Classical Mechanics and Statistical/Thermodynamics

August 2016

Possibly Useful Information

Handy Integrals:

$$\begin{aligned}
\int_0^\infty x^n e^{-\alpha x}\,dx &= \frac{n!}{\alpha^{n+1}} \\
\int_0^\infty e^{-\alpha x^2}\,dx &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \\
\int_0^\infty x\; e^{-\alpha x^2}\,dx &= \frac{1}{2\alpha} \\
\int_0^\infty x^2\; e^{-\alpha x^2}\,dx &= \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \\
\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\;dx &= \sqrt{\frac{\pi}{b}}e^{-a^2/4b}
\end{aligned}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\mathrm{Li}_p(z) \equiv \sum_{n=1}^\infty \frac{z^n}{n^p}$$

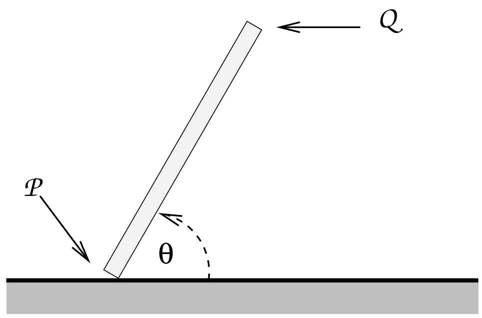
$$\mathrm{Li}_p(1) = \zeta(p)$$

Physical constants:

$$\begin{aligned}
\hbar &= 1.05457\times 10^{-34}\mathrm{m}^2\mathrm{kgs}^{-1} \\
m_{\mathrm{electron}} &= 9.109\times 10^{-31}\mathrm{kg} \\
k_B &= 1.38\times 10^{-23}\mathrm{m}^2\mathrm{kg}\cdot\mathrm{s}^{-2}\mathrm{K}^{-1}
\end{aligned}$$

Classical Mechanics

1. A uniform stick of length ℓ and mass m is initially held upright at an angle $\theta(t=0) = \theta_0$ to the horizontal with one end on the floor. The stick is released and falls to the floor so that θ changes in time. As it falls, it pivots around its contact point $\mathcal{P}$ with the floor without sliding, due to the coefficient of static friction μ, between the stick and the floor. You should ignore the thickness of the stick.



   (a) Draw the free body diagram for this problem. (1 point)

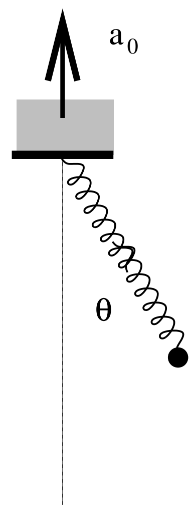
   (b) Calculate the moment of inertia of the stick with respect to the contact point, $\mathcal{P}$. (1 point)

   (c) Determine the tangential velocity of the end of the stick (point $\mathcal{Q}$) as a function of θ, assuming that the pivot point does not slide. (2 points)

   (d) Show that if $0 < \theta_0 < \pi/2$, the frictional force at the pivot changes sign (direction) as a function of θ, and calculate the angle at which it does so as a function of m, ℓ, g and/or θ_0. (3 points)

   (e) Show that in the limit $\theta_0 \to \pi/2$ there is an angle θ_c at which the stick will always slide, for any finite value of μ, and calculate θ_c. (3 points)

2. Consider a pendulum othat consists of a mass, m, suspended by a massless spring of equilibrium length ℓ_0 and spring constant k. The spring is located on Earth, in a uniform gravitational field g pointing downward. The pendulum swings only in the x-z plane, and the point of support of the pendulum accelerates upward with a constant acceleration a_0.



(a) Find the Lagrangian in terms of the generalized coordinates consisting of the length of the sprint, $\ell(t)$, and the angle $\theta(t)$ it makes with respect to the vertical. (2 points)

(b) Find the equations of motion for the generalized co-ordinates. (2 points)

(c) Find the Hamiltonian for the system from the Lagrangian. (2 points)

(d) Derive Hamilton's equations of motion for the system. (2 points)

(e) Assume that $\ell(t)$ and $\theta(t)$ undergo small, slow oscillations. What are the periods of the swinging motion and the oscillation of the spring? (2 points)

3. Consider a relativistic particle of rest mass m_0 moving in a given potential V, described by the Hamiltonian

$$H = \sqrt{p^2c^2 + m_0^2c^4} + V(\mathbf{r}).$$

   (a) Write down Hamilton's equations. (2 points)
   (b) From these obtain an expression for the momentum in terms of the velocity. (2 points)
   (c) Using the result of (b) above, obtain an explicit expression for the rate of change of the momentum in terms of the velocity $\mathbf{v}$ and the acceleration $\dot{\mathbf{v}}$. (2 points)
   (d) Derive the corresponding Lagrangian. Be sure to write the Lagrangian as a function of velocity and position. (2 points)
   (e) What is the energy expressed in terms of the velocity rather than the momentum? (1 point)
   (f) If the potential is rotationally invariant, what are the corresponding constants of the motion? Is the Hamiltonian among these? Why or why not? (1 point)

   Note that you may find it convenient to introduce the quantity $\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Statistical Mechanics

4. It can be shown that the Helmholtz free energy for a photon gas is given by:

$$F(T, V, N) = -\frac{1}{3}\sigma V T^4$$

where σ is the Stefan-Boltzmann constant. Using this relation, answer the following:

(a) What are the equations of state (that is, P, S, and μ as a function of T, V and N)? (3 points)

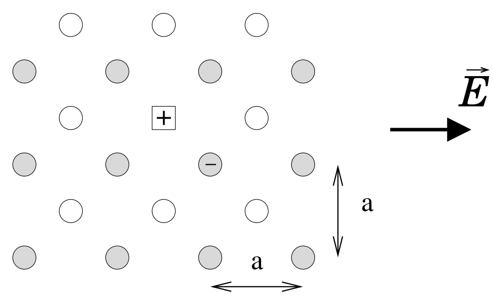
(b) Consider a Carnot cycle using a photon gas as its working fluid. The cycle is driven by one hot and one cold temperature reservoir, with temperatures T_h and T_c respectively. Draw the cycle in the P-V plane. **Caution:** This is **not** an ideal gas! Think carefully about the steps in a Carnot cycle and use your results from above to determine what the cycle will look like. (2 points)

(c) Solve for the heat exchanged in each leg of your Carnot cycle. Your answer may depend upon T_h, T_c, and any other variables you might choose in defining your cycle. (3 points)

(d) Using these values for the heat exchanged, calculate the efficiency of a Carnot cycle that uses a photon gas as its working fluid. If you cannot calculate it, devise a careful argument for its value. (2 points)

5. Consider a two-dimensional model of a solid at absolute temperature T that contains a small number N of electron donor atoms, represented by a square in the figure below. These atoms replace a small fraction of the number of ordinary atoms of the solid, and rapidly ionize. The donated electron always sits on one of the four atoms immediately and diagonally adjacent to the donor atom. While the positively charged donor ion is fixed in place, the electron is free to move between any one of the four lattice sites surrounding the positive ion. The lattice spacing is a. Neglect any interaction between impurities on different sites, and assume that all donor atoms are ionized in this fashion.

   In this problem a uniform electric field will be applied in the x-direction, polarizing the system.



   (a) Calculate the mean electric polarization, i.e. mean electric dipole moment per unit volume, in the presence of a uniform electrical field applied along the x direction, so that $\vec{E} = E_0\,\hat{\imath}$. (2 point)

   (b) Calculate the entropy per unit volume as a function of temperature. (3 points).

   (c) What is the entropy per unit volume at very low temperature ($k_BT \ll ea\mathcal{E}$)? Explain why this must be the case based on the physics of the problem. (If you cannot solve part (b) above, you can still determine the correct anwer based on principles of symmetry). (2.5 points).

   (d) What is the entropy per unit volume at high low temperature ($k_BT \gg ea\mathcal{E}$)? Explain why this must be the case based on the physics of the problem. (If you cannot solve part (b) above, you can still determine the correct anwer based on principles of symmetry). (2.5 points).

6. Consider a set of spinless free bosonic gas atoms each of mass m moving in three dimensions. The state of an atom is given by its momentum $\vec{p}$, and a variable σ which can be either 0 or 1. The energy for an atom is given by

$$E(\vec{p}, \sigma) = \frac{p^2}{2m} + \sigma\,\Delta$$

where $\Delta > 0$, and $\sigma \in \{0, 1\}$.

(a) If $\Delta = 0$, then we have a degeneracy of two for every energy eigenstate. What is the Bose-Einstein transition temperature for the system in this limit and how does it compare to a similar gas without the σ degree of freedom? (4 points)

(b) Write a formal expression for the partition function in the grand canonical ensemble when $\Delta > 0$. Show that it factors into a product of a partition function for the ground state atoms, and a partition for the excited state atoms. (This expression will involve a product over states that you cannot simplify.) (1 point)

(c) Calculate $\bar{N} \equiv \langle N \rangle$ for this system. You should get an expression in terms of T, Z, z (or μ) and Δ. (2 points)

(d) Determine the critical density at which the transition occurs as a function of T, m, and Δ, and expand it to lowest order in $\Delta/k_B T$. Is the critical density increased or decreased as Δ is increased from zero? Why? (3 points)