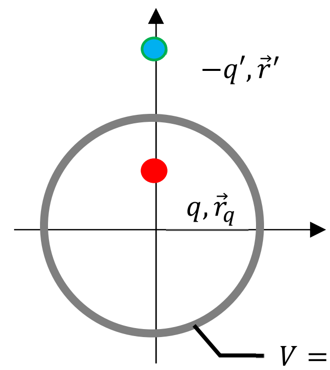
Physics 5573, Spring 2022
Test 2, Maxwell and Conductors

1) A grounded spherical conductor has a charge q inside at the position $\vec{r}_q$. You wish to determine the properties of this system everywhere inside the sphere.



Note: This problem is exactly the same as the problem done in class: Two charges placed relative to a sphere such that the potential on the sphere is everywhere equal to zero. Images.

A) How would you setup your coordinates to simplify the solution to this problem? Draw a picture.

Put both charges on the z-axis, at $\vec{r}_q = z_q\,\hat{z},\ \vec{r}' = z'\hat{z}$.

B) What general approach would you take to the solution to this problem? Explain, briefly, why you would take this approach to the problem?

Images. This is from experience in solving this problem previously and recognizing it as such.

C) What would be the general form of the solution for the potential inside the sphere, considering your approach to the problem? This doesn't mean you should solve the problem, just that you should give an expression that it would be possible to solve.

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\vec{r}-\vec{r}_q|} - \frac{q'}{|\vec{r}-\vec{r}'|}\right)$$

Note that I've explicitly included the sign of the image charge, $-q'$.

D) What approach would you take to go from the general solution from (C) to obtaining a complete solution to the problem? Give some details of how you would do this, but you don't need to do all the algebra to obtain a result.

There are two unknowns in the setup of this solution, q', and z'. Two equations are needed to solve for these. The simplest approach would be to use the points on the z-axis: $\vec{r} = \pm R\,\hat{z}$. This would require:

$$\frac{q}{|\pm R - z_q|} = \frac{q'}{|\pm R - z'|}$$

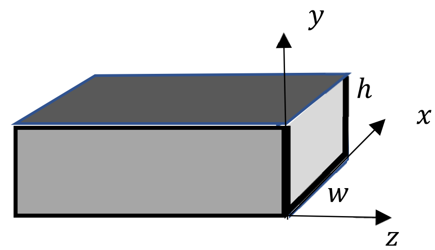
E) What questions would you ask yourself, what physics would you explore, and what calculations would you make to check your answer to this problem?

Check that potential is zero everyone on the sphere: $\phi(|\vec{r}| = R) = 0$

Calculate $\vec{E}$ everywhere in the sphere, $\vec{E}(\vec{r}) = -\vec{\nabla}\,\phi(\vec{r})$ and check that it is perpendicular to the sphere everyone on the (inside) surface.

Use $\vec{E}(R)$ on the sphere to calculate the surface charge density, and check that the total charge is equal to $-q'$.

2) A very long (assume infinite) rectangular box has a height h and a width w have the left, right, and bottom walls all grounded ($V = 0$). The top wall of the box has an electric potential, $V_{Top}(x)$, where "x" is the coordinate across the box, perpendicular to the "very long" direction.



A) Using cartesian coordinates as shown, the x direction is defined in the question and the z-direction is the "infinite" direction of the box. Because this problem is translationally invariant in z, the solution will be independent of z.

B) Solve Laplace's equation in Cartesian coordinates and separation of variables:

$$\partial_x^2\, \phi(x,y) + \partial_y^2\, \phi(x,y) = 0, \qquad \phi(x,y) = X(x)\, Y(y)$$

$$\frac{1}{X(x)}\, \partial_x^2\, X(x) + \frac{1}{Y(y)} \partial_y^2\, Y(y) = 0$$

$$\partial_x^2\, X(x) = -c^2\, X(x), \qquad \partial_y^2\, Y(y) = c^2\, Y(y)$$

C) The general solutions are products of:

$$X(x) = A_c \sin(c\, x) + B_c \cos(c\, x)\,, \qquad Y(y) = F_c \sinh(c\, y) + G_c \cosh(c\, y)$$

D) Using the fact that $X(0) = X(w) = 0$ and $Y(0) = 0$

$$B_c = 0, c = n\frac{\pi}{w}, G_c = 0$$

The general form for the potential is:

$$\phi(x,y) = \sum_n A_n \sin\left(n\frac{\pi}{w}x\right) \sinh\left(n\frac{\pi}{w}\, y\right)$$

The coefficients A_n can be determined from the potential on the top plate:

$$V_{Top}(x) = \sum_n A_n \sin\left(n\frac{\pi}{w}x\right) \sinh\left(n\frac{\pi}{w}\, h\right)$$

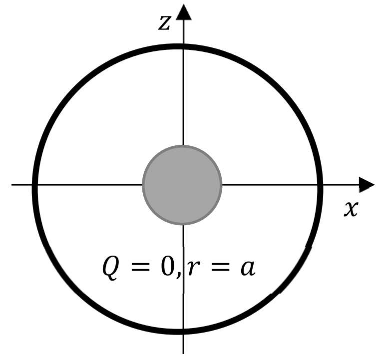
Considering the Fourier expansion:

$$\int_0^w \sin\left(m\frac{\pi}{w}\, x\right)\, V_{Top}(x) = \sum_n A_n \sinh\left(n\frac{\pi}{w}\, h\right) \int_0^w \sin(m\frac{\pi}{w}\, x) \sin\left(n\frac{\pi}{w}x\right)\, dx$$

$$\int_0^w \sin\left(m\frac{\pi}{w}\, x\right)\, V_{Top}(x) = \sum_n A_n \sinh\left(n\frac{\pi}{w}\, h\right) \left(\frac{w}{2}\right)\, \delta_{m,n}$$

E) Check that the electric field is perpendicular to the surfaces of the conducting box and that the potential in the center of the box is something like an average of the top and bottom potential.

3) Consider two concentric spheres. The outer sphere is an insulator and has a radius $r = b$. The inner sphere is a solid conducting sphere with $r = a$ and $Q = 0$ (no net charge). You want to distribute charge on the outer sphere so that the potential is $V(b,\theta) = V_0 \cos^2\theta$.



$V(b,\theta) = V_0 \cos^2\theta\,, r = b$

A) The definition of $V(b,\theta)$ gives the coordinates. Choose the z direction as the direction where $\theta = 0$.

B) With spherical boundaries, and azimuthal symmetry, we will do an expansion in Legendre Polynomials.

C) The form for the potential is:

$$\phi(r,\theta) = \sum_l \left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

The boundary conditions on the two spheres can be used to solve for the coefficients.

D) The potential on the inner sphere is a constant, so for all terms except $l = 0$,

$$l \neq 0, A_l\, a^l + \frac{B_l}{a^{l+1}} = 0, \;\; B_l = A_l\, a^{2l+1}$$

The potential on the outer sphere, $\cos^2\theta$, is a linear combination of P_0 and P_2 so for all other terms:

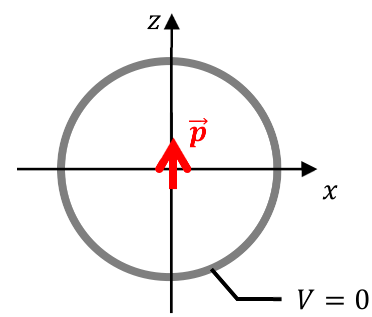
$$l \neq 0, 2, \qquad A_l\, b^l + \frac{B_l}{b^{l+1}} = 0, \;\; B_l = A_l\, b^{2l+1}$$

This means $A_l = B_l = 0$ for $l \neq 0, 2$ and $B_2 = A_2\, a^5$.

The other $l = 0, 2$ terms will be found from the potential on the outer sphere, and the requirement that $Q = 0$ on the inner sphere. The charge on the inner sphere can be found by calculating the electric field, using this to determine the surface charge, and setting the coefficients so that the charge is zero.

E) Again use the electric field to determine the charge densities on the surfaces.

4) A point dipole $\vec{p} = p\,\hat{z}$ is placed at the center of a grounded, conducting sphere of radius R. Solve for the electric properties inside the sphere.



A) The total potential is a sum of the dipole plus the sphere:

$$\phi_T(r,\theta) = \phi_p(r,\theta) + \phi_s(r,\theta)$$

Write the potential $\phi_p(r,\theta)$ in spherical coordinates. Remember

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

The potential of the dipole is:

$$\phi_P(r,\theta) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}$$

B) Give a general expression for $\phi_s(r,\theta)$ as an expansion (sum) in spherical coordinates. What are the boundary conditions that ϕ_s must satisfy?

The potential due to the sphere is:

$$\phi_s(r,\theta) = \sum_l \left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

The potential needs to be well behaved as $r \to 0$, so

$$\phi_s(r,\theta) = \sum_l A_l\, r^l\, P_l(\cos\theta)$$

On the surface of the sphere, the total potential is zero, meaning:

$$\phi_s(R,\theta) = -\phi_p(R,\theta) = -\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{R^2}$$

C) Solve for $\phi_s(r,\theta)$ and $\phi_T(r,\theta)$. Show your work.

$$\phi_s(R,\theta) = \sum_l A_l\, R^l\, P_l(\cos\theta) = -\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{R^2} = -\frac{1}{4\pi\epsilon_0}\frac{p}{R^2}\,P_1(\cos\theta)$$

Only the $l = 1$ term will remain,

$$A_1 = -\frac{1}{4\pi\epsilon_0}\frac{p}{R^3}$$

$$\phi_s(r,\theta) = -\frac{1}{4\pi\epsilon_0}\frac{p}{R^3}\,r\cos\theta$$

$$\phi_T(r,\theta) = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\left(1 - \frac{r^3}{R^3}\right)$$

D) Solve for the electric field inside the sphere. Show that your solution gives the correct boundary conditions for the electric field.

The electric field is:

$$\vec{E}(r,\theta) = -\vec{\nabla}\,\phi(r,\theta)$$

$$\vec{E}(r,\theta) = -\left(\hat{r}\,\partial_r + \frac{\hat{\theta}}{r}\,\partial_\theta\right)\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^2} - \frac{r}{R^3}\right)\cos\theta$$

$$\vec{E}(r,\theta) = \frac{p}{4\pi\epsilon_0}\left(\frac{2}{r^3} + \frac{1}{R^3}\right)\cos\theta\ \hat{r} + \frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\sin\theta\ \hat{\theta}$$

The tangential component of the field at the surface of the sphere is:

$$\hat{\theta}\cdot\vec{E}(R,\theta) = \frac{p}{4\pi\epsilon_0}\left(\frac{1}{R^3} - \frac{1}{R^3}\right)\sin\theta = 0$$

As required for a conductor.

E) What is the surface charge density on the sphere?

The surface charge on the conductor is related to the normal component of the electric field on the surface. The normal is $\hat{n} = -\hat{r}$.

$$\sigma(\theta) = \epsilon_0\left(-\hat{r}\cdot\vec{E}(R,\theta)\right)$$

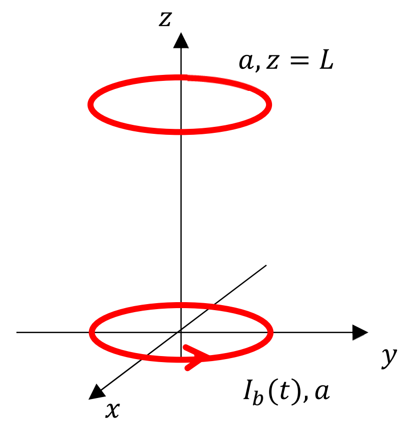
$$\sigma(\theta) = -\frac{p}{4\pi}\left(\frac{2}{R^3} + \frac{1}{R^3}\right)\cos\theta = -\frac{3}{4\pi R^3}\,p\cos\theta$$

This shows there is positive charge at the top of the sphere $\left(0 \le \theta < \frac{\pi}{2}\right)$, negative charge at the bottom of the sphere $\left(\frac{\pi}{2} < \theta \le \pi\right)$, and the integral of the surface charge is zero. This agrees with the physics:

Negative charge on the sphere will move to points of higher potential due to the dipole, and positive charge on the sphere will move to points of lower (negative) potential due to the dipole.

There is no net charge on the dipole, the electric field for $r > R$ but inside the metal of the conducting sphere must be zero, so there must be no net charge inside the sphere.

5) A pair of identical Helmholtz coils are centered on the z-axis. The coils have radius $r = a$, resistance R, and are separated by a distance $L \gg a$.



- Assume L is small enough that we can neglect the effects of the finite speed of light (retardation effects).
- Assume self-induction, the EMF in a coil due to the change in the magnetic field created by that coil, can be neglected

The current in the bottom coil is counterclockwise and increasing:

$$I_b(t) = I_0 \left(1 - e^{-\frac{t}{\tau}}\right)$$

A) What is the time-dependent magnetic moment of the bottom coil, $\vec{m}_b(t)$.

$$\vec{m}_b(t) = \pi\, a^2\, I_b(t)\, \hat{z} = A\, I_b(t)\, \hat{z}$$

B) Treating the bottom coil as a magnetic dipole, what is the magnetic field in the center of the top coil, $\vec{B}_b(z = L, t)$?

$$\vec{B}_b(\vec{r} = L\,\hat{z}, t) = \frac{\mu_0}{4\pi}\frac{1}{L^3}\left(3\,(\vec{m}(t)\cdot\hat{z})\,\hat{z} - \vec{m}(t)\right)$$

$$\vec{B}_b(\vec{r} = L\,\hat{z}, t) = 2\frac{\mu_0}{4\pi}\frac{\vec{m}(t)}{L^3} = \frac{\mu_0}{2\pi}\frac{A\, I_b(t)}{L^3}\,\hat{z}$$

C) Assume the field through the top loop is approximately a constant over the area of the loop at any time t (of course is changes in time):

$$\vec{B}_{bottom}(\vec{r}, t)\cdot\hat{n}_{top} \approx \text{Constant(t)}$$

Use Faraday's law to determine the EMF and current in the top coil. Include the direction of the current.

Faraday's Law gives:

$$EMF = \oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\iint \vec{B}_{bottom}(\vec{r}_{top}, t)\cdot\hat{z}\, dS$$

Approximating the field as constant in space over the area of the top loop, the right-hand side of the equation is just the field in the middle of the loop times the area, $\vec{B}_{bottom}$ being in the z-direction. This gives:

$$EMF = -\frac{d}{dt}\frac{\mu_0}{2\pi}\frac{A^2\, I_b(t)}{L^3}$$

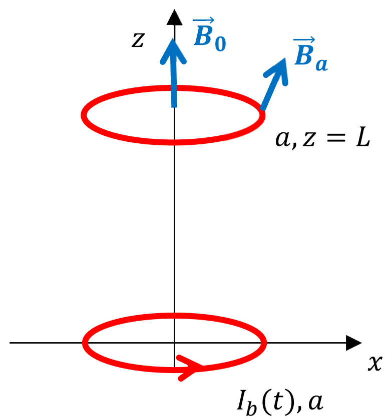
$$EMF = \frac{\mu_0}{2\pi}\frac{A^2}{L^3}\left(-\frac{d}{dt}\, I_0\left(1 - e^{-\frac{t}{\tau}}\right)\right) = -\frac{\mu_0}{2\pi}\frac{A^2}{L^3}\frac{I_0}{\tau}\, e^{-\frac{t}{\tau}}$$

$$I_{top}(t) = \frac{EMF}{R} = = -\frac{\mu_0}{2\pi}\frac{A^2}{L^3}\frac{I_0}{\tau\, R}\, e^{-\frac{t}{\tau}}$$

The current in the top loop is a maximum initially when the current in the bottom loop is changing fastest and goes to zero at long times when the current in the bottom loop goes to a constant final value.

To consider the direction: The normal to the top loop being $\hat{n} = \hat{z}$ determines the positive and negative directions around the loop for the EMF. The negative EMF and current here means the current is in the $-\hat{\phi}$ direction, $\hat{\phi}$ being the usual azimuthal direction around the z-axis.

D) Check the assumption made in Part C that the magnetic field through the top loop is approximately constant. You might, for example, determine how much the field changes from the middle to the edge of the coil as a function of the small parameter $\frac{a}{L}$.



To determine the variation of the magnetic field across the top loop, we need to consider the field due to the bottom loop in the center, $\vec{B}_0$, and the field at the edge of the loop, $\vec{B}_a$.

In fact, because the EMF depends on the flux through the loop, $\propto \vec{B} \cdot \hat{n}$, we only need to consider the component of the field in the z-direction.

Keeping the dipole approximation for the field due to the bottom loop, we have:

$$\vec{B}_{bottom}(x, z = L) = \frac{\mu_0}{4\pi} \frac{1}{(x^2 + L^2)^{\frac{3}{2}}} \left(3\ (\vec{m} \cdot \hat{r})\, \hat{r} - \vec{m}\right)$$

$$\hat{r} = \cos\theta\, \hat{z} + \sin\theta\, \hat{x}, \qquad \cos\theta = \frac{L}{\sqrt{x^2 + L^2}}, \qquad \sin\theta = \frac{x}{\sqrt{x^2 + L^2}},$$

$$\hat{z} \cdot \vec{B}_{bottom}(x, z = L) = \frac{\mu_0}{4\pi} \frac{1}{(x^2 + L^2)^{\frac{3}{2}}} \left(3\ m\cos^2\theta - m\right)$$

$$\hat{z} \cdot \vec{B}_{bottom}(x, z = L) = \frac{\mu_0}{4\pi} \frac{m}{(x^2 + L^2)^{\frac{3}{2}}} \left(3\cos^2\theta - 1\right)$$

$$\hat{z} \cdot \vec{B}_{bottom}(x, z = L) = \frac{\mu_0}{4\pi} \frac{m}{(x^2 + L^2)^{\frac{3}{2}}} \left(3\frac{L^2}{x^2 + L^2} - 1\right) = \frac{\mu_0}{4\pi} \frac{m\,(2L^2 - x^2)}{(x^2 + L^2)^{\frac{5}{2}}}$$

$$\hat{z} \cdot \vec{B}_{bottom}(x, z = L) = \frac{\mu_0}{4\pi} \frac{m}{L^3} \frac{\left(2 - \frac{x^2}{L^2}\right)}{\left(1 + \frac{x^2}{L^2}\right)^{\frac{5}{2}}}$$

Considering the points $x = 0$ and $x = a$:

$$B_0 = 2\frac{\mu_0}{4\pi}\frac{m}{L^3}$$

$$B_a = \frac{\mu_0}{4\pi}\frac{m}{L^3}\frac{\left(2-\frac{a^2}{L^2}\right)}{\left(1+\frac{a^2}{L^2}\right)^{\frac{5}{2}}} \approx \frac{\mu_0}{4\pi}\frac{m}{L^3}\left(2-\frac{a^2}{L^2}\right)\left(1-\frac{5}{2}\frac{a^2}{L^2}\right) \approx \frac{\mu_0}{4\pi}\frac{m}{L^3}\left(2-6\,\frac{a^2}{L^2}\right)$$

The relative change in the field across the loop is (the $\frac{\mu_0}{4\pi}\frac{m}{L^3}$ factor cancels out):

$$\frac{B_0 - B_a}{B_0} = \frac{2-\left(2-6\frac{a^2}{L^2}\right)}{2} = 3\frac{a^2}{L^2}$$