PHYS 5013 November 12, 2019
Exam Sample Two Mullen

Name: ______________________________

Each of the following problems is designed to test your understanding and integration of the material we have studied. Please note that:

- You may use your notes or anything in your own handwriting.
- The problems are broken up into small parts; in no case do you need the result of earlier questions to proceed, so if you get stuck, just move on.
- Explanations are as important as having the right answer. You must not only state the right answer, but make it clear how you derived it.
- If you don't know how to do the entire problem, try to communicate what you do understand.
- Make sure your writing is readable!

You will have up to two hours for this exam. Cheating will be punished by the most gruesome method I can devise. And I can be pretty inventive. Don't cheat. This exam is broken into three parts.

Part I The first part consists of several true/false questions, for a total of 40 points. If you mark a statement as false you must explain why it is false or give a counter example in the space provided.

Part II The second part involves a short essay. Please write enough to convince me that you understand the material. It is worth 20 points.

Part III The third part involves detailed problem solving, and is worth 60 points. Please read the questions carefully, and show all your work.

Part I

______ 1. The set of numbers $\{2, 4, 6, 8\}$ form a group under the operation of multiplication mod 10. For example, $(2 \times 8) \bmod 10 = 16 \bmod 10 = 6$.

a. True b. False

______ 2. The equation $\mathcal{A}x = b$ has a solution only if b has no component in the null-space of $\mathcal{A}$.

a. True b. False

______ 3. The eigenvectors of a Hermitian operator are always orthogonal.

a. True b. False

______ 4. A Hermitian matrix can always be diagonalized by a similarity transformation.

a. True b. False

______ 5. Degenerate eigenvalues can still be degenerate after applying a perturbation.

a. True b. False

______ 6. The eigenvectors of a matrix form a group under addition.

a. True b. False

______ 7. If $\{\vec{x}_{i,n}^{(0)}\}$ are a set of degenerate eigenvectors sharing the eigenvalue $\lambda_i^{(0)}$, then the first order correction to any one eigenvector $\vec{x}_{i,n}^{(1)}$ will be orthogonal to every vector in the subspace spanned by this set.

a. True b. False

______ 8. The space of 2×2 matrices excluding the zero matrix forms a group under addition.

a. True b. False

______ 9. Given linear operators $\mathcal{A}$ and $\mathcal{B}$, if $\mathcal{A}\mathcal{B} = \mathbb{1}$ then $\mathcal{B}\mathcal{A} = \mathbb{1}$.

a. True b. False

______ 10. All isometric operators are self-adjoint..

a. True b. False

Part II

11. What is the solvability condition for linear systems and how have we used it in solving problems?

Part III

12. Consider the Schrodinger equation for a particle in an infinite governed by the following time independent Schrodinger equation:

$$H\psi = \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_0 \cosh \lambda x \right\} \psi(x) = E\,\psi(x)$$

with $-\infty \leq x \leq \infty$ and $V_0 > 0$.

a. Switch to the dimensionless distance variable $s = x/\ell$ and energy $\epsilon = E/E_0$ and state your choice of ℓ and E_0 to simplify the above eigenvalue equation.

b. Given the normalized variational wavefunction $\phi(s)$:

$$\phi(s) = \left(\frac{\pi}{\alpha}\right)^{1/4} e^{-\alpha s^2/2}$$

find an expression that you must minimize to get the ϕ that best approximates the groundstate.

c. From the solution above, derive an equation that you need to solve to get a bound on the groundstate energy.

d. Write an expression that when implemented in MATHEMATICA would solve this equation for you. If you use some other software, give an expression in that software that would solve this equation for you.

e. Find an approximate value for the groundstate energy as a function of V_0 when V_0 is very large, explaining what "large" means in this problem.

13. **Degenerate Perturbation Theory:** Consider the matrix

$$\mathcal{A} = \mathcal{A}_0 + \epsilon \mathcal{A}_1$$

where

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

the matrix A has eigenvalues λ_i with eigenvectors x_i, so that

$$\mathcal{A} x_i = \lambda_i x_i.$$

The eigenvalues and normalized eigenvectors of $\mathcal{A}_0$ are:

$$\lambda_1 = 0 \qquad x_{1,1}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad x_{1,2}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 2 \qquad x_{2,1}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_{2,2}^{(0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

a. Calculate the first order correction to λ_2, only.

b. Calculate the second order correction to the state with the largest eigenvalue (up to first order).

c. Given the above states, construct a new perturbation matrix $\mathcal{A}_2$ so that the sole effect of the perturbation $\epsilon \mathcal{A}_2$ is to change the eigenvalue of $x_{2,2}^{(0)}$ to $2 + \epsilon$, with no effect on any other state. (*Hint:* Think about projection matrices.)

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{a x - b x^2}\,dx = \sqrt{\frac{\pi}{b}}e^{a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad\qquad \sum_{n=1}^\infty (-1)^{n+1}\frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad\qquad f_p(-1) = -\zeta(p)$$

$$\begin{array}{cc}
\zeta(1) = \infty & \zeta(-1) = -\frac{1}{12} = 0.0833333 \\
\zeta(2) = \frac{\pi^2}{6} = 1.64493 & \zeta(-2) = 0 \\
\zeta(3) = 1.20206 & \zeta(-3) = \frac{1}{120} = 0.0083333 \\
\zeta(4) = \frac{\pi^4}{90} = 1.08232 & \zeta(-4) = 0
\end{array}$$