$$\nabla f(x,z) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(z) \\ \vdots \\ \frac{\partial f}{\partial x_n}(z) \end{bmatrix}.$$

Affine Function: Inner product with an additional scalar value

$$f(x) = a^T x + b.$$

Linear Functions: Linear functions are of the form

$$\text{Superposition: } f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$
$$\text{Homogeneity: } f(\alpha x) = \alpha f(x)$$
$$\text{Additivity: } f(x+y) = f(x) + f(y).$$

Superposition is a combination of homogeneity and additivity.

Linear Regression: Models the relationship between a dependent variable and one or more independent variables, aiming to estimate coefficients that minimize the error between predicted and observed values

$$\hat{y} = x^T\beta + v$$

where x are the regressors, $\hat{y}$ is the prediction, β is an n-vector and v is a scalar.

Norm Of Vector: Measurement of how large a vector or a collection or vectors are

$$\|x\| = x^T x = \sqrt{x_1^2 + \cdots + x_n^2}$$
$$\|x+y\| = \sqrt{\|x\|^2 + 2x^T y + \|y\|^2}$$
$$\|(a,b,c)\| = \sqrt{\|a\|^2 + \|b\|^2 + \|c\|^2}.$$

Triangle Inequality: The norm of a sum of two vectors is no more than the sum of their norms

$$\|x+y\| \leq \|x\| + \|y\|.$$

Root Mean Square (RMS): Typical value of $\|x_i\|$

$$\mathbf{rms}(x) = \frac{\|x\|}{\sqrt{n}}.$$

Chebyshev Inequality: Provides an upper bound on the probability that a random variable deviates from its mean by a certain multiple of its standard deviation

$$\frac{k}{n} \leq \left(\frac{\mathbf{rms}(x)}{a}\right)^2$$

where k is the number of entries in x with absolute value of at least a.

Distance: Calculates the Euclidean distance between two vectors

$$\mathbf{dist}(a,b) = \|a-b\|.$$

Average: Calculates the average value of a vector

$$\mathbf{avg}(x) = \frac{x_1 + \cdots + x_n}{n}.$$

Nearest Neighbor: The vector that is the closest vector to another given vector

$$||x - z_j|| \leq ||x - z_i||, \;\; i = 1, \ldots, n.$$

Standard Deviation: A measure of the dispersion of the values within the vector

$$\mathbf{std}(x) = \frac{\|x - (\mathbf{1}^T x/n)\|}{\sqrt{n}}.$$

J-Clust: J_C is a method of quantifying how good a clustering of data points is for a given cluster

$$J_C = \sum_{i=1}^{N}\sum_{j=1}^{k} \frac{1}{N}\mathbf{min}(\|x_i - z_j\|^2)$$

where z_j is calculated by

$$z_j = \frac{1}{|G_j|}\sum_{i\in G_j}^{n} x_i.$$

K-Means Algorithm: The k-means algorithm clusters data for a given data set into k clusters.

1. *Partition the vectors into k groups:* For each vector $i = 1, \ldots, N$, assign x_i to the group associated with the nearest representative
2. *Update representatives:* For each group $j = 1, \ldots, k$, set z_j to be the mean of the vectors in group j.

Linear Independence: Linear independence and dependence can be summarized with

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

where for a collection of vectors to be linearly independent or dependent one of the following statements must be true.

1. *Linearly Independent:* All values of β must be zero.
2. *Linearly Dependent:* There exists at least one β that is non zero.

Exam 2 Notes

Matrix Types: Matrix sizes are defined in terms of their rows (m) by columns (n):

- **Square Matrix:** $m = n$.
- **Tall Matrix:** $m > n$.
- **Wide Matrix:** $m < n$.

2D Rotation Matrix: The 2D rotation matrix is:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

3D Rotation Matrices: The 3D rotation matrices are:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Incidence Matrix: An incidence matrix is of the form:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

The rows and columns of an incidence matrix represent:

- *Columns:* Represent the edges of the graph.
- *Rows:* Represent the vertices of the graph.

The values of the elements indicate:

- 1: Edge j points to node i.
- -1: Edge j points from node i.
- 0: Otherwise.

Dirichlet Energy: Measures the potential differences across edges:

$$\mathcal{D}(v) = \|A^T\|^2 = \sum_{(k,l)\in\text{Edges}} (v_l - v_k)^2$$

Matrix Multiplication: Matrix multiplication is performed by:

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \end{aligned}$$

The main properties of matrix multiplication are:

- *Associativity:* $(AB)C = A(BC)$
- *Scalar Multiplication:* $\gamma(AB) = (\gamma A)B$
- *Distributivity:*
  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$
- *Transpose Product:* $(AB)^T = B^T A^T$

Some Matrix Representations: Some matrix representations are:

- *Column Interpretation:*

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_n \end{bmatrix}$$

- *Row Interpretation:*

$$AB = \begin{bmatrix} (B^T a_1)^T \\ (B^T a_2)^T \\ \vdots \\ (B^T a_m)^T \end{bmatrix}$$

- *Gram Matrix:*

$$G = A^T A$$

- *QR Factorization:*

$$A = QR$$

Matrix Inverses: The rules for if a matrix is invertible:

- Matrix must be square.
- The columns are linearly independent.
- The rows are linearly independent.
- The matrix has a left inverse.
- The matrix has a right inverse.

For a 2×2 matrix the inverse is:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3×3 matrix the inverse is:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} \begin{Vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \\ a_{23} & a_{21} \\ a_{33} & a_{31} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{Vmatrix} & \begin{Vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \\ a_{11} & a_{13} \\ a_{31} & a_{33} \\ a_{12} & a_{11} \\ a_{32} & a_{31} \end{Vmatrix} & \begin{Vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{13} & a_{11} \\ a_{23} & a_{21} \\ a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \end{bmatrix}$$