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Quantum Mechanics 1

PHYS 5393 HOMEWORK ASSIGNMENT #6

PROBLEMS: {2.2, 2.3, 2.4, 2.6, Q-1}

Due: October 12, 2021 By: 5 PM

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Problem 1: 2.2

Look again at the Hamiltonian of Chapter 1, problem 1.13. Suppose the typist made an error and wrote $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{H}} = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2|.$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using this illegal Hamiltonian of this kind. (You may assume $H_{11} = H_{22} = 0$ for simplicity).

This is not a Hermitian operator:

$$\tilde{H} \neq \tilde{H}^{\dagger} : \quad \tilde{H} = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2| , \quad \tilde{H}^{\dagger} = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|2\rangle\langle 1|$$

This can be showed even further:

$$\tilde{H} = \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix} , \quad \tilde{H}^{\dagger} = \begin{pmatrix} H_{11} & 0 \\ H_{12} & H_{22} \end{pmatrix}$$

If we expand this to the time evolution operator

$$\tilde{\mathcal{U}}(t) = \exp\left(\frac{-i\tilde{H}t}{\hbar}\right) = 1 - \frac{i}{\hbar}\tilde{H}t$$

If we check this for being unitary:

$$\tilde{\mathcal{U}}^{\dagger}(t)\tilde{\mathcal{U}}(t) = (1 + i/\hbar\, \tilde{H}^{\dagger}t)(1 - i/\hbar\, \tilde{H}t) = 1 + i(\tilde{H}^{\dagger} - \tilde{H})t , \quad \text{since } \tilde{H}^{\dagger} \neq \tilde{H}, \quad \tilde{\mathcal{U}}^{\dagger}(t)\tilde{\mathcal{U}}(t) \neq 1$$

This in turn leads to the state vectors no longer being normalized

$$\tilde{H}^2 = H_{12}^2|1\rangle\langle 2|1\rangle\langle 2| = 0 \implies \tilde{H}^n = 0 \text{ if } n = \text{integer and } n \geq 2$$

Problem 1: 2.2 Review

Procedure:

- Write the illegal Hamiltonian in matrix form.
- Proceed to calculate the Hermitian adjoint of this matrix.
- Show that this illegal Hamiltonian is not Hermitian and thus an illegal Hamiltonian.
- Expand out the generator function for the translation operator.
- Show that because of the lack of Hermiticity that the translation operator is no longer unitary.
- Show that the state vectors are no longer normalized with this Hamiltonian.

Key Concepts:

- This illegal Hamiltonian is not Hermitian.
- This lack of Hermiticity will cause the time translation operator to no longer be unitary.
- Because the time translation operator is not unitary, the state vectors will not be normalized.

Variations:

- The Hamiltonian that is given to us can be different.
  - If it can be shown to be Hermitian, then the rest of the problem is done by showing how the time translation operator will still be unitary and how the state vectors are normalized.
  - If the Hamiltonian is still not Hermitian, then the same process is laid out in front with showing that the time evolution operator is still not Hermitian and thus the state vectors are not normalized.

Problem 2: 2.3

An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $S \cdot \hat{n}$ with eigenvalue $\hbar/2$, where $\hat{n}$ is a unit vector, lying in the xz-plane, that makes an angle β with the z-axis.

(a) Obtain the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.

Eigenstate : $|\alpha\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle$, $\mathcal{U}(t)|\alpha,0\rangle = e^{-i\omega t/2}\cos(\beta/2)|+\rangle + e^{i\omega t/2}\sin(\beta/2)|-\rangle$

$$P = |\langle S_x:+|\alpha,0\rangle|^2 = \frac{1}{2}\left| e^{-i\omega t/2}\cos(\beta/2)\langle +|+\rangle + e^{i\omega t/2}\sin(\beta/2)\langle -|-\rangle \right|^2$$

$$P = \frac{1}{2}\left(\cos^2(\beta/2) + \sin^2(\beta/2) + \cos(\beta/2)\sin(\beta/2)\left(e^{i\omega t} + e^{-i\omega t}\right)\right)$$

$$P = \frac{1}{2}\left(1 + \cos(\beta/2)\sin(\beta/2)\left(2\cos(\omega t)\right)\right) = \frac{1}{2}\left(1 + 2\sin(\beta)\cos(\omega t)\right)$$

$$\boxed{P = \frac{1}{2}\left(1 + 2\sin(\beta)\cos(\omega t)\right)}$$

(b) Find the expectation value of S_x as a function of time.

$\langle S_x\rangle = \langle\alpha|S_x|\alpha\rangle$: $\langle S_x(t)\rangle = \langle\alpha(t)|\tilde{S}_x|\alpha(t)\rangle$: $|\alpha(t)\rangle = e^{-i\omega t/2}\cos(\beta/2)|+\rangle + e^{i\omega t/2}\sin(\beta/2)|-\rangle$

$$\langle\tilde{S}_x(t)\rangle = \frac{\hbar}{2}\left(e^{i\omega t/2}\cos(\beta/2)\langle +| + e^{-i\omega t/2}\sin(\beta/2)\langle -|\right)\left(e^{i\omega t/2}\sin(\beta/2)|+\rangle + e^{-i\omega t/2}\cos(\beta/2)|-\rangle\right)$$

$$\langle\tilde{S}_x(t)\rangle = \frac{\hbar}{2}\left(\left(e^{i\omega t} + e^{-i\omega t}\right)\cos(\beta/2)\sin(\beta/2)\right) = \frac{\hbar}{2}\cdot 2\cos(\omega t)\cos(\beta/2)\sin(\beta/2)$$

$$\boxed{\langle\tilde{S}_x(t)\rangle = \frac{\hbar}{2}\sin(\beta)\cos(\omega t)}$$

(c) For your own peace of mind show that your answers make good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

When $\beta = 0$, $\langle S_x(t)\rangle = 0$: when $\beta = \pi/2$, $\langle S_x(t)\rangle = \frac{\hbar}{2}\cos(\omega t)$. This means when $\beta = 0$ we are on the z-axis and the expectation value for x should be zero. When $\beta = \pi/2$ we are in the xy-plane and thus our expectation value should be dependent upon time, oscillating between $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$.

Problem 2: 2.3 Review

Procedure:

- Write out the eigenstate that corresponds to the electron in the magnetic field in the positive z-direction.
- Write out the state for the electron in the S_x state.
- To find $|\alpha, 0\rangle$, use the time evolution operator on the initial $|\alpha\rangle$ ket.
- Use the probability calculation of $|\langle S_x : +|\alpha, 0\rangle|^2$ and simplify the result.
- Proceed to use the expectation value equation $\langle\alpha|S_x|\alpha\rangle$ and simplify the result where $|\alpha\rangle$ is $|\alpha, 0\rangle$ where the 0 corresponds to the initial time when the measurement is taken.
- Finally, plug in the angles for β in the expectation value equation to show what one should expect as a measurement for the expectation value.

Key Concepts

- In the context of this problem, the eigenstate corresponds to $|\alpha\rangle$ in the probability equation.
- The state in which we are looking for the probability is S_x.
- We must use the time evolution operator on the state α to find what the probability as a function of time will be.
- Because we can find $|\alpha, t\rangle$, we can find probabilities that are dependent on time and expectation values that are dependent on time.
- When we plug in $\beta = 0$ into the expectation value equation, we find that are particle is purely in the z-direction and thus we should have a zero expectation value.
- Conversely, when we plug in $\beta = \pi/2$, we find ourselves in the xy-plane and thus our expectation value will be a function of time.

Variations

- The eigenstate of our electron can change.
  - This can change the $|\alpha\rangle$ ket to something other value.
- The electron can be in a different state.
  - This electron for instance can be in a different direction, e.g. S_y or some arbitrary direction. This would change the calculation of the probability and expectation value but not the overall process.
- We can be asked to evaluate our expectation value for different angles to see if the final answer makes sense.

Problem 3: 2.4

Derive the neutrino oscillation probability (2.65) and use it, along with the data in Figure 2.2, to estimate the values $\triangle m^2 c^4$ (in units of eV^2) and θ.

$$|\nu_e\rangle = \cos(\alpha)|\nu_1\rangle - \sin(\alpha)|\nu_2\rangle \,, \quad \mathcal{U}(t,t_0) = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right] \,, \quad E = pc\left(1 + \frac{m^2c^2}{2p^2}\right)$$

$$P = |\langle \nu_e | \nu_e(t)\rangle|^2 : \quad |\nu_e(t)\rangle = \mathcal{U}(t,t_0)|\nu_e\rangle = \cos(\alpha)e^{-iE_1 t/\hbar}|\nu_1\rangle - \sin(\alpha)e^{-iE_2 t/\hbar}|\nu_2\rangle$$

$$P = |\cos^2(\alpha)e^{\overbrace{-iE_1 t/\hbar}^{x_1}} - \sin^2(\alpha)e^{\overbrace{-iE_2 t/\hbar}^{x_2}}|^2 = \cos^4(\alpha) + \sin^4(\alpha) + 2\cos^2(\alpha)\sin^2(\alpha)(e^{x_2} + e^{x_1})$$

$$P = \cos^2(\alpha)(1-\sin^2(\alpha)) + \sin^2(\alpha)(1-\cos^2(\alpha)) + 2\cos^2(\alpha)\sin^2(\alpha)(e^{x_2}+e^{x_1})$$

$$P = 1 - 2\cos^2(\alpha)\sin^2(\alpha) + 2\cos^2(\alpha)\sin^2(\alpha)(e^{x_2}+e^{x_1}) = 1 - 2\cos^2(\alpha)\sin^2(\alpha)(1-\cos(x_2-x_1))$$

$$P = 1 - \sin^2(2\alpha)\sin^2\left(\frac{\Delta m^2 c^4 L}{4E\hbar}\right) \checkmark$$

minimum $P = 0.4 \quad \therefore \quad 1 - \sin^2(2\alpha) = 0.4 \longrightarrow \sin(2\alpha) = \sqrt{0.6} \quad \alpha = \sin^{-1}\left(\frac{\sqrt{0.6}}{2}\right) = 25.38^\circ$

$$\lambda = 50\,\frac{\text{km}}{\text{MeV}} \; : \; \lambda = 2\pi \cdot \frac{4\hbar c}{\Delta m^2} \quad \therefore \quad \Delta m = \sqrt{\frac{8\pi\hbar c}{\lambda}} \; : \; \hbar c = 200\times10^{-18}\,\text{MeV}\cdot\text{km}$$

$$\Delta m^2 = \frac{8\pi \cdot 200\times10^{-18}\,\text{MeV}\cdot\text{km}}{50\,\text{km}/\text{MeV}} = 1.005\times10^{-16}(\text{MeV})^2 = 1.01\times10^{-4}(\text{eV})^2$$

$$\Delta m^2 c^4 = (1.01\times10^{-4}(\text{eV})^2)\cdot(3.0\times10^{6}\,\text{m/s})^4 = 8.14\times10^{21}(\text{eV})^2(\text{m/s})^4$$

$$\boxed{\Delta m^2 c^4 = 8.14\times10^{21}(\text{eV})^2(\text{m/s})^4}$$

Problem 3: 2.4 Review

Procedure:

- Use the probability equation $|\langle S_\eta : \pm|\alpha\rangle|^2$ where $|\alpha\rangle$ and $\langle S_\eta : \pm|$ is the state of the neutrino.
- Use the time evolution operator on $|\alpha\rangle$ to create a state that is dependent on time.
- Proceed to calculate the probability oscillation with the above probability equation and determine what the probability of the state will be with respect to time.
- Once the calculation has been completed, use values from the graph to show what the quantity of $\triangle m^2c^4$.

Key Concepts:

- We use the probability equation of two states to determine what the neutrino oscillation probability will be.
- To create a state that is dependent upon time, we need to use the time evolution operator. This in turn can be used to calculate what the probability oscillation will be.
- We can read values off the graph to in turn determine what the quantity $\triangle m^2c^4$ will be.

Problem 4: 2.6

Consider a particle in one dimension whose Hamiltonian is given by

$$\tilde{\mathbf{H}} = \frac{\tilde{\mathbf{p}}^2}{2m} + \mathbf{V}(x).$$

By calculating $[[\tilde{\mathbf{H}}, \tilde{\mathbf{x}}], \tilde{\mathbf{x}}]$ prove

$$\sum_{a'} |\langle a"|\tilde{\mathbf{x}}|a'\rangle|^2 (E_{a'} - E_{a"}) = \frac{\hbar^2}{2m}$$

where $|a'\rangle$ is an energy eigenket with eigenvalue $E_{a'}$.

$$\langle a''|\tilde{x}|a'\rangle = \langle a'|\tilde{x}^\dagger|a''\rangle^* : \text{ Since } \tilde{x} = \tilde{x}^\dagger : \langle a''|\tilde{x}|a'\rangle = \langle a'|\tilde{x}|a''\rangle^*$$

$$|\langle a''|\tilde{x}|a'\rangle|^2 = |\langle a''|\tilde{x}|a'\rangle\langle a''|\tilde{x}|a'\rangle| = |\langle a''|\tilde{x}|a'\rangle\langle a'|\tilde{x}|a''\rangle|$$

$$[[H,x],x] = [H,x]x - x[H,x] = Hx^2 - xHx - xHx + x^2H = Hx^2 - 2xHx + x^2H$$

Calculate expectation value of $[[H,x],x] \to D$

$$\langle\alpha|D|\alpha\rangle = \langle\alpha|[[H,x],x]|\alpha\rangle = \langle\alpha|Hx^2 - 2xHx + x^2H|\alpha\rangle$$

$$\langle\alpha|Hx^2|\alpha\rangle - 2\langle\alpha|xHx|\alpha\rangle + \langle\alpha|x^2H|\alpha\rangle \quad : \text{ if } \langle\alpha| = \langle a'' \text{ then}$$

$$\langle a''|Hx^2|a''\rangle - 2\langle a''|xHx|a''\rangle + \langle a''|x^2H|a''\rangle$$

$$E_{a''}\langle a''|x^2|a''\rangle - 2\langle a''|xHx|a''\rangle + \langle a''|x^2|a''\rangle E_{a''}$$

Apply completeness relation :

$$2E_{a''}\langle a''|x^2|a''\rangle - 2\sum_{a'}\langle a''|xH|a'\rangle\langle a'|x|a''\rangle = 2E_{a''}\langle a''|x^2|a''\rangle - 2E_{a'}\langle a''|x^2|a''\rangle$$

$$2(E_{a''} - E_{a'})\langle a''|x^2|a''\rangle : \text{ Because } \tilde{x} = \tilde{x}^\dagger \to \langle a''|x^2|a''\rangle = \sum_{a'}\langle a''|x|a'\rangle\langle a'|x|a''\rangle$$

$$2(E_{a''} - E_{a'})\langle a''|x^2|a''\rangle = \sum_{a'} 2(E_{a''} - E_{a'})\langle a''|x|a'\rangle\langle a''|x|a'\rangle = \sum_{a'} 2(E_{a''} - E_{a'})|\langle a''|x|a'\rangle|^2$$

$$\text{w/ } \langle\alpha|[[H,x],x]|\alpha\rangle = -\frac{i\hbar}{2}\langle\alpha|[p,x]|\alpha\rangle = -\frac{i\hbar}{m}\cdot -i\hbar\langle\alpha|\alpha\rangle = -\frac{\hbar^2}{m}$$

$$\sum_{a'} 2(E_{a''} - E_{a'})|\langle a''|x|a'\rangle|^2 = -\frac{\hbar^2}{m} \quad \therefore \quad \sum_{a'}|\langle a''|x|a'\rangle|^2(E_{a'} - E_{a''}) = \frac{\hbar^2}{2m} \checkmark$$

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Problem 4: 2.6 Review

Procedure:

- Begin by calculating the commutator $[[\tilde{\mathbf{H}}, \tilde{\mathbf{x}}], \tilde{\mathbf{x}}]$.
- Calculated the expectation value of this commutator.
- Use the mathematical formalism of this expanded commutator to write out three separate expectation values.
- Apply the Hamiltonian on the eigenstates to determine the eigenvalues.
- Use a discrete completeness relation to determine the last eigenvalue.
- Regroup terms and use the Hermiticity of the $\tilde{\mathbf{x}}$ to rewrite the statement.
- Proceed to use the identity $[[\tilde{\mathbf{H}}, \tilde{\mathbf{x}}], \tilde{\mathbf{x}}] = \frac{-i\hbar}{2}[\tilde{\mathbf{p}}, \tilde{\mathbf{x}}]$ to show that the other side of the equation is equal to the desired result.

Key Concepts:

- We need to use commutator identities to simplify our math.
- Because we are using a bounded energy, the completeness relation that we use is discrete.
- Using completeness relations and commutator identities, we can prove the following if we use the Hermiticity of our operators to do so.

Problem 5: Q-1

Suppose the state vectors $|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a unitary operator $\tilde{\mathbf{U}}$ with eigenvalues λ and λ', respectively. What relation must λ and λ' satisfy if $|\alpha\rangle$ is not orthogonal to $|\beta\rangle$?

$$\langle\alpha|\alpha\rangle = \langle\alpha|\mathcal{U}^{\dagger}\mathcal{U}|\alpha\rangle = \lambda^{*}\lambda = 1 \Rightarrow \lambda = e^{i\varphi_1}$$

$$\langle\beta|\beta\rangle = \langle\beta|\mathcal{U}^{\dagger}\mathcal{U}|\beta\rangle = \lambda'^{*}\lambda' = 1 \Rightarrow \lambda = e^{i\varphi_2}$$

$$\langle\beta|\alpha\rangle = \langle\beta|\mathcal{U}^{\dagger}\mathcal{U}|\alpha\rangle = \lambda'^{*}\lambda'\langle\beta|\alpha\rangle \Rightarrow \lambda'\lambda = 1 \Rightarrow \lambda = \lambda' = e^{i\varphi_1}$$

The eigenvalues of the two eigenkets must be equal in general

Problem 5: Q-1 Review

Procedure:

- Calculate $\langle\alpha|\alpha\rangle = \langle\alpha|\mathcal{U}^\dagger\mathcal{U}|\alpha\rangle$, $\langle\beta|\beta\rangle = \langle\beta|\mathcal{U}^\dagger\mathcal{U}|\beta\rangle$ and $\langle\beta|\alpha\rangle = \langle\beta|\mathcal{U}^\dagger\mathcal{U}|\alpha\rangle$.
- Show that the multiplication of the eigenvalues must equal 1 for the states to not be orthogonal.

Key Concepts:

- For the states to not be orthogonal, the inner products cannot me equal to zero.
- If the inner products are not going to be zero, then the eigenvalues must be the same.

Variations:

- We can be given a different condition, such as what is the condition for the eigenvalues if these states are orthogonal.
  - This would be somewhat redundant because the values of the eigenvalues wouldn't matter if the states ended up being orthogonal.
- We could be asked to show what the relation between eigenvalues would be if the two states were normalized.
  - We would have to show that the product of the two eigenvalues would have to be equal to one. This would mean they have to be inverses of one another or the dual compliment of one another.