E & M I

Workshop 5 – Rotating Charged Sphere, Solutions

A standard problem in magnetostatics (and therefore possibly will show up on qualifiers) is calculating the magnetic field due to a rotating sphere with a constant surface charge density. This problem is done in a lot of places, but it will be good to make sure you can do this yourself.

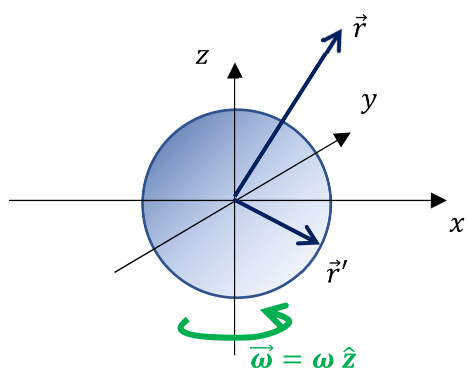
The sphere is centered at $\vec{r} = 0$, has a radius R, a surface charge density σ, and is rotating around the z axis with and angular speed ω. The angular velocity is:

$$\vec{\omega} = \omega\,\hat{z}$$

The Right-Hand-Rule says that this corresponds to a counter-clockwise rotation when looking down on the sphere from the positive z axis.

As a start, we want to solve for the vector potential $\vec{A}(\vec{r})$ for points outside the sphere.

a) Draw a picture that can be used to define the coordinates you will be using for this problem.



b) Write an expression for the current density for the spinning sphere, $\vec{J}(\vec{r}')$, where $\vec{r}'$ will be the position of currents to be integrated.

Write the magnitude of $\vec{J}(\vec{r}')$ in terms of spherical coordinates, r', θ', ϕ', and write the direction of the vector in two different ways, using $\hat{r}', \hat{\theta}', \hat{\phi}'$ and using $\hat{x}', \hat{y}', \hat{z}'$. Check to make sure your expression has the correct units and the magnitude and direction of $\vec{J}$ are correct at various points on the sphere.

Note: It might be useful to remember that $\vec{v} = \vec{\omega} \times \vec{r}$.

The charge is all on the surface of the sphere, so $\vec{J}(\vec{r}')$ is non-zero only for $\vec{r}' = R$. Each small area of the sphere is a charge $dQ = \sigma\, dA$ and is moving with a velocity:

$$\vec{v}(\vec{r}') = \omega\,\hat{z} \times \vec{r}' = \omega\, R \sin\theta'\;\hat{\phi}'$$

This means each point is moving about the z-axis azimuthally with the equator moving the fastest and the poles not moving.

This gives the current density:

$$\vec{J}(\vec{r}') = \sigma\,\omega\,R\,\delta(r'-R)\sin\theta'\;\hat{\phi}'$$

When doing integrals over spherical coordinates of vectors, we need to take into account the fact that the unit vectors $\hat{r}', \hat{\theta}', \hat{\phi}'$ depend on the position $\vec{r}'$.

For example, relevant to this case, for the position $\vec{r}' = R\,\hat{x}$, $\hat{\phi}' = \hat{y}$ while for the position $\vec{r}' = R\,\hat{y}$, $\hat{\phi}' = -\hat{x}$. This gives:

$$\vec{J}(\vec{r}') = \sigma\,\omega\,R\,\delta(r'-R)\sin\theta'\;(-\sin\phi'\;\hat{x} + \cos\phi'\;\hat{y})$$

c) Using the definition for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint \vec{J}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\,d^3r'$$

And the multipole expansion:

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}\;Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Write down the multipole expansion for the vector potential. This should still include sums over l and m and the volume integral over $\vec{r}'$. Remember to include the vector direction(s) of the current density.

Plugging in the expression for the current density:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\iiint d^3r'\,r'^l\,Y_{lm}^*(\theta',\phi')\sigma\,\omega\,R\,\delta(r'-R)\sin\theta'\;\hat{\phi}'$$

$$\vec{A}(\vec{r}) = \mu_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\sigma\,\omega}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}R^{l+3}\iint d\Omega'\;Y_{lm}^*(\theta',\phi')\sin\theta'\,(-\sin\phi'\;\hat{x} + \cos\phi'\;\hat{y})$$

Next, simplify this equation by using the orthonormality of the spherical harmonics:

$$\iint \sin\theta \, d\theta \, d\phi \;\; Y^*_{l',m'}(\theta,\phi) \, Y_{l,m}(\theta,\phi) = \delta_{l,l'} \, \delta_{m,m'}$$

A table of the spherical harmonics is at: https://en.wikipedia.org/wiki/Table_of_spherical_harmonics

but (here's a hint) all you'll really need for this problem are:

$$Y_{1,0}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta$$

$$Y_{1,1}(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\,\pi}} \cdot \sin\theta \cdot e^{i\phi}, \quad Y_{1,-1}(\theta,\phi) = -Y^*_{1,1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\,\pi}} \cdot \sin\theta \cdot e^{-i\phi}$$

d) Write your expression for $\vec{J}(\vec{r}\,')$ in terms of spherical harmonics, showing that you only need the $l = 1$ terms to do this.

The angular dependence of $\vec{J}$ is:

$$\sin\theta' \; (-\sin\phi' \;\; \hat{x} + \cos\phi' \;\; \hat{y}\,) = \frac{1}{2}\sin\theta' \left(i\left(e^{i\phi} - e^{-i\phi}\right)\hat{x} + \left(e^{i\phi} + e^{-i\phi}\right)\hat{y}\right)$$

$$= \sqrt{\frac{2\,\pi}{3}} \left(-i\left(Y_{1,1} + Y_{1,-1}\right)\hat{x} + \left(-Y_{1,1} + Y_{1,-1}\right)\hat{y}\right)$$

As noted, we'll only have $l = 1$ terms in $\vec{J}$ which means that there will only be $l = 1$ terms in the result for $\vec{A}(\vec{r})$.

e) Use the orthonormality of the spherical harmonics to complete the integral for $\vec{A}$. Show that your result gives a vector potential of the form:

$$\vec{A}(\vec{r}) = A_\phi(\vec{r}) \, \hat{\phi}$$

From above, and letting $l = 1$, we'll only get a couple of terms in $Y_{lm}(\theta,\phi)$, the field coordinates:

$$\vec{A}(\vec{r}) = \mu_0 \sum_{m=-1}^{1} \frac{\sigma\,\omega}{3} \frac{Y_{1m}(\theta,\phi)}{r^2} R^4 \sqrt{\frac{2\,\pi}{3}} \times$$

$$\iint d\Omega' \;\; Y^*_{1m}(\theta',\phi')\left(-i\left(Y_{1,1} + Y_{1,-1}\right)\hat{x} + \left(-Y_{1,1} + Y_{1,-1}\right)\hat{y}\right)$$

$$\vec{A}(\vec{r}) = \mu_0 \sum_{m=-1}^{1} \frac{\sigma\,\omega}{3} \frac{Y_{1m}(\theta,\phi)}{r^2} R^4 \sqrt{\frac{2\,\pi}{3}} \left(-i\left(\delta_{m1} + \delta_{m-1}\right)\hat{x} + \left(-\delta_{m1} + \delta_{m-1}\right)\hat{y}\right)$$

$$\vec{A}(\vec{r}) = \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r^2} \frac{\sin\theta}{2} \;\; \left(-i\left(-e^{i\phi} + e^{-i\phi}\right)\hat{x} + \left(e^{i\phi} + e^{-i\phi}\right)\hat{y}\right)$$

$$\vec{A}(\vec{r}) = \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r^2} \sin\theta \ \left(-\sin\phi\ \hat{x} + \cos\phi\ \hat{y}\,\right)$$

$$\vec{A}(\vec{r}) = \ \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r^2} \sin\theta\ \hat{\phi} = A_\phi(r,\theta)\hat{\phi}$$

Note: Using an expression like we used above for the velocity,

$$\vec{\omega} \times \vec{r} = \omega\, r \sin\phi\ \hat{\phi}$$

$$\vec{A}(\vec{r}) = \mu_0 \frac{\sigma\, R^4}{3} \frac{\vec{\omega} \times \vec{r}}{r^3}$$

f) Using:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B = \frac{1}{r\sin\theta}\left(\partial_\theta \sin\theta\, A_\phi - \partial_\phi\, A_\theta\right)\hat{r} + \left(\frac{1}{r\sin\theta}\ \partial_\phi\, A_r - \frac{1}{r}\ \partial_r\, r\, A_\phi\right)\hat{\theta} + \frac{1}{r}\left(\partial_r\, r\, A_\theta - \partial_\theta\, A_r\right)\hat{\phi}$$

Solve for the magnetic field of the spinning sphere.

Because we only have a ϕ component of $\vec{A}$, most of these terms are zero. This gives us:

$$B_r = \frac{1}{r\sin\theta}\ \partial_\theta\, \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r^2} \sin^2\theta = \frac{2}{3}\, \mu_0\, \sigma\, \omega\ \frac{R^4}{r^3} \cos\theta$$

$$B_\theta = -\frac{1}{r}\ \partial_r\, \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r} \sin\theta = \mu_0 \frac{\sigma\,\omega}{3} \frac{R^4}{r^3} \sin\theta$$

$$B_\phi = 0$$

This gives:

$$\vec{B}(\vec{r}) = \mu_0 \frac{\sigma\,\omega\, R^4}{3\, r^3}\ \left(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta}\right)$$

g) Show that your result implies that the spinning sphere is a pure magnetic dipole,

$$\vec{m} = m\,\hat{z}$$

What is the magnitude, m?

Hint: For a vector $\vec{r} = (r,\theta,\phi)$ the unit vectors give: (You might draw a diagram showing this)

$$\hat{z}\cdot\hat{r} = \cos\theta\,, \qquad \hat{z}\cdot\hat{\theta} = -\sin\theta$$

Consider the magnetic field due to a point dipole:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}\cdot\hat{r})\,\hat{r}}{r^3} - \frac{\vec{m}}{r^3}\right)$$

Writing:

$$\vec{m} = m\cos\theta\,\hat{r} - m\sin\theta\ \hat{\theta}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\frac{m\cos\theta\ \hat{r}}{r^3} - \frac{m\cos\theta\ \hat{r} - m\sin\theta\ \hat{\theta}}{r^3}\right) = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\ \hat{\theta}\right)$$

Comparing to the result above:

$$m = \frac{4\pi}{3}\ R^4\,\sigma$$

Just for Kicks) If we are interested in what happens inside the sphere, where $r < R$, we need to use: (There are two solutions to the r-dependence for Laplace, $r > r'$ and $r < r'$.

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty}\ \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\,\frac{r^l}{R^{l+1}}\ Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

Solve for the magnetic field inside the sphere.

Let's do this using:

$$\vec{J}(\vec{r}') = \sqrt{\frac{2\,\pi}{3}}\ \sigma\,\omega\,R\,\delta(r'-R)\big(-i\left(Y_{1,1}+Y_{1,-1}\right)\hat{x} + \left(-Y_{1,1}+Y_{1,-1}\right)\hat{y}\,\big)$$

Again, we'll only have the $l = 1$ terms in the spherical harmonic expansion:

$$\vec{A}(\vec{r}) = \mu_0 \sum_{m=-1}^{1} \frac{\sigma\,\omega\,R}{3}\,\frac{Y_{1m}(\theta,\phi)\,r}{R^2}\sqrt{\frac{2\,\pi}{3}}\int r'^2\,dr'\ \delta(r'-R)\,\times$$

$$\iint d\Omega'\ Y_{1m}^*(\theta',\phi')\big(-i\left(Y_{1,1}+Y_{1,-1}\right)\hat{x} + \left(-Y_{1,1}+Y_{1,-1}\right)\hat{y}\big)$$

$$\vec{A}(\vec{r}) = \mu_0 \sum_{m=-1}^{1} \frac{\sigma\,\omega\,R}{3}\ r\,Y_{1m}(\theta,\phi)\sqrt{\frac{2\,\pi}{3}}\ \left(-i\left(\delta_{m1}+\delta_{m-1}\right)\hat{x} + \left(-\delta_{m1}+\delta_{m-1}\right)\hat{y}\right)$$

$$\vec{A}(\vec{r}) = \mu_0\frac{\sigma\,\omega\,R}{3}\ r\ \sin\theta\ \left(-\sin\phi\ \hat{x} + \cos\phi\ \hat{y}\right)$$

$$\vec{A}(\vec{r}) = \mu_0\frac{\sigma\,\omega\,R}{3}\ r\ \sin\theta\ \hat{\phi} = \mu_0\frac{\sigma\,R}{3}\ \vec{\omega}\times\vec{r}$$

The magnetic field is:

$$B_r = \frac{1}{r\sin\theta}\ \partial_\theta\,\mu_0\frac{\sigma\,\omega\,R}{3}\ r\ \sin^2\theta = \frac{2}{3}\ \mu_0\,\sigma\,\omega\,R\cos\theta$$

$$B_\theta = -\frac{1}{r}\ \partial_r\,\mu_0\frac{\sigma\,\omega\,R}{3}\ r^2\ \sin\theta = -\frac{2}{3}\ \mu_0\,\sigma\,\omega\,R\sin\theta$$

Giving:

$$\vec{B}(\vec{r}) = \frac{2}{3}\ \mu_0\,\sigma\,\omega\,R\left(\cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}\right) = \frac{2}{3}\ \mu_0\,\sigma\,\omega\,R\,\hat{z}$$

Inside the sphere, the magnetic field is a constant pointing in the $\hat{z}$ direction.