University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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Discrete Probability

Introduction to Discrete Probability

Intro to Discrete Probability

Discrete Probability shares many similarities with Combinatorics

First developed over 300 years ago to analyze gambling games

Since become fundamental in countless CS applications

- Study of Complexity of Algorithms
- Artificial Intelligence
- Game Design
- Data Science
- Machine Learning
- Bioinformatics

Intro to Discrete Probability

In Discrete Probability we are concerned with the probability of a finite event occurring

Simply, the probability of an event in an experiment is the ratio of the number of ways the event can occur divided by the number of possible outcomes

Example: Suppose the experiment is to roll a die and the event is that an even number is rolled. Find the probability.

Intro to Discrete Probability

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Example: Suppose the experiment is to roll a die and the event is that an even number is rolled. Find the probability.

There are six possible outcomes: $1, 2, 3, 4, 5, 6$

There are three possible ways to roll and even number: $2, 4, 6$

Therefore the probability that an even number is rolled is $\frac{3}{6} = \frac{1}{2}$

Intro to Discrete Probability

Def: An **experiment** is a procedure that yields one of a given set of possible outcomes.

Def: The **sample space** of the experiment is the set of possible outcomes.

Def: An **event** is a subset of the sample space.

Coin Flip Example: Suppose that the experiment is flipping 3 coins and we want to know the probability that all three coins show the same side of the coin

Sample Space: The set of all possible combos of three flips

Event: The set of all combos where all three coins are the same

Intro to Discrete Probability

Def: If S is a finite nonempty sample space of **equally likely** outcomes, and E is an event, that is, a subset of S, then the **probability** of E, written $p(E)$, is

$$p(E) = \frac{|E|}{|S|}$$

Coin Flip Example: Listing all possible combinations

$$S =$$

$$E =$$

So the probability of the event is $p(E) = \dfrac{|E|}{|S|} =$

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Coin Flip Example: Listing all possible combinations

$$\begin{aligned} S = \{&(HHH), (HHT), (HTH), (HTT), \\ &(THH), (THT), (TTH), (TTT)\} \end{aligned}$$

$$E = \{(HHH), (TTT)\}$$

So the probability of the event is $p(E) = \dfrac{|E|}{|S|} = \dfrac{2}{8} = \dfrac{1}{4}$

Intro to Discrete Probability

Note that from the definition $p(E) = \dfrac{|E|}{|S|}$ it is clear that

$$0 \le p(E) \le 1$$

Dice Sum Example: What is the probability that when two dice are rolled, the numbers on the two dice sum to 7?

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Dice Sum Example: What is the probability that when two dice are rolled, the numbers on the two dice sum to 7?

Without explicitly listing the entire sample space we know that it contains $6 \times 6 = 36$ possible outcomes

Of those 36 possible rolls there are 6 ways to make 7

$$(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$$

So the probability of summing to 7 is $p(E) = \dfrac{6}{36} = \dfrac{1}{6}$

Intro to Discrete Probability

Example: Consider a lottery that awards a large prize to anyone that correctly chooses a set of 6 distinct numbers out of a possible n numbers. What is the probability of winning a lottery that selects 6 numbers out of 40?

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Note that we don't care about the order of the selection and repetitions are not allowed.

There is exactly 1 way to select the correct combination of numbers

The number of possible combinations is $C(40, 6)$ or

$$C(40, 6) = \frac{40!}{34!6!} = 3,838,380$$

So the probability of winning is $\dfrac{1}{3,838,380} \approx 0.00000026$

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Example: In Powerball contestants select 5 white balls and 1 Powerball where the white balls are picked at random from a drum that holds 69 balls numbered from 1 to 69. The Powerball number is a single ball that is picked from a second drum that has 26 numbers ranging from 1 to 26. What is the probability of winning the Jackpot?

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There is again 1 way to choose the correct numbers

There are $C(69, 5)C(26, 1)$ possible ways to choose the balls

$$C(69, 5)C(26, 1) = \frac{69!}{61!5!}\frac{26!}{25!1!} = 292,201,338$$

So the probability of winning is $\dfrac{1}{292,201,338} \approx 0.000000003422$

Intro to Discrete Probability

Example: What is the probability that a person is dealt a flush in a 5-card poker game? A flush is 5 cards of the same suit, but not in sequential order.

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Strategy: Compute the number of ways to draw 5 cards of the same suit and subtract the number of those that are straights

The number of combos of the same suit is given by

$$C(4,1)C(13,5) = \frac{4!}{3!1!}\frac{13!}{11!5!} = 5148$$

The number of straights of the same suit is

$$4 \times |\{(A\ 2\ 3\ 4\ 5), (2\ 3\ 4\ 5\ 6), \ldots, (10\ J\ Q\ K\ A)\}| = 40$$

So the total number of flushes is $5148 - 40 = 5108$

Intro to Discrete Probability

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Strategy: Compute the number of ways to draw 5 cards of the same suit and subtract the number of those that are straights

The number of possible hands is $C(52, 5) = 2,598,960$

So the probability of drawing a flush is

$$p(\text{Flush}) = \frac{5,108}{2,598,960} \approx 0.001965$$

Intro to Discrete Probability

Probabilities of Complements and Unions of Events

Often times we want to find the probability of an event **not** occurring

Theorem: Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complement of E, is given by

$$p(\bar{E}) = 1 - p(E)$$

Proof: Note that $|\bar{E}| = |S| - |E|$

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$$p(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - p(E)$$

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Example: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

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Let E be the event that at least one of the bits is 1

Then $\bar{E}$ is the event that all of the bits are 0

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Intro to Discrete Probability

We can also find the probability of the union of two events

Theorem: Let E_1 and E_2 be two events in the sample space S

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: Recall that $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$

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$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_1|}{|S|} + \frac{|E_1 \cap E_2|}{|S|} \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \end{aligned}$$

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By the Union rule, this is the sum of the probability of an Ace and the probability of a Diamond, minus the probability that the card is the Ace of Diamonds

In math:

$$\begin{aligned} p(\mathrm{A} \cup \diamond) &= p(\mathrm{A}) + p(\diamond) - p(\mathrm{A} \cap \diamond) \\ &= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} \\ &= \frac{4}{13} \end{aligned}$$

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