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Quantum Mechanics 2

PHYS 5403 Homework Assignment 3

Problems: $\{1, 2, 3, 4\}$

Due: March 2, 2022 at 5:00 PM

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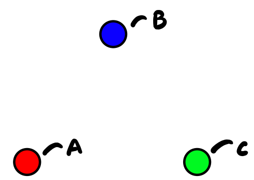
Problem 1:

The group of permutations of three particles is isomorphic to the group of geometric transformations that keep an equilateral triangle invariant (rotations of $\theta = 2\pi/3$ and reflections that leave one corner of the triangle invariant).

(a) Using this correspondence, find that the set of 6 unitary 2×2 matrices which represent the $3! = 6$ permutations. Show that the determinant of the matrices give the parity of the permutation.

We are going to treat these three particles like they are the points of an equilateral triangle. This will look like

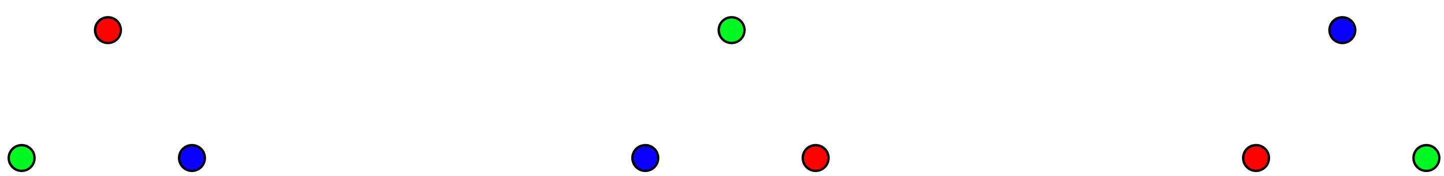


We now want to examine the 6 different permutations for this system. The two different types of permutations we have are

$$\text{Rotation} \rightarrow R_{1,2,\dots} \quad , \quad \text{Reflection} \rightarrow Q_{1,2,\dots}$$

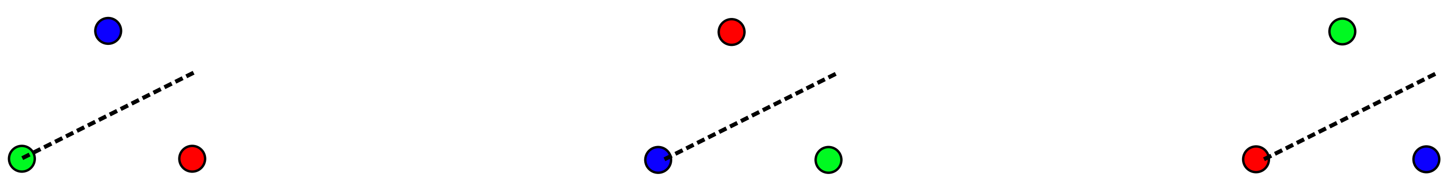
Looking at the rotations first we have

Rotation 1 : $\tilde{R}_1$ $\theta = 2\pi/3$ Rotation 2 : $\tilde{R}_2$ $\theta = 4\pi/3$ $\theta = 2\pi$ Rotation 3 : $\tilde{R}_3$



Where if we examine rotation 3, it is the identity. We now wish to see what the reflections will look like. This then looks like

Reflection 1 : $\tilde{Q}_1$ Reflection 2 : $\tilde{Q}_2$ Reflection 3 : $\tilde{Q}_3$



The dotted axis is the axis of which the reflection is occurring. We calculate two-dimensional rotations with

$$\tilde{R}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Problem 1: Continued

This then means that our rotation matrices are

$$\boxed{\tilde{R}_1(\theta=2\pi/3) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad \tilde{R}_2(\theta=4\pi/3) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad \tilde{R}_3(\theta=2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Taking the determinant of each matrix we see

$$\text{Det}(R_1) = \frac{1}{4}(-1(-1)-(-\sqrt{3})\sqrt{3})) = \frac{1}{4}(4) = 1 \checkmark, \quad \text{Det}(R_2) = \frac{1}{4}(-1(-1)-\sqrt{3}(-\sqrt{3})) = \frac{1}{4}(4) = 1 \checkmark, \quad \text{Det}(R_1) = 1 \checkmark$$

we can see that this makes sense with what we should expect for parity.

Now looking at the reflections, if $\tilde{Q}_3$ is performed twice it is equal to $\tilde{R}_3$, namely

$$\tilde{Q}_3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{Q}_3^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & cb+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can then further see

$$a^2+bc=1, \quad ab+bd=0, \quad ac+dc=0, \quad cb+d^2=1 \Rightarrow ab=-bd \therefore a=-d$$

This then means $\tilde{Q}_3$ is

$$\tilde{Q}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det(\tilde{Q}_3) = -1 \checkmark$$

we can then further see that $\tilde{Q}_2$ is a reflection of $\tilde{Q}_3$ and then a rotation of $\tilde{R}_1$

$$\tilde{Q}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \det(\tilde{Q}_2) = -1 \checkmark$$

And lastly, $\tilde{Q}_1$ is a reflection of $\tilde{Q}_2$ and then a rotation of $\tilde{R}_1$

$$\tilde{Q}_1 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad \det(\tilde{Q}_3) = -1 \checkmark$$

We can then say our reflection matrices are

$$\boxed{\tilde{Q}_1 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad \tilde{Q}_2 = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \tilde{Q}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Problem 1: Continued

(b) Consider three indistinguishable spinless particles. Write down the ket $|\alpha\rangle$ of the 3 particle state in terms of products of single particle states. Assume now that the particles are physically located at the corners of an equilateral triangle, which rotates around axis perpendicular to the triangle plane (z axis). Considering the statistics, write down the allowed quantum numbers for the total angular momentum J_z.

Since we are treating each individual point as if it were its own state we will have three. Mathematically this will will look like

$$|\alpha\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle$$

For us this will be something like

$$\boxed{|\alpha\rangle(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{\sqrt{6}}\left(|\varphi_1\rangle|\varphi_2\rangle|\varphi_3\rangle + |\varphi_1\rangle|\varphi_3\rangle|\varphi_2\rangle + |\varphi_2\rangle|\varphi_3\rangle|\varphi_1\rangle + |\varphi_2\rangle|\varphi_1\rangle|\varphi_3\rangle + |\varphi_3\rangle|\varphi_2\rangle|\varphi_1\rangle + |\varphi_3\rangle|\varphi_1\rangle|\varphi_2\rangle\right)}$$

We now focus on determining what values of m we are allowed to have. We first begin by applying a rotation on our ket

$$\tilde{D}(\theta = 2\pi/3)|\alpha\rangle = c|\alpha\rangle \Rightarrow e^{\left(-i/\hbar \tilde{J}_z \cdot \frac{2\pi}{3}\right)}|\alpha\rangle = c|\alpha\rangle$$

Letting $|\alpha\rangle \to |j,m\rangle$ we now have

$$e^{\left(-i/\hbar \tilde{J}_z \cdot 2\pi/3\right)}|j,m\rangle = e^{\left(-im\cdot 2\pi/3\right)}|j,m\rangle$$

Our rotation is equal to two transpositions therefore we can say our rotation is a product of permutations and thus

$$P_{jk}|\alpha\rangle = |\alpha\rangle \Rightarrow \tilde{D}(\theta)|\alpha\rangle = |\alpha\rangle \therefore c = 1$$

For the above to be true we say,

$$e^{\left(-im\cdot 2\pi/3\right)}|\alpha\rangle = e^{\left(-i2\pi n\right)}|\alpha\rangle \therefore \cancel{-i}m\cdot\frac{\cancel{2\pi}}{3} = \cancel{-i}\cdot\cancel{2\pi}n \therefore m = 3n$$

We can then say with confidence

$$\boxed{m = 3n, \quad -j \leq 3n \leq j}$$

(c) Suppose we have now three indistinguishable spin 1 particles and the orbital part of the three particle state is anti-symmetric. Write down the spin part of the state in terms of products of single particle kets.

For the orbital part to be anti-symmetric, and for Bosons to be symmetric under particle exchange, the spin part must be anti-symmetric

Problem 1: Continued

Taking this into acount, the spin part should look something like

$$\boxed{|\xi\rangle(s_1,s_2,s_3) = \frac{1}{\sqrt{6}}\left(|s_1\rangle|s_2\rangle|s_3\rangle - |s_1\rangle|s_3\rangle|s_2\rangle + |s_2\rangle|s_3\rangle|s_1\rangle - |s_2\rangle|s_1\rangle|s_3\rangle + |s_3\rangle|s_1\rangle|s_2\rangle - |s_3\rangle|s_2\rangle|s_1\rangle\right)}$$

Where odd permutations of the original state are negative and even permutations are positive. The values of $S_j = 0, \pm 1$ for any j.

Problem 1: Review

Procedure:

- – Since each of these rotations are of $2\pi/3$ we simply put in $2n$ multiples of this in for θ in
$$\tilde{\mathbf{D}}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
to get our rotation matrices
  – We then take the determinant of each matrix to show the expected parity
  – Find the reflection matrices with
$$\tilde{\mathbf{Q}}_{2,1} = \tilde{\mathbf{Q}}_{3,2}\tilde{\mathbf{R}}_1$$
where we simply due matrix multiplication to find these
- – We calculate a three product particle state with
$$|\alpha\rangle\,(\phi_1, \phi_2, \phi_3) = |\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$$
  – We then rotate this state by $\theta = 2\pi/3$ in the z direction and we choose to have the c value be 1
  – Proceed to calculate what value of m that tells us the allowed quantum numbers for J_z
- – Proceed to calculate the anti-symmetric spin state with
$$|\xi\rangle\,(S_1, S_2, S_3) = |S_1\rangle \otimes |S_2\rangle \otimes |S_3\rangle$$
where $-1 \leq S_i \leq 1$ for any i

Key Concepts:

- – Because our rotation is by $2\pi/3$ this is the value for θ that is plugged in to express each subsequent rotation
  – We achieve reflections beyond $\tilde{\mathbf{Q}}_3$ by first performing a separate reflection and rotation of $\tilde{\mathbf{R}}_1$
  – This can be thought of as permutations of reflections or rotations to achieve specific orientations
- – We find our three particle state by taking tensor products between each individual state
  – Because we want our rotation of our state to be invariant we can create an expression that allows us to find an allowed range for the magnetic quantum numbers
- – Since this is a Spin 1 particle we are dealing with Bosons
  – Bosons are symmetric under particle exchange, where as Fermions are not
  – If the orbital part is anti-symmetric, we know the spin part must also be anti-symmetric for the total state to be symmetric
  – We then calculate the tensor product of these three particles to find the representation

Variations:

- – We could be given a different shape, leading to a different orientation
    * This would then lead to a different angle of rotation and thus everything after it would change with the main procedure staying the same
- – We could have our system rotated by a different amount
    * This would possibly change the equation that leads to the relationship for the magnetic quantum number
- – We could be dealing with a different type of particle
    * This would cause the symmetry to be different thus requiring each individual part to be either symmetric or anti-symmetric

Problem 2:

(a) Suppose N identical non-interacting spin 5/2 particles are subjected to the potential of a 1 dimensional harmonic oscillator. Compute the total energy of the ground state.

The Hamiltonian for our system looks like

$$\tilde{\mathcal{H}}(\tilde{P},x) = \sum_{i=1}^{N} \frac{\tilde{P}_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2$$

The energy levels for a 1D Harmonic oscillator are calculated with

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Since we are working with N particles, we must write E_n as

$$E_n^N = \sum_{n=0}^{N} \left(n + \frac{1}{2}\right)\hbar\omega$$

But we have to assume theres a chance we can have less than 6 particles in the ground state. To take care of this we add

$$N \bmod(6) \times \hbar\omega\,(n + 1/2)$$

To our energy equation. We then have a total groundstate energy that looks like

$$\boxed{E_0 = 6\sum_{n=0}^{\tilde{n}-1} \hbar\omega(n + 1/2) + N \bmod(6) \times \hbar\omega(\tilde{n} + 1/2)}$$

(b) Two identical non-interacting spin 1/2 particles occupy the energy levels of a 2D harmonic oscillator,

$$\mathcal{H} = \sum_{i=i}^{2} \left(\frac{P_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2\right)$$

where P_i is the momentum of the particles and x_i their coordinates ($i = 1, 2$). Each particle has a wavefunction of the form

$$\psi_n(x_i, s_i) = \phi_n(x^i)\chi(s_i),$$

where $\phi_n(x)$ is the orbital part (indexed by the energy level $n \in \mathbb{N}$) and $\chi(s)$ the spin part. Write down the 2-particle wave functions for the ground state and first excited states and their respective energies.

Since our two particle system consists of Fermions, we know that the two-particle wave function must be fully anti-symmetric. We then write out our two-particle wave function as

$$\Psi_T(\vec{x}_1, s_1, \vec{x}_2, s_2) = \frac{1}{\sqrt{2}}\left(\Psi_1(\vec{x}_1, s_1) - \Psi_2(\vec{x}_2, s_2)\right) = \frac{1}{\sqrt{2}}\left(\phi_1(\vec{x}_1)\chi(s_1) - \phi_2(\vec{x}_2)\chi(s_2)\right) \quad (**)$$

Problem 2: Continued

Where the total wave function is a combination of anti-symmetric and symmetric pieces, namely

$$\psi_T(\vec{x}_1,s_1,\vec{x}_2,s_2) = (\varphi^A(\vec{x}_1,\vec{x}_2)\chi^S(s_1,s_2)) \quad \text{or} \quad \psi_T(\vec{x}_1,s_1,\vec{x}_2,s_2) = (\varphi^S(\vec{x}_1,\vec{x}_2)\chi^A(s_1,s_2))$$

We are choosing to represent the total spin state like

$$|\chi_T\rangle \doteq |s,m\rangle \quad w/ \quad -|s| \le m \le s \quad \Rightarrow \quad m_i = \pm\frac{1}{2} \ \& \ s=0,1$$

This means we can write the orbital parts as

$$\varphi_T^A(\vec{x}_1,\vec{x}_2) = \frac{1}{\sqrt{2}}(\varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2) - \varphi_1(\vec{x}_2)\varphi_2(\vec{x}_1)), \quad \varphi_T^S(\vec{x}_1,\vec{x}_2) = \frac{1}{\sqrt{2}}(\varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2) + \varphi_1(\vec{x}_2)\varphi_2(\vec{x}_1))$$

As for the spin part, we first examine our triplet which is symmetric $(s=1)$

$$\chi_T^S(s_1,s_2) = \begin{cases} \chi_1(-1/2)\chi_2(-1/2) & m=-1, s=1 \\ \frac{1}{\sqrt{2}}(\chi_1(1/2)\chi_2(-1/2) + \chi_1(-1/2)\chi_2(1/2)) & m=0, s=1 \\ \chi_1(1/2)\chi_2(1/2) & m=1, s=1 \end{cases}$$

We do the same for a singlet which is anti-symmetric $(s=0)$

$$\chi_T^A(s_1,s_2) = \frac{1}{\sqrt{2}}(\chi_1(1/2)\chi_2(-1/2) - \chi_1(-1/2)\chi_2(1/2)) \qquad m=0, s=0$$

Since we are interested in the ground state, we set $s=0$ which forces us to use the anti-symmetric spin piece and the symmetric orbital piece. We then say our total two particle wave function is

$$\boxed{\psi_T(\vec{x}_1,s_1,\vec{x}_2,s_2) = \frac{1}{2}(\varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2) + \varphi_1(\vec{x}_2)\varphi_2(\vec{x}_1))(\chi_1(1/2)\chi_2(-1/2) - \chi_1(-1/2)\chi_2(1/2))}$$

Since we are in 2D, the ground state energy will be

$$E_{G.S} = \hbar\omega\left(n_1+\frac{1}{2}\right)^{(2)} + \hbar\omega\left(n_2+\frac{1}{2}\right)^{(2)} = \frac{\hbar\omega}{2}(2) + \frac{\hbar\omega}{2}(2) = \hbar\omega + \hbar\omega = 2\hbar\omega$$

Twice that for a 1D system. As expected.

Problem 2: Continued

Since we are looking at the first excited state, S will go from 0 → 1 and therefore we must now have an anti-symmetric orbital piece and a symmetric spin piece. This then means our total wave function is

$$\boxed{\Psi_T(\vec{x}_1,s_1,\vec{x}_2,s_2) = \frac{1}{\sqrt{2}}\left(\varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2) - \varphi_1(\vec{x}_2)\varphi_2(\vec{x}_1)\right) \times \begin{cases} \chi_1(-1/2)\chi_2(-1/2) & m=-1 \\ \frac{1}{\sqrt{2}}\left(\chi_1(1/2)\chi_2(-1/2) + \chi_1(-1/2)\chi_2(1/2)\right) & m=0 \\ \chi_1(1/2)\chi_2(1/2) & m=1 \end{cases}}$$

We then find the energy for this state to be

$$E_{ex} = E_{gr} + \hbar\omega\left(n_1 + \frac{1}{2}\right) + \hbar\omega\left(n_2 + \frac{1}{2}\right) = 2\hbar\omega + (n_1 + n_2 + 1)\hbar\omega$$

We then set $n_1 = n_2 = 0$ and our excited energy should be

$$\boxed{E_{ex} = 3\hbar\omega}$$

Problem 2: Review

Procedure:

- – Begin by writing out the Hamiltonian for a simple Harmonic Oscillator

$$\tilde{\mathcal{H}}(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}) = \sum_{i=1}^{N} \frac{\tilde{\mathbf{p}}_i^2}{2m} + \frac{1}{2}m\omega^2 x_i^2$$

  – We calculate the ground state energy of our 1D Harmonic Oscillator with

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

  – Proceed to calculate the ground state energy for N number of particles by adding a mod function into

- – Because we are working with Fermions the two-particle wave function will be fully anti-symmetric, proceed to write out a two particle wave function

$$\psi_T(\vec{x}_1, S_1, \vec{x}_2, S_2) = \phi^{A,S}(\vec{x}_1)\chi^{S,A}(S_2)$$

  where ϕ is the orbital part and χ is the spin part

  – Because our triplet is symmetric and ($S = 1$), there will be three separate possible spin states for m ranging between -1 and 1

  – Because a singlet is anti-symmetric and ($S = 0$) there is only one possible spin state with $m = 0$

  – Proceed to write out all the combinations of the total wave function

  – To find the excited energy just simply set $n_1 = n_2 = 0$

Key Concepts:

- – Because we are working with a 1D Simple Harmonic Oscillator we use the equation for the ground state energy in compliance with mathematical formalism to find what the ground state energy is for N number of particles

- – The total wave function for Fermions is anti-symmetric, for Bosons it is symmetric

  – Depending on the value of s it will tell us the value of m since $-|s| \leq m \leq s \implies m_i = \pm 1/2 \implies s = 0, 1$

  – If the orbital part is symmetric, the spin part is anti-symmetric $\implies s = 0$

  – If the orbital part is anti-symmetric, the spin part is symmetric $\implies s = 1$

  – For a total wave function to be symmetric, both parts must be anti-symmetric or symmetric

  – For a total wave function to be anti-symmetric, both parts must be of mixed symmetries

Variations:

- – We could be given a different type of particle

    * This would affect the total number of particles that can occupy the ground state as well but would not affect anything after that

- – We could be given a different system with a different number of particles

    * This would only affect specific values of m and s but not the overall procedure

Problem 3:

Consider a one dimensional lattice of localized spins $\mathbf{S}(m)$, where m labels the lattice site. The spin operators for spin $s = 1/2$ have mixed commutation relations, so spin operators do not describe neither fermions nor bosons.

(a) Consider the spin operator $\mathbf{S}(m)$ for site m with components $S_x(m), S_y(m)$ and $S_z(m)$. Defining the ladder operators

$$a_m = S_x(m) - iS(m)$$
$$a_m^\dagger = S_x(m) + iS_y(m),$$

where $a_m^\dagger a_m = S_z(m) + \frac{1}{2}$ (assume $\hbar = 1$), show that they follow the commutation relations:

(i) Bose type for different sites

$$\left[a_i^\dagger, a_j\right] = \left[a_i^\dagger, a_j^\dagger\right] = \left[a_i, a_j\right] = 0$$

for $i \neq j$.

We first examine how these spin operators act when $i \neq j$. We start with

$$\tilde{S}_a(i)|j\rangle = |0\rangle$$

Since $i \neq j$, where $S_a(i)$ means spin operator of direction "a" at lattice site i. So for us we know

$$\tilde{S}_\alpha(m)|n\rangle = |0\rangle$$

We now examine the relationship that tells us

$$a_m^\dagger a_m = \tilde{S}_z(m) + \frac{1}{2} \Rightarrow (a_m^\dagger a_m)^\dagger = \left(\tilde{S}_z(m) + \frac{1}{2}\right)^\dagger = \left(\tilde{S}_z^\dagger(m) + \frac{1}{2}\right) = a_m a_m^\dagger$$

We can use the above to say if we act each individual spin operator on their eigenstates what we should expect

$$\tilde{S}_z(m)|n\rangle = n\,\delta_{mn}|n\rangle \,,\quad \tilde{S}_y(m)|n\rangle = A_{mn}\delta_{mn}|n\rangle \,,\quad \tilde{S}_x(m)|n\rangle = B_{mn}\delta_{mn}|n\rangle$$

Where we are looking at site m in directions (x,y,z) for the state $|n\rangle$. We can then use these relationships to say. To make life easier I will use c, D instead of i, j. We start with looking at $[a_i^\dagger, a_j] = 0$,

$$[a_c^\dagger, a_D]|k\rangle = (a_c^\dagger a_D - a_D a_c^\dagger)|k\rangle = \left((\tilde{S}_x(c) + i\tilde{S}_y(c))(\tilde{S}_x(D) - i\tilde{S}_y(D)) - (\tilde{S}_x(D) - i\tilde{S}_y(D))(\tilde{S}_x(c) + i\tilde{S}_y(c))\right)|k\rangle$$

$$= \left(\tilde{S}_x(c)\tilde{S}_x(D) - i\tilde{S}_x(c)\tilde{S}_y(D) + i\tilde{S}_y(c)\tilde{S}_x(D) + \tilde{S}_y(c)\tilde{S}_y(D) - \tilde{S}_x(D)\tilde{S}_x(c) + i\tilde{S}_x(D)\tilde{S}_y(c) - i\tilde{S}_y(D)\tilde{S}_x(c) - \tilde{S}_y(D)\tilde{S}_y(c)\right)|k\rangle$$

The above consists of several mixed terms, I will show how this works a pair at a time

$$(\tilde{S}_x(c)\tilde{S}_x(D) - \tilde{S}_x(D)\tilde{S}_x(c))|k\rangle = \tilde{S}_x(c)\tilde{S}_x(D)|k\rangle - \tilde{S}_x(D)\tilde{S}_x(c)|k\rangle$$

Problem 3: Continued

$$= A_{DK}\delta_{DK}\tilde{S}_x(C)|K\rangle - A_{CK}\delta_{CK}\tilde{S}_x(D)|K\rangle$$

$$= A_{DK}B_{CK}\delta_{DK}\delta_{CK}|K\rangle - A_{CK}B_{DK}\delta_{CK}\delta_{DK}|K\rangle$$

$$= A_{DK}B_{CK}(\underbrace{\delta_{DK}\delta_{CK} - \delta_{CK}\delta_{DK}}_{=0 \text{ iff } C\neq D})|K\rangle = A_{DK}B_{CK}(0)|K\rangle = 0 \checkmark$$

$$= (\cancel{\tilde{S}_x(C)\tilde{S}_x(D)} - \cancel{i\tilde{S}_x(C)\tilde{S}_y(D)} + \cancel{i\tilde{S}_y(C)\tilde{S}_x(D)} + \cancel{\tilde{S}_y(C)\tilde{S}_y(D)} - \cancel{\tilde{S}_x(D)\tilde{S}_x(C)} + \cancel{i\tilde{S}_x(D)\tilde{S}_y(C)} - \cancel{i\tilde{S}_y(D)\tilde{S}_x(C)} - \cancel{\tilde{S}_y(D)\tilde{S}_y(C)})|K\rangle$$

We can then say for the mixed terms that are anti-symmetric that their difference is zero. Therefore we can say

$$\boxed{[a_i^+, a_j] = 0}$$

We now look at $[a_i^+, a_j^+] = 0$,

$$[a_C^+, a_D^+]|K\rangle = (a_C^+ a_D^+ - a_C^+ a_D^+)|K\rangle = ((\tilde{S}_x(C) + i\tilde{S}_y(C))(\tilde{S}_x(D) + i\tilde{S}_y(D)) - (\tilde{S}_x(D) + i\tilde{S}_y(D))(\tilde{S}_x(C) + i\tilde{S}_y(C)))|K\rangle$$

$$= (\tilde{S}_x(C)\tilde{S}_x(D) + i\tilde{S}_x(C)\tilde{S}_y(D) + i\tilde{S}_y(C)\tilde{S}_x(D) - \tilde{S}_y(C)\tilde{S}_y(D) - \tilde{S}_x(D)\tilde{S}_x(C) - i\tilde{S}_x(D)\tilde{S}_y(C) - i\tilde{S}_y(D)\tilde{S}_x(C) + \tilde{S}_y(D)\tilde{S}_y(C))|K\rangle$$

Again with the mixed terms we can see that

$$= (\cancel{\tilde{S}_x(C)\tilde{S}_x(D)} + \cancel{i\tilde{S}_x(C)\tilde{S}_y(D)} + \cancel{i\tilde{S}_y(C)\tilde{S}_x(D)} - \cancel{\tilde{S}_y(C)\tilde{S}_y(D)} - \cancel{\tilde{S}_x(D)\tilde{S}_x(C)} - \cancel{i\tilde{S}_x(D)\tilde{S}_y(C)} - \cancel{i\tilde{S}_y(D)\tilde{S}_x(C)} + \cancel{\tilde{S}_y(D)\tilde{S}_y(C)})|K\rangle$$

We can further say

$$\boxed{[a_i^+, a_j^+] = 0}$$

We look at the last relationship $[a_i, a_j] = 0$

$$[a_C, a_D]|K\rangle = (a_C a_D - a_C a_D)|K\rangle = ((\tilde{S}_x(C) - i\tilde{S}_y(C))(\tilde{S}_x(D) - i\tilde{S}_y(D)) - (\tilde{S}_x(D) - i\tilde{S}_y(D))(\tilde{S}_x(C) - i\tilde{S}_y(C)))|K\rangle$$

$$= (\cancel{\tilde{S}_x(C)\tilde{S}_x(D)} - \cancel{i\tilde{S}_x(C)\tilde{S}_y(D)} - \cancel{i\tilde{S}_y(C)\tilde{S}_x(D)} - \cancel{\tilde{S}_y(C)\tilde{S}_y(D)} - \cancel{\tilde{S}_x(D)\tilde{S}_x(C)} + \cancel{i\tilde{S}_x(D)\tilde{S}_y(C)} + \cancel{i\tilde{S}_y(D)\tilde{S}_x(C)} + \cancel{\tilde{S}_y(D)\tilde{S}_y(C)})|K\rangle$$

These again have the anti-symmetric terms that are zero so we can say

$$\boxed{[a_i, a_j] = 0}$$

Problem 3: Continued

(ii) Fermi type for spin operators on the same site,

$$\left\{a_i, a_i^\dagger\right\} = \mathbb{I},$$

and $a_i^2 = (a_i^\dagger)^2 = 0$.

We start with writing out the anti-commutator

$$\{a_i, a_i^+\} = (a_i a_i^+ + a_i^+ a_i) = ((\tilde{S}_x(i) - i\tilde{S}_y(i))(\tilde{S}_x(i) + i\tilde{S}_y(i)) + (\tilde{S}_x(i) + i\tilde{S}_y(i))(\tilde{S}_x(i) - i\tilde{S}_y(i)))$$

$$= (\tilde{S}_x^2(i) + \cancel{i\tilde{S}_x(i)\tilde{S}_y(i)} - \cancel{i\tilde{S}_y(i)\tilde{S}_x(i)} + \tilde{S}_y^2(i) + \tilde{S}_x^2(i) - \cancel{i\tilde{S}_x(i)\tilde{S}_y(i)} + \cancel{i\tilde{S}_y(i)\tilde{S}_x(i)} + \tilde{S}_y^2(i))$$

$$= (2\tilde{S}_x^2(i) + 2\tilde{S}_y^2(i)) = 2(\tilde{S}_x^2(i) + \tilde{S}_y^2(i)) = 2(\tilde{S}^2(i) - \tilde{S}_z^2(i))$$

We can then go on to see if we apply this to a ket

$$\{a_i, a_i^+\}|i\rangle = 2(\tilde{S}^2(i) - \tilde{S}_z^2(i))|i\rangle = 2(\tilde{S}^2(i)|i\rangle - \tilde{S}_z^2(i)|i\rangle) \qquad (*)$$

We know that $(s = 1/2)$ and because of this

$$\tilde{S}^2|s\rangle = s(s+1)|s\rangle \quad , \quad \tilde{S}_z^2|s\rangle = (s)^2|s\rangle$$

Therefore we can say $(*)$ will become

$$\{a_i, a_i^+\}|i\rangle = 2\left(\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)^2\right)|i\rangle = 2\left(\frac{1}{2}\left(\frac{3}{2}\right) - \frac{1}{4}\right)|i\rangle = 2\left(\frac{1}{2}\right)|i\rangle = \mathbb{I}|i\rangle$$

From this we can say

$$\boxed{\{a_i, a_i^+\} = \mathbb{I}}$$

(b) Consider now a spin chain with periodic boundary conditions. Show that the operators

$$C_i = \exp\left[i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right] a_i$$

$$C_i^\dagger = a_i^\dagger \exp\left[-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right]$$

follow Fermi statistics. This transformation is called *Wigner-Jordan transformation*. Write down the inverse transformation. Show that

$$C_i^\dagger C_i = a_i^\dagger a_i,$$

and

$$C_i^\dagger C_{i+1} = a_i^\dagger a_{i+1}.$$

Problem 3: Continued

We begin first by taking an anti-commutator of c_i and $c_i^\dagger$,

$$\{c_i, c_i^\dagger\} = (c_i c_i^\dagger + c_i^\dagger c_i)$$

$$= \left(\exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) a_i a_i^\dagger \exp\left(-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) + a_i^\dagger \exp\left(-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) \exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) a_i \right)$$

$$= \left(a_i a_i^\dagger \exp\left(i\pi \sum_{j=1}^{i-1} (\cancelto{0}{a_j^\dagger a_j - a_j^\dagger a_j})\right)\right) + a_i^\dagger a_i \exp\left(i\pi \sum_{j=1}^{i-1} (\cancelto{0}{a_j^\dagger a_j - a_j^\dagger a_j})\right) = a_i a_i^\dagger + a_i^\dagger a_i$$

$$= \{a_i, a_i^\dagger\} = \mathbb{1} \Rightarrow c_i c_i^\dagger = a_i a_i^\dagger \ \& \ c_i^\dagger c_i = a_i^\dagger a_i \Rightarrow \{c_i, c_j^\dagger\} = \delta_{ij} \ \therefore \ \underline{\text{Fermionic}}$$

We can then use the relationships above to derive the inverse transformation

$$\boxed{c_i^\dagger c_i - \frac{1}{2} = \hat{S}_z(i)}$$

We can then move on to showing what the relation evaluates to

$$c_i^\dagger c_{i+1} = a_i^\dagger \exp\left(-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) \exp\left(i\pi \sum_{j=1}^{i} a_j^\dagger a_j\right) a_{i+1}$$

$$= a_i^\dagger a_{i+1} \exp\left(-i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) \exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j\right) \exp\left(i\pi (a_j^\dagger a_j)\right)$$

$$= a_i^\dagger a_{i+1} \cancelto{1}{\exp\left(i\pi \sum_{j=1}^{i-1} a_j^\dagger a_j - a_j^\dagger a_j\right)} \cancelto{1}{\exp\left(i\pi (a_j^\dagger a_j)\right)} = a_i^\dagger a_{i+1}$$

This of course means this Fermionic and follows with

$$\boxed{c_i^\dagger c_i = a_i^\dagger a_i \ , \ c_i^\dagger c_{i+1} = a_i^\dagger a_{i+1}}$$

(c) Transform the Heisenberg spin exchange Hamiltonian

$$\mathcal{H} = J \sum_{m=1}^{N} \mathbf{S}(m) \cdot \mathbf{S}(m+1)$$

into a Hamiltonian of fermions. Interpret the result.

We begin first by expanding this sum and and then substituting in terms to simplify

Problem 3: Continued

Our Hamiltonian becomes

$$\mathcal{H} = J\sum_{m=1}^{N}\left(S_x(m)\cdot S_x(m+1) + S_y(m)\cdot S_y(m+1) + S_z(m)\cdot S_z(m+1)\right)$$

$$= J\sum_{m=1}^{N}\left(S_x(m)\cdot S_x(m+1) + S_y(m)\cdot S_y(m+1) + (C_m^{+}C_m - 1/2)(C_{m+1}^{+}C_{m+1} - 1/2)\right)$$

$$= J\sum_{m=1}^{N}\left(S_x(m)\cdot S_x(m+1) + S_y(m)\cdot S_y(m+1) + C_m^{+}C_mC_{m+1}^{+}C_{m+1} - \frac{1}{2}C_m^{+}C_m - \frac{1}{2}C_{m+1}^{+}C_{m+1} + \frac{1}{4}\right)$$

$$= J\sum_{m=1}^{N}\left(S_x(m)\cdot S_x(m+1) + S_y(m)\cdot S_y(m+1) + a_m^{+}a_ma_{m+1}^{+}a_{m+1} - \frac{1}{2}a_m^{+}a_m - \frac{1}{2}a_{m+1}^{+}a_{m+1} + \frac{1}{4}\right)$$

We then use the common definitions

$$S_x(m) = \frac{a_m + a_m^{+}}{2}, \quad S_y(m) = \frac{a_m - a_m^{+}}{2i}$$

This then turns the Hamiltonian into,

$$\boxed{\mathcal{H} = J\sum_{m=1}^{N}\left(\frac{1}{2}(a_ma_{m+1}^{+} + a_m^{+}a_{m+1}) + a_m^{+}a_ma_{m+1}^{+}a_{m+1} - \frac{1}{2}a_m^{+}a_m - \frac{1}{2}a_{m+1}^{+}a_{m+1} + \frac{1}{4}\right)}$$

We have created a Hamiltonian that is completely dependent upon creation and annihilation operators. Doing this we have created an operator that shares an eigenstate with the traditional Hamiltonian just consistent upon fermionic rules.

Problem 3: Review

Procedure:

- – Begin with the identity
$$\tilde{\mathbf{S}}_\alpha(i)\,|j\rangle = |0\rangle$$
which tell us
$$\tilde{\mathbf{S}}_x(j)\,|n\rangle = A\delta_{j,n}\,|n\rangle \quad , \quad \tilde{\mathbf{S}}_y(j)\,|n\rangle = B\delta_{j,n}\,|n\rangle \quad , \quad \tilde{\mathbf{S}}_{zs}(j)\,|n\rangle = n\delta_{j,n}\,|n\rangle$$
  – Proceed to use the above commutator relationships to prove the identities given to us
  – Use the relationship
$$\tilde{\mathbf{S}}^2\,|S\rangle = S(S+1)\,|S\rangle \quad , \quad \tilde{\mathbf{S}}_z^2\,|S\rangle = S^2\,|S\rangle$$
to prove the anti-commutator relationship
- – Use the definitions for operators given to us and take an anti-commutator of them to show that it is equal to a δ function and thus Fermionic
  – Proceed to take a product of the operators given to us to show the desired relationship
- – Take a dot product of the spin operators and then substitute in other relationships
$$\tilde{\mathbf{S}}_x(m) = \frac{a_m + a_m^\dagger}{2} \quad , \quad \tilde{\mathbf{S}}_y(m) = \frac{a_m - a_m^\dagger}{2i}$$
to create a Hamiltonian that is consistent solely upon creation and annihilation operators

Key Concepts:

- – The top equation reads spin operator in direction α at lattice site i for state j
  – We use mathematical formalism to prove the relationships asked of us by taking these commutators and anti-commutators on states
- – We use anti-commutators and products of operators to prove relationships
- – Using the relationships given to us, after taking a dot product to express our Hamiltonian in terms of spin operators we can then further substitute in relationships to create a Hamiltonian that is only comprised of creation and annihilation operators

Variations:

- – This part of the problem is proving identities
  * The only way this changes is if we are given a different definition for a_m and $a_m^\dagger$ where the same broad procedure would apply
- – We once again are proving identities
  * This part can only change if the definition of the operators are changed
- – We could be given a different definition of the Hamiltonian
  * We then would use the same procedure but with a different definition

Problem 4:

Consider $(a^\dagger, a)$ as the creation/annihilation operators for bosons in a single mode. Coherent states are defined as the eigenstates of the annihilation operator

$$a\,|z\rangle = z\,|z\rangle\,,$$

where z is a complex number $\langle z|z\rangle = 1$ is normalized.

(a) Writing $|z\rangle$ as a generic superposition of $\{|n\rangle\}$, states in the form:

$$|z\rangle = \sum_{z=0}^{\infty} \phi_n\,|n\rangle\,,$$

where

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,,$$

show that

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right)\exp\left(za^\dagger\right)|0\rangle\,.$$

We first begin with $|z\rangle$ and expanding in a complete set for $|n\rangle$

$$|z\rangle = \sum_n |n\rangle\langle n|z\rangle = \sum_n |n\rangle\langle z|n\rangle^\dagger = \sum_n |n\rangle\langle z|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle^\dagger = \sum_n \frac{1}{\sqrt{n!}}\langle 0|a^n|z\rangle|n\rangle$$

$$= \sum_n \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle|n\rangle = \langle 0|z\rangle\sum_n \frac{z^n}{\sqrt{n!}}|n\rangle$$

Since $|z\rangle$ is normalized we can use this to determine,

$$\langle z|z\rangle = \sum_m\sum_n \langle z|m\rangle\langle m|n\rangle\langle n|z\rangle = \sum_m\sum_n \langle z|m\rangle\langle z|n\rangle^\dagger\langle m|n\rangle$$

$$= \sum_m\sum_n \langle z|\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle\langle z|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle^\dagger\langle m|n\rangle$$

$$= \sum_m\sum_n \langle z|\frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle\langle 0|\frac{(a)^n}{\sqrt{n!}}|z\rangle\langle m|n\rangle = \sum_m\sum_n \frac{z^m}{\sqrt{m!}}\langle z|0\rangle\frac{z^n}{\sqrt{n!}}\langle 0|z\rangle\langle m|n\rangle$$

$$= |\langle 0|z\rangle|^2\sum_n \frac{z^{2n}}{n!} = |\langle 0|z\rangle|^2\exp(z^2)$$

We know the above is equal to one so we can say

$$|\langle 0|z\rangle|^2\exp(z^2) = 1 \Rightarrow |\langle 0|z\rangle|^2 = \exp(-z^2) \Rightarrow \langle 0|z\rangle = \exp(-1/2\cdot z^2)$$

We can then take the final part of the last line and we find $|z\rangle$ to be

$$|z\rangle = \exp(-1/2|z|^2)\sum_n \frac{z^n}{\sqrt{n!}}\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle = \exp(-1/2|z|^2)\sum_n \frac{(za^\dagger)^n}{n!}|0\rangle = \exp(-1/2|z|^2)\exp(za^\dagger)|0\rangle$$

Problem 4: Continued

We Finally have $|z\rangle$ to be

$$\boxed{|z\rangle = \exp(-\tfrac{1}{2}|z|^2)\exp(za^\dagger)|0\rangle}$$

(b) Show equivalently that

$$|z\rangle = \exp\Big(za^\dagger - z^*a\Big)|0\rangle \equiv D(z)|0\rangle\,,$$

where $D(z)$ is a unitary operator.

Starting with the above we use the common identity

$$\exp(\tilde{x})\exp(\tilde{y}) = \exp(\tilde{x}+\tilde{y}+\tfrac{1}{2}[\tilde{x},\tilde{y}])$$

Using this for our example we have

$$\exp(za^\dagger - z^*a) = \exp(za^\dagger)\exp(-z^*a) = \exp(za^\dagger - z^*a - \tfrac{|z|^2}{2}\overset{1}{\cancel{[a^\dagger,a]}})$$

We now need to apply this to $|0\rangle$ and show us it returns $|z\rangle$

$$\exp(za^\dagger)\exp(-z^*a)\exp(-z^2/2)|0\rangle = \exp(-z^2/2)\exp(za^\dagger)\sum_n \frac{-(za)^n}{n!}|0\rangle$$
$$= \exp(-z^2/2)\exp(za^\dagger)\exp(0)|0\rangle$$
$$= \exp(-z^2/2)\exp(za^\dagger)|0\rangle = |z\rangle \checkmark$$

Where we have used the fact that $(a)^m|0\rangle = 0$. Therefore this is a unitary rotation operator

$\square$

(c) Compute the overlap of two coherent states, $\langle z|z'\rangle$.

We begin by defining $|z'\rangle$,

$$|z'\rangle = \exp(-\tfrac{1}{2}|z'|^2)\exp(z'a^\dagger)|0\rangle$$

We now calculate $\langle z|z'\rangle$

$$\langle z|z'\rangle = \langle 0|\exp(z^*a)\exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)\exp(z'a^\dagger)|0\rangle$$
$$= \langle 0|\sum_m \frac{(z^*a)^m}{m!}\exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)\sum_m \frac{(z'a^\dagger)^m}{m!}|0\rangle$$
$$= \langle 0|\sum_m \frac{(z^*z')^m}{m!}\exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)|0\rangle = \exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)\exp(z^*z')$$

Problem 4: Continued

we then calculate the overlap with

$$|\langle z|z'\rangle|^2 = \exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)\exp(z^*z') * \exp(-\tfrac{1}{2}|z|^2)\exp(-\tfrac{1}{2}|z'|^2)\exp(zz'^*)$$

$$= \exp(-|z|^2)\exp(-|z'|^2)\exp(z^*z' + zz'^*) = \exp(-|z|^2 - |z'|^2 + z^*z' + zz'^*)$$

Therefore our overlap is

$$\boxed{|\langle z|z'\rangle|^2 = \exp(zz'^* + z^*z' - |z|^2 - |z'|^2)}$$

(d) Find the probability distribution P_n of finding the state $|n\rangle$ occupied in $|z\rangle$.

To find the probability distribution for state $|n\rangle$ occupied in $|z\rangle$ we use

$$P_n = |\langle n|z\rangle|^2$$

we can then proceed to calculating $\langle n|z\rangle$ and then taking the modulus squared of it

$$\langle n|z\rangle = \langle n| e^{\frac{-|z|^2}{2}} e^{za^\dagger}|0\rangle = e^{-|z|^2/2}\langle n|\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = e^{-|z|^2/2}\frac{z^n}{\sqrt{n!}}$$

we can now find the probability to be

$$\boxed{P_n = |\langle n|z\rangle|^2 = e^{-|z|^2}\frac{z^{2n}}{(n!)}}$$

Problem 4: Review

Procedure:

- - Begin by expanding $|z\rangle$ in a complete set
  - Take and inner product of $|z\rangle$ and derive a new expression
  - Solve for $\langle 0|z\rangle$ from the above and put this back into the expression for $|z\rangle$ after expanding in a complete set
- - Start with the common identity
  $$e^{\tilde{\mathbf{x}}}e^{\tilde{\mathbf{y}}} = e^{\tilde{\mathbf{x}}+\tilde{\mathbf{y}}+1/2[\tilde{\mathbf{x}},\tilde{\mathbf{y}}]}$$
  and apply this to $|0\rangle$
- - Define a new state
  $$|z'\rangle = e^{-1/2|z'|^2}e^{z'a^\dagger}|0\rangle$$
  - Calculate the overlap with
  $$|\langle z|z'\rangle|^2$$
- - Calculate the probability with
  $$P_n = |\langle n|z\rangle|^2$$

Key Concepts:

- - Given that our initial state is a superposition of $|n\rangle$ we can express it in the form that we are asked to prove
- - We can then show that this new operator acts like a rotation operator by returning $|z\rangle$ if it is applied on our state
- - Having a definition of $|z\rangle$ and $|z'\rangle$ we can calculate the overlap of these two states with the correct equation
- - We can then calculate the probability with our given equation

Variations:

- - Since we are proving identities, we cannot change the problem without creating a brand new problem
    * Anything that changes in part (a) will create change through part (d) where the same procedure is used but for a different starting point