Problem:

The molecules of a hypothetical ideal gas have internal energy levels that are equally spaced so that the n^{th} energy eigenvalue is $E_n = \varepsilon n$, where $n = 0, 1, 2, \ldots$
The degeneracy of the n^{th} level is $n+1$.

Task: Calculate the contribution of the internal energy states to the thermal energy.

We have an ideal gas (= non-interacting gas).
We start with the single-particle partition function.
Consider only internal contibution:

$$Q_{int,1} = \sum_{n=0}^{\infty} (n+1)\, e^{-\beta \varepsilon n} \qquad \left(\beta = \frac{1}{kT}\right)$$

$$\underset{x = e^{-\beta\varepsilon}}{=} \sum_{n=0}^{\infty} (n+1)\, x^n$$

$$\underset{m = n+1}{=} \sum_{m=1}^{\infty} m\, x^{m-1}$$

$$= \frac{d}{dx}\left(\sum_{m=1}^{\infty} x^m\right)$$

the constant does not contribute $\rightarrow$ $$= \frac{d}{dx}\left(\sum_{m=1}^{\infty} x^m + \underbrace{1}_{\text{constant}}\right)$$

$$= \frac{d}{dx}\Big(\underbrace{\sum_{m=0}^{\infty} x^m}_{\frac{1}{1-x}}\Big)$$

$$= -\frac{1}{(1-x)^2}(-1)$$

$$x = e^{-\beta\varepsilon} \rightarrow \quad = (1-x)^{-2}$$

$$= (1-e^{-\beta\varepsilon})^{-2}$$

$$\Rightarrow \boxed{Q_{int,1} = (1-e^{-\beta\varepsilon})^{-2}}$$

We want thermal energy, i.e., ~~thermal~~ expectation value of $\mathcal{H}$.

$$U_{int,1} = \frac{1}{Q_{int,1}} \sum_{n=0}^{\infty} \varepsilon n (n+1) e^{-\beta\varepsilon n}$$

$$= \frac{1}{Q_{int,1}} \sum_{n=0}^{\infty} \left\{ -\frac{\partial}{\partial\beta}\left[(n+1)e^{-\beta\varepsilon n}\right]\right\}$$

$$= \frac{-1}{Q_{int,1}} \frac{\partial}{\partial\beta}\Big(\underbrace{\sum_{n=0}^{\infty}(n+1)e^{-\beta\varepsilon n}}_{\text{from before: } Q_{int,1}}\Big)$$

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$$\Rightarrow \quad U_{int,1} = - \frac{1}{Q_{int,1}} \frac{\partial}{\partial \beta} Q_{int,1}$$

$$= - \frac{\partial}{\partial \beta} \log Q_{int,1}$$

Plug in:

$$U_{int,1} = - \frac{\partial}{\partial \beta} \log \left((1 - e^{-\beta \varepsilon})^{-2} \right)$$

$$= + 2 \frac{1}{1 - e^{-\beta \varepsilon}} (- e^{-\beta \varepsilon})(- \varepsilon)$$

$$\boxed{U_{int,1} = + 2 \varepsilon \frac{1}{e^{\beta \varepsilon} - 1}}$$

So: internal thermal energy of single particle

$$U_{int,1} = 2 \varepsilon (e^{\beta \varepsilon} - 1)^{-1}$$

For N particles: $\boxed{U_{int,N} = 2 N \varepsilon (e^{\beta \varepsilon} - 1)^{-1}}$

Of course: we also have "motional" energy $\rightsquigarrow$ need to add $\frac{3}{2} N k T$