E & M I

Workshop 10 – Dielectrics, 4/13/2022

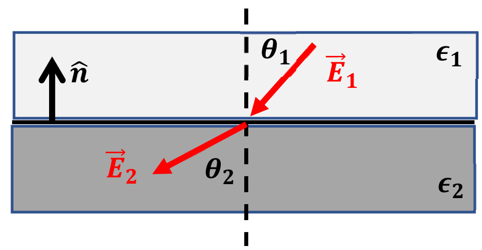
1) Boundary Conditions

A) In class on Monday, we introduced the concept of the polarization field of a material, the electric displacement, the dielectric, and the polarizability:

$$\vec{D} = \epsilon_0\,\vec{E} + \vec{P} = \epsilon\,\vec{E}, \qquad \vec{P} = \epsilon_0\,\chi_e\,\vec{E}, \qquad \epsilon = \epsilon_0\,(1+\chi_e)$$

The simplest model for ϵ and χ_e is that they are "Linear, Isotropic, and Homogeneous". Explain, briefly, what these terms mean in the context of dielectrics.

Consider the interface between two dielectric materials, ϵ_1 and ϵ_2. $\hat{n}$ is the normal to the interface and there is no free charge on this surface. There are electric fields $\vec{E}_1$ and $\vec{E}_2$ in the dielectric materials.



From Maxwell's equations and the definition of $\vec{D}$ the boundary conditions on the fields can be written:

$$\vec{\nabla}\cdot\vec{D} = 0 \Leftrightarrow \hat{n}\cdot\left(\vec{D}_1 - \vec{D}_2\right) = \sigma_{free} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \Leftrightarrow \hat{n}\times\left(\vec{E}_1 - \vec{E}_2\right) = 0$$

B) Define the normal and parallel component fields as, for example:

$$\vec{E} = \vec{E}_n + \vec{E}_p, \qquad \hat{n}\cdot\vec{E}_p = 0, \qquad \hat{n}\times\vec{E}_n = 0$$

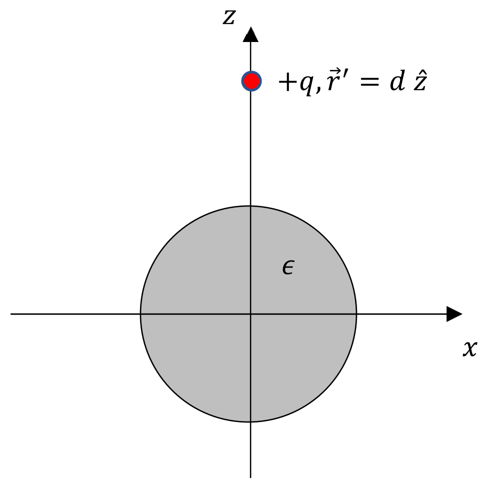
Using the boundary conditions above, determine the boundary condition for the parallel fields $\vec{E}_{1p}$ and $\vec{E}_{2p}$.

C) Using the boundary conditions above, determine the boundary condition for the normal fields $\vec{E}_{1n}$ and $\vec{E}_{2n}$. (Consider $\vec{D}_{1n}$ and $\vec{D}_{2n}$.)

D) Defining θ_1 and θ_2 as the angles of the field relative to the normal, derive a relationship between $\tan\theta_1$ and $\tan\theta_2$, giving the change in direction of the electric field in the two dielectrics.

For the picture above, which is larger, ϵ_1 or ϵ_2?

2) Dielectric Sphere and a charge



Consider an insulating sphere of radius R and dielectric constant ϵ and a point charge $+q$ on the z-axis with a position $\vec{r}\,' = d\,\hat{z}$.

Due to the spherical boundary conditions and the azimuthal symmetry of the problem, we know we can use an expansion in Legendre Polynomials. For any azimuthally symmetric function satisfying Laplace's Equation:

$$F(r,\theta) = \sum_l \left(a_l\, r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

And for the point charge:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\theta)$$

Where $r_< = r, \text{for } r < r'$, $r_< = r', \text{for } r > r'$, $r_> = r, \text{for } r > r'$, and $r_> = r', \text{for } r < r'$.

A) Define the potentials:

$\phi_{SI}(r,\theta)$ = The potential due to the sphere for $r < R$ (inside the sphere).
$\phi_{SO}(r,\theta)$ = The potential due to the sphere for $r > R$ (outside the sphere).
$\phi_{qI}(r,\theta)$ = The potential due to the charge for $r < d$.
$\phi_{qII}(r,\theta)$ = The potential due to the charge for $r > d$.

Write down Legendre polynomial expansions for the total potential $\phi_T = \phi_S + \phi_q$ (sum of the sphere potential and potential of q) both inside and outside the sphere, for

$r < d$. Do not include terms that must be zero for ϕ_T to remain finite everywhere.

B) Using the fact that ϕ_T is continuous at the surface of the sphere, find a relation between the coefficients a_l and b_l in the Legendre expansions ϕ_{SI} and ϕ_{SO}.

C) Using the boundary condition that $\hat{r}\cdot\vec{D}$ is continuous at the surface of the sphere (and that $\vec{D} = \epsilon\,\vec{E} = -\epsilon\,\vec{\nabla}\phi$), solve for the coefficients a_l and b_l in ϕ_{SI} and ϕ_{SO}.

D) The force on the point charge due to the dipole is:

$$F_q = -q\,\partial_r\,\phi_{SO}(r,0)|_{r=d}$$

Solve for the force due to the dipole, $l = 1$, term in the expansion for ϕ_{SO}.

E) This force decreases for large d as d^{-5}. The force on a point charge due to a permanent dipole $\vec{p}$ depends on the distance as d^{-3}.

Explain why the force on the point charge due to the dielectric sphere is smaller at large d (d^{-5}) than for a permanent dipole (d^{-3}) for large d.