Assignment 4, Problem 1:

(a) all the magnetic moments are independent

$$\Rightarrow Q_N = (Q_1)^N$$

$$Q_1 = \sum_{\sigma=\pm 1} e^{-\beta \mu B \sigma} = e^{-\beta \mu B} + e^{\beta \mu B}$$

$$= 2 \cosh(\beta \mu B)$$

$$\Rightarrow Q_N = \left(2 \cosh(\beta \mu B)\right)^N$$

$$-\frac{\partial}{\partial \beta} \log Q_N = -N \frac{\sinh(\beta \mu B)}{\cosh(\beta \mu B)} \mu B$$

$$\boxed{\langle U \rangle = -N \mu B \tanh(\beta \mu B)}$$

(b)

$$S = -\left(\frac{\partial A}{\partial T}\right)_{V,N} \quad \text{and} \quad A = -\frac{1}{\beta} \log Q_N = -kT \log Q_N$$

$$\Rightarrow S = -\frac{\partial}{\partial T}\left(-kT \log Q_N\right) = k \frac{\partial}{\partial T}\left(T \log Q_N\right)$$

$$= k \log Q_N + kT \frac{\partial}{\partial T} \log Q_N$$

$$= \underbrace{k \log Q_N}_{kN \log(2\cosh(\beta\mu B))} + kT \underbrace{\frac{\partial \beta}{\partial T}}_{-\frac{1}{kT^2}} \underbrace{\frac{\partial}{\partial \beta} \log Q_N}_{N\mu B \tanh(\beta\mu B)}$$

So: $$\boxed{S = kN\left[\log\left(2\cosh(\beta\mu B)\right) - \frac{\mu B}{kT} \tanh(\beta\mu B)\right]}$$

Or: $$\boxed{S = kN\left[\log\left(2\cosh(\beta\mu B)\right) - \beta\mu B \tanh(\beta\mu B)\right]}$$

Assignment 4, Problem 2:

(a) We want to calculate the specific heat C_V.

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \quad \text{and} \quad U = -\frac{\partial}{\partial \beta} \log Q_N$$

$$Q_N = \frac{1}{N!\, h^N} \underbrace{\left(\int e^{-\beta \frac{p^2}{2m}}\, dp\right)^N}_{\sqrt{\frac{2m\pi}{\beta}}} \cdot \underbrace{\left(\int e^{-\beta \varepsilon_0 \left|\frac{x}{a}\right|^n}\, dx\right)^N}_{\int e^{-\beta \varepsilon_0 a^{-n} |x|^n}\, dx}$$

let $z = \beta \varepsilon_0 a^{-n} x^n \rightsquigarrow x = \left(\frac{z}{\beta \varepsilon_0 a^{-n}}\right)^{1/n}$

$$\Rightarrow dz = n\,(x^{n-1})\, \beta \varepsilon_0 a^{-n}\, dx$$

or $$\frac{a^n}{\beta \varepsilon_0 n} \underbrace{x^{-n+1}}_{= \left(\frac{z}{\beta \varepsilon_0 a^{-n}}\right)^{\frac{-n+1}{n}}} dz = dx$$

$$\Rightarrow dx = \frac{a^n}{\beta \varepsilon_0 n} \frac{a^{1-n}}{(\beta \varepsilon_0)^{(1-n)/n}} \; z^{(1-n)/n} \; dz$$

$$= \frac{a}{n\,(\beta \varepsilon_0)^{1/n}} \; z^{(1-n)/n} \, dz$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\beta \varepsilon_0 a^{-n} |x|^n} dx = 2 \frac{a}{n(\beta \varepsilon_0)^{1/n}} \underbrace{\int_0^{\infty} e^{-z} \mathbf{z}^{\frac{1-n}{n}} dz}_{\Gamma\left(\frac{1}{n}\right)}$$

the factor of 2 comes from changing the limits from $-\infty, \infty$ to $0, \infty$

$$\Rightarrow Q_N = \frac{1}{N!\,h^N} \left(\frac{2m\pi}{\beta}\right)^{N/2} \left(\frac{2a}{n(\beta\varepsilon_0)^{1/n}}\right)^N \left(\Gamma\left(\frac{1}{n}\right)\right)^N$$

factor of $\beta^{-N/2} \cdot \beta^{-N/n} = \beta^{(-nN-2N)/2n}$

$= \beta^{-N\left(\frac{1}{2}+\frac{1}{n}\right)}$

$$\frac{\partial}{\partial \beta} \log Q_N = -N\left(\frac{1}{2}+\frac{1}{n}\right)\frac{1}{\beta}$$

$$\Rightarrow \langle \mathcal{H} \rangle = U = N\left(\frac{1}{2}+\frac{1}{n}\right) kT$$

$$\Rightarrow \boxed{C_V = \left(\frac{\partial U}{\partial T}\right)_V = Nk\left(\frac{1}{2}+\frac{1}{n}\right)}$$

The power of x enters into the proportionality constant. If the experimentalist measures C_V/N, then the power of n should be visible as deviation from $\frac{1}{2}$.

(b)

$$\text{We need } \int e^{-\beta \varepsilon_0 a^{-n} \rho^n} \rho \, d\rho$$

φ integration will give a factor of 2π

$$\text{let } z = +\beta \varepsilon_0 a^{-n} \rho^n \rightsquigarrow \left(\frac{z}{\beta \varepsilon_0 a^{-n}}\right)^{1/n} = \rho$$

$$\Rightarrow dz = \beta \varepsilon_0 a^{-n} n \rho^{n-1} d\rho$$

$$\Rightarrow \rho \, d\rho = \underbrace{\rho^{2-n}}_{\left(\frac{z a^n}{\beta \varepsilon_0}\right)^{\frac{2-n}{n}}} \frac{a^n}{\beta \varepsilon_0 n} dz$$

$$\Rightarrow \rho \, d\rho = \frac{a^2}{n (\beta \varepsilon_0)^{2/n}} z^{\frac{2-n}{n}} dz$$

$$\Rightarrow \int_0^\infty e^{-\beta \varepsilon_0 a^{-n} \rho^n} \rho \, d\rho = \frac{a^2}{n(\beta\varepsilon_0)^{2/n}} \Gamma\left(\frac{2}{n}\right)$$

$$\Rightarrow Q_N = \frac{1}{N! h^{2N}} \left(\frac{2m\pi}{\beta}\right)^N (2\pi)^N \left(\frac{a^2}{n(\beta\varepsilon_0)^{2/n}} \Gamma\left(\frac{2}{n}\right)\right)^N$$

scales as $\beta^{-N} \beta^{-2N/n}$

$\sim \beta^{-N(1+\frac{2}{n})}$

$$\Rightarrow \frac{\partial}{\partial\beta} \log Q_N = -N\left(1+\frac{2}{n}\right)\frac{1}{\beta} = -N\left(1+\frac{2}{n}\right) kT$$

and $U = N\left(1+\frac{2}{n}\right) kT = 2N\left(\frac{1}{2}+\frac{1}{n}\right) kT$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \underbrace{2N\left(\frac{1}{2}+\frac{1}{n}\right)}_{\text{two times the 1D result}}$$

Assignment 4, Problem 3:

$$\mathcal{H} = \sum_{i=1}^{N} c\,|\vec{p}_i|$$

Let's work in the canonical ensemble:

Pressure $P = -\left(\frac{\partial A}{\partial V}\right)_T$

Internal energy $\boxed{U = -\frac{\partial}{\partial \beta} \log Q_N}$

We have $Q_N = e^{-\beta A} \Rightarrow -\beta A = \log Q_N$

$$\Rightarrow A = -\frac{1}{\beta} \log Q_N$$

$\Rightarrow$ Pressure $\boxed{P = +\frac{1}{\beta} \frac{\partial}{\partial V} \log Q_N}$

So, let's calculate the partition function Q_N:

$$Q_N = \frac{1}{N!\,h^{3N}} \int e^{-c \sum_{i=1}^{N} |\vec{p}_i|} \, d^{3N}p \, d^{3N}q$$

$$= \frac{V^N}{N!\,h^{3N}} \underbrace{\left(\int e^{-c|\vec{p}\,|} \, d^3p \right)^N}$$

the integration over $\vec{q}$ can be done readily

we have N identical integrals over three-dimensional vector $\vec{p}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta c|\vec{p}|}\, d^3p = 4\pi\int_0^{\infty} e^{-\beta c|\vec{p}|}\, p^2\, dp$$

let $\beta c p = x \Rightarrow \beta c\, dp = dx$

$$p^2 = \frac{x^2}{\beta^2 c^2}$$

$$= \frac{1}{\beta c^3} 4\pi \underbrace{\int_0^{\infty} e^{-x} x^2\, dx}_{\substack{T(3) = 2 \\ \text{(integral table)}}} = \frac{4\pi}{\beta^3 c^3}$$

Thus: $Q_N = \dfrac{V^N}{N!\, h^{3N}} (4\pi)^N \dfrac{1}{c^{3N}} \dfrac{1}{\beta^{3N}}$

$\dfrac{\partial}{\partial V} \log Q_N = \dfrac{N}{V} \Rightarrow P = \dfrac{1}{\beta}\dfrac{N}{V} = kT\dfrac{N}{V} = kTn$

density

$-\dfrac{\partial}{\partial \beta} \log Q_N = +3N\dfrac{1}{\beta} = +3NkT$

$\Rightarrow \dfrac{E}{N} = +3kT$

Assignment 4, Problem 4:

a) $E_{sp} = \left(n + \frac{1}{2}\right)\hbar\omega$, where $n = 0, 1, 2, \ldots$

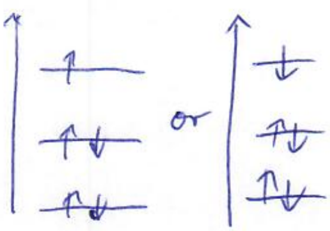
$k = m\omega^2$ or $\omega = \sqrt{\frac{k}{m}}$

(i) spin-0 bosons: $E = 5 \cdot \frac{1}{2}\hbar\omega = \frac{5}{2}\hbar\omega$

degeneracy = 1

(ii) spin-$\frac{1}{2}$ fermions: $E = \left(2\cdot\frac{1}{2} + 2\cdot\frac{3}{2} + \frac{5}{2}\right)\hbar\omega$

$= \left(1 + 3 + \frac{5}{2}\right)\hbar\omega = 6.5\,\hbar\omega$



degeneracy = 2

(iii) spin-$\frac{1}{2}$ bosons:

$E = 5 \cdot \frac{1}{2}\hbar\omega = \frac{5}{2}\hbar\omega$

degeneracy = 6

or ↑↑↓↓↓

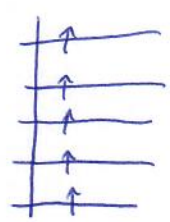
or ↑↓↓↓↓

or ↓↓↓↓↓

or ↑↑↑↑↓

or ↑↑↑↑↑

(iv) spin-0



$E = \left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \frac{9}{2}\right)\hbar\omega = \frac{25}{2}\hbar\omega$

$= 12.5\,\hbar\omega$

degeneracy = 1

(v) spin-$\frac{5}{2}$ fermions: $\quad E = \frac{5}{2}\hbar\omega$

$$\underbrace{-\tfrac{5}{2}, -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}}_{\text{6 different spin projections}}$$

degeneracy = 6

(there are 6 different ways to choose 5 different objects out of 6)

b) $E_{sp} = (n_x + n_y + 1)\hbar\omega$, where $n_x = 0, 1, \ldots$
$n_y = 0, 1, \ldots$

(i) $\underbrace{E = 5\hbar\omega}_{n_x = n_y = 0}$ and deg = 1

(ii)

$\uparrow\downarrow \quad \uparrow \quad (0,1), (1,0)$
$\uparrow\downarrow \quad (0,0)$

$E = (1+1+2+2+2)\hbar\omega = 8\hbar\omega$

degeneracy = 4

or $\uparrow\downarrow \quad \downarrow$ / $\uparrow\downarrow$ or $\uparrow \quad \uparrow\downarrow$ / $\uparrow\downarrow$ or $\downarrow \quad \uparrow\downarrow$ / $\uparrow\downarrow$

(iii) $E = 5\hbar\omega$, deg = 6 } as in a)

(iv) $E = (1 + 2\cdot 2 + 2\cdot 3)\hbar\omega = 11\hbar\omega$, degeneracy = 3

$\uparrow \; \uparrow \; — \qquad \uparrow \; — \; \uparrow \qquad — \; \uparrow \; \uparrow \quad (2,0), (0,2), (1,1)$
$\uparrow \; \uparrow \qquad$ or $\qquad \uparrow \; \uparrow \qquad$ or $\qquad \uparrow \; \uparrow \quad (0,1), (1,0)$
$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \quad (0,0) = (n_x, n_y)$

(v) $E = 5\hbar\omega$ and deg. = 6

(c) possible: (i), (ii), (v)
impossible: (iii), (iv)

} bosons: integer spin
fermions: half-integer spin