E & M I
Homework 2

1) Multipole expansion:

Consider the second charge distribution considered in the Multipole Expansion workshop:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

a) Using the multipole expansions for the potential $\phi(\vec{r})$, calculate the third (quadrupole) term in for $\phi(\vec{r})$ for this charge distribution.

b) Write your result in spherical coordinates, r, θ, ϕ. Explain why your result doesn't depend on ϕ. In this, quadrupole, approximation, for what directions in space is the potential equal to zero? (Draw a picture.)

c) Using spherical coordinates, calculate the electric field of this charge distribution in the quadrupole approximation. Sketch the field in the x-z plane.

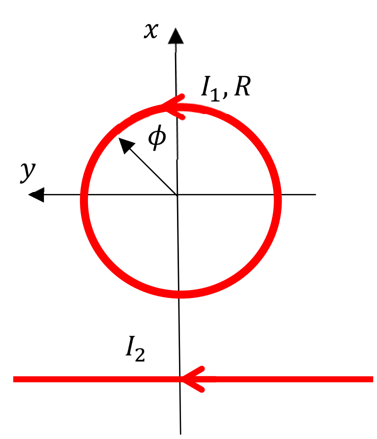
2) Charge-Dipole interaction:

Consider a point charge Q at the origin $\vec{r} = 0$ and an electric point dipole $\vec{p}$ at a position $\vec{r}_p$ (not at the origin). Calculate:

a) The potential energy of the Q & $\vec{p}$ system, by:
   i. Calculate the potential energy of $\vec{p}$ in the electric potential of Q
   ii. Calculate the potential energy of Q in the electric potential of $\vec{p}$.
   iii. Show these are equal.

b) The forces on $\vec{p}$ and Q, by:
   i. Calculate the force on $\vec{p}$ due to the field of Q
   ii. Calculate the force on Q due to the field of $\vec{p}$
   iii. Compare the results.

3) Magnetic Force on a Loop:



Consider the two currents shown: (i) a circular loop of radius R with counterclockwise current I_1 centered at the origin in the x-y plane and (ii) a very long straight wire with current I_2 parallel to the y-direction at $z = 0, x = -d$.

We're going to find solutions for the force on the loop, first by integration and then by considering the potential energy.

a) Predict the direction of the total force on the loop due to the magnetic field of the long-straight wire. Explain your prediction.

b) What is the magnetic field $\vec{B}_2(\vec{r})$ (magnitude and direction) due to the long wire everywhere in the x-y plane?

c) The force on the loop due to the magnetic field of the wire is:

$$\vec{F}_{21} = I_1 \oint d\vec{l} \times \vec{B}_2$$

It should be clear that we want to do this integral over the circle by integrating over $d\phi$.

i) Derive expressions for $\vec{B}_2(\vec{r})$ and $d\vec{l}$ in terms of R, ϕ, d, and $d\phi$ (for points on the current loop). Write these in terms of the $\hat{x}$ and $\hat{y}$ components.

Check your answer at a few simple points (such as for the angles $\phi = 0, \phi = \frac{\pi}{2}, \dots$)

iii) Write out an integral that gives the force on the loop. You should simplify this as much as possible, but you don't need to solve it, as it's somewhat messy.

Does the direction agree with your prediction?

d) Another approach to this problem is to determine the potential energy of the loop due to the magnetic field of the wire, U_{Loop}, and then use that the force is $F = -\vec{\nabla}\, U_{Loop}$. In this case you're interested in the change in energy as the loop moves relative to the wire

$$F_x = -\partial_d\, U_{Loop}$$

i) A current loop is equivalent to a sheet of small current loops, each a magnetic dipole (Fig. 4.7 in the textbook). Using small loops, $d\vec{S} = dA\,\hat{n}$, gives magnetic dipoles

$$d\vec{m} = I\, dA\, \hat{n}$$

The potential energy of the magnetic dipoles

$$dU(\vec{r}) = -d\vec{m} \cdot \vec{B}_2(\vec{r})$$

Write a surface integral for the potential energy of the loop. Use polar coordinates.

ii) Write the field $\vec{B}_2(\vec{r})$ everywhere inside the loop using polar coordinates. (This is a simple extension to part c above.)

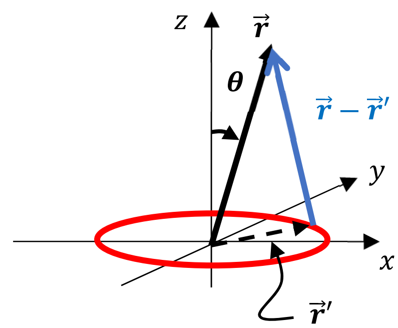
iii) Solve your integral to determine U_{Loop} and take the derivative to get F_y. Does your result make sense? It might be useful to know that:

$$\int_0^{2\pi} \frac{d\phi}{d + r\cos\phi} = \frac{2\pi}{\sqrt{d^2 - r^2}}$$

iv) Consider your result for the force on the loop in the limit that $d \gg R$. Show that this is equivalent to the force on a magnetic point dipole due to the magnetic field of the wire. Remember that the force on a dipole is

$$\vec{F} = \nabla\,(\,\vec{m} \cdot \vec{B}\,)$$

4) Back to the Ring:

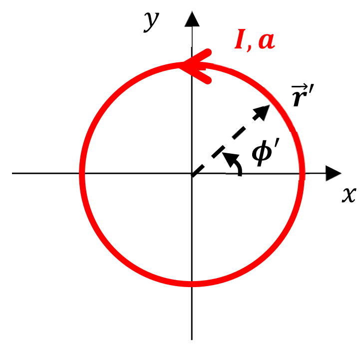


Consider one more time the current-carrying ring considered in class. The ring is in the x-y plane with radius a and a counter-clockwise current I. (Bottom picture)

As before, we want to calculate the magnetic field at a point

$$\vec{r} = r\cos(\theta)\ \hat{z} + r\sin(\theta)\ \hat{x}$$

In this case we'll look at this problem using a multipole expansion in spherical harmonics. We'll use the result:



$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{a^l}{r^{l+1}}\ Y_{lm}^*(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

a) Write down (or look up from class) and expression for the current density of the ring, $\vec{J}(\vec{r}')$ in terms of the spherical coordinates r', θ', ϕ'.

Remember that for a 1D distribution you'll need two δ-functions.

b) Using the definition for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \vec{J}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\ d^3r'$$

Write down the multipole (spherical harmonics) expansion for the vector potential. This should still include sums over l and m and the volume integral over $\vec{r}'$. Remember to include the vector direction(s) of the current density.

c) Perform the integrals over dr' and $d\theta'$ to determine an expression for $\vec{A}(\vec{r})$ just in terms of an integral on $d\phi'$. Of course, you'll still have the sums over l and m.

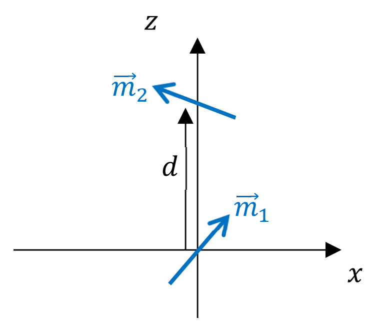
d) Show that the $l = 0$ term in the expansion is zero.

e) Calculate the $l = 1$ term in the multipole expansion. This will include the sum over $m = 0, \pm 1$.

f) Calculate the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Compare your results to those found in class, including the results for $\vec{B}(\vec{r} = z\,\hat{z})$ and for $r \gg a$.

For Fun) Try calculating the $l = 2$ term in the multipole expansion.

5) Magnetic Dipole Interactions in a Magnetic Field:



NOTE: In this problem, you can assume for all your answers that $\vec{m}_1$ and $\vec{m}_2$ are either parallel or anti-parallel. I'm fairly sure that this has to be true, but there are still a couple of cases where I need a more complete proof. If you want to prove this, please do. If not, you can assume it.

Two identical magnetic dipoles are shown, $\vec{m}_1$ at the origin and $\vec{m}_2$ at $\vec{r} = d\,\hat{z}$. The magnetic dipoles are free to rotate in the x-z plane (they don't have components in $\hat{y}$). To simplify the solution to this problem, define a quantity related to the magnetic field due to the dipoles:

$$B_d = \frac{\mu_0}{4\pi}\frac{m}{d^3}, \quad |\vec{m}_1| = |\vec{m}_2| = m$$

There is a uniform, constant magnetic field $\vec{B}$ in the x-z plane.

a) Write down an expression for the total potential energy of the two dipoles interacting with each other and the magnetic field.

b) First, consider the case where $\vec{B} = 0$. What are the are the configurations of the dipoles that give the lowest potential energy (there are two)? Show that adding a magnetic field $\vec{B} = B_{ext}\,\hat{z}$ will result in one configuration having the lowest energy.

c) If instead the magnetic field is $\vec{B} = B_{ext}\,\hat{x}$, the lowest energy configuration of the dipoles will depend on the magnitude of the field.

i) If $B_{ext} \ll B_d$ what do you expect the lowest-energy configuration of the dipoles to be? If $B_{ext} \gg B_d$ what do you expect the lowest-energy configuration of the dipoles to be? Explain.

ii) Assuming $\vec{m}_1$ and $\vec{m}_2$ remain parallel, determine the lowest energy configuration of the two dipoles as a function of the magnitude B_{ext}.

Show that there is a "critical" value, B_c, for the external field where the lowest-energy configuration changes abruptly to the dipoles being aligned with $\vec{B}$.