Question 1:

$\boxed{\text{Assignment 6 Solns}}$

a) Given the radial symmetry of the potential, $V = V(r)$, we can write the relevant Lagrangian as:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{Ce^{-\alpha r}}{r}$$

As $\theta$ is a cyclic co-ordinate, $l = mr^2\dot{\theta} = \text{const}$ is a conserved quantity.

The EOM for $r$ is:

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + C(1+\alpha r)\frac{e^{-\alpha r}}{r^2} = 0$$

$\hookrightarrow$

$$\ddot{r} = \frac{l^2}{m^2r^3} - \frac{C}{m}(1+\alpha r)\frac{e^{-\alpha r}}{r^2}$$

or

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{C(1+\alpha r)e^{-\alpha r}}{r^2}$$

Looking at our EOM through the window of "F = Ma", we appear to have an effective force,

$$F_{eff} = \frac{l^2}{mr^3} - \frac{C(1+\alpha r)e^{-\alpha r}}{r^2}$$

w/ effective potential,

$$V_{eff}(r) = \frac{1}{2}\frac{l^2}{mr^2} - \frac{Ce^{-\alpha r}}{r}$$

To sketch the potential we in principle need to know the relative values of $l, m, C, \alpha$. We could do this, or $\Rightarrow$ Do some rearrangement of the constants:

$$\frac{V_{eff}}{C} = \frac{l^2}{2Cmr^2} - \frac{e^{-\alpha r}}{r}$$

Define $\quad \bar{l}^2 = \dfrac{l^2}{2Cm} \quad$ + $\quad \bar{r} = \alpha r$

$$\Rightarrow \quad \frac{V_{eff}}{C} = \frac{\bar{l}^2\alpha^2}{\bar{r}^2} - \frac{e^{-\bar{r}}\alpha}{\bar{r}}$$

or,

$$\frac{V_{eff}}{C\alpha} = \frac{\bar{l}^2\alpha}{\bar{r}^2} - \frac{e^{-\bar{r}}}{r}$$

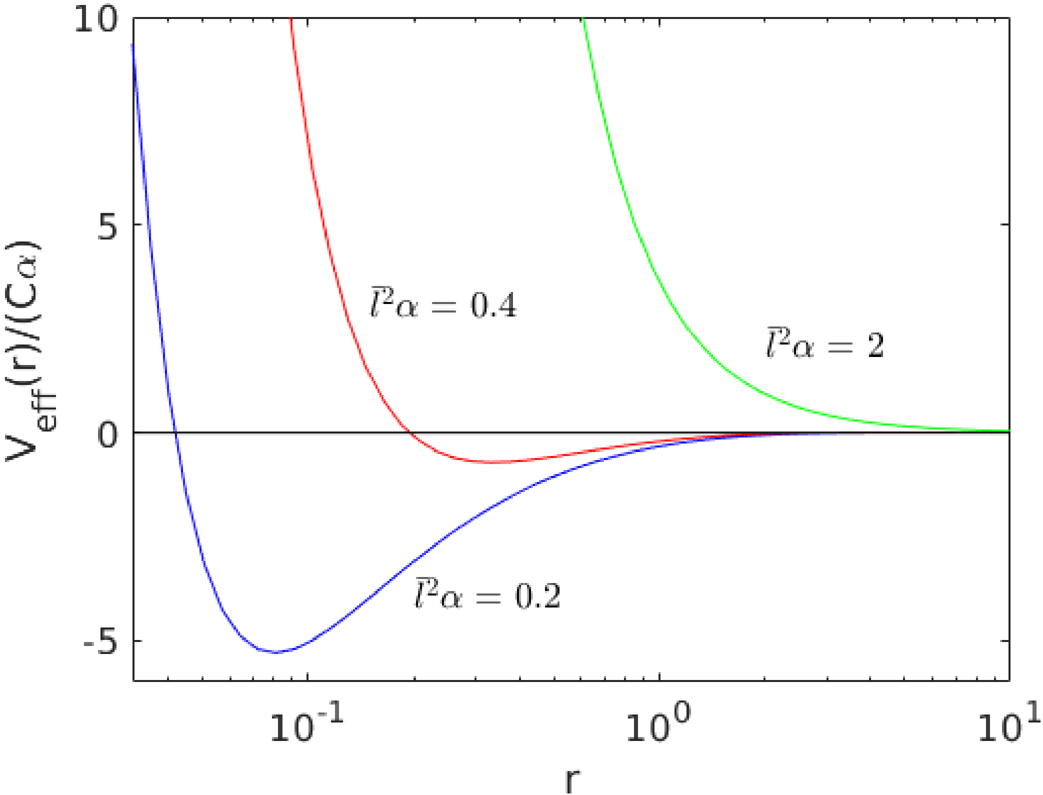
We plot this quantity in the attached plot for various values of $\bar{l}^2\alpha$.

The important features are:

i) $r \to \infty$ $\quad V_{eff} \to 0$ (all terms decay)

ii) $r \to 0$ $\quad V_{eff} \to \infty$ as $1/r^2$ (like Coulomb example in class)

iii) The intermediate behavior depends on $\bar{l}^2\alpha$. For $\bar{l}^2\alpha$ "small" → there is a large region of attractive potential. For $\bar{l}^2\alpha$ "large" this region becomes very small. This will be clarified more in b)!



$V_{eff}(r)/(C\alpha)$
10
5
0
-5
$\bar{l}^2\alpha = 0.4$
$\bar{l}^2\alpha = 2$
$\bar{l}^2\alpha = 0.2$
$10^{-1}$
$10^{0}$
$10^{1}$
r

b) There could be two types of generic motion:

i) bound motion

ii) scattering (unbound) motion.

The precise energies for which each will occur inherently depend on the values of the constants in the potential. However, there is one qualitative feature we can look for:

⇒ Bound motion requires the potential to have a local minimum

(ie. must have at least 2 turning points)

This is equivalent to requiring there be a solution of $\frac{\partial V_{eff}}{\partial r} = 0$ for $r \geq 0$.

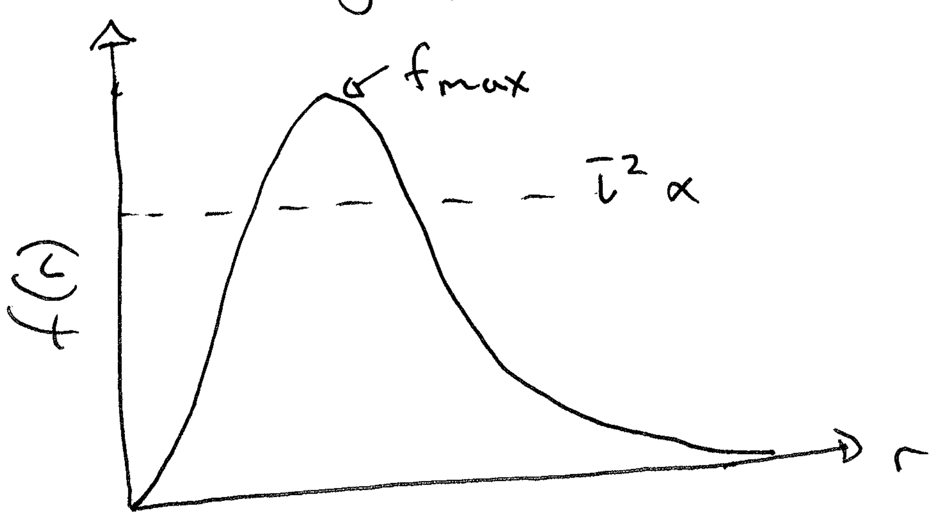
From a) we can adopt the rescaled effective potential,

$$\frac{\partial}{\partial r}\left(\frac{V_{eff}}{C\alpha}\right) = \frac{\partial}{\partial r}\left(\frac{\bar{l}^2\alpha}{r^2} - \frac{e^{-r}}{r}\right) = 0 \quad \left[\begin{array}{c}\text{adopting} \\ \bar{r}=r \\ \text{for notation}\end{array}\right]$$

& for $r>0$ this becomes:

$$\boxed{\bar{l}^2\alpha = e^{-r}(r+1)\,r} = f(r)$$

We cannot analytically solve this eqn, but we look at it graphically:




There exist two interesting cases/solutions.

i) if $\bar{l}^2\alpha < f_{max}$ :

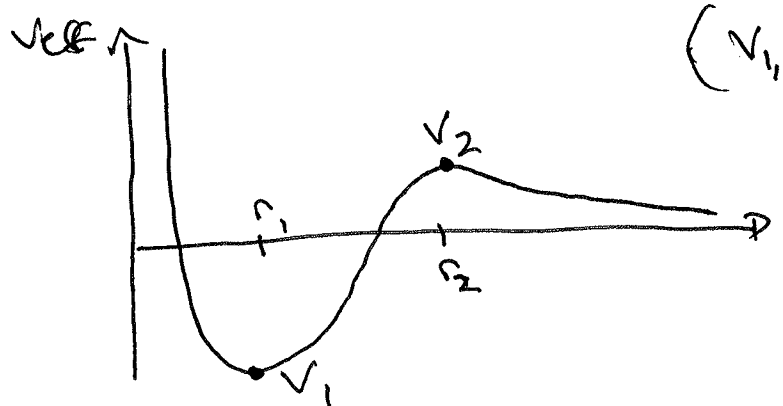
The potential has 2 critical points $r_1$ & $r_2$ (two solutions of $f(r) = \bar{l}^2\alpha$) → one minimum & one maximum → classify them by looking at the slope $\frac{df}{dr}$. Clearly:

$r_1 < r_2$ → $r_1$ is minimum & $r_2$ maximum.

ii) $f_{max} < \bar{l}^2\alpha$ → no min/max exist

Case ii) has an easy consequence: no bound motion can exist → only scattering motion for all $E_0$.

Case i) is more complex → let's sketch potential, ($V_{1,2} = V_{eff}(r_{1,2})$)



Now, if:

1. $E_0 > V_2$ : scattering (unbound motion)

2. $0 < E_0 < V_2$ : Both scattering & ~~unbound~~ bound motion exists. It will depend on whether the initial position of the particle is,

   $r > r_2$ (scattered)

   $r < r_2$ (bound)

3. $E_0 < 0$ : bound motion

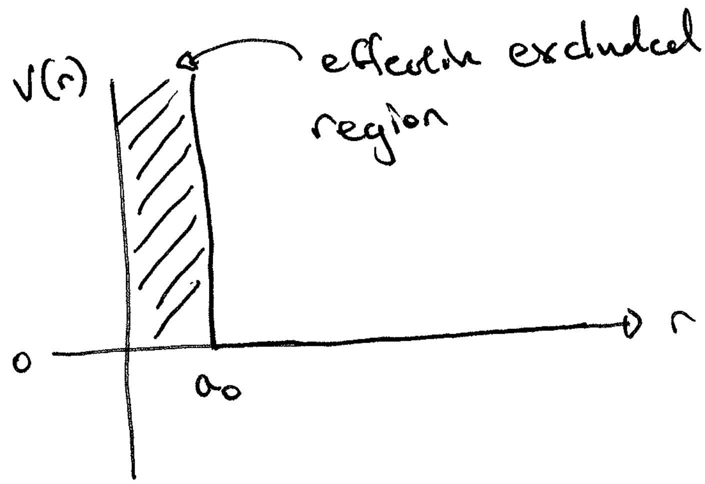
4. $E_0 = V_1$ : circular bound motion as $\dot{r} = 0$ !

## Question 2:

a) Hard-core potential,

$$V(r) = \begin{cases} \infty & r \lesssim a_0 \\ 0 & r > a_0 \end{cases}$$

Sketch:




$\Rightarrow$ equivalent to an impenetrable sphere of radius $a_0$.

We want to compute:

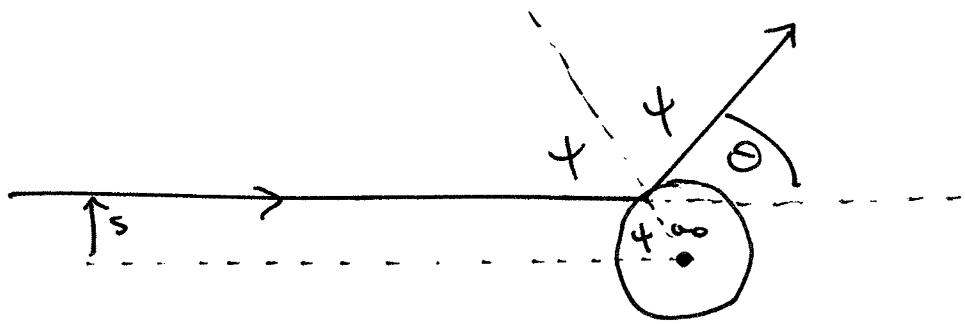
i) $$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$$ scattering cross-section

ii) $$\sigma_T(\theta) = 2\pi \int_0^{\pi} \sigma(\theta)\sin\theta \, d\theta$$ total cross-section

There are 2 possible ways to obtain i):

* graphical/geometric approach.
* brute force by definition of $\Theta$

Let's start w/ the former. We use the analogy of the hard sphere ~~to~~ to sketch the scattering event:




For $s > a_0$ there will be no scattering $(\theta = 0)$, so let's explicitly assume $s \leq a_0$. Similar to our discussion in class we note that:

$$2\psi + \theta = \pi \quad ①$$

$$s = a_0 \sin\psi \quad ②$$

Substituting ① → ② :

$$s = a_0 \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= a_0 \cos\left(\frac{\theta}{2}\right) \quad ③$$

and thus,

$$\frac{ds}{d\theta} = -\frac{a_0}{2} \sin\left(\frac{\theta}{2}\right) \quad ④$$

3 + 4 enable us to directly compute the scattering cross section,

$$\sigma(\theta) = \frac{\left(a_0 \cos\left(\frac{\theta}{2}\right)\right)}{\sin\theta} \cdot \left| -\frac{a_0}{2} \sin\left(\frac{\theta}{2}\right) \right|$$

$$= \frac{a_0^2}{2} \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \qquad \left( 0 \leq \theta \leq \pi \Rightarrow \sin\left(\frac{\theta}{2}\right) \geq 0 \right)$$

$$= \frac{a_0^2}{2} \frac{\frac{1}{2}\sin\theta}{\sin\theta} = \boxed{\frac{a_0^2}{4}}.$$

Then, the total cross section is,

$$\sigma_T = 2\pi \int_0^{\pi} \sigma(\theta) \sin\theta \, d\theta$$

$$= \pi a_0^2$$

Looking at our results,

* $\sigma(\theta)$ is independent of $\theta$, which makes sense because our scattering problem always 'looks the same' for the hard core sphere potential
* $\sigma_T$ corresponds to the area of a disk in 2D, which is what our potential looks like geometrically!

Our alternative derivation of the cross-sections uses the definition of the scattering angle (see lectures):

$$\Theta = \pi - 2\int_{r_{min}}^{\infty} \frac{dr'}{r'\sqrt{\frac{1}{s^2} - \frac{V(r')}{Es^2} - \frac{1}{r'^2}}}$$

Clearly, $r_{min} = a_0$ from our defn of $V$ or the hard sphere interpretation. Then, as $V \to 0$ for $r > a_0$,

$$\Theta = \pi - 2\int_{a_0}^{\infty} \frac{dr'}{r'\sqrt{1/s^2 - 1/r'^2}}$$

$\downarrow$ substitute $u = 1/r$

$$= \pi - 2\int_0^{1/a_0} \frac{s\,du}{\sqrt{1 - s^2u^2}}$$

$$= \pi - 2\,\mathrm{asin}\left(s/a_0\right)$$

We invert this to obtain,

$$s = a_0 \sin\left(\pi/2 - \frac{\theta}{2}\right)$$

$$= a_0 \cos\left(\frac{\theta}{2}\right)$$

(identical to our previous derivation)

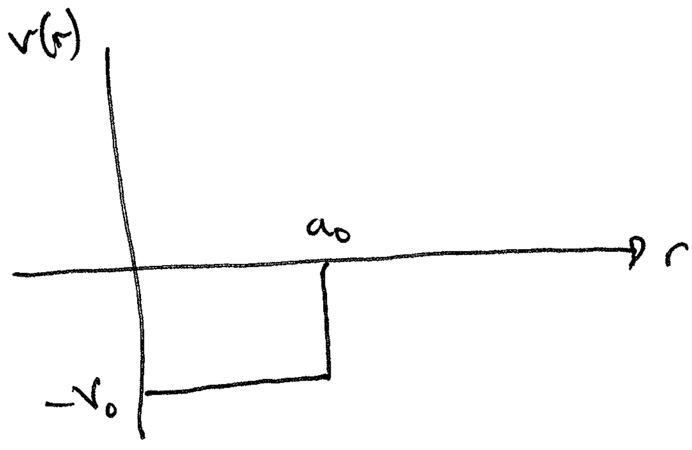
&

$$\frac{ds}{d\theta} = -\frac{a_0}{2} \sin\left(\frac{\theta}{2}\right)$$

From this point our solution for $\sigma(\theta)$ & $\sigma_T$ follows identically to our geometric approach.

b) Soft-core potential,

sketch:




We will approach the problem by first writing some general expressions relating the impact parameter to the angular momentum.

For a particle w/ mass $m$ outside the attractive potential, $r > a_0$:

$$L_{out} = m v_{out} s \qquad \left( v_{out} \perp s \text{ for incoming beam} \right)$$

and it has energy determined solely by the kinetic contribution,

$$E_{out} = \frac{1}{2} m v_{out}^2 \Rightarrow v_{out} \equiv \sqrt{\frac{2E_{out}}{m}}$$

Then, we can rewrite,

$$L_{out} = \sqrt{2mE_{out}}\, s$$

Alternatively, when the particles is inside the attractive potential, $r \leq a_0$:

$$L_{in} = m v_{in} s_{in}$$ note we do not assume $s_{in} = s$ ! *

& $$E_{in} = \frac{1}{2} m v_{in}^2 - V_0$$ new contribution from attractive potential.

Rearranging from $E_{in}$ we obtain,

$$v_{in} = \sqrt{\frac{2(E_{in}+V)}{m}}$$

and thus,

$$l_{in} = s_{in}\sqrt{2m(E_{in}+V_0)}$$

* In fact, as we will momentarily show, $s_{in} \not= s$, as $s_{in}$ is defined as perpendicular to the deflected path of the particle in the attractive potential!

Angular momentum is conserved in this problem, so we can equate $l_{in} = l_{out}$ to obtain a relation between $s$ and $s_{in}$:

$$l_{in} = l_{out}$$

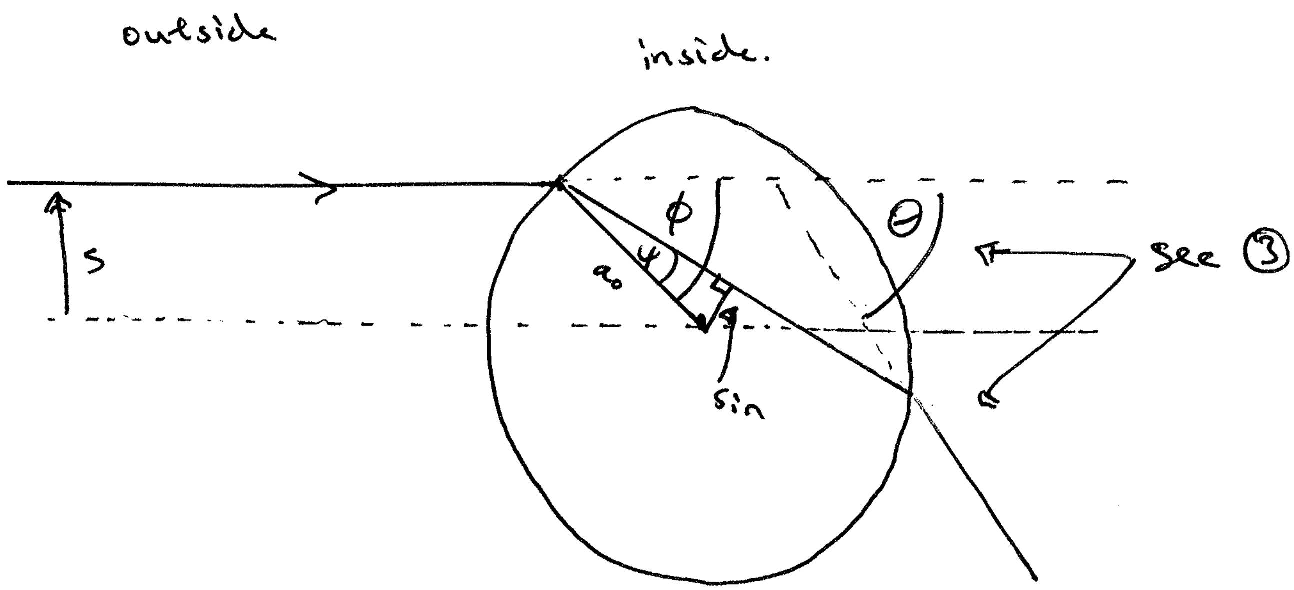
$$\downarrow$$

$$s_{in}\sqrt{2m(E_{in}+V_0)} = s\sqrt{2mE_{out}}$$

Energy conservation, $E_{in} = E_{out} = E$, then lets us obtain

$$s = s_{in}\sqrt{\frac{E+V_0}{E}} = n\, s_{in} \quad \text{w/ } n = \sqrt{1+\frac{V_0}{E}}$$

To proceed we should sketch the trajectory of our particle, so that we can identify our scattering angle $\theta$ etc.




(Note how $s_{in}$ ⊥ to deflected trajectory)

From the diagram we identify:

$$s = a_0 \sin\phi \qquad ①$$

$$s_{in} = a_0 \sin\psi \qquad ②$$

and the scattering angle $\theta$ is obtained as,

$$\theta = 2(\phi - \psi) \qquad ③$$

Where does ③ come from? We obtain it by noting two features:

First, the potential inside & outside the sphere is <u>constant</u> (i.e., $-V_0$ & $0$ respectively). This means that the force due to the potential, $F \sim -\nabla V$, is zero everywhere, except at the <u>boundary</u> between the regions. This is consistent w/ our figure & calculation of $\sin$, where the particles trajectory abruptly deviates at the surface $r = a_0$.

Second, our particle will traverse the boundary between the exterior ($V=0$) & interior ($V=-V_0$) potentials twice. Hence the total scattering angle as $t \to \infty$ ($r \to \infty$) will be <u>twice</u> the deflection at one surface. Thus,

$$\Theta = 2 \times \underbrace{(\phi - \psi)}_{\text{deflection at one surface}}$$

2 deflections.

[For more detailed discussion of this see the <u>link</u> <u>on canvas</u>]

Proceeding w/ eqs 1 – 3,

$$a_0 \sin\psi = a_0 \sin\left(\phi - \frac{\theta}{2}\right)$$

and thus,

$$s = a_0 n \sin\left(\phi - \frac{\theta}{2}\right)$$

We rewrite the trig term using a difference identity,

$$s = a_0 n \left[ -\sin\left(\frac{\theta}{2}\right)\cos\phi + \cos\left(\frac{\theta}{2}\right) \overbrace{\sin\phi}^{\equiv s/a_0} \right]$$

$$= -a_0 n \sin\left(\frac{\theta}{2}\right)\cos\phi + ns\cos\left(\frac{\theta}{2}\right)$$

$$\downarrow$$

$$\left[1 - n\cos\left(\frac{\theta}{2}\right)\right]s = a_0 n \sin\left(\frac{\theta}{2}\right)\cos\phi$$

Squaring both sides & using $\cos^2\phi \to 1 - \sin^2\phi = 1 - s^2/a_0^2$, we get (after some more rearranging),

$$s^2\left[1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right] = a_0^2 n^2 \sin^2\left(\frac{\theta}{2}\right)$$

$$\hookrightarrow \quad s = \frac{a_0 n \sin\left(\frac{\theta}{2}\right)}{\sqrt{\cancel{1} - 2n\cos\left(\frac{\theta}{2}\right) + n^2}}$$

or, dividing through top & bottom by $n$,

$$S = \frac{a_0 \sin\left(\frac{\theta}{2}\right)}{\sqrt{\frac{1}{n^2} - \frac{2}{n}\cos\left(\frac{\theta}{2}\right) + 1}}$$

which is the required result.

Question 3:

a) For the repulsive for $F = k/r^3$ $(k>0)$

we have the associated potential,

$$V = \frac{k}{2r^2} \qquad \left( -\frac{\partial V}{\partial r} = k/r^3 \checkmark \right)$$

We can re-express this in terms of $u = 1/r$ as,

$$V(u) = \frac{k}{2} u^2$$

Using the supplied formula for the scattering angle, it remains to solve:

$$\Theta = \pi - 2 \int_0^{u_{max}} \frac{s\,du}{\sqrt{1 - \frac{ku^2}{2E} - s^2u^2}}$$

First, we need to compute $u_{max}$:

$$u_{max} = \frac{1}{r_{min}}$$ → $r_{min}$ is closest distance of approach for particle subject to ~~V(r~~ effective potential $V_{eff}(r)$.

We obtain the effective potential in the usual way by including the angular momentum contribution,

$$V_{eff}(r) = \underbrace{\frac{L^2}{2mr^2}}_{\text{angular momentum correction.}} + \underbrace{\frac{k}{2r^2}}_{V(r)}$$

Then, $r_{min}$ is found by computing the solution of:

$$V_{eff}(r_{min}) = E$$

which is,

$$r_{min}^2 = \frac{L^2}{2Em} + \frac{k}{2E} \quad \text{or} \quad r_{min} = \sqrt{\frac{L^2 + mk}{2Em}}$$

Our solution equivalently gives $u_{max} = \sqrt{\dfrac{2Em}{L^2 + mk}}$

Returning to the integral, we can write it in the form,

$$\Theta = \pi - 2\int_0^{u_{max}} \frac{s\,du}{\sqrt{1-au^2}}$$

w/ $a = \dfrac{k}{2E} + s^2$

Introducing $\bar{u} = \sqrt{a}\,u$,

$$\Theta = \pi - 2\int_0^{\sqrt{a}\,u_{max}} du \; \frac{s}{\sqrt{a}} \frac{1}{\sqrt{1-\bar{u}^2}}$$

$$= \pi - \frac{2s}{\sqrt{2}}\left[ \arcsin(\bar{u}) \Big|_0^{\sqrt{a}\,u_{max}} \right]$$

yielding,

$$\Theta = \pi - \frac{2\sqrt{2E}\;s}{\sqrt{k+2Es^2}} \arcsin\left[\sqrt{\frac{m(k+2Es^2)}{L^2+mk}}\right]$$

Finally we get $\theta \doteq \Theta(s,E)$ by eliminating the angular momentum,

$$l = s\sqrt{2mE} \rightarrow l^2 = s^2 \cdot 2mE$$

which means the argument of the arcsin becomes,

$$\sqrt{\frac{mk + 2Es^2m}{l^2 + mk}} = \sqrt{\frac{mk + 2Ems^2}{mk + 2mEs^2}} = 1$$

$$\therefore \quad \Theta = \pi - \frac{2\sqrt{2E}\,s}{\sqrt{k + 2Es^2}} \cdot \frac{\pi}{2}$$

$$= \pi\left(1 - \frac{s\sqrt{2E}}{\sqrt{k + 2Es^2}}\right).$$

b) For the potential $V(u) = \frac{1}{2}ku^2$ we have the associated force, $-\frac{\partial V}{\partial u} = ku$

Plugging this into our differential eqn (Eq 8) we have:

$$\frac{d^2u}{d\theta^2} + u = -\frac{mk}{l^2}u$$

or, rearranging:

$$\frac{d^2u}{d\theta^2} = -\left(1 + \frac{mk}{L^2}\right)u$$

Clearly, this is an EOM for a harmonic oscillator w/ frequency $\gamma = \sqrt{1 + \frac{mk}{L^2}}$. The general solution is,

$$u(\theta) = \alpha\cos(\gamma\theta) + \beta\sin(\gamma\theta)$$

when $\alpha$ + $\beta$ are some constants to be determined.

c) We take the particle to be incoming from $r = -\infty$ @ $t = -\infty$ such that:

* $u = 0$ @ $t = -\infty$
* $\theta_i = \pi$ @ $t = -\infty$

Then,

$$0 = \alpha\cos(\gamma\pi) + \beta\sin(\gamma\pi)$$

$$\Rightarrow \underline{\alpha = -\beta\tan(\gamma\pi)} \quad (i)$$

Moreover, as $t \to \infty$ we again have $r \to \infty$, and by definition $\theta_f = \theta$. Thus,

$$0 = \alpha \cos(\gamma\theta) + \beta \sin(\gamma\theta)$$

$$= -\beta \tan(\gamma\pi)\cos(\gamma\theta) + \beta\sin(\gamma\theta)$$

Multiplying through by $\cos(\gamma\pi)$,

$$0 = -\beta \sin(\gamma\pi)\cos(\gamma\theta) + \beta\sin(\gamma\theta)\cos(\gamma\pi)$$

Using a trig identity the RHS can be rewritten as,

$$0 = \beta \sin(\gamma(\theta - \pi))$$

which implies (for $\alpha, \beta \neq 0$) $\quad \gamma = \dfrac{\pi}{\theta - \pi} \quad$ (ii)

d) Defining $x = \dfrac{\theta}{\pi} \longrightarrow \gamma = \dfrac{1}{x-1}$

But from (b) we also have $\gamma^2 = 1 + \dfrac{mk}{L^2}$, so:

$$\frac{1}{(x-1)^2} = 1 + \frac{mk}{L^2}$$

$$\frac{1}{(x-1)^2} - 1 = \frac{mk}{L^2}$$

Plugging in $l^2 = 2mEs^2$ (by definition):

$$\frac{1}{(x-1)^2} - 1 = \frac{mk}{2mEs^2}$$

$$\downarrow$$

$$\frac{1-(x^2-2x+1)}{(x-1)^2} = \frac{k}{2Es^2}$$

$\downarrow$ rearranging,

$$s^2 = -\frac{k}{2E}\frac{(x-1)^2}{x(x-2)} \qquad (*)$$

For the differential cross section we proceed by using (from $*$):

$$2s\,ds = \frac{k}{2E}\frac{2(1-x)}{x^2(x-2)^2}\,dx \qquad \left(\text{from } \frac{d}{dx}(s^2) = 2s\frac{ds}{dx}\right)$$

We can plug this expression into

$$\sigma\, d\theta = \frac{s\,ds}{\sin\theta} \qquad \text{w/} \left\{ \sin\theta = \sin(\pi x) \right.$$

(we are ignoring absolute values here, but can check...)

$$= \frac{k}{2E}\frac{(1-x)\,dx}{x^2(2-x)^2\sin(\pi x)} \quad \square.$$