COLLEGE OF ARTS AND SCIENCES
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Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #3

PROBLEMS: $\{1, 2, 3, 4\}$

Due: September 16, 2021 By: 11:59 PM

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PROFESSOR
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Problem 1:

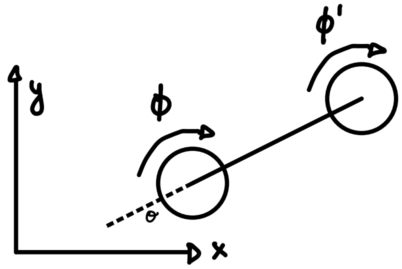
Consider the situation of Fig. 1. Two wheels with radius R are mounted on an axle of length l but can rotate independently. The axle-wheels contraption is allowed to roll (without slippage) on a 2D plane defined by the Cartesian co-ordinates x and y. Taking ϕ and ϕ' to be the angle of rotation of each wheel about the axis defined by the axle, θ to be the angle the axle makes with respect to the x-axis of the 2D plane, show that the system has: i) two *nonholonomic* constraint equations,

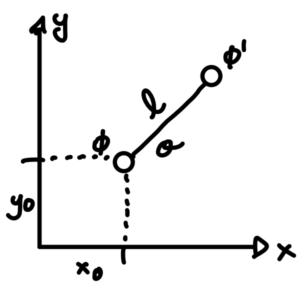
$$\begin{aligned} \cos(\theta)dx + \sin(\theta)dy &= 0, \\ \sin(\theta)dx - \cos(\theta)dy &= R(d\phi + d\phi') \end{aligned} \tag{1}$$

where the co-ordinates (x, y) correspond to the center of the axle, and ii) one *holonomic* constraint

$$\theta + \frac{R}{l}(\phi - \phi') = \text{const.} \tag{2}$$

Hint: For ii) consider the motion of the relative vector between the two wheels.

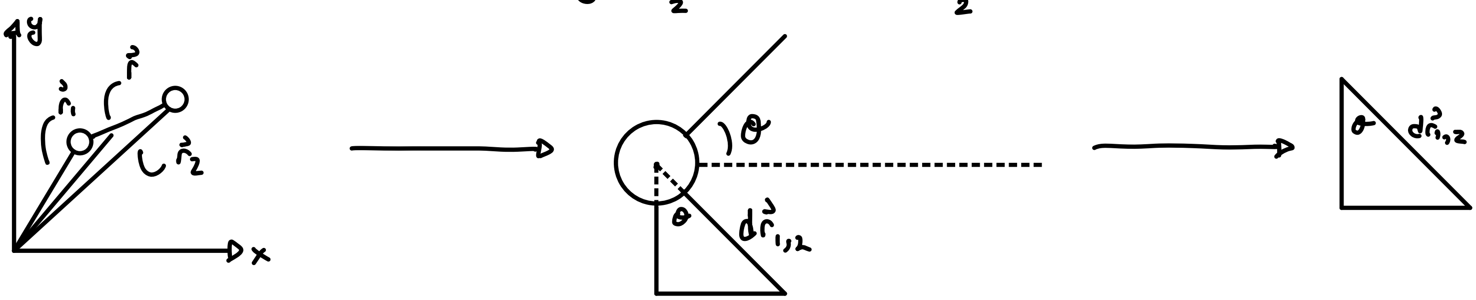




$$x = x_0 + \frac{l}{2}\cos\theta \quad , \quad y = y_0 + \frac{l}{2}\sin\theta$$

$$dx = -\frac{l}{2}\sin\theta\, d\theta \quad , \quad dy = \frac{l}{2}\cos\theta\, d\theta$$

$$\cos\theta\, dx + \sin\theta\, dy = -\frac{l}{2}\cos\theta\sin\theta\, d\theta + \frac{l}{2}\cos\theta\sin\theta\, d\theta = 0 \checkmark$$



$$\vec{r} = \frac{\vec{r}_1 + \vec{r}_2}{2} \quad : \quad d\vec{r}_1 = \frac{R}{2}d\varphi\sin\theta\,\hat{x} - \frac{R}{2}d\varphi\cos\theta\,\hat{y} \quad : \quad d\vec{r}_x = \frac{R}{2}d\varphi\sin\theta + \frac{R}{2}d\varphi'\sin\theta \equiv dx$$

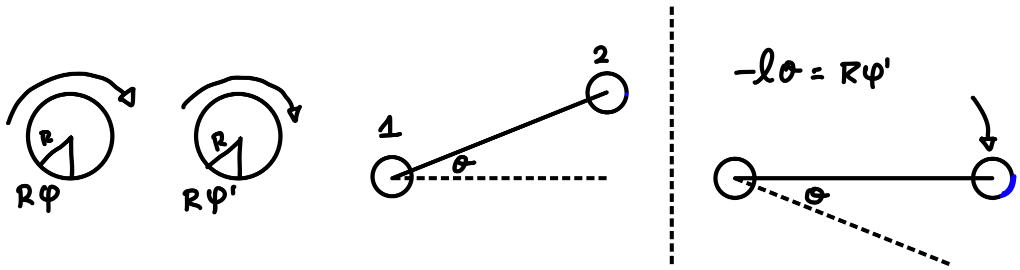
$$d\vec{r}_2 = \frac{R}{2}d\varphi'\sin\theta\,\hat{x} - \frac{R}{2}d\varphi'\cos\theta\,\hat{y} \quad : \quad d\vec{r}_y = -\frac{R}{2}d\varphi\cos\theta - \frac{R}{2}d\varphi'\cos\theta \equiv dy$$

$$\frac{\sin\theta}{2}(Rd\varphi\sin\theta + Rd\varphi'\sin\theta) - \frac{\cos\theta}{2}(-Rd\varphi\cos\theta - Rd\varphi'\cos\theta)$$

$$\frac{\sin^2\theta}{2}(Rd\varphi + Rd\varphi') + \frac{\cos^2\theta}{2}(Rd\varphi + Rd\varphi') = \frac{(Rd\varphi + Rd\varphi')}{2}\cancel{(\sin^2\theta + \cos^2\theta)}$$

$$\sin\theta\, dx - \cos\theta\, dy = \frac{R}{2}(d\varphi + d\varphi') \checkmark$$

Problem 1: Continued



Axle moves arc length $= -l\theta$

Wheel " " $= R\varphi'$

$$R\varphi^{(\prime)} = \pm l\theta$$

From the above we can say when wheel 1 is held still and 2 can move freely

$$\theta = -\frac{R}{l}\varphi' \quad (*)$$

And if the converse scenario happens

$$\theta = \frac{R}{l}\varphi \quad (**)$$

We can then say

$$\theta + \frac{R}{l}\varphi - \frac{R}{l}\varphi' = \theta + \frac{R}{l}(\varphi - \varphi') = \text{const} \checkmark$$

Since the addition of (*) and (**) is a constant.

Problem 1: Review

Procedure:

- Define a co-ordinate system for the axles and use this to find x, dx, y, and dy. Proceed to use these and show that first equation in (1) can be derived.
- Define a vector $\vec{r}$ and use the same procedure for the first equation in (1) to find the other constraint equation.
- Proceed to use relationships for rotation, add them together to get the other constraint equation.

Key Concepts:

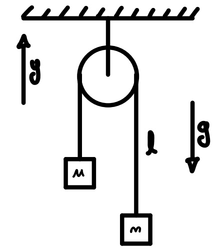
- Holonomic constraints are constraints that are equal to constants or involve inequalities.
- Nonholonomic constraints are definite and equal to specific values.

Variations:

- We can be given a different system.
  - This would create different constraints because of different geometries.

Problem 2:

This question involves two parts, but both make use of d'Alembert's principle explicitly.
Part 1: Consider the Atwood machine illustrated in Fig. 2(a).



(a) Write down d'Alembert's principle for the system.

d'Alembert's principle takes a dynamic system and makes it static.

$$(-Mg - M\ddot{y}_M)\delta_M + (-mg + m\ddot{y}_m)\delta_m = 0 \;:\; y_M + y_m - l = 0 \;:\; \ddot{y}_M = -\ddot{y}_m \;:\; \delta_M = -\delta_m$$

$$\boxed{(-Mg - M\ddot{y}_m)\delta_m - (-mg + m\ddot{y}_m)\delta_m = 0}$$

(b) Use your answer to (a) to show that the motion of the system is entirely governed by

$$\ddot{y}_m = \frac{M-m}{M+m}g, \tag{3}$$

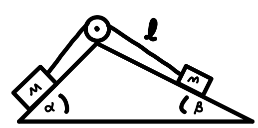
where y_m is the position of the mass block of weight m.

$$-Mg - M\ddot{y}_m = -mg + m\ddot{y}_m \;,\; -M\ddot{y}_m - m\ddot{y}_m = Mg - mg \;,\; M\ddot{y}_m + m\ddot{y}_m = mg - Mg$$

$$\boxed{\ddot{y}_m = \frac{(m-M)}{(M+m)}g \;,\; \ddot{y}_M = \frac{(M-m)}{(M+m)}g}$$

(c) Assume now that the masses are allowed to rest on the sides of a fixed wedge, as per Fig. 2(b). Show that the equation of motion becomes,

$$\ddot{L}_m = \frac{m\sin(\beta) - M\sin(\alpha)}{M+m}g. \tag{4}$$



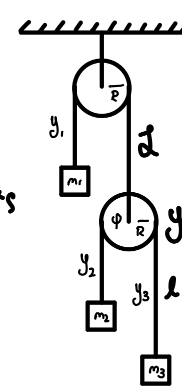
where L_m is the distance of the block parallel along the relevant surface of the wedge.

$$(-Mg\sin\alpha - M\ddot{y}_M)\delta_M + (-mg\sin\beta + m\ddot{y}_m)\delta_m = 0 \;:\; y_M + y_m - l = 0 \;:\; \ddot{y}_M = -\ddot{y}_m \;:\; \delta_M = -\delta_m$$

$$-Mg\sin\alpha - M\ddot{y}_m = -mg\sin\beta + m\ddot{y}_m \;:\; mg\sin(\beta) - Mg\sin(\alpha) = M\ddot{y}_m + m\ddot{y}_m$$

$$\boxed{\ddot{y}_m = \frac{m\sin(\beta) - M\sin(\alpha)}{M+m}g \;,\; \ddot{y}_M = \frac{M\sin(\alpha) - m\sin(\beta)}{M+m}}$$

Part 2: Consider the mass-pulley system illustrated in Fig. 2(c).



(d) Identify a set of generalized co-ordinates and the constraints in the system.

L - coordinates , N - moving objects , D - Directions of movement , m - constraints

L = ND - m

$\mathcal{L}$ - Length of top rope , l - Length of bottom rope

Problem 2: Continued

y_1 - Length of rope from m_1 to first contact of pulley

y_2 - Length of rope from m_2 to first contact of pulley

y_3 - Length of rope from m_3 to first contact of pulley

y_p - Length of rope from pulley

$$d = y_1 + y_p \ , \ l = y_2 + y_3 + 2y_p \ : \ \ddot{y}_1 = -\ddot{y}_p \ , \ 2\ddot{y}_p = \ddot{y}_2 + \ddot{y}_3 = -2\ddot{y}_1$$

(e) Using the co-ordinates defined in (d), compute the equations of motion for the system.

Using the constraints from part d.), and d'Alembert's principle : δ's are independent

$$(m_1\ddot{y}_1 - m_1 g)\delta y_1 + (m_2\ddot{y}_2 - m_2 g)\delta y_2 + (m_3\ddot{y}_3 - m_3 g)\delta y_3 = 0$$

$$\delta_1 = -\frac{\delta_2 + \delta_3}{2} \ : \ (m_1\ddot{y}_1 - m_1 g) \cdot -\frac{(\delta_2 + \delta_3)}{2} + (m_2\ddot{y}_2 - m_2 g)\delta y_2 + (m_3\ddot{y}_3 - m_3 g)\delta y_3 = 0$$

$$(m_1 g - m_1\ddot{y}_1)\frac{(\delta_2 + \delta_3)}{2} + (m_2\ddot{y}_2 - m_2 g)\delta y_2 + (m_3\ddot{y}_3 - m_3 g)\delta y_3 = 0$$

$$\frac{m_1}{2} g\delta y_2 + \frac{m_1}{2} g\delta y_3 - \frac{m_1}{2}\ddot{y}_1\delta y_2 - \frac{m_1}{2}\ddot{y}_1\delta y_3 + (m_2\ddot{y}_2 - m_2 g)\delta y_2 + (m_3\ddot{y}_3 - m_3 g)\delta y_3 = 0$$

$$\left[\frac{m_1}{2}(g - \ddot{y}_1) + m_2(\ddot{y}_2 - g)\right]\delta y_2 + \left[\frac{m_1}{2}(g - \ddot{y}_1) + m_3(\ddot{y}_3 - g)\right]\delta y_3 = 0$$

$$\ddot{y}_1 = \frac{-(\ddot{y}_2 + \ddot{y}_3)}{2} \ (*) \ , \ \frac{m_1}{2}(g - \ddot{y}_1) + m_2(\ddot{y}_2 - g) = 0 \ (**) \ , \ \frac{m_1}{2}(g - \ddot{y}_1) + m_3(\ddot{y}_3 - g) = 0 \ (***)$$

Putting $(*)$ into $(**)$: $\frac{m_1}{2}\left(g + \frac{\ddot{y}_2 + \ddot{y}_3}{2}\right) + m_2(\ddot{y}_2 - g) = 0$, $\frac{m_1}{2} g + \frac{m_1\ddot{y}_2}{4} + \frac{m_1\ddot{y}_3}{4} + m_2\ddot{y}_2 - m_2 g = 0$

$$\ddot{y}_2\left(\frac{m_1}{4} + m_2\right) + \frac{m_1}{2} g - m_2 g + \frac{m_1}{4}\ddot{y}_3 = 0 \ : \ \ddot{y}_2\left(\frac{m_1}{4} + m_2\right) = m_2 g - \frac{m_1}{2} g - \frac{m_1}{4}\ddot{y}_3$$

$$\ddot{y}_2 = m_2 g - \frac{m_1}{2}(g + \ddot{y}_3/2) \Big/ \frac{m_1 + 4m_2}{4} = (4m_2 g - 2m_1(g + \ddot{y}_3/2))/\cancel{4} / (m_1 + 4m_2)/\cancel{4}$$

$$\ddot{y}_2 = 4m_2 g - 2m_1\left(\frac{2g + \ddot{y}_3}{2}\right) / 4m_2 + m_1 = 4m_2 g - 2m_1 g - m_1\ddot{y}_3 / (4m_2 + m_1)$$

$$\ddot{y}_2 = 2g(2m_2 - m_1) - m_1\ddot{y}_3 / (4m_2 + m_1)$$

$$\ddot{y}_2 = \frac{2g(2m_2 - m_1) - m_1\ddot{y}_3}{4m_2 + m_1} \ (4*)$$

Putting $(*)$ into $(***)$: $\frac{m_1}{2}(g - \ddot{y}_1) + m_3(\ddot{y}_3 - g) = 0$, $m_3(\ddot{y}_3 - g) = \frac{m_1}{2}(\ddot{y}_1 - g)$, $\ddot{y}_3 = \frac{1}{2}\frac{m_1}{m_3}(\ddot{y}_1 - g) + g$

$$\ddot{y}_3 = \frac{1}{2}\frac{m_1}{m_3}\ddot{y}_1 - \frac{1}{2}\frac{m_1}{m_3} g + g = \frac{1}{2}\frac{m_1}{m_3}\left(-\frac{\ddot{y}_2 + \ddot{y}_3}{2}\right) - \frac{1}{2}\frac{m_1}{m_3} g + g = -\frac{1}{4}\frac{m_1}{m_3}\ddot{y}_2 - \frac{1}{4}\frac{m_1}{m_3}\ddot{y}_3 - \frac{1}{2}\frac{m_1}{m_3} g + g$$

Problem 2: Continued

$$\ddot{y}_3 + \frac{1}{4}\frac{m_1}{m_3}\ddot{y}_3 = g - \frac{1}{2}\frac{m_1}{m_3}g - \frac{1}{4}\frac{m_1}{m_3}\ddot{y}_2 \;,\; \ddot{y}_3\left(1 + \frac{1}{4}\frac{m_1}{m_3}\right) = g\left(1 - \frac{1}{2}\frac{m_1}{m_3}\right) - \frac{1}{4}\frac{m_1}{m_3}\ddot{y}_2$$

$$\ddot{y}_3\left(\frac{4m_3 + m_1}{4m_3}\right) = \frac{4m_3 g - 2m_1 g - m_1\ddot{y}_2}{4m_3} \;,\; \ddot{y}_3 = \frac{4m_3 g - 2m_1 g - m_1\ddot{y}_2}{4m_3 + m_1}$$

$$\ddot{y}_3 = \frac{2g(2m_3 - m_1) - m_1\ddot{y}_2}{4m_3 + m_1} \quad \text{(5*)}$$

Putting (4*) into (5*): $\ddot{y}_3 = \dfrac{\overbrace{2g(2m_3 - m_1)}^{\alpha}}{\underbrace{4m_3 + m_1}_{\gamma}} - \dfrac{m_1}{4m_3 + m_1}\,\dfrac{\overbrace{2g(2m_2 - m_1) - m_1\ddot{y}_3}^{\beta}}{\underbrace{4m_2 + m_1}_{\lambda}}$

$$\ddot{y}_3 = \frac{\alpha}{\gamma} - \frac{m_1}{\gamma}\cdot\frac{\beta - m_1\ddot{y}_3}{\lambda} = \frac{\alpha}{\gamma} - \left(\frac{m_1}{\gamma}\cdot\frac{\beta - m_1\ddot{y}_3}{\lambda}\right) = \frac{\alpha}{\gamma} - \left(\frac{m_1\beta - m_1^2\ddot{y}_3}{\gamma\lambda}\right) = \frac{\lambda\alpha - m_1\beta + m_1^2\ddot{y}_3}{\gamma\lambda}$$

$$\ddot{y}_3 - \frac{m_1^2\ddot{y}_3}{\gamma\lambda} = \ddot{y}_3\left(1 - \frac{m_1^2}{\gamma\lambda}\right) = \ddot{y}_3\left(\frac{\gamma\lambda - m_1^2}{\gamma\lambda}\right) = \frac{\lambda\alpha - m_1\beta}{\gamma\lambda} \;:\; \ddot{y}_3 = \frac{\lambda\alpha - m_1\beta}{\gamma\lambda - m_1^2}$$

$$\ddot{y}_3 = \frac{(4m_2 + m_1)(4gm_3 - 2gm_1) - (4gm_1m_2 - 2gm_1^2)}{(4m_3 + m_1)(4m_2 + m_1) - m_1^2} = \frac{16gm_2m_3 - 8gm_1m_2 + 4gm_1m_3 - \cancel{2gm_1^2} - 4gm_1m_2 + \cancel{2gm_1^2}}{16m_3m_2 + 4m_3m_1 + 4m_2m_1 + \cancel{m_1^2} - \cancel{m_1^2}}$$

$$\ddot{y}_3 = \frac{16gm_2m_3 - 12gm_1m_2 + 4gm_1m_3}{16m_3m_2 + 4m_3m_1 + 4m_2m_1} = \frac{4gm_2m_3 - 3gm_1m_2 + gm_1m_3}{4m_2m_3 + m_1m_2 + m_1m_3} = \frac{m_2g(4m_3 - 3m_1) + gm_1m_3}{m_2(4m_3 + m_1) + m_1m_3}$$

$$\ddot{y}_3 = \frac{g(m_2(4m_3 - 3m_1) + m_1m_3)}{m_2(4m_3 + m_1) + m_1m_3} \quad \text{(6*)}$$

Putting (5*) into (4*): $\ddot{y}_2 = \dfrac{\overbrace{2g(2m_2 - m_1)}^{\alpha}}{\underbrace{4m_2 + m_1}_{\gamma}} - \dfrac{m_1}{4m_2 + m_1}\,\dfrac{\overbrace{2g(2m_3 - m_1) - m_1\ddot{y}_2}^{\beta}}{\underbrace{4m_3 + m_1}_{\lambda}}$

$$\ddot{y}_2 = \frac{\alpha}{\gamma} - \frac{m_1}{\gamma}\cdot\frac{\beta - m_1\ddot{y}_2}{\lambda} = \frac{\alpha}{\gamma} - \left(\frac{m_1}{\gamma}\cdot\frac{\beta - m_1\ddot{y}_2}{\lambda}\right) = \frac{\alpha}{\gamma} - \left(\frac{m_1\beta - m_1^2\ddot{y}_2}{\gamma\lambda}\right) = \frac{\lambda\alpha - m_1\beta + m_1^2\ddot{y}_2}{\gamma\lambda}$$

$$\ddot{y}_2 - \frac{m_1^2\ddot{y}_2}{\gamma\lambda} = \ddot{y}_2\left(1 - \frac{m_1^2}{\gamma\lambda}\right) = \ddot{y}_2\left(\frac{\gamma\lambda - m_1^2}{\gamma\lambda}\right) = \frac{\lambda\alpha - m_1\beta}{\gamma\lambda} \;:\; \ddot{y}_2 = \frac{\lambda\alpha - m_1\beta}{\gamma\lambda - m_1^2}$$

$$\ddot{y}_2 = \frac{(4m_3 + m_1)(4gm_2 - 2gm_1) - (4gm_1m_3 - 2gm_1^2)}{(4m_2 + m_1)(4m_3 + m_1) - m_1^2} = \frac{16gm_2m_3 - 8gm_1m_3 + 4gm_1m_2 - \cancel{2gm_1^2} - 4gm_1m_3 + \cancel{2gm_1^2}}{16m_2m_3 + 4m_1m_2 + 4m_1m_3 + \cancel{m_1^2} - \cancel{m_1^2}}$$

$$\ddot{y}_2 = \frac{16gm_2m_3 - 12gm_1m_3 + 4gm_1m_2}{16m_2m_3 + 4m_1m_2 + 4m_1m_3} = \frac{4gm_2m_3 - 3gm_1m_3 + gm_1m_2}{4m_2m_3 + m_1m_2 + m_1m_3} = \frac{m_3g(4m_2 - 3m_1) + gm_1m_2}{m_2(4m_3 + m_1) + m_1m_3}$$

$$\ddot{y}_2 = \frac{g(m_3(4m_2 - 3m_1) + m_1m_2)}{m_2(4m_3 + m_1) + m_1m_3} \quad \text{(7*)}$$

Putting (6*) and (7*) into (*): $\ddot{y}_1 = \dfrac{-1}{2}\left(\dfrac{g(m_3(4m_2 - 3m_1) + m_1m_2)}{m_2(4m_3 + m_1) + m_1m_3} + \dfrac{g(m_2(4m_3 - 3m_1) + m_1m_3)}{m_2(4m_3 + m_1) + m_1m_3}\right)$

Problem 2: Continued

$$\ddot{y}_1 = -\frac{1}{2}\left(\frac{4gm_2m_3 - 3gm_1m_3 + gm_1m_2 + 4gm_2m_3 - 3gm_1m_2 + gm_1m_3}{m_2(4m_3+m_1) + m_1m_3}\right)$$

$$\ddot{y}_1 = -\frac{1}{2}\left(\frac{8gm_2m_3 - 2gm_1m_3 - 2gm_1m_2}{m_2(4m_3+m_1)+m_1m_3}\right) = \frac{gm_1m_2 + gm_1m_3 - 4gm_2m_3}{m_2(4m_3+m_1)+m_1m_3} = \frac{g(m_1(m_2+m_3)-4m_2m_3)}{m_2(4m_3+m_1)+m_1m_3}$$

$$\ddot{y}_1 = \frac{g(m_1(m_2+m_3)-4m_2m_3)}{m_2(4m_3+m_1)+m_1m_3} \quad (8*)$$

$$\boxed{\ddot{y}_1 = \frac{g(m_1(m_2+m_3)-4m_2m_3)}{m_2(4m_3+m_1)+m_1m_3}, \quad \ddot{y}_2 = \frac{g(m_3(4m_2-3m_1)+m_1m_2)}{m_2(4m_3+m_1)+m_1m_3}, \quad \ddot{y}_3 = \frac{g(m_2(4m_3-3m_1)+m_1m_3)}{m_2(4m_3+m_1)+m_1m_3}}$$

(f) What is the acceleration of each mass? From your equations, identify conditions on the masses such that m_1 would be stationary? Does your result make sense?

$$\ddot{y}_1 = \frac{g(m_1(m_2+m_3)-4m_2m_3)}{m_2(4m_3+m_1)+m_1m_3}, \quad \ddot{y}_2 = \frac{g(m_3(4m_2-3m_1)+m_1m_2)}{m_2(4m_3+m_1)+m_1m_3}, \quad \ddot{y}_3 = \frac{g(m_2(4m_3-3m_1)+m_1m_3)}{m_2(4m_3+m_1)+m_1m_3}$$

For m_1 to be stationary, $\ddot{y}_1 = 0$. $g(m_1(m_2+m_3)-4m_2m_3) = 0$, $m_1 = \frac{4m_2m_3}{m_2+m_3}$

$$\boxed{m_1 = \frac{4m_2m_3}{m_2+m_3}}$$

This makes sense because m_1 would have to be greater than m_2 and m_3 for it to not move in any matter to counteract the weight of the other two masses.

Problem 2: Review

Procedure:

- Begin by defining generalized co-ordinates and d'Alembert's principle

$$\sum_i (F_i + m_i\ddot{q})\delta m_i = 0$$

  for each scenario outlined in this problem.

- Identify that the virtual work are independent of one another and this can be used to solve the problem.
- Do some algebra and solve for the accelerations of each mass.
- Determine when the masses would be stationary by setting the accelerations would be zero.

Key Concepts:

- Virtual works are independent of one another and this can be used to solve for accelerations of each mass.
- You can always determine the number of generalized co-ordinates with

$$L = ND - m$$

  where L is generalized co-ordinates, N is moving objects, D is directions of movement, and m are the total number of constraints in the system.

- We can determine mass relations in the three pulley system by setting accelerations equal to zero and then solving for a mass.

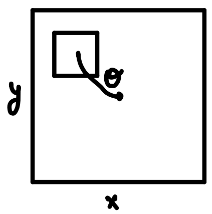
Variations:

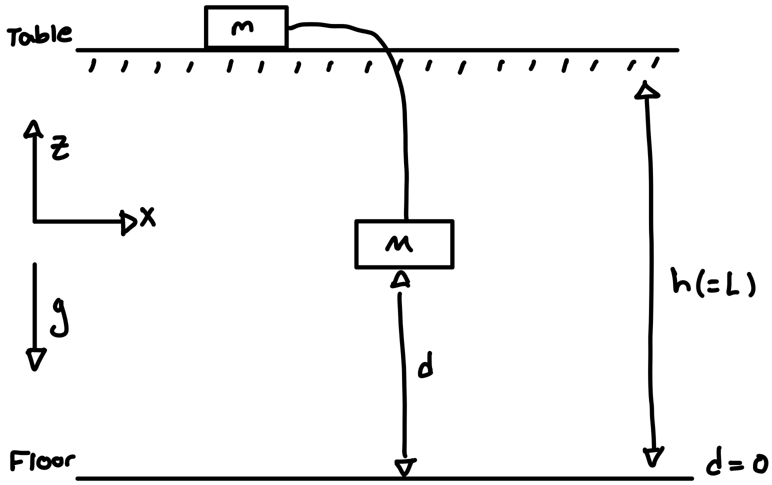
- We can be given different systems to analyze
  - We would use the same procedure but with different systems.

Problem 3:

Consider a pair of blocks of mass M and m connected by a massless string. The former mass lies on top of a table while the latter mass hangs below, in a configuration shown in Fig. 4. You may assume the motion of the hanging block is restricted to be only in the vertical direction (e.g., along $\hat{z}$ only), while the block on the table is restricted to move in a 2D plane (e.g., the $\hat{x}-\hat{y}$ plane defined by the table's surface). Moreover, assume the string is precisely the same length as the height of the table above the ground.

(a) Use the Lagrange formalism to write down the equations of motion for appropriate generalized co-ordinates.





Assumption: We will assume that the remaining length on the table is r. We will also say that the distance of mass M from the bottom of the ground is z. With the length of the string as L we can relate this as: $r = z$

We will generalize our co-ordinates to: r, θ, z

$$L = T - U : \quad T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}M(\dot{z}^2), \quad U = Mgz \quad \text{w/} \quad r = z, \ \dot{r} = \dot{z}$$

$$x = r\cos\theta, \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \quad \dot{x}^2 = (\dot{r}^2\cos^2\theta - 2\dot{r}\dot{\theta}r\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta)$$

$$y = r\sin\theta, \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \quad \dot{y}^2 = (\dot{r}^2\sin^2\theta + 2\dot{r}\dot{\theta}r\cos\theta\sin\theta + r^2\dot{\theta}^2\cos^2\theta)$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\cos^2\theta \cancel{- 2\dot{r}\dot{\theta}r\cos\theta\sin\theta} + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta \cancel{+ 2\dot{r}\dot{\theta}r\cos\theta\sin\theta} + r^2\dot{\theta}^2\cos^2\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\overbrace{(\cos^2\theta + \sin^2\theta)}^{1} + r^2\dot{\theta}^2\overbrace{(\cos^2\theta + \sin^2\theta)}^{1} = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M(\dot{r}^2) - Mgr$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 : \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r} + M\dot{r}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} + M\ddot{r} = (M+m)\ddot{r}, \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - Mg$$

$$mr\dot{\theta}^2 - Mg - (M+m)\ddot{r} = 0 \quad : \quad \ddot{r}(M+m) = mr\dot{\theta}^2 - Mg$$

$$\boxed{\ddot{r} = \frac{mr\dot{\theta}^2 - Mg}{M+m}}$$

Problem 3: Continued

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \ : \ \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \ , \ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2mr\dot{\theta}\dot{r} + mr^2\ddot{\theta} \ , \ \frac{\partial L}{\partial \theta} = 0$$

$$2mr\dot{\theta}\dot{r} + mr^2\ddot{\theta} = 0 \ : \ \cancel{m}r^{\cancel{2}}\ddot{\theta} = -2\cancel{m}\cancel{r}\dot{\theta}\dot{r} \ : \ \ddot{\theta} = -2\dot{\theta}\dot{r}/r$$

$$\boxed{\ddot{\theta} = -2\dot{\theta}\dot{r}/r}$$

(b) Discuss the physical interpretation of the equations derived in (a) and identify any relevant conserved quantities.

The acceleration due to r is dependent upon M and m where as the acceleration due to θ is not dependent upon either mass. The conserved quantities in this problem are energy. If this system was non-conservative we would not be able to use Euler-Lagranges method.

We can also say the movement in radial direction is preserved since there is no explicit dependence upon x, y, or z in our Lagrangian.

The lower block is stationary when the force due to rotation of upper block balances with gravity.

Problem 3: Review

Procedure:

- Begin by identifying generalized co-ordinates.
- Write out the Lagrangian in terms of x, y, and z.
- Convert from cartesian to a different co-ordinate system.
- Extremize the action (Lagrangian) with the Euler Lagrange equation

  $$\frac{\partial L}{\partial q_i} - \frac{d}{dp}\frac{\partial L}{\partial \dot{q}_i} = 0$$

  where q_i are our dependent variables and p are our independent variables.
- Solve for the equations of motion for r and θ.
- Comment on the physical interpretation of these equations.

Key Concepts:

- We can use Euler Lagrange formalism to extremize a Lagrangian and solve for equations of motion.

Variations:

- With Lagrangian questions we can be given a different system and this changes the entire problem but not the procedure.
- We can be asked to solve for different scenarios and interpret different scenarios.
  - We would use the same procedure and then use our equations of motion to solve for different scenarios.

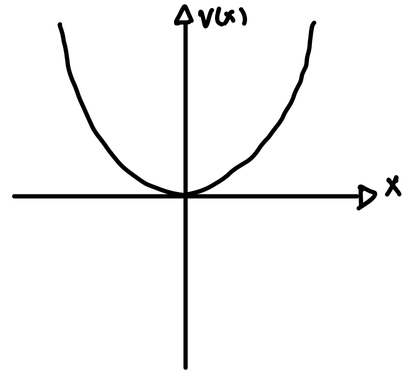
Problem 4:

This question serves as a useful introduction to some notions of *statistical distributions* in classical phase space. We will investigate two simple examples.

First, consider a massive particle in 1D subject to a harmonic potential with frequency $\omega = m = 1$.

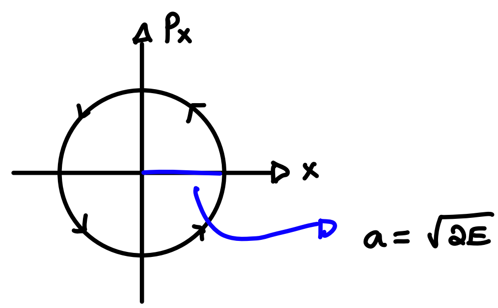
(a) Sketch a phase portrait of the system that gives a sufficient description of the general dynamics.

$$U(x) = \frac{1}{2} m \omega^2 x^2 \quad \text{w/} \quad m = \omega = 1 \quad \therefore \quad U(x) = \frac{1}{2} x^2, \quad K(x) = \frac{1}{2} x^2$$



$$E = KE + PE = \frac{1}{2} p_x^2 + \frac{1}{2} x^2 \; : \; E = \frac{1}{2} p_x^2 + \frac{1}{2} x^2 \; : \; 1 = \frac{p_x^2}{2E} + \frac{x^2}{2E} \; : \; 1 = \frac{p_x^2}{a^2} + \frac{x^2}{a^2}$$

$a = \sqrt{2E}$



Circle
$a^2 = p_x^2 + x^2$

(b) Consider an ensemble of points (e.g., a set of particles with a variety of initial conditions) that all fall within some region bounded by the circle $(x - x_0)^2 + (p - p_0)^2 = a$ where the radius satisfies $0 < a < \sqrt{x_0^s + p_0^2}/2$. The area of phase-space (typically referred to as the *phase space volume*) occupied by the ensemble is that of a circle πa^2. Use physical reasoning (i.e., 'hand-wavy' arguments) based off your solution to (a) to explain why the area of phase-space occupied by the ensemble is conserved in time. Your solution does not need to be qualitative.

From the above, $a = \sqrt{2E}$ and is only dependent upon the energy of the system. Because a is only dependent upon the energy, it will not change unless the energy of the system changes. Moreover, since the energy of the system is conserved in time, and the area is implicitly dependent upon the energy, we can say that the area of the phase-space is conserved in time. The distance between trajectories will not remain constant with time.

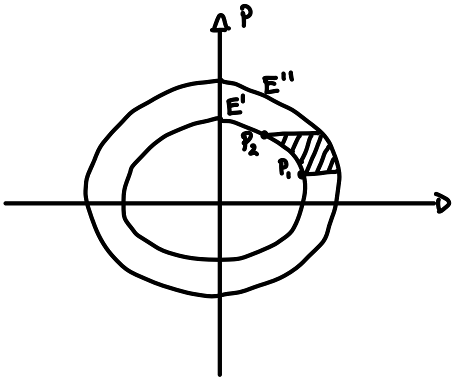
Now, instead consider a massive particle in 1D subject to gravity. A phase-portrait for a single-particle is shown in Fig. 4.

(c) Consider an ensemble of points in the phase-space confined to an area defined by the constraints $p_{\leq} p \leq p_2$ and $E' \leq E \leq E$" where p is the momentum and E the energy of the particle. Compute the area of phase space occupied by the ensemble.

$$E = \frac{p^2}{2m} + mgq, \quad \therefore \quad q(p) = \frac{1}{mg}\left(E - p^2/2m\right)$$

$$\frac{1}{mg} \int_{p_1}^{p_2} \int_{q_1(p)}^{q_2(p)} dq \, dp = \frac{1}{mg} (p_2 - p_1) \left[(E'' - p^2/2m) - (E' - p^2/2m) \right]$$

Problem 4: Continued



$$A = \frac{1}{mg}(E'' - E')(P_2 - P_1)$$

$$\boxed{A = \frac{1}{mg}(E'' - E')(P_2 - P_1)}$$

(d) By solving the equations of motion to yield $q(t)$ and $p(t)$, show/ argue that the phase space area enclosed by the evolving space occupied by the ensemble.

$$L = T - U \; : \; T = \frac{1}{2}m\dot{q}^2 \; , \; U = mgq \; : \; L = \frac{1}{2}m\dot{q}^2 - mgq \; : \; \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{\partial L}{\partial q} = -mg \; , \; \frac{\partial L}{\partial \dot{q}} = m\dot{q} \; , \; \frac{d}{dt} = m\ddot{q} \; : \; -m\ddot{q} - mg = 0 \; : \; \ddot{q} = -g$$

$$\ddot{q} = -g \rightarrow q(t) = q_0 + \dot{q}t - \frac{1}{2}gt^2 \; : \; p(t) = m \cdot \dot{q} = m\dot{q} - mgt$$

$$q(t) = q_0 + \dot{q}t - \frac{1}{2}gt^2 \; , \; p(t) = m\dot{q} - mgt$$

The ensemble of this system can be described by the area, because the area of the phase space is dependent upon q and p, and p is dependent upon energy, with energy remaining constant, the area of our phase space will remain constant over time and thus our ensemble is preserved.

The result that the phase-space volume is constant in time is a result of Liouville's theorem, which plays a crucial role in the relating deterministic classical mechanics to the more tractable framework of statistical mechanics. The latter allows us to describe the thermodynamic properties of macroscopic systems composed of *many* microscopic particles. We will revisit Liouville's theorem when we address Hamiltonian mechanics.

Problem 4: Review

Procedure:

- Begin by writing out the energy for a harmonic oscillator and sketching a phase portrait of this scenario.
- Comment on this ensemble and how it evolves in time.
- Calculate the area of the phase space by integrating between two curves.
- Solve for the equations of motion by exrtemizing the Lagrangian.

Key Concepts:

- The phase-space area will only change if there is a change in energy.

Variations:

- We can be asked to examine a different particle potentially with a different potential.
  - We would use the same formalism but with a different system.