Physics 5393
Solutions to Exam II

1. Consider an eigenket with continuous eigenvalues that represent a rotation angle in a plane about an axis

$$\tilde{\mathbf{\Phi}} \left|\phi\right\rangle = \phi \left|\phi\right\rangle.$$

Now consider a unitary operator that performs an angular rotation

$$\tilde{\mathbf{U}}(\Delta\phi) \left|\phi\right\rangle = \left|\phi + \Delta\phi\right\rangle.$$

(a) In addition to the operator being unitary, it must be continuous from the unit operator, reversible, and successive rotations must also yield a rotation. Give a possible form for an infinitesimal rotation operator and confirm that it satisfies all four criteria. In addition, if there are any conditions that the criteria above impose on components of the infinitesimal operator make sure that you state these.

In order for the operator to be continuous from the identity operator (unit operator), it must satisfy the following relation $\lim_{\delta\phi\to 0} \tilde{\mathbf{U}}(\delta\phi) \to \tilde{\mathbf{1}}$. In this case, a possible form is

$$\tilde{\mathbf{U}}(\delta\phi) = \tilde{\mathbf{1}} - i\tilde{\mathbf{K}}\delta\phi.$$

For the operator to be unitarity, it must satisfy the following

$$\tilde{\mathbf{U}}^\dagger(\delta\phi)\tilde{\mathbf{U}}(\delta\phi) = \tilde{\mathbf{1}}.$$

Applying this condition to our operator and keeping only terms to first order in $\delta\phi$ we find

$$\tilde{\mathbf{U}}^\dagger(\delta\phi)\tilde{\mathbf{U}}(\delta\phi) = \tilde{\mathbf{1}} + i(\tilde{\mathbf{K}}^\dagger - \tilde{\mathbf{K}})\delta\phi,$$

which can only be satisfied if the operator $\tilde{\mathbf{K}}$ is Hermitian $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^\dagger$. Note that the unitarity condition automatically causes the operator to be reversible since $\tilde{\mathbf{U}}^\dagger$ reverses the transformation $\tilde{\mathbf{U}}^\dagger = \tilde{\mathbf{1}} - i\tilde{\mathbf{K}}(-\delta\phi)$. The final condition that successive rotation lead to a rotation can be shown as follows

$$\tilde{\mathbf{U}}(\delta\phi_1)\tilde{\mathbf{U}}(\delta\phi_2) = \tilde{\mathbf{1}} - i\tilde{\mathbf{K}}(\delta\phi_1 + \delta\phi_2),$$

again keeping only terms to first order in $\delta\phi$. Therefore, it satisfies all conditions if the generator $(\tilde{\mathbf{K}})$ is Hermitian.

(b) The infinitesimal operator deduced above comprises an operator that generates the rotations that is denoted here as $\tilde{\mathbf{K}}_z$. Calculate the commutation relation $\left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{K}}_z\right]$ starting with the calculation of the following commutator $\left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{U}}(\delta\phi)\right]$.

Starting as suggested in the statement of the problem and applying the operator on an eigenket of rotation

$$\left.\begin{array}{l} \tilde{\mathbf{\Phi}}\tilde{\mathbf{U}}(\delta\phi) \left|\phi\right\rangle = (\phi + \delta\phi) \left|\phi + \delta\phi\right\rangle \\ \tilde{\mathbf{U}}(\delta\phi)\tilde{\mathbf{\Phi}} \left|\phi\right\rangle = \phi \left|\phi + \delta\phi\right\rangle \end{array}\right\} \quad \Rightarrow \quad \left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{U}}(\delta\phi)\right] \left|\phi\right\rangle = \delta\phi \left|\phi + \delta\phi\right\rangle \approx \delta\phi \left|\phi\right\rangle.$$

Using this commutator, the desired commutation relation follows

$$\delta\phi = \left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{U}}(\delta\phi)\right] = \tilde{\mathbf{\Phi}}\tilde{\mathbf{U}}(\delta\phi) - \tilde{\mathbf{U}}(\delta\phi)\tilde{\mathbf{\Phi}} = \tilde{\mathbf{\Phi}}(\tilde{\mathbf{1}} - i\delta\phi\tilde{\mathbf{K}}) - (\tilde{\mathbf{1}} - i\delta\phi\tilde{\mathbf{K}})\tilde{\mathbf{\Phi}} = i\delta\phi\left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{K}}\right],$$

which leads to

$$\left[\tilde{\mathbf{\Phi}}, \tilde{\mathbf{K}}\right] = -i.$$

(c) Starting with the infinitesimal angular translation operator, derive an explicit form for the operator $\tilde{\mathbf{K}}_z$ acting on an arbitrary state $|\alpha\rangle$ in the rotation angle basis $\left\langle \phi \middle| \tilde{\mathbf{K}}_z \middle| \alpha \right\rangle$.

The first step is to apply the infinitesmal rotation operator on the state vector and then expand in a complete set

$$\begin{aligned}
\left(\tilde{\mathbf{1}} - i\delta\phi\tilde{\mathbf{K}}\right) |\alpha\rangle =& \tilde{\mathbf{U}}(\delta\phi)\, |\alpha\rangle \\
=& \int \tilde{\mathbf{U}}(\delta\phi)\, |\phi'\rangle\langle\phi'|\, |\alpha\rangle\, d\phi' \\
=& \int |\phi' + \delta\phi\rangle\langle\phi'|\, |\alpha\rangle\, d\phi' \\
=& \int |\phi'\rangle\, \langle\phi' - \delta\phi\,|\alpha\rangle\, d\phi' \\
=& \int |\phi'\rangle \left(1 - \delta\phi\frac{\partial}{\partial\phi'}\right) \langle\phi'\,|\alpha\rangle\, d\phi'.
\end{aligned}$$

Comparing the two sides, the generator in the angle representation is

$$\tilde{\mathbf{K}} \doteq -i\frac{\partial}{\partial\phi}.$$

2. B particles and anti-particles can be produced at particle accelerators in a variety of states that are associated with observables that may or may not commute. The two most common states are those associated with the Hamiltonian and are specified by the eigenvalue equations

$$\begin{aligned}
\tilde{\mathbf{H}}\, |B_1\rangle &= m_1\, |B_1\rangle \\
\tilde{\mathbf{H}}\, |B_2\rangle &= m_2\, |B_2\rangle\,,
\end{aligned}$$

where m_i is the mass associated with the state and $m_1 \neq m_2$. In addition, they can be specified by their "*flavor*" through the eigenvalue equations

$$\begin{aligned}
\tilde{\mathbf{F}}\, \left|B^0\right\rangle &= +1\, \left|B^0\right\rangle \\
\tilde{\mathbf{F}}\, \left|\bar{B}^0\right\rangle &= -1\, \left|\bar{B}^0\right\rangle.
\end{aligned}$$

Furthermore, the eigenkets are related to each other as follows

$$\begin{aligned}
|B_1\rangle &= \frac{1}{\sqrt{2}}\Big[\, \left|B^0\right\rangle + \left|\bar{B}^0\right\rangle \Big] \\
|B_2\rangle &= \frac{1}{\sqrt{2}}\Big[\, \left|B^0\right\rangle - \left|\bar{B}^0\right\rangle \Big].
\end{aligned}$$

For those parts below that need an initial state, assume that at time $t = 0$ the system is in the $\left|B^0\right\rangle$ state.

(a) Express the time evolution operator in a representation using the eigenstates of the $\tilde{\mathbf{F}}$ operator and write the matrix associated with this operator.

The representation in the $\tilde{\mathbf{F}}$ basis is

$$\begin{aligned}
e^{-i\tilde{\mathbf{H}}t/\hbar} = & \left|B^0\right\rangle\!\left\langle B^0\right|\, e^{-i\tilde{\mathbf{H}}t/\hbar}\, \left|B^0\right\rangle\!\left\langle B^0\right| \\
&+ \left|B^0\right\rangle\!\left\langle B^0\right|\, e^{-i\tilde{\mathbf{H}}t/\hbar}\, \left|\bar{B}^0\right\rangle\!\left\langle \bar{B}^0\right| \\
&+ \left|\bar{B}^0\right\rangle\!\left\langle \bar{B}^0\right|\, e^{-i\tilde{\mathbf{H}}t/\hbar}\, \left|B^0\right\rangle\!\left\langle B^0\right| \\
&+ \left|\bar{B}^0\right\rangle\!\left\langle \bar{B}^0\right|\, e^{-i\tilde{\mathbf{H}}t/\hbar}\, \left|\bar{B}^0\right\rangle\!\left\langle \bar{B}^0\right|
\end{aligned}$$

Since the eigenstates $\left|B^0\right\rangle$ and $\left|\bar{B}^0\right\rangle$ are not eigenstates of the Hamiltonian, they must be expressed in terms of $\left|B_1\right\rangle$ and $\left|B_2\right\rangle$

$$\left.\begin{aligned} \left|B_1\right\rangle &= \frac{1}{\sqrt{2}}\Big[\,\left|B^0\right\rangle + \left|\bar{B}^0\right\rangle\Big] \\ \left|B_2\right\rangle &= \frac{1}{\sqrt{2}}\Big[\,\left|B^0\right\rangle - \left|\bar{B}^0\right\rangle\Big] \end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned} \left|B^0\right\rangle &= \frac{1}{\sqrt{2}}\Big[\,\left|B_1\right\rangle + \left|B_2\right\rangle\Big] \\ \left|\bar{B}^0\right\rangle &= \frac{1}{\sqrt{2}}\Big[\,\left|B_1\right\rangle - \left|B_2\right\rangle\Big]. \end{aligned}\right.$$

Hence, the matrix elements are

$$\begin{aligned} \left\langle B^0 \left| e^{-i\tilde{\mathbf{H}}t/\hbar} \right| B^0 \right\rangle &= \frac{1}{2}\Big(\left\langle B_1\right| + \left\langle B_2\right|\Big)e^{-i\tilde{\mathbf{H}}t/\hbar}\Big(\left|B_1\right\rangle + \left|B_2\right\rangle\Big) = \frac{1}{2}\left(e^{-im_1t/\hbar} + e^{-im_2t/\hbar}\right) \\ \left\langle B^0 \left| e^{-i\tilde{\mathbf{H}}t/\hbar} \right| \bar{B}^0 \right\rangle &= \frac{1}{2}\Big(\left\langle B_1\right| + \left\langle B_2\right|\Big)e^{-i\tilde{\mathbf{H}}t/\hbar}\Big(\left|B_1\right\rangle - \left|B_2\right\rangle\Big) = \frac{1}{2}\left(e^{-im_1t/\hbar} - e^{-im_2t/\hbar}\right) \\ \left\langle \bar{B}^0 \left| e^{-i\tilde{\mathbf{H}}t/\hbar} \right| B^0 \right\rangle &= \frac{1}{2}\Big(\left\langle B_1\right| - \left\langle B_2\right|\Big)e^{-i\tilde{\mathbf{H}}t/\hbar}\Big(\left|B_1\right\rangle + \left|B_2\right\rangle\Big) = \frac{1}{2}\left(e^{-im_1t/\hbar} - e^{-im_2t/\hbar}\right) \\ \left\langle \bar{B}^0 \left| e^{-i\tilde{\mathbf{H}}t/\hbar} \right| \bar{B}^0 \right\rangle &= \frac{1}{2}\Big(\left\langle B_1\right| - \left\langle B_2\right|\Big)e^{-i\tilde{\mathbf{H}}t/\hbar}\Big(\left|B_1\right\rangle - \left|B_2\right\rangle\Big) = \frac{1}{2}\left(e^{-im_1t/\hbar} + e^{-im_2t/\hbar}\right) \end{aligned}$$

(b) Calculate the probability that the system is in the state $\left|\bar{B}^0\right\rangle$ at some arbitrary time $t > 0$?
In order to derive the time evolution of the initial state, it must be expressed in the eigenkets of the Hamiltonian. The initial state can be derived from the expressions given in the statement of the problem, which yield

$$\begin{aligned} \left|B^0\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|B_1\right\rangle + \left|B_2\right\rangle\right] \\ \left|\bar{B}^0\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|B_1\right\rangle - \left|B_2\right\rangle\right], \end{aligned}$$

where we also specify the $\left|\bar{B}^0\right\rangle$ state as it will be used in obtaining the solution. The time evolved state is then

$$\left|\alpha, t\right\rangle = \boldsymbol{\mathcal{U}}(t)\left|B^0\right\rangle = \frac{1}{\sqrt{2}}\left[\left|B_1\right\rangle e^{-im_1t/\hbar} + \left|B_2\right\rangle e^{-m_2t/\hbar}\right].$$

The probability of this state evolving to a $\left|\bar{B}^0\right\rangle$ at a later time is

$$\left|\left\langle \bar{B}^0 \left|\alpha, t\right\rangle\right.\right|^2 = \left|\frac{1}{2}\left(1 - e^{i\Delta mt/\hbar}\right)e^{-im_2t/\hbar}\right|^2 = \frac{1}{2}\left[1 - \cos\left(\frac{\Delta mt}{\hbar}\right)\right] = \sin^2\left(\frac{\Delta mt}{2\hbar}\right)$$

with $\Delta m = m_2 - m_1$.

(c) What is the probability that a measurement of the mass yields a value of m_2 for $t > 0$?
The time dependent state of the system is given by

$$\left|\alpha, t\right\rangle = \boldsymbol{\mathcal{U}}(t)\left|B^0\right\rangle = \frac{1}{\sqrt{2}}\left[\left|B_1\right\rangle e^{-im_1t/\hbar} + \left|B_2\right\rangle e^{-m_2t/\hbar}\right].$$

The probability of being in the mass eiegenstate $\left|B_2\right\rangle$ is

$$\left|\left\langle B_1 \left|\alpha; t\right\rangle\right.\right|^2 = \frac{1}{2}.$$