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Quantum Mechanics 2

PHYS 5403 Homework Assignment 2

Problems: {1, 2, 3}

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Problem 1:

A wavefunction is written in the momentum representation $\psi(\tilde{\mathbf{p}}) \equiv \langle \tilde{\mathbf{p}} | \psi \rangle$.

(a) Using the Wigner definition of parity, $\pi\psi(\tilde{\mathbf{p}}) = \langle \tilde{\mathbf{p}} | \pi | \psi \rangle$, calculate the inverted wavefunction in momentum space.

We start with the Wigner definition of parity

$$\tilde{\pi}\psi(\vec{p}) = \langle \vec{p} | \tilde{\pi} | \psi \rangle$$

Which for us is

$$\tilde{\pi}\psi(\vec{p}) = \int \langle \vec{p} | \vec{x} \rangle \langle \vec{x} | \tilde{\pi} | \psi \rangle \, d\vec{x} = \int \frac{e^{-i\vec{p}\cdot\vec{x}}}{(2\pi)^3} \psi(\vec{x}) \, d\vec{x} = \psi(-\vec{p})$$

Where we expanded in a complete set for position. The inverted wave function is then

$$\boxed{\psi(-\vec{p}) = \frac{1}{(2\pi)^3} \int e^{-i\vec{p}\cdot\vec{x}/\hbar} \psi(\vec{x}) \, d\vec{x}}$$

(b) A charged particle with charge q sits in a 1D quantum well with eigenstates $|n\rangle$, $n = 1, 2, \ldots$ A small potential $U(x) = -qE_0x^m$ due to a weak electric field is then introduced, where m is a positive integer and $x = 0$ at the center of the well. using the properties of the parity operator, derive the parity selection rule for the matrix element

$$\langle n' | U(x) | n \rangle .$$

The eigenstates for a particle in a box $|n\rangle$ where $n = 1 \ldots \infty$ are

$$\tilde{\pi} |n\rangle = (-1)^{n+1} |n\rangle$$

If we have a potential of the form, $U(x) = \alpha x^m$ where $(\alpha \equiv -qE_0)$

$$-x = \tilde{\pi} x \tilde{\pi}^{-1}$$

Then we can say

$$\tilde{\pi} x^m \tilde{\pi}^{-1} = (\tilde{\pi} x \tilde{\pi}^{-1})^m = (-1)^m x^m$$

where we then have

$$\langle n' | U(x) | n \rangle = (\langle n' | \tilde{\pi}^{-1})(\tilde{\pi} U(x) \tilde{\pi}^{-1})(\tilde{\pi} | n \rangle) = (-1)^m (-1)^{n+n'+2} \langle n' | U(x) | n \rangle$$

We can then deduce that

$$\boxed{m + n' + 2 \Rightarrow \text{Must be even}}$$

Problem 1: Review

Procedure:

- – Start with the Wigner definition
$$\tilde{\pi}\psi(\tilde{\mathbf{p}}) = \langle \tilde{\mathbf{p}}|\tilde{\pi}|\psi\rangle \quad \text{and} \quad \tilde{\pi}\psi(\tilde{\mathbf{p}}) = \psi(-\tilde{\mathbf{p}})$$
  – Expand in a complete set and write the momentum representation of this new wavefunction
- – Use eigenvalue of a particle in a box $|M\rangle$ relation
$$\tilde{\pi}\,|M\rangle = (-1)^{M+1}\,|M\rangle$$
  on the potential that is given to us
  – Proceed to use the identity
$$\tilde{\pi}^{-1}\tilde{\pi} = \mathbb{I}$$
  in the matrix element we are given
  – Deduce what conditions will create a non zero matrix element

Key Concepts:

- – The Wigner definition of parity is quite simply
$$\tilde{\pi}\psi(\tilde{\mathbf{p}}) = \psi(-\tilde{\mathbf{p}})$$
  which is the inverted wave function that we are asked to solve for
- – We can use the eigenstates for a particle in a box along with the identity
$$-x = \tilde{\pi}x\tilde{\pi}^{-1}$$
  which tells us
$$\tilde{\pi}x^M\tilde{\pi}^{-1} = (-1)^M x^M$$

Variations:

- – We could be asked to write this out in position space instead of momentum space
    - * We then would handle the math differently but it would follow the same procedure
- – We could be observing a different system
    - * This would only really change if the potential now had an operator term in it

Problem 2:

(a) If $|\hat{n},-\rangle$ is a two component eigenstate of the spin projection $\tilde{\mathbf{S}}\cdot\hat{n} = \frac{1}{2}\vec{\sigma}\cdot\hat{n}$, with eigenstate $-\hbar/2$, show that application of the time reversal operator on this state, namely $i\sigma_y\tilde{\mathbf{K}}|\hat{n},-\rangle$, results in a state with the spin reversed.

We first define

$$|\hat{n},\pm\rangle = e^{i\beta\tilde{S}_z/\hbar}e^{i\alpha\tilde{S}_y/\hbar}|\pm\rangle$$

Where $|\pm\rangle$ are eigenstates of S_z. $\hat{n}$ is parametrized by Euler Angles α and β, for a Spin-1/2 particle we know

$$\vec{S}\cdot\hat{n} = \frac{1}{2}\vec{\sigma}\cdot\hat{n}$$

we can then write $|\hat{n},\pm\rangle$

$$|\hat{n},\pm\rangle = [\tilde{\mathbb{1}}\cos(\beta/2) - i\tilde{\sigma}_z\sin(\beta/2)]\times[\tilde{\mathbb{1}}\cos(\alpha/2) - i\tilde{\sigma}_y\sin(\alpha/2)]|\pm\rangle$$

we can then see how the time reversal operator works

$$\tilde{T}|\hat{n},\pm\rangle = -i\tilde{\sigma}_y\tilde{K}[\tilde{\mathbb{1}}\cos(\beta/2) - i\tilde{\sigma}_z\sin(\beta/2)]\times[\tilde{\mathbb{1}}\cos(\alpha/2) - i\tilde{\sigma}_y\sin(\alpha/2)]|\pm\rangle$$
$$= e^{i\beta\tilde{S}_z/\hbar}e^{i\alpha\tilde{S}_y/\hbar}(-i\tilde{\sigma}_y\tilde{K})|\pm\rangle = \pm|\hat{n},\mp\rangle$$

we can then clearly see we have reversed our spin ✓

■

(b) A spin 1 particle has the Hamiltonian

$$\mathcal{H} = \alpha S_z^2 + \beta(S_x^2 - S_y^2).$$

Is the Hamiltonian invariant under time reversal symmetry? Prove your answer using the properties of the time reversal symmetry operator $\mathcal{T}$.

In terms of the time reversal operator, spin has the relationship

$$\tilde{T}\tilde{S}\tilde{T}^{-1} = -\tilde{S}, \quad \tilde{T}\tilde{S}_i^2\tilde{T}^{-1} = (\tilde{T}S_i\tilde{T}^{-1})^2 = \tilde{S}_i^2$$

For our Hamiltonian we see

$$\tilde{T}\tilde{\mathcal{H}}\tilde{T}^{-1} = \alpha\tilde{T}\tilde{S}_z^2\tilde{T}^{-1} + \beta(\tilde{T}\tilde{S}_x^2\tilde{T}^{-1} - \tilde{T}\tilde{S}_y^2\tilde{T}^{-1}) = \alpha\tilde{S}_z^2 + \beta(\tilde{S}_x^2 - \tilde{S}_y^2) = \tilde{\mathcal{H}}$$

Which shows us that our Hamiltonian is invariant.

■

Problem 2: Continued

(c) Calculate the exact eigenstates of the Hamiltonian in part (b), and show that those states obey the same symmetry you found for the Hamiltonian under the time reversal symmetry.

Lets first begin by writing Sx, Sy, and Sy in 3×3 matrix representations

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Where we calculate these with

$$\langle S, M | \tilde{S}_i | S, M' \rangle = S_i\sqrt{(1-M')(2+M)}\,\delta_{M,M'+1}$$

Where $\tilde{S}_i$ is a spin in a direction i (This must be written in terms of $\tilde{S}_+$ and $\tilde{S}_-$) we can then $\tilde{S}_i^2$ are

$$\tilde{S}_z^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \tilde{S}_z^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

With these the Hamiltonian is then

$$\mathcal{H} = \alpha S_z^2 + \beta(S_x^2 \cdot S_y^2) = \left(\hbar^2\alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\beta\hbar^2}{2}\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right)\right) = \hbar^2\begin{pmatrix} \alpha & 0 & \beta \\ 0 & 0 & 0 \\ \beta & 0 & \alpha \end{pmatrix}$$

We begin calculating eigenvalues

$$\det(\mathcal{H} - \lambda\mathbb{1}) = \det\left[\begin{pmatrix} \hbar^2\alpha - \lambda & 0 & \beta \\ 0 & -\lambda & 0 \\ \beta & 0 & \hbar^2\alpha - \lambda \end{pmatrix}\right] = -\lambda(\alpha^2\hbar^4 - 2\alpha\hbar^2\lambda - \beta^2\hbar^4 + \lambda^2) = 0$$

Solving for λ we find

$$\alpha^2\hbar^4 - 2\alpha\hbar^2\lambda - \beta^2\hbar^4 + \lambda^2 = 0 \implies \lambda = \hbar^2(\alpha \pm \beta), \ \lambda = 0$$

Now we solve for the eigenstates,

$$\mathcal{H}|\psi\rangle = \lambda|\psi\rangle \implies \hbar^2\begin{pmatrix} \alpha & 0 & \beta \\ 0 & 0 & 0 \\ \beta & 0 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix} \implies \begin{aligned} \hbar^2(\alpha x + \beta z) &= \lambda(x) \\ 0 &= 0 \\ \hbar^2(\beta x + \alpha z) &= \lambda(z) \end{aligned}$$

Problem 2: Continued

Solving for x, y, and z we find

$$\hbar^2\alpha x + \hbar^2\beta z = \lambda x \quad , \quad \hbar^2\beta x + \hbar^2\alpha z = \lambda z$$

From the above we deduce that

$$z = \frac{\alpha^2\hbar^2 - 2\alpha\hbar^2\lambda - \beta^2\hbar^4 + \lambda^2}{\alpha\hbar^2 - \lambda} \; , \; y = 0 \; , \; x = \frac{\alpha^2\hbar^4 - 2\alpha\hbar^2\lambda - \beta^2\hbar^4 + \lambda^2}{\hbar^2}$$

The value of x and z is of course dependent upon λ. The eigenstates are then

$$|\lambda_\pm\rangle = \frac{1}{\sqrt{2}}\left[|1,1\rangle \pm |1,-1\rangle\right] \quad w/ \quad |\lambda_0\rangle = |1,0\rangle$$

We now act the time reversal operator on the eigenstate

$$\tilde{T}|j,m\rangle = (-1)^m|j,-m\rangle$$

That we choose. We choose to do this on $|\pm\rangle$

$$\tilde{T}|\pm\rangle = \tilde{T}\frac{1}{\sqrt{2}}(|1,1\rangle \pm |1,-1\rangle) = \frac{1}{\sqrt{2}}(-|1,-1\rangle \pm |1,1\rangle) = \pm|\mp\rangle$$

where we can clearly see that since these are eigenstates of Z these states obey the same symmetry under time reversal for our Hamiltonian

$\blacksquare$

Problem 2: Review

Procedure:

- – Start with the state
  $$|\hat{n},\pm\rangle = e^{i\beta\tilde{\mathbf{S}}_z/\hbar}e^{i\alpha\tilde{\mathbf{S}}_y/\hbar}|\pm\rangle$$
  and write it in terms of the Pauli spin matrices
  $$|\hat{n},\pm\rangle = -i\tilde{\sigma}_y\tilde{\mathbf{K}}\Big[\mathbb{I}\cos{(\beta/2)} - i\tilde{\sigma}_z\sin{(\beta/2)}\Big] \times \Big[\mathbb{I}\cos{(\alpha/2)} - i\tilde{\sigma}_y\sin{(\alpha/2)}\Big]|\pm\rangle$$
  – Proceed to calculate and show
  $$\tilde{\mathbf{T}}|\hat{n},\pm\rangle = \pm|\hat{n},\mp\rangle$$
  by using relationships between Pauli spin matrices
- – Use the relationship
  $$\tilde{\mathbf{T}}\tilde{\mathbf{S}}\tilde{\mathbf{T}}^{-1} = -\tilde{\mathbf{S}} \quad , \quad \tilde{\mathbf{T}}\tilde{\mathbf{S}}_i^2\tilde{\mathbf{T}}^{-1} = \tilde{\mathbf{S}}_i^2$$
  on our Hamiltonian and show that it is invariant
- – Calculate the Spin 1 matrix elements with
  $$\langle S,M|\tilde{\mathbf{S}}_i|S,M'\rangle = S_i\sqrt{(1-M')(2+M)}\delta_{M,M'+1}$$
  where S_i are the eigenvalues for the respective directions. Proceed to calculate the matrix representation of $\tilde{\mathbf{S}}_i^2$
  – Proceed to calculate the Hamiltonian in matrix form, then its eigenvalues and eigenstates
  – Calculate the time reversal of the eigenstates of the Hamiltonian

Key Concepts:

- – When we act a time reversal operator on the given spin state it will reverse the spin of said particle
- – Using the relationships in (b), since the Hamiltonian consists of spin operators the Hamiltonian in this case is invariant under time reversal symmetry
- – We constructed a Hamiltonian whose eigenstates are shown to obey the same time reversal symmetry that we should expect
  – Since our eigenstates in the z direction, we can think of this time reversal operator as in flipping the direction of a spin 1 particle

Variations:

- – We could be given a different initial eigenstate
  * This would change what the initial $|\hat{n},\pm\rangle$ state was but everything after that should follow the same procedure
- – We could be given a different Hamiltonian that has a non $\tilde{\mathbf{S}}^2$ term
  * This would cause the Hamiltonian to be invariant due to the relationship that we know for the time reversal operator
- – If our Hamiltonian in (b) was different
  * This would change the eigenvalues and eigenstates but other than that it would be the same primary procedure

Problem 3:

Consider the time reversal symmetry operator $\mathcal{T} = \mathcal{U}\mathcal{K}$ acting in angular momentum states $|j, m\rangle$, where $\mathcal{U}$ is a unitary operator and $\mathcal{K}$ is the conjugation.

(a) Using the properties of the time reversal symmetry operator $\mathcal{T}$ and of the conjugation operator $\mathcal{K}$, calculate:

(i) $\mathcal{J}\tilde{\mathbf{J}}_z\mathcal{U}$

(ii) $\mathcal{U}\tilde{\mathbf{J}}_{\pm}\mathcal{U}$

(iii) $\mathcal{U}\tilde{\mathbf{J}}^2\mathcal{U}$

Express your answer in angular momentum operators only. Find whether each of those angular momentum operators commute or anticommute with $\mathcal{U}$.

Using the definition of a unitary operator,

$$\tilde{\mathcal{U}}\tilde{\mathcal{U}}^{\dagger} = \mathbb{1} \quad \therefore \quad \tilde{\mathcal{U}}^{-1} = \tilde{\mathcal{U}}^{\dagger} \Rightarrow \tilde{\mathcal{U}} = (\tilde{\mathcal{U}}^{\dagger})^{-1}$$

We can write out these evaluations as

$$\hat{\mathcal{U}}\tilde{A}\hat{\mathcal{U}} = \tilde{\mathcal{U}}\tilde{A}(\hat{\mathcal{U}}^{\dagger})^{-1} = (\tilde{K}^{-1}\tilde{T})\tilde{A}((\hat{K}^{-1}\tilde{T})^{\dagger})^{-1} = (\tilde{K}^{-1}\tilde{T})\tilde{A}(((\tilde{K}^{-1})^{\dagger})^{-1}(\tilde{T}^{\dagger})^{-1})$$

Since $\tilde{T}$ is anti-unitary, the adjoint is $(\tilde{T}\tilde{T}^{\dagger} = \tilde{T}^{\dagger}\tilde{T} = \mathbb{1})$. Using this we rewrite the last line as

$$\tilde{\mathcal{U}}\tilde{A}\tilde{\mathcal{U}} = (\tilde{K}^{-1}\hat{T})\tilde{A}(((\tilde{K}^{-1})^{\dagger})^{-1}(-\tilde{T})^{-1}) = -(\tilde{K}^{-1}\tilde{T})\tilde{A}(((-\tilde{K})^{\dagger})^{-1}(\tilde{T})^{-1}) = (-\tilde{K}\tilde{T})\tilde{A}((K^{\dagger})^{-1}\tilde{T}^{-1})$$

$$= -(\tilde{K}\tilde{T})\tilde{A}((-\tilde{K})^{-1}\tilde{T}^{-1}) = (\tilde{K}\tilde{T})\tilde{A}(\tilde{K}^{-1}\tilde{T}^{-1}) = -(\tilde{K}\tilde{T})\tilde{A}(\tilde{K}\tilde{T}^{-1}) = (\tilde{K}\tilde{T})\tilde{A}(\tilde{T}^{-1}\tilde{K})$$

The final result is

$$\tilde{\mathcal{U}}\tilde{A}\tilde{\mathcal{U}} = \tilde{K}(\tilde{T}\tilde{A}\tilde{T}^{-1})\tilde{K} \qquad (*)$$

Where the final result was obtained with the aide of the relations ; $\tilde{T}^{\dagger} = \tilde{T}^{-1} = -\tilde{T}$ and $\tilde{K}^{\dagger} = \tilde{K}^{-1} = -\tilde{K}$. We now evaluate $(*)$ with the knowledge of

$$\tilde{T}\vec{x}\tilde{T} = \vec{x} \text{ (commute)}, \quad \tilde{T}\vec{P}\tilde{T} = -\vec{P} \text{ (Anti-commute)}, \quad \tilde{K}\tilde{\Omega}\tilde{K} = \tilde{\Omega}^{*}$$

We now evaluate (i) $\Rightarrow$ (iii)

(i) $\tilde{J}_z$: $\tilde{\mathcal{U}}\tilde{J}_z\tilde{\mathcal{U}} = \tilde{K}(\tilde{T}\tilde{J}_z\tilde{T}^{-1})\tilde{K} = \tilde{K}(-\tilde{J}_z)\tilde{K} = (-\tilde{J}_z)^{*} = (-\tilde{J}_z) \Rightarrow$ Anti-Commute

(ii) $\tilde{J}_{\pm}$: $\tilde{\mathcal{U}}\tilde{J}_{\pm}\tilde{\mathcal{U}} = \tilde{K}(\tilde{T}\tilde{J}_{\pm}\tilde{T}^{-1})\tilde{K} = \tilde{K}(\tilde{T}(\tilde{J}_x \pm i\tilde{J}_y)\tilde{T}^{-1})\tilde{K} = \tilde{K}(\tilde{T}\tilde{J}_x\tilde{T}^{-1} \pm i\tilde{T}\tilde{J}_y\tilde{T}^{-1})\tilde{K}$

$$= \underbrace{\tilde{K}(\tilde{T}\tilde{J}_x\tilde{T}^{-1})\tilde{K}}_{-\tilde{J}_x} + \underbrace{\tilde{K}(\pm i\tilde{T}\tilde{J}_y\tilde{T}^{-1})\tilde{K}}_{\mp\tilde{J}_y} = -\tilde{J}_x \mp i\tilde{J}_y \Rightarrow \text{Anti-Commute}$$

Problem 3: Continued

(iii) $\tilde{J}^2$: $\tilde{\mathcal{U}}\tilde{J}^2\tilde{\mathcal{U}} = \tilde{K}(\tilde{T}(\tilde{J}_x^2+\tilde{J}_y^2+\tilde{J}_z^2)T^{-1})\tilde{K} = \tilde{K}(\tilde{T}\tilde{J}_x^2\tilde{T}^{-1}+\tilde{T}\tilde{J}_y^2\tilde{T}^{-1}+\tilde{T}\tilde{J}_z^2T^{-1})\tilde{K}$

$$= \underbrace{\tilde{K}(\tilde{T}\tilde{J}_x^2\tilde{T}^{-1})\tilde{K}}_{\tilde{J}_x^2} + \underbrace{\tilde{K}(\tilde{T}\tilde{J}_y^2\tilde{T}^{-1})\tilde{K}}_{\tilde{J}_y^2} + \underbrace{\tilde{K}(\tilde{T}\tilde{J}_z^2\tilde{T}^{-1})\tilde{K}}_{\tilde{J}_z^2} = \tilde{J}_x^2+\tilde{J}_y^2+\tilde{J}_z^2$$

$$= \tilde{J}_x^2+\tilde{J}_y^2+\tilde{J}_z^2 \Rightarrow \text{Commute}$$

Therefore finally we have

$$\boxed{\tilde{J}_z \Rightarrow (-\tilde{J}_z)\text{, Anti-commute : } \tilde{J}_\pm \Rightarrow (-\tilde{J}_\pm)\text{, Anti-commute : } \tilde{J}^2 \Rightarrow (\tilde{J}^2)\text{, Commute}}$$

(b) Using your result in (a), calculate the selection rule for the matrix elements $\langle j,m'|\mathcal{U}|j,m\rangle$.

For each individual component of $\tilde{J}$, $\tilde{T}$ will Anti-commute. The total angular momentum will commute. This means our selection rules will look like

$$\tilde{\mathcal{U}}|j,m\rangle = \tilde{J}_z\tilde{T}|j,m\rangle = -\tilde{T}\tilde{J}_z|j,m\rangle = -(m)\tilde{T}|j,m\rangle \Rightarrow \tilde{T}|j,m\rangle \propto |j,-m\rangle$$

We can see from the above for (i) the selection rule is that $m'=-m$ for the matrix elements to be non-zero. For (ii)

$$\tilde{\mathcal{U}}|j,m\rangle = \tilde{J}_\pm\tilde{T}|j,m\rangle = \tilde{J}_x\tilde{T}|j,m\rangle \pm i\tilde{J}_y\tilde{T}|j,m\rangle = C_\pm\tilde{T}|j,m\pm1\rangle$$

This means the matrix elements will be non-zero if and only if $m'=m\pm1$. Any other instance will create a zero matrix element. Finally (iii) is

$$\tilde{\mathcal{U}}|j,m\rangle = \tilde{J}^2\tilde{T}|j,m\rangle = j\tilde{T}|j,m\rangle \Rightarrow \tilde{T}|j,m\rangle \propto |j,-m\rangle$$

For the total angular momentum the selection rule will once again be $m'=-m$ and the matrix element will be j. So finally we have

$$\boxed{\text{(i) } m'=-m\text{ , }-(m)\text{ , (ii) } m'=m\pm1\text{, } C_\pm\text{ , (iii) } m'=m\pm1\text{, } j}$$

(c) Now show that

$$\frac{\langle j,m|\mathcal{U}|j,-m'\rangle}{\langle j,m|\mathcal{U}|j,-m\rangle} = i^{2(m'-m)}.$$

Choosing $\langle j,m|\mathcal{U}|j,-m\rangle = (i)^{2m}$, then find that

$$\mathcal{T}|j,m\rangle = (-1)^m|j,-m\rangle.$$

Problem 3: Continued

We start with the definition

$$\langle j,m|\tilde{u}|j,-m\rangle = (i)^{2m} \qquad (*)$$

We can go on to say that

$$\langle j,m|\tilde{u}|j,-m'\rangle = (i)^{2m'} \quad \& \quad \langle j,m|\tilde{u}|j,-m\rangle = (i)^{2m}$$

This then means we can show

$$\frac{\langle j,m|\tilde{u}|j,-m'\rangle}{\langle j,m|\tilde{u}|j,-m\rangle} = \frac{(i)^{2m'}}{(i)^{2m}} = (i)^{2m'-2m} = (i)^{2(m'-m)}$$

Going back to (*)

$$\langle j,m|\tilde{u}|j,-m\rangle = (i)^{2m} \Rightarrow \tilde{T}|j,m\rangle = (-1)^m|j,-m\rangle \quad w/ \quad \tilde{T}|r'\rangle = |-r'\rangle$$

Depending on the magnetic quantum number $(-1)^m$ can either be positive or negative. We then use the previous result to show

$$\tilde{T}|j,m\rangle = (-1)^m|j,-m\rangle$$

$\blacksquare$

(d) Find the time reversed state of the rotated ket $\mathcal{D}(R)|j,m\rangle$.

We start by applying the time reversal operator to our state with a rotation

$$\tilde{T}\tilde{D}(R)|j,m\rangle = \tilde{T}e^{i\varphi\vec{J}\cdot\hat{n}/\hbar}|j,m\rangle = e^{i\varphi\vec{J}\cdot\hat{n}/\hbar}\tilde{T}|j,m\rangle = (-1)^m D(R)|j,-m\rangle$$

Where we used, $\tilde{T}D(R) = D(R)\tilde{T}$

(e) Starting from the ket $\mathcal{D}(R)\mathcal{T}|j,m\rangle$, show that

$$\mathcal{D}^{(j)*}_{m',m}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R).$$

Hint: Use the commutator $[\mathcal{D}(R),\mathcal{T}]$ in your derivation.

We first start with the definition of

$$\tilde{T}|j,m\rangle = (-1)^m|j,-m\rangle$$

We begin by calculating the matrix elements

Problem 3: Continued

$$\langle j,-m'|\tilde{D}(R)\tilde{T}|j,m\rangle = e^{i\delta}(-1)^m\langle j,-m'|\tilde{D}(R)|j,-m\rangle = e^{i\delta}(-1)^m\tilde{D}^{(j)}_{-m',-m}(R)$$

$$= \langle j,-m'|\tilde{T}\sum_{m''}|j,m''\rangle\langle j,m''|\tilde{D}(R)|j,m\rangle$$

$$= \sum_{m''}\tilde{D}(R)^{(j)*}_{m'',m}\langle j,-m'|\tilde{T}|j,m''\rangle = \sum_{m''}\tilde{D}(R)^{(j)*}_{m'',m}\langle j,-m'|j,-m''\rangle$$

$$= \sum_{m''}\tilde{D}^{(j)*}_{m'',m}(R)\,e^{i\delta}(-1)^{m''}\delta_{-m'',-m'} = (-1)^{m-m'}\tilde{D}^{(j)*}_{-m',m}(R) \checkmark$$

$\blacksquare$

(f) If a system is time reversal invariant and has no degeneracy in the energy spectrum $\mathcal{H}|E\rangle = E|E\rangle$, show that

$$\langle E|\tilde{\mathbf{L}}|E\rangle = 0.$$

For the system to be time reversal invariant with no degeneracy we see that

$$\tilde{\mathcal{H}}|E\rangle = E|E\rangle \Rightarrow \tilde{\mathcal{H}}\tilde{T}|E\rangle = E\tilde{T}|E\rangle \Rightarrow |\tilde{E}\rangle = \tilde{T}|E\rangle \Rightarrow |\tilde{E}\rangle = e^{i\delta}|E\rangle$$

Knowing the above, we can say

$$\langle E|\tilde{L}|E\rangle = \langle\tilde{E}|\tilde{T}\tilde{L}\tilde{T}^{-1}|\tilde{E}\rangle = -\langle\tilde{E}|\tilde{L}|\tilde{E}\rangle = -e^{i\delta}\langle E|\tilde{L}|E\rangle e^{i\delta} = -\langle E|\tilde{L}|E\rangle$$

In order for the above to be correct, $\langle E|\tilde{L}|E\rangle = 0$ ✓

$\blacksquare$

Problem 3: Review

Procedure:

- – Begin with the relationship
$$\tilde{\mathbf{U}}\tilde{\mathbf{A}}\tilde{\mathbf{U}} = \tilde{\mathbf{K}}(\tilde{\mathbf{T}}\tilde{\mathbf{A}}\tilde{\mathbf{T}}^{-1})\tilde{\mathbf{K}}$$
and further
$$(\tilde{\mathbf{T}}\tilde{\mathbf{J}}_i\tilde{\mathbf{T}}^{-1}) = -\tilde{\mathbf{J}}_i \qquad \Longrightarrow \text{Anti-Commute}$$
and lastly
$$(\tilde{\mathbf{T}}\tilde{\mathbf{J}}_i\tilde{\mathbf{T}}^{-1})^2 = \tilde{\mathbf{J}}_i \qquad \Longrightarrow \text{Commute}$$
to answer which commute and don't
- – Use a unitary operator consisting of $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{T}}$ on the angular momentum eigenstate for each selection rule that we are seeking
- – Start with the definition
$$\langle j, m|\tilde{\mathbf{U}}|j, -m\rangle = (i)^{2m}$$
and expand to show the required relationships
- – Use the rotation operator after the time reversal operator to find the new rotated time reversed key
- – Start with the relationship
$$\tilde{\mathbf{T}}\,|j, m\rangle = (-1)^m\,|j, -m\rangle$$
and then calculate the rotated time reversed matrix element
- – Use the typical energy eigenvalue equation
$$\mathcal{H}\,|E\rangle = E\,|E\rangle$$
and then the expectation value at the beginning of the question

Key Concepts:

- – Commuting with $\mathcal{U}$ means the operator is left unchanged after time reversal and anti-commuting means it is changed
- – Using the unitary operator on the angular momentum eignestate we can determine what our values of m and m' have to be for the matrix elements to be non zero
- – Using the definition of how to calculate matrix elements of $\mathcal{U}$ we can determine the relationship that is being asked of us to prove
- – We can switch the operation of time reversal and rotation to determine how our state will evolve after rotating
- – Using the rules of time reversal symmetry on the angular momentum eigenstates we can calculate the matrix elements of a rotated time reversed state to determine our relationship
- – Using the standard Hamiltonian eigenvalue equation we can show that the matrix element of $\tilde{\mathbf{L}}$ is zero

Variations:

- – Since all of these are proving identities the entire problem would have to change to create variations
  - ∗ Aspects of this problem can be used in other problems