

E & M Qualifier

January 14, 2016

To insure that the your work is graded correctly you MUST:

1. use only the reference material supplied (Schaum's Guides),
2. use only the blank answer paper provided,
3. write only on one side of the page,
4. put your alias (**NOT YOUR REAL NAME**) on every page,
5. start each problem by stating your units e.g., SI or Gaussian,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
 - (a) the first number is the problem number,
 - (b) the second number is the page number for **that** problem (start each problem with page number 1),
 - (c) the third number is the total number of pages you used to answer **that** problem,
 - (d) try to answer every problem, but if you don't please include a single numbered page stating that you have skipped that problem.
7. **DO NOT** staple your exam when done. Paper clips will be provided.

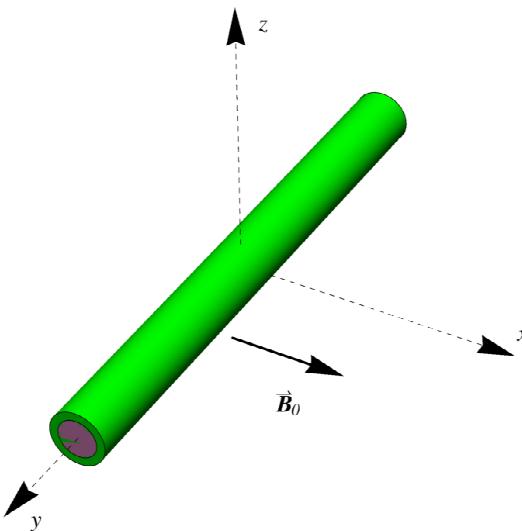
1. Consider a Lorentz frame K containing no polarizable materials in which there is a magnetic induction $\mathbf{B} = B^x \hat{\mathbf{x}} + B^y \hat{\mathbf{y}} + B^z \hat{\mathbf{z}}$ but no electric field.
- (a) [1 pt] For the above magnetic induction, write down the 4-dimensional electromagnetic field tensor $F^{\alpha\beta}$ in frame K as a matrix.
 - (b) [1 pt] Write down a homogeneous Lorentz boost Λ^α_β in the y-direction from frame K to another frame K' which is moving with velocity $\mathbf{v} = v_0 \hat{\mathbf{y}}$ as seen by observers that are at rest in frame K.
 - (c) [2 pt] Apply the boost Λ^α_β to $F^{\alpha\beta}$ to find $F'^{\alpha\beta}$, the field strength tensor as seen in the moving frame K' .
 - (d) [2 pt] What are the electric field components E'^x , E'^y , and E'^z and the magnetic induction components B'^x , B'^y , and B'^z in frame K'?
 - (e) [4 pt] Consider explicitly a \mathbf{B} field in the K frame caused by an **uncharged** infinitely long and thin wire centered on the y-axis $(x, z) = (0, 0)$ which carries a current I in the $+y$ direction. Assume that no polarizable materials are present, i.e., assume $\epsilon_r = 1$ and $\mu_r = 1$. What are $\mathbf{B}'(x', y', z')$ and $\mathbf{E}'(x', y', z')$ in the K' frame, written as functions of the K'-coordinates? Where does \mathbf{E}' point?

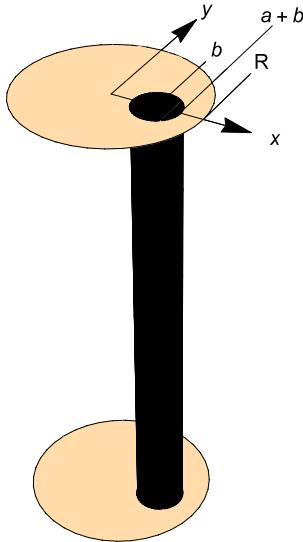
2. Consider a very long hollow cylinder made of iron that is placed with its axis perpendicular to a uniform external magnetic induction $\mathbf{B}_0 = B_0 \hat{\mathbf{x}}$. Assume the inner radius of the hollow cylinder is a and the outer radius is b . Also assume the permeability μ of the iron is a constant. The goal of this problem is to calculate the magnetic induction \mathbf{B} inside the hollow region ($0 \leq \rho \equiv \sqrt{x^2 + y^2} < a$).

- (a) [3 pt] Starting with Maxwell's equations for static \mathbf{B} and \mathbf{H} fields and assuming that there is no free current density, $\mathbf{J}_f = 0$, prove that the field \mathbf{H} can be written as the negative gradient of a magnetic scalar potential Φ_M that satisfies the Poisson equation with an appropriate source term. For this particular problem the Poisson equation reduces to the Laplace equation except at the cylinder's boundaries.
- (b) [3 pt] Derive the appropriate boundary conditions to be satisfied by the scalar potential Φ_M and the magnetic field \mathbf{H} at $\rho = a$ and $\rho = b$.
- (c) [4 pt] Solve for the \mathbf{H} field in the interior region $\rho < a$. Hint: solve the Laplace equation for Φ_M in the three regions $0 \leq r < a$, $a < r < b$, and $b < r < \infty$, and appropriately match these solutions at the cylinder's boundaries. Show that for large μ , (i.e., when $\mu \rightarrow \infty$) the iron provides complete shielding from the magnetic field, i.e., $\mathbf{H} \rightarrow 0$ for $\rho < a$.

Hint:

$$\nabla^2 \Phi_M = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi_M}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi_M}{\partial \phi^2} + \frac{\partial^2 \Phi_M}{\partial z^2}.$$





3. A very long straight conductor has a circular cross section of radius R and carries a current I . Inside the conductor, there is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor and a distance b from it ($a + b < R$). The goal of this problem is to show that the magnetic induction $\mathbf{B}(x, y)$ inside the hole is uniform and to calculate its value. Assume the wire of radius R is centered on the z axis, i.e., at $(x, y) = (0, 0)$ and the cylindrical hole of radius a is centered at $(x, y) = (b, 0)$. Assume the current I is uniformly distributed in the conducting material.

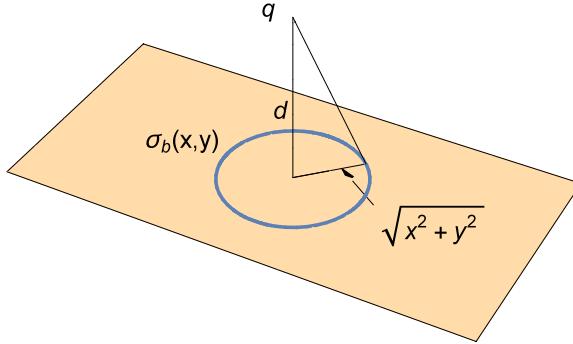
(a) [3 pts]

Ignoring the hole, use Amperé's Law to find the magnetic induction, $\mathbf{B}_R(x, y)$, inside a homogeneous cylindrical wire of radius R that carries a uniform current density $J_R = I_R/\pi R^2$ in the $+z$ direction.

(b) [4 pts] Ignoring the current in the wire of radius R assume an imaginary wire of radius a located at $(x, y) = (b, 0)$ carries a current density $J_a = I_a/\pi a^2$ in the $-z$ direction. Use Ampere's Law to find the magnetic induction, $\mathbf{B}_a(x, y)$, inside the imaginary wire of radius a caused by J_a .

(c) [3 pts]

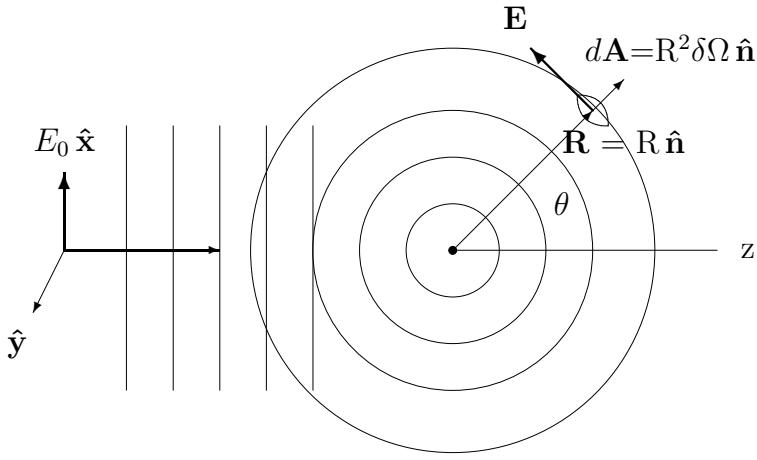
By adjusting the two current densities to have the same magnitude, and superimposing the two magnetic inductions, find the resultant $\mathbf{B}(x, y)$ field inside the hole in the original conductor that carries a current I described at the beginning of this problem.



4. Consider a large flat interface at $z = 0$ between a dielectric and free space. The region where $z < 0$ is filled with a uniform linear dielectric material with a relative permittivity ϵ_r (equivalently a dielectric constant ϵ_r). If the only free charge present is a point charge $q > 0$ situated a distance d from the origin at $\mathbf{r}_q = (0, 0, d)$, where $d > 0$, answer the following 5 questions.

To answer them you should look at the electric field as a sum of two fields, a coulomb part \mathbf{E}_q caused by the point charge q and a second part \mathbf{E}_b caused by the bound surface charge $\sigma_b(x, y)$ located on the $z = 0$ interface.

- (a) [2 pts] Write two expressions for the z component of the total electric field $E^z = E_q^z + E_b^z$, one just above the dielectric's surface and one just below the dielectric's surface. The E_b^z part is directly related to σ_b by Gauss's law.
- (b) [3 pts] Use the two electric fields from part (a) and the continuity of the normal part of the displacement vector ϵE^z to solve for $\sigma_b(x, y)$ as a function of the known coulomb field $E_q^z(x, y, 0)$.
- (c) [3 pts] Calculate the electric field at the position of the charge q caused by the bound surface charge σ_b . You simply have to integrate a superposition of coulomb fields. From symmetry the resultant field points in the $\pm z$ direction.
- (d) [2 pts] Show that this resultant bound charge field at $(0, 0, d)$ can be interpreted as the field of a single image charge q' located at point $\mathbf{r}_{q'} = (0, 0, -d)$. What is the value of q' ?



5. In this question a monochromatic linearly polarized plane wave is scattered by a free electron. If the initial speed of the particle is non-relativistic (i.e., $\beta \ll 1$) and the frequency of the plane wave satisfies $h\nu \ll m_e c^2$, then the electron is accelerated by the plane wave's electric field in accord with Newton's 2nd law, but its speed remains non-relativistic. Due to its acceleration, the electron emits radiation in all directions thus scattering the original plane wave. See the figure.

- (a) [2 pts] Assume the plane wave travels in the z-direction and is polarized in the x-direction as shown in the figure. Compute the acceleration, $\dot{\beta}(t) = \dot{\mathbf{v}}(t)/c$, of the electron caused by the plane wave's electric field.
- (b) [3 pts] Compute the electric field \mathbf{E} , the magnetic induction \mathbf{B} , and the Poynting vector \mathbf{S} of the radiated wave. { Hint: In Gaussian units $\mathbf{E}_G = q[\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\beta})]/(cR)|_{ret}$, $\mathbf{B}_G = \hat{\mathbf{n}} \times \mathbf{E}$, and $\mathbf{S}_G = (c/4\pi)\mathbf{E} \times \mathbf{H}$. In SI units $\mathbf{E}_{SI} = (1/4\pi\epsilon_0)\mathbf{E}_G$, $\mathbf{B}_{SI} = (1/c)\mathbf{B}_G$, and $\mathbf{S}_{SI} = \mathbf{E} \times \mathbf{H}$. }
- (c) [3 pts] Use your results to compute the differential scattering cross section

$$\frac{d\sigma(\theta, \phi)}{d\Omega} = \frac{<\mathbf{S} \cdot d\mathbf{A}>}{|<\mathbf{S}_0>|\delta\Omega}.$$

In the above $<>$ stands for a time average and $|<\mathbf{S}_0>|$ is the magnitude of the time averaged Poynting vector of the incoming plane wave. The detector area element $d\mathbf{A}$ subtends a solid angle $\delta\Omega$ at the radiating electron and is typically of the form

$$d\mathbf{A} = R^2 \delta\Omega \hat{\mathbf{n}}.$$

- (d) [2 pts] Integrate your differential cross section over all (θ, ϕ) directions to obtain the total Thompson cross section σ_T .

6. (a) [2 pts] In a homogeneous, linear and isotropic conducting material whose electromagnetic properties (at low frequencies) are described by constant (and real) values of the permittivity, permeability, and conductivity respectively ϵ , μ , and σ , show that Maxwell's equations require that the electric field satisfy the telegraph equation

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0, \quad (\text{SI})$$

$$\nabla^2 \mathbf{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial^2 t} - \frac{4\pi\sigma\mu}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0. \quad (\text{Gaussian})$$

- (b) [3 pts] Given a linearly polarized plane wave of angular frequency ω whose electric field is of the form

$$\mathbf{E}(z, t) = \text{Real} \left\{ E_0 e^{i(kz - \omega t)} \right\} \hat{\mathbf{x}},$$

evaluate k^2 as a function of ϵ , μ , σ , and ω .

- (c) [2 pts] Find the real and imaginary parts of k assuming $\sigma \gg \omega\epsilon$.
 (d) [3 pts] Using your results from (c) find the skin depth δ of the conductor. The skin depth is defined by the depth at which the wave's amplitude decreases by e^{-1} , i.e.,

$$\frac{|\mathbf{E}(z + \delta, t)|}{|\mathbf{E}(z, t)|} = \frac{1}{e}$$