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Quantum Mechanics 1

PHYS 5393 HOMEWORK ASSIGNMENT #4

PROBLEMS: {1.13, 1.18, 1.25, 1.28}

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Problem 1: 1.13

A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}\Big[|1\rangle\langle 2| + |2\rangle\langle 1|\Big]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

Observables acting on states will reveal physical quantities to be measured.

$$\hat{G} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{12} & H_{22}-\lambda \end{pmatrix} : \det(\hat{G}) = (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 = 0$$

To find the eigenvalues we will do the following w/ the characteristic equation:

$$0 = (H_{11}-\lambda)(H_{22}-\lambda) - H_{12}^2 = H_{11}H_{22} - \lambda H_{11} - \lambda H_{12} + \lambda^2 - H_{12}^2 : \lambda H_{11} + \lambda H_{12} - \lambda^2 = H_{11}H_{22} - H_{12}^2$$

$$\lambda(H_{11} + H_{12} - \lambda) = H_{11}H_{22} - H_{12}^2 \quad \text{using mathematica} : \quad \lambda = \frac{1}{2}\left(H_{11} + H_{22} \pm \sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}\right)$$

$$\boxed{\lambda = \frac{1}{2}\left(H_{11} + H_{22} \pm \sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}\right)}$$

If $H_{12} = 0$ then $\lambda = \frac{1}{2}\left(H_{11} + H_{22} \pm (H_{11} - H_{22})\right) = H_{11}$ or H_{22}

$$\hat{G} = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} H_{11}-\lambda & 0 \\ 0 & H_{22}-\lambda \end{pmatrix} : \det(\hat{G}) = (H_{11}-\lambda)(H_{22}-\lambda) = 0 : \lambda = H_{11}, H_{12} \checkmark$$

Using the following relationship:

$$(S\cdot\hat{n})|\hat{n};\pm\rangle = \frac{\hbar}{2}|n;\pm\rangle$$

And

$$|\hat{n};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle \quad , \quad |\hat{n};-\rangle = \cos\frac{\beta}{2}|+\rangle - e^{i\alpha}\sin\frac{\beta}{2}|-\rangle$$

W/ $\alpha = 0$: we shift $\beta \to \beta + \pi$ to change orientation of the spin

$$\boxed{|\lambda_+\rangle = \cos\frac{\beta}{2}|1\rangle + \sin\frac{\beta}{2}|2\rangle \quad , \quad |\lambda_-\rangle = -\sin\frac{\beta}{2}|1\rangle + \cos\frac{\beta}{2}|2\rangle}$$

W/ $H \doteq A\mathbb{1} + B\sigma_z + C\sigma_x$, $A = \frac{H_{11}+H_{22}}{2}$, $B = \frac{H_{11}-H_{22}}{2}$, $C = H_{12}$ and $\beta = \tan^{-1}(C/B)$

Problem 1: 1.13 Review

Procedure:

- Calculate the eigenvalues of the matrix $\tilde{\mathbf{H}}$.
- Use the relationship
$$(\tilde{\mathbf{S}} \cdot \hat{n}) |\hat{n}; \pm\rangle = \frac{\hbar}{2} |n; \pm\rangle$$
and solve for the eigenstates of $|\hat{n}; +\rangle$ and $|\hat{n}; -\rangle$.
- Apply a shift of $\beta \to \beta + \pi$ and conclude the final results.

Key Concepts:

- We use the standard eigenvalue eigenket formalism to find the energy eigensates of this Hamiltonian.

Variations:

- We can be given a different Hamiltonian.
  - We would use the same procedure to deduce the results that we are looking for.

Problem 2: 1.18

Two Hermitian operators anticommute:

$$\{\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\tilde{\mathbf{A}} = 0.$$

Is it possible to have a simultaneous (that is, common) eigenket of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$? Prove or illustrate your assertion.

Simultaneous means if an observable acts on one of two states, it will not affect the other state.

$$\tilde{A}|a',b'\rangle = a|a',b'\rangle \ , \ \tilde{B}|a',b'\rangle = b|a',b'\rangle \ , \ \{\tilde{A},\tilde{B}\} = \tilde{A}\tilde{B} + \tilde{B}\tilde{A} = 0$$

To prove this we will have to test $\{\tilde{A},\tilde{B}\}|a',b'\rangle$

$$\{\tilde{A},\tilde{B}\}|a',b'\rangle = [\tilde{A}\tilde{B} + \tilde{B}\tilde{A}]|a',b'\rangle = \tilde{A}\tilde{B}|a',b'\rangle + \tilde{B}\hat{A}|a',b'\rangle$$

$$\tilde{A}\tilde{B}|a',b'\rangle = a\tilde{B}|a',b'\rangle = ab|a',b'\rangle : \ \tilde{B}\tilde{A}|a',b'\rangle = b\tilde{A}|a',b'\rangle = ba|a',b'\rangle$$

Scalar products are commutative therefore $a \cdot b = b \cdot a$, and thus

This then yields the result: $\{\tilde{A},\tilde{B}\}|a',b'\rangle = ab|a',b'\rangle + ab|a',b'\rangle = 2ab|a',b'\rangle$

The only way for $\{\tilde{A},\tilde{B}\} = 0$ would be for one of the eigenvalues (at least one) to be equal to zero. i.e. a or b equal to zero.

Problem 2: 1.18 Review

Procedure:

- Begin by applying an arbitrary simultaneous state to the anti commutator of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$.
- Carry out the algebra and show that the only way the above is true is if $ab = ba = 0$. This means one of the eigenvalues is zero.

Key Concepts:

- The only way the above is true is if one of the eigenvalues is zero.
- Because of the above the only way that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ can have a simultaneous eigenstate is if one of the states is the null vector.

Variations:

- We could be asked to prove this for a commutation relation instead.
  - We would use the same procedure and deduce what the eigenvalues would have to be relative to one another.

Problem 3: 1.25

Consider a three-dimensional ket space. If a certain set of orthonormal kets, say $|1\rangle$, $|2\rangle$, and $|3\rangle$, are used as the base kets, the operators $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are represented by

$$\tilde{\mathbf{A}} \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad , \quad \tilde{\mathbf{B}} \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

(a) Obviously $\tilde{\mathbf{A}}$ exhibits a degenerate spectrum. Does $\tilde{\mathbf{B}}$ also exhibit a degenerate spectrum?

$$\hat{B} - \lambda I = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{pmatrix} = \hat{B}^{\square}$$

$$\det(\hat{B}^{\square}) = (b-\lambda)(\lambda^2 - (-ib)(ib)) = 0 \quad \therefore \quad \lambda = b \text{ or } \lambda^2 - b^2 = 0 \quad \therefore \quad \lambda = \pm b$$

Because $\hat{B}$ has eigenvalues that are repeated, $\hat{B}$ exhibits a degenerate spectrum

(b) Show that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ commute.

$$\hat{A}\hat{B} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} \; : \; \hat{B}\hat{A} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$$\hat{A}\hat{B} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = \hat{A}\hat{B}$$

$$\boxed{\hat{A} \text{ and } \hat{B} \text{ commute}}$$

(c) Find a new set of orthonormal kets which are simultaneous eigenkets of both $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$. Specify the eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

The basis set for both $\hat{A}$ and $\hat{B}$ are $|1\rangle$, $|2\rangle$, and $|3\rangle$. We can see immediately that one simultaneous eigenket is $|1\rangle = |a,b\rangle$.

We can then see that for when $\lambda = -a$ and $\lambda = b$, this basis set is $\frac{1}{\sqrt{2}}(|2\rangle + i|3\rangle)$.

Conversely when $\lambda = -a$ and $\lambda = -b$, the basis set is $\frac{1}{\sqrt{2}}(|2\rangle - i|3\rangle)$.

$$\boxed{\lambda = a,b \rightarrow |1\rangle \, , \; \lambda = -a,b \rightarrow \frac{1}{\sqrt{2}}(|2\rangle + i|3\rangle) \, , \; \lambda = -a,-b \rightarrow \frac{1}{\sqrt{2}}(|2\rangle - i|3\rangle)}$$

No, these are simultaneous eigenkets.

Problem 3: 1.25 Review

Procedure:

- Find the eigenvalues of $\tilde{\mathbf{B}}$ and show that they are degenerate.
- Show that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ commute with one another.
- Determine a new set of orthonormal kets by reading off the eigenvalues from the matrices above.

Key Concepts:

- A spectrum is degenerate if eigenvalues are repeated.
- Commutative operators follow $\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\tilde{\mathbf{A}}$.

Variations:

- We can be given different matrices.
  - We would then use the same procedure to determine what is being asked.

Problem 4: 1.28

Construct the transformation matrix that connects the $\tilde{\mathbf{S}}_z$ diagonal basis to the $\tilde{\mathbf{S}}_x$ diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle \langle a^{(r)}| .$$

$$|S_x : \pm\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|S_x : +\rangle\langle +| + \frac{1}{\sqrt{2}}|S_x : -\rangle\langle -|$$

$$\boxed{\hat{U} = \frac{1}{\sqrt{2}}\left(|S_x : +\rangle\langle +| + |S_x : -\rangle\langle -|\right) \doteq \sum_r |b^{(r)}\rangle\langle a^{(r)}|}$$

Problem 4: 1.28 Review

Procedure:

- Write out $\tilde{\mathbf{S}}_x$ in a bra ket form and then a matrix form.
- Write the corresponding bra ket relationship to determine the representation for $\tilde{\mathbf{U}}$.
- Show that this form is consistent with operator given in the problem statement.

Key Concepts:

- We can write a transformation matrix that will connect one direction of the Spin 1/2 operators to another that is a diagonal basis.

Variations:

- We can be asked to do this for a different direction.
  - We then would have to write the transformed direction in the basis of the new direction, and then follow the same procedure.