Physics 5393
Solutions to Exam III

1. Consider the following Hamiltonian

$$\tilde{\mathbf{H}} = \left(\tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} + \frac{1}{2} \right) \hbar\omega.$$

Next, define the operator $\tilde{\mathbf{N}} = \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}}$ through the eigenvalue equation

$$\tilde{\mathbf{N}} \left| n \right\rangle = n \left| n \right\rangle,$$

but unlike the simple harmonic oscillator, the operators $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}^\dagger$ statisfy the following anti-commutation relations

$$\begin{aligned} \left\{ \tilde{\mathbf{a}}^\dagger, \tilde{\mathbf{a}} \right\} &= \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} + \tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger = \tilde{\mathbf{1}} \\ \left\{ \tilde{\mathbf{a}}, \tilde{\mathbf{a}} \right\} &= \tilde{\mathbf{a}} \tilde{\mathbf{a}} + \tilde{\mathbf{a}} \tilde{\mathbf{a}} = 0 \\ \left\{ \tilde{\mathbf{a}}^\dagger, \tilde{\mathbf{a}}^\dagger \right\} &= \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}}^\dagger + \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}}^\dagger = 0 \end{aligned}$$

(a) Calculate

$$\begin{aligned} \tilde{\mathbf{a}} \left| n \right\rangle &= ? \\ \tilde{\mathbf{a}}^\dagger \left| n \right\rangle &= ? \\ \tilde{\mathbf{N}}^2 \left| n \right\rangle &= ? \end{aligned}$$

To determine the effect of the $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}^\dagger$ operators on the eigenstates of $\tilde{\mathbf{N}}$, we apply the operator $\tilde{\mathbf{N}}$ on the kets $\tilde{\mathbf{a}} \left| n \right\rangle$ and $\tilde{\mathbf{a}}^\dagger \left| n \right\rangle$. Let's begin with $\tilde{\mathbf{a}} \left| n \right\rangle$

$$\tilde{\mathbf{N}} \left(\tilde{\mathbf{a}} \left| n \right\rangle \right) = \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \tilde{\mathbf{a}} \left| n \right\rangle = \left(1 - \tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger \right) \tilde{\mathbf{a}} \left| n \right\rangle = \tilde{\mathbf{a}} \left(1 - \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right) \left| n \right\rangle = (1-n) \tilde{\mathbf{a}} \left| n \right\rangle,$$

where the anti-commutation relation is used in the second equality. This implies that $\tilde{\mathbf{a}} \left| n \right\rangle$ behaves as $c \left| 1-n \right\rangle$. The coefficient c can be calculated using the inner product of $\tilde{\mathbf{a}} \left| n \right\rangle$ with its dual $\left\langle n \right| \tilde{\mathbf{a}}^\dagger$

$$\left. \begin{aligned} \left\langle n \left| \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right| n \right\rangle &= n \\ \left\langle n \left| \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right| n \right\rangle &= |c|^2 \left\langle 1-n \,|\, 1-n \right\rangle = |c|^2 \end{aligned} \right\} \quad \Rightarrow \quad c = \sqrt{n},$$

where we have assumed that the eigenkets are properly normalized, and we take the value of c as a real positive number.
We repeat the procedure for $\tilde{\mathbf{a}}^\dagger \left| n \right\rangle$

$$\tilde{\mathbf{N}} \left(\tilde{\mathbf{a}}^\dagger \left| n \right\rangle \right) = \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger \left| n \right\rangle = \tilde{\mathbf{a}}^\dagger \left(1 - \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right) \left| n \right\rangle = (1-n) \tilde{\mathbf{a}}^\dagger \left| n \right\rangle,$$

where we again make use of the anti-commutation relation in the second equality. This ket also behaves as $\tilde{\mathbf{a}}^\dagger \left| n \right\rangle = c \left| 1-n \right\rangle$, where the coefficient c is derived as follows

$$\left. \begin{aligned} \left\langle n \left| \tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger \right| n \right\rangle &= \left\langle n \left| 1 - \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right| n \right\rangle = 1-n \\ \left\langle n \left| \tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger \right| n \right\rangle &= |c|^2 \left\langle 1-n \,|\, 1-n \right\rangle = |c|^2 \end{aligned} \right\} \quad \Rightarrow \quad c = \sqrt{1-n},$$

Hence,

$$\tilde{\mathbf{a}}^\dagger \left| n \right\rangle = \sqrt{1-n} \left| 1-n \right\rangle.$$

The final expression is derived as follows

$$\begin{aligned} \tilde{\mathbf{N}}^2 \left| n \right\rangle &= \left(\tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right) \left(\tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right) \left| n \right\rangle \\ &= \tilde{\mathbf{a}}^\dagger \left(\tilde{\mathbf{a}} \tilde{\mathbf{a}}^\dagger \right) \tilde{\mathbf{a}} \left| n \right\rangle \\ &= \tilde{\mathbf{a}}^\dagger \left(\tilde{\mathbf{1}} - \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \right) \tilde{\mathbf{a}} \left| n \right\rangle \\ &= \tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}} \left| n \right\rangle = \tilde{\mathbf{N}} \left| n \right\rangle, \end{aligned}$$

where the second and third anti-commutation relations are used to show $\tilde{\mathbf{a}}^\dagger \tilde{\mathbf{a}}^\dagger = \tilde{\mathbf{a}} \tilde{\mathbf{a}} = 0$ to eliminate the second term in the second to the last equation above.

(b) Give as quantitative an argument as possible for why n is a positive integer and therefore why $\tilde{\mathbf{a}}\,|0\rangle = 0$. Also, show that there is an upper cutoff on the number of eigenstates. What is the total number of eigenstates of $\tilde{\mathbf{N}}$ and what are their eigenvalues?

To deduce this result, use the property of an inner product of a ket with its dual being greater than or equal to zero

$$\langle \alpha\,|\alpha\rangle \geq 0.$$

This can be applied to each of the two kets independently $\tilde{\mathbf{a}}\,|n\rangle$ and $\tilde{\mathbf{a}}^\dagger\,|n\rangle$

$$\begin{aligned}\left\langle n\left|\tilde{\mathbf{a}}^\dagger\tilde{\mathbf{a}}\right|n\right\rangle &= \left\langle n\left|\tilde{\mathbf{N}}\right|n\right\rangle = n \geq 0\\ \left\langle n\left|\tilde{\mathbf{a}}\tilde{\mathbf{a}}^\dagger\right|n\right\rangle &= \left\langle n\left|1-\tilde{\mathbf{N}}\right|n\right\rangle = 1-n \geq 0,\end{aligned}$$

where the anti-commutation relation was used on the second expression. The condition is then $0 \leq n \leq 1$. In order for the condition $\tilde{\mathbf{a}}\,|0\rangle = 0$ to be satisfied, n must also be an integer.

2. A neutron, which has a spin of ½, is traveling along the $+y$ axis traversing a Stern Gerlach apparatus with its magnetic field along the $+z$ axis. The apparatus selects neutrons with spins aligned along the $+z$ axis. These neutrons then traverse a second Stern Gerlach apparatus with it's axis rotated about the y axis by an angle θ relative to the z axis. *In order to receive credit on this problem, you are required to start with the appropriate operator and derive it's effect on the prepared state. Recall that there are several techniques that were discussed in class and the textbook for doing this.*

(a) Calculate the state of the neutrons after traversing the second apparatus in the S_z basis $\{\,|\pm\rangle\}$ along with the probability of having their spin up $|+\rangle$.

From the statement of the problem, the initial state $\left|\tilde{\mathbf{S}}_z,+\right\rangle$ is rotated about the y axis by an angle θ relative to the z axis. The final rotated state can be represented as

$$|\alpha\rangle = e^{-i\tilde{\mathbf{S}}_y\theta/\hbar}\left|\tilde{\mathbf{S}}_z,+\right\rangle.$$

The most straightforward method for solving this problem is to expand the state $\left|\tilde{\mathbf{S}}_z,+\right\rangle$ in the S_y basis

$$\left.\begin{aligned}\left|\tilde{\mathbf{S}}_y,+\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\tilde{\mathbf{S}}_z,+\right\rangle + i\left|\tilde{\mathbf{S}}_z,-\right\rangle\right]\\ \left|\tilde{\mathbf{S}}_y,-\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\tilde{\mathbf{S}}_z,+\right\rangle - i\left|\tilde{\mathbf{S}}_z,-\right\rangle\right]\end{aligned}\right\} \quad\Rightarrow\quad \left\{\begin{aligned}\left|\tilde{\mathbf{S}}_z,+\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|\tilde{\mathbf{S}}_y,+\right\rangle + \left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\\ \left|\tilde{\mathbf{S}}_z,-\right\rangle &= \frac{-i}{\sqrt{2}}\left[\left|\tilde{\mathbf{S}}_y,+\right\rangle - \left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\end{aligned}\right.$$

This leads to the following rotated state

$$\begin{aligned}|\alpha\rangle &= \frac{1}{\sqrt{2}}e^{-i\tilde{\mathbf{S}}_y\theta/\hbar}\left[\left|\tilde{\mathbf{S}}_y,+\right\rangle + \left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\\ &= \frac{1}{\sqrt{2}}\left[e^{-i\theta/2}\left|\tilde{\mathbf{S}}_y,+\right\rangle + e^{i\theta/2}\left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\\ &= \frac{1}{\sqrt{2}}\left\{\left[\left|\tilde{\mathbf{S}}_y,+\right\rangle + \left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\cos\left(\frac{\theta}{2}\right) + i\left[-\left|\tilde{\mathbf{S}}_y,+\right\rangle + \left|\tilde{\mathbf{S}}_y,-\right\rangle\right]\sin\left(\frac{\theta}{2}\right)\right\}\\ &= \cos\left(\frac{\theta}{2}\right)\left|\tilde{\mathbf{S}}_z,+\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\tilde{\mathbf{S}}_z,-\right\rangle.\end{aligned}$$

The final step is to calculate the probability of being in the $\left|\tilde{\mathbf{S}}_z,+\right\rangle$ state

$$\left|\langle S_z,+\,|\alpha\rangle\right|^2 = \cos^2\left(\frac{\theta}{2}\right).$$

(b) Now calculate the expectation value of S_z for the neutron after traversing the second apparatus. The expectation value is given by

$$\langle S_z \rangle = \left\langle \alpha \left| \tilde{\mathbf{S}}_z \right| \alpha \right\rangle = \left\langle \tilde{\mathbf{S}}_z, + \right| e^{-i\tilde{\mathbf{S}}_y\theta/\hbar} \tilde{\mathbf{S}}_z e^{-i\tilde{\mathbf{S}}_y\theta/\hbar} \left| \tilde{\mathbf{S}}_z, + \right\rangle .$$

Using the result from above, the expectation value can be expressed as follows

$$\left.\begin{aligned} |\alpha\rangle &= \cos\left(\frac{\theta}{2}\right) \left|\tilde{\mathbf{S}}_z, +\right\rangle + \sin\left(\frac{\theta}{2}\right) \left|\tilde{\mathbf{S}}_z, -\right\rangle \\ \langle\alpha| &= \cos\left(\frac{\theta}{2}\right) \left\langle\tilde{\mathbf{S}}_z, +\right| + \sin\left(\frac{\theta}{2}\right) \left\langle\tilde{\mathbf{S}}_z, -\right| \end{aligned}\right\} \quad \Rightarrow \quad \frac{\hbar}{2}\cos^2\left(\frac{\theta}{2}\right) - \frac{\hbar}{2}\sin^2\left(\frac{\theta}{2}\right) = \frac{\hbar}{2}\cos\theta .$$