Domain wall based argument for absence of a phase transition in one dimension:

$T = 0 \rightsquigarrow$ all magnetic moments pointing in one direction minimizes the energy.

↳ thus, at $T=0$ we have a vanishing entropy: $S=0$ (just one configuration)

Consider $T > 0$ and look at excitations that are created by flipping all the magnetic moments to the right of some side:

$$\underbrace{\uparrow\ \uparrow\ \uparrow\ \ \uparrow\ \uparrow\ \uparrow}_{T=0} \qquad\qquad \uparrow\ \uparrow\ \uparrow \Big\{ \downarrow\ \downarrow\ \downarrow$$

domain wall

the energy cost for creating a domain wall is $2J$

(let's assume absence of external magnetic field)

$$\overset{1}{\uparrow}\cdot\overset{2}{\uparrow}\cdot\overset{3}{\uparrow}\cdot\overset{4}{\uparrow}\cdot\overset{5}{\uparrow}\cdot\overset{6}{\uparrow}\cdot\overset{7}{\uparrow}\cdot\overset{8}{\uparrow}$$

possible spot to put domain wall

Since there are ~~sites~~ $N-1$ possible ways of creating a domain wall, the entropy increases by $\Delta S = k \log(N-1)$

$$\Rightarrow \begin{cases} \Delta A = \Delta U - T\Delta S \\ = \underbrace{2J}_{\Delta E = J-(-J)} - kT \log(N-1) \end{cases}$$

$\rightsquigarrow$ the creation of a domain wall lowers the Helmholtz

free energy, provided $T > 0$
and $N \to \infty$
($\Delta A < 0$ as $N \to \infty$)

$\Rightarrow$ More domain walls will be created until the spins are completely randomized and the net magnetization is zero.

$\leadsto$ $M = 0$ for $T > 0$ in the $N \to \infty$ limit.

Let's look at the analogous argument for the two-dimensional Ising model (again, assume absence of magnetic field).

$$\begin{array}{ccccc}
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\end{array}$$

5×5 lattice ($L = 5$)
(open BC's)

domain wall

difference in energy $= 10J$ (it's a little easier to use open boundary condition)

Energy cost associated with creating a domain wall : $2JL$, where L is the lattice size.

Domain wall can be at any of the L columns : entropy is of order $\log L$

$\Rightarrow \Delta A \sim 2JL - kT \log L$

$\rightsquigarrow \Delta A > 0$ as $L \to \infty$ (assuming $J > 0$)

Creating a domain wall increases the free energy and thus most of the ~~spins~~ magnetic moments will remain up and the magnetization remains finite for sufficiently small T.

$\Rightarrow \boxed{\mathcal{M} > 0 \text{ for } T > 0}$ (but T sufficiently small)

$\rightsquigarrow$ 2D Ising model should exhibit ferromagnetic phase

$\rightsquigarrow \mathcal{M}$ goes to zero at $T = T_c$

(for $T > T_c$: disordered phase)

The solution to the 2D Ising model is not straight forward ...

We will use mean-field approach:

Ising model (as before)

$$\mathcal{H} = -\sum_{(i,j)} J' \mu_{i,z} \mu_{j,z} - \sum_i B \mu_{i,z}$$

we are breaking up the double sum

$$= + J \sum_j^q s_j \left(-\sum_i s_i\right) + H\left(-\sum_i s_i\right)$$

this should only go over nearest neighbors!

$$= -\left(J \sum_j^q s_j + H\right)\left(\sum_i s_i\right)$$

So far, we have not made any approximations.

Rewrite slightly:

$$\mathcal{H} = -\sum_i \underbrace{\left(J \sum_j^q s_j + H\right)}_{} s_i$$

still no approximation

interpret as effective field that the i^{th} magn. moment sees (due to external field + interactions)

Define $H_{eff}^{(i)} = J \sum_{j}^{z} s_j + H$

Importantly: In the magnetic moment system w/o approximation (i.e., in the Ising model), the effective field depends on the orientation of the i^{th} magnetic moment:
the reason is that the orientation of the neighboring ~~spins~~ magnetic moments, depends on the orientation of the i^{th} magn. ~~spin~~ moment (hence the superscript "(i)" on H_{eff}).

In the simplest version of the mean-field theory, we write

$$H_{eff} = J q m + H$$

← this is the mean value of all $H_{eff}^{(i)}$

$$H_{eff} = J \sum_{j}^{q} \bar{s}_j + H$$

m is defined to be dimensionless.

m as of yet ~~not~~ known.

# of nearest neighbors

$$\underbrace{m = \bar{s}_j = \langle S_j \rangle}_{\text{magnetization}}$$

Mean-field theory ignores the deviations of $H_{eff}^{(i)}$ from H_{eff} !

H_{eff} is independent of the orientation of the i^{th} magnetic moment.

$\rightsquigarrow$ this is how we're constructing H_{eff}

So: The system of N interacting magnetic moments has been reduced to a system of one magnetic moment interacting with an effective field, which depends on all the other magnetic moments.

Partition function Q_{MF} for one magnetic moment in the effective field H_{eff} is:

$$Q_{MF} = \sum_{s=\pm 1} \exp\left(-\beta s H_{eff}\right)$$

$$= 2\cosh\left(\beta\left(J q m + H\right)\right)$$

Helmholtz free energy A:

$$A = -kT \log Q_{MF}$$

$$= -kT \log\left(2\cosh\left(\beta(Jqm + H)\right)\right)$$

Magnetization m:

m was defined to be dimensionless
→ hence derivative with respect to
H and not B

$$m = kT \frac{\partial}{\partial H} \log Q_{MF}$$

$$\Rightarrow \quad m = \tanh\left(\beta(Jqm + H)\right)$$

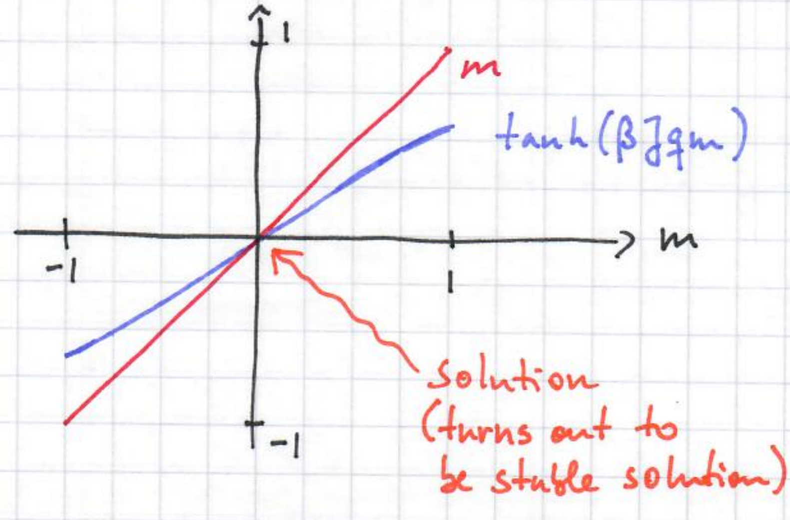
this is a transcendental equation that determines m:

recall, we had set $\bar{s}_j = m$ but we didn't know what value m has.

Now, we have an equation for the magnetization that depends on the magnetization.

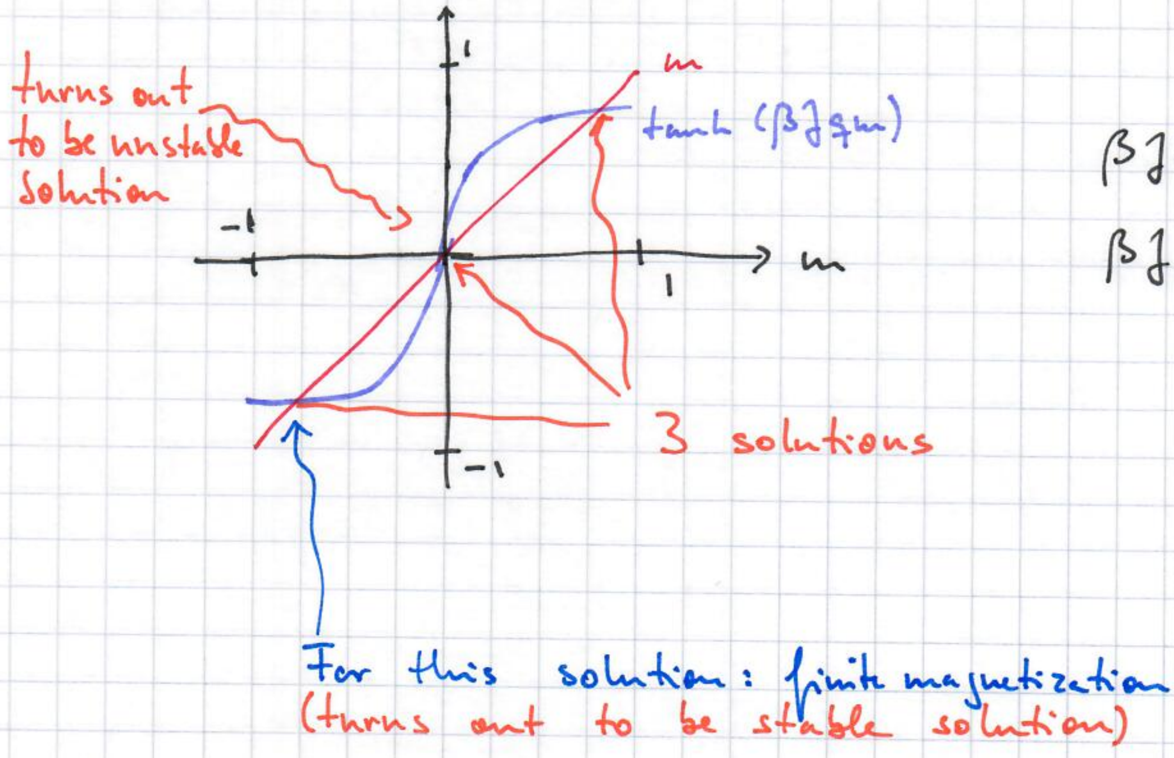
The equation needs to be solved self-consistently: we construct an effective field consistent with the solution.

Let's visualize this graphically (assume $H=0$):



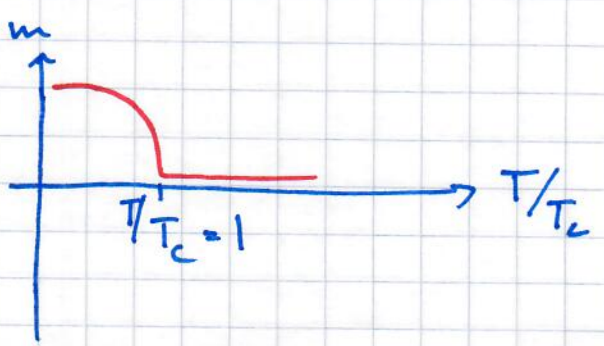
$\beta J q$ positive

$\beta J q = 0.8$



$\beta J q$ positive

$\beta J q \sim 1.5$



The "additional" solutions exist for

$\beta J q \geq 1$

$\Rightarrow kT \leq Jq$

$\Rightarrow \boxed{k T_c = J q}$ within mean-field theory

Square lattice: $q = 4$
(2D)

$$\Rightarrow kT_c = 4J$$

exact solution: $kT_c = 2.269 J$
(due to Onsager)
↳ Chapter 15

Note: Mean-field theory predicts completely wrong behavior for 1D system!

↳ quite frequently, mean-field theory fails in 1D.

$\underbrace{\text{low dimensions} \rightsquigarrow \text{enhanced fluctuations}}_{\text{rule of thumb}}$

Major shortcoming of the specific mean-field result derived:

- We only have a dependence on nearest neighbors and not (!) on the dimensionality of the system.

How do we know how to "label" the solutions?

Stable versus unstable ...

Near T_c, m is small:

$$\Rightarrow \tanh(x) \overset{\text{Taylor expansion}}{\approx} x - \frac{x^3}{3} + \ldots \quad \text{where } x = \frac{Jqm}{kT}$$

recall: $m = \tanh(\beta J q m)$

$$\rightsquigarrow m \approx \frac{Jqm}{kT} - \frac{1}{3}\left(\frac{Jqm}{kT}\right)^3$$

Solutions: $m = 0$ (Solution 1)

$$m = \pm\sqrt{3}\left(\frac{kT}{Jq}\right)^{3/2}\left(\frac{Jq}{kT} - 1\right)^{1/2}$$

(Solutions 2/3)

Intuitively, we know that Solution 1 needs to be chosen when $T > T_c$ and Solutions 2/3 when $T < T_c$.

Formally: Would take Solution 1 and Solutions 2/3 and calculate Helmholtz free energy for both.
Choose solution that yields the smaller Helmholtz free energy (this is the stable solution).

Let's continue to work near T_c and let's look at the $T \lesssim T_c$ solution:

$$m \approx \pm\sqrt{3}\left(\frac{kT}{Jq}\right)^{3/2}\left(\frac{Jq}{kT}-1\right)^{1/2}$$

Same

Rewrite using $T_c = \frac{Jq}{k}$ $\left[m \approx \pm\sqrt{3}\,\frac{kT}{Jq}\left(1-\frac{kT}{Jq}\right)^{1/2}\right]$

$$\Rightarrow m \approx \pm\sqrt{3}\left(\frac{T}{T_c}\right)\left(\frac{T_c - T}{T_c}\right)^{1/2}$$

near $T = T_c$
$T < T_c$

m approaches zero as a power law as T approaches T_c from below

$$m \propto \varepsilon^{1/2}, \text{ where } \underbrace{\varepsilon = \frac{T_c - T}{T_c}}_{\text{reduced temperature}}$$

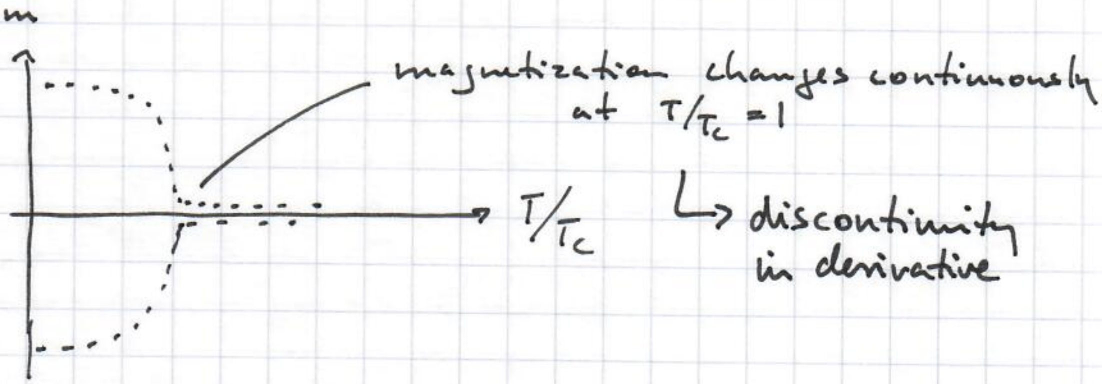
magnetization $\hat{=}$ order parameter

$$\propto \varepsilon^{\bar{\beta}}, \text{ where } \bar{\beta} = \frac{1}{2}$$

critical exponent

(exact solution $\bar{\beta} = 1/8$) in 2D

So far, phase transition investigated for $H=0$ case:



For finite H, discontinuity in derivative of magnetization goes away (it "gets smeared out").

A few concepts from Chapters 16 and 17

For the Ising model in 2D, we identified a thermal phase transition at $T = T_c$

for $H = 0$ (no external field):

$kT_c = Jq$ ($q = 4$ for square lattice)

obtained within mean-field theory

exact (due to Onsager, see Ch. 15):

$kT_c = 2.269\, J$

Let $t = \dfrac{T - T_c}{T_c}$ $\Rightarrow$ $t \to 0$ means that we are approaching critical point

Order parameter has, in general, regular and singular part (part that diverges or whose derivative diverges):

There exist other critical exponents.

For example: $\xi \sim |t|^{-\nu}$

correlation length

Near the critical point, ξ is the only relevant length scale

ξ measures the "spatial memory".

this is also refered to as "scaling hypothesis"

The concept of universality implies that widely different systems, with critical temperatures differing by orders of magnitude share the same critical exponents

But ξ diverges at $t=0$

$\Rightarrow$ no scale left at $t=0$.

If the system has no characteristic scale / length, then it is invariant under a scale transformation.

⇝ as a consequence, things look the same when we change the "~~length~~ length scale resolution".

This is the idea behind renormalization group theory. We can coarse grain the physics, "eliminating" microscopic details.

Landau theory of phase transition

Phenomenological expression for the free energy ...

Landau theory assumes: phase transition can be described / characterized by order parameter

Ising model: order parameter $\hat{=}$ magnetization

$$m = 0 \quad \text{for } T > T_c$$
$$m \neq 0 \quad \text{for } T < T_c$$

Magnetization m small near phase transition.

Look at Gibbs free energy $G = G(T, P, N)$

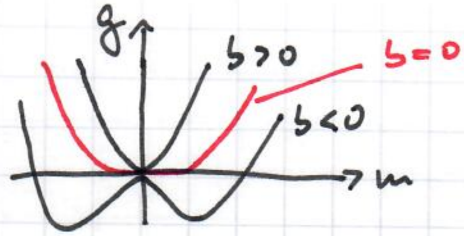
$$= E - TS + PV$$

Gibbs free energy per unit volume g:

$$g(T, m) = a(T) + \frac{b(T)}{2} m^2 + \frac{c(T)}{4} m^4 - Hm$$

assumption: g is analytic fct. of m } turns out to be wrong (but never mind...)

Want to minimize g:



$$\frac{\partial g}{\partial m} = b m + c m^3 = 0 \quad \leftarrow \text{assume } H = 0!$$

Solution: $m = 0 \rightsquigarrow$ yields minimal g if b and c are positive

$$m^2 = -\frac{b}{c}$$

$\downarrow$

assume $b = b_0 (T - T_c)$ & $c > 0$

$$\Rightarrow m = \pm \left(\frac{b_0}{c}\right)^{1/2} (T_c - T)^{1/2}$$

$$= \pm \left(\frac{b_0 T_c}{c}\right)^{1/2} \varepsilon^{1/2}$$

for $T \le T_c$

$$\varepsilon = \frac{T_c - T}{T_c}$$

this is in agreement with the mean-field result !

$$S = -\frac{\partial G}{\partial T}$$

entropy

$$\Rightarrow \quad s = -\frac{\partial g}{\partial T} = -a' - \frac{b'}{2} m^2 - \frac{b}{2}(m^2)' - \frac{c}{4}(m^4)'$$

prime derivative with respect to T

Specific heat

$$C = T\frac{ds}{dT} = -Ta'' - Tb'(m^2)' - \frac{cT}{4}(m^4)''$$

$$\Rightarrow \quad C = \begin{cases} -Ta'' & \text{for } T \to T_c^+ \quad (m=0) \\ -Ta'' + \dfrac{Tb_0^2}{2c} & \text{for } T \to T_c^- \end{cases}$$

jump in the specific heat at the critical point (again, in agreement w/ mean-field theory)