University of Colorado **Boulder**

Department of Computer Science
CSCI 2824: Discrete Structures
Chris Ketelsen

Propositional Logic (Lecture 3)

Propositional Equivalence

Propositional Logic

Last Time:

- New connectives: **conditional** and **biconditional**
- Necessary and Sufficient conditions
- Solving Knight and Knaves puzzles with truth tables

More on Conditionals

Remember the conditional: *If* p *then* q is symbolized as $p \to q$

- The occurrence of p implies of the occurrence of q
- The conditional is True except when $p = T$ and $q = F$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

More on Conditionals

There are **three** conditionals closely related to $p \to q$

- The **converse**: $q \to p$
- The **inverse**: $\neg p \to \neg q$
- The **contrapositive**: $\neg q \to \neg p$

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Question: How are these related to $p \to q$? Are they the same? Different? What do you think?

Example: "If it's raining then the grass is wet"

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Example: "If it's raining then the grass is wet"

Converse: "If the grass is wet then it's raining"

More on Conditionals

There are **three** conditionals closely related to $p \rightarrow q$

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- The **inverse**: $\neg p \rightarrow \neg q$
- The **contrapositive**: $\neg q \rightarrow \neg p$

Question: How are these related to $p \rightarrow q$? Are they the same? Different? What do you think?

Example: "If it's raining then the grass is wet"

Inverse: "If it's not raining then the grass is not wet"

More on Conditionals

There are **three** conditionals closely related to $p \rightarrow q$

- The **converse**: $q \rightarrow p$
- The **inverse**: $\neg p \rightarrow \neg q$
- The **contrapositive**: $\neg q \rightarrow \neg p$

Question: How are these related to $p \rightarrow q$? Are they the same? Different? What do you think?

Example: "If it's raining then the grass is wet"

Contrapositive: "If the grass is not wet then it isn't raining"

More on Conditionals

The **contrapositive** is the only one of the three that feels the same as the conditional. Let's use a truth table to check, just to be sure.

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F	T			
T	F	F	T	F			
F	T	T	F	T			
F	F	T	T	T			

More on Conditionals

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p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

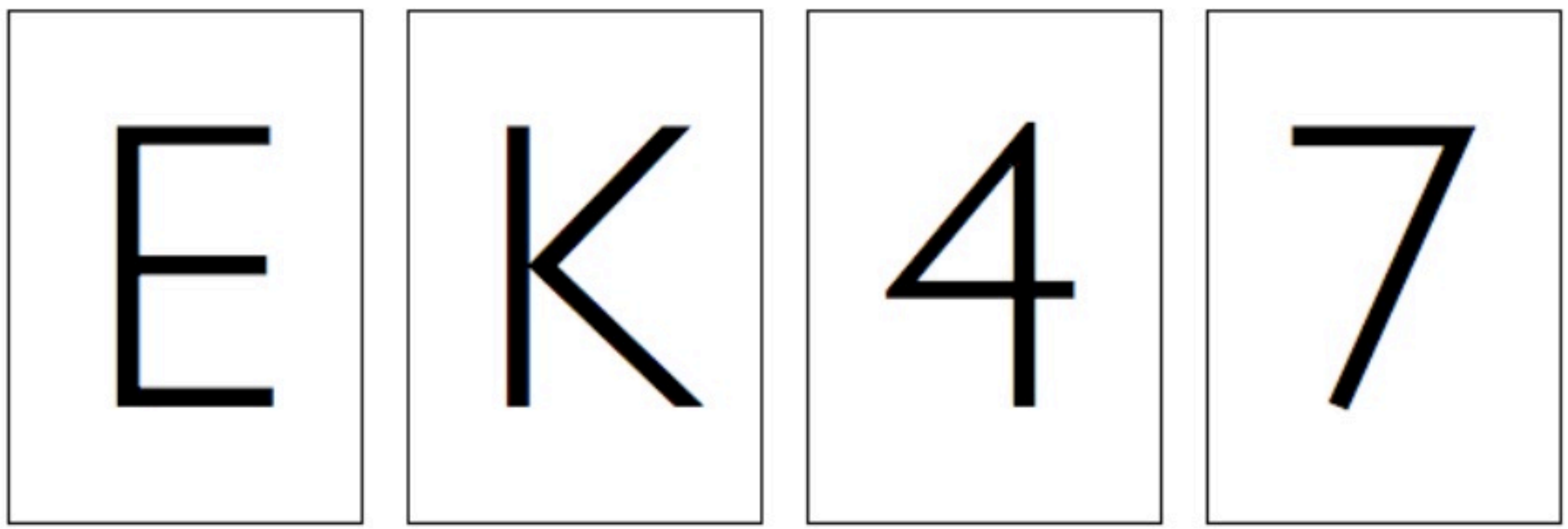
Confirmed! The conditional $p \to q$ and it's contrapositive $\neg q \to \neg p$ are logically equivalent.

Definition: Two compound propositions r and s are called **logically equivalent** if they have the same truth values for every combination of truth values of their constituent parts. We write $r \equiv s$.

The Wason Selection Task

Consider the following four cards that have letters on one side and numbers on the other side. Suppose I tell you the following rule:

If a card has an odd number then its letter is a vowel



Questions: Name the cards (and only those cards) that you need to turn over to verify that the given rule is true.

The Wason Selection Task

If a card has an odd number then its letter is a vowel

E	K	4	7

Answer:

- Need to turn over the **7** to check for a vowel
- **ALSO** need to turn over the **K** to check for an even

That is, we need to check the CONTRAPOSITIVE statement!

The Wason Selection Task

- Experiment designed by Peter Wason in the late 60s.
- Demonstrates that while if-then relations are intuitive to us, contrapositives are not.
- Only **4%** of the general population gets it correct!

Logical Equivalence

Great, so $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$

Besides doing better on silly experiments, **WHO CARES?!**

Mathematical arguments (and proving cool things like program correctness) are based on the progression from assumptions to interesting results through a path of logical equivalences

To prove a statement like "If p then q" you might assume p to be true and try to prove that q is true

But sometimes it might be way easier to assume q to be false, and prove that p is false

Since $p \rightarrow q \equiv \neg q \rightarrow \neg p$, both methods are equivalent

Tautologies and Contradictions

Some Special Cases:

Example: Tomorrow it will either snow, or it will not snow.

What can you say about this proposition?

Tautologies and Contradictions

Some Special Cases:

Example: Tomorrow it will either snow, or it will not snow.

What can you say about this proposition?

Yes, it's silly. But it's also a compound proposition that is **true** regardless of the truth value of it's constituent parts.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Such a compound proposition is called a **Tautology**

Tautologies and Contradictions

Example: Here's another tautology $p \to p$

Example: Here's one with three constituent propositions:

$$((p \to q) \land (q \to r)) \to (p \to r)$$

Question: If we wanted to check this one with a truth table, how many rows would it have to have?

Tautologies and Contradictions

Example: Here's another tautology $p \to p$

Example: Here's one with three constituent propositions:

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

Question: If we wanted to check this one with a truth table, how many rows would it have to have?

Answer: We'd have to have a row for each possible combination of truth values of p, q, and r. Since there are three propositions, that would be $2^3 = 8$ truth value combinations.

A truth table with n constituent propositions needs 2^n rows

Tautologies and Contradictions

Can you think of another example of a tautology?

Maybe one with just p and q?

Tautologies and Contradictions

Can you think of another example of a tautology?

One thing we could do is take two compound propositions that we know are logically equivalent, and join them together with a biconditional

Example: $(p \to q) \leftrightarrow (\neg q \to \neg p)$

p	q	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Note: Compound propositions r and s are logically equivalent if $r \leftrightarrow s$ is a tautology

Tautologies and Contradictions

Some Special Cases:

Question: What's the opposite of a tautology?

Tautologies and Contradictions

Some Special Cases:

Question: What's the opposite of a tautology?

Answer: A compound proposition that is **False** for all possible truth values is called a **Contradiction.**

Example: It is snowing and It is not snowing ($p \wedge \neg p$)

More Examples: Take any tautology and negate the whole thing

Definition: A compound proposition that is neither a tautology nor a contradiction is called a **contingency**

More Logical Equivalences

What's another way to express the following

It is not the case Sam is both a man and a lawyer

More Logical Equivalences

What's another way to express the following

It is not the case Sam is both a man and a lawyer

How about:

Either Sam is not a man or Sam is not a lawyer

These two propositions are $\neg(p \wedge q)$ and $\neg p \vee \neg q$ and they're logically equivalent

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

More Logical Equivalences

What's another way to express the following

It is not the case that Sandy is either a man or a lawyer

More Logical Equivalences

What's another way to express the following

It is not the case that Sandy is either a man or a lawyer

How about:

Sandy is not a man and Sandy is not a lawyer

These two propositions are $\neg(p \lor q)$ and $\neg p \land \neg q$ and they're logically equivalent

Exercise for You: Work out the truth table to show the logical equivalence

More Logical Equivalences

The two previous facts give us a powerful logical manipulation that we can perform on compound propositions

Think of them as distribution rules. If you distribute a negation into a conjunction/disjunction you get a disjunction/conjunction of the negated propositions

Together these two manipulations are called **DeMorgan's Law**:

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Are there other simple manipulations like DeMorgan's Law?

More Logical Equivalences

Totally. Here's another useful one: $p \to q \equiv \neg p \lor q$

Intuition:

- If it's raining the grass is wet
- Either it's not raining or the grass is wet

Exercise for You: Work out the truth table to show the equivalence

Here's another intuitive pair (called **Distribution**):

- $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
- $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

And oh so many more ...

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

More Logical Equivalences

- $p \rightarrow q \equiv \neg p \vee q$ Relation by Implication
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$ Contraposition
- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Def of Biconditional
- $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alt Def of Exclusive Or

There are additional equivalences in Tables 7 & 8 in the textbook that follow directly from those in Table 6 and the ones on this slide.

Proving Logical Equivalences

Let's say that you think that two compound propositions are logically equivalent. You could

- Prove it with a truth table
- Use equivalence rules to go from one to the other!

Example: Show that $p \to q \equiv \neg q \to \neg p$ (w/o a truth table)

Proving Logical Equivalences

Let's say that you think that two compound propositions are logically equivalent. You could

- Prove it with a truth table
- Use equivalence rules to go from one to the other!

Example: Show that $p \to q \equiv \neg q \to \neg p$ (w/o a truth table)

$$
\begin{aligned}
p \to q &\equiv \neg p \vee q && \text{(relation by implication [RBI])}\\
&\equiv q \vee \neg p && \text{(commutativity)}\\
&\equiv \neg\neg q \vee \neg p && \text{(double negation)}\\
&\equiv \neg q \to \neg p && \text{(RBI)}
\end{aligned}
$$

Logical Equivalence

Example: Show that $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to q) \vee (p \to r) \equiv$

Logical Equivalence

Example: Show that $(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$$\begin{aligned}
(p \to q) \lor (p \to r) &\equiv (\neg p \lor q) \lor (\neg p \lor r) \quad (RBI) \\
&\equiv \neg p \lor q \lor \neg p \lor r \quad \text{(associativity)} \\
&\equiv (\neg p \lor \neg p) \lor (q \lor r) \quad \text{(commutativity)} \\
&\equiv \neg p \lor (q \lor r) \quad \text{(idemptotency)} \\
&\equiv p \to (q \lor r) \quad (RBI)
\end{aligned}$$

That was so much faster than an 8-row truth table!

Logical Equivalence

Example: Show that $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

Propositional Satisfiability

OK, one more concept and an interesting application

Def: A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a **contradiction**)

Example: Show that $(p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q)$ is satisfiable

Propositional Satisfiability

OK, one more concept and an interesting application

Def: A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a **contradiction**)

Example: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable

To show that a proposition is satisfiable we just need to find **one** combo of truth values that makes the proposition true

1. The first two conjuncts tell us that p and q must have the same truth values
2. The third conjunct then tells us that p and q must be false

So the prop is satisfiable and $p = F$ and $q = F$ is a solution

Propositional Satisfiability

Example: Show that the following proposition is **not** satisfiable

$$(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$$

Propositional Satisfiability

Example: Show that the following proposition is **not** satisfiable

$$(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$$

To show that a proposition is unsatisfiable we need a logical argument that says there is no set of truth values that makes the proposition true

1. The first two conjuncts tell us that $p = F$
2. Then the third conjunct tells us that $q = T$
3. But the fourth conjunct is then false, a contradiction

So we conclude that the proposition is unsatisfiable

Propositional Satisfiability

Example: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$$