Homework # 5

(1)

$$H_0 = \hbar\omega(a^\dagger a + 1/2)$$

with

$$V(t) = \lambda\left[f(t)a + f^*(t)a^\dagger\right]$$

a)

$$a(t) = e^{iH_0t/\hbar}\, a\, e^{-iH_0t/\hbar}$$

$$= e^{i\omega\hat{N}t}\, a\, e^{-i\omega\hat{N}t}$$

Since:

$$e^{-B} A e^{B} = \sum_{n=0}^{\infty} \frac{1}{n!}[A,B]_n$$

$$= A + [A,B] + \frac{1}{2!}[[A,B],B] + \dots$$

and $[a,\hat{N}] = a$, then

$$a(t) = e^{\overbrace{i\omega t \hat{N}}^{-B}} \, a \, e^{\overbrace{-i\omega t \hat{N}}^{B}}$$

$$= a + (-i\omega t)\, a + \frac{1}{2!}(-i\omega t)^2 a$$

$$= a^{-i\omega t}.$$

b)

The transition probability between $|0\rangle$ ~~and~~ at $t=0$ and $|n\rangle$ at time t is:

$$|\langle 0| U(t,0) |n\rangle|^2.$$

In perturbation theory,

$$U(t,0) = 1 - \frac{i}{\hbar}\int_0^t dt' \, V_D(t)$$

$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, V_D(t') V_D(t'') + \ldots$$

where $V_D(t) = \lambda \left[f(t)\, a(t) \; f(t)\, a^{\dagger}(t) \right]$

For $n=1$, in leading order,

$$\lambda \langle 0 | \left(f(t)\, a(t) + f^*(t)\, a^\dagger(t) \right) | 1 \rangle$$

$$= \lambda f(t)\, e^{-i\omega t}.$$

$$|\langle 0 | U(t,0) | 1 \rangle|^2 = \frac{\lambda^2}{\hbar^2} \Big| \overbrace{\int_0^t dt'\, f(t)\, e^{-i\omega t'}}^{f(\omega) \text{ for } t \to \infty} \Big|^2$$

$$\xrightarrow{t \to \infty} \frac{\lambda^2}{\hbar^2} |f(\omega)|^2,$$

since $\langle 0 | 1 \rangle = 0$.

For $n=2$, the first order correction is zero. Going to second order,

$$\lambda^2 \langle 0 | \left(f(t')\, a(t') + f^*(t')\, a^\dagger(t') \right)$$
$$\times \left(f(t'')\, a(t'') + f^*(t'')\, a^\dagger(t'') \right) | 2 \rangle$$

$$= \sqrt{2}\, \lambda^2\, f(t')\, f(t'')\, e^{-i\omega(t'+t'')}.$$

$$P_{0,2} = \frac{2\lambda^4}{\hbar^4} \left| \int_0^\infty dt' f(t') e^{-i\omega t'} \int_0^{t'} dt'' f(t'') e^{-i\omega t''} \right|^2$$

c)

If $\quad V(t) = \lambda x^3 e^{-\gamma t}$,

since $\quad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$, we have:

$$\langle 0 | (a(t) + a^\dagger(t))^3 | 3 \rangle = \sqrt{6}\, e^{-3i\omega t} .$$

$\therefore$

$$P_{0,3} = |\langle 0 | U(t,0) | 3 \rangle|^2$$

$$= \left| -\frac{i}{\hbar} \sqrt{6} \int_0^t dt' \lambda e^{-3i\omega t' - \gamma t'} \right|^2$$

$$= \frac{6\lambda^2}{\hbar^2} \left| \frac{1 - e^{-(3i\omega + \gamma)t}}{3\omega - i\gamma} \right|^2$$

$$\xrightarrow{t \to \infty} \frac{6\lambda^2}{\hbar^2} \frac{1}{9\omega^2 + \gamma^2} .$$

A

2

For a three level system,

$$\begin{pmatrix} \epsilon_1 & 0 & \Delta(t) \\ 0 & \epsilon_2 & \Delta(t) \\ \Delta^*(t) & \Delta^*(t) & \epsilon_3 \end{pmatrix}$$

where $\Delta(t) = \Delta e^{i\omega t}$, the transition $|1\rangle \rightarrow |2\rangle$ between times $t=0$ and t is:

$$|\langle 1 | U(t,t_0) | 2 \rangle|^2$$

Since $\langle 1 | V | 2 \rangle = 0$, we need to go to next order to get the leading term. The only possibility for:

$$\sum_i \langle 1|V|i\rangle\langle i|V|2\rangle$$

is $\langle 1|V(t')|3\rangle\langle 3|V(t'')|2\rangle$

$$= \Delta(t')\,\Delta^*(t'')\, e^{it'\omega_{13}}\; e^{it''\omega_{32}}$$

$$= \Delta^2\, e^{i(\omega+\omega_{13})t'}\; e^{i(-\omega+\omega_{32})t''}$$

where:

$$\omega_{ij} = \frac{\epsilon_i - \epsilon_j}{\hbar}$$

$\therefore$

$$|\langle 1|U(t,0)|2\rangle|^2$$

$$= \frac{\Delta^4}{\hbar^4}\cdot\left|\int_0^t dt'\, e^{i(\omega+\omega_{13})t'}\int_0^{t'} dt''\, e^{i(-\omega+\omega_{32})t''}\right|^2$$

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$$= \frac{\Delta^4}{\hbar^4} \left| \int_0^t dt' \, e^{i(\omega+\omega_{13})t'} \, \frac{1 - e^{i(\omega_{32}-\omega)t'}}{\omega_{32}-\omega} \right|^2$$

$$= \frac{\Delta^4}{\hbar^4} \cdot \left| \int_0^t dt' \, \frac{e^{i(\omega+\omega_{13})t'} - e^{i\omega_{12}t'}}{\omega_{32}-\omega} \right|^2$$

$$= \frac{\Delta^4}{\hbar^4} \cdot \left| \frac{e^{i(\omega+\omega_{13})t} - 1}{\omega+\omega_{13}} - \frac{e^{i\omega_{12}t} - 1}{\omega_{12}} \right|^2 .$$

$$\frac{1}{(\omega_{32}-\omega)^2}$$

This transition between states $|1\rangle$ and $|2\rangle$ is mediated by the state $|3\rangle$,

$$|1\rangle \longrightarrow |3\rangle \longrightarrow |2\rangle$$ and requires two transitions (second order).

(3) (1)

For a particle in a box, with potential

$$V_0(x) = \begin{cases} 0, & |x| \leq L/2 \\ \infty, & |x| \geq L/2, \end{cases}$$

the wavefunction is:

$$\psi_n(x) = \frac{1}{\sqrt{2L}} \cdot \left[e^{in\pi x/L} + (-1)^{n+1} e^{-in\pi x/L} \right]$$

with $n = 1 \ldots \infty$ and energy levels,

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2.$$

For a perturbation of the form:

$$V(x) = -e\mathcal{E}x,$$

with $\mathcal{E}$ an external electric field, the first order correction to the energy is:

$$E_n^{(1)} = -e\mathcal{E}\langle \psi_n | x | \psi_n \rangle$$

$$= -e\mathcal{E}\int_{-L/2}^{L/2} dx \, |\psi_n(x)|^2 x = 0$$

by symmetry. The leading correction is:

$$E_n^{(2)} = \frac{2m}{\hbar^2}\left(\frac{L}{\pi}\right)^2 e^2 \mathcal{E}^2 \sum_{l \neq n} \frac{|\langle \psi_n | x | \psi_l \rangle|^2}{n^2 - l^2}$$

for $n=1$, $\langle \psi_1 | x | \psi_l \rangle = 0$ for l odd ∴

$$\langle \psi_1 | x | \psi_{2\nu} \rangle = \frac{4iL}{\pi^2}(-1)^\nu \left[\frac{1}{4\nu^2-1} + \frac{2}{(4\nu^2-1)^2}\right]$$

$$\Rightarrow |\langle \psi_1 | x | \psi_{2\nu} \rangle|^2 = \frac{16L^2}{\pi^4} \cdot \underbrace{\left[\frac{1}{(4\nu^2-1)^2} + \frac{4}{(4\nu^2-1)^3} + \frac{4}{(4\nu^2-1)^4}\right]}_{I(\nu)}$$

$$E_1^{(2)} = \frac{32m}{\hbar^2}\cdot\left(\frac{L}{\pi}\right)^4 \frac{e^2\mathcal{E}^2}{\pi^2}\sum_{\ell=1}^{\infty}\frac{I(\nu)}{4\nu^2-1}$$

$$= -\frac{32m}{\hbar^2}\cdot\frac{L^4}{\pi^6}\, e^2\mathcal{E}^2 \underbrace{\left(\frac{1}{2}-\frac{21\pi^2}{768}-\frac{\pi^4}{768}\right)}_{\alpha}$$

The leading correction to the $|\psi_1\rangle$ state is:

$$|\psi_1^{(1)}\rangle = -\frac{2m}{\hbar^2}\left(\frac{L}{\pi}\right)^2\cdot e\mathcal{E}\sum_{\nu=1}^{\infty}\frac{\langle\psi_{2\nu}|x|\psi_1\rangle|\psi_{2\nu}\rangle}{1-4\nu^2}$$

$$= \frac{8im}{\hbar^2}\frac{L^3}{\pi^4}\, e\mathcal{E}\,\cancel{(-1)^\nu}\times$$

$$\sum_{\nu=1}^{\infty}(-1)^\nu\left[\frac{1}{(4\nu^2-1)^2}+\frac{2}{(4\nu^2-1)^3}\right]|\psi_{2\nu}\rangle$$

The probability of finding the electron in $n = 2$ is:

$$|\langle \psi_2 | \tilde{\psi}_1 \rangle|^2 = |\langle \psi_2 | \psi_1^{(1)} \rangle|^2$$

$$= \frac{25}{9^3} \cdot 8^2 \left(\frac{m L^3}{\hbar^2 \pi^4} \right)^2$$

b)

If $\mathcal{E}(t) = \mathcal{E}_0 e^{-t/\tau}$,

then

$$\langle 1|U(0,\infty)|2\rangle \cong -\frac{i}{\hbar}\int_0^\infty dt\, e^{i\omega_{12}t - t/\tau}$$

$$\times(-1)\, e\mathcal{E}_0 \langle 1|x|2\rangle$$

$$= -\frac{i}{\hbar}\cdot\frac{1}{i\omega_{12}-\tau^{-1}}\cdot e\mathcal{E}\langle 1|x|2\rangle$$

$\therefore$

$$|\langle 1|U(0,\infty)|2\rangle|^2 = \frac{\tau^2}{(\omega_{12}\tau)^2+1}(e\mathcal{E})^2|\langle 1|x|2\rangle|^2$$

$$= \frac{16(11)^2}{9^2}\times\frac{L^2}{\pi^4}\frac{\tau^2}{\left(3\hbar\pi^2/2mL^2\right)^2\tau^2+1}$$