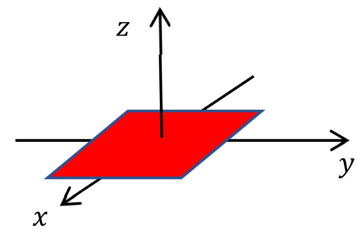
E & M I
Workshop 1 – Fun with Gauss, 1/26/2022

In class on Monday, we solved for the electric potential and field of a uniformly charged sphere the "hard way". We can, of course, do this problem using Gauss' Law without much thought at all. However, we need to be very careful about problems that we *think* we can do without much thought.

In this workshop, we'll take a close look at using Gauss' Law as a calculational tool.

1) Using Symmetry properties:



Consider a large, thin, uniform slab of positive charge lying in the x-y plane. We'll model this as having an infinite area, zero thickness, and a charge per unit area σ. The electric field can be written in the most general form as:

$$\vec{E}(x, y, z) = E_x(x, y, z)\hat{x} + E_y(x, y, z)\hat{y} + E_z(x, y, z)\hat{z}$$

We can use the symmetry properties of the physical problem (the slab) to greatly simplify this expression.

a) Translations and Rotations:

Consider the coordinate translations: $(x, y, z) \to (x + a, y, z)$ and $(x, y, z) \to (x, y + a, z)$; And the rotations: $\vec{r} \to R_{\hat{z},\phi}\,\vec{r}$ where $R_{\hat{z},\phi}$ is a rotation about the z-axis by an angle ϕ.

i) How will these transformations change the problem?

ii) What does this imply about the electric field, $\vec{E}(x, y, z)$? Use these symmetries to rewrite a somewhat simplified expression for the electric field. Explain your result.

b) Inversions:

Consider the inversions in the x and y coordinates: $\hat{x} \to -\hat{x}$ and $\hat{y} \to -\hat{y}$

i) How will these inversions change the problem?

ii) Show that this results in a very simple, but expected, form for the electric field, $\vec{E}(\vec{r})$. Explain your results clearly. Using a picture to visualize the inversions is highly encouraged.

c) Scaling:

Consider the scaling of the coordinates: $\vec{r} \to \alpha\,\vec{r}$ where α is any number.

How will scaling the coordinate change the problem?

Show that this simplifies $\vec{E}(\vec{r})$ to something *really* simple.

2) The "Gaussian Slab":

I'm sure you have all done this many times, but let's solve for the infinite slab's electric field using Gauss' Law and the Divergence Theorem:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\iiint_V \vec{\nabla} \cdot \vec{F}(\vec{r}) d^3x = \oiint_{S=\delta V} \vec{F}(\vec{r}) \cdot \hat{n}\, dS \quad \text{for a vector field } \vec{F}(\vec{r}) \text{ in a volume } V$$

a) Draw a picture(s) and explain how you will apply the two expressions above to solve for the electric field of the slab. Be sure to describe the volume V and the surface S you will use in the integrals above.

b) Write out the integrals needed to determine the electric field. Be clear how you are doing the surface integral.

c) Solve for the electric field due to the slab. Explain why this model is an approximation and not physical.

3) Spherical symmetry and Gauss:

The general form for the electric field in spherical coordinates is, of course:

$$\vec{E}(r,\theta,\phi) = E_r(r,\theta,\phi)\,\hat{r} + E_\theta(r,\theta,\phi)\,\hat{\theta} + E_\phi(r,\theta,\phi)\,\hat{\phi}$$

a) Using symmetry arguments like those in Question 1 above for a spherically symmetric charge density:

$$\rho(\vec{r}) = \rho(r), \qquad r = \vec{r} \cdot \hat{r}$$

provide a convincing argument to write down a much simpler form for the electric field. Remember, you need to be convincing and complete!

b) Using Gauss' Law and the Divergence Theorem, determine an integral (a 1D integral) that can be solved for the electric field, $E_r(r)$, for any symmetric charge distribution, $\rho(r)$.

4) A Classical "Hydrogen Atom":

If we consider the electron's probability density in a hydrogen atom as a static, classical charge density (I know, quite a stretch) we can make some classical predictions about the atom. We'll start here by finding the electric field due to the charge distribution and will take this a bit farther in a homework assignment.

Consider a charge distribution of the form:

$$\rho(r) = \rho_0\, e^{-\alpha r}$$

The charge distribution has a total charge Q and $\alpha = \frac{2}{a_0}$ where a_0 is the length scale in the problem. Of course, for a hydrogen atom, $Q = -e$ and a_0 is the Bohr radius, but we'll use Q and a_0 in this problem.

a) Determine what ρ_0 needs to be in terms of Q and a_0. Be sure to explain how you determined your answer.

Hint: There is a standard trick for doing these sorts of integrals:

$$\int x^n\, e^{-\alpha x}\, dx = (-1)^n \frac{\partial^n}{\partial \alpha^n} \int e^{-\alpha x}\, dx$$

Of course, if you're not in a test (or qualifier) you can always pull up Mathematica, Wolfram Alpha, or whatever python library you might use.

b) Using your result from Question 3b, calculate the electric field everywhere for the charge cloud.

c) As always, we need to check our results. Show that your answer to 4b is correct in the limit that r gets large.

NOTE: This means more than finding what happens if $r = \infty$. Check the functional behavior in the limit as $r \to \infty$.

d) Check that your result to 4b also is well behaved in the limit as $r \to 0$. Remember that this charge distribution is NOT due to a point charge, just the charged cloud.

5) What the heck, we probably won't get here but consider a "Fuzzy Slab":

Going back to the Slab, determine the electric field due to an infinite charged slab lying in the x-y plane with a charge distribution:

$$\rho(\vec{r}) = \rho(z) = \rho_0\, e^{-\alpha |z|}$$

Note that this result disagrees with our symmetry arguments from Questions 1a, 1b, and 1c. Which of the symmetry arguments doesn't work for this case and why?