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Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #2

PROBLEMS: $\{1, 2, 3, 4\}$

Due: September 10, 2021 By: 1:30 PM

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PROFESSOR
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Problem 1:

Consider a classical particle subject to the effective 1D potential,

$$V(x) = \frac{x^2}{2} - \frac{x_0\sqrt{1+gx^2}}{1+gx_0^2}, \tag{1}$$

where $x_0, g > 0$ are constants.

(a) Determine the extrema of $V(x)$ and comment on their nature as a function of g and x_0.

$$\frac{\partial V}{\partial x} = x - \frac{1}{2}\frac{2gx_0x}{\sqrt{1+gx^2}\sqrt{1+gx_0^2}} = x - \frac{gx_0x}{((1+gx^2)(1+gx_0^2))^{1/2}}$$

$$\frac{\partial V}{\partial x} = 0 \;:\; x - \frac{gx_0x}{((1+gx^2)(1+gx_0^2))^{1/2}} = 0 \;:\; x = \frac{gx_0x}{((1+gx^2)(1+gx_0^2))^{1/2}}$$

$$1 = \frac{gx_0}{((1+gx^2)(1+gx_0^2))^{1/2}} \;:\; ((1+gx^2)(1+gx_0^2))^{1/2} = gx_0 \;:\; (1+gx^2)(1+gx_0^2) = g^2x_0^2$$

$$1+gx^2 = \frac{g^2x_0^2}{(1+gx_0^2)} \;:\; gx^2 = \frac{g^2x_0^2}{(1+gx_0^2)} - 1 \;:\; gx^2 = \frac{g^2x_0^2 - 1 - gx_0^2}{(1+gx_0^2)}$$

$$x^2 = \frac{1}{g}\frac{g^2x_0^2 - 1 - gx_0^2}{(1+gx_0^2)} \;:\; x = \pm\sqrt{\frac{1}{g}\frac{gx_0(gx_0 - x_0) - 1}{1+gx_0^2}}$$

$$\boxed{x = \pm\sqrt{\frac{gx_0^2(g-1) - 1}{g(1+gx_0^2)}}\;,\; x = 0}$$

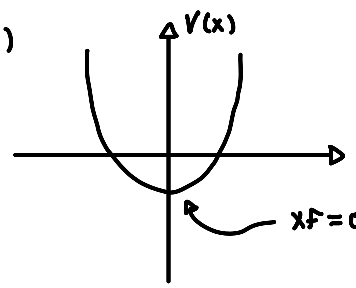
$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_f} : 1 - \frac{gx_0}{\sqrt{1+gx_0^2}} \;:\; \frac{\partial^2 V}{\partial x^2} > 0 \text{ for } g < \frac{x_0 + \sqrt{4+x_0^2}}{2x_0} \quad \frac{\partial^2 V}{\partial x^2} < 0 \text{ Otherwise.}$$

$$\text{W/ } x_f = 0,\; g < \frac{x_0+\sqrt{4+x_0^2}}{2x_0} \;\therefore\; \text{is stable}$$

$$g \geq \frac{x_0+\sqrt{4+x_0^2}}{2x_0},\; x_f = 0 \text{ is unstable},\; x_f = \pm\sqrt{\ldots} \text{ are stable}$$

(b) Using your answer to (a) as a guide, sketch/plot the potential for various values of g and assuming $x_0 = 1$. You may use any numerical tools at your disposal if that is useful.

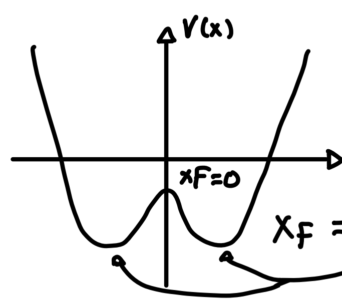
i.)



$$g < \frac{x_0+\sqrt{4+x_0^2}}{2x_0}$$

$x_f = 0$ $\quad V(0) = \dfrac{-x_0}{\sqrt{1+gx_0^2}}$

ii.)



$$g > \frac{x_0+\sqrt{4+x_0^2}}{2x_0}$$

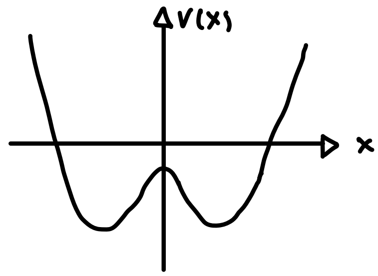
$$x_F = \pm\sqrt{\frac{gx_0^2(g-1) - 1}{g(1+gx_0^2)}}$$

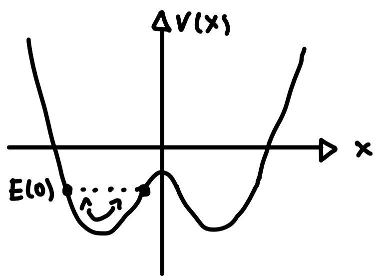
Problem 1: Continued

For $x_0 = 1$, the change between i.) & ii.) happens @ $g = \frac{1}{2} + \frac{\sqrt{5}}{2}$

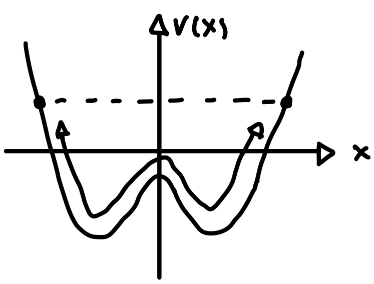
(c) Imagine that the particle is placed at the point $x(0) = x_0 >$ and is initially at rest. Without explicitly solving the dynamics, comment on the expected behavior of $x(t)$ as a function of g and x_0.

Case 1 : w/ $g > \frac{x_0 + \sqrt{4 + x_0^2}}{2x_0}$



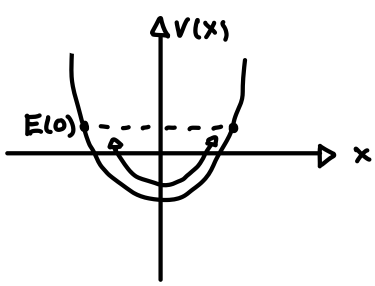


If a particle is placed at x_0 such that the energy at this location is less than the height of the dividing barrier $V(0)$ $\longrightarrow$ Then the particle will be confined to a well.



If the particle is released at a point of $E(0)$ that is higher than the dividing barrier, the particle will oscillate between both of the wells.

Case 1 : w/ $g < \frac{x_0 + \sqrt{4 + x_0^2}}{2x_0}$



For $x(0) = x_0 \neq 0$ $\rightarrow$ The particle just oscillate between a single local minimum

Problem 1: Review

Procedure:

- Begin by finding the extrema of equation (1) with the first derivative and solving for position x.
- Check the stability of these extrema points with

$$\frac{\partial^2 V}{\partial x^2}\bigg|_{x=x_f}$$

  by checking the sign of the above equation.
- Plot the potentials for multiple values, particularly for the extrema from part (a).
- Evaluate what will happen to a particle when it is placed at specific points in part (b).

Key Concepts:

- Using the rules of calculus we can find the extrema of our potential and we can proceed to make plots of those potentials.
- When we have plots of our potentials, we can infer what will happen to our particle as it is placed at specific potential levels.
- Using the equation defined above we can determine when an extrema is stable and when it is unstable.

Variations:

- We can be given a different potential.
  - This would change our extrema as well as the plots.

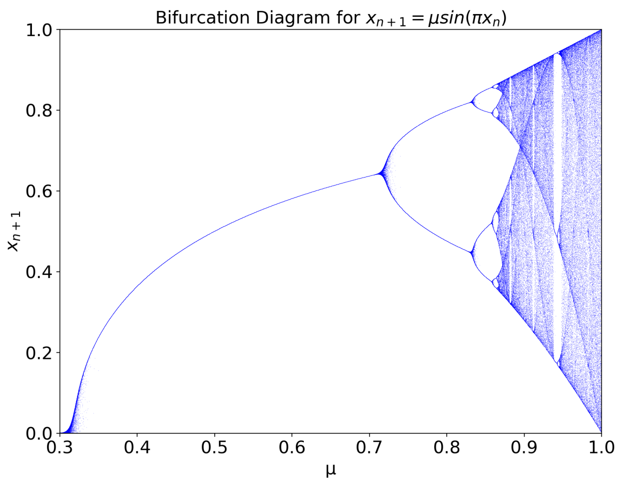
Problem 2:

Consider the nonlinear iterative map,

$$x_{n+1} = \mu \sin(\pi x_n) \tag{2}$$

where $x \in [0, 1]$ and $0 \leq \mu \leq 1$.

(a) Construct a bifurcation diagram of the map with $x_0 = 0.4$. To do this you will need to write a short code (in whatever language you choose) and evaluate the iterative map for a range of $0 < \mu < 1$. An example of what to expect is shown in Fig. 1. Note: While you should compute values $x_0, x_1, x_2, ..., x_N$, in the bifurcation diagram you should only plot the last handful of values of x_n you evaluate. For example, for $N = 10^3$ one might only plot $x_{900}, ..., x_{1000}$ in the bifurcation diagram.



(b) This map can feature up to two fixed points depending on the value of μ. Determine the fixed points (you may leave one of them as a transcendental equation - it will be obvious which this is!) and thus the special value μ_0 that delineates maps featuring two fixed points ($\mu > \mu_0$ from those with only one $\mu < \mu_0$).

A fixed point occurs where $x_{n+1} = x_n$, or where $f(x_n) = x_n$

$$x_{n+1} = f(x_n) = \mu \sin(\pi x_n)$$

The first fixed point is rather trivial, w/ $x_n = 0$, $x_{n+1} = 0$. Therefore,

$$\boxed{x_a^* = 0}$$

The second fixed point is a transcendental equation : $x^* = \mu \sin(\pi x^*)$

$$\boxed{x_b^* = \mu \sin(\pi x^*)}$$

To solve for the value of μ, we will take a derivative of the transcendental

$$f(x^*) = x^* \ : \ x^* = \mu \sin(\pi x^*) \ : \ \frac{\partial f}{\partial x^*} = 1 = \pi \mu \cos(\pi x^*)$$

$$1 = \pi \mu \cos(\pi x^*) \ , \ \frac{1}{\pi} = \mu \cos(\pi x^*)$$

Problem 2: Continued

$\cos(\pi x^*) \longrightarrow$ This term will at most be 1, using this we can deduce that

$$\frac{1}{\mu\pi} \leq 1 \text{ , and thus } \mu \geq 1/\pi$$

$$\boxed{\mu_0 \geq 1/\pi}$$

(c) Assume $\mu > \mu_0$. Perform a stability analysis of the fixed points of the map by studying the derivative of the function that defines the map, e.g., $x_{n+1} = f(x_n)$ with $f(x) = \mu \sin(\pi x)$. For the critical point defined by a transcendental equation you will find the stability depends on the value of μ. You may use numerical methods to determine for what values μ this point is stable/ unstable.

$$\text{unstable if } \frac{\partial x_{n+1}}{\partial x_n} \geqslant 1 \text{ , and stable if } \frac{\partial x_{n+1}}{\partial x_n} < 1$$

The first fixed point x_a^* is trivial, it is stable because $\frac{\partial x_a^*}{\partial x^*} = 0 < 1 \therefore$ stable

For the second fixed point, $\frac{\partial x_b}{\partial x^*} = \pi\mu\cos(\pi x^*)$. The stability of this is discussed below.

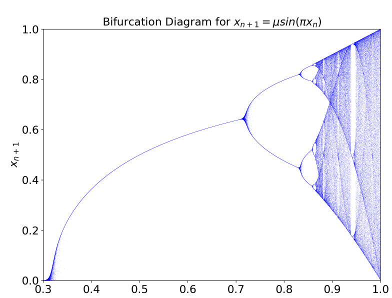
$$\text{With } x_b = \mu\sin(\pi x_b) \text{ and } \frac{\partial x_b}{\partial x_b} = \pi\mu\cos(\pi x_b)$$

We can obtain the following results numerically:

$$\mu = 0.72 \;\&\; x_b = 0.64$$

If $\mu < 0.72$ then our fixed point is stable. If $\mu > 0.72$ then our fixed point is unstable.

(d) Explain how your results for (b) and (c) relate to the features of the bifurcation diagram in Fig. 1.



For $\mu < \mu_0$ the single fixed point is stable. If $\mu_0 < \mu < 0.72$ then x_b is stable. If $\mu > 0.72$ then x_b is unstable.

Problem 2: Continued

```
# import libraries
import numpy as np
import matplotlib.pyplot as plt

def logistic(m, x): #equation given to us
    return m * np.sin(np.pi*x)

n = 1000000  # total number of points
m = np.linspace(0.0, 1.0, n) # range of mu
iterations = 100 # times function is iterated through

x = np.random.uniform(0.4, 1.0, n)  # initial Value for x_0, with
range
for _ in range(iterations):  # iterating through function
    x = logistic(m, x) # updating x_n+1

fig = plt.figure(figsize=(10, 7.5)) # figure size
ax = plt.axes() # plot axes

ax.set_title(f"Bifurcation Diagram for $x_{'{n+1}'} = \u03BC
sin(\u03C0x_n)$", fontsize=18) # set set title
ax.set_xlabel('\u03BC', fontsize=18) # x label of graph
ax.set_ylabel('$x_{n+1}$', fontsize=18) # y label of graph
ax.tick_params(axis='both', which='major', labelsize=18) # add ticks
to graph
ax.set_xlim(0.3, 1.0) # set x bounds of graph
ax.set_ylim(0.0, 1.0) # set y bounds of graph

ax.plot(m, x, ',b', alpha=.5) # plot graph
plt.show() # show plot
```

Problem 2: Review

Procedure:

- Begin by writing code for a bifurcation diagram for equation (2).
- Solve for the fixed points of equation (2), this is where $x_{n+1} = f(x_n)$.
- Determine the stability of these fixed points with

$$\frac{\partial x_{n+1}}{\partial x_n} \geq 1 \quad \rightarrow \quad \text{Unstable}, \quad \frac{\partial x_{n+1}}{\partial x_n} < 1 \quad \rightarrow \quad \text{Stable}$$

  and then proceed to solve for the value of μ in the transcendental equation.
- Comment on the features of the bifurcation diagram.

Key Concepts:

- Bifurcation diagrams help portray chaotic behavior.
- Fixed points occur where $x_{n+1} = f(x_n)$ and the stability of these points can be determined with the above equation.
- Where the bifurcation diagram starts to split is where chaotic behavior stars to occur.

Variations:

- We can be given a different iterative map equation, equation (2).
  - This would change our code and everything that follows after part (a), with the same procedure.

Problem 3:

Consider the map defined by:

$$x_{n+1} = \begin{cases} 2\alpha x_n, & \text{if } x_n < 1/2 \\ 2\alpha(1-x_n), & \text{if } x_n \geq 1/2 \end{cases}$$

where $0 \leq \alpha \leq 1$. For what values of α is the map chaotic?

For an iterative map, the Lyapunov exponent is as follows

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left[\left| \frac{\partial F}{\partial x} \right|_{x=x_i} \right]$$

The map is said to be chaotic if $\lambda > 0$.

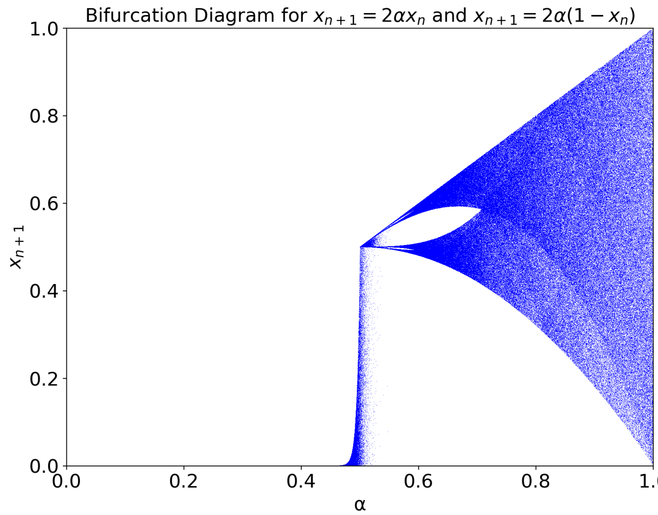
$$\frac{\partial x_{n+1}}{\partial x_n} = \begin{cases} 2\alpha & \text{if } x_n < 1/2 \\ -2\alpha & \text{if } x_n \geq 1/2 \end{cases}$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \log(2\alpha)$$

$$= \lim_{n \to \infty} \frac{n-1}{n} \log(2\alpha)$$

$$= \log(2\alpha)$$

When $\alpha > 1/2$, the map is chaotic. When $\alpha \leq$ the map is not chaotic.



Problem 3: Review

Procedure:

- Begin by calculating the Lyapunov exponent with

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left[\left|\frac{\partial f}{\partial x}\right|_{x=x_i}\right].$$

- Determine the value for α for which the map is chaotic.

Key Concepts:

- Maps are chaotic when the Lyapunov exponent is greater than zero.
- For certain values of α our map will be chaotic.

Variations:

- We can be given a different map.
  - This would change some of the components but it would be the same procedure as before.

Problem 4:

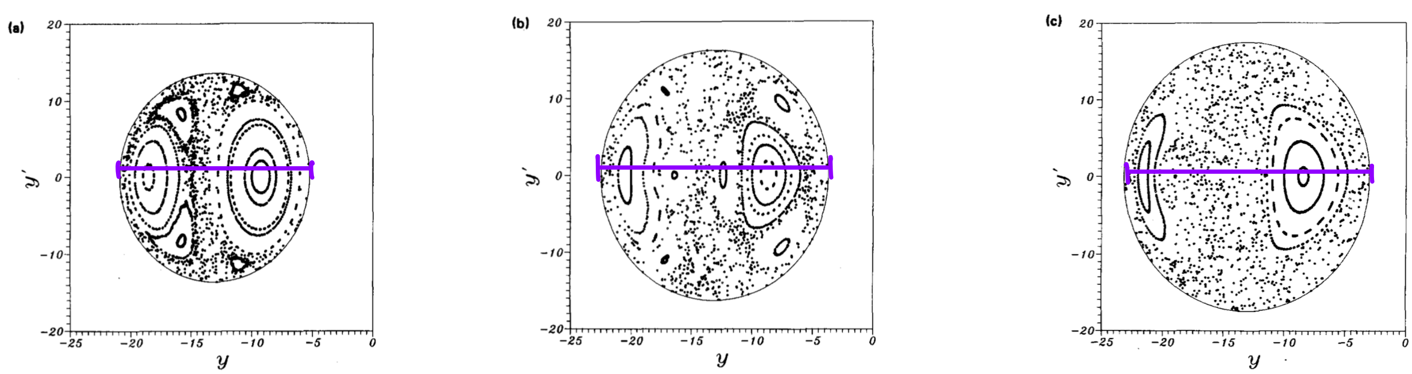
For this question, you need to read the paper "Deterministic chaos in the elastic pendulum: A simple laboratory for nonlinear dynamics", R. Cuerno, A. F. Ranada, and J. J. Ruiz-Lorenzo, Am J. Phys. 60, 73 (1992). You can access it at `https://appt.scitation.org/doi/10.1119/1.17047`. To download the pdf you will need to be on campus, use the University VPN or access the journal via the library website.

(a) Give a one paragraph summary of the manuscript including their methods and main results.

The article goes over different methods and techniques of solving a mechanical system of an elastic pendulum. The motion of this pendulum was characterized with four methods; Poincaré section, maximum Lyapunov exponenent, correlation function, and a power spectrum. With these four methods the method of numerical integration that was used in this article was a Runge-Kutta 4 method with using milne methods as well. The author of this article discovered the idea that most systems in nature are nonintegrable (cannot be solved) and possess very complicated patterns of evolution. When energy in the system is high, trajectories are less chaotic and more "normal". When energy is low the trajectories tend to be more chaotic.

(b) Outline the procedure used to generate Fig. 3 (e.g., what "recipe" did the authors follow to obtain this figure). Discuss and comment on what is plotted in each panel of Fig. 3 (e.g., what information is being conveyed about the system and what are the authors trying to report).

The procedure that was used to generate Fig. 3 was a poincare map. The authors are trying to demonstrate the level of chaos that is present for each individual map. For each individual map the continuous curve marks the boundary of the energetically allowed region for the selected surface. The purple line increases as energy increases.



E/m = -20 J/kg E/m = 20 J/kg E/m = 40 J/kg

The continuous curves in the poincare map indicate non-chaotic motion where as the dots in each map indicate chaos. As the E/m value increases the dots and the curves become more spread out. This procedure is done by keeping track of when a system crosses a specific plane designated by the creator of the map. Each individual dot represents when this plane was crossed.

Problem 4: Review

Procedure:

- Read the paper and answer the questions.

Key Concepts:

- Poincare maps can be used to portray chaotic behavior.

Variations:

- We can have a different paper to read.
  - This would change the topic of discussion.