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Math Methods in Physics

PHYS 5013 Homework Assignment #2

Problems: $\{1, 2, 3, 4, 5, 6, 7, 8\}$

Due: September 13, 2021 By 10:30 AM

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Problem 1:

Use a Lagrange multiplier approach to find the extremal points (x, y) of the function

$$f(x,y) = 2x^2 + \frac{1}{2}y^2 - xy$$

subject to the constraint

$$g(x,y) = 4x^2 + y^2 - 4 = 0.$$

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y)$$

$$L = 2x^2 + \frac{1}{2}y^2 - xy - \lambda(4x^2 + y^2 - 4)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad , \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = 0 \quad , \quad \frac{\partial L}{\partial x} = 4x - y - 8x\lambda \quad , \quad y = 4x(1-2\lambda)$$

$$\frac{\partial L}{\partial \dot{y}} = 0 \quad , \quad \frac{\partial L}{\partial y} = y - x - 2y\lambda \quad , \quad x = y(1-2\lambda)$$

w/ $x = y(1-2\lambda)$: $\cancel{y} = 4 \cdot \cancel{y}(1-2\lambda)^2$: $1/4 = (1-2\lambda)^2$: $\pm\frac{1}{2} = 1-2\lambda$: $\lambda = 1/2(1 \pm 1/2)$

W/ $y = 4x(1-2\lambda)$: $x = 4x(1-2\lambda)^2$: $1/4 = (1-2\lambda)^2$: $\pm 1/2 = 1-2\lambda$: $\lambda = 1/2(1 \pm 1/2)$

$4x^2 + 16x^2(1-2\lambda)^2 - 4 = 0$: $x^2 + 4x^2(1-2\lambda)^2 - 1 = 0$: $x = \pm\sqrt{\frac{1}{(1+4(1-2\lambda)^2)}}$

$\lambda = 1/4$, $y = 4x(1-1/2) = 2x$: $\lambda = 3/4$, $y = 4x(1-3/2) = -2x$

$$\boxed{\begin{array}{ll} \lambda = 1/4 , & (1/\sqrt{2}, \sqrt{2}) \\ \lambda = 1/4 , & (-1/\sqrt{2}, -\sqrt{2}) \\ \lambda = 3/4 , & (1/\sqrt{2}, \sqrt{2}) \\ \lambda = 3/4 , & (-1/\sqrt{2}, -\sqrt{2}) \end{array}}$$

Problem 1: Review

Procedure:

- Use the Lagrange multiplier approach: $h = f - \lambda g$.
- Use the Euler Lagrange equation to find an equation for y and for x.
- Solve for the values of λ.
- Insert values of λ into equation for $y(x)$.
- Determine (x, y) for different values of λ.

Key Concepts:

- Lagrange multipliers can be used to extremize two functions.
- Values of λ will determine extremal points of both functions.

Variations:

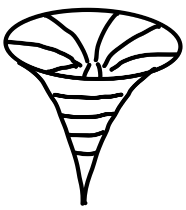
- F and G can be changed.
  - This would change the final answer but not the overall procedure.

Problem 2:

A bug moves on a surface with cylindrical symmetry. The surface is given by $z = \log \rho$ where ρ is the distance of the surface from the z-axis. As an intelligent bug, it wants to move the shortest distance between two points.

(a) Calculate the differential equation of such a curve, between any two points on the surface. (You can colve this using a local constraint, but that's the hard way.)

$$z = \ln(\rho) \; : \; ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \; : \; dz = \frac{d\rho}{\rho} \; : \; ds^2 = d\rho^2 + \rho^2 d\theta^2 + \frac{d\rho^2}{\rho^2}$$



we wish to minimize this line element : $ds^2 = d\rho^2 + \rho^2 d\theta^2 + \frac{d\rho^2}{\rho^2}$

$$K[\rho(\theta)] = \int ds = \int \sqrt{\left(1 + \frac{1}{\rho^2}\right)\rho'^2 + \rho^2} \; d\theta$$

$$f = \sqrt{\left(1 + \frac{1}{\rho^2}\right)\rho'^2 + \rho^2} \; , \; \alpha = \left(1 + \frac{1}{\rho^2}\right) \; : \; \rho' \frac{\partial f}{\partial \rho'} - f = c \; : \; \frac{\partial f}{\partial \rho'} = \frac{\alpha \rho'}{f} \; , \; \rho' \frac{\partial f}{\partial \rho'} = \frac{\alpha \rho'^2}{f}$$

$$\rho' \frac{\partial f}{\partial \rho'} - f = c \; : \; \frac{\alpha \rho'^2}{f} - f = c \; : \; \frac{\alpha \rho'^2 - f^2}{f} = c \; : \; \frac{\alpha \rho'^2 - \alpha \rho'^2 - \rho^2}{\rho} = c \; : \; \frac{-\rho^2}{f} = c$$

$$\frac{-\rho^2}{c} = f \; : \; \frac{\rho^4}{c^2} = \left(1 + \frac{1}{\rho^2}\right)\rho'^2 + \rho^2 \;\; \therefore \;\; \rho'^2 = \left(\frac{\rho^4}{c^2} - \rho^2\right) \Big/ \left(1 + \frac{1}{\rho^2}\right)$$

$$\rho' = \sqrt{\left(\frac{\rho^6}{c^2} - \rho^4\right) \Big/ (\rho^2 + 1)} \; : \; \frac{d\rho}{d\theta} = \sqrt{\left(\frac{\rho^6}{c^2} - \rho^4\right) \Big/ (\rho^2 + 1)}$$

$$\boxed{\frac{d\rho}{d\theta} = \sqrt{\left(\frac{\rho^6}{c^2} - \rho^4\right) \Big/ (\rho^2 + 1)}}$$

(b) Solve for either the function $\rho(\theta)$ or $\theta(\rho)$, which ever you find easier. (You may need to consult a good integral table).

$$\theta = \int \sqrt{\frac{(\rho^2 + 1)}{\rho^4 \left(\frac{\rho^2}{c^2} - 1\right)}} \; d\rho = \frac{1}{c} \left\{ \mathcal{E}(\arcsin(c/\rho) - a) - \mathcal{E}(\arcsin(c/\rho_0) - a) \right\}$$

$$\boxed{\theta = \frac{\mathcal{E}}{c} \left\{ \arcsin(c/\rho) - \arcsin(c/\rho_0) \right\}}$$

Problem 2: Review

Procedure:

- Extremize the functional : $ds = \sqrt{d\rho^2 + \rho^2 d\theta^2 + dz^2}$.
- Differentiate z with respect to ρ and substitute into ds.
- Factor our $d\theta^2$ from the functional.
- Extremize the functional with respect to ρ' where $\rho' \equiv d\rho/d\theta$.
- Use the first derivative Euler Lagrange since the function is not explicitly dependent upon the independent variable.
- Solve for $d\rho/d\theta$.
- Rearrange $d\rho/d\theta$ where $\theta = [......]d\rho$ and integrate numerically.

Key Concepts:

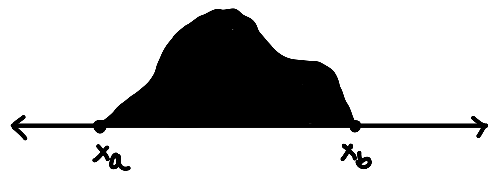
- Extremizing the ds functional will create a shortest path for our bug.
- Use cylindrical co-ordinates because they are the easiest to work with in this problem.
- Since there is now explicit independent variable dependence, we can use the first order Euler Lagrange equation.

Variations:

- The surface can change thus changing the functional and the ds element.
  - If the function changes, then we would use the same procedure but with a different function.

Problem 3:

Consider a curve $y(x)$ where $y(x_a) = 0$, and the total length of the curve is L. Find the curve that gives you the maximum area enclosed between $y(x)$ and the x-axis. (Assume that $L < \pi(x_b - x_a)/2$). There is a simple reason why the problem changes when L is too larger. Can you see why?



$$f(x,y) = \frac{1}{2}\int_{x_a}^{x_b}(xy' - x'y)\,dt \; : \; g(x,y) = \int_{x_a}^{x_b}\sqrt{x'^2+y'^2}\,dt \quad w/ \quad y(x_a) = y(x_b) = 0$$

$$L(x,y,\lambda) = f(x,y) - \lambda g(x,y) \; : \; L(x,y,\lambda) = \frac{1}{2}(x\dot{y} - \dot{x}y) - \lambda\left(\sqrt{x'^2+y'^2}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} \; : \; \frac{\partial L}{\partial \dot{y}} = \frac{x}{2} - \frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \; , \; \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\dot{x}}{2} - \frac{d}{dt}\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \; , \; \frac{\partial L}{\partial y} = \frac{-\dot{x}}{2}$$

$$\frac{\dot{x}}{2} - \frac{d}{dt}\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} + \frac{\dot{x}}{2} = 0 \; , \quad \dot{x} = \lambda\frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \; , \quad \dot{y} = \lambda\frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$(x-x_0)^2 + (y-y_0)^2 = \lambda^2\,\frac{\cancel{\dot{x}^2+\dot{y}^2}}{\cancel{\dot{x}^2+\dot{y}^2}} \; : \; (x-x_0)^2 + (y-y_0)^2 = \lambda^2$$

$$\boxed{(x-x_0)^2 + (y-y_0)^2 = \lambda^2}$$

If L becomes too large, the function becomes double valued while the radius of the semi circle will explode to infinity.

Problem 3: Review

Procedure:

- Set f to be the maximum area function : $\int_{x_a}^{x_b}(xy' - x'y)\,dt$.
- Set g to be the ds element : $\sqrt{x'^2 + y'^2}\,dt$.
- Use the method of Lagrange multipliers : $h = f - \lambda g$.
- Use the Euler Lagrange equation on the Lagrange multiplier equation.
- Use the professors trick to show
$$\dot{x} = \lambda \frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \quad \text{and} \quad \dot{y} = \lambda \frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}.$$
- Reduce and solve.

Key Concepts:

- We use the method of Lagrange multipliers to extremize this surface that is to be constrained under the function.
- Use the dependent f and g to create a Lagrange multiplier equation.
- Use the Euler Lagrange equation to extremize the functional.
- The professors trick allows us to bypass difficult differentiation.

Variations:

- The end points of the function, x_a, x_b can yield different values when plugged into the equation for $y(x)$.
  - We would use the same procedure up to this point.
- The length of L can change.
  - This would in turn change boundary conditions.

Problem 4:

Often constrained problems yield difficult integral problems or differential equations to solve. In each case below, apply the calculus of variations and derive a final expression for the curve specified as an integral, that is:

$$x - x_a = \int_{y_a}^{y} F(y)\, dy$$

where $F(y)$ is a known function of y. (This is called *reducing the problem to quadrature*.)

(a) Determine the curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) on a path of fixed length but along which a frictionless particle would slide with the shortest time. (That is, similar to the brachistochrone but on a path of fixed length).

$$f(x,y) = \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{ds}{v} \ , \ g(x,y) = \sqrt{1+x'^2} \ : \ \mathcal{L}(x,y) = \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} - \lambda\sqrt{1+x'^2}$$

$$\int \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} - \lambda\sqrt{1+x'^2}\, dy$$

$$\frac{\partial L}{\partial x} - \frac{d}{dy}\frac{\partial L}{\partial x'} = 0 \ : \ \frac{\partial L}{\partial x} = 0 \ , \ \frac{\partial L}{\partial x'} = \frac{x'}{\sqrt{2gy}\cdot\sqrt{1+x'^2}} - \lambda\frac{x'}{\sqrt{1+x'^2}}$$

$$\frac{d}{dy}\left(\frac{x'}{\sqrt{2gy}\cdot\sqrt{1+x'^2}} - \lambda\frac{x'}{\sqrt{1+x'^2}}\right) = 0 \ : \ \frac{x'}{\sqrt{2gy}\cdot\sqrt{1+x'^2}} - \lambda\frac{x'}{\sqrt{1+x'^2}} \longrightarrow \text{const}$$

we can argue that the derivative of a const is 0 $\therefore$ the terms in parentheses is a const

$$\frac{x'}{\sqrt{2gy}\cdot\sqrt{1+x'^2}} - \lambda\frac{x'}{\sqrt{1+x'^2}} = c \ : \ \frac{x'}{\sqrt{1+x'^2}}\left(\frac{1}{\sqrt{2gy}} - \lambda\right) = c \ : \ \frac{1}{\sqrt{2gy}} - \lambda \longrightarrow \alpha$$

$$\frac{x'^2}{1+x'^2}\cdot\alpha^2 = c^2 \ : \ x'^2\alpha^2 = c^2 + c^2x'^2 \ : \ x'^2\alpha^2 - x'^2c^2 = c^2 \ : \ x'^2 = \frac{c^2}{\alpha^2 - c^2} \ : \ x' = \sqrt{\frac{c^2}{\alpha^2-c^2}}$$

$$\frac{dx}{dy} = \sqrt{\frac{c^2}{\alpha^2-c^2}} \ : \ \int dx = \int\sqrt{\frac{c^2}{\alpha^2-c^2}}\, dy \ : \ x_b - x_a = \int_{y_a}^{y_b}\sqrt{\frac{c^2}{\alpha^2-c^2}}\, dy$$

$$\alpha = \frac{1}{\sqrt{2gy}} - \lambda \qquad \boxed{x_b - x_a = \int_{y_a}^{y_b}\sqrt{\frac{c^2}{\alpha^2-c^2}}\, dy}$$

(b) Determine the shortest length curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) along which a frictionless particle would slide with a fixed time.

$$f(x,y) = \sqrt{1+x'^2} \ , \ g(x,y) = \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{ds}{v} \ : \ \mathcal{L}(x,y) = \sqrt{1+x'^2} - \lambda\frac{\sqrt{1+x'^2}}{\sqrt{2gy}}$$

$$\int \sqrt{1+x'^2} - \lambda\frac{\sqrt{1+x'^2}}{\sqrt{2gy}}\, dy$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial x'} = 0 \ : \ \frac{\partial \mathcal{L}}{\partial x} = 0 \ , \ \frac{\partial \mathcal{L}}{\partial x'} = \frac{x'}{\sqrt{1+x'^2}} - \frac{\lambda x'}{\sqrt{2gy}\sqrt{1+x'^2}} \ : \ \frac{d}{dy}\left(\frac{x'}{\sqrt{1+x'^2}} - \frac{\lambda x'}{\sqrt{2gy}\sqrt{1+x'^2}}\right)$$

Problem 4: Continued

$$\frac{d}{dy}\left(\frac{x'}{\sqrt{1+x'^2}} - \frac{\lambda x'}{\sqrt{2gy}\sqrt{1+x'^2}}\right) = 0 \quad : \quad \frac{x'}{\sqrt{1+x'^2}} - \frac{\lambda x'}{\sqrt{2gy}\sqrt{1+x'^2}} \longrightarrow \text{const}$$

we can argue that the derivative of a const is 0 ∴ the terms in parentheses is a const

$$\frac{x'}{\sqrt{1+x'^2}} - \frac{\lambda x'}{\sqrt{2gy}\sqrt{1+x'^2}} = c \quad : \quad \frac{x'}{\sqrt{1+x^2}}\left(1 - \frac{\lambda}{\sqrt{2gy}}\right) = c \quad : \quad 1 - \frac{\lambda}{\sqrt{2gy}} \longrightarrow \alpha$$

$$\frac{x'^2}{1+x'^2}\alpha^2 = c^2 \quad : \quad x'^2\alpha^2 = c^2 + c^2x'^2 \quad : \quad x'^2\alpha^2 - c^2x'^2 = c^2 \quad : \quad x'^2(\alpha^2 - c^2) = c^2$$

$$x'^2 = \frac{c^2}{\alpha^2 - c^2} \quad : \quad x' = \sqrt{\frac{c^2}{\alpha^2-c^2}} \quad : \quad \frac{dx}{dy} = \sqrt{\frac{c^2}{\alpha^2-c^2}} \quad : \quad \int_{x_a}^{x_b} dx = \int_{y_a}^{y_b}\sqrt{\frac{c^2}{\alpha^2-c^2}}\,dy$$

$$\alpha = 1 - \frac{\lambda}{\sqrt{2gy}} \qquad \boxed{x_b - x_a = \int_{y_a}^{y_b}\sqrt{\frac{c^2}{\alpha^2-c^2}}\,dy}$$

(c) A solid object can be defined by rotating a curve $\rho(z)$ about the z-axis to determine a surface of revolution. The top and bottom surfaces of the object would be disks parallel to the $x-y$ plane. Determine the curve $\rho(z)$ connecting the points $\rho(z_a) = \rho_a$ to $\rho(z_b) = \rho_b$ so that it minimizes the surface area of the object but has a fixed volume and moment of inertia. Assume that the density of the object has a fixed mass/ volume, denoted by the constant α.

$$V = \pi\int \rho^2\,dz \quad : \quad I = \frac{MR^2}{2}, \quad I = \frac{\alpha\pi}{2}\int \rho^4\,dz \quad : \quad A = 2\pi\int \rho\sqrt{1+\rho'^2}\,dz$$

$$\mathcal{L}(\rho,\rho',z,z') = \rho\sqrt{1+\rho'^2} - \lambda_1\rho^2 - \lambda_2\rho^4$$

$$\rho'\frac{\partial\mathcal{L}}{\partial\rho'} - \mathcal{L} = C \quad : \quad \frac{\partial\mathcal{L}}{\partial\rho'} = \frac{\rho\rho'}{\sqrt{1+\rho'^2}} \quad : \quad \rho'\frac{\partial\mathcal{L}}{\partial\rho'} = \frac{\rho\rho'^2}{\sqrt{1+\rho'^2}} \longrightarrow \alpha$$

$$\rho'\frac{\partial\mathcal{L}}{\partial\rho'} - \mathcal{L} = \frac{\rho\rho'^2}{\sqrt{1+\rho'^2}} - \rho\sqrt{1+\rho'^2} + \lambda_1\rho^2 + \lambda_2\rho^4 = C \quad : \quad \frac{-\rho + (\lambda_1\rho^2 + \lambda_2\rho^4)\alpha}{\alpha} = \alpha C$$

using mathematica

$$\boxed{\frac{d\rho}{dz} = \pm\sqrt{\frac{-((C-\rho(\rho(\lambda_2\rho^2+\lambda_1)-1)))(C-\rho(1+\lambda_1\rho+m\rho^3))}{(C-\rho^2(\lambda_1+\lambda_2\rho^2))^2}}}$$

Problem 4: Review

Procedure:

- Use the Lagrange multiplier approach to extremize a functional which is subject to a constraint.
- We want to extremize time $\therefore f = t = ds/v$ with $ds = \sqrt{1+x'^2}\, dy$ and $v = \sqrt{2gy}$.
- Use $g = \sqrt{1+x'^2}$, plug f and g into the Lagrange multiplier equation, $h = f - \lambda g$.
- Extremize h with respect to the correct independent variables.
- Because $\partial L/\partial x = 0$, $d/dt(\partial L/\partial x')$ yields $\partial L/\partial x' = c$.
- Solve for x' and solve the differential equation.
- Reverse f and g for part (b), then repeat steps once again.
- Minimize the surface area $\therefore f$ represents the surface area, constraints are volume and moment of inertia.
- This function will then have two constraints $\therefore$ two $\lambda' s$.
- Use the first derivative trick and extremize the functional.
- Solve for the differential equation.

Key Concepts:

- With Lagrange multipliers, $f \rightarrow$ surface, distance, time etc. to be extremized. Then this means the constraints are then defined as g.
- $\partial L/\partial q_i =$ constant.
- Situations with multiple constraints will have multiple $\lambda' s$.

Variations:

- The surface can be changed.
  - This would change some small details but not the overall procedure.
- The object rolling in (c) can be different.
  - This then changes the constraints of the functional.

Problem 5:

Consider the variational problem:

$$I = \int \left(f(x, y_1, y_1', y_2, y_2') - \lambda(x) g(y_1, y_2) \right) dx$$

where f is the optimization function and g is a local constraint.

(a) If f is just a square root such as $\sqrt{1 + y_1'^2}$, it would be easier to minimize f^2. Does this give you the same curve?

$$\mathcal{L}(x, y_1, y_2, y_1', y_2') = f(x, y_1, y_1', y_2, y_2') - \lambda(x) g(y_1, y_2)$$

w/ $f = \sqrt{1+y_1'^2}$: $$\frac{\partial \mathcal{L}}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_{1,2}'} = \frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial F}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0$$

$$\frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial F}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0 \longrightarrow F$$

w/ $f^2 = 1 + y_1'^2$: $$\frac{\partial \mathcal{L}}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_{1,2}} = \frac{\partial f^2}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial f^2}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0$$

$$2f \cdot \frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx} 2f \frac{\partial f}{\partial y_{1,2}'} - \frac{d}{dx}\frac{\partial f}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0 \longrightarrow G$$

If we compare F w/ G we see that there are extra terms in G that will not simplify down to F. Therefore these will not yield the same curve.

(b) If g is just a square root, such as $\sqrt{y_1^2 + y_2^2}$, it would be easier to replace g with g^2. Does this give you the same curve?

$$\mathcal{L}(x, y_1, y_2, y_1', y_2') = f(x, y_1, y_1', y_2, y_2') - \lambda(x) g(y_1, y_2)$$

w/ $g = \sqrt{y_1^2 + y_2^2}$: $$\frac{\partial \mathcal{L}}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_{1,2}'} = \frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial F}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0$$

$$\frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial F}{\partial y_{1,2}'} - \lambda(x)\frac{\partial g}{\partial y_{1,2}} = 0 \longrightarrow F$$

w/ $g^2 = y_1^2 + y_2^2$: $$\frac{\partial \mathcal{L}}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y_{1,2}'} = \frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial f}{\partial y_{1,2}} - \lambda(x) \cdot 2g \cdot \frac{\partial g}{\partial y_{1,2}} = 0$$

$$\frac{\partial f}{\partial y_{1,2}} - \frac{d}{dx}\frac{\partial f}{\partial y_{1,2}} - \lambda(x) \cdot 2g \cdot \frac{\partial g}{\partial y_{1,2}} = 0 \longrightarrow G$$

If we compare F w/ G we can assume that $\lambda(x) \cdot 2g \to \lambda(x)$ because $\lambda(x)$ is an arbitrary function and thus we can say F and G will yield the same curve.

Problem 5: Review

Procedure:

- Use the Euler Lagrange equation on the functional, once with $F = f$ and once with $F = f^2$.
- Proceed to show that these do not yield the same curve.
- Repeat with $G = g$ and $G = g^2$.
- Proceed to show that there is no difference between the two curves.

Key Concepts:

- When we square the function we are extremizing, this will yield a different curve.
- When we square our constraint equation, this will not yield a different curve.

Variations:

- The function being extremized can change.
  - This should still yield a different curve.
- The constraint equation can change.
  - This shouldn't affect the overall curve.

Problem 6:

Consider a 3D crystal with a surface defined by a function $z(x,y)$. We seek to minimize the total surface free energy:

$$F[z] = \int dx\, dy \left\{ \alpha(x,y)\sqrt{1+z_x^2+z_y^2} \right\}$$

where $\alpha(z_x, z_y)$ is the direction dependent surface tension, and we have introduced the compact but confusing notation, $z_x \equiv \partial z/\partial x$ and $z_y \equiv \partial z/\partial y$. We wish to enforce the constraint of a constant volume to the crystal where the volume is given by:

$$V[z] = \int dx\, dy\, z$$

via a Lagrange multiplier, λ.

(a) What is the Euler-Lagrange equation for the system?

$$\mathcal{L}(x,y,z,\lambda) = F[z] - \lambda V[z] = \alpha(x,y)\sqrt{1+z_x^2+z_y^2} - \lambda z$$

$$\frac{\partial L}{\partial z} - \frac{d}{dx}\frac{\partial L}{\partial z_x} - \frac{d}{dy}\frac{\partial L}{\partial z_y} = 0$$

$$\frac{\partial L}{\partial z} = -\lambda \;,\; \frac{\partial L}{\partial z_x} = \frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} \;,\; \frac{d}{dx}\frac{\partial L}{\partial z_x} = \frac{d}{dx}\frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} \; : \; \frac{d}{dx}\frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} = -\lambda$$

$$\frac{\partial L}{\partial z_y} = \frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} \;,\; \frac{d}{dy}\frac{\partial L}{\partial z_y} = \frac{d}{dy}\frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} \; : \; \frac{d}{dy}\frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} = -\lambda$$

$$-\lambda - \frac{d}{dx}\frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} - \frac{d}{dy}\frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} = 0 \; : \; -\frac{d}{dx}\frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} - \frac{d}{dy}\frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} = \lambda$$

$$\boxed{-\frac{d}{dx}\frac{\alpha(x,y) z_x}{(1+z_x^2+z_y^2)^{1/2}} - \frac{d}{dy}\frac{\alpha(x,y) z_y}{(1+z_x^2+z_y^2)^{1/2}} = \lambda}$$

(b) Show that in the isotropic case, where $\alpha(z_x, z_y) = \alpha_0$, that the shape that minimizes the surface energy for fixed volume is a sphere. You may do this either by direct substitution to verify the solution

$$z(x,y) = \sqrt{R^2 - x^2 - y^2}$$

or by changing to polar co-ordinates $z(\rho, \phi)$ in the original functional, and invoking cylindrical symmetry, so that your 2D problem becomes effectively a one dimensional problem for $z(\rho)$. In either case, determine the relation between λ and the radius of the sphere.

$$-\frac{d}{dx}\frac{\alpha_0 z_x}{(1+z_x^2+z_y^2)^{1/2}} - \frac{d}{dy}\frac{\alpha_0 z_y}{(1+z_x^2+z_y^2)^{1/2}} = \lambda$$

$$z_x = \frac{-x}{\sqrt{R^2-x^2-y^2}} \;,\; z_y = \frac{-y}{\sqrt{R^2-x^2-y^2}} \; : \; z_x^2 = \frac{x^2}{R^2-x^2-y^2} \;,\; z_y^2 = \frac{y^2}{R^2-x^2-y^2}$$

$$z_x^2 + z_y^2 = \frac{x^2+y^2}{R^2-x^2-y^2} \; : \; 1 + z_x^2 + z_y^2 = \frac{R^2 \cancel{-x^2} \cancel{-y^2} \cancel{+x^2} \cancel{+y^2}}{R^2-x^2-y^2} = \frac{R^2}{R^2-x^2-y^2}$$

Problem 6: Continued

$$\frac{z_x}{(1+z_x^2+z_y^2)^{1/2}} = \frac{-x}{\sqrt{R^2-x^2-y^2}} \Big/ \frac{R}{\sqrt{R^2-x^2-y^2}} = -\frac{x}{R} \quad : \quad \frac{z_y}{(1+z_x^2+z_y^2)^{1/2}} = \frac{-y}{R}$$

$$-\frac{d}{dx}\frac{-a_0 x}{R} - \frac{d}{dy}\frac{-a_0 y}{R} = \lambda \quad : \quad \frac{a_0}{R} + \frac{a_0}{R} = \lambda \quad : \quad \frac{2a_0}{R} = \lambda$$

$R = \frac{2a_0}{\lambda}$ w/ $R = \sqrt{x^2+y^2+z^2}$ $\longrightarrow$ This is a sphere !

$$\boxed{R = \frac{2a_0}{\lambda}}$$

Problem 6: Review

Procedure:

- Define a functional with constraints $\rightarrow$ Lagrange multiplier method.
- Add extra terms into the Euler Lagrange equation because of multiple independent variables.
- Use Euler Lagrange equations on the functional to create the Equations of Motion.
- Solve for the constraint in terms of other variables.
- Proceed to use the $z(x, y)$ equation in the Equation of Motion.
- Show that the equation will simplify down to a sphere.

Key Concepts:

- Two independent variables just get added together in the Euler Lagrange equation.
- Multiple dependent variables will cause multiple Euler Lagrange equations.
- Constraints can be dependent upon multiple independent variables.
- Use Lagrange multipliers to extremize functions with constraints.

Variations:

- The constraint equation can change.
  - This can add independent variables to the system, or change the function.
- The function to be extremized can be slightly changed.
  - Multiple dependent variables can be added $\rightarrow$ multiple Euler Lagrange equations.

Problem 7:

Variational Derivation of Poisson Equation:

The energy in an electrostatic field in the presence of charges is:

$$E = \int \left\{ \frac{1}{8\pi} (\nabla \phi(\vec{r}))^2 - \rho(\vec{r})\phi(\vec{r}) \right\} d\vec{r}$$

Show that the minimum energy configuration of the potential $\phi(\vec{r})$ satisfies the equation:

$$\vec{\nabla}^2(\vec{r}) = -4\pi\rho(\vec{r}).$$

$$f = \frac{1}{8\pi}(\nabla\phi(\vec{r}))^2 - \rho(\vec{r})\phi(\vec{r}) \quad : \quad \frac{\partial f}{\partial \phi(\vec{r})} - \frac{d}{dr}\frac{\partial f}{\partial \nabla\phi(\vec{r})} = 0$$

$$\frac{\partial f}{\partial \nabla \phi(\vec{r})} = \frac{1}{4\pi}\nabla\phi(\vec{r}) \quad , \quad \frac{d}{dr}\cdot\frac{\partial f}{\partial \nabla\phi(\vec{r})} = \frac{\nabla^2\phi(\vec{r})}{4\pi} \quad , \quad \frac{\partial f}{\partial \phi(\vec{r})} = -\rho(\vec{r})$$

$$-\rho(\vec{r}) - \frac{\nabla^2\phi(\vec{r})}{4\pi} = 0 \quad : \quad \frac{\nabla^2\phi(\vec{r})}{4\pi} = -\rho(\vec{r}) \quad : \quad \nabla^2\phi(\vec{r}) = -4\pi\rho(\vec{r})$$

$$\boxed{\nabla^2\phi(\vec{r}) = -4\pi\rho(\vec{r})}$$

Thus the *minimal energy* configuration for the field is also the one given by the Poisson equation.

The above problem is rather simple. I am assigning it to you for the notes below, so that you see *why* it works. It is a common technique in quantum field theory, so it's worth knowing the background.

Background: The expression for the energy above avoids counting the "self-energy", so that a charge does not feel the force of the electric field it creates. To see this assume that we have a set of point charges q_i each with potential $\phi_i(\vec{r})$. Then the total potential is the sum of the individual contributions:

$$\phi(\vec{r}) = \sum_i \phi_i(\vec{r}).$$

The energy of the charges (in some units) can be written as

$$E = \frac{1}{8\pi}\int \left(\vec{\mathcal{E}}(\vec{r})\right)^2 d\vec{r}$$

(including the self-energy (Jackson, *Classical Electrodynamics*, Pg. 46)) which we can write as

$$2E = \int \left\{ \frac{1}{8\pi} \sum_{i \neq j} (\nabla\phi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r})) \right\} d\vec{r}$$

where the factor of 2 comes from double counting in the sum over i and j. The energy can also be written as:

$$E = \int \left\{ \sum_{i \neq j} q_i \delta(\vec{r} - \vec{r}_i)\phi_j(\vec{r}) \right\} d\vec{r}$$

where $\delta(\vec{r} - \vec{r}_i)$ is a function sharply peaked at $\vec{r}_1$, the location of the point charge, and there is no double counting. If we remove the restriction $i \neq j$, then both expressions pick up a self-energy term. However, if we subtract the two expressions, these cancel out, and we are left with simply the total energy:

Problem 7: Continued

$$\int \left\{ \frac{1}{8\pi} \sum_{i,j} (\bigtriangledown \phi_i(\vec{r}) \cdot \bigtriangledown \phi_j(\vec{r})) - q_i \delta(\vec{r} - \vec{r}_i) \phi_j(\vec{r}) \right\} = 2E + E_s - E - E_s = E.$$

If we take the continuum limit and replace the discrete distribution by a continuous one, then we obtain the expression at the top of the page.

This "background" discussion does not contain any questions. I am including it merely because the above formulation is routinely invoked in some field theory courses without explanation.

Problem 7: Review

Procedure:

- Extremize the functional with the Euler Lagrange equation.
- Independent variable in this case will be $\vec{r}$.
- Dependent variable in this case is $\vec{\nabla}(\phi(\vec{r}))$.

Key Concepts:

- Functions can be extremized with the Euler Lagrange equation.

Variations:

- This problem cannot be changed have it remain the same type of problem since it is a derivation.

Problem 8:

Obtain access to MATHEMATICA on some computer. As proof of having access, if N is your OU ID number, calculate the prime factors of $N^2 + 1$ by typing in the command:

$$\textbf{FactorInteger}[N * N + 1]$$

and then hold down the "shift" key and the "return" key at the same time to execute the command.

```
In[1]:= FactorInteger[113 556 911 * 113 556 911 + 1]
Out[1]= {{2, 1}, {2113, 1}, {265 621, 1}, {11 487 757, 1}}
```

Problem 8: Review

Procedure:

- Download MATHEMATICA and follow the instructions in the problem statement.

Key Concepts:

- MATHEMATICA can be used to find the prime factors of an integer.

Variations:

- This problem is too simple to be changed.