COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #1

PROBLEMS: {1, 2, 3}

Due: September 1, 2021

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Robert Lewis-Swan

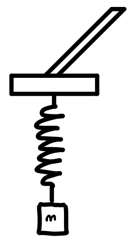
Problem 1:

Consider a mass m attached by a massless spring to a piston, as illustrated in Fig. 1. The attachment point of the spring to the piston can be described by the co-ordinate X while the mass is described by co-ordinate x.

(a) If the motion of the mass is subject to a damping force $F_d = -m\nu\dot{x}$, show that it is described by the second-order differential equation

$$\ddot{x} + \nu\dot{x} + \omega_0^2 x = F_0(t), \tag{1}$$

where $\omega_0 = \sqrt{k/m}$ and $F_0 = \omega_0^2 X(t)$.



Newton's second law : $F = d\vec{p}/dt = m\ddot{x}$

Damped harmonic oscillator : $F = \underbrace{-m\nu\dot{x}}_{\text{Damping}} \underbrace{- kx}_{\text{Spring}} = m\ddot{x}$: $m\ddot{x} + m\nu\dot{x} + kx = 0$

Damped driven harmonic oscillator : $m\ddot{x} + m\nu\dot{x} + kx = k \cdot x_0 e^{\alpha t}\cos(\omega t)$

$\ddot{x} + \nu\dot{x} + \frac{k}{m}x = \frac{k}{m}x_0 e^{\alpha t}\cos(\omega t)$ w/ $\omega_0 = \sqrt{k/m}$ & $X(t) = x_0 e^{\alpha t}\cos(\omega t)$

$\ddot{x} + \nu\dot{x} + \omega_0^2 x = \omega_0^2 X(t)$ w/ $\omega_0^2 X(t) = F_0$

$$\boxed{\ddot{x} + \nu x + \omega_0^2 x = F_0(t)}$$

(b) Assume the piston is initialized at $X(0) = 0$ and is driven according to,

$$X(t) = X_0 e^{\alpha t}\cos(\omega t). \tag{2}$$

Determine the particular solution of Eq. (1) in this case.

$\ddot{x} + \nu\dot{x} + \omega_0^2 x = F_0(t)$: $x_p(t) = De^{\alpha t}\cos(\omega t - \delta)$

$\dot{x}(t) = D\alpha e^{\alpha t}\cos(\omega t - \delta) - D\omega e^{\alpha t}\sin(\omega t - \delta) = De^{\alpha t}(\alpha\cos(\omega t - \delta) - \omega\sin(\omega t - \delta))$

$\ddot{x}(t) = \alpha De^{\alpha t}(\alpha\cos(\omega t - \delta) - \omega\sin(\omega t - \delta)) + De^{\alpha t}(-\alpha\omega\sin(\omega t - \delta) - \omega^2\cos(\omega t - \delta))$

$\ddot{x}(t) = De^{\alpha t}((\alpha(\alpha\cos(\omega t - \delta) - \omega\sin(\omega t - \delta)) - (\alpha\omega\sin(\omega t - \delta) + \omega^2\cos(\omega t - \delta)))$

$D\cancel{e^{\alpha t}}((\alpha(\alpha\cos(\omega t - \delta) - \omega\sin(\omega t - \delta)) - (\alpha\omega\sin(\omega t - \delta) + \omega^2\cos(\omega t - \delta)))$

$+$

$\nu D\cancel{e^{\alpha t}}(\alpha\cos(\omega t - \delta) - \omega\sin(\omega t - \delta))$

$+$

$\omega_0^2 D\cancel{e^{\alpha t}}\cos(\omega t - \delta) = \omega_0^2 x_0 \cancel{e^{\alpha t}}\cos(\omega t)$

Problem 1: Continued

$$D\left(\alpha^2\cos(\omega t-\delta)-\alpha\omega\sin(\omega t-\delta)-\alpha\omega\sin(\omega t-\delta)-\omega^2\cos(\omega t-\delta)\right)+D\left(\alpha\nu\cos(\omega t-\delta)-\omega\nu\sin(\omega t-\delta)\right)+D\omega_0^2\cos(\omega t-\delta)$$
$$=$$
$$\omega_0^2x_0\cos(\omega t)$$

$$D\left(\cos(\omega t-\delta)(\alpha^2-\omega^2)-2(\alpha\omega\sin(\omega t-\delta))\right)+D\nu\left(\alpha\cos(\omega t-\delta)-\omega\sin(\omega t-\delta)\right)+D\omega_0^2\cos(\omega t-\delta)=\omega_0^2x_0\cos(\omega t)$$

$$D\left(\cos(\omega t-\delta)(\alpha^2-\omega^2)-2(\alpha\omega\sin(\omega t-\delta))+\nu\alpha\cos(\omega t-\delta)-\nu\omega\sin(\omega t-\delta)+\omega_0^2\cos(\omega t-\delta)\right)=\omega_0^2x_0\cos(\omega t)$$

$$D\left(\cos(\omega t-\delta)\underbrace{(\alpha^2-\omega^2+\nu\alpha+\omega_0^2)}_{\to\gamma}-\sin(\omega t-\delta)\underbrace{(2\alpha\omega+\nu\omega)}_{\to\lambda}\right)=\omega_0^2x_0\cos(\omega t)$$

$$D\left[\cos(\omega t-\delta)\gamma-\sin(\omega t-\delta)\lambda\right]=\omega_0^2x_0\cos(\omega t)$$

$$D\left[\gamma\left(\cos(\omega t)\cos(\delta)+\sin(\omega t)\sin(\delta)\right)-\lambda\left(\sin(\omega t)\cos(\delta)-\cos(\omega t)\sin(\delta)\right)\right]=\omega_0^2x_0\cos(\omega t)$$

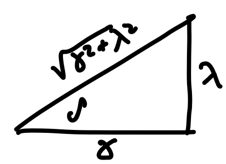
$$D\left[\gamma\cos(\omega t)\cos(\delta)+\gamma\sin(\omega t)\sin(\delta)-\lambda\sin(\omega t)\cos(\delta)+\lambda\cos(\omega t)\sin(\delta)\right]-\omega_0^2x_0\cos(\omega t)=0$$

$$\cos(\omega t)\overbrace{\left(D\gamma\cos(\delta)+D\lambda\sin(\delta)-\omega_0^2x_0\right)}^{\textcircled{1}}+\sin(\omega t)\overbrace{\left(D\gamma\sin(\delta)-D\lambda\cos(\delta)\right)}^{\textcircled{2}}=0$$

Cos(ωt) and sin(ωt) are both linearly independent thus ① and ② are equal to zero. From this we can determine D and δ.

$$D\gamma\sin(\delta)-D\lambda\cos(\delta)=0 \quad : \quad \gamma\sin(\delta)-\lambda\cos(\delta)=0 \quad : \quad \text{Tan}(\delta)=\lambda/\gamma \quad \therefore \quad \delta=\text{Tan}^{-1}(\lambda/\gamma)$$

$$\boxed{\delta=\text{Tan}^{-1}\left(\frac{2\alpha\omega+\nu\omega}{\alpha^2-\omega^2+\nu\alpha+\omega_0^2}\right)}$$



$$D\gamma\cos(\delta)+D\lambda\sin(\delta)=\omega_0^2x_0 \quad : \quad D(\gamma\cos(\delta)+\lambda\sin(\delta))=\omega_0^2x_0 \quad : \quad D=\frac{\omega_0^2x_0}{\gamma\cos(\delta)+\lambda\sin(\delta)}$$

$$\gamma=\alpha^2-\omega^2+\nu\alpha+\omega_0^2$$

$$\boxed{x_p(t)=\frac{\omega_0^2x_0e^{\alpha t}\cos(\omega t)}{\gamma\cos(\delta)+\lambda\sin(\delta)}}$$

$$\lambda=2\alpha\omega+\nu\omega=\omega(2\alpha+\nu)$$

(c) What is the resonance frequency ω_R? Does its existence depend on the sign of α?

Resonant Frequency occurs at maximum amplitude

$$D=\frac{\omega_0^2x_0}{\gamma\cos(\delta)+\lambda\sin(\delta)}=\frac{\omega_0^2x_0}{\gamma\cdot\frac{\gamma}{\sqrt{\gamma^2+\lambda^2}}+\lambda\cdot\frac{\lambda}{\sqrt{\gamma^2+\lambda^2}}}=\frac{\omega_0^2x_0\sqrt{\gamma^2+\lambda^2}}{\gamma^2+\lambda^2}$$

$$D=\frac{\omega_0^2x_0\sqrt{\gamma^2+\lambda^2}}{\gamma^2+\lambda^2}=\frac{\omega_0^2x_0}{(\gamma^2+\lambda^2)^{1/2}}$$

Problem 1: Continued

$$\frac{\partial \gamma^2}{\partial w} = 2\gamma \cdot (-2w) \quad : \quad \frac{\partial \lambda^2}{\partial w} = 2\lambda(2\alpha + \nu)$$

$$D = w_0^2 x_0 (\gamma^2 + \lambda^2)^{-1/2} \quad : \quad \frac{\partial D}{\partial w} = \frac{-1}{2} w_0^2 x_0 (\gamma^2 + \lambda^2)^{-3/2} \cdot (-4\gamma w + 2\lambda(2\alpha + \nu))$$

$$\frac{(-\lambda(2\alpha + \nu) + 2\gamma w)\, w_0^2 x_0}{(\gamma^2 + \lambda^2)^{3/2}} = 0 \quad : \quad 2\gamma w - \lambda(2\alpha + \nu) = 0 \quad \therefore \quad 2\gamma w = \lambda(2\alpha + \nu)$$

$$2(\alpha^2 - w^2 + \nu\alpha + w_0^2)\cancel{w} = \cancel{w}(2\alpha + \nu)^2 \quad : \quad 2\alpha^2 - 2w^2 + 2\nu\alpha + 2w_0^2 = 4\alpha^2 + 4\alpha\nu + \nu^2$$

$$\cancel{\alpha^2} - w^2 + \cancel{\nu\alpha} + w_0^2 = \overset{1}{\cancel{2}}\alpha^2 + \overset{1}{\cancel{2}}\alpha\nu + \nu^2/2 \quad : \quad -w^2 + w_0^2 = \alpha^2 + \alpha\nu + \nu^2/2$$

$$w^2 = w_0^2 - \alpha^2 - \alpha\nu - \nu^2/2 \quad : \quad w^2 = w_0^2 - \alpha(\alpha + \nu) - \nu^2/2$$

$$w = (w_0^2 - \alpha(\alpha + \nu) - \nu^2/2)^{1/2}$$

$$\boxed{w = \pm\sqrt{w_0^2 - \alpha(\alpha + \nu) - \nu^2/2}}$$

The sign of α does matter for the resonant frequency.

Problem 1: Review

Procedure:

- Begin by writing out Newton's second law with a spring and damping force that is not in equilibrium.
- Show that Newton's second law can be rearranged to show equation (1).
- Proceed to make a guess for the particular solution

$$x_p(t) = De^{\alpha t}\cos\left(\omega t - \delta\right)$$

  and insert it into the differential equation with equation (2) as well.
- Solve for the phase δ and then put this result back into the particular solution.
- To find the resonant frequency, maximize the amplitude and solve for ω.

Key Concepts:

- We use Newton's second law to arrive at equation (1).
- With the help of equation (2) and the particular solution guess we can find a complete solution to our differential equation.
- Our particular solution involves a phase δ to account for the sine term in our differential equation.
- Resonant frequency occurs at the maximum amplitude, where the amplitude is found in our particular solution of our differential equation.

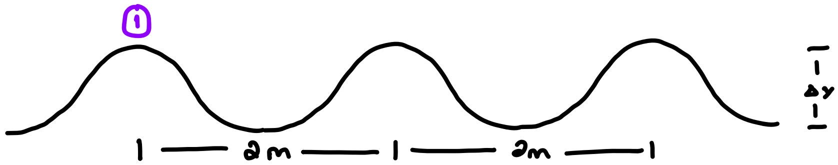
Variations:

- We can obviously be given a different differential equation.
  - This would change what our phase and amplitude would be for our equation.
- We can be given a different driving force.
  - Thus changing our particular solution and everything that follows after it.

Problem 2:

When a car drives along a bumpy road, periodic ripples in the road surface can force the wheels to oscillate on the suspension (e.g., springs).

(a) If the spacing between ripples on the road is about 2 m, at what speed will the cars suspension be driven into resonance?



Will treat the spring of each car as an undamped driven oscillator

$$\ddot{x} + \omega_0^2 x = F_0 \cos(\omega t)$$

x is the position / length of spring to the equilibrium

ω_0 is the natural oscillation frequency of the spring

F_0 is the force due to ripples in the road

ω frequency of effective force due to ripples

We need to first estimate what the spring constant "k" will be. When 4 adults enter the car the spring will compress by Δx:

$$m_p g = k\Delta x \longrightarrow k = \frac{m_p g}{\Delta x}$$

We then can relate our spring constant with ω_0

$$\omega_0 = \sqrt{\frac{4k}{M_{car}}}$$

We can then relate the driven frequency as

$$\omega = \frac{2\pi}{\lambda_{Ripple}} \cdot V_{car}$$

In the absence of damping we can say,

$$\omega = \omega_0 : \quad \frac{2\pi}{\lambda_{ripple}} \cdot V_{car} = \sqrt{\frac{4k}{M_{car}}}, \quad V_{car} = \frac{\lambda_{ripple}}{2\pi}\sqrt{\frac{4 m_p g}{M_{car}\,\Delta x}}$$

w/ $M_{car} = 800$ kg, $m_p = 80$ kg, $\Delta x = 0.02$ m

$$\boxed{V_{car} = 4.7 \text{ m/s}}$$

Problem 2: Continued

(b) Estimate a realistic damping constant provided by the shock absorbers so that the car's suspension does not catastrophically fall apart.

Note: This question has no unique quantitative solution. You will be marked primarily on your approach and the application of physical principles. However, your answer should provide reasonable estimates for relevant parameters to enable you to arrive at a concrete solution. Things to ponder include: How high are the ripples in the road? How far does a car drop in height when 4 adults are inside?

We can then use that

$$D = \frac{x_0 \omega_0^2}{2\beta} \frac{1}{\sqrt{\omega_0^2 - \beta^2}}$$

Solving this for β yields

$$\beta = \frac{\omega_0}{\sqrt{2}} \sqrt{1 \pm \sqrt{D^2 - x_0^2}}$$

We can then use our values from a.) w/ $x_0 \approx 0.05$ m

$$\boxed{\beta = 3.6 \ s^{-1}}$$

Problem 2: Review

Procedure:

- Begin by assuming that the spring will oscillate like an undamped driven oscillator.
- Proceed to solve for the spring constant with the mass of the people as m.
- Solve for the natural frequency and driven frequency, set them equal, and solve for the velocity of the var.
- Proceed to solve for β from the equation
$$D = \frac{x_0\omega_0^2}{2\beta}\frac{1}{\sqrt{\omega_0^2-\beta^2}}.$$

Key Concepts:

- We can model the motion of this spring as an undamped driven oscillator.
- The resonant frequency occurs when the natural frequency is equal to driven frequency.
- We can then solve for the damping constant by using the equation for amplitude.

Variations:

- The distance in the spacing of the ripples can be changed.
  - This would change some of the values that were being plugged into the equations, but not the overall process.
- We can be asked for other values.
  - We would use the same formalism but search for different values.

Problem 3:

Consider the nonlinear damped-driven system

$$\ddot{x} + (x^2 - \dot{x}^2 - 1)\dot{x} + x = 0, \tag{3}$$

that describes an harmonic oscillator with $m = \omega = 1$.

(a) Find an expression for the change in energy $\dot{E}$.

$$E = T + U : T = \frac{1}{2}m\dot{x}^2,\ U = \frac{1}{2}Kx^2 \therefore E = \frac{1}{2}Kx^2 + \frac{1}{2}m\dot{x}^2 : \dot{E} = Kx\cdot\dot{x} + m\dot{x}\ddot{x}$$

$$\ddot{x} = -x - (x^2 + \dot{x}^2 - 1)\dot{x} : \dot{E} = x\dot{x} + \dot{x}(-x - (x^2 + \dot{x}^2 - 1)\dot{x})$$

$$\boxed{\dot{E} = x\dot{x} + \dot{x}(-x - (x^2 + \dot{x}^2 - 1)\dot{x})}$$

(b) For the polar co-ordinates (r, θ) given by the transformation $x = r\cos(\theta)$ and $\dot{x} = r\sin(\theta)$, derive the equations of motion:

$$\begin{aligned} \dot{r} &= r(1 - r^2)\sin^2(\theta) \\ \dot{\theta} &= (1 - r^2)\sin(\theta)\cos(\theta) - 1. \end{aligned} \tag{4}$$

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0 : \underline{r^2 = x^2 + \dot{x}^2} : 2r\cdot\dot{r} = 2x\cdot\dot{x} + 2\dot{x}\cdot\ddot{x} : r\cdot\dot{r} = x\cdot\dot{x} + \dot{x}\ddot{x} : \ddot{x} = \frac{r\dot{r} - x\cdot\dot{x}}{\dot{x}}$$

$$\frac{r\cdot\dot{r} - r^2\cos\theta\sin\theta}{r\sin\theta} + (r^2\cos^2\theta + r^2\sin^2\theta - 1)r\sin\theta + r\cos\theta = 0$$

$$r\cdot\dot{r} - \cancel{r^2\cos\theta\sin\theta} + (r^2 - 1)r^2\sin^2\theta + \cancel{r^2\cos\theta\sin\theta} = 0$$

$$r\cdot\dot{r} + (r^2 - 1)r^2\sin^2\theta = 0 : \dot{r} + (r^2 - 1)r\sin^2\theta = 0 : \dot{r} = r(1 - r^2)\sin^2\theta \checkmark$$

$$\boxed{\dot{r} = r(1 - r^2)\sin^2\theta}$$

$$\frac{\dot{x}}{x} = \tan\theta : \dot{x}x^{-1} = \tan\theta : \frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} = \frac{\dot{\theta}}{\cos^2(\theta)} \therefore \ddot{x} = \frac{\dot{x}^2}{x} + \frac{\dot{\theta}}{\cos^2(\theta)}x = \frac{r^2\sin^2\theta}{r\cos\theta} + \frac{r\dot{\theta}}{\cos\theta}$$

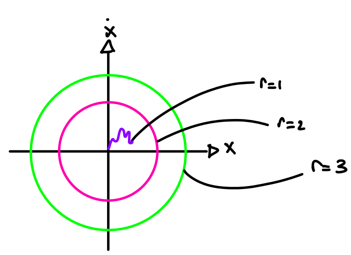
$$\frac{r^2\sin^2\theta + r^2\dot{\theta}}{r\cos\theta} + (r^2\cos^2\theta + r^2\sin^2\theta - 1)r\sin\theta + r\cos\theta = 0 : \frac{r^2(\dot{\theta} + \sin^2\theta)}{r\cos\theta} + (r^2 - 1)r\sin\theta + r\cos\theta = 0$$

$$r^2(\dot{\theta} + \sin^2\theta) + (r^2 - 1)r^2\sin\theta\cos\theta + r^2\cos^2\theta = 0 : \dot{\theta} + \sin^2\theta + (r^2 - 1)\sin\theta\cos\theta + \cos^2\theta = 0$$

$$\dot{\theta} + (r^2 - 1)\sin\theta\cos\theta + 1 = 0 : \dot{\theta} = (1 - r^2)\sin\theta\cos\theta - 1$$

$$\boxed{\dot{\theta} = (1 - r^2)\sin\theta\cos\theta - 1}$$

(c) Construct a rough phase portrait of the system in terms of the co-ordinates $(x, \dot{x})$. Your diagram should include at least three trajectories that characterize motion in the system, one of which is an attractor. Hint: Look at your solutions in (a) and (b) for insight.



Problem 3: Review

Procedure:

- Begin by writing out the energy as $E = T + U$ and then calculating a time derivative of the energy.
- Proceed to use the equations for x and $\dot{x}$ plug them into equation (3). Manipulate both equations to get one for $\dot{r}$ and $\dot{\theta}$.
- Construct a phase portrait for multiple values of r and plot this.

Key Concepts:

- We can find an expression for a change in energy by taking the time derivative of the energy.
- Using the polar co-ordinate transformation we can find the equations in (4).
- Phase portraits portray how the velocity and position of a particle are related with one another.

Variations:

- We can be given a different damped-driven system, equation (3).
  - We then follow the same procedure to arrive at the same answers.