COLLEGE OF ARTS AND SCIENCES
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DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Math Methods in Physics

PHYS 5013 HOMEWORK ASSIGNMENT #3

PROBLEMS: $\{1, 2, 3, 4, 5, 6\}$

Due: September 20, 2021 By 10:30 AM

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PROFESSOR
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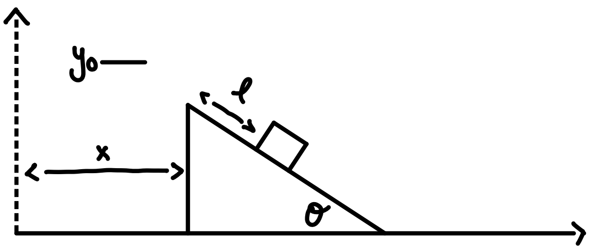
Problem 1:

Taken from the 2017 Classical and Statistical Mechanics Qualifier:

A wedge of mass M and angle θ moves frictionlessly along the x-axis. A small mass m a distance l from the top of the wedge moves frictionlessly alogn the wedge. Other than gravity and the normal force on the wedge from the ground, there are no external forces on the system.



(a) (2 pts) Find the kinetic energy of the system in terms of the generalized co-ordinates x and l.

Block m location : $x_1' = x + l\cos\theta$, $y_1' = y_0 - l\sin\theta$: Block M location , $x_2' = x$

$\dot{x}_1' = (\dot{x} + \dot{l}\cos\theta)$, $\dot{y}_1' = -\dot{l}\sin\theta$: $(\dot{x}_1')^2 = \dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2\cos^2\theta$, $(\dot{y}_1')^2 = \dot{l}^2\sin^2\theta$: $(\dot{x}_2')^2 = \dot{x}^2$

$$T = \frac{1}{2}m(\dot{x}_1'^2 + \dot{y}_1'^2) + \frac{1}{2}M(\dot{x}_2'^2) = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2\cos^2\theta + \dot{l}^2\sin^2\theta) + \frac{1}{2}M(\dot{x}^2)$$

$$\boxed{T = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2) + \frac{1}{2}M(\dot{x}^2)}$$

(b) (1 pt) Find the Lagrangian.

$L = T - U$: $U = mg(y_0 - l\sin\theta)$: * $y_0 \to$ Highest point of block on top of wedge

$$\boxed{L = T - U = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{l}\cos\theta + \dot{l}^2) + \frac{1}{2}M(\dot{x}^2) - mg(y_0 - l\sin\theta)}$$

(c) (2 pts) Find the equations of motion for the generalized co-ordinates and the ratio $\mu = m/(m+M)$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \ , \ \frac{\partial L}{\partial l} - \frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = 0$$

$\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{l}\cos\theta + M\dot{x}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + m\ddot{l}\cos\theta + M\ddot{x}$, $-m\ddot{x} - m\ddot{l}\cos\theta - M\ddot{x} = 0$

$(M+m)\ddot{x} = -m\ddot{l}\cos\theta$, $\ddot{x} = -m\ddot{l}\cos\theta/(M+m)$, $\ddot{x} = -\mu\ddot{l}\cos\theta$

$\frac{\partial L}{\partial l} = mg\sin\theta$, $\frac{\partial L}{\partial \dot{l}} = m\dot{x}\cos\theta + m\dot{l}$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = m\ddot{x}\cos\theta + m\ddot{l}$, $mg\sin\theta - m\ddot{x}\cos\theta - m\ddot{l} = 0$

$m\ddot{l} = mg\sin\theta - m\ddot{x}\cos\theta$, $\ddot{l} = g\sin\theta - \ddot{x}\cos\theta$

Problem 1: Continued

$$\ddot{x} = \mu(\ddot{x}\cos\theta - g\sin\theta)\cos\theta = \mu\ddot{x}\cos^2\theta - \mu g\sin\theta\cos\theta \;:\; \ddot{x} - \mu\ddot{x}\cos^2\theta = \ddot{x}(1-\mu\cos^2\theta) = -\mu g\sin\theta\cos\theta$$

$$\ddot{x} = \mu g\sin\theta\cos\theta/(\mu\cos^2\theta - 1) = \mu g\sin(2\theta)/2(\mu\cos^2\theta - 1)$$

$$\ddot{l} = g\sin\theta + \mu\ddot{l}\cos^2\theta \;:\; \ddot{l} - \mu\ddot{l}\cos^2\theta = g\sin\theta \;:\; \ddot{l}(1-\mu\cos^2\theta) = g\sin\theta \;:\; \ddot{l} = g\sin\theta/(1-\mu\cos^2\theta)$$

$$\boxed{\ddot{x} = \frac{\mu g\sin(2\theta)}{2(\mu\cos^2\theta - 1)} \;,\; \ddot{l} = \frac{g\sin\theta}{1-\mu\cos^2\theta}}$$

(d) (1 pt) Is there a speed of the wedge in which the acceleration of the mass is up the wedge? If so, find it. If not, prove mathematically or explain why.

$\ddot{x} = -\mu\ddot{l}\cos\theta$, if $\ddot{x} > 0$ then $\ddot{l} < 0$ and conversely

For the block to move up the wedge, $\ddot{l} < 0$ $\therefore$ $\ddot{x}$ must be greater than 0

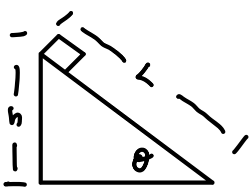
This will never happen due to the term $(\mu\cos^2\theta - 1)$ never being > 0

$\boxed{\text{This cannot happen}}$

(e) (2 pts) Integrate the equations of motion and find how long it takes for the particle to slide off if it starts at a height, h above the ground with both wedge and particle at rest.

$$\ddot{x} = \frac{\mu\sin(2\theta)}{2(\mu\cos^2\theta - 1)}g \;,\; w/\; \frac{\mu\sin(2\theta)}{2(\mu\cos^2\theta - 1)} = \alpha \;,\; x(t) = x_0 + \dot{x}t + \frac{\alpha}{2}gt^2$$

$$\ddot{l} = \frac{\sin\theta}{1-\mu\cos^2\theta}g \;,\; w/\; \frac{\sin\theta}{1-\mu\cos^2\theta} = \beta \;,\; l(t) = l_0 + \dot{l}t + \frac{\beta}{2}gt^2$$



$$t_0 = 0,\; l_0 = 0 \;:\; t_1 = t,\; l_1 = L \quad w/\; \dot{l}_0 = 0,\; l_0 = 0$$

$$L = \frac{\beta}{2}gt^2 \;\therefore\; \boxed{t = \sqrt{\frac{2L}{\beta g}}} \qquad w/\; L = \frac{h}{\sin\theta}$$

(f) (1 pt) Show that in the limit of $M \to \infty$ this agrees with the expected result.

$$\lim_{M\to\infty} \frac{m}{M+m} = 0 \;\therefore\; \mu \to 0 \;,\; \ddot{x} = 0 \to x(t) = x_0 + \dot{x}t \;\; w/\; \dot{x} = 0 \;\; x(t) = x_0 \;:\; \text{Stationary}$$

$$\ddot{l} = \sin\theta g \to l(t) = \cancel{l_0}^{\,0} + \cancel{\dot{l}t}^{\,0} + \frac{1}{2}\sin\theta g t^2 = \frac{1}{2}\sin\theta g t^2$$

$$L = \frac{1}{2}\sin\theta g t^2 \;\therefore\; t = \sqrt{\frac{2L}{g\sin\theta}} \;:\; \lim_{M\to\infty} \beta \to \sin\theta \;\therefore\; \lim_{M\to\infty}\sqrt{\frac{2L}{\beta g}} \to \sqrt{\frac{2L}{g\sin\theta}}$$

$$\boxed{t = \sqrt{\frac{2L}{g\sin\theta}}}$$

Problem 1: Continued

(g) (1 pt) How far did the wedge move during this time?

$$x(t) = x_0 + \frac{\alpha}{2} g \cdot \sqrt{\frac{2L}{\beta g}}^{\,2} = x_0 + \frac{\alpha L}{\beta} \quad : \quad \frac{\alpha}{\beta} = \frac{\mu \sin(2\theta)}{2(\mu\cos^2\theta - 1)} \cdot \frac{1 - \mu\cos^2\theta}{\sin\theta} = -\frac{\mu \sin(2\theta)}{2\sin\theta}$$

$$\Delta x = -\frac{\mu \sin(2\theta)}{2\sin\theta} L = \frac{-\mu \cdot \cancel{2}\cancel{\sin\theta}\cos\theta}{\cancel{2}\cancel{\sin\theta}} L = -\mu\cos\theta L$$

$$\boxed{\Delta x = -\mu\cos\theta L}$$

Problem 1: Review

Procedure:

- Use the procedure for finding Equations of Motion with the Euler Lagrange formalism.
- Answer all other questions with the use of the Equations of Motion.

Key Concepts:

- We can use the Euler Lagrange formalism to find Equations of Motion for multiple systems.

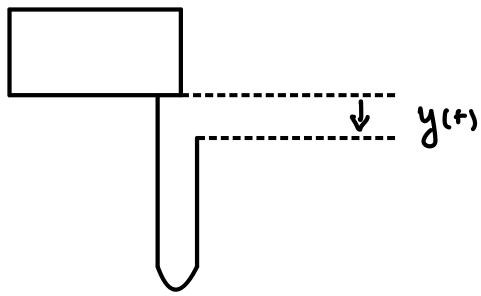
Variations:

- The entire system can change.
  - Thus changing the generalized co-ordinates, and the resulting equations of motion.
- We can be asked other questions pertaining to the Equations of Motion.
  - We would then consult the Equations of Motion and deduce how they can be used to answer the questions.

Problem 2:

Taken from the Spring 2020 Classical and Statistical Mechanics Qualifier:

A heavy rope of mass m and length L has uniform density and is attached at one end to the edge of a (fixed) balcony. It is initially at rest with its free end at a height equal to the point of support so that $y(t=0)=0$. The end of the rope is released and it falls downward due to the force of gravity. You should treat the rope as ideal, so that the portion at the left hand side is at rest, and all motion is elastic.



(a) (2 pts) Determine the position of the center of mass of the rope as a function of y.

$$C_m = \sum_i \frac{m_i y_i}{M} \; : \; M = \rho \cdot L \;\therefore\; C_m = \frac{\rho \cdot L_L \cdot L_L(y)}{M} + \frac{\rho \cdot L_R \cdot L_R(y)}{M} \; : \; m_L = \rho \cdot L_L \, , \; m_R = \rho \cdot L_R$$

$$L_L(y) = \frac{(L+y)}{2} \, , \; L_R(y) = \frac{(L-y)}{2} \; : \; C_m = \frac{\rho \cdot L_L \cdot L_L/2 + \rho \cdot L_R \cdot (L_R/2 + y)}{\rho L}$$

$$C_m = \frac{1}{L}\left(\frac{(L+y)}{2} \cdot \frac{(L+y)}{4} + \frac{(L-y)}{2} \frac{(L+3y)}{4}\right) = \frac{L^2 + 2yL + y^2 + L^2 + 3yL - yL - 3y^2}{8L} = \frac{L^2 + 2yL - y^2}{4L}$$

$$\boxed{C_m(y) = \frac{L^2 + 2yL - y^2}{4L}}$$

(b) (2 pts) Using the above expression for the center of mass, write down the Lagrangian for the rope.

$$L = T - U \; : \; T = \frac{1}{2} m (\dot{y}^2) \, , \; U = mgh \; : \; U = -\frac{1}{4} \rho g (L^2 + 2yL - y^2) \, , \; T = \frac{\rho}{4}(L-y)\dot{y}^2$$

$$\boxed{\mathcal{L} = \frac{\rho}{4}(L-y)\dot{y}^2 + \frac{\rho}{4} g (L^2 + 2yL - y^2)}$$

(c) (2 pts) Determine the speed at which the end of the rope falls as a function of y.

$$\cancelto{0}{K_0} + U_0 = K_1 + U_1 \; : \; U_0 = K_1 + U_1 \; : \; U_0 = mgh = \rho \cdot L \cdot g \cdot h = \rho \cdot \frac{L}{2} \cdot g \cdot \frac{L}{2} = \frac{\rho g L^2}{4}$$

$$\cancel{\frac{\rho}{4}}(L-y)\dot{y}^2 - \cancel{\frac{\rho}{4}} g (L^2 + 2yL - y^2) = -\cancel{\frac{\rho}{4}} g L^2 : \; (L-y)\dot{y}^2 = -gL^2 + gL^2 + 2gyL - gy^2$$

$$\dot{y}^2 = \frac{2gyL - gy^2}{L-y} = \frac{g(2yL - y^2)}{L-y} \;\therefore\; \dot{y} = \pm \sqrt{\frac{g(2yL - y^2)}{L-y}}$$

$$\boxed{\dot{y} = \pm \sqrt{\frac{g(2yL - y^2)}{L-y}}}$$

Problem 2: Continued

(d) (2 pts) Determine the acceleration of the end of the rope as a function of height.

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \; : \; \frac{\partial L}{\partial \dot{y}} = \frac{\rho}{2}(L-y)\cdot\dot{y} \; , \; \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\rho}{2}L\cdot\ddot{y} - \frac{\rho}{2}\dot{y}^2 - \frac{\rho}{2}y\ddot{y} = \frac{\rho}{2}\left((L-y)\ddot{y} - \dot{y}^2\right)$$

$$\frac{\partial L}{\partial y} = -\frac{\rho}{4}\dot{y}^2 + \frac{\rho}{4}g(2L-2y) \; : \; -\frac{\rho}{4}\dot{y}^2 + \frac{\rho}{4}g(2L-2y) - \frac{\rho}{4}\left(2(L-y)\ddot{y} - 2\dot{y}^2\right) = 0$$

$$-\dot{y}^2 + g(2L-2y) = 2(L-y)\ddot{y} - 2\dot{y}^2 \; : \; \dot{y}^2 + 2g(L-y) = 2(L-y)\ddot{y}$$

$$\frac{2gyL - gy^2}{L-y} - 2g(L-y) = 2(L-y)\ddot{y} \; : \; \ddot{y} = \frac{g(2yL-y^2)}{2(L-y)^2} + g$$

$$\boxed{\ddot{y} = \frac{g(2yL-y^2)}{2(L-y)^2} + g}$$

(e) (2 pts) How does your answer compare to g? Explain your result.

$\ddot{y} = \frac{gy(2L-y)}{2(L-y)^2} + g$, @ $y=0$ the acceleration will be g. As y grows the acceleration of y will increase and become greater than g up to the point where $L=y$ where at that point in time it becomes infinitely larger than g.

Problem 2: Review

Procedure:

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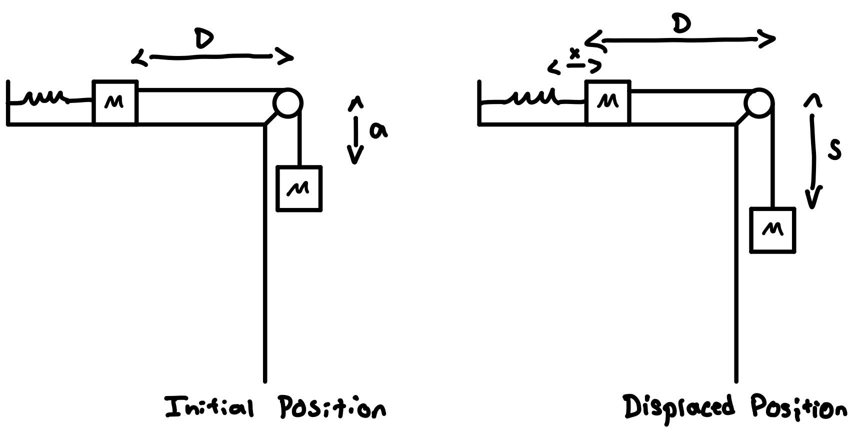
Problem 3:

Taken from the 2018 Classical and Statistical Mechanics Qualifier:

Two blocks of mass M are connected by a rope of length l and mass $m = \lambda l$, where λ is the linear density of the rope. One block slides fricitionlessly on a horizontal surface attached to the wall by spring with spring constant k. The rope attached to the other side, goes over an ideal, massless pulley and is attached to the second block which hangs freely.

The figure on the left is the initial condition, where D is the distance between the sliding mass and the edge of the surface when the spring exerts no force (it is neither compressed nor stretched) and the entire system is at rest. In this initial arrangement the length of the rope supporting the suspended block is at rest. In this initial arrangement the length of the rope supporting the suspended block is $a = l - D$. (Note that this is not an equilibrium position.) The figure on the right shows the system a short time after the blocks are released from rest, so that the spring is now stretched and the blocks are moving. We define x as the displacement of the block on the horizontal surface from the neutral position of the spring and $s = x + a$ is the length of the rope supporting the suspended block. (Initially, $s = a$.)



(a) (2 pts) Find the potential energy.

$$U = U_{spring} + U_{mg} \quad : \quad m = \lambda l, \quad a = l - D, \quad s = x + a$$

$$U_{spring} = \frac{1}{2}kx^2 = \frac{1}{2}k(s-a)^2, \quad U_{mg_B} = Mgh = Mgs, \quad U_{mg_R} = m \cdot g \cdot \frac{s}{2} = \frac{\lambda}{2}gs^2 \quad w/ \ m = \lambda \cdot s$$

$$\boxed{U = \frac{1}{2}k(s-a)^2 - Mgs - \frac{\lambda g s^2}{2}}$$

(b) (2 pts) Find the kinetic energy and Lagrangian.

$$T = \frac{1}{2}M(\dot{x}^2) + \frac{1}{2}M(\dot{x}^2) + \frac{1}{2}m(\dot{x}^2) = \frac{1}{2}(2M+m)(\dot{x}^2) \quad : \quad \dot{x} = \dot{s} \quad : \quad T = \frac{1}{2}(2M+m)(\dot{s}^2)$$

$$\boxed{T = \frac{1}{2}(2M+m)(\dot{s}^2)}$$

(c) (2 pts) Find the equation of motion for s using the Lagrangian.

$$L = \frac{1}{2}(2M+m)(\dot{s}^2) - \frac{1}{2}k(s-a)^2 + Mgs + \frac{\lambda}{2}gs^2 \quad : \quad \frac{\partial L}{\partial s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = 0 : \frac{\partial L}{\partial s} = -k(s-a) + Mg + \lambda g s$$

$$\frac{\partial L}{\partial \dot{s}} = (2M+m)(\dot{s}) \quad : \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = (2M+m)(\ddot{s}) \quad -k(s-a) + Mg + \lambda g s - (2M+m)(\ddot{s}) = 0$$

$$\boxed{\ddot{s} = \frac{g(M+\lambda s) - k(s-a)}{(2M+m)}}$$

Problem 3: Continued

(d) (2 pts) Find the equilibrium position for the system.

Equilibrium occurs @ $\ddot{s} = 0$ or when $\dot{s}$ is maximum

$$0 = \frac{g(M+\lambda s) - k(s-a)}{2M+m} \quad : \quad g(M+\lambda s) = k(s-a) \quad : \quad \frac{k}{g}(s-a) = M + \lambda s \quad : \quad \frac{k}{g}s - \frac{k}{g}a = M + \lambda s$$

$$\frac{k}{g}s - \lambda s = M + \frac{k}{g}a \quad : \quad s\left(\frac{k}{g} - \lambda\right) = M + \frac{k}{g}a \quad : \quad s = \left(M + \frac{k}{g}a\right)\left(\frac{k}{g} - \lambda\right)^{-1}$$

$$\boxed{s = \frac{Mg + ka}{k - \lambda g}}$$

(e) (2 pts) Find the frequency of oscillation for the system about the equilibrium position.

$$(m+2M)\ddot{s} + (k - \lambda g)s - ka - mg = 0 \quad : \quad \text{Let } s = u + \frac{ka + mg}{k - mg}$$

$$(m+2M)\ddot{u} + (k-\lambda g)\left(u + \frac{ka+mg}{k-mg}\right) - ka - mg = 0, \quad (m+2M)\ddot{u} + (k - \lambda g)u = 0$$

$$\boxed{\omega = \sqrt{\frac{k - \lambda g}{m + 2M}}}$$

Problem 3: Review

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Problem 4:

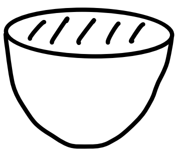
Taken from the 1996 Classical and Statistical Mechanics Qualifier:

A point particle of mass m moves under the influence of gravity ($V(x, y, z) = mgz$). The particle itself is constrained to stay on a frictionless surface given by

$$z = \alpha(x^2 + y^2)$$

where $\alpha > 0$.

(a) (2 pts) Derive the equations of motion for x, y, and z by minimizing the appropriately constrained action, using the method of Lagrange multipliers.



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad U = mgz \; : \; L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$h = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz + \lambda(t)(x^2 + y^2 - z/\alpha)$$

$$\frac{d}{dt}\left(\frac{\partial h}{\partial \dot{x}}\right) - \frac{\partial h}{\partial x} = 0, \quad \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{y}}\right) - \frac{\partial h}{\partial y} = 0, \quad \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{z}}\right) - \frac{\partial h}{\partial z} = 0$$

$$\frac{\partial h}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial h}{\partial x} = 2x\lambda \; : \; m\ddot{x} - 2x\lambda = 0$$

$$\frac{\partial h}{\partial \dot{y}} = m\dot{y}, \quad \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{y}}\right) = m\ddot{y}, \quad \frac{\partial h}{\partial y} = 2y\lambda \; : \; m\ddot{y} - 2y\lambda = 0$$

$$\frac{\partial h}{\partial \dot{z}} = m\dot{z}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}, \quad \frac{\partial h}{\partial z} = -mg - \frac{\lambda}{\alpha} \; : \; m\ddot{z} + mg + \frac{\lambda}{\alpha} = 0$$

$$\boxed{m\ddot{x} = 2x\lambda, \quad m\ddot{y} = 2y\lambda, \quad m\ddot{z} + mg + \lambda/\alpha = 0}$$

(b) (2 pts) Consider the class of trajectories (i.e. solutions) for which $z = \text{constant} \equiv z_0$. Calculate the force required to maintain the constraint for these trajectories.

$$\text{w/ } \ddot{z} = 0 \; : \; mg = -\lambda/\alpha \; \therefore \; \lambda = -\alpha mg \; \therefore \; \ddot{x} = -2\alpha gx, \quad \ddot{y} = -2\alpha gy$$

$$\boxed{m\ddot{x} = -2m\alpha gx, \quad m\ddot{y} = -2m\alpha gy}$$

(c) Consider the class of trajectories for which $y = 0$. (These are different trajectories than those in part (b).) Assume that the particle reaches a maximum height z_1:

(i) (2 pts) Using conservation of energy derive an expression for $\dot{z} \equiv dz/dt$ as a function of z, and z_1. (That is, remove the dependence on any other spatial co-ordinate.)

$$U_i + K_i = U_f + \cancel{K_f}^{0} \; : \; K_i = U_f - U_i \; : \; T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2), \quad U = mgz_{(i)}, \quad z = \alpha x^2$$

$$\dot{z} = 2\alpha x \cdot \dot{x} \quad \text{w/ } x = \sqrt{\frac{z}{\alpha}} \; : \; \dot{z} = 2\alpha\sqrt{\frac{z}{\alpha}}\,\dot{x} \; \therefore \; \dot{x} = \frac{\dot{z}}{2\alpha}\sqrt{\frac{\alpha}{z}} \; : \; \dot{x}^2 = \frac{\dot{z}^2}{4\alpha} \cdot \frac{1}{z}$$

$$\frac{1}{2}m\left(\frac{\dot{z}^2}{4\alpha} \cdot \frac{1}{z} + \dot{z}^2\right) = mgz_1 - mgz \; : \; \frac{\dot{z}^2 \cancel{m}}{2}\left(\frac{1}{4\alpha z} + 1\right) = \cancel{m}g(z_1 - z)$$

Problem 4: Continued

$$\boxed{\dot{z} = \pm\sqrt{2g(z_1 - z)\left(\frac{1}{4\alpha z} + 1\right)^{-1}}}$$

(ii) (2 pts) From the expression derived in (i) and the equations of motion you calculated above, determine the force in the z-direction required to maintain the constraint.

$$\dot{z} = \sqrt{2g(z_1 - z)\left(\frac{4\alpha z}{1+4\alpha z}\right)} = \sqrt{2gz_1\left(\frac{4\alpha z}{1+4\alpha z}\right) - 2gz\left(\frac{4\alpha z}{1+4\alpha z}\right)}$$

$$\frac{d}{dt}\left(\frac{4\alpha z}{1+4\alpha z}\right) = \frac{4\alpha\dot{z}}{1+4\alpha z} - \frac{16\alpha^2 z\dot{z}}{(1+4\alpha z)^2} = \frac{4\alpha\dot{z} + 16\alpha^2 z\dot{z} - 16\alpha^2 z\dot{z}}{(1+4\alpha z)^2} = \frac{4\alpha\dot{z}}{(1+4\alpha z)^2}$$

$$\ddot{z} = \frac{1}{2}\frac{1}{\sqrt{2g(z_1 - z)\left(\frac{4\alpha z}{1+4\alpha z}\right)}}\cdot\left[\frac{8g\alpha\dot{z}z_1}{(1+4\alpha z)^2} - \frac{8g\alpha\dot{z}z}{(1+4\alpha z)} - \frac{8g\alpha z\dot{z}}{(1+4\alpha z)^2}\right]$$

$$\ddot{z} = \frac{1}{2}\frac{1}{\sqrt{2g(z_1 - z)\left(\frac{4\alpha z}{1+4\alpha z}\right)}}\left[\frac{8g\alpha\dot{z}}{(1+4\alpha z)}\left(\frac{z_1 - z}{(1+4\alpha z)} - z\right)\right]$$

$$\ddot{z} = \frac{1}{m}(\lambda - mg) \quad \therefore \quad \lambda = m\ddot{z} + mg$$

$$\boxed{\lambda = \frac{1}{2}\frac{m}{\sqrt{2g(z_1 - z)\left(\frac{4\alpha z}{1+4\alpha z}\right)}}\left[\frac{8g\alpha\dot{z}}{(1+4\alpha z)}\left(\frac{z_1 - z}{(1+4\alpha z)} - z\right)\right] + mg}$$

(iii) (2 pts) Calculate the total force required to maintain the constraint for this class of trajectory.

$$\dot{x} = \frac{\dot{z}}{2\alpha}\sqrt{\frac{\alpha}{z}}, \quad \dot{x} = \frac{\dot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}}, \quad \ddot{x} = \frac{\ddot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}} - \frac{\dot{z}}{4\sqrt{\alpha}}\cdot\sqrt{\frac{1}{(\dot{z})^3}} = \frac{\ddot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}} - \frac{1}{4\sqrt{\alpha}}\sqrt{\frac{1}{\dot{z}}}$$

$$\ddot{x} = -\frac{2\lambda\alpha x}{m} \ : \ -\frac{2\lambda\alpha x}{m} = \frac{\ddot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}} - \frac{1}{4\sqrt{\alpha}}\sqrt{\frac{1}{\dot{z}}} \ : \ \frac{2\lambda\alpha x}{m} = \frac{1}{4\sqrt{\alpha}}\sqrt{\frac{1}{\dot{z}}} - \frac{\ddot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}}$$

$$\boxed{\lambda = \frac{m}{2\alpha x}\left[\frac{1}{4\sqrt{\alpha}}\sqrt{\frac{1}{\dot{z}}} - \frac{\ddot{z}}{2\sqrt{\alpha}}\sqrt{\frac{1}{z}}\right]}$$

w/ $\ddot{z}$ & $\dot{z}$ equal to the above.

Problem 4: Review

Procedure:

- Use the procedure for finding Equations of Motion with the Euler Lagrange formalism.
- Answer all other questions with the use of the Equations of Motion.

Key Concepts:

- We can use the Euler Lagrange formalism to find Equations of Motion for multiple systems.

Variations:

- The entire system can change.
  - Thus changing the generalized co-ordinates, and the resulting equations of motion.
- We can be asked other questions pertaining to the Equations of Motion.
  - We would then consult the Equations of Motion and deduce how they can be used to answer the questions.

Problem 5:

Many solid state systems are crystals, and the problems involving them have a rotational symmetry where the system is rotated by $\theta = \pi/2$ (four-fold symmetry) or $\theta = \pi/3$ (six-fold rotational symmetry). Is there a conserved quantity associated with this symmetry by Noether's Theorem? If so, what is it? If not, why not?

Emmy Noether's Theorem states : If there is a symmetry in a system , there is a conserved quantity in that system.

From the statement above, we can see that there are symmetries in this system therefore there is a conserved quantity. More preisely we can see that if there is a

Spacial translation Symmetry $\longrightarrow$ conserved total momentum

Rotational Symmetry $\longrightarrow$ conserved angular momentum

Time translation Symmetry $\longrightarrow$ Conserved energy

But the above statements are only true if the translations and rotations are incremental and not large. This problem only has Symmetries for $\theta = \pi/2$ and $\theta = \pi/3$ therefore there is no conserved quantity with this symmetry.

$\boxed{\text{No Conserved Quantity}}$

Problem 5: Review

Procedure:

- Recognize there is not an infinitesimal symmetry $\therefore$ no conserved quantity.

Key Concepts:

- If there is an infinitesimal symmetry there will be a conserved quantity.
  - Spacial translation $\rightarrow$ conserved total angular momentum.
  - Rotational translation $\rightarrow$ conserved angular momentum.
  - Time translation $\rightarrow$ conserved energy.

Variations:

- This symmetry could be continuous $\therefore$ leading to a conserved quantity.

Problem 6:

Consider a two particle interacting system in 3D with the Lagrangian:

$$L = \frac{1}{2}(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2) - v(\vec{r}_1 - \vec{r}_2)$$

(a) Show that the transformation:

$$\vec{r}_i \to \vec{r}_i + \epsilon\vec{v}_0 t$$

where $\vec{v}_0$ is a constant vector, leaves the Lagrangian unchanged to order ϵ up to a total derivative.

Lets call $\varepsilon \vec{v}_0 t \to \varepsilon(t)$ $\therefore$ $\vec{r}_i \to \vec{r}_i + \varepsilon(t)$: $\vec{r}_1 = \vec{r}_1 + \varepsilon(t)$, $\vec{r}_2 = \vec{r}_2 + \varepsilon(t)$

$$S[\vec{r}(t)] \int_{t_1}^{t_2} L(\vec{r}(t), \dot{\vec{r}}(t))\, dt \; : \; S[\vec{r}(t) + \varepsilon(t)] = S[\vec{r}(t)] + \delta S + \mathcal{O}(\varepsilon^2)$$

IF $\vec{r}(t)$ is stationary then $\delta S = 0$: $S = \int_{t_1}^{t_2} \frac{1}{2}(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2) - V(\vec{r}_1 - \vec{r}_2)\, dt$

$$S = \int_{t_1}^{t_2} \frac{1}{2}\left(m_1(\dot{\vec{r}}_1^2 + 2\dot{\vec{r}}_1\dot{\varepsilon}(t) + \dot{\varepsilon}(t)^2) + m_2(\dot{\vec{r}}_2^2 + 2\dot{\vec{r}}_2\dot{\varepsilon}(t) + \dot{\varepsilon}(t)^2)\right) - V(\vec{r}_1 + \varepsilon(t) - \vec{r}_2 - \varepsilon(t))\, dt$$

$$S = \int_{t_1}^{t_2} \frac{1}{2}(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2) - V(\vec{r}_1 - \vec{r}_2)\, dt + \int_{t_1}^{t_2} m_1\dot{\vec{r}}_1\dot{\varepsilon}(t) + m_2\dot{\vec{r}}_2\varepsilon(t)\, dt + \mathcal{O}(\varepsilon(t)^2)$$

$$S = \int_{t_1}^{t_2} \frac{1}{2}(m_1\dot{\vec{r}}_1^2 + m_2\dot{\vec{r}}_2^2) - V(\vec{r}_1 - \vec{r}_2)\, dt \to S[\vec{r}(t)]$$

From $S[\vec{r}(t)]$ we can see that L is unchanged under the transformation

$\blacksquare$

(b) Calculate the conserved quantity associated with this symmetry of the action.

$$\sum_{i=1}^{n} \frac{\partial F}{\partial \dot{y}_i}\eta_i + \left(f - \sum_{i=1}^{n} \dot{y}_i \frac{\partial f}{\partial \dot{y}_i}\right)\xi = \text{const} \quad \xi = 0 \;\therefore\; \sum_{i=1}^{n} \frac{\partial F}{\partial \dot{y}_i}\eta_i = \text{const}$$

$$\int_{t_1}^{t_2} m_1\dot{\vec{r}}_1\dot{\varepsilon}(t) + m_2\dot{\vec{r}}_2\varepsilon(t)\, dt \to \delta S \; : \; \dot{\varepsilon}(t)^2 \to \mathcal{O}(\varepsilon(t)^2)$$

$$\delta S = 0 \text{ if } \varepsilon(t_1) = \varepsilon(t_2) : \delta S = (m_1\dot{\vec{r}}_1\cancelto{0}{\varepsilon + m_2\dot{\vec{r}}_2\varepsilon})\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \varepsilon(m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2)\, dt$$

$$0 = \int_{t_1}^{t_2} \varepsilon(m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2)\, dt \text{ For any } \varepsilon(t) \;\therefore\; m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0 \; : \; \frac{d}{dt}(m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2) = m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2$$

$$\boxed{\text{The linear momentum is conserved}}$$

(c) What is this new conserved quantity? Is it useful?

This new conserved quantity is linear momentum. It is useful because we can extrapolate information about our system with knowing about this conserved quantity.

While the above problem is posed in three dimensions, you can work in one dimension and generalize.

Problem 6: Review

Procedure:

- Perform a translation in the $\vec{r}$ direction on the action.
- Proceed to show that the functional is unchanged.
- Using symmetries and Noether's equation, show that linear momentum is conserved.
- Discuss why there is a conserved quantity.

Key Concepts:

- Invariations of the action under infinitesimal translations show that there is a conserved quantity.

Variations:

- The functional can change.
  - This would lead to a different end result with the same procedure.
- The translations can change.
  - Thus altering our functional.