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Quantum Mechanics 2

PHYS 5403 Homework Assignment 7

Problems: {1, 2, 3}

Due: May 6, 2022 at 5:00 PM

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Problem 1:

(a) Based on the properties of the Dirac algebra only (which do not depend on the representation), prove that β and α_i matrices must have even dimensionality.

we know that the Dirac Matrices α^i, β anti-commute, namely

$$\{\alpha^i, \beta\} = 0$$

This then means we can say

$$\alpha^i \beta + \beta \alpha^i = 0$$

which leads to

$$\alpha^i = -\beta \alpha^i \beta$$

Taking the trace of this we have

$$Tr(\alpha^i) = Tr(-\beta\alpha^i\beta) = T(-\alpha^i\beta^2) = Tr(-\alpha^i)$$

The only thing that is equal to the negative of itself is zero $\therefore$

$$Tr(\alpha^i) = 0$$

The above relation tells us

$$Tr(\beta) = Tr(\alpha^i) = 0$$

This means the matrices mentioned above must have eigenvalues of ± 1, this can only happen with an even dimensionality.

(b) Show explicitly that the Dirac algebra cannot be satisfied in a representation with dimension $d = 2$. Hint: assume $\alpha^i = \sigma_i$ as Pauli matrices and show that one cannot find a matrix for β that satisfies the Dirac algebra.

Starting with $d=2$, we begin with the assumption

$$\alpha^1 = \sigma_x \quad , \quad \alpha^2 = \sigma_y \quad , \quad \alpha^3 = \sigma_z$$

where with these assumptions we can say

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \Rightarrow Tr(\sigma_i) = 0 \quad , \quad Tr(\sigma_i^2) = \mathbb{1}$$

However it is impossible to find a fourth 2×2 matrix (β) which anti-commutes with the other three.

Problem 1: Continued

If we write $\{\alpha^i, \beta\}$ with $i=1,2,3$ we then have

$$\alpha^1\beta + \beta\alpha^1 = 0 \Rightarrow \alpha^1\beta = -\beta\alpha^1$$
$$\alpha^2\beta + \beta\alpha^2 = 0 \Rightarrow \alpha^2\beta = -\beta\alpha^2$$
$$\alpha^3\beta + \beta\alpha^3 = 0 \Rightarrow \alpha^3\beta = -\beta\alpha^3$$

Where $\alpha^1 = \sigma_x$, $\alpha^2 = \sigma_y$, $\alpha^3 = \sigma_z$ we can immediately see

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This immediately tells us that $d=2$ is not valid due to there only being 3 anti-commuting matrices.

(c) Find the explicit representation for (β, α^i), where

$$\alpha^3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

with $\mathbb{I}$ the 2×2 identity matrix.

We know that α^3 will look something like

$$\alpha^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The representations of $\beta, \alpha^1, \alpha^2$ and α^3 must satisfy

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij}$$
$$\{\alpha^i, \beta\} = 0$$
$$\beta^2 = \mathbb{I}$$

Along with the matrices being traceless. We are going to choose the Pauli matrices to be in the off diagonals of α^1 and α^2.

Problem 1: Continued

Let's choose the representation,

$$\alpha^i = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline \sigma^i & 0 \end{array}\right) \qquad i = 1,2,3$$

These matrices can be multiplied in Blocks,

$$\alpha^i \alpha^j = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline \sigma^i & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & \sigma^j \\ \hline \sigma^j & 0 \end{array}\right) = \left(\begin{array}{c|c} \sigma^i\sigma^j & 0 \\ \hline 0 & \sigma^i\sigma^j \end{array}\right)$$

This then tells us that α^1 and α^2 are

$$\alpha^1 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} \; , \; \alpha^2 = \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{pmatrix}$$

Using these with the commutation relationships we can say

$$\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}\beta = -\beta\begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} \; , \; \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{pmatrix}\beta = -\beta\begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}\beta = -\beta\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}$$

Using these relationships we can say β is

$$\beta = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&1 \\ -1&0&0&0 \\ 0&1&0&0 \end{pmatrix}$$

This means our matrices are then

$$\boxed{\alpha^1 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix} \; , \; \alpha^2 = \begin{pmatrix} 0&0&0&-i \\ 0&0&i&0 \\ 0&-i&0&0 \\ i&0&0&0 \end{pmatrix} \; , \; \alpha^3 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix} , \; \beta = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&1 \\ -1&0&0&0 \\ 0&1&0&0 \end{pmatrix}}$$

Where the above all follow the rules of Dirac Algebra.

Problem 1: Review

Procedure:

- – Use the rules of Dirac Algebra to show that the traces of α^i and β are zero
- – Start with the assumption that the dimension is $d = 2$ and show that this is not possible
  - ∗ This is done by showing that the matrices in the dimension of 2×2 will not satisfy the rules of Dirac Algebra
- – Taking the representation for α^3 that is given to us, we find a representation of for the rest of α^i and β that will satisfy the rules of Dirac Algebra

Key Concepts:

- – Because of the matrices α^i and β are traceless, we can deduce that the eigenvalues of these matrices bust be ± 1
  - ∗ This insinuates that the dimensionality of the matrices must be even
- – The Dirac Algebra rules are

$$\begin{aligned} \{\boldsymbol{\alpha_i}, \boldsymbol{\alpha_j}\} &= 2\delta^{ij} \\ \{\boldsymbol{\alpha_i}, \boldsymbol{\beta}\} &= 0 \\ \boldsymbol{\beta^2} &= \mathbb{I} \end{aligned}$$

- – Taking the representation of for α^i as

$$\alpha^i \doteq \{\sigma_x, \sigma_y, \sigma_z\} \quad \text{with } \ i = 1, 2, 3$$

  it can be shown that $\{\boldsymbol{\alpha_i}, \boldsymbol{\beta}\} \neq 0$

- – Taking the representation that for α^3 to be what is given to us we then choose the other matrices for α^1 to be of the form

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

  where with the correct selection of α^i we can show that the Dirac Algebra rules are obeyed

Variations:

- – This part is a proof so it cannot be changed unless the entire problem is changed
- – We could be asked to prove that the dimensionality would start at $d = 4$
  - ∗ This would require us to use the rules of Dirac Algebra to show that this dimension would be valid
- – We could be given a different initial representation for α^3
  - ∗ This would requires us to assemble other matrices that follow the rules of Dirac Algebra

Problem 2:

The normalization of the solutions of the Dirac equation with positive and negative energy satisfy

$$\bar{\psi}\psi = \pm 1,$$

where + corresponds to the positive energy states and − to the negative energy ones, with

$$\bar{\psi} = \psi^\dagger \gamma^0$$

at the Dirac adjoint.

(a) Find the eigenergies and the normalized eigenfunctions that solve the Dirac equation for a free particle

$$(i\gamma^\mu \partial_\mu - m)\psi(\mathbf{x}, t) = 0.$$

The standard Dirac equation is

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi \Rightarrow \left(\frac{\hbar c}{i}\alpha^k \partial_k + \beta m c^2\right)\psi = \mathcal{H}\psi$$

The equation for a free particle becomes

$$(i\gamma^\mu \partial_\mu - m)\psi(x,t) = 0 \qquad (*)$$

Where we know $\gamma^0 \doteq \beta$, $\gamma^i = \beta\alpha^i$ w/ $\partial_\mu = (\partial/\partial t, -\vec{\nabla})$. This means $(*)$ becomes

$$\left(i\gamma^0\partial_0 - i\beta\alpha^i\partial_i - m\right)\psi(x,t) = 0 \qquad (**)$$

Where our wave function is a 4-component spinor like

$$\psi(x,t) = e^{-iEt}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$$

This means $(**)$ becomes

$$\left(i\gamma^0\partial_0 - i\beta\alpha^i\partial_i - m\right)\psi(x,t) = \left(i\beta\frac{\partial}{\partial t} - i\beta\alpha^i\partial_i - m\right)e^{-iEt}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$$

$$\left(i\beta\frac{\partial}{\partial t}e^{-iEt}\right)\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix} = \left(i\beta\alpha^i\partial_i + m\right)e^{-iEt}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$$

Problem 2: Continued

$$(\beta E \cancel{e^{-iEt}})\psi(x) = (i\beta\alpha^i\partial_i + m)\cancel{e^{-iEt}}\psi(x) \Rightarrow \beta E\psi(x) = (i\beta\alpha^i\partial_i + m)\psi(x)$$

The solution to Dirac's Equation is a plane wave with momentum p, $\therefore$ $\psi(x)$ is

$$\psi(x,t) = e^{ip\cdot x}\psi$$

where ψ is

$$\psi \doteq e^{-iEt}\begin{pmatrix}\varphi \\ \chi\end{pmatrix}$$

Putting this into the Dirac equation we find

$$E\begin{pmatrix}\varphi \\ \chi\end{pmatrix} = (m\beta + \alpha^i p_i)\begin{pmatrix}\varphi \\ \chi\end{pmatrix} = \begin{pmatrix} m & \sigma^i p_i \\ \sigma^i p_i & -m\end{pmatrix}\begin{pmatrix}\varphi \\ \chi\end{pmatrix} \Rightarrow \begin{array}{l} E\varphi = m\varphi + \sigma^i p_i\chi \\ E\chi = \sigma^i p_i\varphi - m\chi\end{array}$$

Solving we find

$$\varphi(E-m) = \sigma^i p_i\chi \quad \therefore \quad \varphi = \frac{\sigma^i p_i}{E-m}\chi \,,\; \chi = \frac{\sigma^i p_i}{(E+m)}\varphi$$

Putting this into the other equation will give

$$E = \pm\sqrt{m^2+p^2}$$

Which are our eigen energies. To calculate the eigen functions we compute

$$\sigma^i p_i = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z\end{pmatrix}$$

This then means χ and φ are

$$\chi = \frac{1}{(E+m)}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} + \frac{1}{(E+m)}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix}$$

$$= \frac{1}{(E+m)}\begin{pmatrix}p_z \\ p_x + ip_y\end{pmatrix} + \frac{1}{(E+m)}\begin{pmatrix}p_x - ip_y \\ -p_z\end{pmatrix} = \frac{1}{(E+m)}\left(\begin{pmatrix}p_z \\ p_x + ip_y\end{pmatrix} + \begin{pmatrix}p_x - ip_y \\ -p_z\end{pmatrix}\right)$$

Problem 2: Continued

If we then want to find the positive eigenfunctions only we have

$$\boxed{\psi_1(x,t) = e^{(i(P\cdot x - \sqrt{m^2+P^2})t)} \begin{pmatrix} 1 \\ 0 \\ \frac{P_z}{E+m} \\ \frac{P_x+iP_y}{E+m} \end{pmatrix}, \psi_2(x,t) = e^{(i(P\cdot x - \sqrt{m^2+P^2})t)} \begin{pmatrix} 0 \\ 1 \\ \frac{-P_z}{E+m} \\ \frac{P_x-iP_y}{E+m} \end{pmatrix}}$$

(b) Show that the orbital angular momentum **L** of a free Dirac particle is not a constant of the motion. Use the fact that

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k,$$

where ϵ_{ijk} is the Levi-Civita tensor. Defining the spin operator as $\Sigma \equiv \mathbb{I} \otimes \sigma$, where $\mathbb{I}$ is the 2×2 identity matrix and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, show that the total angular momentum

$$\mathbf{J} = \mathbf{L} + \frac{\hbar}{2}\Sigma$$

is conserved.

The Hamiltonian of a free particle is ($\hbar = c = 1$)

$$\mathcal{H} = \alpha^i p_i + m\beta$$

For the orbital angular momentum to not be a constant of motion we need $[L_i, \mathcal{H}] = 0$. We then have

$$[L_i, \mathcal{H}] = [L_i, \alpha^j P_j] + [L_i, m\beta]$$

$$= [L_i, \alpha^j P_j] + \overset{0}{[L_i, \cancel{\alpha^i} P_i]} + [L_i, \alpha^k P_k]$$

$$= i\hbar\epsilon_{ijk}\alpha^j P_k + i\hbar\epsilon_{ikj}\alpha^k P_j = i\hbar\epsilon_{ijk}(\alpha^j P_k - \alpha^k P_j) \checkmark$$

We then need $[J, \mathcal{H}] = 0$ for it to be conserved. We can then say

$$[J, \mathcal{H}] = [L, \mathcal{H}] + \frac{\hbar}{2}[\sigma^i, \mathcal{H}]$$

$$= i\hbar\epsilon_{ijk}(\alpha^j P_k - \alpha^k P_j) + \frac{\hbar}{2}([\sigma^i, \alpha^i P_i] + \overset{0}{\cancel{[\sigma^i, m\beta]}})$$

$$= i\hbar\epsilon_{ijk}(\alpha^j P_k - \alpha^k P_j) + \frac{\hbar}{2}([\sigma^i, \alpha^i P_i] + [\sigma^i, \alpha^j P_j] + [\sigma^i, \alpha^k P_k])$$

Problem 2: Continued

$$= i\hbar\epsilon_{ijk}(\alpha^j P_k - \alpha^k P_j) + \frac{i\hbar}{2}(2\epsilon_{ijk}\alpha^k P_j + 2\epsilon_{ikj}\alpha^j P_k)$$

$$= i\hbar\epsilon_{ijk}(\alpha^j P_k - \alpha^k P_j) + \frac{i\hbar}{2}(2\epsilon_{ijk}\alpha^k P_j - 2\epsilon_{ijk}\alpha^j P_k)$$

$$= i\hbar\cancel{\epsilon_{ijk}}(\cancel{\alpha^j P_k} - \cancel{\alpha^k P_j}) + i\hbar\cancel{\epsilon_{ijk}}(\cancel{\alpha^k P_j} - \cancel{\alpha^j P_k}) = 0 \checkmark$$

We can see that angular momentum is conserved

(c) Now show that the operators $\mathbf{p}\cdot\Sigma$ and $\mathbf{p}\cdot\mathbf{L}$ are each one constants of the motion. the operator

$$\frac{\mathbf{p}\cdot\Sigma}{|\mathbf{p}|}$$

is called *helicity.*

Starting with $p\cdot\Sigma$ we have

$$[P\cdot\Sigma, \mathcal{H}] = P[\Sigma,\mathcal{H}] + [P,\mathcal{H}]\Sigma$$

$$= P_x[\Sigma_x,\mathcal{H}] + P_y[\Sigma_y,\mathcal{H}] + P_z[\Sigma_z,\mathcal{H}]$$

$$\overset{0}{[P_x,\cancel{\mathcal{H}}]}\Sigma_x + \overset{+0}{[P_y,\cancel{\mathcal{H}}]}\Sigma_y + \overset{0}{[P_z,\cancel{\mathcal{H}}]}\Sigma_z$$

We then know that the following terms

$$P_x[\Sigma_x,\mathcal{H}] + P_y[\Sigma_y,\mathcal{H}] + P_z[\Sigma_z,\mathcal{H}] = 0$$

Therefore we can then say $p\cdot\Sigma$ is a constant of motion. We then see that

$$[P\cdot L, \mathcal{H}] = \alpha^i[P_iL_i, P_i] + \alpha^j[P_jL_j,P_j] + \alpha^k[P_kL_k,P_k]$$

$$= \alpha^i P_i[\cancel{L_i},\overset{0}{P_i}] + \alpha^j P_j[\cancel{L_j},\overset{0}{P_j}] + \alpha^k P_k[\cancel{L_k},\overset{0}{P_k}] = 0$$

We can see that $P\cdot L$ is also a constant of motion.

Problem 2: Continued

(d) Calculate the equation of motion for the position operator $\mathbf{x}$ of a free Dirac particle. Show that the velocity operator $\mathbf{v} \equiv \frac{d}{dt}\mathbf{x}$ is not a constant of the motion, unlike the momentum $\mathbf{p}$.

To calulate the equation of motion we use

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}[H_H, A_H(t)] + \left(\frac{\partial A_S}{\partial t}\right)_H$$

This means for $x(t)$ we have

$$\frac{d}{dt}X(t) = \frac{i}{\hbar}[\mathcal{H}_H, X_H(t)] + \left(\frac{\partial X_S}{\partial t}\right)_H$$

The equation of motion for the position operator is then calculated with

$$V(t) = \frac{i}{\hbar}[\alpha^i P_i + m\beta, X_H(t)] + \cancel{\left(\frac{\partial X_S}{\partial t}\right)_H}^{\,0} = \frac{i}{\hbar}[\alpha^i P_i + m\beta, X_H(t)]$$

$$= \frac{i}{\hbar}[\alpha^i P_i, X_H(t)] + \frac{i}{\hbar}\cancel{[m\beta, X_H(t)]}^{\,0}$$

$$= \frac{i}{\hbar}\left(\alpha^i P_i X_H(t) - X_H(t)\alpha^i P_i\right) = \frac{i}{\hbar}(-i\hbar\mathbb{1})\alpha^i = \alpha^i\mathbb{1}$$

This means our equation of motion is then

$$\boxed{V(t) = \alpha^i\mathbb{1}}$$

We can then show

$$[\mathcal{H}, V(t)] = [\alpha^i P_i + m\beta, \alpha^i] = [\alpha^i P_i, \alpha^i] + \cancel{[m\beta, \alpha^i]}^{\,0}$$

$$= \alpha^i P_i \alpha^i - \alpha^{i2} P_i \neq 0$$

Because the commutator is non zero, V(t) is not a constant of motion

Problem 2: Review

Procedure:

- – Starting with the standard Dirac Equation

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi \quad \rightarrow \quad i\hbar\frac{\partial\psi}{\partial t} = \Bigg(c\alpha^i p_i + \beta mc^2\Bigg)\psi$$

  where we have written our Hamiltonian like $\mathcal{H} = c\alpha^i p_i + \beta mc^2$ we can get to the Free Particle form like

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\psi = 0$$

  where $\gamma^\mu = \beta\alpha^\mu$ where we have $\mu \in 1, 2, 3$ and $\alpha^0 = 1$ and the rest are as previously defined
  - Proceed to expand the expression for $\mathcal{H}$ with $\psi(x,t)$ as a 4 component spinor and show that the solution to the Dirac Equation for the free particle is a plane wave
  - Redefining ψ, we use $\mathcal{H}\psi = E\psi$ to show that the desired expression can be obtained
- – Using the definition for the Hamiltonian of $\mathcal{H} = c\alpha^i p_i + \beta mc^2$ we can show $[\boldsymbol{L}_i, \mathcal{H}] \neq 0$ and thus not a constant of motion
  - Proceed to show $[\boldsymbol{J}, \mathcal{H}] = 0$ to show that angular momentum is conserved
- – Take commutators of the quantities that are given to us to show that they will evaluate to be zero
    * The rules for why some of these quantities are zero is a little hand wavey but is none the less correct
- – Using the definition for the Heisenberg picture we can find the equation of motion with

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}[\mathcal{H}_H, A_H(t)] + \Bigg(\frac{\partial A_S}{\partial t}\Bigg)_H$$

  where the subscripts H and S indicate the Heisenberg and Schrodinger pictures respectively
  - Take a commutator again to show that $\boldsymbol{V}(t)$ is not a constant of motion

Key Concepts:

- – Expanding the equation for a free particle in the Dirac equation we are required to define our wave function as a four component spinor of the form

$$\psi(x,t) = e^{-iEt}\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$$

  to show that the solution to the Dirac equation is a plane wave
  - For the spin component of our wave function

$$\begin{pmatrix}\phi\\ \chi\end{pmatrix} \quad \rightarrow \quad \phi = \begin{pmatrix}\uparrow_+\\ \downarrow_+\end{pmatrix} \quad , \quad \chi = \begin{pmatrix}\uparrow_-\\ \downarrow_-\end{pmatrix}$$

  This tells us what type of spin particle we have and what type of energy
- – True for parts (b) through (d), quantities that commute with a Hamiltonian are considered to be constants of motion

Variations:

- – We could be given a different situation for what to solve our Dirac equation for
    * This would require the same broad procedure but now with a different situation

Problem 3:

(a) Show that the Dirac equation for a central potential $V(r)$ can be written in the form

$$\chi = \frac{c}{E - V(r) + mc^2}(\sigma \cdot \mathbf{p})\phi$$

where the total wavefunction is a four component spinor

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

in the bi-spinor representation.

The Dirac equation takes the form of

$$H\Psi = E\Psi$$

Where our Hamiltonian is

$$\mathcal{H} = \left(c\alpha^i P_i + mc^2\beta + V(r)\right)$$

Which when we put this in we get

$$\mathcal{H} = c\begin{pmatrix} 0 & \sigma^i P_i \\ \sigma^i P_i & 0 \end{pmatrix} + mc^2\begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix} + V(r)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} mc^2\beta + V(r)\mathbb{1} & c\sigma^i P_i \\ c\cdot\sigma^i P_i & -mc^2\beta + V(r)\mathbb{1} \end{pmatrix}$$

This means our Dirac Equation becomes

$$\begin{pmatrix} mc^2\beta + V(r)\mathbb{1} & c\sigma^i P_i \\ c\cdot\sigma^i P_i & -mc^2\beta + V(r)\mathbb{1} \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = E\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

This gives two equations with two unknowns

$$(mc^2\beta + V(r)\mathbb{1})\varphi + (c\sigma^i P_i)\chi = E\varphi$$

$$(c\cdot\sigma^i P_i)\varphi + (-mc^2\beta + V(r)\mathbb{1})\chi = E\chi$$

$$\hookrightarrow \sigma^i P_i \doteq \sigma \cdot P$$

We now solve this system of equations

$$mc^2\beta\varphi + V(r)\varphi + c(\sigma\cdot P)\chi = E\varphi \quad , \quad c(\sigma\cdot P)\varphi - mc^2\beta\chi + V(r)\chi = E\chi$$

$$c(\sigma\cdot P)\chi = (E - mc^2\beta - V(r))\varphi \quad , \quad c(\sigma\cdot P)\varphi = (E - V(r) + mc^2\beta)\chi$$

Problem 3: Continued

Where we can simply show now from the last line

$$\boxed{\chi = \frac{c}{(E - V(r) + mc^2\beta)}(\sigma \cdot p)\varphi}$$

(b) Assume that ϕ describes an s-wave orbital with spin $\downarrow$ of the form

$$\phi(\mathbf{r}, t) = R(r)\exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}0\\1\end{pmatrix}.$$

Calculate χ explicitly and show that it describes a p-wave function with spin $s = 1/2$ and *orbital* angular momentum $l = 1$. Hint: express χ in terms of spherical harmonics and spinors.

Taking the previous result for χ, we have

$$\chi = \frac{c}{(E - V(r) + mc^2\beta)}(\sigma \cdot p) R(r)\exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}0\\1\end{pmatrix} \qquad (*)$$

Taking $(\sigma \cdot p)$ we have

$$(\sigma \cdot p) = \sigma^1 P_1 + \sigma^2 P_2 + \sigma^3 P_3 = \begin{pmatrix}0&1\\1&0\end{pmatrix}P_x + \begin{pmatrix}0&-i\\i&0\end{pmatrix}P_y + \begin{pmatrix}1&0\\0&-1\end{pmatrix}P_z = \begin{pmatrix}P_z & P_x - iP_y\\ P_x + iP_y & -P_z\end{pmatrix}$$

This means $(*)$ becomes

$$\chi = \frac{c}{(E - V(r) + mc^2\beta)} R(r)\exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}P_z & P_x - iP_y\\ P_x + iP_y & -P_z\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{c}{(E - V(r) + mc^2\beta)} R(r)\exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}P_x - iP_y\\ -P_z\end{pmatrix}$$

χ is then finally

$$\boxed{\chi(r,t) = \frac{c}{(E - V(r) + mc^2\beta)} R(r)\exp\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix}P_x - iP_y\\ -P_z\end{pmatrix}}$$

The first component of our column vector

$$P_x - iP_y$$

Problem 3: Continued

Can be written in terms of Spherical Harmonics, with $\ell=1$ and $m=-1$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,e^{-i\varphi}\sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot\frac{P_x - iP_y}{P}$$

Conversely for P_z, $\ell=1$, $m=0$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\frac{P_z}{P}$$

We can then say

$$P_x - iP_y = 2\sqrt{\frac{2\pi}{3}}\,P\,Y_1^{-1}(\theta,\varphi) \quad , \quad P_z = 2\sqrt{\frac{\pi}{3}}\,P\,Y_1^{0}(\theta,\varphi)$$

This means for a particle with positive energy and spin $\downarrow$

$$\boxed{\Psi(r,t) = \frac{c}{(E - V(r) + mc^2\beta)}\,R(r)\exp\left(-\frac{iEt}{\hbar}\right) 2\sqrt{\frac{\pi}{3}}\,P\begin{pmatrix} 0 \\ 1 \\ \sqrt{2}\,Y_1^{-1}(\theta,\varphi) \\ -Y_1^{0}(\theta,\varphi)\end{pmatrix}}$$

Where one can see that this is clearly a p-wave function with $s=\frac{1}{2}$, $\ell=1$

(c) Using your previous result, show that $\chi(\mathbf{r},t)$ describes a $j=1/2$ state with $m=-1/2$, where j and m are the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$ quantum number. Hint: use a table of Clebsch-Gordan coefficients.

$\chi(r,t)$ can be written as

$$\chi(r,t) = \frac{c}{(E - V(r) + mc^2\beta)}\,R(r)\exp\left(-\frac{iEt}{\hbar}\right) 2\sqrt{\frac{\pi}{3}}\,P\begin{pmatrix} \sqrt{2}\,Y_1^{-1}(\theta,\varphi) \\ -Y_1^{0}(\theta,\varphi)\end{pmatrix}$$

We then know that our particle is a spin $\frac{1}{2}$ particle oriented down, this means

$$\text{w/ } \ell=1 \Rightarrow m \in [-1,1] \;,\; m \neq 1 \rightarrow \text{From } \chi(r,t) \Rightarrow m = -1, 0$$

For $m=0$

$$|\ell - s| \leq j \leq |\ell + s| \Rightarrow |\tfrac{1}{2}| \leq j \leq |\tfrac{3}{2}|$$

Problem 3: Continued

If $j = 1/2$, this means $s = 1/2$, we then can say

$$m \in \{-1/2, 1/2\}$$

This then means we have a state with the quantum numbers

$$j = 1/2 \,, \; l = 1 \,, \; m = -1/2$$

we then take our $\chi(r,t)$ and expand in a complete of total angular momentum

$$\chi(r,t) = \sum_{J,M} |J,M\rangle\langle J,M| \chi(r,t)$$

$$= \sum_{J,M} |J,M\rangle \underbrace{\langle J,M| [\text{stuff}] \left(\sqrt{2}\, Y_1^{-1}(\theta,\varphi)|\uparrow\rangle - Y_1^{0}(\theta,\varphi)|\downarrow\rangle\right)}_{\text{C.G Coefficients}}$$

Since we have both $m_s = -1/2$, $m_s = 1/2$

$$\chi(r,t) = \sum_{J,M} |J,M\rangle\langle J,M| [\text{stuff}] \left(\sqrt{2}\, Y_1^{-1}(\theta,\varphi)|\uparrow\rangle - Y_1^{0}(\theta,\varphi)|\downarrow\rangle\right)$$

$$= \sum_{J,M} |J,M\rangle [\text{stuff}] \langle J,M| \left(\sqrt{2}\; |1,-1\rangle \otimes |1/2, 1/2\rangle - |1,0\rangle \otimes |1/2,-1/2\rangle\right)$$

$$= \sum_{J,M} |J,M\rangle [\text{stuff}] \left(\sqrt{2}\, \underbrace{\langle 3/2,-1/2|1,-1;1/2,1/2\rangle}_{\cancel{\sqrt{1/3}}} - \underbrace{\langle 3/2,-1/2|1,0;1/2,-1/2\rangle}_{\cancel{\sqrt{2/3}}}\right)$$

$$+$$

$$= \sum_{J,M} |J,M\rangle [\text{stuff}] \left(\sqrt{2}\, \underbrace{\langle 1/2,-1/2|1,-1;1/2,1/2\rangle}_{-\sqrt{2/3}} - \underbrace{\langle 1/2,-1/2|1,0;1/2,-1/2\rangle}_{\sqrt{1/3}}\right)$$

We can then see that the only state that exists is when $j = 1/2$, $m = -1/2$ due to the C.G coefficients not cancelling

Problem 3: Review

Procedure:

- – Start with the standard eigenvalue equation $\mathcal{H}\psi = E\psi$ with the Hamiltonian defined as
$$\mathcal{H} = (c\alpha^i p_i + mc^2\beta + V(r))$$
where we have now included our central potential
  – Take the standard representation for the Dirac matrices α^i and β with the definition of our wave function to show that we can get the desired expression
- – Using the definition for ϕ that was given to us put this into the previous expression for χ and substitute what $(\sigma \cdot p)$ evaluates to
  – Taking the expression for χ we can then write this in terms of Spherical Harmonics
- – Taking the values for l and m from part (b) we can deduce what the range of our quantum numbers can run between by expanding χ in terms of the complete angular momentum basis
  – After expanding in a complete set we then can determine our Clebsh Gordan coefficients and what state is possible

Key Concepts:

- – After writing our Hamiltonian in this new form we can write χ in terms of what is required from us
- – We can write χ in terms of spherical tensors
- – After writing χ in terms of spherical tensors, we can determine what states are possible by calculating the Clebsh Gordan coefficients

Variations:

- – We could be given a different potential
  * This would require us to solve for different equations but it would be the same broad procedure
- – For parts (b) and (c), if we could have a particle with a different spin or angular momentum
  * This would change what our quantum numbers were when calculating the Clebsh Gordan coefficients but it would be the same procedure