MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — December 18, 2012

Prof. Alan Guth

FORMULA SHEET FOR FINAL EXAM

Exam Date: December 19, 2012

*** Some sections below are marked with asterisks, as this section is. The asterisks indicate that you won't need this material for the quiz, and need not understand it. It is included, however, for completeness, and because some people might want to make use of it to solve problems by methods other than the intended ones.

Index Notation:

$$\vec{A} \cdot \vec{B} = A_i B_i\,, \qquad \vec{A} \times \vec{B}_i = \epsilon_{ijk} A_j B_k\,, \qquad \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

Rotation of a Vector:

$$A'_i = R_{ij} A_j\,, \qquad \text{Orthogonality: } R_{ij} R_{ik} = \delta_{jk} \qquad (R^T T = I)$$

Rotation about z-axis by ϕ:

$$R_z(\phi)_{ij} \;=\; \begin{array}{c c} & \begin{array}{ccc} \scriptstyle j=1 & \scriptstyle j=2 & \scriptstyle j=3 \end{array} \\ \begin{array}{c} \scriptstyle i=1 \\ \scriptstyle i=2 \\ \scriptstyle i=3 \end{array} & \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Rotation about axis $\hat{n}$ by ϕ:***

$$R(\hat{n}, \phi)_{ij} = \delta_{ij} \cos\phi + \hat{n}_i \hat{n}_j (1 - \cos\phi) - \epsilon_{ijk} \hat{n}_k \sin\phi\ .$$

Vector Calculus:

Gradient: $(\vec{\nabla}\varphi)_i = \partial_i \varphi\,, \qquad \partial_i \equiv \dfrac{\partial}{\partial x_i}$

Divergence: $\vec{\nabla} \cdot \vec{A} \equiv \partial_i A_i$

Curl: $(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$

Laplacian: $\nabla^2 \varphi = \vec{\nabla} \cdot (\vec{\nabla}\varphi) = \dfrac{\partial^2 \varphi}{\partial x_i \partial x_i}$

Fundamental Theorems of Vector Calculus:

Gradient: $$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a})$$

Divergence: $$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{A}\, \mathrm{d}^3 x = \oint_S \vec{A} \cdot \mathrm{d}\vec{a}$$

where S is the boundary of $\mathcal{V}$

Curl: $$\int_S (\vec{\nabla} \times \vec{A}) \cdot \mathrm{d}\vec{a} = \oint_P \vec{A} \cdot \mathrm{d}\vec{\ell}$$

where P is the boundary of S

Delta Functions:

$$\int \varphi(x)\delta(x-x')\,\mathrm{d}x = \varphi(x')\,, \qquad \int \varphi(\vec r)\delta^3(\vec r-\vec r\,')\,\mathrm{d}^3x = \varphi(\vec r\,')$$

$$\int \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x')\,\mathrm{d}x = -\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=x'}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i)=0$$

$$\vec\nabla\cdot\left(\frac{\vec r-\vec r\,'}{|\vec r-\vec r\,'|^3}\right) = -\nabla^2\frac{1}{|\vec r-\vec r\,'|} = 4\pi\delta^3(\vec r-\vec r\,')$$

$$\partial_i\left(\frac{\hat r_j}{r^2}\right) \equiv \partial_i\left(\frac{x_j}{r^3}\right) = -\partial_i\partial_j\left(\frac{1}{r}\right) = \frac{\delta_{ij}-3\hat r_i\hat r_j}{r^3} + \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\vec r)$$

$$\vec\nabla\cdot\frac{3(\vec d\cdot\hat r)\hat r-\vec d}{r^3} = -\frac{8\pi}{3}(\vec d\cdot\vec\nabla)\delta^3(\vec r)$$

$$\vec\nabla\times\frac{3(\vec d\cdot\hat r)\hat r-\vec d}{r^3} = -\frac{4\pi}{3}\vec d\times\vec\nabla\delta^3(\vec r)$$

Electrostatics:

$\vec F = q\vec E$, where

$$\vec E(\vec r) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{(\vec r-\vec r\,')\,q_i}{|\vec r-\vec r\,'|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{(\vec r-\vec r\,')}{|\vec r-\vec r\,'|^3}\,\rho(\vec r\,')\,\mathrm{d}^3x'$$

ϵ_0 =permittivity of free space $= 8.854\times10^{-12}\ \mathrm{C^2/(N{\cdot}m^2)}$

$$\frac{1}{4\pi\epsilon_0} = 8.988\times10^{9}\ \mathrm{N{\cdot}m^2/C^2}$$

$$V(\vec r) = V(\vec r_0) - \int_{\vec r_0}^{\vec r}\vec E(\vec r\,')\cdot\mathrm{d}\vec\ell\,' = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec r\,')}{|\vec r-\vec r\,'|}\mathrm{d}^3x'$$

$$\vec\nabla\cdot\vec E = \frac{\rho}{\epsilon_0}\,, \qquad \vec\nabla\times\vec E = 0\,, \qquad \vec E = -\vec\nabla V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}\ \text{(Poisson's Eq.)}\,, \qquad \rho = 0 \implies \nabla^2 V = 0\ \text{(Laplace's Eq.)}$$

Laplacian Mean Value Theorem (no generally accepted name): If $\nabla^2 V = 0$, then the average value of V on a spherical surface equals its value at the center.

Energy:

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{\substack{ij\\ i\neq j}}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int\mathrm{d}^3x\,\mathrm{d}^3x'\,\frac{\rho(\vec r)\rho(\vec r\,')}{|\vec r-\vec r\,'|}$$

$$W = \frac{1}{2}\int\mathrm{d}^3x\,\rho(\vec r)V(\vec r) = \frac{1}{2}\epsilon_0\int\left|\vec E\right|^2\mathrm{d}^3x$$

Conductors:

Just outside, $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Pressure on surface: $\frac{1}{2}\sigma|\vec{E}|_{\text{outside}}$

Two-conductor system with charges Q and $-Q$: $Q = CV$, $W = \frac{1}{2}CV^2$

N isolated conductors:

$$V_i = \sum_j P_{ij} Q_j \,, \qquad P_{ij} = \text{elastance matrix, or reciprocal capacitance matrix}$$

$$Q_i = \sum_j C_{ij} V_j \,, \qquad C_{ij} = \text{capacitance matrix}$$

Image charge in sphere of radius a: Image of Q at R is $q = -\dfrac{a}{R}Q$, $r = \dfrac{a^2}{R}$

Separation of Variables for Laplace's Equation in Cartesian Coordinates:

$$V = \begin{Bmatrix} \cos\alpha x \\ \sin\alpha x \end{Bmatrix} \begin{Bmatrix} \cos\beta y \\ \sin\beta y \end{Bmatrix} \begin{Bmatrix} \cosh\gamma z \\ \sinh\gamma z \end{Bmatrix} \qquad \text{where } \gamma^2 = \alpha^2 + \beta^2$$

Separation of Variables for Laplace's Equation in Spherical Coordinates:

Traceless Symmetric Tensor expansion:

$$\nabla^2\varphi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\nabla^2_\theta\,\varphi = 0 \ ,$$

where the angular part is given by

$$\nabla^2_\theta\,\varphi \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\nabla^2_\theta\, C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} = -\ell(\ell+1)C^{(\ell)}_{i_1 i_2 \ldots i_\ell}\hat{n}_{i_1}\hat{n}_{i_2}\ldots\hat{n}_{i_\ell} \ ,$$

where $C^{(\ell)}_{i_1 i_2 \ldots i_\ell}$ is a symmetric traceless tensor and

$\hat{n} = \sin\theta\cos\phi\,\hat{e}_1 + \sin\theta\sin\phi\,\hat{e}_2 + \cos\theta\,\hat{e}_3$.

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty}\left(C^{(\ell)}_{i_1 i_2 \ldots i_\ell} r^\ell + \frac{C'^{(\ell)}_{i_1 i_2 \ldots i_\ell}}{r^{\ell+1}}\right)\hat{r}_{i_1}\hat{r}_{i_2}\ldots\hat{r}_{i_\ell} \ , \quad \text{where } \vec{r} = r\hat{r}$$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) \{ \, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \, \} \, \hat{r}_{i_1} \ldots \hat{r}_{i_\ell}$$

where $\{ \ldots \}$ denotes the traceless symmetric part of $\ldots$.

Special cases:

$$\{ \, 1 \, \} = 1$$

$$\{ \, \hat{z}_i \, \} = \hat{z}_i$$

$$\{ \, \hat{z}_i \hat{z}_j \, \} = \hat{z}_i \hat{z}_j - \tfrac{1}{3} \delta_{ij}$$

$$\{ \, \hat{z}_i \hat{z}_j \hat{z}_k \, \} = \hat{z}_i \hat{z}_j \hat{z}_k - \tfrac{1}{5} \big(\hat{z}_i \delta_{jk} + \hat{z}_j \delta_{ik} + \hat{z}_k \delta_{ij} \big)$$

$$\begin{aligned} \{ \, \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m \, \} = {} & \hat{z}_i \hat{z}_j \hat{z}_k \hat{z}_m - \tfrac{1}{7} \big(\hat{z}_i \hat{z}_j \delta_{km} + \hat{z}_i \hat{z}_k \delta_{mj} + \hat{z}_i \hat{z}_m \delta_{jk} + \hat{z}_j \hat{z}_k \delta_{im} \\ & + \hat{z}_j \hat{z}_m \delta_{ik} + \hat{z}_k \hat{z}_m \delta_{ij} \big) + \tfrac{1}{35} \big(\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk} \big) \end{aligned}$$

Legendre Polynomial / Spherical Harmonic expansion:

General solution to Laplace's equation:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} \, r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)$$

Orthonormality: $\displaystyle \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \, Y^*_{\ell' m'}(\theta, \phi) \, Y_{\ell m}(\theta, \phi) = \delta_{\ell' \ell} \delta_{m' m}$

Azimuthal Symmetry:

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Electric Multipole Expansion:

First several terms:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\, \frac{Q}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{1}{2} \frac{\hat{r}_i \hat{r}_j}{r^3} \, Q_{ij} + \cdots \right] \quad \text{, where}$$

$$Q = \int \mathrm{d}^3 x \, \rho(\vec{r}) \, , \quad p_i = \int d^3 x \, \rho(\vec{r}) \, x_i \quad Q_{ij} = \int d^3 x \, \rho(\vec{r}) (3 x_i x_j - \delta_{ij} |\vec{r}|^2) \, ,$$

$$\vec{E}_{\text{dip}}(\vec{r}) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{\vec{p} \cdot \hat{r}}{r^2} \right) = \frac{1}{4\pi\epsilon_0} \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} - \frac{1}{3\epsilon_0} p_i \delta^3(\vec{r})$$

$$\vec{\nabla} \times \vec{E}_{\text{dip}}(\vec{r}) = 0 \, , \qquad \vec{\nabla} \cdot \vec{E}_{\text{dip}}(\vec{r}) = \frac{1}{\epsilon_0} \rho_{\text{dip}}(\vec{r}) = -\frac{1}{\epsilon_0} \vec{p} \cdot \vec{\nabla} \delta^3(\vec{r})$$

Traceless Symmetric Tensor version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} C^{(\ell)}_{i_1 \ldots i_\ell} \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \ ,$$

where

$$C^{(\ell)}_{i_1 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \rho(\vec{r}) \{ x_{i_1} \ldots x_{i_\ell} \} \, \mathrm{d}^3x \qquad (\vec{r} \equiv r\hat{r} \equiv x_i \hat{e}_i)$$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{r'^{\ell}}{r^{\ell+1}} \{ \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \} \hat{r}'_{i_1} \ldots \hat{r}'_{i_\ell} \ , \qquad \text{for } r' < r$$

$$(2\ell-1)!! \equiv (2\ell-1)(2\ell-3)(2\ell-5)\ldots 1 = \frac{(2\ell)!}{2^\ell \ell!} \ , \ \text{with} \ \ (-1)!! \equiv 1 \ .$$

Reminder: $\{ \ldots \}$ denotes the traceless symmetric part of $\ldots$.

Griffiths version:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int r'^{\ell} \rho(\vec{r}\,') P_\ell(\cos\theta') \, \mathrm{d}^3x$$

where θ' = angle between $\vec{r}$ and $\vec{r}\,'$.

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta') \, , \qquad \frac{1}{\sqrt{1 - 2\lambda x + \lambda^2}} = \sum_{\ell=0}^{\infty} \lambda^\ell P_\ell(x)$$

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^\ell (x^2 - 1)^\ell \, , \qquad \text{(Rodrigues' formula)}$$

$$P_\ell(1) = 1 \qquad P_\ell(-x) = (-1)^\ell P_\ell(x) \qquad \int_{-1}^{1} \mathrm{d}x \, P_{\ell'}(x) P_\ell(x) = \frac{2}{2\ell+1} \delta_{\ell'\ell}$$

Spherical Harmonic version:***

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{q_{\ell m}}{r^{\ell+1}} Y_{\ell m}(\theta, \phi)$$

where $q_{\ell m} = \int Y^*_{\ell m} r'^{\ell} \rho(\vec{r}\,') \, \mathrm{d}^3x'$

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r'^{\ell}}{r^{\ell+1}} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) \ , \qquad \text{for } r' < r$$

Electric Fields in Matter:

Electric Dipoles:

$\vec{p} = \int d^3x\, \rho(\vec{r})\, \vec{r}$

$\rho_{\text{dip}}(\vec{r}) = -\vec{p} \cdot \vec{\nabla}_{\vec{r}}\, \delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}(\vec{p} \cdot \vec{E})$ (force on a dipole)

$\vec{\tau} = \vec{p} \times \vec{E}$ (torque on a dipole)

$U = -\vec{p} \cdot \vec{E}$

Electrically Polarizable Materials:

$\vec{P}(\vec{r})$ = polarization = electric dipole moment per unit volume

$\rho_{\text{bound}} = -\nabla \cdot \vec{P}$, $\sigma_{\text{bound}} = \vec{P} \cdot \hat{n}$

$\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$, $\vec{\nabla} \cdot \vec{D} = \rho_{\text{free}}$, $\vec{\nabla} \times \vec{E} = 0$ (for statics)

Boundary conditions:

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_{\text{free}}$$

$$\vec{E}^{\parallel}_{\text{above}} - \vec{E}^{\parallel}_{\text{below}} = 0 \qquad \vec{D}^{\parallel}_{\text{above}} - \vec{D}^{\parallel}_{\text{below}} = \vec{P}^{\parallel}_{\text{above}} - \vec{P}^{\parallel}_{\text{below}}$$

Linear Dielectrics:

$\vec{P} = \epsilon_0 \chi_e \vec{E}$, χ_e = electric susceptibility

$\epsilon \equiv \epsilon_0(1 + \chi_e)$ = permittivity, $\vec{D} = \epsilon \vec{E}$

$\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1 + \chi_e$ = relative permittivity, or dielectric constant

Clausius-Mossotti equation: $\chi_e = \dfrac{N\alpha/\epsilon_0}{1 - \frac{N\alpha}{3\epsilon_0}}$, where N = number density of atoms or (nonpolar) molecules, α = atomic/molecular polarizability ($\vec{P} = \alpha \vec{E}$)

Energy: $W = \dfrac{1}{2} \displaystyle\int \vec{D} \cdot \vec{E}\, \mathrm{d}^3x$ (linear materials only)

Force on a dielectric: $\vec{F} = -\vec{\nabla} W$ (Even if one or more potential differences are held fixed, the force can be found by computing the gradient with the total charge on each conductor fixed.)

Magnetostatics:

Magnetic Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} , \qquad \text{where } \vec{p} = \gamma m_0 \vec{v} , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{F} = \int I \mathrm{d}\vec{\ell} \times \vec{B} = \int \vec{J} \times \vec{B} \, \mathrm{d}^3x$$

Current Density:

Current through a surface S: $I_S = \int_S \vec{J} \cdot \mathrm{d}\vec{a}$

Charge conservation: $\dfrac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$

Moving density of charge: $\vec{J} = \rho \vec{v}$

Biot-Savart Law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\vec{\ell}' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, \mathrm{d}a'$$

$$= \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \, \mathrm{d}^3x$$

where μ_0 = permeability of free space $\equiv 4\pi \times 10^{-7}$ N/A^2

Examples:

Infinitely long straight wire: $\vec{B} = \dfrac{\mu_0 I}{2\pi r} \hat{\phi}$

Infinitely long tightly wound solenoid: $\vec{B} = \mu_0 n I_0 \, \hat{z}$, where n = turns per unit length

Loop of current on axis: $\vec{B}(0, 0, z) = \dfrac{\mu_0 I R^2}{2(z^2 + R^2)^{3/2}} \hat{z}$

Infinite current sheet: $\vec{B}(\vec{r}) = \dfrac{1}{2} \mu_0 \vec{K} \times \hat{n}$, $\hat{n}$ = unit normal toward $\vec{r}$

Vector Potential:

$$\vec{A}(\vec{r})_{\text{coul}} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, \mathrm{d}^3x' \,, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \,, \qquad \vec{\nabla} \cdot \vec{A}_{\text{coul}} = 0$$

$\vec{\nabla} \cdot \vec{B} = 0$ (Subject to modification if magnetic monopoles are discovered)

Gauge Transformations: $\vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\Lambda(\vec{r})$ for any $\Lambda(\vec{r})$. $\vec{B} = \vec{\nabla} \times \vec{A}$ is unchanged.

Ampère's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \,, \quad \text{or equivalently} \quad \int_P \vec{B} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}$$

Magnetic Multipole Expansion:

Traceless Symmetric Tensor version:

$$A_j(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{\ell=0}^{\infty} \mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} \frac{\{\, \hat{r}_{i_1} \ldots \hat{r}_{i_\ell} \,\}}{r^{\ell+1}}$$

$$\text{where } \mathcal{M}^{(\ell)}_{j;i_1 i_2 \ldots i_\ell} = \frac{(2\ell-1)!!}{\ell!} \int \mathrm{d}^3x J_j(\vec{r}) \{\, x_{i_1} \ldots x_{i_\ell} \,\}$$

Current conservation restriction: $\displaystyle\int \mathrm{d}^3x \operatorname*{Sym}_{i_1 \ldots i_\ell}(x_{i_1} \ldots x_{i_{\ell-1}} J_{i_\ell}) = 0$

where $\displaystyle\operatorname*{Sym}_{i_1 \ldots i_\ell}$ means to symmetrize — i.e. average over all orderings — in the indices $i_1 \ldots i_\ell$

Special cases:

$\ell = 1$: $\displaystyle\int \mathrm{d}^3x\, J_i = 0$

$\ell = 2$: $\displaystyle\int \mathrm{d}^3x\, (J_i x_j + J_j x_i) = 0$

Leading term (dipole): $\displaystyle\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$,

where

$$m_i = -\frac{1}{2} \epsilon_{ijk} \mathcal{M}^{(1)}_{j;k}$$

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2} \int \mathrm{d}^3x\, \vec{r} \times \vec{J} = I\vec{a} ,$$

where $\displaystyle\vec{a} = \int_S \mathrm{d}\vec{a}$ for any surface S spanning P

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} + \frac{2\mu_0}{3} \vec{m}\, \delta^3(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}_{\text{dip}}(\vec{r}) = 0 , \qquad \vec{\nabla} \times \vec{B}_{\text{dip}}(\vec{r}) = \mu_0 \vec{J}_{\text{dip}}(\vec{r}) = -\mu_0 \vec{m} \times \vec{\nabla} \delta^3(\vec{r})$$

Griffiths version:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \oint (r')^\ell P_\ell(\cos\theta') \mathrm{d}\vec{\ell}'$$

Magnetic Fields in Matter:

Magnetic Dipoles:

$$\vec{m} = \frac{1}{2} I \int_P \vec{r} \times \mathrm{d}\vec{\ell} = \frac{1}{2} \int \mathrm{d}^3x\, \vec{r} \times \vec{J} = I\vec{a}$$

$\vec{J}_{\rm dip}(\vec{r}) = -\vec{m} \times \vec{\nabla}_{\vec{r}}\delta^3(\vec{r} - \vec{r}_d)$, where $\vec{r}_d$ = position of dipole

$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$ (force on a dipole)

$\vec{\tau} = \vec{m} \times \vec{B}$ (torque on a dipole)

$U = -\vec{m} \cdot \vec{B}$

Magnetically Polarizable Materials:

$\vec{M}(\vec{r})$ = magnetization = magnetic dipole moment per unit volume

$\vec{J}_{\rm bound} = \vec{\nabla} \times \vec{M}$, $\vec{K}_{\rm bound} = \vec{M} \times \hat{n}$

$\vec{H} \equiv \dfrac{1}{\mu_0}\vec{B} - \vec{M}$, $\vec{\nabla} \times \vec{H} = \vec{J}_{\rm free}$, $\vec{\nabla} \cdot \vec{B} = 0$

Boundary conditions:

$$B^\perp_{\rm above} - B^\perp_{\rm below} = 0 \qquad H^\perp_{\rm above} - H^\perp_{\rm below} = -(M^\perp_{\rm above} - M^\perp_{\rm below})$$
$$\vec{B}^\parallel_{\rm above} - \vec{B}^\parallel_{\rm below} = \mu_0(\vec{K} \times \hat{n}) \qquad \vec{H}^\parallel_{\rm above} - \vec{H}^\parallel_{\rm below} = \vec{K}_{\rm free} \times \hat{n}$$

Linear Magnetic Materials:

$\vec{M} = \chi_m \vec{H}$, χ_m = magnetic susceptibility

$\mu = \mu_0(1 + \chi_m)$ = permeability, $\vec{B} = \mu\vec{H}$

Magnetic Monopoles:

$\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi}\dfrac{q_m}{r^2}\hat{r}$; Force on a static monopole: $\vec{F} = q_m\vec{B}$

Angular momentum of monopole/charge system: $\vec{L} = \dfrac{\mu_0 q_e q_m}{4\pi}\hat{r}$, where $\hat{r}$ points from q_e to q_m

Dirac quantization condition: $\dfrac{\mu_0 q_e q_m}{4\pi} = \dfrac{1}{2}\hbar \times$ integer

Connection Between Traceless Symmetric Tensors and Legendre Polynomials or Spherical Harmonics:

$P_\ell(\cos\theta) = \dfrac{(2\ell)!}{2^\ell(\ell!)^2}\{\, \hat{z}_{i_1} \ldots \hat{z}_{i_\ell} \,\}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$

For $m \geq 0$,

$Y_{\ell m}(\theta, \phi) = C^{(\ell,m)}_{i_1 \ldots i_\ell}\, \hat{n}_{i_1} \ldots \hat{n}_{i_\ell}$,

where $C^{(\ell,m)}_{i_1 i_2 \ldots i_\ell} = d_{\ell m}\, \{\, \hat{u}^+_{i_1} \ldots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \ldots \hat{z}_{i_\ell} \,\}$,

with $d_{\ell m} = \dfrac{(-1)^m(2\ell)!}{2^\ell \ell!}\sqrt{\dfrac{2^m\,(2\ell+1)}{4\pi\,(\ell+m)!\,(\ell-m)!}}$,

and $\hat{u}^+ = \dfrac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$

Form $m < 0$, $Y_{\ell,-m}(\theta, \phi) = (-1)^m Y^*_{\ell m}(\theta, \phi)$

Maxwell's Equations:

$$\text{(i)}\quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(iii)}\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\ ,$$

$$\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(iv)}\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

where $\mu_0\epsilon_0 = \dfrac{1}{c^2}$

Lorentz force law: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

Charge conservation: $\dfrac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$

Maxwell's Equations in Matter:

Polarization $\vec{P}$ and magnetization $\vec{M}$:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}\ , \qquad \vec{J}_b = \vec{\nabla}\times\vec{M}\ , \qquad \rho = \rho_f + \rho_b\ , \qquad \vec{J} = \vec{J}_f + \vec{J}_b$$

Auxiliary Fields:

$$\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}\ , \qquad \vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$$

Maxwell's Equations:

$$\text{(i)}\quad \vec{\nabla}\cdot\vec{D} = \rho_f \qquad \text{(iii)}\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\ ,$$

$$\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(iv)}\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

For linear media:

$$\vec{D} = \epsilon\vec{E}\ , \qquad \vec{H} = \frac{1}{\mu}\vec{B}$$

where ϵ = dielectric constant, μ = relative permeability

$$\vec{J}_d \equiv \frac{\partial\vec{D}}{\partial t} = \text{displacement current}$$

Maxwell's Equations with Magnetic Charge:

$$\text{(i)}\quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho_e \qquad \text{(iii)}\vec{\nabla}\times\vec{E} = -\mu_0\vec{J}_m - \frac{\partial\vec{B}}{\partial t}\ ,$$

$$\text{(ii)}\quad \vec{\nabla}\cdot\vec{B} = \mu_0\rho_m \qquad \text{(iv)}\vec{\nabla}\times\vec{B} = \mu_0\vec{J}_e + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Magnetic Lorentz force law: $\vec{F} = q_m\left(\vec{B} - \dfrac{1}{c^2}\vec{v}\times\vec{E}\right)$

Current, Resistance, and Ohm's Law:

$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$, where σ = conductivity. $\rho = 1/\sigma$ = resistivity

Resistors: $V = IR$, $\quad P = IV = I^2R = V^2/R$

Resistance in a wire: $R = \dfrac{\ell}{A}\rho$, where ℓ = length, A = cross-sectional area, and ρ = resistivity

Charging an RC circuit: $I = \dfrac{V_0}{R} e^{-t/RC}$, $\quad Q = CV_0 \left[1 - e^{-t/RC}\right]$

EMF (Electromotive force): $\mathcal{E} \equiv \oint (\vec{E} + \vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{\ell}$, where $\vec{v}$ is either the velocity of the wire or the velocity of the charge carriers (the difference points along the wire, and gives no contribution)

Inductance:

Universal flux rule: Whenever the flux through a loop changes, whether due to a changing $\vec{B}$ or motion of the loop, $\mathcal{E} = -\dfrac{\mathrm{d}\Phi_B}{\mathrm{d}t}$, where Φ_B is the magnetic flux through the loop

Mutual inductance: $\Phi_2 = M_{21} I_1$, M_{21} = mutual inductance

(Franz) Neumann's formula: $M_{21} = M_{12} = \dfrac{\mu_0}{4\pi} \oint_{P_1} \oint_{P_2} \dfrac{\mathrm{d}\vec{\ell}_1 \cdot \mathrm{d}\vec{\ell}_2}{|\vec{r}_1 - \vec{r}_2|}$

Self inductance: $\Phi = LI$, $\quad \mathcal{E} = -L\dfrac{\mathrm{d}I}{\mathrm{d}t}$; $\quad L$ = inductance

Self inductance of a solenoid: $L = n^2 \mu_0 \mathcal{V}$, where n = number of turns per length, $\mathcal{V}$ = volume

Rising current in an RL circuit: $I = \dfrac{V_0}{R}\left[1 - e^{\frac{R}{L}t}\right]$

Boundary Conditions:

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$
$$E_1^{\perp} - E_2^{\perp} = \frac{1}{\epsilon_0}\sigma \qquad \vec{D}_1^{\parallel} - \vec{D}_2^{\parallel} = \vec{P}_1^{\parallel} - \vec{P}_2^{\parallel}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = -\hat{n} \times \vec{K}_f$$
$$H_1^{\perp} - H_2^{\perp} = M_2^{\perp} - M_1^{\perp} \qquad \vec{B}_1^{\parallel} - \vec{B}_2^{\parallel} = -\mu_0 \hat{n} \times \vec{K}$$

Conservation Laws:

Energy density: $u_{\rm EM} = \frac{1}{2}\left[\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right]$

Poynting vector (flow of energy): $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

Conservation of energy:

Integral form: $\frac{\mathrm{d}}{\mathrm{d}t}\left[U_{\rm EM} + U_{\rm mech}\right] = -\int \vec{S}\cdot \mathrm{d}\vec{a}$

Differential form: $\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$, where $u = u_{\rm EM} + u_{\rm mech}$

Momentum density: $\vec{\wp}_{\rm EM} = \frac{1}{c^2}\vec{S}$; $\frac{1}{c^2}S_i$ is the density of momentum in the i'th direction

Maxwell stress tensor: $T_{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2\right)$

where $-T_{ij} = -T_{ji} =$ flow in j'th direction of momentum in the i'th direction

Conservation of momentum:

Integral form: $\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{{\rm mech},i} + \frac{1}{c^2}\int_{\mathcal{V}} S_i\,\mathrm{d}^3x\right) = \oint_S T_{ij}\,\mathrm{d}a_j$, for a volume $\mathcal{V}$ bounded by a surface S

Differential form: $\frac{\partial}{\partial t}\left(\wp_{{\rm mech},i} + \wp_{{\rm EM},i}\right) = \partial_j T_{ji}$

Angular momentum:

Angular momentum density (about the origin): $\vec{\ell}_{\rm EM} = \vec{r}\times\vec{\wp}_{\rm EM} = \epsilon_0[\vec{r}\times(\vec{E}\times\vec{B})]$

Wave Equation in 1 Dimension:

$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$, where v is the wave velocity

Sinusoidal waves:

$f(z,t) = A\cos\left[k(z-vt)+\delta\right] = A\cos\left[kz-\omega t+\delta\right]$

where

$\omega =$ angular frequency $= 2\pi\nu$	$\nu =$ frequency
$v = \frac{\omega}{k} =$ phase velocity	$\delta =$ phase (or phase constant)
$k =$ wave number	$\lambda = 2\pi/k =$ wavelength
$T = 2\pi/\omega =$ period	$A =$ amplitude

Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$

Complex notation: $f(z,t) = \mathrm{Re}[\tilde{A}e^{i(kz-\omega t)}]$, where $\tilde{A} = Ae^{i\delta}$; "Re" is usually dropped.

Wave velocities: $v = \frac{\omega}{k} =$ phase velocity; $v_{\rm group} = \frac{\mathrm{d}\omega}{\mathrm{d}k} =$ group velocity

Electromagnetic Waves:

Wave Equations: $\nabla^2 \vec{E} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} = 0\ , \quad \nabla^2 \vec{B} - \dfrac{1}{c^2}\dfrac{\partial^2 \vec{B}}{\partial t^2} = 0$

Linearly Polarized Plane Waves:

$\vec{E}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n}\ ,$ where $\tilde{E}_0$ is a complex amplitude, $\hat{n}$ is a unit vector, and $\omega/|\vec{k}| = v_{\text{phase}} = c$.

$\hat{n}\cdot\vec{k} = 0$ (transverse wave)

$$\vec{B} = \frac{1}{c}\,\hat{\boldsymbol{k}} \times \vec{E}$$

Energy and Momentum:

$$u = \epsilon_0 E_0^2 \underbrace{\cos^2(kz - \omega t + \delta)}_{\text{averages to } 1/2}\ , \quad (\vec{k} = k\,\hat{z})$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = uc\,\hat{z}\ , \qquad I \text{ (intensity)} = \left\langle |\vec{S}| \right\rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\vec{\wp}_{\text{EM}} = \frac{1}{c^2}\,\vec{S} = \frac{u}{c}\,\hat{z}$$

Electromagnetic Waves in Matter:

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \text{index of refraction}$$

$$v = \text{phase velocity} = \frac{c}{n}$$

$$u = \frac{1}{2}\left[\epsilon|\vec{E}|^2 + \frac{1}{\mu}|\vec{B}|^2\right]$$

$$\vec{B} = \frac{n}{c}\,\hat{\boldsymbol{k}} \times \vec{E}$$

$$\vec{S} = \frac{1}{\mu}\vec{E}\times\vec{B} = \frac{uc}{n}\,\hat{z}$$

Reflection and Transmission at Normal Incidence:

Boundary conditions:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\ ,$$

$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}\ .$$

Incident wave ($z < 0$):

$$\vec{E}_I(z,t) = \tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\,\hat{e}_x$$

$$\vec{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0,I}\, e^{i(k_1 z - \omega t)}\,\hat{e}_y\ .$$

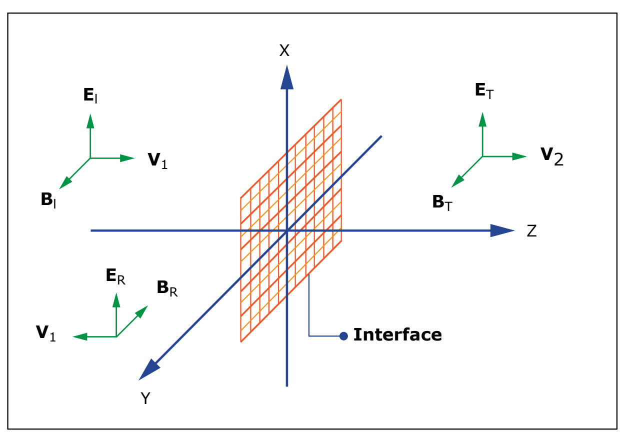

Image by MIT OpenCourseWare.

Transmitted wave ($z > 0$):

$$\vec{E}_T(z,t) = \tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{e}_x$$

$$\vec{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0,T}\, e^{i(k_2 z - \omega t)}\, \hat{e}_y \ .$$

Reflected wave ($z < 0$):

$$\vec{E}_R(z,t) = \tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{e}_x$$

$$\vec{B}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0,R}\, e^{i(-k_1 z - \omega t)}\, \hat{e}_y \ .$$

ω must be the same on both sides, so

$$\frac{\omega}{k_1} = v_1 = \frac{c}{n_1} \ , \qquad \frac{\omega}{k_2} = v_2 = \frac{c}{n_2}$$

Applying boundary conditions and solving, approximating $\mu_1 = \mu_2 = \mu_0$,

$$\tilde{E}_{0,R} = \frac{n_1 - n_2}{n_1 + n_2}\,\tilde{E}_{0,I} \qquad E_{0,T} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{0,I}$$

Electromagnetic Potentials:

The fields: $\vec{B} = \vec{\nabla} \times \vec{A}$, $\quad \vec{E} = -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t}$

Gauge transformations: $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$, $\quad V' = V - \dfrac{\partial \Lambda}{\partial t}$

Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0 \quad \Longrightarrow \quad \nabla^2 V = -\dfrac{1}{\epsilon_0}\rho$ (but $\vec{A}$ is complicated)

Lorentz gauge: $\vec{\nabla} \cdot \vec{A} = -\dfrac{1}{c^2}\dfrac{\partial V}{\partial t} \quad \Longrightarrow$

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \ , \qquad \Box^2 \vec{A} = -\mu_0 \vec{J} \ , \qquad \text{where } \Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$\Box^2$ = D'Alembertian

Retarded time solutions (Lorentz gauge):

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\, \frac{\rho(\vec{r}\,',t_r)}{|\vec{r} - \vec{r}\,'|} \ , \qquad \vec{A}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\, \frac{\vec{J}(\vec{r}\,',t_r)}{|\vec{r} - \vec{r}\,'|}$$

where

$$t_r = t - \frac{|\vec{r} - \vec{r}\,'|}{c}$$

Liénard-Wiechert Potentials (potentials of a point charge):

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\hat{\boldsymbol{\mathscr{r}}}\right)}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{q\vec{v}_p}{|\vec{r}-\vec{r}_p|\left(1-\frac{\vec{v}_p}{c}\cdot\hat{\boldsymbol{\mathscr{r}}}\right)} = \frac{\vec{v}_p}{c^2}V(\vec{r},t)$$

where $\vec{r}_p$ and $\vec{v}_p$ are the position and velocity of the particle at the retarded time t_r, and

$$\vec{\boldsymbol{\mathscr{r}}} = \vec{r}-\vec{r}_p\ , \qquad \boldsymbol{\mathscr{r}} = |\vec{r}-\vec{r}_p|\ , \qquad \hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{r}-\vec{r}_p}{|\vec{r}-\vec{r}_p|}$$

Fields of a point charge (from the Liénard-Wiechert potentials):

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\vec{r}-\vec{r}_p|}{\left(\vec{u}\cdot(\vec{r}-\vec{r}_p)\right)^3}\left[(c^2-v_p^2)\vec{u}+(\vec{r}-\vec{r}_p)\times(\vec{u}\times\vec{a}_p)\right]$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{\boldsymbol{\mathscr{r}}}\times\vec{E}(\vec{r},t)$$

where $\vec{u} = c\,\hat{\boldsymbol{\mathscr{r}}} - \vec{v}_p$

Radiation:

Radiation from an oscillating electric dipole along the z axis:

$$p(t) = p_0\cos(\omega t)\ , \quad p_0 = q_0 d$$

Approximations: $d \ll \lambda \ll r$,

$$V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\,\hat{z}$$

$$\vec{E} = -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\theta}\ , \qquad \vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{r}\times\vec{E}(\vec{r},t)$$

Poynting vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2\hat{r}$

Intensity: $I = \left\langle\vec{S}\right\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{r}\ , \qquad$ using $\left\langle\cos^2\right\rangle = \frac{1}{2}$

Total power: $\langle P\rangle = \int\left\langle\vec{S}\right\rangle\cdot d\vec{a} = \frac{\mu_0 p_0^2\omega^4}{12\pi c}$

Magnetic Dipole Radiation:

Dipole moment: $\vec{m}(t) = m_0 \cos(\omega t)\,\hat{z}$, at the origin

$$\vec{E} = -\frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\phi}\ , \qquad \vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{r}\times\vec{E}(\vec{r},t)$$

Compared to the electric dipole radiation, $p_0 \to \dfrac{m_0}{c}$, $\qquad \hat{\theta} \to -\hat{\phi}$

General Electric Dipole Radiation:

$$\vec{E}(\vec{r},t) = \frac{\mu_0}{4\pi r}[(\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}\,]\ , \qquad \vec{B}(\vec{r},t) = \frac{1}{c}\,\hat{r}\times\vec{E}(\vec{r},t) = -\frac{\mu_0}{4\pi r c}[\hat{r}\times\ddot{\vec{p}}]$$

Multipole Expansion for Radiation:

The electric dipole radiation formula is really the first term in a doubly infinite series. There is electric dipole, quadrupole, ... radiation, and also magnetic dipole, quadrupole, ... radiation.

Radiation from a Point Particle:

When the particle is at rest at the retarded time,

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2|\vec{r}-\vec{r}\,'|}[\ \hat{\boldsymbol{\imath}}\times(\ \hat{\boldsymbol{\imath}}\times\vec{a}_p)]$$

Poynting vector: $\vec{S}_{\text{rad}} = \dfrac{1}{\mu_0 c}|\vec{E}_{\text{rad}}|^2\ \hat{\boldsymbol{\imath}} = \dfrac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\dfrac{\sin^2\theta}{\boldsymbol{\imath}^2}\right)\ \hat{\boldsymbol{\imath}}$

where θ is the angle between $\vec{a}_p$ and $\hat{\boldsymbol{\imath}}$.

Total power (Larmor formula): $P = \dfrac{\mu_0 q^2 a^2}{6\pi c}$

(valid for $\vec{v}_p = 0$ or $|\vec{v}_p| \ll c$)

Liénard's Generalization if $\vec{v}_p \neq 0$:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right) = \underbrace{\frac{\mu_0 q^2}{6\pi m_0^2 c}\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau}}_{\text{For relativists only}}$$

Radiation Reaction:

Abraham-Lorentz formula:

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\vec{a}}$$

The Abraham-Lorentz formula is guaranteed to give the correct average energy loss for periodic or nearly periodic motion, but one would like a formula that works under general circumstances. The Abraham-Lorentz formula leads to runaway solutions which are clearly unphysical. The problem of radiation reaction for point particles in classical electrodynamics apparently remains unsolved.

More Information about Spherical Harmonics:***

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ \ +1}{4\pi}\frac{(\ \ -m)!}{(\ \ +m)!}}\, P_\ell^m(\cos\theta)e^{im\phi}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, which can be defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell\ \ !}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

Legendre Polynomials:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

SPHERICAL HARMONICS $Y_{lm}(\theta, \varphi)$

l = 0

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

l = 1

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

l = 2

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}$$
$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\varphi}$$
$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

l = 3

$$Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta e^{3i\varphi}$$
$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta e^{2i\varphi}$$
$$Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta(5\cos^2\theta - 1)e^{i\varphi}$$
$$Y_{30} = \sqrt{\frac{7}{4\pi}}\left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right)$$

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Vector Identities:

Triple Products

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Products Rules

$$\nabla (f g) = f (\nabla g) + g (\nabla f)$$

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

$$\nabla (f \mathbf{A}) = f (\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

$$\nabla (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \times (f \mathbf{A}) = f (\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$$

$$\nabla (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} (\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$$

Second Derivatives

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$