E & M Qualifier

January, 2020

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

The magnitude of the electric field on the axis of a uniformly-charged ring and a uniformly-charged disk are given by

$$E_{ring} = \frac{Qz}{4\pi\epsilon_0(z^2+R^2)^{3/2}} \tag{1}$$

$$E_{disk} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) \tag{2}$$

where z is the distance from the center of the ring or disk, R is the radius of the ring or disk, Q is the total charge of the ring, and σ is the charge density of the disk.

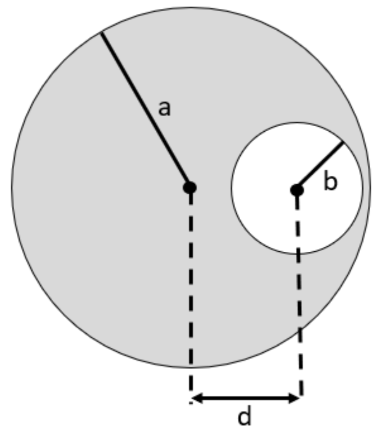
(a) Use Gauss's Law to determine the electric field from an infinite sheet of charge with uniform charge density. Explicitly show that your result is consistent with the electric field on the axis of a disk when the disk is either very large or you measure the field very close to the disk. [2 points]

(b) Assuming that the electric potential an infinite distance from the disk is zero, determine the electric potential at the center of a uniform disk of charge. [3 points]

(c) Show that you can use explicit integration of infinitesimal rings to determine the magnitude of the electric field on the axis of a solid disk with uniform charge density. [5 points]

Problem 2: Magnetostatics

Consider a long, cylindrical conductor of radius a containing a long, cylindrical hole of radius b. The axes of the cylinder and hole are parallel and are a distance d apart. A current I is uniformly distributed through the conductor (the darkened part of the diagram below).

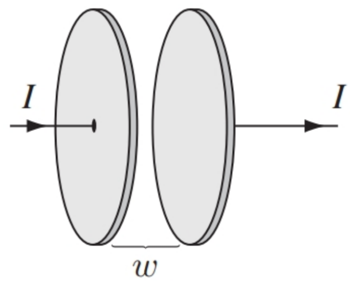


(a) Use superposition to determine the magnitude of the magnetic field at the center of the hole. [7 points]

(b) Briefly discuss two special cases: $b = 0$ and $d = 0$. What is the magnitude of the magnetic field for those two cases, and why do those magnitudes make sense? [3 points]

Problem 3: Maxwell's Equations

Consider a constant current I_0 flowing into a capacitor comprised of two circular parallel conducting plates of radius a and separation $w \ll a$ as shown in the figure. Assume that at $t = 0$ the net charge on the capacitor plates is zero. *You can ignore edge effects and assume that the wires leading to the capacitor are straight and infinitely long.*



(a) Calculate the surface charge and current densities as functions of time in terms of the parameters given. [3 points]

(b) Calculate the electric and magnetic fields in the gap between the plates in the quasi-static approximation. [4 points]

(c) The current is switched off at $t = 0$ (not the same t as before), with the current decreasing as

$$I(t) = \begin{cases} \dfrac{(t_0 - t)}{t_0} I_0 & 0 < t < t_0 \\ 0 & t > t_0. \end{cases}$$

Calculate the electric field for values of $r > a$ in the quasi-static approximation. Also give an estimate of the distance from the wire within which your calculated electric field is valid. *Note: The electric field can only be calculated up to a term that is independent of the coordinates but it can depend on t.* [3 points]

Problem 4: Electromagnetic Plane Waves

(a) Write down Maxwell's equations within a conducting medium of conductivity σ (Remember that $\vec{J} = \sigma\vec{E}$). [1 point]

(b) Use Maxwell's equations to find the wave equation for $\vec{E}$ and $\vec{B}$ for this case. [2 points]

(c) A plane wave of low frequency ω ($\omega \ll \sigma/\epsilon_0$) is propagating in the z-direction inside the conducting medium. Let $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$ and $\vec{B} = \vec{B}_0 e^{i(kz-\omega t)}$, where $\vec{E}_0$, $\vec{B}_0$, and k are complex. Use your result of part (b) to find the value of k. What is the meaning of the real and imaginary parts of k? [3 points]

(d) Calculate the ratio of the amplitude of the two fields and their phase difference. What does the phase difference mean? [2 points]

(e) Calculate the time-averaged Poynting vector. [2 points]

Problem 5: Radiation

For an electric dipole with an oscillating frequency ω and dipole moment $\vec{p} = p_o \cos(\omega t)\,\hat{z}$, the vector and scalar potentials using the Lorenz gauge are:

$$\vec{A}(\vec{r}, t) = \frac{\mu_o p_o c k}{4\pi} \sin(kr - \omega t)\,\hat{z}$$

$$V(\vec{r}, t) = \frac{p_o k}{4\pi\epsilon_o} \frac{cos\theta}{r} \sin(kr - \omega t)$$

when the distance from the dipole (centered at the origin) is large compared to the dipole size and the wavelength, and the wavelength is large compared to the dipole size.

(a) Explain the directional dependences of the vector and scalar potentials. Hint: recall how the potentials depend on the charge density and current density. [2 points]

(b) Determine the magnetic field in the radiation zone, where $kr \gg 1$. [2.5 points]

(c) Determine the electric field in the radiation zone, where $kr \gg 1$. [2.5 points]

(d) Calculate the time-averaged Poynting vector in the radiation zone. [2 points]

(e) Calculate the time-averaged power radiated by the electric dipole. [1 point]

Problem 6: Relativity

The equation of motion for a charged particle in an electromagnetic field (in Gaussian units) is given by $\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v}/c \times \vec{B})$ in three vector notation.

(a) What is the appropriate manifestly covariant form of the above equation of motion? Display the components of the covariantized EM field $F_{\mu\nu}$ [2 points]

(b) Express the four-velocity v^{μ} in component form. Show that its "length-squared" is invariant in all inertial reference frames. [2 points]

(c) For a particle moving in the $x - y$ plane with $\vec{E} = 0$ but with a uniform magnetic field $\vec{B} = B\hat{k}$, describe qualitatively the motion of the charged particle. Include a sketch of the $\vec{B}$ field and the path of motion. [2 points]

(d) Write the zeroth equation of the covariant Lorentz force law for the charged particle in a uniform magnetic field. What does it tell us? [2 points]

(e) Write the space part of the Lorentz force law for a particle in a uniform magnetic field. Extract an expression for the precession frequency $\vec{\omega}_B$ of the particle in terms of the magnetic field strength $\vec{B}$. [2 points]