Physics 5403 Exam #2
Spring 2022

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Time: 3 hours

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1 Harmonic Oscillator

A quantum harmonic oscillator is described by the unperturbed Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where p is the momentum, x is the position operator and ω is the frequency. Consider the perturbation

$$V = Am\omega^2 x^2,$$

with A a real constant.

(a) Compute the perturbed energies of the states in *second* order in perturbation theory. Show that the result is consistent with the exact solution of the problem.

(b) Assume that $A \equiv A(t)$ has explicit time dependence of the form:

$$A_0 e^{i\omega t} e^{-\tau t}$$

for $t \geq 0$, with $\tau \to 0_+$. Suppose an electron is in the ground state at $t = 0$. Compute the probability of transition from the ground state to the next *available* excited state. Compute your result in lowest order of perturbation theory where the result is non-zero.

2 Quantum rotor

A quantum rotor has the Hamiltonian

$$\frac{L^2}{2\mu a^2} |l, m\rangle = \frac{\hbar^2}{2\mu a^2} l(l+1) |l, m\rangle,$$

where $\mathbf{L}$ is the total angular momentum, with $l \in 0, 1, 2, ..., \infty$, μ is the mass of the rotor and a is a constant. Suppose a perturbation of the form

$$V(\mathbf{r}) = V_0 yz \tag{1}$$

is turned on, with $\mathbf{r} = (x, y, z)$ the position operator.

(a) Write down the potential $V(\mathbf{r})$ in terms of a spherical tensor of rank 2.

(b) Calculate the perturbed energy levels and perturbed kets of the *first degenerate* excited states using perturbation theory. You don't have to calculate any integrals.

3 Scattering Potential

A free particle with energy $E = \hbar^2 k^2/(2m)$ and mass m is scattered by a local potential $V(\mathbf{x})$.

(a) Starting from the Lipmann-Schwinger equation

$$|\psi^+_{\mathbf{k}}\rangle = |\mathbf{k}\rangle + \frac{1}{E - \hat{\mathcal{H}}_0 + i0^+} \hat{V} \, |\psi^+_{\mathbf{k}}\rangle \, ,$$

where $\hat{\mathcal{H}}_0$ is the unperturbed Hamiltonian, $|\psi^+\rangle$ is the scattered state and $|\mathbf{k}\rangle$ the free particle one, with momentum $\mathbf{k}$, write down the integral equation for the wave function of the scattered state $\psi^+(\mathbf{k}') \equiv \langle \mathbf{k}' | \psi^+ \rangle$ in the *momentum* representation.

(b) Show that the Lipmann-Schwinger equation is equivalent to the integral equation

$$f^{(+)}(\mathbf{k}, \mathbf{k}') = -2\pi^2 \left(\frac{2m}{\hbar^2} \right) V(\mathbf{k}' - \mathbf{k}) + \frac{2m}{\hbar^2} \int \frac{d\mathbf{q}}{k^2 - q^2 + i0_+} V(\mathbf{k}' - \mathbf{q}) f^{(+)}(\mathbf{q}, \mathbf{k})$$

where

$$f^{(+)}(\mathbf{k}, \mathbf{k}') = -2\pi^2 \frac{2m}{\hbar^2} \langle \mathbf{k}' | V | \psi^+ \rangle \tag{2}$$

is the scattering amplitude, and $V(\mathbf{q})$ is the Fourier transform of $V(\mathbf{x})$.

(c) Show explicitly that the optical theorem

$$\mathrm{Im} f^{(+)}(\mathbf{k}, \mathbf{k}) = \frac{k}{4\pi} \sigma_T$$

is valid in the lowest order in V where the result applies, where σ_T is the *total* scattering cross section. Hint: write σ_T in the first Born approximation and compute $\mathrm{Im} f^+(\mathbf{k}, \mathbf{k})$ in the *second* Bord approximation. Use the fact that

$$\mathrm{Im} \frac{1}{a + i0_+} = -\pi \delta(a), \qquad \delta(f[x] - f[b]) = \frac{1}{|f'(b)|} \delta(x - b)$$

(d) Assume now the scattering potential

$$V(\mathbf{x}) = g e^{-\mu |\mathbf{x}|}.$$

Derive from Eq. (2) the differential scattering cross section in the first Born approximation.