E & M I
Workshop 3 – Dipoles, Solutions

The simplest model for net neutral charge distributions, the leading term at large distances, is the dipole. This is an approximation useful for considering neutral molecules with a net $\pm$ charge separation.

The electric potential due to the dipole moment, or second term in the multipole expansion is:

$$\phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\vec{p} = \sum_q q\,\vec{r}_q \ \ \text{(point charges)} \quad \vec{p} = \int d^3r_q\,\rho(\vec{r}_q)\,\vec{r}_q \ \ \text{(charge density } \rho(\vec{r}_q)\text{)}$$

Calculating the electric field that corresponds to the dipole potential, we got the result:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi^1(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\,(\vec{p}\cdot\vec{r})\frac{\vec{r}}{r^2} - \vec{p}\right) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(\,3\,(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p})$$

1) Calculating Dipoles:

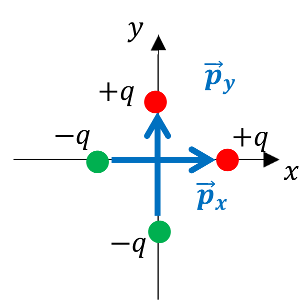
i) Consider the charge distribution with four charges:

q at (x = a, y = 0, z = 0), -q at (x = -a, y = 0, z = 0), q at (x = 0, y = a, z = 0), -q at (x = 0, y = -a, z = 0)

Draw a picture showing the charges. Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer.

The basic structure of a dipole is positive and negative charges separated by some distance. It's a vector pointing from the negative charge to the positive charge.

This charge distribution looks like two equal dipoles, $\vec{p}_x$ in the x-direction and $\vec{p}_y$ in the y-direction. The total dipole is the sum of these two, giving a dipole at a 45° angle in the +x-+y direction.



ii) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

The dipole in this case will be:

$$\vec{p} = \sum q\,\vec{r}_q = q\,(a\,\hat{x}) - q\,(-a\,\hat{x}) + q\,(a\,\hat{y}) - q\,(-a\,\hat{y})$$

$$\vec{p} = 2\,q\,a\,(\hat{x}+\hat{y}) = \vec{p}_x + \vec{p}_y$$

As predicted.

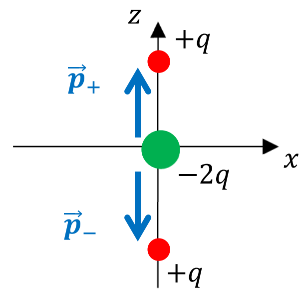
iii) Consider the charge distribution with 3 charges:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

Draw a picture showing the charges. Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer.

In this case, the charge distribution looks like to dipoles along the z-axis, one in the +z direction and the other in the -z direction.

The total will be zero.



iv) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

The dipole in this case will be:

$$\vec{p} = \sum q\, \vec{r}_q = q\,(a\,\hat{z}) + q\,(-a\,\hat{z}) - 2q\,(0) = 0$$

As predicted.

2) Non-Uniformly Charged Line:

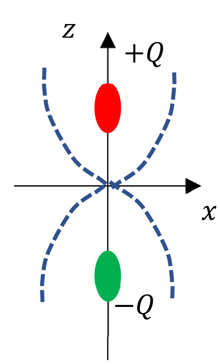
Consider a charged line of length $2L$ on the z-axis that extends from $z = -L$ to $z = L$. The linear charge density on the line varies as a function of z as:

$$\lambda(z) = \lambda_0 \sin\left(\frac{\pi}{2L}\, z\right)$$

i) Draw a picture and describe what you would expect the dipole moment $\vec{p}$ to be for this charged line. Include the direction of the dipole and an estimate of the magnitude of the dipole.

This charge distribution looks like positive charge distributed on the +z axis and negative charge distributed on the -z axis. The magnitude of this charge is:

$$Q = \int_0^L \lambda_0 \sin\left(\frac{\pi}{2L}\, z\right)\, dz = -\lambda_0\, \frac{2L}{\pi} \cos\left(\frac{\pi}{2L}\, z\right)\Big|_0^L = \lambda_0\, \frac{2L}{\pi}$$



The “separation” between the charges will be somewhere between L and $2L$, probably closer to $2L$ because the charge is peaked at the ends of the distribution. An estimate for the dipole would be:

$$\vec{p} \approx \lambda_0\, L^2\, \hat{z}$$

ii) Solve for the dipole moment, $\vec{p}$. Compare your result to your expected result from above and explain any differences. (Use any resource you want to do the integral.)

Starting from the general expression:

$$\vec{p} = \int d^3r_q\, \rho(\vec{r}_q)\, \vec{r}_q$$

$$\rho(\vec{r}_q) = \lambda(z)\, \delta(\mathrm{x})\delta(y)$$

$$\vec{p} = \hat{z}\, \lambda_0 \int_{-L}^{L} \sin\left(\frac{\pi}{2L}\, z\right)\, z\, dz$$

Integrating by parts:

$$\vec{p} = \hat{z}\, \lambda_0 \left(-\frac{2L}{\pi}\, z\, \cos\left(\frac{\pi}{2L}\, z\right) \Big|_{-L}^{L} + \frac{2L}{\pi} \int_{-L}^{L} \cos\left(\frac{\pi}{2L}\, z\right)\, dz \right)$$

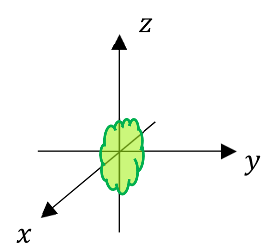
$$\vec{p} = \hat{z}\, \lambda_0 \left(\frac{8\, L^2}{\pi^2} \right) = \frac{8}{\pi^2}\, \lambda_0\, L^2\, \hat{z}$$

As $\frac{8}{\pi^2} \approx 0.8$, our estimate is very close.

Dipole Interactions: Potential Energy, Force, and Torque

You likely have seen the result that the potential energy of a dipole $\vec{p}$ at a position $\vec{r}$ in an electric field $\vec{E}(\vec{r})$ is

$$U_p = -\vec{p} \cdot \vec{E}(\vec{r})$$



Consider a “cloud” of charges with a distribution $\rho(\vec{r})$ confined in a very small region around $\vec{r} = 0$. Let the total charge be zero: $\int \rho(\vec{r})\, d^3r = 0$

There is an external potential $\phi(\vec{r})$ due to other sources for the potential.

The potential energy of the charged cloud is:

$$U = \int \rho(\vec{r})\, \phi(\vec{r})\, d^3r$$

If the cloud is confined to a small region relative to changes in $\phi(\vec{r})$, we can approximate the potential:

$$\phi(\vec{r}) = \phi(0) + \vec{r} \cdot \vec{\nabla}\phi(\vec{r})|_{r=0} + \frac{1}{2}\, (\vec{r} \cdot \vec{\nabla})(\vec{r} \cdot \vec{\nabla})\, \phi(\vec{r})|_{r=0} + \cdots$$

1) Finding the Potential Energy

i) Using the definition of the dipole potential energy and the expansion of the potential, show that the dipole potential energy is given by the second term in the expansion of the potential.

$$U = \int \rho(\vec{r})\, \phi(\vec{r})\, d^3r$$

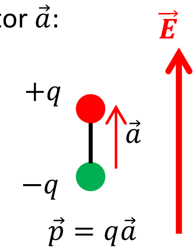
$$U = \int \rho(\vec{r})\phi(0)\, d^3r + \int \rho(\vec{r})\, \vec{r} \cdot \vec{\nabla}\phi(\vec{r})|_{r=0}\, d^3r + \cdots$$

$$U = Q\,\phi(0) + \int \rho(\vec{r})\, \vec{r}\, d^3r \cdot \vec{\nabla}\phi(\vec{r})|_{r=0} + \cdots = Q\,\phi(0) - \vec{p} \cdot \vec{E}(0) + \cdots$$

This gives

$$U_p = -\vec{p} \cdot \vec{E}$$

ii) The standard picture of a dipole is two charges $\pm q$ separated by a (small) vector $\vec{a}$:



By considering such a dipole in a constant electric field $\vec{E}$, explain why the potential energy makes physical sense. For example, what orientation will give the lowest energy and why?

The positive charge lowers its energy by moving in the direction of the electric field, and the negative lowers its potential energy by moving in the direction opposite to the electric field:

$$\Delta U = -q \int \vec{E} \cdot d\vec{l}$$

This means the lowest (negative) energy is for $\vec{p}$ parallel to $\vec{E}$ and the largest (positive) energy is for $\vec{p}$ anti-parallel to $\vec{E}$.

2) Force and Torque on a Dipole

The relation between energy and force gives the force on a dipole:

$$\vec{F} = -\vec{\nabla}U_p = \vec{\nabla}(\vec{p} \cdot \vec{E}) = (\vec{p} \cdot \vec{\nabla})\, \vec{E}$$

i) Showing this last relation, consider the double cross product: $(\vec{\nabla} \times \vec{E} = 0)$

$$\vec{p} \times (\vec{\nabla} \times \vec{E}) = \hat{e}_i\, \epsilon_{ijk}\, p_j\, \epsilon_{klm}\, \partial_l\, E_m$$

$$= \hat{e}_i\, \epsilon_{kij}\, \epsilon_{klm}\, p_j\, \partial_l\, E_m$$

$$= \hat{e}_i\, (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, p_j\, \partial_l\, E_m$$

$$= \hat{e}_i\, \partial_i\, p_j\, E_j - p_j\, \partial_j\, \hat{e}_i\, E_i$$

$$\vec{p} \times (\vec{\nabla} \times \vec{E}) = 0 = \vec{\nabla}\,(\vec{p} \cdot \vec{E}) - (\vec{p} \cdot \vec{\nabla})\, \vec{E} \;\Rightarrow\; \vec{\nabla}(\vec{p} \cdot \vec{E}) = (\vec{p} \cdot \vec{\nabla})\, \vec{E}$$

ii) The torque on a dipole can be determined by changes in energy due to rotations in $\vec{p}$. For a small rotation of the dipole by an angle $d\theta$ around a direction $\hat{n}$, the change in energy is given by the work done by the torque:

$$dU = -\vec{\tau} \cdot \hat{n}\, d\theta = -\vec{E} \cdot d\vec{p}$$

$$d\vec{p} = d\theta\, \hat{n} \times \vec{p}$$

$$\vec{\tau} \cdot \hat{n}\, d\theta = \vec{E} \cdot (\hat{n} \times \vec{p})\, d\theta$$

For another quick vector algebra problem, **show**: $\vec{E} \cdot (\hat{n} \times \vec{p}) = \hat{n} \cdot (\vec{p} \times \vec{E})$

$$\vec{E} \cdot (\hat{n} \times \vec{p}\,) = E_i\, \epsilon_{ijk}\, n_j\, p_k$$

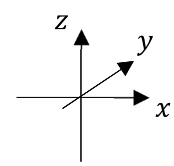
$$= n_j\, \epsilon_{jki}\, p_k\, E_i = \hat{n} \cdot (\,\vec{p} \times \vec{E}\,)$$

This gives:

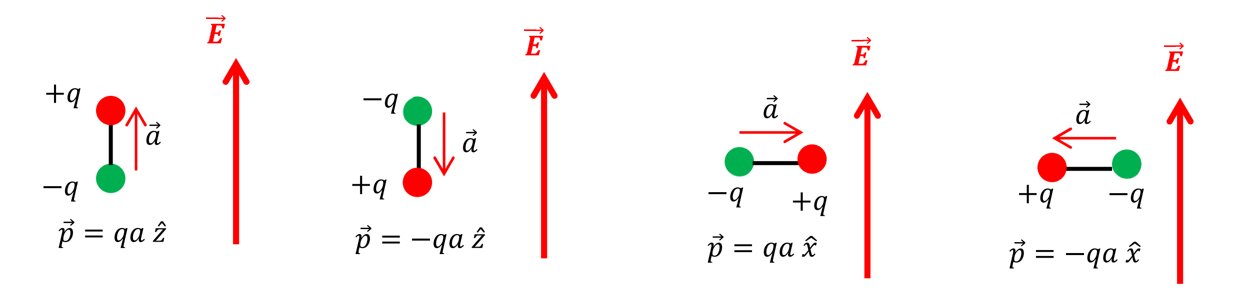
$$\vec{\tau} \cdot \hat{n}\, d\theta = \left(\vec{p} \times \vec{E}\,\right) \cdot \hat{n}\, d\theta$$

$$\vec{\tau} = \vec{p} \times \vec{E}$$

Again, using the standard model of a dipole, demonstrate that this expression for the torque makes physical sense for a dipole in a constant field $\vec{E}$.



Consider four basic cases. Define $\vec{E} = E\,\hat{z}$



$\vec{p} \times \vec{E} \propto \hat{z} \times \hat{z} = 0$	$\vec{p} \times \vec{E} \propto -\hat{z} \times \hat{z} = 0$	$\vec{p} \times \vec{E} \propto \hat{x} \times \hat{z} = -\hat{y}$	$\vec{p} \times \vec{E} \propto -\hat{x} \times \hat{z} = \hat{y}$
Stable Equilibrium	Unstable Equilibrium	Counter-Clockwise torque	Clockwise torque