Solutions to Homework 4
Physics 5393

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P-1.13 A two-state system is characterized by the Hamiltonian

$$\tilde{\mathbf{H}} = H_{11}\, |1\rangle\langle 1| + H_{22}\, |2\rangle\langle 2| + H_{12}\Big[\, |1\rangle\langle 2| + |2\rangle\langle 1|\,\Big]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$.

The straightfoward method is to use standard matrix methods to solve for the eigenvalues and then derive an appropriate set of eigenvectors. Here the problem is solved using an alternate method that takes advantage of the fact that the Hamiltonian is written in a form that has terms that resemble the spin operators

$$\begin{aligned}
\tilde{\mathbf{S}}_z &= \frac{\hbar}{2}\Big(\, |+\rangle\langle +| - |-\rangle\langle -|\,\Big) \equiv \frac{\hbar}{2}\tilde{\sigma}_z \\
\tilde{\mathbf{S}}_x &= \frac{\hbar}{2}\Big(\, |+\rangle\langle -| + |-\rangle\langle +|\,\Big) = \frac{\hbar}{2}\tilde{\sigma}_x \\
\tilde{\mathbf{1}} &= \Big(\, |+\rangle\langle +| + |-\rangle\langle -|\,\Big) \\
\Rightarrow\quad \tilde{\mathbf{H}} &\doteq \frac{H_{11}+H_{22}}{2}\tilde{\mathbf{1}} + \frac{H_{11}-H_{22}}{2}\tilde{\sigma}_z + H_{12}\tilde{\sigma}_x \\
\tilde{\mathbf{H}} &= \frac{H_{11}+H_{22}}{2}\tilde{\mathbf{1}} + + \cos\beta\,\tilde{\sigma}_z + \sin\beta\,\tilde{\sigma}_x,
\end{aligned}$$

where the last form follows problem 1.11. The eigenvalues for this Hamiltonian are

$$\lambda = \frac{H_{11}+H_{22}}{2} \pm \left[\left(\frac{H_{11}-H_{22}}{2}\right)^2 + H_{12}^2\right]^{1/2}.$$

The eigenvectors can be derived as in problem 1.11 with $\alpha = 0$.

$$\begin{aligned}
|\lambda_+\rangle &= \cos\frac{\beta}{2}\ |1\rangle + \sin\frac{\beta}{2}\ |2\rangle \\
|\lambda_-\rangle &= \cos\frac{\beta}{2}\ |2\rangle - \sin\frac{\beta}{2}\ |1\rangle
\end{aligned}$$

with $\tan\beta = 2H_{12}/(H_{11}-H_{22})$.

P-1.18 Two Hermitian operators anti-commute:

$$\Big\{\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\Big\} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\tilde{\mathbf{A}} = 0 \tag{1}$$

Is it possible to have a simultaneous (that is, common) eigenket of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$? Prove or illustrate your assertion.

For this to be possible, the following condition must be satisfied

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}}\,|a,b\rangle = -\tilde{\mathbf{B}}\tilde{\mathbf{A}}\,|a,b\rangle \quad \Rightarrow \quad ab = -ab.$$

This requires that the eigenvalues a and/or b must be equal to zero.

P-1.23 Consider a three-dimensional ket space. If a certain set of orthonormal kets—say, $|1\rangle$, $|2\rangle$, $|3\rangle$—are used as the base kets, the $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are represented by

$$\tilde{\mathbf{A}} \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \qquad \tilde{\mathbf{B}} \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

a) Obviously $\tilde{\mathbf{A}}$ exhibits a degenerate spectrum. Does $\tilde{\mathbf{B}}$ also exhibit a degenerate spectrum?
To determine if $\tilde{\mathbf{B}}$ has degenerate eigenvalues, we calculate its eigenvalues using the characteristic equation for $\tilde{\mathbf{B}}$

$$\det\left[\tilde{\mathbf{B}} - \lambda\tilde{\mathbf{1}}\right] = (b-\lambda)(\lambda^2 - b^2) = 0 \quad \Rightarrow \quad \lambda = b,\ \pm b.$$

Therefore, $\tilde{\mathbf{B}}$ has degenerate eigenvalues.

b) Show that $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ commute.
We must show that $\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \tilde{\mathbf{B}}\tilde{\mathbf{A}}$

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = \tilde{\mathbf{B}}\tilde{\mathbf{A}}.$$

Hence they commute.

c) Find a new set of orthonormal kets that are simultaneous eigenkets of both $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$. Specify the eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?
We make the following association of eigenvalue to eigenket for the operator $\tilde{\mathbf{A}}$

$$\begin{aligned} a &\to |1\rangle \\ -a &\to |2\rangle \\ -a &\to |3\rangle \end{aligned}$$

Notice, since b is on the diagonal, it can be associated with the eigenvalue a and therefore the simultaneous eigenvector is $|a,b\rangle = |1\rangle$. This leaves the 2×2 subspace to arrange. This subspace has eigenvalues $-a$ for $\tilde{\mathbf{A}}$ and $\pm b$ for $\tilde{\mathbf{B}}$. The eigenvectors will then be linear combinations of $|2\rangle$ and $|3\rangle$

$$\begin{aligned} |-a,b\rangle &= |2\rangle + s_1\,|3\rangle \\ |-a,-b\rangle &= |2\rangle + s_2\,|3\rangle\,, \end{aligned}$$

where we select the coefficient of $|2\rangle$ to be one since both coefficients are not independent. The problem is most easily solved in a matrix representation using the eigenvalue equations and the matrices given earlier

$$\begin{aligned} \tilde{\mathbf{B}}\,|-a,b\rangle &= b\,|-a,b\rangle & s_1 &= i \\ \tilde{\mathbf{B}}\,|-a,-b\rangle &= -b\,|-a,-b\rangle & s_2 &= -i,. \end{aligned}$$

The simultaneous eigenkets are therefore

$$\begin{aligned} |-a,b\rangle &= \frac{1}{\sqrt{2}}\left[|2\rangle + i\,|3\rangle\right] \\ |-a,-b\rangle &= \frac{1}{\sqrt{2}}\left[|2\rangle - i\,|3\rangle\right] \end{aligned}$$

Note, this choice is not unique, therefore, the eigenvalues do not completely specify the eigenvectors.

P-1.28 Construct the transformation matrix that connects the $\tilde{\mathbf{S}}_z$ diagonal basis to the $\tilde{\mathbf{S}}_x$ diagonal basis. Show that your result is consistent with the general relation

$$\tilde{\mathbf{U}} = \sum_i |b_i\rangle\langle a_i|\,.$$

The eigenstates of $\tilde{\mathbf{S}}_x$ expressed in the eigenstates of $\tilde{\mathbf{S}}_z$ are

$$\begin{aligned} \left|\tilde{\mathbf{S}}_x;+\right\rangle &= \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right] \\ \left|\tilde{\mathbf{S}}_x;-\right\rangle &= \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right]. \end{aligned}$$

The change of basis operator in Diarc notation can be expressed in the basis $|a_i\rangle = \left|\tilde{\mathbf{S}}_z;\pm\right\rangle$ as follows

$$\tilde{\mathbf{U}} = \sum_{ij} |a_i\rangle\langle a_i|\,\tilde{\mathbf{U}}\,|a_j\rangle\langle a_j|\,,$$

where the matrix elements are

$$\tilde{\mathbf{U}}_{ij} = \left\langle a_i \middle| \tilde{\mathbf{U}} \middle| a_j \right\rangle = \langle a_i | b_j\rangle$$

with $i = 1 = +$ and $i = 2 = -$ and $|b_j\rangle = \left|\tilde{\mathbf{S}}_x;\pm\right\rangle$. Finally, combining the various equations together, the matrix representation of the change of basis operator is

$$\langle a_i | b_j\rangle = \left\langle \tilde{\mathbf{S}}_z;i \middle| \tilde{\mathbf{U}} \middle| \tilde{\mathbf{S}}_z;j \right\rangle = \left\langle \tilde{\mathbf{S}}_z;i \middle| \tilde{\mathbf{S}}_x;j \right\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Returning to the operator in Dirac form, leads to the following

$$\tilde{\mathbf{U}} = \sum_{ij} |a_i\rangle\langle a_i|\,\tilde{\mathbf{U}}\,|a_j\rangle\langle a_j| \sum_{ij} |a_i\rangle\langle a_i|\;|b_j\rangle\langle a_j| = \sum_j |b_j\rangle\langle a_j|\,,$$

Which leads to the desired result.