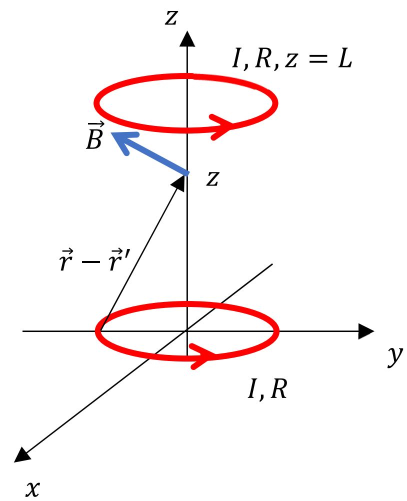
E & M I

Workshop 6 – Helmholtz Coils, Solution

We have done several calculations of magnetic fields, but we haven't really applied our results. That's the goal of today's workshop.



1) Helmholtz coils are used to create reasonably uniform magnetic fields in labs, or sometimes to cancel out external unwanted fields. This consists of two current-carrying loops (or multiple loops), parallel to each other and centered on the same axis.

In this case, consider loop of radius R in the x-y plane, centered at the origin, and carrying a current I, and a second, identical loop centered on the z-axis at $z = L$.

a) Solve for (or write down) an expression for the total magnetic field of the two loops on the z-axis, $B_z(z)$, for $0 \leq z \leq L$.

Considering the field due to one ring, the one at the origin, the symmetry of the problem indicates the field on the z-axis will be in the z-direction.

Considering the magnetic field due to the point on the loop shown, we have:

$$d\vec{l} = R\, d\phi\, \hat{x}, \qquad \vec{r} - \vec{r}' = z\, \hat{z} + R\, \hat{y}, \qquad |\vec{r} - \vec{r}'| = \sqrt{R^2 + z^2}$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{R\, d\phi}{(R^2 + z^2)^{3/2}} \left(\hat{x} \times (z\, \hat{z} + R\, \hat{y})\right)$$

The z-component is:

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{R^2\, d\phi}{(R^2 + z^2)^{3/2}}\, \hat{z}$$

Integrating over $d\phi$ just gives a factor of 2π because the integrand doesn't depend on ϕ. This gives:

$$B_1(z) = \frac{\mu_0}{2\pi} \frac{I\, \pi R^2}{(R^2 + z^2)^{\frac{3}{2}}}\, \hat{z} = \frac{\mu_0}{2\pi} \frac{\vec{m}}{(R^2 + z^2)^{\frac{3}{2}}}, \qquad \vec{m} = I\, \pi R^2\, \hat{z}$$

The field just depends on the distance from the center of the coil. This means the total field due to the two coils is:

$$B_T(z) = \frac{\mu_0\, \vec{m}}{2\pi} \left(\frac{1}{(R^2 + z^2)^{\frac{3}{2}}} + \frac{1}{(R^2 + (L - z)^2)^{\frac{3}{2}}} \right)$$

b) For experiments a uniform field is sometimes desired. Solve for the value of z where $\partial_z B_z = 0$. The answer should be obvious but show how you got your answer and explain the results.

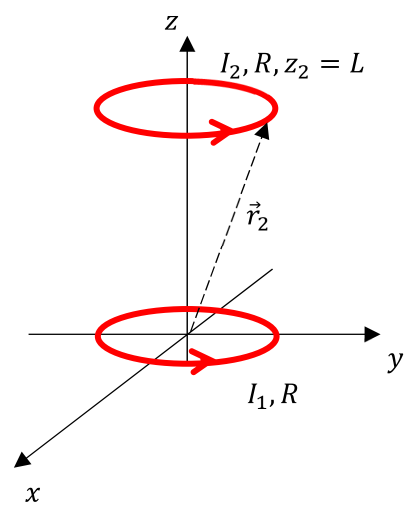
$$\partial_z B_T(z) = \frac{\mu_0 \vec{m}}{2\pi}\left(-\frac{3}{2}\right)\left(\frac{2z}{(R^2+z^2)^{\frac{5}{2}}} - \frac{2(L-z)}{(R^2+(L-z)^2)^{\frac{5}{2}}}\right)$$

$$\partial_z B_T(z) = 0 \Rightarrow \frac{z}{(R^2+z^2)^{\frac{5}{2}}} = \frac{(L-z)}{(R^2+(L-z)^2)^{\frac{5}{2}}}$$

This is obviously true if $z = \frac{L}{2}$.

Again, the symmetry of the problem indicates this must be the case. The coils are identical so there's no reason the maximum would be closer to one than the other.

2) In designing the Helmholtz coil pair, we need to be careful that the structure is mechanically stable. The two coils will exert a magnetic force on each other. Here we'll approximate the magnetic force on the top ring due to the magnetic field of the bottom ring. We'll distinguish the rings by labeling the bottom ring "1" and the top ring "2".



a) Write an integral that will give the force on the top ring, current I_2 and positions of the ring $\vec{r}_2$, due to the magnetic field of the bottom ring, $\vec{B}_1$.

Your answer should include something that looks like $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$

For the integral around the top coil, $d\vec{l}_2 = R\, d\phi_2\, \hat{\phi}_2$

$$\vec{F}_2 = I_2 \oint d\vec{l}_2 \times \vec{B}_1(\vec{r}_2) = I_2\, R \oint d\phi_2\, \hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$$

Doing this integral exactly is rather messy, so let's make the approximation that the field due to the bottom ring can be replaced by the magnetic dipole approximation. Remember that the field and vector potential of a magnetic dipole are:

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1\cdot\hat{r}_2)\,\hat{r}_2}{r_2^3} - \frac{\vec{m}_1}{r_2^3}\right), \quad \vec{A}_1(\vec{r}_2) = \frac{\mu_0}{4\pi}\frac{\vec{m}_1\times\hat{r}_2}{r_2^2}, \quad \vec{m}_1 = I_1\,\pi R^2\,\hat{z}$$

b) As we have seen in similar problems, we need to be careful with the vector nature of our integrand when doing integrals such as in part (a). This is due, in part, to the fact that the directions of spherical coordinates $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ all change for different positions $\vec{r}$. There are a couple of ways to handle this.

Consider the dipole field from the bottom coil:

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\ (3\ (\hat{z}\cdot\hat{r}_2)\hat{r}_2 - \hat{z}) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\ (3\cos\theta_2\ \hat{r}_2 - \hat{z})$$

We'll use:

$$\hat{r}_2 = \cos\phi_2 \sin\theta_2\ \hat{x} + \sin\phi_2 \sin\theta_2\ \hat{y} + \cos\theta_2\ \hat{z}$$

$$\hat{\theta}_2 = \cos\phi_2 \cos\theta_2\ \hat{x} + \sin\phi_2 \cos\theta_2\ \hat{y} - \sin\theta_2\ \hat{z}$$

$$\hat{\phi}_2 = -\sin\phi_2\ \hat{x} + \cos\phi_2\ \hat{y}$$

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\ (\,3\cos\theta_2 \sin\theta_2\ (\cos\phi_2\ \hat{x} + \sin\phi_2\ \hat{y}) + (3\cos^2\theta_2 - 1)\ \hat{z})$$

(i) Write the cross product, $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$ in terms of the Cartesian unit vectors of $\hat{x}, \hat{y},$ and $\hat{z}$ and do the cross product. This should be a function of r_2, θ_2, ϕ_2.

Using Cartesian Coordinates:

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\ (\,3\cos\theta_2 \sin\theta_2\ (\cos\phi_2\ \hat{x} + \sin\phi_2\ \hat{y}) + (3\cos^2\theta_2 - 1)\ \hat{z})$$

$$\begin{aligned}\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\ &(3\cos\theta_2 \sin\theta_2\ (-\cos^2\phi_2 - \sin^2\phi_2)\ \hat{z} \\ &+ (3\cos^2\theta_2 - 1)\ (\cos\phi_2\ \hat{x} + \sin\phi_2\ \hat{y})\end{aligned}$$

$$\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2) = -\frac{\mu_0\, m_1}{4\pi\, r_2^3}\left(3\cos\theta_2 \sin\theta_2\ \hat{z} + (1 - 3\cos^2\theta_2)\ (\cos\phi_2\ \hat{x} + \sin\phi_2\ \hat{y})\right)$$

(ii) Write the cross product, $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$ in terms of the spherical unit vectors of $\hat{r}_2, \hat{\theta}_2,$ and $\hat{\phi}_2$ and do the cross product. This again should be a function of r_2, θ_2, ϕ_2.

Using:

$$\hat{z} = \cos\theta_2\ \hat{r}_2 - \sin\theta_2\ \hat{\theta}_2$$

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\left(2\cos\theta_2\ \hat{r}_2 + \sin\theta_2\ \hat{\theta}_2\right)$$

$$\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\left(2\cos\theta_2\ \hat{\theta}_2 + \sin\theta_2\ (-\hat{r}_2)\right)$$

Re-writing this in Cartesian coordinates to check answers and to do the integral:

$$\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2) = \frac{\mu_0\, m_1}{4\pi\, r_2^3}\left((2\cos^2\theta_2 - \sin^2\theta_2)(\cos\phi_2\, \hat{x} + \sin\phi_2\, \hat{y}) - 3\cos\theta_2 \sin\theta_2\, \hat{z}\right)$$

But:

$$2\cos^2\theta_2 - \sin^2\theta_2 = 3\cos^2\theta_2 - 1$$

Giving the same result as above.

c) Solve for the force on the top loop and show that it will only have a z-component. You should be able to use either of the expressions for the integrand found in part (b).

Doing the integral, we note that:

$$\oint d\phi \cos\phi = \oint d\phi \sin\phi = 0$$

So

$$\vec{F}_2 = -\frac{\mu_0\, m_1}{4\pi\, r_2^3} \oint d\phi_2\, 3\cos\theta_2 \sin\theta_2\ \hat{z} = -\frac{\mu_0}{2}\frac{I_2\, R\, m_1}{r_2^3}\ 3\cos\theta_2 \sin\theta_2\ \hat{z}$$

Considering the picture above:

$$r_2 = \sqrt{L^2 + R^2}, \qquad \cos\theta_2 = \frac{L}{r_2}, \qquad \sin\theta_2 = \frac{R}{r_2}$$

$$\vec{F}_2 = -\frac{3}{2}\, m_1\, I_2 R^2\, \frac{L}{r_2^5}\ \hat{z} = -\frac{3\,\mu_0}{2\,\pi}\, m_1\, m_2 \frac{L}{(L^2+R^2)^{5/2}}\ \hat{z}$$

$$m_2 = I_2\, \pi R^2$$

d) An even simpler model for this problem is to treat BOTH loops as dipoles. In this case, the potential energy due to the interaction between the dipoles is:

$$U = -\vec{m}_2 \cdot \vec{B}_1(\vec{r}_2)$$

Using this approximation, explain the physics behind the direction of the force found in part (c) makes physical sense.

$$U = -\frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1\cdot\hat{r}_2)(\vec{m}_2\cdot\hat{r}_2)}{r_2^3} - \frac{\vec{m}_1\cdot\vec{m}_2}{r_2^3}\right)$$

In the point-dipole approximation here, $\vec{m}_1$, $\vec{m}_2$, and $\hat{r}_2$ are all in the z-direction, and $r_2 = L$ giving:

$$U = -\frac{\mu_0}{2\pi\, L^3}\, m_1\, m_2$$

We can lower the energy by making L smaller, meaning the force will be attractive. This is also not too surprising if we consider that the magnetic force between two long straight wires with the currents in the same direction is attractive.

The dipole-dipole force will be:

$$\vec{F}_{dd} = -\nabla U = -\partial_L\, U\, \hat{z} = -\frac{3}{2\pi}\frac{\mu_0\, \vec{m}_1\, \vec{m}_2}{L^4}\ \hat{z}$$

From above:

$$\vec{F}_2 = -\frac{3\,\mu_0}{2\,\pi}\, m_1\, m_2 \frac{L}{(L^2+R^2)^{\frac{5}{2}}}\ \hat{z} \rightarrow -\frac{3\,\mu_0}{2\,\pi}\frac{m_1\, m_2}{L^4}\ \hat{z}\ L \gg R$$

3) It is also of interest to understand the energy of the Helmholtz coil pair. We have found a couple of different ways to calculate the magnetic energy, including the interaction between a current density and vector potential:

$$W_{21} = \iiint \vec{J_2}(\vec{r}_2) \cdot \vec{A}_1(\vec{r}_2)\, d^3 r_2$$

Note: This is the energy the top loop current due to the bottom loop, so there isn't a factor of $\frac{1}{2}$ out front which is there to avoid double-counting interactions.

Using the same dipole approximation as above for the bottom loop, calculate the interaction potential energy above.

First consider the vector potential from the bottom loop, approximated as a magnetic dipole:

$$, \quad \vec{A}_1(\vec{r}_2) = \frac{\mu_0}{4\pi} \frac{\vec{m}_1 \times \hat{r}_2}{r_2^2} = \frac{\mu_0\, m_1}{4\pi\, r_2^2}\, \hat{z} \times \hat{r}_2$$

From what we've done above:

$$\hat{z} = \cos\theta_2\ \hat{r}_2 - \sin\theta_2\ \hat{\theta}_2$$

$$\hat{z} \times \hat{r}_2 = -\sin\theta_2\ \hat{\theta}_2 \times \hat{r}_2 = \sin\theta_2\ \hat{\phi}_2$$

The current density can be written:

$$\vec{J}_2(\vec{r}_2) = I_2\, \hat{\phi}_2\, \delta\big(r_2 - (L^2 + R^2)\big) \frac{\delta\left(\theta_2 - \tan^{-1}\left(\frac{R}{L}\right)\right)}{r_2}$$

Giving:

$$W_{21} = \frac{\mu_0\, m_1\, I_2}{4\pi} \iiint d\phi_2 \sin\theta_2\, d\theta_2\, r_2^2\, dr_2\ \delta\big(r_2 - (L^2 + R^2)\big) \frac{\delta\left(\theta_2 - \tan^{-1}\left(\frac{R}{L}\right)\right)}{r_2} \frac{\sin\theta_2}{r_2^2}$$

$$W_{21} = \frac{\mu_0\, m_1\, I_2}{4\pi} \frac{2\pi}{\sqrt{L^2 + R^2}} \sin^2\left(\tan^{-1}\left(\frac{R}{L}\right)\right) = \frac{\mu_0\, m_1\, I_2}{2\pi} \frac{\pi}{\sqrt{L^2 + R^2}} \frac{R^2}{L^2 + R^2}$$

But $m_2 = I_2\, \pi\, R^2$

$$W_{21} = \frac{\mu_0\, m_1\, m_2}{2\pi} \frac{1}{(L^2 + R^2)^{3/2}}$$

In the limit $L \gg R$ this gives:

$$W_{21} = \frac{\mu_0}{2\pi\, L^3}\, m_1\, m_2 = -U$$

Note: I changed notation here from what was handed out in class. Going back to the derivations, what is here called W_{21} is the work needed to bring the loops together, not the potential energy.