# Solutions to Homework 9
# Physics 5393

## Sakurai

P-1.8 Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator $\tilde{\mathbf{A}}$. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of $\tilde{\mathbf{A}}$? Justify your answer.

Only under the condition that the two eigenkets are degenerate

$$\tilde{\mathbf{A}}\left(|i\rangle + |j\rangle\right) = a_{ij}\left(|i\rangle + |j\rangle\right).$$

P-3.4 Consider the $2 \times 2$ matrix defined by

$$\mathbf{U} = \frac{a_0 + i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}}{a_0 - i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}}$$

where $a_0$ is a real number and $\vec{\mathbf{a}}$ is a three-dimensional vector with real components.

a) Prove that $\mathbf{U}$ is unitary and unimodular.

Keep in mind that both the numerator and denominator are matrices and therefore $\mathbf{U}$ must be a matrix. Given this, the matrix can be better expressed as follows

$$\mathbf{U} = \mathbf{A}\left(\mathbf{A}^\dagger\right)^{-1},$$

where $\mathbf{A} = a_0 + i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}$ and $\mathbf{A} = a_0 + i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}$. To prove unitary, we must calculate

$$\begin{aligned}
\mathbf{U}\mathbf{U}^\dagger &= \left[\mathbf{A}\left(\mathbf{A}^\dagger\right)^{-1}\right]\left[\mathbf{A}\left(\mathbf{A}^\dagger\right)^{-1}\right]^\dagger \\
&= \left[\mathbf{A}\left(\mathbf{A}^\dagger\right)^{-1}\right]\left[(\mathbf{A})^{-1}\mathbf{A}^\dagger\right] \\
&= \mathbf{A}\left(\mathbf{A}\mathbf{A}^\dagger\right)^{-1}\mathbf{A}^\dagger,
\end{aligned}$$

where use is made of $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{B}\mathbf{A})^{-1}$; this follows from matrix multiplication being associative

$$(\mathbf{A}\mathbf{B})(\mathbf{A}\mathbf{B})^{-1} = \mathbf{I} \quad \Rightarrow \quad \mathbf{A}\left(\mathbf{B}\mathbf{B}^{-1}\right)\mathbf{A}^{-1} = \mathbf{I}$$

The term $\mathbf{A}\mathbf{A}^\dagger$ can be simplified as follows

$$\begin{aligned}
\mathbf{A}\mathbf{A}^\dagger &= \left[a_0 + i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}\right]\left[a_0 - i\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}}\right] \\
&= \left[a_0^2\mathbf{I} + (\vec{\boldsymbol{\sigma}} \cdot \vec{\mathbf{a}})^2\right] \\
&= \left[a_0^2 + a^2\right]\mathbf{I} \equiv \alpha\,\mathbf{I},
\end{aligned}$$

where $a^2 = a_1^2 + a_2^2 + a_3^2$ and its inverse is

$$\left(\mathbf{A}\mathbf{A}^\dagger\right)^{-1} = \frac{1}{\alpha}\,\mathbf{I}.$$

The remaining part of the proof is

$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{A}\left(\frac{1}{\alpha}\,\mathbf{I}\right)\mathbf{A}^\dagger = \mathbf{A}\mathbf{A}^\dagger\left(\frac{1}{\alpha}\,\mathbf{I}\right) = \frac{\alpha}{\alpha}\,\mathbf{I} = \mathbf{I}.$$

To determine if the matrix is unimodular (determinant is one), we expand $\mathbf{A}$ and $\mathbf{A}^\dagger$

$$\mathbf{A} = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \qquad \mathbf{A}^\dagger = \begin{pmatrix} a_0 - ia_3 & -ia_1 + a_2 \\ -ia_1 - a_2 & a_0 + ia_3 \end{pmatrix}.$$

The determinant of both matrices is $\alpha$. The determinant of an inverse is

$$\det\left[\mathbf{A}^{-1}\right] = \frac{1}{\det\left[\mathbf{A}\right]}.$$

Therefore $\det[\mathbf{U}] = 1$.

b) In general, a $2 \times 2$ unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for $\mathbf{U}$ in terms of $a_0$, $a_1$, $a_2$, and $a_3$.

The first step is to write the matrix in a form that can be easily manipulated. This can be done by multiplying by $\mathbf{I}$

$$\begin{aligned} \mathbf{U} = \mathbf{A}(\mathbf{A}^\dagger)^{-1} = \mathbf{A}\left[\mathbf{A}\mathbf{A}^{-1}\right](\mathbf{A}^\dagger)^{-1} = \mathbf{A}^2\left(\mathbf{A}^\dagger\mathbf{A}\right)^{-1} \\ = \frac{1}{\alpha^2}\begin{pmatrix} a_0^2 - a^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & a_0 - a^2 - 2ia_0a_3 \end{pmatrix}. \end{aligned}$$

From here, we use the expression for a unitary unimodular matrix given in the textbook

$$\begin{aligned} \cos\left(\frac{\phi}{2}\right) &= \mathsf{Re}(a) = \frac{a_0^2 - a^2}{\alpha^2} &\Rightarrow\quad& \sin\left(\frac{\phi}{2}\right) = \sqrt{1 - \cos^2(\phi/2)} = \frac{2a_0|a|}{\alpha^2} \\ -n_y \sin\left(\frac{\phi}{2}\right) &= \mathsf{Re}(b) = \frac{2a_0a_2}{\alpha^2} &\Rightarrow\quad& n_y = -\frac{a_2}{|a|} \\ -n_x \sin\left(\frac{\phi}{2}\right) &= \mathsf{Im}(b) = \frac{2a_0a_1}{\alpha^2} &\Rightarrow\quad& n_x = -\frac{a_1}{|a|} \\ -n_z \sin\left(\frac{\phi}{2}\right) &= \mathsf{Im}(a) = \frac{2a_0a_3}{\alpha^2} &\Rightarrow\quad& n_z = -\frac{a_3}{|a|}. \end{aligned}$$

P-3.5 The spin dependent Hamiltonian of an electron-positron system in the presence of a uniform in the $z$-direction can be written as

$$\tilde{\mathbf{H}} = A\tilde{\mathbf{S}}^{(e^-)} \cdot \tilde{\mathbf{S}}^{(e^+)} + \frac{eB}{mc}\left(\tilde{\mathbf{S}}_z^{(e^-)} - \tilde{\mathbf{S}}_z^{(e^+)}\right).$$

Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.

a) Is this an eigenfunction of $\tilde{\mathbf{H}}$ in the limit $A \to 0$, $eB/mc \neq 0$? If it is, what is the energy eigenvalue? If not, what is the expectation value of $\tilde{\mathbf{H}}$?

The Hamiltonian for this case is

$$\tilde{\mathbf{H}} = \frac{eB}{mc}\left[\tilde{\mathbf{S}}_z^{(e^-)} - \tilde{\mathbf{S}}_z^{(e^+)}\right].$$

If we apply it to the spin function $\chi_+^{(e^-)}\chi_-^{(e^+)}$, we find

$$\frac{eB}{mc}\left[\tilde{\mathbf{S}}_z^{(e^-)} - \tilde{\mathbf{S}}_z^{(e^+)}\right]|{+-}\rangle = \frac{eB}{mc}\left[\frac{\hbar}{2} - \left(-\frac{\hbar}{2}\right)\right]|{+-}\rangle = \frac{eB\hbar}{mc}|{+-}\rangle,$$

where $|{+-}\rangle = \chi_+^{(e^-)}\chi_-^{(e^+)}$ is an eigenstate of this Hamiltonian.

b) Solve the same problem for $A \neq 0$, $eB/mc \to 0$.

To solve this part of the problem, use Eq. 3.8.19 in the textbook

$$\tilde{\mathbf{S}}^{(e^-)}\cdot\tilde{\mathbf{S}}^{(e^+)} = \tilde{\mathbf{S}}_z^{(e^-)}\tilde{\mathbf{S}}_z^{(e^+)} + \frac{1}{2}\tilde{\mathbf{S}}_+^{(e^-)}\tilde{\mathbf{S}}_-^{(e^+)} + \frac{1}{2}\tilde{\mathbf{S}}_-^{(e^-)}\tilde{\mathbf{S}}_+^{(e^+)}.$$

Applying this on the spin function, we find

$$\left[\tilde{\mathbf{S}}^{(e^-)}\cdot\tilde{\mathbf{S}}^{(e^+)}\right]|{+-}\rangle = \left[-\frac{\hbar^2}{4}|{+-}\rangle + 0 + \frac{\hbar^2}{2}|{-+}\rangle\right] = \frac{\hbar^2}{4}\left[2\,|{-+}\rangle - |{+-}\rangle\right].$$

The expectation value is

$$\left\langle +- \middle| \tilde{\mathbf{H}} \middle| +- \right\rangle = -A\frac{\hbar^2}{4}.$$

P-3.9 What is the meaning of the following equation?

$$\mathbf{U}^{-1}\tilde{\mathbf{A}}_k\mathbf{U} = \sum_l \mathbf{R}_{kl}\tilde{\mathbf{A}}_l,$$

where the three components of $\tilde{\mathbf{A}}$ are matrices. From this equation show that matrix elements $\left\langle m \middle| \tilde{\mathbf{A}}_k \middle| n \right\rangle$ transform like vectors.

The equation states that the rotated operators $\tilde{\mathbf{A}}_k$ are linear combinations of the unrotated operators

$$\tilde{\mathbf{A}}_k' = \sum_l \mathbf{R}_{kl}\tilde{\mathbf{A}}_l.$$

The matrix elements are given by

$$\left\langle m \middle| \tilde{\mathbf{A}}_k' \middle| n \right\rangle = \sum_l \mathbf{R}_{kl}\left\langle m \middle| \tilde{\mathbf{A}}_l \middle| n \right\rangle,$$

therefore, the martix elements transform like components of a vector

$$V_k' = \sum \mathbf{R}_{kl}V_l.$$

P-3.38 The $j = 1$ rotation operator.

b) Show that for $j = 1$ only, it is legitimate to replace $e^{-i\tilde{\mathbf{J}}_y\beta/\hbar}$ by

$$\tilde{\mathbf{1}} - i\left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)\sin\beta - \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^2(1-\cos\beta)$$

We start by expanding the operator

$$\exp\left(\frac{-i\tilde{\mathbf{J}}_y\beta}{\hbar}\right) = \left[1 - \frac{1}{2!}\left(\frac{\tilde{\mathbf{J}}_y\beta}{\hbar}\right)^2 + \frac{1}{4!}\left(\frac{\tilde{\mathbf{J}}_y\beta}{\hbar}\right)^4 - \cdots\right]$$
$$- i\left[\left(\frac{\tilde{\mathbf{J}}_y\beta}{\hbar}\right) - \frac{1}{3!}\left(\frac{\tilde{\mathbf{J}}_y\beta}{\hbar}\right)^3 + \cdots\right].$$

By brute force, one can show

$$\frac{\tilde{\mathbf{J}}_y}{\hbar} = \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^3 \quad\Rightarrow\quad \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^n = \begin{cases} \left(\dfrac{\tilde{\mathbf{J}}_y}{\hbar}\right) & n = \text{odd} \\ \left(\dfrac{\tilde{\mathbf{J}}_y}{\hbar}\right)^2 & n = \text{even}. \end{cases} \tag{1}$$

Therefore,

$$\exp\left(\frac{-i\tilde{\mathbf{J}}_y\beta}{\hbar}\right) = 1 - \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^2 + \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^2\left[1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \cdots\right]$$
$$- i\left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)\left[\beta - \frac{\beta^3}{3!} + \cdots\right],$$

and finally

$$\exp\left(\frac{-i\tilde{\mathbf{J}}_y\beta}{\hbar}\right) = 1 - \left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)^2[1 - \cos\beta] - i\left(\frac{\tilde{\mathbf{J}}_y}{\hbar}\right)\sin\beta.$$

c) Using (b), prove

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

The operator $\tilde{\mathbf{J}}_y$ is given in part (a) and its square is given by

$$\left(\tilde{\mathbf{J}}_y\right)^2 = \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

Combining all the pieces, we arrive at the desired result

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$