Physics 5403 Homework #5
Spring 2022

Instructor Bruno Uchoa

Due date: April 6, 2022

March 29, 2022

1 Harmonic Oscillator

A one dimensional quantum oscillator with frequency ω has the unperturbed Hamiltonian

$$\mathcal{H} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right),$$

where a and $a^\dagger$ are creation and annihilation operators. At $t = 0$, we turn on a time dependent perturbation

$$V(t) = \lambda \left[f(t)a + f^*(t)a^\dagger \right],$$

where $f(t)$ is some integrable function, such that $f(t \to \infty) = 0$.

a) Find the time dependence of the creation and annihilation operators in the interaction picture. Use the fact that

$$\mathrm{e}^{-B} A \mathrm{e}^{B} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, B]_n = A + [A, B] + \frac{1}{2!} [[A, B], B] + \dots .$$

where $[A, B]_{n+1} \equiv [[A, B]_n, B]$ and $[A, B]_0 \equiv A$. This relation is known as the Baker-Hausdorff identity.

b) At $t = 0$ the quantum oscillator is in the ground state $|0\rangle$. Using leading order of perturbation theory, find the probability for the transition $|0\rangle \to |n\rangle$ at $t \to \infty$ for $n = 1$ and 2.

c) Suppose now that instead of the perturbed potential (1), we turn on a potential of the form:

$$V(t) = \lambda x^3 \mathrm{e}^{-\tau t}$$

at $t = 0$ ($\tau > 0$). Find the transition probability to the third excited state, $|0\rangle \to |3\rangle$ in perturbation theory at $t \to \infty$.

2 Three level system

Consider a system of three levels with the Hamiltonian

$$\begin{pmatrix} \epsilon_1 & 0 & \Delta(t) \\ 0 & \epsilon_2 & \Delta(t) \\ \Delta^*(t) & \Delta^*(t) & \epsilon_3 \end{pmatrix}$$

where

$$\Delta(t) = \Delta e^{i\omega t},$$

with Δ real. Find the transition probability between levels ϵ_1 and ϵ_2 in leading order of perturbation theory where the result is non-trivial, when $|\Delta(t)| \ll |\epsilon_i - \epsilon_j|$, with $i, j = 1, 2, 3$, and $i \neq j$. Interpret your result.

3 Particle in a box

A non-relativistic electron with energy dispersion

$$E_k = \frac{k^2}{2m}$$

is confined to a 1-dimensional square cavity of size L centered at $x = 0$.

a) Write the wavefunctions of the particle in the box and their corresponding energy levels.

b) If the system is perturbed by a weak electric field $\mathcal{E}_0$ with potential $V(x) = -\mathcal{E}_0 x$, calculate the first non zero correction to the energy of the ground state. Hint: use the fact that:

$$\sum_{n=1}^{\infty} \left[\frac{1}{(4n^2-1)^3} + \frac{4}{(4n^2-1)^4} + \frac{4}{(4n^2-1)^5} \right] = \frac{1}{2} - \frac{\pi^2}{64} \left(\frac{7}{4} + \frac{\pi^2}{12} \right).$$

c) Using your result in b), find the corresponding correction to the ground state *ket*. Assume now that the particle is prepared in that state. Find the probability of measuring the particle in the first excited state of the unperturbed system.

d) Suppose now the electric field is time dependent,

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-t/\tau},$$

and is turned on at $t = 0$ ($\tau > 0$). If the particle is in the ground state at $t < 0$, find the probability of a transition to the first excited level at times $t \gg \tau$.