E & M I

Workshop 4 – Magnetic Fields and Forces, Solutions

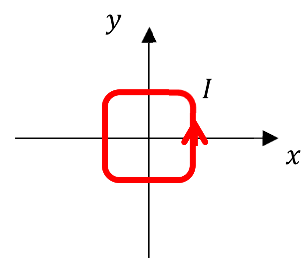
Monday's class was our start with magnetic fields and forces, considering "static" situations. The argument was made that the experimental similarities between electric and magnetic forces, and the fact that there are no magnetic "charges" (monopoles), the fundamental source of magnetic fields is a dipole and we can write down the field due to a magnetic dipole in complete analogy with the electric dipole:

$$\vec{B}_m(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\,\frac{(\vec{m}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$

where $\vec{m}$ is the magnetic dipole. The potential energy, force, and torque on a dipole are:

$$U = -\vec{m}\cdot\vec{B}, \qquad \vec{F} = \vec{\nabla}\left(\vec{m}\cdot\vec{B}\right), \qquad \vec{\tau} = \vec{m}\times\vec{B}$$

In the book and briefly in class (detailed notes soon), the argument was made that a square current loop with current I and side length L is a magnetic dipole $\vec{m} = I\,L^2\,\hat{n} = I\,A\,\hat{n}$. $\hat{n}$ is the normal to the loop using the right-hand rule for the current. This should approach a point-dipole as the area of the current loop gets small.



$\vec{m} = I\,L^2\,\hat{z}$

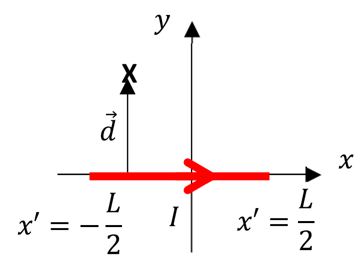
1) Square Loop Magnetic Field and Magnetic Dipoles

Let's test this equivalence to a dipole for a square loop.

At the end of class we considered the Biot-Savart law for a current loop magnetic field:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,I\oint d\vec{l}'\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

where $\vec{r}'$ and $d\vec{l}'$ refer to points on the current loop. Doing this integral for a current of length L on the x-axis and centered at $x = 0$ for a point a displacement $\vec{d}$ perpendicular to the current at the point $\vec{r} = x\,\hat{x} + d\,\hat{y}$:



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\,I\int_{-\frac{L}{2}}^{\frac{L}{2}} dx'\,\hat{x}\times\frac{(x-x')\hat{x} + d\,\hat{y}}{((x-x')^2+d^2)^{\frac{3}{2}}} = (\hat{I}\times\hat{d})\,\frac{\mu_0}{4\pi}\frac{I}{d}\,(\sin\theta_+ - \sin\theta_-)$$

Where $\hat{I}$ is the direction of the current, $\hat{d}$ is the perpendicular direction from the wire to the field point $\vec{d}$, $\hat{I}\times\hat{d}$ is the direction of the field, and:

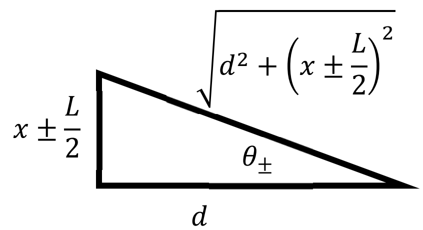
$$\sin\theta_+ = \sin\left(\tan^{-1}\left(\frac{x+\frac{L}{2}}{d}\right)\right), \qquad \sin\theta_- = \sin\left(\tan^{-1}\left(\frac{x-\frac{L}{2}}{d}\right)\right)$$

a) Show that:

$$\sin\theta_+ = \frac{x+\frac{L}{2}}{\sqrt{d^2+\left(x+\frac{L}{2}\right)^2}}, \qquad \sin\theta_- = \frac{x-\frac{L}{2}}{\sqrt{d^2+\left(x-\frac{L}{2}\right)^2}}$$

Drawing triangles to covert $\tan^{-1}(stuff)$ into algebra:

$$\tan^{-1}\left(\frac{x\pm\frac{L}{2}}{d}\right)$$



This gives the results above.

b) Checking the algebra (okay, checking my algebra) show that you get the known (?) results for:

i. The field at a point perpendicular to the middle of the current ($x = 0$), and

For $x = 0$,

$$\sin\theta_+ = \frac{L}{2\sqrt{d^2+\frac{L^2}{4}}}, \sin\theta_- = -\sin\theta_+, \sin\theta_+ - \sin\theta_- = 2\sin\theta_+$$

$$B = \left(\hat{I}\times\hat{d}\right)\frac{\mu_0}{4\pi}\frac{I}{d}\left(2\sin\theta_+\right) = \left(\hat{I}\times\hat{d}\right)\frac{\mu_0}{4\pi}\frac{I}{d}\frac{L}{\sqrt{d^2+\frac{L^2}{4}}}$$

This is an integral everyone should be able to do. It's something that is done in intro physics.

ii. The field for a very long current, $L \gg d$ and $L \gg x$.
(Compare to the results of Ampere: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$)

In this case, $\sin\theta_+ = 1, \sin\theta_- = -1$ and so

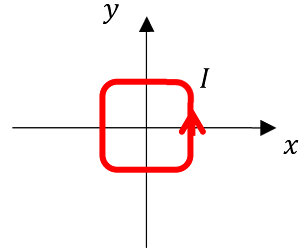
$$B = \left(\hat{I}\times\hat{d}\right)\frac{\mu_0}{4\pi}\frac{I}{d}\left(2\sin\theta_+\right) = \left(\hat{I}\times\hat{d}\right)\frac{\mu_0}{2\,\pi}\frac{I}{d}$$

Ampere's law for an infinite wire gives:

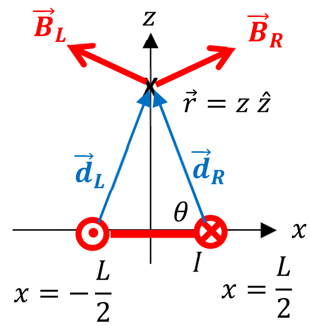
$$\oint \vec{B}\cdot d\vec{l} = B(r)\,2\,\pi\,r = \mu_0\,I_{enclosed}$$

$$B(r) = \frac{\mu_0}{2\,\pi}\frac{I}{r}$$

c) Next, let's take our square loop "dipole" and calculate its magnetic field. Consider a square loop in the x-y plane, centered at the origin as shown in the figures. We'll determine the magnetic field at points on the z-axis, $\vec{B}(\vec{r}), \vec{r} = z\,\hat{z}$.
The bottom figure is an edge-on view of the loop looking in the +y direction. The current is going into the page on the right and out of the page on the left.



$\vec{m} = I\,L^2\,\hat{z}$



i) What will be the direction of the magnetic field on the z-axis? Describe how you will solve for the magnetic field? What symmetries might you use to simplify the calculation?
Draw a picture (I would suggest something like the bottom figure) to show and explain how you will calculate the field.

The loop is symmetric about the z-axis for reflections in x and y. Therefore, the field must be in the z-direction. The fields due to the wires at $x = \pm\frac{L}{2}$ show this. The x-components cancel resulting in a field in the z-direction.

We can just calculate the z-component of the field due to one wire, and multiply by 4.

ii) Calculate $\vec{B}(z\,\hat{z})$ for any z. Show your work.

The result from above for the field a distance d from the center of the wire can be used. Consider the leg of the loop at $x = \frac{L}{2}$.

$$B = (\hat{I} \times \hat{d})\,\frac{\mu_0}{4\pi}\frac{I}{d}\,\frac{L}{\sqrt{d^2 + \frac{L^2}{4}}}$$

For this one wire:

$$\hat{I} = \hat{y}, \hat{d} = \sin\theta\;\hat{z} - \cos\theta\;\hat{x}, \hat{I} \times \hat{d} = \sin\theta\;\hat{x} + \cos\theta\;\hat{z}$$

$$B_{z1} = \frac{\mu_0}{4\pi}\frac{I}{d}\,\frac{L}{\sqrt{d^2 + \frac{L^2}{4}}}\cos\theta = \frac{\mu_0}{4\pi}\frac{I}{d}\,\frac{L}{\sqrt{d^2 + \frac{L^2}{4}}}\,\frac{L}{2d}$$

$$B_{z1} = \frac{\mu_0}{8\,\pi}\frac{(I\,L^2)}{d^2}\,\frac{1}{\sqrt{d^2 + \frac{L^2}{4}}}, \qquad d^2 = z^2 + \frac{L^2}{4}$$

This gives the total field:

$$B_z = 4\,B_{z1} = \frac{\mu_0}{2\,\pi}\,\frac{(I\,L^2)}{\left(z^2 + \frac{L^2}{4}\right)\sqrt{z^2 + \frac{L^2}{2}}}$$

iii) Show that your result approaches the point-magnetic-dipole field $\vec{B}_m(\vec{r})$ above in the limit $z \gg L$.

The result from above, for the $z \gg L$ limit is:

$$B_z = \frac{\mu_0}{2\,\pi}\frac{(I\,L^2)}{z^3}$$

For a point magnetic dipole,

$$\vec{B}_m(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}\cdot\vec{r})\,\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$

In this case:

$$\vec{m} = I\,L^2\,\hat{z}, \qquad \vec{r} = z\,\hat{z}$$

Giving

$$\vec{B}_m(z) = \frac{\mu_0}{4\pi}\left(3\frac{(I\,L^2\,z)\,z\,\hat{z}}{z^5} - \frac{I\,L^2\,\hat{z}}{z^3}\right) = \frac{\mu_0}{4\pi}\left(2\frac{(I\,L^2)}{z^3}\right)$$

The same result as above.