Exam #1

1

a) For $l = 1,\ s = 1/2$, $\therefore$

$$\frac{1}{2} \leq J \leq 3/2 \Rightarrow -J \leq m \leq J.$$

Defining:

$$|ls; Jm\rangle \equiv |Jm\rangle$$

then:

$$\boxed{|3/2, 3/2\rangle = |1\,1\rangle\,|1/2\,1/2\rangle}$$

since: $$J_-|JM\rangle = \hbar\sqrt{(J \mp m)(J \pm m + 1)}\,|J, m-1\rangle$$

then:

$$J_- |3/2\ 3/2\rangle = \hbar\sqrt{3}\,|3/2, 1/2\rangle$$

$$= (J_{e,-} + J_{s,-})\,|1\,1\rangle|1/2, 1/2\rangle$$

$$= \hbar\left(\sqrt{2}\,|1\,0\rangle|1/2\ 1/2\rangle + |1\,1\rangle|1/2\ -1/2\rangle\right)$$

$$\therefore \boxed{|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}\,|1\,0\rangle|1/2\ 1/2\rangle + \frac{1}{\sqrt{3}}|1\,1\rangle|1/2\ -1/2\rangle}$$

In the same way,

$$J_- |3/2\ 1/2\rangle = \hbar\, 2\, |3/2, -1/2\rangle$$

$$= (J_{e-} + J_{s-})\Big[\sqrt{2/3}\,|1\,0\rangle|1/2\ 1/2\rangle$$

$$+ \frac{1}{\sqrt{3}}|1\,1\rangle|1/2\ -1/2\rangle\Big]$$

$$= \hbar\Big[\sqrt{2/3}\,\sqrt{2}\,|1\,-1\rangle|1/2\ 1/2\rangle$$

$$+ \sqrt{2/3}\,|1\,0\rangle|1/2\ -1/2\rangle$$

$$+ \sqrt{2/3}\, |1\,0\rangle |1/2\, -1/2\rangle \Big]$$

$$\therefore \quad \boxed{|3/2, -1/2\rangle = \frac{1}{\sqrt{3}} |1\, -1\rangle |1/2\; 1/2\rangle + \sqrt{2/3}\, |1\,0\rangle |1/2\, -1/2\rangle}$$

Finally,

$$\boxed{|3/2\, -3/2\rangle = |1, -1\rangle |1/2\, -1/2\rangle}$$

By orthogonality,

$$\boxed{|1/2\; 1/2\rangle = \frac{\sqrt{2}}{\sqrt{3}} |1\,1\rangle |1/2\, -1/2\rangle - \sqrt{\frac{1}{3}}\, |1\,0\rangle |1/2\; 1/2\rangle}$$

and finally,

$$\boxed{\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\sqrt{\tfrac{2}{3}}\,|1,-1\rangle\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \tfrac{1}{\sqrt{3}}\,|1,0\rangle\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle}$$

b)

For two identical electrons in the $l=0$ state, the orbital part of the total wavefunction is symmetric $\therefore$ the total spin state is a singlet (anti-symmetric) with total spin $S=0$ and $M_S=0$.

2

$$Q \equiv e\langle \alpha, j, m=j | 3z^2-1 | \alpha j m=j \rangle$$

a)

For $\begin{cases} x = \sin\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \cos\theta \end{cases}$, $\hat{n} = (x, y, z)$

$$Y_2^0(\theta,\phi) \longrightarrow T_0^2(\hat{n}) = \underbrace{\sqrt{\frac{5}{16\pi}}}_{\alpha}(3z^2-1)$$

$$\therefore \boxed{(3z^2-1) = \alpha^{-1} T_0^2}$$

In the same way,

$$T_{\pm 2}^2 = \sqrt{\frac{3}{2}}\,\alpha\,(x \pm iy)^2$$

$$\Rightarrow \quad T^2_{+2} - T^2_{-2} = \sqrt{\tfrac{3}{2}}\,\alpha\, 2i\, xy$$

$$\therefore \quad \boxed{xy = \sqrt{\tfrac{2}{3}}\,\frac{1}{2\alpha i}\left(T^2_{+2} - T^2_{-2}\right)}$$

Also,

$$T^2_{+2} + T^2_{-2} = 2\sqrt{\tfrac{3}{2}}\,\alpha\left(x^2 - y^2\right)$$

$$\therefore \quad \boxed{x^2 - y^2 = \sqrt{\tfrac{2}{3}}\,\frac{1}{2\alpha}\left(T^2_{+2} + T^2_{-2}\right).}$$

Finally,

$$T^2_{+1} \pm T^2_{-1} = -\sqrt{6}\,\alpha\, 2xz$$

$$\Rightarrow \quad \boxed{xz = +\frac{1}{2\sqrt{6}\,\alpha}\left(T^2_{-1} - T^2_{+1}\right)}$$

c

b)

$$e\langle \alpha, j m' | (x^2 - y^2) | \alpha j, m = j\rangle$$

$$= \frac{e}{\sqrt{6}\,\alpha} \langle \alpha, j, m' | \left(T^2_{+2} + T^2_{-2}\right) | \alpha j, m = j\rangle$$

$$= \frac{e}{\sqrt{6}\,\alpha} \overset{0}{\cancel{\langle j 2; j, 2 | j 2; j m'\rangle}} \times \frac{\langle \alpha j \| T^{(2)} \| \alpha j\rangle}{\sqrt{2j+1}}$$

$$+ \frac{e}{\sqrt{6}\,\alpha} \langle j 2; j, -2 | j 2; j m'\rangle \times \underbrace{\frac{\langle \alpha j \| T^{(2)} \| \alpha j\rangle}{\sqrt{2j+1}}}_{c}$$

Also,

$$Q = \frac{e\langle \alpha j j | T^2_0 | \alpha, j, j\rangle}{\alpha}$$

$$= \frac{e}{\alpha} \langle j\,2; j\,0 | j\,2; j\,j \rangle \cdot e$$

$\therefore$

$$e \langle \alpha\, j\, m' | (x^2 - y^2) | \alpha\, j\, m = j \rangle$$

$$= \frac{e}{\sqrt{6}\,\alpha} \frac{\langle j\,2; j, -2 | j\,2; j\, m' \rangle}{\langle j\,2; j\,0 | j\,2; j\,j \rangle}\, Q$$

③

a) $|\psi_i\rangle = a_i^{\dagger}|0\rangle$

$\langle\psi_i|H|\psi_j\rangle = \langle\psi_i|\omega(a_1^{\dagger}a_2 + a_2^{\dagger}a_1|\psi_j\rangle$

$= \omega\langle\psi_i|a_1^{\dagger}a_2|\psi_j\rangle$

$+ \omega\langle\psi_i|a_2^{\dagger}a_1|\psi_j\rangle$

$= \omega\left(\delta_{j,2}\delta_{i,1} + \delta_{j,1}\delta_{i,2}\right)$

$\therefore$

$$H = \omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in the $|\psi_i\rangle$ basis. The energy levels are: $\pm\omega$, with eigenkets

$$|\pm\rangle = \left(|\varphi_1\rangle \pm |\varphi_2\rangle\right)\frac{1}{\sqrt{2}}$$

$$= \frac{\left(a_1^\dagger \pm a_2^\dagger\right)}{\sqrt{2}}|0\rangle .$$

b)

$$|\phi\rangle = A \sum_{ij} c_i c_j a_i^\dagger a_j^\dagger |0\rangle$$

$$\langle\phi|\phi\rangle = |A|^2 \sum_{\substack{ij\\kl}} c_i c_j c_k c_l \langle 0| a_k a_l a_i^\dagger a_j^\dagger |0\rangle$$

Since:

$$a_k a_l a_i^\dagger a_j^\dagger = a_k\left(a_i^\dagger a_l + \delta_{il}\right) a_j^\dagger$$

$$= a_k a_i^\dagger a_l a_j^\dagger + a_k a_j^\dagger \delta_{il}$$

$$= a_k a_i^\dagger (a_j^\dagger a_l + \delta_{jl}) + a_k a_j^\dagger \delta_{il}$$

$$= a_k a_i^\dagger a_j^\dagger a_l + a_k a_i^\dagger \delta_{jl} + a_k a_j^\dagger \delta_{il}$$

$$\therefore \quad \langle 0 | a_k a_l a_i^\dagger a_j^\dagger | 0 \rangle = \delta_{ik}\delta_{jl} + \delta_{kj}\delta_{il}$$

$$\Rightarrow \langle \phi | \phi \rangle = |A|^2 \sum_{\substack{ij \\ kl}} c_i c_j c_k c_l \times (\delta_{ik}\delta_{lj} + \delta_{kj}\delta_{il})$$

$$= 2 \sum_{ij} c_i^2 c_j^2 = 1$$

$$= 2\Big(\sum_i c_i^2\Big)^2 = 1.$$

$$\therefore \quad |A|^2 = \frac{1}{2\left(\sum_i c_i^2\right)^2} \Rightarrow |A| = \frac{1}{\sqrt{2}\sum_i c_i^2}.$$