Quantum Mechanics Qualifying Exam

January 2019

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Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Spherical Harmonics

$$Y_0^0(\theta,\phi) = \frac{1}{2\sqrt{\pi}}$$

$$Y_1^0(\theta,\phi) = \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\cos\theta \qquad Y_1^{\pm1}(\theta,\phi) = \mp\tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\sin\theta e^{\pm i\phi} \qquad Y_1^{\pm1}(\theta,\phi) = \mp\tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0(\theta,\phi) = \tfrac{1}{3}\sqrt{\tfrac{5}{\pi}}(3\cos^2\theta - 1) \qquad Y_2^{\pm1}(\theta,\phi) = \mp\tfrac{1}{2}\sqrt{\tfrac{15}{2\pi}}\sin\theta\cos\theta e^{\pm i\phi} \qquad Y_2^{\pm2}(\theta,\phi) = \tfrac{1}{4}\sqrt{\tfrac{15}{2\pi}}\sin^2\theta e^{\pm 2i\phi}$$

Normalization:

$$\int_0^{2\pi} d\phi \int_0^\pi Y_\ell^m(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi)^* \sin\theta\, d\theta = \delta_{m,m'}\delta_{\ell,\ell'}$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$

$\mu_0 = 4\pi \times 10^{-7}$T m/A

electronic mass $m_e = 9.11 \times 10^{-31}$kg

Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K

speed of light: $c = 3.00 \times 10^8$m/s

Gravitaional Constant: $6.67 \times 10^{-11}\mathrm{J}\cdot\mathrm{m}\cdot\mathrm{kg}^{-2}$

$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$

electronic charge $e = 1.60 \times 10^{-19}$C

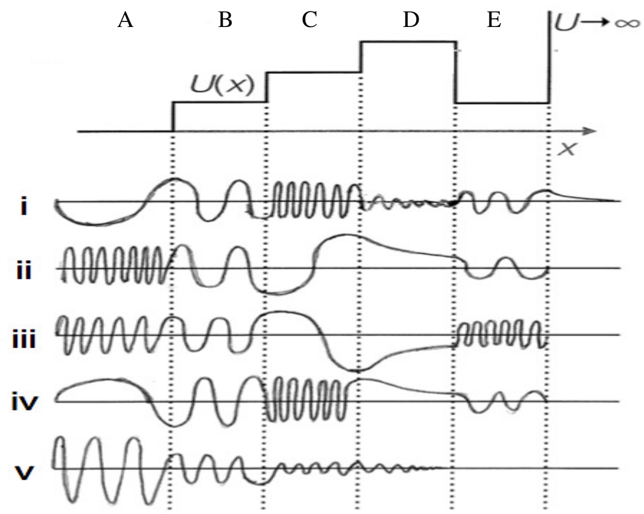
Atomic mass unit: 1.66×10^{-27}kg.

Planck's constant: $\hbar = 1.054 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$

Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

1. Piecewise constant potentials:

   (a) Consider the following potential $U(x)$, and the 5 wave functions (i–v) sketched below it.



   For each of the five wave functions shown, specify whether it represents a plausible solution to the Schrodinger equation with potential $U(x)$ for some initially incoming wave. If not, explain why. [3 points]

   (b) A particle in a 1D infinite square well (extending from $x = 0$ to $x = L$) is initially put into the following state:

$$\Psi(x,0) = \begin{cases} 0, & \text{for } x < 0 \\ C\sin^3(\pi x/L), & \text{for } 0 < x < L \\ 0, & \text{for } L < x \end{cases}$$

      i. Calculate the wave function $\Psi(x,t)$ for $t > 0$ (an answer in the form of a series is acceptable). [2 points]

      ii. What is the expectation value of the particle's energy? [2 points]

   (c) A particle in a 1D infinite square well (extending from $x = 0$ to $x = L$) is initially put into the following state

$$\Psi(x,0) = \begin{cases} 0, & \text{for } x < 0 \\ C\delta\left(x - \frac{L}{2}\right), & \text{for } 0 < x < L \\ 0, & \text{for } L < x \end{cases}$$

   Calculate the wave function $\Psi(x,t)$ for $t > 0$ (an answer in the form of a series is acceptable). [3 points]

2. **Hydrogen Atom Measurements**

This problem explores an ensemble of hydrogen atoms. Only the Coulomb interaction between proton and electron will be included, neglecting spin-orbit interactions and other perturbations. The state of the atoms is defined in the usual basis:

$$\begin{aligned}
\hat{H}|n,l,m_z,\sigma_z\rangle &= -\frac{Ryd}{n^2}|n,l,m_z,\sigma_z\rangle \\
\hat{L}^2|n,l,m_z,\sigma_z\rangle &= l(l+1)\hbar^2|n,l,m_z,\sigma_z\rangle \\
\hat{L}_z|n,l,m_z,\sigma_z\rangle &= m_z\hbar|n,l,m_z,\sigma_z\rangle \\
\hat{S}_z|n,l,m_z,\sigma_z\rangle &= \sigma_z\frac{\hbar}{2}|n,l,m_z,\sigma_z\rangle .
\end{aligned}$$

($n,\ l,\ m_z$ are integers and $\sigma_z = \pm 1$.)

Assume that initially the atoms in the ensemble are all in the state:

$$|\Psi_0\rangle = \sqrt{\frac{2}{3}}\left[|1,0,0,-1\rangle - \frac{1}{2}|2,1,1,1\rangle - \frac{1}{2}|2,1,-1,1\rangle\right]. \tag{1}$$

(a) If you measure the energy of the atoms in the ensemble, what will be values that will be measured and their probabilities? What is the average value and uncertainty of the energy of the atoms in the ensemble? (2 points)

(b) The total angular momentum in the z-direction is given by $\hat{J}_z = \hat{L}_z + \hat{S}_z$. Repeat question (a) for $\hat{J}_z$ rather than the energy. (2 points)

(c) Consider performing the energy measurement on the ensemble of atoms described in part (a) and creating a sub-ensemble made up of those atoms for which the result of the energy measurement was $E = -Ryd/4$. If this is an ideal measurement, what is the new state of the atoms in this sub-ensemble, $|\Psi_1\rangle$? (1 point)

Now suppose we wanted to measure the x-component of the angular momentum of the atoms in the two ensembles. The eigenstates of $\hat{L}_x$ are given by

$$\begin{aligned}
|l=0,m_x=0\rangle &= |l=0,m_z=0\rangle \\
|l=1,m_x=\pm 1\rangle &= \frac{1}{2}\left[|l=1,m_z=1\rangle \pm \sqrt{2}|l=1,m_z=0\rangle + |l=1,m_z=-1\rangle\right] \\
|l=1,m_x=0\rangle &= \frac{1}{\sqrt{2}}\left[|l=1,m_z=1\rangle - |l=1,m_z=-1\rangle\right]
\end{aligned}$$

(d) If we start with the ensemble of atoms in the state $|\Psi_0\rangle$, what are the possible results for a measurement of $\hat{L}_x$ and the corresponding probabilities? (2 points)

(e) If we start with the sub-ensemble of atoms in the state $|\Psi_1\rangle$, what are the possible results for a measurement of $\hat{L}_x$ and the corresponding probabilities? Explain the differences between the answers to (d) and (e). (3 points)

3. A spin half particle with spin operator $\vec{s}$ is placed in a magnetic field that is pointing in the $z-$direction: $\vec{B} = B\hat{z}$. The magnetic moment of the particle is $\vec{\mu} = \gamma\vec{s}$ and the Hamiltonian of the system is given by $H = -\vec{\mu} \cdot \vec{B}$. Take the magnetic field to be time-independent.

   The basis states for describing the wave function of the particle are denoted by $|+\rangle$ and $|-\rangle$ which are the spin up and spin down eigenstates of the spin along the $z-$direction s_z.

   (a) At time $t = 0$ the $x-$component of the particle's spin was measured and the value was $+\hbar/2$. This measurement puts the particle into a state $|\psi(t = 0)\rangle$. Find $|\psi(t = 0)\rangle$ in terms of $|+\rangle$ and/or $|-\rangle$. (1 point)

   (b) Let the state evolve with time. Find $|\psi(t)\rangle$ at the time t. (2 points)

   (c) What is the average value of s_y (the $y-$component of the spin) at time t? (3 points)

   (d) Suppose at time t the spin of the particle is measured in the direction $\hat{z}\cos\theta + \hat{x}\sin\theta$. If the measurement yields the value $-\hbar/2$, what quantum state does the particle jump to after the measurement is over? Your answer should involve θ, $|+\rangle$, and $|-\rangle$. (4 points)

4. You have a particle in a 2D harmonic oscillator potential so that it can be described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) + \frac{1}{2}k\left(X^2 + Y^2\right)$$

with eigenenergies $E = (n_x + n_y + 1)\hbar\omega_0$ and $\omega_0 = \sqrt{k/m}$. This is usually written in terms of the dimensionless variables $x = X/\ell$ and $y = Y/\ell$ where $\ell \equiv \sqrt{\hbar/\sqrt{km}}$:

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\left(x^2 + y^2\right)$$

Your friend happens to use a rotated coordinate system:

$$\begin{aligned} x' &= x\,\cos\alpha - y\,\sin\alpha \\ y' &= x\,\sin\alpha + y\,\cos\alpha \end{aligned}$$

where $0 < \alpha < \pi/2$.

(a) Show that the Hamiltonian in your friend's co-ordinate system will also be a 2D harmonic oscillator. (2 points)

(b) You put the particle in the state $(n_x,\, n_y) = (1,\, 0)$ with an energy $2\hbar\omega_0$. Will your friend observe the same energy eigenvalue as you did? Will they observe the same *average* energy? Explain or prove your result. (1 point)

(c) Will your friend say that the particle is in an eigenstate? Explain. (1 point)

(d) What is the probability that your friend will say that the particle is in state:

   i. $(n'_x,\, n'_y) = (0,\, 0)$?

   ii. $(n'_x,\, n'_y) = (1,\, 0)$?

   iii. $(n'_x,\, n'_y) = (0,\, 1)$?

   You should give a sound argument for or calculate your result to get credit - don't just write down a guess. (3 points)

(e) If you start with a particle in the state $(n_x,\, n_y) = (1,\, 1)$, what is the probability your friend will observe it in $(n'_x,\, n'_y) = (0,\, 2)$? (3 points)

The eigenfunctions for the 1D harmonic oscillator in the same dimensionless units are:

$$\begin{aligned} \phi_0(z) &= \left(\frac{1}{\sqrt{\pi}}\right)^{1/2} e^{-z^2/2} \\ \phi_1(z) &= \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} z\,e^{-z^2/2} \\ \phi_2(z) &= \left(\frac{1}{2\sqrt{\pi}}\right)^{1/2} (2z^2 - 1)e^{-z^2/2} \end{aligned}$$

5. **Perturbation Theory**

Consider a spin-1/2 particle in the presence of a static magnetic field along the z and x directions

$$\vec{B} = B_z \hat{e}_z + B_x \hat{e}_x$$

(a) Show that the Hamiltonian is

$$\hat{H} = \hbar\omega_o \hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$$

where $\hbar\omega_o = \mu_B B_z$ and $\hbar\Omega = 2\mu_B B_x$. [1 pt]

(b) If B_x=0, the eigenvectors are $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ with eigenvalues $\pm\hbar\omega_o$ respectively. Now turn on a weak x field with $B_x \ll B_z$. Use perturbation theory to find the new eigenvectors and eigenvalues to lowest order in B_x/B_z. [3 pts]

(c) This problem can be solved exactly. Find the eigenvectors and eigenvalues for arbitrary values of B_z and B_x. Show that these agree with your results in part b) by taking appropriate limits. [4 pts]

(d) Plot the energy eigenvalues as a function of B_z for fixed B_x. Label the eigenvectors on the curves when B_z=0 and when $B_z \to \pm\infty$. [2 pts]

6. **Raising and lowering operators:**

The quantum harmonic oscillator Hamiltonian can be written

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

where we can write $\hat{a}$ and $\hat{a}^\dagger$ in terms of the dimensionless position $\hat{x}$ and momentum $\hat{p}$ operators:

$$\begin{aligned} \hat{a} &= (\hat{x} + i\hat{p}) \\ \hat{a}^\dagger &= (\hat{x} - i\hat{p}) \end{aligned}$$

so that $\left[\hat{a},\, \hat{a}^\dagger\right] = \hat{1}$. The operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy

$$\begin{aligned} \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}$$

The energy eigenstates of the system can be denoted by $|n\rangle$, and $\hat{H}|n\rangle = \hbar\omega_0(n+\frac{1}{2})|n\rangle$. Define the operator $\hat{\mathcal{S}}(\alpha) = \exp\left\{\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right\}$ where α is a complex number, and the state $|\alpha\rangle = \hat{\mathcal{S}}(\alpha)|0\rangle$ where $|0\rangle$ is the groundstate of the harmonic oscillator. (*Note:* do not confuse energy eigenstate "kets" $|n\rangle$ where n is an integer with these "kets" $|\alpha\rangle$ where α is a complex number.)

(a) Calculate $\langle\hat{x}\rangle$ in any state $|n\rangle$. (1 point)

(b) Given two operators $\hat{A}$ and $\hat{B}$, it can be shown that

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[A,B]}$$

so long as $\hat{A}$ and $\hat{B}$ commute with their own commutator. Use this to show (2 points):

$$\hat{\mathcal{S}}(\alpha)|0\rangle = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}|0\rangle$$

(c) What is the probability a system in state $|\alpha\rangle$ will be measured to be in state $|n\rangle$? (1 point)

(d) Show that $|\alpha\rangle$ is an eigenstate of $\hat{a}$ and calculate its eigenvalue. (2 points)

(e) Calculate $\langle\hat{x}\rangle$ in the state $|\alpha\rangle$. (2 points)

(f) Given a second complex number β and the state $|\beta\rangle = \hat{\mathcal{S}}(\beta)|0\rangle$, what is the probability that a system in state $|\alpha\rangle$ will be measured to be in state $|\beta\rangle$? (2 points)