University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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# Sets and Functions

## Countable and Uncountable Sets

# Countable and Uncountable Sets

So far we've discussed sets like $A = \{a, b, c\}$ where, e.g. $|A| = 3$

Such a set is said to be a finite set or have finite cardinality

But we've not talked much about the cardinality of sets like $\{n \in \mathbb{N} \mid n^2\}$ which clearly has an infinite number of elements

OK, so shouldn't the cardinality of $B = \{n \in \mathbb{N} \mid n^2\}$ be $|B| = \infty$ and we're done?

# Countable and Uncountable Sets

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Note quite, it turns out that it's useful to break up just **how infinite** a set is into two classes. Roughly they are described as follows:

**Countably Infinite**: We could count each member of the set if we had infinite time

**Uncountable**: We could never even list each element of the set even in infinite time

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Note quite, it turns out that it's useful to break up just **how infinite** a set is into two classes. Roughly they are described as follows:

**Countably Infinite**: We could count each member of the set if we had infinite time

**Uncountable**: We could never even list each element of the set even in infinite time

**Def**: An infinite set $A$ is called **countable** or **countably infinite** if there is a one-to-one map between the natural numbers and each element of $A$. A set $A$ is called **uncountable** if it is infinite but not countable.

# Countable and Uncountable Sets

**Example**: Show that the set of positive even integers is countable

We want to find a one-to-one map between the positive even integers $\{2, 4, 5, 8, 10, \ldots\}$ and the natural numbers $\{0, 1, 2, \ldots\}$

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We want to find a one-to-one map between the positive even integers $\{2, 4, 5, 8, 10, \ldots\}$ and the natural numbers $\{0, 1, 2, \ldots\}$

This one's pretty straightforward. We have

$$
\begin{array}{ccc}
\mathbb{N} & & \text{Evens} \\
\hline
0 & \Leftrightarrow & 2 \\
1 & \Leftrightarrow & 4 \\
2 & \Leftrightarrow & 6 \\
 & \vdots &
\end{array}
$$

**Better Yet**: Define function relationship $f(n) = 2n + 2$

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We want to find a one-to-one map between the natural numbers and $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

| $\mathbb{N}$ | | Integers |
|---|---|---|
| 0 | $\Leftrightarrow$ | 0 |
| 1 | $\Leftrightarrow$ | −1 |
| 2 | $\Leftrightarrow$ | 1 |
| 3 | $\Leftrightarrow$ | −2 |
| 4 | $\Leftrightarrow$ | 2 |
| | $\vdots$ | |

**Better Yet:** Define function $f(n) = (-1)^n \lfloor (n+1) / 2 \rfloor$

# Countable and Uncountable Sets

**Example:** The positive rational numbers are countable

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

# Countable and Uncountable Sets

**Example**: The real numbers are uncountable. Let's look at just $[0, 1]$

$$
\begin{array}{cccccccccccccc}
0. & 1 & 2 & 3 & 4 & 3 & 5 & 3 & 4 & 5 & 3 & 0 & 8 & \cdots \\
0. & 9 & 8 & 0 & 8 & 0 & 8 & 0 & 9 & 0 & 9 & 0 & 9 & \cdots \\
0. & 7 & 5 & 0 & 0 & 3 & 8 & 4 & 2 & 3 & 4 & 0 & 8 & \cdots \\
0. & 0 & 8 & 2 & 3 & 4 & 0 & 8 & 2 & 4 & 3 & 0 & 8 & \cdots \\
0. & 5 & 9 & 8 & 2 & 3 & 6 & 1 & 5 & 3 & 8 & 9 & 4 & \cdots \\
0. & 8 & 9 & 2 & 4 & 7 & 8 & 2 & 3 & 4 & 6 & 5 & 9 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Let's suppose we can list them all, and **look for a contradiction**

# Countable and Uncountable Sets

**Example**: The real numbers are uncountable. Let's look at just $[0, 1]$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0. | 1 | 2 | 3 | 4 | 3 | 5 | 3 | 4 | 5 | 3 | 0 | 8 | ⋯ |
| 0. | 9 | 8 | 0 | 8 | 0 | 8 | 0 | 9 | 0 | 9 | 0 | 9 | ⋯ |
| 0. | 7 | 5 | 0 | 0 | 3 | 8 | 4 | 2 | 3 | 4 | 0 | 8 | ⋯ |
| 0. | 0 | 8 | 2 | 3 | 4 | 0 | 8 | 2 | 4 | 3 | 0 | 8 | ⋯ |
| 0. | 5 | 9 | 8 | 2 | 3 | 6 | 1 | 5 | 3 | 8 | 9 | 4 | ⋯ |
| 0. | 8 | 9 | 2 | 4 | 7 | 8 | 2 | 3 | 4 | 6 | 5 | 9 | ⋯ |
| | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |

Let's suppose we can list them all, and **look for a contradiction**

**Contradiction**: We'll **construct** a number that can't be in the list

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**Example**: The real numbers are uncountable. Let's look at just $[0, 1]$

$$
\begin{array}{cccccccccccccc}
0. & 1 & 2 & 3 & 4 & 3 & 5 & 3 & 4 & 5 & 3 & 0 & 8 & \cdots \\
0. & 9 & 8 & 0 & 8 & 0 & 8 & 0 & 9 & 0 & 9 & 0 & 9 & \cdots \\
0. & 7 & 5 & 0 & 0 & 3 & 8 & 4 & 2 & 3 & 4 & 0 & 8 & \cdots \\
0. & 0 & 8 & 2 & 3 & 4 & 0 & 8 & 2 & 4 & 3 & 0 & 8 & \cdots \\
0. & 5 & 9 & 8 & 2 & 3 & 6 & 1 & 5 & 3 & 8 & 9 & 4 & \cdots \\
0. & 8 & 9 & 2 & 4 & 7 & 8 & 2 & 3 & 4 & 6 & 5 & 9 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Let's suppose we can list them all, and **look for a contradiction**

**Contradiction**: We'll **construct** a number that can't be in the list

**Strategy**: Set the $k^{\text{th}}$ digit of our new number based on the $k^{\text{th}}$ digit of the $k^{\text{th}}$ number in list according to a rule

# Countable and Uncountable Sets

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$$\begin{array}{cccccccccccccc}
0. & 1 & 2 & 3 & 4 & 3 & 5 & 3 & 4 & 5 & 3 & 0 & 8 & \cdots \\
0. & 9 & 8 & 0 & 8 & 0 & 8 & 0 & 9 & 0 & 9 & 0 & 9 & \cdots \\
0. & 7 & 5 & 0 & 0 & 3 & 8 & 4 & 2 & 3 & 4 & 0 & 8 & \cdots \\
0. & 0 & 8 & 2 & 3 & 4 & 0 & 8 & 2 & 4 & 3 & 0 & 8 & \cdots \\
0. & 5 & 9 & 8 & 2 & 3 & 6 & 1 & 5 & 3 & 8 & 9 & 4 & \cdots \\
0. & 8 & 9 & 2 & 4 & 7 & 8 & 2 & 3 & 4 & 6 & 5 & 9 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

**Rule**:

- If $k^{\text{th}}$ digit of the $k^{\text{th}}$ number is a 3, our number's is a 5
- If $k^{\text{th}}$ digit of the $k^{\text{th}}$ number is not a 3, our number's is a 3

# Countable and Uncountable Sets

**Example**: The real numbers are uncountable. Let's look at just $[0, 1]$

$$
\begin{array}{cccccccccccccc}
0. & 1 & 2 & 3 & 4 & 3 & 5 & 3 & 4 & 5 & 3 & 0 & 8 & \cdots \\
0. & 9 & 8 & 0 & 8 & 0 & 8 & 0 & 9 & 0 & 9 & 0 & 9 & \cdots \\
0. & 7 & 5 & 0 & 0 & 3 & 8 & 4 & 2 & 3 & 4 & 0 & 8 & \cdots \\
0. & 0 & 8 & 2 & 3 & 4 & 0 & 8 & 2 & 4 & 3 & 0 & 8 & \cdots \\
0. & 5 & 9 & 8 & 2 & 3 & 6 & 1 & 5 & 3 & 8 & 9 & 4 & \cdots \\
0. & 8 & 9 & 2 & 4 & 7 & 8 & 2 & 3 & 4 & 6 & 5 & 9 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

**Rule**:

- If $k^{\text{th}}$ digit of the $k^{\text{th}}$ number is a 3, our number's is a 5
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$$m = 0\,.\;3\;3\;3\;5\;5\;3\;\cdots$$

# Countable and Uncountable Sets

**Claim**: Our constructed number, $m$, can't already be in the list

**Argument**:

1. $m$ isn't the $1^{\text{st}}$ number b/c their $1^{\text{st}}$ digits don't match
2. $m$ isn't the $2^{\text{nd}}$ number b/c their $2^{\text{nd}}$ digits don't match
3. $m$ isn't the $3^{\text{rd}}$ number b/c their $3^{\text{rd}}$ digits don't match

and so on and so on ..

Thus we've constructed an $m$ that can't be in the list

This is our contradiction that proves that the real numbers in $[0, 1]$ are uncountable

This proof is called **Cantor's Diagonal Argument**