Solutions to Homework 5
Physics 5393

Sakurai

P-1.33 In the main text we discussed the effect of $\mathscr{J}(dx')$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle\tilde{\mathbf{x}}\rangle$ and $\langle\tilde{\mathbf{p}}\rangle$ under infinitesimal translations. Using (1.6.25), (1.6.45), and $|\alpha\rangle \to \mathscr{J}(dx')\,|\alpha\rangle$ only, prove $\langle\tilde{\mathbf{x}}\rangle \to \langle\tilde{\mathbf{x}}\rangle + dx'$, $\langle\tilde{\mathbf{p}}\rangle \to \langle\tilde{\mathbf{p}}\rangle$ under infinitesimal translations.

To solve this problem, start with the following commutation relation

$$\left[\tilde{\mathbf{x}}, \mathscr{J}(dx')\right] = dx'$$

and apply the adjoint infinitesimal translation operator on it

$$\mathscr{J}^\dagger\left[\tilde{\mathbf{x}}, \mathscr{J}\right] = \mathscr{J}^\dagger\tilde{\mathbf{x}}\mathscr{J} - \tilde{\mathbf{x}} \quad\Rightarrow\quad \mathscr{J}^\dagger\tilde{\mathbf{x}}\mathscr{J} = \tilde{\mathbf{x}} + dx'.$$

Next calculate the expectation value

$$\left\langle \mathscr{J}^\dagger\tilde{\mathbf{x}}\mathscr{J}\right\rangle = \left\langle \mathscr{J}^\dagger\tilde{\mathbf{x}}\mathscr{J}\right\rangle + dx' \quad\Rightarrow\quad \langle\tilde{\mathbf{x}}\rangle \to \langle\tilde{\mathbf{x}}\rangle + dx'.$$

Now repeat this for an infinitesimal translation of the expectation value of the momentum operator. In this case the two operators commute

$$\left[\tilde{\mathbf{p}}, \mathscr{J}(dx')\right] = 0.$$

Therefore, $\langle\tilde{\mathbf{p}}\rangle \to \langle\tilde{\mathbf{p}}\rangle$; the expectation value of the momentum operator does not change under translations.

P-1.34 Starting with a momentum operator *pop* having eigenstates $|p'\rangle$, define an infinitesimal boost operator $\tilde{\mathscr{B}}(dp')$ that changes one momentum eigenstate into another, that is

$$\tilde{\mathscr{B}}(d\vec{\mathbf{p}}')\,\left|\vec{\mathbf{p}}'\right\rangle = \left|\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\right\rangle$$

Show that the form $\tilde{\mathscr{B}}(d\vec{\mathbf{p}}') = 1 + i\tilde{\vec{\mathbf{W}}}\cdot d\vec{\mathbf{p}}'$, where $\tilde{\vec{\mathbf{W}}}$ is Hermitian, satisfies the unitarity, associative, and inverse properties that are appropriate for $\tilde{\mathscr{B}}(d\vec{\mathbf{p}}')$. Use dimensional analysis to express $\tilde{\vec{\mathbf{W}}}$ in terms of the position operator $\tilde{\vec{\mathbf{x}}}$, and show that the result satisfies the canonical commutation relations $\left[\tilde{\vec{\mathbf{x}}}_i, \tilde{\vec{\mathbf{p}}}_j\right] = -\hbar\delta_{ij}$. Derive an expression for the matrix element $\left\langle \tilde{\vec{\mathbf{p}}}' \middle| \tilde{\vec{\mathbf{x}}}' \middle| \alpha \right\rangle$ in terms of a derivative with respect to $\vec{\mathbf{p}}'$ of $\langle\vec{\mathbf{p}}'\,|\alpha\rangle$.

The solution to this problem follows the outline used to derive the properties of the translation operator. In the following discussion, only terms up to first order in $d\vec{\mathbf{p}}'$ are kept and $\tilde{\vec{\mathbf{W}}}$ is take to be Hermitian.

- **Unitary operator:**

$$\tilde{\mathscr{B}}^\dagger(d\vec{\mathbf{p}}')\tilde{\mathscr{B}}(d\vec{\mathbf{p}}') = \left(1 + i\tilde{\vec{\mathbf{W}}}\cdot d\vec{\mathbf{p}}'\right)\left(1 - i\tilde{\vec{\mathbf{W}}}\cdot d\vec{\mathbf{p}}'\right) = 1.$$

- **Associative:**
$$\tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\tilde{\mathcal{B}}(d\vec{\mathbf{p}}'') = \left(1 - i\tilde{\vec{\mathbf{W}}}\cdot d\vec{\mathbf{p}}'\right)\left(1 - i\tilde{\vec{\mathbf{W}}}\cdot d\vec{\mathbf{p}}''\right) = 1 - i\tilde{\vec{\mathbf{W}}}\cdot\left(d\vec{\mathbf{p}}' + d\vec{\mathbf{p}}''\right) = \tilde{\mathcal{B}}(d\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'').$$

- **Inverse:**
$$\tilde{\mathcal{B}}(-d\vec{\mathbf{p}}')\tilde{\mathcal{B}}(d\vec{\mathbf{p}}') = 1.$$

From dimensional analysis, the operator must have dimensions of an inverse momentum. Hence, using the de Broglie relation $W = \text{Length}/\hbar$ and $\tilde{\mathcal{B}}(d\vec{\mathbf{p}}') = 1 + i\tilde{\vec{\mathbf{x}}}\cdot d\vec{\mathbf{p}}'/\hbar$.

The commutation relation is derived as for the position translation case by calculating the commutator $\left[\tilde{\vec{\mathbf{p}}}, \tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\right]$ two different ways and equating them. The first approach is to apply the operators directly on the eigenket
$$\begin{aligned}\left[\tilde{\vec{\mathbf{p}}}, \tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\right]\left|\vec{\mathbf{p}}'\right\rangle &= \left(\tilde{\vec{\mathbf{p}}}\tilde{\mathcal{B}}(d\vec{\mathbf{p}}') - \tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\tilde{\vec{\mathbf{p}}}\right)\left|\vec{\mathbf{p}}'\right\rangle \\ &= \left(\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\right)\left|\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\right\rangle - \vec{\mathbf{p}}'\left|\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\right\rangle \\ &= d\vec{\mathbf{p}}'\left|\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\right\rangle \\ &\approx d\vec{\mathbf{p}}'\left|\vec{\mathbf{p}}'\right\rangle\end{aligned}$$

The second method is to use the form of the infinitesimal momentum operator
$$\begin{aligned}\left[\tilde{\vec{\mathbf{p}}}, \tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\right]\left|\vec{\mathbf{p}}'\right\rangle &= \tilde{\vec{\mathbf{p}}}'\left(1 + i\tilde{\vec{\mathbf{x}}}\cdot d\vec{\mathbf{p}}'/\hbar\right)\left|\vec{\mathbf{p}}'\right\rangle - \left(1 + i\tilde{\vec{\mathbf{x}}}\cdot d\vec{\mathbf{p}}'/\hbar\right)\tilde{\vec{\mathbf{p}}}'\left|\vec{\mathbf{p}}'\right\rangle \\ &= \frac{i}{\hbar}\left[\tilde{\vec{\mathbf{p}}}, \tilde{\vec{\mathbf{x}}}\cdot d\vec{\mathbf{p}}'\right]\left|\vec{\mathbf{p}}'\right\rangle.\end{aligned}$$

Equating the two equations
$$dp'_i = \frac{i}{\hbar}\left(\tilde{\mathbf{p}}_i\sum_j \tilde{\mathbf{x}}_j\, dp'_j - \sum_j \tilde{\mathbf{x}}_j\, dp'_j\tilde{\mathbf{p}}_i\right) \quad\Rightarrow\quad [\tilde{\mathbf{x}}_j, \tilde{\mathbf{p}}_i] = i\hbar\delta_{ij},$$

where the commutator is derived in the last step by equating the coefficients of dp_j on the two sides of the equation.

To derive the position operator in momentum space, we follow the procedure used for the momentum operator in coordinate space
$$\left\langle\vec{\mathbf{p}}'\left|\tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\right|\alpha\right\rangle = \left\langle\vec{\mathbf{p}}' + d\vec{\mathbf{p}}'\,|\alpha\right\rangle = \left\langle\vec{\mathbf{p}}'\,|\alpha\right\rangle + \sum_j\frac{\partial\left\langle\vec{\mathbf{p}}'\,|\alpha\right\rangle}{\partial p'_j}\, dp'_j.$$

We can also use the explicit form of the momentum translation operator
$$\left\langle\vec{\mathbf{p}}'\left|\tilde{\mathcal{B}}(d\vec{\mathbf{p}}')\right|\alpha\right\rangle = \left\langle\vec{\mathbf{p}}'\,|\alpha\right\rangle + \frac{i}{\hbar}\sum_j\left\langle\vec{\mathbf{p}}'\,|x_j|\,\alpha\right\rangle dp'_j$$

Equating the two equations, we arrive at the position operator in momentum space
$$\frac{i}{\hbar}\left\langle\vec{\mathbf{p}}'\,|x_j|\,\alpha\right\rangle = \frac{\partial\left\langle\vec{\mathbf{p}}'\,|\alpha\right\rangle}{\partial p'_j} \quad\Rightarrow\quad \left\langle\vec{\mathbf{p}}'\left|\tilde{\vec{\mathbf{x}}}\right|\alpha\right\rangle = -i\hbar\vec{\boldsymbol{\nabla}}_{\vec{\mathbf{p}}'}\left\langle\vec{\mathbf{p}}'\,|\alpha\right\rangle.$$

P-1.35 The Gaussian wave packet.

(a) Verify (1.271a) and (1.271b) for the expectation value of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{p}}^2$ from the Gaussian wave packet (1.267).

These are relatively straightforward integrals to calculate. We start with the momentum expectation value

$$\begin{aligned}\langle\tilde{\mathbf{p}}\rangle &= \int_{-\infty}^{\infty} dx' \,\left\langle \alpha \left| x' \right.\right\rangle \left\langle x' \left|\tilde{\mathbf{p}}\right| \alpha \right\rangle = \int_{-\infty}^{\infty} dx' \,\left\langle \alpha \left| x' \right.\right\rangle \left(-i\hbar \frac{d}{dx'}\right) \left\langle x' \left| \alpha \right.\right\rangle \\ &= \frac{\hbar k}{d\sqrt{\pi}} \int_{-\infty}^{\infty} dx' \, \exp\left(\frac{-x'^2}{d^2}\right) = \hbar k,\end{aligned}$$

where

$$\left\langle x' \left| \alpha \right.\right\rangle = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right] \exp\left[ikx' - \frac{x'^2}{2d^2}\right]$$

and use was made of

$$e^{-ikx'}\left(-i\hbar\frac{d}{dx'}\right)e^{ikx'} = \hbar k.$$

The expectation value of $\left\langle\tilde{\mathbf{p}}^2\right\rangle$ is derived as follows

$$\begin{aligned}\left\langle\tilde{\mathbf{p}}^2\right\rangle &= -\hbar^2 \int_{-\infty}^{\infty} \left\langle \alpha \left| x' \right.\right\rangle \left[\frac{d^2}{dx'^2}\right] \left\langle x' \left| \alpha \right.\right\rangle \, dx' \\ &= -\frac{\hbar^2}{d\sqrt{\pi}} \int_{-\infty}^{\infty} \left[-\frac{1}{d^2} + \left(-ik - \frac{x'}{d^2}\right)^2\right] \exp\left(-\frac{x'^2}{d^2}\right) \, dx' \\ &= \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{d^5\sqrt{\pi}} \int_{-\infty}^{\infty} x'^2 \exp\left(-\frac{x'^2}{d^2}\right) \, dx' \\ &= \hbar^2\left[\frac{1}{d^2} + k^2\right] - \frac{\hbar^2}{2d^2} = \frac{\hbar^2}{2d^2} + \hbar^2 k^2\end{aligned}$$

(b) Evaluate the expectation value of $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{p}}^2$ using the momentum-space wave function (1.7.42).

In this case, we use the momentum representation of the Gaussian wave packet

$$\left\langle p \left| \alpha \right.\right\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2}\right].$$

The expectation value $\langle\tilde{\mathbf{p}}\rangle$ is

$$\begin{aligned}\langle\tilde{\mathbf{p}}\rangle = \left\langle \alpha \left|\tilde{\mathbf{p}}\right| \alpha \right\rangle &= \int_{-\infty}^{\infty} \left\langle \alpha \left|\tilde{\mathbf{p}}\right| p' \right\rangle \left\langle p' \left| \alpha \right.\right\rangle \, dp' = \int_{-\infty}^{\infty} p' \left|\left\langle p' \left| \alpha \right.\right\rangle\right|^2 \, dp' \\ &= \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} p' \exp\left[-\frac{(p' - \hbar k)^2 d^2}{\hbar^2}\right] \, dp' \\ &= \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} (q - \hbar k) \exp\left[-\frac{q^2 d^2}{\hbar^2}\right] \, dq = \hbar k,\end{aligned}$$

where a change of variables was performed to arrive at the final integral; $q = (p' - \hbar k)$.
The expectation value of $\left\langle \tilde{\mathbf{p}}^2 \right\rangle$ is derived as follows

$$\left\langle \tilde{\mathbf{p}}^2 \right\rangle = \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} (q - \hbar k)^2 \exp\left[-\frac{q^2 d^2}{\hbar^2} \right] \, dq = \frac{\hbar^2}{2d^2} + \hbar^2 k^2,$$

where we only show the last steps as all the others are the same as in the calculation of $\langle \tilde{\mathbf{p}} \rangle$ except for the replacement of $\tilde{\mathbf{p}}$ with $\tilde{\mathbf{p}}^2$.

P-1.36 Momentum wavefunctions

(a) Prove the following:

i. $\langle p' |\tilde{\mathbf{x}}| \alpha \rangle = -i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle$

We proceed in two steps. First, we expand the expression in a complete set

$$\left\langle p' |\tilde{\mathbf{x}}| \alpha \right\rangle = \int \left\langle p' |\tilde{\mathbf{x}}| p'' \right\rangle \left\langle p'' | \alpha \right\rangle \, dp''.$$

Next, we evaluate the first term, which we then apply to the second term.
In order to extract the position operator, we expand the inner product using the completeness relation

$$\begin{aligned} \left\langle p' |\tilde{\mathbf{x}}| p'' \right\rangle &= \int \left\langle p' |\tilde{\mathbf{x}}| x' \right\rangle \left\langle x' | p'' \right\rangle \, dx' = \int x' \left\langle p' | x' \right\rangle \left\langle x' | p'' \right\rangle \, dx' \\ &= \frac{1}{2\pi\hbar} \int x' \exp\left[-i \frac{(p' - p'')x'}{\hbar} \right] \, dx' \\ &= \frac{i}{2\pi} \frac{\partial}{\partial p'} \int \exp\left[-i \frac{(p' - p'')x'}{\hbar} \right] \, dx' = i\hbar \frac{\partial}{\partial p'} \delta(p' - p''). \end{aligned}$$

Finally, we apply this to the second term in the original integral

$$\begin{aligned} \left\langle p' |\tilde{\mathbf{x}}| \alpha \right\rangle &= \int \left\langle p' |\tilde{\mathbf{x}}| p'' \right\rangle \left\langle p'' | \alpha \right\rangle \, dp'' \\ &= i\hbar \int \frac{\partial}{\partial p''} \delta(p' - p'') \left\langle p'' | \alpha \right\rangle \, dp'' = i\hbar \frac{\partial}{\partial p'} \left\langle p' | \alpha \right\rangle. \end{aligned}$$

ii. $\langle \beta |\tilde{\mathbf{x}}| \alpha \rangle = \int dp' \phi^*_\beta(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p')$

To prove this identity, we apply the completeness relation and then the result of previous part of the problem. First the application of the completeness relation

$$\langle \beta |\tilde{\mathbf{x}}| \alpha \rangle = \int \, dp' \left\langle \beta | p' \right\rangle \left\langle p' |\tilde{\mathbf{x}}| \alpha \right\rangle = \int \, dp' \phi^*_\beta(p') \left[i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p') \right].$$

(b) What is the physical significance of

$$\mathcal{J}(\Xi) \equiv \exp\left(\frac{i\Xi\tilde{\mathbf{x}}}{\hbar} \right),$$

where $\tilde{\mathbf{x}}$ is the position operator and Ξ is some number of dimension of momentum? Justify your answer.

Given its similarity to the position translation operator, this one is a momentum translation operator. This can be seen in that the roles of position and momentum are reversed in the two operators.

To show that this makes sense, let's apply this operator, $\mathcal{J}(\Xi)$, on an eigenket of momentum to produce a new ket and then apply the momentum operator to calculate the "new" momentum as follows

$$\begin{aligned}\tilde{\mathbf{p}}\left[\mathcal{J}(\Xi)\left|p'\right\rangle\right] &= \{\mathcal{J}(\Xi)\tilde{\mathbf{p}} + [\tilde{\mathbf{p}}, \mathcal{J}(\Xi)]\}\left|p'\right\rangle \\ &= \left\{p'\mathcal{J}(\Xi) - i\hbar\frac{\partial}{\partial x}\mathcal{J}(\Xi)\right\}\left|p'\right\rangle \\ &= (p' - \Xi)\left[\mathcal{J}(\Xi)\left|p'\right\rangle\right],\end{aligned}$$

where the result of Problem 1.29 is used to calculate the commutation relation. This then shows the momentum operator acting on the ket $\mathcal{J}(\Xi)\left|p'\right\rangle$ has eigenvalues of a momentum translated by Ξ. *This last part can also be carried out using the infinitesimal version of the operator.*

Additional Problems

Q-1 Consider the Hamiltonian H of a particle in one-dimensional problem defined by:

$$H = \frac{1}{2m}\tilde{\mathbf{P}}^2 + V(\tilde{\mathbf{X}})$$

where $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{X}}$ are operators defined in section E of chapter 2 and which satisfy the relation: $[\tilde{\mathbf{X}}, \tilde{\mathbf{P}}] = i\hbar$. The eigenvectors of H are denoted by $\left|\phi_n\right\rangle$: $H\left|\phi_n\right\rangle = E_n\left|\phi_n\right\rangle$, where n is a discrete index.

a) Show that:

$$\left\langle\phi_n\left|\tilde{\mathbf{P}}\right|\phi_{n'}\right\rangle = \alpha\left\langle\phi_n\left|\tilde{\mathbf{X}}\right|\phi_{n'}\right\rangle$$

where α is a coefficient which depends on the difference between E_n and $E_{n'}$. Calculate α:

Start with $[\tilde{\mathbf{X}}, \tilde{\mathbf{H}}]$;

$$\begin{aligned}[\tilde{\mathbf{X}}, \tilde{\mathbf{H}}] &= \frac{1}{2m}[\tilde{\mathbf{X}}, \tilde{\mathbf{P}}^2] \\ &= \frac{1}{2m}\left\{\tilde{\mathbf{P}}[\tilde{\mathbf{X}}, \tilde{\mathbf{P}}] + [\tilde{\mathbf{X}}, \tilde{\mathbf{P}}]\tilde{\mathbf{P}}\right\} \\ &= \frac{1}{2m}\left\{2i\hbar\tilde{\mathbf{P}}\right\} \\ &= i\frac{\hbar}{m}\tilde{\mathbf{P}}.\end{aligned}$$

Now calculate, the representation of the commutator in the $\left|\phi_n\right\rangle$ basis:

$$\begin{aligned}\left\langle\phi_n\left|[\tilde{\mathbf{X}}, \tilde{\mathbf{H}}]\right|\phi_{n'}\right\rangle &= \left\langle\phi_n\left|\tilde{\mathbf{X}}\tilde{\mathbf{H}} - \tilde{\mathbf{H}}\tilde{\mathbf{X}}\right|\phi_{n'}\right\rangle \\ &= (E_{n'} - E_n)\left\langle\phi_n\left|\tilde{\mathbf{X}}\right|\phi_{n'}\right\rangle\end{aligned}$$

From the expansion of the commutator, the following is given:

$$\left\langle\phi_n\left|P\right|\phi_{n'}\right\rangle = \frac{im}{\hbar}(E_{n'} - E_n)\left\langle\phi_n\left|\tilde{\mathbf{X}}\right|\phi_{n'}\right\rangle$$

b) From this, deduce, using the completeness relation, the equation:

$$\sum_{n'} (E_n - E_{n'})^2 \left| \left\langle \phi_n \left| \tilde{\mathbf{X}} \right| \phi_{n'} \right\rangle \right|^2 = \frac{\hbar^2}{m^2} \left\langle \phi_n \left| \tilde{\mathbf{P}}^2 \right| \phi_n \right\rangle$$

The proof follows:

$$\begin{aligned}
\frac{\hbar^2}{m^2} \left\langle \phi_n \left| \tilde{\mathbf{P}}^2 \right| \phi_n \right\rangle &= \frac{\hbar^2}{m^2} \sum_{n'} \left\langle \phi_n \left| \tilde{\mathbf{P}} \right| \phi_{n'} \right\rangle \left\langle \phi_{n'} \left| \tilde{\mathbf{P}} \right| \phi_n \right\rangle \\
&= \left(\frac{\hbar^2}{m^2} \right) \left(\frac{im}{\hbar} \right)^2 \left[(E_{n'} - E_n) \left\langle \phi_n \left| \tilde{\mathbf{X}} \right| \phi_{n'} \right\rangle \right] \left[(E_n - E_{n'}) \left\langle \phi_{n'} \left| \tilde{\mathbf{X}} \right| \phi_n \right\rangle \right] \\
&= \sum_{n'} (E_n - E_{n'})^2 \left| \left\langle \phi_n \left| \tilde{\mathbf{X}} \right| \phi_{n'} \right\rangle \right|^2
\end{aligned}$$