An extra example: spiralling football.

* Let's consider the rotational motion of a football thrown by a quarterback.

* We assume → no applied torque (ignore gravity)

A simple treatment is given by Euler's eqns (defined w.r.t. body-fixed axes):

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\,\omega_2 \omega_3 = 0$$

$$I_2 \omega_2 + (I_1 - I_3)\,\omega_1 \omega_3 = 0$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\,\omega_1 \omega_2 = 0$$

A football has cylindrical symmetry such that

$$I_1 = I_2 = I_\perp \neq I_3$$

Consequence
⇒ $\dot{\omega}_3 = 0$ → ω_3 is a constant.

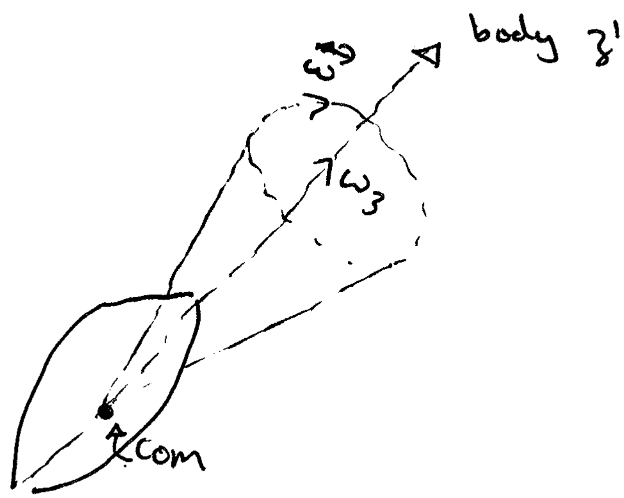
This enables us to obtain $\omega_1(t)$ & $\omega_2(t)$ in a straightforward manner.

i.e. $\omega_1(t) = C\sin(\Omega t + \delta)$ w/ $C = (\omega_{1i}^2 + \omega_{2i}^2)^{1/2}$

$\omega_2(t) = C\cos(\Omega t + \delta)$ $\delta = \operatorname{atan}\left(\frac{\omega_{1i}}{\omega_{2i}}\right)$

$\omega_3 = \omega_{3i}$

In the body-fixed frame:



$\vec{\omega}$ precesses in $x'y'$ plane about z'

[In space fixed frame football 'wobbles']

Note also → if no torques acting on football
→ angular moment $\vec{L}$ = const in space frame. (inertial)

What is the role of precession (wobble, of football?) and it's nature?

Proceed by expressing $\vec{L}$ in body-fixed frame:

$$\left.\begin{aligned} L_1 &= I_\perp \omega_1 = l \sin\theta \sin\psi \\ L_2 &= I_\perp \omega_2 = l \sin\theta \cos\psi \\ L_3 &= I_3 \omega_3 = l \cos\theta \end{aligned}\right\}$$

1)

(we assume $\vec{L}$ is along space fixed z)

Seperately we have instantaneous angular velocity,

$$\vec{\omega} = \begin{pmatrix} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{pmatrix}$$

2)

Note $\underline{\dot{\theta} = 0}$ (see diagram!)

The rate of precession:

$$\omega_p = \dot{\phi}$$

With $\dot{\theta} = 0$ → clearly we have:

$$L_1 \Rightarrow \quad I_\perp \dot{\phi} \sin\theta \sin\psi = l \sin\theta \sin\psi$$

$$\rightarrow \omega_p = l / I_\perp$$

Evaluating L with initial conditions then yields,

(*)

$$\omega_p = \omega_{3i}\left[\left(\frac{I_3}{I_\perp}\right)^2 + \frac{(\omega_{1i}^2+\omega_{2i}^2)}{\omega_{3i}^2}\right]^{1/2}$$

And clearly,

$$\cos\theta = L_3/L = \left[1+\left(\frac{I_\perp}{I_3}\right)^2\frac{(\omega_{1i}^2+\omega_{2i}^2)}{\omega_{3i}^2}\right]^{-1/2}$$

Important consequences:

i) $\theta \longrightarrow 0$ for $\omega_{3i} \gg \sqrt{\omega_{1i}^2+\omega_{2i}^2}$

i.e. fast spinning pass

ii) $\dfrac{\omega_p}{\omega_{3i}} \longrightarrow I_3/I_\perp$

$\boxed{\text{depends only on properties of ball}}$

wobble to spin ratio

(typically $\sim 3/5$ for current footballs)

$$I_3 \propto \int dV\, x^2+y^2$$

$$I_\perp \propto \frac{1}{2}\int dV\, 2r^2 - z^2$$

∴ want narrow ball!

(*) compute L using $\omega_1(t)$ & $\omega_2(t)$ & $\omega_3(t)$ 2||e

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$= I_\perp^2 C^2 + I_3^2\omega_3^2$$

$$= I_\perp^2\omega_3^2\left(\left(\frac{I_3}{I_\perp}\right)^2 + C^2/\omega_3^2\right)$$

$$\rightarrow \frac{L}{I_\perp} = \omega_3\left(\left(\frac{I_3}{I_\perp}\right)^2 + C^2/\omega_3^2\right)^{1/2} \quad \checkmark$$