# 5163, Homework Assignment 7 due on Friday, 04/01/2022, at 6pm (to be uploaded to Canvas)

This homework set consists of four problems.

Problem 1:
A system is composed of a large number $N$ of one-dimensional quantum harmonic oscillators whose angular frequencies are distributed over the range $\omega_a \leq \omega \leq \omega_b$ with a frequency distribution function $D(\omega) = A\omega^{-1}$, where $A$ is a real constant. Let us assume that the quantum oscillators can be treated as distinguishable quantum particles.
(a) Calculate the specific heat per quantum oscillator at temperature $T$.
Hint: It is convenient to use the canonical ensemble.
(b) Evaluate your result from part (a) in the high temperature limit (clearly define what "high $T$" and "low $T$" mean).
(c) Make a plot of your result from part (a) and compare with the single-frequency case.

Problem 2:
The "baloneyon" is an imaginary fermion with spin-1/2 and the relationship $E = B|\vec{p}|^4$ between the energy $E$ and the momentum $\vec{p}$ (as an aside, such dispersion curves can be engineered to a very good approximation using cold atoms),

$$E = B|\vec{p}|^4. \tag{1}$$

Consider a non-interacting gas of baloneyons in two spatial dimensions.
(a) What units does $B$ have?
(b) Determine the Fermi energy of a non-interacting gas of baloneyons as a function of the particle density.
(c) Explicitly check the units of your result obtained in part (b).
(d) Provide a physical interpretation of the Fermi energy.

Problem 3:
Consider a single electron with mass $m$, intrinsic spin $\frac{1}{2}\hbar\hat{\vec{\sigma}}$, and spin magnetic moment $\hat{\vec{\mathcal{M}}}_s$, where

$$\hat{\vec{\sigma}} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}. \tag{2}$$

Using the eigen states $|\uparrow\rangle$ and $|\downarrow\rangle$ of $\hat{\sigma}_z$ as basis, we have the following matrix representations:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

$$\sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \tag{4}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The spin of the electron has two possible orientations, up and down, with respect to an applied magnetic field $\vec{B}$. Letting the B-field point along the negative $z$-direction ($\vec{B} = -B_z\hat{e}_z$ with $B_z = |\vec{B}|$), the quantum mechanical Hamiltonian $\hat{\mathcal{H}}$ takes the form

$$\hat{\mathcal{H}} = -\hat{\vec{\mathcal{M}}}_s \cdot \vec{B} = \mu_B\hat{\vec{\sigma}} \cdot \vec{B} = -\mu_B B_z\hat{\sigma}_z, \tag{6}$$

where

$$\mu_B = \frac{e\hbar}{2mc}. \tag{7}$$

Use the canonical ensemble to treat this problem.
Express the density matrix $\hat{\rho}$ in terms of the eigen states $|\uparrow\rangle$ and $|\downarrow\rangle$ of $\hat{\sigma}_z$ and calculate the thermal expectation value $\langle\hat{\sigma}_z\rangle$.

Problem 4:
Consider a three-dimensional free particle in a box of length $L$. Assume periodic boundary conditions. Use the canonical ensemble to treat this problem.
(a) Find a compact expression for the density matrix $\hat{\rho}$ in the coordinate representation, i.e., find a compact expression for the quantity $\langle\,\vec{r}\,|\,\hat{\rho}\,|\,\vec{r}\,'\,\rangle$; "compact" means that the expression should not contain any (infinite) sums.
Hint: Consider converting the sum over $\vec{k}$ into an integral.
(b) Evaluate and interpret the quantity $\langle\,\vec{r}\,|\,\hat{\rho}\,|\,\vec{r}\,\rangle$.
(c) Calculate $\langle\hat{\mathcal{H}}\rangle$.