E & M I

Workshop 4 – Magnetic Fields and Forces, 2/16/2022

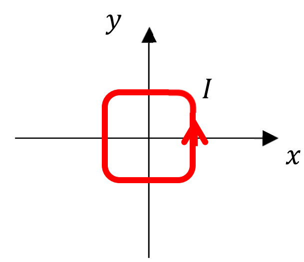
Monday's class was our start with magnetic fields and forces, considering "static" situations. The argument was made that the experimental similarities between electric and magnetic forces, and the fact that there are no magnetic "charges" (monopoles), the fundamental source of magnetic fields is a dipole and we can write down the field due to a magnetic dipole in complete analogy with the electric dipole:

$$\vec{B}_m(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\,\frac{(\vec{m}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$

where $\vec{m}$ is the magnetic dipole. The potential energy, force, and torque on a dipole are:

$$U = -\vec{m}\cdot\vec{B}, \qquad \vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right), \qquad \vec{\tau} = \vec{m}\times\vec{B}$$

In the book and briefly in class (detailed notes soon), the argument was made that a square current loop with current I and side length L is a magnetic dipole $\vec{m} = I\,L^2\,\hat{n} = I\,A\,\hat{n}$. $\hat{n}$ is the normal to the loop using the right-hand rule for the current. This should approach a point-dipole as the area of the current loop gets small.



$\vec{m} = I\,L^2\,\hat{z}$

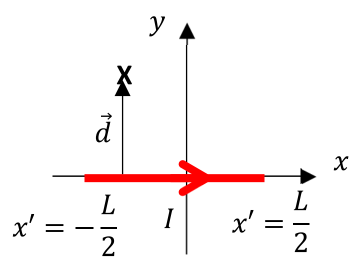
1) Square Loop Magnetic Field and Magnetic Dipoles

Let's test this equivalence to a dipole for a square loop.

At the end of class we considered the Biot-Savart law for a current loop magnetic field:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,I\oint d\vec{l}'\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

where $\vec{r}'$ and $d\vec{l}'$ refer to points on the current loop. Doing this integral for a current of length L on the x-axis and centered at $x = 0$ for a point a displacement $\vec{d}$ perpendicular to the current at the point $\vec{r} = x\,\hat{x} + d\,\hat{y}$:



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\hat{x}\times\frac{(x-x')\hat{x} + d\,\hat{y}}{\left((x-x')^2+d^2\right)^{\frac{3}{2}}} = \left(\hat{I}\times\hat{d}\right)\frac{\mu_0}{4\pi}\frac{I}{d}\left(\sin\theta_+ - \sin\theta_-\right)$$

Where $\hat{I}$ is the direction of the current, $\hat{d}$ is the perpendicular direction from the wire to the field point $\vec{d}$, $\hat{I}\times\hat{d}$ is the direction of the field, and:

$$\sin\theta_+ = \sin\left(\tan^{-1}\left(\frac{x+\frac{L}{2}}{d}\right)\right), \qquad \sin\theta_- = \sin\left(\tan^{-1}\left(\frac{x-\frac{L}{2}}{d}\right)\right)$$

a) Show that:

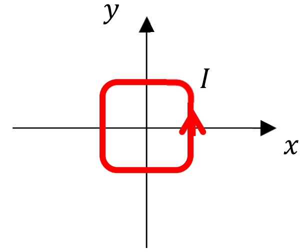
$$\sin\theta_+ = \frac{x+\frac{L}{2}}{\sqrt{d^2+\left(x+\frac{L}{2}\right)^2}}, \qquad \sin\theta_- = \frac{x-\frac{L}{2}}{\sqrt{d^2+\left(x-\frac{L}{2}\right)^2}}$$

b) Checking the algebra (okay, checking my algebra) show that you get the known (?) results for:

i. The field at a point perpendicular to the middle of the current ($x = 0$), and

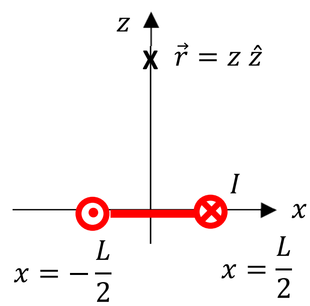
ii. The field for a very long current, $L \gg d$ and $L \gg x$.
(Compare to the results of Ampere: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$)

c) Next, let's take our square loop "dipole" and calculate its magnetic field. Consider a square loop in the x-y plane, centered at the origin as shown in the figures. We'll determine the magnetic field at points on the z-axis, $\vec{B}(\vec{r}), \vec{r} = z\,\hat{z}$.



$\vec{m} = I\,L^2\,\hat{z}$

The bottom figure is an edge-on view of the loop looking in the +y direction. The current is going into the page on the right and out of the page on the left.



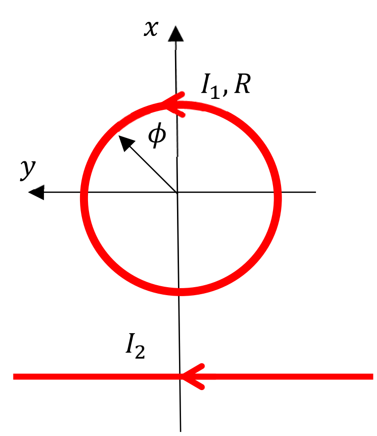
i) What will be the direction of the magnetic field on the z-axis? Describe how you will solve for the magnetic field? What symmetries might you use to simplify the calculation?
Draw a picture (I would suggest something like the bottom figure) to show and explain how you will calculate the field.

ii) Calculate $\vec{B}(z\,\hat{z})$ for any z. Show your work.

iii) Show that your result approaches the point-magnetic-dipole field $\vec{B}_m(\vec{r})$ above in the limit $z \gg L$.

2) Magnetic Force on a Loop:

Consider the two currents shown: (i) a circular loop of radius R with counterclockwise current I_1 centered at the origin in the x-y plane and (ii) a very long straight wire with current I_2 parallel to the y-direction at $z = 0, x = -d$.



We're going to find solutions for the force on the loop, first by integration and then by considering the potential energy.

a) What is the direction of the total force on the loop due to the magnetic field of the long-straight wire. Explain your prediction.

b) What is the magnetic field $\vec{B}_2(\vec{r})$ (magnitude and direction) due to the long wire everywhere in the x-y plane?

c) The force on the loop due to the magnetic field of the wire is:

$$\vec{F}_{21} = I_1 \oint d\vec{l} \times \vec{B}_2$$

It should be clear that we want to do this integral over the circle by integrating over $d\phi$.

i) Determine an expression for the $d\vec{l}$ in terms of R, ϕ, and $d\phi$. This is, of course, a vector. Write it in terms of the $\hat{x}$ and $\hat{y}$ components.

ii) Rewrite the magnetic field, $\vec{B}_2(\vec{r})$, for points on the loop in terms of R, ϕ, and d. Hint: Find the distance from the wire to points on the loop in terms of R, ϕ, and d. Check your answer at a few simple points (such as for the angles $\phi = 0, \phi = \frac{\pi}{2}, \ldots$)

iii) Write out an integral that gives the force on the loop. You should simplify this as much as possible, but you don't need to solve it, as it's somewhat messy.

Does the direction agree with your prediction?

d) Another approach to this problem is to determine the potential energy of the loop due to the magnetic field of the wire, U_{Loop}, and then use that the force is $F = -\vec{\nabla}\, U_{Loop}$, or in this case the change in energy as the loop moves relative to the wire (or wire moves relative to the loop)

$$F_y = -\partial_d\, U_{Loop}$$

i) We found that a current loop is equivalent to a sheet of small current loops, each a magnetic dipole. (Fig. 4.7 in the textbook and related text.)

Breaking the loop into small areas $d\vec{S} = dA\,\hat{n}$, corresponding to small magnetic dipoles

$$d\vec{m} = I\, dA\, \hat{n}$$

Each with a potential energy

$$dU(\vec{r}) = -d\vec{m} \cdot \vec{B}_2(\vec{r})$$

Write a surface integral for the potential energy of the loop. Use polar coordinates.

ii) What is the field $\vec{B}_2(\vec{r})$ everywhere inside the loop? Use polar coordinates. (This is a simple extension to part C-ii.)

iii) Solve your integral to determine U_{Loop} and take the derivative to get F_y. Does your result make sense? It might be useful to know that:

$$\int_0^{2\pi} \frac{d\phi}{d + r\cos\phi} = \frac{2\pi}{\sqrt{d^2 - r^2}}$$