Homework Assignment #4

Math Methods

Homework Due: Monday, September 28th

Instructions:

Reading: Please re-read the rest of Chapter 1. There is no reading quiz this week.
Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

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1. A rocket is fired horizontally off of a rooftop. As it leaves the rooftop it has an initial horizontal velocity v_0 and a constant horizontal acceleration a_0 in addition to the acceleration, g, downward due to gravity. What is the shape of its trajectory? (Hyperbola? Parabola? Straight line? Or something else?) *Hint:* This question can be answered without any need for calculation, if you think geometrically).

2. Byron and Fuller, Chapter 1, problem 1

3. Show that $\epsilon_{ijk}\epsilon_{ijk} = 6$.

4. Demonstrate algebraically whether or not the cross product is associative. That is, verify or falsify the following:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

   If they are not in general equal, are they at least of equal magnitude?

5. Consider a two dimensional system where the vector $\vec{x}$ is given by

$$\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

   and the x-coordinate is transformed into a different, non-orthogonal coordinates as:

$$\begin{aligned} x_1' &= \frac{1}{\sqrt{2}}(x_1 + x_2) \\ x_2' &= x_2 \end{aligned}$$

   The gradient of the scalar function is:

$$\vec{\nabla}\phi(\vec{x}) = \partial_1\phi\,\hat{e}_1 + \partial_2\phi\,\hat{e}_2$$

   Find the components of the gradient in the primed co-ordinates and show that it transforms as a covariant vector.

6. Show that for an orthogonal transformation, there is no distinction between a contravariant and a covariant vector.

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2. A rocket is fired horizontally off of a rooftop. As it leaves the rooftop it has an initial horizontal velocity v_0 and a constant horizontal acceleration a_0 in addition to the acceleration, g, downward due to gravity. What is the shape of its trajectory? (Hyperbola? Parabola? Straight line? Or something else?) *Hint:* This question can be answered without any need for calculation, if you think geometrically).

In our original co-ordinate system we have

$$x(t) = x_0 + v_{0,x}\,t + \frac{1}{2}a_0\,t^2 \tag{1}$$

$$y(t) = y_0 + v_{0,y}\,t - \frac{1}{2}g\,t^2 \tag{2}$$

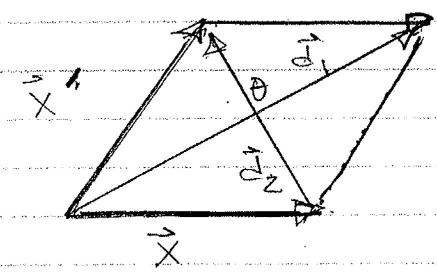
Or in terms of vectors:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2 \tag{3}$$

Now we simply rotate our system to the $\vec{r}\,'$ coordinate system so that $\hat{k}' \parallel \vec{a}$ in our new system. In that coordinate system we are back to a simple projectile motion problem albeit with different components for the initial velocity in the x' and y' directions. However, that change will not alter the nature of the trajectories: in the primed system they will be parabolas. Therefore, the trajectory in our original problem will be a rotated parabola, which is not a hyperbola or any other geometric figure.

HW 1 Solutions

(1) Show that the diagonals of a rhombus are perpendicular.



Denote the two sides of the rhombus by the vectors $\vec{x}$ and $\vec{x}'$. The diagonals are:

$$\vec{d}_1 = \vec{x}' + \vec{x}$$

$$\vec{d}_2 = \vec{x}' - \vec{x}$$

Then $\vec{d}_1 \cdot \vec{d}_2 = (\vec{x}' + \vec{x}) \cdot (\vec{x}' - \vec{x})$

$$= |\vec{d}_1|\,|\vec{d}_2| \cos\theta = \vec{x}' \cdot \vec{x}' - \vec{x} \cdot \vec{x}$$

$$= |\vec{x}'|^2 - |\vec{x}|^2$$

But the length of the sides of a rhombus are equal so $|\vec{x}'| = |\vec{x}|$ and

$$\vec{d}_1 \cdot \vec{d}_2 = 0$$

But $|\vec{d}_1| \neq 0$ and $|\vec{d}_2| \neq 0$ so we must have

$$\cos\theta = 0 \Rightarrow \vec{d}_1 \perp \vec{d}_2$$

(3) Show $\epsilon_{ijk}\epsilon_{ijk} = 6$

From the text we know

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Since

$$\epsilon_{klm} = -\epsilon_{mlk}$$

then

$$\epsilon_{ijk}\epsilon_{mlk} = \delta_{jl}\delta_{im} - \delta_{il}\delta_{jm}$$

We wish to force $i = m$ and $j = l$ —

$$\therefore \epsilon_{ijk}\epsilon_{ijk} = \epsilon_{ijk}\epsilon_{mlk}\delta_{im}\delta_{jl}$$

$$= (\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm})\delta_{im}\delta_{jl}$$

$$= \delta_{im}\delta_{im}\delta_{jl}\delta_{jl} - \delta_{ij}\delta_{ji}$$

Where $\delta_{ij}\delta_{ij} = \delta_{ii} = \sum_{i=1}^{3}\delta_{ii} = 3$.

We have the product of <u>two</u> such sums in the first term so

$$\epsilon_{ijk}\epsilon_{ijk} = (3)(3) - 3 = 6.$$

2) Demonstrate algebraically if

$$\vec{a} \times (\vec{b} \times \vec{c}) \overset{?}{=} (\vec{a} \times \vec{b}) \times \vec{c}$$

Let's calculate the m'th component of each side:

$$(LHS)_m = a_l (b_i c_j \epsilon_{ijk}) \epsilon_{lkm}$$

$$(RHS)_m = (a_i b_j \epsilon_{ijk}) c_l \epsilon_{klm}$$

Using: $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$$
\begin{aligned}
(LHS)_m &= a_l b_i c_j \; \epsilon_{ijk} \epsilon_{lkm} \\
&= -a_l b_i c_j \; \epsilon_{ijk} \epsilon_{klm} \\
&= -a_l b_i c_j \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \\
&= -a_i b_i c_m + a_j c_j b_m \\
&= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}
\end{aligned}
$$

$$
\begin{aligned}
(RHS)_m &= a_i b_j c_l \; \epsilon_{ijk} \epsilon_{klm} \\
&= a_i b_j c_l \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \\
&= a_i c_i b_m - b_j c_j a_m \\
&= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}
\end{aligned}
$$

The lengths are obviously not the same.
To be precise

$$|LHS|^2 = a_i c_i b_j b_j + a_i b_i c_j c_j - 2 a_i c_i a_j b_j b_k c_k$$

$$|RHS|^2 = a_i c_i b_j b_j + b_i c_i a_j a_j - 2 a_i c_i b_j c_j a_k b_k$$

$$|LHS|^2 - |RHS|^2 = a_i b_i c_j c_j - b_i c_i a_j a_j = (\vec{a}\cdot\vec{b})|\vec{c}|^2 - (\vec{b}\cdot\vec{c})|\vec{a}|^2$$

Note in passing:

$$LHS - RHS = (\vec{b}\cdot\vec{c})\,\vec{a} - (\vec{a}\cdot\vec{b})\,\vec{c} = \vec{b}\times(\vec{a}\times\vec{c}).$$

#4

$$x_1' = \frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2$$

$$x_2' = x_2$$

The above gives our transformation matrix,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Therefor our transformation a_{ij} is

$$a_{ij} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix}$$

What are the unit vectors, $\hat{e}_1'$ & $\hat{e}_2'$?

$$\vec{x} = \vec{x}'$$

$$x_j \hat{e}_j = x_i' \hat{e}_i'$$

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)\hat{e}_1' + x_2 \hat{e}_2'$$

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 = x_1 \left(\frac{1}{\sqrt{2}} \hat{e}_1'\right) + x_2 \left(\frac{\hat{e}_1'}{\sqrt{2}} + \hat{e}_2'\right)$$

So $$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix}$$

So our basis vectors transform as

$$\begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

Check:

$$x_i' \hat{e}_i' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2\right) \\ x_2 \end{bmatrix} \begin{bmatrix} \sqrt{2}\, \hat{e}_1 \\ -\hat{e}_1 + \hat{e}_2 \end{bmatrix}$$

$$= (x_1 + x_2)\, \hat{e}_1 + x_2 (-\hat{e}_1 + \hat{e}_2)$$

$$= x_1 \hat{e}_1 + \cancel{x_2 \hat{e}_1} - \cancel{x_2 \hat{e}_1} - x_2 \hat{e}_2$$

$$= x_1 \hat{e}_1 + x_2 \hat{e}_2 = x_j \hat{e}_j$$

which checks!

How do the components of $\vec{\nabla}\phi$ change?

$$\vec{\nabla}\phi \cdot \vec{ds} = \frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2$$

$$= \frac{\partial\phi}{\partial x_1'} dx_1' + \frac{\partial\phi}{\partial x_2'} dx_2'$$

but if

$$\begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$$

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix}$$

So $$\vec{\nabla}\phi \cdot \vec{ds} = \frac{\partial\phi}{\partial x_1}\left(\sqrt{2}\, dx_1' - dx_2'\right) + \frac{\partial\phi}{\partial x_2} dx_2'$$

$$= \left(\sqrt{2}\frac{\partial\phi}{\partial x_1}\right) dx_1' + \left(-\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2}\right) dx_2'$$

$$= \left(\frac{\partial\phi}{\partial x_1'}\right) dx_1' + \left(\frac{\partial\phi}{\partial x_2'}\right) dx_2'$$

So $$\begin{pmatrix} \frac{\partial\phi}{\partial x_1'} \\ \frac{\partial\phi}{\partial x_2'} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\frac{\partial\phi}{\partial x_1} \\ -\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \end{pmatrix}$$

which is how $\hat{e}_i$ transforms

#5 If we use the notation of Byron & Fuller then all indices are lower indices. A contravariant vector transforms as

$$x_j' = a_{jk} x_k$$

Then the change of a scalar function ϕ when we move $d\vec{s}$ is the same in all co-ordinate systems. If we write $\frac{\partial \phi}{\partial x_i}$ as $\partial_i \phi$ then

$$d\phi = \partial_j \phi \, dx_j = \partial_j' \phi' \, dx_j'$$

Assume $\nabla \phi$ transforms as $\mathbb{b}$, so that

$$\partial_j' \phi' = b_{jl} \partial_l \phi$$

Then

$$\partial_j \phi \, dx_j = \partial_j' \phi \, \partial x_j'$$

$$= b_{jl} \partial_l \phi \quad a_{jk} \, dx_k$$

$$= b_{jl} a_{jk} \partial_l \phi \, dx_k .$$

This can only hold true if $b_{jl} a_{jk} = \delta_{lk}$.

Then

$$\mathbb{b}^T \mathbb{a} = \underline{\mathbb{1}}.$$

$$\mathbb{b}^T = \mathbb{a}^{-1}$$

But for an orthogonal matrix

$$\mathcal{U}^T = \mathcal{U}^{-1}$$

So for orthogonal transformations there is no distinction between contra- & co-variant vectors.

It is common to write contravariant indices with superscripts & co-variant indices as subscripts.

For example

$$d\phi = \partial^\mu \phi \, dx_\mu .$$

Only a expressions that have summed pairs of super & subscripts are true scalars for that set of transformations —