CSPB 2824 - Stade - Discrete Structures

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Started on	Saturday, 2 December 2023, 1:00 PM
State	Finished
Completed on	Saturday, 2 December 2023, 1:38 PM
Time taken	37 mins 55 secs
Marks	28.00/28.00
Grade	**10.00** out of 10.00 (**100**%)

Question **1**

Correct

Mark 2.00 out of 2.00

Suppose we run an experiment in which we toss a coin 3 times. The sample space for this experiment is $S =$ {HHH, THH, HTH, HHT, TTT, HTT, THT, TTH}

Let A be the event "the coin comes up Tails exactly two times" and B be the event "the first flip results in Tails". Mark the outcomes below that belong to the event $A \cap B$.

- [ ] HHH
- [ ] THH
- [ ] HTH
- [ ] HHT
- [ ] TTT
- [ ] HTT
- [x] THT ✔
- [x] TTH ✔

Your answer is correct.

The correct answers are: THT, TTH

Correct

Marks for this submission: 2.00/2.00.

Question **2**

Correct

Mark 4.00 out of 4.00

Suppose you flip a fair coin 6 times. Answer the following questions. You should input your answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

What is the probability that the coin comes up Heads all 6 times?

0.016

✔

What is the probability that the coin comes up Heads at least 5 times?

0.109

✔

What is the probability that the coin comes up Heads all 6 times?

Solution: There are $2^6 = 64$ ways to flip a coin 6 times. Of those 64 ways, only 1 of them results in HHHHHH. The the probability is $\frac{1}{64} \approx 0.016$.

What is the probability that the coin comes up Heads at least 5 times?

Solution: The outcomes for this event are HHHHHH, THHHHH, HTHHHH, HHTHHH, HHHTHH, HHHHTH, HHHHHT. There are 7 equally likely outcomes in the event, thus the probability is $\frac{7}{64} \approx 0.109$.

Correct

Marks for this submission: 4.00/4.00.

Question 3

Correct

Mark 2.00 out of 2.00

What is the probability that a five-card poker hand (drawn from a standard 52-card deck) contains at least one ace? You should input your answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

0.341

Solution: It's easier to compute the probability that no aces appear and then use the complement rule. There are 48 cards in the deck that *aren't* aces, so the number of ways to choose $5$5 non-aces is $C(48,5)$

Then the probability of picking no aces is $p(E) = C(48,5)/C(52,5)$.

Finally, the probability of picking at least one ace is

$p(\bar{E}) = 1 - C(48,5)/C(52,5) \approx 0.341$

Correct

Marks for this submission: 2.00/2.00.

Question **4**

Correct

Mark 6.00 out of 6.00

Suppose a family has two children. Let E be the event that the family's children are of different sexes and F be the event that the family has at least one girl. Assume that the probability of a child being of either sex is equally likely. Answer the following questions.

You should input answers to numerical questions in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

Compute $p(E) =$

0.50

Compute $p(E \mid F) =$

0.667

Are the events E and F independent? No

Solution:

There are 4 outcomes in the sample space: {BB, BG, GB, GG}

The outcomes in the event E are {BG, GB}

Thus, we have $p(E) = 2/4 = 0.5$

The outcomes in the event F are {BG, GB, GG} and the outcomes in the event $E \cap F$ are {BG, GB}. Thus, we have $p(E \mid F) = p(E \cap F)/p(F) = (1/2)/(3/4) = 2/3 \approx 0.667$

Since $p(E \mid F) \neq p(E)$ the event E and F are **NOT** independent.

Correct

Marks for this submission: 6.00/6.00.

Question **5**

Correct

Mark 4.00 out of 4.00

Suppose you have 6 dice in a bag. You draw a single die and roll it.

- One die is a standard fair die, numbered 1-6
- Two dice are have an extra pip drawn in the middle of the side with 4 pips, such that there is no 4 on these two dice, but there are two 5's (that is, these dice are numbered {1, 2, 3, 5, 5, 6})
- Three dice are loaded such that it is twice as likely that an even number is rolled than an odd

What is the probability that you roll a 4?

0.139

✔

What is the probability that you roll a 5?

0.194

✔

You should input your answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

Solution:

Let F be the event that you pull a fair die out and roll it.

Let M be the event that you pull one of the dice with two 5s but no 4s out and roll it

Let L be the event that you pull one of the three even-loaded die out and roll it

Let R_k be the event that you roll a k

We are going to need the probability of rolling a 4 (R_4) subject to the condition that we know which type of die we're rolling.

$p(R_4 \mid F) = \frac{1}{6}$

$p(R_4 \mid M) = 0$

$p(R_4 \mid L)$ requires a bit more work. Let's focus just on the loaded dice for a second.

We know $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$

We also know that $p(1) = p(3) = p(5)$ and $p(2) = p(4) = p(6) = 2p(1)$

So $1 = 3p(1) + 3p(2) = 3p(1) + 6p(1) = 9p(1)$

Which means $p(1) = \frac{1}{9}$ (which is the probability for any of the odd numbers on these loaded dice)

and $p(2) = \frac{2}{9}$ (which is the probability for any of the even numbers on these loaded dice)

That means $p(R_4 \mid L) = \frac{2}{9}$

The Law of Total Probability (LTP) gives:

$$
\begin{aligned}
p(R_4) &= p(R_4 \mid F)p(F) + p(R_4 \mid M)p(M) + p(R_4 \mid L)p(L) \\
&= \frac{1}{6} \cdot \frac{1}{6} + 0 \cdot \frac{2}{6} + \frac{2}{9} \cdot \frac{3}{6} \\
&= \frac{5}{36} + \frac{1}{9}
\end{aligned}
$$

Similarly, for the probability of rolling a 5:

$$\begin{aligned} p(R_5) &= p(R_5 \mid F)p(F) + p(R_5 \mid M)p(M) + p(R_5 \mid L)p(L) \\ &= \frac{1}{6} \cdot \frac{1}{6} + \frac{2}{6} \cdot \frac{2}{6} + \frac{1}{9} \cdot \frac{3}{6} \\ &= \frac{1}{36} + \frac{4}{36} + \frac{2}{36} \\ &= \frac{7}{36} \end{aligned}$$

Correct

Marks for this submission: 4.00/4.00.

Question **6**

Correct

Mark 6.00 out of 6.00

Please enter any numeric answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

Part A.

Suppose you have a fair coin. You flip the coin two times. Let T_1 be the event that the first flip results in Tails. Let T_2 be the event that the second flip results in Tails.

Are the events T_1 and T_2 independent? Yes ✔

Which **one** of the following calculations would allow you to make this conclusion? (b) ✔

(a) Comparing $p(T_2 \cap T_1)$ against $p(T_2)$
(b) Comparing $p(T_2 \mid T_1)$ against $p(T_2)$
(c) Comparing $p(T_2)$ against $p(T_1)$
(d) Comparing $p(T_2 \mid T_1)$ against $p(T_1 \mid T_2)$

Now suppose that you have three coins in your pocket. Two of the coins are fair, but the third coin has tails on both sides. You reach into your pocket without looking, pull out a coin, and you flip this coin two times.

Part B.

Without any knowledge of the result of the first flip, what is the probability of getting Tails on the second flip? That is, what is $p(T_2)$?

0.667

✔

If you observe that the result of the first flip is Tails, what is the probability that the second flip will also result in Tails?

0.75

✔

Are the events T_1 and T_2 independent? No ✔

Solution:

Part A:

Because we know that we are flipping a fair coin, we know that the events T_1 and T_2 are independent.

The definition of independence (see lecture slides for 7.2: Basic Probability Theory) tells us that T_1 and T_2 are independent if $p(T_2 \mid T_1) = p(T_2)$

Alternatively, you could check $p(T_1 \mid T_2)$ against $p(T_1)$, but answer (b) is the only option that clearly fits with the definition of independence.

Part B:

Now we do not know whether or not we flip the fair coin or the two-tails coin. So we need to use the Law of Total Probability (LTP) to calculate the probability of getting a tails, conditioned on which coin we flip.

Let F be the event that we flip a fair coin and let $\overline{F}$ be the event that we do not (i.e., we flip the two-tails coin).

$$\begin{aligned} p(T_2) &= p(T_2 \mid F)p(F) + p(T_2 \mid \overline{F})p(\overline{F}) \\ &= \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

But if we know that our first flip is tails, we are now looking for $p(T_2 \mid T_1)$. We use the definition of conditional probability to find:

$$p(T_2 \mid T_1) = \frac{p(T_2 \cap T_1)}{p(T_1)}$$

From the first part of Part B, we know $p(T_1) = \frac{2}{3}$, because with no knowledge of the other flip, this expression also applies to the first flip.

Then we can use the LTP to find the numerator:

$$\begin{aligned} p(T_2 \cap T_1) &= p(T_2 \cap T_1 \mid F)p(F) + p(T_2 \cap T_1 \mid \overline{F})p(\overline{F}) \\ &= \frac{1}{4} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{1}{2} \end{aligned}$$

Putting the numerator and denominator together, we find:

$$p(T_2 \mid T_1) = \frac{1/2}{2/3} = \frac{3}{4}$$

Finally, because $p(T_2 \mid T_1) = 0.75 \neq 0.667 = p(T_2)$, the events are **not** independent. Note that this differs from Part A in that here we **do not know which coin is flipped**, whereas in Part A we do.

Correct

Marks for this submission: 6.00/6.00.

Question **7**

Correct

Mark 4.00 out of 4.00

Suppose again that you have three coins in your pocket. Two of the coins are fair, but the third coin has tails on both sides. You reach into your pocket without looking, pull out a coin, and you flip this coin two times.

Please enter any numeric answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

If you observe that both flips are Tails, what is the probability that you picked the two-Tailed coin out of your pocket?

0.667

✔

Suppose instead that you flip the coin n times and observe n consecutive Tails. What is the smallest integer n required for you to conclude that you've chosen the two-Tailed coin with probability greater than 0.9?

5

✔

Solution:

Let F be the event that you pick a fair coin out of your pocket and B be the event that you picked the biased coin out of your pocket. We want to know $p(B \mid TT)$ where TT represents the event that both flips were tails. Using Bayes' Thm, we have

$$p(B \mid TT) = p(TT \mid B)p(B)/p(TT) =$$

$$= p(TT \mid B)p(B)/(p(TT \mid F)p(F) + p(TT \mid B)p(B))$$

Assuming that you're equally likely to pick any coin out of your pocket, we have $p(B) = 1/3$ and $p(F) = 2/3$. Plugging things in, we have

$$p(B \mid TT) = 1(1/3)/((1/4)(2/3) + (1)(1/3)) = 2/3 \approx 0.667$$

For n flips the calculation is

$$p(B \mid nT's) = 1(1/3)/((1/2)^n(2/3) + (1)(1/3))$$

The first time this probability exceeds 0.9 is when $n = 5$, which gives

$$p(B \mid 5T's) = 1(1/3)/((1/2)^5(2/3) + (1)(1/3)) \approx 0.941$$

Correct

Marks for this submission: 4.00/4.00.

Question **8**

Not answered

Not graded

Bayesian Spam Filters (optional question)

Suppose we have received 5 emails and categorized each of them as either **spam** or **ham** (any email that is not spam is ham). The results are displayed below in the table. Use this information to answer the following questions. You should input your answers in decimal form, rounded to the nearest thousandth. *For example, if you find 0.0148 as an answer, you should enter "0.015".*

Email 1	Email 2	Email 3	Email 4	Email 5
spam	**spam**	**spam**	**ham**	**ham**
buy	money	nigeria	money	fly
car	profit	bank	bank	home
nigeria	home	check	home	nigeria
profit		wire	car	

Part A.

Based on the data above, estimate the probability of an arbitrary email being **spam**, $p(\text{spam})$.

Based on the data above, estimate the probability of an arbitrary email being **ham**, $p(\text{ham})$.

Part B.

Given that you know an email is **spam**, estimate the probability that it contains the word "nigeria". (That is, estimate $p(\text{nigeria} \mid \text{spam})$)

Given that you know an email is **ham**, estimate the probability that it contains the word "nigeria". (That is, estimate $p(\text{nigeria} \mid \text{ham})$)

Part C.

As discussed in class, we can categorize an email by comparing $p(\text{ham} \mid \text{email})$ to $p(\text{spam} \mid \text{email})$, and classifying the given email according to which quantity is larger. In general, these conditional probabilities are estimated using Bayes' Rule as

$$p(\text{category} \mid \text{email}) = \frac{p(\text{email} \mid \text{category})p(\text{category})}{p(\text{email})}$$

We note that for both the **spam** and **ham** computations, the denominator in this fraction is the same, and thus we can instead classify the email according to which numerator is larger.

Compute the estimated numerator for $p(\text{spam} \mid \text{nigeria})$.

Compute the estimated numerator for $p(\text{ham} \mid \text{nigeria})$.

Part D.

Based on your work above, would our spam filter classify an email containing solely the word "nigeria" as **ham** or **spam**?

Note: Make sure to double-check your answers for Part C before making a selection!

Choose...

Solution:

Part A:

There are 3 spam emails and 5 emails total, so our estimate is $p(\text{spam}) = \frac{3}{5}$

There are 2 ham emails and 5 emails total, so our estimate is $p(\text{spam}) = \frac{2}{5}$

Part B:

"Given that we know an email is spam" means that we restrict our attention to only the 3 spam emails. Thus our estimate of $p(\text{nigeria} \mid \text{spam})$ is the number of spam emails with "nigeria" in them, divided by the total number of spam emails. That gives $2/3$

Similarly, we find an estimate for $p(\text{nigeria} \mid \text{ham})$ of $1/2$

Part C:

Bayes' Theorem gives $p(\text{spam} \mid \text{nigeria}) = \frac{p(\text{nigeria} \mid \text{spam})p(\text{spam})}{p(\text{nigeria})}$, so the numerator we must calculate is $p(\text{nigeria} \mid \text{spam})p(\text{spam})$

Turns out, we have these quantities from Parts A and B, for both spam and ham! We find:

$$p(\text{nigeria} \mid \text{spam})p(\text{spam}) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$$

and

$$p(\text{nigeria} \mid \text{ham})p(\text{ham}) = \frac{1}{2} \cdot \frac{2}{5} = \frac{1}{5}$$

Part D:

The numerator for $p(\text{ham} \mid \text{nigeria})$ is larger than the numerator for $p(\text{spam} \mid \text{nigeria})$, and the denominators are the same ($p(\text{nigeria})$). So we would classify this email as **spam**.