Networks and Graphs Workbook.

This workbook explores basic ideas of Networks and Graphs and is for you to explore these ideas.

I have been trying to get a good PDF copy from the publisher, and it has not arrived, so we will use this PDF hack this semester. The workbook is from Jacob's Mathematics a Human Endeavor - *third edition. I highly recommend this book especially if you will take CSPB2824.*

- Please complete all the exercises on the pages provided (some pages are left out intentionally).
  You may use an online drawing tool to sketch on the pictures as needed - yes you MUST sketch on the pictures - or print out the pages and use a pencil.
- Please add notes and thoughts in the margins.
- Many answers are short - a number or yes/no.
- This will be graded as a workbook - so scribbles, diagrams, sketches etc.. are all good!. We are looking for evidence that it was thoughtfully completed.
- Expect this to take 2 -3 hours to complete.
- Submit ALL pages as a PDF.
- If you are doing this by hand with printed copies, you may photograph 4 pages at a time and include 4 photos on each page of a PDF - notice this will ONLY work if you print ONE SIDED. Please plan ahead. Yes, include the 'text only' pages for clarity.

LESSON

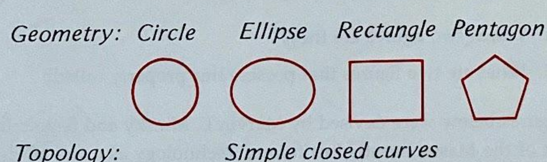
1 The Mathematics of Distortion

Beneath the city of London is an elaborate network of subways. Called the Underground, it consists of 279 stations connected by 410 kilometers of tracks.

Although the map above is an accurate representation of the subway lines and stations in central London, it is not the map used by passengers. For someone traveling on the Underground, all that matters is the way in which the stations are connected by the various lines. The map on the facing page, which shows these connections in a more straight-forward way, is the one posted in each Underground station.

It is easy to see how the second map was made. The artist treated the lines of the original map as if they were elastic bands and stretched and bent them into simpler shapes. The result is a figure having some properties different from the original one and some properties that have remained unchanged. The properties that have remained unchanged are *topological.*

Topology, sometimes called "the mathematics of distortion," deals with very basic properties of geometric figures: properties that remain unchanged no matter how those figures are stretched or bent. Look, for example, at the figures below.



From the standpoint of geometry, each figure is different and has its own name. From the standpoint of topology, they are all alike. The figures are examples of *simple closed curves.*

A **simple closed curve** is a curve on which it is possible to start at any point and move continuously around the curve, passing through every other point exactly once before returning to the starting point.

The figures are also *topologically equivalent.*

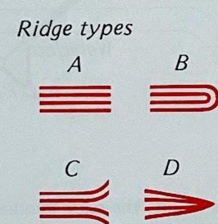
Figures are **topologically equivalent** if they can be stretched and bent into the same shape without connecting or disconnecting any points.

Because topology deals with very basic ideas, its influence on other areas of mathematics has been immense. Even though it is a comparatively young branch of mathematics, it has proved to have many practical applications and, because of its many surprises, is often a source of much enjoyment.

Exercises

Set I

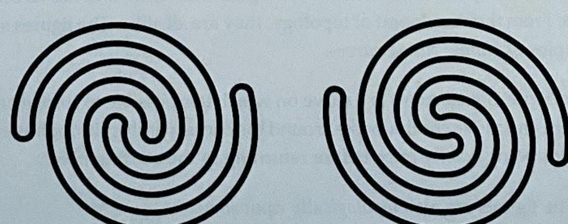
The figures below are from a book on fingerprints.



Two of these figures can be stretched and bent into the same shape without connecting or disconnecting any points.

1. Which two figures are they?
2. What are two figures that possess this property called?

The figures below were devised by Marvin L. Minsky and Seymour A. Papert of the Massachusetts Institute of Technology as a test of visual perception.



3. Place tracing paper over this page and trace each figure. Are the figures topologically equivalent?
4. What does each one consist of?

The first neon sign made its appearance at the Paris Motor Show in 1910. The symbols that are the easiest to make in neon are those that are topologically equivalent to a line segment. Examples are shown at the right.

Use the forms of the capital letters shown below to answer the following questions.

ABCDEFGHIJK
LMNOPQRSTUVWXYZ

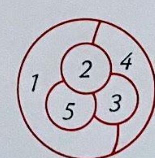
5. Twelve of these letters are topologically equivalent to a line segment. Which letters are they?

Which letters are topologically equivalent to each of the following shapes?

6. (Two letters) 7. (Two letters) 8. (Two letters) 9. (Two letters) 10. (Four letters)

The following topological problem was presented in a lecture by the German mathematician August Ferdinand Moebius in 1840.*

> A king with five sons stated in his will that after his death his land was to be divided into five regions so that each region shared some of its border (more than just a point) with each of the others. Can the will be carried out?

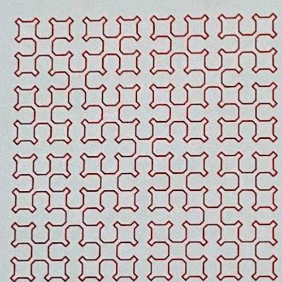


Look at the adjoining map.

11. Does region 1 share some of its border with each of the other regions?
12. Does region 2 share some of its border with each of the other regions?
13. Does region 3 share some of its border with each of the other regions?
14. Does the map solve the problem? Explain why or why not.
15. Can you draw a different map that solves the problem?

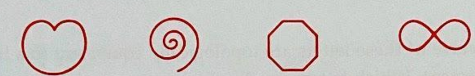
Famous Problems of Mathematics, by Heinrich Tietze (Graylock Press, 1965).

Set II



Even though this figure looks quite complex, in topology it is called a *simple closed curve.*

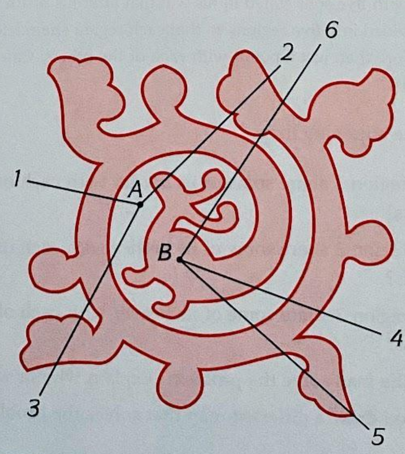
1. Use the definition of simple closed curve in this lesson to explain why.
2. Which of the figures below are simple closed curves?



A basic idea of topology is that a simple closed curve in a plane divides the plane into exactly two regions: an inside and an outside. This fact is called the *Jordan Curve Theorem,* after the nineteenth-century French mathematician Camille Jordan.

3. Which of the four figures in exercise 2 divide the plane into exactly two regions?

This simple closed curve appeared on a medieval wall tile.*

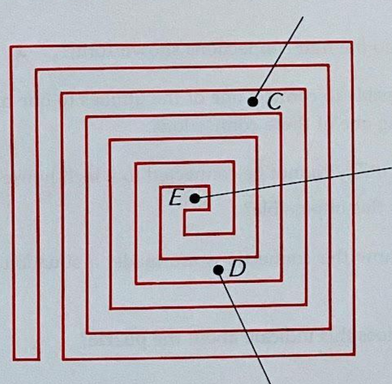


**Handbook of Regular Patterns,* by Peter S. Stevens (MIT Press, 1980).

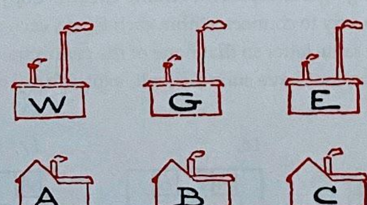
4. Is point A inside or outside the curve?
5. How many times does each straight line from point A to the outside cross the curve?
6. Is point B inside or outside the curve?
7. How many times does each straight line from point B to the outside cross the curve?

Whether a point is *inside* or *outside* a simple closed curve can be found from the number of times that a straight line joining that point to the outside of the curve crosses the curve.

8. How?
9. Determine whether each of the points C, D, and E is inside or outside the curve below.

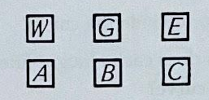


An old puzzle titled "Water, Gas, and Electricity" is to draw lines connecting the three utilities to each of three houses without any of the lines crossing each other.*

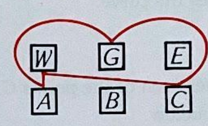


Amusements in Mathematics, by Henry Ernest Dudeney (Nelson and Sons, 1917; Dover, 1958).

10. Copy the figure below showing the utilities and houses and try to connect them as indicated.



The figure below illustrates a partial solution to the puzzle.



11. What do the four connections shown form?

It is now impossible to connect one of the utilities to one of the houses without crossing one of these connections.

12. Which utility cannot be connected to which house?

13. Why is this impossible?

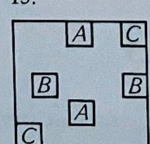
Regardless of how the connections are made, a situation of this sort always arises.

14. What does this indicate about the puzzle?

Topology is used in the design of the electric circuits used in microelectronics. The connections between the parts of the circuits cannot cross each other or the circuits will short.

The following figures represent electric circuits. Copy each one and then try to find a way to connect, within each figure, each pair of squares labeled with the same letter so that none of the connections cross. If you think a figure does not have such a circuit, explain why not.

15.



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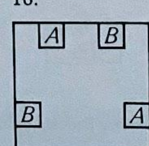

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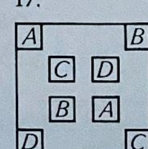
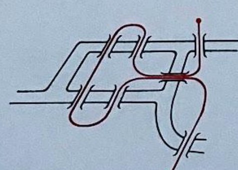


From *Topographia Prussiae et Pomerelliae* by M. Zeiller. Frankfurt. c. 1650.

LESSON

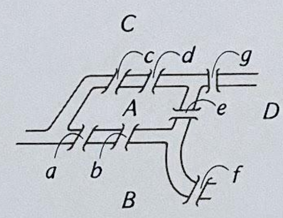
2 The Seven Bridges of Königsberg: An Introduction to Networks



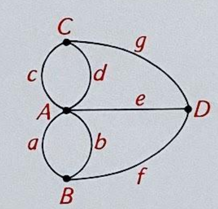
One of the problems that led to the development of topology is about seven bridges in the city of Königsberg in old Germany. The drawing above, from a book published in about 1650, shows that the center of Königsberg was on an island in the middle of a river. The river flowed around the island from the left and, on the right, separated into two branches. Seven bridges made it possible to travel from one part of the city to another.

The citizens of Königsberg wondered whether it was possible to travel around the city and cross each of the seven bridges exactly once. Everyone who tried it ended up either skipping or recrossing at least one bridge. Most people came to the conclusion that a path crossing every bridge exactly once did not exist, but they did not know why.

Eventually, the problem of the Königsberg bridges came to the attention of the Swiss mathematician Leonhard Euler. Euler found the puzzle interesting and, in an article published in 1736, proved that it could not be solved. He began by representing each area of land with a capital letter and each bridge with a small letter. The map can be simplified even further by representing the four areas of land by four points and the seven bridges by seven lines connecting the points. This is shown in the diagram at the right, which we will call a *network*.

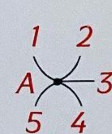


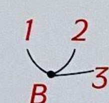
A **network** is a figure consisting of points, called *vertices,** connected by lines, called *edges.*



The problem of the Königsberg bridges is equivalent to the problem of drawing this network without retracing any edge or taking the pencil off the paper. Euler showed that whether or not this problem has a solution depends on the *degree* of each vertex of the network.

The **degree** of a vertex of a network is the *number* of edges that meet at it.





In the network for the bridge problem, five edges meet at vertex A and so the degree of vertex A is 5. Three edges meet at each of the vertices B, C, and D, and so the degree of each of these vertices is 3. Euler's results showed that this network, which has four vertices of an odd degree, cannot be drawn with one continuous stroke of a pencil.

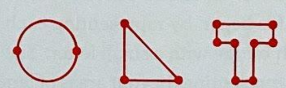
Exercises

Set I

A network can be "traveled" if it is possible to draw it without retracing any edge or taking your pencil off the paper. It is easy to see that the first network below can be traveled by starting at any vertex. The second network can be traveled only if you start on one of the vertices marked with an arrow.

**Vertices* is the plural of *vertex.*

Each of the networks below is topologically equivalent to a simple closed curve.



1. Do you think that every network topologically equivalent to a simple closed curve can be traveled?

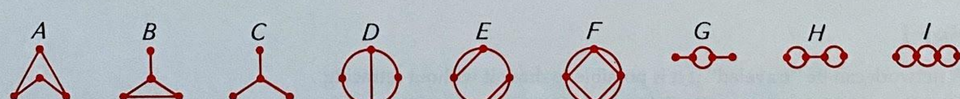
All the vertices of such networks have the same degree.

2. What is meant by the *degree* of a vertex?
3. What is the degree of each vertex of the three networks above?

Each of these networks is topologically equivalent to a line segment.

4. Do you think that every network topologically equivalent to a line segment can be traveled?
5. How many vertices of degree 1 do such networks have?
6. What is the degree of the other vertices in such networks?

Each of these networks has four vertices. Notice that the degree of each vertex of network A is 2, so it has 4 "even" vertices.



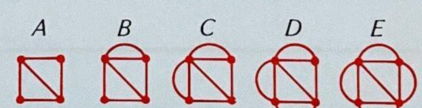
7. Copy and complete the following table for these networks.

Network	Number of even vertices	Number of odd vertices	Can the network be traveled?
A	4	0	Yes
B			
C			

(Continue the table to show all nine networks.)

8. What do all the vertices of the networks that cannot be traveled have in common?

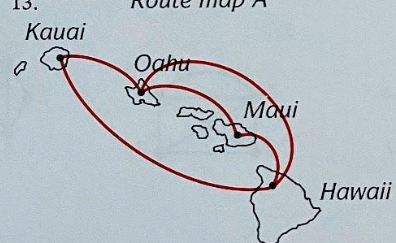
Each of the following networks also has four vertices.



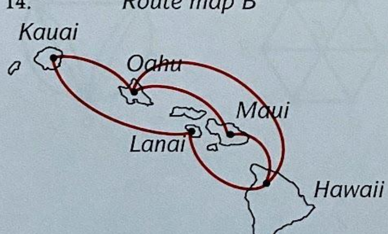
9. Which networks have two even vertices and two odd vertices?
10. Which have four even vertices?
11. Which have four odd vertices?
12. Which network(s) cannot be traveled?

The maps below show some different possible nonstop flights between the Hawaiian Islands. Place your paper over each map and try to travel each flight network. (Whether or not you are able to do this may depend on where you start.)

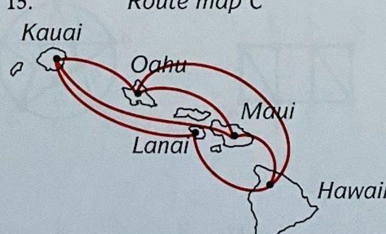
13. *Route map A*



14. *Route map B*



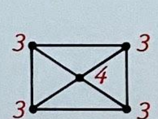
15. *Route map C*



16. How does route map B differ from route map A?
17. How does route map C differ from route map B?
18. Which maps show flight networks that can be traveled?

19. Does adding a vertex to a network seem to have any effect on whether it can be traveled?
20. Does adding an edge to a network seem to have any effect on whether it can be traveled?

Set II

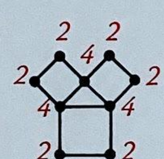


In 1809, the French mathematician Louis Poinsot noted that the figure shown at the left cannot be drawn in one continuous stroke of a pencil.

1. What do the numbers at the vertices of this network represent?
2. How many vertices of even degree does it have?
3. How many vertices of odd degree does it have?

Place tracing paper over the networks below and try to travel each one. (Some of the networks can be traveled only if you start from certain vertices.) Keep a record of those that can be traveled and those that cannot.

4.



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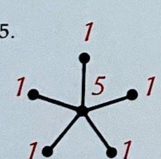

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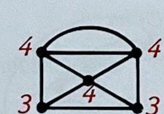

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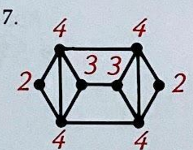

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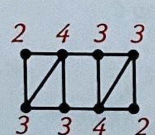

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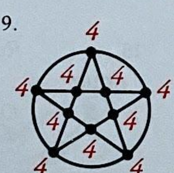

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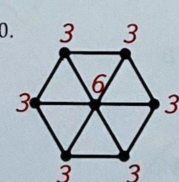

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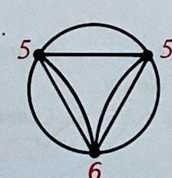

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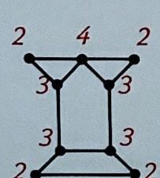

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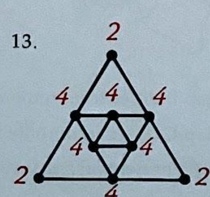


14. Copy and complete the following table for these networks.

Network	Number of even vertices	Number of odd vertices	Can the network be traveled?
4	9	0	Yes
5			

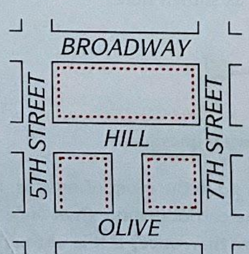
(Continue the table to show all ten networks.)

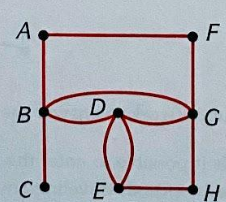
Of the ten networks, six can be traveled. If your table shows fewer than six, try again to draw each of the networks that you have marked "no."

On the basis of the information in your table, does it seem that a network can be traveled if

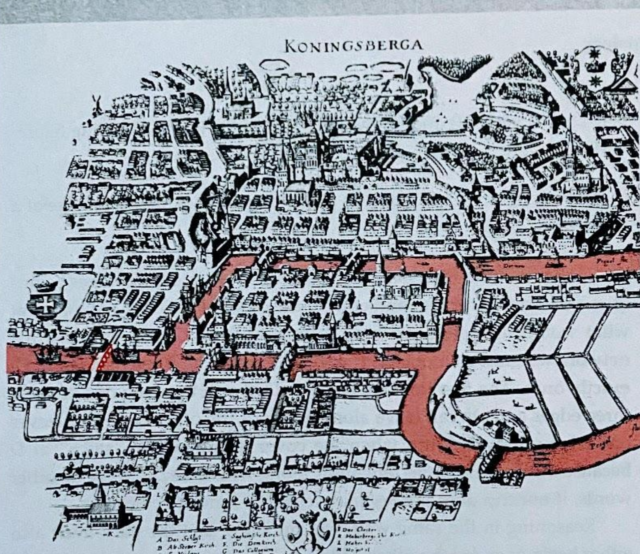
15. all of the vertices are even?
16. it has more even vertices than odd vertices?
17. it has two odd vertices?
18. it has more than two odd vertices?
19. Write a conclusion about whether a network can or cannot be traveled based on your answers to exercises 15–18.

The brown dots on the map at the left below show the locations of parking meters along several city streets. The person who has the job of collecting money from these meters has to travel each edge of the network shown at the right of the map.





20. Place tracing paper over the network and try to travel it.
21. Where do you think would be the best places for the route to begin and end?
22. In what way do these vertices of the network differ from the rest of the vertices?



After *Topographia Prussiae et Pomerelliae* by M. Zeiller. Frankfurt. c. 1650.

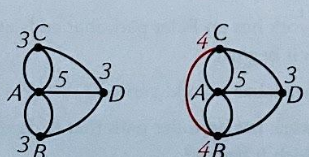
LESSON

3

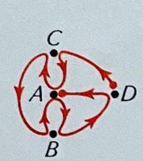
Euler Paths

It was in 1736 that Leonhard Euler proved that it was impossible to walk through the city of Königsberg and cross each of its seven bridges exactly once. In 1875, an eighth bridge was built. It is not difficult to show that the addition of this new bridge made it possible to travel the network.

Networks representing the city before and after the new bridge was built are shown below. The first network contains four odd vertices.



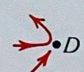
Adding an edge between vertices B and C to form the second network changes two of these vertices to even. One way to travel the new network



is to start at vertex D and follow the path shown in the adjoining figure, ending at vertex A. Such a path is called an *Euler path.*

> An **Euler path** is a continuous path that passes along every edge of a network exactly once.

Notice that the Euler path for the network with eight bridges begins at one of the odd vertices and ends at the other one. To see why, look at what would happen if we did not start at vertex D. In traveling the network we must go along each edge, including the three that end at D, exactly once. The first time we come to D, it will be along one of these three edges and we can leave along either of the other two. This leaves one edge untraveled and when we cover it we will be "stuck" at D because there are no edges remaining along which we can leave. In other words, if our trip does not *begin* at vertex D, it must *end* there.

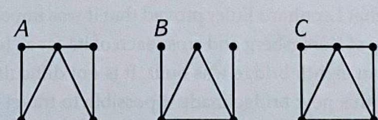


Reasoning in the same way, we can show that the path must also either begin or end at the other odd vertex, A. Because a path has exactly one beginning and one ending, it follows that it must begin at one of the odd vertices and end at the other. It also follows that a network with more than two odd vertices cannot be traveled in a single trip. Euler proved that every network having no more than two odd vertices has an Euler path.

Exercises

Set I

Two of the networks below have Euler paths and one does not.

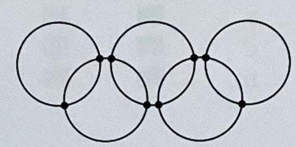


1. What is meant by *Euler path?*
2. Which network has an Euler path that ends at the same vertex from which it begins?
3. Which network has no odd vertices?
4. Which network has an Euler path that ends at a different vertex from which it begins?
5. Which network has two odd vertices?
6. Which network does not have an Euler path?
7. How many odd vertices does it have?

On the basis of your answers to exercises 2–7, write rules for identifying each of the following. State each rule as a complete sentence.

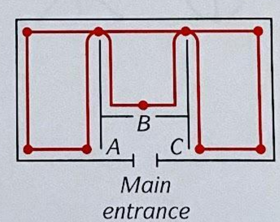
8. A network that has an Euler path that returns to its starting point.
9. A network that has an Euler path that does not return to its starting point.
10. A network that does not have an Euler path.

This figure is the symbol for the Olympics.

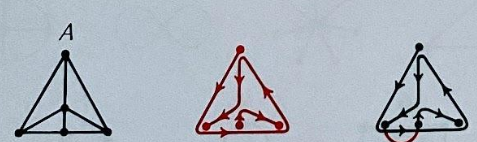


11. Does it have an Euler path?
12. How can you tell without trying to draw it?
13. Place tracing paper over the symbol and try to travel it in a single trip.
14. Does whether or not you are able to travel the symbol depend on the vertex at which you start?

This figure represents the floor plan of a small art museum consisting of a foyer and three exhibit rooms. Pictures are hung on all four walls of each exhibit room.

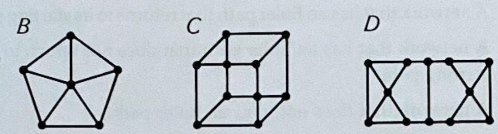


15. Is it possible to start at door A and walk along each of the walls with the paintings exactly once? If so, which door—A, B, or C—would you end at?
16. Is it possible to start at door B and walk along each of the walls with the paintings exactly once? If so, which door—A, B, or C—would you end at?
17. How can you tell from the vertices of the network whether or not the walks described in exercises 15 and 16 are possible?



To draw network A above, two trips are necessary. In other words, the figure can be drawn without retracing any edge if the pencil is removed from the paper once. One way of doing this is shown in the second figure. The leap of the pencil is equivalent to adding an edge. By adding the edge in color, as in the third figure, we can connect two odd vertices and thus make both of them even.

18. Find the fewest number of trips necessary to travel each of the networks below by drawing them.

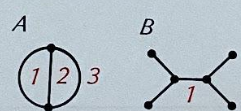


Network	Number of odd vertices	Least number of trips
A	4	2
B		
C		
D		

19. Copy and complete the table at the left for these networks.
20. State a rule for how the number of trips necessary to travel a network is related to the number of odd vertices that it has.

Network A at the left separates the paper into three regions: two are inside the network and the third is the region outside. Network B has just one region: the one outside.

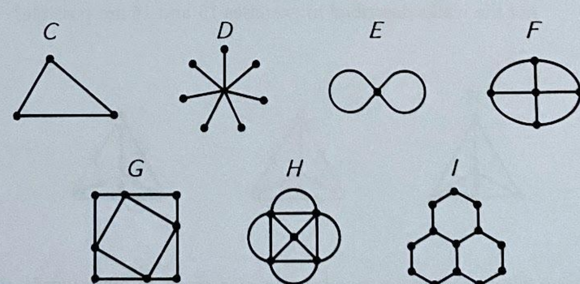
The number of regions of a network is related to the numbers of vertices and edges that it contains.



21. Copy and complete the following table for the nine networks at the left and below.

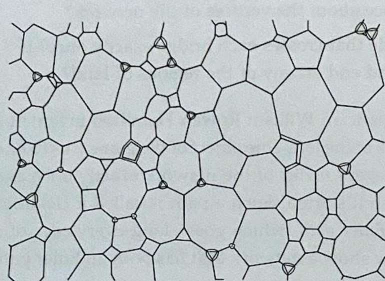
Network	Number of regions	Number of vertices	Number of edges
A	3	2	3
B	1	6	5

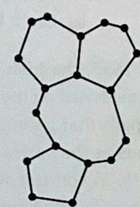
(Continue the table to show all nine networks.)



22. Write a formula relating the number of regions, R, in a network, to the number of vertices, V, and the number of edges, E. (*Hint:* Compare $R + V$ with E.)

This photograph shows cells of soap film between two sheets of glass. Four of the cells are pictured in the network at the right.



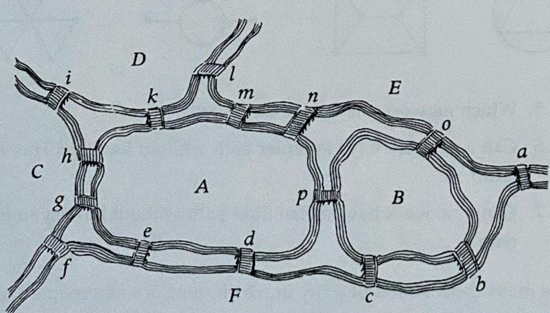


Institut fur leichte Flachentragwerke, Stuttgart, Professor Frei Otto.

23. Count the numbers of regions, vertices, and edges of this network.
24. Does the formula you wrote for exercise 22 fit this network? Show why or why not.

Set II

In his article on the Königsberg bridges, Euler considered another bridge problem, which is illustrated below.*



Two islands, A and B, are surrounded by water that leads to four rivers. Fifteen bridges cross the rivers and the water surrounding the islands. Is it possible to make a trip that crosses each bridge exactly once?

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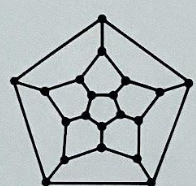
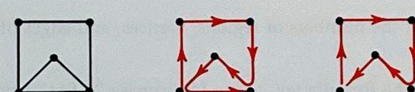
*Euler's article is included in *Mathematics: An Introduction to Its Spirit and Use,* with introduction by Morris Kline (W. H. Freeman and Company, 1979).

1. Draw a network in which six vertices represent the regions of land and 15 edges represent the bridges.
2. What do you notice about the vertices of the network?
3. Can a trip be made that crosses each bridge exactly once? If so, can it begin and end on any of the regions of land?

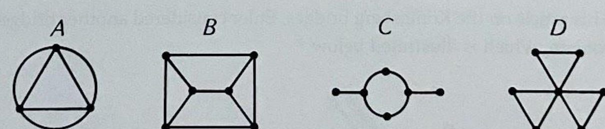


In 1857, the Irish mathematician William Rowan Hamilton invented a game based on the network at the left. One object of the game was to find a path that goes through every *vertex* of the network exactly once and ends at the vertex at which it started. Such a path is called a *Hamilton path,* in contrast with an *Euler path,* which goes along every *edge* of a network. The figures below show a network that has both an Euler path and a Hamilton path.



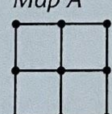
Network *An Euler path* *A Hamilton path*

4. Which of the networks below have Euler paths?



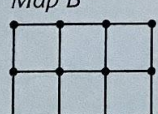
5. Which networks have Hamilton paths?
6. Can a network have an Euler path without having a Hamilton path?
7. Can a network have a Hamilton path without having an Euler path?

Map A



These maps show streets of a city in which there is a newspaper rack at every intersection.

Map B



8. Would someone responsible for keeping the racks filled with newspapers be more interested in an Euler path or a Hamilton path connecting them?
9. Why?
10. Does either map have such a path? If so, sketch the path(s).

Look again at the network for Hamilton's game.

11. Does it have an Euler path? Explain why or why not.
12. Place tracing paper over the figure and see if you can discover a Hamilton path for it.

Set III

This advertisement of the Bell Telephone Laboratories concerns finding the shortest Hamilton path for a given network. According to the advertisement, this problem is "important in many areas of modern business and technology, where 'shortest path' may really mean the least hook-up wire, travel time, or transmission power."

Report from

BELL LABORATORIES

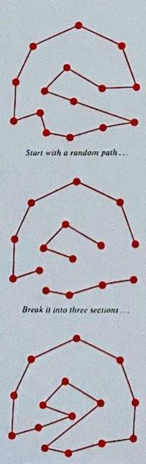
Found: A rapid route to the shortest path

The critical feature of Shen Lin's method is its speed; it makes many good approximations in a reasonable time and selects the best.

To make one approximation, the computer chooses a "starting path" at random. It removes three links of this path (thus breaking it into three sections – see figures) and connects the sections differently to see if a shorter path results. If not, it systematically removes other combinations of three links in the original random path, until all combinations have been tried. But, whenever such a reconnection *does* produce a shorter path, it takes this as a *new* starting path, and begins the series of breaks again. One "approximation" is completed when no further improvement results from such breaking and reconnection.

In the same way – beginning each time with a new and different "starting path" – many additional approximations are found. They usually have some path sections in common; it simplifies the problem to assume that these are part of the absolute minimum path. So, they are routinely incorporated into every new starting path and no longer broken. This speeds computation and the time that's saved is used to find even more approximations.

In general, using a high-speed digital computer, 100 approximations take about $0.75n^3$ milliseconds (n = number of points). For a typical 40-point problem, experiments indicate that about one out of 16 approximations will be the actual minimum solution; for 60 points, about one out of 64. So, if we find 300 approximations in a 60-point problem (roughly eight minutes on a computer) there is a high probability that one of these is the shortest possible.

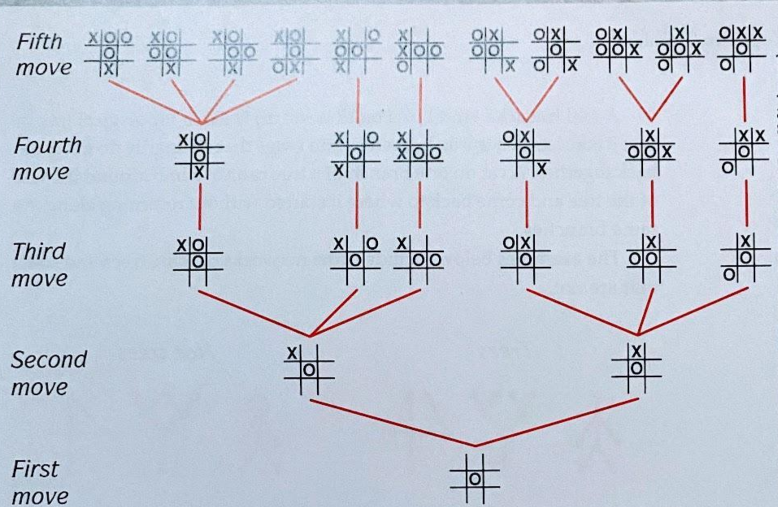


What is the shortest path through a number of points, touching each just once and ending at the starting point? This "traveling salesman problem" is important in many areas of modern business and technology, where "shortest path" may really mean the least hook-up wire, travel time, or transmission power.

It might seem that the problem could be solved by measuring all paths and taking the shortest but, even with a computer, this is a colossal task. At a million paths per second, for instance, it would take several billion years to compute and compare all paths in a 25-point problem! Shortcut methods have been devised, but they are still too slow when, say, 60 points are involved. In practice, approximate solutions (almost-shortest paths) are found largely through the educated judgments of engineers looking at graphs or maps . . . or for certain limited problems, through special computer programs.

Now, mathematician Shen Lin of Bell Telephone Laboratories has developed a new way of getting good approximate solutions to problems of up to 145 points. Because his method is fast, it is possible to find many such approximations. It is then easy to pick the shortest of these. Often (see left), this is the absolute minimum. If not, it is at least short enough for most engineering purposes.

Bell Telephone Laboratories
Research and Development Unit of the Bell System



Adapted from *Human Information Processing*, 2nd ed., p. 561, Peter H. Lindsay and Donald A. Norman, Harcourt Brace Jovanovich, 1977.

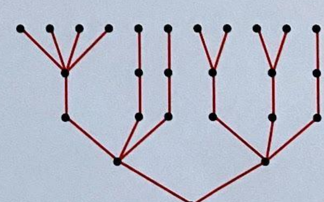
LESSON

4

Trees

One of the simplest of games is tic-tac-toe. Because of its simplicity, a child who plays the game for awhile usually learns how to avoid losing.

Although there are nine spaces in which to make the first move, the number of reasonable choices for successive moves rapidly becomes smaller. For example, if the first player chooses the center space, the second player has only two choices that are actually different: either a corner space or a side space. Possibilities for the first five plays in the game are shown in the diagram above.*



The basic structure of this diagram is shown in the network at the right. Mathematicians call such a network a *tree*.

A **tree** is a network that does not contain any simple closed curves.

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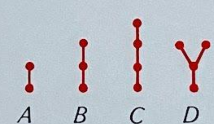
*Adapted from a figure in *Human Information Processing*, by Peter H. Lindsay and Donald A. Norman (Harcourt Brace Jovanovich, 1977.)

A real tree (the kind birds build nests in) is like a topological tree in that it consists of a trunk, branches, and twigs that ordinarily do not grow back together. A cat on one branch of a tree cannot climb around the rest of the tree and come back to where it started without returning along the same branches.

The examples below include some networks that are trees and some that are not.

There is just one tree that contains two vertices and one tree that contains three. They are shown as figures A and B below. There are two different trees that contain four vertices, and they are shown as figures C and D.



The longest path on any of these trees is clearly on the one labeled C: it consists of three edges.

> The **diameter** of a tree is the largest number of edges that a path on the tree can have.

The diameters of the four trees above are 1, 2, 3, and 2, respectively.

Tree networks appear in a wide variety of subjects. You have already seen trees used in mathematics in solving problems in counting and in probability.* Mapmakers draw complicated trees to represent rivers and their tributaries. The map on the next page shows the Mississippi River and some of the other rivers that flow into it. Genealogists use trees to show the relationships between members of different generations of a family. The process used by the postal service to sort mail can be represented by a tree: the ZIP code identifies its main branches. Tree diagrams are drawn by chemists to show the arrangements of atoms in molecules.

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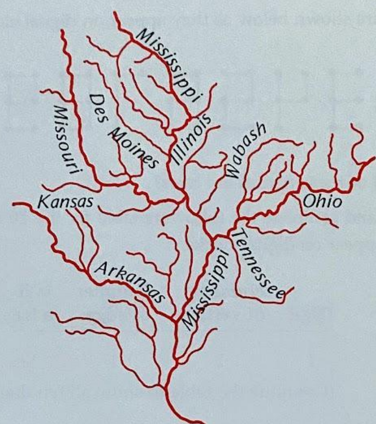
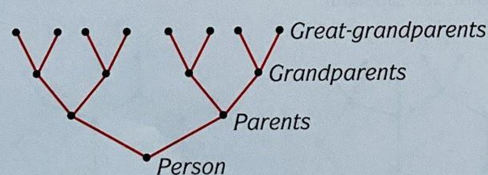
*See, for example, the trees on pages 403 and 458.



EXERCISES

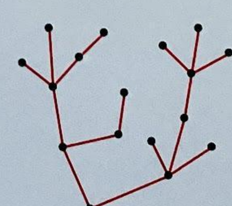
SET I

Four generations of a person's family tree are shown in this figure.



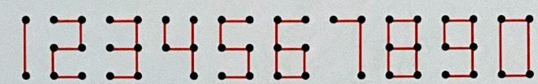
1. How many vertices does this family tree have?
2. How many edges does it have?

This figure illustrates a typical arrangement of stems of a cactus of the genus *Hatiora*.



3. How many vertices does this cactus tree have?
4. How many edges does it have?
5. How is the number of edges in each of the trees for exercises 1–4 related to the number of vertices?

The 10 digits are shown below as they appear on digital clocks.



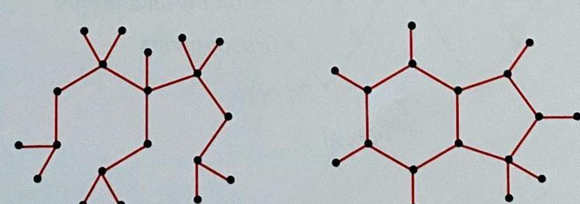
6. Which of these figures are trees?
7. Copy and complete the following table for the 10 digits as they appear on digital clocks.

Digit	Number of vertices	Number of edges	Is it a tree?
1	2	1	Yes

(Continue the table to show all ten digits)

8. Does the pattern you noticed in exercise 5 apply to all of the digits whose digital clock symbols are trees?
9. Write a formula for the pattern, letting V represent the number of vertices of a tree and E represent the number of edges.
10. Try to draw an example of a tree for which this formula is not true.

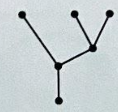
These networks show the arrangement of the atoms in molecules of nitroglycerin and saccharin.



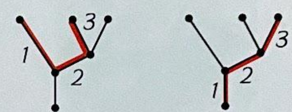
Nitroglycerin *Saccharin*

11. Is the formula you wrote for exercise 9 true for the nitroglycerin molecule? Show why or why not.
12. Is it true for the saccharin molecule? Show why or why not.
13. How many edges of the network for the saccharin molecule would have to be removed in order for it to become a tree?
14. If these edges were removed, would the formula be true for the resulting tree?

The diameter of a tree is the largest number of edges that a path on the tree can have. For example, the diameter of this tree is 3 because no path on it has more than three edges.



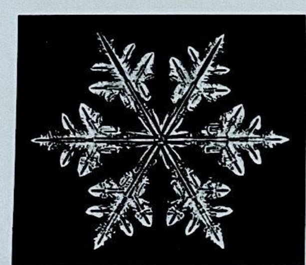
Tree

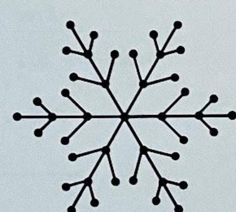


Example paths

15. Are there any paths for this tree other than the ones shown in the examples that also contain three edges?

Many snowflakes have the shapes of trees. Below is a photograph of one taken through a microscope, together with a tree diagram of its structure.



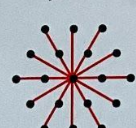


16. What is the diameter of this tree?

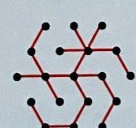
The trees below are adapted from figures in *Patterns in Nature* by Peter S. Stevens.* Notice that each tree has the same number of vertices.

"Spiral"

"Explosion"



"Branching"



17. How many vertices does each tree have?
18. What is the diameter of "Spiral"?
19. What is the diameter of "Explosion"?
20. What is the diameter of "Branching"?

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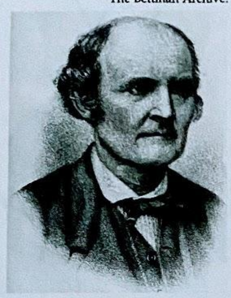
*Little, Brown, 1974.

The Bettman Archive.

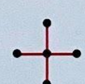


Arthur Cayley.

Set II

In 1875, the English mathematician Arthur Cayley wrote a paper on trees that proved to be particularly useful in chemistry.

In it, he counted the numbers of possible trees having different numbers of vertices. All of the different trees having five or fewer vertices are shown in the table below. An example of a molecule having each structure is also shown.

	Tree	Example molecule
A		Oxygen
B		Water
C		Hydrogen peroxide
D		Ammonia
E		Calcium hydroxide
F		Nitric acid
G		Carbon tetrafluoride

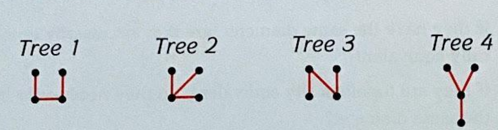
Notice from the table that there is only one tree having two vertices.

How many different trees are there that have

1. three vertices?
2. four vertices?
3. five vertices?

Although the tree shown here may seem different from any of the trees in the table above, it is considered to be the same as tree B. This is because it and tree B both have three vertices and are topologically equivalent (they can be stretched and bent into the same shape).

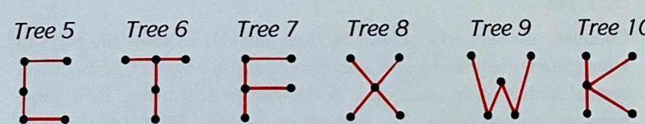
Each of the following trees has four vertices.



Which of these trees are topologically equivalent to

4. tree C?
5. tree D?

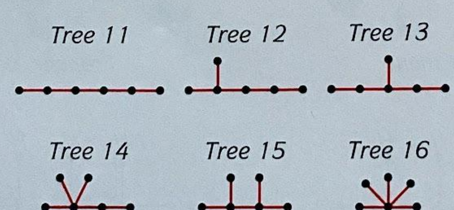
Each of the following trees has five vertices.



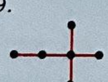
Which of these trees are topologically equivalent to

6. tree E?
7. tree F?
8. tree G?

Cayley showed that there are six different trees that have six vertices.



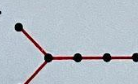
To which of these six trees is each of the following trees topologically equivalent?

9.


10.


11.


12.


13.


14.