E & M I

Workshop 8 – Plane Waves, 3/28/2022

We won't spend much time considering time dependence and waves in E&M I, but we'll consider the basic derivations and properties of waves to build some mental pictures.

Today, in particular, you should be working in groups of 3.

1) Picturing waves:

(This is an exercise developed at Oregon State University. It has been given to groups of physics and astronomy faculty at the AAPT/APS/AAS New Faculty Workshop with some very interesting results and discussions.)

Each group will be given sheets of paper with a set of Coordinate Axes like the one attached below, and a vector $\vec{k}$. The vector $\vec{k}$ has the units of inverse length.

A) On your paper, draw lines for the positions $\vec{r}$ that satisfy:

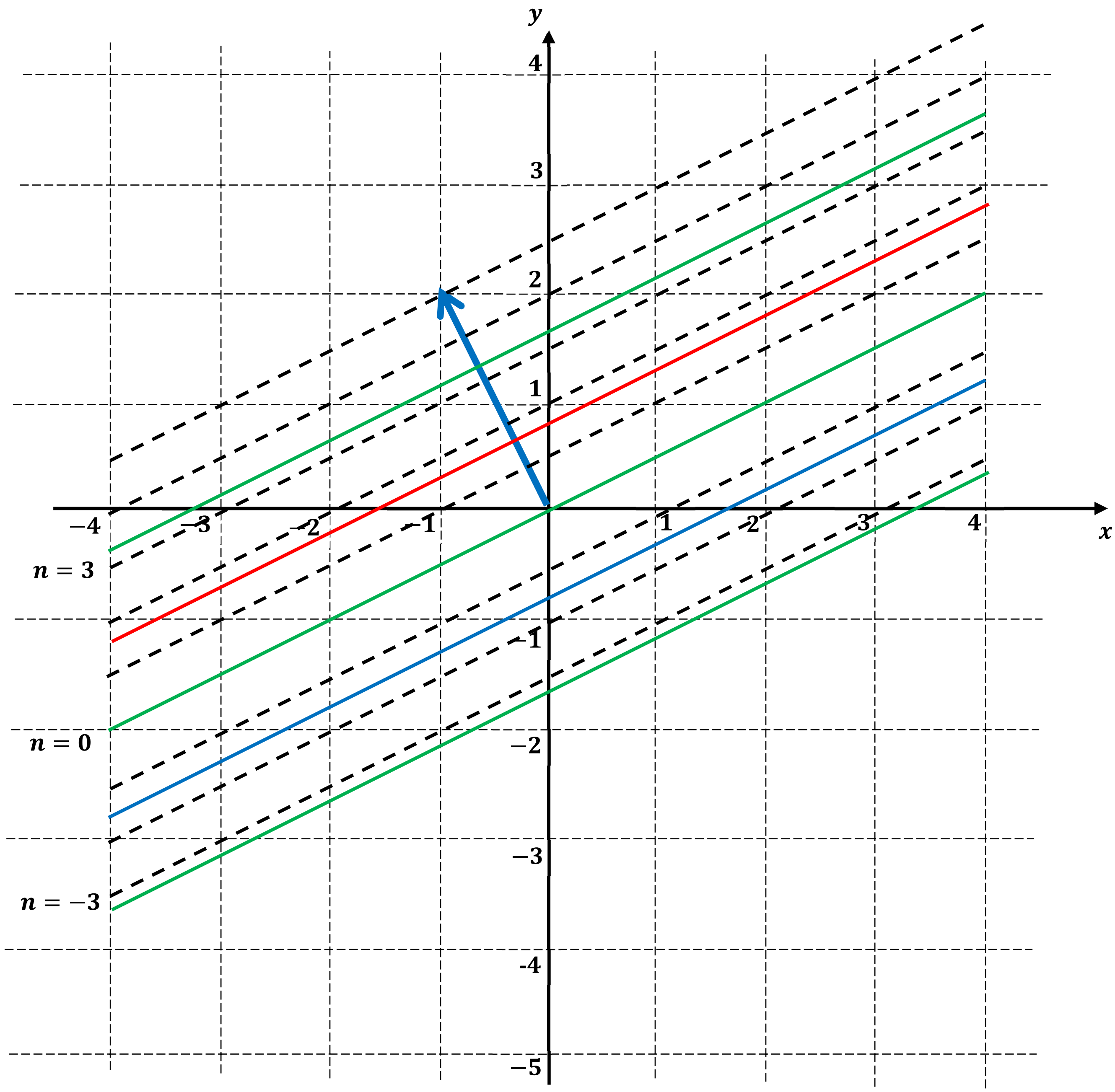
$$\vec{k} \cdot \vec{r} = n, \qquad n = \text{integers}$$

There will be one line for each integer n. Of course, you don't have to do EVERY integer, but you should describe how you drew each of these lines.

$$\vec{k} \cdot \vec{r} = n \Rightarrow k_x\, x + k_y\, y = n$$

$$y = \frac{n}{k_y} - \frac{k_x}{k_y}\, x$$

This is a series of straight lines with slope $-\frac{k_x}{k_y}$ and x-intercept $\frac{n}{k_y}$. For example, below $k_x = -1, k_y = 2$ giving the lines shown.



B) On the same paper, draw a representation of the function:

$$f_{\vec{k}}(\vec{r}) = A\sin\left(\vec{k}\cdot\vec{r}\right)$$

> The function $f_{\vec{k}}(\vec{r}) = 0$ when $\vec{k}\cdot\vec{r} = n\,\pi$. These are the green lines shown above $(n = 0, n = \pm\pi, \dots)$
>
> The function has maxima when $\vec{k}\ \cdot\vec{r} = \frac{\pi}{2}, 5\frac{\pi}{2}, 9\frac{\pi}{2}, \dots \pm (4n+1)\frac{\pi}{2}$
>
> The function has minima when $\vec{k}\ \cdot\vec{r} = 3\frac{\pi}{2}, 7\frac{\pi}{2}, 11\frac{\pi}{2}, \dots \pm (4n+3)\frac{\pi}{2}$
>
> $f_{\vec{k}}(\vec{r})$ will be an oscillating sin function along any vector parallel to $\vec{k}$.

C) Finally, consider the behavior of the function:

$$f_{\vec{k},\alpha} = A\sin(\vec{k}\cdot\ \vec{r} - \alpha\,t)$$

t is, of course, time. You can either represent this on your picture by indicating what happens to the function with time and/or describe this behavior in your answers.

> In this case, the lines for $\vec{k}\cdot\vec{r} = n$ will move as a function of time. For example, consider the $n = 0$ constant "phase" line. With time this is given by:
>
> $$\vec{k}\cdot\vec{r} = \alpha\,t$$
>
> The $n = 0$ line will move in the direction of $\vec{k}$, following the positive n lines in the $t = 0$ picture above.

2) Exploring Plane Waves:

In class last week, we kind of waved our hands a bit and came up with a description of wave properties for E & M fields in a vacuum. Here we'll go a little further.

A) We did this in class, but I want to make sure that you can do this. Starting with the vacuum Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\,\partial_t \vec{E}, \qquad \frac{1}{c^2} = \mu_0\,\epsilon_0$$

Derive the wave equations for $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. You don't have to go into great detail, as it's in the notes and any textbook, but you should be sure that you can recreate this quickly and efficiently.

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\partial_t \vec{\nabla} \times \vec{B}$$

$$\hat{x}_i\,\epsilon_{ijk}\,\partial_j\,\epsilon_{klm}\,\partial_l\,E_m = -\partial_t\left(\frac{1}{c^2}\partial_t\,\vec{E}\right)$$

$$\hat{x}_i\left(\delta_{il}\,\delta_{jm} - \delta_{im}\delta_{\mathrm{jl}}\right)\partial_j\partial_l\,E_m = -\frac{1}{c^2}\,\partial_t^2\,\vec{E}$$

$$\hat{x}_i\left(\partial_i\,\vec{\nabla}\cdot\vec{E} - \nabla^2\,E_i\right) = -\frac{1}{c^2}\,\partial_t^2\,\vec{E}$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E} = -\frac{1}{c^2}\,\partial_t^2\vec{E}$$

$$\nabla^2\,\vec{E} - \frac{1}{c^2}\,\partial_t^2\,\vec{E} = 0$$

In exactly the same way,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\partial_t\,\vec{\nabla} \times \vec{E}$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \nabla^2\vec{B} = -\frac{1}{c^2}\,\partial_t^2\vec{B}$$

$$\nabla^2\,\vec{B} - \frac{1}{c^2}\,\partial_t^2\,\vec{B} = 0$$

B) Consider solutions to the wave equations of the form:

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega\, t)}, \qquad \vec{B}(\vec{r},t) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega\, t)}$$

Where $\vec{E}_0$ and $\vec{B}_0$ are complex, constant vectors, and $\vec{k}$ and ω are real constants.

Show that these are solutions to the wave equations. Use the wave equations and Maxwell's equations to determine the relations between $\vec{E}_0, \vec{B}_0, \vec{k},$ and ω.

Using these forms:

$$\nabla^2\, \vec{E} = \vec{E}_0\, e^{i\omega t}\, \partial_j \partial_j\, e^{i\, k_l\, x_l} = \vec{E}_0\, e^{i\omega t}\, \partial_j (i\, k_j)\, e^{i\, k_l\, x_l} = -k^2\, \vec{E}$$

$$\nabla^2\, \vec{B} = -k^2\, \vec{B}$$

$$\partial_t^2\, \vec{E} = \vec{E}_0\, e^{i\vec{k}\cdot\vec{r}}\, \partial_t^2\, e^{i\omega t} = -\omega^2 \vec{E}, \qquad \partial_t^2\, \vec{B} = -\omega^2 \vec{B}$$

The wave equations become:

$$\nabla^2\, \vec{E} - \frac{1}{c^2}\, \partial_t^2\, \vec{E} = \left(-k^2 + \frac{\omega^2}{c^2}\right) \vec{E} = 0$$

$$\nabla^2\, \vec{B} - \frac{1}{c^2}\, \partial_t^2\, \vec{B} = \left(-k^2 + \frac{\omega^2}{c^2}\right) \vec{B} = 0$$

$$k = \frac{\omega}{c}$$

We also have:

$$\vec{\nabla} \cdot \vec{E} = \partial_j\, E_{0j}\, e^{i(\vec{k}\cdot\vec{r}-\omega\, t)} = i\, k_j\, E_{0j} e^{i(\vec{k}\cdot\vec{r}-\omega\, t)} = 0$$

$$\vec{k} \cdot \vec{E}_0 = 0, \text{ and } \vec{k} \cdot \vec{B}_0 = 0$$

Finally, we have:

$$\vec{\nabla} \times \vec{E} = -\partial_t\, \vec{B}$$

$$\hat{x}_j\, \epsilon_{jlm}\, \partial_l\, E_{0m} e^{i(\vec{k}\cdot\vec{r}-\omega\, t)} = -\vec{B}_0 e^{i(\vec{k}\cdot\vec{r})}\, \partial_t\, e^{i\,\omega\, t}$$

$$\hat{x}_j\, \epsilon_{jlm}\, (i\, k_l)\, E_{0m} e^{i(\vec{k}\cdot\vec{r}-\omega\, t)} = -\vec{B}_0 e^{i(\vec{k}\cdot\vec{r})}\, (-i\,\omega)\, e^{i\,\omega\, t}$$

$$\vec{k} \times \vec{E} = \omega\, \vec{B}$$

$$\vec{k} \times \vec{E} = k\, c\, \vec{B}, \;\; \hat{k} \times \vec{E} = c\, \vec{B}$$

And

$$\hat{k} \times \vec{B} = -\frac{1}{c}\, \vec{E}$$

C) Let's consider a more specific example of these solutions for this part. Let:

$$\vec{k} = k\,\hat{z}, \qquad \vec{E}_0 = E_0\,\hat{x}, \qquad E_0^* = E_0$$

Solve for the energy density of the wave, averaged over one period of the oscillation.

Using this case:

$$\vec{E} = E_0\,\hat{x}\,e^{i(k\,z-\omega t)}$$

$$\vec{B} = \frac{1}{c}\,\hat{z}\times E_0\,\hat{x}\,e^{i(k\,z-\omega t)} = \frac{E_0}{c}\,\hat{y}\,e^{i(k\,z-\omega t)}$$

Remember that:

$$u(t) = \frac{\epsilon_0}{2}\left(\left|\vec{E}(t)\right|^2 + c^2\left|\vec{B}(t)\right|^2\right), \qquad \langle\,u\,\rangle = \frac{1}{T}\int_0^T u(t)\,dt, \qquad T = \frac{2\pi}{\omega}$$

Doing this using the absolute values gives:

$$u(t) = \frac{\epsilon_0}{2}\left(E_0^2 + c^2\frac{E_0^2}{c^2}\right) = \epsilon_0\,E_0^2$$

D) Using the same example as in (C), calculate the Poynting vector for the propagating waves:

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$$

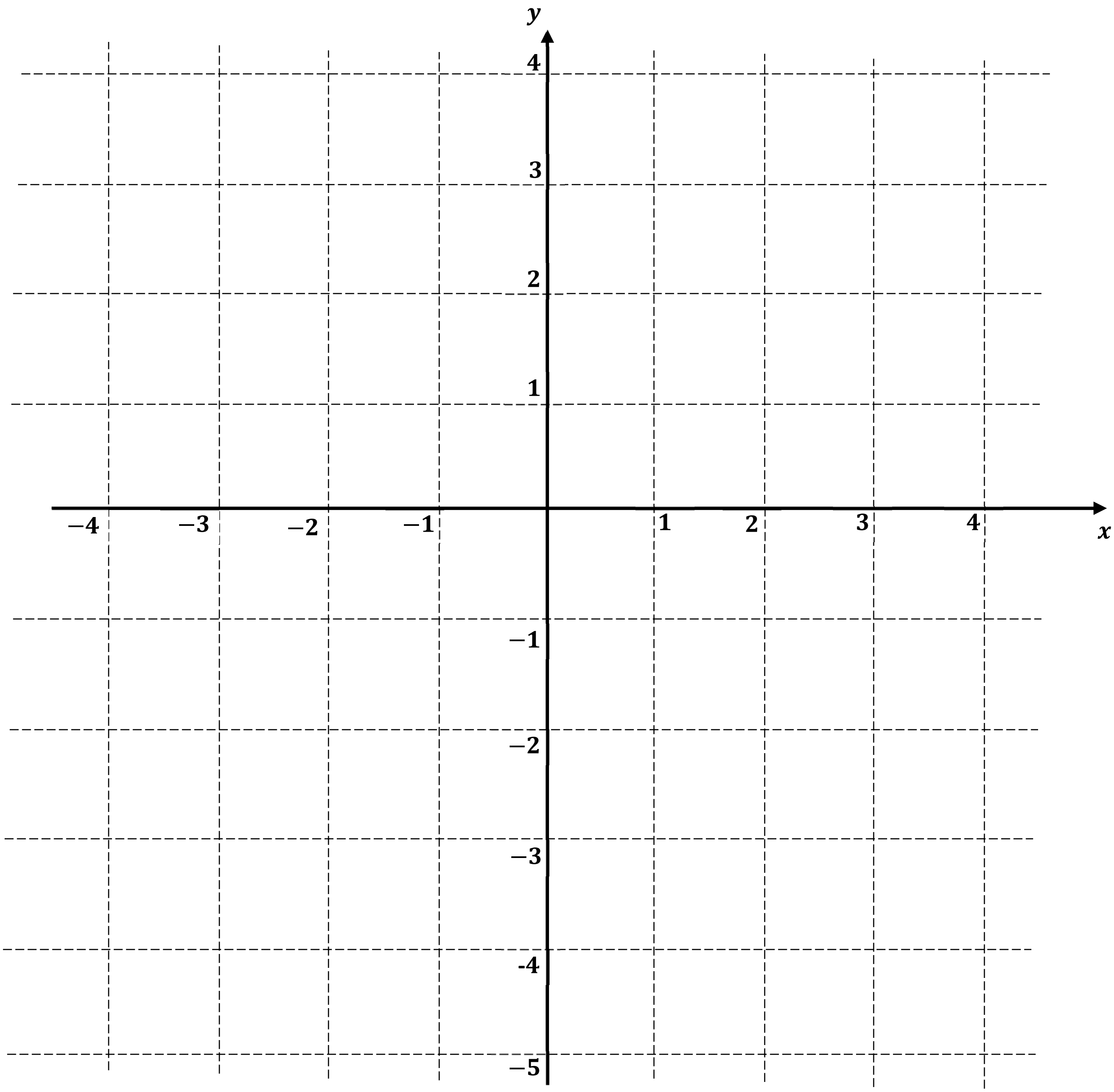
I need to check this question because we must be careful with using complex expressions for the fields. This is written in various forms:

$$\vec{S} = \frac{1}{\mu_0}\left(Re(\vec{E})\times Re(\vec{B})\right)$$

and

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E}^*\times\vec{B}\right)$$

I want to consider the differences and the best way to approach this.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