PHYS5153 Assignment 4

Due: 1:30pm on 09/24/2021.
Marking: Total of 10 marks (weighting of each question is indicated).
Fine print: Solutions should be presented legibly (handwritten or LaTeX is equally acceptable) so that the grader can follow your line of thinking and any mathematical working should be appropriately explained/described. If you provide only equations you will be marked zero. If you provide equations that are completely wrong but can demonstrate some accompanying logical reasoning then you increase your chances of receiving more than zero. If any of your solution has relied on a reference or material other than the textbook or lectures, please note this and provide details.

Question 1 (4 marks)

Consider a double pendulum, as illustrated in Fig. **??**.

(a) Find expressions for the potential and kinetic energies in terms of coordinates φ_1 and φ_2.

(b) Using your answer to (a), write down the Lagrangian and derive the corresponding Lagrange's equations of motion when $l_1 = l_2 = l$ and $m_1 = m_2 = m$.

(c) Compare and relate your results in (a) and (b) to those obtained in the paper at https://doi.org/10.1119/1.16860 (look at Appendix B).

In the remainder of the question we will look into regimes of different dynamics for the double pendulum. You may assume $l_1 = l_2 = l$ and $m_1 = m_2 = m$ throughout.

(d) Consider the case where the upper pendulum is pinned in place, i.e., $\dot{\varphi}_1 = 0$. Comment on what the equations of motion [from (b)] reduce to (i.e., what they physically correspond to).

(e) Alternatively, we can assume that the motion of each pendulum remains small and invoke a *small-angle approximation. This generically entails expanding trigonometric functions to their lowest order, e.g.,* $\sin(\theta) \approx \theta$ *and* $\cos\theta \approx 1$. *Under this approximation, solve for the possible frequencies assuming an ansatz* $\varphi_j(t) = A_j e^{i\omega t}$ *for* $j = 1, 2$ *(i.e., solve for the eigenfrequencies of the motion).*

(f) Your answer to part (e) will suggest that in a certain limit the motion of the double pendulum is exactly solvable. In this light, contrast to the results reported in Fig. 9 of the paper at https://doi.org/10.1119/1.16860. What is the key result being communicated via this figure? What does it suggest about the general motion of the double pendulum, when we drop the prior approximations?

Question 2 (2 marks)

The following is a recap problem for variational calculus.

Consider a bead constrained to slide (without friction) along a wire in 2D. The wire follows an arbitrary path between the start point $(x, y) = (x_1, y_1)$ to the end point (x_2, y_2). We will use variational calculus to determine what the fastest path between these points is assuming the bead is subject only to the force of gravity along $+y$.

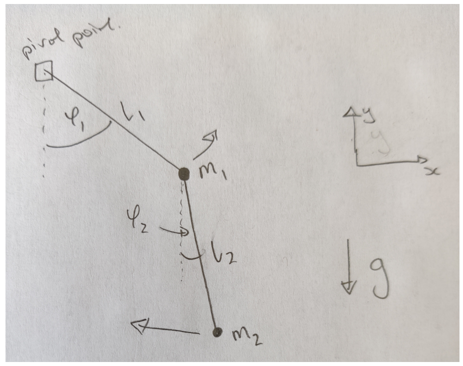

Figure 1: Double pendulum for Q1

(a) Assuming the bead is initially at rest, show that the time taken to travel between the start and end points is given by the integral,

$$t = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \sqrt{\frac{1+x'^2}{y}}, \tag{1}$$

where $x' = dy/dx$.

(b) To find the optimal path one should determine the stationary value of the integral ($\delta t = 0$) by using the Euler equation,

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0, \tag{2}$$

for $F = \sqrt{(1+x'^2)/y}/\sqrt{2g}$. Do this and show that the optimal path is a cycloid,

$$\begin{aligned} x &= \frac{c^2}{4g}\left(\theta - \sin\theta\right), \\ y &= \frac{c^2}{4g}\left(1 - \cos\theta\right) \end{aligned} \tag{3}$$

where θ simply defines the parametrization of the path and we have assumed that $(x_1, y_1) = (0, 0)$. *Note: You may find it useful to invoke the latter expression for $y(\theta)$ as a variable change when solving an integral that arises.*

Question 3 (2 marks)

Consider a generalized mechanics where the Lagrangian can be written as a function $L = L(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$ so that it additionally involves terms dependent on $\ddot{\mathbf{q}}$. Use Hamilton's principle to show that the corresponding Euler-Lagrange equation for such a system will be,

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0. \tag{4}$$

Question 4 (2 marks)

Consider a small bead of mass m that is constrained to slide frictionlessly around a hoop of radius R. The hoop rotates at fixed angular frequency ω about its vertical axis of symmetry, but is otherwise motionless (e.g., fixed in space).

(a) Assuming the system is subject to gravity, derive the equations describing the motion of the bead.

(b) Examine the equations obtained in (a) and obtain the possible scenarios for which the bead can be stationary (with respect to its angular motion about the hoop). Compare and contrast the different scenarios in terms of the value of ω.