4.6 Cryptography

Introduction

Number theory plays a key role in cryptography, the subject of transforming information so that it cannot be easily recovered without special knowledge. Number theory is the basis of many classical ciphers, first used thousands of years ago, and used extensively until the 20th century. These ciphers encrypt messages by changing each letter to a different letter, or each block of letters to a different block of letters. We will discuss some classical ciphers, including shift ciphers, which replace each letter by the letter a fixed number of positions later in the alphabet, wrapping around to the beginning of the alphabet when necessary. The classical ciphers we will discuss are examples of private key ciphers where knowing how to encrypt allows someone to also decrypt messages. With a private key cipher, two parties who wish to communicate in secret must share a secret key. The classical ciphers we will discuss are also vulnerable to cryptanalysis, which seeks to recover encrypted information without access to the secret information used to encrypt the message. We will show how to cryptanalyze messages sent using shift ciphers.

Number theory is also important in public key cryptography, a type of cryptography invented in the 1970s. In public key cryptography, knowing how to encrypt does not also tell someone how to decrypt. The most widely used public key system, called the RSA cryptosystem, encrypts messages using modular exponentiation, where the modulus is the product of two large primes. Knowing how to encrypt requires that someone know the modulus and an exponent. (It does not require that the two prime factors of the modulus be known.) As far as it is known, knowing how to decrypt requires someone to know how to invert the encryption function, which can only be done in a practical amount of time when someone knows these two large prime factors. In this chapter we will explain how the RSA cryptosystem works, including how to encrypt and decrypt messages.

The subject of cryptography also includes the subject of cryptographic protocols, which are exchanges of messages carried out by two or more parties to achieve a specific security goal. We will discuss two important protocols in this chapter. One allows two people to share a common secret key. The other can be used to send signed messages so that a recipient can be sure that they were sent by the purported sender.

Classical Cryptography

One of the earliest known uses of cryptography was by Julius Caesar. He made messages secret by shifting each letter three letters forward in the alphabet (sending the last three letters of the alphabet to the first three). For instance, using this scheme the letter B is sent to E and the letter X is sent to A. This is an example of **encryption**, that is, the process of making a message secret.

To express Caesar's encryption process mathematically, first replace each letter by an element of $\mathbf{Z}_{26}$, that is, an integer from 0 to 25 equal to one less than its position in the alphabet. For example, replace A by 0, K by 10, and Z by 25. Caesar's encryption method can be represented by the function f that assigns to the nonnegative integer p, $p \leq 25$, the integer $f(p)$ in the set $\{0, 1, 2, \ldots, 25\}$ with

$$f(p) = (p + 3) \bmod 26.$$

In the encrypted version of the message, the letter represented by p is replaced with the letter represented by $(p + 3) \bmod 26$.

EXAMPLE 1 What is the secret message produced from the message "MEET YOU IN THE PARK" using the Caesar cipher?

Solution: First replace the letters in the message with numbers. This produces

12 4 4 19 24 14 20 8 13 19 7 4 15 0 17 10.

Now replace each of these numbers p by $f(p) = (p + 3)$ **mod** 26. This gives

15 7 7 22 1 17 23 11 16 22 10 7 18 3 20 13.

Translating this back to letters produces the encrypted message "PHHW BRX LQ WKH SDUN." ◀

To recover the original message from a secret message encrypted by the Caesar cipher, the function f^{-1}, the inverse of f, is used. Note that the function f^{-1} sends an integer p from $\mathbf{Z}_{26}$, to $f^{-1}(p) = (p - 3)$ **mod** 26. In other words, to find the original message, each letter is shifted back three letters in the alphabet, with the first three letters sent to the last three letters of the alphabet. The process of determining the original message from the encrypted message is called **decryption**.

There are various ways to generalize the Caesar cipher. For example, instead of shifting the numerical equivalent of each letter by 3, we can shift the numerical equivalent of each letter by k, so that

$$f(p) = (p + k) \textbf{ mod } 26.$$

Such a cipher is called a *shift cipher*. Note that decryption can be carried out using

$$f^{-1}(p) = (p - k) \textbf{ mod } 26.$$

Here the integer k is called a **key**. We illustrate the use of a shift cipher in Examples 2 and 3.

EXAMPLE 2 Encrypt the plaintext message "STOP GLOBAL WARMING" using the shift cipher with shift $k = 11$.

Solution: To encrypt the message "STOP GLOBAL WARMING" we first translate each letter to the corresponding element of $\mathbf{Z}_{26}$. This produces the string

18 19 14 15 6 11 14 1 0 11 22 0 17 12 8 13 6.

We now apply the shift $f(p) = (p + 11)$ **mod** 26 to each number in this string. We obtain

3 4 25 0 17 22 25 12 11 22 7 11 2 23 19 24 17.

Translating this last string back to letters, we obtain the ciphertext "DEZA RWZMLW HLCXTYR." ◀

EXAMPLE 3 Decrypt the ciphertext message "LEWLYPLUJL PZ H NYLHA ALHJOLY" that was encrypted with the shift cipher with shift $k = 7$.

Solution: To decrypt the ciphertext "LEWLYPLUJL PZ H NYLHA ALHJOLY" we first translate the letters back to elements of $\mathbf{Z}_{26}$. We obtain

11 4 22 11 24 15 11 20 9 11 15 25 7 13 24 11 7 0 0 11 7 9 14 11 24.

Next, we shift each of these numbers by $-k = -7$ modulo 26 to obtain

4 23 15 4 17 8 4 13 2 4 8 18 0 6 17 4 0 19 19 4 0 2 7 4 17.

Finally, we translate these numbers back to letters to obtain the plaintext. We obtain "EXPERIENCE IS A GREAT TEACHER." ◀

We can generalize shift ciphers further to slightly enhance security by using a function of the form

$$f(p) = (ap + b) \bmod 26,$$

where a and b are integers, chosen so that f is a bijection. (The function $f(p) = (ap + b) \bmod 26$ is a bijection if and only if $\gcd(a, 26) = 1$.) Such a mapping is called an *affine transformation*, and the resulting cipher is called an *affine cipher*.

EXAMPLE 4 What letter replaces the letter K when the function $f(p) = (7p + 3) \bmod 26$ is used for encryption?

Solution: First, note that 10 represents K. Then, using the encryption function specified, it follows that $f(10) = (7 \cdot 10 + 3) \bmod 26 = 21$. Because 21 represents V, K is replaced by V in the encrypted message. ◀

We will now show how to decrypt messages encrypted using an affine cipher. Suppose that $c = (ap + b) \bmod 26$ with $\gcd(a, 26) = 1$. To decrypt we need to show how to express p in terms of c. To do this, we apply the encrypting congruence $c \equiv ap + b \pmod{26}$, and solve it for p. To do this, we first subtract b from both sides, to obtain $c - b \equiv ap \pmod{26}$. Because $\gcd(a, 26) = 1$, we know that there is an inverse $\overline{a}$ of a modulo 26. Multiplying both sides of the last equation by $\overline{a}$ gives us $\overline{a}(c - b) \equiv \overline{a}ap \pmod{26}$. Because $\overline{a}a \equiv 1 \pmod{26}$, this tells us that $p \equiv \overline{a}(c - b) \pmod{26}$. This determines p because p belongs to $\mathbf{Z}_{26}$.

CRYPTANALYSIS The process of recovering plaintext from ciphertext without knowledge of both the encryption method and the key is known as **crytanalysis** or **breaking codes**. In general, cryptanalysis is a difficult process, especially when the encryption method is unknown. We will not discuss cryptanalysis in general, but we will explain how to break messages that were encrypted using a shift cipher.

If we know that a ciphertext message was produced by enciphering a message using a shift cipher, we can try to recover the message by shifting all characters of the ciphertext by each of the 26 possible shifts (including a shift of zero characters). One of these is guaranteed to be the plaintext. However, we can use a more intelligent approach, which we can build upon to cryptanalyze ciphertext resulting from other ciphers. The main tool for cryptanalyzing ciphertext encrypted using a shift cipher is the count of the frequency of letters in the ciphertext. The nine most common letters in English text and their approximate relative frequencies are E 13%, T 9%, A 8%, O 8%, I 7%, N 7%, S 7%, H 6%, and R 6%. To cryptanalyze ciphertext that we know was produced using a shift cipher, we first find the relative frequencies of letters in the ciphertext. We list the most common letters in the ciphertext in frequency order; we hypothesize that the most common letter in the ciphertext is produced by encrypting E. Then, we determine the value of the shift under this hypothesis, say k. If the message produced by shifting the ciphertext by $-k$ makes sense, we presume that our hypothesis is correct and that we have the correct value of k. If it does not make sense, we next consider the hypothesis that the most common letter in the ciphertext is produced by encrypting T, the second most common letter in English; we find k under this hypothesis, shift the letters of the message by $-k$, and see whether the resulting message makes sense. If it does not, we continue the process working our way through the letters from most common to least common.

Mathematicians make the best code breakers. Their work in World War II changed the course of the war.

EXAMPLE 5 Suppose that we intercepted the ciphertext message ZNK KGXRE HOXJ MKZY ZNK CUXS that we know was produced by a shift cipher. What was the original plaintext message?

Solution: Because we know that the intercepted ciphertext message was encrypted using a shift cipher, we begin by calculating the frequency of letters in the ciphertext. We find that the most common letter in the ciphertext is K. So, we hypothesize that the shift cipher sent the plaintext letter E to the ciphertext letter K. If this hypothesis is correct, we know that $10 = 4 + k$ **mod** 26, so $k = 6$. Next, we shift the letters of the message by -6, obtaining THE EARLY BIRD GETS THE WORM. Because this message makes sense, we assume that the hypothesis that $k = 6$ is correct. ◀

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BLOCK CIPHERS Shift ciphers and affine ciphers proceed by replacing each letter of the alphabet by another letter in the alphabet. Because of this, these ciphers are called **character** or **monoalphabetic ciphers**. Encryption methods of this kind are vulnerable to attacks based on the analysis of letter frequency in the ciphertext, as we just illustrated. We can make it harder to successfully attack ciphertext by replacing blocks of letters with other blocks of letters instead of replacing individual characters with individual characters; such ciphers are called **block ciphers**.

We will now introduce a simple type of block cipher, called the **transposition cipher**. As a key we use a permutation σ of the set $\{1, 2, \ldots, m\}$ for some positive integer m, that is, a one-to-one function from $\{1, 2, \ldots, m\}$ to itself. To encrypt a message we first split its letters into blocks of size m. (If the number of letters in the message is not divisible by m we add some random letters at the end to fill out the final block.) We encrypt the block $p_1 p_2 \ldots p_m$ as $c_1 c_2 \ldots c_m = p_{\sigma(1)} p_{\sigma(2)} \ldots, p_{\sigma(m)}$. To decryt a ciphertext block $c_1 c_2 \ldots c_m$, we transpose its letters using the permutation σ^{-1}, the inverse of σ. Example 6 illustrates encryption and decryption for a transposition cipher.

EXAMPLE 6 Using the transposition cipher based on the permutation σ of the set $\{1, 2, 3, 4\}$ with $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 2$,

(a) Encrypt the plaintext message PIRATE ATTACK.

(b) Decrypt the ciphertext message SWUE TRAE OEHS, which was encrypted using this cipher.

Solution: (a) We first split the letters of the plaintext into blocks of four letters. We obtain PIRA TEAT TACK. To encrypt each block, we send the first letter to the third position, the second letter to the first position, the third letter to the fourth position, and the fourth letter to the second position. We obtain IAPR ETTA AKTC.

(b) We note that σ^{-1}, the inverse of σ, sends 1 to 2, sends 2 to 4, sends 3 to 1, and sends 4 to 3. Applying $\sigma^{-1}(m)$ to each block gives us the plaintext: USEW ATER HOSE. (Grouping together these letters to form common words, we surmise that the plaintext is USE WATER HOSE.) ◀

CRYPTOSYSTEMS We have defined two families of ciphers: shift ciphers and affine ciphers. We now introduce the notion of a cryptosystem, which provides a general structure for defining new families of ciphers.

DEFINITION 1

A *cryptosystem* is a five-tuple $(\mathcal{P}, \mathcal{C}, \mathcal{K}, \mathcal{E}, \mathcal{D})$, where $\mathcal{P}$ is the set of plaintext strings, $\mathcal{C}$ is the set of ciphertext strings, $\mathcal{K}$ is the *keyspace* (the set of all possible keys), $\mathcal{E}$ is the set of encryption functions, and $\mathcal{D}$ is the set of decryption functions. We denote by E_k the encryption function in $\mathcal{E}$ corresponding to the key k and D_k the decryption function in $\mathcal{D}$ that decrypts ciphertext that was encrypted using E_k, that is $D_k(E_k(p)) = p$, for all plaintext strings p.

We now illustrate the use of the definition of a cryptosystem.

EXAMPLE 7 Describe the family of shift ciphers as a cryptosytem.

Solution: To encrypt a string of English letters with a shift cipher, we first translate each letter to an integer between 0 and 25, that is, to an element of $\mathbf{Z}_{26}$. We then shift each of these integers by a fixed integer modulo 26, and finally, we translate the integers back to letters. To apply the defintion of a cryptosystem to shift ciphers, we assume that our messages are already integers, that is, elements of $\mathbf{Z}_{26}$. That is, we assume that the translation between letters and integers is outside of the cryptosystem. Consequently, both the set of plaintext strings $\mathcal{P}$ and the set of ciphertext strings $\mathcal{C}$ are the set of strings of elements of $\mathbf{Z}_{26}$. The set of keys $\mathcal{K}$ is the set of possible shifts, so $\mathcal{K} = \mathbf{Z}_{26}$. The set $\mathcal{E}$ consists of functions of the form $E_k(p) = (p + k) \bmod 26$, and the set $\mathcal{D}$ of decryption functions is the same as the set of encrypting functions where $D_k(p) = (p - k) \bmod 26$. ◀

The concept of a cryptosystem is useful in the discussion of additional families of ciphers and is used extensively in cryptography.

Public Key Cryptography

All classical ciphers, including shift ciphers and affine ciphers, are examples of **private key cryptosystems**. In a private key cryptosystem, once you know an encryption key, you can quickly find the decryption key. So, knowing how to encrypt messages using a particular key allows you to decrypt messages that were encrypted using this key. For example, when a shift cipher is used with encryption key k, the plaintext integer p is sent to

$$c = (p + k) \bmod 26.$$

Decryption is carried out by shifting by $-k$; that is,

$$p = (c - k) \bmod 26.$$

So knowing how to encrypt with a shift cipher also tells you how to decrypt.

When a private key cryptosystem is used, two parties who wish to communicate in secret must share a secret key. Because anyone who knows this key can both encrypt and decrypt messages, two people who want to communicate securely need to securely exchange this key. (We will introduce a method for doing this later in this section.) The shift cipher and affine cipher cryptosystems are private key cryptosystems. They are quite simple and are extremely vulnerable to cryptanalysis. However, the same is not true of many modern private key cryptosystems. In particular, the current US government standard for private key cryptography, the Advanced Encryption Standard (AES), is extremely complex and is considered to be highly resistant to cryptanalysis. (See [St06] for details on AES and other modern private key cryptosystems.) AES is widely used in government and commercial communications. However, it still shares the property that for secure communications keys be shared. Furthermore, for extra security, a new key is used for each communication session between two parties, which requires a method for generating keys and securely sharing them.

To avoid the need for keys to be shared by every pair of parties that wish to communicate securely, in the 1970s cryptologists introduced the concept of **public key cryptosystems**. When such cryptosystems are used, knowing how to send an encrypted message does not help decrypt messages. In such a system, everyone can have a publicly known encryption key. Only the decryption keys are kept secret, and only the intended recipient of a message can decrypt it, because, as far as it is currently known, knowledge of the encryption key does not let someone recover the plaintext message without an extraordinary amount of work (such as billions of years of computer time).

The RSA Cryptosystem

M.I.T. is also known as the 'Tute.

In 1976, three researchers at the Massachusetts Institute of Technology—Ronald Rivest, Adi Shamir, and Leonard Adleman—introduced to the world a public key cryptosystem, known as the **RSA system**, from the initials of its inventors. As often happens with cryptographic discoveries, the RSA system had been discovered several years earlier in secret government research in the United Kingdom. Clifford Cocks, working in secrecy at the United Kingdom's Government Communications Headquarters (GCHQ), had discovered this cryptosystem in 1973. However, his invention was unknown to the outside world until the late 1990s, when he was allowed to share classified GCHQ documents from the early 1970s. (An excellent account of this earlier discovery, as well as the work of Rivest, Shamir, and Adleman, can be found in [Si99].)

Unfortunately, no one calls this the Cocks cryptosystem.

In the RSA cryptosystem, each individual has an encryption key (n, e) where $n = pq$, the modulus is the product of two large primes p and q, say with 200 digits each, and an exponent e that is relatively prime to $(p - 1)(q - 1)$. To produce a usable key, two large primes must be found. This can be done quickly on a computer using probabilistic primality tests, referred to earlier in this section. However, the product of these primes $n = pq$, with approximately 400 digits, cannot, as far as is currently known, be factored in a reasonable length of time. As we will see, this is an important reason why decryption cannot, as far as is currently known, be done quickly without a separate decryption key.

RSA Encryption

To encrypt messages using a particular key (n, e), we first translate a plaintext message M into sequences of integers. To do this, we first translate each plaintext letter into a two-digit number, using the same translation we employed for shift ciphers, with one key difference. That is, we include an initial zero for the letters A through J, so that A is translated into 00, B into 01, . . . , and J into 09. Then, we concatenate these two-digit numbers into strings of digits. Next, we divide this string into equally sized blocks of $2N$ digits, where $2N$ is the largest even number such that the number $2525 \ldots 25$ with $2N$ digits does not exceed n. (When necessary, we pad the plaintext message with dummy Xs to make the last block the same size as all other blocks.)

After these steps, we have translated the plaintext message M into a sequence of integers $m_1, m_2, \ldots, m_k$ for some integer k. Encryption proceeds by transforming each block m_i to a ciphertext block c_i. This is done using the function

$$C = M^e \bmod n.$$

(To perform the encryption, we use an algorithm for fast modular exponentiation, such as Algorithm 5 in Section 4.2.) We leave the encrypted message as blocks of numbers and send these to the intended recipient. Because the RSA cryptosystem encrypts blocks of characters into blocks of characters, it is a block cipher.

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CLIFFORD COCKS (BORN 1950) Clifford Cocks, born in Cheshire, England, was a talented mathematics student. In 1968 he won a silver medal at the International Mathematical Olympiad. Cocks attended King's College, Cambridge, studying mathematics. He also spent a short time at Oxford University working in number theory. In 1973 he decided not to complete his graduate work, instead taking a mathematical job at the Government Communications Headquarters (GCHQ) of British intelligence. Two months after joining GCHQ, Cocks learned about public key cryptography from an internal GCHQ report written by James Ellis. Cocks used his number theory knowledge to invent what is now called the RSA cryptosystem. He quickly realized that a public key cryptosystem could be based on the difficulty of reversing the process of multiplying two large primes. In 1997 he was allowed to reveal declassified GCHQ internal documents describing his discovery. Cocks is also known for his invention of a secure identity based encryption scheme, which uses information about a user's identity as a public key. In 2001, Cocks became the Chief Mathematician at GCHQ. He has also set up the Heilbronn Institute for Mathematical Research, a partnership between GCHQ and the University of Bristol.

Example 8 illustrates how RSA encryption is performed. For practical reasons we use small primes p and q in this example, rather than primes with 200 or more digits. Although the cipher described in this example is not secure, it does illustrate the techniques used in the RSA cipher.

EXAMPLE 8 Encrypt the message STOP using the RSA cryptosystem with key (2537, 13). Note that $2537 = 43 \cdot 59$, $p = 43$ and $q = 59$ are primes, and

$$\gcd(e, (p-1)(q-1)) = \gcd(13, 42 \cdot 58) = 1.$$

Solution: To encrypt, we first translate the letters in STOP into their numerical equivalents. We then group these numbers into blocks of four digits (because $2525 < 2537 < 252525$), to obtain

$$1819 \quad 1415.$$

We encrypt each block using the mapping

$$C = M^{13} \bmod 2537.$$

Computations using fast modular multiplication show that $1819^{13} \bmod 2537 = 2081$ and $1415^{13} \bmod 2537 = 2182$. The encrypted message is 2081 2182. ◀

RSA Decryption

The plaintext message can be quickly recovered from a ciphertext message when the decryption key d, an inverse of e modulo $(p-1)(q-1)$, is known. [Such an inverse exists because $\gcd(e, (p-1)(q-1)) = 1$.] To see this, note that if $de \equiv 1 \pmod{(p-1)(q-1)}$, there is an integer k such that $de = 1 + k(p-1)(q-1)$. It follows that

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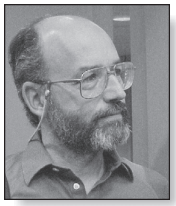
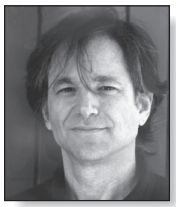
$$C^d \equiv (M^e)^d = M^{de} = M^{1+k(p-1)(q-1)} \pmod{n}.$$

RONALD RIVEST (BORN 1948) Ronald Rivest received a B.A. from Yale in 1969 and his Ph.D. in computer science from Stanford in 1974. Rivest is a computer science professor at M.I.T. and was a cofounder of RSA Data Security, which held the patent on the RSA cryptosystem that he invented together with Adi Shamir and Leonard Adleman. Areas that Rivest has worked in besides cryptography include machine learning, VLSI design, and computer algorithms. He is a coauthor of a popular text on algorithms ([CoLeRiSt09]).



ADI SHAMIR (BORN 1952) Adi Shamir was born in Tel Aviv, Israel. His undergraduate degree is from Tel Aviv University (1972) and his Ph.D. is from the Weizmann Institute of Science (1977). Shamir was a research assistant at the University of Warwick and an assistant professor at M.I.T. He is currently a professor in the Applied Mathematics Department at the Weizmann Institute and leads a group studying computer security. Shamir's contributions to cryptography, besides the RSA cryptosystem, include cracking knapsack cryptosystems, cryptanalysis of the Data Encryption Standard (DES), and the design of many cryptographic protocols.



LEONARD ADLEMAN (BORN 1945) Leonard Adleman was born in San Francisco, California. He received a B.S. in mathematics (1968) and his Ph.D. in computer science (1976) from the University of California, Berkeley. Adleman was a member of the mathematics faculty at M.I.T. from 1976 until 1980, where he was a coinventor of the RSA cryptosystem, and in 1980 he took a position in the computer science department at the University of Southern California (USC). He was appointed to a chaired position at USC in 1985. Adleman has worked on computer security, computational complexity, immunology, and molecular biology. He invented the term "computer virus." Adleman's recent work on DNA computing has sparked great interest. He was a technical adviser for the movie *Sneakers*, in which computer security played an important role.

By Fermat's little theorem [assuming that $\gcd(M, p) = \gcd(M, q) = 1$, which holds except in rare cases, which we cover in Exercise 28], it follows that $M^{p-1} \equiv 1 \pmod{p}$ and $M^{q-1} \equiv 1 \pmod{q}$. Consequently,

$$C^d \equiv M \cdot (M^{p-1})^{k(q-1)} \equiv M \cdot 1 = M \pmod{p}$$

and

$$C^d \equiv M \cdot (M^{q-1})^{k(p-1)} \equiv M \cdot 1 = M \pmod{q}.$$

Because $\gcd(p, q) = 1$, it follows by the Chinese remainder theorem that

$$C^d \equiv M \pmod{pq}.$$

Example 9 illustrates how to decrypt messages sent using the RSA cryptosystem.

EXAMPLE 9 We receive the encrypted message 0981 0461. What is the decrypted message if it was encrypted using the RSA cipher from Example 8?

Solution: The message was encrypted using the RSA cryptosystem with $n = 43 \cdot 59$ and exponent 13. As Exercise 2 in Section 4.4 shows, $d = 937$ is an inverse of 13 modulo $42 \cdot 58 = 2436$. We use 937 as our decryption exponent. Consequently, to decrypt a block C, we compute

$$M = C^{937} \bmod 2537.$$

To decrypt the message, we use the fast modular exponentiation algorithm to compute $0981^{937} \bmod 2537 = 0704$ and $0461^{937} \bmod 2537 = 1115$. Consequently, the numerical version of the original message is 0704 1115. Translating this back to English letters, we see that the message is HELP. ◀

RSA as a Public Key System

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Why is the RSA cryptosystem suitable for public key cryptography? First, it is possible to rapidly construct a public key by finding two large primes p and q, each with more than 200 digits, and to find an integer e relatively prime to $(p - 1)(q - 1)$. When we know the factorization of the modulus n, that is, when we know p and q, we can quickly find an inverse d of e modulo $(p - 1)(q - 1)$. [This is done by using the Euclidean algorithm to find Bézout coefficients s and t for d and $(p - 1)(q - 1)$, which shows that the inverse of d modulo $(p - 1)(q - 1)$ is $s \bmod (p - 1)(q - 1)$.] Knowing d lets us decrypt messages sent using our key. However, no method is known to decrypt messages that is not based on finding a factorization of n, or that does not also lead to the factorization of n.

Factorization is believed to be a difficult problem, as opposed to finding large primes p and q, which can be done quickly. The most efficient factorization methods known (as of 2010) require billions of years to factor 400-digit integers. Consequently, when p and q are 200-digit primes, it is believed that messages encrypted using $n = pq$ as the modulus cannot be found in a reasonable time unless the primes p and q are known.

Although no polynomial-time algorithm is known for factoring large integers, active research is under way to find new ways to efficiently factor integers. Integers that were thought, as recently as several years ago, to be far too large to be factored in a reasonable amount of time can now be factored routinely. Integers with more than 150 digits, as well as some with more than 200 digits, have been factored using team efforts. When new factorization techniques are found,

it will be necessary to use larger primes to ensure secrecy of messages. Unfortunately, messages that were considered secure earlier can be saved and subsequently decrypted by unintended recipients when it becomes feasible to factor the $n = pq$ in the key used for RSA encryption.

The RSA method is now widely used. However, the most commonly used cryptosystems are private key cryptosystems. The use of public key cryptography, via the RSA system, is growing. Nevertheless, there are applications that use both private key and public key systems. For example, a public key cryptosystem, such as RSA, can be used to distribute private keys to pairs of individuals when they wish to communicate. These people then use a private key system for encryption and decryption of messages.

Cryptographic Protocols

So far we have shown how cryptography can be used to make messages secure. However, there are many other important applications of cryptography. Among these applications are **cryptographic protocols**, which are exchanges of messages carried out by two or more parties to achieve a particular security goal. In particular, we will show how cryptography can be used to allow two people to exchange a secret key over an insecure communication channel. We will also show how cryptography can be used to send signed secret messages so that the recipient can be sure that the message came from the purported sender. We refer the reader to [St05] for thorough discussions of a variety of cryptographic protocols.

KEY EXCHANGE We now discuss a protocol that two parties can use to exchange a secret key over an insecure communications channel without having shared any information in the past. Generating a key that two parties can share is important for many applications of cryptography. For example, for two people to send secure messages to each other using a private key cryptosystem they need to share a common key. The protocol we will describe is known as the **Diffie-Hellman key agreement protocol**, after Whitfield Diffie and Martin Hellman, who described it in 1976. However, this protocol was invented in 1974 by Malcolm Williamson in secret work at the British GCHQ. It was not until 1997 that his discovery was made public.

Suppose that Alice and Bob want to share a common key. The protocol follows these steps, where the computations are done in $\mathbf{Z}_p$.

(1) Alice and Bob agree to use a prime p and a primitive root a of p.

(2) Alice chooses a secret integer k_1 and sends a^{k_1} **mod** p to Bob.

(3) Bob chooses a secret integer k_2 and sends a^{k_2} **mod** p to Alice.

(4) Alice computes $(a^{k_2})^{k_1}$ **mod** p.

(5) Bob computes $(a^{k_1})^{k_2}$ **mod** p.

At the end of this protocol, Alice and Bob have computed their shared key, namely

$$(a^{k_2})^{k_1} \textbf{ mod } p = (a^{k_1})^{k_2} \textbf{ mod } p.$$

To analyze the security of this protocol, note that the messages sent in steps (1), (2), and (3) are not assumed to be sent securely. We can even assume that these communications were in the clear and that their contents are public information. So, p, a, a^{k_1} **mod** p, and a^{k_2} **mod** p are assumed to be public information. The protocol ensures that k_1, k_2, and the common key $(a^{k_2})^{k_1}$ **mod** $p = (a^{k_1})^{k_2}$ **mod** p are kept secret. To find the secret information from this public information requires that an adversary solves instances of the discrete logarithm problem,

because the adversary would need to find k_1 and k_2 from a^{k_1} **mod** p and a^{k_2} **mod** p, respectively. Furthermore, no other method is known for finding the shared key using just the public information. We have remarked that this is thought to be computationally infeasible when p and a are sufficiently large. With the computing power available now, this system is considered unbreakable when p has more than 300 decimal digits and k_1 and k_2 have more than 100 decimal digits each.

DIGITAL SIGNATURES Not only can cryptography be used to secure the confidentiality of a message, but it also can be used so that the recipient of the message knows that it came from the person they think it came from. We first show how a message can be sent so that a recipient of the message will be sure that the message came from the purported sender of the message. In particular, we can show how this can be accomplished using the RSA cryptosystem to apply a **digital signature** to a message.

Suppose that Alice's RSA public key is (n, e) and her private key is d. Alice encrypts a plaintext message x using the encryption function $E_{(n,e)}(x) = x^e$ **mod** n. She decrypts a ciphertext message y using the decryption function $D_{(n,e)} = x^d$ **mod** n. Alice wants to send the message M so that everyone who receives the message knows that it came from her. Just as in RSA encryption, she translates the letters into their numerical equivalents and splits the resulting string into blocks $m_1, m_2, \ldots, m_k$ such that each block is the same size which is as large as possible so that $0 \leq m_i \leq n$ for $i = 1, 2, ..., k$. She then applies her *decryption function* $D_{(n,e)}$ to each block, obtaining $D_{n,e}(m_i)$, $i = 1, 2, \ldots, k$. She sends the result to all intended recipients of the message.

When a recipient receives her message, they apply Alice's encryption function $E_{(n,e)}$ to each block, which everyone has available because Alice's key (n, e) is public information. The result is the original plaintext block because $E_{(n,e)}(D_{(n,e)}(x)) = x$. So, Alice can send her message to as many people as she wants and by signing it in this way, every recipient can be sure it came from Alice. Example 10 illustrates this protocol.

EXAMPLE 10 Suppose Alice's public RSA cryptosystem key is the same as in Example 8. That is, $n = 43 \cdot 59 = 2537$ and $e = 13$. Her decryption key is $d = 937$, as described in Example 9. She wants to send the message "MEET AT NOON" to her friends so that they are sure it came from her. What should she send?

Solution: Alice first translates the message into blocks of digits, obtaining 1204 0419 0019 1314 1413 (as the reader should verify). She then applies her decryption transformation $D_{(2537,13)}(x) = x^{937}$ **mod** 2537 to each block. Using fast modular exponentiation (with the help of a computational aid), she finds that 1204^{937} **mod** $2537 = 817$, 419^{937} **mod** $2537 = 555$, 19^{937} **mod** $2537 = 1310$, 1314^{937} **mod** $2537 = 2173$, and 1413^{937} **mod** $2537 = 1026$.

So, the message she sends, split into blocks, is 0817 0555 1310 2173 1026. When one of her friends gets this message, they apply her encryption transformation $E_{(2537,13)}$ to each block. When they do this, they obtain the blocks of digits of the original message which they translate back to English letters. ◀

We have shown that signed messages can be sent using the RSA cryptosystem. We can extend this by sending signed secret messages. To do this, the sender applies RSA encryption using the publicly known encryption key of an intended recipient to each block that was encrypted using sender's decryption transformation. The recipient then first applies his private decryption transformation and then the sender's public encryption transformation. (Exercise 32 asks for this protocol to be carried out.)