Homework # 5

a) $$\psi_\sigma(\vec{x}) = \frac{1}{\sqrt{V}} \sum_k e^{i\vec{k}\cdot\vec{x}}\, a_{k\sigma}$$

$$K = -\frac{\hbar^2}{2m}\frac{1}{V}\sum_{\sigma,k,k'} \int d^d x\, a^\dagger_{k\sigma} a_{k'\sigma}\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} \, (-k'^2)$$

$$= \frac{\hbar^2}{2m}\frac{1}{V}\sum_{\sigma,\vec{k},\vec{k}'} k'^2 \underbrace{\int d^d x\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{V\delta_{k,k'}} a^\dagger_{k\sigma} a_{k'\sigma}$$

$$= \sum_{\sigma,k} \frac{\hbar^2 k^2}{2m} a^\dagger_{k\sigma} a_{k\sigma}$$

b) $$\vec{P} = -i\hbar \int d^d\vec{x}\, \psi^\dagger(\vec{x}) \nabla \psi(\vec{x})$$

$$= \frac{1}{V}\sum_{\sigma k k'} a^\dagger_{k\sigma} a_{k'\sigma} \underbrace{\int d^d\vec{x}\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{V\delta_{\vec{k},\vec{k}'}} \times (\hbar \vec{k}')$$

$$= \sum_{k,\sigma} \hbar \vec{k}\, a^\dagger_{k\sigma} a_{k\sigma}$$

c)

$$|FS\rangle = \prod_{\alpha \in N} c^{+}_{\alpha} |0\rangle$$

The total Energy is

$$\langle FS | K | FS \rangle = \sum_{\sigma, k} \frac{\hbar^2 k^2}{2m} \underbrace{\langle FS | a^{+}_{k\sigma} c_{k\sigma} | FS \rangle}_{\Theta(k_F - k)}$$

$$= V \left(\frac{1}{V} \sum_{\sigma, k} \right) \frac{\hbar^2 k^2}{2m} \Theta(k_F - k)$$

$$= \frac{V}{(2\pi)^2} \int_0^{k_F} d^2k \, \frac{\hbar^2 k^2}{m}$$

$$= \frac{V \hbar^2 k_F^4}{8\pi \, m}.$$

The total momentum is:

$$\langle FS|\vec{P}|FS\rangle = \sum_{\sigma,\vec{k}} \hbar\vec{k}\, \overbrace{\langle FS|a^{\dagger}_{k\sigma}a_{k\sigma}|FS\rangle}^{\Theta(k_F-k)}$$

$$= \frac{V}{(2\pi)^2}\int_0^{k_F} d^2k\ \hbar\vec{k}$$

$$= \frac{V\hbar}{(2\pi)^2}\int_0^{k_F} dk\, k^2 \int_0^{2\pi} (\cos\theta, \sin\theta)\, d\theta$$

$$= 0$$

The total momentum in the Fermi surface is zero (system is time reversal invariant and it's center of mass is at rest: <u>no</u> currents) d)

$$\vec{S} = \int d^d x \; \hat{\psi}^{\dagger}(\vec{x})\, \vec{\sigma}\, \hat{\psi}(\vec{x})$$

$$= \int d^d x \sum_{\alpha\beta} \psi_{\alpha}(\vec{x})\, \sigma_{\alpha\beta}\, \psi_{\beta}(\vec{x})$$

$$= \frac{1}{V} \sum_{\alpha\beta} \sum_{k k'} \sigma_{\alpha\beta}\, a^{\dagger}_{k\alpha}\, a_{k'\beta} \int d^d x \; e^{i(\vec{k}' - \vec{k}) \cdot \vec{x}}$$

$$= \sum_{\alpha\beta} \sum_{k} a^{\dagger}_{k\alpha}\, \sigma_{\alpha\beta}\, a_{k\beta}$$

Hence:

$$H_B = -\gamma_B \vec{S} \cdot \vec{B}$$

$$= -\gamma_B \vec{B} \cdot \sum_{k,\alpha\beta} a^{\dagger}_{k\alpha}\, \sigma_{\alpha\beta}\, a_{k\beta}$$

Assuming $\vec{B} = B\hat{z}$,

$$H_B = -\mu_B B \sum_{\alpha\beta,k} a^{+}_{k\alpha} \sigma^{z}_{\alpha\beta} a_{k\beta}$$

$$= -\mu_B B \sum_{k} \left(a^{+}_{k\uparrow} a_{k\uparrow} - a^{+}_{k\downarrow} a_{k\downarrow} \right)$$

The total Energy is:

$$\langle FS | K + H_B | FS \rangle = \langle K \rangle_{FS} + \langle H_B \rangle_{FS}$$

$$\langle H_B^{\uparrow} \rangle_{FS} = -\mu_B B \sum_{k} \langle a^{+}_{k\uparrow} a_{k\uparrow} \rangle_{FS}$$

$$= -\mu_B B \frac{V}{(2\pi)^2} \int_0^{k_F} d\vec{k}\ \Theta(k - k_F)$$

$$= -\frac{\mu_B B V}{4\pi} k_F^2 .$$

$$\langle H_B^{\downarrow} \rangle_{FS} = + \frac{\mu_B B V}{4\pi} k_F^2$$

The total energy for spin is:

$$E_{\uparrow,\downarrow} = \frac{V}{8\pi}\frac{\hbar^2 k_F^4}{m} \mp \frac{\mu_B B V}{4\pi} k_F^2$$

∴ The ground state for electrons with spin "up" have lower energy than for spin "down". The system then reequilibrate with two Fermi surfaces, one for up spins and another for down, such that $E_{\uparrow, k_{F\uparrow}} = E_{\downarrow, k_{F\downarrow}}$.

(4)

(2)

a)

For $\begin{cases} x = \sin\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \cos\theta \end{cases}$, $\hat{n} = (x, y, z)$

$$Y_2^{\pm 2}(\theta, \phi) \longrightarrow T^2_{\pm 2} = \sqrt{\tfrac{3}{2}} \propto (x \pm iy)^2$$

$$\therefore \quad x^2 - y^2 = \underline{\text{const}} \cdot \left(T^2_{+2} + T^2_{-2} \right)$$

as can be seen from the definition of spherical Harmonics.

b) From the Wigner-Eckart theorem,

$$\Delta_{mm'} \equiv \langle n,1,m'|(x^2-y^2)|n,1,m\rangle$$

$$= \underline{\text{const}}\ \langle n,1,m'|(T^2_2 + T^2_{-2})|n,1,m\rangle$$

$$\propto \langle 21;1,m'|21;2m\rangle \ \frac{\langle n,1\|T^2\|n1\rangle}{\sqrt{5}}$$

$$+ \langle 21;1m'|21,-2m\rangle \times \frac{\langle n,1\|T^2\|n1\rangle}{\sqrt{5}}$$

$\therefore$ $\Delta_{m,m'}$ is non-zero for $m=1,\ m'=-1$ or
$m=-1,\ m'=1,$

since: $\begin{cases} m' = 2+m \text{ or} \\ m' = -2+m. \end{cases}$

$\therefore$

$$\hat{V} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} m=-1 & 0 & 1 \end{array} & \\ \begin{pmatrix} 0 & 0 & \Delta \\ 0 & 0 & 0 \\ \Delta & 0 & 0 \end{pmatrix} & \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \end{array}$$

in the $|n,1,m\rangle$ basis.

(c)

c) Since the three levels are degenerate, then

$$\begin{vmatrix} \delta\epsilon & 0 & -\Delta \\ 0 & \delta\epsilon & 0 \\ -\Delta & 0 & \delta\epsilon \end{vmatrix} = 0$$

$\therefore \ \delta\epsilon = 0, \pm\Delta.$

For $\delta\epsilon = 0$, the eigenket is

$$|0\rangle = |n,1,0\rangle.$$

For $\delta\epsilon = \pm\Delta$:

$$|\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|n,1,1\rangle \pm |n,1,-1\rangle\right)$$

D) If the degeneracy of the levels is lifted by

$$H_Z = \mu_B B m ,$$

then the unperturbed energy is:

$$E^0_m = \mu_B B m ,$$

with $\mu_B B \gg \Delta$.

~~Det~~:

$$\Delta E_m = \sum_{m' \neq m} \frac{|\langle n 1 m | \hat{V} | n 1, m' \rangle|^2}{E^0_m - E^0_{m'}}$$

$$\Delta E_0 = 0$$

$$\Delta E_1 = \frac{|\Delta|^2}{E^0_1 - E^0_{-1}} = \frac{|\Delta|^2}{2\mu_B B}$$

$$\Delta E_{-1} = \frac{|\Delta|^2}{E^0_{-1} - E^0_1} = -\frac{|\Delta|^2}{2\mu_B B}$$

where

$$E_m = E_m^0 + \Delta E_m .$$

(3)

a)

For a harmonic oscillator with energy levels:

$$E_n^0 = \hbar\omega(n + \tfrac{1}{2}),$$

the corrected ket of the ground state is:

$$|\bar{0}\rangle = |0\rangle + \lambda \sum_{n=1}^{\infty} \frac{\langle n|\sin(kx)|0\rangle}{E_0^0 - E_n^0} |n\rangle$$

$$= |0\rangle - \frac{\lambda}{\hbar\omega} \sum_{n=1}^{\infty} \frac{\langle n|\sin(kx)|0\rangle}{n} |n\rangle.$$

Since:

$$e^{A+B} = e^A e^B e^{-[A,B]/2}$$

then:

$$\langle n|e^{ikx}|0\rangle = \langle n|\exp\Big[ik\underbrace{\sqrt{\tfrac{\hbar}{2m\omega}}(a+a^\dagger)}_{x}\Big]|0\rangle$$

Defining $\beta \equiv \sqrt{\frac{\hbar}{2m\omega}}$, then

$$\langle n|e^{ikx}|0\rangle = \langle n|e^{ik\beta a^{+}} e^{ik\beta a}|0\rangle e^{-\beta^2 k^2/2} \quad (*)$$

$$= \langle n|e^{ik\beta a^{+}}|0\rangle e^{-\beta^2 k^2/2}$$

$$= \langle n| \sum_{n'=0}^{\infty} \frac{1}{n'!} (ik\beta a^{+})^{n'} |0\rangle e^{-\beta^2 k^2/2}$$

$$= \frac{1}{\sqrt{n!}} (ik\beta)^{n} e^{-\beta^2 k^2/2}.$$

Hence,

$$|\bar{0}\rangle = |0\rangle - \frac{\lambda}{2i\hbar\omega} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(ik\beta)^n}{\sqrt{n!}} \cdot e^{-\beta^2 k^2/2} |n\rangle$$

$$+ \frac{\lambda}{2i\hbar\omega} \sum_{n=1}^{\infty} \frac{1}{n} \frac{(-ik\beta)^n}{\sqrt{n!}} e^{-\beta^2 k^2/2} |n\rangle$$

(*) $[a, a^{+}] = 1$.

$$= |0\rangle - \frac{\lambda}{\hbar\omega} e^{-\beta^2 \kappa^2/2} \times$$

$$\times \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)} \frac{1}{\sqrt{(2\nu+1)!}} (\kappa\beta)^{2\nu+1} \, |2\nu+1\rangle$$

b)

In leading order,

$$\langle\bar{0}|x|\bar{0}\rangle = \langle\bar{0}|x|0\rangle + \langle 0|x|\bar{0}\rangle + O(\lambda^2)$$

$\therefore$

$$\langle\bar{0}|x|\bar{0}\rangle \simeq -\frac{2\lambda}{\hbar\omega} e^{-\beta^2\kappa/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu+1} \frac{(\kappa\beta)^{2\nu+1}}{\sqrt{(2\nu+1)!}} \langle 0|x|2\nu+1\rangle$$

$$= -\frac{2\lambda}{\hbar\omega} e^{-\beta^2\kappa/2} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu+1} \frac{\kappa^{2\nu+1}}{\sqrt{(2\nu+1)!}} \beta^{2(\nu+1)} \times$$

$$\times \langle 0|a+a^\dagger|2\nu+1\rangle$$

$$= -\frac{2\lambda}{\hbar\omega} e^{-\beta^2\kappa^2/2} \kappa\beta^2$$

$$= \frac{-\lambda\kappa}{m\omega^2} e^{-\hbar\kappa^2/4m\omega}$$

c)

If $\delta V(x) \simeq \lambda k x$, then

$$\Delta E_n = \lambda k \cancelto{0}{\langle n | x | n \rangle}$$

$$+ \lambda^2 k^2 \sum_{n' \neq n} \frac{|\langle n | x | n' \rangle|^2}{E_n^0 - E_{n'}^0}$$

$\therefore$

$$\Delta E_0 = -\frac{\lambda^2 k^2}{\hbar \omega} \sum_{n' \neq 0} \frac{|\langle 0 | a + a^\dagger | n' \rangle|^2}{n'} \frac{\hbar}{2m}$$

$$= -\frac{\lambda^2 k^2}{2m\omega}.$$