E & M Qualifier

January 10, 2019

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

Consider a dielectric sphere of radius R_d and dielectric constant ϵ_r surrounded by a conducting shell of inner radius R_d and outer radius R_c. The dielectric sphere has an embedded free volume charge density ρ_f. Calculate

(a) the displacement fields [3 points],

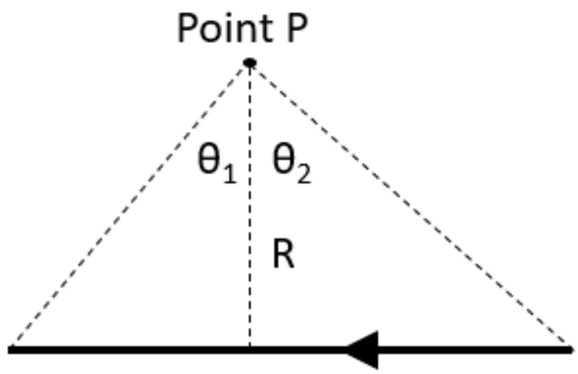
(b) the electric fields [2 points] and

(c) the polarization [2 points] everywhere in terms of the quantities given.

(d) Finally, calculate the potential at the center of the sphere relative to a point at infinity. [3 points]

Problem 2: Magnetostatics

Consider a short length of wire carrying a current I to the left, as shown in the diagram below.



(a) Use the Biot-Savart Law to show that the magnitude of the magnetic field at point P is $B = \frac{\mu_0 I}{4\pi R}(\sin\theta_1 + \sin\theta_2)$ [3 points].

(b) Calculate the vector potential $\vec{A}$ at point P [1 point].

(c) Use your result from part (b) to calculate the magnetic field at point P. Show that your result is consistent with part (a). [3 points]

(d) Use Ampere's Law to find the magnetic field at point P. [2 points]

(e) Your result to part (d) is inconsistent with your answers from parts (a) and (c). Why didn't that application of Ampere's Law work out correctly? [1 point]

Problem 3: Maxwell equations

Use $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \epsilon\mathbf{E}$, where the permittivity ϵ and permeability μ are both constant, scalar, real, and positive. For parts *a*) and *b*), make no further assumptions. For parts *c*)-*f*), assume an isotropic, homogenous, non-dispersive, dielectric medium with no free charges and no free currents ($\rho = 0$, $\mathbf{J} = 0$).

(a) Write down the four macroscopic Maxwell equations in a medium (involving $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, ρ, and $\mathbf{J}$) in differential form. Are these four equations enough to describe the interaction of EM-fields and matter? If not, state which other equation is needed, and why. [2 points]

(b) Explicitly derive the continuity equation from Maxwell's equations. What is the physical meaning of the continuity equation? [1 point]

(c) From Maxwell's equations, derive a wave equation for the electric field $\mathbf{E}(x, y, z, t)$, and a wave equation for the magnetic field $\mathbf{B}(x, y, z, t)$. Your two wave equations should be written in terms of $\mathbf{E}$, $\mathbf{B}$ and their derivatives alone, e.g. they should not contain $\mathbf{D}$, $\mathbf{H}$ or their derivatives. [1 point]

(d) Identify the speed of propagation in your two wave equations. What is the expression for the speed, c, as a function of ϵ and μ? What is it in the vacuum-limit where μ and ϵ take on their vacuum values μ_0 and ϵ_0? [1 point]

(e) Explicitly show that plane waves $\mathbf{E} = \mathbf{E_0} \exp\left(i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right)$, and $\mathbf{B} = \mathbf{B_0} \exp\left(i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right)$ are solutions of the wave equations for $\mathbf{E}$ and $\mathbf{B}$. [2 points]

(f) Use Maxwell's equations to show explicitly that $\mathbf{k}$ must be perpendicular to both $\mathbf{E_0}$ and $\mathbf{B_0}$, and that $\mathbf{E_0}$ and $\mathbf{B_0}$ must be perpendicular to each other. What is the physical meaning of these relations? [3 points]

Problem 4: EM waves

Consider an infinitely long plane wave of electromagnetic radiation with an electric field described by the equation $\mathbf{E}(z,t) = E_o \cos(kz - \omega t)\,\hat{\mathbf{x}}$. The wave is propagating in a non-magnetic dielectric medium with a refractive index of $n(k) = 1 + Ak^2$ where $A = (200/2\pi)^2$ nm^2. Assume that there are no free charges or free currents.

(a) In the equation for $\mathbf{E}(z,t)$, how are ω and k related to $n(k)$? [1 point]

(b) What is the equation for the magnetic field $\mathbf{B}(z,t)$ of the electromagnetic wave? Be explicit about how the amplitude, direction, and phase of $\mathbf{B}$ are related to those of $\mathbf{E}$. [2 points]

(c) What is the intensity of the electromagnetic wave? How much force per unit area is transported by the electromagnetic wave? [2 points]

(d) Plot the dispersion relation, ω versus k, for wavelengths between 400 nm and 800 nm. How does it differ from the dispersion relation for electromagnetic radiation in a vacuum? [2 points]

(e) Now suppose that the wave is a pulse of finite length (11 micrometers long) and appears yellow (wavelength of 600 nm). What is the velocity of the pulse in units of meters per second? How does the length of the pulse change as it travels? [3 points]

Problem 5: Radiation

The solution to the Maxwell equations in the Lorenz gauge is given by

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \int dt' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta[t' - (t - |\vec{x}-\vec{x}'|/c)]. \tag{1}$$

Assume source functions $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$ localized in space at the origin of the co-ordinate system and having harmonic time dependence $e^{-i\omega t}$. The co-ordinates $\vec{x}$ locate the observation points while $\vec{x}'$ locate the source points.

(a) Find an expression for $\vec{A}(\vec{x})$ for all space at all times (*i.e.* integrate out the time dependence). [3 points]

(b) In the near radiation zone with $r \ll \lambda$ with $k = \omega/c = 2\pi/\lambda$, then find a simplified expression for $\vec{A}(\vec{x})$. Why is this also called the static zone? [3 points]

(c) In the far or radiation zone, then $r \gg \lambda$. By expanding $|\vec{x}-\vec{x}'|$ in terms of $r \equiv |\vec{x}|$, $\hat{n} \equiv \vec{x}/|\vec{x}|$ and $\vec{x}'$, find a simplified expression for $\vec{A}(\vec{x})$. (*Hint*: $\sqrt{1-2\epsilon} \simeq 1-\epsilon$ for small ϵ.) Can you identify the outgoing spherical wave factor? [3 points]

(d) Keeping the first term in the expansion of the exponential and using the identity $\int \vec{J} d^3x = -\int \vec{x}'(\vec{\nabla}'\cdot\vec{J})d^3x'$, the equation of continuity $\nabla\cdot\vec{J} + \partial\rho/\partial t = 0$ and the definition of the electric dipole moment $\vec{p} = \int \vec{x}'\rho(\vec{x}')d^3x'$, find an expression for how $\vec{A}$ varies with $\vec{p}$ in the far zone (*i.e.* electric dipole radiation). [1 point]

Problem 6: Relativity

Recall in Gaussian units that

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2}$$

and the Hodge dual tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \equiv \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{3}$$

(a) Show that the four-volume element $d^4x \equiv dtdxdydz$ is invariant under a boost from the lab frame to the moving frame (with co-ordinates d^4x') in the x-direction. [1 point]

(b) Express Maxwell equations in terms of the electromagnetic field tensor $F^{\mu\nu}$ and four-current $j^\mu \equiv (\rho, \vec{J})$. (State which system of units you are using *e.g.* Gaussian, SI, Lorentz-Heaviside,$\cdots$). [3 points]

(c) Express the Lorentz force equation in terms of particle four-momentum p^μ, $F^{\mu\nu}$ and four-velocity u^μ. [3 points]

(d) Express $F^{\mu\nu}F_{\mu\nu}$ and $F^{\mu\nu}\tilde{F}_{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$. If we have a pure $\vec{E}$ field but $\vec{B} = 0$ in one frame, can we find another frame where $\vec{E} = 0$ and $\vec{B} \neq 0$? [3 points]