E & M I
Workshop 3 – Dipoles, 2/9/2022

The simplest model in electrostatics is, of course, that of a point charge. If, however, the net charge of an object is zero, a dipole model is the next simplest and important. This is an approximation useful for considering neutral molecules with a net $\pm$ charge separation (a permanent dipole) or neutral atoms or materials that develop a dipole in response to an electric field. We'll see this in discussions of dielectrics.

From Monday, the electric potential due to the dipole moment, or second term in the multipole expansion is:

$$\phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\vec{p} = \sum_q q\,\vec{r}_q \quad \text{(point charges)} \qquad \vec{p} = \int d^3r_q\,\rho(\vec{r}_q)\,\vec{r}_q \quad \text{(charge density } \rho(\vec{r}_q)\text{)}$$

Calculating the electric field that corresponds to the dipole potential, we got the result:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\phi^1(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\,\hat{e}_i\,\partial_i\,\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\,\partial_i\,\frac{p_j\,x_j}{(x_k\,x_k)^{3/2}}$$

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\left(\frac{1}{r^3}\,\partial_i\,p_j x_j + p_j\,x_j\,\left(-\frac{3}{2}\right)\frac{\partial_i\,(x_k\,x_k)}{(x_k\,x_k)^{\frac{5}{2}}}\right)$$

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\left(\frac{1}{r^3}\,p_j\,\delta_{ij} - \frac{3}{2}\,p_j\,x_j\,\frac{2\,x_k\,\delta_{ik}}{r^5}\right)$$

$$\vec{E}(\vec{r}) = \frac{-1}{4\pi\epsilon_0}\hat{e}_i\,\left(\frac{p_i}{r^3} - 3\,\vec{p}\cdot\vec{r}\,\frac{x_i}{r^5}\right)$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(3\,(\vec{p}\cdot\vec{r})\frac{\vec{r}}{r^2} - \vec{p}\right) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(\,3\,(\vec{p}\cdot\hat{r})\,\hat{r} - \vec{p})$$

1) Calculating Dipoles:

i) Consider the charge distribution with four charges:

q at (x = a, y = 0, z = 0), -q at (x = -a, y = 0, z = 0), q at (x = 0, y = a, z = 0), -q at (x = 0, y = -a, z = 0)

Draw a picture showing the charges. Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer.

ii) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

iii) Consider the charge distribution with 3 charges:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

Draw a picture showing the charges. Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer.

iv) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

2) Non-Uniformly Charged Line:

Consider a charged line of length $2L$ on the z-axis that extends from $z = -L$ to $z = L$. The linear charge density on the line varies as a function of z as:

$$\lambda(z) = \lambda_0 \sin\left(\frac{\pi}{2L} z\right)$$

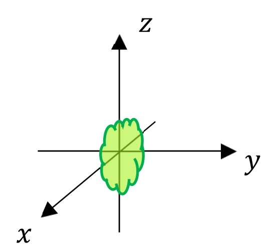
i) Draw a picture and describe what you would expect the dipole moment $\vec{p}$ to be for this charged line. Include the direction of the dipole and an estimate of the magnitude of the dipole.

ii) Solve for the dipole moment, $\vec{p}$. Compare your result to your expected result from above and explain any differences. (Use any resource you want to do the integral.)

Dipole Interactions: Potential Energy, Force, and Torque

You likely have seen the result that the potential energy of a dipole $\vec{p}$ at a position $\vec{r}$ in an electric field $\vec{E}(\vec{r})$ is

$$U_p = -\vec{p} \cdot \vec{E}(\vec{r})$$



Consider a "cloud" of charges with a distribution $\rho(\vec{r})$ confined in a very small region around $\vec{r} = 0$. Let the total charge be zero: $\int \rho(\vec{r})\, d^3r = 0$

There is an external potential $\phi(\vec{r})$ due to other sources for the potential.

The potential energy of the charged cloud is:

$$U = \int \rho(\vec{r})\, \phi(\vec{r})\, d^3r$$

If the cloud is confined to a small region relative to changes in $\phi(\vec{r})$, we can approximate the potential:

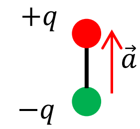
$$\phi(\vec{r}) = \phi(0) + \vec{r} \cdot \vec{\nabla}\phi(\vec{r})|_{r=0} + \frac{1}{2}\left(\vec{r} \cdot \vec{\nabla}\right)\left(\vec{r} \cdot \vec{\nabla}\right) \phi(\vec{r})|_{r=0} + \cdots$$

1) Finding the Potential Energy

i) Using the definition of the dipole potential energy and the expansion of the potential, show that the dipole potential energy is given by the second term in the expansion of the potential.

ii) The standard picture of a dipole is two charges $\pm q$ separated by a (small) vector $\vec{a}$:

By considering such a dipole in a constant electric field $\vec{E}$, explain why the potential energy makes physical sense. For example, what orientation will give the lowest energy and why?



$\vec{p} = q\vec{a}$

2) Force and Torque on a Dipole

The relation between energy and force gives the force on a dipole:

$$\vec{F} = -\vec{\nabla} U_p = \vec{\nabla}(\vec{p}\cdot\vec{E})$$

This can also be written as the directional derivative:

$$\vec{\nabla}(\vec{p}\cdot\vec{E}) = (\vec{p}\cdot\vec{\nabla})\vec{E}$$

i) As practice in doing some vector calculus, prove this last relation.

To show this, expand the double cross product:

$$\vec{p}\times(\vec{\nabla}\times\vec{E})$$

Remember that:

$$\vec{A}\times\vec{B} = \hat{e}_i\,\epsilon_{ijk}\,A_j\,B_k,\;\; \epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

and $\vec{\nabla}$ can be treated as a vector although it must operate on everything on its right.
Show all your work.

ii) The torque on a dipole can be determined by changes in energy due to rotations in $\vec{p}$. For a small rotation of the dipole by an angle $d\theta$ around a direction $\hat{n}$, the change in energy is given by the work done by the torque:

$$dU = -\vec{\tau}\cdot\hat{n}\,d\theta = -\vec{E}\cdot d\vec{p}$$

$$d\vec{p} = d\theta\,\hat{n}\times\vec{p}$$

$$\vec{\tau}\cdot\hat{n}\,d\theta = \vec{E}\cdot(\hat{n}\times\vec{p})\,d\theta$$

For another quick vector algebra problem, **show**: $\vec{E}\cdot(\hat{n}\times\vec{p}) = \hat{n}\cdot(\vec{p}\times\vec{E})$

This gives:

$$\vec{\tau} = \vec{p}\times\vec{E}$$

Again, using the standard model of a dipole, demonstrate that this expression for the torque makes physical sense for a dipole in a constant field $\vec{E}$.

3) Charge-Dipole interaction:

Consider a charge Q at the origin $\vec{r} = 0$ and a dipole $\vec{p}$ at a position $\vec{r}_p$ (not at the origin). Using the results above, calculate:

i) The potential energy of the Q & $\vec{p}$ system.

ii) Calculate the force on the dipole due to the charge Q. Compare this to the force on the charge Q due to the dipole and explain your result.

4) Dipole-Dipole interaction:

Consider a dipole $\vec{p}_1$ at the origin $\vec{r} = 0$ and a dipole $\vec{p}_2$ at a position $\vec{r}_2$ (not at the origin). Using the results above, calculate:

i) The potential energy of the two-dipole system.

ii) The torque on $\vec{p}_2$ due to $\vec{p}_1$.

Let the dipole at the origin be $\vec{p}_1 = p_1\,\hat{z}$.

iii) Explore and explain the dependence of the dipole-dipole energy on the directions of $\vec{p}_2$, and $\vec{r}_2$.

iv) Explore and explain the effect of the torque on $\vec{p}_2$, as it depends on the directions of $\vec{p}_2$, and $\vec{r}_2$.