Chapter 5

Linear independence

In this chapter we explore the concept of linear independence, which will play an important role in the sequel.

5.1 Linear dependence

A collection or list of n-vectors $a_1, \ldots, a_k$ (with $k \geq 1$) is called *linearly dependent* if

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

holds for some $\beta_1, \ldots, \beta_k$ that are not all zero. In other words, we can form the zero vector as a linear combination of the vectors, with coefficients that are not all zero. Linear dependence of a list of vectors does not depend on the ordering of the vectors in the list.

When a collection of vectors is linearly dependent, at least one of the vectors can be expressed as a linear combination of the other vectors: If $\beta_i \neq 0$ in the equation above (and by definition, this must be true for at least one i), we can move the term $\beta_i a_i$ to the other side of the equation and divide by β_i to get

$$a_i = (-\beta_1/\beta_i)a_1 + \cdots + (-\beta_{i-1}/\beta_i)a_{i-1} + (-\beta_{i+1}/\beta_i)a_{i+1} + \cdots + (-\beta_k/\beta_i)a_k.$$

The converse is also true: If any vector in a collection of vectors is a linear combination of the other vectors, then the collection of vectors is linearly dependent.

Following standard mathematical language usage, we will say "The vectors $a_1, \ldots, a_k$ are linearly dependent" to mean "The list of vectors $a_1, \ldots, a_k$ is linearly dependent". But it must be remembered that linear dependence is an attribute of a *collection* of vectors, and not individual vectors.

Linearly independent vectors. A collection of n-vectors $a_1, \ldots, a_k$ (with $k \geq 1$) is called *linearly independent* if it is not linearly dependent, which means that

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0 \tag{5.1}$$

only holds for $\beta_1 = \cdots = \beta_k = 0$. In other words, the only linear combination of the vectors that equals the zero vector is the linear combination with all coefficients zero.

As with linear dependence, we will say "The vectors $a_1, \ldots, a_k$ are linearly independent" to mean "The list of vectors $a_1, \ldots, a_k$ is linearly independent". But, like linear dependence, linear independence is an attribute of a collection of vectors, and not individual vectors.

It is generally not easy to determine by casual inspection whether or not a list of vectors is linearly dependent or linearly independent. But we will soon see an algorithm that does this.

Examples.

- A list consisting of a single vector is linearly dependent only if the vector is zero. It is linearly independent only if the vector is nonzero.
- Any list of vectors containing the zero vector is linearly dependent.
- A list of two vectors is linearly dependent if and only if one of the vectors is a multiple of the other one. More generally, a list of vectors is linearly dependent if any one of the vectors is a multiple of another one.
- The vectors
$$a_1 = \begin{bmatrix} 0.2 \\ -7.0 \\ 8.6 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} -0.1 \\ 2.0 \\ -1.0 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0.0 \\ -1.0 \\ 2.2 \end{bmatrix}$$
are linearly dependent, since $a_1 + 2a_2 - 3a_3 = 0$. We can express any of these vectors as a linear combination of the other two. For example, we have $a_2 = (-1/2)a_1 + (3/2)a_3$.
- The vectors
$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad a_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
are linearly independent. To see this, suppose $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$. This means that
$$\beta_1 - \beta_3 = 0, \qquad -\beta_2 + \beta_3 = 0, \qquad \beta_2 + \beta_3 = 0.$$
Adding the last two equations we find that $2\beta_3 = -0$, so $\beta_3 = 0$. Using this, the first equation is then $\beta_1 = 0$, and the second equation is $\beta_2 = 0$.
- The standard unit n-vectors $e_1, \ldots, e_n$ are linearly independent. To see this, suppose that (5.1) holds. We have
$$0 = \beta_1 e_1 + \cdots + \beta_n e_n = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix},$$
so we conclude that $\beta_1 = \cdots = \beta_n = 0$.

Linear combinations of linearly independent vectors. Suppose a vector x is a linear combination of $a_1, \ldots, a_k$,

$$x = \beta_1 a_1 + \cdots + \beta_k a_k.$$

When the vectors $a_1, \ldots, a_k$ are linearly independent, the coefficients that form x are *unique*: If we also have

$$x = \gamma_1 a_1 + \cdots + \gamma_k a_k,$$

then $\beta_i = \gamma_i$ for $i = 1, \ldots, k$. This tells us that, in principle at least, we can find the coefficients that form a vector x as a linear combination of linearly independent vectors.

To see this, we subtract the two equations above to get

$$0 = (\beta_1 - \gamma_1)a_1 + \cdots + (\beta_k - \gamma_k)a_k.$$

Since $a_1, \ldots, a_k$ are linearly independent, we conclude that $\beta_i - \gamma_i$ are all zero.

The converse is also true: If each linear combination of a list of vectors can only be expressed as a linear combination with one set of coefficients, then the list of vectors is linearly independent. This gives a nice interpretation of linear independence: A list of vectors is linearly independent if and only if for any linear combination of them, we can infer or deduce the associated coefficients. (We will see later how to do this.)

Supersets and subsets. If a collection of vectors is linearly dependent, then any superset of it is linearly dependent. In other words: If we add vectors to a linearly dependent collection of vectors, the new collection is also linearly dependent. Any nonempty subset of a linearly independent collection of vectors is linearly independent. In other words: Removing vectors from a collection of vectors preserves linear independence.

5.2 Basis

Independence-dimension inequality. If the n-vectors $a_1, \ldots, a_k$ are linearly independent, then $k \leq n$. In words:

A linearly independent collection of n-vectors can have at most n elements.

Put another way:

Any collection of $n+1$ or more n-vectors is linearly dependent.

As a very simple example, we can conclude that any three 2-vectors must be linearly dependent. This is illustrated in figure 5.1.

We will prove this fundamental fact below; but first, we describe the concept of basis, which relies on the independence-dimension inequality.