COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Statistical Mechanics

CH. 11 FERMI SYSTEMS LECTURE NOTES

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Doerte Blume

4-13-22

We begin our examination of Fermi - Systems by using the canonical ensemble.

Bose - Einstein condensates

Bose- Einstein condensates is a condition under low Temperature where all the occupied states want to have the same momentum.

In the context of Fermi systems we are going to look at both the non-relativistic and relativistic cases

We calculate the ensemble average of our allowed occupied states with

$$\langle n_{\vec{k}} \rangle = -\frac{1}{\beta}\frac{\partial}{\partial \varepsilon_{\vec{k}}} \log(Q) \quad w/ \quad Q = \sum_{\vec{k}} \log(1+ze^{-\beta\varepsilon_{\vec{k}}})$$

Where our number of particles with no spin degeneracy can be calculated with

$$N = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

Going back to our Ideal Gas

$$\frac{P}{kT} = \frac{1}{\lambda^3}\left(4\pi\int_0^\infty \log\left(1+ze^{-\frac{\beta \vec{p}^2}{2m}}\right)p^2\,dp\right)$$

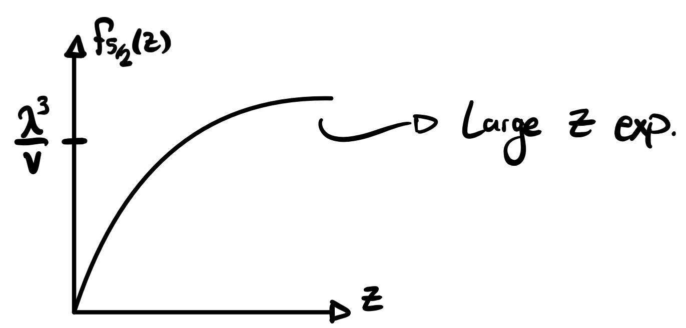
If we call

$$f_{5/2}(z) = \frac{4}{\sqrt{\pi}}\int_0^\infty \log(1+ze^{-x^2})\,x^2\,dx = \sum_{\ell=1}^{\infty}\frac{(-1)^{\ell+1}z^\ell}{\ell^{5/2}}$$

We can then in general say

$$\frac{P}{kT} = \frac{1}{\lambda^3} f_{5/2}(z)$$

If we plot $f_{5/2}(z)$ we see



We can then look at Small z (very High Temperature) and observe what happens to our ideal gas law

$$\frac{PV}{KT} \approx 1 + \frac{1}{2^{5/2}}\frac{\lambda^3}{V} + \dots \quad n_0 \approx N$$

Where in the expansion above the term λ^3/V is the correction due to quantum. In the classical limit $\lambda \to 0$.

So far, we have been looking at spin 0 Fermions (which doesn't actually exist). So, instead we have the spin be $s = 1/2 \therefore m_s = \pm 1/2$. But we throw out $m_s = -1/2$ and we then have a degeneracy of

$$g = 2s + 1$$

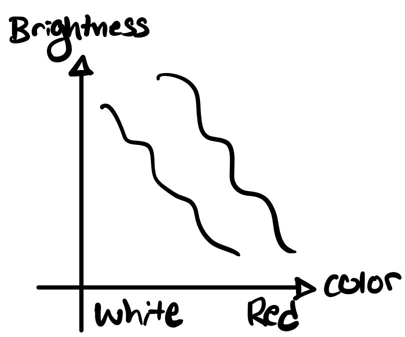
This then means our total number of particles can be calculated with

$$N = g \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

4-18-22

We continue our examination of Fermionic Quantum Systems

We begin by examining a White Dwarf. A white Dwarf is a star that has essentially "collapsed" in on itself and is extremely dense. These stars tend to follow



These stars have the properties

$$\rho \approx 10^7 \frac{g}{cm^3}, \quad m \approx 10^{33} g, \quad \text{radius} \approx 5 \cdot 10^6 m, \quad \text{core } T \approx 10^7 K$$

These stars are kept together due to a Quantum Mechanical process, Specifically they are kept together because of the degeneracy pressure in the stars.

If we wish to find the ground state energy of our electrons in this star we use

$$E_0 = 2 \sum_{|\vec{k}| < k_F} \varepsilon_{\vec{k},s}$$

Where the 2 out front is due to the particles being spin up or down.

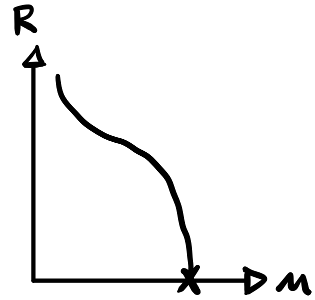
The total energy of our system is

$$E_{TOT} = E_0 + E_G$$

where E_G is the energy from gravitation. We then find the critical radius w/

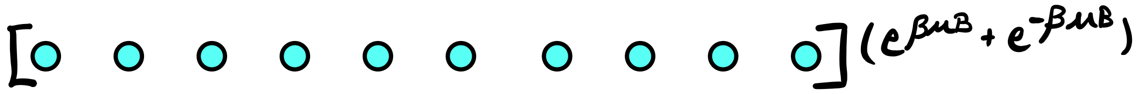
$$\frac{\partial E_{TOT}}{\partial R} = 0 \Rightarrow R_{cr}$$

This then means graphically it looks like



Where we have found the mass at which our system collapses.

We now look at magnetic behavior of an ideal Fermi Gas. There is an external magnetic field being applied and our system looks like



Where the above are magnetic moments of the nuclei. If we wish to calculate the magnetization it is done by

$$\mathcal{M} = \frac{1}{V}\left(\frac{\partial}{\partial \beta}\left(\frac{\log(Q_N)}{V}\right)\right)_{T,V,N} = \frac{1}{V}\left\langle -\frac{\partial \hat{\mathcal{H}}}{\partial B}\right\rangle$$

where $\mathcal{H}$ is our N-particle Hamiltonian. The susceptibility can be calculated with

$$\chi = \frac{\partial \mathcal{M}}{\partial \beta}$$

If $\chi > 0$, we call our system a paramagnet. If $\chi < 0$, we call our system a diamagnet.

We then have a Hamiltonian that takes into account our magnetic field that will look like

$$\hat{\mathcal{H}} = \frac{\hat{\vec{p}}^2}{2m} - \hat{\mu}\cdot\vec{B} + (\vec{A} - \text{dep.})$$

The single particle Hamiltonian takes the form

$$\hat{\mathcal{H}}_{single} = \frac{\hat{P}^2}{2m} - \hat{\vec{\mu}} \cdot \vec{B}$$

Intrinsic Magnetic moment

Where the magnetic moment is then

$$\hat{\vec{\mu}} = -g_s \mu_B \frac{1}{\hbar} \hat{\vec{S}}$$

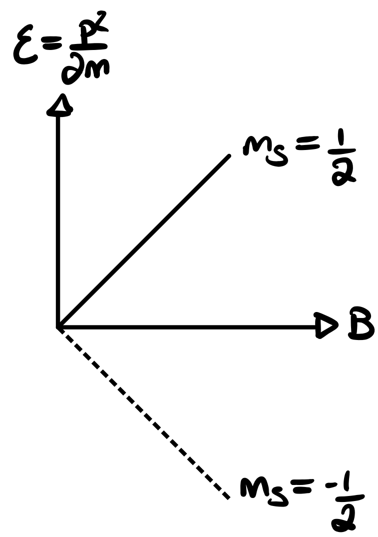
Taking this into account our single particle Hamiltonian is then

$$\hat{\mathcal{H}} = \frac{\hat{\vec{P}}^2}{2m} + \mu_B B \hat{e}_z$$

This tells us that

$$\varepsilon = \frac{p^2}{2m} + \mu_B B$$

Looking at this graphically we have



We can then find our partition function with

$$Q_N = \sum_{n_k^{\pm}} e^{-\beta E_n}$$

4-20-22

Previously we defined energy in terms of occupation numbers

$$E_n = \sum_k \left[\underbrace{\left(\frac{\hbar^2 \vec{k}^2}{2m} - \mu_B B \right) n_k^{+}}_{m_s = -\frac{1}{2}} + \underbrace{\left(\frac{\hbar^2 \vec{k}^2}{2m} + \mu_B B \right) n_k^{-}}_{m_s = \frac{1}{2}} \right]$$

We then know that our partition function is

$$Q_N = \sum{}' e^{-\beta E_n}$$

We then started looking at Paramagnetism

$$\hat{\mathcal{H}}_{N=1} = \frac{\vec{p}^2}{2m} - \vec{\mu}\cdot\vec{B}$$

With this paramagnetism we can then say

$$2\mu B = \mu_0 \underbrace{\left(\frac{1+r}{2}N\right)}_{\bar{N}_+} - \mu_0 \underbrace{\left(\frac{1-r}{2}N\right)}_{\bar{N}_-}$$

Where the above is our chemical potential energy where $r \in [0,1]$. We now expand our chemical potential

$$\mu_0(xN) = \left(\frac{3xN}{4\pi V}\right)^{2/3} \frac{h^2}{2m}$$

Where we then do

$$\chi = \frac{\partial \mathcal{M}_0}{\partial B} = \frac{\partial}{\partial B}\left(\frac{2\mu_B^2 BN}{V \frac{\partial \mu_0(x,N)}{\partial x}}\right)\Bigg|_{x=0}$$

Where we have defined

$$\mathcal{M} = \frac{1}{V}\mu_B(\bar{N}_+ - \bar{N}_-)$$

We then can say

$$2\mu_B B = \frac{\partial \mu_0(x,N)}{\partial x}\Bigg|_{x=\frac{1}{2}}, \quad r = \frac{2\mu_B B}{\frac{\partial \mu_0(x,N)}{\partial x}}\Bigg|_{x=\frac{1}{2}}$$