University of Colorado **Boulder**

Department of Computer Science
CSCI 2824: Discrete Structures
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Predicate Logic, Quantifiers, and Rules of Inference (Lecture 1):

Predicates and Quantifiers

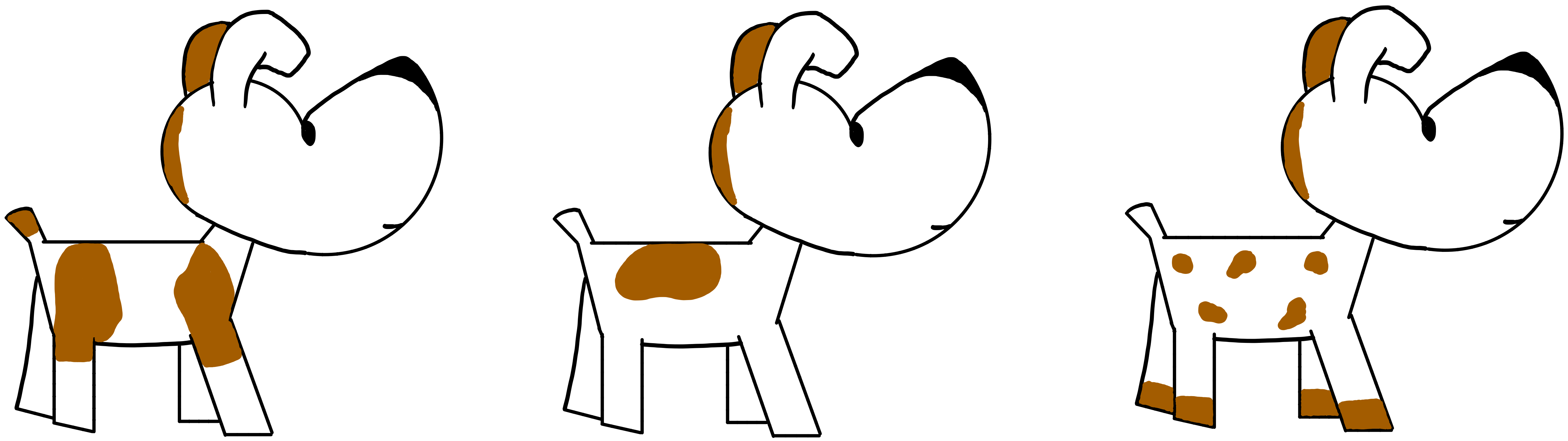
Propositional Logic

Last Time:

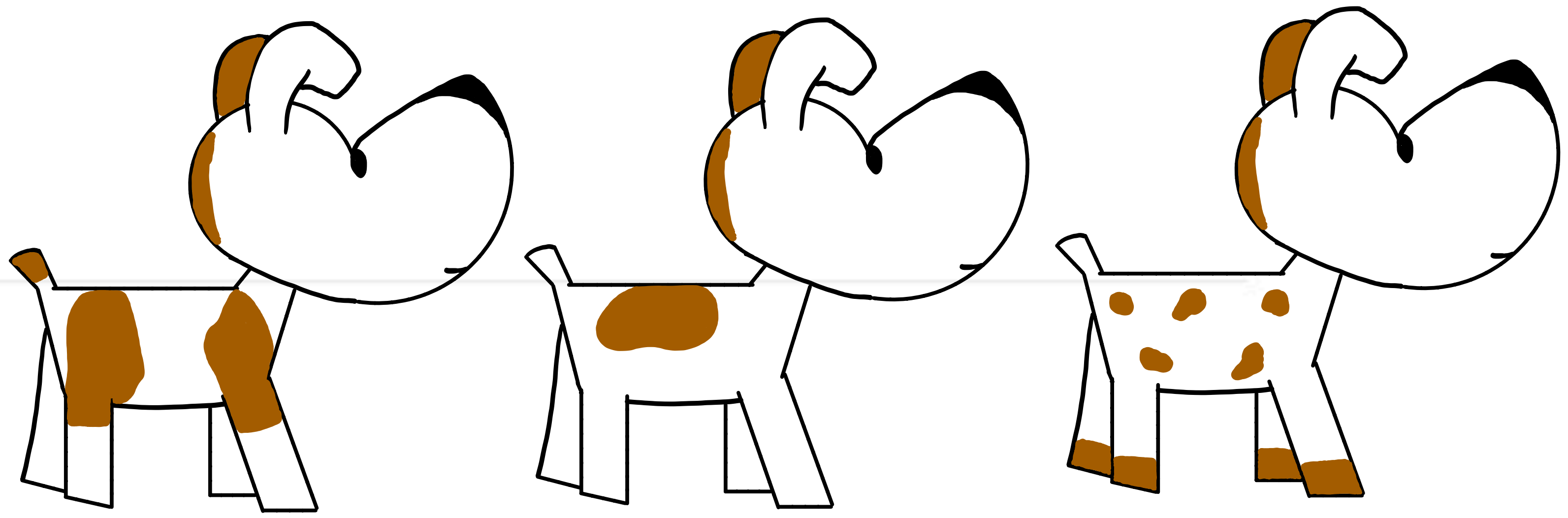
- Finished propositional logic
- Looked at the satisfiability problem

Predicate Logic

The problem with propositional logic is that it cannot explain some very useful classes of language statements



Predicate Logic



Example:

- All of my dogs love peanut butter
- Tukey is one of my dogs

Would be nice if I could conclude that Tukey loves peanut butter

But propositional logic does not easily allow this

Example: With propositional logic would have to represent "All of my dogs love peanut butter" with something like

$$Tukey\ loves\ PB \wedge Fisher\ loves\ PB \wedge Bayes\ loves\ PB$$

But what if I want to talk about all of the dogs at the dog park?

Predicate Logic

Two new constructs to help us: **Predicates** and **Quantifiers**

Consider the statement $x > 3$

- x is a variable or a placeholder
- > 3 is a **predicate**

Let $P(x)$ represent $x > 3$

We call $P(x)$ a propositional function. When we assign a value to x then $P(x)$ becomes a proposition and has a truth value

Example: $P(4)$ is true, while $P(1)$ is false

Predicate Logic

Propositional functions can have multiple variables

Example: Let $Q(x, y)$ represent $x + 1 = y$

- What is the truth value of $Q(1, 2)$?
- What is the truth value of $Q(3, 2)$?

Example: Let $R(x, y, z)$ represent $x^2 + y^2 = z^2$

- What is the truth value of $R(1, 1, 1)$?
- What is the truth value of $R(3, 4, 5)$?

Predicates in Computer Science

We see predicates in computer science all the time

If - Then Statements:

- **if** $x > 17$ **then** STOP

While Loops:

- **while** $i < 10$ **do** $i = i + 1$

Error Checking:

- **assert**(len(passwd) > 0)

Predicate Logic

When using predicates we have to think about what values we input

Example:

- All of my dogs love peanut butter
- Tukey is one of my dogs

Example: Let $P(x)$ represent "Likes peanut butter"

- $P(Tukey)$ is a true statement
- What is $P(7)$?

The set of values we intend to plug in is called the **domain of discourse** or commonly just the **domain**

The Universal Quantifier

OK, so say we fix the domain $\{Tukey, Fisher, Bayes\}$

How do we say something like "All my dogs love peanut butter"?

Remember $P(x)$ represent "loves peanut butter"

We introduce the **universal quantifier**: $\forall x$

- $\forall x\ P(x)$ means "For all x in my domain, $P(x)$"

Example: $\forall x\ P(x)$ represents "All my dogs love peanut butter"

Note that with the quantifier $\forall x\ P(x)$ becomes a proposition again

The Universal Quantifier

Question: When is the statement $\forall x \; P(x)$ true?

The Universal Quantifier

Question: When is the statement $\forall x\ P(x)$ true?

Answer: It's true if $P(x)$ is true for all x in my domain

Example: Let the domain be the integers. $\forall x\ (x^2 \geq 0)$ is true

This is certainly true for any integer x

Question: When is the statement $\forall x\ P(x)$ false?

The Universal Quantifier

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Example: Let the domain be the integers. $\forall x\ (x^2 \geq 0)$ is true

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Question: When is the statement $\forall x\ P(x)$ false?

Answer: It's false if there is some value of x that makes $P(x)$ false

Example: Let the domain be the integers. $\forall x\ (x^2 > x)$ is false

The Universal Quantifier

Long-Term Takeaway:

- To **disprove** a universal proposition, all you need is **one** specific counterexample that makes it not work
- To **prove** a universal proposition **one** specific example does **nothing**! Usually you have to work a lot harder ...
- **NEVER** try to prove a universal statement by choosing a specific example and showing $P(x)$ to be true

The Existential Quantifier

OK, that's how we talk about **all things**

What if we want to say something about **something**

Many mathematical statements claim that there an element of the domain that has a certain property

For this, we introduce the **existential quantifier:** $\exists x$

- $\exists x\ P(x)$ means "there exists an x in my domain, $P(x)$"

Example: $\exists x\ P(x)$ could mean "There exists a dog that loves peanut butter"

The Existential Quantifier

Question: When is the statement $\exists x\ P(x)$ true?

The Existential Quantifier

Question: When is the statement $\exists x\ P(x)$ true?

Answer: It's true if $P(x)$ is true for at least one x in my domain

Example: Let the domain be the integers. $\exists x\ (x^2 > x)$ is true

Let $x = 2$ for example

Question: When is the statement $\exists x\ P(x)$ false?

The Existential Quantifier

Question: When is the statement $\exists x\, P(x)$ true?

Answer: It's true if $P(x)$ is true for at least one x in my domain

Example: Let the domain be the integers. $\exists x\, (x^2 > x)$ is true

Let $x = 2$ for example

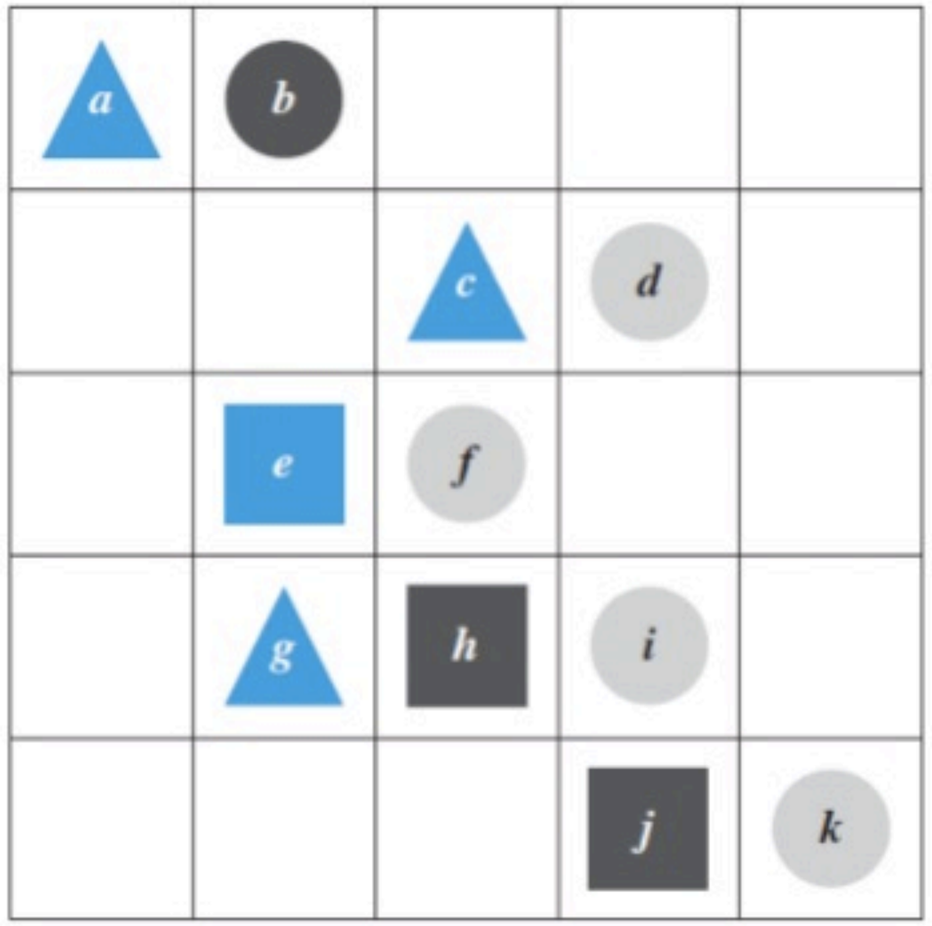
Question: When is the statement $\exists x\, P(x)$ false?

Answer: It's false if there is no value of x that makes $P(x)$ true

Example: Let the domain be the integers. $\exists x\, (x = x + 1)$ is false

Tarski's World

Example: Determine the truth values



- $\forall x$ (Triangle(x) $\rightarrow$ Blue(x))
- $\exists y$ (Square(y) $\wedge$ RightOf(d,y))
- $\forall x$ (Blue(x) $\rightarrow$ Triangle(x))
- $\exists y$ (Square(y) $\wedge$ Gray(y))

Special Cases

Special Cases: The Empty Domain

- If the domain is empty then $\forall x\ P(x)$ is true

Example: "All monkeys on Venus are neon green". (This is called a **vacuously** true statement)

- If the domain is empty then $\exists x\ P(x)$ is false

Since nothing qualifies as an x there certainly can't be one that exists to make $P(x)$ true

Scope of Quantifiers

Quantifiers have the narrowest scope of all logical operators

Example: $\forall x\ (P(x) \land Q(x))$ is not the same as $\forall x\ P(x) \land Q(x)$

Without the parentheses $\forall x$ only applies to $P(x)$

Note that the second example is not even a proposition anymore because we must give a value to the second x before it can have a truth value

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value no matter which predicates you use and which domain they're defined over

Example: Are these equivalent?

$$\forall x\,(P(x) \land Q(x)) \;\equiv\; \forall x\,P(x) \land \forall x\,Q(x)$$

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Example: Are these equivalent?

$$\forall x \, (P(x) \land Q(x)) \equiv \forall x \, P(x) \land \forall x \, Q(x)$$

Intuition: Let the domain be my dogs and $P(x)$ represent "like peanut butter" and $Q(x)$ represent "has a short tail".

What do you think?

Equivalences in Predicate Logic

It turns out that it is true. To prove it, let's look at the universal quantifier a new way. Let our domain be $\{a, b, c\}$. Then

$$\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge P(c)$$

Then $\forall x\, (P(x) \wedge Q(x))$ becomes

$$(P(a) \wedge Q(a)) \wedge (P(b) \wedge Q(b)) \wedge (P(c) \wedge Q(c))$$

and $\forall x\, P(x) \wedge \forall x\, Q(x)$ becomes

$$(P(a) \wedge P(b) \wedge P(c)) \wedge (Q(a) \wedge Q(b) \wedge Q(c))$$

By the associate law of conjunctions for propositions we can rearrange the order so that they're the same

Equivalences in Predicate Logic

OK, so you can distribute an ∀ across conjunctions

Can you do the same thing with disjunctions?

Example: Are these equivalent?

$$\forall x\ (P(x) \vee Q(x)) \ \equiv\ \forall x\ P(x) \vee \forall x\ Q(x)$$

Equivalences in Predicate Logic

OK, so you can distribute an ∀ across conjunctions

Can you do the same thing with disjunctions?

Example: Are these equivalent?

$$\forall x\,(P(x) \vee Q(x)) \;\equiv\; \forall x\, P(x) \vee \forall x\, Q(x)$$

Nope!

Counterexample: Let the domain be the set of integers, $P(x)$ represent "is odd" and $Q(x)$ represent "is even".

- First statement says "all integers are either even or odd"
- Second says "either all integers are even or all are odd"

Equivalences in Predicate Logic

Example: Are these equivalent?

$$\exists x\, (P(x) \lor Q(x)) \;\equiv\; \exists x\, P(x) \lor \exists x\, Q(x)$$

What do you think?

Equivalences in Predicate Logic

$$\exists x\,(P(x) \vee Q(x)) \;\equiv\; \exists x\,P(x) \vee \exists x\,Q(x)$$

It turns out that it is true. To see this, let our domain again be $\{a, b, c\}$, and think of the existential quantifier as

$$\exists x\,P(x) \equiv P(a) \vee P(b) \vee P(c)$$

Then $\exists x\,(P(x) \vee Q(x))$ becomes

$$(P(a) \vee Q(a)) \vee (P(b) \vee Q(b)) \vee (P(c) \vee Q(c))$$

and $\exists x\,P(x) \vee \exists x\,Q(x)$ becomes

$$(P(a) \vee P(b) \vee P(c)) \vee (Q(a) \vee Q(b) \vee Q(c))$$

By the associate law of disjunctions for propositions we can rearrange the order so that they're the same

Equivalences in Predicate Logic

Example: Are these equivalent?

$$\exists x\,(P(x) \land Q(x)) \;\equiv\; \exists x\,P(x) \land \exists x\,Q(x)$$

What do you think?

Equivalences in Predicate Logic

Example: Are these equivalent?

$$\exists x\,(P(x) \land Q(x)) \;\equiv\; \exists x\, P(x) \land \exists x\, Q(x)$$

What do you think?

NOPE!

Exercise For You: Think of a counterexample!

So we now have

- Distribution of $\forall$ over conjunctions
- Distribution of $\exists$ over disjunctions

Negations in Predicate Logic

Question: What is the negation of a universal: $\neg \forall x \, P(x)$?

Example: What is the negation of the following?

- "All CSCI 2824 students have taken Computer Systems"

Negations in Predicate Logic

Question: What is the negation of a universal: $\neg\forall x\ P(x)$?

Example: What is the negation of the following?

- "All CSCI 2824 students have taken Computer Systems"

How about

- "There exists a CSCI 2824 student who has not taken Computer Systems"

Let our domain be the set of CSCI 2824 students and $P(x)$ represent "has taken Computer Systems"

The negated statement is then $\exists x\ \neg P(x)$

Negations in Predicate Logic

So pushing a $\neg$ through a $\forall$ turns it into a $\exists$

Question: What is the negation of an existential: $\neg \exists x \ P(x)$?

Example: What is the negation of the following?

- "There exists a CSCI 2824 student who has taken Algorithms"

How about

- "All CSCI 2824 students have **not** taken Algorithms"

The negated statement is then $\forall x \ \neg P(x)$

So pushing a $\neg$ through a $\exists$ turns it into a $\forall$

DeMorgan's Law for Quantifiers

Collectively, these are called **DeMorgan's Law for Quantifiers**

- $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$
- $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

And then we had the distribution laws

- $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$
- $\exists x\, (P(x) \lor Q(x)) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$

With these rules, and the logical equivalences we found for regular propositions, we can prove all kinds of equivalences of quantifier propositions

Equivalences in Predicate Logic

Example: Prove that $\neg \forall x \, (P(x) \rightarrow Q(x)) \equiv \exists x \, (P(x) \wedge \neg Q(x))$

Equivalences in Predicate Logic

Example: Prove that $\neg\forall x\,(P(x) \to Q(x)) \equiv \exists x\,(P(x) \land \neg Q(x))$

$$
\begin{aligned}
\neg\forall x\,(P(x) \to Q(x)) &\equiv \exists x\,\neg(P(x) \to Q(x)) && \text{(DeMorgan)}\\
&\equiv \exists x\,\neg(\neg P(x) \lor Q(x)) && \text{(RBI)}\\
&\equiv \exists x\,(\neg\neg P(x) \land \neg Q(x)) && \text{(DeMorgan)}\\
&\equiv \exists x\,(P(x) \land \neg Q(x)) && \text{(Double-Neg)}
\end{aligned}
$$

Equivalences in Predicate Logic

Example: Prove that $\neg \forall x\, (P(x) \rightarrow Q(x)) \equiv \exists x\, (P(x) \wedge \neg Q(x))$

$$
\begin{aligned}
\neg \forall x\, (P(x) \rightarrow Q(x)) &\equiv \exists x\, \neg (P(x) \rightarrow Q(x)) \quad \text{(DeMorgan)} \\
&\equiv \exists x\, \neg (\neg P(x) \vee Q(x)) \quad \text{(RBI)} \\
&\equiv \exists x\, (\neg\neg P(x) \wedge \neg Q(x)) \quad \text{(DeMorgan)} \\
&\equiv \exists x\, (P(x) \wedge \neg Q(x)) \quad \text{(Double-Neg)}
\end{aligned}
$$

Let's read this one out loud and see if it makes sense:

- It's not the case that for every x, if $P(x)$ then $Q(x)$
- There exists an x such that $P(x)$ and not $Q(x)$

Predicate Logic

Example: Determine whether

$$\forall x\,(P(x) \rightarrow Q(x)) \equiv \forall x\,P(x) \rightarrow \forall x\,Q(x)$$

is a valid equivalence.

English Translations

Example: Translate the following into symbols

- "Every student in CSCI 2824 has passed Calculus 1"

English Translations

Translating from English into Logical Expressions

Example: Translate the following into symbols

- "Every student in CSCI 2824 has passed Calculus 1"

Let's write it in more quantifier-y logic verbiage

- "For every student in CSCI 2824, that student has passed Calculus 1"

If we let our domain be all students in CSCI 2824

and $C(x)$ represent "has taken Calculus 1"

Then this becomes: $\forall x \; C(x)$

English Translations

Translating from English into Logical Expressions

Example: Translate the following into symbols

- "Every student in CSCI 2824 has passed Calculus 1"

But what if we want to use a bigger domain, like all students at CU?

Let $D(x)$ represent "is a CSCI 2824 student"

Then we can translate the sentence as $\forall x\ (D(x) \rightarrow C(x))$

Note that it is **NOT** this: $\forall x\ (D(x) \wedge C(x))$

This says that **every** student at CU is in 2824 and has passed Calculus 1

English Translations

Example: Let the domain be the set of all CU students, and translate:

- "Every student in CSCI 2824 is either taking Data Structures, or has already passed it"

English Translations

Example: Let the domain be the set of all CU students, and translate:

- "Every student in CSCI 2824 is either taking Data Structures, or has already passed it"

- Let $P(x)$ represent "has passed Data Structures"
- Let $T(x)$ represent "is taking Data Structures"

Then we have $\forall x\ (D(x) \rightarrow (P(x) \vee T(x))$

English Translations

Example: Let the domain be the set of all CU students, and translate:

- "Every student in CSCI 2824 is either taking Data Structures, or has already passed it"

- Let $P(x)$ represent "has passed Data Structures"
- Let $T(x)$ represent "is taking Data Structures"

Then we have $\forall x \; (D(x) \rightarrow (P(x) \vee T(x))$

Ehh... OK it should probably be $\forall x \; (D(x) \rightarrow (P(x) \oplus T(x))$

Most people that pass DS don't take it again just for fun