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# Electrodynamics 1

PHYS 5573 Homework Assignment 2

Problems: {1, 2, 3, 4, 5}

Due: March 4, 2022 at 5:00 PM


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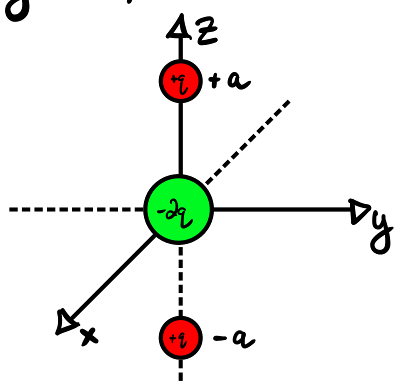
## Problem 1:

Consider the second charge distribution considered in the Multipole Expansion workshop:

$$q \text{ at } (x=0, y=0, z=a), \;\; q \text{ at } (x=0, y=0, z=-a), \;\; -2q \text{ at } (x=0, y-0, z=0)$$

(a) Using the multipole expansions for the potential $\phi(\vec{r})$, calculate the third (quadrupole) term in for $\phi(\vec{r})$ for this charge distribution.

Our charge distribution looks something like,




Where the dipole moment is $\vec{p} = 0$. If we expand our potential in a Power Series we will see

$$\varphi(\vec{r}) = \varphi(0) + \vec{r}\cdot\vec{\nabla}\varphi(\vec{r})|_{r=0} + \frac{1}{2}(\vec{r}\cdot\vec{\nabla})(\vec{r}\cdot\vec{\nabla})\varphi(\vec{r})|_{r=0} + \ldots\ldots$$

we then wish to solve for the third term which follows after what is seen above. The multipole expansion for the potential again was

$$\frac{1}{|\vec{r}-\vec{r}_q|} = \sum_n \frac{1}{n!}(-1)^n(\vec{r}_q\cdot\nabla)^n\frac{1}{r}$$

If we expand this out we see

$$\frac{1}{|\vec{r}-\vec{r}_q|} = \frac{1}{r} + \frac{x_{qi}\,x_i}{r^3} + \frac{1}{2}\left(3\frac{(x_{qi}\,x_i)(x_{qj}\,x_j)}{r^5} - \frac{r_q^2}{r^3}\right) + \mathcal{O}\left(\frac{1}{r^4}\right)$$

we are only interested in the third term

$$\frac{1}{2}\left(3\frac{(x_{qi}\cdot x_i)(x_{qj}\cdot x_j)}{r^5} - \frac{r_q^2}{r^3}\right)$$

For our system we have the following definitions

$$\vec{x}_{q_1} = 0\hat{x} + 0\hat{y} + a\hat{z}\,, \quad \vec{x}_{q_2} = 0\hat{x} + 0\hat{y} + 0\hat{z}\,, \quad \vec{x}_{q_3} = 0\hat{x} + 0\hat{y} - a\hat{z}$$

## Problem 1: Continued

This then turns into

$$= \frac{1}{2}\left(\frac{3(a\hat{z}\cdot z\hat{z})(a\hat{z}\cdot z\hat{z})}{r^5} - \frac{a^2}{r^3}\right) + \frac{1}{2}\left(\frac{3(-a\hat{z}\cdot z\hat{z})(-a\hat{z}\cdot z\hat{z})}{r^5} - \frac{a^2}{r^3}\right) + \frac{1}{2}\left(\cancel{\frac{3(0)(0)}{r^5}} - \cancel{\frac{0}{r^3}}\right)^{0}$$

$$= \frac{1}{2}\left(\frac{3a^2z^2}{r^5} - \frac{a^2}{r^3}\right) + \frac{1}{2}\left(\frac{3a^2z^2}{r^5} - \frac{a^2}{r^3}\right) = a^2\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)$$

we of course recognize that

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

knowing this we say our third term in our potential is

$$\boxed{\theta^3(\vec{r}) = \frac{a^2}{(x^2+y^2+z^2)^{3/2}}\left(\frac{3z^2}{(x^2+y^2+z^2)} - 1\right)}$$

(b) Write your result in spherical coordinates, $r, \theta, \phi$. Explain why your result doesn't depend on $\phi$. In this, quadrupole, approximation, for what directions in space is the potential equal to zero? (Draw a picture.)

we first start with some definitions for x, y, z in spherical co-ordinates

$$x = r\sin(\theta)\cos(\varphi), \quad y = r\sin(\theta)\sin(\varphi), \quad z = r\cos(\theta)$$

This then means

$$x^2+y^2+z^2 = r^2(\sin^2(\theta)\underbrace{(\cos^2(\varphi)+\sin^2(\varphi))}_{1} + \cos^2(\theta)) = r^2\underbrace{(\sin^2(\theta)+\cos^2(\theta))}_{1} = r^2$$

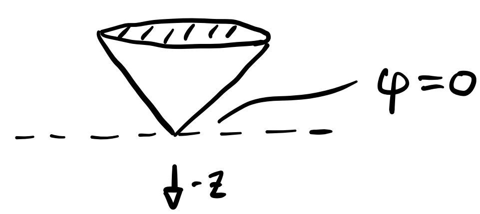
with this we can say in spherical co-ordinates we have

$$\boxed{\theta^3(\vec{r}) = \frac{a^2}{r^3}\left(3\cos^2(\theta) - 1\right)}$$

Because our charges lie along the z-axis and $\varphi$ is perpendicular to that axis we see that there is no dependence on it. The potential will be zero when

$$3\cos^2(\theta) - 1 = 0 \Rightarrow \theta = \cos^{-1}(\sqrt{3}/3)$$

Which graphically looks like



## Problem 1: Continued

(c) Using spherical coordinates, calculate the electric field of this charge distribution in the quadrupole approximation. Sketch the field in the $x - z$ plane.

We can calculate the electric field with

$$\vec{E} = -\vec{\nabla}\varphi(\vec{r})$$

Where of course we have our spherical gradient,

$$\vec{\nabla}(r,\theta,\varphi) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

This then means our electric field is

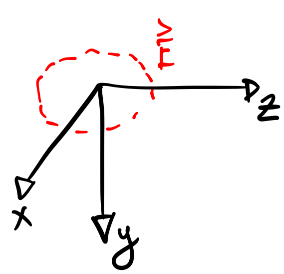
$$\vec{E} = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \varphi}\hat{\varphi}\right)\frac{a^2}{r^3}\left(3\cos^2(\theta) - 1\right)$$

$$= a^2\left(\frac{\partial}{\partial r}\frac{1}{r^3}(3\cos^2(\theta)-1)\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\frac{1}{r^3}(3\cos^2(\theta)-1)\hat{\theta}\right)$$

$$= a^2\left(\frac{1}{r^4}(1-3\cos^2(\theta))\hat{r} - \frac{1}{r^4}(6\cos(\theta)\sin(\theta))\hat{\theta}\right) = \frac{a^2}{r^4}\left((1-3\cos^2(\theta))\hat{r} - (6\cos(\theta)\sin(\theta))\hat{\theta}\right)$$

This means our Electric Field will be

$$\boxed{\vec{E}(r,\theta,\varphi) = \frac{1}{4\pi\varepsilon_0}\frac{a^2}{r^4}\left((1-3\cos^2(\theta))\hat{r} - (6\cos(\theta)\sin(\theta))\hat{\theta}\right)}$$



# Problem 1: Review

**Procedure:**

- – The third term that we are going to use is

$$\frac{1}{2}\Big(3\frac{(x_{q,i}x_i)(x_{q,j}x_j)}{r^5} - \frac{r_q^2}{r^3}\Big)$$

  which is calculated with

$$\frac{1}{|\vec{r} - \vec{r}_q|} = \sum_n (-1)^n(\vec{r}_q \cdot \vec{\nabla})^n \frac{1}{r}$$

  where we construct vectors for each charge and put them in the formula

- – Use the cartesian to spherical transformations and put them in the expression found in (a)
- – Calculate the Electric Field with

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

**Key Concepts:**

- – We use the multipole expansion to approximate the potential of our quadrupole
- – Here we use cartesian to spherical transformations to write our potential in terms of $r$ and $\theta$
- – We can calculate the Electric Field by knowing the potential and taking the negative gradient of it

**Variations:**

- – We could be asked to look a different term in this expansions
    - ∗ We would simply apply the same procedure as we did for the third

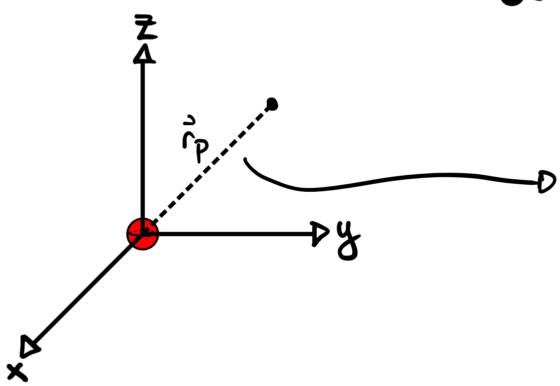
## Problem 2:

Consider a point charge $Q$ at the origin $\vec{r} = 0$ and an electric point dipole $\vec{p}$ at a position $\vec{r}_p$ (not at the origin). Calculate:

(a) The potential energy of the $Q$ & $\vec{p}$ system, by:

- (i) Calculate the potential energy of $\vec{p}$ in the electric potential of $Q$
- (ii) Calculate the potential energy of $Q$ in the electric potential of $\vec{p}$
- (iii) Show these are equal

To calculate the potential energy of $Q$ and $\vec{p}$ we use




$$U = \sum_i q_i \varphi_i$$

$$\vec{r}_p = x\hat{x} + y\hat{y} + z\hat{z}$$

We first look at part (i), the electric potential of $Q$ is simply

$$\varphi(r) = \frac{-1}{4\pi\varepsilon_0}\frac{Q}{r}$$

To find the electric potential energy of a dipole and a point particle we use

$$U = \vec{E}(r) \cdot \vec{p}$$

We find the Electric Field of the point particle to be

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

This then means the electric potential energy for $\vec{p}$ from $Q$ would be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot(qr)\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}\cos(\alpha) \checkmark$$

We then now find the electric potential of our dipole (From Griffiths)

$$\varphi(r,\alpha) = \frac{\hat{r}\cdot\hat{p}}{4\pi\varepsilon_0 r^2} = \frac{q\cos(\alpha)}{4\pi\varepsilon_0 r}$$

## Problem 2:

This then means our electric potential energy would be

$$U = q\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}\cos(\alpha)\ \checkmark$$

From this we can see they are clearly equal and the potential energy is

$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r}\cos(\alpha)}$$

(b) The forces on $\vec{p}$ and $Q$, by:

(i) Calculate the force on $\vec{p}$ due to the field of $Q$

(ii) Calculate the force on $Q$ due to the field of $\vec{p}$

(iii) Compare the results

We can calculate the forces with

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

Where $\vec{p}$ is our dipole moment and $\vec{E}$ is of the point particle. With this we have

$$\vec{F} = \left(P_x\hat{x} + P_y\hat{y} + P_z\hat{z}\right)\cdot\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\frac{Q}{4\pi\varepsilon_0}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(P_x\frac{\partial}{\partial x} + P_y\frac{\partial}{\partial y} + P_z\frac{\partial}{\partial z}\right)\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(P_x\frac{\partial}{\partial x}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}} + P_y\frac{\partial}{\partial y}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}} + P_z\frac{\partial}{\partial z}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{3/2}}\right)$$

We are going to do this for one direction at a time

$$F_x = \frac{Q}{4\pi\varepsilon_0}\left\{P_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}\right] - P_y\left[\frac{3xy}{(x^2+y^2+z^2)^{5/2}}\right] - P_z\left[\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\right]\right\}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{P_x}{(x^2+y^2+z^2)^{3/2}} - \frac{3x}{(x^2+y^2+z^2)^{5/2}}(P_x x + P_y y + P_z z)\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{P_x}{r^3} - \frac{3x}{r^5}(P_x x + P_y y + P_z z)\right]$$

## Problem 2: Continued

This means the total Force would be

$$\vec{F}_{DC} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\left[\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}\right]$$

If we do this the other way around we can use

$$\vec{F} = q\vec{E}$$

The Electric Field for a dipole can be written as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3[\vec{p}\cdot(-\hat{r})](-\hat{r}) - \vec{p}\right] = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

This then means the force is $F = QE$

$$\vec{F}_{CD} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

This then means the forces for both are

$$\boxed{\vec{F}_{DC} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\left[\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}\right], \quad \vec{F}_{CD} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]}$$

If we inspect these Forces, we can see that they are pointing in the <u>opposite directions</u> of one another.

# Problem 2: Review

**Procedures:**

- – Calculate the Electric Potential energy of a dipole with
$$U = \vec{E}(\vec{r}) \cdot \vec{p}$$
where $\vec{E}(\vec{r})$ is that from the point charge and $\vec{p}$ is the dipole moment
- – We calculate the force from a dipole with
$$\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$
where $\vec{p}$ is our dipole moment and $\vec{E}$ is the electric field of the point particle

**Key Concepts:**

- – The potential energy of $\vec{p}$ due to the potential of $Q$ is the same as if we calculate the potential energy of $Q$ due to the potential of $\vec{p}$
- – The force on $\vec{p}$ due to the field of $Q$ will be equal but opposite to that of the force on $Q$ due to the field of $\vec{p}$

**Variations:**

- – The only real change that can be made to this problem is if the dipole is placed somewhere else in space
  * We of course can have different conceptual questions asked but would require other information to answer those questions

## Problem 3:

Consider the two currents shown: (i) a circular loop of radius $R$ with counterclockwise current $I_1$ centered at the origin in the $x-y$ plane and (ii) a very long straight wire with current $I_2$ parallel to the $y$-direction at $z=0$, $x=-d$.

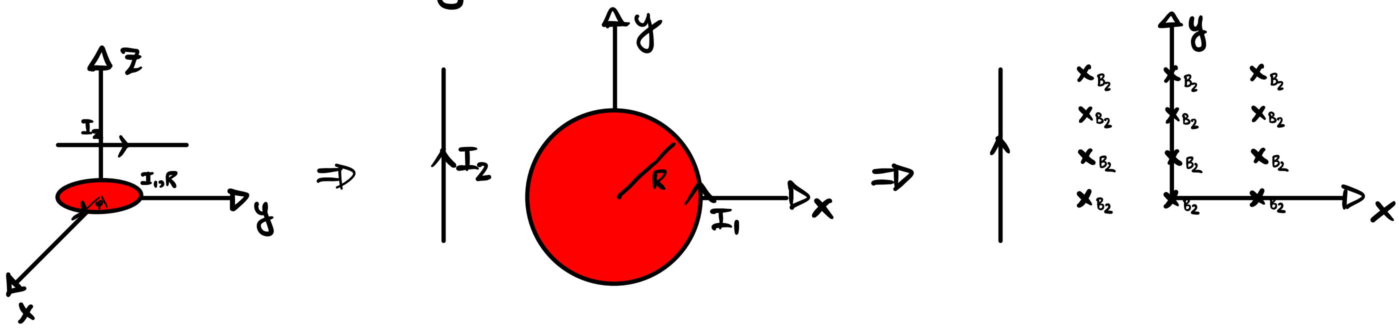
We're going to find solutions for the force on the loop, first by integration and then by considering the potential energy.

(a) Predict the direction of the total force on the loop due to the magnetic field of the long-straight wire. Explain your prediction.

We know that force from a current carrying wire is

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

We can think of this visually as




With the above pictures and using the Biot-Savart Law I predict it should point radially inward due to the orthogonality of the magnetic field and the current in the loop.

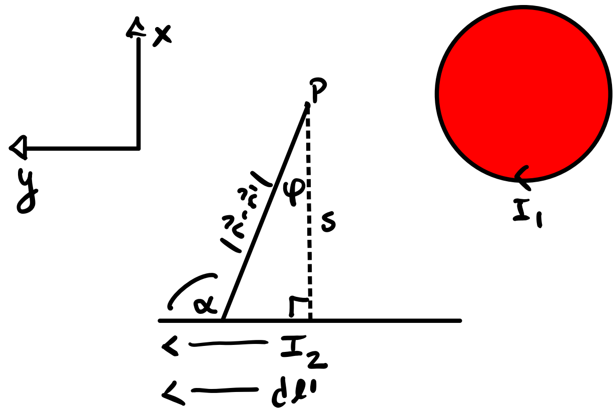
This should lead to a net force in the +x-direction

(b) What is the magnetic field $\vec{B}_2(\vec{r})$ (magnitude and direction) due to the long wire everywhere in the $x-y$ plane?

The magnetic field of a steady line current can be calculated with

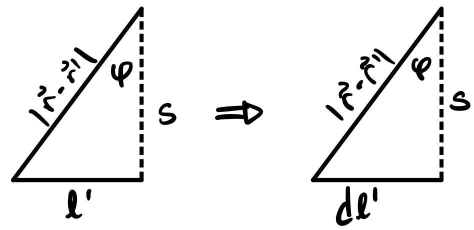
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times (\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2}\, d\ell' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times (\hat{r}-\hat{r}')}{|\vec{r}-\vec{r}'|^2} \qquad (*)$$

If we view our wire of current like so



## Problem 3: Continued

Where we can essentially say we are looking at a distance s away at point P. Using the Biot-Savart Law we see that




Where we want to look at solving for the hypotenuse of our triangle. We can use geometry and magnitudes of cross products to say

$$|\vec{r}-\vec{r}'| = d\ell' \sin(\alpha) \quad , \quad |\vec{r}-\vec{r}'| = d\ell' \cos(\varphi) \quad \therefore \quad d\ell' \sin(\alpha) = d\ell' \cos(\varphi)$$

We know that $\ell' = s\tan(\varphi)$ $\therefore$

$$d\ell' = s \cdot \sec^2(\varphi)\, d\varphi$$

We can then go on to say with knowing $s = |\vec{r}-\vec{r}'|\cos(\varphi)$, $(*)$ then turns into

$$\vec{B} = \frac{\mu_0}{4\pi} I_2 \int \frac{d\vec{\ell}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2} = \frac{\mu_0 I_2}{4\pi} \int \left(\frac{\cos^2(\varphi)}{s^2}\right)\left(\frac{s}{\cos^2(\varphi)}\right) \cos(\varphi)\, d\varphi \; (-\hat{z})$$

$$= \frac{\mu_0}{4\pi s} I_2 \int_{\varphi_1}^{\varphi_2} \cos(\varphi)\, d\varphi\, (-\hat{z}) = \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_2) - \sin(\varphi_1))(-\hat{z})$$

So finally our magnetic field will be

$$\boxed{\vec{B} = \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_1) - \sin(\varphi_2))(\hat{z})}$$

(c) The force on the loop due to the magnetic field of the wire is:

$$\vec{F}_{21} = I_1 \oint d\vec{l} \times \vec{B}_2$$

It should be clear that we want to do this integral over the circle by integrating over $d\phi$.

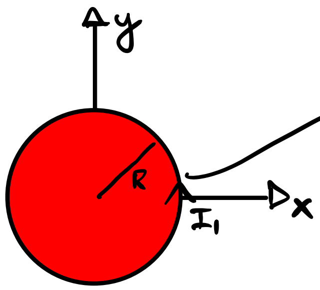
(i) Derive expressions for $\vec{B}_2(\vec{r})$ and $d\vec{l}$ in terms of $R, \phi, d$ and $d\phi$ (for points on the current loop). Write these in terms of the $\hat{x}$ and $\hat{x}$ components.

Check your answer at a few simple points (such as for the angles $\phi = 0, \phi = \frac{\pi}{2}, ...$)

We first begin by writing out $d\ell'$ for our current loop

## Problem 3: Continued

We are interested in using $d\ell$ for the loop and $\vec{B}_2(\vec{r})$ from the wire




$$d\vec{\ell} = R\, d\varphi\, (\hat{\varphi}) \quad , \quad \vec{B}_2(\vec{r}) = \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_1) - \sin(\varphi_2))(\hat{z})$$

$$\hat{\varphi} = -\sin(\varphi)\,\hat{x} + \cos(\varphi)\,\hat{y}$$

We then move on to writing $d\vec{\ell}$ in $\hat{x}$ and $\hat{y}$,

$$d\vec{\ell} = R\,d\varphi\,(-\sin(\varphi)\,\hat{x} + \cos(\varphi)\,\hat{y}) = -R\sin(\varphi)\,d\varphi\,\hat{x} + R\cos(\varphi)\,d\varphi\,\hat{y}$$

When $\varphi = 0$, $d\vec{\ell} = R\,d\varphi\,\hat{y}$ ✓ , $\varphi = \pi/2$ , $d\vec{\ell} = -R\,d\varphi\,\hat{x}$ ✓

We can then with confidence say

$$\boxed{\vec{B}_2(\vec{r}) = \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_1) - \sin(\varphi_2))(\hat{z}) \ , \ d\vec{\ell} = -R\sin(\varphi)\,d\varphi\,(\hat{x}) + R\cos(\varphi)\,d\varphi\,(\hat{y})}$$

(ii) Write out an integral that gives the force on the loop. You should simplify this as much as possible, but you don't need to solve it, as it's somewhat messy.

Does the direction agree with your prediction?

We can now calculate $d\vec{\ell} \times \vec{B}_2(\vec{r})$

$$d\vec{\ell} \times \vec{B}_2(\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin(\varphi)\,d\varphi & R\cos(\varphi)\,d\varphi & 0 \\ 0 & 0 & \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_1) - \sin(\varphi_2)) \end{vmatrix}$$

$$d\vec{\ell} \times \vec{B}_2(\vec{r}) = \frac{\mu_0 I_2}{4\pi}\frac{R}{s}\cos(\varphi)(\sin(\varphi_1) - \sin(\varphi_2))\,d\varphi\,(\hat{x}) + \frac{\mu_0 I_2}{4\pi}\frac{R}{s}\cos(\varphi)(\sin(\varphi_1) - \sin(\varphi_2))\,d\varphi\,(\hat{y})$$

The force on the loop is then $(s = d + R\cos(\varphi))$

$$\boxed{\vec{F}_{21} = \frac{\mu_0 I_1 I_2 R}{4\pi}\oint \frac{\cos(\varphi)}{s}(\sin(\varphi_1) - \sin(\varphi_2))(\hat{x} + \hat{y})\,d\varphi}$$

Which is not what I originally predicted. I think I wasn't considering a non-uniform field when I made my prediction

## Problem 3: Continued

(d) Another approach to this problem is to determine the potential energy of the loop due to the magnetic field of the wire, $U_{\text{Loop}}$, and then use that the force is $F = -\vec{\nabla} U_{\text{Loop}}$. In this case you're interested in the change in energy as the loop moves relative to the wire

$$F_x = \partial_d U_{\text{Loop}}$$

(i) A current loop is equivalent to a sheet of small current loops, each a magnetic dipole (Fig. 4.7 in the textbook). Using small loops, $d\vec{S} = dA\,\hat{n}$, gives magnetic dipoles

$$d\vec{m} = I\,dA\,\hat{n}$$

The potential energy of the magnetic dipoles

$$dU(\vec{r}) = -d\vec{m} \cdot \vec{B}_2(\vec{r})$$

Write a surface integral for the potential energy of the loop. Use polar coordinates.

Recycling the magnetic field due to this wire we have

$$\vec{B}_2(\vec{r}) = \frac{\mu_0 I_2}{4\pi s}(\sin(\varphi_1) - \sin(\varphi_2))(\hat{z})$$

Where the magnetic dipoles of the loop will be aligned in the $(\hat{z})$ direction (This is due to the normal vector of the loop) and we see that $d\vec{m}$ is then,

$$d\vec{m} = I\,dA\,\hat{n} = I_1 r\,dr\,d\varphi\,(\hat{z})$$

This then means our potential energy will be

$$\boxed{U = \frac{\mu_0 I_1 I_2}{2\pi}(\sin(\varphi_1) - \sin(\varphi_2)) \int_0^{2\pi}\int_0^R \frac{r}{d + r\cos(\varphi)}\,dr\,d\varphi}$$

(ii) Write the field $\vec{B}_2(\vec{r})$ everywhere inside the loop using polar coordinates. (This is a simple extension to part c above.)

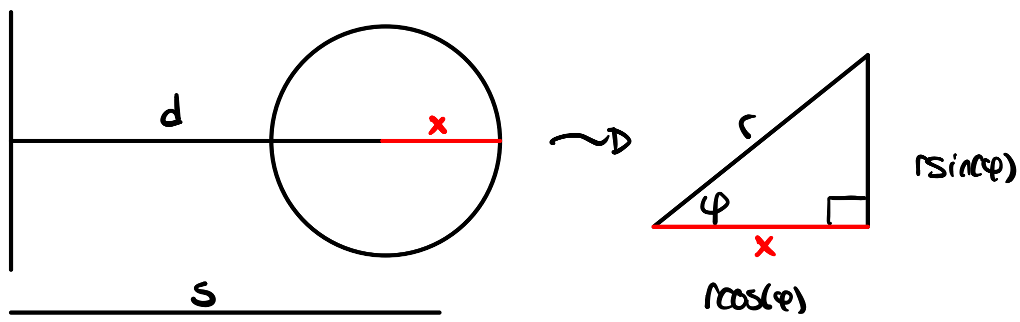
Going back to the magnetic field, to be in the loop we impose the constraint

$$d - R < s < d + R$$

Where $s$ is an arbitrary distance from the wire to somewhere inside the loop, $R$ is the radius of the loop, and $d$ is the distance from the wire to the center of the loop.

We keep $d$ and $s$ but need to define an arbitrary distance away from the center in all directions. We do this with

## Problem 3: Continued




$\Rightarrow \quad s = d + r\cos(\varphi)$

$0 < r \leq R$

we can go on to say our magnetic field is

$$\boxed{\vec{B} = \frac{\mu_0 I_2}{2\pi}\frac{(\sin(\varphi_1)-\sin(\varphi_2))}{(d+r\cos(\varphi))}(\hat{z})}$$

(iii) Solve your integral to determine $U_{\text{Loop}}$ and take the derivative to get $F_y$. Does your result make sense? It might be useful to know that:

$$\int_0^{2\pi}\frac{d\phi}{d+r\cos(\phi)} = \frac{2\pi}{\sqrt{d^2+r^2}}$$

with our new definition for $\vec{B}_2(\vec{r})$ in the loop we find the potential energy to be

$$U = \frac{\mu_0 I_1 I_2}{4\pi}(\sin(\varphi_1)-\sin(\varphi_2))\int_0^R\int_0^{2\pi}\frac{r}{d+r\cos(\varphi)}\,d\varphi\,dr$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}(\sin(\varphi_1)-\sin(\varphi_2))\int_0^R\frac{2\pi r}{\sqrt{d^2-r^2}}\,dr = \mu_0 I_1 J_2\left(d-\sqrt{d^2-R^2}\right)$$

we then move on to writing $U$ in terms of $y$. Since we have defined $y = R\sin(\varphi)$, we can then say

The force in the $y$-direction is then $F = -\nabla U$

$$\boxed{F_x = \mu_0 I_1 I_2\left(\frac{d}{\sqrt{d^2-R^2}}-1\right)}$$

This makes sense because as $y$ grows in either direction the force will go to 0 and that makes sense with what we should expect. It should also be greatest in the middle

## Problem 3: Continued

(iv) Consider your result for the force on the loop in the limit that $d >> R$. Show that this is equivalent to the force on a magnetic point dipole due to the magnetic field of the wire. Remember that the force on a dipole is

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$

Going back to $F_x$, if $d >> R$ then $F_x$

$$\lim_{d>>R} \mu_0 I_1 I_2 \left( \frac{d}{\sqrt{d^2-R^2}} - 1 \right) = \frac{\mu_0 I_1 I_2}{2\pi d^2} \pi R^2 = \frac{\mu_0 I_2 \mu_1}{2\pi d^2}$$

The dipole approximation then gives

$$F = \vec{\nabla}(\vec{\mu}_1 \cdot \vec{B}) = \vec{\nabla}\left(\mu_1 \hat{z} \cdot (\mu_0 I_2 / 2\pi d)(-\hat{z})\right) = \frac{\mu_0 I_2 \mu_1}{2\pi d^2}$$

Where we can see these are equivalent

# Problem 3: Review

**Procedure:**

- – Draw a picture of the system and make a prediction of what you can expect the force to be
- – Calculate the magnetic field of a steady line current with

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l'} \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^2}$$

- – Calculate the force with

$$\vec{F}_{21} = I_1 \oint d\vec{l} \times \vec{B}_2$$

- – Use the expression for potential energy given to us and calculate force from this energy

**Key Concepts:**

- – Due to the symmetry of our system and the Biot-Savart Law we can deduce that the magnetic field in the $y$ direction will cancel and we should have a net magnetic field in the $+x$ direction
- – Using the Biot-Savart Law we can calculate our magnetic field by constructing vectors and performing the desired integral
- – We use the current of loop and the magnetic field that is produced by the wire to calculate the force on the loop
- – This problem allows us to calculate the same quantity by determining the potential energy of the loop due to the wire and then taking a negative gradient of the potential energy to find the force

**Variations:**

- – We could be given a different shape or geometry
    - * We use the same broad procedure with subtle differences between each problem

## Problem 4:

Consider one more time the current-carrying ring considered in class. The ring is in the $x-y$ plane with radius $a$ and a counter-clockwise current $I$.

As before, we want to calculate the magnetic field at a point

$$\vec{r} = r\cos(\theta)\,\hat{z} + r\sin(\theta)\,\hat{x}$$

In this case we'll look at this problem using a multipole expansion in spherical harmonics. We'll use the result:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{a^l}{r^{l+1}}\,Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

(a) Write down (or look up from class) and expression for the current density of the ring, $\vec{J}(\vec{r}')$ in terms of the spherical coordinates $r', \theta', \phi'$.

Remember that for a 1D distribution you'll need two $\delta$-functions.

The current density from class was

$$\boxed{\vec{J}(\vec{r}') = \frac{I}{a}\,\delta(r'-a)\,\delta(\cos(\theta'))\;\hat{\varphi}}$$

(b) Using the definition for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\int\int \vec{J}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\,d^3r'$$

Write down the multipole (spherical harmonics) expansion for the vector potential. This should still include sums over $l$ and $m$ and the volume integral over $\vec{r}'$. Remember to include the vector direction(s) of the current density.

The multipole expansion of our vector potential is

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int\int\int \delta(r'-a)\,\delta(\cos(\theta'))\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{a^{l-1}}{r^{l+1}}\,Y^*_{l,m}(\theta',\varphi')\,Y_{l,m}(\theta,\varphi')\,r'^2\sin(\theta')\,dr'\,d\theta'\,d\varphi'\;(\hat{\varphi})}$$

(c) Perform the integrals over $dr'$ and $d\theta'$ to determine an expression for $\vec{A}(\vec{r})$ just in terms of an integral on $d\phi'$. Of course, you'll still have the sums over $l$ and $m$.

Performing this integral we have

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int\int \delta(\cos(\theta'))\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{a}{r}\right)^{l+1} Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)\,\sin(\theta')\,d\theta'\,d\varphi'\;(\hat{\varphi})$$

We can see we have a vector potential now in a $d\varphi'$ form with

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{a}{r}\right)^{l+1} Y^*_{lm}(\pi/2,\varphi')\,Y_{lm}(\theta,\varphi)\,d\varphi'\;(\hat{\varphi})}$$

## Problem 4: Continued

(d) Show that $l = 0$ term in the expansion is zero

If we set $l=0$ we have spherical harmonics of the form

$$Y_{00}^*(\tfrac{\pi}{2},\varphi') = \frac{1}{2}\sqrt{\frac{1}{\pi}} \quad , \quad Y_{00}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

This then changes our integral to

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} I \frac{a}{r}\int d\varphi' \,(\hat{\varphi}) = \frac{\mu_0}{4\pi} I \frac{a}{r}\int_0^{2\pi} -\sin(\varphi')\,\hat{x} + \cos(\varphi')\,\hat{y}\; d\varphi'$$

$$= \frac{\mu_0}{4\pi} I \frac{a}{r}\left(\cos(\varphi')\,\hat{x} + \cancel{\sin(\varphi')}^{\,0}\,\hat{y}\right)\Big|_0^{2\pi} = 0 \checkmark$$

This of course means the $l=0$ term is zero ✓

(e) Calculate the $l = 1$ term in the multipole expansion. This will include the sum over $m = 0, \pm 1$.

Calculating the $l=1$ expansion we have

$$l=1 : \quad \sum_{l=0}^{1}\sum_{m=-1}^{1} \frac{4\pi}{2l+1}\frac{a^l}{r^{l+1}} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

$$\frac{4\pi}{r}\left(\left(Y_{0,0}^*(\theta',\varphi')Y_{0,0}(\theta,\varphi)\right) + \frac{a}{3r}\left(Y_{1,-1}^*(\theta',\varphi')Y_{1,-1}(\theta,\varphi) + Y_{1,0}^*(\theta',\varphi')Y_{1,0}(\theta,\varphi) + Y_{1,-1}^*(\theta',\varphi')Y_{1,1}(\theta,\varphi)\right)\right) \qquad (*)$$

The respective Spherical Harmonics are

$$Y_{0,0}^*(\theta',\varphi') = \frac{1}{2}\sqrt{\frac{1}{\pi}} \quad , \quad Y_{0,0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_{1,-1}^*(\theta',\varphi') = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\varphi'}\sin(\theta') \quad , \quad Y_{1,-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\varphi}\sin(\theta)$$

$$Y_{1,0}^*(\theta',\varphi') = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta') \quad , \quad Y_{1,0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta)$$

$$Y_{1,1}(\theta',\varphi') = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\varphi'}\sin(\theta') \quad , \quad Y_{1,1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\varphi}\sin(\theta)$$

Putting these into $(*)$ our expansion becomes

$$= \frac{4\pi}{r}\left(\frac{1}{4\pi} + \frac{a}{3r}\left(\frac{3}{8\pi}e^{i(\varphi'-\varphi)}\sin(\theta')\sin(\theta) + \frac{3}{4\pi}\cos(\theta')\cos(\theta) + \frac{3}{8\pi}e^{i(\varphi-\varphi')}\sin(\theta')\sin(\theta)\right)\right)$$

## Problem 4: Continued

$$= \frac{1}{r} + \frac{a}{r^2}\left(\sin(\theta')\sin(\theta)\cos(\varphi'-\varphi) + \cos(\theta')\cos(\theta)\right)$$

So the $l=1$ term in the multipole expansion is

$$l=1,\ m=0,\pm1 \Rightarrow \quad \frac{1}{r}\left(1+\frac{a}{r}\left(\sin(\theta')\sin(\theta)\cos(\varphi'-\varphi) + \cos(\theta')\cos(\theta)\right)\right)$$

we then go ahead and integrate this using common integrals for spherical Harmonics and we find

$$\boxed{A^{(1)}(\vec{r}) = \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin(\theta)\,\hat{\theta}}$$

(f) Calculate the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$. Compare your results to those found in class, including the results for $\vec{B}(\vec{r} = z\,\vec{z})$ and for $r >> a$.

For Fun, try calculating the $l = 1$ term in the multipole expansion.

we now calculate the magnetic field using $\vec{B} = \vec{\nabla}\times\vec{A}$. The curl of $\vec{A}$ is then

$$B_r = \frac{1}{r\sin\theta}\partial_\theta \sin(\theta)\, A^{(1)}(\varphi) = \frac{1}{r\sin\theta}\partial_\theta \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin^2\theta = \frac{\mu_0 I}{2}\frac{a^2}{r^3}\cos\theta$$

$$B_\theta = -\frac{1}{r}\partial_r\, r\, A^{(1)}(\varphi) = -\frac{1}{r}\partial_r \frac{\mu_0 I}{4}\frac{a^2}{r}\sin\theta = \frac{\mu_0 I}{4}\frac{a^2}{r^3}\sin\theta$$

We now have a magnetic field of the form

$$\boxed{\vec{B}(\hat{r},\hat{\theta}) = \frac{\mu_0 \mathcal{M}}{4\pi r^3}\left[2\cos(\theta)\,\hat{r} + \sin(\theta)\,\hat{\theta}\right]}$$

If we are only on $z$, then $\theta = 0$ and $|\vec{r}| = z$ $\therefore$

$$\boxed{\vec{B}(z) = \frac{\mu_0 \mathcal{M}}{4\pi z^3}(\hat{z})}$$

# Problem 4: Review

**Procedure:**

- – Write down the current density from class
- – Use the definition of vector potential with the previous current density to write an expression for it

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \int \int \vec{J}(\vec{r}) \frac{1}{|\vec{r} - \vec{r}'|} \, d^3 r'$$

- – Evaluate the $r'$ and $\theta'$ integrals leaving just $\phi'$ left
- – Set $l = 0$ and evaluate the integral from part (c)
- – Set $l = 1$ and solve for this vector potential
- – Calculate the magnetic field from the vector potential with

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

**Key Concepts:**

- – This is simply an expression
- – This is simply just extending the vector potential with certain values of $l$ and $m$
- – When we evaluate the integrals we must also evaluate the delta function and substitute these values in for the spherical harmonics
- – This is just showing that the vector potential when $l = 0$ is zero
- – Here we see when we set $l = 1$ we have a non zero vector potential that is originally dependent upon spherical harmonics
- – We can calculate the magnetic field by calculating the curl of vector potential

**Variations:**

- – Once again if the geometry changes
    - ∗ We use the same procedure but with slightly different geometry

## Problem 5:

NOTE: In this problem, you can assume for all your answers that $\vec{m}_1$ and $\vec{m}_2$ are either parallel or anti-parallel. I'm fairly sure that this has to be true, but there are still a couple of cases where I need a more complete proof. If you want to prove this, please do. If not, you can assume it.

Two identical magnetic dipoles are shown, $\vec{m}_1$ at the origin and $\vec{m}_2$ at $\vec{r} = d\,\hat{z}$. The magnetic dipoles are free to rotate in the $x-y$ plane (they don't have components in $\hat{y}$). To simplify the solution to this problem, define a quantity related to the magnetic field due to the dipoles:

$$B_d = \frac{\mu_0}{4\pi}\frac{m}{d^3}, \quad |\vec{m}_1| = |\vec{m}_2| = m$$

There is a uniform, constant magnetic field $\vec{B}$ in the $x-z$ plane.

(a) Write down an expression for the total potential energy of the two dipoles interacting with each other and the magnetic field.

Since dipoles do not have charge, we calculate the potential energy of a dipole to be

$$U = -\vec{m}\cdot\vec{B}$$

where $\vec{m}$ is a dipole moment and $\vec{B}$ is the magnetic field. We can calculate the magnetic field due to a dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3(\hat{n}\cdot\vec{m})\hat{n} - \vec{m}}{r^3}$$

The expression for the potential energy is

$$U = -\vec{m}_2\cdot\vec{B}_1(\vec{r}_2) = -\vec{m}_2\cdot\frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1\cdot\hat{r}_2)\hat{r}_2}{r_2^3} - \frac{\vec{m}_1}{r_2^3}\right) = \frac{\mu_0}{4\pi}\frac{1}{r_2^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r}_2)(\vec{m}_2\cdot\hat{r}_2)\right)$$

But we know that $|r_2| = d$, $\hat{r}_2 = \hat{z}$ therefore

$$U = \frac{\mu_0}{4\pi d^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3m_{1z}m_{2z}\right)$$

The potential energy is then finally

$$\boxed{U = \frac{\mu_0}{4\pi d^3}\left(m_{1x}m_{2x} - 2m_{1z}m_{2z}\right) - (\vec{m}_1 + \vec{m}_2)\cdot\vec{B}}$$

(b) First, consider the case where $\vec{B} = 0$. What are the are the configurations of the dipoles that give the lowest potential energy (there are two)? Show that adding a magnetic field $\vec{B} = B_{\text{ext}}\,\hat{z}$ will result in one configuration having the lowest energy.

If we take the expression for potential energy we can clearly see that the quantity

$$(\vec{m}_1\cdot(\vec{m}_2\cdot\hat{z})\hat{z} + \vec{m}_2\cdot(\vec{m}_1\cdot\hat{z})\hat{z})$$

## Problem 5: Continued

will be greatest if the dipoles are aligned in the same direction. This then means we will have the smallest potential energy if

$$\boxed{\text{Both point in } (\pm z) \text{ direction}}$$

If we add another magnetic field in the $\hat{z}$ direction we will see that the electric potential energy will decrease due to another negative term being summed.

(c) If instead the magnetic field is $\vec{B} = B_{\text{ext}}\,\hat{x}$, the lowest energy configuration of the dipoles will depend on the magnitude of the field.

(i) If $B_{\text{ext}} << B_d$ what do you expect the lowest-energy configuration of the dipoles to be? If $B_{\text{ext}} >> B_d$ what do you expect the lowest-energy configuration of the dipoles to be? Explain.

If $B_{ext} << B_d$ its almost like it doesn't exist to us. So if it's like it doesn't exist to us we can say the lowest energy configuration will come from the dipoles pointing in the same direction.

If $B_d << B_{ext}$, its almost like $B_d$ doesn't exist so this means we only have a magnetic field in the x. So we would want our dipoles to both be aligned in the positive x-direction.

(ii) Assuming $\vec{m}_1$ and $\vec{m}_2$ remain parallel, determine the lowest energy configuration of the two dipoles as a function of the magnitude $B_{\text{ext}}$.

we still want our dipoles to both be aligned in the same direction but in line with the external magnetic field. Therefore the total potential energy will be

$$U = -\vec{m}_i \cdot \vec{B}_{ext} - \vec{m}_i \cdot \vec{B}_j(\vec{r}_i) = -\vec{m}_1 \cdot \vec{B}_{ext} - \vec{m}_2 \cdot \vec{B}_{ext} - \vec{m}_1 \cdot \vec{B}_2(\vec{r}_1) - \vec{m}_2 \cdot \vec{B}_1(\vec{r}_2)$$

The potential energy then becomes

$$U = -\vec{m}_1 \cdot \vec{B}_{ext} - \vec{m}_2 \cdot \vec{B}_{ext} - [\ \dots\ ]$$

Where the term in brackets is the term for potential energy that was previously found. This then means that if $\vec{m}_1$ and $\vec{m}_2$ are parallel to $\vec{B}_{ext}$ we will have the minimum potential energy.

# Problem 5: Review

**Procedure:**

- – Calculate the potential energy of a dipole with
$$U = -\vec{m} \cdot \vec{B}$$
and the magnetic field of a dipole with
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\hat{n} \cdot \vec{m})\hat{n} - \vec{m}}{r^3}$$
- – Evaluate where the magnitude of this quantity will be greatest (Where the vectors are aligned)
- – Evaluate the expressions to see what happens in the limiting cases

**Key Concepts:**

- – The potential energy of a dipole is calculated by taking the dot product of said dipole in the presence of a magnetic field
- – Here we notice that in order for this quantity to not decrease in magnitude, the cosine term in our dot product must be one that then says we must be aligned or anti aligned with one another
- – Essentially the same as part (b)

**Variations:**

- – We would have to be given a completely different system in order for this problem to change