Solutions to Homework 8
Physics 5393

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P-2.15 The simple harmonic oscillator and the translation operator.

a) Write down the wave function (in coordinate space) for the state specified in P-2.12 at $t = 0$

$$\exp(-i\tilde{\mathbf{p}}a/\hbar)\,|0\rangle\,.$$

You may use

$$\langle x'\,|0\,\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}}\exp\left(-\frac{x'^2}{2x_0^2}\right) \quad \text{with} \quad x_0^2 \equiv \frac{\hbar}{m\omega}.$$

The operator $\exp(-i\tilde{\mathbf{p}}a/\hbar)$ is the previously introduced translation operator $\mathcal{J}(a)$. Applied on the dual position eigenket, it yields

$$\mathcal{J}(a)\,\big|x'\,\big\rangle = \big|x' - a\,\big\rangle\,.$$

Therefore,

$$\big\langle x'\,\big|e^{-i\tilde{\mathbf{p}}a}\big|\,0\big\rangle = \big\langle x' - a\,|0\,\big\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}}\exp\left(-\frac{(x'-a)^2}{2x_0^2}\right).$$

b) Obtain a simple expression for the probability that the state is found in the ground state at $t = 0$. Does this probability change for $t > 0$?

The probability that the system is in the ground state is

$$\left|\big\langle 0\,\big|e^{-i\tilde{\mathbf{p}}a}\big|\,0\big\rangle\right|^2 = \left|\int_{-\infty}^{\infty} dx'\,\big\langle 0\,|x'\,\big\rangle\,\big\langle x'\,\big|e^{-i\tilde{\mathbf{p}}a}\big|\,0\big\rangle\right|^2.$$

The explicit form of the integral is

$$\frac{1}{\pi^{1/2}x_0}\int_{-\infty}^{\infty} dx'\,\exp\left(-\frac{(x'-a)^2 + x'^2}{2x_0^2}\right).$$

The integral can be calculated by completing the squares

$$(x'-a)^2 + x'^2 = 2\left(x'^2 - ax' + \frac{a^2}{2}\right) = 2\left(x' - \frac{a}{2}\right)^2 + \frac{a^2}{2},$$

and performing a change of variables $y = x' - a/2$. The integral is therefore

$$\frac{1}{\pi^{1/2}x_0}e^{-a^2/4x_0^2}\int_{-\infty}^{\infty} dy\,\exp\left(-\frac{y^2}{2x_0^2}\right) = e^{-a^2/4x_0^2}.$$

Hence, the probability is

$$\mathcal{P} = \left|e^{-a^2/4x_0^2}\right|^2 = e^{-a^2/2x_0^2}.$$

Since the quantity being calculated has only a time dependence in the eigenstates

$$|0;t\,\rangle = e^{-iE_0t/\hbar}\,|0\rangle \qquad \text{and} \qquad \langle 0;t| = e^{-iE_0t/\hbar}\,\langle 0|\,,$$

the time dependence cancels and the probability is independent of time.

P-2.16 Consider a one-dimensional simple harmonic oscillator.

a) Using:

$$\left.\begin{matrix}\tilde{\mathbf{a}}\\ \tilde{\mathbf{a}}^\dagger\end{matrix}\right\} = \sqrt{m\omega 2\hbar}\left(\tilde{\mathbf{x}} \pm \frac{i\tilde{\mathbf{p}}}{m\omega}\right), \qquad = \left.\begin{matrix}\tilde{\mathbf{a}}\,|n\rangle\\ \tilde{\mathbf{a}}^\dagger\,|n\rangle\end{matrix}\right\} = \begin{cases}\sqrt{n}\,|n-1\rangle\\ \sqrt{n-1}\,|n+1\rangle\end{cases}$$

evaluate $\langle m\,|\tilde{\mathbf{x}}|\,n\rangle$, $\langle m\,|\tilde{\mathbf{p}}|\,n\rangle$, $\langle m\,|\{\tilde{\mathbf{x}},\tilde{\mathbf{p}}\}|\,n\rangle$, $\langle m\,|\tilde{\mathbf{x}}^2|\,n\rangle$, and $\langle m\,|\tilde{\mathbf{p}}^2|\,n\rangle$.

First solve for $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{p}}$

$$\tilde{\mathbf{x}} = \sqrt{\frac{\hbar}{2m\omega}}(\tilde{\mathbf{a}}^\dagger + \tilde{\mathbf{a}})$$
$$\tilde{\mathbf{p}} = i\sqrt{\frac{\hbar m\omega}{2}}(\tilde{\mathbf{a}}^\dagger - \tilde{\mathbf{a}}).$$

Next apply the operators on an eigenstate

$$\tilde{\mathbf{x}}\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right]$$
$$\tilde{\mathbf{p}}\,|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right].$$

Finally, using the information above, the various matrix elements are given

$$\langle m\,|\tilde{\mathbf{x}}|\,n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right]$$
$$\langle m\,|\tilde{\mathbf{p}}|\,n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right]$$
$$\langle m\,|\{\tilde{\mathbf{x}},\tilde{\mathbf{p}}\}|\,n\rangle = i\hbar\left[\sqrt{(n+1)(n+2)}\,\delta_{n+2,m} - \sqrt{n(n-1)}\,\delta_{n-2,m}\right]$$
$$\langle m\,|\tilde{\mathbf{x}}^2|\,n\rangle = \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\,\delta_{n-2,m} + (2n+1)\,\delta_{n,m} + \sqrt{n(n-1)}\,\delta_{n-2,m}\right]$$
$$\langle m\,|\tilde{\mathbf{p}}^2|\,n\rangle = -\frac{\hbar m\omega}{2}\left[\sqrt{(n-1)(n+2)}\,\delta_{n+2,m} - (2n+1)\,\delta_{nm} + \sqrt{n(n-1)}\,\delta_{n-2,m}\right]$$

b) Check that the virial theorem holds fo the expectation values of the kinetic and potential energy taken with respect to an energy eigenstate.

Recall that the virial theorem is

$$\left\langle\frac{\tilde{\mathbf{p}}^2}{m}\right\rangle = \left\langle\tilde{\mathbf{x}}\frac{dV(\tilde{\mathbf{x}})}{dx}\right\rangle.$$

To show that it is satisfied use the matrix elements calculated above to find

$$\left\langle\frac{\tilde{\mathbf{p}}^2}{m}\right\rangle = \frac{1}{m}\langle n\,|\tilde{\mathbf{p}}^2|\,n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)$$
$$\left\langle\tilde{\mathbf{x}}\frac{dV(\tilde{\mathbf{x}})}{dx}\right\rangle = m\omega^2\langle n\,|\tilde{\mathbf{x}}^2|\,n\rangle = \hbar\omega\left(n+\frac{1}{2}\right).$$

Therefore satisfied.

P-2.19 Consider again a one-dimensional simple harmonic oscillator. Do the following algebraically—that is, without using the wave functions.

a) Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.
We construct the linear combination as follows

$$|\alpha\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,.$$

The expectation value is then

$$\langle\alpha\,|\tilde{\mathbf{x}}|\,\alpha\rangle = \cos^2\theta\,\langle 0\,|\tilde{\mathbf{x}}|\,0\rangle + \sin^2\theta\,\langle 1\,|\tilde{\mathbf{x}}|\,1\rangle + \ \cos\theta\sin\theta\,\langle 1\,|\tilde{\mathbf{x}}|\,0\rangle + \cos\theta\sin\theta\,\langle 0\,|\tilde{\mathbf{x}}|\,1\rangle\,.$$

To evaluate each term, we use the definition of the position operator in terms of the creation and annihilation operators

$$\tilde{\mathbf{x}} = \sqrt{\frac{\hbar}{2m\omega}}\left(\tilde{\mathbf{a}} + \tilde{\mathbf{a}}^\dagger\right).$$

Applying the creation and annihilation operators, leads to

$$\langle\alpha\,|\tilde{\mathbf{x}}|\,\alpha\rangle = 2\cos\theta\sin\theta\sqrt{\frac{\hbar}{2m\omega}} \quad\Rightarrow\quad \theta_{\max} = \frac{\pi}{4} \quad\Rightarrow\quad \sqrt{\frac{\hbar}{2m\omega}},$$

this is determined by calculating the derivative relative to θ and equating to zero; the standard procedure of calculating a maximum.

b) Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$, using (i) the Schrödinger picture and (ii) the Heisenberg picture.
We assume that the system at $t = 0$ is in the state

$$|\alpha, 0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

In the Schrödinger picture, the state vector evolves as follows

$$|\alpha, 0; t\rangle = e^{-i\tilde{\mathbf{H}}t/\hbar}\,|\alpha, 0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\,e^{-i\omega/2} + |1\rangle\,e^{-i3\omega t/2}\right),$$

where the two lowest energy are

$$\begin{aligned}\tilde{\mathbf{H}}\,|0\rangle &= \frac{1}{2}\hbar\omega\,|0\rangle\\ \tilde{\mathbf{H}}\,|1\rangle &= \frac{3}{2}\hbar\omega.\end{aligned}$$

Schrödinger picture: In this approach, the expectation value is calculated as follows

$$\frac{1}{\sqrt{2}}\left(\langle\alpha, 0; t\,|\tilde{\mathbf{x}}|\,\alpha, 1; t\rangle + \langle\alpha, 1; t\,|\tilde{\mathbf{x}}|\,\alpha, 0; t\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t} + e^{+i\omega t}\right) = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

Heisenberg picture: In this approach, the operator carries the time dependence

$$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(0)\cos\omega t + \frac{\tilde{\mathbf{p}}(0)}{m\omega}\sin\omega t,$$

as derived in class and the textbook. The expectation value of $\tilde{\mathbf{x}}(0)$ was calculated in part (a), therefore

$$\langle x(0)\rangle \cos\omega t = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t.$$

The expectation value of $\tilde{\mathbf{p}}(0)$ can be calculated using the creation and annihilation operators

$$\tilde{\mathbf{p}} = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\tilde{\mathbf{a}} + \tilde{\mathbf{a}}^\dagger\right).$$

Applying these operators, leads to

$$\langle \tilde{\mathbf{p}}(0)\rangle = 0.$$

Hence, the expectation value is

$$\langle \tilde{\mathbf{x}}(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t,$$

which is the same as for the Schrödinger picture as expected.

c) Evaluate $\left\langle (\Delta x)^2\right\rangle$ as a function of time using either picture.

This will be calculated in the Schrödinger picture. The general form of the expectation value to be calculated is

$$\left\langle (\Delta\tilde{\mathbf{x}})^2\right\rangle = \left\langle \tilde{\mathbf{x}}^2\right\rangle - \langle\tilde{\mathbf{x}}\rangle^2.$$

The second terms has already been calculated in part (b), therefore, we only need to calculate $\left\langle x^2\right\rangle$. This is calculated as follows

$$\begin{aligned}\left\langle x^2\right\rangle &= \left\langle \alpha, 0; t\left|\tilde{\mathbf{x}}^2\right|\alpha, 0; t\right\rangle \\ &= \frac{1}{2}\left\langle 0\left|\tilde{\mathbf{x}}^2\right|0\right\rangle + \frac{1}{2}\left\langle 1\left|\tilde{\mathbf{x}}^2\right|0\right\rangle e^{i\omega t} + \frac{1}{2}\left\langle 0\left|\tilde{\mathbf{x}}^2\right|1\right\rangle e^{-i\omega t} + \frac{1}{2}\left\langle 1\left|\tilde{\mathbf{x}}^2\right|1\right\rangle.\end{aligned}$$

To evaluate the inner products, we use relation between the annihilation and creation operators, and the $\tilde{\mathbf{x}}$ operator applied twice on the eigenkets

$$\begin{aligned}\tilde{\mathbf{x}}^2\left|0\right\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\tilde{\mathbf{x}}\left|1\right\rangle = \frac{\hbar}{2m\omega}\left[\left|0\right\rangle + \sqrt{2}\left|2\right\rangle\right] \\ \tilde{\mathbf{x}}^2\left|1\right\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\tilde{\mathbf{x}}\left[\left|0\right\rangle + \sqrt{2}\left|2\right\rangle\right] = \frac{\hbar}{2m\omega}\left[\left|1\right\rangle + 2\left|1\right\rangle + \sqrt{6}\left|3\right\rangle\right] = \frac{\hbar}{2m\omega}\left[3\left|1\right\rangle + \sqrt{6}\left|3\right\rangle\right].\end{aligned}$$

Applying these relations, the expectation value is

$$\left\langle \tilde{\mathbf{x}}^2\right\rangle = \frac{\hbar}{2m\omega}\left[\frac{1}{2} + \frac{3}{2}\right] = \frac{\hbar}{m\omega}.$$

Therefore,

$$\left\langle (\Delta\tilde{\mathbf{x}})^2\right\rangle = \frac{\hbar}{m\omega}\left[1 - \frac{1}{2}\cos^2\omega t\right].$$

Additional Problem

P-1 The wave function at $t = 0$ for a particle in a harmonic oscillator potential, $V(\tilde{\mathbf{x}}) = \frac{1}{2}m\omega^2\tilde{\mathbf{x}}^2$, is of the form

$$\psi(x,0) = Ae^{-(\alpha x)^2/2}\left[\cos\beta\, H_0(\alpha x) + \frac{\sin\beta}{2\sqrt{2}}\, H_2(\alpha x)\right],$$

where β and A are real constants, $\alpha^2 \equiv \sqrt{\frac{m\omega}{\hbar}}$, and Hermite polynomials are normalized so that

$$\int_{-\infty}^{+\infty} e^{-\alpha^2x^2}\left(H_n(\alpha x)\right)^2\, dx = \frac{\sqrt{\pi}}{\alpha}2^n n!.$$

a) Derive an expression for $\psi(x,t)$ that is properly normalized.

The wavefunction that is given can be expanded in the eigenfunctions of the simple harmonic oscillator, which are the following

$$\psi_n(x) = \sqrt{\frac{\alpha}{2^n n!\sqrt{\pi}}}\, e^{-\alpha^2x^2/2}H_n(\alpha x) \quad \Rightarrow \quad \begin{cases} \psi_0(x) = \sqrt{\frac{\alpha}{2\sqrt{\pi}}}\, e^{-\alpha^2x^2/2}H_n(\alpha x) \\ \psi_2(x) = \sqrt{\frac{\alpha}{8\sqrt{\pi}}}\, e^{-\alpha^2x^2/2}H_n(\alpha x), \end{cases}$$

where the two eigenstates that contribute, due to orthogonality, are specified explicitly. The wavefunction at $t = 0$ is hence given by

$$\psi(x,0) = a_0\psi_0(x) + a_2\psi_2(x),$$

where the coefficients a_0 and a_2 are derived from the orthogonality of the eigenstates

$$a_0 = \int_{-\infty}^{\infty} \psi_0(x)\psi(x,0)\, dx = A\left(\frac{\pi}{\alpha^2}\right)^{1/4}\cos\beta$$

$$a_2 = \int_{-\infty}^{\infty} \psi_2(x)\psi(x,0)\, dx = A\left(\frac{\pi}{\alpha^2}\right)^{1/4}\sin\beta.$$

The time dependent wavefunction is therefore

$$\psi(x,t) = A\left(\frac{\pi}{\alpha^2}\right)^{1/4}\left[\cos\beta\psi_0(x)e^{-iE_0t/\hbar} + \sin\beta\psi_2(x)e^{-iE_2t/\hbar}\right]$$

which after properly normalizing is

$$\psi(x,t) = \cos\beta\psi_0(x)e^{-iE_0t/\hbar} + \sin\beta\psi_2(x)e^{-iE_2t/\hbar}$$

b) What are the possible results of a measurement of the energy of the particle in this state and what are the relative probabilities of getting these values?

The possible energies and probabilities are:

$$E_0 = \tfrac{1}{2}\hbar\omega \quad P_0 = \cos^2\beta$$

$$E_2 = \tfrac{5}{2}\hbar\omega \quad P_2 = \sin^2\beta$$

c) What is $\langle\tilde{\mathbf{x}}\rangle$ at $t = 0$? How does it change with time?

Since the expectation value is an odd function (asymmetric about the origin), the integral is zero

$$\langle x\rangle = 0.$$

Since it is zero at $t = 0$, the expectation value remains zero for all time.