Assignment 10 Solutions

Question 1:

a) See your lecture notes for details!

① The central force is spherically symmetric, and so is the associated potential,

$$V(r) = -a/r^2$$

② The symmetry of the force/potential is inherited by the Lagrangian describing the particle's motion.

$$\boxed{\text{Spherical symmetry} \Rightarrow \text{conservation of total angular momentum.}}$$

③ By definition, our motion must be in a plane perpendicular to the angular momentum vector $\vec{L}$,

$$\vec{r} \cdot \vec{L} = m\vec{r} \cdot (\vec{r} \times \dot{\vec{r}})$$
$$= m\dot{\vec{r}} \cdot (\vec{r} \times \vec{r}) = 0 \quad \therefore \vec{r} \perp \vec{L}$$

posn vector of particle

b) Using polar co-ordinates to describe motion in the plane, the Lagrangian is:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - V(r)$$

The equations of motion are:

i) $$\frac{d}{dt}\left(mr^2\dot{\varphi}\right) = 0 \Rightarrow l = mr^2\dot{\varphi} \text{ is conserved}$$

magnitude of angular momentum.

ii) $$m\ddot{r} - mr\dot{\varphi} + \frac{\partial V}{\partial r} = 0$$

replace w/ $\dot{\varphi} = l/mr^2$

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial V}{\partial r} = 0$$

or

$$m\ddot{r} = F_{eff}(r) \quad \text{w/ } F_{eff} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

and $$V_{eff} = \frac{l^2}{2mr^2} + V(r)$$

such that $F_{eff} = -\frac{\partial V_{eff}}{\partial r}$!

c)

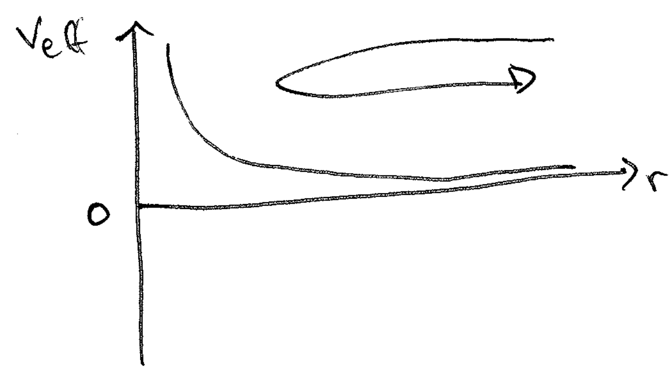
$$V_{eff} = \frac{\frac{L^2}{2m} - a}{r^2}$$

$\propto \frac{1}{r^2}$

sign depends on relative magnitude of $\frac{L^2}{2m}$ & a.

i)

$$\frac{L^2}{2m} - a > 0$$



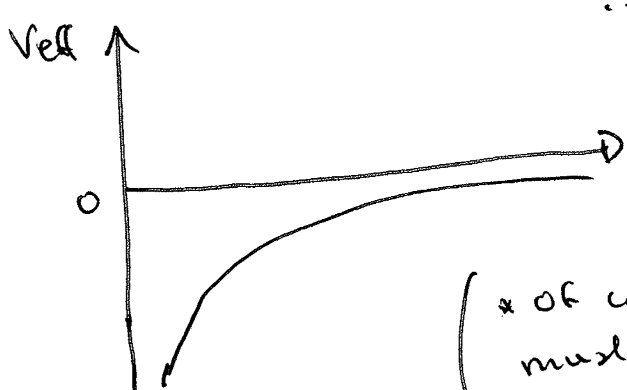
scattered motion for any $E_0 > 0$

ii)

$$\frac{L^2}{2m} - a < 0$$

$\Rightarrow$ divergent attractive potential as $r \to 0$

$\therefore$ particle is attracted to $r=0$ for any initial energy.



(* of course, physically our model must break down as $r \to 0$...)

iii) A circular orbit requires $F_{eff} = 0$.

$\Rightarrow$ Solve for $\frac{\partial V_{eff}}{\partial r} = 0$

$\Rightarrow$ Only solution is trivial case w/

$$\frac{l^2}{2m} - a = 0 \quad \text{or} \quad V_{eff} = 0 \;!$$

d) We have that total mechanical energy can be written as,

$$E = T + V = \frac{m\dot{r}^2}{2} + V_{eff}(r)$$

and l is conserved.

Hence, rearranging we can obtain,

$$\dot{r} = \pm \sqrt{\frac{2}{m}(E - V_{eff})} \,.$$

Separately,

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dr} \cdot \frac{dr}{dt}$$

or,

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}}$$

$$= \pm \frac{L}{mr^2} \cdot \frac{1}{\sqrt{\frac{2}{m}\left(E - V_{eff}\right)}}$$

$$\Rightarrow \quad \varphi - \varphi_0 = \pm \int_{r_0}^{r} dr' \, \frac{L}{mr'^2} \frac{1}{\sqrt{\frac{2}{m}\left(E - V_{eff}(r')\right)}} .$$

Question 2:

a) See your lecture notes!

b) This system of equations is known for exhibiting an Andronov-Hopf bifurcation.

Let's write down the equations in polar co-ordinates:

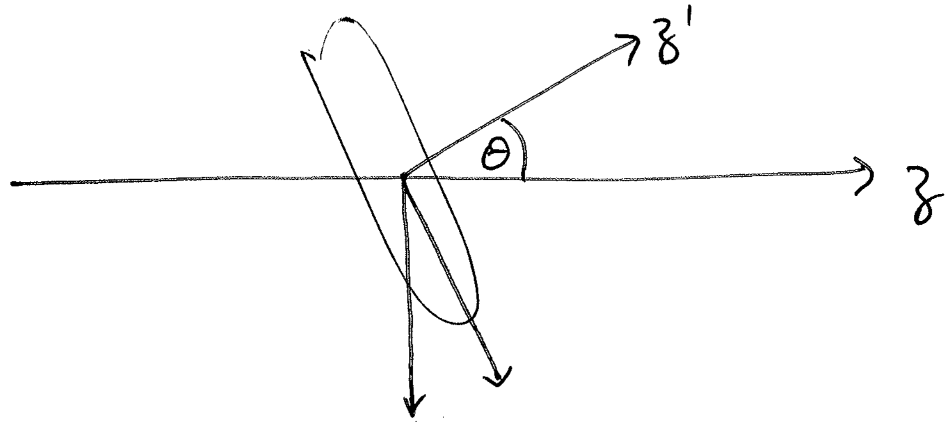
$$\dot{r} = ar - Cr^3 = ar\left(1 - \frac{C}{a} r^2\right)$$

&

$$\dot{\theta} = b$$

Question 3:

a)



Working w/ body-fixed z' along the normal to the disc, we compute the principal moment of inertia:

$$I_z = \int dV \, \rho_0 (x^2+y^2) \qquad (= I_{zz})$$

$\rho_0 = \frac{m}{\pi R^2}$ = mass density

$$= \rho_0 \int_0^R dr \int_0^{2\pi} d\theta \; r^2 \, r dr$$

[disc is of negligible thickness along z']

$$= 2\pi \rho_0 \, R^4/4 = \frac{mR^2}{2}$$

The remaining moments are equal, $I_1 = I_2$. There are two ways to obtain them.

$$\text{i)}\quad I_1 = I_2 = \overbrace{\int dV\, y^2 + \cancel{z^2}}^{I_1} \xrightarrow{} 0 \text{ as disc is negligibly thin}$$

$$= \rho_0 \int_0^{2\pi} d\theta \int_0^R dr\, r^3 \sin\theta \quad \text{~ evaluate using known integral or formula sheet.}$$

$$= \frac{mR^2}{4}$$

$$\text{ii)}\quad I_1 + I_2 = \int dV\, x^2 + y^2 + 2\cancel{z^2} \to 0$$

$$= I_3$$

$$\therefore I_1 = I_2 = I_3/2 = \frac{mR^2}{4}.$$

b) This is the same as the barbell problem we studied in Assignment 7. The key part to realize is that θ is fixed, & there is no rotation about body-fixed $\hat{z}$ (the disc is rigidly attached). Depending on how you choose your axes you should obtain something resembling,

$$\vec{\omega} = \dot{\phi} \begin{pmatrix} \sin\phi \sin\theta \\ \cos\phi \sin\theta \\ \cos\theta \end{pmatrix}$$

w/ the $\sin\phi$ & $\cos\phi$ terms depending on axes choice.

c) The kinetic energy is solely due to rotation (as the origin of our body-fixed axes is at the com). Thus, we evaluate it using body-fixed quantities,

$\vec{\omega}$ in body-fixed axes.

$$T = T_{rot} = \frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]$$

Using that $I_1 = I_2 = \underline{I_d}$ we have:

$$T = \frac{I_d}{2}\dot{\phi}^2\sin^2\theta + \frac{I_3}{2}\dot{\phi}^2\cos^2\theta$$

d) We want $\dot{\vec{\omega}} = 0$ for the motion to be preserved (this is the same as the barbell in A7!).

Here w/ a fixed rotation frequency Ω,

$$\vec{\omega} \to \Omega\begin{pmatrix}\sin\phi\sin\theta\\ \cos\phi\sin\theta\\ \cos\theta\end{pmatrix} \Rightarrow \sqrt{\omega_1^2+\omega_2^2} = \Omega\sin\theta$$

Euler's equations give us:

$$(I_3 - I_2)\,\omega_2\omega_3 = N_1$$

$$(I_1 - I_3)\,\omega_1\omega_3 = N_2$$

& $(I_2 - I_1)\,\omega_1\omega_2 = N_3$

$I_1 = I_2 \Rightarrow N_3 = 0$

& $|\vec{N}|^2 = \sqrt{N_1^{\,2} + N_2^{\,2}} = \sqrt{\left(\frac{mR^2}{4}\omega_2\omega_3\right)^2 + \left(\frac{mR^2}{4}\omega_1\omega_3\right)^2}$

$= \frac{mR^2}{4}\sqrt{\omega_1^2+\omega_2^2}\,|\omega_3|$

$= \frac{mR^2}{4}\Omega^2 \cos\theta \sin\theta \qquad (\text{for } 0<\theta<\pi/2)$