Physics 5403 Homework #3
Spring 2022

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1 Identical particles I

The group of permutations of three particles is isomorphic to the group of geometric transformations that keep an equilateral triangle invariant (rotations of $\theta = 2\pi/3$ and reflections that leave one corner of the triangle invariant).

a) Using this correspondence, find that the set of 6 unitary 2×2 matrices which represent the $3! = 6$ permutations. Show that the determinant of the matrices give the parity of the permutation.

b) Consider three indistinguishable spinless particles. Write down the ket $|\alpha\rangle$ of the 3-particle state in terms of products of single particle states. Assume now that the particles are physically located at the corners of an equilateral triangle, which rotates around axis perpendicular to the triangle plane (z axis). Considering the statistics, write down the allowed quantum numbers for the total angular momentum J_z.

c) Suppose we have now three indistinguishable spin 1 particles and the orbital part of the three particle state is anti-symmetric. Write down the spin part of the state in terms of products of single particle kets.

2 Identical particles II

a) Suppose N identical non-interacting spin 5/2 particles are subjected to the potential of a 1 dimensional harmonic oscillator. Compute the total energy of the ground state.

b) Two identical non-interacting spin 1/2 particles occupy the energy levels of a 2D harmonic oscillator,

$$\mathcal{H} = \sum_{i=1}^{2} \left(\frac{P_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 \right),$$

where P_i is the momentum of the particles and x_i their coordinates ($i = 1, 2$). Each particle has a wavefunction of the form

$$\psi_n(x_i, s_i) = \phi_n(x_i)\chi(s_i),$$

where $\phi_n(x)$ is the orbital part (indexed by the energy level $n \in \mathbb{N}$) and $\chi(s)$ the spin part. Write down the 2-particle wave functions for the ground state and first excited states and their respective energies.

3 Jordan-Wigner transformation

Consider a one dimensional lattice of localized spins $\mathbf{S}(m)$, where m labels the lattice site. The spin operators for spin $s = 1/2$ have mixed commutation relations, so spin operators do not describe neither fermions nor bosons.

a) Consider the spin operator $\mathbf{S}(m)$ for site m with components $S_x(m)$,$S_y(m)$ and $S_z(m)$. Defining the ladder operators

$$\begin{aligned} a_m &= S_x(m) - iS(m) \\ a_m^\dagger &= S_x(m) + iS_y(m), \end{aligned}$$

where $a_m^\dagger a_m = S_z(m) + \frac{1}{2}$ (assume $\hbar = 1$), show that they follow the commutation relations:

i) Bose type for different sites

$$\left[a_i^\dagger, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = [a_i, a_j] = 0$$

for $i \neq j$.

ii) Fermi type for spin operators on the same site,

$$\left\{a_i, a_i^\dagger\right\} = 1,$$

and $a_i^2 = (a_i^\dagger)^2 = 0$.

b) Consider now a spin chain with periodic boundary conditions. Show that the operators

$$\begin{aligned} C_i &= \exp\left[i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right] a_i \\ C_i^\dagger &= a_i^\dagger \exp\left[-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right] \end{aligned}$$

follow Fermi statistics. This transformation is called *Wigner-Jordan transformation.* Write down the inverse transformation. Show that

$$C_i^\dagger C_i = a_i^\dagger a_i,$$

and

$$C_i^\dagger C_{i+1} = a_i^\dagger a_{i+1}.$$

c) Transform the Heisenberg spin exchange Hamiltonian

$$\mathcal{H} = J \sum_{m=1}^{N} \mathbf{S}(m) \cdot \mathbf{S}(m+1)$$

into a Hamiltonian of fermions. Interpret the result.

4 Coherent states

Consider $(a^\dagger, a)$ as the creation/annihilation operators for bosons in a single mode. Coherent states are defined as the eigenstates of the annihilation operator

$$a|z\rangle = z|z\rangle,$$

where z is a complex number and $\langle z|z\rangle = 1$ is normalized.

a) Writing $|z\rangle$ as a generic superposition of $\{|n\rangle\}$, states in the form:

$$|z\rangle = \sum_{n=0}^{\infty} \phi_n |n\rangle,$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle,$$

show that

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\exp\left(za^\dagger\right)|0\rangle.$$

b) Show equivalently that

$$|z\rangle = \exp\left(za^\dagger - z^* a\right)|0\rangle \equiv D(z)|0\rangle,$$

where $D(z)$ is a unitary operator.

c) Compute the overlap of two coherent states, $\langle z|z'\rangle$.

d) Find the probability distribution P_n of finding the state $|n\rangle$ occupied in $|z\rangle$.