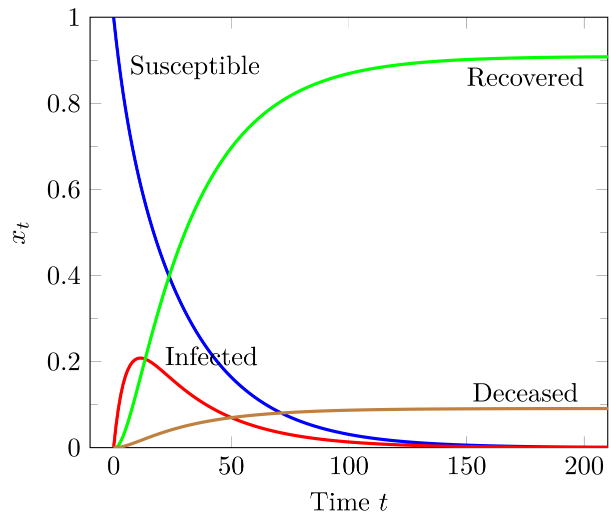

Figure 9.5 Simulation of epidemic dynamics.

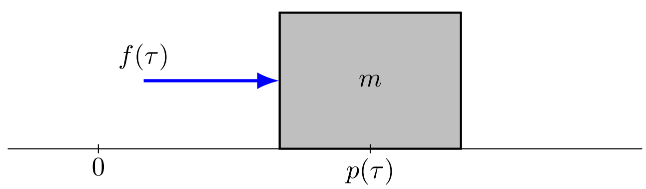

Figure 9.6 Mass moving along a line.

9.4 Motion of a mass

Linear dynamical systems can be used to (approximately) describe the motion of many mechanical systems, for example, an airplane (that is not undergoing extreme maneuvers), or the (hopefully not too large) movement of a building during an earthquake. Here we describe the simplest example: A single mass moving in 1-D (*i.e.*, a straight line), with an external force and a drag force acting on it. This is illustrated in figure 9.6. The (scalar) position of the mass at time τ is given by $p(\tau)$. (Here τ is continuous, *i.e.*, a real number.) The position satisfies Newton's law of motion, the differential equation

$$m\frac{d^2p}{d\tau^2}(\tau) = -\eta\frac{dp}{d\tau}(\tau) + f(\tau),$$

where $m > 0$ is the mass, $f(\tau)$ is the external force acting on the mass at time τ, and $\eta > 0$ is the drag coefficient. The right-hand side is the total force acting on the mass; the first term is the drag force, which is proportional to the velocity and in the opposite direction.

Introducing the velocity of the mass, $v(\tau) = dp(\tau)/d\tau$, we can write the equation above as two coupled differential equations,

$$\frac{dp}{d\tau}(\tau) = v(\tau), \qquad m\frac{dv}{d\tau}(\tau) = -\eta v(\tau) + f(\tau).$$

The first equation relates the position and velocity; the second is from the law of motion.

Discretization. To develop an (approximate) linear dynamical system model from the differential equations above, we first discretize time. We let $h > 0$ be a time interval (called the 'sampling interval') that is small enough that the velocity and forces do not change very much over h seconds. We define

$$p_k = p(kh), \qquad v_k = v(kh), \qquad f_k = f(kh),$$

which are the continuous quantities 'sampled' at multiples of h seconds. We now use the approximations

$$\frac{dp}{d\tau}(kh) \approx \frac{p_{k+1} - p_k}{h}, \qquad \frac{dv}{d\tau}(kh) \approx \frac{v_{k+1} - v_k}{h}, \tag{9.5}$$

which are justified since h is small. This leads to the (approximate) equations (replacing $\approx$ with $=$)

$$\frac{p_{k+1} - p_k}{h} = v_k, \qquad m\frac{v_{k+1} - v_k}{h} = f_k - \eta v_k.$$

Finally, using state $x_k = (p_k, v_k)$, we write this as

$$x_{k+1} = \begin{bmatrix} 1 & h \\ 0 & 1 - h\eta/m \end{bmatrix} x_k + \begin{bmatrix} 0 \\ h/m \end{bmatrix} f_k, \quad k = 1, 2, \ldots,$$

which is a linear dynamical system of the form (9.2), with input f_k and dynamics and input matrices

$$A = \begin{bmatrix} 1 & h \\ 0 & 1 - h\eta/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ h/m \end{bmatrix}.$$

This linear dynamical system gives an approximation of the true motion, due to our approximation (9.5) of the derivatives. But for h small enough, it is accurate. This linear dynamical system can be used to simulate the motion of the mass, if we know the external force applied to it, *i.e.*, u_t for $t = 1, 2, \ldots$.

The approximation (9.5), which turns a set of differential equations into a recursion that approximates it, is called the *Euler method*, named after the mathematician Leonhard Euler. (There are other, more sophisticated, methods for approximating differential equations as recursions.)

Example. As a simple example, we consider the case with $m = 1$ (kilogram), $\eta = 1$ (Newtons per meter per second), and sampling period $h = 0.01$ (seconds). The external force is

$$f(\tau) = \begin{cases} 0.0 & 0.0 \leq \tau < 0.5 \\ 1.0 & 0.5 \leq \tau < 1.0 \\ -1.3 & 1.0 \leq \tau < 1.4 \\ 0.0 & 1.4 \leq \tau. \end{cases}$$

We simulate this system for a period of 2.5 seconds, starting from initial state $x_1 = (0, 0)$, which corresponds to the mass starting at rest (zero velocity) at position 0. The simulation involves iterating the dynamics equation from $k = 1$ to $k = 250$. Figure 9.7 shows the force, position, and velocity of the mass, with the axes labeled using continuous time τ.

9.5 Supply chain dynamics

The dynamics of a supply chain can often be modeled using a linear dynamical system. (This simple model does not include some important aspects of a real supply chain, for example limits on storage at the warehouses, or the fact that demand fluctuates.) We give a simple example here.

We consider a supply chain for a single divisible commodity (say, oil or gravel, or discrete quantities so small that their quantities can be considered real numbers). The commodity is stored at n warehouses or storage locations. Each of these locations has a target (desired) level or amount of the commodity, and we let the n-vector x_t denote the *deviations* of the levels of the commodities from their target levels. For example, $(x_5)_3$ is the actual commodity level at location 3, in period 5, minus the target level for location 3. If this is positive it means we have more than the target level at the location; if it is negative, we have less than the target level at the location.

The commodity is moved or transported in each period over a set of m transportation links between the storage locations, and also enters and exits the nodes through purchases (from suppliers) and sales (to end-users). The purchases and sales are given by the n-vectors p_t and s_t, respectively. We expect these to be positive; but they can be negative if we include returns. The net effect of the purchases and sales is that we add $(p_t - s_t)_i$ of the commodity at location i. (This number is negative if we sell more than we purchase at the location.)

We describe the links by the $n \times m$ incidence matrix A^{sc} (see §7.3). The direction of each link does not indicate the direction of commodity flow; it only sets the *reference direction* for the flow: Commodity flow in the direction of the link is considered positive and commodity flow in the opposite direction is considered negative. We describe the commodity flow in period t by the m-vector f_t. For example, $(f_6)_2 = -1.4$ means that in time period 6, 1.4 units of the commodity are moved along link 2 in the direction opposite the link direction (since the flow is negative). The n-vector $A^{\text{sc}} f_t$ gives the net flow of the commodity into the n locations, due to the transport across the links.