COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Electrodynamics 1

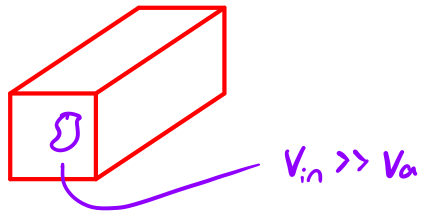
CH. 15 ELECTROSTATICS OF DIELECTRICS LECTURE NOTES

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Bruce Mason

4-11-22

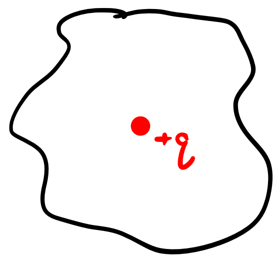
Consider some volume of material



If we then look at the tiny purple spot

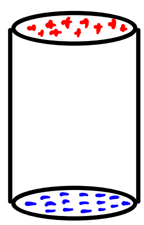
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad , \quad \vec{\nabla}\times\vec{E} = 0$$

We then take a closer look at this purple spot



Where if we place a charge q in our material it will attract all the negative charges and will polarize the atoms around it.

If we look at this like it is a cylinder



This then means we now have

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}(\rho_{bound} + \rho_{free})$$

If we want to find the potential of these dipoles we have

$$\varphi_p(\vec{r}) = \int \frac{\vec{P}(\vec{r})\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r' = \int \vec{P}(\vec{r}\,')\cdot\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

Where we then have

$$\vec{\nabla}'\cdot\left(\frac{\vec{P}}{|\vec{r}-\vec{r}\,'|}\right) = \partial_i \frac{P_i}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|\vec{r}-\vec{r}\,'|}\partial_i P_i + P_i\,\partial_i\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}\,'|} + \vec{P}\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}\,'|}$$

We can then also say

$$\vec{P}\cdot\vec{\nabla}'\frac{1}{|\vec{r}-\vec{r}'|} = \vec{\nabla}\cdot\frac{\vec{P}}{|\vec{r}-\vec{r}'|} - \vec{\nabla}\cdot\frac{\vec{P}}{|\vec{r}-\vec{r}'|}$$

This means our potential becomes

$$\varphi_p(\vec{r}) = \int \vec{\nabla}\cdot\left(\frac{\vec{P}}{|\vec{r}-\vec{r}'|}\right) - \int \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}'|} = \int \frac{\vec{P}\cdot\hat{n}}{|\vec{r}-\vec{r}'|}\,ds - \int \frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r}'|}$$

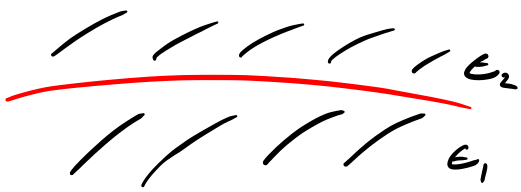
We now introduce the idea of Auxiliary field

$$\rho_b = -\vec{\nabla}\cdot\vec{P}\ ,\quad \vec{D} = \epsilon_0\vec{E} + \vec{P}\ ,\quad \vec{\nabla}\cdot\vec{D} = \epsilon_0\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \rho_{Free} + \rho_b - \rho_b$$

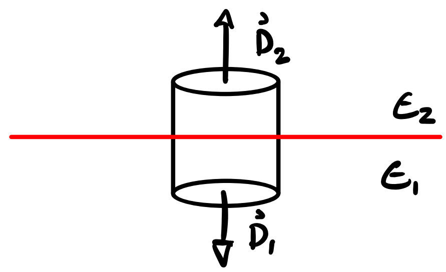
This then tells us that our Auxiliary Fied will Follow

$$\vec{\nabla}\cdot\vec{D} = \rho_{Free}$$

Now lets look at the surface between two materials



Looking at Gauss' Law for this surface



We can then say from Gauss' Law

$$\vec{D}_2\cdot\hat{n} - \vec{D}_1\cdot\hat{n} = \sigma_{Free}$$

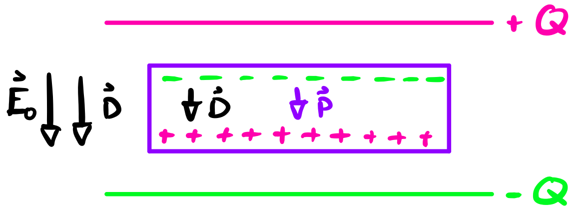
So we can then say

$$\vec{E}_{2T} = \vec{E}_{1T}$$

About our Electric Field. We then say in the presence of a dielectric we know

$$\vec{E} = \frac{\vec{D}-\vec{P}}{\epsilon_0}$$

If we then choose to look at this diagrammatically we will have



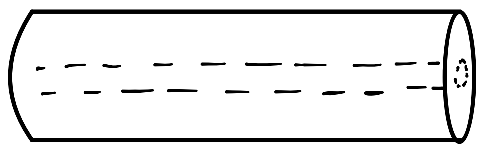
Observing this we see

$$\vec{E} = \vec{D} - \vec{P} = (\vec{D}\downarrow) - (\vec{P}\downarrow) < \vec{E}_0$$

That our electric field decreases in the dielectric

Example

Looking at a capacitor



We define capacitance as $C = Q/V$. We use Gauss' Law to first find E

$$E(r)\cdot 2\pi r l = \frac{Q}{\varepsilon_0 L} l \Rightarrow E(r) = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}$$

And we now find V with

$$V = \varphi(b) - \varphi(a) = -\int \vec{E}\cdot d\vec{l} = \frac{Q}{2\pi L\varepsilon_0}\ln(b/a)$$

This then tells us that the capacitance is

$$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

We now want to do the same but with a dielectric. We first define

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E}$$

Where $\chi_e \rightarrow$ susceptibility ("Likelyhood of becoming a dipole"). In terms of $\vec{D}$ we know that

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0}, \quad \vec{D} = \frac{Q}{2\pi L}\frac{1}{r} \Rightarrow \vec{E} = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}, \quad V = \frac{Q}{2\pi\varepsilon_0 L}\ln(b/a)$$

This then means our capacitance is

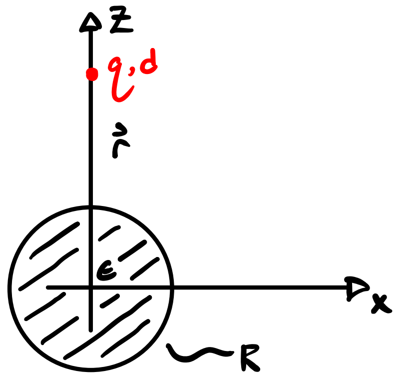
$$C = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

4-18-22

To recap, our relationship for Dielectrics is

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

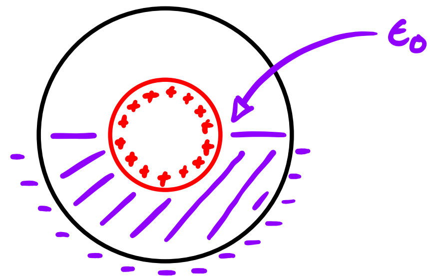
If we then look at a sphere with a charge above it



We can find the force due to this sphere with

$$\vec{F} = \frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{p}}{r^3}$$

If we look at a cylinder from the side



We can then state the following

$$D_{\perp out} - D_{\perp in} = \sigma' \quad , \quad D_{\perp +} = \sigma'_+ \quad , \quad D_{\perp -} = \sigma'_-$$

If we wish to find the Electric Field it is

$$E(r) = \frac{Q}{\pi L(\epsilon+\epsilon_0)} \frac{1}{r}$$

This then means our potential is then

$$V = \Delta\varphi = \int \vec{E}\cdot d\vec{r} = \frac{Q}{\pi L(\epsilon+\epsilon_0)} \ln(b/a)$$

We now look at four separate charges as seen below

The energy that is required to assemble these charges is

$$u = \frac{\epsilon_0}{2}\int |E|^2 \, d^3r$$

We then can calculate the work with

$$\delta w = \sum_i \delta q_i \, \varphi_i \, ds_i$$

We then have a boundary condition that says

$$\hat{n}\cdot(\vec{D}_{IN} - \vec{D}_{OUT}) = \sigma' \Rightarrow \hat{n}\cdot\vec{D}_{OUT} = \sigma'_i \Rightarrow \delta\sigma'_i = -\hat{n}\cdot\delta\bar{P}_{i\,out}$$

This then means

$$\delta w = -\int \varphi(\vec{r})\,\delta\vec{D}(\vec{r})\cdot\hat{n}\, ds = -\int \vec{\nabla}\cdot(\varphi\,\delta\vec{D})\, d^3r$$
$$= -\int (\nabla\varphi)\cdot\delta\bar{D}\, d^3r - \int \varphi(\vec{\nabla}\cdot\delta\vec{D})\, d^3r = \int \vec{E}\cdot\delta\bar{D}\, d^3r$$

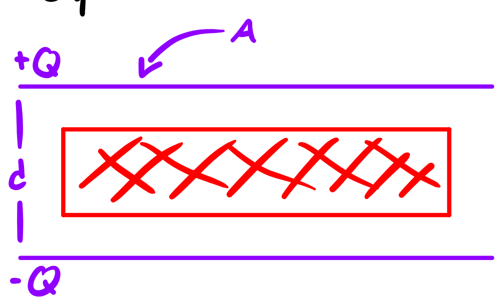
We can go on to say

$$\vec{D} = \epsilon\vec{E} \quad \delta(\vec{D}\cdot\vec{E}) = \delta D\cdot E + \vec{D}\cdot\delta\vec{E} + \epsilon\,\delta\vec{E}\cdot\vec{E}$$

The potential energy of a Dielectric can be calculated with

$$u = \frac{1}{2}\int \vec{E}\cdot\bar{D}\, d^3r$$

Looking at a parallel plate capacitor



The difference between potentials is

$$\Delta V = \int \vec{E} \cdot d\vec{l} = Ed$$

In a vacuum we have

$$E = \frac{\sigma'}{\epsilon_0} = \frac{Q}{A\epsilon_0} \Rightarrow \Delta V = \frac{Q}{A\epsilon_0} d \Rightarrow C = \frac{\epsilon_0 A}{d}$$

With the Dielectric we know

$$D = \sigma' = \epsilon E \Rightarrow E = \frac{Q}{\epsilon A} \Rightarrow \Delta V = \frac{Q}{\epsilon A} d$$

The potential energy is then

$$U = \frac{\epsilon_0}{2} E^2 A d = \frac{\epsilon_0}{2} \frac{Q^2}{A^2 \epsilon_0^2} A d = \frac{1}{2} \frac{Q^2}{\epsilon_0} \frac{d}{A} \Rightarrow U_0 = \frac{1}{2} \frac{Q^2}{C_0}$$

We can go on to say further

$$U = \frac{1}{2} C_0 V^2$$

We know $\Delta U > 0 \rightarrow$ we have to push the Dielectric out