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Quantum Mechanics 1

Ch. 3 The Theory of Angular Momentum Lecture Notes

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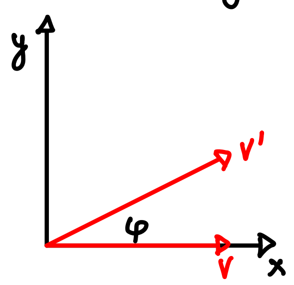
Consider a system



$$\begin{pmatrix} \text{Rotation} \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix}$$

Rotation about η axis $\doteq R_\eta$

$$R_z = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \quad R_y = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

Rotations are not commutative!

If we expand $\sin\varphi \to \varphi$, and $\cos\varphi \to 1-\varphi^2/2$: R_x & R_y become

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\delta\varphi^2/2 & -\delta\varphi \\ 0 & \delta\varphi & 1-\delta\varphi^2/2 \end{pmatrix}, \quad R_y = \begin{pmatrix} 1-\delta\varphi^2/2 & 0 & \delta\varphi \\ 0 & 1 & 0 \\ -\delta\varphi & 0 & 1-\delta\varphi^2/2 \end{pmatrix} \to \text{Infinitesimal rotation's}$$

$$R_x(\delta\varphi)R_y(\delta\varphi) - R_y(\delta\varphi)R_x(\delta\varphi) = R_z(\delta\varphi) - \mathbb{1}$$

$$\to \begin{pmatrix} 0 & -\delta\varphi^2 & 0 \\ \delta\varphi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \text{Non-commutating of order } \delta\varphi^2$$

Two rotations about same axis, these will commute

For spin we write these as, $|S_z:+\rangle$ and $|S_z:-\rangle$ etc.....

The rotation operator: $D(R)$ where R is some angle like φ, θ, etc....

For a state to rotate we use: $|\alpha\rangle_R = \tilde{D}(R)|\alpha\rangle$

$\tilde{\mathcal{U}}(\delta\varphi) = \tilde{\mathbb{1}} - i\tilde{G}\,\delta\varphi \to$ Infitesimal rotation operator

G is defined as, $\tilde{G} \doteq \tilde{J}/\hbar$: $\hbar$ units of angular momentum, $\tilde{J}$ units of $\hbar$

$\tilde{D}(\hat{n},\delta\varphi) = \tilde{\mathbb{1}} - i\dfrac{\tilde{\vec{J}}\cdot\hat{n}}{\hbar}\delta\varphi \longrightarrow$ This is our rotation operator in infintesimal form

$\lim\limits_{N\to\infty}\left(\mathbb{1} - i\dfrac{\tilde{\vec{J}}\cdot\hat{n}}{\hbar}\dfrac{\varphi}{N}\right)^N = e^{-i\tilde{\vec{J}}\cdot\hat{n}\varphi/\hbar} \longrightarrow$ Finite rotation operator

$e^{-i\tilde{\vec{J}}\cdot\hat{n}\varphi/\hbar}$ expanded out gives us $\mathbb{1} - i\dfrac{\tilde{\vec{J}}\cdot\hat{n}}{\hbar}\varphi - \dfrac{(\tilde{\vec{J}}\cdot\hat{n})^2\varphi^2}{2\hbar^2}$

Identities :

$R\mathbb{1} = R \Rightarrow \tilde{D}(R)\mathbb{1} = \tilde{D}(R)$

$R_1 R_2 = R_3 \Rightarrow \tilde{D}(R_1)\tilde{D}(R_2) = \tilde{D}(R_3) \longrightarrow$ Successive

$R R^{-1} = \mathbb{1} \Rightarrow \tilde{D}(R)\tilde{D}^{-1}(R) = \tilde{\mathbb{1}} \longrightarrow$ Inverse

$(R_1 R_2)R_3 = R_1(R_2 R_3) \Rightarrow (\tilde{D}(R_1)\tilde{D}(R_2))\tilde{D}(R_3) = \tilde{D}(R_1)(\tilde{D}(R_2)\tilde{D}(R_3)) \longrightarrow$ Associative

$R_x R_y - R_y R_x = R_z - \mathbb{1}$, $\tilde{D}_x(\delta\varphi)\tilde{D}_y(\delta\varphi) - \tilde{D}_y(\delta\varphi)\tilde{D}_x(\delta\varphi) = \tilde{D}_z(\delta\varphi^2) - \tilde{\mathbb{1}} \longrightarrow$ commutation

$$\left(\tilde{\mathbb{1}} - \frac{i\tilde{J}_x\delta\varphi}{\hbar} - \frac{\tilde{J}_x^2\delta\varphi^2}{2\hbar^2}\right)\left(\tilde{\mathbb{1}} - \frac{i\tilde{J}_y\delta\varphi}{\hbar} - \frac{\tilde{J}_y^2\delta\varphi^2}{2\hbar^2}\right) \longrightarrow (*)$$

$$\left(\tilde{\mathbb{1}} - \frac{i\tilde{J}_y\delta\varphi}{\hbar} - \frac{\tilde{J}_y^2\delta\varphi^2}{2\hbar^2}\right)\left(\tilde{\mathbb{1}} - \frac{i\tilde{J}_x\delta\varphi}{\hbar} - \frac{\tilde{J}_x^2\delta\varphi^2}{2\hbar^2}\right) \longrightarrow (**)$$

$(*) - (**) = \frac{\delta\varphi^2}{\hbar^2}(\tilde{J}_y\tilde{J}_x - \tilde{J}_x\tilde{J}_y) = \tilde{\mathbb{1}} - \frac{i\tilde{J}_z}{\hbar}\delta\varphi^2 - \tilde{\mathbb{1}}$: $(\tilde{J}_x\tilde{J}_y - \tilde{J}_y\tilde{J}_x) = i\hbar\tilde{J}_z$

In general we have : $[\tilde{J}_i, \tilde{J}_j] = i\hbar\epsilon_{ijk}\tilde{J}_k$

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Now we want to see how this applies, starting with a spin-1/2 operator:

$\tilde{S}_x = \hbar/2(|+\rangle\langle-| + |-\rangle\langle+|)$, $\tilde{S}_y = i\hbar/2(|-\rangle\langle+| - |+\rangle\langle-|)$, $\tilde{S}_z = \hbar/2(|+\rangle\langle+| - |-\rangle\langle-|)$

Now we want to calculate an expectation value:

$\langle\tilde{S}_x\rangle_R = {}_R\langle\alpha|\tilde{S}_x|\alpha\rangle_R = \langle\alpha|\tilde{D}_z^{-1}(\varphi)\tilde{S}_x\tilde{D}_z(\varphi)|\alpha\rangle \Rightarrow (\langle\alpha|\tilde{D}_z^{-1}(\varphi))\tilde{S}_x(\tilde{D}_z(\varphi)|\alpha\rangle)$

or $\langle\alpha|(\tilde{D}_z^{-1}(\varphi)\tilde{S}_x\tilde{D}_z(\varphi))|\alpha\rangle$ w/ $D_z(\varphi) = e^{-i\tilde{S}_z\varphi/\hbar}$

$\tilde{D}_z^{-1}(\varphi)\tilde{S}_x\tilde{D}_z(\varphi) = \frac{\hbar}{2}e^{i\tilde{S}_z\varphi/\hbar}(|+\rangle\langle-| + |-\rangle\langle+|)e^{-i\tilde{S}_z\varphi/\hbar}$

$= \frac{\hbar}{2}\left[e^{i\varphi/2}|+\rangle\langle-|e^{i\varphi/2} + e^{-i\varphi/2}|-\rangle\langle+|e^{-i\varphi/2}\right] = \frac{\hbar}{2}\left[e^{i\varphi}|+\rangle\langle-| + e^{-i\varphi}|-\rangle\langle+|\right]$

$= \frac{\hbar}{2}\left[[|+\rangle\langle-| + |-\rangle\langle+|]\cos\varphi + [|+\rangle\langle-| - |-\rangle\langle+|]i\sin(\varphi)\right]$

$\tilde{S}_x \xrightarrow{R} \tilde{S}_x\cos(\varphi) - \tilde{S}_y\sin(\varphi) \longrightarrow \tilde{S}_x$ rotated about z-axis

$\langle\tilde{S}_x\rangle_R = \langle\tilde{S}_x\rangle\cos(\varphi) - \langle\tilde{S}_y\rangle\sin(\varphi)$, Now we want to look at $\tilde{D}(z)|\alpha\rangle$

$\tilde{D}(z)|\alpha\rangle = e^{-i\tilde{S}_z\varphi/\hbar}|\alpha\rangle = e^{-i\tilde{S}_z\varphi/\hbar}\left[|+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle\right]$

$= e^{-i\varphi/2}|+\rangle\langle+|\alpha\rangle + e^{i\varphi/2}|-\rangle\langle-|\alpha\rangle \longrightarrow$ How state changes under rotation

If we rotate φ by 2π : $e^{i\varphi/2} \to e^{i(\varphi+2\pi)/2} \to e^{i\varphi/2}e^{i\pi} \to -e^{i\varphi/2}$, flips sign !

We now wish to look at spin precession

The time evolution operator is : $\mathcal{U}(t) = e^{-i\tilde{H}t/\hbar}$, The Hamiltonian $\tilde{H} = \frac{e}{mc}\tilde{S}\cdot\vec{B}_z = \omega \cdot \tilde{S}_z$

$\omega = \frac{eB_z}{mc}$ $\therefore$ $\mathcal{U}(t) = e^{-i\omega t\tilde{S}_z/\hbar}$ $\longrightarrow$ can rewrite to show rotation operator

$D(\omega t) = e^{-i\omega t\tilde{S}_z}$, $\langle\tilde{S}_x(t)\rangle \to \langle S_x\rangle\cos(\omega t) - \langle S_y\rangle\sin(\omega t)$

Nuetron Experiment : $e^{-i\omega T/2}$, $T \to$ Time to travel through magnetic field

what is necessary for $\omega T = 4\pi$? $\omega T = g_n\frac{eBT}{mc} \rightsquigarrow g_n\frac{e\Delta BT}{mc} = 4\pi$

$T = \hbar k = mv$ w/ $v = \ell/T$ $\therefore$ $T = m\ell/\hbar k$ $\rightsquigarrow$ $\frac{g_n e\Delta B\ell}{\hbar ck} = 4\pi$, $\Delta B = \frac{4\pi\hbar c}{g_n e\ell k}$

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$e^{iS_z\varphi/\hbar}\tilde{S}_x e^{-iS_z\varphi/\hbar}$ $\longrightarrow$ Performing rotation around z for x , $\tilde{S}_x\cos(\varphi) - \tilde{S}_y\sin(\varphi)$

$$e^{iA\varphi}\tilde{B}e^{-iA\varphi} = \tilde{B} + (i\varphi)[\tilde{A},\tilde{B}] + \frac{(i\varphi)^2}{2!}[\tilde{A},[\tilde{A},\tilde{B}]] + \frac{(i\varphi)^3}{3!}[\tilde{A},[\tilde{A},[\tilde{A},\tilde{B}]]]$$

we get this by doing : $(\mathbb{1} + i\varphi\tilde{A} + \frac{1}{2}(i\varphi)^2\tilde{A}^2)\tilde{B}(\mathbb{1} - i\varphi\tilde{A} + \frac{1}{2}(i\varphi)^2\tilde{A}^2)$

$$\tilde{B} + i\varphi(\tilde{A}\tilde{B} - \tilde{B}\tilde{A}) + \frac{(i\varphi)^2}{2!}(\hat{A}^2\tilde{B} + \tilde{B}\tilde{A}^2 - 2\tilde{A}\tilde{B}\tilde{A}) = \tilde{B} + i\varphi[\tilde{A},\tilde{B}] + \frac{(i\varphi)^2}{2}(\tilde{A}(\tilde{A}\tilde{B}-\tilde{B}\tilde{A}) + (\tilde{B}\tilde{A}-\tilde{A}\tilde{B})\tilde{A})$$

$$\tilde{B} + i\varphi[\tilde{A},\tilde{B}] + \frac{(i\varphi)^2}{2}(\tilde{A}(\tilde{A}\tilde{B}-\tilde{B}\tilde{A}) - (\tilde{A}\tilde{B}-\tilde{B}\tilde{A})\tilde{A}) = \tilde{B} + i\varphi[\hat{A},\tilde{B}] + \frac{(i\varphi)^2}{2}[\tilde{A},[\tilde{A},\tilde{B}]]$$

Therefore we have $e^{iS_z\varphi/\hbar}\tilde{S}_x e^{-iS_z\varphi/\hbar}$

$$= \tilde{S}_x + \left(\frac{i\varphi}{\hbar}\right)[\tilde{S}_z,\tilde{S}_x] + \frac{1}{2!}\left(\frac{i\varphi}{\hbar}\right)^2[\tilde{S}_z,[\tilde{S}_z,\tilde{S}_x]] + \frac{1}{3!}\left(\frac{i\varphi}{\hbar}\right)^3[\tilde{S}_z,[\tilde{S}_z,[\tilde{S}_z,\tilde{S}_x]]]$$

w/ $[\tilde{S}_z,\tilde{S}_x] = i\hbar\tilde{S}_y$, $[\tilde{S}_z,\tilde{S}_y] = -i\hbar\tilde{S}_x$

$$= \tilde{S}_x\underbrace{(\mathbb{1} - \varphi^2/2! + \dots)}_{\text{cos expansion}} - \tilde{S}_y\underbrace{(\varphi - \varphi^3/3! + \dots)}_{\text{Sine expansion}} = \tilde{S}_x\cos(\varphi) - \tilde{S}_y\sin(\varphi)$$

Now we wish to talk about Pauil spin operators. we first start off with two states : $|+\rangle, |-\rangle$

$$|+\rangle = |+\rangle\langle+|+\rangle + |-\rangle\langle-|+\rangle \doteq \begin{pmatrix}\langle+|+\rangle\\ \langle-|+\rangle\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \chi_+$$

$$|-\rangle = |-\rangle\langle-|-\rangle + |+\rangle\langle+|-\rangle \doteq \begin{pmatrix}\langle+|-\rangle\\ \langle-|-\rangle\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \chi_-$$

$$|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle = \begin{pmatrix}\langle+|\alpha\rangle\\ \langle-|\alpha\rangle\end{pmatrix} \Rightarrow \chi$$

$$\tilde{S}_i = |+\rangle\langle+|\tilde{S}_i|+\rangle\langle+| + |+\rangle\langle+|\tilde{S}_i|-\rangle\langle-| + |-\rangle\langle-|\tilde{S}_i|+\rangle\langle+| + |-\rangle\langle-|\tilde{S}_i|-\rangle\langle-|$$

$$\tilde{S}_i \doteq \begin{pmatrix} \langle + | \tilde{S}_i | + \rangle & \langle + | \tilde{S}_i | - \rangle \\ \langle - | \tilde{S}_i | + \rangle & \langle - | \tilde{S}_i | - \rangle \end{pmatrix}$$

$$\therefore \tilde{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z \;,\; \tilde{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \;,\; \tilde{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

Looking at the commutations of these matrices we have

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \;,\; \{\sigma_i, \sigma_j\} = 2\delta_{ij} \Rightarrow \sigma_i^2 = \mathbb{1}$$

Other properties of these are

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = 0 \;,\; \sigma_1\sigma_2 - \sigma_2\sigma_1 = 2i\sigma_3 \;,\; \sigma_1\sigma_2 = i\sigma_3 \Rightarrow \sigma_1\sigma_2 = -\sigma_2\sigma_1 \;,\; \sigma_i = \sigma_i^{+}$$

$$\det(\sigma_i) = -1 \;,\; \mathrm{tr}(\sigma_i) = 0 \;,\; (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})\mathbb{1} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \quad (*)$$

Proving $(*)$ we have :

$$\sum_i \sigma_i a_i \sum_i \sigma_j b_j = \sum_{ij}(\sigma_i\sigma_j)(a_i b_j) \;,\; \sigma_i\sigma_j = \frac{1}{2}(\{\sigma_i,\sigma_j\} + [\sigma_i,\sigma_j])$$

$$= \frac{1}{2}\sum_{ij}(\{\sigma_i,\sigma_j\} + [\sigma_i,\sigma_j])a_i b_j = \frac{1}{2}\sum_{ij}\delta_{ij}a_i b_j + i\epsilon_{ijk}\sigma_k a_i b_j = (\vec{a}\cdot\vec{b})\mathbb{1} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{-i\vec{S}\cdot\hat{n}\varphi/\hbar} = e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} = \left(\mathbb{1} - \frac{(\vec{\sigma}\cdot\hat{n})^2}{2!}\left(\frac{\varphi}{2}\right)^2 + \frac{(\vec{\sigma}\cdot\hat{n})^4}{4!}\left(\frac{\varphi}{2}\right)^4 + \cdots\right) + i\left(\vec{\sigma}\cdot\hat{n} - \frac{(\vec{\sigma}\cdot\hat{n})^3}{3!} + \cdots\right)$$

$$(\vec{\sigma}\cdot\hat{n})^2 = (\overset{1}{\cancel{\hat{n}\cdot\hat{n}}})\mathbb{1} + \sigma(\overset{0}{\cancel{\hat{n}\times\hat{n}}}) = \mathbb{1}$$

$$e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} = \mathbb{1}\left(1 - \frac{1}{2!}\left(\frac{\varphi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\varphi}{2}\right)^4 + \cdots\right) + \mathbb{1}\left(\frac{\varphi}{2} - \frac{1}{3!}\left(\frac{\varphi}{2}\right)^3\right)\vec{\sigma}\cdot\hat{n}$$

$$= \cos(\varphi/2) - i\sin(\varphi/2)(\vec{\sigma}\cdot\hat{n})$$

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The Pauli matrices are : $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The commutation of these are : $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$: $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The anti-commutation of thes are : $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ $\quad \chi = a_+\chi_+ + a_-\chi_-$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\mathbb{1} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \;,\; e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} = \mathbb{1}\cos(\varphi/2) - i\sin(\varphi/2)(\vec{\sigma}\cdot\hat{n}) \quad (*)$$

w/ $\hat{n} = n_x\hat{\imath} + n_y\hat{\jmath} + n_z\hat{k}$: Example, $\sigma_x n_x + \sigma_y n_y + \sigma_z n_z$, expanding $(*)$

$$e^{-i\vec{\sigma}\cdot\hat{n}\varphi/2} = \begin{pmatrix} \cos(\varphi/2) - in_z\sin(\varphi/2) & (-in_x - n_y)\sin(\varphi/2) \\ (in_x - n_y)\sin(\varphi/2) & \cos(\varphi/2) + in_z\sin(\varphi/2) \end{pmatrix}$$

Now looking at spin up and spin down

$\chi \rightarrow \left(\quad\right)\left(\;\right) = e^{-i\vec{\sigma}'\cdot\hat{n}\varphi/2}$, w/ $\chi^{+}\sigma'_{k}\chi \Rightarrow$ Matrix element expectation value

$\chi^{+}\sigma'_{k}\chi = \sum_{l} R_{kl}\,\chi^{+}\sigma'_{l}\chi \longrightarrow$ will rotate as Euclidean vector

w/ $R_{kl} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$: $\chi^{+}\sigma'_{l}\chi \rightarrow \chi^{+} e^{i\vec{\sigma}'\cdot\hat{n}\varphi/2}\,\sigma'_{l}\,e^{-i\vec{\sigma}'\cdot\hat{n}\varphi/2}\chi$, $\chi \doteq |\alpha\rangle$

we now look at an example:

$e^{-i\vec{\sigma}'\cdot\hat{n}\varphi/2} = \mathbb{1}\cos(\varphi/2) - i(\vec{\sigma}'\cdot\hat{n})\sin(\varphi/2)$, $e^{i\vec{\sigma}'\cdot\hat{n}\varphi/2} = \mathbb{1}\cos(\varphi/2) + i(\vec{\sigma}'\cdot\hat{n})\sin(\varphi/2)$

$$e^{i\sigma'_3\varphi/2}\,\sigma'_1\,e^{-i\sigma'_3\varphi/2} = \sigma'_1\cos^2(\varphi/2) - i\sigma'_1\sigma'_3\cos(\varphi/2)\sin(\varphi/2)$$
$$+\, i\sigma'_3\sigma'_1\cos(\varphi/2)\sin(\varphi/2) + \sigma'_3\sigma'_1\sigma'_3\sin^2(\varphi/2)$$

$$e^{i\sigma'_3\varphi/2}\,\sigma'_1\,e^{-i\sigma'_3\varphi/2} = \sigma'_1\cos^2(\varphi/2) + i[\sigma'_3,\sigma'_1]\cos(\varphi/2)\sin(\varphi/2) + \sigma'_3\sigma'_1\sigma'_3\sin^2(\varphi/2)$$
$$= \sigma'_1\cos^2(\varphi/2) - 2\sigma'_2\cos(\varphi/2)\sin(\varphi/2) + \sigma'_3\sigma'_1\sigma'_3\sin^2(\varphi/2)$$

$\sigma'_3\sigma'_1 + \sigma'_1\sigma'_3 = 0$, $\sigma'_3\sigma'_1 = -\sigma'_1\sigma'_3$ $\therefore$ $\sigma'_3\sigma'_1\sigma'_3 = -\sigma'_1\sigma'_3\sigma'_3 = -\sigma'_1$, $\{\sigma'_3,\sigma'_3\} = 2\mathbb{1}$

$$= \sigma'_1\cos^2(\varphi/2) - 2\sigma'_2\cos(\varphi/2)\sin(\varphi/2) - \sigma'_1\sin^2(\varphi/2)$$
$$= \sigma'_1(\cos^2(\varphi/2) - \sin^2(\varphi/2)) - \sigma'_2(2\cos(\varphi/2)\sin(\varphi/2))$$

$e^{i\sigma'_3\varphi/2}\,\sigma'_1\,e^{-i\sigma'_3\varphi/2} = \sigma'_1\cos(\varphi) - \sigma'_2\sin(\varphi) \longrightarrow$ operator rotates like Euclidean vector

This tells us operators and states will rotate differently, states pick up phase

we are now going to take a state and rotate it

β about y axis and then α about z axis

$$\chi_n = \left[\mathbb{1}\cos(\alpha/2) - i(\sigma'_3\sin(\alpha/2))\right]\left[\mathbb{1}\cos(\beta/2) - i(\sigma'_2\sin(\beta/2))\right]\begin{pmatrix}1\\0\end{pmatrix}$$

$$\chi_n = \begin{pmatrix} \cos(\alpha/2) - i\sin(\alpha/2) & 0 \\ 0 & \cos(\alpha/2) + i\sin(\alpha/2) \end{pmatrix}\begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$\chi_n = \begin{pmatrix} \cos(\beta/2)e^{-i\alpha/2} \\ \sin(\beta/2)e^{i\alpha/2} \end{pmatrix}$, $\chi_n = e^{-i\alpha/2}\begin{pmatrix} \cos(\beta/2) \\ \sin(\beta/2)e^{i\alpha} \end{pmatrix}$

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Rotation Matrices for Euler Angles : $\underset{z'}{R(\gamma)}\underset{y'}{R(\beta)}\underset{z}{R(\alpha)}$ (*)

$R_{y'}(\beta) = R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)$, $R_{z'}(\gamma) = R_{y'}(\beta)R_z(\gamma)R_{y'}^{-1}(\beta) \longrightarrow$ Sub into (*)

$$R_{z'}(\gamma)R_{y'}(\beta)R_z(\alpha) = [R_{y'}(\beta)R_z(\gamma)R_{y'}^{-1}(\beta)]R_{y'}(\beta)R_z(\alpha)$$

$$= R_{y'}(\beta)\underbrace{R_z(\gamma)R_z(\alpha)}_{\text{Flip these}} = [R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)]R_z(\alpha)R_z(\gamma)$$

$$R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)$$

$$R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) \Rightarrow D(\alpha,\beta,\gamma) = D_z(\alpha)D_y(\beta)D_z(\gamma)$$

Example: Spin 1/2

$$D(\alpha,\beta,\gamma) = e^{-i\sigma_3\alpha/2}e^{-i\sigma_2\beta/2}e^{-i\sigma_3\gamma/2}, \quad D_z(\alpha) = e^{-i\sigma_3\alpha/2}, D_y(\beta) = e^{-i\sigma_2\beta/2}, D_z(\gamma) = e^{-i\sigma_3\gamma/2}$$

$$e^{-i\vec{\sigma}\cdot\hat{n}\,\varphi/2} = \begin{pmatrix} \cos(\varphi/2) - in_z\sin(\varphi/2) & (in_x - n_y)\sin(\varphi/2) \\ (-in_x + n_y)\sin(\varphi/2) & \cos(\varphi/2) + in_z\sin(\varphi/2) \end{pmatrix}$$

$$e^{-i\vec{\sigma}_2\beta/2} = \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix}, \quad n_x = n_z = 0, \quad n_y = 1$$

$$e^{-i\sigma_3\alpha/2} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}, \quad n_z = 1, \quad n_y = n_x = 0$$

$\therefore$ we can write D as

$$D(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}$$

Commutation relations for $\tilde{J}$: $[\tilde{J}_i, \tilde{J}_j] = i\hbar\epsilon_{ijk}\tilde{J}_k$, $[\tilde{J}_x, \tilde{J}_y] = i\hbar J_z$

$$\tilde{J}^2 = \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2$$

Commutes with $\tilde{J}_z$

$$[\tilde{J}^2, \tilde{J}_z] = (\tilde{J}_x^2 + \tilde{J}_y^2 + \cancel{\tilde{J}_z^2})\tilde{J}_z - \tilde{J}_z(\tilde{J}_x^2 + \tilde{J}_y^2 + \cancel{\tilde{J}_z^2})$$

$$= \tilde{J}_x(\tilde{J}_x\tilde{J}_z) - (\tilde{J}_z\tilde{J}_x)\tilde{J}_x + \tilde{J}_y(\tilde{J}_y\tilde{J}_z) - (\tilde{J}_z\tilde{J}_y)\tilde{J}_y$$

, next add zero twice

$$= \tilde{J}_x(\tilde{J}_x\tilde{J}_z) - (\tilde{J}_z\tilde{J}_x)\tilde{J}_x + \tilde{J}_y(\tilde{J}_y\tilde{J}_z) - (\tilde{J}_z\tilde{J}_y)\tilde{J}_y + \tilde{J}_y\tilde{J}_z\tilde{J}_y - \tilde{J}_y\tilde{J}_z\tilde{J}_y + \tilde{J}_x\tilde{J}_z\tilde{J}_x - \tilde{J}_x\tilde{J}_z\tilde{J}_x$$

$$= \tilde{J}_x[\tilde{J}_x, \tilde{J}_z] + [\tilde{J}_x, \tilde{J}_z]\tilde{J}_x + \tilde{J}_y[\tilde{J}_y, \tilde{J}_z] + [\tilde{J}_y, \tilde{J}_z]\tilde{J}_y$$

$$= \tilde{J}_x(\cancel{-i\hbar}\tilde{J}_y) + (i\hbar\cancel{\tilde{J}_y})J_x + \tilde{J}_y(\cancel{i\hbar}\tilde{J}_x) + (i\hbar\cancel{\tilde{J}_x})\tilde{J}_y = 0$$

Lets look at $\tilde{J}^2$ on a state

$$\tilde{J}^2|a,b\rangle = a|a,b\rangle, \quad \tilde{J}_z|a,b\rangle = b|a,b\rangle$$

Now lets look at the S.H.O

$\tilde{a} = \frac{\tilde{x}}{\sqrt{2}\sigma_x} + \frac{i\tilde{P}_x}{\sqrt{2}\sigma_{px}}$, $\tilde{a}^+ = \frac{\tilde{x}}{\sqrt{2}\sigma_x} - \frac{i\tilde{P}_x}{\sqrt{2}\sigma_{px}}$, we then define the following:

$\tilde{J}_\pm = \tilde{J}_x \pm i\tilde{J}_y$, now the commutation the relationships are:

$[\tilde{J}^2, \tilde{J}_\pm] = 0$, $[\tilde{J}_z, \tilde{J}_\pm] = \pm\hbar\tilde{J}_\pm$, $[\tilde{J}_+, \tilde{J}_-] = 2\hbar\tilde{J}_z$

Now we wish to show $\tilde{J}^2$ acts as raising, lowering operator

$\tilde{J}^2[\tilde{J}_\pm|a,b\rangle] = a[\tilde{J}_\pm|a,b\rangle]$

$\tilde{J}_z[\tilde{J}_\pm|a,b\rangle] = (\tilde{J}_\pm J_z \pm \hbar\tilde{J}_\pm)|a,b\rangle = (b\pm\hbar)(J_z|a,b\rangle) = (b\pm\hbar)|a,b\pm\hbar\rangle$

$= (\tilde{J}_\pm b \pm \hbar\tilde{J}_\pm)|a,b\rangle = (b\pm\hbar)(\tilde{J}_\pm|a,b\rangle)$, w/ $\tilde{J}_\pm|a,b\rangle = C_\pm|a,b\rangle$

11-17-21

The useful commutation relations : $[\tilde{J}_i, \tilde{J}_j] = i\hbar\epsilon_{ijk}\tilde{J}_k$, $[\tilde{J}^2, \tilde{J}_z] = 0$

The eigenvalue equations are : $\tilde{J}^2|a,b\rangle = a|a,b\rangle$, $\tilde{J}_z|a,b\rangle = b|a,b\rangle$

$\tilde{J}_\pm = \tilde{J}_x \pm i\tilde{J}_y$, $[\tilde{J}^2, \tilde{J}_\pm] = 0$, $[\tilde{J}_z, \tilde{J}_\pm] = \pm\hbar\tilde{J}_\pm$, $[\tilde{J}_+, \tilde{J}_-] = 2\hbar\tilde{J}_z$

Now looking at $\tilde{J}^2[\tilde{J}_\pm|a,b\rangle] = a[\tilde{J}_\pm|a,b\rangle]$

$\tilde{J}_z[\tilde{J}_\pm|a,b\rangle] = (\tilde{J}_\pm\tilde{J}_z \pm \hbar\tilde{J}_\pm)|a,b\rangle = (b\pm\hbar)[\tilde{J}_z|a,b\rangle]$ $\therefore$ $\tilde{J}_\pm|a,b\rangle = C_\pm|a,b\pm\hbar\rangle$

Now we want to find $C_\pm$ and b

$\tilde{J}_-|a,b\rangle \overset{DC}{\Longleftrightarrow} \langle a,b|\tilde{J}_+$, $\tilde{J}_+|a,b\rangle \overset{DC}{\Longleftrightarrow} \langle a,b|\tilde{J}_-$

$\langle a,b|\tilde{J}_+\tilde{J}_-|a,b\rangle \geq 0$, $\langle a,b|\tilde{J}_-\tilde{J}_+|a,b\rangle \geq 0$ $\longrightarrow$ Inner product must be ≥ 0

$\tilde{J}_+\tilde{J}_- + \tilde{J}_-\tilde{J}_+ = 2(\tilde{J}_x^2 + \tilde{J}_y^2) = 2(\tilde{J}^2 - \tilde{J}_z^2)$ $\therefore$ we now have

$2\langle a,b|\tilde{J}^2 - \tilde{J}_z^2|a,b\rangle \geq 0 \Rightarrow a - b^2 \geq 0$ $\therefore$ $a \geq b^2$ $\longrightarrow$ Zcomp. $\leq$ Total

$\tilde{J}_+|a,b_{max}\rangle = 0$ compared to $\tilde{J}_-[\tilde{J}_+|a,b_{max}\rangle] = 0 \Rightarrow \tilde{J}_-(0) = 0$

$\tilde{J}_-\tilde{J}_+ = \tilde{J}_x^2 + \tilde{J}_y^2 + i(\tilde{J}_x\tilde{J}_y - \tilde{J}_y\tilde{J}_x) = \tilde{J}_x^2 + \tilde{J}_y^2 + i[\tilde{J}_x, \tilde{J}_y] = \tilde{J}^2 - \tilde{J}_z^2 - \hbar\tilde{J}_z$, now

$\tilde{J}_-\tilde{J}_+|a,b_{max}\rangle = a - b_{max}^2 - \hbar b_{max}|a,b_{max}\rangle = 0$ $\therefore$ $a = b_{max}(b_{max}+\hbar)$, now $\tilde{J}_-$

$\tilde{J}_-|a,b_{min}\rangle = 0$, $\tilde{J}_+\tilde{J}_-|a,b_{min}\rangle = 0$ $\therefore$ $a = b_{min}(b_{min}-\hbar) \Rightarrow b_{min} = -b_{max}$

From the above we have a relationship for b, $-b_{max} \leq b \leq b_{max}$

$J_+^n|a,b_{min}\rangle \Rightarrow b_{max} = b_{min} + n\hbar = -b_{max} + n\hbar = b_{max} = \frac{n\hbar}{2}$

we then say a is : $a = j(j+1)\hbar^2$ w/ $j = \frac{b_{max}}{\hbar}$ and $b = m\hbar$

we now have, $\tilde{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$: $\tilde{J}_z|j,m\rangle = m\hbar|j,m\rangle$ w/ $-j \leq m \leq j$

Now we want to Find the normalization Factors

$$\tilde{J}_+|j,m\rangle = C^+_{j,m}|j,m+1\rangle \quad , \quad \tilde{J}_-|j,m\rangle = C^-_{j,m}|j,m-1\rangle$$

$$\langle j,m|\tilde{J}_-\tilde{J}_+|j,m\rangle = |C^+_{jm}|^2 = [j(j+1) - m^2 - m]\hbar^2$$

$$C^+_{j,m} = \hbar\sqrt{(j-m)(j+m+1)} \rightarrow \tilde{J}^+ \quad , \quad C^-_{j,m} = \hbar\sqrt{(j+m)(j-m+1)} \rightarrow \tilde{J}^-$$

Now we look for matrix elements of spin

$$\tilde{D}^{(j)}_{m'm}(R) = \langle j,m'|e^{-i\vec{J}\cdot\hat{n}\varphi}|j,m\rangle$$

$$\tilde{D}^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle j,m'|e^{-i\tilde{J}_z\alpha/\hbar}e^{-i\tilde{J}_y\beta/\hbar}e^{-i\tilde{J}_z\gamma/\hbar}|j,m\rangle$$

$$= e^{-i(m\alpha+m'\gamma)}\langle j,m'|e^{-i\tilde{J}_y\beta/\hbar}|j,m\rangle$$

$$e^{-i\tilde{J}_y\beta/\hbar} = [1 - 1/2!(\tilde{J}_y\beta/\hbar)^2 + 1/4!(\tilde{J}_y\beta/\hbar)^4 + \dots] + i[(\tilde{J}_y\beta/\hbar) - 1/3!(\tilde{J}_y\beta/\hbar)^3 + \dots]$$

$$\tilde{J}_y = \frac{(\tilde{J}_+ - \tilde{J}_-)}{2i} \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad \frac{J^n}{\hbar} = \begin{cases} \frac{J_y}{\hbar} & n \text{ odd} \\ \left(\frac{J_y}{\hbar}\right)^2 & n \text{ even} \end{cases}$$

$$e^{-i\tilde{J}_y\beta/\hbar} \Rightarrow \mathbb{1} - \left(\frac{J_y}{\hbar}\right)^2(1-\cos(\beta)) - i\frac{J_y}{\hbar}\sin(\beta)$$

11-29-21

Everything we know about angular momentum

$$[\tilde{J}_i, \tilde{J}_j] = i\hbar\epsilon_{ijk}\tilde{J}_k \quad , \quad \tilde{J}^2 = \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z^2 \quad , \quad [\tilde{J}^2, \tilde{J}_i] = 0 \quad , \quad \tilde{J}_\pm = \tilde{J}_x \pm i\tilde{J}_y$$

$$[\tilde{J}^2, \tilde{J}_\pm] = 0 \quad , [\tilde{J}_z, \tilde{J}_\pm] = \pm\hbar\tilde{J}_\pm \quad , [\tilde{J}_+, \tilde{J}_-] = 2\hbar\tilde{J}_z \quad , \tilde{D}(\hat{n},\varphi) = e^{-i\vec{J}\cdot\hat{n}\varphi/\hbar} = \mathbb{1} - i\vec{J}\cdot\hat{n}\varphi/\hbar$$

$$\tilde{J}^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle \quad , \quad \tilde{J}_z|j,m\rangle = m\hbar|j,m\rangle, \quad m = -j, j+1, j+2, \dots, j$$

Angular momentum classically is, $\vec{L} = \vec{r} \times \vec{p}$

we now take a look at $[\tilde{L}_x, \tilde{L}_y]$ w/ $\tilde{L}_x = \tilde{y}\tilde{P}_z - \tilde{z}\tilde{P}_y$, $\tilde{L}_y = \tilde{z}\tilde{P}_x - \tilde{x}\tilde{P}_z$, now

$$[\tilde{L}_x, \tilde{L}_y] = [\tilde{y}\tilde{P}_z - \tilde{z}\tilde{P}_y, \tilde{z}\tilde{P}_x - \tilde{x}\tilde{P}_z] = [\tilde{y}\tilde{P}_z, \tilde{z}\tilde{P}_x - \tilde{x}\tilde{P}_z] - [\tilde{z}\tilde{P}_y, \tilde{z}\tilde{P}_x - \tilde{x}\tilde{P}_z]$$

$$[\tilde{L}_x, \tilde{L}_y] = [\tilde{y}\tilde{P}_z, \tilde{z}\tilde{P}_x] - \overset{0}{\cancel{[\tilde{y}\tilde{P}_z, \tilde{x}\tilde{P}_z]}} - \overset{0}{\cancel{[\tilde{z}\tilde{P}_y, \tilde{z}\tilde{P}_x]}} + [\tilde{z}\tilde{P}_y, \tilde{x}\tilde{P}_z]$$

$$[\tilde{L}_x, \tilde{L}_y] = \tilde{y}\tilde{P}_x[\tilde{P}_z, \tilde{z}] + \tilde{x}\tilde{P}_y[\tilde{z}, \tilde{P}_z] = i\hbar(\tilde{x}\tilde{P}_y - \tilde{y}\tilde{P}_x) = i\hbar\tilde{L}_z$$

Now we take the rotation operator :

$D(\delta\varphi, \hat{z}) = \tilde{\mathbb{1}} - i\delta\varphi/\hbar\, \tilde{L}_z = (\tilde{\mathbb{1}} - i\delta\varphi/\hbar\, \tilde{L}_z)|x, y, z\rangle$

12-1-21

If we look at a translation operator: $(\tilde{\mathbb{1}} - i\tilde{p}_x \delta x)|x\rangle = |x + \delta x\rangle$

$$(\tilde{\mathbb{1}} - i\tilde{p}_x \delta x)|\alpha\rangle = (\tilde{\mathbb{1}} - i\tilde{p}_x \delta x)\int |x'\rangle\langle x'|\alpha\rangle\, dx' = \int |x' + \delta x\rangle\langle x'|\alpha\rangle\, dx' = \int |x'\rangle\langle x' - \delta x|\alpha\rangle\, dx'$$

$$\langle x|(\tilde{\mathbb{1}} - i\tilde{p}_x \delta x)|\alpha\rangle = \int \langle x|x'\rangle\langle x' - \delta x|\alpha\rangle\, dx' = \langle x - \delta x|\alpha\rangle$$ $\Rightarrow$ Expect similar on final

Now we want to look at rotations:

$$\left[\tilde{\mathbb{1}} - \frac{i\tilde{P}_y}{\hbar}(\tilde{x}\delta\varphi) + \frac{i\tilde{P}_x}{\hbar}(\tilde{y}\delta\varphi)\right]|x', y', z'\rangle = |x' - y'\delta\varphi, y' + x'\delta\varphi, z'\rangle$$

If we look at this in matrix form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \approx \begin{pmatrix} 1 & -\delta\varphi & 0 \\ \delta\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x' - y'\delta\varphi \\ y' + x'\delta\varphi \\ z' \end{pmatrix}$$

This is how it acts on a state:

$$\langle x', y', z'|\mathbb{1} - \frac{i\tilde{L}_z\delta\varphi}{\hbar}|\alpha\rangle = \langle x' + y'\delta\varphi, y' - x'\delta\varphi, z'|\alpha\rangle$$

We now want to transform co-ordinates: $\langle x', y', z'|\alpha\rangle \to \langle r, \theta, \varphi|\alpha\rangle$

This then affects the state like: $\langle r, \theta, \varphi|\tilde{\mathbb{1}} - \frac{i\tilde{L}_z\delta\varphi}{\hbar}|\alpha\rangle = \langle r, \theta, \varphi - \delta\varphi|\alpha\rangle$

$\langle r, \theta, \varphi - \delta\varphi|\alpha\rangle = \langle r, \theta, \varphi|\alpha\rangle - \delta\varphi \frac{\partial}{\partial\varphi}\langle r, \theta, \varphi|\alpha\rangle$ $\longrightarrow$ Taylor series expansion

$$\langle r, \theta, \varphi|\tilde{\mathbb{1}} - \frac{i\tilde{L}_z\delta\varphi}{\hbar}|\alpha\rangle = \langle r, \theta, \varphi|\alpha\rangle - \frac{i\delta\varphi}{\hbar}\langle r, \theta, \varphi|\tilde{L}_z|\alpha\rangle$$

$$\Rightarrow \langle r, \theta, \varphi|\tilde{L}_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle r, \theta, \varphi|\alpha\rangle$$

We want to look at what happens if the generator is in the x-direction:

$$(\tilde{\mathbb{1}} - \frac{i\delta\varphi_x}{\hbar}\tilde{L}_x)|x', y', z'\rangle = \left[\tilde{\mathbb{1}} - \frac{i\delta\varphi_x}{\hbar}(\tilde{y}\tilde{P}_z - \tilde{z}\tilde{P}_y)\right]|x', y', z'\rangle$$
$$= |x', y' - z'\delta\varphi_x, z' + y'\delta\varphi_x\rangle$$

We now want to determine the matrix element

$$\langle x', y', z'|\tilde{\mathbb{1}} - \frac{i\delta\varphi_x}{\hbar}L_x|\alpha\rangle = \langle x', y' + z'\delta\varphi_x, z' - y'\delta\varphi_x|\alpha\rangle \equiv f(x', y', z')$$

From rectangular to spherical co-ordinates

$$\langle r, \theta, \varphi|\tilde{\mathbb{1}} - \frac{i\delta\varphi_x}{\hbar}\tilde{L}_x|\alpha\rangle = \langle r, \theta - \delta\theta, \varphi - \delta\varphi|\alpha\rangle$$

We now ask ourself how does δz change

$$\delta z = \begin{cases} \delta(r\cos\theta) = r \to \delta(\cos\theta) = -r\sin\theta\,\delta\theta \\ y\,\delta\varphi_x = r\sin\theta\sin\varphi\,\delta\varphi_x \end{cases} \Rightarrow \delta\varphi = -\sin\varphi\,\delta\varphi_x$$

$$\delta x = 0 = \delta(r\sin\theta\cos\varphi) = r\cos\theta\cos\varphi\,\delta\theta - r\sin\theta\sin\varphi\,\delta\varphi = 0 \Rightarrow \delta\varphi = \cot\theta\cot\varphi\,\delta\theta$$

$\delta\varphi$ will then become : $\delta\varphi = -\cos\varphi\cot\theta\,\delta\varphi_x$, This then means our rotation will be

$$\langle r,\theta,\varphi|\,\tilde{\mathbb{1}} - \frac{i\delta\varphi_x}{\hbar}\tilde{L}_x|\alpha\rangle = \langle r,\theta + \sin\varphi\,\delta\varphi_x, \varphi + \cot\theta\cos\varphi\,\delta\varphi_x|\alpha\rangle$$

We now want to determine L_x

$$\langle r,\theta,\varphi|\alpha\rangle - \frac{i\delta\varphi_x}{\hbar}\langle r,\theta,\varphi|\tilde{L}_x|\alpha\rangle = \langle r,\theta,\varphi|\alpha\rangle + \delta\varphi_x\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\langle r,\theta,\varphi|\alpha\rangle$$

$$\therefore\ \langle r,\theta,\varphi|L_x|\alpha\rangle = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)\langle r,\theta,\varphi|\alpha\rangle$$

$$\langle r,\theta,\varphi|L_y|\alpha\rangle = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)\langle r,\theta,\varphi|\alpha\rangle$$

$$\langle r,\theta,\varphi|L_z|\alpha\rangle = -i\hbar\frac{\partial}{\partial\varphi}\langle r,\theta,\varphi|\alpha\rangle$$

12-6-21

The angular momentum operators are :

$$\tilde{L}_x \doteq i\hbar(\sin(\varphi)\,\partial/\partial\theta + \cot(\theta)\cos(\varphi)\,\partial/\partial\varphi),\ \tilde{L}_y \doteq -i\hbar(\cos(\varphi)\,\partial/\partial\theta - \cot(\theta)\sin(\varphi)\,\partial/\partial\varphi),\ \tilde{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

The ladder operator is then :

$$\tilde{L}_\pm = \tilde{L}_x \pm i\tilde{L}_y \doteq -i\hbar e^{\pm i\varphi}(\pm i\,\partial/\partial\theta - \cot\theta\,\partial/\partial\varphi)$$

The total angular momentum squared is :

$$\tilde{L}^2 = \tilde{L}_z^2 + (\tilde{L}_+\tilde{L}_- + \tilde{L}_-\tilde{L}_+)/2 \doteq -\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\partial/\partial\theta)\right]$$

The Spherical Harmonics derivation :

$$\tilde{L}_z|l,m\rangle = m\hbar|l,m\rangle\ ,\ -i\hbar\frac{\partial}{\partial\varphi}\Phi(\varphi) = m\hbar\Phi(\varphi) \Rightarrow \Phi(\varphi) \propto e^{im\varphi}$$

Where we use ladder operators to derive the spectrum

$$\tilde{L}_+|l,l\rangle = 0 \text{ or } \tilde{L}_-|l,-l\rangle = 0$$

$$\left(i\frac{\partial}{\partial\theta} - \cot(\theta)\frac{\partial}{\partial\varphi}\right)\Theta(\theta)\Phi(\varphi) = 0$$

$$\left(\frac{\partial}{\partial\theta} - l\cot(\theta)\right)\Theta(\theta) = 0$$

$$\Theta(\theta) \propto \sin^l(\theta)$$

The eigenfunction for the angular momentum is

$$Y_\ell^{m=\ell}(\theta,\varphi) = c_\ell e^{i\ell\varphi}\sin^\ell(\theta)$$

The normalization is then

$$\iint Y_{\ell'}^{m'}(\theta,\varphi)\, Y_\ell^{m}(\theta,\varphi)\, d\Omega = \delta_{m'm}\delta_{\ell'\ell}$$

we now look at the rotation Operator again, but with Spherical Harmonics. Start with a State:

$$|\hat{n}\rangle = \tilde{D}(R)|\hat{z}\rangle$$

A rotation from the z-axis to a direction along n axis is given by θ and φ.

$$|\hat{n}\rangle = \tilde{D}(\alpha=\varphi,\beta=\theta,\gamma=0)|\hat{z}\rangle = \sum_{\ell,m}\tilde{D}(\alpha=\varphi,\beta=\theta,\gamma=0)|\ell,m\rangle\langle\ell,m|\hat{z}\rangle$$

where we look at $\langle\ell,m'|\hat{n}\rangle$

$$\langle\ell,m'|\hat{n}\rangle = \sum_m \langle\ell,m'|\tilde{D}(\alpha=\varphi,\beta=\theta,\gamma=0)|\ell,m\rangle\langle\ell,m|\hat{z}\rangle = \sum_m \tilde{D}^{(\ell)}_{m';m}(R)\langle\ell,m|\hat{z}\rangle$$

where we have

$$\langle\ell,m'|\hat{n}\rangle = Y_\ell^{m'\dagger}(\theta,\varphi) \Rightarrow \langle\ell,m|\hat{z}\rangle = Y_\ell^m(\theta=0,\varphi) = Y_\ell^0(0,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}$$

where in general we have

$$\langle\ell,m'|\hat{n}\rangle = \tilde{D}^{(\ell)}_{m',0}(\theta,\varphi,0)\sqrt{\frac{2\ell+1}{4\pi}}\ ,\quad \tilde{D}^{(\ell)}_{m'0}(\theta,\varphi,0) = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^{m\dagger}(\theta,\varphi)$$