University of Colorado **Boulder**

Department of Computer Science

CSCI 2824: Discrete Structures

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Propositional Logic (Lecture 2)

Conditionals and Biconditionals

Propositional Logic

Last Time:

- **Propositions** are declarative statements with **truth values**
- Combine propositions with **negations** and **connectives**
- Combinations of propositions are also propositions
- **Truth tables** give us truth values of combined propositions

p	$\neg p$
T	F
F	T

Propositional Logic

Last Time:

- **Propositions** are declarative statements with **truth values**
- Combine propositions with **negations** and **connectives**
- Combinations of propositions are also propositions
- **Truth tables** give us truth values of combined propositions

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

Conditionals

Two new connectives: **conditional** ($\rightarrow$) and **biconditional** ($\leftrightarrow$)

Conditionals: Describe *if-then* relationships

Example: *If I have the day off then I go to the gym*

p = "I have the day off"

q = "I go to the gym"

In symbols: $p \rightarrow q$

Given the four possible combinations of truth values of p and q what are the associated truth values of the conditional $p \rightarrow q$?

Conditionals

Think of the conditional $p \to q$ as defining a **rule**. Then think of cases when rule is valid or rule is broken.

$p = T, q = T$: *I have the day off, I go to the gym*

Rule is clearly satisfied, so $p \to q$ is _____

$p = F, q = F$: *I don't have the day off, I don't go to the gym*

Rule is not broken, so $p \to q$ is _____

What about:

$p = T, q = F$: *I have the day off, I don't go to the gym*

Conditionals

Think of the conditional $p \to q$ as defining a **rule**. Then think of cases when rule is valid or rule is broken.

$p = T, q = F$: *I have the day off, I don't go to the gym*

In this case the rule is broken, so $p \to q$ is _____

What about the last case:

$p = F, q = T$: *I don't have the day off, I go to the gym*

Conditionals

Think of the conditional $p \to q$ as defining a **rule**. Then think of cases when rule is valid or rule is broken.

$p = T, q = F$: *I have the day off, I don't go to the gym*

In this case the rule is broken, so $p \to q$ is **False**

What about the last case:

$p = F, q = T$: *I don't have the day off, I go to the gym*

The rule doesn't state that the only time that I go to the gym is when I have the day off. Maybe I sometimes go to the gym after work

This doesn't break the rule, so $p \to q$ is _____

Conditionals

Def: Let p and q be propositions. The **conditional** if p then q, denoted by $p \to q$, is false when p is true but q is false, and true otherwise.

The truth-table for the conditional looks as follows:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditionals in English Language

Example: Translate the following sentence into symbols

If it snows then I wear boots to work

Conditionals in English Language

Example: Translate the following into symbols

I wear a tie to work only if I teach that day

Be careful here. Which expression is the one that its occurrence guarantees the occurrence of the other?

Conditionals in English Language

Example: Translate the following into symbols

I wear a tie to work only if I teach that day

Be careful here. Which expression is the one that its occurrence guarantees the occurrence of the other?

This says that the only way that you'll see me in a tie, is if I teach that day. **My wearing a tie implies that I must be teaching**.

Thus if $p =$ *I wear a tie* and $q =$ *I teach* then we have $p \to q$

Conditionals in English Language

Example: Translate the following into symbols

I wear a tie to work only if I teach that day

This says that the only way that you'll see me in a tie, is if I teach that day. **My wearing a tie implies that I must be teaching**.

Thus if $p =$ *I wear a tie* and $q =$ *I teach* then we have $p \rightarrow q$

The Crux: In the "only if" case you have to recognize that just because "if" is spatially closer to "I teach that day" does not mean that it is the if-part of the conditional

Biconditionals

Two new connectives: **conditional** ($\rightarrow$) and **biconditional** ($\leftrightarrow$)

Biconditionals: Describe *if-and-only-if* relationships

Example: *A polygon is a triangle if and only if it has exactly 3 sides*

p = "A polygon is a triangle"

q = "A polygon has exactly 3 sides"

In symbols: $p \leftrightarrow q$

Biconditionals

Def: Let p and q be propositions. The **biconditional** p if and only if q, denoted by $p \leftrightarrow q$, is true when p and q have the same truth value, and false otherwise.

The truth table for the biconditional looks like

p	q	$p \leftrightarrow q$
T	T	
T	F	
F	T	
F	F	

Biconditionals

Example: Determine the truth values of $(p \to q) \wedge (q \to p)$

Strategy: Determine truth values of constituent propositions first

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T			
T	F			
F	T			
F	F			

Biconditionals

Do the truth values for $(p \to q) \wedge (q \to p)$ look familiar?

p	q	$(p \to q) \wedge (q \to p)$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditionals

They're exactly the same as for the biconditional $p \leftrightarrow q$

p	q	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Negation of the Biconditional

Question: What's another way to write $\neg(p \leftrightarrow q)$?

Negation of the Biconditional

Question: What's another way to write $\neg(p \leftrightarrow q)$?

p	q	$\neg(p \leftrightarrow q)$	$p \oplus q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

Application: Logic Puzzles

On the island there are two types of people: Knights who always tell the truth, and Knaves who always lie.

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight". Use a truth table to determine what type of people A and B are.

Let $p =$ *A is a knight* and $q =$ *B is a knight*

How could you represent A's statement symbolically?

Application: Logic Puzzles

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight". Use a truth table to determine what type of people A and B are.

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Application: Logic Puzzles

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight". Use a truth table to determine what type of people A and B are.

New Strategy: We'd like to know the combinations of truth values of p and q that ensure that statements made by A and B are consistent with their nature as Knights of Knaves. I.e. we don't want A to be a Knight but utter a False statement.

In this example, one way to accomplish this is to test that p (the statement that A is a Knight) is equivalent in truth value to the statement that he uttered (i.e. $\neg p \vee q$)

We can do that by testing the proposition $p \leftrightarrow (\neg p \vee q)$

Application: Logic Puzzles

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight". Use a truth table to determine what type of people A and B are.

Again, we'll find the truth tables for the constituent parts first

p	q	$\neg p$	$\neg p \vee q$	$p \leftrightarrow (\neg p \vee q)$
T	T			
T	F			
F	T			
F	F			

Application: Logic Puzzles

Example: On the island you encounter two people, who we'll call A and B. A tells you that "I am a Knave or B is a Knight". Use a truth table to determine what type of people A and B are.

Again, we'll find the truth tables for the constituent parts first

p	q	$\neg p$	$\neg p \vee q$	$p \leftrightarrow (\neg p \vee q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

Conclusion: A and B are **both** Knights.

Application: Logic Puzzles

Example: You encounter two people, who we'll call A and B. Person A tells you that "B is a Knight" and Person B tells you that "The two of us are of opposite types". Use a truth table to determine the types of people that A and B are.

This one is a little trickier, because A and B **both** make statements. You'll have to think a bit to come up with the test proposition that ensures that the nature of both people are consistent with their statements.

Application: Logic Puzzles

Example: You encounter two people, who we'll call A and B. Person A tells you that "B is a Knight" and Person B tells you that "The two of us are of opposite types". Use a truth table to determine the types of people that A and B are.

Solution: Let $p =$ *A is a Knight* and $q =$ *B is a Knight*

A's statement that *B is a Knight* is again q

B's statement that they are of opposites can be written as $p \oplus q$

The fact that their statements are consistent with their types can be written as $p \leftrightarrow q \quad \text{and} \quad q \leftrightarrow (p \oplus q)$

Since we want both propositions to be true at the same time, the final test proposition is their conjunction $(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$

Application: Logic Puzzles

The truth table for the final test proposition can be written as

p	q	$p \leftrightarrow q$	$\neg p$	$p \oplus q$	$q \leftrightarrow (p \oplus q)$	$(p \leftrightarrow q) \wedge (q \leftrightarrow (p \oplus q))$
T	T	T	F	F	F	F
T	F	F	F	T	F	F
F	T	F	T	T	T	F
F	F	T	T	F	T	T

Since the only row that gives a T for the test proposition in the last row, we conclude that both A and B are Knaves.

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **sufficient** that n be divisible by 12 for n to be divisible by 6

Let $r = n$ *is divisible by 12* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **sufficient** that n be divisible by 12 for n to be divisible by 6

Let $r = n$ *is divisible by 12* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

If a number is divisible by 12 then it's definitely divisible by 6

If a number is divisible by 6 it doesn't have to be divisible by 12

Answer: For a sufficient condition, the condition goes at the front of the conditional: $r \rightarrow s$

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **necessary** that n^2 be divisible by 9 for n to be divisible by 6

Let $q = n^2$ *is divisible by 9* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

Application: Necessary and Sufficient Conditions

Example: Let n be a natural number. It is **necessary** that n^2 be divisible by 9 for n to be divisible by 6

Let $q = n^2$ *is divisible by 9* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

If n is divisible by 6 then n^2 is definitely divisible by 9

If n^2 is divisible by 9 the number is **not** for sure divisible by 6

Answer: For a necessary condition, the condition goes at the end of the conditional: $s \rightarrow q$

Application: Necessary and Sufficient Conditions

Example: How could you translate the following into symbols?

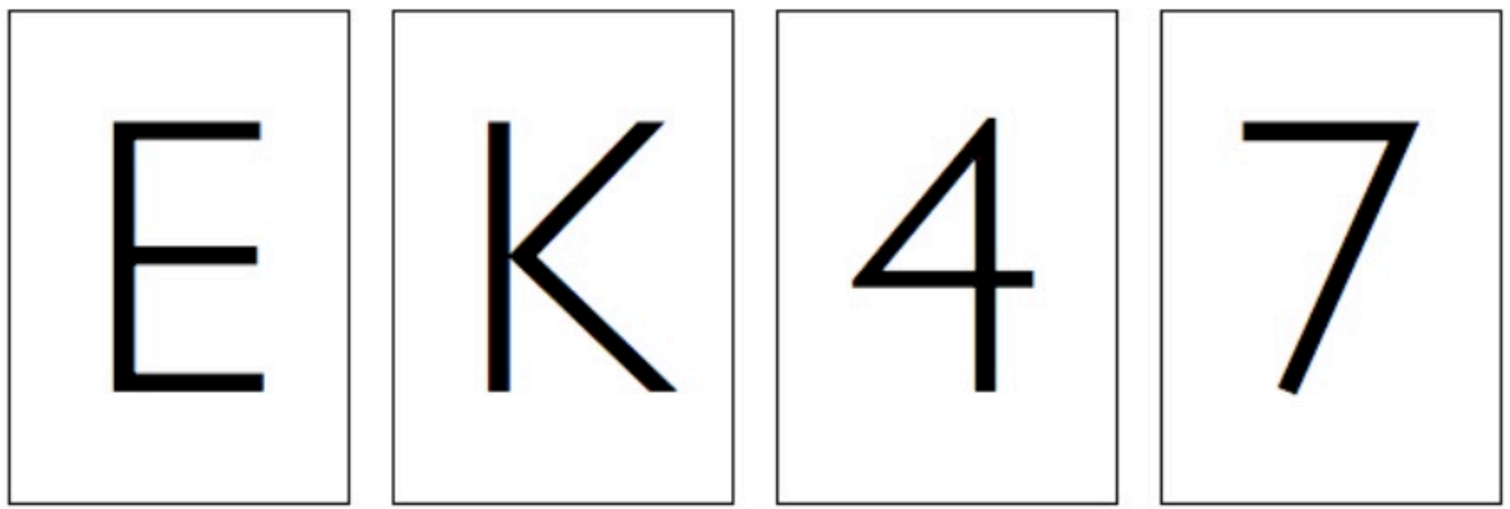
- $p =$ *n is even and divisible by* 3
- $q =$ *n is divisible by* 6

It is both necessary **and** sufficient that n is even and divisible by 3 for n to be divisible by 6

A Psychology Experiment

Consider the following four cards that have letters on one side and numbers on the other side. Suppose I tell you the following rule:

If a card has an odd number then its letter is a vowel



Questions: Name the cards (and only those cards) that you need to turn over to verify that the given rule is true.