Physics 5403 Homework #7
Spring 2022

Instructor Bruno Uchoa

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1 Dirac algebra

a) Based on the properties of the Dirac algebra only (which do not depend on the representation), prove that β and α^i matrices must have *even* dimensionality.

b) Show explicitly that the Dirac algebra cannot be satisfied in a representation with dimension $d = 2$. Hint: assume $\alpha^i = \sigma^i$ as Pauli matrices and show that one cannot find a matrix for β that satisfies the Dirac algebra.

c) Find the explicit representation for (β, α^i), where

$$\alpha^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

with $\mathbf{1}$ the 2×2 identity matrix.

2 Free Dirac particle

The normalization of the solutions of the Dirac equation with positive and negative energy satisfy

$$\bar{\psi}\psi = \pm 1,$$

where $+$ corresponds to the positive energy states and $-$ to the negative energy ones, with

$$\bar{\psi} = \psi^\dagger \gamma^0$$

the Dirac adjoint.

a) Find the eigenenergies and the normalized eigenfunctions that solve the Dirac equation for a free particle,

$$(i\gamma^\mu \partial_\mu - m)\Psi(\mathbf{x}, t) = 0.$$

b) Show that the orbital angular momentum $\mathbf{L}$ of a free Dirac particle is *not* a constant of the motion. Use the fact that

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k,$$

where ϵ_{ijk} is the Levi-Civita tensor. Defining the spin operator as $\mathbf{\Sigma} \equiv \mathbf{1} \otimes \boldsymbol{\sigma}$,where $\mathbf{1}$ is the 2×2 identity matrix and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, show that the total angular momentum

$$\mathbf{J} = \mathbf{L} + \frac{\hbar}{2}\mathbf{\Sigma}$$

is conserved.

c) Now show that the operators $\mathbf{p}\cdot\boldsymbol{\Sigma}$ and $\mathbf{p}\cdot\mathbf{L}$ are each one constants of the motion. The operator

$$\frac{\mathbf{p}\cdot\boldsymbol{\Sigma}}{|\mathbf{p}|}$$

is called *helicity*.

d) Calculate the equation of motion for the position operator $\mathbf{x}$ of a free Dirac particle. Show that the velocity operator $\mathbf{v}\equiv\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}$ is not a constant of the motion, unlike the momentum $\mathbf{p}$.

3 Central potential

a) Show that the Dirac equation for a central potential $V(r)$ can be written in the form

$$\chi = \frac{c}{E - V(r) + mc^2}(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi$$

where the total wavefunction is a four component spinor

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix},$$

in the bi-spinor representation.

b) Assume that φ describes an s-wave orbital with spin $\downarrow$ of the form

$$\varphi(\mathbf{r},t) = R(r)\mathrm{exp}\left(-\frac{iEt}{\hbar}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Calculate χ explicitly and show that it describes a p-wave function with spin $s = 1/2$ and *orbital* angular momentum $\ell = 1$. Hint: express χ in terms of spherical harmonics and spinors.

c) Using your previous result, show that $\chi(\mathbf{r},t)$ describes a $j = 1/2$ state with $m = -1/2$, where j and m are the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ quantum number. Hint: use a table of Clebsh-Gordan coefficients.