24.5 Proofs of shortest-paths properties

Throughout this chapter, our correctness arguments have relied on the triangle inequality, upper-bound property, no-path property, convergence property, path-relaxation property, and predecessor-subgraph property. We stated these properties without proof at the beginning of this chapter. In this section, we prove them.

The triangle inequality

In studying breadth-first search (Section 22.2), we proved as Lemma 22.1 a simple property of shortest distances in unweighted graphs. The triangle inequality generalizes the property to weighted graphs.

Lemma 24.10 (Triangle inequality)
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$ and source vertex s. Then, for all edges $(u, v) \in E$, we have

$$\delta(s, v) \leq \delta(s, u) + w(u, v) .$$

Proof Suppose that p is a shortest path from source s to vertex v. Then p has no more weight than any other path from s to v. Specifically, path p has no more weight than the particular path that takes a shortest path from source s to vertex u and then takes edge (u, v).

Exercise 24.5-3 asks you to handle the case in which there is no shortest path from s to v. ■

Effects of relaxation on shortest-path estimates

The next group of lemmas describes how shortest-path estimates are affected when we execute a sequence of relaxation steps on the edges of a weighted, directed graph that has been initialized by INITIALIZE-SINGLE-SOURCE.

Lemma 24.11 (Upper-bound property)
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let the graph be initialized by INITIALIZE-SINGLE-SOURCE(G, s). Then, $v.d \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once $v.d$ achieves its lower bound $\delta(s, v)$, it never changes.

Proof We prove the invariant $v.d \geq \delta(s, v)$ for all vertices $v \in V$ by induction over the number of relaxation steps.

For the basis, $v.d \geq \delta(s, v)$ is certainly true after initialization, since $v.d = \infty$ implies $v.d \geq \delta(s, v)$ for all $v \in V - \{s\}$, and since $s.d = 0 \geq \delta(s, s)$ (note that $\delta(s, s) = -\infty$ if s is on a negative-weight cycle and 0 otherwise).

For the inductive step, consider the relaxation of an edge (u, v). By the inductive hypothesis, $x.d \geq \delta(s, x)$ for all $x \in V$ prior to the relaxation. The only d value that may change is $v.d$. If it changes, we have

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) && \text{(by the inductive hypothesis)} \\ &\geq \delta(s, v) && \text{(by the triangle inequality)} , \end{aligned}$$

and so the invariant is maintained.

To see that the value of $v.d$ never changes once $v.d = \delta(s, v)$, note that having achieved its lower bound, $v.d$ cannot decrease because we have just shown that $v.d \geq \delta(s, v)$, and it cannot increase because relaxation steps do not increase d values. ■

Corollary 24.12 (No-path property)
Suppose that in a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, no path connects a source vertex $s \in V$ to a given vertex $v \in V$. Then, after the graph is initialized by INITIALIZE-SINGLE-SOURCE(G, s), we have $v.d = \delta(s, v) = \infty$, and this equality is maintained as an invariant over any sequence of relaxation steps on the edges of G.

Proof By the upper-bound property, we always have $\infty = \delta(s, v) \leq v.d$, and thus $v.d = \infty = \delta(s, v)$. ■

Lemma 24.13
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $(u, v) \in E$. Then, immediately after relaxing edge (u, v) by executing RELAX(u, v, w), we have $v.d \leq u.d + w(u, v)$.

Proof If, just prior to relaxing edge (u, v), we have $v.d > u.d + w(u, v)$, then $v.d = u.d + w(u, v)$ afterward. If, instead, $v.d \leq u.d + w(u, v)$ just before the relaxation, then neither $u.d$ nor $v.d$ changes, and so $v.d \leq u.d + w(u, v)$ afterward. ■

Lemma 24.14 (Convergence property)
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, let $s \in V$ be a source vertex, and let $s \leadsto u \to v$ be a shortest path in G for

some vertices $u, v \in V$. Suppose that G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps that includes the call RELAX(u, v, w) is executed on the edges of G. If $u.d = \delta(s, u)$ at any time prior to the call, then $v.d = \delta(s, v)$ at all times after the call.

Proof By the upper-bound property, if $u.d = \delta(s, u)$ at some point prior to relaxing edge (u, v), then this equality holds thereafter. In particular, after relaxing edge (u, v), we have

$$\begin{aligned} v.d &\leq u.d + w(u, v) && \text{(by Lemma 24.13)} \\ &= \delta(s, u) + w(u, v) \\ &= \delta(s, v) && \text{(by Lemma 24.1)} . \end{aligned}$$

By the upper-bound property, $v.d \geq \delta(s, v)$, from which we conclude that $v.d = \delta(s, v)$, and this equality is maintained thereafter. ■

Lemma 24.15 (Path-relaxation property)
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $s \in V$ be a source vertex. Consider any shortest path $p = \langle v_0, v_1, \ldots, v_k \rangle$ from $s = v_0$ to v_k. If G is initialized by INITIALIZE-SINGLE-SOURCE(G, s) and then a sequence of relaxation steps occurs that includes, in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$ after these relaxations and at all times afterward. This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of the edges of p.

Proof We show by induction that after the ith edge of path p is relaxed, we have $v_i.d = \delta(s, v_i)$. For the basis, $i = 0$, and before any edges of p have been relaxed, we have from the initialization that $v_0.d = s.d = 0 = \delta(s, s)$. By the upper-bound property, the value of $s.d$ never changes after initialization.

For the inductive step, we assume that $v_{i-1}.d = \delta(s, v_{i-1})$, and we examine what happens when we relax edge (v_{i-1}, v_i). By the convergence property, after relaxing this edge, we have $v_i.d = \delta(s, v_i)$, and this equality is maintained at all times thereafter. ■

Relaxation and shortest-paths trees

We now show that once a sequence of relaxations has caused the shortest-path estimates to converge to shortest-path weights, the predecessor subgraph G_π induced by the resulting π values is a shortest-paths tree for G. We start with the following lemma, which shows that the predecessor subgraph always forms a rooted tree whose root is the source.

Lemma 24.16
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, let $s \in V$ be a source vertex, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the graph is initialized by INITIALIZE-SINGLE-SOURCE(G, s), the predecessor subgraph G_π forms a rooted tree with root s, and any sequence of relaxation steps on edges of G maintains this property as an invariant.

Proof Initially, the only vertex in G_π is the source vertex, and the lemma is trivially true. Consider a predecessor subgraph G_π that arises after a sequence of relaxation steps. We shall first prove that G_π is acyclic. Suppose for the sake of contradiction that some relaxation step creates a cycle in the graph G_π. Let the cycle be $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_k = v_0$. Then, $v_i.\pi = v_{i-1}$ for $i = 1, 2, \ldots, k$ and, without loss of generality, we can assume that relaxing edge (v_{k-1}, v_k) created the cycle in G_π.

We claim that all vertices on cycle c are reachable from the source s. Why? Each vertex on c has a non-NIL predecessor, and so each vertex on c was assigned a finite shortest-path estimate when it was assigned its non-NIL π value. By the upper-bound property, each vertex on cycle c has a finite shortest-path weight, which implies that it is reachable from s.

We shall examine the shortest-path estimates on c just prior to the call RELAX(v_{k-1}, v_k, w) and show that c is a negative-weight cycle, thereby contradicting the assumption that G contains no negative-weight cycles that are reachable from the source. Just before the call, we have $v_i.\pi = v_{i-1}$ for $i = 1, 2, \ldots, k-1$. Thus, for $i = 1, 2, \ldots, k-1$, the last update to $v_i.d$ was by the assignment $v_i.d = v_{i-1}.d + w(v_{i-1}, v_i)$. If $v_{i-1}.d$ changed since then, it decreased. Therefore, just before the call RELAX(v_{k-1}, v_k, w), we have

$$v_i.d \geq v_{i-1}.d + w(v_{i-1}, v_i) \qquad \text{for all } i = 1, 2, \ldots, k-1 . \tag{24.12}$$

Because $v_k.\pi$ is changed by the call, immediately beforehand we also have the strict inequality

$$v_k.d > v_{k-1}.d + w(v_{k-1}, v_k) .$$

Summing this strict inequality with the $k-1$ inequalities (24.12), we obtain the sum of the shortest-path estimates around cycle c:

$$\begin{aligned} \sum_{i=1}^{k} v_i.d &> \sum_{i=1}^{k} (v_{i-1}.d + w(v_{i-1}, v_i)) \\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i) . \end{aligned}$$

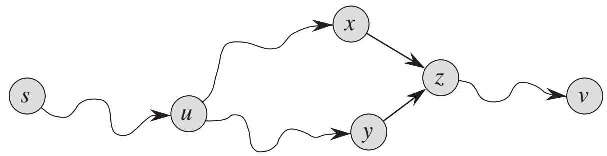

Figure 24.9 Showing that a simple path in G_π from source s to vertex v is unique. If there are two paths p_1 ($s \rightsquigarrow u \rightsquigarrow x \to z \rightsquigarrow v$) and p_2 ($s \rightsquigarrow u \rightsquigarrow y \to z \rightsquigarrow v$), where $x \neq y$, then $z.\pi = x$ and $z.\pi = y$, a contradiction.

But

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d ,$$

since each vertex in the cycle c appears exactly once in each summation. This equality implies

$$0 > \sum_{i=1}^{k} w(v_{i-1}, v_i) .$$

Thus, the sum of weights around the cycle c is negative, which provides the desired contradiction.

We have now proven that G_π is a directed, acyclic graph. To show that it forms a rooted tree with root s, it suffices (see Exercise B.5-2) to prove that for each vertex $v \in V_\pi$, there is a unique simple path from s to v in G_π.

We first must show that a path from s exists for each vertex in V_π. The vertices in V_π are those with non-NIL π values, plus s. The idea here is to prove by induction that a path exists from s to all vertices in V_π. We leave the details as Exercise 24.5-6.

To complete the proof of the lemma, we must now show that for any vertex $v \in V_\pi$, the graph G_π contains at most one simple path from s to v. Suppose otherwise. That is, suppose that, as Figure 24.9 illustrates, G_π contains two simple paths from s to some vertex v: p_1, which we decompose into $s \rightsquigarrow u \rightsquigarrow x \to z \rightsquigarrow v$, and p_2, which we decompose into $s \rightsquigarrow u \rightsquigarrow y \to z \rightsquigarrow v$, where $x \neq y$ (though u could be s and z could be v). But then, $z.\pi = x$ and $z.\pi = y$, which implies the contradiction that $x = y$. We conclude that G_π contains a unique simple path from s to v, and thus G_π forms a rooted tree with root s. ■

We can now show that if, after we have performed a sequence of relaxation steps, all vertices have been assigned their true shortest-path weights, then the predecessor subgraph G_π is a shortest-paths tree.

Lemma 24.17 (Predecessor-subgraph property)
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, let $s \in V$ be a source vertex, and assume that G contains no negative-weight cycles that are reachable from s. Let us call INITIALIZE-SINGLE-SOURCE(G, s) and then execute any sequence of relaxation steps on edges of G that produces $v.d = \delta(s, v)$ for all $v \in V$. Then, the predecessor subgraph G_π is a shortest-paths tree rooted at s.

Proof We must prove that the three properties of shortest-paths trees given on page 647 hold for G_π. To show the first property, we must show that V_π is the set of vertices reachable from s. By definition, a shortest-path weight $\delta(s, v)$ is finite if and only if v is reachable from s, and thus the vertices that are reachable from s are exactly those with finite d values. But a vertex $v \in V - \{s\}$ has been assigned a finite value for $v.d$ if and only if $v.\pi \neq$ NIL. Thus, the vertices in V_π are exactly those reachable from s.

The second property follows directly from Lemma 24.16.

It remains, therefore, to prove the last property of shortest-paths trees: for each vertex $v \in V_\pi$, the unique simple path $s \overset{p}{\leadsto} v$ in G_π is a shortest path from s to v in G. Let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$. For $i = 1, 2, \ldots, k$, we have both $v_i.d = \delta(s, v_i)$ and $v_i.d \geq v_{i-1}.d + w(v_{i-1}, v_i)$, from which we conclude $w(v_{i-1}, v_i) \leq \delta(s, v_i) - \delta(s, v_{i-1})$. Summing the weights along path p yields

$$\begin{aligned}
w(p) &= \sum_{i=1}^{k} w(v_{i-1}, v_i) \\
&\leq \sum_{i=1}^{k} (\delta(s, v_i) - \delta(s, v_{i-1})) \\
&= \delta(s, v_k) - \delta(s, v_0) && \text{(because the sum telescopes)} \\
&= \delta(s, v_k) && \text{(because } \delta(s, v_0) = \delta(s, s) = 0\text{)} \ .
\end{aligned}$$

Thus, $w(p) \leq \delta(s, v_k)$. Since $\delta(s, v_k)$ is a lower bound on the weight of any path from s to v_k, we conclude that $w(p) = \delta(s, v_k)$, and thus p is a shortest path from s to $v = v_k$. ■

Exercises

24.5-1
Give two shortest-paths trees for the directed graph of Figure 24.2 (on page 648) other than the two shown.