CSPB 2824 - Stade - Discrete Structures

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Started on	Tuesday, 31 October 2023, 4:12 PM
State	Finished
Completed on	Tuesday, 31 October 2023, 4:13 PM
Time taken	39 secs
Grade	**10.00** out of 10.00 (**100**%)

Question **1**

Correct

Mark 1.00 out of 1.00

Match

$\sum_{n=1}^{8} n$ 36 ✔

$\sum_{n=4}^{6} n$ 15 ✔

$\sum_{n=8}^{8} n$ 8 ✔

$\sum_{n=1}^{3} n$ 6 ✔

$\sum_{n=3}^{3} n$ 3 ✔

Your answer is correct.

Correct

Marks for this submission: 1.00/1.00.

Question **2**

Correct

Mark 1.00 out of 1.00

Match

$\sum_{n=1}^{8} (2n-1)$	64	✔
$\sum_{n=4}^{6} (2n-1)$	27	✔
$\sum_{n=8}^{8} (2n-1)$	15	✔
$\sum_{n=1}^{3} (2n-1)$	9	✔
$\sum_{n=3}^{3} (2n-1)$	5	✔

Your answer is correct.

Correct

Marks for this submission: 1.00/1.00.

Question **3**

Correct

Mark 1.00 out of 1.00

Match

Let $a_n = (n-1)^2$

$\sum_{n=1}^{8}(n+a_n)$	176	✔
$\sum_{n=4}^{6}(n+a_n)$	65	✔
$\sum_{n=8}^{8}(n+a_n)$	57	✔
$\sum_{n=1}^{3}(n+a_n)$	11	✔
$\sum_{n=3}^{3}(n+a_n)$	7	✔
a_3	4	✔
a_1	0	✔
a_8	49	✔

Your answer is correct.

Let a_n = (n-1)^2$$

Correct

Marks for this submission: 1.00/1.00.

Question **4**

Correct

Mark 2.00 out of 2.00

Which of the following is a solution to this recurrence?

$a_n = -2a_{n-1},\ a_0 = 1$

FIRST write out the terms of a_n

Select one:

- $a_n = 2^n$

- (selected) $a_n = (-2)^n$ ✔
- \(a_n = -2n + 1\)
- \(a_n = (-1)^n\)

Your answer is correct.

Solution:

Writing out the first handful of terms, we might notice a pattern emerge. (Note that if you do not see the pattern emerge in these types of problems, using an approach such as **iteration** (Rosen 7 ed., Section 2.4) will also work.)

\(i=0 \rightarrow a_0 = 1\)

\(i=1 \rightarrow a_1 = -2 a_0 = -2\)

\(i=2 \rightarrow a_2 = -2 a_1 = 4\)

\(i=3 \rightarrow a_3 = -2 a_2 = -8\)

\(i=4 \rightarrow a_4 = -2 a_3 = 16\)

and so on...

First, we notice that the even iterates are all positive and the odd iterates are all negative. This suggests that \((-1)^n\) is a component of the answer.

Next, we notice that the absolute value (part besides the minus sign) of each term is a power of 2.

Finally, note that the power of two for each term is just the index of that term, \(i\).

So we have: \(a_n = (-1)^n 2^n\), or \(a_n = (-2)^n\)

Correct

Marks for this submission: 2.00/2.00.

Question **5**

Correct

Mark 2.00 out of 2.00

Which of the following is a solution to this recurrence relation?

$a_n = (n+1) a_{n-1}$, $a_0 = 2$

FIRST write out the terms of a_n

Select one:

- (selected) $a_n = (n+1)! \cdot 2$ ✔
- $a_n = 2^{n+1}$
- $a_n = n! + 2$
- $a_n = n! \cdot 2^{n+1}$

Solution:

Writing out the first handful of terms, we might notice a pattern emerge. The following makes use of the **iteration** strategy (Rosen 7th ed., Sec. 2.4, Example 10).

$i=0 \rightarrow a_0 = 2$ (given)

$i=1 \rightarrow a_1 = 2 \cdot a_0 = 2 \cdot 2$

$i=2 \rightarrow a_2 = 3 \cdot a_1 = 3 \cdot 2 \cdot 2$

$i=3 \rightarrow a_3 = 4 \cdot a_2 = 4 \cdot 3 \cdot 2 \cdot 2$

$i=4 \rightarrow a_4 = 5 \cdot a_3 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2$

and so on...

We can rewrite the $i=4$ term in a more suggestive manner:

$a_4 = (5 \cdot 4 \cdot 3 \cdot 2) \cdot 2$,

which makes it much more noticeable that this is equivalent to

$a_4 = 5! \cdot 2$

So in general, we have $a_n = (n+1)! \cdot 2$.

We can plug this into the given recursion to confirm:

$(n+1) a_{n-1} = (n+1) ((n-1+1)! \cdot 2) = (n+1)\cdot n! \cdot 2 = (n+1)! \cdot 2 = a_n$,

as desired.

Correct

Marks for this submission: 2.00/2.00.

Question **6**

Correct

Mark 1.00 out of 1.00

This is an MIT version of a classic problem that can come up in a technical job interview.

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In a new reality TV series called Surveyevor, a group of contestants is placed on a small island. Before the series begins, each contestant agrees to have a small purple or red tattoo, in the shape of an eye, applied to the middle of his or her forehead. In all, there are $p \geq 1$ purple eyes and $r \geq 0$ red eyes. However, none of the contestants knows the color of his or her third eye, nor how many total purple and red eyes there are. Furthermore, there are no mirrors and no one is allowed to discuss the tattoos ever. Therefore, everyone knows the colors of everyone else's third eye, but not their own.

Good thing, because a contestant who learns that he or she has a purple eye must leave the island at the end of the show that day, and is therefore no longer eligible to win the 1 million cash prize at the end of the show! The contestants live in uneasy ignorance for several weeks. As time goes on, however, most of them lose their fear of being exiled, adapt to island living, and even make friends with one another. Things are going quite well for the islanders, but as you might suppose, the television audience grows bored, and the show's ratings plummet. When the network threatens to cancel the series, the producer decides she needs to do something, fast: on the next show, to the surprise of the happy islanders, the producer herself appears and convenes a meeting. Very loudly, she proclaims, "I see that at least one person here has a purple eye."

Assuming that all the contestants are master logicians, what happens?

Solution:

All the purple-tattooed contestants leave the island at the end of the pth day.

Let's use induction to prove that the conclusion is correct.

n is the number days that have past.

p is the number of contestants with a purple eye tattoo.

We clarify this proposition P(n) by stating:

- If $p < n$, all purple eyed remain.
- If $p=n$, all purple eyed leave on that day.
- If $p > n$, all purple eyed have left.

* "P()" is the proposition, "p" is the number of purple eyed.

What are some other key insights we need to be clear on before doing the problem?

Select one or more:

- [x] Induction is on the number of days, n. ✔
- [x] At least one person has a purple eye tattoo and no one can see their own tattoo. ✔
- [x] A person must leave when they realize they have a purple eye. ✔
- [x] Everyone on the island has a red or purple eye tattoo ✔
- [x] Everyone on the island is a master of logic. ✔
- [ ] Induction is on the number of purple eyes, p.

Your answer is correct.

Correct

Marks for this submission: 1.00/1.00.

Question 7

Correct

Mark 1.00 out of 1.00

To prove,

First we consider the base case n=1

Is our proposition true on day 1?

We need to consider the 3 cases regarding the number off purple eyes from above.

(remember,p is fixed, the induction is on n, the number of days)

- If $p < n$, all purple eyed remain.
- If $p=n$, all purple eyed leave on that day.
- If $p > n$, all purple eyed have left.

Select one or more:

- [x] Suppose $p > 1$. Consider events on day 1 from the perspective of a purple-eyed islander. We know that someone has a purple eye, and she can indeed see at least one other person with a purple eye. Therefore, the facts available to her are consistent with her having either a purple or red eye. So she survives the day. Part 1 base case proved! ✔
- [x] Suppose $p = 1$. The single purple-eyed islander sees no one else with a purple eye, concludes that he must have a purple eye, and leaves the island at the end of the show. No one else leaves because *everyone else does see* the purple-eyed islander, and they have no reason at this point to think they too are purple-eyed. Base case part 2, proved! ✔
- [x] This statement is vacuously true, because the if-part ($p < 1$) is false; the problem statement says that $p \geq 1$. Base case part 3, proved! ✔
- [ ] The contestants are confused and just leave anyway.

Your answer is correct.

Correct

Marks for this submission: 1.00/1.00.

Question **8**

Correct

Mark 1.00 out of 1.00

Now, for the induction part.

Now suppose that P(n) is true where $n \geq 0$. We must verify the three parts of P(n + 1)

We assume for a fixed n number of days, it is true that all purple eyed (p is always the number of purple eyed) will leave on the pth day.

We must show, that for day n+1, all of our propositions are true. p stays the same, it is fixed.

Select one or more:

- [x] a.
  Suppose $p < n + 1$. Then either $p = n$ (in which case all the purple-eyed contestants left the island on day n by part 2 of P(n)) or else $p < n$ (in which case all the purpleeyed contestants were already gone on day n by part 3 of P(n)). In either case, all the purple-eyed people are already out of luck. ✔

- [x] b.
  By the principle of induction, P(n) is true for all $n \geq 0$, and the theorem follows. ✔

- [x] c.
  Therefore P(n) implies P(n + 1) for all $n \geq 0$. ✔

- [x] d.
  Suppose $p = n + 1$. Then $p > n$, so all the purple-eyed contestants survived the preceding day by part 1 of P(n). Thus, on day n + 1 each purple-eyed islander knows $p > n$, but sees only n other people with a purple eye. Thus, each purple-eyed islander realizes that she has a purple eye and leaves the island at the end of the show. ✔

- [x] e.
  Suppose $p > n + 1$. Then $p > n$ so all the purple-eyed contestants survived the preceding day by part 1 of P(n). Furthermore, each purple-eyed islander can see at least $n + 1 > n$ other purple-eyed people, so the observation that everyone survives is consistent with she herself having either a purple or a red eye by P(n) as well. Thus, each purple-eyed islander survives the day. ✔

Your answer is correct.

Correct

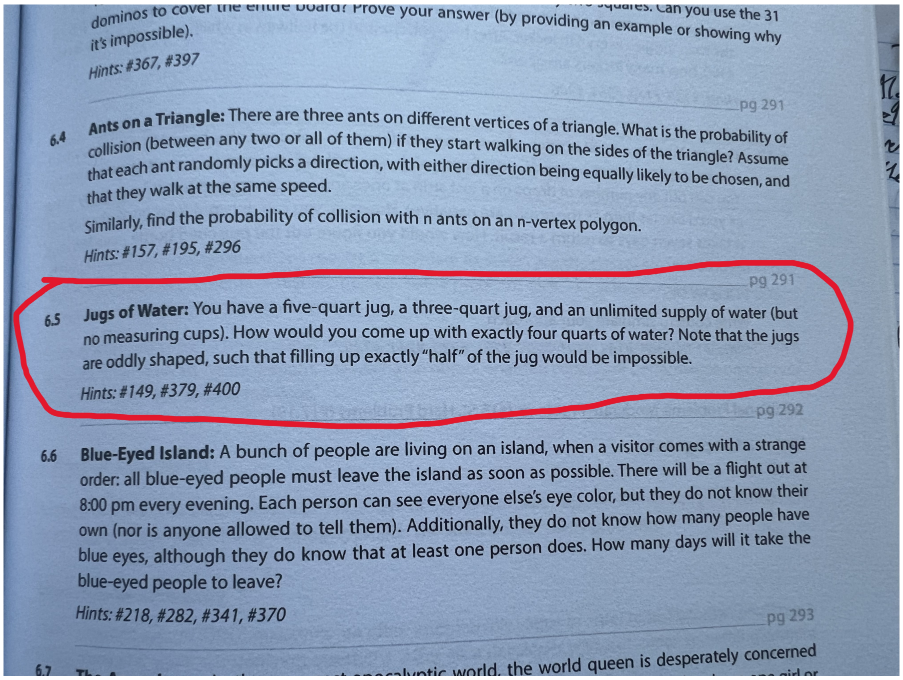
Marks for this submission: 1.00/1.00.

Information

We saw on Piazza, a similar question was posted as a code interview type question.

Can you describe a less formal or more inductive (not induction!) solution?

Perhaps using a first person point of view?



dominos to cover the entire board? Prove your answer (by providing an example or showing why it's impossible).

Hints: #367, #397

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6.4 **Ants on a Triangle:** There are three ants on different vertices of a triangle. What is the probability of collision (between any two or all of them) if they start walking on the sides of the triangle? Assume that each ant randomly picks a direction, with either direction being equally likely to be chosen, and that they walk at the same speed.

Similarly, find the probability of collision with n ants on an n-vertex polygon.

Hints: #157, #195, #296

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6.5 **Jugs of Water:** You have a five-quart jug, a three-quart jug, and an unlimited supply of water (but no measuring cups). How would you come up with exactly four quarts of water? Note that the jugs are oddly shaped, such that filling up exactly "half" of the jug would be impossible.

Hints: #149, #379, #400

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6.6 **Blue-Eyed Island:** A bunch of people are living on an island, when a visitor comes with a strange order: all blue-eyed people must leave the island as soon as possible. There will be a flight out at 8:00 pm every evening. Each person can see everyone else's eye color, but they do not know their own (nor is anyone allowed to tell them). Additionally, they do not know how many people have blue eyes, although they do know that at least one person does. How many days will it take the blue-eyed people to leave?

Hints: #218, #282, #341, #370

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6.7 ... world, the world queen is desperately concerned

Question **9**

Not answered

Not graded

REMINDER - THIS IS AN OPTIONAL QUESTION.

Write a Python function **mega_digit(N, k)** that takes positive integer arguments N and k.

First it calculates a special number P that is the number N repeated k times.

Then, it returns the "mega digit" of P.

For example, if $N=123$ and $k=3$, then your function should return the "mega digit" of $P=123123123$.

What is a "mega digit" you ask? Great question! The "mega digit" of a number is defined recursively as follows:

- If P is a single digit ($1 \leq P \leq 9$), then the mega digit of P is equal to P.
- If P is more than one digit ($P \geq 10$), then the mega digit of P is equal to the mega digit of the sum of the digits of P.

Example 1: Let $N = 12$, $k = 2$. Then

$P = 1212$, and $\texttt{mega_digit}(1212) = \texttt{mega_digit}(1+2+1+2) = \texttt{mega_digit}(6) = 6$

Example 2: Let $N = 321$, $k = 3$. Then

$P = 321321321$, and

$\texttt{mega_digit}(321321321) = \texttt{mega_digit}(3+2+1+3+2+1+3+2+1)$

$= \texttt{mega_digit}(18) = \texttt{mega_digit}(1+8) = \texttt{mega_digit}(9) = 9$

Notes:

- There are two visible test cases worth 0.5 points each, and five hidden test cases worth 1 point each. The integers N and k will satisfy $1 \leq N \leq 10^{16}$, and $1 \leq k \leq 10^6$.
- The hidden test cases increase in the size of the inputs (but stay within the bounds given above). Getting the last 2 or 3 points in particular require at least a modest amount of effort to be invested in making your code efficient.

Hints:

- You may want to write helper function that sums up the digits of a number n.
- You may want to avoid calculating the number P explicitly. Using the definition of "mega digit" should allow you to take a short cut instead.

For example:

Test	Result
`print(mega_digit(12,2))`	6
`print(mega_digit(321,3))`	9

Answer: (penalty regime: 0 %)