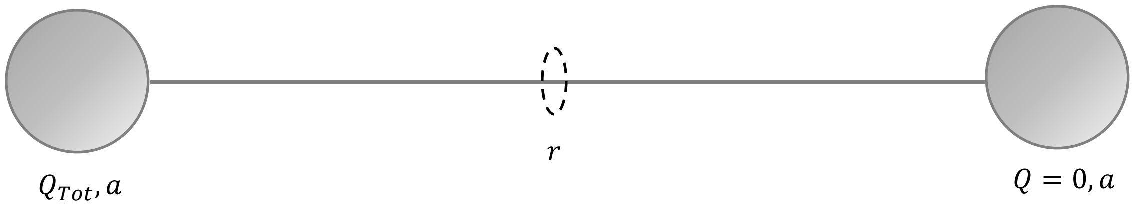
E & M I

Workshop 7 – Maxwell & Faraday, 3/23/2022

In today's workshop, we'll consider an argument for Maxwell's addition to Ampere's Law. There will be some hand-waving, neglecting some issues of time-dependence, but it gives interesting results and also gives some practice with electrostatics and Ampere's Law.



Consider two conducting spheres, both with radius a, that are very far apart> we can treat them as basically isolated spheres. Initially the left sphere has a charge Q_{Tot} and the right sphere is uncharged.

A) Using the results from electrostatics, explain/justify briefly why (for electro-STATICS) that:

i) The electric field for a charged sphere will be zero inside and perpendicular to the sphere at the surface (and outside).

The force on charges is $\vec{F} = q\,\vec{E}$ or the current density is $\vec{J} = \sigma\,\vec{E}$. For electrostatics, charges aren't moving so the field must be zero.

ii) All the charge will be on the surface of the sphere.

Because $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{E} = 0$ everywhere inside the conductor, the charge density everywhere inside the conductor must be zero.

On the surface, however, we've found that the perpendicular electric field can be discontinuous across a surface charge density. This means it's possible to have charge on the surface, $\vec{E} = 0$ inside and $\vec{E} \cdot \hat{n} \neq 0$ outside and satisfy Gauss' Law.

iii) The electric potential is a constant inside and on the surface and equal to

$$\phi(R) = k\frac{Q}{a}, \qquad k = \frac{1}{4\pi\epsilon_0}$$

Inside the conductor,

$$\vec{E} = 0 = -\nabla\phi$$

This means $\phi =$ constant inside, or for any points inside

$$\phi(\vec{r}) - \phi(\vec{r}\,') = -\int_{\vec{r}}^{\vec{r}\,'} \vec{E} \cdot d\vec{l} = 0$$

The charge on the sphere will be uniform so it will look like a point charge.

Consider what happens if we connect the two spheres with a conducting wire. Assume that the wire is thin enough that the net charge on the wire will be much smaller than Q_{Tot}.

B) Explain what will happen to the charge in the system when the spheres are connected. What will be the charge distribution a very long time after the spheres are connected? Why?

For simplicity, let's assume $Q_{tot} > 0$.
Initially the left sphere will have a higher potential and the right sphere will have $\phi = 0$.
The charge will lower its energy by moving from the left sphere to the right sphere.

The charge will stop flowing (electrostatics) when the potential everywhere on the two conductors (and wire) is a constant. This means that in the final situation,

$$Q_R = Q_L = \frac{1}{2}\, Q_{Tot}$$

The wire has a resistance R that is large enough so that the charge remains uniformly distributed on the spheres (the current is not too large). Remember Ohm's Law:

$$I(t) = \frac{\Delta V(t)}{R} = \frac{\phi_L(\,t) - \phi_R(t)}{R}$$

$\phi_L(t)$ and $\phi_R(t)$ are the potentials of the left and right spheres.

C) Write down a differential equation for the time-dependent charge on the right sphere, $Q_R(t)$.

Considering the charge on the right spheres:

$$I(t) = \frac{d}{dt}\, Q_R(t), \qquad Q_L(t) = Q_{Tot} - Q_R(t), \qquad \phi = \frac{k}{a}\, Q$$

Giving:

$$\frac{d}{dt}\, Q_R(t) = \frac{k}{a\, R}\big(Q_{Tot} - 2\, Q_R(t)\big)$$

Or

$$\frac{d}{dt}\, Q_R(t) + \frac{2\, k}{a\, R}\, Q_R(t) = \frac{k}{a\, R}\, Q_{Tot}$$

The boundary conditions are:

$$Q_R(0) = 0, \;\; Q_R(t \to \infty) = \frac{1}{2}\, Q_{Tot}$$

D) Solve the differential equation from (C) for the charge $Q_R(t)$ and the current in the wire $I(t)$. Show that these have the correct values at $t = 0$ (when the spheres are connected) and $t \to \infty$.

The general solution to this differential equation is:

$$Q_R(t) = A + B\, e^{-\gamma t}$$

Solving:

$$-\gamma\, B\, e^{-\gamma t} + \frac{2\,k}{a\,R}\,(A + B\, e^{-\gamma t}) = \frac{k}{a\,R}\, Q_{Tot}$$

$$\left(2 - \gamma\frac{aR}{k}\right) B\, e^{-\gamma t} = Q_{Tot} - 2\,A$$

This means:

$$\gamma = \frac{2\,k}{a\,R}, \qquad A = \frac{1}{2}\, Q_{Tot}$$

$$Q_R(t) = \frac{1}{2}\, Q_{Tot} + B\, e^{-\gamma\, t}$$

The boundary condition, $Q_R(0) = 0$ gives:

$$Q_R(t) = \frac{1}{2} Q_{Tot}\,(1 - e^{-\gamma\, t})$$

Note: Checking units –

$$k\frac{Q}{a} = Volt \Rightarrow k = \frac{V\,m}{C}, \qquad R = \frac{V}{A} = \frac{V\,s}{C}$$

$$\frac{k}{a\,R} = \frac{V\,m}{C}\,\frac{1}{m}\,\frac{C}{V\,s} = \frac{1}{s}$$

Next consider Ampere's Law using a small "Amperian" loop of radius r around the center of the wire as shown.

E) Consider Ampere's Law in the magneto-static approximation:

$$\vec{\nabla} \times \vec{B} = \mu_0\,\vec{J} \quad \Rightarrow \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 \iint \vec{J} \cdot \hat{n}\, dS = \mu_0\, I_{enclosed}$$

Using this result and the simplest surface for the loop, a flat disk bounded by the loop, calculate the "quasi-static" approximation to the magnetic field, $\vec{B}(t)$ at the center of the wire.

The current is:

$$I(t) = \frac{d}{dt}\, Q_R(t) = \frac{\gamma}{2}\, Q_{Tot}\, e^{-\gamma\, t}$$

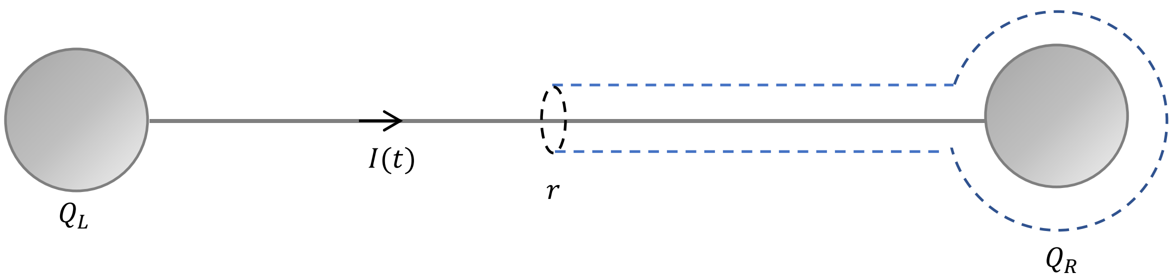
By the symmetry here, the magnetic field should only depend on the distance from the long wire and wrap around the wire, $\vec{B}(\vec{r}) = B(r)\,\hat{\phi}$

$$\oint \vec{B} \cdot d\vec{l} = 2\,\pi\,r\,B(r) = \mu_0\,I(t)$$

$$B(r,t) \approx \frac{\mu_0}{2\,\pi\,r}\,\frac{\gamma}{2}\,Q_{Tot}\,e^{-\gamma\,t}$$

Stoke's Theorem, and Ampere's Law, should work for ANY surface with the boundary being the loop given. You might see where this is going…

Consider a surface that is in the shape of a "flask", with the open end being the amperian loop, a long cylindrical "neck" around the wire, and a spherical bulb centered on, the right sphere:



F) What is the current through this new surface? This is a "demonstration" that the magnetostatic Ampere's Law is insufficient.

In this case there is no current through the surface bounding the loop, so that Ampere's Law seems to indicate $\vec{B} = 0$.

G) Determine the electric field due to the right sphere (don't over-think this) and show that you will get the same result as part E using:

$$\vec{\nabla} \times \vec{B} = \mu_0\,\vec{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\,\vec{E} \;\Rightarrow\; \oint \vec{B} \cdot d\vec{l} = \mu_0\left(\iint \vec{J} \cdot \hat{n}\,dS + \epsilon_0\frac{\partial}{\partial t}\iint \vec{E} \cdot \hat{n}\,dS\right)$$

Considering the E-field of the right sphere:

$$\vec{E}(r,t) = k\,\frac{Q_R(t)}{r^2}\,\hat{r}$$

And the flux of the electric field for a sphere of radius r is:

$$\iint \vec{E} \cdot \hat{r}\,dS = |\vec{E}|\,4\pi r^2 = \frac{Q_R(t)}{\epsilon_0}$$

And

$$\epsilon_0 \frac{\partial}{\partial t} \iint \vec{E} \cdot \hat{r}\, dS = \frac{\partial}{\partial t}\, Q_R(t) = I(t)$$

This gives the same result as above:

$$\vec{\nabla} \times \vec{B} = \mu_0\, I(t)$$

$$B(r,t) \approx \frac{\mu_0}{2\,\pi\, r} \frac{\gamma}{2}\, Q_{Tot}\, e^{-\gamma\, t}$$

Bonus Question: What is the problem with how we did part E, considering Maxwell?

The problem here is that there is a time-dependent electric field in the wire causing the current. This means in:

$$\vec{\nabla} \times \vec{B} = \mu_0\, \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\, \vec{E}$$

Both the first and second terms on the right will be non-zero.