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Classical Mechanics

CH. 4 THE KINEMATICS OF RIGID BODY MOTION LECTURE NOTES

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Rigid Body Motion (ch. 4)

Rigid Body: Ensemble of point particles w/ constrained inter-particle distance

$r_{ij} = C_{ij} \ \forall_{i,j}$

N particles in 3D $\Rightarrow$ $3N$ co-ordinates / degrees of freedom

N^2 constraints $\rightarrow$ $\frac{N(N-1)}{2}$ "constraints"

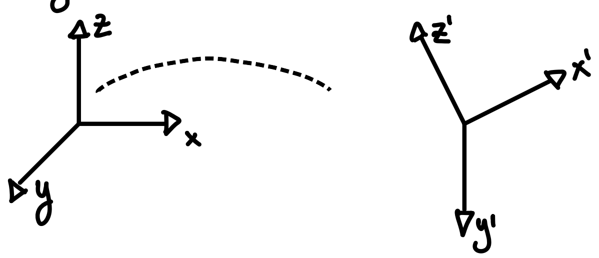
Actually $\Rightarrow$ 6 co-ordinates

* 3 co-ordinates to characterize position of body
* 3 co-ordinates to characterize orientation (rotations) of body

Frames / co-ordinate systems

$\Rightarrow$ Space-fixed x, y, z

$\Rightarrow$ Body-fixed Frame



$\rightarrow$ 3 Euler Angles

Orthogonal Transformations (4.2)

$\vec{r}' = A\vec{r}$ (e.g. rotation in 3D)

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ \& \ \vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} : \ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} : \ x \rightarrow x_1, \ y \rightarrow x_2 \dots \ x_i' = \sum_{i=1}^{3} a_{ij} x_j$$

Definition: An orthogonal linear transformation satisfies $\sum_{i=1}^{3} a_{ij} a_{ik} = \delta_{jk}$

$\Rightarrow$ 6 unique conditions

$A^{-1} = A^T$ $\rightarrow$ Hermition

Example: Rotations

$$R_3^{\alpha} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (R_3^{\alpha})^T = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (R_3^{\alpha})^{-1} = (R_3^{\alpha})^T$$

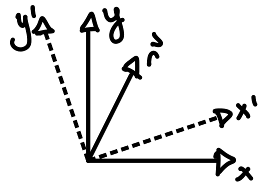
$$(R_3^\sigma)^T R_3^\sigma = \mathbb{1} \quad , \quad (R_3^\sigma)^{-1} R_3^\sigma = \mathbb{1}$$

Note: Can interpret relations as active or passive rotations

Active → Co-ordinate frame is static & vector is transformed

Passive → Co-ordinate frame rotates & vector is fixed

Notation → Passive rotation.



$$A\vec{r} = (\vec{r})'$$

Some formal properties:

i) $AB \neq BA$: matrix multiplication is non-commutative

ii) $(AB)C = A(BC)$: Matrix multiplication is associative

iii) |Determinant| is unity i.e $\det(A) = \pm 1$

$\det(AB) = \det(A)\det(B) = \pm 1$, rotations → $\det(A) = +1$

Euler-Angles (4.4)

⇒ θ, φ, ψ

Definition: Angles which* define three successive rotations about some axes (to be defined....) such that:

$$x, y, z \longrightarrow x', y', z'$$

Euler Angles → Define orientation of body-fixed axes w.r.t space-fixed

For any single rotation about an arbitrary axis, we can always decompose it into:

$A = BCD$, $\vec{r}\,' = A\vec{r}$, $\vec{r}\,' = (BCD)\vec{r}$, $\vec{r} = (D^{-1}C^{-1}B^{-1})\vec{r}\,'$

We associate: $B \to \psi$ angle , $C \to \theta$ angle , $D \to \varphi$ angle

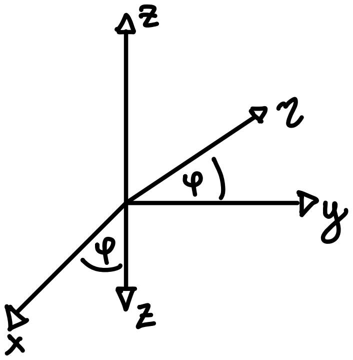
Choose B, C, D along specific axes, ⟶ Convention "$\underset{B}{z}\,\underset{C}{x}\,\underset{D}{z}$"

⇒ Cannot have consecutive rotations along the same axis

(see P.152) Fig 4.7

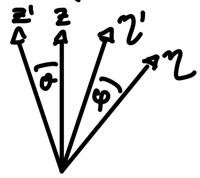
① Rotate counter-clockwise about z (space-fixed z-axis) by φ

$$D = (R_z^{\varphi}) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



② Rotate counter-clockwise about z (intermediate "x-axis") by θ

$$C = (R_x^{\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$



③ Rotate about z' (body-fixed z' axis) counter-clockwise by ψ

$$B = (R_z^{\psi}) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Finite</u> <u>Rotations</u> (4.7)

⇒ Euler angles → characterize <u>motion</u> of a rigid body

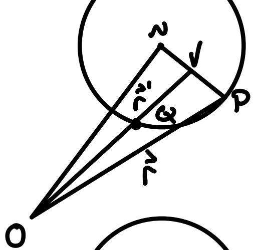
$A(t)$ → rotational dynamics $\theta(t), \psi(t), \varphi(t)$

<u>Euler's Rotation Theorem</u>: The general displacement of a rigid body w/ one point fixed is a rotation

<u>Interpretation</u>: motion is describable by a single rotation w/ axis passing through the fixed point

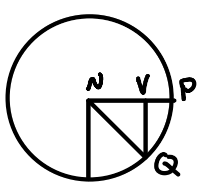
<u>Implication</u>: Do everything as one rotation (complementary for Euler angles)

<u>Define</u>: Rotation angle Φ Like to define mapping between $\vec{r}$ & $\vec{r}\,'$ in terms of Φ



⇒ Rotation maps $\vec{r} \to \vec{r}\,'$ ($P \to Q$)

$$\vec{r}\,' = \overrightarrow{ON} + \overrightarrow{NV} + \overrightarrow{VQ}$$



$$\hat{r} = \frac{\overrightarrow{ON}}{|\overrightarrow{ON}|}$$

A few tricks

i) $\vec{ON} = \hat{n}(\hat{n}\cdot\vec{r})$

ii) $|\vec{NV}| = |\vec{NQ}|\cos(\Phi) = |\vec{NP}|\cos(\Phi)$

$\Rightarrow$ Also, $\vec{NP} = |\vec{OP}| - |\vec{ON}| = \vec{r} - \hat{n}(\hat{n}\cdot\vec{r})$

$\vec{NV} = |\vec{NP}|\cos(\Phi)\,\hat{\vec{NP}} = \cos(\Phi)\vec{NP} = \cos(\Phi)[\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})]$

iii.) Same for, $\vec{VQ} = \sin(\Phi)\,\vec{r}\times\hat{n}$

i.) $\to$ iii.) : $\vec{r}\,' = \hat{n}(\hat{n}\cdot\vec{r}) + [\vec{r} - \hat{n}(\hat{n}\cdot\vec{r})]\cos(\Phi) + [\vec{r}\times\hat{n}]\sin(\Phi)$

Final Result,

$$\vec{r}\,' = \vec{r}\cos(\Phi) + \hat{n}(\hat{n}\cdot\vec{r})(1-\cos(\Phi)) + (\vec{r}\times\hat{n})\sin(\Phi)$$

$\Rightarrow$ Rotation formula for a clockwise rotation about $\hat{n}$ w/ angle Φ translate:

$$\cos(\Phi/2) = \cos((\varphi+\psi)/2)\cos(\theta/2)$$

Infinitesimal Rotations (4.8)

$\Rightarrow$ No vector associated w/ rotations

$\Rightarrow$ Fundamental problem $\vec{a} \to \vec{r} \to \vec{r}\,'$, $\vec{b} \to \vec{r}\,' \to \vec{r}\,''$, $\vec{a}+\vec{b} \to \vec{r} \to \vec{r}\,'' = \vec{b}+\vec{a}$

Take $\Phi \to d\Phi$

$\vec{r}\,' = \vec{r} + (\vec{r}\times\hat{n})\,d\Phi$ <— Infinitesimal rotation, $d\vec{r} = \vec{r}\,' - \vec{r} = \vec{r}\times d\vec{\Omega}$

Passive : $d\vec{r} = -\vec{r}\times d\hat{\Omega}$: $(d\Phi \to -d\Phi)$

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N particles, $\dot{\vec{r}}_j$: $r_{ij}' = c_{ij}$

* 3 position co-ordinates

* 3 orientation co-ordinates
  ↳ Euler angles <=> rotations

Convention : $z\,x\,z$

$\phi \to$ Rotation about space-fixed z-axis

$\theta \to$ Rotation about intermediate x-axis

$\psi \to$ Rotation about body-fixed z'-axis

$xyz \to$ Space , x', y', z' body

Single relation : Angle Φ about $\hat{n}$

$\Rightarrow$ Infinitesimal rotations : $\Phi \to d\Phi$, $\vec{r} \to \vec{r}\,'$, $d\vec{r} = \vec{r} \times d\Omega$ w/ $d\hat{\Omega} = d\Phi\,\hat{n}$

<u>Rate of change of a Vector (4.9)</u>

$\left(\frac{d\vec{G}}{dt}\right)$: $(d\vec{G})_{space} \neq (d\vec{G})_{Body}$ $\therefore$ $(d\vec{G})_{space} = (d\vec{G})_{Body} + (d\vec{G})_{rotation}$

If $\vec{G}$ is pinned inside a rigid body , $(d\vec{G})_{Body} = 0$ $\Rightarrow$ $(d\vec{G})_{space} = (d\vec{G})_{Rotation}$

$(d\vec{G})_{Body} = 0 \Rightarrow (d\vec{G})_{space} = (d\vec{G})_{rotation}$ $d\vec{\Omega} \times \vec{G}$, $(d\vec{G})_{space} = \vec{\omega} \times \vec{G}$

$\vec{\omega} = \frac{d\vec{\Omega}}{dt}$ $\to$ Instantaneous angular velocity , $\to$ lies along instantaneous axis of rot.

$\vec{\omega}$ can be expressed as, $\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$

$\vec{\omega}_\phi \to \dot{\phi}$ Along space-fixed z-axis , $\vec{\omega}_\theta \to \dot{\theta}$ along intermediate x-axis

$\vec{\omega}_\psi \to$ Along body fixed z'-axis, $\vec{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}$

Reverse of discussion of Euler angles.

① $\vec{\omega}_\psi = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$ <- Body-fixed z'

② $\vec{\omega}_\theta = B \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$, $B = R_z^\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

③ $\vec{\omega}_\phi = R_x^\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{pmatrix}$

Together : $\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$