COLLEGE OF ARTS AND SCIENCES
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The UNIVERSITY of OKLAHOMA

Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #8

PROBLEMS: {1, 2, 3}

Due: November 5, 2021 By 6:00 PM

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PROFESSOR
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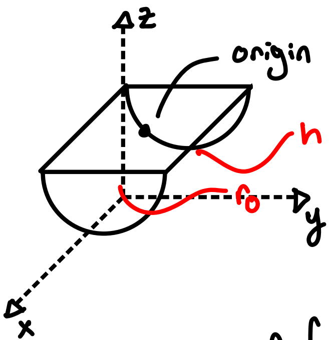
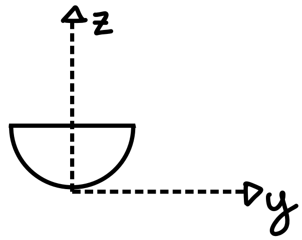
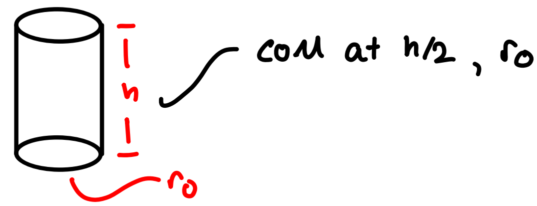
Problem 1:

Consider a half-cylinder of radius r_0 and mass m that is allowed to roll on a horizontal plane, as in Fig. 1. Take the angle ϕ to correspond to the deviation of the upper flat plane from horizontal.

(a) Compute the moment of inertia tensor with respect to the half-cylinder's center-of-mass.







$$x_{cm} = \frac{\rho}{m}\int x\,dv\,,\quad y_{cm} = \frac{\rho}{m}\int y\,dv\,,\quad z_{cm} = \frac{\rho}{m}\int z\,dv \qquad dv \rightarrow r,\theta,x$$

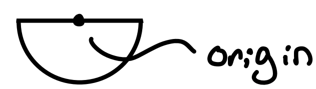
$$x \in [-h/2, h/2]\,,\quad \theta \in [\pi, 2\pi]\,,\quad r \in [0, r_0] \quad \text{w/} \quad y = r\sin(\theta) \;\&\; z = r\cos(\theta)$$

$$x_{cm} = \frac{\rho}{m}\int_{h/2}^{-h/2}\int_{\pi}^{2\pi}\int_0^{r_0} x r\,dr\,d\theta\,dx = 0$$

$$y_{cm} = \frac{\rho}{m}\int_{h/2}^{-h/2}\int_{\pi}^{2\pi}\int_0^{r_0} r^2\cos(\theta)\,dr\,d\theta\,dx = 0$$

$$z_{cm} = \frac{\rho}{m}\int_{h/2}^{-h/2}\int_{\pi}^{2\pi}\int_0^{r_0} r^2\sin(\theta)\,dr\,d\theta\,dx = \frac{\rho}{m}\cdot\frac{2hr_0^3}{3} = \frac{\cancel{m}}{V}\cdot\frac{1}{\cancel{m}}\cdot\frac{2hr_0^3}{3} = \frac{2}{\cancel{h}\pi \cancel{r_0^2}}\,\frac{2\cancel{h}r_0^{\cancel{3}}}{3} = \frac{4r_0}{3\pi}$$

$\text{CoM} = (0, 0, 4r_0/3\pi)$, Place origin at flat edge and at r_0 :



$$I'_{xx} = \int_V \rho(r)(r^2 - x^2)\,dv\,,\quad I'_{yy} = \int_V \rho(r)(r^2 - y^2)\,dv\,,\quad I'_{zz} = \int_V \rho(r)(r^2 - z^2)\,dv$$

$$I'_{xx} = \frac{2m}{h\pi r_0^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\pi}^{2\pi}\int_0^{r_0} r^3\,dr\,d\theta\,dx = \frac{2m}{\pi r_0^2 h}\cdot\frac{\pi r_0^4 h}{4} = \frac{m r_0^2}{2}$$

$$I'_{yy} = \frac{2m}{h\pi r_0^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\pi}^{2\pi}\int_0^{r_0}(x^2 + r^2\sin^2(\theta))\,r\,dr\,d\theta\,dx = \frac{2m}{\pi r_0^2 h}\cdot\frac{\pi r_0^2 h(3r_0^2 + 4h^2)}{24} = \frac{m(3r_0^2 + 4h^2)}{12}$$

$$I'_{zz} = \frac{2m}{h\pi r_0^2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\pi}^{2\pi}\int_0^{r_0}(x^2 + r^2\cos^2(\theta))\,r\,dr\,d\theta\,dx = \frac{2m}{\pi r_0^2 h}\cdot\frac{\pi r_0^2 h(3r_0^2 + 4h^2)}{24} = \frac{m(3r_0^2 + 4h^2)}{12}$$

$$I'_{xx} = \frac{m r_0^2}{2}\,,\quad I'_{yy} = \frac{m(3r_0^2 + 4h^2)}{12}\,,\quad I'_{zz} = \frac{m(3r_0^2 + 4h^2)}{12}$$

Steiners parallel axis theorem

$$I_{jk} = I'_{jk} - M(|R|^2\delta_{jk} - R_j R_k)$$

$$I_{xx} = \frac{m r_0^2}{2} - m\left((4r_0/3\pi)^2 - 0(0)\right) = \frac{m r_0^2}{2} - m\left((16r_0^2/9\pi^2)\right) = \frac{m r_0^2}{18\pi^2}(9\pi^2 - 32)$$

Problem 1: Continued

$$I_{yy} = \frac{m(3r_0^2+4h^2)}{12} - m((4r_0/3\pi)^2 - 0(0)) = = \frac{m}{36\pi^2}(r_0^2(9\pi^2-64)+12\pi^2h^2)$$

$$I_{zz} = \frac{m(3r_0^2+4h^2)}{12} - m((4r_0/3\pi)^2 - (4r_0/3\pi)^2) = \frac{m(3r_0^2+4h^2)}{12}$$

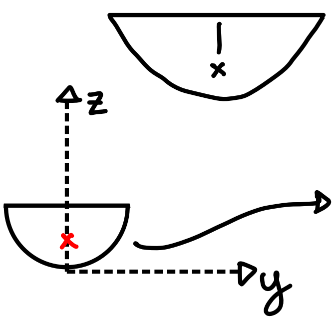
$$I_{xx} = \frac{mr_0^2}{18\pi^2}(9\pi^2-32) \,,\quad I_{yy} = \frac{m}{36\pi^2}(r_0^2(9\pi^2-64)+12\pi^2h^2) \,,\quad I_{zz} = \frac{m(3r_0^2+4h^2)}{12}$$

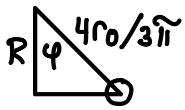
$$\boxed{I = \frac{m}{6\pi^2}\begin{pmatrix} r_0^2(9\pi^2-32)/3 & 0 & 0 \\ 0 & (r_0^2(9\pi^2-64)+12\pi^2h^2)/6 & 0 \\ 0 & 0 & \pi^2(3r_0^2+4h^2)/2 \end{pmatrix}}$$

(b) Adapt your result from (a) to show that the Lagrangian for the half-cylinder can be written as,

$$L = \frac{1}{2}\left[\frac{3}{2} - \frac{8}{3\pi}\cos(\phi)\right] mr_0^2\dot{\phi}^2 + \frac{4mgr_0}{3\pi}\cos(\phi). \tag{1}$$

Hint: You might find it easier to use the contact point of the half-cylinder with the horizontal plane as your point of reference.





$\text{w/ } R = -\frac{4r_0}{3\pi}\cos(\varphi) \quad \therefore \quad h = -\frac{4r_0}{3\pi}\cos(\varphi)$

arc length = horiz. dist.

$I \equiv I_{xx}$ COM

$\bar{z} \rightsquigarrow z = r_0 - \frac{4r_0}{3\pi}\cos(\varphi) \,,\quad y = r_0\varphi - \frac{4r_0}{3\pi}\sin(\varphi) \,,\quad R = \frac{4r_0}{3\pi}$

$$L = T - U \,,\quad T = T_T + T_R \;:\; T_T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \,,\quad T_R = \frac{1}{2}I\omega^2 \,,\quad \omega = \dot{\varphi}$$

$$\dot{x} = 0 \,,\quad \dot{y} = r_0\dot{\varphi} - R\dot{\varphi}\cos(\varphi) \,,\quad \dot{y}^2 = r_0^2\dot{\varphi}^2 - 2r_0R\dot{\varphi}^2\cos(\varphi) + R^2\dot{\varphi}^2\sin^2(\varphi)$$

$$\dot{z} = R\dot{\varphi}\sin(\varphi) \,,\quad \dot{z}^2 = R^2\dot{\varphi}^2\sin^2(\varphi) \,,\quad \dot{x}^2+\dot{y}^2+\dot{z}^2 = r_0^2\dot{\varphi}^2 - 2r_0R\dot{\varphi}^2\cos(\varphi) + R^2\dot{\varphi}^2$$

$$T_R = \frac{1}{2}\frac{mr_0^2}{18\pi^2}(9\pi^2-32)\dot{\varphi}^2 = \frac{mr_0^2\dot{\varphi}^2}{36\pi^2}(9\pi^2-32) = mr_0^2\dot{\varphi}^2(1/4 - 8/(9\pi^2))$$

$$T_T = \frac{1}{2}m(r_0^2\dot{\varphi}^2 - 8r_0^2/3\pi\cos(\varphi) + 16r_0^2\dot{\varphi}^2/9\pi^2) \,,\quad T_R = mr_0^2\dot{\varphi}^2(1/4 - 8/(9\pi^2))$$

$$T = mr_0^2\dot{\varphi}^2(1/2 - 8/6\pi\cos(\varphi) + \cancel{8/9\pi^2} + 1/4 - \cancel{8/9\pi^2}) = mr_0^2\dot{\varphi}^2(3/4 - 8/6\pi\cos(\varphi))$$

$$L = T - U = \frac{1}{2}(3/2 - 8/3\pi\cdot\cos(\varphi))mr_0^2\dot{\varphi}^2 + 4mgr_0/3\pi\cdot\cos(\varphi) \checkmark$$

∎

Problem 1: Continued

(c) Show that when the half-cylinder is near equilibrium its motion is equivalent to that of a pendulum of effective length

$$l = \left(\frac{9\pi}{8} - 2\right) r_0. \tag{2}$$

$L = \frac{1}{2}(3/2 - 8/3\pi \cdot \cos(\varphi)) m r_0^2 \dot{\varphi}^2 + 4 m g r_0 / 3\pi \cdot \cos(\varphi)$, Equilibrium $\Rightarrow \varphi << 1$

$$\text{Taylor}(\cos(\varphi)) = 1 - \frac{\varphi^2}{2} \quad \therefore \quad L = \frac{1}{2}\left(\frac{3}{2} - \frac{8}{3\pi}\left(1 - \frac{\varphi^2}{2}\right)\right) m r_0^2 \dot{\varphi}^2 + \frac{4mgr_0}{3\pi}\left(1 - \frac{\varphi^2}{2}\right)$$

$$\frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0 , \quad \frac{\partial L}{\partial \dot{\varphi}} = \left(\frac{3}{2} - \frac{8}{3\pi}\left(1 - \frac{\varphi^2}{2}\right)\right) m r_0^2 \dot{\varphi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \left(\frac{3}{2} - \frac{8}{3\pi}\left(1 - \frac{\varphi^2}{2}\right)\right) m r_0^2 \ddot{\varphi} + \frac{8\varphi m r_0^2 \dot{\varphi}^2}{3\pi} , \quad \frac{\partial L}{\partial \varphi} = \frac{4\varphi}{3\pi} m r_0^2 \dot{\varphi}^2 - \frac{4\varphi}{3\pi} m g r_0$$

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} \; : \; \frac{4\varphi}{3\pi} m r_0 (r_0 \dot{\varphi}^2 - g) = \left(\frac{3}{2} - \frac{8}{3\pi}\left(1 - \frac{\varphi^2}{2}\right)\right) m r_0^2 \ddot{\varphi} + \frac{8\varphi}{3\pi} m r_0^2 \dot{\varphi}^2$$

$$\overset{0}{\cancel{\frac{4\varphi}{3\pi} m r_0^2 \dot{\varphi}^2}} - \frac{4\varphi}{3\pi} m g r_0 = \frac{3}{2} m r_0^2 \ddot{\varphi} - \frac{8}{3\pi} m r_0^2 \ddot{\varphi} + \overset{0}{\cancel{\frac{4\varphi^2}{3\pi} m r_0^2 \ddot{\varphi}}} + \overset{0}{\cancel{\frac{8\varphi}{3\pi} m r_0^2 \dot{\varphi}^2}} \quad \text{no terms} > \mathcal{O}(2)$$

$$\frac{4\varphi}{3\pi} \cancel{m} g \cancel{r_0} = \frac{3}{2} \cancel{m} r_0^{\cancel{2}} \ddot{\varphi} - \frac{8}{3\pi} \cancel{m} r_0^{\cancel{2}} \ddot{\varphi} , \quad \frac{4\varphi}{3\pi} g = \frac{3}{2} r_0 \ddot{\varphi} - \frac{8}{3\pi} r_0 \ddot{\varphi} , \quad \varphi g = \frac{9\pi}{8} r_0 \ddot{\varphi} - 2 r_0 \ddot{\varphi}$$

$$\varphi g = \ddot{\varphi}\left(\frac{9\pi}{8} - 2\right) r_0 \quad \therefore \quad \ddot{\varphi} = \varphi g / \left(\frac{9\pi}{8} - 2\right) r_0$$

$\ddot{\varphi} = \varphi \dfrac{g}{\left(\frac{9\pi}{8} - 2\right) r_0} \longrightarrow$ Analagous to $\ddot{\theta} = \frac{g}{l}\theta$ For small angle pendulum $\therefore$

$$\boxed{l = \left(\frac{9\pi}{8} - 2\right) r_0}$$

Problem 1: Review

Procedure:

- Calculate the volume of the solid.
- Calculate the center of mass using
$$q_{COM} = \frac{\rho}{m}\int_V q_i \, dV.$$
- Place the origin along the principle axis and at the center of the co-ordinate system.
- Proceed to calculate the moment of inertia w.r.t the origin, not the center of mass.
- Then proceed to calculate the moment of inertia w.r.t to the center of mass with Steiner's parallel axis theorem.
- Define the position of the center of mass in the y and z directions and calculate the Lagrangian. Make sure to include a rotational and translational term for the kinetic energy.
- Show that the above result can be obtained.
- Lastly, perform a Taylor expansion on the cos terms in the Lagrangian and then use the Euler Lagrange formalism to find the EOM for the generalized co-ordinates.
- Throw out any terms above second order and compare the result with the EOM for a simple pendulum.

Key Concepts:

- Use Steiner's parallel axis theorem to determine the moment of inertia w.r.t the center of mass of our solid.
- When we Taylor expand our cos terms we can show that the EOM of the center of mass of this solid is that of a simple pendulum.
- We only use the moment of inertia in the x direction because it is perpendicular to the motion of the oscillation.

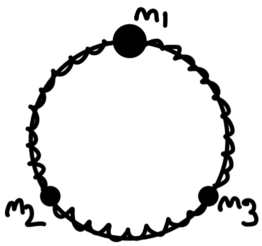
Variations:

- We can be given a different solid.
  - Thus changing our calculations but not our over all procedure.
- We can be asked qualitative questions about the motion of our cylinder.
  - Resort to looking at equations derived along the way in the process.

Problem 2:

Consider a system of 3 masses, two with mass m and one with mass $2m$ constrained to slide (frictionlessly) about a ring of radius r_0. All three masses are linked by 3 identical springs with associated spring constant k.

(a) Compute the normal modes of the system and comment briefly on the natrue of each mode.



The normal nodes of this system will be the eigenvectors of the following matrix equation

$$V - w^2 T = 0$$

The potential energy of this system is dependent upon the springs in our system:

$$V = \frac{1}{2}k\left((x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_2 - x_3)^2\right)$$

$$V = \frac{1}{2}k\left(x_1^2 - 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_3 + x_3^2 + x_2^2 - 2x_2x_3 + x_3^2\right)$$

$$V = \frac{1}{2}k\left(2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3\right) \quad w/ \quad x_1x_2 = -x_2x_1$$

$$V = \frac{1}{2}\begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix}$$

The kinetic energy of this system

$$T = \frac{1}{2}\left(2mx_1^2 + mx_2^2 + mx_3^2\right)$$

$$T = \frac{1}{2}\begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$V - w^2 T = \frac{1}{2}\begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix} - \frac{w^2}{2}\begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = 0 \ : \ \begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix} - w^2\begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = 0$$

$$V - w^2 T = \begin{pmatrix} 2k - w^2 2m & -k & -k \\ -k & 2k - w^2 m & -k \\ -k & -k & 2k - w^2 m \end{pmatrix} = 0$$

Using Mathematica the eigenvalues are

$$\lambda_1 = \frac{3k}{m}, \ \lambda_2 = \frac{2k}{m}, \ \lambda_3 = 0$$

Problem 2: Continued

And using Mathematica the eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, the normal modes of this system are:

$$\boxed{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

For the first normal mode, the 2m mass remains at rest and the other masses oscillate in opposite directions.

For the second normal mode, the 2m mass moves in the opposite direction of the smaller masses.

The third normal mode has all of the masses oscillate in the same direction.

```
In[∘]:= ClearAll["Global'*"]
        A = {{k, -k/2, -k/2}, {-k, 2 k, -k}, {-k, -k, 2 k}};
        MatrixForm[A]
Out[∘]//MatrixForm=
```

$$\begin{pmatrix} k & -\frac{k}{2} & -\frac{k}{2} \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix}$$

```
In[∘]:= Eigenvalues[A]
In[∘]:= {3 k, 2 k, 0}
        Eigenvectors[A]
Out[∘]= {3 k, 2 k, 0}
Out[∘]= {{0, -1, 1}, {-1, 1, 1}, {1, 1, 1}}
```

(b) Assuming the three masses are initially prepared near their equilibrium condition. The heavier mass is perturbed slightly away from this configuration while the others are held in place. Assuming the masses are simultaneously released, solve for the ensuing motion. Which of the normal modes participate?

In this situation, all three masses must be in motion. This automatically eliminates the normal mode

$$\vec{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad w^2 = \frac{3k}{m}$$

Because the zero in the first row represents the motion of the 2m mass.

Problem 2: Continued

This means the remaining normal modes are

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \; \omega^2 = \frac{2k}{m} \quad : \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \omega^2 = 0$$

The ansatz solution is as Follows

$$\eta_i = c_i \vec{a}_i e^{-i\omega_i t}$$

For our remaing normal modes this ansatz solution is:

$$A = c_2 \vec{a}_2 e^{-i\omega_2 t} + c_3 \vec{a}_3 e^{-i\omega_3 t} = c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{2k}{m}}t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0}$$

$$\boxed{A = c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{-i\sqrt{\frac{2k}{m}}t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

Problem 2: Review

Procedure:

- Write out the potential energy in terms that can be expressed in a matrix. Then put them in a matrix.
- Write out the kinetic energy in terms that can be expressed in a matrix. Then put them in a matrix.
- Solve for the eigenvalues ω^2 from the equation $\mathbf{V} - \omega^2\mathbf{T} = 0$.
- Obtain the eigenvalues and the eigenvectors. The eigenvectors are in turn the normal modes to the system.
- Discuss the motion of these particles qualitatively.
- Use the ansatz
$$\eta_i = c_i\vec{a}_i e^{-i\omega_i t}$$
where $\vec{a}_i$ are the normal modes and ω_i are the eigenvalues that correspond to their respective normal modes.
- Write out this ansatz.

Key Concepts:

- The eigenvalues that we are searching for are ω^2.
- The eigenvectors of this system are the normal modes that we are searching for.
- We use the ansatz to write out our system of normal modes that will participate in the motion that is described in part (b).
- The signs in our normal modes tell us what direction our beads are moving on our hoop.

Variations:

- We can be given a different system that we wish to examine.
  - This changes the potential and kinetic energies of our system and essentially the entire problem.
- We can be given a different scenario to examine for the ansatz in (b).
  - This would change our ansatz by excluding or including other normal modes that used to not participate.

Problem 3:

The remainder of this assignment will be focused on the paper Appl. Phys. B **66**, 181 (1998) by Daniel James. This work discusses the formation of crystals of trapped ions, which is a platform that is being pursued to realize a quantum computer by many prominent academic research groups (Maryland, Innsbruck), national labs (NIST) and companies (Honeywell, IonQ).

(a) Read up to the end of Sec. 2 of the manuscript. In just a few sentences, summarize the key points of the work.

The manuscript talks about what a Quantum Computer is, how it works, and what they are useful for. In this manuscript, the device they are using to create a Quantum computer is a cold trapped ion system which are coupled together with a Coulomb force between them. The first section speaks on the equilibrium positions of the cold trapped ions and the motion of these ions due to the potential that is present. Section 2 talks about the motion of these ions away from their equilibrium positions. Section 2 also discusses the Lagrangian, eigenvalues and eigenvectors, normal modes, Hamiltonian, and various other values of this system.

(b) Explain what the terms in Eq. (1) are describing.

$$V = \sum_{m=1}^{N} \frac{1}{2} M \nu^2 x_m(t)^2 + \sum_{\substack{n,m=1 \\ m \neq n}}^{N} \frac{Z^2 e^2}{8\pi\epsilon_0} \frac{1}{|x_n(t) - x_m(t)|}$$

$V \to$ Potential energy of each ion chain, $M \to$ Mass of each ion, $e \to$ Electron charge
$\nu \to$ Trap Frequency, $\epsilon_0 \to$ Permitivity of free space, $Z \to$ Degree of ionization of ions
$x_m(t) \to$ The position of the m^{th} ion, $x_n(t) \to$ The position of the n^{th} ion

$\sum_{m=1}^{N} \frac{1}{2} M \nu^2 x_m(t)^2 \longrightarrow$ Kinetic energy : $\frac{Z^2 e^2}{8\pi\epsilon_0} \frac{1}{|x_n(t) - x_m(t)|} \to$ Coulomb potential

(c) Derive Eq. (5).

We will differentiate (1) w.r.t $x_m^{(0)}$. First, if

$$x_m(t) > x_n(t) \longrightarrow (x_m(t) - x_n(t)) \; \& \; n \in [1, m-1]$$

$$x_n(t) > x_m(t) \longrightarrow (x_n(t) - x_m(t)) \; \& \; n \in [m+1, N]$$

Then we make a substitution

$$V \Bigg|_{\substack{x_m(t) = x_m^{(0)} \\ x_n(t) = x_n^{(0)}}} = \sum_{m=1}^{N} \frac{1}{2} M \nu^2 x_m^{(0)2} + \sum_{\substack{n,m=1 \\ m \neq n}}^{N} \frac{Z^2 e^2}{8\pi\epsilon_0} \frac{1}{|x_n^{(0)} - x_m^{(0)}|}$$

Taking the derivative w.r.t $x_m^{(0)}$,

$$\frac{\partial V}{\partial x_m^{(0)}} = M \nu^2 x_m^{(0)} + \frac{Z^2 e^2}{8\pi\epsilon_0} \left[\sum_{m+1}^{N} \frac{1}{(x_m^{(0)} - x_n^{(0)})^2} - \sum_{1}^{m-1} \frac{1}{(x_m^{(0)} - x_n^{(0)})^2} \right] = 0$$

Problem 3: Continued

Divide by $\mu\nu^2$

$$\frac{\partial V}{\partial x_m^{(0)}} = x_m^{(0)} + \frac{Z^2e^2}{8\pi\epsilon_0 \mu \nu^2}\left[\sum_{m+1}^{N} \frac{1}{(x_m^{(0)} - x_n^{(0)})^2} - \sum_{1}^{m-1} \frac{1}{(x_m^{(0)} - x_n^{(0)})^2}\right] = 0$$

With $l^3 = \dfrac{Z^2e^2}{8\pi\epsilon_0 \mu \nu^2}$

$$\frac{\partial V}{\partial x_m^{(0)}} = x_m^{(0)} - \sum_{1}^{m-1} \frac{l^3}{(x_m^{(0)} - x_n^{(0)})^2} + \sum_{m+1}^{N} \frac{l^3}{(x_m^{(0)} - x_n^{(0)})^2} = 0$$

Divide by l and substitute $u_m = x_m^{(0)}/l$ & $u_n = x_n^{(0)}/l$

$$\frac{\partial V}{\partial x_m^{(0)}} = u_m - \sum_{1}^{m-1} \frac{1}{(u_m - u_n)^2} + \sum_{m+1}^{N} \frac{1}{(u_m - u_n)^2} = 0$$

■

(d) Look at the results of Table 1 for the equilibrium positions of the ions. Focusing only on N even, can you explain the trends in:

(i) The spacing between the two central ions,

As the number of ions is increased, the spacing between the central ions is decreasing.

(ii) The position of the outermost ions.

As the number of ions is increased, the spacing between the outermost ions is increasing.

Problem 3: Review

Procedure:

- Read the paper and summarize the results.
- Discuss the terms found in the equation that we are asked to examine.
- Derive the equation that is asked by using the previous equations and rules of derivatives for absolute value functions.
- Discuss the qualitative parts of the paper.

Key Concepts:

- We can learn stuff by reading papers.

Variations:

- We can be asked to examine different parts of the paper.