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Classical Mechanics

CH. 11 CLASSICAL CHAOS LECTURE NOTES

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8-23-21

Chaos :

* Phase-Space
* Phase-Portrait
* Harmonic Oscillator : damped + driven
* Attractors, limit cycles + fixed points
* "Chaotic" $\to$ Lyapenov exponents
  * $\hookrightarrow$ KAM Theorem
* Iterative maps / bifurcation
* Poincare maps / Surfaces

Simple Harmonic Motion

* Physical examples :
  i) Mass on a spring
  (Hooke's Law $F = -k\Delta x$)
  ii) Pendulum w/ Small-angle approximation
  iii) Quantum Mechanics

SHO : $\ddot{x} = -\omega_0^2 x$

$\hookrightarrow$ Frequency of motion

$F = m\ddot{x} = -kx \quad \therefore \quad \omega_0^2 = k/m$: Force has some associated potential,

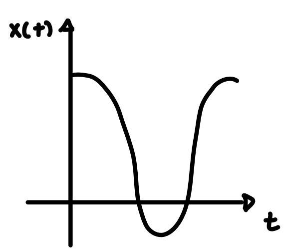
$$F = -\frac{\partial V}{\partial x} \to V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2$$

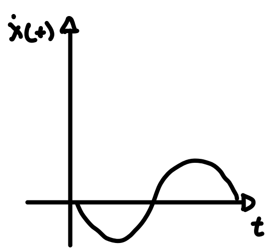
Solution : $x(t) = a_0 \sin(\omega t) + b_0\cos(\omega t) = A_0 \cos(\omega_0 t - \delta)$

Energy is conserved :

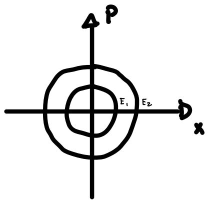
$$E = KE + PE = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{A_0^2 k}{2}\sin^2(\omega_0 t - \delta) + \frac{A_0^2}{2}k\cos^2(\omega_0 t - \delta) = \frac{A_0^2 k}{2}$$

Phase Space





$(x, p) \Rightarrow$ Position-momentum phase space : $p = m\dot{x}$



$\bar{x} = \sqrt{k/2}\, x \quad , \quad \bar{p} = \sqrt{1/2m}\, p$

$E_2 > E_1 \quad : \quad \frac{p^2}{2m} + \frac{kx^2}{2} = E \quad : \quad \bar{p}^2 + \bar{x}^2 = E$

$$p(t) = m\dot{x}(t) = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \quad : \quad t_2 > t_1 \ (=0) \quad : \quad x(t_2) - x(t_1) > 0 \longrightarrow p(t_2) > 0$$

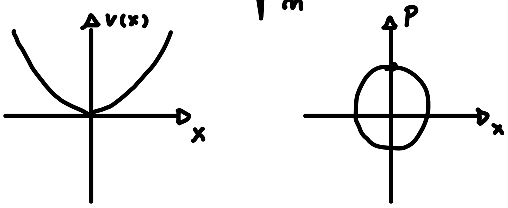
Phase Portrait

=> A collection of trajectories / orbits in phase-space is called a phase portrait and provides a dense illustration of a system's allowed dynamics

Generic Recipe : (Energy is conserved)

$$E = T + V = \frac{m\dot{x}^2}{2} + V(x) \quad : \quad \dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(x)} \quad : \quad p(x) = \pm\sqrt{2m\left[E - V(x)\right]}$$

HO: $V(x) = \frac{1}{2}kx^2$



Damping

$$m\ddot{x} + b\dot{x} + \overset{k = m\omega_0^2}{kx} = 0$$

$\quad$ Viscous damping $\propto \dot{x}$

8-25-21

Recap :

* Harmonic motion : $\ddot{x} = -\omega_0^2 x$, Restoring force : $F = -kx$, Associated potential : $V(x) = \frac{1}{2}kx^2$

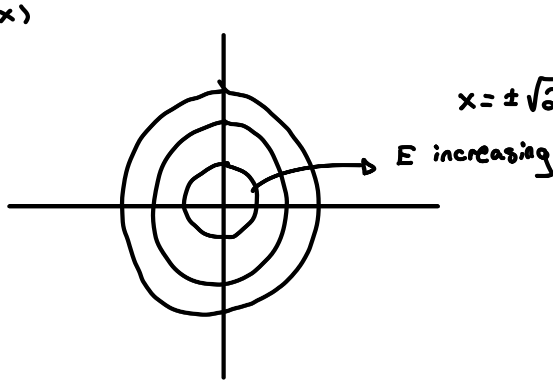
* Generic Solution : $x(t) = A\sin(\omega t - \delta)$, $E = T + V = \frac{A_0^2}{2}k$

* Phase Space => Dynamics in terms of trajectories in (x, p)

* Identifying periodic motion, look for closed trajectories orbit

* Phase-portraits $E = p^2/2m + V(x)$

$x = \pm\sqrt{2m(E - V(x))}$



Damping

$m\ddot{x} + b\dot{x} + kx = 0 \quad w/ \ \omega_0^2 = k/m$

$\hookrightarrow$ viscous damping $\propto \dot{x}$

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$: $\beta = b/2m \longrightarrow$ damping parameter

Solutions

$x(t) = e^{rt} \Rightarrow$ Final $x(t)$ should be real

$r^2 x + 2\beta r x + \omega_0^2 x = 0 \longrightarrow r^2 + 2\beta r + \omega_0^2 = 0 \quad \therefore \; r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

① Sign of $\beta^2 - \omega_0^2$ matters

② $\beta \in \mathbb{R}$

③ 2 Solutions

Solution $\rightarrow$ Combination of 2 linearly independent solutions :

$$x(t) = e^{-\beta t}\left[A_1 e^{-i\sqrt{\omega_0^2-\beta^2}\,t} + A_2 e^{i\sqrt{\omega_0^2-\beta^2}}\right]$$

$\hookrightarrow$ Decaying $\qquad \hookrightarrow$ oscillatory

1) Underdamped : $\beta^2 < \omega_0^2$

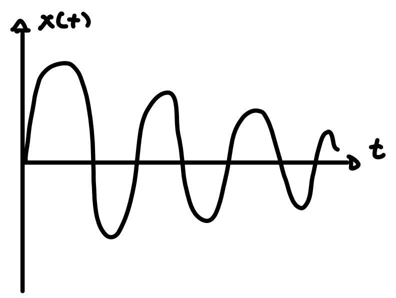
2) Critical damping : $|\beta| = |\omega_0|$

3) Over damping : $\beta^2 > \omega_0^2$

1) Underdamped

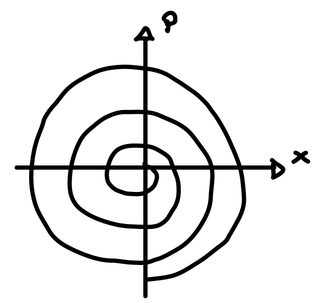
Set $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$: $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$: $A \in \mathbb{R}$

Typical Dynamics :



$x(t)$, $p = m\dot{x}$

decaying spiral $\Rightarrow$ $(x,p) = (0,0)$ is called an attractor



$\Rightarrow$ Fixed point of motion that attracts trajectories

$\Rightarrow$ This case : Independent of initial conditions

damping $\rightarrow$ energy is not conserved :

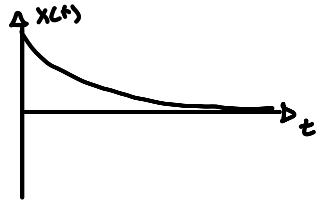
$$E = \frac{m\dot{x}^2}{2} + \frac{m\omega_0^2 x}{2} \rightarrow E(t) = \frac{m}{2}\left(-\beta A e^{-\beta t}\cos(\omega_1 t - \delta) - Ae^{-\beta t}\sin(\omega_1 t - \delta)\right)^2 + \frac{1}{2}m\omega_0^2\left(Ae^{-\beta t}\cos(\omega_1 t - \delta)\right)^2$$

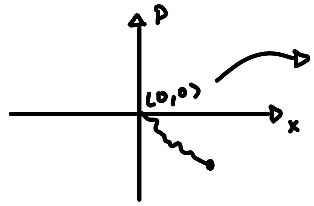
$$E(t) = e^{-2\beta t}\left[\cdots\cdots\right]$$

3) Over damping

$x(t) = A_1 e^{-\beta t} e^{-\omega_1 t} + e^{-\beta t} e^{\omega_1 t}$: Here $\omega_1 = \sqrt{\beta^2 - \omega_0^2} \in \mathbb{R}$

Dynamics : pure decay





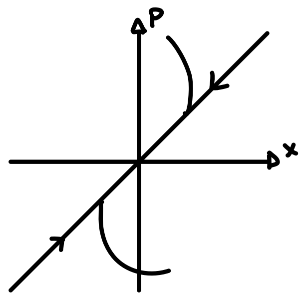
Remains the attractor

2) Critical damping : Assignment Q2

$\omega_1 \to 0$, $x(t) = (A_1 + A_2) e^{-\beta t}$ $\Rightarrow$ Repeated real roots

$x(t) = t e^{-\beta t}$: Final solution $x(t) = A e^{-\beta t} + B t e^{-\beta t}$

Dynamics :



Degenerate nodal sink

Driven - damped oscillator :

$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$ Rewrite as, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t)$ $A = F_0/m$

Homogeneous + Particular solution , $\beta = \frac{b}{2m}$

$\downarrow$ $x_H(t)$ $\downarrow$ $x_p(t)$

$x_p(t) = D\cos(\omega t - \delta) \to$ trig identities : $D = \frac{A}{[(\omega_0^2 - \omega^2)^2 + (2\omega\beta)^2]^{1/2}}$: $\delta = \arctan\left[\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right]$

$x(t) = x_H(t) + x_p(t)$: $\lim_{t\to\infty} x(t) = x_p(t)$

$\Rightarrow$ $x_H(t)$ encodes transient dynamics that are sensitive to initial conditions

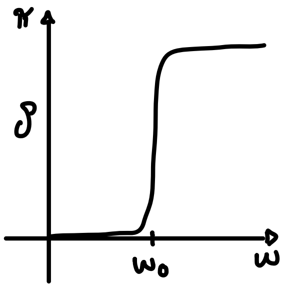
$\Rightarrow$ $x_p(t)$ is the long-time ($t \gg 1/\beta$) steady-state response, insensitive to initial conditions.
$\to$ Modulated force leads to periodic response of oscillator at the driving frequency.

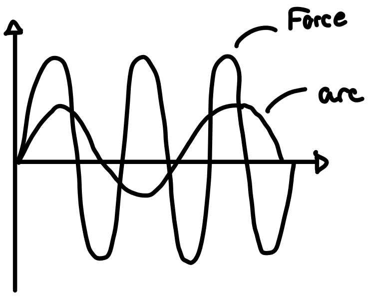
Features of interest :

① Driving @ $\omega = \omega_0$: commonly called resonance $\Rightarrow E = \text{const}$

$$E = \frac{1}{2} m D^2 \left[\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)\right] = \frac{1}{2} m D^2 \omega_0^2 \text{ @ } \omega = \omega_0$$

② Phase lag $\to \delta$



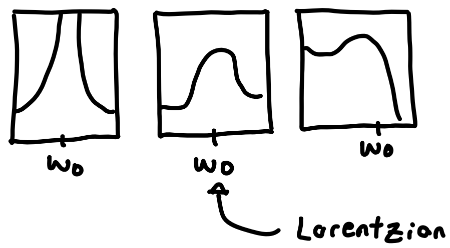


③ The amplitude "response" is maximal @ $\omega = \pm\sqrt{\omega_0^2 - 2\beta^2}$ [$\Rightarrow$ solving $\frac{\partial D}{\partial \omega} = 0$]

$\Rightarrow$ For $\beta \to 0$: D diverges

$\Rightarrow$ $\beta \neq 0$: broadening

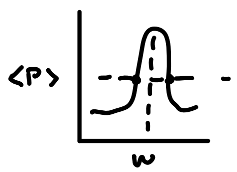
$0 \longrightarrow \beta \longrightarrow$



Subtlety:

P = rate of work = $F\dot{x}$: $\langle P\rangle_{cycle} = \frac{1}{T}\int_0^{T = 2\pi/\omega} P(t)\,dt = \frac{F_0}{2} D\omega \sin(\delta)$: $\frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2+(2\omega\beta)^2}$

$\langle P\rangle_{max} \to \omega = \omega_0$



- $\frac{\langle P\rangle_{max}}{2} \to P$

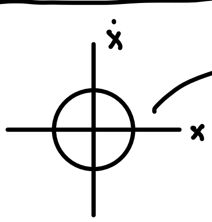
8-30-21

Recap:

* Damped oscillator
  - $\Rightarrow$ under, over-damped, critical
  - $\Rightarrow$ Independent of initial conditions $\Rightarrow (0,0) \longrightarrow$ Attractor
* Damped - driven oscillator
  - $\Rightarrow$ Stabilized oscillations @ long time:

    $x(t) = x_c(t) + x_p(t)$

Why do we care:



$\to$ attractor fixed pt $\to$ orbit

Van der Pol Oscillator:

$\Rightarrow$ Electrical engineering
$\Rightarrow$ Biology
$\Rightarrow$ Quantum sensing

Equation of motion:

$$m\ddot{x} - \varepsilon(x_0^2 - x^2)\dot{x} + m\omega_0^2 x = F_0\cos(\omega_f t)$$

Goldstein:

$$m\ddot{x} - \varepsilon(1 - x^2) + kx = F_0\cos(\omega_D t) \;:\; x \to x/x_0,\; m \to m/x_0,\; \varepsilon \to \varepsilon x_0,\; k \to m\omega_0^2/x_0,\; F_0 \to F_0/x_0^2$$

Kick The Tyres:

* $\varepsilon \to 0$: driven oscillator

* $F \to 0$, $\varepsilon \to 0$: Simple harmonic oscillator

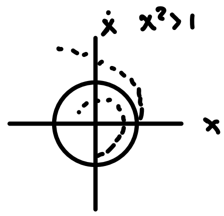
* $F = 0$, $\varepsilon \neq 0$: nonlinear damped oscillator

* $F \neq 0$, $\varepsilon \neq 0$

$F = 0, \varepsilon \neq 0$: $x_0 = 1$

$x^2 < 1 \Rightarrow$ damping < 0

$x^2 > 1 \Rightarrow$ damping > 0



$\to$ Limit Cycle
Attractor $\to$ Generalization of Fixed pt

Definition: Limit Cycle

$\Rightarrow$ Closed trajectory (orbit) where at least one another trajectory flow into it as $t \to \infty$ or $t \to -\infty$

* Stable limit cycle:

 a) A trajectory on the limit cycle will not leave.

 b) Small perturbation of the trajectory will flow back to the limit cycle as $t \to \infty$

$\Rightarrow$ Generalization of a fixed point to an orbit w/ periodic motion

* Unstable limit cycle:

 $\Rightarrow$ Nearby trajectories "spiral" into it as $t \to -\infty$

 $\Rightarrow$ Many (most) complex Systems exhibit unstable limit cycles

VDP Oscillator w/ : $F=0$ & $\varepsilon \neq 0$ but "small"

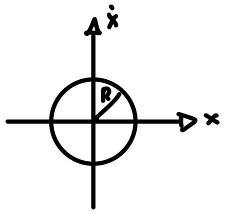
$\Rightarrow$ Obtain analytic solution for a stable limit cycle

$\Rightarrow$ Limit cycle $\longrightarrow$ Circular orbit

$\Rightarrow$ "Method of averaging"

EOM : $k=m=1$

$\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0$ for $F_0 = 0$: $\varepsilon \ll 1 \rightarrow$ "small"



$x(t) = a(t)\cos(t + \psi(t))$ (i)

$\dot{x}(t) = -a(t)\sin(t+\psi(t))$ (ii)

$\Rightarrow$ Choose (ii) so that orbit in phase-space is approx circular.

$\frac{d}{dt}$(i) = (ii) $\Rightarrow$ $\dot{a}$ & $\dot{\psi}$: (i) & (ii) $\Rightarrow$ (*) [use $\ddot{x}$ obtained for (ii)]

We obtain two coupled equations and we solve for $\dot{a}$ & $\dot{\psi}$

$\dot{a} = -\varepsilon a\left[a^2\cos^2(t+\psi(t))\right]\sin^2(t+\psi(t))$ (iii)

$\dot{\psi} = -\varepsilon\left[a^2\cos^2(t+\psi(t))-1\right]\sin(t+\psi(t))\times\cos(t+\psi(t))$ (iv) : Exact up to here

Method of averaging, $\dot{a}, \dot{\psi}$ are "small" $\Rightarrow$ Replace cos( ) & sin( ) with "average" values

$\dot{a} \approx -\varepsilon a \frac{1}{2\pi}\int_0^{2\pi} d\phi \, \sin^2\phi\,(a^2\cos^2\phi - 1)$: $\dot{\psi} \approx -\varepsilon \frac{1}{2\pi}\int_0^{2\pi} d\phi\,(a^2\cos^2\phi - 1)\sin\phi\cos\phi$

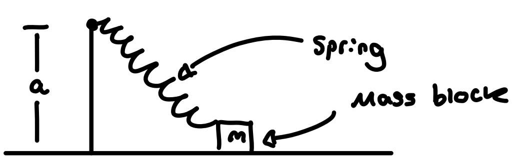
$\dot{a} \approx \frac{\varepsilon a}{8}(4-a^2)$ @ $a=2$, $\dot{a}=0$: $\dot{\psi} \approx 0$

$x(t) = 2\cos(t+\psi_0)$

$\frac{a^2(2-a)}{2+a} = e^{-\varepsilon(t+t_0)}$: $a<2$: $t_0 = 0$

<u>Go back a little :</u> Fixed pts

Example :



$F_s = -k\Delta L \rightarrow L_{eq} = L$

$F_s = -k\left(\sqrt{a^2+x^2} - L\right)$

$V(x) = \frac{k}{2}\left(\sqrt{a^2+x^2} - 1\right)^2$: $m\ddot{x} = F_x = -\frac{\partial V}{\partial x} = -kx\left(1 - L/\sqrt{a^2-x^2}\right)$

$F_x = 0 \longrightarrow$ Fixed points of motion : Solutions $x = 0$ or $x = \pm\sqrt{\ell^2 - a^2}$

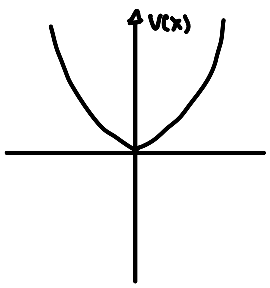
$x = 0$: Spring will push/pull vertically only $\longrightarrow F_x = 0$

$\ell > a$: unstable , $\ell < a$: stable : $x = \pm\sqrt{\ell^2 - a^2}$ will exist

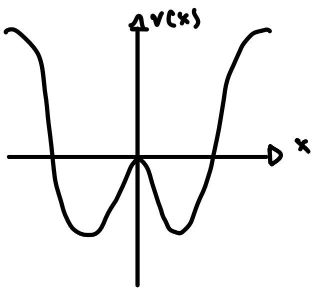
Particle in a potential picture:

$\Rightarrow$ "Marble" sitting in $V(x)$: $KE = \frac{m\dot{x}^2}{2}$

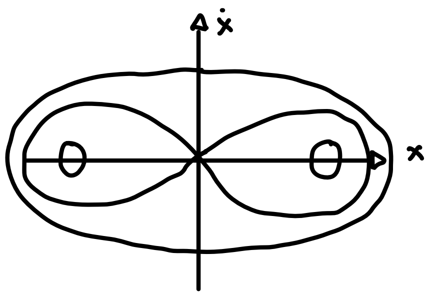
i) $\ell < a$ stable fixed point at $x=0$



ii) $\ell > a$ 3 fixed points. 1 unstable @ $x=0$ 2 stable @ $x = \sqrt{\ell^2 - a^2}$



$p(x) = \pm\sqrt{2m(E - V(x))}$



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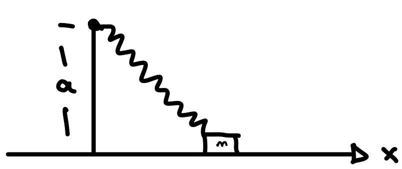
Recap

Limit cycles : Attractor $\Rightarrow$ Nearby trajectory flows into as $t \to \pm\infty$, + stable , - unstable

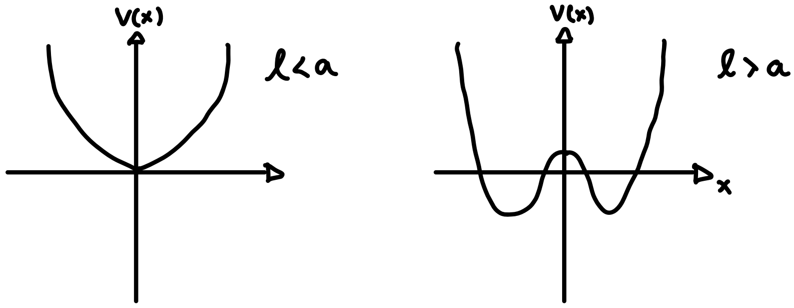
Fixed points :



$F = m\ddot{x} \Rightarrow$ Fixed points have $F_x = 0$, $(\ddot{x} = 0)$, $x = 0$ or $x = \pm\sqrt{\ell^2 - a^2}$

Graphical

$$\frac{-\partial V(x)}{\partial x} = F_x \Rightarrow V(x) = \frac{k}{2}\left(\sqrt{a^2+x^2} - l^2\right)^2$$



2^{nd} Derivative test:

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_f} > 0 \quad \text{Stable (minimum)} \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_f} < 0 \quad \text{Unstable (maximum)}$$

Chaotic trajectories (11.4)

* Everything we talked about so far: Regular dynamics

* Chaotic dynamics: most real systems $\Rightarrow$ property of deterministic dynamics

Properties of chaotic dynamics

$\Rightarrow$ Exponentially sensitive to initial conditions

$\Rightarrow$ Mixing

$\Rightarrow$ Dense orbits

$\Big\}$ Pg. 492

$\Rightarrow$ Feature of nonlinear dynamics

Dense - (quasiperiodic) "orbits"

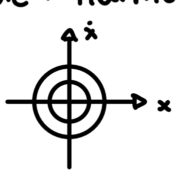
$\Rightarrow$ Trajectories do not close & pass through all available phase space

Mixing: Define regions I_1 & I_2 that are part of the available phase space.

$\Rightarrow$ Any trajectory that passes through $I_1 \rightarrow$ Passes through I_2

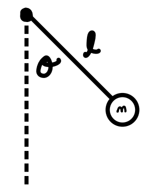
Sensitivity to initial conditions

Example: Harmonic oscillator, $\ddot{x} = -\omega_0^2 x$

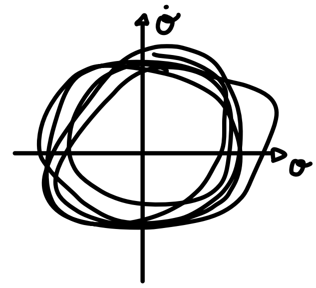


$|\delta\vec{x}(t)| = \text{const.} \longrightarrow \sqrt{(x_1-x_2)^2} = \text{const} \longrightarrow$ Linear System

Example: Driven pendulum w/ damping.



$$\ddot{\theta} + b\dot{\theta} + \sin\theta = F\cos(\omega t)$$



Poincare - Bendixson Theorem: "chaos requires 3 variables + nonlinearity"

$$\dot{x} = \sigma y - \sigma x \;,\; \dot{y} = \rho x - x\xi - y \;,\; \dot{\xi} = xy - \beta\xi$$

Iterative processes & chaos (11.8)

Iterative map: $x_{n+1} = f(x_n) \longleftarrow$ Rule that connects $x_0 \to x_1 \to x_2$

Examples:

$\Rightarrow$ Logistic map [logistic equation]

$\Rightarrow$ Quadratic map

$\Rightarrow$ Tent map

$\Rightarrow$ Gauss iterated map (nonlinear map)

Rich Behavior

$\Rightarrow$ Fixed points (stable / unstable)

$\Rightarrow$ Limit cycles (Periodic)

$\Rightarrow$ Bifurcations

$\Rightarrow$ Chaos

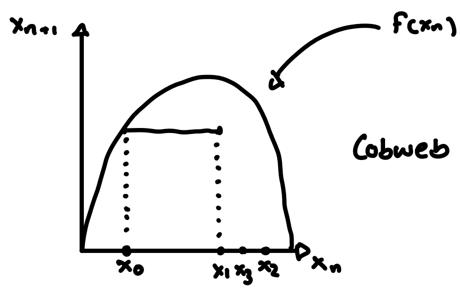
Example: Logistic Map

Rule: $x_{n+1} = a x_n(1 - x_n)$: $x_n \in [0,1]$, $a \in [0,4]$

Demographic model (population dynamics)

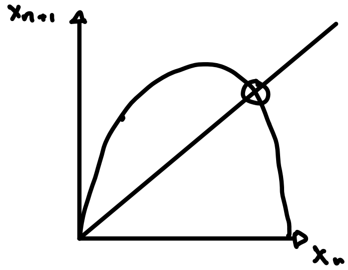
* Reproduction: $x_n \ll 1 \to x_{n+1} = a x_n$

* Starvation: $x_n \to$ "large"

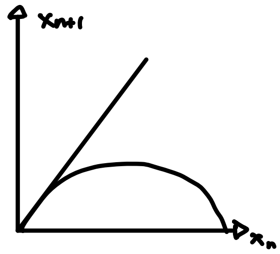


Cobweb Plot

Fixed point : $x_f = f(x_f)$



$a \to 0$



Fixed Points of Logistic Map

$x_f = a\, x_f (1 - x_f)$

1) $x_f = 0$

2) $1 = a(1 - x_f) \Rightarrow x_f = 1 - 1/a$

$a \in [0,4]$ & $x_n \in [0,1]$ for $a \geq 1$

Stability of fixed points

Stable : $x_0 = x_f + \delta$, $x_1 = x_f + \delta'$ $\quad \delta' < \delta$

Small perturbations flows back to x_f

Unstable : Small perturbation flows further away

Logistic map :

i) $x_f = 0$: $x_0 = \delta$, $x_1 = a\delta(1-\delta) \approx a\delta$: $a < 1$ stable $\quad a > 1$ unstable

ii) $x_f = 1 - 1/a$: $x_0 = 1 - 1/a + \delta$, $x_1 = a(1 - 1/a + \delta)(\delta - 1/a) \approx 1 - 1/a + \delta(2-a)$

$a < 3$ Stable $\left[\, |2-a| < 1 \,\right]$, $a > 3$ unstable $\left[\, |2-a| > 1 \,\right]$

Generic Recipe :

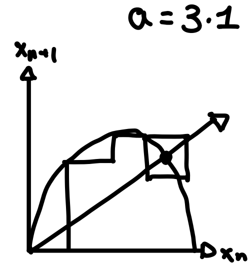
x_f is $x_{n+1} = f(x_n) \Rightarrow$ stable if $\left| \frac{\partial f}{\partial x} \right|_{x_f} < 1$, unstable if $\left| \frac{\partial f}{\partial x} \right|_{x_f} > 1$

$x_{n+1} \approx x_f + \left. \frac{\partial f}{\partial x} \right|_{x = x_f} \delta + O(\delta^2)$

a: 0 ------------|------------|------------ 4

$a < 1$	$1 < a < 3$	$a > 3$
$x_f = 0$ Stable	$x_f = 1 - 1/a$: stable $x_f = 0$: unstable	no stable Fixed pts.

$a = 3.1$



Limit Cycle Between

$x = 0.558$

$x = 0.765$

Pitchfork Bifurcation

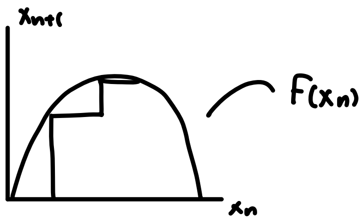
9-6-21

Chaos :

Exponential sensitivity to initial conditions $\longrightarrow$ Lyapunov exponent

- Iterative maps : $x_{n+1} = f(x_n)$
- Logistic maps : $x_{n+1} = \alpha x_n (1 - x_n)$

Cobweb Plots



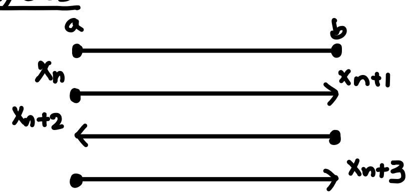
Fixed points : unstable , stable

$$x_f + \delta , \quad \left| \frac{dF}{dx} \right|_{x = x_f} < 1$$

Logistic model : fixed points

@ $x_f = 0$, stable for $a < 1$
$x_f = 1 - 1/a$, stable for $a < 3$

Limit Cycles



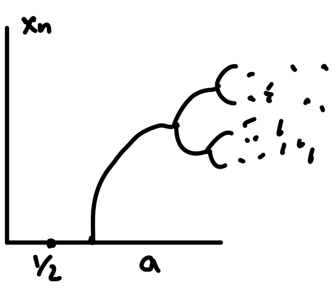
Limit Cycle of 2 points "Pitchfork"

Bifurcation Diagram

→ want to visualize the "long-time" behavior our map

→ Identify Fixed points , limit cycle , chaos

Typical Figure



i) x_0 in available domain : Logistic $\rightarrow x \in [0,1]$

ii) $\{ x_0 , x_1 , x_2 \ldots \ldots x_{1001} , x_N \}$

"trajectory"

$a < 1$: 0
$a > 1$: $1 - 1/a$
$a > 3$:

Lyapunov Exponent

→ Quantifier of chaos of butterfly effect

→ Sensitivity to initial conditions

$$\Delta x(t) \sim \Delta x(0)\, e^{\lambda t}$$

λ → Lyapunov exponent

$\Delta x(0)$: $x - x'$, $x' = x + \varepsilon$

$\lambda > 0$ Chaotic dynamics , $\lambda < 0$ no chaos

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log\left(\frac{\Delta x(t)}{\Delta x(0)}\right) \quad : \quad \text{Chaos} \longrightarrow \lambda > 0$$

↳ $t > 1/2$

Generic (numeric recipe)

1) set initial conditions $x(0)$ + $x'(0) = x(0) + \varepsilon$

2) Integrate EOM to obtain $x(t)$ & $x'(t)$, $\Delta x(t) = x'(t) - x(t)$

"Direct approach"

Lyapunov exponent for iterative map : $\varepsilon_n = \varepsilon_0 e^{n\lambda}$

$\varepsilon_n = |x_n - x_n'|$, $\varepsilon_0 = |x_0 - x_0'|$, n = time , t , $\lambda > 0$ Chaos : $\lambda \leq 0$ no chaos

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log\left(\frac{\varepsilon_n}{\varepsilon_0}\right)$$

Better approach : consider x_0

$$x_0' = x_0 + \varepsilon_0 \quad , \quad \varepsilon_0 = |x_0 - x_0'|$$

Take $n = 1$ $\quad [x_{n+1} = f(x_n)] \longrightarrow$ map

$$x_1 = f(x_0) \quad , \quad x_1' = f(x_0') = f(x_0 + \varepsilon_0)$$

$x_1' = x_1 + \varepsilon_1$, $[\varepsilon_1 = |x_1 - x_1'|]$

$$\varepsilon_1 = x_1' - x_1 = f(x_0 + \varepsilon_0) - f(x_0) = \varepsilon_0 [f(x_0 + \varepsilon_0) - f(x_0)]/\varepsilon_0 \longrightarrow \left.\frac{df}{dx}\right|_{x = x_0}$$

$$\varepsilon_1 = \varepsilon_0 \left|\frac{df}{dx}\right|_{x = x_0} \quad \text{for } \varepsilon_0 \to 0$$

Now $n = 2$

$$x_2 = f(x_1) = f(f(x_0))$$

$x_2' = f(f(x_0 + \varepsilon_0))$

$$\varepsilon_2 = x_2' - x_2 = \varepsilon_0 \left|\frac{df}{dx}\right|_{x=x_1} \left|\frac{df}{dx}\right|_{x=x_0}$$

$$\varepsilon_n = \varepsilon_0 \left|\frac{df}{dx}\right|_{x=x_{n-1}} \cdots\cdots \left|\frac{df}{dx}\right|_{x=x_0}$$

Claim : $\varepsilon_n = \varepsilon_0 e^{n\lambda}$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\varepsilon_n}{\varepsilon_0}\right) = \lim_{n\to\infty} \frac{1}{n} \log\left(\left|\frac{df}{dx}\right|_{x=x_{n-1}} \cdots\cdots \left|\frac{df}{dx}\right|_{x=x_0} \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left(\left|\frac{df}{dx}\right|_{x=x_j} \right)$$

$\longrightarrow \{x_0, \ldots\ldots, x_{n-1}\}$

Direct Approach : alternative $\longrightarrow$ tangent space

Poincare sections

plot $\vec{x}(t) \longrightarrow$ hard to tell if it is chaotic
$\vec{x}'(t) \longrightarrow$ still hard

Use phase portrait approach

poincare sections : 4D phase-space : ex (x, y, p_x, p_y) $\quad p_x = m\dot{x}$

Recipe for a poincare section

1) Pick an initial condition , $(x(0), y(0), p_x(0), p_y(0))$
$\downarrow$
Initial energy $E_0 =$ const : $E(x, y, p_x, p_y)$

$E(x, y, p_x, p_y) = E_0$: Eliminate $p_x = p_x(x, y, p_x, E_0)$

2) Consider a plane set by choosing $x=0$, plane exists (y, p_y)

$\rightarrow$ Every time the trajectory $(y(t), p_y(t), x(t))$ cuts through the $x=0$ plane we plot the coordinates

" Visually "