Vector entry labels. In applications such as the ones described above, each entry of a vector has a meaning, such as the count of a specific word in a document, the number of shares of a specific stock held in a portfolio, or the rainfall in a specific hour. It is common to keep a separate list of labels or tags that explain or annotate the meaning of the vector entries. As an example, we might associate the portfolio vector $(100, 50, 20)$ with the list of ticker symbols (AAPL, INTC, AMZN), so we know that assets 1, 2, and 3 are Apple, Intel, and Amazon. In some applications, such as an image, the meaning or ordering of the entries follow known conventions or standards.

1.2 Vector addition

Two vectors *of the same size* can be added together by adding the corresponding elements, to form another vector of the same size, called the *sum* of the vectors. Vector addition is denoted by the symbol $+$. (Thus the symbol $+$ is overloaded to mean scalar addition when scalars appear on its left- and right-hand sides, and vector addition when vectors appear on its left- and right-hand sides.) For example,

$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}.$$

Vector subtraction is similar. As an example,

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}.$$

The result of vector subtraction is called the *difference* of the two vectors.

Properties. Several properties of vector addition are easily verified. For any vectors a, b, and c of the same size we have the following.

- Vector addition is *commutative*: $a + b = b + a$.
- Vector addition is *associative*: $(a + b) + c = a + (b + c)$. We can therefore write both as $a + b + c$.
- $a + 0 = 0 + a = a$. Adding the zero vector to a vector has no effect. (This is an example where the size of the zero vector follows from the context: It must be the same as the size of a.)
- $a - a = 0$. Subtracting a vector from itself yields the zero vector. (Here too the size of 0 is the size of a.)

To show that these properties hold, we argue using the definition of vector addition and vector equality. As an example, let us show that for any n-vectors a and b, we have $a + b = b + a$. The ith entry of $a + b$ is, by the definition of vector

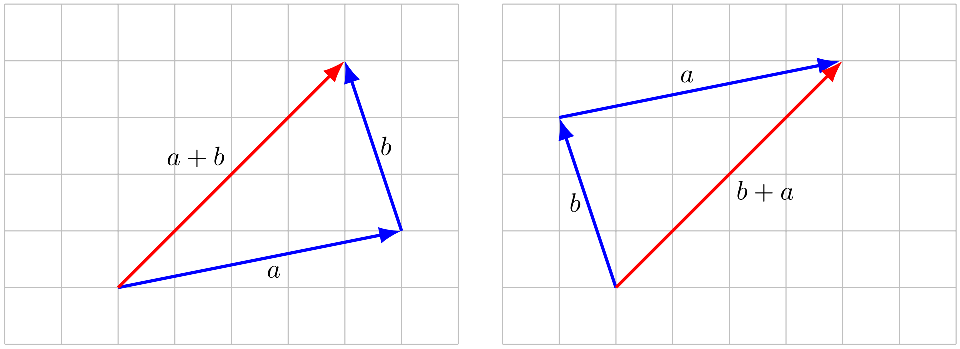

Figure 1.6 *Left.* The lower blue arrow shows the displacement a; the displacement b, shown as the shorter blue arrow, starts from the head of the displacement a and ends at the sum displacement $a + b$, shown as the red arrow. *Right.* The displacement $b + a$.

addition, $a_i + b_i$. The ith entry of $b + a$ is $b_i + a_i$. For any two numbers we have $a_i + b_i = b_i + a_i$, so the ith entries of the vectors $a + b$ and $b + a$ are the same. This is true for all of the entries, so by the definition of vector equality, we have $a + b = b + a$.

Verifying identities like the ones above, and many others we will encounter later, can be tedious. But it is important to understand that the various properties we will list can be derived using elementary arguments like the one above. We recommend that the reader select a few of the properties we will see, and attempt to derive them, just to see that it can be done. (Deriving all of them is overkill.)

Examples.

- *Displacements.* When vectors a and b represent displacements, the sum $a+b$ is the net displacement found by first displacing by a, then displacing by b, as shown in figure 1.6. Note that we arrive at the same vector if we first displace by b and then a. If the vector p represents a position and the vector a represents a displacement, then $p+a$ is the position of the point p, displaced by a, as shown in figure 1.7.

- *Displacements between two points.* If the vectors p and q represent the positions of two points in 2-D or 3-D space, then $p - q$ is the displacement vector from q to p, as illustrated in figure 1.8.

- *Word counts.* If a and b are word count vectors (using the same dictionary) for two documents, the sum $a+b$ is the word count vector of a new document created by combining the original two (in either order). The word count difference vector $a - b$ gives the number of times more each word appears in the first document than the second.

- *Bill of materials.* Suppose $q_1, \ldots, q_N$ are n-vectors that give the quantities of n different resources required to accomplish N tasks. Then the sum n-vector $q_1 + \cdots + q_N$ gives the bill of materials for completing all N tasks.

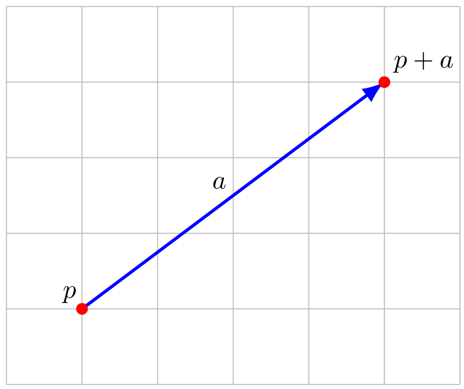

Figure 1.7 The vector $p + a$ is the position of the point represented by p displaced by the displacement represented by a.

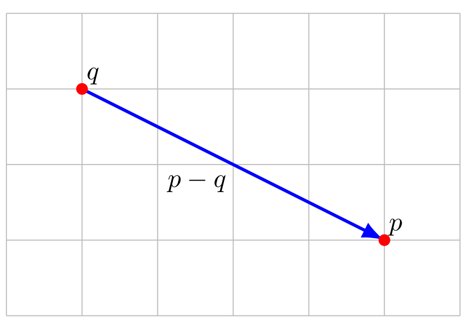

Figure 1.8 The vector $p - q$ represents the displacement from the point represented by q to the point represented by p.

- *Market clearing.* Suppose the n-vector q_i represents the quantities of n goods or resources produced (when positive) or consumed (when negative) by agent i, for $i = 1, \ldots, N$, so $(q_5)_4 = -3.2$ means that agent 5 consumes 3.2 units of resource 4. The sum $s = q_1 + \cdots + q_N$ is the n-vector of total net surplus of the resources (or shortfall, when the entries are negative). When $s = 0$, we have a closed market, which means that the total quantity of each resource produced by the agents balances the total quantity consumed. In other words, the n resources are *exchanged* among the agents. In this case we say that the *market clears* (with the resource vectors $q_1, \ldots, q_N$).

- *Audio addition.* When a and b are vectors representing audio signals over the same period of time, the sum $a + b$ is an audio signal that is perceived as containing both audio signals combined into one. If a represents a recording of a voice, and b a recording of music (of the same length), the audio signal $a+b$ will be perceived as containing both the voice recording and, simultaneously, the music.

- *Feature differences.* If f and g are n-vectors that give n feature values for two items, the difference vector $d = f - g$ gives the difference in feature values for the two objects. For example, $d_7 = 0$ means that the two objects have the same value for feature 7; $d_3 = 1.67$ means that the first object's third feature value exceeds the second object's third feature value by 1.67.

- *Time series.* If a and b represent time series of the same quantity, such as daily profit at two different stores, then $a+b$ represents a time series which is the total daily profit at the two stores. An example (with monthly rainfall) is shown in figure 1.9.

- *Portfolio trading.* Suppose s is an n-vector giving the number of shares of n assets in a portfolio, and b is an n-vector giving the number of shares of the assets that we buy (when b_i is positive) or sell (when b_i is negative). After the asset purchases and sales, our portfolio is given by $s + b$, the sum of the original portfolio vector and the purchase vector b, which is also called the *trade vector* or *trade list*. (The same interpretation works when the portfolio and trade vectors are given in dollar value.)

Addition notation in computer languages. Some computer languages for manipulating vectors define the sum of a vector and a scalar as the vector obtained by adding the scalar to each element of the vector. This is not standard mathematical notation, however, so we will not use it. Even more confusing, in some computer languages the plus symbol is used to denote concatenation of arrays, which means putting one array after another, as in $(1, 2) + (3, 4, 5) = (1, 2, 3, 4, 5)$. While this notation might give a valid expression in some computer languages, it is not standard mathematical notation, and we will not use it in this book. In general, it is very important to distinguish between mathematical notation for vectors (which we use) and the syntax of specific computer languages or software packages for manipulating vectors.

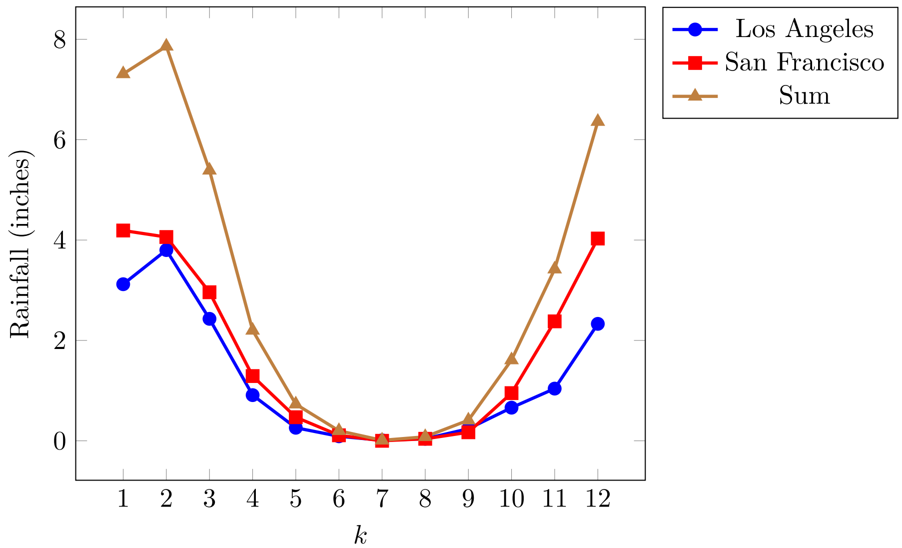

Figure 1.9 Average monthly rainfall in inches measured in downtown Los Angeles and San Francisco International Airport, and their sum. Averages are 30-year averages (1981–2010).

1.3 Scalar-vector multiplication

Another operation is *scalar multiplication* or *scalar-vector multiplication*, in which a vector is multiplied by a scalar (*i.e.*, number), which is done by multiplying every element of the vector by the scalar. Scalar multiplication is denoted by juxtaposition, typically with the scalar on the left, as in

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}.$$

Scalar-vector multiplication can also be written with the scalar on the right, as in

$$\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix}(1.5) = \begin{bmatrix} 1.5 \\ 13.5 \\ 9 \end{bmatrix}.$$

The meaning is the same: It is the vector obtained by multiplying each element by the scalar. A similar notation is $a/2$, where a is a vector, meaning $(1/2)a$. The scalar-vector product $(-1)a$ is written simply as $-a$. Note that $0\,a = 0$ (where the left-hand zero is the scalar zero, and the right-hand zero is a vector zero of the same size as a).

Properties. By definition, we have $\alpha a = a\alpha$, for any scalar α and any vector a. This is called the *commutative property* of scalar-vector multiplication; it means that scalar-vector multiplication can be written in either order.