E & M I

Workshop 2 – Multipole Expansions, Solutions

Last week in class (zoom) and the reading we looked at the concept of multipole expansions of the electric potential in regions outside of a charge distribution. This is, basically, expanding the potential in powers of r, the distance from the charges to the point where the potential is being calculated.

We can write the potential as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_q \frac{q}{\left|\vec{r}-\vec{r}_q\right|}\,, \qquad \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r_q\,\frac{\rho(\vec{r}_q)}{\left|\vec{r}-\vec{r}_q\right|}$$

The first being for point charges, the second being for continuous charge distributions.

The important result here is that:

$$\nabla^2\,\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = 4\,\pi\,\delta^3(\vec{r}-\vec{r}_q)$$

Doing the series expansion for $r \gg r_q$ gives:

$$\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = \sum_n (-1)^n\,\left(\vec{r}_q\cdot\nabla\right)^n\frac{1}{r}$$

The first three terms are:

$$\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = \frac{1}{r}+\frac{x_{qi}\,x_i}{r^3}+\frac{1}{2}\left(3\,\frac{\left(x_{qi}\,x_i\right)\left(x_{qj}\,x_j\right)}{r^5}-\frac{r_q^2}{r^3}\right)+O\left(\frac{1}{r^4}\right)$$

In writing this, x_i are the Cartesian coordinates of $\vec{r}$, x_{qi} are the Cartesian Coordinates of $\vec{r}_q$, and as always we sum over repeated indices $i, j, k, \ldots$

We also considered Laplace's equation in spherical coordinates and expanded in powers of $\frac{1}{r}$.

Using: $\phi(\vec{r}) = R(r)\,F(\theta,\phi) = R(r)F(\Omega)$ Laplace's equation becomes:

$$\frac{1}{R(r)}\,\partial_r\,r^2\,\partial_r\,R(r)+\frac{1}{F(\Omega)}\,\nabla_\Omega^2\,F(\Omega) = 0$$

$$\nabla_\Omega^2 = \frac{1}{\sin(\theta)}\,\partial_\theta\,\sin(\theta)\,\,\partial_\theta+\frac{1}{\sin^2(\theta)}\,\partial_\phi^2$$

Writing $R(r)$ and $F(\Omega)$ as polynomial expansions in r and $\hat{n} = \frac{\vec{r}}{r}$ (which depends on θ and ϕ) we have:

$$\phi(\vec{r}) = \sum_\ell\left(A_\ell\,r^\ell+\frac{B_\ell}{r^{(\ell+1)}}\right)\,F_\ell(\Omega)$$

The functions $F_\ell(\Omega)$ will be related to either the Legendre Polynomials (for problems with no ϕ dependence or the Spherical Harmonics in more general cases.

1) Legendre Polynomials:

One result from the expansion in spherical coordinates is that for $\vec{r} > \vec{r}_q$:

$$\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = \sum_{\ell=0} \frac{r_q^\ell}{r^{\ell+1}}\ P_\ell(\cos\theta), \ \ \cos\theta = \frac{\vec{r}\cdot\vec{r}_q}{r\,r_q}$$

Compare the first three terms of this expansion to the Taylor's Series expansion shown above. Using this comparison, determine the polynomials $P_0(x), P_1(x)$, and $P_2(x)$.

(Of course, in this case the functions will be in terms of $x = \cos(\theta)$.

Check to see that your results for the Legendre Polynomials is correct. (Give the reference you used to check your results.)

Comparing these:

$$\frac{1}{r} + \frac{x_{qi}\,x_i}{r^3} + \frac{1}{2}\left(3\frac{\left(x_{qi}\,x_i\right)\left(x_{qj}\,x_j\right)}{r^5} - \frac{r_q^2}{r^3}\right) = \frac{1}{r}\ P_0(\cos\theta) + \frac{r_q}{r^2} P_2(\cos\theta) + \frac{r_q^2}{r^3}\ P_3(\cos\theta)$$

Consider this term by term:

$$\frac{1}{r} = \frac{1}{r}\ P_0(\cos\theta) \Rightarrow P_0(\cos\theta) = 1$$

$$\frac{\vec{r}_q\cdot\vec{r}}{r^3} = \frac{r_q\,r\cos\theta}{r^3} = \frac{r_q}{r^2}\ P_2(\cos\theta)\ \Rightarrow\ P_2(\cos\theta) = \cos\theta$$

$$\frac{1}{2}\left(3\frac{\left(x_{qi}\,x_i\right)\left(x_{qj}\,x_j\right)}{r^5} - \frac{r_q^2}{r^3}\right) = \frac{1}{2}\left(3\frac{\left(r_q\,r\cos\theta\right)^2}{r^5} - \frac{r_q^2}{r^3}\right) = \frac{r_q^2}{r^3}\ P_3(\cos\theta)$$

$$\Rightarrow P_3(\cos\theta) = \frac{1}{2}(\,3\cos^2\theta - 1)$$

This agrees with the result for the Legendre Polynomials, as can be seen, for example, on Wolfram Alpha:
https://www.wolframalpha.com/input?i=legendre+polynomial

2) Dipole Field:

Consider the second, "dipole", ($\ell = 1$) term in the expansion of $\frac{1}{\left|\vec{r}-\vec{r}_q\right|}$ when used in the expressions for the potential:

$$\phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\sum_q q\,x_{qi}\right)\frac{x_i}{r^3}, \qquad \phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\int d^3r_q\ \rho\left(\vec{r}_q\right)x_{qi}\right)\frac{x_i}{r^3}$$

The term in the parentheses in the equations above is the "Dipole Moment", a vector that depends on the charges and their positions (or the charge density).

In the book, the author labels these $\vec{d}$, but we'll use the more common notation, $\vec{p}$.

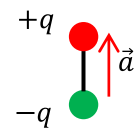
i) Write the dipole potential in terms of $\vec{p}$ and $\vec{r}$ (instead of in terms of the components shown above).

$$\phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

ii) For a dipole $\vec{p} = p\,\hat{z}$, for what directions in space does the potential equal zero?

The dipole potential will be zero for all $\vec{r}$ perpendicular to $\vec{p}$, where $\vec{p}\cdot\vec{r} = 0$.

For $\vec{p} = p\,\hat{z}$ (and the dipole at the origin) the potential will be zero for all points in the x-y plane, $z = 0$.



$\vec{p} = q\vec{a}$

If you think about a standard dipole in the $\hat{z}$ direction, a charge $+q$ at $z = \ +a/2$, a charge $-q$ at $x = \ -a/2$, and $\vec{p} = q\,a\,\hat{z}$, it makes sense that $\phi = 0$ on the x-y plane because each point on the x-y plane is equal distant from the two charges $\pm q$. The potentials add to zero.

iii) Using the dipole potential, calculate the electric field due to just the dipole moment of a charge distribution, $\vec{p}$.
This was left as an exercise in the book, but it's a good idea to do this.

Calculating the electric field that corresponds to the dipole potential, we got the result:

$$\vec{E}(\vec{r}) = \ -\vec{\nabla}\phi^1\,(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\,\hat{e}_i\,\partial_i\,\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\,\partial_i\,\frac{p_j\,x_j}{(x_k\,x_k)^{3/2}}$$

$$\vec{E}(\vec{r}) \ = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\left(\frac{1}{r^3}\,\partial_i\,p_j x_j + p_j\,x_j\,\left(-\frac{3}{2}\right)\frac{\partial_i\,(x_k\,x_k)}{(x_k\,x_k)^{\frac{5}{2}}}\right)$$

$$\vec{E}(\vec{r}) \ = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\left(\frac{1}{r^3}\,p_j\,\delta_{ij} - \frac{3}{2}\,p_j\,x_j\,\frac{2\,x_k\,\delta_{ik}}{r^5}\right)$$

$$\vec{E}(\vec{r}) \ = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\,\left(\frac{p_i}{r^3} - 3\,\vec{p}\cdot\vec{r}\,\frac{x_i}{r^5}\right)$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\,(\vec{p}\cdot\vec{r})\frac{\vec{r}}{r^2} - \,\vec{p}\right) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(\,3\,(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p})$$

iv) Using your result from above, draw a sketch showing the electric field due to a dipole $\vec{p} = p\,\hat{z}$. Draw this in the x-z plane. Your sketch should show the E-field vectors at a few representative places with the vector lengths giving the approximate magnitudes.

Repeat for a dipole $\vec{p} = p\,\hat{x}$.

Coming Soon!!!