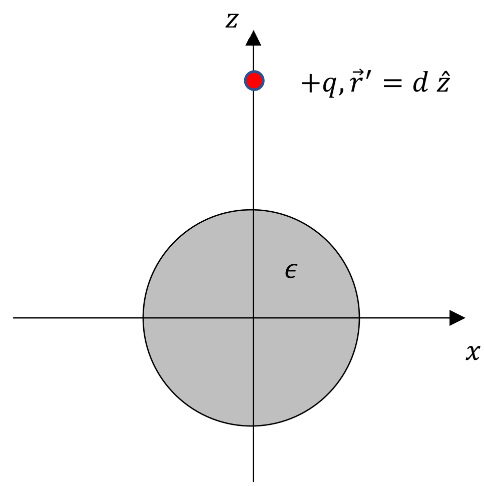
E & M I
Homework 5: Dielectric and Magnetic Materials
And Qualifiers…

Problem 1) Dielectric Sphere and a charge:



This is a workshop problem the only one group got very far in solving. It's worthwhile to do this, as it's a good example of how to deal with the boundary conditions for a dielectric.

Consider an insulating sphere of radius R and dielectric constant ϵ and a point charge $+q$ on the z-axis with a position $\vec{r}\,' = d\,\hat{z}$.

Due to the spherical boundary conditions and the azimuthal symmetry of the problem, we know we can use an expansion in Legendre Polynomials. For any azimuthally symmetric function satisfying Laplace's Equation:

$$F(r,\theta) = \sum_l \left(a_l\, r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

And for the point charge:

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\theta)$$

Where $r_< = r, \text{for } r < r'$, $r_< = r', \text{for } r > r'$, $r_> = r, \text{for } r > r'$, and $r_> = r', \text{for } r < r'$.

A) Define the potentials:

$\phi_{SI}(r,\theta)$ = The potential due to the sphere for $r < R$ (inside the sphere).
$\phi_{SO}(r,\theta)$ = The potential due to the sphere for $r > R$ (outside the sphere).
$\phi_{qI}(r,\theta)$ = The potential due to the charge for $r < d$.
$\phi_{qII}(r,\theta)$ = The potential due to the charge for $r > d$.

Write down Legendre polynomial expansions for the total potential $\phi_T = \phi_S + \phi_q$ (sum of the sphere potential and potential of q) both inside and outside the sphere, for

$r < d$. Do not include terms that must be zero for ϕ_T to remain finite everywhere.

B) Using the fact that ϕ_T is continuous at the surface of the sphere, find a relation between the coefficients a_l and b_l in the Legendre expansions ϕ_{SI} and ϕ_{SO}.

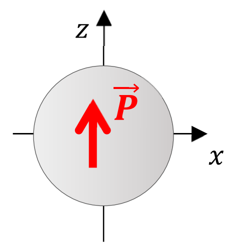
C) Using the boundary condition that $\hat{r} \cdot \vec{D}$ is continuous at the surface of the sphere (and that $\vec{D} = \epsilon\,\vec{E} = -\epsilon\,\vec{\nabla}\phi$), solve for the coefficients a_l and b_l in ϕ_{SI} and ϕ_{SO}.

D) The force on the point charge due to the dipole is:

$$F_q = -q\, \partial_r\, \phi_{SO}(r,0)|_{r=d}$$

Solve for the force due to the dipole, $l = 1$, term in the expansion for ϕ_{SO}.

2) Electric Dipole Sphere



A “ferroelectric” sphere of radius R has a constant polarization field $\vec{P}(\vec{r}) = P_0\,\hat{z}$. In this problem, we’ll find the fields inside and outside of the sphere.

The properties of electric fields and electric materials are given by:

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{D} = \rho_{free}, \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\left(\rho_{free} + \rho_{bound}\right), \qquad \vec{D} = \epsilon_0 \vec{E} + \vec{P},$$

$$\vec{D} = \epsilon\,\vec{E}, \qquad \rho_b = -\vec{\nabla} \cdot \vec{P}, \qquad \sigma_b = \hat{n} \cdot \vec{P}\,, \qquad \vec{D}_{\perp,in} - \vec{D}_{\perp,out} = \sigma_{free}, \qquad \vec{E}_{\parallel,in} = \vec{E}_{\parallel,out}$$

$\vec{D}$ is created by free charges, $\vec{E}$ by free and “bound” charges from macroscopic dipole moments, and the fields must also satisfy the boundary conditions. We’ll solve for $\vec{E}$ using the bound charges, as there are no free charges in the problem.

A) Using the definitions above, determine the bound charges:

$$\rho_b = \vec{\nabla} \cdot \vec{P}, \qquad \sigma_b = \hat{n} \cdot \vec{P}$$

Show that $\rho_b = 0$ while the bound surface charge has a distribution very similar to what we found on a conducting sphere in a uniform electric field (Workshop 9 & Hw 4).

B) Using the bound surface charge, write an integral that can be solved for the potential $\phi(\vec{r})$. This should be an application of Coulomb’s law.

C) Solve your integral from (B). A couple of hints that will do most of the work:

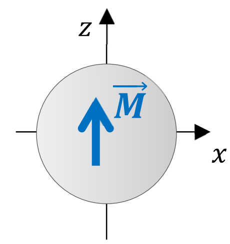
$$\frac{\cos(\theta')}{|\vec{r} - \vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\, P_l(\hat{r}' \cdot \hat{r}) \cos(\hat{r}' \cdot \hat{r})$$

$$\iint d\Omega'\; P_l(\hat{r}' \cdot \hat{r}_1)\, P_m(\hat{r}' \cdot \hat{r}_2) = \frac{4\pi}{2\,l + 1}\, P_l(\hat{r}_1 \cdot \hat{r}_2)\, \delta_{lm}$$

Where: $r_<$ is the smaller of r and r', $r_>$ is the smaller of r and r', and the 2D integral over $d\Omega'$ is over the solid angle of $\vec{r}'$, or $d\Omega' = \sin\theta'\; d\theta'\; d\phi'$.

D) Solve for the fields $\vec{E}$ and $\vec{D}$ both outside and inside the sphere. You’ll get something very much like the results from Problem 1. Show that, again, the boundary conditions are met.

E) Draw pictures of the $\vec{E}$ and $\vec{D}$ fields. How do these compare with the fields from Problem 1?



3) Magnetic Sphere:

Interestingly, the methods used in Problem 2 can also solve for the fields due to a magnetized sphere.

The properties of magnetic fields and magnetic materials are given by:

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}, \qquad \vec{\nabla} \times \vec{B} = \mu_0(\vec{J}_{free} + \vec{J}_m), \quad \vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{B} = \mu_0\,(\vec{H} + \vec{M}),$$

$$\vec{B} = \mu\,\vec{H}, \qquad \vec{J}_m = \vec{\nabla} \times \vec{M}, \qquad \vec{K}_m = \hat{n} \times \vec{M}$$

$$\vec{B}_{\perp,in} = \vec{B}_{\perp,out}, \qquad \vec{H}_{\parallel,in} - \vec{H}_{\parallel,out} = I_{free}$$

$\vec{H}$ is created by free currents, $\vec{B}$ by free and "bound" currents from macroscopic magnetic moments, and the fields must also satisfy the boundary conditions at the surface of the materials.

To do this, we'll use the concepts of the "Magnetic Scalar Potential" and "Magnetic Bound Charges":

If there are no free currents, the curl of $\vec{H}$ is zero, so we can define a potential ψ_m:

$$\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}\,\psi_m$$

Considering the divergence equation:

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0 \Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$

And the boundary condition $(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0$ related the field $\vec{B}$ outside and inside the magnetic material requires:

$$(\vec{H}_2 + \vec{M}_2) \cdot \hat{n} = \ (\vec{H}_1 + \vec{M}_1) \cdot \hat{n} \Rightarrow (\vec{H}_2 - \vec{H}_1) \cdot \hat{n} = \hat{n} \cdot \vec{M}$$

This means the solution for $\vec{H}$ is the same as for and electrostatics problem with the analogy of a scalar potential and "magnetic charges" (a problem-solving device, not real magnetic charges):

$$\vec{\nabla} \times \vec{H} = 0, \qquad \vec{\nabla} \cdot \vec{H} = \rho_m, \qquad \nabla^2\psi_m = \rho_m, \qquad \rho_m = -\vec{\nabla} \cdot \vec{M}, \qquad \sigma_m = \hat{n} \cdot \vec{M}$$

Considering this analogy and your solution to Problem 2 above, solve for the field $\vec{H}$ and $\vec{B}$ for the magnetized sphere.

Note: The solution to Workshop 12, Problem 1 will be (is) posted for comparison.

Problem 4: Write and solve a problem that you think could be on the E&M Qualifier.