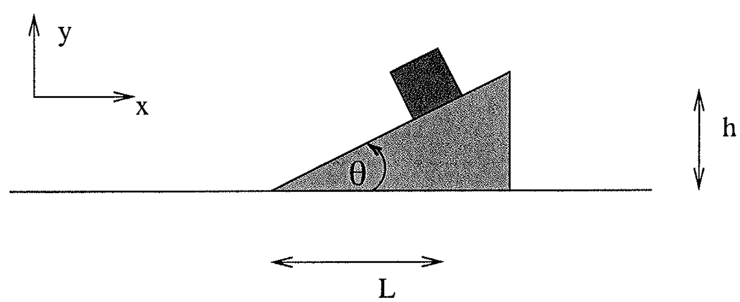
Classical Mechanics

1. A block of mass m_1 sits atop a triangular wedge of mass m_2, which is itself on a frictionless plane, as shown. The two are initially at rest, and the block is a height h above the surface of the plane, a horizontal distance L from the bottom edge of the wedge. The wedge has an opening angle θ, as shown.



   (a) Assume that there is no friction between the block and the wedge. The block slides down the wedge. What are the velocities (measured with respect to the fixed inertial reference frame denoted by the x and y axes shown) of the block and wedge just as the block reaches the lower edge of the wedge? (3 points).

   (b) Now replace the block by a ball of radius R (and mass m_1). The ball rolls down the wedge without slipping. What are the velocities of the ball and wedge just as the ball reaches the lower edge of the wedge? (3 points).

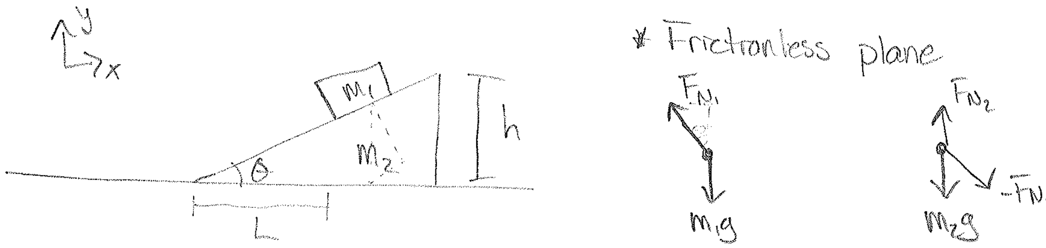
   (c) Return to the block problem, but now assume that the coefficients of static and kinetic friction between the block and the wedge are μ (they have the same value). What is μ_{min}, the minimum value of μ for which the system is stable? (1 point).

   (d) If $\mu < \mu_{\text{min}}$, calculate the minimum **horizontal** force that can be applied to the wedge such that the block will not accelerate down the wedge. (3 points).

   Note: you can neglect the finite size of the block in your calculation, and you are asked for the velocities before the block or ball make contact with the frictionless plane.

Jan 2008

Classical #1



a) * Assume no friction b/w block + wedge

* Find velocities when block reaches bottom of wedge

- Forces

  * Block $= \langle -F_{N_1}\sin\theta, -m_1 g + F_{N_1}\cos\theta \rangle$

  * Wedge $= \langle F_N \sin\theta, -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

  $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = mgh$

- Kinematics

  $r_{1,f} = r_{1,i} + \frac{1}{2}a_i t^2$

  $v_{1,f}^2 = 2a\Delta r \qquad v_1 = a_1 t$

$r_{1,f} = \langle x_{f,1}, 0 \rangle$ * Note: $-m_2 g - F_{N_1}\cos\theta + F_{N_2} = 0$

$\Rightarrow \cancelto{0}{y_{1,f}} = y_{1,i} + \cancelto{0}{v_{1,i}t} + \frac{1}{2}a_{1,y}t^2$

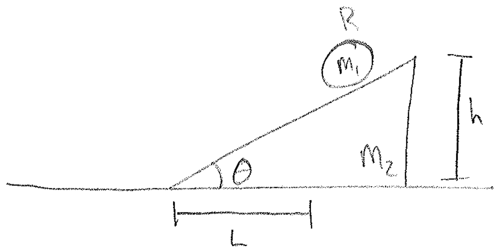
$0 = h + \frac{1}{2}a_{1,y}t^2$

$\left(\frac{-2hm_1}{F_{N_1}\cos\theta - m_1 g}\right)^{1/2} = t$

$\Rightarrow v_1 = at$

$v_1 = \left(\frac{-2h}{F_{N_1}\cos\theta - m_1 g}\right)\langle -F_{N_1}\sin\theta, F_{N_1}\cos\theta - m_1 g \rangle, \quad v_2 = \left(\frac{-2hm_1}{m_2(F_{N_1}\cos\theta - m_1 g)}\right)\langle F_{N_1}\sin\theta, 0 \rangle$

b) * Block is now ball rolling w/o slipping (ie. $V_{tangent} = V_{cm}$)



* Frictionless Plane

- Forces

* Ball = $\langle -F_{N_1}\sin\theta,\ -m_1 g + F_{N_1}\cos\theta \rangle$

* Wedge = $\langle F_{N_1}\sin\theta,\ -m_2 g - F_{N_1}\cos\theta + F_{N_2} \rangle$

- Energy

$$m_1 g h = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} m_2 v_2^2 \quad , \quad I = \tfrac{2}{5} m_1 R^2$$

- Kinematics

$$\vec{r}_f = \vec{r}_i + \tfrac{1}{2}\vec{a}t^2$$
$$\vec{\theta}_f = \vec{\theta}_i + \tfrac{1}{2}\vec{\alpha}t^2$$

$>$ $v = \omega R$, $a = \alpha R$

$$\vec{v}_f^{\,2} = 2\vec{a}\,\Delta\vec{r}$$
$$\vec{v}_f = \vec{a}t$$

2. Consider a point particle of mass m constrained to move on a parabola in the x-z plane, i.e.,

$$z = \frac{\alpha}{2} x^2.$$

Assume the constraint force is frictionless and gravity acts vertically ($F_z = -mg$).

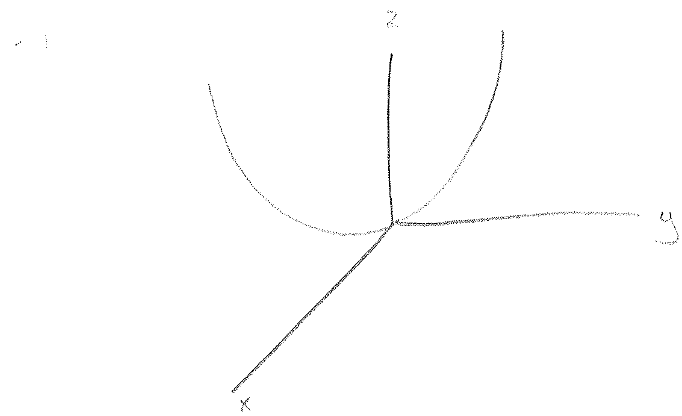
   (a) Use Lagrangian mechanics to write a second order differential equation for $x(t)$. (2 points)

   (b) Find a first integral of this equation (any way you can) and evaluate the constant of integration using the maximum value x_{max} reached by x. (4 points)

   (c) Assume that the particle is pulled a short distance from the origin and allowed to oscillate. Calculate the period in the limit of small oscillations, $\epsilon \equiv \alpha x_{\rm max} << 1$. (4 points)

Jan 2008

Classical # 2

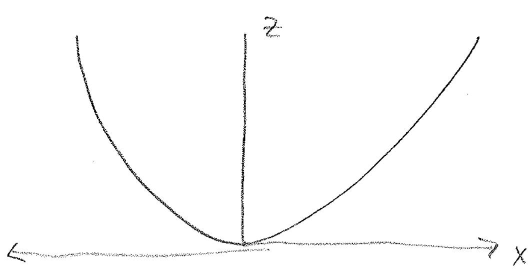


* Particle of mass m constrained to move on parabola in x-z plane

$\Rightarrow z = \frac{\alpha}{2} x^2$

* Constraint force is frictionless

$F_z = -mg$



a) $\mathcal{L} = T - U$

$T = \frac{1}{2} m v^2$ $\qquad U = mgz$

$= \frac{1}{2} m (\dot{x}^2 + \dot{z}^2)$ $\qquad = mg \frac{\alpha}{2} x^2$

$= \frac{1}{2} m (\dot{x}^2 + (\alpha \dot{x})^2)$

$= \frac{1}{2} m \dot{x}^2 (1 + \alpha)$

$\Rightarrow \mathcal{L} = \frac{1}{2}(m\dot{x}^2(1+\alpha) - mg\alpha x^2)$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

$$mg\alpha x = \frac{\partial}{\partial t}\left(m\dot{x}(1+\alpha)\right)$$

$$mg\alpha x = m(1+\alpha)\ddot{x}$$

$$g\alpha x = (1+\alpha)\ddot{x}$$

$$x = \frac{1+\alpha}{g\alpha}\ddot{x}$$

b)

3. **Angular momentum and the Rungé-Lenz vector:** Given a point particle of mass m, trajectory $\vec{r}(t)$, and momentum $\vec{p}(t)$, we can define the angular momentum

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

and the Rungé-Lenz vector

$$\vec{\mathcal{A}} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

We consider the explicit case of a $1/r$ potential, so that

$$H = \frac{p^2}{2m} - \frac{1}{r}$$

(a) Prove that the Poisson bracket of H and $\vec{L}$ is zero, that is:

$$\left\{H, \vec{L}\right\} = 0.$$

(3 points).

(b) Prove that the Poisson bracket of H and $\vec{\mathcal{A}}$ is zero, that is:

$$\left\{H, \vec{\mathcal{A}}\right\} = 0.$$

(3 points)

(c) What do your results in parts (a) and (b) imply about the behavior of $\vec{\mathcal{A}}$ and $\vec{L}$? (1 point)

(d) Evaluate $\vec{r} \cdot \vec{\mathcal{A}} = r\mathcal{A}\cos\theta$, using the explicit form for $\vec{\mathcal{A}}$ above. Use this to calculate the orbital motion of the particle (that is, a relationship between r and θ as the particle moves about its orbit). (3 points)

Jan 2008

Classical #3

$\vec{L} \equiv \vec{r} \times \vec{p}$

$\vec{A} = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$

$H = \frac{p^2}{2m} - \frac{1}{r}$

a) Prove Poisson bracket, $\{H, L\} = 0$

$$H = \frac{p^2}{2m} - \frac{1}{r} = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$L = \vec{r} \times p = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \langle yp_z - zp_y, -(xp_z - zp_x), xp_y - yp_x \rangle$$

$$\{H, L\} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial L}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial L}{\partial q_i} \right)$$
$$= \frac{\partial H}{\partial x}\frac{\partial L}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial L}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial L}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial L}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial L}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial L}{\partial z}$$

$\frac{\partial H}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$

$\frac{\partial L}{\partial p_x} = \langle 0, -z, -y \rangle$ $\qquad$ $\frac{\partial L}{\partial x} = \langle 0, -p_z, p_y \rangle$

$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$

$\frac{\partial L}{\partial p_y} = \langle -z, 0, x \rangle$ $\qquad$ $\frac{\partial L}{\partial y} = \langle p_z, 0, -p_x \rangle$

$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$ $\qquad$ $\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$

$\frac{\partial L}{\partial p_z} = \langle y, -x, 0 \rangle$ $\qquad$ $\frac{\partial L}{\partial z} = \langle -p_y, p_x, 0 \rangle$

$$= \frac{x}{(\ )^{3/2}}\langle 0, -z, -y \rangle - \frac{p_x}{m}\langle 0, -p_z, p_y \rangle +$$
$$\frac{y}{(\ )^{3/2}}\langle -z, 0, x \rangle - \frac{p_y}{m}\langle p_z, 0, -p_x \rangle +$$
$$\frac{z}{(\ )^{3/2}}\langle y, -x, 0 \rangle - \frac{p_z}{m}\langle -p_y, p_x, 0 \rangle$$

$$\langle 0, 0, 0 \rangle \checkmark$$

b) Show $\{H, A\} = 0$

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$A = \frac{1}{m}\vec{p} \times \vec{L} - \hat{r}$$

$$= \frac{1}{m}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_x \end{vmatrix} - \langle x, y, z \rangle \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{1}{m}\langle p_y(xp_y - yp_x) - p_z(zp_x - xp_z),\ p_z(yp_z - zp_y) - p_x(xp_y - yp_x),\ p_x(zp_x - xp_z) - p_y(yp_z - zp_y) \rangle - \langle x, y, z \rangle$$

$$\{H, \hat{A}\} = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial A}{\partial q_i}$$

$$= \frac{\partial H}{\partial x}\frac{\partial A}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial A}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial A}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial A}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial A}{\partial p_z} - \frac{\partial H}{\partial p_z}\frac{\partial A}{\partial z}$$

$$\frac{\partial H}{\partial x} = -\left(-\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x\right) = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$\frac{\partial A}{\partial x} = \frac{1}{m}\langle p_y^2 + p_z^2,\ -p_xp_y,\ -p_xp_z \rangle - \langle 1, 0, 0 \rangle$$

$$\frac{\partial A}{\partial p_x} = \frac{1}{m}\langle -yp_y - zp_z,\ -xp_y + 2yp_x,\ 2zp_x - xp_z \rangle$$

$$\frac{\partial H}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\frac{\partial A}{\partial y} = \frac{1}{m}\langle -p_xp_y,\ p_z^2 + p_x^2,\ -p_yp_z \rangle - \langle 0, 1, 0 \rangle$$

$$\frac{\partial A}{\partial p_y} = \frac{1}{m}\langle 2xp_y - yp_x,\ -zp_z - xp_x,\ -yp_z + 2zp_y \rangle$$

$$\frac{\partial H}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\frac{\partial A}{\partial z} = \frac{1}{m}\langle -p_xp_z,\ -p_yp_z,\ p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle$$

$$\frac{\partial A}{\partial p_z} = \frac{1}{m}\langle -zp_x + 2xp_z,\ 2yp_z - zp_y,\ -xp_x - yp_y \rangle$$

$$\Rightarrow \{H, A\}_3 = \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle -y p_y - z p_z ,\ -x p_y + 2y p_x ,\ 2z p_x - x p_z \rangle$$

$$- \frac{p_x}{m} \left(\frac{1}{m} \langle p_y^2 + p_z^2 ,\ -p_x p_y ,\ -p_x p_z \rangle - \langle 1, 0, 0 \rangle \right)$$

$$+ \frac{y}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2x p_y - y p_x ,\ -z p_z - x p_x ,\ -y p_z + 2z p_y \rangle$$

$$- \frac{p_y}{m} \left(\frac{1}{m} \langle -p_x p_y ,\ p_z^2 + p_x^2 ,\ -p_y p_z \rangle - \langle 0, 1, 0 \rangle \right)$$

$$+ \frac{z}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{1}{m} \langle 2x p_z - z p_x ,\ 2y p_z - z p_y ,\ -x p_x - y p_y)$$

$$- \frac{p_z}{m} \left(\frac{1}{m} \langle -p_x p_z ,\ -p_y p_z ,\ p_x^2 + p_y^2 \rangle - \langle 0, 0, 1 \rangle \right)$$

$$= \frac{1}{m r^3} \langle x(-y p_y - z p_z) + y(2x p_y - y p_x) + z(2x p_z - z p_x) ,\ x(-x p_y + 2y p_x) + y(-z p_z - x p_x) + z(2y p_z - z p_y)$$
$$,\ x(2z p_x - x p_z) + y(-y p_z + 2z p_y) + z(-x p_x - y p_y) \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle p_x(p_y^2 + p_z^2) + p_y(-p_x p_y) + p_z(-p_x p_z) ,\ p_x(-p_x p_y) + p_y(p_z^2 + p_x^2) + p_z(-p_y p_z)$$
$$,\ p_x(-p_x p_z) + p_y(-p_y p_z) + p_z(p_x^2 + p_y^2) \rangle$$

$$= \frac{1}{m r^3} \langle -xy p_y - xz p_z + \cancel{2} xy p_y - y^2 p_x + 2xz p_z - z^2 p_x ,\ -x^2 p_y + \cancel{2} yx p_x - \cancel{yz p_z} - \cancel{xy p_x} + \cancel{2} yz p_z - z^2 p_y ,$$
$$\cancel{2} xz p_x - x^2 p_z - y^2 p_z - \cancel{2yz p_y} - \cancel{xz p_x} - \cancel{yz p_y} \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle - \frac{1}{m^2} \langle \cancel{p_x p_y^2} + \cancel{p_x p_z^2} - \cancel{p_x p_y^2} - \cancel{p_x p_z^2} ,\ -\cancel{p_x^2 p_y} + \cancel{p_z^2 p_y} + \cancel{p_x^2 p_y} - \cancel{p_y p_z^2} ,$$
$$,\ -\cancel{p_x^2 p_z} - \cancel{p_y^2 p_z} + \cancel{p_x^2 p_z} + \cancel{p_y^2 p_z} \rangle$$

$$= \frac{1}{m r^3} \langle xy p_y + xz p_z - y^2 p_x - z^2 p_x ,\ xy p_x + yz p_z - x^2 p_y - z^2 p_y ,\ xz p_x + yz p_y - x^2 p_z - y^2 p_z \rangle$$

$$+ \frac{1}{m} \langle p_x, p_y, p_z \rangle$$

c) If $\{H, L\}$ and $\{H, A\}$ are both 0, then both $\vec{L}$ and $\vec{A}$ are constants of the motion.

d) $\vec{r} \cdot \vec{A} = rA\cos\theta$

$$\vec{r} \cdot \left(\frac{1}{m}\vec{p} \times \vec{L} - \hat{r}\right) = \frac{1}{m}\left(\vec{r} \cdot (\vec{p} \times \vec{L})\right) - \vec{r} \cdot \hat{r}$$

$$= \frac{1}{m}\vec{L} \cdot (\vec{r} \times \vec{p}) - r$$

$$= \frac{1}{m}L^2 - r$$

$$\frac{L^2}{m} - r = rA\cos\theta$$

$$\frac{L^2}{m} = r(A\cos\theta + 1)$$

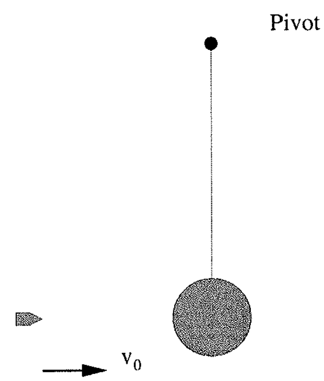
$$\frac{L^2}{m} \cdot \frac{1}{A\cos\theta + 1} = \vec{r}$$

Classical Mechanics and Statistical/Thermodynamics

August 2008

Classical Mechanics

1. **The ballistic pendulum:** Consider a pendulum with a bob of mass m connected to a frictionless pivot by an ideal massless rigid rod of length ℓ. A projectile of mass ϵm $(0 < \epsilon << 1)$ moving horizontally at speed v_0 hits the center of the bob, as shown. When it strikes, it becomes imbedded in the bob.



   (a) What is the minimum initial speed of the projectile such that the pendulum will make a full rotation? (2 points)

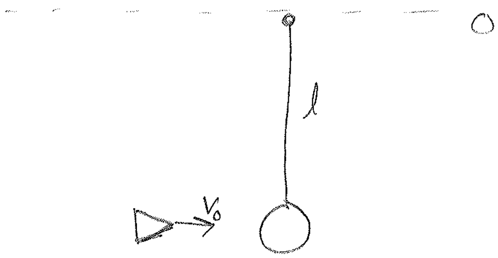
   (b) The rod is replaced by an ideal massless non-rigid string. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

   (c) Now assume that projectile rebounds elastically from the bob in the horizontal direction. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (2 points)

   (d) Finally, assume that the projectile passes completely through the pendulum bob, in a time $t << \sqrt{\ell/g}$. After it exits, it carries with it some of the original mass of the bob, such that the exiting projectile now has a mass $2\epsilon m$ and moves at a speed $3v_0/4$. What is the minimum initial speed of the projectile such that the pendulum will make a full revolution without the string going slack? (3 points)

Aug 2008

Classical #1



$m_b = \epsilon m$

$m_p = m$

a) * Collision (Inelastic)

$$\cancel{\tfrac{1}{2} m \epsilon v_0^2 = \tfrac{1}{2} m (1+\epsilon) v^2}$$

$$\cancel{\frac{\epsilon}{1+\epsilon} v_0^2 = v^2}$$

$$\epsilon m v_0 = (1+\epsilon) m v$$

$$\frac{\epsilon}{1+\epsilon} v_0 = v$$

* Conservation of energy

$$\tfrac{1}{2} m (1+\epsilon)\left(\frac{\epsilon}{1+\epsilon} v_0\right)^2 - mg\ell = mg\ell + \tfrac{1}{2} m \cancelto{0}{(1+\epsilon)} v^2$$

$$\tfrac{1}{2} m \frac{\epsilon^2}{1+\epsilon} v_0^2 = 2mg\ell + \tfrac{1}{2} m \cancelto{0}{(1+\epsilon)} v^2$$

$$\frac{\epsilon^2}{1+\epsilon} v_0^2 = 4mg\ell$$

2. The isotropic harmonic oscillator.

(a) Write the Lagrangian for a point mass m moving under the influence of an isotropic 3-dimensional harmonic oscillator potential

$$V(x, y, z) = \frac{k}{2}(x^2 + y^2 + z^2).$$

There is no external gravitational field. (1 point)

(b) Using the Lagrange equations of motion show that angular momentum is conserved. i.e.,

$$\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

Because the Lagrangian is invariant under rotations about the origin, you can choose coordinates so that motion is constrained to the x-y plane, i.e., the angular momentum points in the z direction. (3 points)

(c) For 2-dimensional motion in the x-y plane choose cylindrical polar coordinates and proceed to solve the Lagrange equations of motion. You can leave the solution for $r(t)$ as an integral of the form $t = \int f(r)dr$. (Don't forget to use conservation of energy, E_0.) (3 points)

(d) Compute the minimum and maximum values or the radial coordinate r as functions of the constants m, E_0, k, L^z. (3 points)

Aug 2008

Classical #2

a) Write Lagrangian for a point mass under the influence of the following potential:

$V(x,y,z) = \frac{k}{2}(x^2+y^2+z^2)$ * No external gravity

$$\mathcal{L} = T - V$$
$$= \frac{1}{2} m v^2 - \frac{1}{2} k (x^2+y^2+z^2)$$
$$= \frac{1}{2} m (\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2} k (x^2+y^2+z^2)$$

b) Use Lagrange equations of motion to show angular momentum conservation

i.e. $\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = 0$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}} \qquad \frac{\partial \mathcal{L}}{\partial z} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$

$$\Rightarrow -kx = \frac{\partial}{\partial t}(m\dot{x}) \qquad \Rightarrow -ky = \frac{\partial}{\partial t}(m\dot{y}) \qquad \Rightarrow -kz = \frac{\partial}{\partial t}(m\dot{z})$$

$$-kx = m\ddot{x} \qquad -ky = m\ddot{y} \qquad -kz = m\ddot{z}$$

$$\vec{r} \times m\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} = \langle ym\dot{z} - zm\dot{y},\ xm\dot{z} - zm\dot{x},\ xm\dot{y} - ym\dot{x} \rangle$$

$$= m\langle y\dot{z} - z\dot{y}, x\dot{z} - z\dot{x}, x\dot{y} - y\dot{x} \rangle$$

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = m\langle \dot{y}\dot{z} + y\ddot{z} - (\dot{z}\dot{y} + z\ddot{y}), \dot{x}\dot{z} + x\ddot{z} - (\dot{z}\dot{x} + z\ddot{x}), \dot{x}\dot{y} + x\ddot{y} - (\dot{y}\dot{x} + y\ddot{x}) \rangle$$

$$= m\langle y\ddot{z} - z\ddot{y}, x\ddot{z} - z\ddot{x}, x\ddot{y} - y\ddot{x} \rangle$$

* substituting from above

$$= m\left(\frac{-k}{m}\right)\langle yz - zy, xz - zx, xy - yx \rangle$$

$$= -k\langle 0, 0, 0 \rangle$$
$$= 0$$

$$\therefore \frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d}{dt}\vec{L} = 0 \checkmark$$

c) For 2-D motion in x-y plane, use cylindrical polar coordinates to solve Lagrange eqns of motion. Leave $r(t)$ as an integral of form $t = \int f(r) dr$

* Hint: use conservation of energy E_0

- In cylindrical polar: $x = r\cos\varphi$ $\quad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\quad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\varphi}^2) - \tfrac{1}{2}k\dot{r}^2 \quad (2\text{-D},\ z=0)$$

$$E_0 = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \tfrac{1}{2}kr^2$$

$$\frac{\partial\mathcal{L}}{\partial r} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{r}} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\varphi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right) = 0$$

$$0 = kr + mr\dot{\varphi}^2 - m\ddot{r} \qquad 0 = -2mr\dot{r}\dot{\varphi} - mr^2\ddot{\varphi}$$

* Solving E_0 for $\dot{\varphi}$ yields

$$E_0 = \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\varphi}^2 + \tfrac{1}{2}kr^2$$

$$\frac{2E_0}{m} = \dot{r}^2 + r^2\dot{\varphi}^2 + \frac{k}{m}r^2$$

$$\frac{2E_0}{m} - \frac{k}{m}r^2 - \dot{r}^2 = r^2\dot{\varphi}^2$$

$$\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2} = \dot{\varphi}^2$$

$$\Rightarrow 0 = kr + mr\left(\frac{2E_0}{mr^2} - \frac{k}{m} - \frac{\dot{r}^2}{r^2}\right) + \tfrac{1}{2}kr^2$$

$$= \cancel{kr} + \frac{2E_0}{r} - \cancel{kr} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$0 = \frac{2E_0}{r} - \frac{m\dot{r}^2}{r} + \tfrac{1}{2}kr^2$$

$$\frac{m\dot{r}^2}{r} = \frac{2E_0}{r} + \tfrac{1}{2}kr^2$$

$$\dot{r}^2 = 2mE_0 + \tfrac{1}{2}\frac{k}{m}r^3$$

$$\Rightarrow t = \int\left(2mE_0 + \tfrac{1}{2}\tfrac{k}{m}r^3\right)^{1/2} dr$$

3. Consider a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis. The potential energy is of the form:

$$V(r,z) = -m\left(\frac{k}{r} + g\,z\right)$$

where m is the particle's mass, k and g are constants, and r is the standard radial coordinate:

$$r \equiv \sqrt{x^2+y^2+z^2}$$

You are to examine the problem in *cylindrical parabolic coordinates* defined by

$$\begin{aligned} \zeta &\equiv r+z \\ \eta &\equiv r-z \\ \phi &\equiv \arctan y/x \end{aligned}$$

In these coordinates we may write the Cartesian coordinates as:

$$\begin{aligned} x &= \sqrt{\zeta\eta}\cos\phi \\ y &= \sqrt{\zeta\eta}\sin\phi \\ z &= \tfrac{1}{2}(\zeta-\eta) \end{aligned}$$

(a) Show that the kinetic energy, T, is given by:

$$T = \frac{m}{8}\left[\left(1+\frac{\zeta}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\zeta}\right)\dot{\zeta}^2\right] + \frac{m}{2}\zeta\eta\,\dot{\phi}^2$$

in these coordinates. (2 points)

(b) What are the canonical momenta, p_ζ, p_η, and p_ϕ, expressed in cylindrical parabolic coordinates? (2 points)

(c) Use Hamilton-Jacobi theory to find the constants of the motion. *Hint:* While the total energy E does not separate in these coordinates, $E(\zeta+\eta)$ can be used to produce a quantity that **does** seperate. (3 points)

(d) What is Hamilton's characteristic function associated with ϕ? (1 point)

(e) Express Hamilton's characteristic functions associated with ζ, η as definite integrals. (2 points)

Aug 2002

Classical #3

For a particle attracted by a fixed gravitating body while also in a uniform gravitational field oriented along the z-axis $\left[V(r,z) = -m\left(\frac{k}{r} + gz\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$ and using cylindrical parabolic coordinates:

$$\xi = r + z \qquad\qquad x = \sqrt{\xi\eta}\cos\varphi$$
$$\eta = r - z \qquad \Rightarrow \qquad y = \sqrt{\xi\eta}\sin\varphi$$
$$\varphi = \arctan(y/x) \qquad\qquad z = \tfrac{1}{2}(\xi - \eta)$$

a) Show that the kinetic energy, T, is: $T = \frac{m}{8}\left[\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\varphi}^2$

$$T = \tfrac{1}{2}m\dot{v}^2$$
$$= \tfrac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

* from above,

$$\dot{x} = \tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\cos\varphi\,\dot{\xi} + \tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\cos\varphi\,\dot{\eta} - \xi^{1/2}\eta^{1/2}\sin\varphi\,\dot{\varphi}$$
$$\dot{x}^2 = \tfrac{1}{4}\xi^{-1}\eta\cos^2\varphi\,\dot{\xi}^2 + \tfrac{1}{4}\xi\eta^{-1}\cos^2\varphi\,\dot{\eta}^2 + \xi\eta\sin^2\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\cos^2\varphi\,\dot{\xi}\dot{\eta} - \eta\cos\varphi\sin\varphi\,\dot{\xi}\dot{\varphi}$$
$$- \xi\cos\varphi\sin\varphi\,\dot{\eta}\dot{\varphi}$$
$$\dot{y} = \tfrac{1}{2}\xi^{-1/2}\eta^{1/2}\sin\varphi\,\dot{\xi} + \tfrac{1}{2}\xi^{1/2}\eta^{-1/2}\sin\varphi\,\dot{\eta} + \xi^{1/2}\eta^{1/2}\cos\varphi\,\dot{\varphi}$$
$$\dot{y}^2 = \tfrac{1}{4}\xi^{-1}\eta\sin^2\varphi\,\dot{\xi}^2 + \tfrac{1}{4}\xi\eta^{-1}\sin^2\dot{\eta}^2 + \xi\eta\cos^2\varphi\,\dot{\varphi}^2 + \tfrac{1}{2}\cancel{\xi}\sin^2\varphi\,\dot{\xi}\dot{\eta} + \eta\sin\varphi\cos\varphi\,\dot{\xi}\dot{\varphi}$$
$$+ \xi\sin\varphi\cos\varphi\,\dot{\eta}\dot{\varphi}$$
$$\dot{z} = \tfrac{1}{2}(\dot{\xi} - \dot{\eta})$$
$$\dot{z}^2 = \tfrac{1}{4}(\dot{\xi}^2 - 2\dot{\xi}\dot{\eta} + \dot{\eta}^2)$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \tfrac{1}{4}\xi^{-1}\eta\dot{\xi}^2 + \tfrac{1}{4}\xi\eta^{-1}\dot{\eta}^2 - \xi\eta\sin^2\varphi\,\dot{\varphi}^2 + \xi\eta\cos^2\varphi\,\dot{\varphi}^2 + \cancel{\tfrac{1}{2}\dot{\xi}\dot{\eta}} + \tfrac{1}{4}\dot{\xi}^2 - \cancel{\tfrac{1}{2}\dot{\xi}\dot{\eta}} + \tfrac{1}{4}\dot{\eta}^2$$
$$= \tfrac{1}{4}\tfrac{\eta}{\xi}\dot{\xi}^2 + \tfrac{1}{4}\tfrac{\xi}{\eta}\dot{\eta}^2 - \xi\eta\sin^2\varphi\,\dot{\varphi}^2 + \xi\eta\cos^2\varphi\,\dot{\varphi}^2 + \tfrac{1}{4}\dot{\xi}^2 + \tfrac{1}{4}\dot{\eta}^2$$
$$= \tfrac{1}{4}\left[\dot{\xi}^2\left(\tfrac{\eta}{\xi}+1\right) + \dot{\eta}^2\left(\tfrac{\xi}{\eta}+1\right)\right] + \xi\eta\dot{\varphi}^2\left(\cancel{\cos^2\varphi + \sin^2\varphi}\right)$$

$$\Rightarrow T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \tfrac{m}{8}\left[\left(1+\tfrac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1+\tfrac{\eta}{\xi}\right)\dot{\xi}^2\right] + \tfrac{m}{2}\xi\eta\dot{\varphi}^2$$

b) Find the canonical momenta in cylindrical parabolic coordinates

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$\Rightarrow \mathcal{L} = T - V$$

$$V = -m\left(\frac{k}{r} + gz\right), \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r = \sqrt{\xi\eta\cos^2\varphi + \xi\eta\sin^2\varphi + \tfrac{1}{4}(\xi - \eta)^2}$$

$$= \sqrt{\xi\eta + \tfrac{1}{4}\xi^2 - \tfrac{1}{2}\xi\eta + \tfrac{1}{4}\eta^2}$$

$$= \sqrt{\tfrac{1}{4}\xi^2 + \tfrac{1}{2}\xi\eta + \tfrac{1}{4}\eta^2}$$

$$= \sqrt{\tfrac{1}{4}(\xi^2 + 2\xi\eta + \eta^2)}$$

$$= \tfrac{1}{2}(\xi + \eta)$$

$$\Rightarrow V = -m\left(\frac{2k}{\xi + \eta} + g\left(\frac{\xi + \eta}{2}\right)\right)$$

$$\Rightarrow \mathcal{L} = T - V$$

$$= \frac{m}{8}\left[\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}^2 + \left(1 + \frac{\eta}{\xi}\right)\dot{\xi}^2\right] + \frac{m}{2}\xi\eta\dot{\varphi}^2 + m\left[\frac{2k}{\xi + \eta} + \frac{g(\xi + \eta)}{2}\right]$$

$$P_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$= m\xi\eta\dot{\varphi}$$

$$P_\eta = \frac{\partial \mathcal{L}}{\partial \dot{\eta}}$$

$$= \frac{m}{4}\left(1 + \frac{\xi}{\eta}\right)\dot{\eta}$$

$$P_\xi = \frac{\partial \mathcal{L}}{\partial \dot{\xi}}$$

$$= \frac{m}{4}\left(1 + \frac{\eta}{\xi}\right)\dot{\xi}$$

c) $H = \sum_i p_i \dot{q}_i - \mathcal{L}$

$$H = p_\varphi \dot{\varphi} + p_\xi \dot{\xi} + p_\eta \dot{\eta} - \mathcal{L}$$

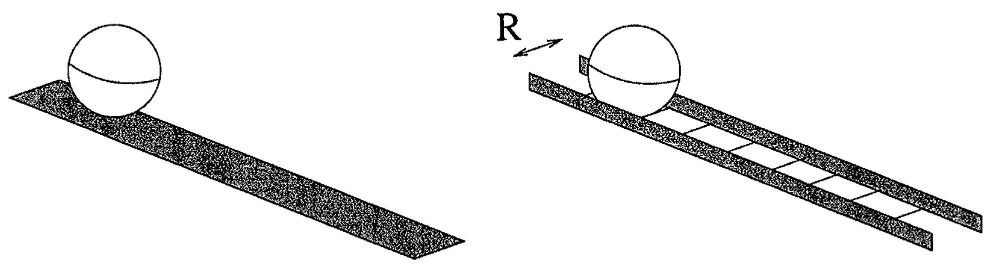
$$= \frac{m}{2}\xi\eta\dot{\varphi}^2 + \frac{m}{8}\left(1+\frac{\eta}{\xi}\right)\dot{\xi}^2 + \frac{m}{8}\left(1+\frac{\xi}{\eta}\right)\dot{\eta}^2 - m\left[\frac{2k}{\xi+\eta} + \frac{g(\xi+\eta)}{2}\right]$$

Classical Mechanics and Statistical/Thermodynamics

August 2009

Classical Mechanics

1. Rolling spheres: Given that the moment of inertia of a sphere of mass m and radius R is $(2/5)mR^2$, please answer the following.



   (a) A sphere of radius R and mass m rolls without slipping down an inclined plane on to a horizontal table (left figure). The condition of "rolling without slipping" forces a relationship between v, the speed of the center of mass of the sphere, and ω, the angular velocity of the sphere about its center of mass. What is this relationship? (0.5 pt.)

   (b) Calculate the speed of the sphere at the bottom of the ramp if the center of mass of the sphere has dropped a distance h when it just touches the table. Assume that $h >> R$. (1.5 pt.)

   (c) The ramp is now replaced by two narrow rails separated by a distance R (right figure). Again the ball rolls downward without slipping, supported by the two rails. In this case, what the relationship between v and ω? (1 pt.)

   (d) In this second case, calculate the speed of the sphere at the bottom if the center of mass has dropped a distance h. (2 pts).

   (e) After the ball reaches the bottom of the rails (part b) it continues to move on the horizontal table. It will either be rolling too fast or too slow to roll without slipping. Which will it be? You must prove your result. (1 pt.)

   (f) Friction between the sphere and the plane will adjust the speed of the sphere until it can again roll without slipping. If the magnitude of the force of friction between the sphere and the plane is

μmg, determine the speed of the ball when it again rolls without slipping. (If you did not solve part (b) above, assume the sphere is moving at speed v_0 without rolling and determine its speed when it rolls without slipping). (4 pts.)

Aug 2009

Classical #1

a) $v = \omega r$

b) $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh$

$$\Rightarrow mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

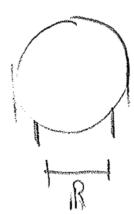
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$$

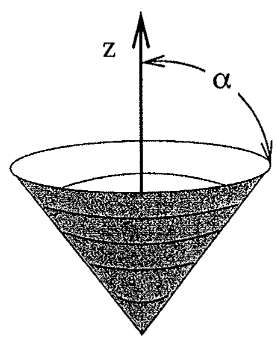
$$gh = \frac{7}{10}v^2$$

$$\Rightarrow v = \sqrt{\frac{10gh}{7}}$$

c)



2. A point particle of mass m travels on the frictionless inner surface of an inverted cone. The cone is oriented so its symmetry axis is parallel to the z-axis, with an opening angle α between the z-axis and the surface of the cone. The force of gravity points in the negative z-direction.



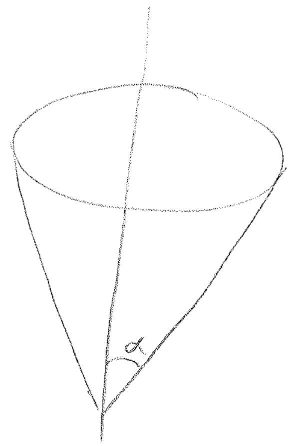
(a) Write the Lagrangian for the problem in cylindrical coordinates. (1 pt.)

(b) Assume the particle is moving in a uniform circular orbit at distance d from the cone tip, measured along the surface of the cone. Determine the angular frequency of the system. (3 pts.)

(c) The opening angle of the cone is abruptly decreased by $\Delta\alpha << \alpha$. This is done in a manner that does **not** impart an impulse or do work on the particle. (Imagine that the cone is instantaneously stretched so that its tip moves slightly downward, but the particle is not displaced during the stretching). Describe the subsequent motion of the particle in this limit. Express your answer in terms of ρ_0, the original radius of the circular orbit, m, α, $\Delta\alpha$, and g. Explain any approximations you are making in deriving your result. (6 pts.)

Aug 2009

Classical #2



* Cone is frictionless
* Particle travels on inner surface of cone
* Gravity points in $-\hat{z}$ direction

a) Write the Lagrangian in cylindrical coordinates

* In cylindrical coordinates: $x = r\cos\varphi$ $\quad \dot{x} = \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}$

$y = r\sin\varphi$ $\quad \dot{y} = \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}$

$z = z$ $\quad \dot{z} = \dot{z}$

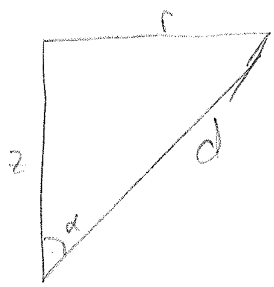
$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \tfrac{1}{2}m\left[(\dot{r}\cos\varphi - r\sin\varphi\dot{\varphi})^2 + (\dot{r}\sin\varphi + r\cos\varphi\dot{\varphi})^2 + \dot{z}^2\right] - mgz$$

$$= \tfrac{1}{2}m\left[\dot{r}^2\cancel{\cos^2\varphi} - \cancel{2r\cos\varphi\sin\varphi\dot{r}\dot{\varphi}} + r^2\cancel{\sin^2\varphi}\dot{\varphi}^2 + \dot{r}^2\cancel{\sin^2\varphi} + \cancel{2r\sin\varphi\cos\varphi\dot{r}\dot{\varphi}} + r^2\cancel{\cos^2\varphi}\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$

$$= \tfrac{1}{2}m\left[\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right] - mgz$$

b) Assuming a uniform circular orbit at a distance d from the tip of the cone, determine the angular frequency of the system



$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m\left[\dot{d}^2\cos^2\alpha + d\cos^2\alpha\,\dot{\varphi}^2 + \dot{d}^2\sin^2\alpha\right] - mgd\sin\alpha$$

$$= \tfrac{1}{2}m\left[\dot{d}^2 + d\cos^2\alpha\,\dot{\varphi}^2\right] - mgd\sin\alpha$$

$r = d\cos\alpha$ $\quad \dot{r} = \dot{d}\cos\alpha$

$z = d\sin\alpha$ $\quad \dot{z} = \dot{d}\sin\alpha$

$\uparrow$

α is const.

b) $\frac{\partial \mathcal{L}}{\partial d} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{d}}$

$\frac{1}{2} m \cos^2\alpha\, \dot{\varphi}^2 - mg\sin\alpha = \frac{\partial}{\partial t}\left[m\dot{d}\right]$

$\frac{1}{2} m \cos^2\alpha\, \dot{\varphi}^2 - mg\sin\alpha = m\ddot{d}$

$\frac{1}{2} \cos^2\alpha\, \dot{\varphi}^2 - g\sin\alpha = \ddot{d}$

$\Rightarrow \dot{\varphi} = \left(\frac{\ddot{d} + g\sin\alpha}{\frac{1}{2}\cos^2\alpha}\right)^{1/2}$

* but b/c of circular motion, $\dot{d} = 0$

$\Rightarrow \dot{\varphi} = \left(2g\frac{\sin\alpha}{\cos^2\alpha}\right)^{1/2}$

$\dot{\varphi} = (2g\tan\alpha\sec\alpha)^{1/2}$

c) How does orbit change as $\alpha \rightarrow \alpha - \Delta\alpha$

$\mathcal{L} = \frac{1}{2}m$

$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$

$0 = \frac{\partial}{\partial t}\left[md\cos^2\alpha\, \dot{\varphi}\right]$

$0 = m\cos^2\alpha\,(\dot{d}\dot{\varphi} + d\ddot{\varphi})$

$\dot{d}\dot{\varphi} = -d\ddot{\varphi}$

$\dot{\varphi} = -\frac{d\ddot{\varphi}}{d}$

3. Consider the Lagrangian for a 1D system with generalized coordinate q:

$$L(q, \dot{q}, t) = e^{\lambda t/m}\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega_0^2}{2}q^2\right] \tag{1}$$

In the above expression, m is a mass, ω_0 is a frequency, and λ is a positive and dimensionless constant.

(a) Derive the equation of motion for the system. (1 pt.)

(b) What is the canonical momentum, p? (1 pt.)

(c) Calculate the Hamiltonian. (3 pts.)

(d) We wish to make a canonical transformation $(q, p) \to (Q, P)$ using the generating function

$$F_2(q, P, t) = e^{\lambda t/2m} q\, P$$

What is the new coordinate and canonical momentum in terms of the old? (2 pts.)

(e) Show that the canonically transformed Hamiltonian is not time dependent. (3 pts.)

Aug 2009

Classical #3

$$\mathcal{L}(q,\dot{q},t) = e^{\lambda t/m}\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega_0^2}{2}q^2\right]$$

m is mass
ω_0 is a frequency
λ is positive, dimensionless const.

a) Derive equation of motion

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$-e^{\lambda t/m} m\omega_0^2 q = \frac{\partial}{\partial t}\left[e^{\lambda t/m} m\dot{q}\right]$$

$$-m\omega_0^2 q \cancel{e^{\lambda t/m}} = m\ddot{q}\cancel{e^{\lambda t/m}} + \cancel{m}\dot{q}\frac{\lambda}{\cancel{m}}\cancel{e^{\lambda t/m}}$$

$$-m\omega_0^2 q = m\ddot{q} + \lambda\dot{q}$$

$$\Rightarrow 0 = m\omega_0^2 q + \lambda\dot{q} + m\ddot{q}$$

b) Find the canonical momentum

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$= m\dot{q}\,e^{\lambda t/m}$$

c) Find the Hamiltonian

$$H = \Sigma_i p\dot{q} - \mathcal{L}$$

$$= m\dot{q}^2 e^{\lambda t/m} - e^{\lambda t/m}\left[\frac{m}{2}\dot{q}^2 - \frac{m\omega_0^2}{2}q^2\right]$$

$$= \frac{m}{2}\dot{q}^2 e^{\lambda t/m} + \frac{m\omega_0^2}{2}q^2 e^{\lambda t/m}$$

d)

Mechanics and Statistical Mechanics Qualifying Exam
Spring 2010

Problem 1: (10 Points)

A child of mass m is playing on a swing hanging from a support by a uniform chain of length L and negligible mass. In this question, you will explore the behavior of the swing in a number of situations.

a. Determine the equation of motion for the system in polar coordinates. **(2 Points)**

b. Consider small oscillations about the equilibrium position. What is the equation of motion for these conditions? **(1 Points)**

c. What is the oscillation frequency for the conditions described in part (b.)? **(2 Points)**

d. By starting at a sufficiently large speed at the bottom of the swing ($\theta = 0°$) the child can go 'over the top' ($\theta = 180°$). If the chain remains maximally extended at the top of the loop, what is the minimum velocity the child must have at the bottom of the loop ($\theta = 0$)? θ is the angle that the chain forms with the vertical. **(2 Points)**

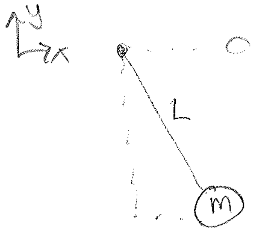
e. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (d.)? **(1 Point)**

f. If the chain is replaced by a rigid rod of negligible mass, what is the minimum velocity of the child at the bottom required to go over the top? **(1 Point)**

g. What is the minimum force applied to the child by the swing, that the child experiences at the bottom of the loop in part (f.)? **(1 Point)**

Jan 2010

Classical #1



a) Determine equation of motion in polar coordinates

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2}mL^2\dot{\varphi}^2 - mgl\sin\varphi$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{r}}$$

$$\frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$-mgL\cos\varphi = \frac{\partial}{\partial t}(mL^2\dot{\varphi})$$

$$-mgL\cos\varphi = mL^2\ddot{\varphi}$$

$$\Rightarrow \ddot{\varphi} = \frac{-g}{L}\cos\varphi$$

b) Consider small oscillations about equilibrium. What is the equation of motion?

* Make small angle approximation

$$\Rightarrow \mathcal{L} = \frac{1}{2}mL^2\dot{\varphi}^2 - mgl\varphi$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

$$-mg\cancel{L} = \frac{\partial}{\partial t}[mL^2\dot{\varphi}]$$

$$-\cancel{m}g\cancel{L} = \cancel{m}L^2\ddot{\varphi}$$

$$\ddot{\varphi} = \frac{-g}{L}$$

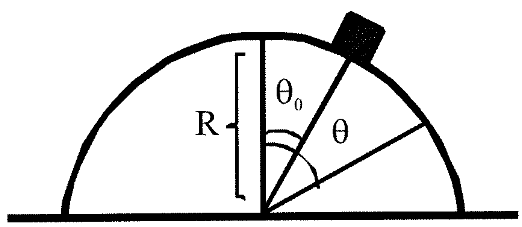
$\Rightarrow$ RADIAL + ANGULAR MOTION

Lagrangian Mechanics

why constraint?

Problem 2: (10 Points)

A hemisphere of radius R rests on the ground. A particle of mass m starts from rest on the sphere at an angle of θ_0 from the vertical that passes through the center of the sphere. Express answers in terms of R, θ_0 and the acceleration due to gravity near the surface of the earth, g.



a. The particle is released and slides without friction. At what angle, θ, measured relative to the vertical, does the particle leave the surface of the sphere? **(4 Points)**

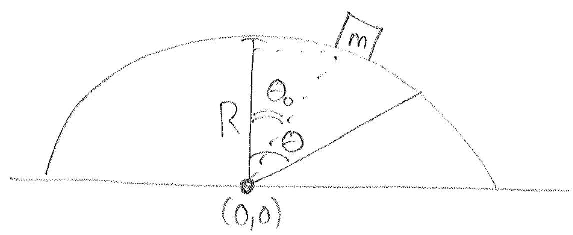
b. What is the angle θ when $\theta_0 = 0$? **(1 Points)**

c. Assume the particle was released with $\theta_0 = 0$. Once the particle leaves the sphere, how long does it take it to hit the ground? **(3 Points)**

d. How far from the center of the sphere is the particle when it hits the ground? **(2 Points)**

Jan 2010

Classical #2

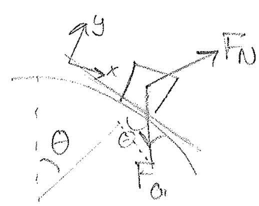


* Normal force 0 when block leaves surface of hemisphere

a) Block is released & slides w/o friction. At what θ (relative to vertical) does block leave the surface of the hemisphere?

$x = R\sin\theta$

$y = R\cos\theta$



$\langle \parallel, \perp \rangle$

$\Rightarrow F_N = \langle 0, F_N \rangle$

$F_G = mg\langle \sin\theta, -\cos\theta \rangle$

$\Rightarrow ma_x = mg\sin\theta$

$ma_y = F_N + mg\cos\theta$

* but $a_y = \frac{v^2}{R}$

$-m\frac{v^2}{R} = -mg\cos\theta_c$

$\frac{v^2}{gR} = \cos\theta_c$

or

$v = \sqrt{gr\cos\theta_c}$

$mgy = \frac{1}{2}mv^2 + mgy$

$mg\cos\theta_0 = \frac{1}{2}mv^2 + mg\cos\theta_c$

$\cancel{m}gR\cos\theta_0 = \frac{1}{2}\cancel{m}gR\cos\theta_c + \cancel{m}gR\cos\theta_c$

$\cos\theta_0 = \frac{3}{2}\cos\theta_c$

$\Rightarrow \theta_c = \cos^{-1}\left(\frac{2}{3}\cos\theta_0\right)$ ✓

b) What is θ_c if $\theta_0 = 0$?

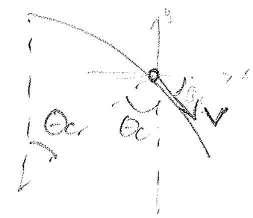
$$\theta_c = \cos^{-1}\left(\tfrac{2}{3}\cos\theta_0\right)$$

$$\theta_c = \cos^{-1}\left(\tfrac{2}{3}\right)$$

c) If particle released at $\theta = 0$, how long does it take to hit the ground after it leaves the surface of the hemisphere?

$y_i = R\cos\theta_c \qquad v_i = v\sin\theta_c \qquad a = -g$

$y_f = 0 \qquad v_f = ?$



$$0 = -\tfrac{1}{2}gt^2 - v\sin\theta_c + R\cos\theta_c$$

$$\Rightarrow \quad t = \frac{v\sin\theta_c \pm \sqrt{v^2\sin^2\theta_c - 2gR\cos\theta_c}}{g}$$

$$t = \frac{v\sin\theta_c \pm \sqrt{v^2\sin^2\theta_c - 2v^2}}{g}$$

$$t = \frac{v\sin\theta_c \pm v\sqrt{\sin^2\theta_c - 2}}{g}$$

Mechanics and Statistical Mechanics Qualifying Exam
Fall 2010

Problem 2 (10 Points):

An isolated uniform sphere of mass m and radius R is rotating with angular velocity ω_0 about an axis running through the sphere. Through only internal forces, the radius increases linearly to 2 R in a time τ, while maintaining uniform density and spherical symmetry.

a. At time τ, what is the angular velocity of the sphere? **(2 Points)**

b. Find an expression for the angular velocity as a function of time. **(1 Points)**

c. When the system reaches 2 R it immediately reverses and its radius linearly decreases to R over the period τ to 2τ. By what angle $\Delta\phi$ is the object behind in its rotation compared to a situation where the sphere does not expand between 0 and 2τ? **(4 Points)**

d. Consider the case where the radius of the sphere expands exponentially with some time constant τ_e. How much does the sphere rotate compared to the case where there is no expansion as t$\rightarrow \infty$? **(3 Points)**

Aug 2010

Classical #2

Initially: mass $= m$

radius $= R$

density: $\rho = \frac{m}{\frac{4}{3}\pi R^3}$

At $t = \tau$: mass: $8m$

radius $= 2R$

density: $\rho = \frac{m}{\frac{4}{3}\pi (2R)^3}$

a) At time $t = \tau$, what is the angular velocity of the sphere

* Conservation of angular momentum

$$L = m\omega r^2 \quad (L = mvr,\ v = \omega r)$$

$$m\omega_0 R^2 = \cancel{8} m \omega_f\, 2R^2$$

$$\frac{\omega_0}{4} = \omega_f$$

b) Find an expression for $\omega(t)$

$t = 0$ $\quad$ $t = \tau$

$r = R$ $\quad$ $r = 2R$ $\quad \Rightarrow r(t) = \frac{R}{\tau}t + R$

$$mR^2\omega_0 = \left(m\left(\frac{R}{\tau}t + R\right)\right)^2 \omega_f$$

$$\omega_0 = \omega_f \left(\frac{t}{\tau} + 1\right)^2$$

$$\Rightarrow \omega_f = \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2}$$

c) If the expansion reverses at $r = 2R$ and returns to its initial state at $t = 2\tau$, by what angle $\Delta\varphi$ is the sphere behind an identical sphere that didn't expand

$$2\omega_0\tau - \left[\int_0^{\tau} \frac{\omega_0}{\left(\frac{t}{\tau}+1\right)^2} + \int_{\tau}^{2\tau} \frac{\omega_0}{\left(\frac{-t}{\tau}+1\right)^2}\right]$$

$$2\omega_0\tau - \left[\left[\tau\omega_0\left(\frac{t}{\tau}+1\right)^{-1}\right]_0^{\tau} + -\omega_0\left(\frac{-t}{\tau}+1\right)^{-1}\tau\Big|_{\tau}^{2\tau}\right]$$

$$2\omega_0\tau - \left[\tau\omega_0\left(\frac{1}{2} - 1\right) - \tau\omega_0(-1-0)\right]$$

$$2\omega_0\tau - \left[-\frac{\tau\omega_0}{2} + \frac{\tau\omega_0}{1}\right]$$

$$2\omega_0\tau - \boxed{\frac{\tau\omega_0}{2}}$$

$$\Rightarrow \Delta\varphi = \frac{3\tau\omega_0}{2}$$

d) What if the sphere exponentially w/ time constant τ_e. How much does sphere rotate compared to the case where there is no expansion ($t \to \infty$)

*Now $r = Re^{t/\tau_e}$

$$\cancel{m}R^2 \omega_0 = m\left(Re^{t/\tau_e}\right)^2 \omega_f$$

$$\Rightarrow \omega_f = \omega_0 e^{-2t/\tau_e}$$

$$2\omega_0 t - \left[\int_0^t \omega_0 e^{-2t/\tau_e} dt\right.$$

$$2\omega_0 t - \left[-\frac{\tau_e}{2}\omega_0 e^{-2t/\tau_e}\right|_0^t$$

$$2\omega_0 t - \frac{\tau_e}{2}\omega_0 e^{-2t/\tau} - \frac{\tau_e}{2}\omega_0$$

$$2\omega_0 t - \frac{\tau_e}{2}\omega_0\left(e^{-2t/\tau} - 1\right) = \Delta\varphi$$

Problem 3 (10 Points):

Consider the following Lagrangian

$$L = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right) e^{2\gamma t}$$

assuming that $\omega > \gamma$ for the questions that follow.

a. Determine the Hamiltonian associated with this Lagrangian. **(3 Points)**

b. Find a transformation to new phase space variables that make H independent of time and show that these form a canonical transformation by determining a generating function of the form $F_2(q, P, t)$. **(4 Points)**

c. Using the equations of motion for the transformed Hamiltonian $K(Q, P, t)$, solve for $Q(t)$ and transform back to get $q(t)$. **(3 Points)**

Aug 2010

Classical #3

Given: $\mathcal{L} = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)e^{2\gamma t}$

a) Determine the Hamiltonian associated w/ the above Lagrangian

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

$$= e^{2\gamma t} m\dot{q} \qquad \Rightarrow \dot{q} = \frac{p}{m}e^{-2\gamma t}$$

$$\Rightarrow H = m\dot{q}^2 e^{2\gamma t} - e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right)$$

$$= e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\right)$$

$$= e^{2\gamma t}\left(\frac{1}{2m}p^2 e^{-4\gamma t} + \frac{1}{2}m\omega^2 q^2\right)$$

$$= \frac{1}{2m}p^2 e^{-2\gamma t} + \frac{1}{2}m\omega^2 q^2 e^{2\gamma t}$$

b) Find a transformation to new phase space variables that make H time independent and show that these form a canonical transformation by determining a $F_2(q, P, t)$ generating function.

* For an $F_2(q, P, t)$: $\quad p = \frac{\partial F_2(q, P)}{\partial q} \qquad Q = \frac{\partial F_2(q, P)}{\partial P}$

$\Rightarrow$ Want: $H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q$

$$\Rightarrow Q = qe^{\gamma t}$$

$$P = pe^{-\gamma t}$$

$$\Rightarrow F_2(q, P, t) = Pqe^{\gamma t}$$

$$\Rightarrow K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

c) Use the Hamiltonian eqns. of motion for K to solve for Q and transform it back to $q(t)$

$$\dot{P} = \frac{\partial K}{\partial Q} = m\omega^2 Q$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{m}$$

$$\dot{P} = m\ddot{Q} \qquad \Rightarrow m\ddot{Q} = \omega^2 m Q$$

$$\ddot{Q} = \omega^2 Q \qquad \Rightarrow Q = Ae^{-i\omega t} + Be^{i\omega t}$$

$$\Rightarrow q = e^{-\gamma t}Q$$

$$= e^{-\gamma t}\left(Ae^{-i\omega t} + Be^{i\omega t}\right)$$

Classical Mechanics and Statistical/Thermodynamics

August 2011

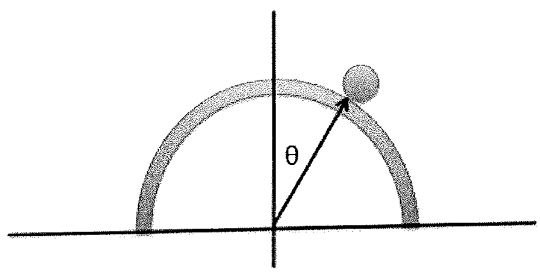
Classical Mechanics

1. A solid uniform marble with mass m and radius r starts from rest on top of a hemisphere with radius R. It will start to roll to the right, and eventually fly off the hemisphere.

   (a) Assume that the marble rolls without slipping at all times. Calculate θ_1, the angle with respect to the vertical at which the marble loses contact with the hemisphere. (3pts).

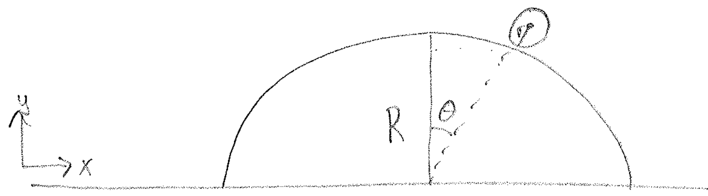
   (b) Where will the marble hit the ground, as measured from the center of the hemisphere? You may use the variable θ_1 in your answer. (If you do not solve part (a), you can still attempt this problem by writing your answer in terms of this variable.) (3pts).

   (c) Now assume that the force of friction between marble and the hemisphere is μN, where N is the normal force between the marble and the hemisphere. Calculate the angle θ_2 at which the marble will no longer roll without slipping. (4pts).



Aug 2011

Classical #1



a) Assume the marble rolls w/o slipping. Find θ_1, where marble looses contact w/ the hemisphere

* Normal force is 0 when marble leaves surface

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy$$

$x = (R+r)\sin\theta$

$y = (R+r)\cos\theta$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mr^2\left(\frac{v}{r}\right)^2 + mg(R+r)\cos\theta_1$$

$$mg(R+r) = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 + mg(R+r)\cos\theta_1$$

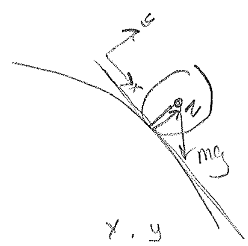
$$g(R+r) = \frac{3}{5}v^2 + g(R+r)\cos\theta_1$$

$$g(R+r) = \frac{3}{5}gR\cos\theta_1 + g(R+r)\cos\theta_1$$

$$\frac{g(R+r)}{\frac{3}{5}g + g(R+r)} = \cos\theta_1$$

$$\frac{R+r}{\frac{3}{5} + R + r} = \cos\theta_1$$

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{R+r}{R+r+3/5}\right)$$



x, y

$\langle \parallel, \perp \rangle$

$F_N = \langle 0, N \rangle$

$F_G = mg\langle \sin\theta, -\cos\theta \rangle$

$\Rightarrow ma_x = mg\sin\theta$

$ma_y = N + mg\cos\theta$

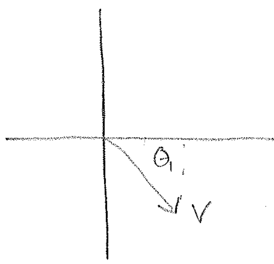
* at θ_1, $N = 0$, $a_y = \frac{v^2}{R}$

$ma_y = mg\cos\theta_1$

$\frac{v^2}{R} = g\cos\theta_1$

$\Rightarrow v = \sqrt{gR\cos\theta_1}$

b) Where will the marble hit the ground, measured from the center of the hemisphere?



$V_x = V\cos\theta$

$V_y = V\sin\theta$

$y_i = (R+r)\cos\theta_1 \qquad V_i = V$

$y_f = 0 \qquad a = g$

$$0 = -\frac{1}{2}gt^2 + vt + (R+r)\cos\theta_1$$

$$0 = -\frac{1}{2}gt^2 - \sqrt{gR\cos\theta_1}\sin\theta_1 t + (R+r)\cos\theta_1$$

$$\Rightarrow \quad t = \frac{+\sqrt{gR\cos\theta_1}\sin\theta_1 \pm \sqrt{gR\cos\theta_1\sin^2\theta_1 - 4\left(-\frac{1}{2}g\right)(R+r)\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\sin\theta_1 \pm \sqrt{gR\cos\theta_1(\sin^2\theta_1 + 2) + 2rg\cos\theta_1}}{-g}$$

$$= \frac{\sqrt{gR\cos\theta_1}\sin\theta_1 - \sqrt{gR\cos\theta_1(\sin^2\theta_1 + 2) + 2rg\cos\theta_1}}{-g}$$

(need (−) root to make overall time positive)

$x_i = (R+r)\sin\theta_1 \qquad v = \sqrt{gR\cos\theta_1}\cos\theta_1$

$x_f = ??$

$$x_f = vt + x_i$$

$$= gR\cos^{3/2}\theta_1 t + (R+r)\sin\theta_1$$

$$= gR\cos^{3/2}\theta_1\left(\frac{1}{-g}\left[\sqrt{gR\cos\theta_1}\sin\theta_1 - \sqrt{gR\cos\theta_1(\sin^2\theta_1+2) + 2rg\cos\theta_1}\right]\right) + (R+r)\sin\theta_1$$

2. Consider a point particle of mass m moving under the influence of a central force:

$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\,\hat{r}$$

where n is an integer greater than one ($n = 2, 3, \ldots$), the variable r is the distance from the origin of the force ($r \equiv |\vec{r}|$) and $\hat{r}$ is a unit vector in the radial direction. In this problem, we will examine when circular orbits are stable for such a central force.

(a) Calculate potential energy of this force. Choose the zero of the potential to be at infinity ($r = \infty$). (1pt)

(b) Show that the angular momentum about the origin, L, is conserved. (You may use the Newtonian, Lagrangian, or Hamiltonian formulations of the problem). (2pts)

(c) Write an expression for the total energy of the particle E as a function of r, dr/dt, L, k, and n. (1pt)

(d) Assume the particle is moving in a circular orbit about the origin, so that $dr/dt = 0$. Calculate the radius of the orbit and the velocity of the particle as a function of the above variables. (3pts)

(e) When is this circular orbit stable? (Hint: look at dE/dr and d^2E/dr^2.) (3pts)

Aug 2011

Classical #2

* Consider a particle of mass m under the influence of a central force
$$\vec{F}(\vec{r}) = -\frac{k}{r^n}\hat{r} \quad \text{where} \quad n \in [\mathbb{Z} > 1].$$

a) Calculate the potential energy of this force

$$U(\vec{r}) = -\int \vec{F} \cdot d\vec{r}$$
$$= -\int_{\infty}^{r} -\frac{k}{r^n}\hat{r}\, d\vec{r}$$
$$= +\int_{\infty}^{r} \frac{k}{r^n} dr \qquad \text{(assuming spherical coordinates)}$$
$$= \int_{\infty}^{r} k r^{-(n)} dr$$
$$= -k\frac{1}{n+1} r^{-n+1}\Big|_{\infty}^{r}$$
$$= -\frac{k}{n+1} r^{-n+1}$$

b) Show that angular momentum about the origin is conserved

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{k}{n+1}r^{-n+1}, \qquad v = \langle \dot{r}, r\dot{\theta}, r\dot{\varphi}\sin\theta \rangle$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$
$$0 = \frac{d}{dt}(mr^2\sin^2\theta\,\dot{\varphi})$$
$$\Rightarrow mr^2\sin^2\theta\,\dot{\varphi} = \text{const.} = L \quad \rightarrow L \text{ is conserved}$$

c) Write an expression for the total energy of the particle (E) as a function of:
$r, \dot{r}, L, k, n$

$$E = T + U$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2\sin^2\theta} + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{n-1}r^{-n+1}$$
$$* \text{ if } \theta = \frac{\pi}{2}$$
$$= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{n-1}r^{-n+1}$$

d) Assume the particle is moving in a circular orbit about the origin. Find the radius of the orbit and the velocity of the particle as a function of the above variables

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\varphi}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{k}{n+1}r^{-n+1} = \frac{L^2}{2mr^2} + \frac{k}{n-1}r^{-n+1}$$

$$\cancel{\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}}$$

0 b/c r is constant

$$\cancel{mr\sin^2\theta\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = \frac{d}{dt}[m\dot{r}]}$$

$$mr\sin^2\theta\dot{\varphi}^2 + mr\dot{\theta}^2 - kr^n = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\Rightarrow r = \left(\frac{km}{L^2}\right)^{\frac{1}{n-3}}$$

$$\Rightarrow L = mr^2\dot{\varphi} \rightarrow \dot{\varphi} = \frac{L}{mr^2}$$

e) When is the orbit stable?

$$\frac{dE}{dr} = \frac{-L^2}{mr^3} + \frac{k}{r^n}$$

$$\frac{d^2E}{dr^2} = \frac{3L^2}{mr^4} + \frac{-nk}{r^{n+1}}$$

$$= \frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{-4/(n-3)} + nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}} > 0$$

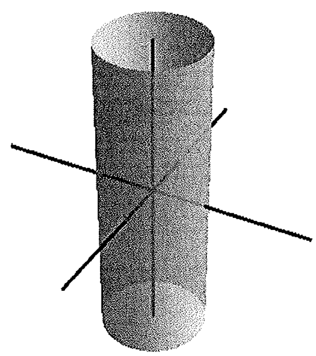
$$\frac{3L^2}{m}\left(\frac{km}{L^2}\right)^{\frac{-4}{n-3}} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)}{n-3}}$$

$$\frac{3L^2}{m} > +nk\left(\frac{km}{L^2}\right)^{\frac{-(n+1)+4}{n-3}}$$

$$\frac{3\cancel{L^2}}{\cancel{m}} > + \frac{\cancel{L^2}nk}{\cancel{km}}$$

$$3 > +n$$

3. A particle of mass m is constrained to move on an infinitely long cylinder of radius a. The center of the cylinder is oriented along the z-axis, as shown. An attractive central potential, $U(r) = U(\sqrt{a^2 + z^2})$, is located at the origin, where r is the radius is spherical coordinates.



(a) Write down the Lagrangian for the problem. (1pt)

(b) From the Lagrangian, explicitly derive the Hamiltonian for the particle. (2pts)

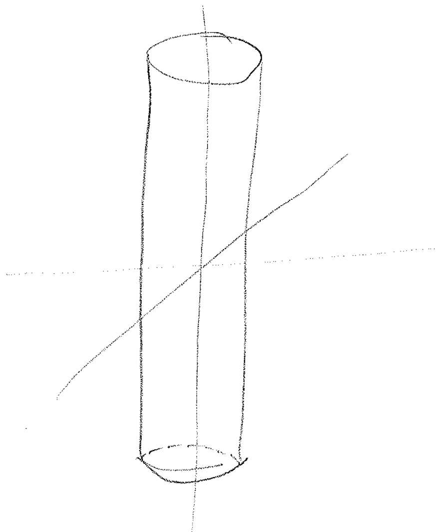
(c) Is angular momentum about the z-axis conserved? Prove your answer. (2pts)

(d) Under what conditions is motion in the z-direction bounded? (2pts)

(e) Assume that the potential is $U(r) = \frac{1}{2}\alpha r^2$. Solve the equations of motion, and reduce the problem to quadrature. (3pts)

Aug 2011

Classical #3



* Particle of mass m constained to move on infinetely long cylinder of radius a; cylinder oriented along z-axis
* Attractive central located at origin,

$$U(r) = U(\sqrt{a^2+z^2}),$$ where r is radius in spherical

a) Find the Lagrangian

$$x = \rho\cos\varphi \qquad \dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}$$
$$y = \rho\sin\varphi \qquad \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi}$$

$$\mathcal{L} = T - U$$
$$= \tfrac{1}{2} m v^2 - U(\sqrt{a^2+z^2})$$
$$= \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(\sqrt{a^2+z^2})$$
$$= \tfrac{1}{2} m(\dot{\rho}^2\cos^2\varphi - 2\rho\cos\varphi\sin\varphi\,\dot{\rho}\dot{\varphi} + \rho^2\sin^2\varphi\,\dot{\varphi}^2 + \dot{\rho}^2\sin^2\varphi + 2\rho\sin\varphi\cos\varphi\,\dot{\rho}\dot{\varphi} + \rho^2\cos^2\varphi\,\dot{\varphi}^2 + \dot{z}^2) - U$$
$$= \tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - U(\sqrt{a^2+z^2})$$

* Assume arbitrary central potential Ar^n

$$= \tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - A(\sqrt{a^2+z^2})^n$$

b) From the Lagrangian, derive the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - \mathcal{L}$$

$$p_\rho = \frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\dot{\rho}$$
$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}$$
$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

$$\Rightarrow H = m\dot{\rho}^2 + m\rho^2\dot{\varphi}^2 + m\dot{z}^2 - \left[\tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - A(\sqrt{a^2+z^2})^n\right]$$
$$= \tfrac{1}{2} m\dot{\rho}^2 + \tfrac{1}{2} m\rho^2\dot{\varphi}^2 + \tfrac{1}{2} m\dot{z}^2 + A(\sqrt{a^2+z^2})^n$$

c) Is angular momentum about z-axis conserved?

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right]$$

$$0 = \frac{d}{dt}\left[m\rho^2\dot{\varphi}\right]$$

$$\Rightarrow m\rho^2\dot{\varphi} = \text{const.} = L \checkmark$$

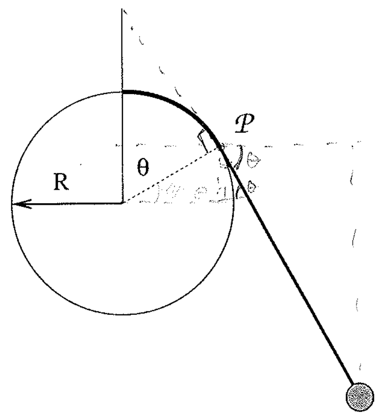
d) Under what conditions is motion in the z direction bounded?

$\Rightarrow$ Motion is bounded when $E_T < U \rightarrow T < 0$

Classical Mechanics and Statistical/Thermodynamics

January 2016

2. A stationary disk of radius R is aligned vertically so that its axis is parallel to the ground. The disk is fixed and does not rotate. A string of length ℓ is attached to the top of the disk, and $\ell > \pi R$. A point mass m is attached to the end of the string and can swing in a vertical plane (left to right in the figure below). As the mass m swings, the point $\mathcal{P}$ where the string just contacts the disk will move. Assume that the string is always taut. The angle between $\mathcal{P}$ and the vertical is θ; it will be the generalized coordinate in this problem.



(a) Determine the x and y position of the point mass as a function of θ, R and ℓ. Use the center of the disk as the origin of your coordinate system. (Hint: Knowing the value of θ determines both the amount of string wrapped on the disk and the angle the straight length of string makes with the vertical.) (1 point)

(b) Treating $\theta(t)$ as the generalized coordinate, determine the kinetic energy of the point mass as a function of m, θ, $\dot{\theta}$, R, and ℓ. (2 points)

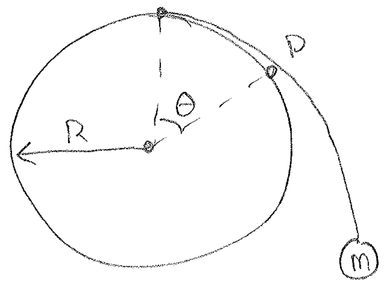
(c) What is the Lagrangian for the system in terms of this generalized coordinate? (2 points)

(d) What are the equations of motion? (1 point)

(e) Assume that the point mass makes small oscillations about some angle θ_0 (which might not be zero). Determine θ_0 and the angular frequency of these oscillations. (4 points).

Spring 2016

Classical #2



* Sting of length ℓ

a) $\vec{r}_m = \langle R\cos(90-\theta),\ R\sin(90-\theta) \rangle$

$= \langle R\sin\theta,\ R\cos\theta \rangle + \langle (\ell - R\theta)\cos\theta, -(\ell - R\theta)\sin\theta \rangle$

$= \langle R\sin\theta + (\ell - R\theta)\cos\theta;\ R\cos\theta - (\ell - R\theta)\sin\theta \rangle$

b) $T = \frac{1}{2} m v^2$

$= \frac{1}{2} m \dot{\vec{r}}^2$

$\dot{\vec{r}} = \langle R\cos\theta\,\dot{\theta} - \ell\sin\theta\,\dot{\theta} - [R\dot{\theta}\cos\theta - R\theta\sin\theta\,\dot{\theta}],\ -R\sin\theta\,\dot{\theta} - \ell\cos\theta\,\dot{\theta} + R\dot{\theta}\sin\theta + R\theta\cos\theta\,\dot{\theta} \rangle$

$\dot{\vec{r}} = \dot{\theta} \langle R\cos\theta - \ell\sin\theta - R\cos\theta + R\theta\sin\theta,\ -R\sin\theta - \ell\cos\theta + R\sin\theta + R\theta\cos\theta \rangle$

$= \dot{\theta} \langle -\ell\sin\theta + R\theta\sin\theta,\ R\theta\cos\theta - \ell\cos\theta \rangle$

$\dot{\vec{r}}^2 = \dot{\theta}^2 [\ell^2\sin^2\theta + R^2\theta^2\sin^2\theta - 2\ell R\theta\sin^2\theta + R^2\theta^2\cos^2\theta + \ell^2\cos^2\theta - 2R\theta\ell\cos^2\theta]$

$= \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta] \implies T = \frac{1}{2} m \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta]$

c) $\mathcal{L} = T - U$

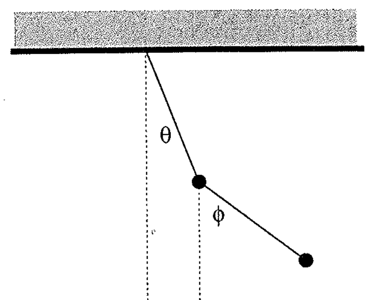
$= \frac{1}{2} m \dot{\theta}^2 [\ell^2 + R^2\theta^2 - 2\ell R\theta] - mg(R\cos\theta - (\ell - R\theta)\sin\theta)$

d) $\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}}$

$\cancel{\frac{1}{2}} m\dot{\theta}^2 R^2\theta - 2\ell R + mgR\sin\theta + mg\ell\cos\theta - R\sin\theta - R\theta\cos\theta = \frac{\partial}{\partial t}[m\dot{\theta}(\ell^2 + R^2\theta^2 - 2\ell R\theta)]$

$m\dot{\theta}^2 R^2\theta - 2\ell R + R\sin\theta(mg - 1) + \cos\theta(mg\ell - R\theta) = m\ell^2\ddot{\theta} + R^2 m\ddot{\theta}\theta^2 + 2R^2 m\dot{\theta}^2\theta - 2\ell R(\dot{\theta}^2 + \ddot{\theta}$

3. Consider a double pendulum, consisting of a mass m suspended from a point with a massless cord of length ℓ, with a second mass m suspended from the first with another massless cord of equal length ℓ. At a given instant, the first mass makes an angle θ with respect to the vertical, while the second mass makes an angle ϕ with respect to the vertical. A uniform gravitational field, with gravitational acceleration g, acts in the vertical direction.



(a) Starting from the description of kinetic and potential energy in Cartesian coordinates, obtain the Lagrangian in terms of the angles θ and ϕ and their time derivatives, $\dot{\theta}$ and $\dot{\phi}$. (2 points)

(b) Now simplify the Lagrangian to the situation when both angles are small, $\theta \ll 1$, $\phi \ll 1$, and obtain the form of two coupled harmonic oscillators. (1 point)

(c) For this system, obtain the mass matrix $\mathbf{M}$ and the spring-constant matrix $\mathbf{K}$, where the Lagragian is written:

$$L = \frac{1}{2}\begin{pmatrix} \dot{\theta}, & \dot{\phi} \end{pmatrix} \mathbf{M} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \theta, & \phi \end{pmatrix} \mathbf{K} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

(2 points)

(d) Show that the normal modes satisfy

$$(\omega^2 \mathbf{M} - \mathbf{K}) \cdot \mathbf{Q} = 0.$$

(1 point)

(e) Determine the characteristic frequencies (eigenfrequencies) ω in terms of the quantity $\omega_0^2 = g/l$. (2 points)

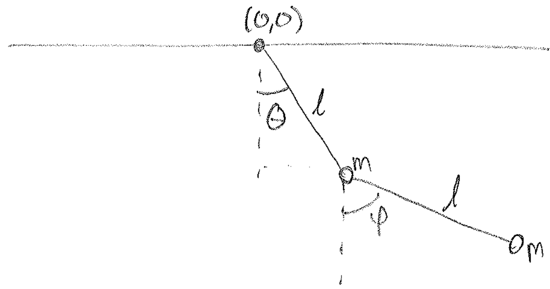
(f) If we write the normal mode vector as

$$\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

determine the ratio q_2/q_1, which characterize the normal modes. (2 points)

Jan 2016

Classical #3



a) Starting w/ cartesian, obtain Lagrangian in terms of $\theta, \dot{\theta}, \varphi, \dot{\varphi}$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2} m (v_1^2 + v_2^2) - mg(y_1 + y_2)$$

$$x_1 = l\sin\theta \qquad y_1 = l\cos\theta$$

$$x_2 = l\sin\theta + l\sin\varphi \qquad y_2 = l\cos\theta + l\cos\varphi$$

$$\dot{x}_1 = l\cos\theta\,\dot{\theta} \qquad \dot{y}_1 = -l\sin\theta\,\dot{\theta}$$

$$\dot{x}_2 = l\cos\theta\,\dot{\theta} + l\cos\varphi\,\dot{\varphi} \qquad \dot{y}_2 = -l\sin\theta\,\dot{\theta} - l\sin\varphi\,\dot{\varphi}$$

$$= \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) - mg(y_1 + y_2)$$

$$= \frac{1}{2} ml^2 [\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\theta}^2 + \cos^2\theta\,\dot{\theta}^2 + 2\cos\theta\cos\varphi\,\dot{\theta}\dot{\varphi} + \cos^2\varphi\,\dot{\varphi}^2 + \sin^2\theta\,\dot{\theta}^2 + 2\sin\theta\sin\varphi\,\dot{\theta}\dot{\varphi} + \sin^2\varphi\,\dot{\varphi}^2]$$

$$= \frac{1}{2} ml^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$

$$= \frac{1}{2} ml^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi)] - mg(2l\cos\theta + l\cos\varphi)$$

b) Apply small angle approx. + get form of coupled harmonic oscillators

* In small angle: $\sin\theta \to \theta$
$\cos\theta \to 1 - \frac{\theta^2}{2}$

$$\Rightarrow \mathcal{L} = \frac{1}{2} ml^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{(\theta-\varphi)^2}{2}) - mgl(2(1 - \frac{\theta^2}{2}) + (1 - \frac{\varphi^2}{2})$$

$$= \frac{1}{2} ml^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}(1 - \frac{1}{2}(\theta^2 - 2\theta\varphi + \varphi^2))] - mgl(2 - \theta^2 + 1 - \frac{1}{2}\varphi^2)$$

$$= \frac{1}{2} ml^2 [2\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\theta}\dot{\varphi}(2 - \theta^2 + 2\theta\varphi - \varphi^2] - mgl(3 - \theta^2 - \frac{1}{2}\varphi^2)$$

c) Find $\overleftrightarrow{M}$ and $\overleftrightarrow{K}$ when the Lagrangian is of the form:

$$\mathcal{L} = \frac{1}{2}(\dot{\theta}, \dot{\varphi})\overleftrightarrow{M}\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix} - \frac{1}{2}(\theta, \varphi)\overleftrightarrow{K}\begin{pmatrix}\theta\\ \varphi\end{pmatrix}$$

$$\Rightarrow \overleftrightarrow{M} = \left[\begin{matrix} ml^2 & \\ & \frac{1}{2}ml^2 \end{matrix}\right.$$

$$\overleftrightarrow{K} = \left[\begin{matrix} \\ \\ \end{matrix}\right.$$