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Quantum Mechanics 1

CH. 1 FUNDAMENTAL CONCEPTS LECTURE NOTES

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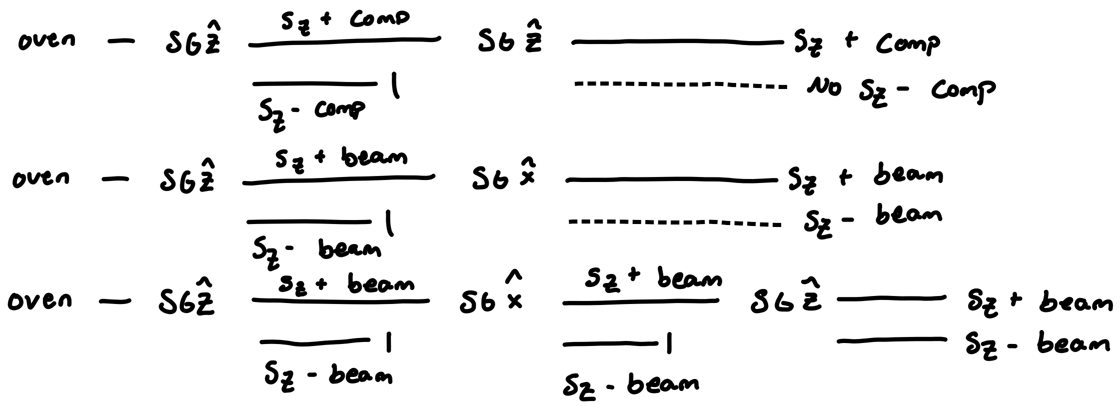
$|S_z +\rangle$ → Spin up : $|S_z -\rangle$ → Spin down



$$|S_z +\rangle = \frac{1}{\sqrt{2}}\left[|S_x +\rangle - |S_x -\rangle\right], \quad |S_z -\rangle = \frac{1}{\sqrt{2}}\left[|S_x +\rangle + |S_x -\rangle\right]$$

$\tilde{A}\,|a_i\rangle = a_i\,|a_i\rangle$: $\tilde{A}\,|\alpha\rangle = c_i\,|\beta\rangle$ ← operators

$\tilde{S}_z\,|S_z +\rangle = \frac{\hbar}{2}\,|S_z +\rangle$: $\tilde{S}_z\,|S_z -\rangle = -\frac{\hbar}{2}\,|S_z -\rangle$ ← Measuring spin

$|S_x \pm\rangle = \frac{1}{\sqrt{2}}\left[|S_z -\rangle \pm |S_z +\rangle\right]$: $\tilde{S}_z\,|S_x \pm\rangle = \frac{1}{\sqrt{2}}\left[-\frac{\hbar}{2}\,|S_z -\rangle \pm \frac{\hbar}{2}\,|S_z +\rangle\right]$

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$|S_z;+\rangle, |S_z;-\rangle$: $|S_x;+\rangle = \frac{1}{\sqrt{2}}[|S_z;-\rangle + |S_z;+\rangle]$, $|S_x;-\rangle = \frac{1}{\sqrt{2}}[|S_z;-\rangle + |S_z;+\rangle]$

$|S_y;+\rangle = \frac{1}{\sqrt{2}}[|S_z;+\rangle + i|S_z;-\rangle]$, $|S_y;-\rangle = \frac{1}{\sqrt{2}}[|S_z;+\rangle - i|S_z;-\rangle]$

$|\alpha\rangle$: → $c|\alpha\rangle$ Constant does not tell you anything specific about the state of the system

$c_1|\alpha\rangle + c_2|\alpha\rangle = |\gamma\rangle$ Superposition of state creates a new state

$\tilde{A}|\alpha\rangle$ Produces new ket $\tilde{A}|\alpha\rangle = c|\gamma\rangle$

$\tilde{A}|a_i\rangle = a_i|a_i\rangle$ $\tilde{A}$ returns same state w/ observable a_i : $|a_i\rangle$ eigenket, a_i eigenvalue

$\tilde{S}|S_z;+\rangle = \hbar/2\,|S_z;+\rangle$ Measure spin in + z, get $\hbar/2$ as observable back

$|\alpha\rangle = \sum c_i |a_i\rangle$

Dual space : $\langle\alpha|$: $\langle\alpha| \overset{DC}{\Leftrightarrow} |\alpha\rangle$ DC → Dual correspondence, allows for definition of inner product

$\langle\alpha|\beta\rangle = c$ Inner product

$c^*\langle\alpha| \overset{DC}{\Leftrightarrow} c|\alpha\rangle$ → $\langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle$

$\langle\alpha|\alpha\rangle \geq 0$: $\langle\alpha|\beta\rangle = 0$ → orthogonal

$|\hat{\alpha}\rangle = \frac{1}{\sqrt{\langle\alpha|\alpha\rangle}}|\alpha\rangle \longrightarrow$ Normalized vector $|\alpha\rangle$, unit vector

$\tilde{X}|\alpha\rangle = |\beta\rangle$ $\tilde{X}$ produces new ket : $\langle\alpha|\tilde{X} = \langle\beta|$ $\tilde{X}$ produces new bra

$\langle\alpha|\tilde{X}^{+} \overset{DC}{\Longleftrightarrow} \tilde{X}|\alpha\rangle$ Hermitian adjoint

$\tilde{X} = \tilde{X}^{+}$ Hermitian operator is equal to Hermitian adjoint

if $\tilde{X} = \tilde{X}^{+}$ then $\langle\alpha|\tilde{X} \overset{DC}{\Longleftrightarrow} \tilde{X}|\alpha\rangle$

$\tilde{X}\tilde{Y} \neq \tilde{Y}\tilde{X}$

$\tilde{X}|\alpha\rangle = c|\beta\rangle$: $\tilde{X} = |\alpha\rangle\langle\beta|$: $\tilde{X}|\gamma\rangle = (|\alpha\rangle\langle\beta|)|\gamma\rangle = |\alpha\rangle(\underbrace{\langle\beta|\gamma\rangle}_{c}) = c|\alpha\rangle$

$\langle\beta|\gamma\rangle^{*}\langle\alpha| \overset{DC}{\Longleftrightarrow} \langle\beta|\gamma\rangle|\alpha\rangle$: $\langle\gamma|\underbrace{(\beta\rangle\langle\alpha|)}_{\tilde{X}^{+}} \overset{DC}{\Longleftrightarrow} \underbrace{(|\alpha\rangle\langle\beta|)}_{\tilde{X}}|\gamma\rangle$

$|\alpha\rangle = (\sum |a_i\rangle\langle a_i|)|\alpha\rangle$: $|\alpha\rangle = \sum |a_i\rangle(\underbrace{\langle a_i|\alpha\rangle}_{c_i})$: $|\alpha\rangle = \sum c_i|a_i\rangle$

$\langle a_i|a_i\rangle = 1$: $\langle a_i|a_j\rangle = \delta_{ij}$

$\langle\alpha|\alpha\rangle = 1$: $\sum_i \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle$: $\sum_i \langle a_i|\alpha\rangle^{*}\langle a_i|\alpha\rangle = \sum_i c_i^{*}c_i = 1$: $\sum_i |c_i|^2 = 1$

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$|\alpha\rangle\langle\beta|\gamma\rangle = |\alpha\rangle c$: For example $\tilde{S}_z|S_z;+\rangle = \frac{\hbar}{2}|S_z;+\rangle$

$\tilde{A} = \tilde{A}^{+}$ Hermitian adjoint = original operator $\longrightarrow$ shows $\tilde{A}$ is hermitian

$|\alpha\rangle = \sum c_i|a_i\rangle$: $\longrightarrow$ $|\alpha\rangle = a|S_z;+\rangle + b|S_z;-\rangle$ $\longrightarrow$ SG Example

$\tilde{A}|a_i\rangle = a_i|a_i\rangle$: $|a_i\rangle \longrightarrow$ Eigenstate, $a_i \longrightarrow$ Eigenvalue

$\langle a_j|\tilde{A} = a_j^{*}\langle a_j|$: Dual correspondance of $\tilde{A}|a_i\rangle = a_i|a_i\rangle$

$\langle a_j|\tilde{A}|a_i\rangle = a_i\langle a_j|a_i\rangle$: $\langle a_j|\tilde{A}|a_i\rangle = a_j^{*}\langle a_j|a_i\rangle$

$\langle a_j|\tilde{A}|a_i\rangle - \langle a_j|\tilde{A}|a_i\rangle = (a_i - a_j^{*})\langle a_j|a_i\rangle = 0$ only satisfied if $a_i = a_i^{*}$ and $a_i \in \mathbb{R}$

If $i=j$: $0 = (a_i - a_j^{*})\langle a_i|a_i\rangle$

If $i \neq j$: $0 = (a_i - a_j)\langle a_j|a_i\rangle \longrightarrow \langle a_j|a_i\rangle$ a_j and a_i are orthogonal

$\langle a_j\,[\,|\alpha\rangle = \sum c_i|a_j\rangle\,]$: $\langle a_j|\alpha\rangle = \sum_i c_i\overbrace{\langle a_i|a_j\rangle}^{\delta_{ij}}$: $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ if $i \neq j$

$\langle a_j|\alpha\rangle = c_j$

$\tilde{\Lambda} = |a_i\rangle\langle a_i|$: $\tilde{\Lambda}|\alpha\rangle = \langle a_i|\alpha\rangle|a_i\rangle$: $|\alpha\rangle = \sum |a_i\rangle\underbrace{\langle a_i|\alpha\rangle}_{c_i}$

$|\alpha\rangle = \sum_i [\,|a_i\rangle\langle a_i|\,]\,|\alpha\rangle \;:\; \sum_i [\,|a_i\rangle\langle a_i|\,] = \tilde{\mathbb{1}}$

$\sum_{ij} |a_i\rangle\langle a_i|\tilde{x}|a_j\rangle\langle a_j| \;:\; \langle a_i|\tilde{x}|a_j\rangle \doteq x_{ij} \;:\; \sum_{ij} X_{ij}\,|a_i\rangle\langle a_j| \;:\; \langle \text{row}|\tilde{x}|\text{column}\rangle$

$$\tilde{X} \doteq \begin{pmatrix} \langle a_1|\tilde{x}|a_1\rangle & \langle a_1|\tilde{x}|a_2\rangle & \cdots \\ \langle a_2|\tilde{x}|a_1\rangle & \langle a_2|\tilde{x}|a_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$|+\rangle\langle +|S_z|+\rangle\langle +| + |-\rangle\langle -|S_z|+\rangle\langle +| + |+\rangle\langle +|S_z|-\rangle\langle -| + |+\rangle\langle -|S_z|-\rangle\langle -|$

$\rightarrow$ Represent as a diagonal

$|\gamma\rangle = \tilde{X}|\alpha\rangle \;:\; \langle a_i|\gamma\rangle = \langle a_i|\tilde{X}|\alpha\rangle \;:\; \langle a_i|\gamma\rangle = \sum_j \langle a_i|\tilde{x}|a_j\rangle\langle a_j|\alpha\rangle$

$$|\alpha\rangle \doteq \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \vdots \end{pmatrix}$$

9-1-21

$\tilde{A} = \sum_{ij} |a_i\rangle\langle a_i|\tilde{A}|a_j\rangle\langle a_j| = \sum_{ij} \langle a_i|\tilde{A}|a_j\rangle|a_i\rangle\langle a_j| \;:\; \langle a_i|\tilde{A}|a_j\rangle = a_j\delta_{ij}$

$\alpha = \sum_i |a_i\rangle\langle a_i||\alpha\rangle \longrightarrow$ State expressed as column vector $:\; \langle a_i|\alpha\rangle \rightarrow$ Matrix elements

$\tilde{A}|\alpha\rangle = \sum_{ij} |a_i\rangle\langle a_i|\tilde{A}|a_j\rangle\langle a_j||\alpha\rangle = \sum_{ij} |a_i\rangle\langle a_i|\tilde{A}|a_j\rangle\langle a_j|\alpha\rangle$

$|\alpha\rangle = \sum_i |a_i\rangle\langle a_i||\alpha\rangle = \sum_i |a_i\rangle\langle a_i|\alpha\rangle \;:\; \langle a_i|\alpha\rangle = \Big(\;\Big),\; \langle\alpha|a_i\rangle = (\quad)$

$|\alpha\rangle = \sum c_i|a_i\rangle = \sum |a_i\rangle\langle a_i|\alpha\rangle \;:\; \langle\alpha|\alpha\rangle = \sum \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle = 1$

$\langle\alpha|\alpha\rangle = \sum |\langle a_i|\alpha\rangle|^2 = 1 = \sum |c_i|^2$

$|\langle a_i|\alpha\rangle|^2 =$ Probability to be in state $a_i \;:\; |\alpha\rangle = \sum c_i|a_i\rangle$

Make measurement $\longrightarrow$ be in same state

$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}\left[\,|S_z;+\rangle \pm |S_z;-\rangle\,\right] \longrightarrow$ Spin x w/ z-Basis

$\langle\alpha|\tilde{A}|\alpha\rangle \equiv \langle\tilde{A}\rangle_\alpha \;:\; \tilde{A} = \sum a_i P_i = \sum a_i |c_i|^2$

$\sum_i \langle\alpha\big[|a_i\rangle\langle a_i|\big]\tilde{A}\big[|a_j\rangle\langle a_j|\big]|\alpha\rangle \Longrightarrow \sum_{ij} \langle\alpha|a_i\rangle\langle a_i|\tilde{A}|a_j\rangle\langle a_j|\alpha\rangle$

$\sum_{ij} \langle\alpha|a_i\rangle\langle a_j|\alpha\rangle a_j\delta_{ij} = \sum_i \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle a_i = \sum_i a_i|\langle a_i|\alpha\rangle|^2 \longrightarrow$ Probability of a_i

$[\tilde{A},\tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A} = 0 \longrightarrow$ Said to be compatible observables

$\langle a_j|\tilde{A}\tilde{B} - \tilde{B}\tilde{A}|a_i\rangle = 0 \;:\; (a_j)\langle a_j|\tilde{B}|a_i\rangle - (a_i)\langle a_j|\tilde{B}|a_i\rangle = 0 \;:\; (a_j - a_i)\langle a_j|\tilde{B}|a_i\rangle = 0$

$i \neq j \Rightarrow \langle a_j|\tilde{B}|a_i\rangle = 0 \;:\; \langle a_j|\tilde{A}|a_i\rangle = \delta_{ij}\,a_i \longrightarrow$ operator written in own basis = diagonal

$\tilde{B}|a_j\rangle = \sum |a_i\rangle\langle a_i|\tilde{B}|a_i\rangle\langle a_i|a_j\rangle = \sum |a_i\rangle\langle a_i|\tilde{B}|a_i\rangle\langle a_i|a_j\rangle = b_j|a_j\rangle$

9-8-21

$[\tilde{A},\tilde{B}]=0 \longrightarrow$ Compatible observables

$[\tilde{A},\tilde{B}]\neq 0 \longrightarrow$ Incompatible observables

$\tilde{A}|a_i\rangle = a_i|a_i\rangle$, $\tilde{B}|a_i\rangle = b_i|a_i\rangle$: $a_i, b_i \longrightarrow$ Eigenstates of $\tilde{A}$ and $\tilde{B}$

$|a_i,b_i\rangle$: $|L,L_x\rangle = |1,-1\rangle$, $|1,0\rangle$, $|1,+\rangle$

$\tilde{A}|a_i,b_i^{(j)}\rangle = a_i \sum c_j|a_i,b_i^{(j)}\rangle$: $\tilde{L}|L,L_x\rangle = \sum c_-|1,-1\rangle + c_0|1,0\rangle + c_+|1,1\rangle$

Measure $\tilde{B}$: $\tilde{B}\sum_j c_j|a_i,b_i^{(j)}\rangle \Rightarrow b_i^{(k)}|a_i,b_i^{(k)}\rangle$ k is one of the j

$[\tilde{A},\tilde{B}]=\tilde{c}$: $[\tilde{A},\tilde{B}]=|a_i\rangle$: $\tilde{A}\tilde{B}|a_i,b_i\rangle = (\tilde{c}+\tilde{B}\tilde{A})|a_i,b_i\rangle = \tilde{c}|a_i,b_i\rangle + a_ib_i|a_i,b_i\rangle$

$|a_i,b_i\rangle = \sum_i |c_j\rangle\langle c_j||a_i,b_i\rangle$

$(\langle\Delta\tilde{A}\rangle)^2(\langle\Delta\tilde{B}\rangle)^2 \geq 1/4\,|\langle[\tilde{A},\tilde{B}]\rangle|^2$: $\Delta\tilde{A} = \tilde{A}-\langle\tilde{A}\rangle\tilde{\mathbb{I}}$, $\Delta\tilde{B} = \tilde{B}-\langle\tilde{B}\rangle\tilde{\mathbb{I}}$

$\langle(\Delta\tilde{A})^2\rangle = \langle\tilde{A}^2 - 2\tilde{A}\langle\tilde{A}\rangle + \langle\tilde{A}^2\rangle\rangle = \langle\tilde{A}^2\rangle - \langle\tilde{A}\rangle^2$

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = |\langle\alpha|\beta\rangle|^2 \longrightarrow$ Schwarz inequality

$\langle\psi|\psi\rangle \geq 0$, $|\psi\rangle = |\alpha\rangle + \lambda|\beta\rangle$: $\langle\psi| = \langle\alpha| + \lambda^*\langle\beta|$

$\langle\psi|\psi\rangle = \langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + \lambda\lambda^*\langle\beta|\beta\rangle \geq 0$

$\frac{d\langle\psi|\psi\rangle}{d\lambda} = \langle\alpha|\beta\rangle + \lambda^*\langle\beta|\beta\rangle = 0$: $\frac{d\langle\psi|\psi\rangle}{d\lambda^*} = \langle\beta|\alpha\rangle + \lambda\langle\beta|\beta\rangle = 0$

$\lambda = \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$, $\lambda^* = \frac{\langle\alpha|\beta\rangle}{\langle\beta|\beta\rangle} \longrightarrow \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

$\tilde{A}|a_i\rangle = a_i|a_i\rangle$: $a_i \longrightarrow$ real : $|\psi\rangle = \sum_i g_i|a_i\rangle$

$\langle\psi|\tilde{A}|\psi\rangle = \sum_i |g_i|^2 a_i$

$\tilde{c} = -\tilde{c}^\dagger \longrightarrow$ Anti Hermitian Operator

$\tilde{c}|c_i\rangle = c_i|c_i\rangle$: $\langle c_j|\tilde{c}^\dagger = c_j^*\langle c_j|$

$\langle c_j|\tilde{c}|c_i\rangle = c_i\langle c_j|c_i\rangle$: $\langle c_j|\tilde{c}^\dagger|c_i\rangle = c_j^*\langle c_j|c_i\rangle$: $-\langle c_j|\tilde{c}|c_i\rangle = c_j^*\langle c_j|c_i\rangle$

$(c_i + c_j^*)\langle c_j|c_i\rangle = 0$: $i=j$: $c_i + c_i^* = 0 \therefore c_i^* = -c_i$

$\Delta\tilde{A}|\rangle$: $\Delta\tilde{B}|\rangle$: $\langle(\Delta\tilde{A})^2\rangle\langle(\Delta\tilde{B})^2\rangle \geq |\langle(\Delta\tilde{A}\Delta\tilde{B})\rangle|^2$: $\Delta\tilde{A}\Delta\tilde{B} = \frac{1}{2}[\Delta\tilde{A},\Delta\tilde{B}] + \frac{1}{2}\{\Delta\tilde{A},\Delta B\}$

$[\tilde{A},\tilde{B}] = -[\tilde{B},\tilde{A}]$ $[\tilde{A},\tilde{B}]^\dagger = -[\tilde{B},\tilde{A}]$ $\{\tilde{A},\tilde{B}\} = \{\tilde{B},\tilde{A}\}$: $\{\tilde{A},\tilde{B}\}^\dagger = \{\tilde{B},\tilde{A}\}$

9-13-21

Incompatible observables : $[\tilde{A},\tilde{B}] \neq 0$

Measure $\tilde{A}$ in the state $|a_2\rangle$, Measure $\tilde{B}$ in the state $|a_2\rangle = \sum_j |b_j\rangle\langle b_j|a_2\rangle$, only the probability is known

Uncertainty Principle

Schwartz inequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

Dispersion $\Delta\tilde{A} = \tilde{A} - \langle\tilde{A}\rangle\tilde{\mathbb{1}}$

$\langle(\Delta\tilde{A})^2\rangle = \langle\tilde{A}^2 - 2\tilde{A}\langle\tilde{A}\rangle + \langle\tilde{A}\rangle^2\rangle = \langle\tilde{A}^2\rangle - \langle\tilde{A}\rangle^2$

Hermitian operator $\{\tilde{A},\tilde{B}\}$ has real eigenvalues

Anti-Hermitian operator $[\tilde{A},\tilde{B}]$ has imaginary eigenvalues

$\{\hat{A},\tilde{B}\}^\dagger = \{\hat{A},\tilde{B}\}$, $[\tilde{A},\tilde{B}]^\dagger = -[\hat{A},\tilde{B}]$

Schwartz Inequality

$\langle(\Delta\tilde{A})^2\rangle\langle(\Delta\tilde{B})^2\rangle \geq |\langle(\Delta\tilde{A}\Delta\tilde{B})\rangle|^2$, $\Delta\tilde{A}\Delta\tilde{B} = \frac{1}{2}[\Delta\tilde{A},\Delta\tilde{B}] + \frac{1}{2}\{\Delta\tilde{A},\Delta\tilde{B}\}$

$\Delta\tilde{A}\Delta\tilde{B} = \frac{1}{2}[\hat{A},\tilde{B}] + \frac{1}{2}\{\Delta\tilde{A},\Delta\tilde{B}\}$

$\langle(\Delta\tilde{A})^2\rangle\langle(\Delta\tilde{B})^2\rangle \geq \frac{1}{4}\langle[\hat{A},\tilde{B}]\rangle^2 + \frac{1}{4}\langle\{\Delta\tilde{A},\Delta\tilde{B}\}\rangle$, $\langle(\Delta\tilde{A})^2\rangle\langle(\Delta\tilde{B})^2\rangle \geq \frac{1}{4}\langle[\tilde{A},\tilde{B}]\rangle^2$

$|\langle c'|a'\rangle|^2_{①} = |\langle b'|a'\rangle|^2|\langle c'|b'\rangle|^2 \longrightarrow$ Stern Gerlach Probability from a to c

$\sum_{b'}\langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle \longrightarrow$ Probability from $a \to c$ all possible values will end up being 1

$|\sum_{b'}\langle c'|b'\rangle\langle b'|a'\rangle|^2 = \sum_{b'}\langle c'|b'\rangle\langle b'|a'\rangle\sum_{b''}\langle a'|b''\rangle\langle b''|c'\rangle$

Change of Basis

$|b_i\rangle = \tilde{u}|a_i\rangle$: $\tilde{u}\tilde{u}^\dagger = \tilde{\mathbb{1}} \to \langle a_i|\tilde{u}^\dagger\tilde{u}|a_i\rangle = 1$

$\tilde{u} = \sum_j |b_j\rangle\langle a_j|$: $\tilde{u}^\dagger = \sum_k |a_k\rangle\langle b_k|$: $\tilde{u}\tilde{u}^\dagger = \sum_{jk}|b_j\rangle\underbrace{\langle a_j|a_k\rangle}_{\delta_{jk}}\langle b_k| = \sum_j |b_j\rangle\langle b_j| = \tilde{\mathbb{1}}$

9-15-21

$|a_i\rangle \longrightarrow |b_i\rangle \longrightarrow$ Change of basis $|b_i\rangle = \tilde{u}|a_i\rangle$

$|b_i\rangle = \left[\sum_j |b_j\rangle\langle a_j|\right]|a_i\rangle \to \tilde{u}^\dagger\tilde{u} = \tilde{\mathbb{1}}$

$$\left(\sum_j |a_j\rangle\langle b_j|\right)\left(\sum_k |b_k\rangle\langle a_k|\right) \longrightarrow \sum_j |a_j\rangle\langle a_j| = \tilde{1}\ , \quad \tilde{u}^\dagger = \tilde{u}^{-1}$$

$$\tilde{u} = \sum_{ji} |a_i\rangle\langle a_i|\,\tilde{u}\,|a_j\rangle\langle a_j| = \sum_{ji} |a_i\rangle\langle a_i|b_j\rangle\langle a_j| = \sum_{ji} (\langle a_i|b_j\rangle)\,|a_i\rangle\langle a_j|$$

$$\tilde{u} \doteq \begin{pmatrix} \langle a_1|b_1\rangle & \langle a_1|b_2\rangle & \langle a_1|b_3\rangle & \cdots \\ \langle a_2|b_1\rangle & \langle a_2|b_2\rangle & \langle a_2|b_3\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \longrightarrow$$ Matrix representation

$$|\alpha\rangle = \sum_i |a_i\rangle\langle a_i|\alpha\rangle\ , \quad \langle b_j|\alpha\rangle = \sum_i \langle b_j|a_i\rangle\langle a_i|\alpha\rangle = \sum_i \langle b_j|\tilde{u}^\dagger|b_i\rangle\langle a_i|\alpha\rangle$$

$$\langle b_i|\alpha\rangle = \begin{pmatrix} \langle b_1|\alpha\rangle \\ \langle b_2|\alpha\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle b_1|u|b_1\rangle & \langle b_1|u|b_2\rangle & \cdots \\ \langle b_2|u|b_1\rangle & \langle b_2|u|b_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \vdots \end{pmatrix} \quad \text{w/} \quad \tilde{u}|a_i\rangle = |b_i\rangle$$

$$\langle b_i|\tilde{x}|b_j\rangle = \sum_{mn} \langle b_i\Big[|a_m\rangle\langle a_m|\tilde{x}|a_n\rangle\langle a_n|\Big]|b_j\rangle = \sum_{mn} \langle a_i|\tilde{u}^\dagger|a_m\rangle\langle a_m|\tilde{x}|a_n\rangle\langle a_n|\tilde{u}|a_j\rangle$$

$\tilde{A}|a_i\rangle = a_i|a_i\rangle$ want to transform to $\tilde{B}|b_i\rangle = b_i|b_i\rangle$ w/ $\tilde{x}_b = \tilde{u}^\dagger\tilde{x}_a\tilde{u}$

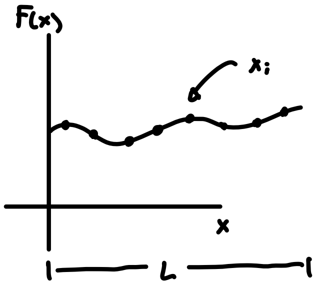
$$\tilde{u}\tilde{A}|a_i\rangle = a_i\tilde{u}|a_i\rangle \ : \ \tilde{u}\tilde{A}\tilde{u}^\dagger\tilde{u}|a_i\rangle = a_i\tilde{u}|a_i\rangle \ : \ \tilde{B}|b_i\rangle = a_i|b_i\rangle$$

Exam Prep

* Basic concepts
  * State vector - How to write in different representations
* Completeness relationship
* Associative theorem
* Eigenvalue equation

9-20-21

$|f_n\rangle \to$ n denotes dimension of space : $|f_n\rangle \doteq \begin{pmatrix} f_n(x_1) \\ f_n(x_i) \\ f_n(x_n \end{pmatrix}$



$$|x_i\rangle \doteq \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow \text{ith}$$

$$\langle x_j|x_i\rangle = \delta_{ij} \ : \ \sum_i |x_i\rangle\langle x_i| = \tilde{1}$$

$$|f_n\rangle = \sum_i |x_i\rangle\langle x_i|f_n\rangle \ : \ \langle x_i|f_n\rangle \to f_n(x_i)$$

$\langle f_n|g_n\rangle = \sum_i \langle f_n|x_i\rangle\langle x_i|g_n\rangle = \sum_i f_n(x_i)g_n(x_i)$ assume g_n and f_n are real functions

$\langle f_n|f_n\rangle = \sum_i f_n(x_i)f_n(x_i)\Delta x_i$: $\Delta x = L/(n-1)$ $\underset{n\to}{\text{dim}}\ \infty \longrightarrow \Delta x = dx$

$\sum_i g_n(x_i)f_n(x_i)$ goes to $\int_0^L g(x)^* f(x)\,dx = \int_0^L \langle g|x\rangle\langle x|f\rangle\,dx$

$\langle x | x' \rangle = 0$ if $x \neq x'$: $\sum |x_i\rangle\langle x_i| = \mathbb{1} = \int_0^L |x'\rangle\langle x'| \, dx'$

$\langle x | f \rangle = \int_0^L \langle x | x' \rangle \langle x' | f \rangle \, dx' = f(x) = \int_0^L \langle x | x' \rangle \langle x' | f \rangle \, dx \longrightarrow \int_{x-\epsilon}^{x+\epsilon} \langle x | x' \rangle \langle x' | f \rangle \, dx'$

$f(x) \left[\int_{x-\epsilon}^{x+\epsilon} \langle x | x' \rangle \, dx' \right] = 1$: $\langle x | x' \rangle = \delta(x, x') = \delta(x - x')$: $\int_a^b \delta(x - x') \, dx' = 1$

$$\delta(x - x') = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases} \quad a \leq x \leq b \, , \quad [\hat{x}, \tilde{y}] = [\tilde{x}, \tilde{z}] = [\tilde{y}, \tilde{z}] = 0$$

Translations

$\mathcal{J}(dx') | x' \rangle = | x' + dx' \rangle$: $\mathcal{J}(dx') | \alpha \rangle = \mathcal{J}(dx') \left[\int_a^b |x'\rangle\langle x'| \right] | \alpha \rangle \, dx = \mathcal{J}(dx') \int_a^b |x'\rangle\langle x' | \alpha \rangle \, dx'$

$\mathcal{J}(dx') | \alpha \rangle = \int_a^b | x' + dx' \rangle \langle x' | \alpha \rangle \, dx'$: $x' \rightarrow x' - dx'$ $\mathcal{J}(dx') | \alpha \rangle = \int_a^b | x' \rangle \langle x' - dx' | \alpha \rangle \, dx'$

$\langle \alpha | \alpha \rangle = \langle \alpha | \underbrace{\mathcal{J}^\dagger(dx') \mathcal{J}(dx')}_{\mathbb{1}} | \alpha \rangle$: $\mathcal{J}(dx'') \mathcal{J}(dx') = \mathcal{J}(dx'' + dx')$: $\mathcal{J}(-dx') = \mathcal{J}^{-1}(dx')$

$dx' \rightarrow 0$ $\mathcal{J}(dx') = \mathcal{J}(0) = 1$: $\mathcal{J}(d\vec{x}) = 1 - i \tilde{\vec{k}} \cdot d\vec{x}'$

$\mathcal{J}^\dagger(dx') \mathcal{J}(dx') = 1 = (1 + i \tilde{\vec{k}}^\dagger \cdot dx')(1 - i \tilde{\vec{k}} \cdot dx') = 1 - i(\tilde{\vec{k}} - \tilde{\vec{k}}^\dagger) \cdot dx' + \mathcal{O}(dx'^2)$

$\tilde{\vec{k}} = \hat{\vec{k}}^\dagger$, $\hat{\vec{k}}$ is Hermitian

9-22-21

Translation operator $\mathcal{J}(dx) | x \rangle = | x + dx \rangle$, Propose $\mathcal{J}(d\vec{x}) = \mathbb{1} - i \vec{k} \cdot d\vec{x}$

unitary operator (maintain normalization)

$$\langle \alpha | \alpha \rangle = \langle \alpha | \mathcal{J}^\dagger(d\vec{x}') \mathcal{J}(d\vec{x}) | \alpha \rangle \Rightarrow \mathcal{J}^\dagger(d\vec{x}') \mathcal{J}(d\vec{x}') = 1 \Rightarrow \tilde{k} = \tilde{k}^\dagger$$

Succesive translations:

$$\mathcal{J}(d\vec{x}') \mathcal{J}(d\vec{x}') = \mathcal{J}(dx'' + dx')$$

Inverse translation:

$$\mathcal{J}(-d\vec{x}') = \mathcal{J}^{-1}(d\vec{x}')$$

No translation

$$\lim_{d\vec{x}' \rightarrow 0} \mathcal{J}(d\vec{x}') = \mathcal{J}(0) = 1$$

Commutation Relation : $[\tilde{x}_i, \tilde{x}_j] = 0$

$$\left[\tilde{\vec{x}}, \tilde{\mathcal{J}}(dx') \right] | \vec{x}' \rangle = \tilde{\vec{x}} \tilde{\mathcal{J}}(dx') | \vec{x}' \rangle - \tilde{\mathcal{J}}(dx') \tilde{\vec{x}} | \vec{x}' \rangle$$

$\tilde{\vec{x}}'|\vec{x}'+d\vec{x}'\rangle = \vec{x}'+d\vec{x}'|\vec{x}'+d\vec{x}'\rangle \quad : \quad \tilde{\mathcal{J}}(dx')\tilde{\vec{x}}|\vec{x}'\rangle = \tilde{\mathcal{J}}(dx')\vec{x}'|\vec{x}'\rangle = \vec{x}'|\vec{x}'+d\vec{x}'\rangle$

$\langle\alpha|\alpha\rangle = 1 \longrightarrow \mathcal{J}(dx)|\alpha\rangle \Leftarrow DC \Rightarrow \langle\alpha|\mathcal{J}^{+}(dx) \quad : \quad [\tilde{\vec{x}}, \tilde{\mathcal{J}}(d\vec{x}')] = d\vec{x}'\,\mathbb{1}$

$-i\tilde{\vec{x}}(\tilde{\vec{k}}\cdot d\vec{x}') + i(\tilde{\vec{k}}\cdot d\vec{x}')\tilde{\vec{x}}' = d\vec{x}'\,\tilde{\mathbb{1}}$

$[\tilde{x}_i, \tilde{k}_j] = i\delta_{ij} \quad , \quad \tilde{\vec{p}} = \hbar\tilde{\vec{k}} \quad : \quad \tilde{\vec{k}} \longrightarrow$ units of m^{-1} from deBroglie wavelength

$[\tilde{x}_i, \tilde{p}_j] = i\hbar\delta_{ij} \Longleftrightarrow \langle\Delta x^2\rangle\langle\Delta p^2\rangle \geq \frac{\hbar^2}{4}$

$\tilde{\mathcal{J}}(d\vec{x}) = \tilde{\mathbb{1}} - i\tilde{\vec{k}}\cdot d\tilde{\vec{x}} \longrightarrow$ Generator function

$$[\tilde{\mathcal{J}}(dy'\hat{y}), \tilde{\mathcal{J}}(dx'\hat{x})]|x,y\rangle = \frac{dx'dy'[P_y, P_x]}{\hbar^2} = 0 \Rightarrow [P_y, P_x] = 0$$

$$\mathcal{J}(\Delta x'\hat{x}) = \left(1 - \frac{i\tilde{P}_x\Delta x}{\hbar}\right) = \left(1 - \frac{i\tilde{P}_x}{\hbar}\frac{\Delta x}{2}\right)\left(1 - \frac{i\tilde{P}_x}{\hbar}\frac{\Delta x}{2}\right) = \left(1 - \frac{i\tilde{P}_x}{\hbar}\frac{\Delta x}{N}\right)^N$$

$$\lim_{N\to\infty}\left(1 - \frac{i\tilde{P}_x}{\hbar}\frac{\Delta x}{N}\right)^N \longrightarrow \tilde{\mathcal{J}}(\Delta x'\hat{x}) = e^{-i\tilde{P}_x\Delta x/\hbar} \quad , \quad \tilde{\mathcal{J}}(\Delta\vec{x}) = e^{-i\tilde{\vec{P}}\cdot\Delta\vec{x}/\hbar}$$

9-27-21

Momentum operator : $\tilde{X}|x'\rangle = x'|x'\rangle$, Normalization : $\int_{-\infty}^{+\infty}\langle x''|x'\rangle\,dx'' = 1$

Dirac delta function : $\langle x''|x'\rangle = \delta(x''-x')$

$$|\alpha\rangle = \int|x'\rangle\langle x'|\,|\alpha\rangle\,dx' = \int|x'\rangle\underbrace{\langle x'|\alpha\rangle}_{\to\,\psi_\alpha(x)}dx' \quad : \quad \langle\alpha|\alpha\rangle = \int\langle\alpha|x'\rangle\langle x'|\alpha\rangle\,dx' = \int|\langle x'|\alpha\rangle|^2dx'$$

$$\langle\alpha|\alpha\rangle = \sum\langle\alpha|a_i\rangle\langle a_i|\alpha\rangle = \sum|\langle a_i|\alpha\rangle|^2 = 1$$

$|\langle x'|\alpha\rangle|^2dx' \longrightarrow$ Probability of being between $x' \to x'+dx'$

$\langle\beta|\alpha\rangle = \int\langle\beta|x'\rangle\langle x'|\alpha\rangle\,dx' = \int\psi_\beta^*(x')\psi_\alpha(x')\,dx' \longrightarrow$ overlap between α & β

With $|\alpha\rangle = \sum|a_i\rangle\langle a_i|\alpha\rangle \Rightarrow \langle x'|\alpha\rangle = \sum\underbrace{\langle x'|a_i\rangle}_{\to\,u_i(x')}\underbrace{\langle a_i|\alpha\rangle}_{\to\,c_i}$ $\quad u_i(x') \to$ Eigenfunctions

$\psi_\alpha(x') = \sum c_i u_i(x') \longrightarrow c_i$ probability , u_i eigenfunction

$\langle\beta|\tilde{A}|\alpha\rangle$ with $\tilde{A}|\alpha\rangle = |\gamma\rangle \quad \therefore \quad \langle\beta|\tilde{A}|\alpha\rangle$ goes to $\langle\beta|\gamma\rangle$

$$\tilde{A} = \tilde{x}^2 \; : \; \langle\beta|\tilde{x}^2|\alpha\rangle = \iint dx''dx'\langle\beta|x''\rangle\langle x''|\tilde{x}^2|x'\rangle\langle x'|\alpha\rangle = \iint dx''dx'\langle\beta|x''\rangle\langle x''|x'\rangle\langle x'|\alpha\rangle\,x'^2$$

$$\therefore \langle\beta|\tilde{x}^2|\alpha\rangle = \iint dx''dx'\,\psi_\beta^*(x')\psi_\alpha(x')\delta(x''-x')\,x'^2 = \int\psi_\beta^*(x')\,x'^2\,\psi_\alpha(x')\,dx'$$

$$\left(1 - \frac{i\tilde{P}\Delta x'}{\hbar}\right)|x'\rangle = |x'+\Delta x'\rangle \quad : \quad \left(1 - \frac{i\tilde{P}\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx'\,\mathcal{J}(\Delta x')|x'\rangle\underbrace{\langle x'|\alpha\rangle}_{\to\text{ wave function}}$$

$$(\ldots)|\alpha\rangle = \int dx'|x'+\Delta x'\rangle\langle x'|\alpha\rangle = \int dx'|x'\rangle\langle x'-\Delta x'|\alpha\rangle \quad : \quad \langle x'-\Delta x'|\alpha\rangle \equiv \psi_\alpha(x'-\Delta x')$$

$$\ldots\; \int dx' |x'\rangle\langle x'-\Delta x'|\alpha\rangle = \int dx' |x'\rangle\left[\langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right]$$

$$= \int dx' |x'\rangle\left[1 - \Delta x' \frac{\partial}{\partial x'}\right]\langle x'|\alpha\rangle \; : \; \tilde{P} = -i\hbar \frac{\partial}{\partial x}$$

9-29-21

$\psi_\alpha(x') = \langle x'|\alpha\rangle$, Analagous to $\tilde{A}|a_i\rangle = a_i|a_i\rangle \; \therefore \; u_i(x') = \langle x'|a_i\rangle$

$\psi_\alpha(x') = \sum_i \langle x'|\alpha\rangle = \sum_i \langle x'|a_i\rangle\langle a_i|\alpha\rangle = \sum u_i(x')\cdot c_i \; : \; \tilde{P} \doteq -i\hbar\frac{\partial}{\partial x} \; : \; u_i \rightarrow$ Eigenfunctions

$|\langle x'|\alpha\rangle|^2 dx' \longrightarrow$ Probability of being at x'

$\tilde{P}|P'\rangle = P'|P'\rangle$, $\langle P'|P''\rangle = \delta(P'-P'') \rightarrow$ For continuous values & δ for discrete

$|\alpha\rangle = \int dp' |p'\rangle\langle p'|\alpha\rangle \; : \; \langle\alpha|\alpha\rangle = \int dp' \langle\alpha|p'\rangle\langle p'|\alpha\rangle$, $\phi_\alpha(p') = \langle p'|\alpha\rangle$

$\langle\alpha|p'\rangle\langle p'|\alpha\rangle dp' = |\varphi_\alpha(p')|^2 dp'$

$\langle x|\tilde{p}|p'\rangle = p'\langle x|p'\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p'\rangle$, $-i\hbar\frac{\partial}{\partial x}\langle x'|p'\rangle = p'\langle x'|p'\rangle$, $\langle x'|p'\rangle = N e^{ip'x'/\hbar}$

$\langle x'|x''\rangle = \delta(x'-x'') = \int dp' \langle x'|p'\rangle\langle p'|x''\rangle = |N|^2 \int dp' e^{ip'x'/\hbar} e^{-ip'x''/\hbar}$

$\int dx' \langle x'|x''\rangle = \int dx' \delta(x'-x'') = 1$

$|N|^2 \int dp' e^{ip'x'/\hbar} e^{-ip'x''/\hbar} = |N|^2 \int dx' \int dp' e^{ip'(x'-x'')/\hbar} = |N^2| \cdot 2\pi\hbar$

$\int_{-P_0}^{P_0} dp' e^{ip'(x'-x'')/\hbar} = \dfrac{2\hbar\sin(P_0\Delta x/\hbar)}{\Delta x}$ w/ $\Delta x \to 0 \approx \dfrac{2\hbar P_0 \Delta x/\hbar}{\Delta x} \approx P_0$ w/ $P = \hbar k = \dfrac{h}{\lambda}$

$\Delta x \to \infty \int dp' \to 0$, $\Delta x \to 0 \int dp' \to P_0$