Solutions to Homework 2
Physics 5393

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P-1.9 Consider a ket space spanned by the eigenkets $\{|a'\rangle\}$ of a Hermitian operator $\tilde{\mathbf{A}}$. There is no degeneracy.

a) Prove that

$$\prod_{a'}(\tilde{\mathbf{A}} - a')$$

is the null operator.

To simplify the notation, we define $a_j \equiv a'$ and rewrite the operator as

$$\prod_j (\tilde{\mathbf{A}} - a_j \tilde{\mathbf{1}}),$$

where the implied identity operator is explicitly given.

To prove that this is the null operator, we apply this operator on an arbitrary ket $|\alpha\rangle$. Then using the completeness relation, this ket is expanded in the eigenkets of the operator $\tilde{\mathbf{A}}$

$$\prod_j (\tilde{\mathbf{A}} - a_j \tilde{\mathbf{1}})\, |\alpha\rangle = \prod_j (\tilde{\mathbf{A}} - a_j \tilde{\mathbf{1}}) \sum_i |a_i\rangle\langle a_i|\, |\alpha\rangle = \sum_i \prod_j (a_i - a_j)\, |a_i\rangle \langle a_i | \alpha\rangle$$

Since the sum is over a complete set, the product will always contain the term $a_j = a_i$, therefore the operator is the null operator.

b) Explain the significance of

$$\prod_{a'' \neq a'} \frac{\tilde{\mathbf{A}} - a''}{a' - a''}.$$

To simplify the notation, we define $a_j \equiv a'$, $a_k \equiv a''$ and rewrite the operator as

$$\prod_{k \neq j} \frac{\tilde{\mathbf{A}} - a_k \tilde{\mathbf{1}}}{a_j - a_k},$$

where the implied identity operator is explicitly given.

In this case, we drop one term in the product $a_k = a_j$. This leaves one term in the sum that will be non-zero

$$\sum_i \prod_{k \neq j} \left(\tilde{\mathbf{A}} - a_k \tilde{\mathbf{1}}\right) |a_i\rangle \langle a_i | \alpha\rangle = \prod_{k \neq j} (a_j - a_k)\, |a_j\rangle \langle a_j | \alpha\rangle .$$

Dividing by $a_j - a_k$ produces the projection operator in the a_j direction.

c) Illustrate (a) and (b) using $\tilde{\mathbf{A}}$ set equal to $\tilde{\mathbf{S}}_z$ of a spin 1/2 system.
The operator $\tilde{\mathbf{A}}$ is replaced with $\tilde{\mathbf{S}}_z$ and its eigenkets $|\pm\rangle$. The first operator is therefore

$$\prod_{a'}(\tilde{\mathbf{A}} - a') = \left(\tilde{\mathbf{S}}_z - \frac{\hbar}{2}\right)\left(\tilde{\mathbf{S}}_z + \frac{\hbar}{2}\right),$$

which is clearly the null operator.
The second operator is

$$\prod_{a'' \neq a'} \frac{\tilde{\mathbf{A}} - a''}{a' - a''} = \begin{cases} \dfrac{\tilde{\mathbf{S}}_z + \hbar/2}{\hbar} \\ \dfrac{\tilde{\mathbf{S}}_z - \hbar/2}{-\hbar}. \end{cases}$$

If we apply these operators on the eigenkets of $\tilde{\mathbf{S}}_z$, the upper operator projects out $|+\rangle$ and the lower operator the $|-\rangle$.

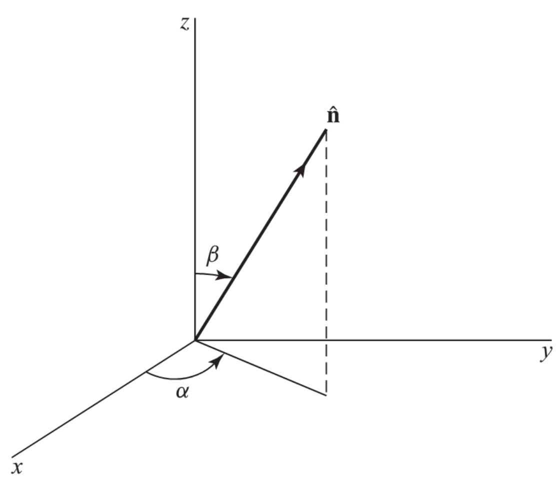
P-1.11 Construct $\left|\tilde{\mathbf{S}}\cdot\hat{\mathbf{n}};+\right\rangle$ such that

$$\tilde{\mathbf{S}}\cdot\hat{\mathbf{n}}\left|\tilde{\mathbf{S}}\cdot\hat{\mathbf{n}};+\right\rangle = \frac{\hbar}{2}\left|\tilde{\mathbf{S}}\cdot\hat{\mathbf{n}};+\right\rangle,$$

where $\hat{\mathbf{n}}$ is characterized by the angles shown in the accompanying figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [Note: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\alpha}\sin\left(\frac{\beta}{2}\right)|-\rangle.$$

But do not just verify that this answer satisfies the above eigenvalue equation. rather, treat the problem as a straightforward eigenvalue problem. Also, do not use rotation operators, which we will introduce later in the book.]



From the figure, the unit vector $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}} = \hat{\mathbf{x}}\sin\beta\cos\alpha + \hat{\mathbf{y}}\sin\beta\sin\alpha + \hat{\mathbf{z}}\cos\beta,$$

therefore the form of the operator is

$$\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}} = \sin\beta\cos\alpha\tilde{\mathbf{S}}_x + \sin\beta\sin\alpha\tilde{\mathbf{S}}_y + \cos\beta\tilde{\mathbf{S}}_z.$$

To proceed, we expand the eigenstate in the $|\pm\rangle$ basis as follows

$$\left|\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}}; +\right\rangle = a\,|+\rangle + b\,|-\rangle$$

with the constraint $|a|^2 + |b|^2 = 1$, to ensure proper normalization. Next, we apply the $\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}}$ operator in the form derived above on the eigenstate in the $|\pm\rangle$ basis. We will proceed by first applying each of the spin operators on the eigenstate. We start with $\tilde{\mathbf{S}}_x$:

$$\begin{aligned}\tilde{\mathbf{S}}_x\left[a\,|+\rangle + b\,|-\rangle\right] \\ &= \tilde{\mathbf{S}}_x\left[a\left(\frac{|S_x;+\rangle - |S_x;-\rangle}{\sqrt{2}}\right) + b\left(\frac{|S_x;+\rangle + |S_x;-\rangle}{\sqrt{2}}\right)\right] \\ &= \frac{\hbar}{2}\left[a\,|-\rangle + b\,|+\rangle\right].\end{aligned}$$

Next $\tilde{\mathbf{S}}_y$:

$$\begin{aligned}\tilde{\mathbf{S}}_y\left[a\,|+\rangle + b\,|-\rangle\right] \\ &= \tilde{\mathbf{S}}_y\left[a\left(\frac{|S_y;+\rangle + |S_y;-\rangle}{\sqrt{2}}\right) + ib\left(\frac{-|S_y;+\rangle + |S_y;-\rangle}{\sqrt{2}}\right)\right] \\ &= \frac{\hbar}{2}\left[ia\,|-\rangle - ib\,|+\rangle\right].\end{aligned}$$

And the last operator $\tilde{\mathbf{S}}_z$

$$\tilde{\mathbf{S}}_z\left[a\,|+\rangle + b\,|-\rangle\right] = \frac{\hbar}{2}\left[a\,|+\rangle - b\,|-\rangle\right].$$

This allows us to apply the $\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}}$ operator on the eigenstate

$$\begin{aligned}\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}}\left[a\,|+\rangle + b\,|-\rangle\right] = \\ \frac{\hbar}{2}\Big[\sin\beta\cos\alpha\left[a\,|-\rangle + b\,|+\rangle\right] + \sin\beta\sin\alpha\left[ia\,|-\rangle - ib\,|+\rangle\right] \\ + \cos\beta\left[a\,|+\rangle - b\,|-\rangle\right]\Big].\end{aligned}$$

We can now equate the form of the eigenstate on the left hand side to that on the right hand side

$$\begin{aligned}\left[a\,|+\rangle + b\,|-\rangle\right] \\ &= \left[\sin\beta\cos\alpha\left[a\,|-\rangle + b\,|+\rangle\right] + \sin\beta\sin\alpha\left[ia\,|-\rangle - ib\,|+\rangle\right] + \cos\beta\left[a\,|+\rangle - b\,|-\rangle\right]\right].\end{aligned}$$

Since the eigenkets are independent of each other, we can equate the coefficients of each pair independently. Therefore, we choose to perform the calculation using the coefficients of $|+\rangle$. We assume that a is real and positive, and use the normalization condition given above

$$\begin{aligned}a &= b\sin\beta\cos\alpha - ib\sin\beta\sin\alpha + a\cos\beta \\ a\left(1 - \cos\beta\right) &= b\sin\beta\left(\cos\alpha - i\sin\alpha\right) \\ a\left(1 - \cos\beta\right) &= be^{-i\alpha}\sin\beta.\end{aligned}$$

We next apply the half angle trigonmetric identities

$$2a\sin^2\left(\frac{\beta}{2}\right) = 2be^{-i\alpha}\cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right).$$

Finally, we calculate the magnitude squared of both sides assuming a is real and b is complex, and simplify

$$\begin{aligned} 4a^2\sin^4\left(\frac{\beta}{2}\right) &= 4|b|^2\cos^2\left(\frac{\beta}{2}\right)\sin^2\left(\frac{\beta}{2}\right) \\ 4a^2\sin^2\left(\frac{\beta}{2}\right) &= 4(1-a^2)\cos^2\left(\frac{\beta}{2}\right) \\ \Rightarrow \quad a = \cos\left(\frac{\beta}{2}\right) & \end{aligned}$$

The coefficient b can be derived by substituting a into the last equation before calculating the magnitude squared to derive

$$b = e^{i\alpha}\sin\left(\frac{\beta}{2}\right).$$

We have therefore arrived at the desired result

$$\left|\tilde{\mathbf{S}}\cdot\hat{\mathbf{n}};+\right\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\alpha}\sin\left(\frac{\beta}{2}\right)|-\rangle.$$

P-1.17 Let $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ be observables. Suppose the simultaneous eigenkets of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$\left[\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\right] = 0? \tag{1}$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

The answer is yes. This can be proved using the completeness relation

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}}\sum_{i,j}|a_i, b_{i,j}\rangle\langle a_i, b_{i,j}| = \sum_{i,j}a_i b_{i,j}\,|a_i, b_{i,j}\rangle\langle a_i, b_{i,j}| = \tilde{\mathbf{B}}\tilde{\mathbf{A}},$$

where in the last step we note that the order of the operators does not matter in deriving the eigenvalues and the notation i, j denotes the degenerate eigenstates of b_i associated with a_i . Also, it is important that the sum be over both a_i and $b_{i,j}$ to ensure the full space is spanned.

P-1.18 Two Hermitian operators anti-commute:

$$\left\{\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\right\} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\tilde{\mathbf{A}} = 0 \tag{2}$$

Is it possible to have a simultaneous (that is, common) eigenket of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$? Prove or illustrate your assertion.

For this to be possible, the following condition must be satisfied

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}}\,|a,b\rangle = -\tilde{\mathbf{B}}\tilde{\mathbf{A}}\,|a,b\rangle \quad \Rightarrow \quad ab = -ab.$$

This requires that the eigenvalues a and/or b must be equal to zero.

Additional Problems

Q-1 The state space of a certain physical system is 3-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$ and $|\psi_1\rangle$ are defined by:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_2\rangle$$

a) Are these kets normalized?

To determine if they are normalized, determine the scalar product. The first equation gives

$$\langle\psi_0|\psi_0\rangle = 1 \tag{3}$$

therefore normalized according to our probablistic interpretation of quantum mechanics. The second equation leads to:

$$\langle\psi_1|\psi_1\rangle = \frac{2}{3} \tag{4}$$

therefore it is not normalized. The proper normalization is

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{\sqrt{2}}|u_2\rangle .$$

b) Calculate the matrices $\tilde{\mathbf{P}}_0$ and $\tilde{\mathbf{P}}_1$ representing, in the above given basis set, the projection operators onto the state $|\psi_0\rangle$ and $|\psi_1\rangle$. Verify that the matrices are Hermitian.

The projection operators are give by $P_i = |\psi_i\rangle\langle\psi_i|$. To extract the matrix elements in the basis set $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$, it must be expressed in this basis set

$$|\psi_i\rangle\langle\psi_i| = \sum_{j,k} |u_j\rangle\langle u_j|\psi_i\rangle\langle\psi_i|u_k\rangle\langle u_k| ,$$

where the matrix elements are given by $\langle u_j|\psi_i\rangle\langle\psi_i|u_k\rangle$. From this expression, the matrix for P_0 is:

$$P_0 = \begin{pmatrix} \frac{1}{2} & \frac{i}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{i}{2\sqrt{2}} & \frac{1}{4} & -\frac{i}{4} \\ \frac{1}{2\sqrt{2}} & \frac{i}{4} & \frac{1}{4} \end{pmatrix}$$

and the matrix for P_1:

$$P_1 = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Both matrices satisfy the condition for being Hermitian.