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Math Methods in Physics

CH. 1 VECTORS IN CLASSICAL PHYSICS LECTURE NOTES

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Scalar: Informal starting definition, A quantity wholly described by a single number.

Vector: Quanity with magnitude and direction

In 3D we use 3 non-coplanar vectors as a reference to define our vectors

$\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$

Non-coplanar: Do not lie in a single plane. Algebraically - "Linearly independent"
There does not exist a_1, a_2 & a_3 such that

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$ where $a_1 = a_2 = a_3$

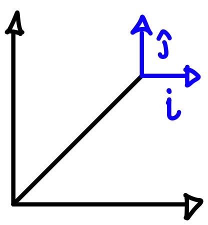
Our lives are easier if we choose

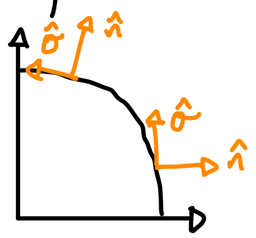
① $\vec{v}_1$, $\vec{v}_2$, & $\vec{v}_3$ are mutually perpendicular / orthogonal.

② $|\vec{v}_1| = |\vec{v}_2| = |\vec{v}_3| = 1 \rightarrow$ normalized

① + ② $\rightarrow$ orthonormal

③ Do not depend on position





If ① + ② + ③ we have a cartesian system.

④ We may choose to have a "right-handed" system of vectors

Multiplication:

"Dot" or "scalar" product

Geometric:

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$, $|\vec{v}| =$ length

Algebraic: Given a cartesian basis with basis vectors $\hat{l}_1, \hat{l}_2, \hat{l}_3$ then

$\hat{l}_1 \cdot \hat{l}_2 = \delta_{ij}$

$\hat{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$, $\hat{y} = y_1\hat{e}_1 + y_2\hat{e}_2 + y_3\hat{e}_3$

$\vec{x} \cdot \vec{y} = \sum_i \sum_j (x_i \hat{e}_i) \cdot (y_j \hat{e}_j) = \sum_i \sum_j x_i y_j \delta_{ij} = \sum_i x_i y_i$

Einstein Summation Notation

"Any repeated index is implicitly summed over"

$\vec{x} = x_i \hat{e}_i \; , \; \vec{y} = y_j \hat{e}_j \; : \; \vec{x} \cdot \vec{y} = x_i \hat{\ell}_i \cdot y_j \hat{\ell}_j = x_i y_i$

Inner product of two vectors $\longrightarrow$ Scalar

Cross or Outer Product

Geometric : If $\vec{A} \times \vec{B} = \vec{C}$ then

① $|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$

② $\vec{A} \perp \vec{C} \; , \; \vec{B} \perp \vec{C}$

③ Direction given by right hand rule

Algebraically

$$C_k = (\vec{A} \times \vec{B})_k = A_i B_j \epsilon_{ijk}$$

$$\epsilon_{ijk} = \text{Levi-Civita} \begin{cases} +1 & \text{if } ijk \text{ is even permutation of } (1,2,3) \\ -1 & \text{if } ijk \text{ is odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Now we can have an ϵ with any number of indices

① $\epsilon_{ijk} \delta_{ik} = 0$

② Any single permutation changes the sign $\epsilon_{ijk\ell} = -\epsilon_{ij\ell k}$

③ $\epsilon_{ijk} \epsilon_{k\ell m} = (\delta_{i\ell} \delta_{jm} - \delta_{j\ell} \delta_{\ell m})$

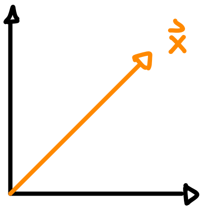
Proof : For any value of k i, ℓ, j & $m \neq k$

Either $i = \ell$ and $j = m$ $\epsilon_{ijk} \epsilon_{kij} = (\epsilon_{ijk})^2 = 1$ or $i = m$ & $j = \ell$

$\epsilon_{ijk} \epsilon_{kji} = -(\epsilon_{ijk})^2 = -1$

Rotations

We want to communicate about a physical vector



We set up a co-ordinate system

$\vec{x} = x_i \hat{e}_i$, what is $x_i = \hat{e}_i \cdot \vec{x}_i$

But our Friends have our co-ordinate system w.r.t ours

$$\vec{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2' + x_3'\hat{e}_3' = x_i'\hat{e}_i'$$

Q: How do we convert from one system to another?

A: We have to figure out how to expand our basis vectors?

$$\hat{e}_i' = (\hat{e}_i' \cdot \hat{e}_j)\hat{e}_j = a_{ij}\hat{e}_j \;, \quad a_{ij} \equiv \hat{e}_i' \cdot \hat{e}_j$$

$$\vec{x} = x_j\hat{e}_j = x_j(\hat{e}_j \cdot \hat{e}_i')\hat{e}_i' = a_{ij}x_j\hat{e}_i' = x_i'\hat{e}_i' = \vec{x} \quad \therefore \quad x_i' = a_{ij}x_j$$

Sometimes we are sloppy & say "the vector $\vec{x}$ transforms....". More properly the components of $\vec{x}$ transforms

The set of numbers a_{ij} are a rotation matrix. Since both basis are orthonormal then

$$\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij} = \underbrace{(\hat{e}_i' \cdot \hat{e}_k)}_{a_{ik}}\hat{e}_k \cdot \underbrace{(\hat{e}_j' \cdot \hat{e}_l)}_{a_{jl}}\hat{e}_l = a_{ik}a_{jl}\delta_{kl} = a_{ik}a_{jk} \text{ when } k = l$$

$$a_{ik}a_{jk} = \delta_{ij} \;:\; (\tilde{a})_{ik}(\tilde{a})^T_{kj} = \mathbb{1} \;, \quad \tilde{a}\tilde{a}^T = \mathbb{1}$$

$\therefore \quad \tilde{a}^T = a^{-1} \rightarrow a$ is an orthogonal matrix

Orthogonal matrices (transformations) preserve dot products $\therefore$ orthogonal transformations preserve length

$\tilde{a}_{jk}\tilde{a}_{kl} \neq \mathbb{1}$ in general

9-27-21

__Scalars__ & __Vectors__ __Re-Visited__

Vector - Student ID HW AVG.

Q: What is a vector?

A: A vector is an object that transforms as a vector

That is, it lies in a space (manifold) with an underlying geometry that connects the different components

A vector transforms under rotation

$$x_i' = a_{ij}x_j$$

Furthermore - we can specify two types of vectors by how they behave under inversion.

Polar vectors change sign under inversion. e.g: $\vec{r}, \vec{v}$

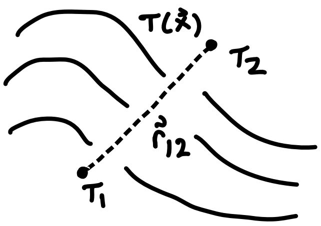
Axial or psuedo vectors do not change sign. e.g: $\vec{L} = \vec{r} \times \vec{p}$, inversion $\rightarrow (-\vec{r}) \times (-\vec{p})$

A scalar is unchanged by rotation ......

Psuedo-Scalars $\rightarrow$ change sign under inversion. A regular scalar does not

Are there other transformations? yes - (dilation (scale))

Why is this important? Consider a physical quantity - a temperature difference



$T_2 - T_1$ does not depend on co-ordinates

$$\Delta T = \vec{\nabla} T \cdot \vec{r}_{12}$$

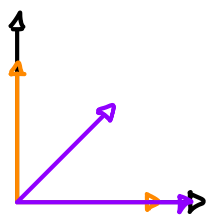
If $\vec{x}_{21} = (1.0\hat{\imath} + 2.0\hat{\jmath})\,m$, we change to $(100.0\hat{\imath} + 200.0\hat{\jmath})$

$\vec{\nabla} T = (20\ K/m\ \hat{\imath} + 30\ K/m\ \hat{\jmath}) \rightsquigarrow (0.20\ K/m\ \hat{\imath} + 0.30\ K/m\ \hat{\jmath})$

In order to understand this better. Lets look at non-orthogonal transformations

Non orthogonal Transformation

Let $\hat{e}_1 = \hat{\imath}$, $\hat{e}_1' = 2\hat{\imath}$: $\hat{e}_2 = \hat{\jmath}$, $\hat{e}_2' = \hat{\imath} + \hat{\jmath}$



In order to go between two co-ordinates

$$\begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} = a_{ij} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

Q: How do our co-ordinates transform?

A: $\vec{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2' = x_1'(2\hat{e}_1) + x_2'(\hat{e}_1 + \hat{e}_2) = (2x_1' + x_2')\hat{e}_1 + (x_2')\hat{e}_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}_{= a_1^T} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad : \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} (a_1^T)^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad : \quad (a_1^T)^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix} = a_1$$

But because $\vec{x}$ exists independent of our co-ordinates

$$\vec{x} = \hat{x}_i' \cdot \hat{e}_i' = (a^T)^{-1}_{ii}\, x_j \cdot a_{ik}\, \hat{e}_k = (a_1^T)^{-1}_{ij}\,(a_{ki}^T) = x_j \hat{e}_k = (a_{ki}^T)(a_{ij}^T)^{-1} x_j \hat{e}_k = x_j \hat{e}_j$$

The transformations of the basis vectors $\hat{e}_i$ and the co-ordinate x_i are not the same. We call quantities that transform as $\hat{e}_i$ co-variant vectors. If they transform as x_i, the co-ordinates we call them contravariant vectors.

So why have you not had to worry about this?

For orthogonal transformations : $a_1 = (a_1^T)^{-1}$

Q_i: why do we care?

A_i: Laws of Physics must hold

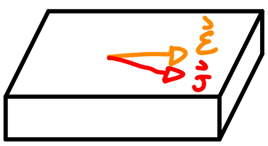
What is a tensor? An object that transforms as a tensor.

Subtle issue: Charge is a scalar, charge density, not a scalar under scale transformations

9-29-21

Tensors

"Tensor" is a scary word to many students. They typically arise when you need to relate two physical vectors that are not parallel.



$\vec{J} = \sigma \vec{\mathcal{E}}$

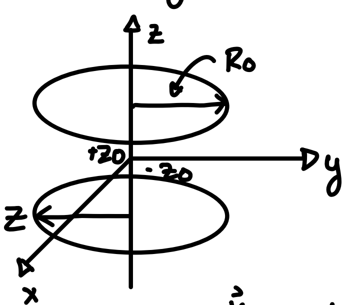
$\sigma \longrightarrow$ conductivity

$\vec{J} \longrightarrow$ charge current density

$J_{kl} = \sigma_k^l \mathcal{E}_l$

The tensior you "all know" is the moment of inertia $\vec{L} = I\vec{\omega}$

$I \longrightarrow$ Tensor



$\vec{r}_1(t) = r_0\cos(\omega t)\hat{i} + r_0\sin(\omega t)\hat{j} + z_0\hat{k}$

$\vec{r}_2(t) = -r_0\cos(\omega t)\hat{i} - r_0\sin(\omega t)\hat{j} - z_0\hat{k}$

$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$, $\vec{p}_i = m_i\vec{v}_i$

$\vec{v}_1 = -\omega r_0\sin(\omega t)\hat{i} - \omega r_0\cos(\omega t)\hat{j}$, $\vec{v}_2 = \omega r_0\sin(\omega t)\hat{i} + \omega r_0\cos(\omega t)\hat{j}$

$\vec{L}_1 = \vec{r}_1 \times \vec{p}_1 = (r_0\cos(\omega t)\hat{i} + r_0\sin(\omega t)\hat{j} + z_0\hat{k}) \times m\omega(-\sin(\omega t)\hat{i} + \cos(\omega t)\hat{j})$

$\vec{L}_1 = m\omega r_0^2(\cos^2(\omega t)\,\hat{i}\times\hat{j} - \sin^2(\omega t)\,\hat{j}\times\hat{i}) + m\omega r_0 z_0(-\sin(\omega t)\,\hat{k}\times\hat{i} + \cos(\omega t)\,\hat{k}\times\hat{j}))$

$\vec{L}_1 = m\omega r_0^2\hat{k} + m\omega R_0 z_0(-\sin(\omega t)\hat{j} - \cos(\omega t)\hat{i}) = m\omega r_0^2\hat{k} - m\omega z_0\hat{r}_\perp$

$\vec{L}_{Tot} = 2(m\omega R_0\hat{k} - m\omega z_0\hat{r}_\perp)$

Case 1: $z_0 = 0$

$\vec{L} = 2m\omega r_0^2\hat{k}$, $\vec{\omega} = \omega\hat{k}$ $\therefore$ $\vec{L} \parallel \vec{\omega} \longrightarrow \vec{L} = I\vec{\omega}$ w/ $I = 2mr_0^2$

If $z_0 \neq 0$ then $\vec{L} \nparallel \vec{\omega}$, If it is to rotate as shown, is $\vec{L}$ constant? No!

$$\frac{d\vec{L}}{dt} \neq 0 = \vec{\tau}$$

We define the moment of inertia tensor $I_{jk} = m_i (x_j^{(i)})(x_k^{(i)})$ where $\vec{x}^{(i)}$ is the location of the ith particle. How does I transform?

$$x_j^{(i)'} = a_{jl} x_l^{(i)} \quad , \quad I_{jk} = m_i x_j^{(i)'} x_k^{(i)'} = m_i a_{jl} x_l^{(i)} a_{km} x_m^{(i)} = m_i x_l^{(i)} x_m^{(i)} (a_{jl} a_{km}) = I_{lm} a_{jl} a_{km}$$

10-1-21

Q: What is a tensor?

A: Something that transforms as a tensor.

For an orthogonal transformation of

$$x_i' = a_{ij} x_j \quad , \quad T_{ij}' = a_{il} a_{jk} T_{lk}$$

What if we have a non-orthogonal transformation?

$x_i' = a_{ij} x_j$ Contravariant vector

$y_i' = (a^T)^{-1} y_i$ Co-variant vector

$b_{ij} = (a^T)^{-1}_{ij}$

We need to know if the index corresponds to a co-variant or contravariant behavior

T^{μ}_{ν} $\mu \to$ contravariant, $\nu \to$ Covariant $\longrightarrow$ General Relativity

Q: B&F specify Cartesian Tensors, what is a cartesian tensor

A: A tensor defined in 2D or 3D with cartesian co-ordinates

Q: What is not a cartesian tensor?

A: Something like special relativity or Spherical tensors

You encounter them in QM when you have coupled angular momentum.

General tensor concepts. Given a cartesian tensor we can always right it as a sum of antisymmetric tensor and a symmetric tensor.

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{\mathbb{S}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{\mathbb{A}}$$

$\mathbb{A}$ is anti-symmetric iff $A_{ij} = -A_{ij}$

An $\mathbb{A}$ 3D has only 3 independent components we can create a psuedo vector from its elements

$$u_i = \frac{1}{2}\epsilon_{ijk} A_{jk}$$

Symmetric matrices have a common representation as "ellipsoids"

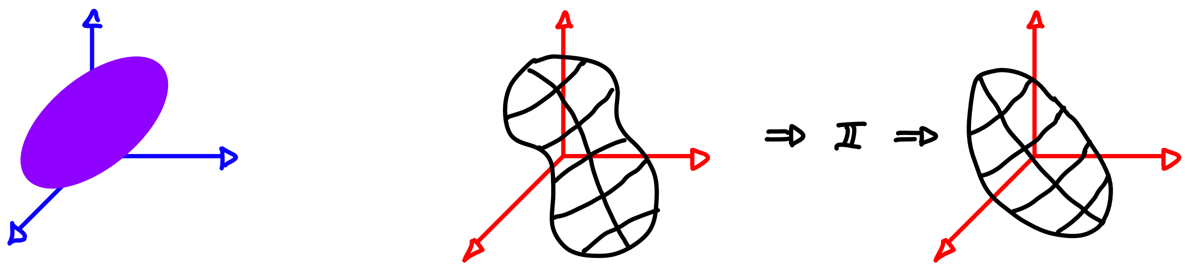
In 3D a tensor $\mathcal{S}$ has 6 independent numbers

A general ellipsoid is given by, $Ax_1^2 + Bx_2^2 + Cx_3^2 + Dx_1x_2 + Ex_2x_3 + Fx_3x_1 = 1$

$$\vec{x}^T \begin{pmatrix} A & D/2 & E/2 \\ D/2 & B & F/2 \\ E/2 & F/2 & C \end{pmatrix} \vec{x} \;:\; \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \mathcal{S} \quad \text{Sign}(\det \mathcal{S})$$

Conductivity, permeability, polarizability, tensors are often plotted or represented by ellipsoids.

For example - given some blob in 3D



Our very important tensor is the metric tensor which relates the length of an infinitesimal vector to the change in its co-ordinates.

$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ For a cartesian system

$$(dx_1, dx_2, dx_3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \longrightarrow \text{Metric tensor } (g)$$

In cylindrical co-ordinates,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \qquad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If my co-ordinates are orthogonal then g is diagonal.

If the small change : $d\vec{s} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3 = h_i dq_i \hat{e}_i \;:\; ds^2 = h_i^2 dq_i^2$

<u>Gradient</u> : The gradient of a scalar field is a vector field. we define $\vec{\nabla}\varphi$ so that the infitesimal change $d\varphi = \vec{\nabla}\varphi \cdot d\vec{s}$

So obviously, $\frac{\partial \varphi}{\partial q_1} = \frac{\varphi(q_1 + dq_1, q_2, q_3) - \varphi(q_1, q_2, q_3)}{dq_1} = \partial_1 \varphi$, $d\varphi = \partial_i \varphi \, dq_i$

But $d\vec{s} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$, $d\varphi = \left(\frac{1}{h_i}\partial_i \varphi\right)\left(h_i dq_i\right) = \left(\frac{1}{h_i}\partial_i \varphi \hat{e}_i\right)\left(h_j dq_j \hat{e}_j\right)$

The notation

" $\nabla\cdot$ " and " $\nabla\times$ " is misleading.

10-4-21

Vector Calculus : (orthogonal co-ordinates)

$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3 dx_3^2$ or $d\vec{s} = h_i dx_i \hat{e}_i$

Example : $x = \rho\cos\sigma$, $y = \rho\sin\sigma$, $z = z$

$\vec{r} = \rho\cos\sigma\,\hat{\imath} + \rho\sin\sigma\,\hat{\jmath} + z\hat{k}$

$d\vec{r} = d\vec{s} = [\cos\sigma\, d\rho + \rho(-\sin\sigma)d\sigma]\hat{\imath} + [\sin\sigma\, d\rho + \rho\cos\sigma\, d\sigma]\hat{\jmath} + dz\,\hat{k}$

$= (\cos\sigma\,\hat{\imath} + \sin\sigma\,\hat{\jmath})d\rho + \rho(-\sin\sigma\,\hat{\imath} + \cos\sigma\,\hat{\jmath})d\sigma + dz\,\hat{k}$

$h_\rho = 1$, $h_\sigma = \rho$, $h_z = 1$: $\hat{e}_\rho = \cos\sigma\,\hat{\imath} + \sin\sigma\,\hat{\jmath}$, $\hat{e}_\sigma = -\sin\sigma\,\hat{\imath} + \cos\sigma\,\hat{\jmath}$, $\hat{e}_z = \hat{k}$

$\vec{\nabla}\varphi = \frac{1}{h_i}\partial_i\varphi\,\hat{e}_i = \frac{1}{h_\rho}\partial_\rho\varphi\,\hat{e}_\rho + \frac{1}{h_\sigma}\partial_\sigma\varphi\,\hat{e}_\sigma + \partial_z\varphi\,\hat{e}_z = \partial_\rho\varphi\,\hat{e}_\rho + \frac{1}{\rho}\partial_\sigma\varphi_\sigma\,\hat{e}_\sigma + \partial_z\varphi\,\hat{e}_z$

Divergence

$\mathrm{div}\,\vec{v}(\vec{r}) = \lim_{d\tau\to 0} \oint \vec{v}(\vec{r}')\cdot d\vec{\delta}'$, $d\tau \equiv$ infintesimal volume , $d\delta \equiv$ infintesimal surface element

$\vec{\nabla}\cdot\vec{v} = \frac{1}{h_1h_2h_3}\left\{\partial_1(V_1h_2h_3) + \partial_2(V_2h_1h_3) + \partial_3(V_3h_1h_2)\right\} = 1$

$h_1 = h_\rho$, $h_2 = h_\sigma$, $h_3 = h_z$

$\frac{1}{\rho}\left\{\partial_\rho(\rho V_\rho) + \partial_\sigma(V_\sigma) + \partial_z(V_z\rho)\right\} = \frac{1}{\rho}\partial_\rho\,\rho V_\rho + \frac{1}{\rho}\partial_\sigma V_\sigma + \partial_z V_z$

Curl $\longrightarrow$ Going aroundedness

It is therefore a vector

$\mathrm{Curl}(\vec{v})\cdot\hat{n} = \lim_{d\sigma\to 0}\oint \vec{v}\cdot d\vec{\lambda}$, $d\sigma$ = small surface , $d\lambda$ = element of the limit boundary at $d\sigma$

$\vec{\nabla}\times\vec{v} = \frac{1}{h_ih_j}\partial_i(h_jV_j)\epsilon_{ijk}\,\hat{e}_k$

$= \frac{1}{h_1h_2}\left[\partial_1(h_2V_2) - \partial_2(h_1V_1)\right]\hat{e}_3 + \frac{1}{h_2h_3}\left[\partial_2(h_3V_3) - \partial_3(h_2V_2)\right]\hat{e}_1 + \frac{1}{h_3h_1}\left[\partial_3(h_1V_1) - \partial_1(h_3V_3)\right]\hat{e}_2$

$h_\rho = h_1 = 1$, $h_\sigma = h_2 = \rho$, $h_3 = h_z = 1$

$\vec{\nabla}\times\vec{v} = \frac{1}{\rho}\left[\partial_\sigma(h_zV_z) - \partial_z(h_\sigma V_\sigma)\right]\hat{e}_\rho + \left[\partial_z(h_\rho V_\rho) - \partial_\rho(h_zV_z)\right]\hat{e}_\sigma + \frac{1}{\rho}\left[\partial_\rho(h_\sigma V_\sigma) - \partial_\sigma(h_\rho V_\rho)\right]\hat{e}_z$

Laplacian

$$"\nabla^2 \varphi" = \frac{1}{h_1 h_2 h_3}\left(\partial_1\left(\frac{h_2 h_3}{h_1}\right)\partial_1 \varphi\right) + \partial_2\left(\frac{h_3 h_1}{h_2}\partial_2 \varphi\right) + \partial_3\left(\frac{h_1 h_2}{h_3}\partial_3 \varphi\right)$$

In Cylindrical : $\frac{1}{\rho}\left[\partial_\rho \rho \partial_\rho \varphi + \partial_\theta \frac{1}{\rho}\partial_\theta \varphi + \partial_z \rho \partial_z \varphi\right] = \frac{1}{\rho}\partial_\rho\left(\rho \partial_\rho \varphi\right) + \frac{1}{\rho^2}\partial_\theta^2 + \partial_z^2 \varphi$