University of Colorado **Boulder**

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CSCI 2824: Discrete Structures
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Discrete Probability
Introduction to Probability Theory

Discrete Probability Theory

So far we've only considered the probability of events where each outcome s in the sample space S was equally likely to occur.

Example: If you roll a fair die each number on the die is equally likely to appear.

In that case we defined the probability of an event E to be

$$p(E) = \frac{|E|}{|S|}$$

Remember, S is the *set* of all possible outcomes and the event E is a *subset* of S

We also noted that the value $p(E)$ satisfied $0 \leq p(E) \leq 1$

Discrete Probability Theory

But what happens if we want to model, say, an unfair coin which is more likely to come up Heads than Tails?

Or a loaded die that is more likely to roll a 3 than any other number?

Let S be the sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome $s \in S$. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

Note: Before we thought of $p(E)$ as a number, but now we think of p as a **function** that maps outcomes to a value between 0 and 1.

Discrete Probability Theory

Def: The function p from the set of all outcomes of the sample space S is called a **probability distribution**

Example: Specify the probability distribution that models the outcomes H and T when a fair coin is flipped.

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There are only 2 outcomes in the sample space: $S = \{H, T\}$

Since $p(H) = p(T)$ and $p(H) + p(T) = 1$ we conclude that

$$p(H) = \frac{1}{2} \quad \text{and} \quad p(T) = \frac{1}{2}$$

Since the sample space $S = \{H, T\}$, specifying these two values completely defines the distribution p

Note: For a finite S, the function p is essentially a look-up table

Discrete Probability Theory

OK, but what happens if the coin is not fair?

Example: Specify the probability distribution that models the outcomes H and T when a biased coin is flipped where the coin is twice as likely to come up H as T.

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We have the following conditions

$$p(H) + p(T) = 1 \quad \text{and} \quad p(H) = 2p(T)$$

Substituting the second equation into the first gives

$$2p(T) + p(T) = 1 \quad \Rightarrow \quad 3p(T) = 1 \quad \Rightarrow \quad p(T) = \frac{1}{3}$$

Then from the first equation we have $p(H) = \dfrac{2}{3}$

Discrete Probability Theory

Since E is a set of possible outcomes, we compute the probability of E by summing the probabilities of the outcomes in E

Def: The probability of an event E is the sum of the probabilities of the outcomes in E. That is

$$p(E) = \sum_{s \in E} p(s)$$

Discrete Probability Theory

Example: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but the other five outcomes are equally likely. What is the probability that an odd number is rolled?

Discrete Probability Theory

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- First we need to specify the probability distribution for the roll of the loaded die
- Then we'll compute the probability of $E = \{1, 3, 5\}$

What do we know?

1. $0 \le p(k) \le 1$ for $1 \le k \le 6$ and $\sum_{k=1}^{6} p(k) = 1$
2. $p(1) = p(2) = p(4) = p(5) = p(6)$
3. $p(3) = 2p(k)$ for $k \neq 3$

Discrete Probability Theory

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Plugging 2 and 3 into the expression that the $p(k)$'s sum to 1

$$1 = p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$$
$$1 = p(1) + p(1) + 2p(1) + p(1) + p(1) + p(1)$$
$$1 = 7p(1)$$

So $p(1) = p(2) = p(4) = p(5) = p(7) = \dfrac{1}{7}$ and $p(3) = \dfrac{2}{7}$

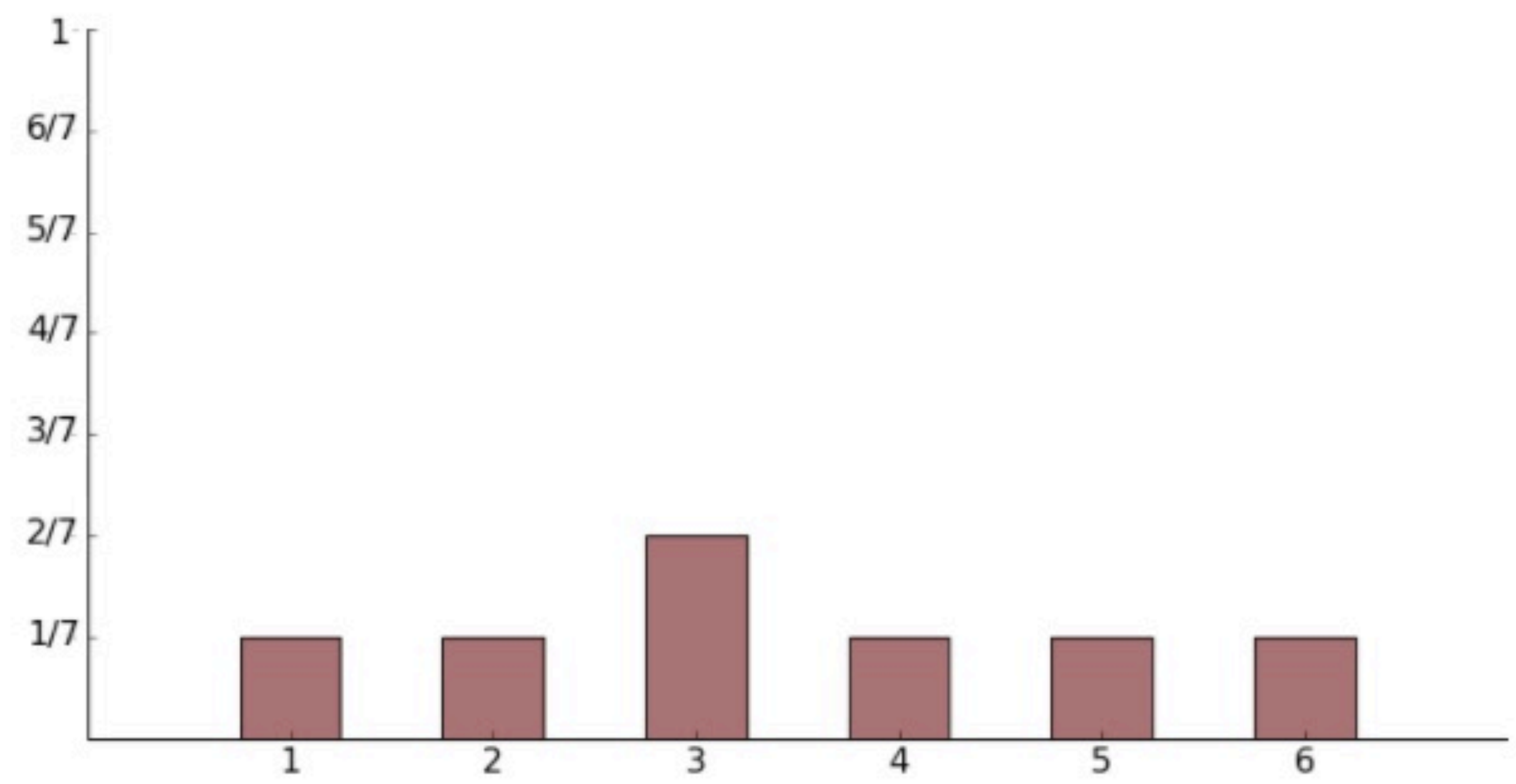
Discrete Probability Theory

Aside: There are two common ways to write down distribution p

We can either describe p as a look-up table

s	1	2	3	4	5	6
$p(s)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

Or we might visualize it with a histogram



Discrete Probability Theory

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We have $p(\{1, 3, 5\}) = p(1) + p(3) + p(5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}$

Follow-Up: For the loaded die in the previous example, what is the probability that the die comes up even?

We could do this in the same way, i.e. summing probabilities over the event $E = \{2, 4, 6\}$, but it's easier than that

$\{2, 4, 6\}$ is the complement of $\{1, 3, 5\}$ in S

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$\{2, 4, 6\}$ is the complement of $\{1, 3, 5\}$ in S, so we use

$$p(\{2,4,6\}) = 1 - p(\{1,3,5\}) = 1 - \frac{4}{7} = \frac{3}{7}$$

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Note that the rules for computing the complements and unions events from the previous section still hold, even in the case that all outcomes are not equally likely

$$\text{Complement:} \quad p(\bar{E}) = 1 - p(E)$$

$$\text{Union:} \quad p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$