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Quantum Mechanics 1

PHYS 5393 HOMEWORK ASSIGNMENT #10

PROBLEMS: $\{3.10, 3.17, 3.20\}$

Due: November 17, 2021 By: 5 PM

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Problem 1: 3.10

Consider a sequence of Euler rotations represented by

$$\mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

$$e^{\left(\frac{-i\vec{\sigma}\cdot\hat{n}\varphi}{2}\right)} = \begin{pmatrix} \cos(\varphi/2) - in_z\sin(\varphi/2) & (-in_x - n_y)\sin(\varphi/2) \\ (-in_x + n_y)\sin(\varphi/2) & \cos(\varphi/2) + in_z\sin(\varphi/2) \end{pmatrix} \longrightarrow \quad (3.63)$$

$$\underbrace{\begin{pmatrix} \cos(\varphi/2) - in_z\sin(\varphi/2) & (-in_x - n_y)\sin(\varphi/2) \\ (-in_x + n_y)\sin(\varphi/2) & \cos(\varphi/2) + in_z\sin(\varphi/2) \end{pmatrix}}_{\hat{A}} = \underbrace{\begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}}_{\hat{B}}$$

we now proceed to take the trace of both matrices to eliminate n_x, n_y, or n_z

$$Tr(\hat{A}) = Tr(\hat{B})$$

$$Tr(\hat{A}) = \cos(\varphi/2) - \cancel{in_z\sin(\varphi/2)} + \cos(\varphi/2) + \cancel{in_z\sin(\varphi/2)} = 2\cos(\varphi/2)$$

$$Tr(\hat{B}) = e^{-i(\alpha+\gamma)/2}\cos(\beta/2) + e^{i(\alpha+\gamma)/2}\cos(\beta/2) = 2\cos((\alpha+\gamma)/2)\cos(\beta/2) \quad \therefore$$

$$\cancel{2}\cos(\varphi/2) = \cancel{2}\cos((\alpha+\gamma)/2)\cos(\beta/2), \quad \varphi/2 = \cos^{-1}(\cos((\alpha+\gamma)/2)\cos(\beta/2))$$

$$\boxed{\varphi = 2\cos^{-1}(\cos((\alpha+\gamma)/2)\cos(\beta/2))}$$

Problem 1: 3.10 Review

Procedure:

- Begin by using equation (3.63) out of the third edition of Sakurai

$$e^{\left(\frac{-i\hat{\sigma}\cdot\hat{n}\phi}{2}\right)} = \begin{pmatrix} \cos(\phi/2) - in_z \sin(\phi/2) & (-in_x - n_y)\sin(\phi/2) \\ (-in_x + n_y)\sin(\phi/2) & \cos(\phi/2) + in_z \sin(\phi/2) \end{pmatrix}$$

  and set this equal to the Euler rotations defined in the problem statement.

- Take the trace of both matrices

$$\text{Tr}(\hat{A}) = a + d \quad \text{where a and d are elements of} \quad \hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  and set them equal to one another.

- Solve for ϕ (or θ, whatever you choose) with the traces of $\mathcal{D}^{(1/2)}$ and $e^{\left(\frac{-i\hat{\sigma}\cdot\hat{n}\phi}{2}\right)}$ being set equal.

Key Concepts:

- Successive Euler angle translations produce a final matrix.
- We can take the trace of of this translation matrix and set it equal to the trace of the matrix for equation (3.63) to solve for ϕ.

Variations:

- We can be given a different translation matrix that is defined in the problem statement.
  - This would change the components for the matrix but would not change the method of which we search for ϕ.

Problem 2: 3.17

An angular-momentum eigenstate $|j, m = m_{\text{max}} = j\rangle$ is rotated by an infinitesimal angle $\mathcal{E}$ about the y-axis. Without using the explicit form of the $d^{(j)}_{m'm}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order $\mathcal{E}^2$.

The probability for this will be

$$P = |\langle jj| D_y(\varepsilon) |jj\rangle|^2$$

where we use the generic rotation equation

$$D_i(\alpha) = e^{-iJ_i\alpha/\hbar}, \quad D_y(\varepsilon) = e^{-iJ_y\varepsilon/\hbar}$$

where we know that

$$e^{-iJ_y\varepsilon/\hbar} = 1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{\hbar^2} + \ldots\ldots$$

we then have,

$$\langle jj| 1 - \frac{iJ_y\varepsilon}{\hbar} - \frac{J_y^2\varepsilon^2}{\hbar^2} |jj\rangle = \langle jj|1|jj\rangle - \frac{i\varepsilon}{\hbar}\langle jj|J_y|jj\rangle - \frac{\varepsilon^2}{\hbar^2}\langle jj|J_y^2|jj\rangle$$

we then use the relationship

$$J_\pm = J_x \pm iJ_y, \quad iJ_y = J_+ - J_x, \quad J_x = J_- + iJ_y \quad \therefore \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$\frac{i\varepsilon}{\hbar}\langle jj|J_y|jj\rangle = \frac{i\varepsilon}{\hbar}\cdot\frac{1}{2i}\langle jj|J_+ - J_-|jj\rangle = \frac{\varepsilon}{2\hbar}\left(\langle jj|\cancel{J_+}|jj\rangle - \langle jj|\cancel{J_-}|jj\rangle\right) = 0$$

$$\frac{\varepsilon^2}{\hbar^2}\frac{1}{4i^2}\langle jj|J_+^2 - J_+J_- - J_-J_+ + J_-^2|jj\rangle = -\frac{\varepsilon^2}{\hbar^2}\frac{1}{4}\left(-\langle jj|\cancel{J_-}J_+|jj\rangle - \langle jj|\cancel{J_+}J_-|jj\rangle\right) = \frac{\varepsilon^2 j}{2\hbar^2}$$

Therefore the probability will be,

$$\boxed{P = \left(1 - \varepsilon^2 j/2\hbar^2\right)^2}$$

Problem 2: 3.17 Review

Procedure:

- To search for the probability we use the equation
$$\mathcal{P} = |\langle\alpha|\tilde{\mathbf{A}}|\alpha\rangle|^2$$
where $|\alpha\rangle \equiv |jj\rangle$ is our state and $\tilde{\mathbf{A}} \equiv \mathcal{D}_y(\epsilon)$ is our rotation by ϵ in the y direction.
- Start by calculating the expectation value for $\mathcal{D}_y(\epsilon)$, and then square that result.
- This rotation operator can be represented as an exponential,
$$\mathcal{D}_i(\theta) = e^{-i\tilde{\mathbf{J}}_i\theta/\hbar}.$$
- Expand out the exponential by using a Taylor Series, and only keep terms to second order.
- Proceed to evaluate the expectation value and solve for $\tilde{\mathbf{J}}_y$ in terms of $\tilde{\mathbf{J}}_+$ and $\tilde{\mathbf{J}}_-$
$$\tilde{\mathbf{J}}_\pm = \tilde{\mathbf{J}}_x \pm i\tilde{\mathbf{J}}_y \quad \rightarrow \quad \tilde{\mathbf{J}}_y = \frac{1}{2i}(\tilde{\mathbf{J}}_+ - \tilde{\mathbf{J}}_-).$$
- Evaluate the expectation value, square it, and only keep values of $\mathcal{O}(2)$.

Key Concepts:

- We can express the rotation of an angular momentum eigenstate in terms of an exponential.
- This exponential can be expanded using a Taylor Series, where we only keep second order terms $\mathcal{O}(2)$.
- We have to solve for $\tilde{\mathbf{J}}_y$ in terms of $\tilde{\mathbf{J}}_+$ and $\tilde{\mathbf{J}}_-$ because we do not explicitly now how $\tilde{\mathbf{J}}_y$ acts on the eigenstates of angular momentum.
- We can proceed to calculate the probability of this rotation, and only keeping second order terms $\mathcal{O}(2)$.

Variations:

- We can be given a rotation that is not in the same direction (y).
  - We then would have an expansion of this operator in the x direction instead of the y and would have to solve for $\tilde{\mathbf{J}}_x$ in terms of $\tilde{\mathbf{J}}_+$ and $\tilde{\mathbf{J}}_-$ instead.
- We could have a different value for m in the momentum eigenstate,
  - This would change the value of the eigenvalues for $\tilde{\mathbf{J}}_+$ and $\tilde{\mathbf{J}}_-$ when acting on the eigenstates.

Problem 3: 3.20

Construct the matrix representations of the operators J_x and J_y for a spin 1 system, in the J_z basis, spanned by the kets $|+\rangle \equiv |1,1\rangle, |0\rangle \equiv |1,0\rangle$, and $|-\rangle \equiv |1,-1\rangle$. Use these matrices to find the three analogous eigenstates for each of the two operators J_x and J_y in terms of $|+\rangle, |0\rangle$, and $|-\rangle$.

First, solve for J_x and J_y in terms of J_+ & J_-

$$J_x = J_\pm \mp iJ_y \,,\quad J_x = J_+ - iJ_y \,,\quad J_x = J_- + iJ_y \,,\quad iJ_y = J_+ - J_x \,,\quad iJ_y = J_x - J_-$$

$$J_+ - J_x = J_x - J_- \quad \therefore \quad J_x = \frac{1}{2}(J_+ + J_-) \quad : \quad J_y = \frac{1}{2i}(J_+ - J_-)$$

$$J_\alpha = \sum_i \sum_j |i\rangle\langle i| J_\alpha |j\rangle\langle j| \,, \quad \text{where} \quad i \in [-1,1] \quad \text{and} \quad j \in [-1,1]$$

$$J_\alpha = |+\rangle\langle +|J_\alpha|+\rangle\langle +| + |+\rangle\langle +|J_\alpha|0\rangle\langle 0| + |+\rangle\langle +|J_\alpha|-\rangle\langle -| + |0\rangle\langle 0|J_\alpha|+\rangle\langle +| +$$
$$|0\rangle\langle 0|J_\alpha|0\rangle\langle 0| + |0\rangle\langle 0|J_\alpha|-\rangle\langle -| + |-\rangle\langle -|J_\alpha|+\rangle\langle +| + |-\rangle\langle -|J_\alpha|0\rangle\langle 0| + |-\rangle\langle -|J_\alpha|-\rangle\langle -|$$

$$J_+|j,m\rangle = \alpha|j,m+1\rangle \,,\quad J_-|j,m\rangle = \beta|j,m-1\rangle \;:\; \alpha = \hbar\sqrt{(j-m)(j+m+1)} \,,\quad \beta = \hbar\sqrt{(j+m)(j-m+1)}$$

$$J_x = |+\rangle\langle +|\overset{0}{\cancel{J_x}}|+\rangle\langle +| + |+\rangle\langle +|\overset{\sqrt{2}/2}{\cancel{J_x}}|0\rangle\langle 0| + |+\rangle\langle +|\overset{0}{\cancel{J_x}}|-\rangle\langle -| + |0\rangle\langle 0|\overset{\sqrt{2}/2}{\cancel{J_x}}|+\rangle\langle +| +$$
$$|0\rangle\langle 0|\overset{0}{\cancel{J_x}}|0\rangle\langle 0| + |0\rangle\langle 0|\overset{\sqrt{2}/2}{\cancel{J_x}}|-\rangle\langle -| + |-\rangle\langle -|\overset{0}{\cancel{J_x}}|+\rangle\langle +| + |-\rangle\langle -|\overset{\sqrt{2}/2}{\cancel{J_x}}|0\rangle\langle 0| + |-\rangle\langle -|\overset{0}{\cancel{J_x}}|-\rangle\langle -|$$

$$J_y = |+\rangle\langle +|\overset{0}{\cancel{J_y}}|+\rangle\langle +| + |+\rangle\langle +|\overset{-i\sqrt{2}/2}{\cancel{J_y}}|0\rangle\langle 0| + |+\rangle\langle +|\overset{0}{\cancel{J_y}}|-\rangle\langle -| + |0\rangle\langle 0|\overset{i\sqrt{2}/2}{\cancel{J_y}}|+\rangle\langle +| +$$
$$|0\rangle\langle 0|\overset{0}{\cancel{J_y}}|0\rangle\langle 0| + |0\rangle\langle 0|\overset{-i\sqrt{2}/2}{\cancel{J_y}}|-\rangle\langle -| + |-\rangle\langle -|\overset{0}{\cancel{J_y}}|+\rangle\langle +| + |-\rangle\langle -|\overset{i\sqrt{2}/2}{\cancel{J_y}}|0\rangle\langle 0| + |-\rangle\langle -|\overset{0}{\cancel{J_y}}|-\rangle\langle -|$$

$$\boxed{J_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \,,\quad J_y \doteq \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}$$

we then use the following equations to find the eigenstates:

$$\tilde{J}_x|1,m\rangle = m|1,m\rangle \,,\quad \tilde{J}_y|1,m\rangle = m|1,m\rangle$$

The eigenvectors are then:

$$\boxed{J_x = \begin{cases} |1,1\rangle = \frac{1}{2}|+\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle \\ |1,0\rangle = -\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ |1,-1\rangle = \frac{1}{2}|+\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle \end{cases} \qquad J_y = \begin{cases} |1,1\rangle = -\frac{1}{2}|+\rangle - \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \\ |1,-1\rangle = -\frac{1}{2}|+\rangle + \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle \end{cases}}$$

Problem 3: 3.20 Review

Procedure:

- We first solve for $\tilde{\mathbf{J}}_x$ and $\tilde{\mathbf{J}}_y$ in terms of $\tilde{\mathbf{J}}_+$ and $\tilde{\mathbf{J}}_-$

$$\tilde{\mathbf{J}}_\pm = \tilde{\mathbf{J}}_x \pm i\tilde{\mathbf{J}}_y \quad \rightarrow \quad \tilde{\mathbf{J}}_x = \frac{1}{2}(\tilde{\mathbf{J}}_+ + \tilde{\mathbf{J}}_-) \quad , \quad \tilde{\mathbf{J}}_y = \frac{1}{2i}(\tilde{\mathbf{J}}_+ - \tilde{\mathbf{J}}_-).$$

- To represent the operator $\tilde{\mathbf{J}}_{x,y}$ as a matrix, expand in a complete set.
- Use the eigenvalue relationships for the raising and lowering operators

$$\tilde{\mathbf{J}}_+ |j,m\rangle = \alpha |j,m+1\rangle \quad \rightarrow \quad \alpha = \hbar\sqrt{(j-m)(j+m+1)}, \quad \tilde{\mathbf{J}}_- |j,m\rangle = \beta |j,m-1\rangle \quad \rightarrow \quad \beta = \hbar\sqrt{(j+m)(j-m+1)}.$$

- Use the raising and lowering operators, their respective eigenvalues, and determine the matrix representation of both $\tilde{\mathbf{J}}_x$ and $\tilde{\mathbf{J}}_y$.
- Once the matrix representation is calculated, find the eigenvalues and their respective eigenstates with

$$\tilde{\mathbf{J}}_{x,y} |1,m\rangle = m |1,m\rangle ,$$

the standard eigenvalue / eigenvector equation.

Key Concepts:

- We cannot find the matrix representations for $\tilde{\mathbf{J}}_x$ and $\tilde{\mathbf{J}}_y$ explicitly, we have to find them in terms of the ladder operators.
- The eigenvalues for the ladder operators can be determined by knowing the initial values of j and m.
- We cannot raise a state past $m = 1$ or lower a state below $m = -1$.
- The eigenstates of angular momentum can be represented as

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |-\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- The eigenvalues for angular momentum are $-1, 0, 1$.
- We can represent any size dimension of a matrix by expanding in a complete set.

Variations:

- There isn't much that can be changed about this problem, we could be asked to represent this operator without knowing the eigenstates.
  - In that case we would have to prove the operator is a ladder operator and then show how it affects a state.
- We could be asked for the z direction representation.
  - In this case we wouldn't have to use the ladder operators since we know how $\tilde{\mathbf{J}}_z$ acts on the eigenstates.