Proof Sheet

Definition of **even**: there exists an integer k such that $\mathbf{n = 2\,k}$
Definition of **odd**: there exists an integer k such that $\mathbf{n = 2k+1}$ (or $2k - 1$).

You may use without proof, any integer must be even or odd.

Prove for all integers n, if n is even, then m = n + 2 is even.

Direct proof:

n	$= 2b$	by definition of even
m	$= n + 2$	given
	$= 2b + 2$	substitution from above
	$= 2(b + 1)$	factoring
	$= 2\,(h)$	let $h = b + 1$

$m = 2\,h$ and hence by the definition of even, m is even

QED

Prove: if n is even, then m = n + 2 is even by contrapositive:

If $m = n + 2$ is not even, then n is not even. (we state the contrapositive)

Or

If $n+2$ is odd, then n is odd.

Note: this is a direct proof

$m = 2k + 1$	m is odd
$m = n + 2$	given
$n = m - 2$	algebra
$= (2\,k + 1) - 2$	substitution from above
$= 2\,k + (1 - 2)$	associative property
$= 2\,k - 1$	algebra
$n = 2\,k - 1$	By definition of odd, n is odd.

QED

Prove an iff version:

We prove for all integers n, n +2 is even <u>if and only if</u> n is even.

Must prove 2 things:

1. If n is even, then m = n + 2 is even. (as above)

n	$= 2k$	by definition of even
m	$= n + 2$	given
	$= 2k + 2$	substitution from above
	$= 2(k + 1)$	factoring
	$= 2(g)$	let $g = k + 1$

$m = 2g$ and hence by the definition of even, m is even.

2. If n+2 is even, then n is even

$n + 2 = 2k$	definition of even.
$n = 2k - 2$	algebra
$= 2(k - 1)$	factoring
$n = 2g$	let $g = k - 1$

Thus n is even by definition

QED

Proof (not the exact same theorem) by example/counterexample:
For existence proofs only, for example, "There exists a prime number n that is not odd"

Consider $n = 2$. QED

Or to disprove a statement, for example, "All prime numbers are odd"

Consider $n = 2$, counterexample. The statement is not true.

Suppose we are trying to prove that if n is even, then $n + 2$ is even

Consider $n = 4$ is even, and see that $4 + 2 = 6$ is even.

Why is this **not** a proof?

Proof by cases:

Trying to show all cases, $n = 2, 4, 6, 8 \ldots$ is not possible. Why?

However, we can prove by cases a more general statement

Generalize statement:

For all n, n and $n + 2$ have the same parity (even or oddness).

Proof by cases:

1. If n is even, then $n + 2$ is even.

(prove this)

2. If n is odd, then $n + 2$ is odd.

(prove this)

Proof by Contradiction

Proof by contradiction seems like much more work than a proof by contradiction. Right? So why would we use it?

Notice we can *only use proof by* ***contrapositive*** *with a conditional - some* kind of if/then proof.

The classic example of proof by contradiction is Euclid's proof that the square root of 2 is irrational. We cannot use a proof by contrapositive or a direct proof. However, we can show that **when we assume the square root of 2 is rational,** we see this is impossible.

Here is a more simple example.

Prove infinity is not a integer (that is, there is no largest integer)

We prove by contradiction

The negation is "there exists a largest integer"

We assume this is true and so if anything bad happens.

There is a largest integer, let's call it z.	premise
So there can be no integer greater than z no (k such that $k > z$)	follows from above
Let's add 1 to z, $z + 1 = b$	by closure of algebra, b is an integer.
Thus $b > z$	by inspection
But we assumed z was the largest integer!!!!	Foul! Contradiction!!!!

Our assumption of the negation cannot be true!

QED Thus the original statement is true.

Now do the same proof that n + 2 is even by contradiction.

This basically, a proof by double negation.

Steps for a proof by contradiction:

1. Say "We prove by contradiction."
2. State the negation of the theorem.
3. Assume the negation, and show that it cannot be true - something must go wrong.
4. When you show that the negation must be false, you have negated the negation and thus proved the theorem.

remember

The negation of If a, then b is NOT If a, then not b.

The negation of If a, then b is

a and not b

Proof: $\neg(a \to b) \equiv \neg(\neg a \vee b)$ *RBI*

$\equiv \neg\neg a \wedge \neg b$ *DeMorgan*

$\equiv a \wedge \neg b$ *double negation*

Let's do 2 equally good versions of contradiction below.

Proof of the if n is even, then m = n + 2 is even by contradiction:

We use a proof by contradiction. Suppose there exists an **n is even AND m = n + 2 is odd**

We use the exact negation above and see what happens.

m	= n + 2	given
	= 2g + 2	n is even
	= 2(g +1)	
	= 2 (y)	y = g +1, this m is even
m	= 2z + 1	m is odd by our given assumption above.

M cannot be both even and odd at the same time. which is a contradiction with our original assumption of the negation, and hence the theorem is proved. (While the method is not more simple for this proof, sometimes contradiction will be easier) QED

Proof of if n is even, then m = n + 2 is even by contradiction:

We use a proof by contradiction:
Suppose there is an n such that **n is even AND m = n + 2 is odd**
We use the exact negation above and see if anything goes wrong.

m	=	n + 2	given
	=	2g + 2	n is even
and			
m	=	2z + 1	m is odd by our given assumption above.

Then since m = m

$2g + 2 = 2z + 1$

$2g + 1 = 2z$ algebra

The right hand side is odd, and left hand side is even. A number cannot be both even and odd at the same time. Notice, we do not know what this number is (it is not n or m), but that does not matter. The fact that no number can be both even and odd is a contradiction.
We could also keep going at the last step and see

$$2g + 1 = 2z$$
$$2g - 2z = -1$$
$$2(g - z) = -1$$
$$2 = -1/(g - z)$$

Since g and z are integers, then g - z is an integer and thus -1/(g-z) is <=1 QED