
13 Red-Black Trees

Chapter 12 showed that a binary search tree of height h can support any of the basic dynamic-set operations—such as SEARCH, PREDECESSOR, SUCCESSOR, MINIMUM, MAXIMUM, INSERT, and DELETE—in $O(h)$ time. Thus, the set operations are fast if the height of the search tree is small. If its height is large, however, the set operations may run no faster than with a linked list. Red-black trees are one of many search-tree schemes that are “balanced” in order to guarantee that basic dynamic-set operations take $O(\lg n)$ time in the worst case.

13.1 Properties of red-black trees

A *red-black tree* is a binary search tree with one extra bit of storage per node: its *color*, which can be either RED or BLACK. By constraining the node colors on any simple path from the root to a leaf, red-black trees ensure that no such path is more than twice as long as any other, so that the tree is approximately *balanced*.

Each node of the tree now contains the attributes *color*, *key*, *left*, *right*, and *p*. If a child or the parent of a node does not exist, the corresponding pointer attribute of the node contains the value NIL. We shall regard these NILs as being pointers to leaves (external nodes) of the binary search tree and the normal, key-bearing nodes as being internal nodes of the tree.

A red-black tree is a binary tree that satisfies the following *red-black properties*:

1. Every node is either red or black.
2. The root is black.
3. Every leaf (NIL) is black.
4. If a node is red, then both its children are black.
5. For each node, all simple paths from the node to descendant leaves contain the same number of black nodes.

Figure 13.1(a) shows an example of a red-black tree.

As a matter of convenience in dealing with boundary conditions in red-black tree code, we use a single sentinel to represent NIL (see page 238). For a red-black tree T , the sentinel $T.nil$ is an object with the same attributes as an ordinary node in the tree. Its *color* attribute is BLACK, and its other attributes—*p*, *left*, *right*, and *key*—can take on arbitrary values. As Figure 13.1(b) shows, all pointers to NIL are replaced by pointers to the sentinel $T.nil$.

We use the sentinel so that we can treat a NIL child of a node x as an ordinary node whose parent is x . Although we instead could add a distinct sentinel node for each NIL in the tree, so that the parent of each NIL is well defined, that approach would waste space. Instead, we use the one sentinel $T.nil$ to represent all the NILs—all leaves and the root's parent. The values of the attributes *p*, *left*, *right*, and *key* of the sentinel are immaterial, although we may set them during the course of a procedure for our convenience.

We generally confine our interest to the internal nodes of a red-black tree, since they hold the key values. In the remainder of this chapter, we omit the leaves when we draw red-black trees, as shown in Figure 13.1(c).

We call the number of black nodes on any simple path from, but not including, a node x down to a leaf the **black-height** of the node, denoted $bh(x)$. By property 5, the notion of black-height is well defined, since all descending simple paths from the node have the same number of black nodes. We define the black-height of a red-black tree to be the black-height of its root.

The following lemma shows why red-black trees make good search trees.

Lemma 13.1

A red-black tree with n internal nodes has height at most $2 \lg(n + 1)$.

Proof We start by showing that the subtree rooted at any node x contains at least $2^{bh(x)} - 1$ internal nodes. We prove this claim by induction on the height of x . If the height of x is 0, then x must be a leaf ($T.nil$), and the subtree rooted at x indeed contains at least $2^{bh(x)} - 1 = 2^0 - 1 = 0$ internal nodes. For the inductive step, consider a node x that has positive height and is an internal node with two children. Each child has a black-height of either $bh(x)$ or $bh(x) - 1$, depending on whether its color is red or black, respectively. Since the height of a child of x is less than the height of x itself, we can apply the inductive hypothesis to conclude that each child has at least $2^{bh(x)-1} - 1$ internal nodes. Thus, the subtree rooted at x contains at least $(2^{bh(x)-1} - 1) + (2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$ internal nodes, which proves the claim.

To complete the proof of the lemma, let h be the height of the tree. According to property 4, at least half the nodes on any simple path from the root to a leaf, not

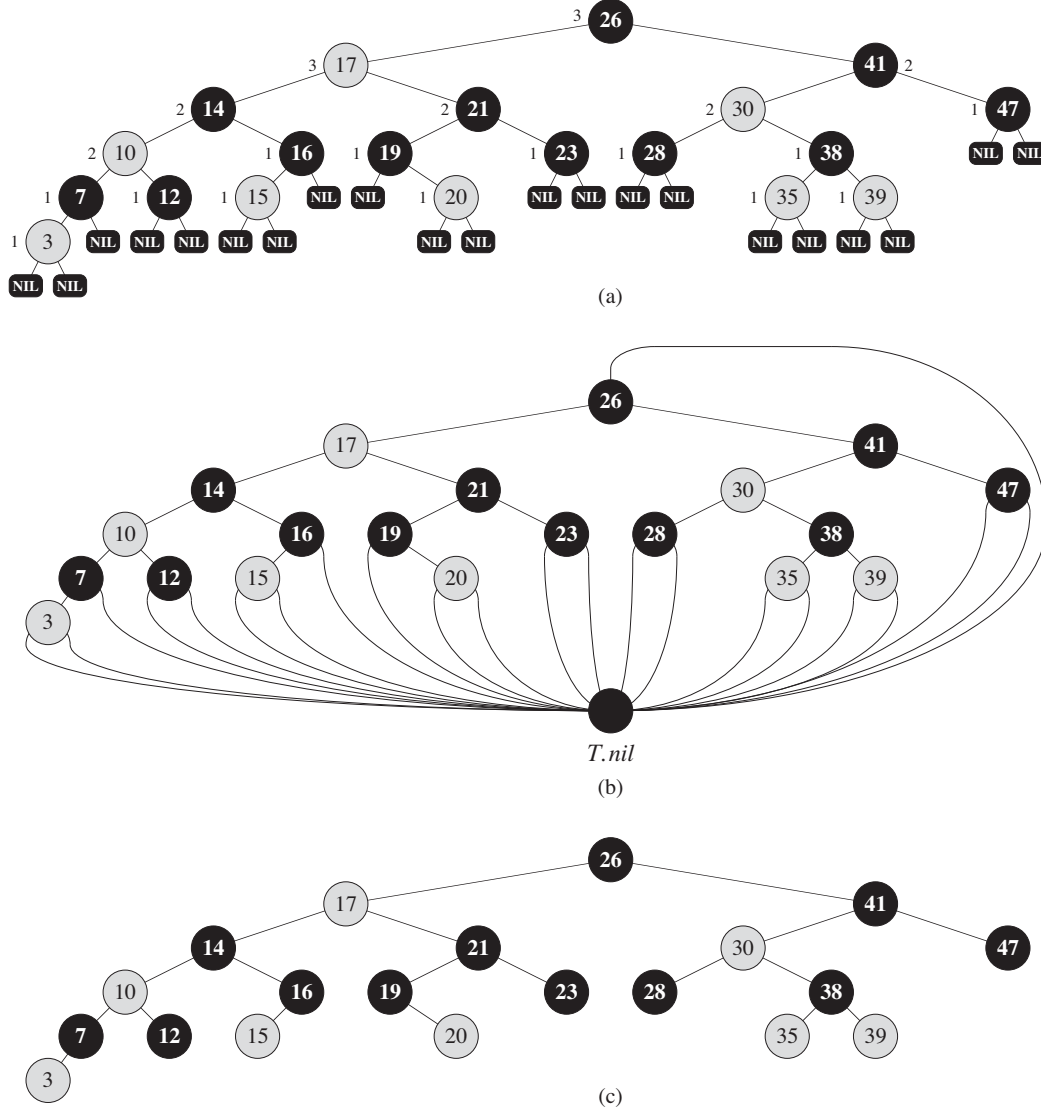


Figure 13.1 A red-black tree with black nodes darkened and red nodes shaded. Every node in a red-black tree is either red or black, the children of a red node are both black, and every simple path from a node to a descendant leaf contains the same number of black nodes. (a) Every leaf, shown as a NIL, is black. Each non-NIL node is marked with its black-height; NILs have black-height 0. (b) The same red-black tree but with each NIL replaced by the single sentinel $T.nil$, which is always black, and with black-heights omitted. The root's parent is also the sentinel. (c) The same red-black tree but with leaves and the root's parent omitted entirely. We shall use this drawing style in the remainder of this chapter.

including the root, must be black. Consequently, the black-height of the root must be at least $h/2$; thus,

$$n \geq 2^{h/2} - 1.$$

Moving the 1 to the left-hand side and taking logarithms on both sides yields $\lg(n + 1) \geq h/2$, or $h \leq 2 \lg(n + 1)$. ■

As an immediate consequence of this lemma, we can implement the dynamic-set operations SEARCH, MINIMUM, MAXIMUM, SUCCESSOR, and PREDECESSOR in $O(\lg n)$ time on red-black trees, since each can run in $O(h)$ time on a binary search tree of height h (as shown in Chapter 12) and any red-black tree on n nodes is a binary search tree with height $O(\lg n)$. (Of course, references to NIL in the algorithms of Chapter 12 would have to be replaced by $T.nil$.) Although the algorithms TREE-INSERT and TREE-DELETE from Chapter 12 run in $O(\lg n)$ time when given a red-black tree as input, they do not directly support the dynamic-set operations INSERT and DELETE, since they do not guarantee that the modified binary search tree will be a red-black tree. We shall see in Sections 13.3 and 13.4, however, how to support these two operations in $O(\lg n)$ time.

Exercises

13.1-1

In the style of Figure 13.1(a), draw the complete binary search tree of height 3 on the keys $\{1, 2, \dots, 15\}$. Add the NIL leaves and color the nodes in three different ways such that the black-heights of the resulting red-black trees are 2, 3, and 4.

13.1-2

Draw the red-black tree that results after TREE-INSERT is called on the tree in Figure 13.1 with key 36. If the inserted node is colored red, is the resulting tree a red-black tree? What if it is colored black?

13.1-3

Let us define a **relaxed red-black tree** as a binary search tree that satisfies red-black properties 1, 3, 4, and 5. In other words, the root may be either red or black. Consider a relaxed red-black tree T whose root is red. If we color the root of T black but make no other changes to T , is the resulting tree a red-black tree?

13.1-4

Suppose that we “absorb” every red node in a red-black tree into its black parent, so that the children of the red node become children of the black parent. (Ignore what happens to the keys.) What are the possible degrees of a black node after all