Question 1:

a)

The Hamilton-Jacobi equation is

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$$

which for the harmonic oscillator gives,

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \quad (*)$$

To proceed, we compute the partial derivatives from $S(q,\alpha,t)$ as given:

$$\frac{\partial S}{\partial q} = \frac{m\omega q}{\tan(\omega t)} - m\omega \frac{\alpha}{\sin(\omega t)}$$

$$\frac{\partial S}{\partial t} = -\frac{m\omega^2}{2}(q^2+\alpha^2)\frac{1}{\sin(\omega t)^2} + \frac{m\omega^2 q\alpha}{\tan(\omega t)\sin(\omega t)}$$

Plugging these into $(*)$ is a bit messy. One way to proceed is to collect terms according to coefficients.

In particular, it is easy to verify that we have:

$$\text{LHS of } (*) = q^2(\ldots) + q\alpha(\ldots) + \alpha^2(\ldots)$$

Looking at each of the bracketed terms:

$$q^2 \Rightarrow \frac{1}{2}m\omega^2\left[\frac{1}{\tan^2(\omega t)} + 1 - \frac{1}{\sin(\omega t)^2}\right]$$

$$= \frac{1}{2}m\omega^2\left[\cot^2(\omega t) + 1 - \operatorname{cosec}^2(\omega t)\right]$$

$$= 0$$

$$q\alpha \Rightarrow m\omega^2\left[\frac{-\cos(\omega t)}{\sin^2(\omega t)} + \frac{\cos(\omega t)}{\sin^2(\omega t)}\right] = 0$$

$$\alpha^2 \Rightarrow \frac{m\omega^2}{2}\left[\frac{1}{\sin^2(\omega t)} - \frac{1}{\sin(\omega t)}\right] = 0$$

Thus all terms on the LHS of (*) vanish.

Hence LHS = RHS, i.e., $S(q,\alpha,t)$ is a valid solution to the Hamilton-Jacobi equation.

b) From our transformation equations we have:

$$Q = \beta = -\frac{\partial S}{\partial \alpha}$$

$$= \frac{m\omega \alpha}{\tan(\omega t)} - m\omega \frac{q\alpha}{\sin(\omega t)}$$

This can be trivially rearranged to obtain,

$$q = -\beta \frac{\sin(\omega t)}{m\omega} + \alpha \cos(\omega t)$$

This is precisely the general solution expected for a harmonic oscillator [you can verify easily by solving the Hamiltonian problem] w/

$$q(0) = \alpha \quad \& \quad p(0) = m\dot{q}(0) = \beta.$$

Question 2:

Let's start w/ the Lagrangian for a particle of mass m subject to gravity along y,

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

Formally, one could derive the associated Hamiltonian, but we will just state it:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

Note that we could immediately solve the problem at this point by noting that:

i) x is cyclic, so: $p_x(t) = p_x(0)$

+ $\dot{x} = \frac{\partial H}{\partial p_x} = p_x/m$

$\Rightarrow x(t) = v_x t + x(0)$

where $v_x \equiv \frac{p_x(0)}{m}$.

ii) $\dot{y} = p_y/m \qquad \dot{p}_y = -mg$

Solving for p_y first: $p_y(t) = -mgt + p_y(0)$
and plugging into $\dot{y}$,

$$\dot{y} = \frac{-mgt + p_y(0)}{m} \quad \Rightarrow \quad y(t) = \frac{-gt^2}{2} + \cancelto{v_y}{\frac{p_y(0)}{m}}\, t + y(0)$$

However, we would like to instead use the Hamilton-Jacobi formalism. For a principal function $S(\vec{q}, \vec{\alpha}, t)$ w/ $\vec{q} = (x, y)$, we have the Hamilton-Jacobi eqn:

$$H\left(x, y; \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) + \frac{\partial S}{\partial t} = 0$$

or,

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + 2m\, gy\right] + \frac{\partial S}{\partial t} = 0$$

To proceed we want to write down the principal function using two things:

i) separation of variables

ii) x is cyclic

This enables us to write:

$$S(\vec{q},\vec{\alpha},t) = \underbrace{p_x x} + W(y,\vec{\alpha}) - \alpha_1 t$$

w/ $\alpha_1 = E$

This comes from x being cyclic & thus $p_x = \text{const.}$

Plugging in our

$$\left(\frac{\partial W}{\partial y}\right)^2 = 2mE - p_x^2 - 2m^2 g y$$

We can obtain $W(y,\alpha)$ by formal integration to obtain (up to a constant):

$$S(\vec{q},\vec{\alpha},t) = p_x x - \frac{(2mE - p_x^2 - 2m^2 g y)^{3/2}}{3m^2 g} - Et$$

Note here that our constants of integration $\vec{\alpha}$ are:

$$\alpha_1 = E$$
$$\alpha_2 = p_x$$

Now, as a result of the Hamilton-Jacobi construction, we have the new canonical variables $\vec{Q}$ + $\vec{P}$ that satisfy:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \qquad + \qquad \dot{P}_i = \frac{-\partial H}{\partial Q_i} = 0$$

such that:

$$P_1 = \alpha_1 = E$$

$$P_2 = \alpha_2 = p_x$$

and, using our transformation equations:

$$\text{(a)} \qquad Q_1 = \beta_1 = \frac{\partial S}{\partial E} = -t - \frac{1}{mg}\left[2mE - p_x^2 - 2m^2gy\right]^{1/2}$$

$$\text{(b)} \qquad Q_2 = \beta_2 = \frac{\partial S}{\partial p_x} = x + \frac{p_x}{m^2 g}\left[2mE - p_x^2 - 2m^2gy\right]^{1/2}$$

We can use Eq (a) to obtain,

$$-mg(t+\beta_1) = \left(2mE - p_x^2 - 2m^2gy\right)^{1/2}$$

This enables us to rewrite (b):

$$\beta_2 \equiv x - \frac{p_x}{m}(t + \beta_1)$$

or

$$x = \frac{p_x}{m}(t + \beta_1) + \beta_2$$

We can similarly rearrange (a) for y:

$$(a) \rightarrow \quad y = \frac{1}{2m^2 g}\left[2mE - p_x^2\right] - \frac{g}{2}(t+\beta_1)^2$$

How do we wrangle these into the final form we want? Clearly we need to make some replacements. First, given conservation of energy:

$$2mE - p_x^2 \equiv p_y(0)^2 + 2m^2 g y(0)$$

Thus:

$$y \equiv y(0) + \frac{v_y^2 - g^2 \beta_1}{2g} - g\beta_1 t - \frac{1}{2} g t^2$$

By identifying:

$$\beta_1 = -v_y/g$$

$$\beta_2 = x(0) + \frac{v_x v_y}{g}$$ (from initial conditions)

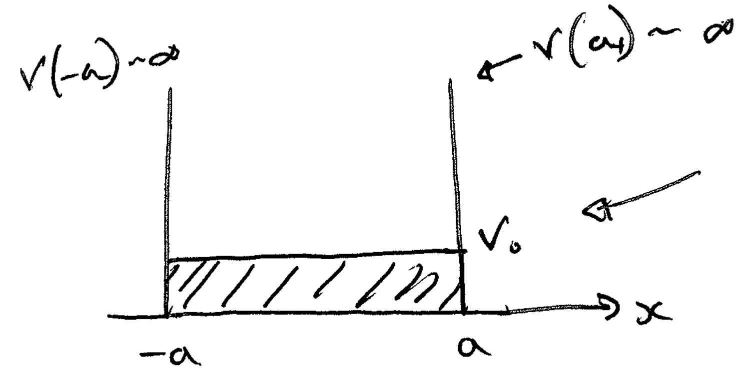
then we finally obtain,

$$x = v_x t + x(0)$$

& $$y = -\frac{1}{2}gt^2 + v_y t + y(0)$$!!

Question 3:

This problem is simply that of a free particle bouncing back and forth between a pair of hard walls.



Clearly need $E > V_0$ for particle.

For $|x| \leq a$: $\quad \frac{p^2}{2m} + V_0 = H = E$

ie., $\quad p = \pm\sqrt{2m(E-V_0)}$

the $+/-$ branches correspond to particle travelling left/right between bouncing off the walls.

To compute the action,

$$J = \oint p \, dx$$

The turning points are clearly $x = \pm a$,

$$\Rightarrow \quad J = 2\int_{-a}^{a} \sqrt{2m(E-V_0)}\;dx$$

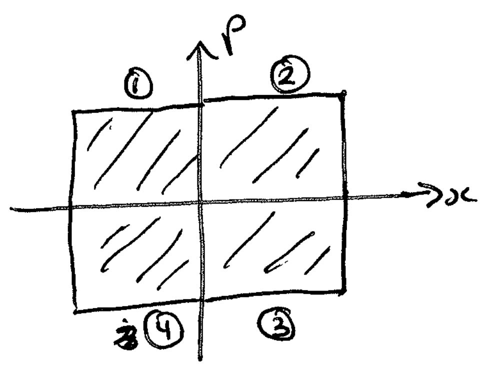
where does the 2 come from:

$$\oint p\,dx = \underbrace{\int_{-a}^{0} +\sqrt{\cdots}\;dx}_{1} + \underbrace{\int_{0}^{a} +\sqrt{}\;dx}_{2}$$

$$+ \underbrace{\int_{a}^{0} -\sqrt{}\;dx}_{3} + \int_{0}^{-a} -\sqrt{}\;dx$$

in phase-space:

$\oint p\,dx =$



Evaluating the integral,

$$J = 4a\sqrt{2m(E-V_0)}$$

We can then rewrite the Hamiltonian in terms of J.

$$J = 4a\sqrt{2m(E-V_0)} \longrightarrow E = \frac{J^2}{32ma^2} + V_0 = H.$$

Then, the period is:

$$\nu = \frac{\partial H}{\partial J} = \frac{J}{16ma^2}$$

But also, $|p(0)| = \frac{J}{4a}$ from our earlier expression.

So,

$$\nu \equiv \frac{|p(0)|}{4ma} \quad \text{or period } T = \frac{4ma}{|p(0)|}$$

This answer makes sense as we alternatively have (from projectile motion):

dist travelled in one period.

$$\text{time} = \frac{\text{distance}}{\text{velocity}} = \frac{4a}{\left(\frac{|p(0)|}{m}\right)}$$

velocity between walls.

$$= \frac{4ma}{|p(0)|} \quad \checkmark$$

Question 4:

a) The kinetic contribution is,

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

$$\dot{y} = l\,\dot{\phi}\sin\phi \qquad \dot{x} = l\left(\dot{\phi} + \dot{\phi}\cos\phi\right)$$

$$\therefore\ T = \frac{ml^2}{2}\left[\left(\dot{\phi}\sin\phi\right)^2 + \left(\dot{\phi} + \dot{\phi}\cos\phi\right)^2\right]$$

$$= ml^2\,\dot{\phi}^2\left[1+\cos\phi\right] \quad \text{or} \quad 2ml^2\,\dot{\phi}^2\cos^2(\phi/2)$$

The potential term due to gravity is,

$$V = mgy = mgl\,(1-\cos\phi)$$

or

$$2mgl\,\sin^2(\phi/2)$$

$$\Rightarrow \quad L = T - V.$$

b) We want to compute the Hamiltonian,

$$H = p_\phi \dot{\phi} - L$$

First,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 4ml^2 \dot{\phi} \cos^2(\phi/2)$$

Thus,

$$H = 4ml^2 \dot{\phi}^2 \cos^2(\phi/2) - 2ml^2 \dot{\phi}^2 \cos^2(\phi/2) + 2mgl \sin^2(\phi/2)$$

$$= 2ml^2 \dot{\phi}^2 \cos^2(\phi/2) + 2mgl \sin^2(\phi/2)$$

$$(T + V)$$

Using,

$$\dot{\phi} = \frac{p_\phi}{4ml^2 \cos^2(\phi/2)}$$

Then,

$$H = \frac{p_\phi^2}{8ml^2 \cos^2(\phi/2)} + 2mgl \sin^2(\phi/2)$$

c) We have that $H = E = T + V$ is conserved

because $\frac{\partial H}{\partial t} = 0$.

Then; from b)

$$E = \frac{p_\phi^2}{8ml^2 \cos^2(\phi/2)} + 2mgl \sin^2 \phi/2$$

$$\Rightarrow \quad p_\phi = \pm \, 8ml^2 E \cos(\phi/2) \sqrt{1 - \frac{2mgl}{E} \sin^2(\phi/2)}$$

d) First to compute the action, we will need the turning points (ϕ_0) of the motion. These are defined by,

$$E = V(\phi_0) \quad \text{in our case,}$$

which gives: $E = 2mgl \sin^2\left(\frac{\phi_0}{2}\right)$

or $\phi_0 = 2 \operatorname{asin}\left[\sqrt{\frac{2mgl}{E}}\right]$.

Then, the action is:

$$J = \oint p_\phi \, d\phi$$

$$= 2 \int_{-\phi_0}^{\phi_0} \sqrt{8mL^2 E} \cos\left(\frac{\phi}{2}\right) \left[1 - \frac{2mgL}{E} \sin^2\left(\frac{\phi}{2}\right)\right]^{1/2} d\phi$$

(see Q2)

Define $u = \sqrt{\frac{2mgL}{E}} \sin(\phi/2)$

such that,

$$du = \frac{1}{2} \sqrt{\frac{2mgL}{E}} \cos(\phi/2) \, d\phi$$

and so,

$$J = 8 \sqrt{\frac{L}{g}} E \int_{-u_0}^{u_0} du \sqrt{1-u^2}$$

Now, $u_0 = \sqrt{\frac{2mgL}{E}} \sin(\phi_0/2)$, but from the prior calculation of turning points, $u_0 = 1$!

$\cancel{u_0 = 1}$.

Then,

$$J = 8\sqrt{\frac{L}{g}}\, E \underbrace{\int_{-1}^{1} du \sqrt{1-u^2}}_{=\pi/2}$$

$$= 4\pi \sqrt{\frac{L}{g}}\, E$$

Re-expressing the Hamiltonian as $H(J)$,

$$H = E = \frac{J}{4\pi}\sqrt{\frac{g}{L}}$$

which enables us to compute the frequency:

$$\nu = \frac{\partial H}{\partial J} = \frac{1}{4\pi}\sqrt{\frac{g}{L}} .$$

Clearly, this result is independent of the initial condition