E & M I

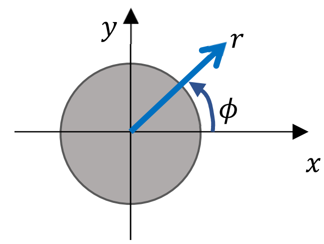
Workshop 11 – 2D Cylindrical Coordinates, 4/20/2022

1) General Solutions

We'll be getting to Cylindrical Coordinates and Bessel Functions in a bit, but let's start with a similar problem but without the z-dependence.

Consider a very long cylindrical object, say a long dielectric cylinder. The natural coordinates to describe this are 2D polar coordinates.

Here I'll use r as the radial coordinate and ϕ as the azimuthal coordinate. (These are the same as spherical coordinates in the x-y plane.)



Because of the large (infinite) extent of the cylinder, symmetry arguments state that solutions to Laplace's equation are independent of z.

If $\Phi(\vec{r})$ is the electric potential, everywhere there is no charge it must satisfy:

$$\nabla^2\, \Phi(\vec{r}) = \frac{1}{r}\partial_r\, r\, \partial_r\, \Phi(\vec{r}) + \frac{1}{r^2}\, \partial_\phi^2\, \Phi(\vec{r}) = 0$$

A) Show that you can use separation of variables to find solutions to $\Phi(\vec{r})$:

$$\Phi(\vec{r}) = R(r)\, \chi(\phi)$$

Derive the differential equations for $R(r)$ and $\chi(\phi)$.

B) The differential equation for $\chi(\phi)$ is of the form:

$$\partial_\phi^2\, \chi_\nu(\phi) = -\nu^2\, \chi_\nu(\phi)$$

Solve for the solutions to this equation and explain why you must have $\nu = n$, an integer.

This being a second order differential equation, you'll have two solutions for each value of n. Show, however, that there is just one physical solution for $n = 0$.

C) Using the results from B, solve for the functions $R_n(r)$ corresponding the values of $n \neq 0$ from the solution to the angular equation. Again, for a second-order differential equation, there will be two solutions for each n.

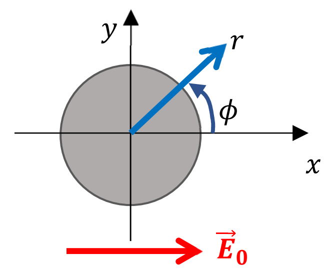
Hint: You might want to start with the simplest possible solutions you can think of.

D) Determine the two solutions for the $n = 0$ radial function $R_0(r)$. Explain why one of these solutions is not physical while the second one is a potential you should recognize.

Hint: Consider the geometry of the problem.

E) Pull together your results from parts B, C, and D to write the most general solution to Laplace's equation for a 2D cylindrical problem.

2) Cylinder in a Constant Field



Next, considering putting the dielectric cylinder in a constant electric field perpendicular to the axis of the cylinder:

$$\vec{E}_0 = E_0\,\hat{x}$$

The cylinder has a radius R and no free charge.

You will use the general solution you found in Question 1 to determine the potential and electric field everywhere.

This is like homework 4 where you considered a conducting sphere in a constant field. See if you can apply the same reasoning as the homework problem to solve the problem in this case.

A) What is the functional form for the total potential $\Phi(\vec{r})$ for $x \to \infty$?

As usual, we'll use two separate potentials inside and outside of the sphere $\Phi_{in}(\vec{r})$ and $\Phi_{out}(\vec{r})$

B) Using part A, explain why there will only be two terms in the expansion for $\Phi_{out}(\vec{r})$. Determine the coefficient (constant) for one of these terms.

C) Considering the limit $r \to 0$, write down a general expansion (sum) for $\Phi_{in}(\vec{r})$.

D) What are the boundary conditions on Φ at $r = R$? Remember that this is a dielectric.

E) Use the boundary conditions to solve for both $\Phi_{in}(\vec{r})$ and $\Phi_{out}(\vec{r})$ everywhere.

F) Solve for the electric field $\vec{E}_{in}(\vec{r})$ and the Polarization $\vec{P}(\vec{r})$ inside the cylinder, and the bound surface charge, σ_b. Remember the definitions:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}, \qquad \vec{D} = \epsilon\,\vec{E}, \qquad \sigma_b = \vec{P}\cdot\hat{n}$$

G) Solve the total electric field outside of the cylinder, $\vec{E}_{Tot}(\vec{r})$.

Show that $\vec{E}_{Tot}$ is the sum of three terms: $\vec{E}_0$, an electric field due to the cylinder that is in the same direction as $\vec{E}_0$ $(\hat{x})$, and a radial electric field due to the cylinder in the $\hat{r}$ direction. Hint: On this last part, it may be useful to write

$$E_0 \cos\theta = \frac{\vec{E}_0 \cdot \vec{r}}{r}$$