axis of the airplane. For an airplane in level flight, heading due South, we have

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.2 Selectors

An $m \times n$ *selector matrix* A is one in which each row is a unit vector (transposed):

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix},$$

where $k_1, \ldots, k_m$ are integers in the range $1, \ldots, n$. When it multiplies a vector, it simply copies the k_ith entry of x into the ith entry of $y = Ax$:

$$y = (x_{k_1}, x_{k_2}, \ldots, x_{k_m}).$$

In words, each entry of Ax is a selection of an entry of x.

The identity matrix, and the *reverser matrix*

$$A = \begin{bmatrix} e_n^T \\ \vdots \\ e_1^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are special cases of selector matrices. (The reverser matrix reverses the order of the entries of a vector: $Ax = (x_n, x_{n-1}, \ldots, x_2, x_1)$.) Another one is the $r\!:\!s$ *slicing matrix*, which can be described as the block matrix

$$A = \begin{bmatrix} 0_{m\times(r-1)} & I_{m\times m} & 0_{m\times(n-s)} \end{bmatrix},$$

where $m = s - r + 1$. (We show the dimensions of the blocks for clarity.) We have $Ax = x_{r:s}$, *i.e.*, multiplying by A gives the $r\!:\!s$ slice of a vector.

Down-sampling. Another example is the $(n/2) \times n$ matrix (with n even)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $y = Ax$, we have $y = (x_1, x_3, x_5, \ldots, x_{n-3}, x_{n-1})$. When x is a time series, y is called the $2\times$ *down-sampled* version of x. If x is a quantity sampled every hour, then y is the same quantity, sampled every 2 hours.

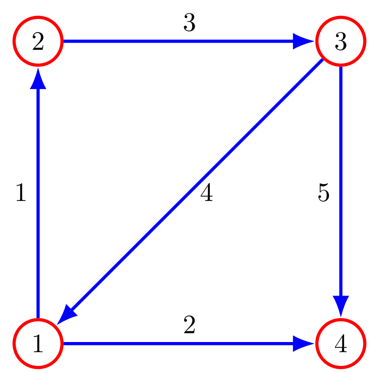

Figure 7.2 Directed graph with four vertices and five edges.

Image cropping. As a more interesting example, suppose that x is an image with $M \times N$ pixels, with M and N even. (That is, x is an MN-vector, with its entries giving the pixel values in some specific order.) Let y be the $(M/2) \times (N/2)$ image that is the upper left corner of the image x, *i.e.*, a cropped version. Then we have $y = Ax$, where A is an $(MN/4) \times (MN)$ selector matrix. The ith row of A is $e_{k_i}^T$, where k_i is the index of the pixel in x that corresponds to the ith pixel in y.

Permutation matrices. An $n \times n$ *permutation matrix* is one in which each column is a unit vector, and each row is the transpose of a unit vector. (In other words, A and A^T are both selector matrices.) Thus, exactly one entry of each row is one, and exactly one entry of each column is one. This means that $y = Ax$ can be expressed as $y_i = x_{\pi_i}$, where π is a permutation of $1, 2, \ldots, n$, *i.e.*, each integer from 1 to n appears exactly once in $\pi_1, \ldots, \pi_n$.

As a simple example consider the permutation $\pi = (3, 1, 2)$. The associated permutation matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying a 3-vector by A re-orders its entries: $Ax = (x_3, x_1, x_2)$.

7.3 Incidence matrix

Directed graph. A *directed graph* consists of a set of *vertices* (or nodes), labeled $1, \ldots, n$, and a set of *directed edges* (or branches), labeled $1, \ldots, m$. Each edge is connected from one of the nodes and into another one, in which case we say the two nodes are connected or adjacent. Directed graphs are often drawn with the vertices as circles or dots, and the edges as arrows, as in figure 7.2. A directed