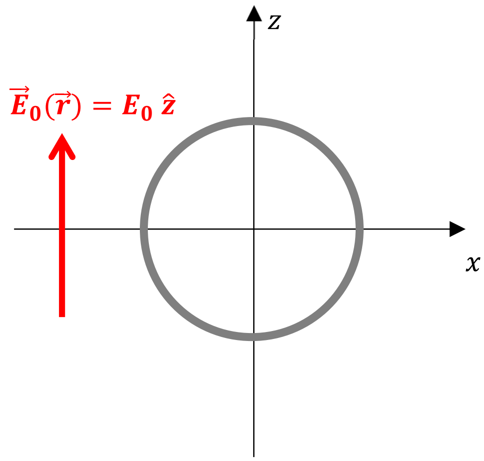
E & M I

Workshop 9 – Spherical Conductor in a Field, 4/4/2022

Today's workshop will consider the fairly standard problem of a spherical conductor placed in a constant, external electric field. We'll consider three different ways to solve this problem to better understand both the physics and the methods for solving this problem.



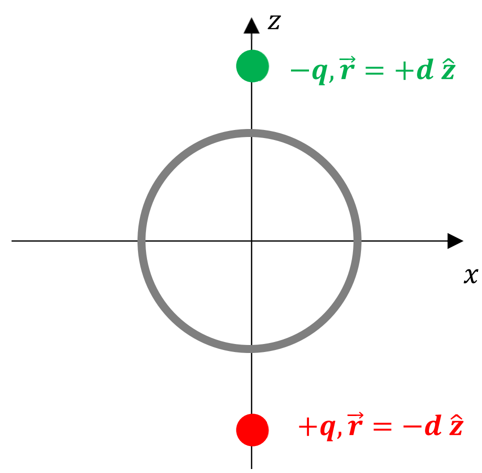
0) Setup:

The diagram shows the metal sphere, centered at the origin, and the external electric field. Let:

Radius of the sphere = R, Charge on the sphere = 0.
The electric field $\vec{E}_0(\vec{r})$ is the result of charges very far away, so we can approximate them as being at infinity. E_0 is a constant.

a) What is the potential, $\phi_0(\vec{r})$, corresponding to the field $\vec{E}(\vec{r})$? Write this in terms of both Cartesian and Spherical coordinates.

b) Predict what will happen to the charge on the sphere. Draw a picture of your prediction and explain your reasoning.



1) Images:

First, consider a slightly different problem. Instead of an external field, the sphere has two point charges equal distance from the center with charges $\pm q$ as shown.

a) Using results from last week*, what are the values and positions of two image charges inside the sphere that will allow a solution for the potential and field everywhere $r > R$? A picture would be a good idea.

b) Show (without doing much work) that the potential on the sphere is the same at the points $(x = 0, z = R), (x = 0, z = -R), (x = R, z = 0), (x = -R, z = 0)$.

c) What is the potential on the sphere, $\phi(R)$?

d) This problem approaches the constant-field problem above if we consider $d \to \infty$ but changing q along with d in such a way that the field due to these two charges, $\vec{E}_{2q}(\vec{r})$, at the origin is a constant:

$$\vec{E}_{2q}(\vec{r} = 0) = E_0\, \hat{z}$$

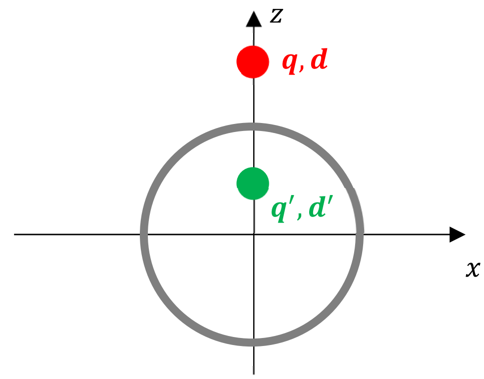
What is the relation between $q, d,$ and E_0 that will keep the field at the origin constant?

e) Considering this limit, $d \to \infty, q \to \infty, E_{2q}(0) = E_0$, show that the images charges found above become a point electric dipole. What is the magnitude of the dipole moment?

* Results from last Wednesday:

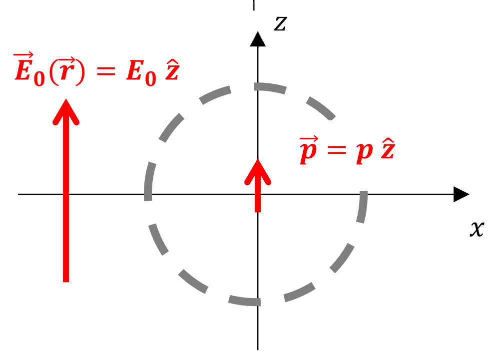
Image-charge solution for a grounded, conducting sphere for radius R and an external charge.

$$q' = -q\frac{R}{d}, \qquad d' = \frac{R^2}{d}$$



2) Dipole

In the textbook, Garg solved this problem by jumping immediately to the idea that the sphere can be modeled by an “image dipole” at the center. This seems to be true from Part 1, but let’s check that this really works.



a) Consider the problem of a point dipole at the origin in a constant electric field. Write down the total potential, due to the sphere and the external field everywhere.

Reminder: For an electric dipole,

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}, \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3}\,(3\,(\vec{p}\cdot\hat{r})\hat{r} - \vec{p})$$

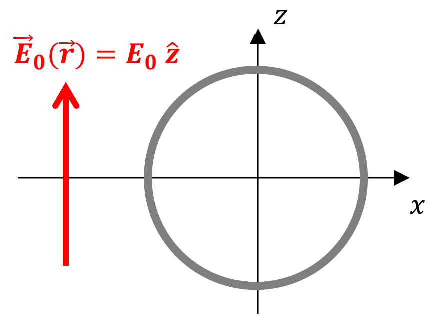
b) What is the value of the image dipole $\vec{p}$ that will give the same result for $\phi(R)$ as Part 1-c above? How does this compare with your answer to Part 1-e?

c) Calculate the total electric field at the surface of the sphere, $\vec{E}_{tot}(|\vec{r}| = R)$ and show that it is perpendicular to the surface of the sphere.

d) What is the surface charge density on the sphere (as a function of θ of course)? Does this match your prediction from the start of the problem?

3) Expansions:

It is also possible to use a general solution to Laplace's equation to solve this problem. For a problem that has a spherical boundary and it independent of the azimuthal angle ϕ, Laplace's equation is solved by:



$$\phi(\vec{r}) = \sum_l \left(a_l\, r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

a) Consider the boundary condition on $\phi(\vec{r})$ in the limit as $|\vec{r}| \to \infty$. Show that, considering this limit, only one term in the sum will be non-zero.

b) Using the $|\vec{r}| \to \infty$ limit, what is one of the remaining coefficients $(a_l \text{ or } b_l)$?

c) Consider the boundary condition for $\phi(R)$ found above. Use this result to calculate $\phi(\vec{r})$. Show that your result is the same as what was found above.