E & M I
Workshop 5 – Rotating Charged Sphere, 2/23/2022

A standard problem in magnetostatics (and therefore possibly will show up on qualifiers) is calculating the magnetic field due to a rotating sphere with a constant surface charge density. This problem is done in a lot of places, but it will be good to make sure you can do this yourself.

The sphere is centered at $\vec{r} = 0$, has a radius R, a surface charge density σ, and is rotating around the z axis with and angular speed ω. The angular velocity is:

$$\vec{\omega} = \omega\,\hat{z}$$

The Right-Hand-Rule says that this corresponds to a counter-clockwise rotation when looking down on the sphere from the positive z axis.

As a start, we want to solve for the vector potential $\vec{A}(\vec{r})$ for points outside the sphere.

a) Draw a picture that can be used to define the coordinates you will be using for this problem.

b) Write an expression for the current density for the spinning sphere, $\vec{J}(\vec{r}\,')$, where $\vec{r}\,'$ will be the position of currents to be integrated.

Write the magnitude of $\vec{J}(\vec{r}\,')$ in terms of spherical coordinates, r', θ', ϕ', and write the direction of the vector in two different ways, using $\hat{r}', \hat{\theta}', \hat{\phi}'$ and using $\hat{x}', \hat{y}', \hat{z}'$. Check to make sure your expression has the correct units and the magnitude and direction of $\vec{J}$ are correct at various points on the sphere.

Note: It might be useful to remember that $\vec{v} = \vec{\omega} \times \vec{r}$.

c) Using the definition for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \vec{J}(\vec{r}\,') \frac{1}{|\vec{r} - \vec{r}\,'|}\, d^3r'$$

And the multipole expansion:

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{R^l}{r^{l+1}}\; Y_{lm}^*(\theta', \phi')\, Y_{lm}(\theta, \phi)$$

Write down the multipole expansion for the vector potential. This should still include sums over l and m and the volume integral over $\vec{r}\,'$. Remember to include the vector direction(s) of the current density.

Next, simplify this equation by using the orthonormality of the spherical harmonics:

$$\iint \sin\theta \, d\theta \, d\phi \;\; Y^*_{l',m'}(\theta,\phi)\, Y_{l,m}(\theta,\phi) = \delta_{l,l'}\, \delta_{m,m'}$$

A table of the spherical harmonics is at: https://en.wikipedia.org/wiki/Table_of_spherical_harmonics

but (here's a hint) all you'll really need for this problem are:

$$Y_{1,0}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta$$

$$Y_{1,1}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\,\pi}} \cdot \sin\theta \cdot e^{i\phi}, \;\; Y_{1,-1}(\theta,\phi) = -Y^*_{1,1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\,\pi}} \cdot \sin\theta \cdot e^{-i\phi}$$

d) Write your expression for $\vec{J}(\vec{r}\,')$ in terms of spherical harmonics, showing that you only need the $l = 1$ terms to do this.

e) Use the orthonormality of the spherical harmonics to complete the integral for $\vec{A}$. Show that your result gives an vector potential of the form:

$$\vec{A}(\vec{r}) = A_\phi(\vec{r})\,\hat{\phi}$$

f) Using:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B = \frac{1}{r\sin\theta}\left(\partial_\theta \sin\theta\, A_\phi - \partial_\phi\, A_\theta\right)\hat{r} + \left(\frac{1}{r\sin\theta}\,\partial_\phi\, A_r - \frac{1}{r}\,\partial_r\, r\, A_\phi\right)\hat{\theta} + \frac{1}{r}\left(\partial_r\, r\, A_\theta - \partial_\theta\, A_r\right)\hat{\phi}$$

Solve for the magnetic field of the spinning sphere.

g) Show that your result implies that the spinning sphere is a pure magnetic dipole,

$$\vec{m} = m\,\hat{z}$$

What is the magnitude, m?

Hint: For a vector $\vec{r} = (r,\theta,\phi)$ the unit vectors give: (You might draw a diagram showing this)

$$\hat{z}\cdot\hat{r} = \cos\theta\,, \qquad \hat{z}\cdot\hat{\theta} = -\sin\theta$$

Just for Kicks) If we are interested in what happens inside the sphere, where $r < R$, we need to use: (There are two solutions to the r-dependence for Laplace, $r > r'$ and $r < r'$.

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r^l}{R^{l+1}}\; Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Solve for the magnetic field inside the sphere.