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Classical Mechanics

CH. 1 SURVEY OF THE ELEMENTARY PRINCIPLES LECTURE NOTES

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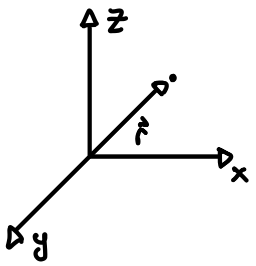
Elementary principles of mechanics

* Conservation theorems
* Constrained motion
* Alemberts principle
* Lagranges equation of motion

Conservation Theorems

* Conserved quantity ⟶ Doesn't change
* (Sometimes) use conserved quantities to reduce / eliminate degrees of freedom
* Help with analytic solutions

$N=1$, (single-particle)



Point-particle in 3D. $\vec{r} = (x, y, z)$ (position vector)

⇒ Defined w.r.t an <u>inertial</u> frame of reference

⟶ No acceleration : Newtons 1^{st} law holds

Three Conservation Theorems:

i) Conservation of linear momentum

$$\sum \vec{F}_i = 0 \longrightarrow \dot{\vec{p}} = 0$$

⟶ Assumption of inertial frame

ii.) Conservation of <u>angular</u> momentum

$$\vec{\tau}_i : \sum_i \vec{N}_i = 0 \longrightarrow \dot{\vec{L}} = 0$$

<u>proof:</u> $\vec{L} = \vec{r} \times \vec{p}$, $\vec{N}_i = \vec{r} \times \vec{F} = \vec{r} \times \dot{\vec{p}}$, $\dot{\vec{L}} = \overset{0}{\cancel{\dot{\vec{r}} \times \vec{p}}} + \underbrace{\vec{r} \times \dot{\vec{p}}}_{\vec{N}_i}$

iii.) Conservation of Energy

⇒ If the forces acting on a particle are <u>conservative</u> then the total energy $E = T + V$ is conserved

Conservative force $\vec{F} = -\vec{\nabla} x(\vec{r})$: $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{\nabla} \times (\vec{r})) = 0$

Conservative : work done by force on particle does not depend on path.

$$W_{1 \to 2} = \int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 m\dot{\vec{v}} \cdot \dot{\vec{v}}\, dt = \int_1^2 \frac{m}{2} \left| |\dot{\vec{v}}|^2 \right| dt = T_2 - T_1$$

Many-particles ($N>1$)

Consider N particles in an inertial frame. $[\vec{r}_i]$

We decompose,

$$\sum_j F_j^{(i)} = \sum_K F_K^{(i)} + F_i^{(e)} = \dot{\vec{p}}_i$$

$\longrightarrow$ j forces on particles i, K interparticle forces

i) External $\Rightarrow$ e.g., harmonic confining potential : $V(\vec{r}_i) = \dfrac{m_i \omega_i^2}{2} \vec{r}_i^{\,2}$

Internal $\Rightarrow$ Interactions between particles : $V(\vec{r}_{ij}) = \frac{1}{2} k_{ij} \vec{r}_{ij}^{\,2}$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$

$\vec{F}_{ji} = -\vec{F}_{ij}$ $\Rightarrow$ "opposite" forces

$\Rightarrow$ True for "conservative" forces

"Proof": $\vec{F}_{ij} = -\vec{\nabla}_{\vec{r}_j} V(\vec{r}_{ij}) = K_{ij}\vec{r}_{ij}$ & $\vec{F}_{ji} = -\vec{\nabla}_{\vec{r}_i} V(\vec{r}_{ij}) = -K_{ij}\vec{r}_{ij}$

Weak law of reaction & action

Also : Strong law, $\vec{F}_{ij}$ lies along vector $\vec{r}_{ij}$ between particles $\vec{F}_{ij} \propto \vec{r}_{ij}$

Define:

$$\vec{R} = \frac{\sum_i m \vec{r}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \vec{r}_i \quad : \quad \vec{P} = M\dot{\vec{R}}$$

Then,

i) Conservation of linear momentum : $\sum_i F_i^{(e)} = 0 \Rightarrow \dot{\vec{P}} = 0$

Sum over external

$\Rightarrow$ Assumption : weak law of reaction & action holds, then, $\vec{F}_{ij} + \vec{F}_{ji} = 0$

ii) Conservation of angular momentum. $\sum_i \vec{N}_i^{(e)} = 0 \Rightarrow \dot{\vec{L}}_{Tot} = 0$

$\vec{L}_{Tot} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i = \sum \vec{L}_i$

$\Rightarrow$ Externally applied torques

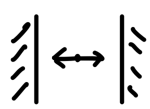
Proof : Assume <u>Strong</u> law holds. $\dot{\vec{L}}_{Tot} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{j\neq i} \vec{r}_i \times \vec{F}_j$

$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_j - \vec{r}_i) \times \vec{F}_{ji}$ (weak) $\propto$ $\vec{r}_{ij} \times \vec{r}_{ij} = 0$ (strong)

iii) Conservation of Energy $\Rightarrow$ Holds true if <u>all</u> internal & external applied forces are conservative

Constraints of Motion

<u>Examples</u> : $\Rightarrow$ Motion of a particle confined within a region



$\Rightarrow$ Particle constrained to follow a parametrized path

$\Rightarrow$ Bead on a wire

$\Rightarrow$ Tricky w/ Newtonian mechanics

$\Rightarrow$ Constraint $\xrightarrow{?}$ reduced degrees of freedom .... Simpler ?

Classification of Constraints

$\Rightarrow$ Two categories.

i) Holonomic : A constraint that can be written as k ($< 3N$) equations.

$$f_i(\vec{r}_N, \ldots\ldots, \vec{r}_{N t}) = 0 \quad , \quad f_k(\ldots\ldots) = 0$$

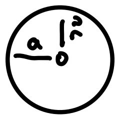
Only depend on position $\vec{r}_i$ & time t [Not velocity $\dot{\vec{r}}_i$]

Example :

$$d = |\vec{r}_1 - \vec{r}_2| \quad , \quad d = \text{const} = d_0 \quad , \quad |\vec{r}_1 - \vec{r}_2| - d_0 = 0$$

ii) Non-Holonomic : Not holonomic : $f_i(\vec{r}, \dot{\vec{r}}, t) = 0$, inequalities

Example : Particle on surface of Sphere



$$|\vec{r}|^2 - a^2 \geq 0 \longrightarrow \text{Excluded region}$$

Further Sub-Categories :

i) Constraint that explicitly depends on t $\Rightarrow$ Rheonomous

ii) Does explicitly depend on t $\Rightarrow$ Scleronomous

Two Comments :

① The Forces of constraint that dictate the dynamics are typically unknown.

We will find they are obtainable from a complete solution of the dynamics.

② Constraints imply that our original set of $3N$ co-ordinates $\vec{r}_i \ldots \vec{r}_N$ are no longer independent.

$$\text{EOM} : m_i \ddot{\vec{r}}_i = \vec{F}_i + \sum_j \vec{F}_{ji} \longrightarrow \text{Coupled equations}$$

Thus ② can be rectified by introducing generalized coordinates

Idea: N particles w/ $3N$ independent coordinates & degrees of freedom

↳ k constraint equations , k equations are used to eliminate k degrees of freedom

Left with $l = 3N-k$ generalized coordinates $q_1, q_2, \ldots q_N$

Note: generalyzed coordinates $\neq$ positions $\Rightarrow$ arbitrary units

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Conservation Theorems

$\Rightarrow$ Linear, angular momenta

$\Rightarrow$ Energy

Constraints

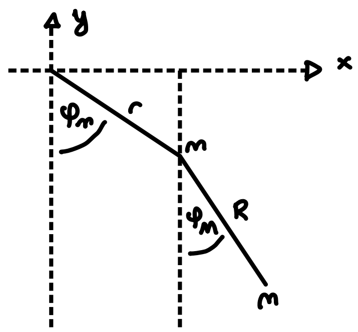
$\Rightarrow$ Holonomic & Nonholonomic

$\Rightarrow$ Forces of constraint
$\Rightarrow$ "unknown"

$\Rightarrow$ Generalized co-ordinates

$3N$ co-ordinates, k constraints, $L = 3N-K$ generalized coordinates

Double Pendulum



(x_m, y_m), (x_M, Y_M)

$\hookrightarrow$ 4 Degrees of Freedom, 2 constraints

$l = 2\cdot2 - 2 = 2$ generalized coordinates

Two constraints:

Rule 1: $f_1(x_m, y_m, x_M, y_M, t) = 0$ w/ $f_1() = x_m^2 + y_m^2 - r^2$

Rule 2: $f_2(\ldots\ldots) = 0$ w/ $f_2() = (x_m - x_M)^2 + (y_m - y_M)^2 - R^2$

In terms of φ_m, φ_M

$x_M = r\sin\varphi_M$, $y_M = -r\cos\varphi_M$

$x_m = x_M + R\sin\varphi_m$, $y_m = y_M - R\cos\varphi_m$

D'Alembert's Principle & Lagranges Equation

Goal: Construct treatment to obtain dynamics w/out explicit knowledge of forces of constraint

Virtual Displacement : $\delta\vec{r}$

An infinitesimal displacement of the system [e.g. co-ords $\vec{r}$] that is consistent with the forces & constraints imposed on the system at fixed time t.

$$\delta\vec{r} \neq d\vec{r}$$

Consider a system in equilibrium $\Rightarrow \vec{F}_i = 0 \qquad \sum_i \vec{F}_i = 0 \,,\ \sum_i \underbrace{\vec{F}_i \cdot \delta\vec{r}_i}_{} = 0$

↳ Virtual work done by a virtual displacement

Decompose applied forces :

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{f}_i$$

$\vec{F}_i^{(a)}$ → Applied force, $\vec{f}_i$ → Force of constraint (unknown)

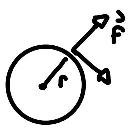
$$\Rightarrow \sum_i F_i^{(a)} \cdot \delta\vec{r}_i + \sum_i \vec{f}_i \cdot \delta\vec{r}_i = 0$$

Consider a system where "net virtual work of constraint" is zero.

$$\sum_i f_i \cdot \delta\vec{r}_i = 0$$

Valid for, e.g, Holonomic Constraints

Example :



$$\vec{f} \cdot \delta\vec{r} = 0 \ , \quad \vec{f} \perp \delta\vec{r}$$

Then,

$$\sum_i F^{(a)} \cdot \delta\vec{r}_i = 0 \longrightarrow \text{Principle of virtual work}$$

↑ In general $\delta\vec{r}_i$ not independent

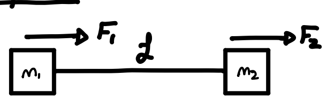
Dynamical case :

First Law $\longrightarrow \vec{F}_i = \dot{\vec{P}}_i \ , \quad \vec{F}_i - \dot{\vec{P}}_i = 0 \ : \ \sum(\vec{F}_i - \dot{\vec{P}}_i)\cdot\delta\vec{r}_i = 0 \ , \ \sum_i (\vec{F}_i^{(a)} \cdot \dot{\vec{P}}_i)\cdot\delta\vec{r}_i + \sum_i \vec{f}_i \delta\vec{r}_i = 0$

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0 \longrightarrow \text{d'Alemberts principle}$$

↳ Equation of motion for system :

Example :



$$(F_1 - m_1\ddot{x}_1)\cdot\delta x_1 + (F_2 - m_2\ddot{x}_2)\cdot\delta x_2 = 0 \ , \quad x_2 - x_1 - d = 0 \ : \ \text{constraint} \ , \ \ddot{x}_2 = \ddot{x}_1 \ : \ \delta x_2 = \delta x_1$$

$$(F_1 - m_1\ddot{x}_1)\cdot\delta x_1 + (F_2 - m_2\ddot{x}_2)\cdot\delta x_1 = 0 \ : \ (m_1 + m_2)\ddot{x}_1 = F_1 + F_2$$

Obtaining Lagranges Equation via d'Alembert's principle

$\Rightarrow$ Math ..... hinges on? $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots\ldots, t)$

Definition : $\dfrac{d\vec{r}_i}{dt} = \sum_i \dfrac{\partial \vec{r}_i}{\partial q_i}\dfrac{dq_i}{dt} + \dfrac{\partial \vec{r}_i}{\partial t}$, $d\vec{r}_i = \sum_i \dfrac{\partial \vec{r}_i}{\partial q_i} dq_i + \dfrac{\partial \vec{r}_i}{\partial t} dt$, $\delta \vec{r} = \sum_i \dfrac{\partial \vec{r}_i}{\partial q_i}\delta q_i$ (*)

$$0 = \underset{①}{\sum_i F_i^{(a)} \cdot \delta \vec{r}_i} - \underset{②}{\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}}$$

(*) $\rightarrow$ ① : $\sum_i F_i^{(a)} \left(\sum_i d\vec{r}_i/dq_i\, \delta q_i \right) = \sum_j Q_j \delta q_j$: $Q_j = \sum_i F_i^{(a)} \cdot \dfrac{\partial \vec{r}_i}{\partial q_j}$ $\longrightarrow$ "Generalized Force"

(*) $\rightarrow$ ② : $= \sum_i m_i \ddot{\vec{r}}_i \cdot \left(\sum_j \partial \vec{r}/\partial q_j\, \delta q_j \right) = \sum_{ij} \left[d/dt \left(m_i \dot{\vec{r}}_i \cdot \partial \vec{r}_i / \partial q_j \right) - m \dot{\vec{r}}_i \cdot d/dt \left(\partial \vec{r}/\partial q_j \right) \right] \delta q_j$

$$\frac{\partial \vec{r}_i}{\partial q} \equiv \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

Thus, ② $\equiv \sum \left[d/dt \left(m\vec{v}_i \cdot \partial \vec{v}_i / \partial \dot{q}_j \right) - m\vec{v}_i \cdot \partial \vec{v}_i / \partial q_j \right] \delta q_j$

$\equiv \dfrac{\partial}{\partial \dot{q}_j}\left(\tfrac{1}{2} m \vec{v}_i^2\right)$ $\qquad \equiv \dfrac{\partial}{\partial q_j}\left(\tfrac{1}{2} m \vec{v}_i^2\right)$

$$① - ② = -\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m \vec{v}_i^2 \right) \right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m \vec{v}_i^2 \right) - Q_j \right\} \delta q_j$$

$$0 = \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j \quad : \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

Now, assume :

i) All applied forces are conservative, a derivative from a potential,

$$F_i^{(a)} = -\vec{\nabla}_i \cdot V \Rightarrow Q_j = \sum_i F^{(a)} \frac{d\vec{r}_i}{dq_j} = -\sum_i (\vec{\nabla} V) \cdot \frac{d\vec{r}}{\partial q_j}$$

Then, $\dfrac{\partial V}{\partial q_i} = \sum_i \dfrac{\partial V}{\partial \vec{r}} \dfrac{\partial \vec{r}_i}{\partial q_j} + \dfrac{\partial V}{\partial t} \cancelto{0}{\dfrac{\partial t}{\partial q_i}} = -Q_j$, Also note : $\dfrac{\partial V}{\partial \dot{q}_j} = 0$

Thus, $-\dfrac{\partial V}{\partial q_j} = \dfrac{d}{dt}\left(\dfrac{\partial}{\partial \dot{q}_j}(T - V) \right) - \dfrac{\partial T}{\partial q_j}$: $0 = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_j} \right) - \dfrac{\partial L}{\partial q_j}$ where $L = T - V \Rightarrow$ Lagrangian

Assumptions we made:

① Holonomic constraint

i) $\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$

ii) $\delta q_1 \ldots\ldots \delta q_i$ are independent

② Potential is a function of $\vec{r}$ & t

③ Conservative Forces

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* d'Alembert's principle : $\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i)\,\delta \vec{r}_i = 0$: $\delta\vec{r}_i$ are not independent

* Lagrange's equation ⟶ generalized coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = 0$$

Example ①: Particle in 2D w/ no other constraints, $z = \text{const}$

Step 1 : Identify generalized coordinates, 2 coordinates : x & y

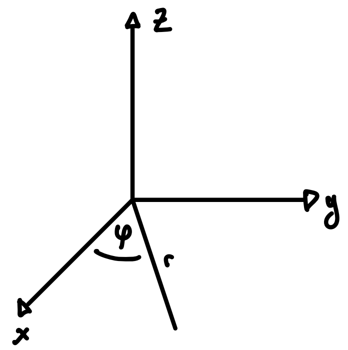
Step 2 : Lagrangian, $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

Step 3 : Generate EOM

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \;,\quad \frac{\partial T}{\partial \dot{y}} = m\dot{y} \;:\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_j \;,\quad m\ddot{x} = Q_x$$

"   " , $m\ddot{y} = Q_y$

Example ②: Bead ~~moving~~ rotating on a wire constrained to a horizontal plane.



* Rotation about z-axis

* Uniform angular velocity, $\dot{\varphi} = \omega$

* Wire imposes constraint: Rotation leads to parametrized path

Polar coordinates, $z = 0$, $x = r\cos\varphi$, $y = r\sin\varphi$ w/ $\dot{\varphi} = \omega$, $\varphi = \omega t$

Generalized coordinates : r

Kinetic Energy : $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left((\dot{r}\cos(\omega t) - r\omega\sin(\omega t))^2 + (\dot{r}\sin(\omega t) + r\omega\cos(\omega t))^2\right)$

$T = \frac{m}{2}(\dot{r}^2 + r^2\omega^2)$, $\dot{r}$ → Translational, ω → Angular : Note: $T \neq T(\dot{q}) = T(q, \dot{q})$

$L = T$ w/ $u = 0$: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$, $\frac{\partial T}{\partial r} = mr\omega^2$, $\frac{\partial T}{\partial \dot{r}} = m\dot{r}$, $\frac{d}{dt}\frac{\partial T}{\partial \dot{r}} = m\ddot{r}$ ∴ $\ddot{r} = r\omega^2$

$r(t) = Ae^{\omega t}$

One Final note: The Lagrangian is not unique

A "correct" Lagrangian satisfies, $\frac{\partial L}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$, $L'(\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t) + F(\vec{q}, \dot{\vec{q}}, t)$

If: $\frac{\partial f}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial P}{\partial \dot{q}_j}\right)$

L' generates the same equations of motion : $L' " \equiv " L$

Example : $f(\vec{q}, \dot{\vec{q}}, t) = dF/dt$ where $F = F(\vec{q}, \dot{\vec{q}}, t)$ is a continuous function w/ continuous 1st & 2nd derivatives.

<u>Check:</u> must satisfy : $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}} \frac{dF}{dt}\right) = \frac{\partial}{\partial q}\frac{dF}{dt} \equiv \frac{d}{dt}\left(\frac{\partial \dot{F}}{\partial \dot{q}}\right) = \frac{\partial \dot{F}}{\partial q} \quad \therefore \quad \frac{\partial \dot{F}}{\partial q} = \frac{\partial \dot{F}}{\partial q}$

<u>Velocity - dependent - potentials</u> (1.5)

$V = U(\vec{q}, \dot{\vec{q}}, t)$, $Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$

Then : Define Lagrangian $L = T - U$ and, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

<u>Example:</u> E & M

$\Rightarrow$ charged particle moving in an EM field

$\hookrightarrow$ Lorentz force : Depends on velocity.

<u>Details:</u> (see also Goldstein)

$L = \frac{1}{2} m \dot{r}^2 - q\Phi + q\vec{A}\cdot\dot{\vec{r}}$, $\Phi \rightarrow$ scalar potential , $\vec{A} \rightarrow$ vector potential

$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$

Lorentz Force :

$F = q(\vec{E} + \dot{\vec{r}} \times B) = q(-\nabla\Phi - \partial A/\partial t + \nabla(\dot{\vec{r}}\cdot\vec{A})$

Lagrangian not unique. Also $\rightarrow$ gauge invariance in E + M.

$\vec{A}' = \vec{A} + \nabla\psi$
$\Phi' = \Phi - \partial\psi/\partial t$

$\hookrightarrow L(\vec{A}, \Phi) \rightarrow L'(\vec{A}', \Phi')$

<u>Last Note:</u> If not all applied forces are conservative,

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}} = Q$$

$\hookrightarrow$ will include terms that involve potential V associated w/ conservative forces.

Summary of "recipe":

1. Determine how many independent coordinates we have to obtain,

   $\vec{q} = q_i, \ldots\ldots q_i$: generalized coordinates

2. Obtain expression for T & V in terms of $\vec{q}$, $\dot{\vec{q}}$ & t

3. $L = T - V$

4. Lagranges equation $\longrightarrow$ EOM