Math details on Fermi-Dirac integrals

let $J = \int_0^\infty \frac{g(\mathcal{E})}{e^{\frac{\mathcal{E}-\mu}{kT}}+1}\, d\mathcal{E}$

For $T \to 0$: $J = \int_0^\mu g(\mathcal{E})\, d\mathcal{E}$

Define $J_1 = J - J_0$

$$\to J_1 = \int_0^\infty \left[\frac{1}{e^{(\mathcal{E}-\mu)/kT}+1} - \theta\left(\frac{\mu-\mathcal{E}}{kT}\right)\right] g(\mathcal{E})\, d\mathcal{E}$$

Let $x = \frac{\mathcal{E}-\mu}{kT} \Rightarrow d\mathcal{E} = kT\, dx$

Thus: $$J_1 = kT \int_{-\mu/kT}^\infty \left[\frac{1}{e^x+1} - \theta(-x)\right] g(kTx+\mu)\, dx$$

Taylor expand...

After a bit of work:

$$J = \int_0^\mu g(\mathcal{E})\, d\mathcal{E} + \frac{\pi^2}{6}\frac{\partial g}{\partial \mu}(kT)^2 + \frac{7\pi^4}{360}\frac{\partial^3 g}{\partial \mu^3}(kT)^4 + \ldots$$

Let's use these math identities in the following problem:

Problem: For a system of fermions with a density of states function $D(\varepsilon)$, determine the dependence of the chemical potential on the temperature for fixed N and $kT \ll \varepsilon_F$.

Solution: The average occupation of a quantum state of energy ε is

$$\frac{1}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1}$$

The number of such quantum states within the energy interval $d\varepsilon$ is $D(\varepsilon)\,d\varepsilon$.

$$\Rightarrow N = \int_0^\infty \frac{D(\varepsilon)}{\exp\left(\frac{\varepsilon-\mu}{kT}\right)+1}\,d\varepsilon$$

To order T^2:

$$N \approx \int_0^\mu D(\varepsilon)\,d\varepsilon + \frac{\pi^2(kT)^2}{6}\frac{\partial D(\mu)}{\partial\mu} + \ldots$$

math trick from previous page

At $T=0$, $\mu = \varepsilon_F$.

At small T, $\mu = \varepsilon_F + \delta\mu$:

$$\Rightarrow N = \int_0^{\varepsilon_F} D(\varepsilon)\, d\varepsilon + D(\varepsilon_F)\,\delta\mu$$

expanding to first order in $\delta\mu$

$$+ \frac{\pi^2 (kT)^2}{6} \left[\frac{\partial D(\varepsilon_F)}{\partial \varepsilon_F} + \frac{\partial^2 D(\varepsilon_F)}{\partial \varepsilon_F^2}\,\delta\mu \right]$$

By definition: $\int_0^{\varepsilon_F} D(\varepsilon)\, d\varepsilon = N$

$$\Rightarrow D(\varepsilon_F)\,\delta\mu = -\frac{\pi^2 (kT)^2}{6} \frac{\partial D(\varepsilon_F)}{\partial \varepsilon_F} + O(\underbrace{T^2\,\delta\mu}_{\text{small}})$$

$$\Rightarrow \boxed{\mu = \varepsilon_F - \frac{\pi^2 (kT)^2}{6} \frac{\frac{\partial D(\varepsilon_F)}{\partial \varepsilon_F}}{D(\varepsilon_F)}}$$

using $\mu = \varepsilon_F + \delta\mu$