5163, Homework Assignment 6
due on Friday, 03/25/2022, at 6pm (to be uploaded to Canvas)

This homework set consists of four problems.

Problem 1:
Consider a non-interacting one-dimensional non-relativistic spinless quantum gas confined to a "rod" of length L. Assuming periodic boundary conditions, calculate the density of states.

Problem 2:
Consider a system consisting of N non-interacting particles each with isospin $I = 3/2$. The energies of the states with different projection quantum numbers m_I (the eigenvalues of $\hat{I}_z$ are $\hbar m_I$; $\hat{I}_z$ denotes the z-component of $\hat{\vec{I}}$) are given by

$$E(m_I = -3/2) = E_1, \tag{1}$$
$$E(m_I = -1/2) = E_2, \tag{2}$$
$$E(m_I = 1/2) = E_3, \tag{3}$$
$$E(m_I = 3/2) = E_3, \tag{4}$$

with

$$E_1 < E_2 < E_3 \tag{5}$$

and

$$\Delta_{12} = E_2 - E_1 \ll \Delta_{23} = E_3 - E_2. \tag{6}$$

You may treat the particles as distinguishable throughout.
(a) Without using the partition function, give the value of the total energy $\langle E \rangle$ at temperatures (i) $T = 0$, (ii) $\Delta_{12} \ll kT \ll \Delta_{23}$, and (iii) $\Delta_{23} \ll T$. Provide a justification for your results. Sketch $\langle E \rangle$ as a function of temperature.
(b) What is the occupation of the four different m_I-states in the $T \to \infty$ limit. Without using the partition function, give a value of the specific heat in the $T \to \infty$ limit. Provide a justification for your results.
(c) Without using the partition function, give the value of the average $\langle \hat{I}_z \rangle$ of the isospin z-component per particle at temperatures (i) $T = 0$, (ii) $\Delta_{12} \ll kT \ll \Delta_{23}$, and (iii) $\Delta_{23} \ll T$. Provide a justification for your results. Sketch $\langle \hat{I}_z \rangle$ as a function of the temperature.
(d) Using the partition function, compute $\langle \hat{I}_z \rangle$ in the $T \to \infty$ limit. How is your result related to the results in part (c)?

Problem 3:
(a) Calculate the density of states $D(\epsilon)$ for a non-interacting three-dimensional gas of spin-0 particles confined to a cube of volume L^3 with periodic boundary conditions.
(b) Calculate $D(\epsilon)$ for a non-interacting three-dimensional gas of spin-0 particles confined to a cube of volume L^3 with hard wall boundary conditions (i.e., boundary conditions such that the single-particle wave function vanishes at the edges of the box).

Problem 4:
This problem considers a quantum mechanical system that contains three non-interacting particles. The spatial degree of the particle can be in one of two states (the spatial degree is for simplicity assumed to be the x-coordinate): $\psi_1(x)$ with single-particle energy E_1 or $\psi_2(x)$ with single-particle energy E_2, where $E_1 < E_2$. In some of the cases considered below, the particles have a spin s. Assume that the energy levels are independent of the projection quantum number m_s.
For each of the following cases, write down (i) the zero-temperature energy, (ii) the degeneracy, and (iii) the normalized zero-temperature wave function(s) of the three-particle quantum system.
(a) Three spinless Boltzmann particles.
(b) Three identical spin-0 bosons.
(c) Three identical spin-1/2 fermions.
(d) Three identical spin-3/2 fermions.