Lab 8

Markov Chains

You have been given a lot of flexibility for your implementation of a predictive paging algorithm for PA4. In this lab, we will introduce *Markov chains*, a stochastic model that you might find useful for deciding which pages to page in/out. Regardless of whether you decide to use a Markov chain for PA4, they are of wide interest in both theory and practice.

Matrix Multiplication

Let A be a $\ell \times m$ matrix and B be a $m \times n$ matrix:

$$A = \begin{pmatrix} a_{1,1}, a_{1,2}, \cdots, a_{1,m} \\ a_{2,1}, a_{2,2}, \cdots, a_{2,m} \\ \vdots \ , \ \vdots \ , \cdots, \ \vdots \\ a_{\ell,1}, a_{\ell,2}, \cdots, a_{\ell,m} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1,1}, b_{1,2}, \cdots, b_{1,n} \\ b_{2,1}, b_{2,2}, \cdots, b_{2,n} \\ \vdots \ , \ \vdots \ , \cdots, \ \vdots \\ b_{m,1}, b_{m,2}, \cdots, b_{m,n} \end{pmatrix}.$$

The multiplication AB gives a $\ell \times n$ matrix defined such that the (i, j)-entry is

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Convince yourself that you essentially already wrote multithreaded code for performing AB. In particular, extend your code to perform AB. What quantity should each thread should be in charge of computing?

Walks and Matrix Multiplication

Recall from your course on algorithms that we can represent a directed graph $G = (V, E)$ as a $n \times n$ adjacency matrix A such that $(A)_{i,j} = 1$ if (i, j) is an arc, and is 0 otherwise.

1. What does the (i, j)-entry of A^2 compute?
2. For any positive integer k, what does the (i, j)-entry of A^k compute?

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A *Markov chain* M can be seen as a generalization of our adjacency matrix. It is defined such that $M_{i,j}$ is the *probability* of going from i to j in a single step. We weight the arcs of our digraph $G = (V, E)$ with non-negative real numbers such that

$$\sum_{j \in V} M_{i,j} = 1 \quad \text{for each } i \in V \quad (\textit{why?})$$

where $M_{i,j} = 0$ if $(i, j) \notin E$. In other words, a $n \times n$ matrix is a Markov chain if its entries are all non-negative and each row sums to 1.

1. For any positive integer k, is M^k a Markov chain?
2. What does the (i, j)-entry of M^k compute?

Finally, think about how you could use a Markov chain as part of a predictive paging algorithm. How do you define M? What are its probabilities? How do we compute them? What do powers of M tell us?