Exam 2 Notes

Matrix Types: Matrix sizes are defined in terms of their rows (m) by columns (n):

- **Square Matrix:** $m = n$.
- **Tall Matrix:** $m > n$.
- **Wide Matrix:** $m < n$.

2D Rotation Matrix: The 2D rotation matrix is:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

3D Rotation Matrices: The 3D rotation matrices are:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Incidence Matrix: An incidence matrix is of the form:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

The rows and columns of an incidence matrix represent:

- *Columns:* Represent the edges of the graph.
- *Rows:* Represent the vertices of the graph.

The values of the elements indicate:

- 1: Edge j points to node i.
- -1: Edge j points from node i.
- 0: Otherwise.

Dirichlet Energy: Measures the potential differences across edges:

$$\mathcal{D}(v) = \|A^T\|^2 = \sum_{(k,l)\in \text{Edges}} (v_l - v_k)^2$$

Matrix Multiplication: Matrix multiplication is performed by:

$$\begin{aligned} AB &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \end{aligned}$$

The main properties of matrix multiplication are:

- *Associativity:* $(AB)C = A(BC)$
- *Scalar Multiplication:* $\gamma(AB) = (\gamma A)B$
- *Distributivity:*
  - $A(B + C) = AB + AC$
  - $(A + B)C = AC + BC$
- *Transpose Product:* $(AB)^T = B^T A^T$

Some Matrix Representations: Some matrix representations are:

- *Column Interpretation:*

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_n \end{bmatrix}$$

- *Row Interpretation:*

$$AB = \begin{bmatrix} (B^T a_1)^T \\ (B^T a_2)^T \\ \vdots \\ (B^T a_m)^T \end{bmatrix}$$

- *Gram Matrix:*

$$G = A^T A$$

- *QR Factorization:*

$$A = QR$$

Matrix Inverses: The rules for if a matrix is invertible:

- Matrix must be square.
- The columns are linearly independent.
- The rows are linearly independent.
- The matrix has a left inverse.
- The matrix has a right inverse.

For a 2×2 matrix the inverse is:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3×3 matrix the inverse is:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} \begin{Vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{Vmatrix} & \begin{Vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{Vmatrix} & \begin{Vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{Vmatrix} \\ \begin{Vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{Vmatrix} & \begin{Vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{Vmatrix} & \begin{Vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{Vmatrix} \\ \begin{Vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{Vmatrix} & \begin{Vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{Vmatrix} & \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \end{bmatrix}$$