Linear dynamical system with input. There are many variations on and extensions of the basic linear dynamical system model (9.1), some of which we will encounter later. As an example, we can add additional terms to the update equation:

$$x_{t+1} = A_t x_t + B_t u_t + c_t, \quad t = 1, 2, \ldots. \tag{9.2}$$

Here u_t is an m-vector called the *input*, B_t is the $n \times m$ *input matrix*, and the n-vector c_t is called the *offset*, all at time t. The input and offset are used to model other factors that affect the time evolution of the state. Another name for the input u_t is *exogenous variable*, since, roughly speaking, it comes from outside the system.

Markov model. The linear dynamical system (9.1) is sometimes called a *Markov* model (after the mathematician Andrey Markov). Markov studied systems in which the next state value depends on the current one, and not on the previous state values $x_{t-1}, x_{t-2}, \ldots$. The linear dynamical system (9.1) is the special case of a Markov system where the next state is a linear function of the current state.

In a variation on the Markov model, called a (linear) K-Markov model, the next state x_{t+1} depends on the current state and $K-1$ previous states. Such a system has the form

$$x_{t+1} = A_1 x_t + \cdots + A_K x_{t-K+1}, \quad t = K, K+1, \ldots. \tag{9.3}$$

Models of this form are used in time series analysis and econometrics, where they are called (vector) *auto-regressive models*. When $K=1$, the Markov model (9.3) is the same as a linear dynamical system (9.1). When $K>1$, the Markov model (9.3) can be reduced to a standard linear dynamical system (9.1), with an appropriately chosen state; see exercise 9.4.

Simulation. If we know the dynamics (and input) matrices, and the state at time t, we can find the future state trajectory $x_{t+1}, x_{t+2}, \ldots$ by iterating the equation (9.1) (or (9.2), provided we also know the input sequence $u_t, u_{t+1}, \ldots$). This is called *simulating* the linear dynamical system. Simulation makes predictions about the future state of a system. (To the extent that (9.1) is only an approximation or model of some real system, we must be careful when interpreting the results.) We can carry out what-if simulations, to see what would happen if the system changes in some way, or if a particular set of inputs occurs.

9.2 Population dynamics

Linear dynamical systems can be used to describe the evolution of the age distribution in some population over time. Suppose x_t is a 100-vector, with $(x_t)_i$ denoting the number of people in some population (say, a country) with age $i-1$ (say, on January 1) in year t, where t is measured starting from some base year, for $i = 1, \ldots, 100$. While $(x_t)_i$ is an integer, it is large enough that we simply consider

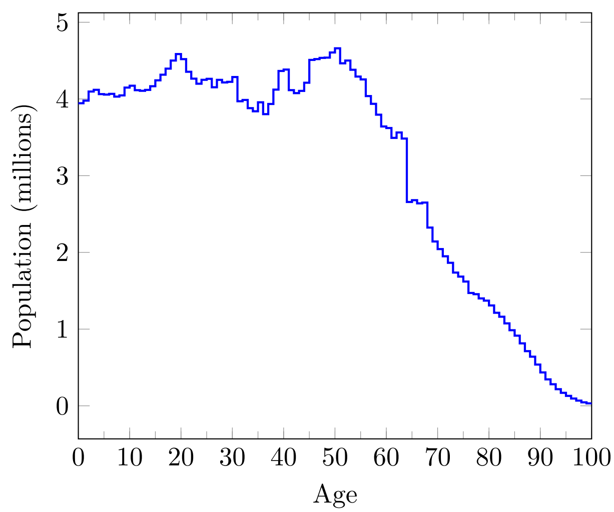

Figure 9.1 Age distribution in the US in 2010. (United States Census Bureau, census.gov).

it a real number. In any case, our model certainly is not accurate at the level of individual people. Also, note that the model does not track people 100 and older. The distribution of ages in the US in 2010 is shown in figure 9.1.

The birth rate is given by a 100-vector b, where b_i is the average number of births per person with age $i-1$, $i = 1, \ldots, 100$. (This is half the average number of births per woman with age $i-1$, assuming equal numbers of men and women in the population.) Of course b_i is approximately zero for $i < 13$ and $i > 50$. The approximate birth rates for the US in 2010 are shown in figure 9.2. The death rate is given by a 100-vector d, where d_i is the portion of those aged $i-1$ who will die this year. The death rates for the US in 2010 are shown in figure 9.3.

To derive the dynamics equation (9.1), we find x_{t+1} in terms of x_t, taking into account only births and deaths, and not immigration. The number of 0-year olds next year is the total number of births this year:

$$(x_{t+1})_1 = b^T x_t.$$

The number of i-year olds next year is the number of $(i-1)$-year-olds this year, minus those who die:

$$(x_{t+1})_{i+1} = (1 - d_i)(x_t)_i, \quad i = 1, \ldots, 99.$$

We can assemble these equations into the time-invariant linear dynamical system

$$x_{t+1} = Ax_t, \quad t = 1, 2, \ldots, \tag{9.4}$$

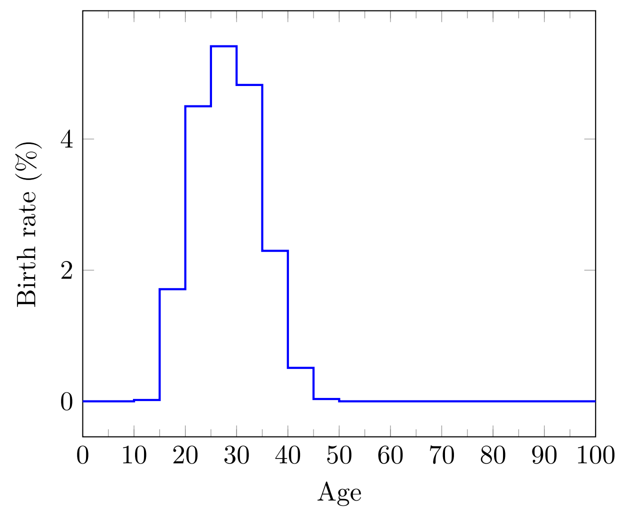

Figure 9.2 Approximate birth rate versus age in the US in 2010. The figure is based on statistics for age groups of five years (hence, the piecewise-constant shape) and assumes an equal number of men and women in each age group. (Martin J.A., Hamilton B.E., Ventura S.J. *et al.*, Births: Final data for 2010. National Vital Statistics Reports; vol. 61, no. 1. National Center for Health Statistics, 2012.)

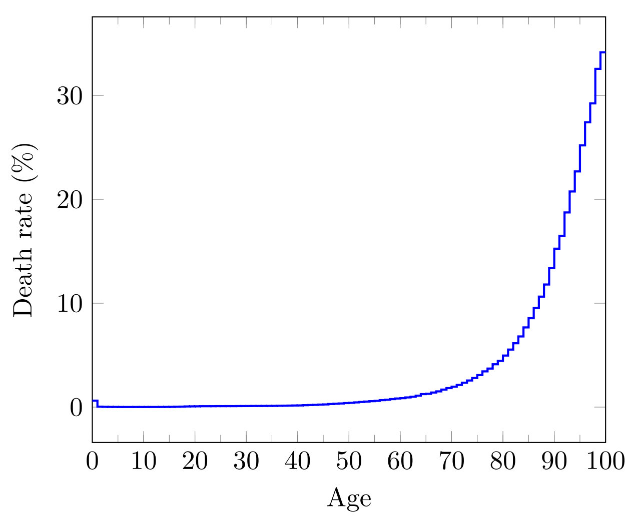

Figure 9.3 Death rate versus age, for ages 0–99, in the US in 2010. (Centers for Disease Control and Prevention, National Center for Health Statistics, wonder.cdc.gov.)

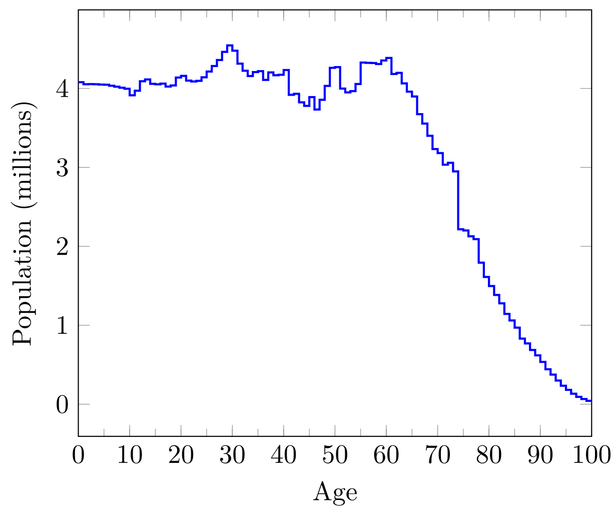

Figure 9.4 Predicted age distribution in the US in 2020.

where A is given by

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{98} & b_{99} & b_{100} \\ 1-d_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-d_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-d_{98} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1-d_{99} & 0 \end{bmatrix}.$$

We can use this model to predict the total population in 10 years (not including immigration), or to predict the number of school age children, or retirement age adults. Figure 9.4 shows the predicted age distribution in 2020, computed by iterating the model $x_{t+1} = Ax_t$ for $t = 1, \ldots, 10$, with initial value x_1 given by the 2010 age distribution of figure 9.1. Note that the distribution is based on an approximate model, since we neglect the effect of immigration, and assume that the death and birth rates remain constant and equal to the values shown in figures 9.2 and 9.3.

Population dynamics models are used to carry out projections of the future age distribution, which in turn is used to predict how many retirees there will be in some future year. They are also used to carry out various 'what if' analyses, to predict the effect of changes in birth or death rates on the future age distribution.

It is easy to include the effects of immigration and emigration in the population dynamics model (9.4), by simply adding a 100-vector u_t:

$$x_{t+1} = Ax_t + u_t,$$

which is a time-invariant linear dynamical system of the form (9.2), with input u_t and $B = I$. The vector u_t gives the net immigration in year t over all ages; $(u_t)_i$ is the number of immigrants in year t of age $i - 1$. (Negative entries mean net emigration.)

9.3 Epidemic dynamics

The dynamics of infection and the spread of an epidemic can be modeled using a linear dynamical system. (More sophisticated nonlinear epidemic dynamic models are also used.) In this section we describe a simple example.

A disease is introduced into a population. In each period (say, days) we count the fraction of the population that is in four different infection states:

- *Susceptible.* These individuals can acquire the disease the next day.
- *Infected.* These individuals have the disease.
- *Recovered* (and immune). These individuals had the disease and survived, and now have immunity.
- *Deceased.* These individuals had the disease, and unfortunately died from it.

We denote the fractions of each of these as a 4-vector x_t, so, for example, $x_t = (0.75, 0.10, 0.10, 0.05)$ means that in day t, 75% of the population is susceptible, 10% is infected, 10% is recovered and immune, and 5% has died from the disease.

There are many mathematical models that predict how the disease state fractions x_t evolve over time. One simple model can be expressed as a linear dynamical system. The model assumes the following happens over each day.

- 5% of the susceptible population will acquire the disease. (The other 95% will remain susceptible.)
- 1% of the infected population will die from the disease, 10% will recover and acquire immunity, and 4% will recover and not acquire immunity (and therefore, become susceptible). The remaining 85% will remain infected.

(Those who have have recovered with immunity and those who have died remain in those states.)

We first determine $(x_{t+1})_1$, the fraction of susceptible individuals in the next day. These include the susceptible individuals from today, who did not become infected, which is $0.95(x_t)_1$, plus the infected individuals today who recovered without immunity, which is $0.04(x_t)_2$. All together we have $(x_{t+1})_1 = 0.95(x_t)_1 + 0.04(x_t)_2$. We have $(x_{t+1})_2 = 0.85(x_t)_2 + 0.05(x_t)_1$; the first term counts those who are infected and remain infected, and the second term counts those who are susceptible and acquire the disease. Similar arguments give $(x_{t+1})_3 = (x_t)_3 + 0.10(x_t)_2$, and $(x_{t+1})_4 = (x_t)_4 + 0.01(x_t)_2$. We put these together to get

$$x_{t+1} = \begin{bmatrix} 0.95 & 0.04 & 0 & 0 \\ 0.05 & 0.85 & 0 & 0 \\ 0 & 0.10 & 1 & 0 \\ 0 & 0.01 & 0 & 1 \end{bmatrix} x_t,$$

which is a time-invariant linear dynamical system of the form (9.1).

Figure 9.5 shows the evolution of the four groups from the initial condition $x_0 = (1, 0, 0, 0)$. The simulation shows that after around 100 days, the state converges to one with a little under 10% of the population deceased, and the remaining population immune.