5. Linear independence

5.1. Linear independence

5.2. Basis

Cash flow replication. Let's consider cash flows over 3 periods, given by 3-vectors. We know from VMLS page 93 that the vectors

$$e_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad l_1 = \begin{bmatrix} 1 \\ -(1+r) \\ 0 \end{bmatrix}, \quad l_2 = \begin{bmatrix} 0 \\ 1 \\ -(1+r) \end{bmatrix}$$

form a basis, where r is the (positive) per-period interest rate. The first vector e_1 is a single payment of 1 in period (time) $t = 1$. The second vector l_1 is loan of \$ 1 in period $t = 1$, paid back in period $t = 1$ with interest r. The third vector l_2 is loan of \$ 1 in period $t = 2$, paid back in period $t = 3$ with interest r. Let's use this basis to replicate the cash flow $c = (1, 2, -3)$ as

$$c = \alpha_1 e_1 + \alpha_2 l_1 + \alpha_3 l_2 = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -(1+r) \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ -(1+r) \end{bmatrix}.$$

From the third component we have $c_3 = \alpha_3(-(1+r))$, so $\alpha_3 = -c_3/(1+r)$. From the second component we have

$$c_2 = \alpha_2(-(1+r)) + \alpha_3 = \alpha_2(-(1+r)) - c_3/(1+r),$$

so $\alpha_2 = -c_2/(1+r) - c_3/(1+r)^2$. Finally, from $c_1 = \alpha_1 + \alpha_2$, we have

$$\alpha_1 = c_1 + c_2/(1+r) + c_3/(1+r)^2,$$

which is the net present value (NPV) of the cash flow c.
Let's check this in Python using an interest rate of 5% per period, and the specific cash flow $c = (1, 2, -3)$.

In [ ]:
```
import numpy as np
r = 0.05
e1 = np.array([1,0,0])
l1 = np.array([1, -(1+r), 0])
l2 = np.array([0,1,-(1+r)])
c = np.array([1,2,-3])
# Coefficients of expansion
alpha3 = -c[2]/(1+r)
alpha2 = -c[1]/(1+r) - c[2]/((1+r)**2)
alpha1 = c[0] + c[1]/(1+r) + c[2]/((1+r)**2) #NPV of cash flow
print(alpha1)
```

```
0.18367346938775508
```

In [ ]:
```
print(alpha1*e1 + alpha2*l1 + alpha3*l2)
```

```
[ 1.,  2., -3.]
```

(Later in the course we'll introduce an automated and simple way to find the coefficients in the expansion of a vector in a basis.)

5.3. Orthonormal vectors

Expansion in an orthonormal basis. Let's check that the vectors

$$a_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad a_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

form an orthonormal basis, and check the expansion of $x = (1, 2, 3)$ in this basis,

$$x = (a_1^T x)a_1 + \ldots + (a_n^T x)a_n.$$

In [ ]:
```
a1 = np.array([0,0,-1])
a2 = np.array([1,1,0])/(2)**0.5
a3 = np.array([1,-1,0])/(2)**0.5
print('Norm of a1 :', (sum(a1**2))**0.5)
```