PHYS 5013 December 7, 2021

Exam Sample Final Mullen

Name: ______________________________

Each of the following problems is designed to test you understanding and integration of the material we have studied. Please note that:

- You may use your notes or anything in your own handwriting.
- The problems are broken up into small parts; in no case do you need the result of earlier questions to proceed, so if you get stuck, just move on.
- Explanations are as important as having the right answer. You must not only state the right answer, but make it clear how you derived it.
- If you don't know how to do the entire problem, try to communicate what you do understand.
- Make sure your writing is readable!

You will have up to two hours for this exam. Cheating will be punished by the most gruesome method I can devise. And I can be pretty inventive. Don't cheat. This exam is broken into three parts.

Part I The first part consists of several true/false questions, for a total of 40 points. If you mark a statement as false you must explain why it is false or give a counter example in the space provided.

Part II The second part involves a short essay. Please write enough to convince me that you understand the material. It is worth 20 points.

Part III The third part involves detailed problem solving, and is worth 60 points. Please read the questions carefully, and show all your work.

Part I

______ 1. The space of 2×2 matrices excluding the zero matrix forms a group under addition.

a. True b. False

______ 2. Fourier transformations can be used to turn any second order equation into an algebraic equation.

a. True b. False

______ 3. The spin of an electron is not represented by a 2D vector because it does not transform as a 2D vector.

a. True b. False

______ 4. The Levi-Civita tensor $\epsilon_{123321} = -1$.

a. True b. False

______ 5. In a non-orthogonal transformation covariant and contravariant vectors will transform differently.

a. True b. False

______ 6. Requiring that a particle travel on the path covering the least distance is an example of a local constraint.

a. True b. False

______ 7. A set of linearly independent vectors is orthonormal.

a. True b. False

______ 8. The null space of an operator is defined as the space spanned by all the degenerate eigenvectors.

a. True b. False

______ 9. The discretization of the second derivative

$$f"(x) = (f(x + \Delta x) + f(x - \Delta x) - 2f(x))\,/\Delta x^2$$

is accurate to order Δx^4.

a. True b. False

______ 10. The form of a Green function depends upon the boundary conditions for the problem.

a. True b. False

______ 11. The equation $\mathcal{A}x = b$ has a solution only if b has no component in the null-space of $\mathcal{A}$.

a. True b. False

______ 12. Every dependent variable in the action generates a separate Euler-Lagrange equation.

a. True b. False

Part II

13. Your younger cousin has just had an undergraduate course on matrices and has learned about linear algebra and eigenvalues. They heard you talk about linear differential operators and basis *functions* and would like to understand how you can treat functions as a vector space, and the difference between vector spaces described by vectors and matrices of fixed size, and vector spaces of continuous functions. Write an essay explaining the similarities and differences between the two problems. (At a minimum you should describe two similarities and two differences).

Part III

14. **Synthesis:** Consider a real function $\phi(r)$. We wish to minimize the quantity

$$I = \int_{-\infty}^{\infty} dx \left\{ \left(\frac{\partial \phi(x)}{\partial x} \right)^2 - \phi(x)^2 \frac{V}{(1+x^2)} \right\}$$

subject to the constraint that: $\int_{-\infty}^{\infty} dx\, (\phi(x))^2 = 1$ and where $V > 0$.

a. Is there a Noether charge associated with the transformation $x \to x + \epsilon$? If so, what is it? If not, why not?

b. Show that this is the case when $\phi(x)$ satisfies:

$$-\frac{\partial^2 \phi}{\partial x^2} - \frac{V}{(1+x^2)} \phi = E\, \phi$$

where E is a constant.

c. Describe how you would solve this problem numerically by discretizing the problem. Discuss how you might choose the size of the "box" in which you solve the problem and how to handle the boundaries, assuming that you are looking for bound states to the potential.

d. We have turned the minimization problem into an eigenvalue problem. Let us find an approximate solution for $\phi(x)$ and an approximation for the minimum value for λ. We will do this by guessing a form for $\phi(x)$

$$\tilde{\phi}(x, a) = \sqrt{\frac{a}{\pi}} \frac{1}{x^2 + a^2}$$

where $\tilde{\phi}(x, a)$ is already normalized over the infinite interval. Derive an integral expression for $\tilde{E}(a) = (\tilde{\phi}, H\phi)$. Do not do or solve this integral!

e. A third way to try to find the function $\phi(x)$ that minimizes I is to choose some number of basis functions $\{f_k(x)\}$ that are orthonormal, and evaluate the matrix $\mathcal{H}$:

$$\mathcal{H}_{j,k} = \int f_j^*(x) \left\{ -\frac{\partial^2}{\partial x}^2 - \frac{V}{1+x^2} \right\} f_k(x)\, dx$$

Given the actual matrix $\mathcal{H}_{j,k}$, how do you find an estimate for the minimum energy? How can you make it more accurate?

f. Now assume that you are given the Green Function $G(x, x')$ for the operator d/dx^2. Write the time independent Schrodinger equation as an integral equation for the unknown function $\phi(x)$.

15. You are given the problem

$$\mathcal{L}_0\, y(x) = f(x)$$

where $\mathcal{L}_0$ is a known linear hermitian operator, $f(x)$ is a known inhomogeneous function. and $y(x)$ is the unknown function you must find. You also know the $\phi_n^{(0)}(x)$, the eigenfunctions of $\mathcal{L}$:

$$\mathcal{L}_0\, \phi_n^{(0)}(x) = \lambda_n^{(0)}\, \phi_n^{(0)}(x)$$

which are non-degenerate.

a. Write down an expression for the Green's function for this problem that will allow you to solve for $f(x)$.

b. Now assume that you wish to solve the problem

$$(\mathcal{L}_0 + \epsilon \mathcal{L}_1)\; y(x) = f(x)$$

where $\mathcal{L}_1$ is also hermitian. Using your result from (a), derive a perturbative expression for the Green's function that is correct to first order in ϵ. (This will involve non-degenerate perturbation theory and a bit of algebra to get the lowest order term).

16. The parabolic coordinates u, v and w are defined in terms of the Cartesian coordinates by:

$$\begin{aligned} x &= \sqrt{uv}\, \cos w \\ y &= \sqrt{uv}\, \sin w \\ z &= \tfrac{1}{2}(u - v) \end{aligned}$$

where $u \geq 0$ and $v \geq 0$. (This system of coordinates in useful in problems of atomic physics when you calculate ionization amplitudes in an external field.)

a. Show that this is an orthogonal coordinate system.

b. Calculate the gradient in this coordinate system.

c. Calculate the divergence in this coordinate system.

d. Calculate the Laplacian in this coordinate system.