Homework Assignment #10

Math Methods

Due: Wednesday, December 1st

Instructions:

Below is the a list of questions and problems from the texbook. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

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1. Byron & Fuller, Chapter 5, problem 2.

2. Use Mathematica or some other symbolic calculational software to calculate the expansion of the following functions by the first eight Legendre polynomials, and plot the expansion and the original function in the interval $-1 < x < 1$.

   (a) $f(x) = |x|$

   (b) $f(x) = \Theta(x)$, (the Heaviside step function).

   Which function is better approximated by the expansion you calculated?

3. Use Mathematica or any similar symbolic manipulation program to Gram-Schmidt orthonormalize the first five polynomials, $\{1, x, x^2, x^3, x^4\}$ is the interval $-\infty < x < \infty$, where the inner product is:

   $$(f(x), g(x)) \equiv \int_{-\infty}^{\infty} f(x)\, g(x)\, e^{-x^2}\, dx$$

   How do your results compare to the Hermite polynomials?

   Hint: You might look at the Mathematica command **Orthogonalize**.

4. **Fourier Analysis:**

   (a) Calculate the discrete Fourier transform of the following functions on the interval $[-\pi, \pi]$, using the sine and cosine basis described on p.241 of the text.

   (*i*)
   $$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

   (*ii*)
   $$f(x) = \frac{|x|}{\pi}$$

   You may do this by hand or by computer.

(b) For both (i) and (ii) use a computer to sum the series:

$$f_n(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

numerically for N =5, 10, 100, and 200, for $-\pi/10 < x < \pi/10$, plotting the results.

(c) Comment on how well the Fourier series can reproduce a discontinuous function, or a function with a discontinuous derivative. How well would you expect it to work on the function:

$$f(x) = \frac{x|x|}{\pi^2}$$

(Parts (a) and (b) worth 10 points, part (c) worth 5 points.)

5. **Simple Fourier Application:** Suppose we have a fourth order differential equation

$$\mathcal{L}\, y(x) = y'''' + \alpha\, y'' + \beta y = x^2 - x$$

defined in the interval $0 \leq x \leq 1$ with the boundary conditions:

$$\begin{aligned} y(0) = &\quad y(1) = 0 \\ y''(0) = &\quad y''(1) = 0 \end{aligned} \tag{1}$$

We may write $y(x)$ in sine series:

$$y(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

(a) The standard fourier expansion differs from the above expression - we have only the sine terms. Why?

(b) Insert the above expression for $y(x)$ into the differential equation, and then take the inner product with $\sin(m\pi x)$, obtaining an algebraic equation for a_m.

(c) Write down the full solution for $y(x)$ in terms of the Fourier sum.

(d) Use the solvability condition to state when the problem will have a solution. Give an example of values for α and β for which the problem will not have a solution.

(Entire problem worth 15 points.)

6. Consider an electron in a box of width L, so that $\psi(0) = \psi(L) = 0$. The time-independent Schrodinger equation is given by

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - e\mathcal{E}\, x \right\} \psi(x) = E\psi(x) \tag{2}$$

(a) Make the change to dimensionless variables. Determine what it means for the electric field to be a "small" perturbation.

(b) In the limit that $\mathcal{E} = 0$, what are the eigenenergies and normalized eigenstates?

(c) In the limit that $\mathcal{E}$ is "small", what is the first order correction to the groundstate energy?

(d) In the limit that $\mathcal{E}$ is "small", what is the second order correction to the groundstate energy?