

18. Show that if A and B are sets $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.
19. Show that if A , B , C , and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.
20. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
21. Show that if A , B , and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.
22. Suppose that A is a countable set. Show that the set B is also countable if there is an onto function f from A to B .
23. Show that if A is an infinite set, then it contains a countably infinite subset.
24. Show that there is no infinite set A such that $|A| < |\mathbb{Z}^+| = \aleph_0$.
25. Prove that if it is possible to label each element of an infinite set S with a finite string of keyboard characters, from a finite list characters, where no two elements of S have the same label, then S is a countably infinite set.
26. Use Exercise 25 to provide a proof different from that in the text that the set of rational numbers is countable. [Hint: Show that you can express a rational number as a string of digits with a slash and possibly a minus sign.]
- *27. Show that the union of a countable number of countable sets is countable.
28. Show that the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.
- *29. Show that the set of all finite bit strings is countable.
- *30. Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a , b , and c are integers, is countable.
- *31. Show that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable by showing that the polynomial function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.
- *32. Show that when you substitute $(3n + 1)^2$ for each occurrence of n and $(3m + 1)^2$ for each occurrence of m in the right-hand side of the formula for the function $f(m, n)$ in Exercise 31, you obtain a one-to-one polynomial function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$. It is an open question whether there is a one-to-one polynomial function $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$.
33. Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality.
34. Show that $(0, 1)$ and \mathbb{R} have the same cardinality. [Hint: Use the Schröder-Bernstein theorem.]
35. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [Hint: Assume that there is such a one-to-one correspondence. Represent a subset of the set of positive integers as an infinite bit string with i th bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with its i th bit equal to the complement of the i th bit of the i th string in the list. Show that this new bit string cannot appear in the list.]
- *36. Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |\mathcal{P}(\mathbb{Z}^+)| = |\mathbb{R}|$. [Hint: Look at the first part of the hint for Exercise 35.]
- *37. Show that the set of all computer programs in a particular programming language is countable. [Hint: A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.]
- *38. Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [Hint: First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2\dots d_n\dots$ the function f with $f(n) = d_n$.]
- *39. We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 37 and 38 to show that there are functions that are not computable.
- *40. Show that if S is a set, then there does not exist an onto function f from S to $\mathcal{P}(S)$, the power set of S . Conclude that $|S| < |\mathcal{P}(S)|$. This result is known as **Cantor's theorem**. [Hint: Suppose such a function f existed. Let $T = \{s \in S \mid s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$.]

2.6 Matrices

Introduction

Matrices are used throughout discrete mathematics to express relationships between elements in sets. In subsequent chapters we will use matrices in a wide variety of models. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models. This section reviews matrix arithmetic that will be used in these algorithms.

DEFINITION 1

A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

EXAMPLE 1

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

We now introduce some terminology about matrices. Boldface uppercase letters will be used to represent matrices.

DEFINITION 2

Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The i th *row* of \mathbf{A} is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The j th *column* of \mathbf{A} is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ \vdots \\ a_{mj} \end{bmatrix}.$$

The (i, j) th *element* or *entry* of \mathbf{A} is the element a_{ij} , that is, the number in the i th row and j th column of \mathbf{A} . A convenient shorthand notation for expressing the matrix \mathbf{A} is to write $\mathbf{A} = [a_{ij}]$, which indicates that \mathbf{A} is the matrix with its (i, j) th element equal to a_{ij} .

Matrix Arithmetic

The basic operations of matrix arithmetic will now be discussed, beginning with a definition of matrix addition.

DEFINITION 3

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The *sum* of \mathbf{A} and \mathbf{B} , denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j) th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because the sum of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

EXAMPLE 2

We have $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}.$

We now discuss matrix products. A product of two matrices is defined only when the number of columns in the first matrix equals the number of rows of the second matrix.

DEFINITION 4

Let \mathbf{A} be an $m \times k$ matrix and \mathbf{B} be a $k \times n$ matrix. The *product* of \mathbf{A} and \mathbf{B} , denoted by \mathbf{AB} , is the $m \times n$ matrix with its (i, j) th entry equal to the sum of the products of the corresponding elements from the i th row of \mathbf{A} and the j th column of \mathbf{B} . In other words, if $\mathbf{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

In Figure 1 the colored row of \mathbf{A} and the colored column of \mathbf{B} are used to compute the element c_{ij} of \mathbf{AB} . The product of two matrices is not defined when the number of columns in the first matrix and the number of rows in the second matrix are not the same.

We now give some examples of matrix products.

EXAMPLE 3 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find \mathbf{AB} if it is defined.



Solution: Because \mathbf{A} is a 4×3 matrix and \mathbf{B} is a 3×2 matrix, the product \mathbf{AB} is defined and is a 4×2 matrix. To find the elements of \mathbf{AB} , the corresponding elements of the rows of \mathbf{A} and the columns of \mathbf{B} are first multiplied and then these products are added. For instance, the element in the $(3, 1)$ th position of \mathbf{AB} is the sum of the products of the corresponding elements of the third row of \mathbf{A} and the first column of \mathbf{B} ; namely, $3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3 = 7$. When all the elements of \mathbf{AB} are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

Matrix multiplication is *not* commutative. That is, if \mathbf{A} and \mathbf{B} are two matrices, it is not necessarily true that \mathbf{AB} and \mathbf{BA} are the same. In fact, it may be that only one of these two products is defined. For instance, if \mathbf{A} is 2×3 and \mathbf{B} is 3×4 , then \mathbf{AB} is defined and is 2×4 ; however, \mathbf{BA} is not defined, because it is impossible to multiply a 3×4 matrix and a 2×3 matrix.

In general, suppose that \mathbf{A} is an $m \times n$ matrix and \mathbf{B} is an $r \times s$ matrix. Then \mathbf{AB} is defined only when $n = r$ and \mathbf{BA} is defined only when $s = m$. Moreover, even when \mathbf{AB} and \mathbf{BA} are

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

FIGURE 1 The Product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$.

both defined, they will not be the same size unless $m = n = r = s$. Hence, if both \mathbf{AB} and \mathbf{BA} are defined and are the same size, then both \mathbf{A} and \mathbf{B} must be square and of the same size. Furthermore, even with \mathbf{A} and \mathbf{B} both $n \times n$ matrices, \mathbf{AB} and \mathbf{BA} are not necessarily equal, as Example 4 demonstrates.

EXAMPLE 4 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$. ◀

Transposes and Powers of Matrices

We now introduce an important matrix with entries that are zeros and ones.

DEFINITION 5

The *identity matrix of order n* is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when \mathbf{A} is an $m \times n$ matrix, we have

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

Powers of square matrices can be defined. When \mathbf{A} is an $n \times n$ matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \quad \mathbf{A}^r = \underbrace{\mathbf{A}\mathbf{A}\mathbf{A} \cdots \mathbf{A}}_{r \text{ times}}.$$

The operation of interchanging the rows and columns of a square matrix arises in many contexts.

DEFINITION 6

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of \mathbf{A} , denoted by \mathbf{A}^t , is the $n \times m$ matrix obtained by interchanging the rows and columns of \mathbf{A} . In other words, if $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

EXAMPLE 5

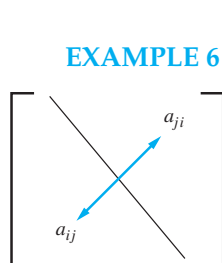
The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Matrices that do not change when their rows and columns are interchanged are often important.

DEFINITION 7

A square matrix \mathbf{A} is called *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Thus $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

Note that a matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the i th row and i th column for some i). This symmetry is displayed in Figure 2.



The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric.

Zero-One Matrices

A matrix all of whose entries are either 0 or 1 is called a **zero-one matrix**. Zero-one matrices are often used to represent discrete structures, as we will see in Chapters 9 and 10. Algorithms using these structures are based on Boolean arithmetic with zero-one matrices. This arithmetic is based on the Boolean operations \wedge and \vee , which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 8

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero-one matrices. Then the *join* of \mathbf{A} and \mathbf{B} is the zero-one matrix with (i, j) th entry $a_{ij} \vee b_{ij}$. The join of \mathbf{A} and \mathbf{B} is denoted by $\mathbf{A} \vee \mathbf{B}$. The *meet* of \mathbf{A} and \mathbf{B} is the zero-one matrix with (i, j) th entry $a_{ij} \wedge b_{ij}$. The meet of \mathbf{A} and \mathbf{B} is denoted by $\mathbf{A} \wedge \mathbf{B}$.

EXAMPLE 7

Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

FIGURE 2 A Symmetric Matrix.

Solution: We find that the join of \mathbf{A} and \mathbf{B} is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of \mathbf{A} and \mathbf{B} is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We now define the **Boolean product** of two matrices.

DEFINITION 9

Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of \mathbf{A} and \mathbf{B} , denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j) th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Note that the Boolean product of \mathbf{A} and \mathbf{B} is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation \vee and with multiplication replaced with the operation \wedge . We give an example of the Boolean products of matrices.

EXAMPLE 8 Find the Boolean product of \mathbf{A} and \mathbf{B} , where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$

We can also define the Boolean powers of a square zero–one matrix. These powers will be used in our subsequent studies of paths in graphs, which are used to model such things as communications paths in computer networks.

DEFINITION 10

Let \mathbf{A} be a square zero–one matrix and let r be a positive integer. The r th *Boolean power* of \mathbf{A} is the Boolean product of r factors of \mathbf{A} . The r th Boolean product of \mathbf{A} is denoted by $\mathbf{A}^{[r]}$. Hence

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be \mathbf{I}_n .

EXAMPLE 9 Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n .

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀

Exercises

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.

- What size is \mathbf{A} ?
- What is the third column of \mathbf{A} ?
- What is the second row of \mathbf{A} ?
- What is the element of \mathbf{A} in the (3, 2)th position?
- What is \mathbf{A}^t ?

2. Find $\mathbf{A} + \mathbf{B}$, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix},$

$\mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}.$

b) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix},$

$\mathbf{B} = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}.$

3. Find \mathbf{AB} if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}.$

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}.$

c) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}.$