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Classical Mechanics

CH. 9 CANONICAL TRANSFORMATION LECTURE NOTES

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11-8-21

Ch. 9: Canonical Transformations

Cyclic co-ordinates: Lagrangian: αq_i that don't appear in $L(\vec{q}, \dot{\vec{q}}, t)$

* generalized momentum was conserved: $\vec{p} = \frac{\partial L}{\partial \dot{\vec{p}}} = \text{constant}$
* related to symmetries

Spherical symmetry $\longrightarrow$ conserved angular momentum

Hamiltonian:

* q_i is cyclic if it doesn't appear in $H(\vec{q}, \vec{p}, t)$
* $\dot{p}_i = -\frac{\partial H}{\partial q_j} = 0 \longrightarrow P_j = \text{constant}$
* can identify conserved quantities via symmetries

Scenario: Consider some physical system for which all co-ordinates $\{q_i\}$ are cyclic

* $H = H(\vec{p}, t) \Rightarrow$ Assum $\frac{\partial H}{\partial t} = 0$, $H = H(\vec{p})$
* All canonical momenta are conserved quantities. $\dot{p}_i = 0 \rightarrow p_i = \alpha_i = \text{const.}$

  $H = H(\alpha_1, \dots, \alpha_n)$

* $q_i = \frac{\partial H}{\partial p_i} = \omega_i$, $\dot{q} = \dot{q}(\alpha, \dots, \alpha_n)$

$\Rightarrow$ Formally solve for all q_i: $q_i(t) = \omega_i(\alpha_1, \dots, \alpha_n)t + \beta_i$

Point: IF,

i) $\frac{\partial H}{\partial t} = 0$

ii) All $\{q_i\}$ are cyclic $\Rightarrow$ Solution of dynamics is fixed

$\Rightarrow$ we have freedom to choose our canonical position and momenta

Idea: what if we always choose canonical position & momenta such that

$H = H(\vec{p})$ etc ....

$\Rightarrow$ Trivial solution of problem

Consider: Central Force problem

i) In cartesian co-ordinates. $(x, y) \rightarrow (p_x, p_y) \Rightarrow$ no cyclic co-ordinates

ii) In polar co-ordinates (r, φ), (P_r, P_φ) : $V(r) \to \varphi$ is cyclic

General Task :

* Identify or transform to a set of canonical co-ordinates, $\{Q_i\}$, $\{P_i\}$ such that

  i) All Q_i are cyclic in Hamiltonian

  ii) P_i & Q_i are canonical

Canonical ? $\Rightarrow$ Hamilton's canonical EOM are conserved

$\Rightarrow$ There exists a Kamiltonian, $K(\vec{Q}, \vec{P}, t)$ such that:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \& \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Alternatively

$(\vec{q}, \vec{p}) \to (\vec{Q}, \vec{P})$ is canonical if :

(*) $[\vec{\varepsilon}, \vec{\varepsilon}]_{\vec{\eta}} = J$ where $\vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$, $\vec{\varepsilon} = \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Poisson Bracket Defined : $[a,b]_{(\vec{c},\vec{d})} = \sum_j \frac{\partial a}{\partial c_j}\frac{\partial b}{\partial d_j} - \frac{\partial a}{\partial d_j}\frac{\partial b}{\partial c_j}$

Unpacking : $[P_i, P_j]_{(\vec{q},\vec{p})} = [Q_i, Q_j]_{(\vec{q},\vec{p})} = 0 \quad \forall\, i,j$ but,

$[Q_i, P_j]_{(\vec{q},\vec{p})} = -[\vec{P}_j, \vec{Q}_j]_{(\vec{q},\vec{p})} = \delta_{ij}$

If $\vec{Q}, \vec{P}$ obey canonical EOM

$\Rightarrow$ Hamilton's principle gives : $\delta\int_{t_1}^{t_2}\left[\sum_i P_i\dot{Q}_i - K(\vec{Q},\vec{P},t)\right]dt = 0$

but, $\vec{q}$ & $\vec{p}$ are also canonical : $\delta\int_{t_1}^{t_2}\left[\sum_i p_i\dot{q}_i - H(\vec{q},\vec{p},t)\right]dt = 0$

Can say :

$$\lambda\left\{\sum_{i=1}^{n} p_i\dot{q}_i - H\right\} = \left(\sum_i P_i\dot{Q}_i - K\right) + \frac{dF}{dt} \leftarrow \text{Associating w/ rescaling co-ordinate}$$

$F(\vec{q}, \vec{Q}, \vec{p}, \vec{P}, t)$

$I \to I + \int_{t_1}^{t_2} dF/dt$, $\delta(I + F(\vec{q},\vec{Q},\vec{p},\vec{P},t_2) - F(\vec{q},\vec{Q},\vec{p},\vec{P},t_1)) = \delta I$

11-15-21

Canonical Transformations

<u>Motivation</u>: If all co-ordinates $\{q_i\}^n$ are <u>cyclic</u>:

$H = H(\vec{P}) \Rightarrow \{P_i\}$ are constants of motion.

Trivial solution: $p_i = \alpha_i$, $q_i = w_i(\alpha_i \ldots)t + \beta_i$, α_i & $\beta_i \rightarrow$ Integration constants

<u>Insight:</u> Judiciously choosing our canonical variables might greatly simplify a physical problem.

<u>Task:</u> Develop generic machinery to transform to best variables $\longrightarrow$ Canonical Variables

$$\underset{H}{\{q_i\} \;\&\; \{P_i\}} \xrightarrow[\text{Canonical Transformation}]{} \underset{K}{\{Q_i\} \;\&\; \{P_i\}}$$

i) $\exists$ a Hamiltonian K: $\dot{Q}_i = \frac{\partial K}{\partial P_i}$ & $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$

ii) Alternatively: use Poisson brackets

$$[\vec{\varepsilon}, \vec{\varepsilon}]_{\vec{\eta}} = J, \quad \vec{\eta} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}, \quad \vec{\varepsilon} = \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$[a,b]_{(\vec{c},\vec{d})} = \sum_j \frac{\partial a}{\partial c_j}\frac{\partial b}{\partial d_j} - \frac{\partial a}{\partial d_j}\frac{\partial b}{\partial c_j}, \quad [Q_i, P_j]_{(\vec{q},\vec{p})} = [q_i, p_j]_{(\vec{Q},\vec{P})} = \delta_{ij}$$

Hamilton's Principle:

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H\right) = \delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K\right) = 0, \quad \left(\sum_i p_i \dot{q}_i - H\right) = \left(\sum_i P_i \dot{Q}_i - K\right) + \frac{dF}{\partial t}$$

F: Generating function, $F(\vec{q}, \vec{Q}, \vec{p}, \vec{P}, t) \Rightarrow$ Function of $4n+1$ variables

$\vec{q}, \vec{p}\ (H)$ $\longrightarrow$ $\vec{Q}, \vec{P}\ (K)$

$2n$ variables $\qquad$ $2n$ variables

$q \rightarrow Q(q,p,t)$, $p \longrightarrow P(q,p,t) \Rightarrow$ Only $(2n+1)$ variables are independent

Freedom $\Rightarrow$ Pick $2n$ variables. $\Rightarrow$ Lets start with $\vec{q}$ & $\vec{Q} \Rightarrow F_1(\vec{q}, \vec{Q}, t) \rightarrow$ Type 1

Revisit, From Hamilton's principle: $\sum_i p_i q_i - H(\vec{q}, \vec{p}, t) = \sum_i P_i Q_i - K(\vec{Q}, \vec{P}, t) + \frac{dF}{dt}$

$$\frac{dF_1}{dt} = \frac{\partial F_1}{\partial t} + \sum_i \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i$$, Plug this back in and do some re-arranging

$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i}\right)\dot{q}_i - \sum_i \left(P_i + \frac{\partial F_1}{\partial Q_i}\right)\dot{Q}_i = H - K + \frac{\partial F_1}{\partial t}$$, want to use that $\vec{q}, \vec{Q}$, and t are independent

$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i}\right)dq_i + \sum_i \left(P_i + \frac{\partial F_1}{\partial Q_i}\right)dQ_i = \left(H - K + \frac{\partial F_1}{\partial t}\right)dt = 0$$

i.) $p_i = \frac{\partial F_1(\vec{q}, \vec{Q}, t)}{\partial q_i}$

ii.) $P_i = -\frac{\partial F_1}{\partial Q_1}(\vec{q},\vec{Q},t)$

iii.) $K = H + \frac{\partial F_1}{\partial t}$

Cases:

Ⓐ: Assuming that you know the generating function $F_1(\vec{q},\vec{Q},t)$

① use i.), $p_i(\vec{q},\vec{Q},t)$

② Invert to obtain $Q_i(\vec{q},\vec{p},t)$

③ Use Q_i w/ ii.) to get $P_i(\vec{q},\vec{Q},t) \Rightarrow P_i(\vec{q},\vec{p},t)$

④ Get $K(\vec{Q},\vec{P},t) = H(q(Q,P,t),p(Q,P,t),t) + \frac{\partial F_1}{\partial t}$

Ⓑ: Have $\vec{Q}(\vec{q},\vec{p},t)$ & $\vec{P}(\vec{q},\vec{p},t) \Rightarrow$ Integrate i) & ii) to formally obtain $F_1 \Rightarrow$ use iii) to obtain K

<u>Simple Example</u>:

<u>Case 1</u>: Ⓐ $F_1(q,Q,t) = qQ$

①-③: i) $p = \frac{\partial F_1}{\partial q} = Q$

ii) $P = -\frac{\partial F_1}{\partial Q} = -q$, $(q,p) \longrightarrow (-P,Q)$

④: $K(Q,P,t) = H(-P,Q,t)$

* Check that canonical equations hold,

$$\dot{q} = \frac{\partial H}{\partial p} \Rightarrow \dot{Q} = -\frac{\partial K}{\partial(-P)} = \frac{\partial K}{\partial P} \quad : \quad \dot{p} = -\frac{\partial H}{\partial q} \Rightarrow -\dot{P} = \frac{\partial K}{\partial Q}, \quad \dot{P} = -\frac{\partial K}{\partial Q}$$

Or, via Poisson bracket:

$$[Q,Q]_{(q,p)} = [P,P]_{(q,p)} = 0$$

$$[Q,P]_{(q,p)} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 0 - 1(-1) = 1 \quad \therefore \text{ canonical iff } = 1$$

$F_1(\vec{q},\vec{Q},t) = \sum_{i=1}^{n} q_i Q_i$ is also canonical

<u>Case 2</u>: $F_1 = -qQ$, $(q,p) \rightarrow (P,-Q)$, prove this is canonical

Question: Is the transformation $(q,p) \longrightarrow (P,Q)$ canonical?

$[Q,P]_{(q,p)} = -1 \quad \therefore$ not canonical

Goldstein pg. 373.

① $F = F_1(\vec{q}, \vec{Q}, t)$

② $F = F_2(\vec{q}, \vec{P}, t) - \sum_i Q_i P_i$

③ $F = F_3(\vec{q}, \vec{Q}, t) + \sum_i q_i p_i$

④ $F = F_4(\vec{p}, \vec{P}, t) + \sum_i q_i p_i - \sum_i Q_i P_i$

<u>Type 2 Generating Function</u>

Plug F back into original equation from Hamilton's principle,

$$\sum_i p_i \dot{q}_i - H = \sum_i \dot{Q}_i P_i - K + \frac{dF}{dt}$$

First, compute total time derivative etc....

$$\sum_i \left(p_i - \frac{\partial F_2}{\partial q_i}\right) dq_i + \left(Q_i - \frac{\partial F_2}{\partial P_i}\right) dP_i + \left(K - H + \frac{dF_2}{dt}\right) dt = 0$$

$\vec{q}, \vec{P}, t$ are independent variables

Type-2 rules:

i) $p_i = \dfrac{\partial F_2}{\partial q_i}$

ii) $Q_i = \dfrac{\partial F_2}{\partial P_i}$

iii) $K = H + \dfrac{\partial F_2}{\partial q}$

Simple Example: $F_2 = qP$, $(q, P) \rightarrow (Q, P)$

11-17-21

$\underset{H}{(q,p)} \longleftrightarrow \underset{K}{(Q,P)}$

* Canonical transformation

* Generating Function F, $(4n+1)\ (q, Q, p, P, t) \longrightarrow (2n+1)$ Independent variables

4 basic categories

$F = F_1(\vec{q}, \vec{Q}, t) = F_2(\vec{q}, \vec{P}, t) - \sum_i Q_i P_i = F_3(\vec{p}, \vec{Q}, t) + \sum_i q_i p_i = F_4(\vec{p}, \vec{P}, t) + \sum_i q_i p_i - Q_i P_i$

For a type-1 generating function:

i) $p = \partial F_2 / \partial q$

ii) $Q = \partial F_2 / \partial P$

iii) $K = H + \partial F_2/\partial t$

Plan Ⓐ: Provided w/ F_1

① obtain $p_i(\vec{q}, \vec{Q}, t)$

② Invert ① $\rightarrow$ obtain $Q_i(\vec{q}, \vec{p}, t)$

③ obtain $P_i(\vec{q}, \vec{Q}(\vec{q}, \vec{p}, t), t) = P_i(\vec{q}, \vec{p}, t)$

④ obtain $K(\vec{Q}, \vec{P}, t) = H(\vec{q}, \vec{p}, t) + \partial F_1/\partial t$

Plan Ⓑ: Given transformation

$q(Q,P,t)$ & $p(Q,P,t) \Rightarrow$

① obtain F_1 by inverting i) & ii)

② [step 4 from Ⓐ] $\Rightarrow$ get K

Illustrative Example: 1D H.O

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \quad , \quad \omega = \sqrt{\frac{k}{m}}$$

want to solve using canonical transformation,

$(q,p) \longrightarrow (Q,P)$, Q is cyclic

$H(q,p) \qquad\qquad K(P)$

Ⓐ: $F_1(q,Q) = \frac{m\omega q^2}{2} \cot(Q)$

Ⓑ: $p = \sqrt{2m\omega P}\cos(Q)$, $q = \sqrt{\frac{2P}{m\omega}} \sin(Q)$

For Type-1:

i) $p = \frac{\partial F_1}{\partial q} = m\omega q \cot(Q)$

ii) $P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2} \sin(Q)^{-2}$

① - ③:

i) $\rightarrow$ $Q = \text{arccot}\left(\frac{p}{m\omega q}\right)$

ii) $\rightarrow$ $P = \frac{m\omega q^2}{2} \frac{1}{\sin[\text{arccot}(P/m\omega q)]^2} = \frac{m\omega q^2}{2} \frac{1}{\sqrt{1 + P/m\omega q}}$

or

ii) $q = \sqrt{\frac{2P}{m\omega}} \sin(Q)$

i) $p = m\omega \cot(Q) \cdot \sqrt{\frac{2P}{m\omega}} \sin(Q) = \sqrt{2m\omega P} \cos(Q)$

④ : $K = H + \cancel{\frac{\partial F}{\partial t}}^{\,0}$

$$K(Q,P) = \frac{\left(\sqrt{2m\omega P}\cos(Q)\right)^2}{2m} + \frac{m\omega^2}{2}\left[\sqrt{\frac{2P}{m\omega}}\sin(Q)\right]^2 = \omega P = K(P)\text{ , } Q \text{ is cyclic}$$

Solving dynamics :

$$\dot{P} = 0 \Rightarrow P(t) = P(0)\text{ , } \dot{Q} = \frac{\partial h}{\partial P} = \omega \Rightarrow Q(t) = \omega t + \beta$$

$\Rightarrow$ Plug back into q & p :

$$q(t) = \sqrt{2m\omega P(0)}\cos(\omega t + \beta)\text{ , } p(t) = \sqrt{\frac{2P(0)}{m\omega}}\sin(\omega t + \beta)$$

B : Reverse situation.

Assume given :

$*$ $\quad p = \sqrt{2m\omega P}\cos(Q)$, $q = \sqrt{\frac{2P}{m\omega}}\sin(Q)$

Type-1 function ,

i) $p = \frac{\partial F_1}{\partial q}$ & ii) $P = -\frac{\partial F_1}{\partial Q}$

First, re-arrange :

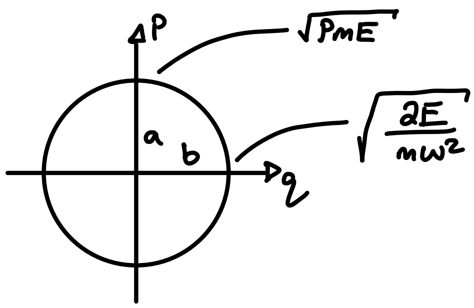
$$p(Q,P,t) \longrightarrow p(q,Q,t)$$

From (*) : $P = \frac{m\omega q^2}{2}\frac{1}{\sin^2(Q)}$ $\quad \therefore \; p = m\omega q\cot(Q)$, $F_1(q,Q) = \frac{m\omega q^2}{2}\cot(Q)$

A subtle point : Canonical transformations preserve phase-space area / volume

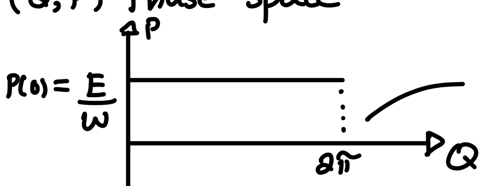
$$\int d\vec{Q}\, d\vec{P} = \int d\vec{q}\, d\vec{p}$$

consider a closed orbit of the oscillator in phase-space



$A = \pi ab = \frac{2\pi E}{\omega}$

(Q, P) Phase Space



$A = \frac{2\pi E}{\omega}$

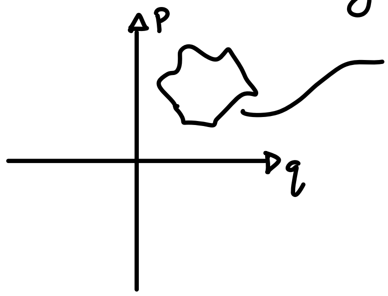
Liouville's Theorem (9.9)

Stat mech ⟶ Phase space

Chunk of stuff : $N \sim 10^{24}$, $V \sim 10^{24}$ "atomic volumes"

Classical stuff : $6N$ co-ordinates , (N particle) , $6N$ (coupled) EOM

Stat mech ⟶ Average behavior instead



Liouville's Theorem : The density of systems in the neighborhood of a given point in phase space is constant in time.

$$\rho(\vec{q}(t), \vec{p}(t), t) = \frac{\#\text{ of states in a volume}}{\text{Volume}} = \frac{dN}{dv}$$ "density of states"

i) $dv = $ constant ? phase space volume is constant in time