Homework #6

1

a) $$G^{(+)}(x,x') = \langle x | \frac{1}{E - H_0 + i0_+} | x' \rangle \frac{\hbar^2}{2m}$$

$$= \frac{\hbar^2}{2m} \int dp\, dp' \, \langle x | p \rangle \langle p | \frac{1}{E - H_0 + i0_+} | p' \rangle$$

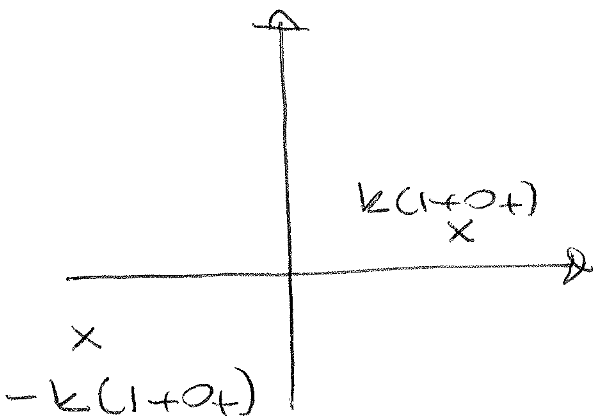
$$\times \langle p' | x' \rangle$$

$$= \int dp\, dp' \, \frac{e^{ipx}}{\sqrt{2\pi}} \, \frac{1}{k^2 - p^2 + i0_+} \, \delta(p - p') \, \frac{e^{-ip'x'}}{\sqrt{2\pi}}$$

$$= \frac{1}{2\pi} \int_0^{\infty} dp \, \frac{e^{ip(x-x')}}{k^2 - p^2 + i0_+}$$

$$= -\frac{1}{2\pi} \int_0^{\infty} dp \, \frac{e^{ip(x-x')}}{(p - p_0)(p + p_0)}$$

where $p_0 \equiv k(1+0_+)$



Now, we choose the contour in the upper half plane for $x > x'$, $\therefore$

$$G^{(+)}(x,x') = -\frac{1}{2\pi}(2\pi i)\cdot\frac{e^{ik(x-x')}}{2k} = \frac{e^{ik(x-x')}}{2ik}$$

for $x < x'$, we choose the lower half plane, where the integrand converges at $p \to \infty$, $\therefore$

$$G^{(+)}(x,x') = -\frac{1}{2\pi}(-2\pi i)\frac{e^{-ik(x-x')}}{-2k}$$

$$= \frac{e^{-ik(x-x')}}{2ik} \quad \therefore$$

$$G^{(+)}(x,x') = \frac{e^{ik|x-x'|}}{2ik}.$$

b)

$$\text{Ta} \quad V(x) = -\gamma \frac{\hbar^2}{2m} \delta(x)$$

$$\psi(x) = \phi(x) - \frac{2m}{\hbar^2} \int dx' \, G^{(+)}_{(x,x')} V(x') \psi(x')$$

$$= \underbrace{\phi(x)}_{\frac{e^{ikx}}{\sqrt{2\pi}}} + \frac{2m}{\hbar^2} \cdot \gamma \frac{\hbar^2}{2m} \cdot \frac{e^{ik|x|}}{2ik} \psi(0)$$

$$\therefore \quad \psi(x) = \frac{e^{ikx}}{\sqrt{2\pi}} + \gamma \frac{e^{ik|x|}}{2ik} \psi(0) \quad (1)$$

For $x=0$,

$$\psi(0) = \frac{1}{\sqrt{2\pi}} + \frac{\gamma}{2ik} \cdot \psi(0)$$

$$\therefore \psi(0) = \frac{1}{\sqrt{2\pi}} \frac{1}{1-\gamma/(2ik)}$$

From (1),

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\left[e^{ikx} + \frac{\gamma}{2ik-\gamma} e^{ik|x|}\right].$$

for $x>0$,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \cdot \left[e^{ikx} + \frac{\gamma}{2ik-\gamma} e^{ikx}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2ik}{2ik-\gamma} e^{ikx} \quad \therefore$$

The transmission coefficient is:

$$T(k) = \frac{2ik}{2ik - \gamma}.$$

For $x < 0$,

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\left[e^{ikx} + \frac{\gamma}{2ik-\gamma} e^{-ikx} \right]$$

$\therefore$ The reflection coefficient is:

$$R(k) = \frac{\gamma}{2ik - \gamma}.$$

c)

Solving the Schroodinger eq.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) - \frac{\gamma\hbar^2}{2m}\delta(x)\,\psi(x) = -E\,\psi(x) \quad (1)$$

the solution is:

$$\psi(x) = A\,e^{-|x|\sqrt{2mE/\hbar^2}} \quad \text{for } x \neq 0.$$

Hence, integrating (1) over the space,

$$\frac{-\hbar^2}{2m}\cdot \psi'(x)\Big|_{x=-\epsilon}^{\epsilon} - \frac{\gamma\hbar^2}{2m}\cdot\psi(0) = -E\int_{-\epsilon}^{\epsilon} dx\,\psi(x)$$

Since $\psi'(x) = -\mathrm{sign}(x)\sqrt{\dfrac{2mE}{\hbar^2}} \cdot \psi(x)$

In the limit of $\varepsilon \to 0$,

$$2\sqrt{\frac{2mE}{\hbar^2}} \cdot \psi(0) - \gamma\,\psi(0) = 0$$

$$\therefore \quad \psi(0)\left[2\sqrt{\frac{2mE}{\hbar^2}} - \gamma\right] = 0$$

$$\Longrightarrow \quad E = \frac{\hbar^2\gamma^2}{8m} = -\frac{\hbar^2 k^2}{2m}$$

$$\therefore \quad \boxed{k = \frac{i\gamma}{2}}$$

which is the condition of resonance of $T(k)$ and $R(k)$.

(2)

Defining $|0\rangle \equiv |100\rangle$ as the ground state of the Hydrogen atom, the scattering amplitude at long distances is:

$$f^{\pm}(\vec{k},\vec{k}') = -2\pi^2 \frac{2m}{\hbar^2} \cdot \langle \pm\vec{k}, 0|V|\vec{k}', 0\rangle$$

within the first Born approximation.

For

$$V(\vec{x},\vec{x}') = \underbrace{-\frac{e^2}{|\vec{x}|}}_{V_1} + \underbrace{\frac{e^2}{|\vec{x}-\vec{x}'|}}_{V_2}$$

we have:

$$f(\vec{k},\vec{k}') = +2\pi^2 \frac{2me^2}{\hbar^2}\left[\langle \vec{k},0| \frac{1}{|\vec{x}|} |\vec{k}',0\rangle \right.$$

$$\left. - \langle \vec{k},0| \frac{1}{|\vec{x}-\vec{x}'|} |\vec{k}',0\rangle \right]$$

The first term gives:

$$\langle \vec{k},0| \frac{1}{|\vec{x}|} |\vec{k},0\rangle = \int d\vec{x}\, d\vec{x}' \, \langle \vec{k},0|\vec{x},\vec{x}'\rangle \frac{1}{|\vec{x}|} \langle \vec{x},\vec{x}'|\vec{k}',0\rangle$$

$$= \int d\vec{x}\, d\vec{x}' \, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} \frac{1}{|\vec{x}|} |\langle \vec{x}'|0\rangle|^2$$

Since:

$$\langle \vec{x}' | 0 \rangle = \frac{2}{a_0^{3/2}} e^{-r'/a_0} \frac{1}{\sqrt{4\pi}}$$

$\therefore$

$$\langle \vec{k}', 0 | \frac{1}{x} | \vec{k}, 0 \rangle = \int d\vec{x} \, \frac{1}{r} \cdot \frac{e^{i\overbrace{(\vec{k}' - \vec{k})}^{\vec{q}} \cdot \vec{x}}}{(2\pi)^3}$$

$$\times \int d\vec{x}' \cdot \frac{4}{a_0^3} \cdot e^{-2r'/a_0} \cdot \frac{1}{4\pi}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dr \, r \cdot \int_{-1}^{1} d\mu \, e^{iqr\mu}$$

$$\times \int_0^\infty dr' \, r'^2 \frac{4}{a_0^3} 4\pi \, e^{-2r'/a_0} \frac{1}{4\pi}$$

D

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dr \, \frac{e^{iqr} - e^{-iqr}}{iqr}$$

$$\times \frac{16\pi}{a_0^3} \underbrace{\int_0^\infty dr' \, r'^2 \, e^{-2r'/a_0}}_{a_0^3/4} \frac{1}{4\pi}$$

$$= \frac{1}{4\pi^2 q^2}$$

The second term gives:

$$\langle \vec{k}, 0 | \frac{1}{|\vec{x} - \vec{x}'|} | \vec{k}, 0 \rangle = \int d\vec{x} d\vec{x}' \frac{1}{|\vec{x} - \vec{x}'|}$$

$$\frac{e^{i\vec{q}\cdot\vec{r}}}{(2\pi)^3} \cdot \frac{4}{a_0^3} \cdot e^{-2r'/a_0} \frac{1}{4\pi}$$

E

Defining $\vec{x}-\vec{x}' \equiv \vec{y}$

$$\langle \vec{k}, 0 | \frac{1}{|\vec{x}-\vec{x}'|} | \vec{k}', 0\rangle = \int d\vec{x}' \, \frac{4}{a_0^3} \cdot e^{-2r'/a_0} e^{i\vec{q}\cdot\vec{x}'}$$

$$\times \int d\vec{y} \cdot \frac{1}{y} \cdot \frac{e^{i\vec{q}\cdot\vec{y}}}{(2\pi)^3} \cdot \frac{1}{4\pi}$$

$$= \frac{2\pi}{(2\pi)^3} \cdot \frac{1}{q^2} \cdot \int_0^\infty dr' \, r'^2 \frac{4}{a_0^3} \cdot e^{-2r'/a_0} \frac{1}{4\pi}$$

$$\times 2\pi \int_{-1}^{1} d\mu \, e^{iqr'\mu}$$

$$= \frac{1}{2\pi q^2} \frac{1}{4\pi} \int_0^\infty dr' \, r'^2 \cdot \frac{4}{a_0^3} \cdot e^{-2r'/a_0} \underbrace{\frac{e^{iqr'} - e^{-iqr'}}{iqr'}}_{\frac{2\sin(qr')}{qr'}}$$

$\boxed{F}$

$$= \frac{1}{2\pi q^2} \cdot \frac{4}{a_0^3} \cdot \left[\frac{8 a_0^3}{\left[4 + a_0^2 q^2\right]^2} \right\} \frac{1}{4\pi}$$

$$= \frac{1}{4\pi^2 q^2} \; \frac{16}{\left[4 + q^2 a_0^2\right]^2}$$

$\therefore$

$$\sigma(q) = |f(q)|^2 = \frac{\cancel{16\pi^2}\, e^4}{\hbar^2} \cdot \frac{1}{q^4} \times$$

$$\left[1 - \frac{16}{\left[4 + q^2 a_0^2\right]^2} \right]^2$$

3

a) Since

$$E - H_0 = E - H + V$$

$$\Longrightarrow \quad G_0^{-1} = G^{-1} + V$$

$$\therefore \quad \underbrace{G_0 G_0^{-1}}_{1} G = G_0 (G^{-1} + V) G$$

$$= G_0 (1 + V G)$$

$$= G$$

b)

$$|k\rangle = (1 + G V)^{-1} |\psi\rangle$$

$$\therefore \quad |\psi\rangle = |k\rangle + G V (1 + G V)^{-1} |\psi\rangle$$

This equation is equivalent to the Lipman-Schwinger Eq. IF:

$$GV(1+GV)^{-1} = G_0 V.$$

Indeed, IF that is true,

$$GV = G_0 V(1+GV)$$

$$= \underbrace{G_0(1+VG)}_{G} V$$

According to result a).

c)

$$\langle \psi_{k'} | \psi_k \rangle = \langle \psi_{k'} | (1 + G V) | k \rangle$$

$$= \langle \psi_{k'} | \left(1 + \frac{1}{E - H + i0_+} V\right) | k \rangle$$

$$= \langle \psi_{k'} | \left(1 + \frac{1}{E - E' + i0_+} V\right) | k \rangle$$

$$= \langle \psi_{k'} | \left(1 + V \frac{1}{E - E' + i0_+}\right) | k \rangle$$

$$= \langle \psi_{k'} | \left(1 - V \frac{1}{E' - H_0 - i0_+}\right) | k \rangle$$

$$= \underbrace{\langle \psi_{k'} | (1 - V G_0^-}_{\langle k' |}) | k \rangle$$

If $|\psi_{k'}\rangle = |k\rangle + G_0^+ V |\psi_k\rangle$

$$\Longrightarrow (1 - G_0^+ V) | \psi_k \rangle = | k \rangle$$

$$\therefore \quad \langle \psi_k | (1 - V G_0^-) = \langle k |$$

Hence

$$\langle \psi_{k'} | \psi_k \rangle = \langle \psi_{k'} | (1 - V G_0^- | k \rangle$$

$$= \langle k' | k \rangle$$

$$= \delta(k - k')$$

d)

$$\langle \psi^-_{k'} | V | k \rangle = \langle k' | (1 + G^- V)^{\dagger} V | k \rangle$$

$$= \langle k' | (1 + V G^+) V | k \rangle$$

$$= \langle k' | V (1 + G^+ V) | k \rangle$$

$$= \langle k' | V | \psi^+_k \rangle$$

A

3

If the spin part of the wavefunction is symmetric, the orbital part is symmetric if the particles are bosons and antisymmetric for fermions.

The differential scattering cross section for identical particles is:

$$\sigma(\theta) = \left| f(\theta) + \sigma f(\pi - \theta) \right|^2$$

where

$$\sigma = \begin{cases} +1, & \text{for bosons} \\ -1, & \text{fermions} \end{cases}$$

In the Born Approximation,

$$f(\theta) = -\frac{2\mu}{\hbar^2} \cdot \frac{1}{q} \int_0^\infty r V(r) \sin(qr) \cdot dr$$

$\boxed{B}$

$$= -\frac{2\mu}{\hbar^2}\frac{1}{q}\int_0^\infty e^{-(r/a)^2}\cdot \sin(qr)\, dr$$

$$= -\frac{2\mu}{\hbar^2}\cdot\cancel{\frac{1}{q}}\left(\cancel{q}\cdot a^3 e^{-\frac{1}{4}a^2q^2}\frac{\sqrt{\pi}}{4}\right)$$

$$= -\frac{2\mu}{\hbar^2}\cdot\frac{\sqrt{\pi}}{4}a^3 e^{-\frac{1}{4}(aq)^2},$$

where $q = 2k\cdot\sin\theta/2$ $\therefore$

$$\sigma(\theta) = \left(\frac{2\mu}{\hbar^2}\right)^2\frac{\pi}{16}\cdot a^6$$

$$\left| e^{-a^2k^2\cdot\sin^2\theta/2} + \sigma e^{-a^2k^2\sin^2\left(\frac{\pi-\theta}{2}\right)}\right|^2$$

where μ is the reduced mass.

$\boxed{D}$

b)

$$f_\ell = \frac{-\pi}{k} T_\ell(E)$$

$$= \frac{-\pi}{k} \langle E, \ell, m | T | E, \ell, m \rangle$$

In the first Born approximation,

$T \approx U$ ∴

$$f_\ell = e^{i\delta_\ell} \sin\delta_\ell \frac{1}{k}$$

$$= \frac{-\pi}{k} \int d^3\vec{x}\, d^3\vec{x}' \langle E, \ell, m | \vec{x} \rangle \langle \vec{x} | U | \vec{x}' \rangle$$

$$\times \langle \vec{x}' | E, \ell, m \rangle$$

$$= -\frac{\pi}{k} \int d^3\vec{x} \langle E \ell m | \vec{x} \rangle U(\vec{x}) \langle \vec{x} | E \ell m \rangle$$

Since:

$$\langle \vec{x} | E, l, m \rangle = \frac{i^l}{\hbar} \sqrt{\frac{2mk}{\pi}}\, j_l(kr)\, Y_l^m(\hat{r})$$

$$f_l \equiv \frac{\pi}{k} \int dr\, r^2 \int d\Omega\, U(r)$$

$$\times \frac{2mk}{\pi \hbar^2} |j_l(rk)|^2\, Y_l^m(\hat{r})\, Y_l^{m*}(\hat{r})$$

$$= -\frac{2m}{\hbar^2} \int dr\, r^2 |j_l(kr)|^2 U(r)$$

Since δ_l is SMALL,

$$f_l \approx \frac{\delta_l}{k} \Longrightarrow$$

$$\delta_l \approx -\frac{2mk}{\hbar^2} \int dr\, r^2 |j_l(kr)|^2 U(r)$$