PHYS5153 Assignment 11

Due: 11:59pm on 12/5/2021.
Marking: Total of 10 marks (weighting of each question is indicated).
Fine print: Solutions should be presented legibly (handwritten or LaTeX is equally acceptable) so that the grader can follow your line of thinking and any mathematical working should be appropriately explained/described. If you provide only equations you will be marked zero. If you provide equations that are completely wrong but can demonstrate some accompanying logical reasoning then you increase your chances of receiving more than zero. If any of your solution has relied on a reference or material other than the textbook or lectures, please note this and provide details.

Question 1 (2 marks)

Consider a harmonic oscillator with Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2. \tag{1}$$

(a) Use the Hamilton-Jacobi equation to show that Hamilton's principal function for this problem is,

$$S(q, \alpha, t) = \frac{m\omega}{2}(q^2 + \alpha^2)\cot(\omega t) - m\omega\alpha q \mathrm{cosec}(\omega t). \tag{2}$$

(b) Demonstrate that $S(q, \alpha, t)$ furnishes the expected solution for the dynamics of the oscillator.

Question 2 (3 marks)

Consider a projectile whose motion is restricted to a 2D plane (defined by x and y co-ordinates). Use the Hamilton-Jacobi equation to confirm that the path of the projectile under the influence of gravity is:

$$\begin{aligned} x(t) &= v_x t + x(0), \\ y(t) &= -\frac{1}{2}gt^2 + v_y t + y(0). \end{aligned} \tag{3}$$

Make sure that your solution includes an explicit form for the Hamilton principal function.

Question 3 (2 marks)

Consider a particle of mass m in a one-dimensional box potential,

$$V(x) = \begin{cases} V_0 & \text{if } |x| \leq a_0 \\ \infty & \text{if } x > a_0 \end{cases}. \tag{4}$$

Use the action-angle formalism to compute the frequency of the particle's periodic motion. Make sure your expression is given in terms of the particle's initial condition. Is your answer consistent with your expectations?

Question 4 (3 marks)

Consider a particle of mass m that is constrained to move along a wire parameterized by

$$\begin{aligned} x(\phi) &= l\left[\phi + \sin(\phi)\right], \\ y(\phi) &= l\left[1 - \cos(\phi)\right], \end{aligned} \tag{5}$$

where l is taken to be a constant. The particle is subject to gravity along y (pointing down).

(a) Compute a Lagrangian describing the particle's motion.

(b) Use your answer to (a) to compute the associated Hamiltonian. Justify that the Hamiltonian is equal to the energy and is conserved (but *do not* assume $H = T + V$ to initially compute the Hamiltonian).

(c) Obtain an expression for the canonical momentum p_ϕ in terms of ϕ and the energy E.

(d) The motion described by (a) and (b) is periodic. Use action-angle variables to show that the frequency of the motion is identical for any initial condition with $\phi \leq \pi$. Hints: You might find the substitution $u = \sqrt{2mgl/E}\,\sin(\phi/2)$ handy to solve an integral that arises.