COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

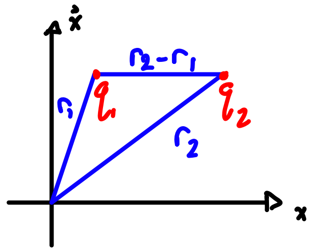
Electrodynamics 1

CH. 3 ELECTROSTATICS IN VACUUM LECTURE NOTES

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Bruce Mason

1-24-22



The Force on these will be : 1 on 2

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 \cdot q_2}{|r_2 - r_1|^2}\hat{e}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{(r_2 - r_1)^2}(r_2 - r_1)$$

The electric field is found with

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|^3}\vec{r}$$

Where we now do

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\frac{\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\frac{x_i\hat{e}_i}{(x_i \cdot x_j)^{3/2}} = \frac{q}{4\pi\epsilon_0}\partial_i\frac{x_i}{(\vec{x}_i \cdot \vec{x}_j)^{3/2}}$$

And then proceed to do the following

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\delta_{ii}(x_i x_i)^{3/2} - x_i\partial_i(x_i x_j)^{3/2}}{(x_j x_j)} = \frac{q}{4\pi\epsilon_0}\frac{3(x_j x_j)^{3/2} - x_i(x_i x_j)^{1/2}\partial x_j \delta_{ij}}{(x_j x_j)^3} = 0$$

Now we look at making r very small

$$\int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \int_{S=\partial V}\vec{E}\cdot\hat{n}\, ds \longrightarrow$$ True for any vector field (well behaved)

$$\vec{E}\cdot\hat{n} = \vec{E}\cdot\hat{r} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}\cdot\hat{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2} \checkmark$$

We know that

$$\vec{\nabla}\cdot\vec{E} = 0 \text{ when } \vec{r} \neq 0 : \int_V \vec{\nabla}\cdot\vec{E}\, d^3x = \frac{q}{\epsilon_0}$$

Delta Functions are then defined as :

$$\delta^3(\vec{r} - \vec{r}') = 0 \quad \vec{r} \neq \vec{r}' , \int \delta^3(\vec{r} - \vec{r}')f(\vec{r}')\, d^3\vec{r}' = f(\vec{r}) \text{ w/ } \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\sum_i \rho(r)\delta^3(\vec{r} - \vec{r}')$$

A common set of equations is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(r)}{c} , \quad \left(\nabla^2 + \frac{1}{c}\frac{\partial^2}{\partial t^2}\right)f(\vec{r}) = 0 , \quad (H - E)\psi = 0$$

which can be solved by using Green's Functions

$$D\psi(\vec{r}) = \delta(\vec{r}) \Rightarrow D_r G(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}')$$

If we now calculate the curl of E:

$$(\vec{\nabla}\times\vec{E})_j = \epsilon_{ijk}\partial_j E_k = \frac{q}{4\pi\epsilon_0}\epsilon_{ijk}\partial_j \frac{x_k}{(x_\ell x_\ell)^{3/2}} = \frac{q}{4\pi\epsilon_0}\epsilon_{ijk}\frac{\partial_i x_k (x_\ell x_\ell)^{3/2} - x_k \partial_j (x_\ell x_\ell)^{3/2}}{(x_\ell x_\ell)^3}$$

$$(\vec{\nabla}\times\vec{E})_j = 0$$

We now look at Stokes' Theorem

$$\int_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, ds = \int \vec{E}\cdot d\vec{l} \quad w/ \quad \int \vec{E}\cdot d\vec{l} = 0 \Rightarrow \int_{r_1}^{r_2}\vec{E}\cdot d\vec{l} = f(r_2) - f(r_1)$$

We can introduce electric potential

$$\vec{E} = -\vec{\nabla}\varphi(\vec{r}) \quad w/ \quad \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}, \quad -\vec{\nabla}\cdot\varphi = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{e}_r$$

Example

$$\rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta(r-R)$$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int_0^\infty r^2\, dr' \int_{-\pi}^{\pi}\sin\theta'\, d\theta' \int_0^{2\pi}\frac{Q}{4\pi R^2}\frac{\delta(r-R)}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi\epsilon_0}\frac{Q}{4\pi}\int_{-1}^{1}du\int_0^{2\pi}\frac{d\varphi'}{\sqrt{r^2+R^2-2rRu}}$$

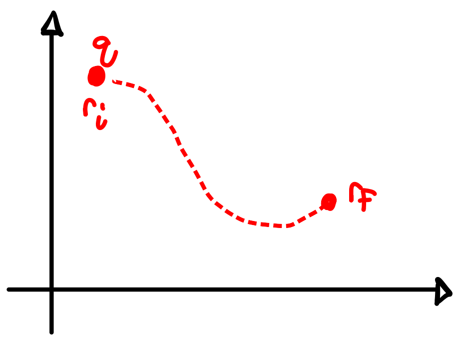
$$= \frac{1}{4\pi\epsilon_0}\frac{Q}{2}\int_{-1}^{1}\frac{du}{\sqrt{r^2+R^2-2rRu}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2rR}\left(|r+R| - |r-R|\right)$$

When $r < R$, $\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$: $r > R$, $\varphi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

" " $E = 0$ " " $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{e}_r$

$r \to R_+$, $\varphi = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$, $E = \frac{J}{\epsilon_0}$: $r \to R_-$, $\varphi = \frac{1}{4\pi\epsilon_0}$, $E = 0$

1-31-21



They change in potential energy is simply

$$\Delta P.E = -\int_i^f \vec{F}\cdot d\vec{l} = -q\int_i^f \vec{E}\cdot d\vec{l} = -q\int_i^f -\vec{\nabla}\varphi\cdot d\vec{l} = q\int d\varphi = q(\varphi(\vec{r}_f) - \varphi(\vec{r}_i))$$

The electric potential energy is

$$U = \sum_a q_a \varphi(\vec{r}_a) = \int \rho(r)\varphi(r) d^3r \ , \ U = \frac{1}{2}\sum_{ab} q_a \frac{q_b}{4\pi\epsilon_0 |\vec{r}_a - \vec{r}_b|} \ , \ U = \frac{1}{8\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

The electric potential and energy can be written as

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r \frac{\rho(r')}{|\vec{r}-\vec{r}'|} \ , \ U = \frac{1}{8\pi\epsilon_0}\int d^3r\, d^3r' \frac{\rho(r)\rho(r')}{|\vec{r}-\vec{r}'|}$$

We now look to solve for $\varphi(\vec{r})$

$$\varphi = \frac{1}{4\pi\epsilon}\int_0^\infty r'^2 dr' \rho(r') \int_{-1}^{1} du \int_0^{2\pi} d\varphi \frac{1}{\sqrt{r^2 + r'^2 - 2rr'u}} \qquad r > r' \Rightarrow 2r'$$

$$\varphi = \frac{2\pi}{4\pi\epsilon_0}\int_0^\infty r'^2 dr' \rho(r') \frac{1}{2rr'}\left(|r+r'| - |r-r'|\right) \ , \ r < r' \Rightarrow 2r$$

$$\varphi(r) = \frac{1}{2\epsilon_0}\left[\int_0^r r'^2 \rho(r') \frac{1}{r} + \int_r^\infty r'^2 dr' \rho(r') \frac{1}{r}\right] = \frac{1}{2\epsilon_0}\frac{1}{r}\int_0^r r'^2 dr' \rho(r') + \frac{1}{2\epsilon_0}\int_r^\infty r' dr' \rho(r')$$

We then find that the electric potential energy is then

$$U = \frac{1}{2}\int d^3r\, \rho(r)[\ldots\ldots]$$

Evaluating this integral we then find

$$U = \frac{4\pi}{2}\int_0^r r^2 dr\, \rho(r)[\ldots\ldots] \ , \ U = \frac{4\pi}{\epsilon_0}\int_0^\infty dr \int_0^r dr' r'^2 r\, \rho(r)\rho(r') = \frac{4\pi}{\epsilon_0}\int_0^\infty dr \int_r^\infty dr' r^2 r' \rho(r)\rho(r')$$

Using one of maxwell's equations we see

$$\nabla^2\varphi = \frac{\rho}{\epsilon_0} \ , \qquad U = \frac{1}{2}\int \varphi(r)(-\epsilon_0\nabla^2\varphi) d^3r = -\frac{\epsilon_0}{2}\int \varphi\nabla^2\varphi\, d^3r$$

Using some math tricks we can evaluate this integral,

$$\vec{\nabla}\cdot(\varphi\vec{\nabla}\varphi) = \partial_i(\varphi\partial_i\varphi) = (\partial_i\varphi)(\partial_i\varphi) + \varphi(\partial_i\partial_i\varphi) = (\vec{\nabla}\varphi)\cdot(\vec{\nabla}\varphi) + \varphi\nabla^2\varphi$$

The electric potential energy then becomes

$$U = -\frac{\epsilon_0}{2}\int \vec{\nabla}\cdot(\varphi\vec{\nabla}\varphi) d^3r - \frac{\epsilon_0}{2}\int(-E^2) d^3r = -\frac{\epsilon_0}{2}\int \cancelto{0}{(\varphi\vec{\nabla}\varphi)}\cdot\hat{n}\, ds + \frac{\epsilon_0}{2}\int E^2 d^3r$$

The electric potential energy then becomes

$$U = \frac{\epsilon_0}{2}\int E^2 d^3r$$

The electric field has energy and other intrinsic properties e.g. momentum etc.

We now look at a sphere of radius R. The charge density is

$$\sigma = Q/4\pi R^2$$

we can then calculate potential and energy of this sphere

$$\varphi(r) = \frac{Q}{4\pi\varepsilon_0 r}, \quad r \geq R \ : \ \vec{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{e}, \quad r \geq R$$

Reference <u>Sec 16</u> to see the following:

$$\vec{\nabla}\cdot\vec{\nabla}(\vec{r}) = \rho(\vec{r}) - \text{"Divy Source"}, \quad \vec{\nabla}\times\vec{\nabla}(\vec{r}) = \bar{\omega}(\vec{r}) - \text{"curly source"}$$

$$V(\vec{r}^{2}) = -\vec{\nabla}\underbrace{g(\vec{r})}_{\text{Scalar potential}} + \vec{\nabla}\underbrace{\times\,\vec{u}(\vec{r})}_{\text{Vector potential}}$$

we then have the following:

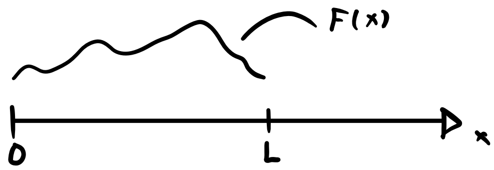
$$g(\vec{r}) = \frac{1}{4\pi}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3r, \quad \vec{u}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\omega}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3r'$$

we wish to solve the Poisson equation : $\nabla^2\varphi = 0$ w/ B.C

First we examine what we call an <u>N-Dimensional</u> <u>vector</u>

$$\vec{A} = \sum_{i=1}^{N} a_i \hat{e}_i$$

Consider a function defined on the x-axis from 0 to L.



we can write F(x) as a Fourier Series:

$$F(x) = b_m + \sum_{n=1}^{\infty}\left(a_n \sin(2n\pi/L\,x) + b_n \cos(2n\pi/L\,x)\right)$$

we calculate a_n and b_n with the following :

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{2n\pi}{L}x\right)F(x)\,dx, \quad b_n = \frac{2}{L}\int_0^L \cos\left(\frac{2n\pi}{L}x\right)F(x)\,dx$$

The dot product is defined as :

$$\langle f, g\rangle = \int_0^L f(x)g(x)\,dx$$

A fourier transform is of the form

$$f(x) = \int_{-\infty}^{+\infty} f_k\left(\frac{1}{\sqrt{2\pi}}e^{-ikx}\right)dk$$

we can choose to find f_k with

$$f_{k'} = \int_{-\infty}^{+\infty} f(x) \left(\frac{1}{\sqrt{2\pi}} e^{ik'x} \right) dx$$

Where we have the following:

$$E_k = \frac{1}{\sqrt{2\pi}} e^{-ikx}, \quad \langle f, g \rangle = \int f^* g \, dx$$

We then find that $f_{k'}$ will be

$$f_{k'} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_k \frac{1}{2\pi} \underbrace{e^{-i(k-k')x}}_{\to \delta(k-k')} dx\, dk$$

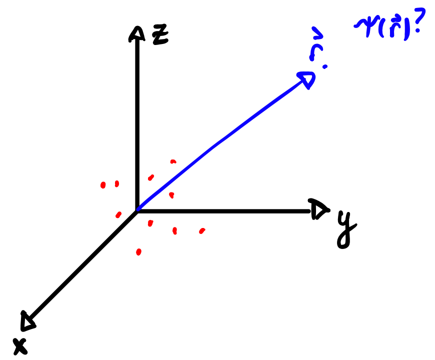
It then can be said

$$\delta(k-k') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(k-k')x} dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\mu|x|} e^{i(k-k')x} dx = -\frac{1}{\pi} \frac{\mu}{x^2+\mu^2}$$

We are interested in solving $\nabla^2 \varphi = 0$, this implies $\nabla^2 F_{abc} = 0$. We then can use this to show

$$\varphi(\vec{r}) = \int C_{abc} F_{abc}(\vec{r})$$

2-2-22



The electric potential for this is simply

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_\alpha \frac{q_\alpha}{|\vec{r}-\vec{r}_\alpha|} \qquad G(\vec{r},\vec{r}_\alpha) = \frac{1}{|\vec{r}-\vec{r}_\alpha|} \longrightarrow (*)$$

We now want to look at Taylor expanding (*)

$$G(\vec{r},\vec{r}_\alpha) = G(\vec{r},0) + \vec{r}_\alpha \cdot \vec{\nabla}_\alpha G(\vec{r},\vec{r}_\alpha) \implies G(\vec{r},\vec{r}_\alpha) = \sum_n \frac{1}{n!} (\vec{r}_\alpha \cdot \vec{\nabla}_\alpha) G$$

We then look at the expansion

$$\frac{1}{|\vec{r}-\vec{r}_\alpha|} = \frac{1}{r} - \vec{r}_\alpha \cdot \vec{\nabla} \frac{1}{r} + \frac{1}{2} (x_{\alpha i} \partial_i \, x_{\alpha j} \partial_j) \frac{1}{r} - \dots\dots$$

Putting this into the potential equation

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\sum_\alpha \frac{q_\alpha}{r} - \sum_\alpha q_\alpha \vec{r}_\alpha \cdot \vec{\nabla} \frac{1}{r} + \frac{1}{2} \sum_\alpha q_\alpha (x_{\alpha i} x_{\alpha j} \partial_i \partial_j) \frac{1}{r} \right]$$

$\hookrightarrow$ Source points

If we get really far away then φ

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_\alpha \frac{q_\alpha}{r} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}$$ — Source Charge

Looking at a dipole

$$\varphi(\vec{r}) = -\frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha \vec{r}_\alpha \cdot \vec{\nabla} \frac{1}{r} = -\frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha x_{\alpha i} \partial_i \frac{1}{(x_j x_j)^{1/2}} = -\frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha x_{\alpha i} \left[-\frac{1}{2} \frac{x_i \partial_j + x_j \partial_i}{(x_j x_j)^{1/2}} \right]$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha \frac{x_{\alpha i} x_i}{r^3} = \frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha \frac{\vec{r}_\alpha \cdot \vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \Big(\underbrace{\sum_\alpha q_\alpha \vec{r}_\alpha}_{\vec{p}} \Big) - \frac{\hat{r}}{r^2}$$

The Quadrapole is then

$$\varphi^{(2)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_\alpha q_\alpha x_{\alpha i} x_{\alpha j} \partial_i \partial_j \frac{1}{r}$$

Working this out we get

$$x_{\alpha i} x_{\alpha j} \partial_i \partial_j \frac{1}{(x_k x_k)^{1/2}} = x_{\alpha i} x_{\alpha i} \left[\frac{3}{2} \frac{\delta_{jk} x_k \delta_{ik} x_k}{(x_k x_k)^{1/2}} \right] - \frac{\delta_{jk} \delta_{ik}}{(x_k x_k)^{3/2}} = 3 \frac{x_{\alpha i} x_i \cdot x_{\alpha j} x_j}{r^5} - \frac{r_\alpha^2}{r^3}$$

$$= 3 \frac{(\vec{r}_\alpha \cdot \hat{r})(\vec{r}_\alpha \cdot \hat{r})}{r^3} - \frac{r_\alpha^2}{r^3} = \frac{1}{r^3} \hat{r}_i \bar{D}_{ij} \hat{r}_j \quad \text{w/} \quad \bar{D}_{ij} = 3 x_{\alpha i} x_{\alpha j} - \delta_{ij} r_\alpha^2$$

The potential is then,

$$\varphi^{(2)}(r) = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \frac{1}{r^3} \left[\hat{r} \cdot \sum_\alpha q_\alpha (3 \vec{r}_\alpha \vec{r}_\alpha - q r_\alpha^2) \right]$$

The results of this potential are, $\varphi^{(n)} \sim \frac{1}{r^{n+1}}$. Now lets look at other co-ordinates

<u>Spherical Co-Ordinates</u>

We have the equations

$$\nabla^2 \varphi(\vec{r}) = 0, \quad \rho(\vec{r}) = 0, \quad \varphi(\vec{r}) = R(r) F(\theta, \varphi)$$

We are looking to solve Laploces equation,

$$\nabla^2 \varphi = \frac{1}{r^2} \partial_r r^2 \partial_r R \cdot F + \overbrace{\frac{1}{r^2 \sin\theta} \partial_\theta \sin\theta \partial_\theta R \cdot F}^{\nabla^2_\Omega} + \frac{1}{r^2 \sin\theta} \partial^2_\varphi R \cdot F = 0$$

This becomes

$$\frac{F}{r^2} \partial_r r^2 \partial_r R + \frac{R}{r^2} \nabla^2_\Omega F = 0, \quad \frac{r^2}{FR} \Big(\underbrace{\frac{1}{R} \partial_r r^2 \partial_r R}_{\substack{\text{only } r \\ -C\Omega}} + \underbrace{\frac{1}{F} \nabla^2_\Omega F}_{\substack{\theta, \varphi \\ C\Omega}} \Big) = 0$$

This then tells us

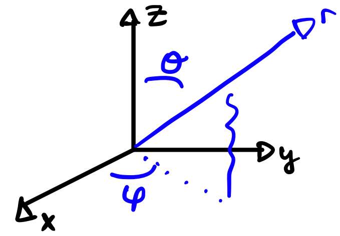
$$\nabla^2_\Omega F(\Omega) = C_\Omega F(\Omega)$$

Where we have an eigenvalue equation where

$$F(\Omega) = C^{(0)} + C_i^{(1)} \hat{r}_i + C_{ij}^{(2)} \hat{r}_{ij} + \dots\dots$$

we now shift from cartesian to spherical

$$\hat{r}_1 = \hat{e}_x = \cos(\theta)\sin(\varphi)\,\hat{e}_r$$
$$\hat{r}_2 = \hat{e}_y = \sin(\theta)\sin(\varphi)\,\hat{e}_r$$
$$\hat{r}_3 = \hat{e}_z = \cos(\varphi)\,\hat{e}_r$$



Looking at the tensor $C_{ij}^{(2)}$

$$F^2(\Omega) = C_{ij}^{(2)}\, \hat{n}_i \cdot \hat{n}_j \quad w/ \quad C_{ij} = c_{ij} - \frac{1}{3} c_{ii}\delta_{ij}$$

Where we have a $F(\hat{r})$

$$F(\hat{r}) = C^0 + C_i^{(1)} \hat{r}_i + C_{ij}^{(2)} \hat{r}_i \hat{r}_j + \dots\dots, \quad F(\vec{r}) = F(r\hat{r}) = C^0 + C_i^{(1)} x_i + C_{il}^{(1)} x_i + \dots\dots$$

we see that Laplaces Equation becomes

$$\nabla^2 F^l(\vec{r}) = \nabla^2 r^l F^l(\hat{n})$$

Continuing on this becomes

$$\nabla^2 r^l F^l(\hat{n}) = F^l\left(1/r^2\, \partial_r r\, \partial_r r^l\right) + r^l \frac{1}{r^2} \nabla^2_\Omega F^{(l)}(\hat{n}) = 0$$
$$= F^l\left(1/r^2\, \partial_r r\, \partial_r r^l\right) = -r^{l-2} \nabla^2 F^{(l)}(\hat{n})$$
$$F^l\left(1/r^2\, \partial_r\, l\, r^{l+1}\right) = -r^{l-2} \nabla^2_\Omega F^l(\hat{n})$$
$$l(l+1)\, r^{l-2} F^l = -r^{l-2} \nabla^2_\Omega F^l(\hat{n})$$

This will simplify to

$$\nabla^2_\Omega F(\hat{n}) = C_\Omega F(\hat{n}), \quad \nabla^2 F^l(\hat{n}) = -l(l+1) F^l(\hat{n})$$

Where the above are the Spherical Harmonics

$$\partial_r r^l \partial_r R^l = l(l+1) R^l \longrightarrow F^l \rightsquigarrow \text{Each to own } F^l$$

The electric potential is then

$$\varphi(\vec{r}) = \sum_l \left(A_l r^l + B_l r^{-(l+1)}\right) P_l \cos(\theta) = \underbrace{\sum_l \left(A_l r^l + B_l r^{-(l+1)}\right)}_{R} \underbrace{\sum_m Y_{lm}(\theta,\varphi)}_{(\theta,\varphi)}$$

For certain cases, we see the potential change

$$A_l = 0 \Rightarrow \varphi(\vec{r}) = \sum_l \frac{B_l}{r^{l+1}} \sum_m Y_{lm}(\theta, \varphi) \quad , \quad B_l = 0 \Rightarrow \varphi(\vec{r}) = \sum_l A_l r^l \sum_m Y_{lm}(\theta, \varphi)$$