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Quantum Mechanics 1

PHYS 5393 Homework Assignment #11

Problems: {3.23, 3.24, 3.25, 3.26}

Due: Never

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Problem 1: 3.23

The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z)f(r).$$

(a) Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the values of l we may obtain when $\mathbf{L}^2$ is measured?

Start with the wave function in spherical co-ordinates

$$\Psi(\vec{x}) = r[\cos\varphi\sin\theta + \sin\varphi\sin\theta + 3\cos\theta]f(r)$$

This in terms of spherical Harmonics is

$$\Psi(r,\theta,\varphi) = \sqrt{\frac{8\pi}{3}}\left[\frac{Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi)}{2} + \frac{Y_1^{-1}(\theta,\varphi) - Y_1^{-1}(\theta,\varphi)}{2i} + \frac{3}{\sqrt{2}}Y_1^0(\theta,\varphi)\right]rf(r)$$

where by inspection we can see that this is an eigenfunction of $\tilde{L}^2$ with eigenvalue $\ell=1$. We can also see it is not an eigenstate of $\tilde{L}_z$.

(b) What are the probabilities for the particle to be found in various m_l states?

To calculate the probabilities of the m_ℓ states, we square the magnitude and divide by the sum of the three:

$$\boxed{m=-1 \Rightarrow P = \frac{1}{11}, \quad m=0 \Rightarrow P = \frac{9}{11}, \quad m=1 \Rightarrow \frac{1}{11}}$$

(c) Suppose it is known somehow that $\psi(x)$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

The wave function is known as

$$\Psi(\vec{x}) = F_\ell(\theta,\varphi)rf(r) \quad w/\ \ell=1$$

Therefore we have,

$$\left[-\frac{\hbar^2}{2m}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)\right]R_{E\ell}(r) = ER_{E\ell}(r) \quad \therefore$$

$$\boxed{V(r)rf(r) = \left[\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)\hbar^2}{2mr^2} + E\right]rf(r)}$$

Problem 1: 3.23 Review

Procedure:

- Begin by writing out the wave function in spherical co-ordinates

$$\psi(\vec{\mathbf{x}}) = r[\cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta]f(r).$$

- Proceed to use equations found in Sakurai to express the Spherical Harmonics from the given values of l and m.
- Inspect the Spherical Harmonic function and identify that this is an eigenfunction of $\tilde{\mathbf{L}}^2$ with eigenvalue $l = 1$ but not an eigenfunction of $\tilde{\mathbf{L}}_z$.
- Calculate the probabilities by squaring the magnitudes of the respective m_l states and then dividing by the sum of all the states.
- Write the wave function as

$$\psi(\vec{\mathbf{x}}) = F_l(\theta, \phi) r f(r) \quad \text{w/} \quad l = 1$$

  and proceed to solve the differential equation.

Key Concepts:

- Because this problem is spherically symmetric we use the wave function in spherical co-ordinates.
- The probabilities of the m_l states are simply the magnitude squared divided by the sum of all the states.
- We can write our wave function in the form of

$$\psi(\vec{\mathbf{x}}) = F_l(\theta, \phi) r f(r)$$

  and proceed to solve our differential equation.

Variations:

- We can be given a different initial wave function.
  - This would change the type of wave function we would use (Like the symmetries and what not) to solve the differential equation.
- We could be asked to solve for different quantities.
  - We then would have to use whatever equations are necessary to answer the question.

Problem 2: 3.24

A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}$$

Interpret this result semiclassically.

We begin by writing the angular momentum operators in terms of the ladder operators

$$\tilde{L}_x = \frac{1}{2}(\tilde{L}_+ + \tilde{L}_-) \Rightarrow \tilde{L}_x^2 = \frac{1}{4}(\tilde{L}_+^2 + \tilde{L}_+\tilde{L}_- + \tilde{L}_-\tilde{L}_+ + \tilde{L}_-^2)$$

$$\tilde{L}_y = \frac{i}{2}(\tilde{L}_+ - \tilde{L}_-) \Rightarrow \tilde{L}_y^2 = \frac{-1}{4}(\tilde{L}_+^2 - \tilde{L}_+\tilde{L}_- - \tilde{L}_-\tilde{L}_+ + \tilde{L}_-^2)$$

We now calculate the expectation values

$$\langle \tilde{L}_x \rangle = \frac{1}{2}\left[\langle l,m|\tilde{L}_+|l,m\rangle + \langle l,m|\tilde{L}_-|l,m\rangle\right] = \frac{1}{2}\left[\langle l,m|l,m+1\rangle + \langle l,m|l,m-1\rangle\right] = 0$$

$$\langle \tilde{L}_y \rangle = \frac{i}{2}\left[\langle l,m|\tilde{L}_+|l,m\rangle - \langle l,m|\tilde{L}_-|l,m\rangle\right] = \frac{i}{2}\left[\langle l,m|l,m+1\rangle - \langle l,m|l,m-1\rangle\right] = 0$$

$$\langle \tilde{L}_x^2 \rangle = \frac{1}{4}\left[\langle l,m|\tilde{L}_+^2|l,m\rangle + \langle l,m|\tilde{L}_+\tilde{L}_-|l,m\rangle + \langle l,m|\tilde{L}_-\tilde{L}_+|l,m\rangle + \langle l,m|\tilde{L}_-^2|l,m\rangle\right]$$

$$= \frac{1}{4}\left[\langle l,m|\tilde{L}_+\tilde{L}_-|l,m\rangle + \langle l,m|\tilde{L}_-\tilde{L}_+|l,m\rangle\right] = \frac{1}{2}\langle l,m|\tilde{L}^2 - \tilde{L}_z^2|l,m\rangle = \frac{1}{2}\left[l(l+1) - m^2\right]\hbar^2$$

$$\langle \tilde{L}_y^2 \rangle = \frac{-1}{4}\left[\langle l,m|\tilde{L}_+^2|l,m\rangle - \langle l,m|\tilde{L}_+\tilde{L}_-|l,m\rangle - \langle l,m|\tilde{L}_-\tilde{L}_+|l,m\rangle + \langle l,m|\tilde{L}_-^2|l,m\rangle\right]$$

$$= \frac{1}{4}\left[\langle l,m|\tilde{L}_+\tilde{L}_-|l,m\rangle + \langle l,m|\tilde{L}_-\tilde{L}_+|l,m\rangle\right] = \frac{1}{2}\langle l,m|\tilde{L}^2 - \tilde{L}_z^2|l,m\rangle = \frac{1}{2}\left[l(l+1) - m^2\right]\hbar^2$$

From the above we can see the result in the problem statement is true.

$\blacksquare$

Problem 2: 3.24 Review

Procedure:

- Write out the angular momentum operator in terms of the ladder operators.
- Calculate the expectation values for $\langle \tilde{\mathbf{L}}_x \rangle$ and $\langle \tilde{\mathbf{L}}_y \rangle$ along with $\langle \tilde{\mathbf{L}}_x^2 \rangle$ and $\langle \tilde{\mathbf{L}}_y^2 \rangle$.

Key Concepts:

- When the ladder operators act on an eigenstate of angular momentum we have the following relationship

$$\tilde{\mathbf{L}}_{\pm} |l, m\rangle \propto c\, |l, m \pm 1\rangle$$

  where c (In this case it is given to us in the problem statement) can be found in the textbook.
- We use the orthogonality of states to simplify these calculations to get an end result.
- Because of the spherical symmetry in this problem we should expect the expectation values of our operators to be the same.

Variations:

- We can be given different eigenvalues.
  - Thus giving us a different result for our expectation values.

Problem 3: 3.25

Suppose a half-integer l-value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2}^{1/2}(\theta,\phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2}^{1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}.$$

Now try to construct $Y_{1/2}^{-1/2}(\theta,\phi)$ (a) by applying L_- to $Y_{1/2}^{1/2}(\theta,\phi)$ and (b) using $L_- Y_{1/2}^{-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory results. (This gives an argument against half-integer l-values for orbital angular momentum.)

We first start with

$$\tilde{L}_- Y_{1/2}^{1/2}(\theta,\varphi) = \sqrt{(s+s_z)(s-s_z+1)}\,\hbar\, Y_{1/2}^{-1/2}(\theta,\varphi) = \hbar\, Y_{1/2}^{-1/2}(\theta,\varphi)$$

We then apply the $\tilde{L}_-$ operator in the position representation

$$-ie^{-i\varphi}\hbar\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right) ce^{i\varphi/2}\sqrt{\sin\theta} = -c\hbar e^{-i\varphi/2}\cot\theta\sqrt{\sin\theta}$$

Applying the lowering operator we get

$$-ie^{-i\varphi}\hbar\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right) ce^{i\varphi/2}\sqrt{\sin\theta} = c\hbar^2\frac{e^{-3i\varphi/2}}{2\sqrt{\sin^3\theta}}\left[\cos\theta - 2\sin^2\theta - \cos^2\theta\right] \neq 0$$

If we then solve the differential equation,

$$-ie^{-i\varphi}\hbar\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right) Y_{1/2}^{-1/2}(\theta,\varphi) = 0 \implies Y_{1/2}^{1/2}(\theta,\varphi) = ce^{-i\varphi/2}\sqrt{\sin\theta}$$

We can see from above this gives a contradiction.

Problem 3: 3.25 Review

Procedure:

- Begin by applying the lowering operator on our Spherical Harmonic.
- Write out $\tilde{\mathbf{L}}_-$ in the position representation.
- Apply this operator and then solve the differential equation.

Key Concepts:

- We cannot have half integers for angular momentum because when the ladder operators are used, they will produce contradicting results.

Variations:

- We could be given a different Spherical Harmonic.
  - It would still produce inconsistencies and thus show us why we cannot have half integers as values for angular momentum.

Problem 4: 3.26

Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m = 0, \pm 1$, and ± 2. (The spherical harmonics for $l = 0, 1$ and 2 given in Section B.5 in Appendix B may be useful.)

We first use Euler angles to calculate the rotation,

$$\tilde{D}(\beta)|2,0\rangle \quad \text{w/} \quad \tilde{D}(\beta) \equiv \tilde{D}(\alpha=0,\beta,\gamma=0)$$

Expand in a complete set,

$$\tilde{D}(\beta)|2,0\rangle = \sum_{m'} |2,m'\rangle\langle 2,m'|\tilde{D}(\beta)|2,0\rangle = \sum_{m'} |2,m'\rangle \tilde{D}^{(l)}_{m';0}(\beta)$$

The rotation is about the z-axis, so

$$\tilde{D}(\beta)|2,0\rangle = \sum_{m'} |2,m\rangle \sqrt{\frac{4\pi}{5}}\, Y_l^{m'*}(\beta,0)$$

The probabilities are then

$$P = |\langle 2,m|\tilde{D}(\beta)|2,0\rangle|^2 = \frac{4\pi}{5}|Y_l^{m'}(\beta,0)|^2$$

The probabilities for $m=0,\pm1,\pm2$ then become

$$\boxed{\begin{aligned} |\langle 2,\pm2|\tilde{D}(\beta)|2,0\rangle|^2 &= \frac{3}{8}\sin^4\beta \\ |\langle 2,\pm1|\tilde{D}(\beta)|2,0\rangle|^2 &= \frac{3}{2}\sin^2(\beta)\cos^2(\beta) \\ |\langle 2,0|\tilde{D}(\beta)|2,0\rangle|^2 &= \frac{1}{4}(3\cos^2(\beta)-1)^2 \end{aligned}}$$

Problem 4: 3.26 Review

Procedure:

- Use the Euler Angle rotation with $\alpha = \gamma = 0$.
- Use this rotation operator on the state $|2,0\rangle$ by expanding in a complete set to arrive at the result

$$\tilde{\mathbf{D}}(\beta)\,|2,0\rangle = \sum_{m'} |2,m\rangle \sqrt{\frac{4\pi}{5}} Y_l^{m'*}(\beta,0).$$

- Proceed to calculate the probabilities with

$$\mathcal{P} = |\,\langle 2,m|\tilde{\mathbf{D}}(\beta)|2,0\rangle\,|^2 = \frac{4\pi}{5}|Y_l^{m'}(\beta,0)|^2.$$

Key Concepts:

- Because this is a rotation about the z axis, this means $\alpha = \gamma = 0$ in the Euler rotation.
- We have to expand in a complete set to see how our operator acts on the state that is given to us.
- We calculate the probability by doing the modulus squared of our state that we want to know with our operator acting on our given state.

Variations:

- We can be asked to find the probabilities for a different state.
  - We would use the same formalism but with a different initial state.