Homework Assignment #2

Math Methods

Homework Due: Monday, September 13th

Instructions:

Note that there is no class one Labor Day, Monday September 6th.

Reading: Reading Quiz # 2 is due on Friday by the start of class and covers the rest of Chapter 2.

Please read Chapter 1. Reading Quiz # 3 on this material is due by the start of class on Wednesday, September 15th.

Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

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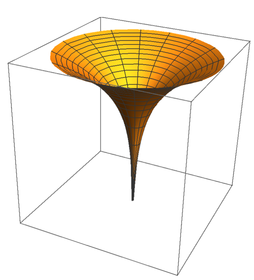
1. Use a Lagrange multiplier approach to find the extremal points (x, y) of the function

$$f(x, y) = 2x^2 + \frac{1}{2}y^2 - xy$$

subject to the constraint

$$g(x, y) = 4x^2 + y^2 - 4 = 0$$

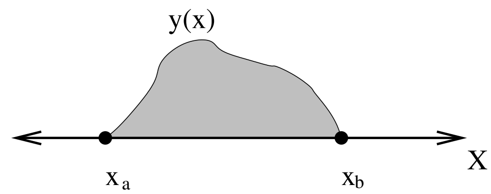
2. A bug moves on a surface with cylindrical symmetry. The surface is given by $z = \log \rho$ where ρ is the distance of the surface from the z-axis. . As an intelligent bug, it wants to move the shortest distance between two points.



   (a) Calculate the differential equation of such a curve. between any two points on the surface. (You can solve this using a local constraint, but that's the hard way.)

   (b) Solve for either the function $\rho(\theta)$ or $\theta(\rho)$, which ever you find easier. (You may need to consult a good integral table).

3. Consider a curve $y(x)$ where $y(x_a) = y(x_b) = 0$, and the total length of the curve is L. Find the curve that gives you the maximum area enclosed between $y(x)$ and the x-axis. (Assume that $L < \pi(x_b - x_a)/2$. There is a simple reason why the problem changes when L is too large. Can you see why?)



4. Often constrained problems yield difficult integral problems or differential equations to solve. In each case below, apply the calculus of variations and derive a final expression for the curve specified as an integral, that is:

$$x - x_a = \int_{y_a}^{y} F(y)\, dy$$

where $F(y)$ is a known function of y. (This is called *reducing the problem to quadrature.*)

   (a) Determine the curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) on a path of fixed length but along which a frictionless particle would slide with the shortest time. (That is, similar to the brachistochrone but on a path of fixed length).

   (b) Determine the shortest length curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) along which a frictionless particle would slide with a fixed time.

   (c) A solid object can be defined by rotating a curve $\rho(z)$ about the z-axis to determine a surface of revolution. The top and bottom surfaces of the object would be disks parallel to the x-y plane. Determine the curve $\rho(z)$ connecting the points $\rho(z_a) = \rho_a$ to $\rho(z_b) = \rho_b$ so that it minimizes the surface area of the object but has a fixed volume and moment of inertia. Assume that the density of the object has a fixed mass/volume, denoted by the constant α.

5. Consider the variational problem:

$$I = \int \left(f(x, y_1, y_1', y_2, y_2') - \lambda(x) g(y_1, y_2)\right)\, dx$$

where f is the optimization function and g is a local constraint.

   (a) If f is just a square root, such as $\sqrt{1 + y_1'^2}$, it would be easier to minimize f^2. Does this give you the same curve?

   (b) If g is just a square root, such as $\sqrt{y_1^2 + y_2^2}$, it would be easier to replace g with g^2. Does this give you the same curve?

6. Consider a 3D crystal with a surface defined by a function $z(x, y)$. We seek to minimize the total surface free energy:

$$F[z] = \int dx\, dy\, \left\{\alpha(x, y)\sqrt{1 + z_x^2 + z_y^2}\right\}$$

where $\alpha(z_x, z_y$ is the direction dependent surface tension, and we have introduced the compact but confusing notation, $z_x \equiv \partial z/\partial x$ and $z_y \equiv \partial z/\partial y$. We wish to enforce the constraint of a constant volume to the crystal where the volume is given by:

$$V[z] = \int dx\, dy\, z$$

via a Lagrange multiplier, λ.

(a) What is the Euler-Lagrange equation for the system?

(b) Show that in the isotropic case, where $\alpha(z_x, z_y) = \alpha_0$, that the shape that minimizes the surface energy for fixed volume is a sphere. You may do this either by direct substitution to verify the solution

$$z(x, y) = \sqrt{R^2 - x^2 - y^2}$$

or by changing to polar coordinates $z(\rho, \varphi)$ in the original functional, and invoking cylindrical symmetry, so that your 2D problem becomes effectively a one dimensional problem for $z(\rho)$. In either case, determine the relation between λ and the radius of the sphere.

7. **Variational derivation of Poisson equation:**

The energy in an electrostatic field in the presence of charges is :

$$E = \int \left\{ \frac{1}{8\pi} \left(\nabla\phi(\vec{r})\right)^2 - \rho(\vec{r})\,\phi(\vec{r}) \right\} d\vec{r}$$

Show that the minimum energy configuration of the potential $\varphi(\vec{r})$ satisfies the equation:

$$\vec{\nabla}^2\phi(\vec{r}) = -4\pi\,\rho(\vec{r})$$

Thus the *minimal energy* configuration for the field is also the one given by the Poisson equation.

The above problem is rather simple. I am assigning it to you for the notes below, so that you see *why* it works. It is a common technique in quantum field theory, so it's worth knowing the background.

Background: The expression for the energy above avoids counting the "self-energy", so that a charge does not feel the force of the electric field it creates. To see this assume that we have a set of point charges q_i each with potential $\phi_i(\vec{r})$. Then the total potential is the sum of the individual contributions:

$$\phi(\vec{r}) = \sum_i \phi_i(\vec{r})$$

The energy of the charges (in some units) can be written as

$$E = \frac{1}{8\pi} \int \left(\vec{\mathcal{E}}(\vec{r})\right)^2 d\vec{r}$$

(including the self-energy (Jackson, *Classical Electrodynamics*, p.46) which we can write as

$$2E = \int \left\{ \frac{1}{8\pi} \sum_{i \neq j} \left(\nabla\phi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r})\right) \right\} d\vec{r}$$

where the factor of 2 comes from double counting in the sum over i and j. The energy can also be written as:

$$E = \int \left\{ \sum_{i \neq j} q_i \, \delta(\vec{r} - \vec{r}_i) \, \phi_j(\vec{r}) \right\} d\vec{r}$$

where $\delta(\vec{r} - \vec{r}_i)$ is a function sharply peaked at $\vec{r}_1$, the location of the point charge, and there is no double counting. If we remove the restriction $i \neq j$, then both expressions pick up a self-energy term. However, if we subtract the two expressions, these cancel out, and we are left with simply the total energy:

$$\int \sum_{i,j} \left\{ \frac{1}{8\pi} \sum_{i,j} \left(\nabla\phi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r}) \right) - q_i \, \delta(\vec{r} - \vec{r}_i) \, \phi_j(\vec{r}) \right\} = 2E + E_S - E - E_S = E$$

If we take the continuum limit and replace the discrete distribution by a continuous one, then we obtain the expression at the top of the page.

This "background" discussion does not contain any questions. I am including it merely because the above formulation is routinely invoked in some field theory courses without explanation.

8. Obtain access to MATHEMACTICA on some computer. As proof of having access, if N is your OU ID number, calculate the prime factors of N^2+1 by typing in the command:

   FactorInteger$[N * N + 1]$

   and then hold down the "shift" key and the "return" key at the same time to execute the command.