Chapter 7

Matrix examples

In this chapter we describe some special matrices that occur often in applications.

7.1 Geometric transformations

Suppose the 2-vector (or 3-vector) x represents a position in 2-D (or 3-D) space. Several important geometric transformations or mappings from points to points can be expressed as matrix-vector products $y = Ax$, with A a 2×2 (or 3×3) matrix. In the examples below, we consider the mapping from x to y, and focus on the 2-D case (for which some of the matrices are simpler to describe).

Scaling. Scaling is the mapping $y = ax$, where a is a scalar. This can be expressed as $y = Ax$ with $A = aI$. This mapping stretches a vector by the factor $|a|$ (or shrinks it when $|a| < 1$), and it flips the vector (reverses its direction) if $a < 0$.

Dilation. Dilation is the mapping $y = Dx$, where D is a diagonal matrix, $D = \mathbf{diag}(d_1, d_2)$. This mapping stretches the vector x by different factors along the two different axes. (Or shrinks, if $|d_i| < 1$, and flips, if $d_i < 0$.)

Rotation. Suppose that y is the vector obtained by rotating x by θ radians counterclockwise. Then we have

$$y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x. \tag{7.1}$$

This matrix is called (for obvious reasons) a *rotation matrix.*

Reflection. Suppose that y is the vector obtained by reflecting x through the line that passes through the origin, inclined θ radians with respect to horizontal. Then we have

$$y = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} x.$$

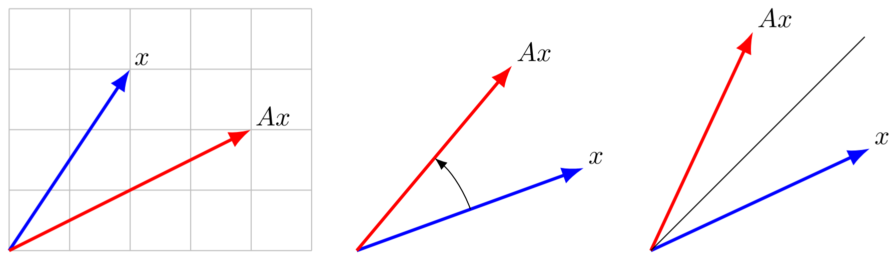

Figure 7.1 From left to right: A dilation with $A = \mathbf{diag}(2, 2/3)$, a counterclockwise rotation by $\pi/6$ radians, and a reflection through a line that makes an angle of $\pi/4$ radians with the horizontal line.

Projection onto a line. The projection of the point x onto a set is the point in the set that is closest to x. Suppose y is the projection of x onto the line that passes through the origin, inclined θ radians with respect to horizontal. Then we have

$$y = \begin{bmatrix} (1/2)(1+\cos(2\theta)) & (1/2)\sin(2\theta) \\ (1/2)\sin(2\theta) & (1/2)(1-\cos(2\theta)) \end{bmatrix} x.$$

Some of these geometric transformations are illustrated in figure 7.1.

Finding the matrix. When a geometric transformation is represented by matrix-vector multiplication (as in the examples above), a simple method to find the matrix is to find its columns. The ith column is the vector obtained by applying the transformation to e_i. As a simple example consider clockwise rotation by 90° in 2-D. Rotating the vector $e_1 = (1, 0)$ by 90° gives $(0, -1)$; rotating $e_2 = (0, 1)$ by 90° gives $(1, 0)$. So rotation by 90° is given by

$$y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x.$$

Change of coordinates. In many applications multiple coordinate systems are used to describe locations or positions in 2-D or 3-D. For example in aerospace engineering we can describe a position using *earth-fixed* coordinates or *body-fixed* coordinates, where the body refers to an aircraft. Earth-fixed coordinates are with respect to a specific origin, with the three axes pointing East, North, and straight up, respectively. The origin of the body-fixed coordinates is a specific location on the aircraft (typically the center of gravity), and the three axes point forward (along the aircraft body), left (with respect to the aircraft body), and up (with respect to the aircraft body). Suppose the 3-vector x^{body} describes a location using the body coordinates, and x^{earth} describes the same location in earth-fixed coordinates. These are related by

$$x^{\text{earth}} = p + Qx^{\text{body}},$$

where p is the location of the airplane center (in earth-fixed coordinates) and Q is a 3×3 matrix. The ith column of Q gives the earth-fixed coordinates for the ith axis of the airplane. For an airplane in level flight, heading due South, we have

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

7.2 Selectors

An $m \times n$ *selector matrix* A is one in which each row is a unit vector (transposed):

$$A = \begin{bmatrix} e_{k_1}^T \\ \vdots \\ e_{k_m}^T \end{bmatrix},$$

where $k_1, \ldots, k_m$ are integers in the range $1, \ldots, n$. When it multiplies a vector, it simply copies the k_ith entry of x into the ith entry of $y = Ax$:

$$y = (x_{k_1}, x_{k_2}, \ldots, x_{k_m}).$$

In words, each entry of Ax is a selection of an entry of x.

The identity matrix, and the *reverser matrix*

$$A = \begin{bmatrix} e_n^T \\ \vdots \\ e_1^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are special cases of selector matrices. (The reverser matrix reverses the order of the entries of a vector: $Ax = (x_n, x_{n-1}, \ldots, x_2, x_1)$.) Another one is the $r\!:\!s$ *slicing matrix*, which can be described as the block matrix

$$A = \begin{bmatrix} 0_{m \times (r-1)} & I_{m \times m} & 0_{m \times (n-s)} \end{bmatrix},$$

where $m = s - r + 1$. (We show the dimensions of the blocks for clarity.) We have $Ax = x_{r:s}$, *i.e.*, multiplying by A gives the $r\!:\!s$ slice of a vector.

Down-sampling. Another example is the $(n/2) \times n$ matrix (with n even)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $y = Ax$, we have $y = (x_1, x_3, x_5, \ldots, x_{n-3}, x_{n-1})$. When x is a time series, y is called the $2\times$ *down-sampled* version of x. If x is a quantity sampled every hour, then y is the same quantity, sampled every 2 hours.