Lecture Set 07

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Angular Momentum – *Algebra*

The algebra that specifies angular momentum, either spin or orbital, is defines by the commutation relations.

- $\left[\tilde{\mathbf{J}}_i, \tilde{\mathbf{J}}_j\right] = i\hbar\epsilon_{ijk}\tilde{\mathbf{J}}_k$
- Define $\tilde{\mathbf{J}}^2 = \tilde{\mathbf{J}}_x^2 + \tilde{\mathbf{J}}_z^2 + \tilde{\mathbf{J}}_z^2$
- $\left[\tilde{\mathbf{J}}^2, \tilde{\mathbf{J}}_i\right] = 0$
- Define $\tilde{\mathbf{J}}_\pm = \tilde{\mathbf{J}}_x \pm i\tilde{\mathbf{J}}_y$
- $\left[\tilde{\mathbf{J}}^2, \tilde{\mathbf{J}}_\pm\right] = 0 \qquad \left[\tilde{\mathbf{J}}_z, \tilde{\mathbf{J}}_\pm\right] = \pm\hbar\tilde{\mathbf{J}}_\pm \qquad \left[\tilde{\mathbf{J}}_+, \tilde{\mathbf{J}}_-\right] = 2\hbar\tilde{\mathbf{J}}_z$
- $\tilde{\boldsymbol{\mathcal{D}}}(\hat{\mathbf{n}}, \phi) = e^{-i\tilde{\vec{\mathbf{J}}}\cdot\hat{\mathbf{n}}\phi/\hbar} \approx \tilde{\mathbf{1}} - i\tilde{\vec{\mathbf{J}}} \cdot \hat{\mathbf{n}}\phi/\hbar$
- $\tilde{\mathbf{J}}^2 \left|j, m\right\rangle = j(j+1)\hbar^2 \left|j, m\right\rangle \qquad \tilde{\mathbf{J}}_z \left|j, m\right\rangle = m\hbar \left|j, m\right\rangle$
  $m = -j, \ldots, j$ integer steps.

Translations

- Translation

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$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x \delta x\right) |\alpha\rangle = \left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x \delta x\right) \int dx' \, |x'\rangle \langle x' | \alpha\rangle$$

Translations

- Translation

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$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right)|\alpha\rangle = \left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right)\int dx'\,|x'\rangle\langle x'|\alpha\rangle$$

$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right)|\alpha\rangle = \int dx'\,|x'+\delta x\rangle\langle x'|\alpha\rangle$$

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$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x \delta x\right) |\alpha\rangle = \int dx' \, |x' + \delta x\rangle \langle x' |\alpha\rangle$$

$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x \delta x\right) |\alpha\rangle = \int dx' \, |x'\rangle \langle x' - \delta x |\alpha\rangle$$

$$\langle x| \left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x \delta x\right) |\alpha\rangle = \int dx' \, \langle x |x'\rangle \langle x' - \delta x |\alpha\rangle$$

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$$\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right)|\alpha\rangle = \int dx'\,|x'\rangle\langle x'-\delta x|\alpha\rangle$$

$$\langle x|\left(\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right)|\alpha\rangle = \int dx'\,\langle x|x'\rangle\langle x'-\delta x|\alpha\rangle$$

$$\Rightarrow \quad \left\langle x\left|\tilde{\mathbf{1}} - i\tilde{\mathbf{p}}_x\delta x\right|\alpha\right\rangle = \langle x-\delta x|\alpha\rangle$$

Rotation

$$\left[\tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{p}}_y}{\hbar}(\tilde{\mathbf{x}}\delta\phi) + \frac{i\tilde{\mathbf{p}}_x}{\hbar}(\tilde{\mathbf{y}}\delta\phi)\right]\left|x', y', z'\right\rangle = \left|x' - y'\delta\phi, y' + x'\delta\phi, z'\right\rangle$$

Rotation

$$\left[\tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{p}}_y}{\hbar}(\tilde{\mathbf{x}}\delta\phi) + \frac{i\tilde{\mathbf{p}}_x}{\hbar}(\tilde{\mathbf{y}}\delta\phi)\right] \left|x', y', z'\right\rangle = \left|x' - y'\delta\phi, y' + x'\delta\phi, z'\right\rangle$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\delta\phi & 0 \\ \delta\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x' - y'\delta\phi \\ y' + x'\delta\phi \\ z' \end{pmatrix}$$

Rotation

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$$\left\langle x', y', z' \left| \tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{L}}_z\delta\phi}{\hbar} \right| \alpha \right\rangle = \left\langle x' + y'\delta\phi, y' - x'\delta\phi, z' \middle| \alpha \right\rangle$$

Rotation

$$\langle x', y', z' \,|\alpha\rangle \to \langle r, \theta, \phi \,|\alpha\rangle$$

$$\Rightarrow \quad \left\langle r, \theta, \phi \left| \tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{L}}_z \delta\phi}{\hbar} \right| \alpha \right\rangle = \langle r, \theta, \phi - \delta\phi \,|\alpha\rangle$$

Rotation

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$$\left.\begin{array}{r} \langle r, \theta, \phi - \delta\phi \,|\alpha\rangle = \langle r, \theta, \phi \,|\alpha\rangle - \delta\phi \dfrac{\partial}{\partial\phi} \langle r, \theta, \phi \,|\alpha\rangle \\ \left\langle r, \theta, \phi \left| \tilde{\mathbf{1}} - \dfrac{i\tilde{\mathbf{L}}_z \delta\phi}{\hbar} \right| \alpha \right\rangle = \langle r, \theta, \phi \,|\alpha\rangle - \dfrac{i\delta\phi}{\hbar} \left\langle r, \theta, \phi \left| \tilde{\mathbf{L}}_z \right| \alpha \right\rangle \end{array}\right\}$$

Rotation

$$\langle x', y', z' \,|\alpha\rangle \to \langle r, \theta, \phi \,|\alpha\rangle$$

$$\Rightarrow \quad \left\langle r, \theta, \phi \left| \tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{L}}_z \delta\phi}{\hbar} \right| \alpha \right\rangle = \langle r, \theta, \phi - \delta\phi \,|\alpha\rangle$$

$$\left.\begin{aligned} \langle r, \theta, \phi - \delta\phi \,|\alpha\rangle &= \langle r, \theta, \phi \,|\alpha\rangle - \delta\phi \frac{\partial}{\partial\phi} \langle r, \theta, \phi \,|\alpha\rangle \\ \left\langle r, \theta, \phi \left| \tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{L}}_z \delta\phi}{\hbar} \right| \alpha \right\rangle &= \langle r, \theta, \phi \,|\alpha\rangle - \frac{i\delta\phi}{\hbar} \left\langle r, \theta, \phi \left| \tilde{\mathbf{L}}_z \right| \alpha \right\rangle \end{aligned}\right\}$$

$$\Rightarrow \quad \left\langle r, \theta, \phi \left| \tilde{\mathbf{L}}_z \right| \alpha \right\rangle = -i\hbar \frac{\partial}{\partial\phi} \langle r, \theta, \phi \,|\alpha\rangle\,.$$

Angular Momentum Operators – *Coordinate Basis*

- $\tilde{\mathbf{L}}_z \doteq -i\hbar\frac{\partial}{\partial\phi}$
- $\tilde{\mathbf{L}}_x \doteq i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$
- $\tilde{\mathbf{L}}_y \doteq -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$
- $\tilde{\mathbf{L}}_\pm = \tilde{\mathbf{L}}_x \pm i\tilde{\mathbf{L}}_y \doteq -i\hbar e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)$
- $\tilde{\mathbf{L}}^2 = \tilde{\mathbf{L}}_z^2 + \left(\tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- + \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+\right)/2 \doteq$
  $-\hbar^2\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]$
  - Used $\left[\tilde{\mathbf{L}}_+, \tilde{\mathbf{L}}_-\right] = 0$

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  $-\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) \right]$
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  - Used $\left[\tilde{\mathbf{L}}_+, \tilde{\mathbf{L}}_-\right] = 0$

Spherical Harmonics – *Derivation*

- Azimuthal coordinate $\tilde{\mathbf{L}}_z \left|\ell, m\right\rangle = m\hbar \left|\ell, m\right\rangle$
  - $-i\hbar\frac{\partial}{\partial\phi}\Phi(\phi) = m\hbar\Phi(\phi) \quad \Rightarrow \quad \Phi(\phi) \propto e^{im\phi}$
  - Depends only on ϕ.
- Use ladder operators to derive the spectrum
  - $\tilde{\mathbf{L}}_+ \left|\ell, \ell\right\rangle = 0$ or $\tilde{\mathbf{L}}_- \left|\ell, -\ell\right\rangle = 0$
  - $\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\Theta(\theta)\Phi(\phi) = 0$
  - $\left(\frac{\partial}{\partial\theta} - l\cot\theta\right)\Theta(\theta) = 0$
  - $\Theta(\theta) \propto \sin^\ell\theta$
- $Y_\ell^{m=\ell}(\theta, \phi) = c_\ell e^{i\ell\phi}\sin^\ell\theta$
- Normalization $\iint Y_{\ell'}^{m'^*}(\theta, \phi)Y_\ell^m(\theta, \phi)\, d\Omega = \delta_{m'm}\delta_{\ell'\ell}$

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  - $\Theta(\theta) \propto \sin^\ell\theta$
- $Y_\ell^{m=\ell}(\theta,\phi) = c_\ell e^{i\ell\phi}\sin^\ell\theta$
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- Normalization $\iint Y_{\ell'}^{m'*}(\theta, \phi)Y_\ell^m(\theta, \phi)\, d\Omega = \delta_{m'm}\delta_{\ell'\ell}$

Spherical Harmonics

$$Y_0^0(\theta,\phi) = \left(\frac{1}{4\pi}\right)^{1/2}$$

$$Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$Y_1^{\pm 1}(\theta,\phi) = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$$

$$Y_2^0(\theta,\phi) = \left(\frac{5}{16\pi}\right)^{1/2} \left(3\cos^2\theta - 1\right)$$

$$Y_2^{\pm 1}(\theta,\phi) = \mp \left(\frac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta e^{\pm i\phi}$$

$$Y_2^{\pm 2}(\theta,\phi) = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm i2\phi}$$

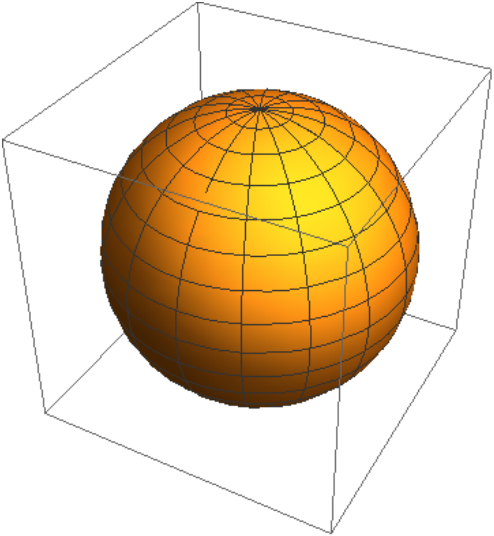
$$Y_3^0(\theta,\phi) = \left(\frac{7}{16\pi}\right)^{1/2} \left(5\cos^3\theta - 3\cos\theta\right)$$

$$Y_3^{\pm 1}(\theta,\phi) = \mp \left(\frac{21}{64\pi}\right)^{1/2} \sin\theta \left(5\cos^2\theta - 1\right) e^{\pm i\phi}$$

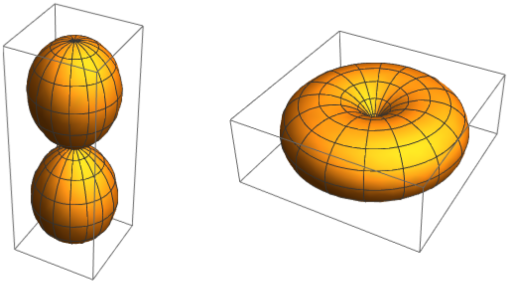
$$Y_3^{\pm 2}(\theta,\phi) = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta \cos\theta e^{\pm i2\phi}$$

$$Y_3^{\pm 3}(\theta,\phi) = \mp \left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm i3\phi}$$

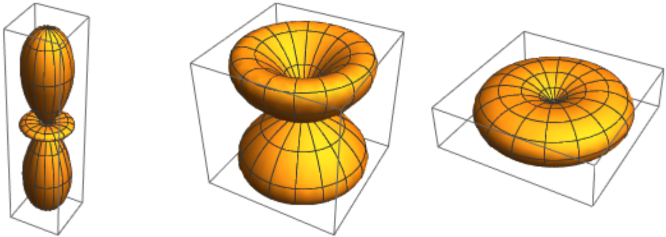
$\ell = 0$ Wavefunction



$\ell = 1$ Wavefunction



$\ell = 2$ Wavefunction



Rotation Operator

- $|\hat{\mathbf{n}}\rangle = \tilde{\mathcal{D}}(R)\,|\hat{\mathbf{z}}\rangle$
  - A rotation from the z axis to a direction along n axis is given by θ and ϕ.
  - $|\hat{\mathbf{n}}\rangle = \tilde{\mathcal{D}}(\alpha=\phi,\beta=\theta,\gamma=0)\,|\hat{\mathbf{z}}\rangle$.
  - $|\hat{\mathbf{n}}\rangle = \sum_{l,m} \tilde{\mathcal{D}}(\alpha=\phi,\beta=\theta,\gamma=0)\,|l,m\rangle\,\langle l,m\,|\hat{\mathbf{z}}\rangle$
  - $\langle l,m'\,|\hat{\mathbf{n}}\rangle =$
    $\sum_m \left\langle l,m'\left|\tilde{\mathcal{D}}(\alpha=\phi,\beta=\theta,\gamma=0)\right|l,m\right\rangle\langle l,m\,|\hat{\mathbf{z}}\rangle =$
    $\sum_m \tilde{\mathcal{D}}^{(l)}_{m'm}(R)\,\langle l,m\,|\hat{\mathbf{z}}\rangle$
  - $\langle l,m'\,|\hat{\mathbf{n}}\rangle = Y_l^{m'*}(\theta,\phi)$
  - $\langle \ell,m\,|\hat{\mathbf{z}}\rangle = Y_\ell^m(\theta=0,\phi) = Y_\ell^0(0,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}$
  - $\langle l,m'\,|\hat{\mathbf{n}}\rangle = \tilde{\mathcal{D}}^{(l)}_{m'0}(\theta,\phi,0)\sqrt{\frac{2l+1}{4\pi}}$
  - $\tilde{\mathcal{D}}^{(l)}_{m0}(\theta,\phi,0) = \sqrt{\frac{4\pi}{2l+1}}\,Y_l^{m*}(\theta,\phi)$

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  - $|\hat{\mathbf{n}}\rangle = \tilde{\mathcal{D}}(\alpha = \phi, \beta = \theta, \gamma = 0)\,|\hat{\mathbf{z}}\rangle$.
  - $|\hat{\mathbf{n}}\rangle = \sum_{l,m} \tilde{\mathcal{D}}(\alpha = \phi, \beta = \theta, \gamma = 0)\,|l, m\rangle \langle l, m\,|\hat{\mathbf{z}}\rangle$
  - $\langle l, m'\,|\hat{\mathbf{n}}\rangle = \sum_m \left\langle l, m' \left| \tilde{\mathcal{D}}(\alpha = \phi, \beta = \theta, \gamma = 0) \right| l, m \right\rangle \langle l, m\,|\hat{\mathbf{z}}\rangle = \sum_m \tilde{\mathcal{D}}^{(l)}_{m'm}(R) \langle l, m\,|\hat{\mathbf{z}}\rangle$
  - $\langle l, m'\,|\hat{\mathbf{n}}\rangle = Y_l^{m'*}(\theta, \phi)$
  - $\langle \ell, m\,|\hat{\mathbf{z}}\rangle = Y_\ell^m(\theta = 0, \phi) = Y_\ell^0(0, \phi) = \sqrt{\frac{2\ell+1}{4\pi}}$
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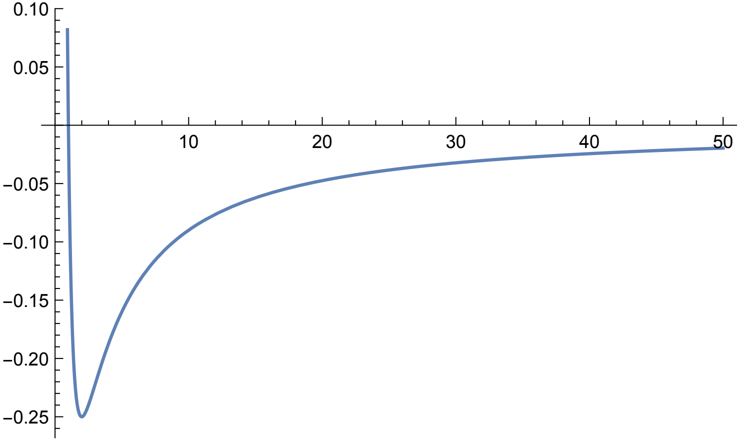
Effective Potential – Coulomb



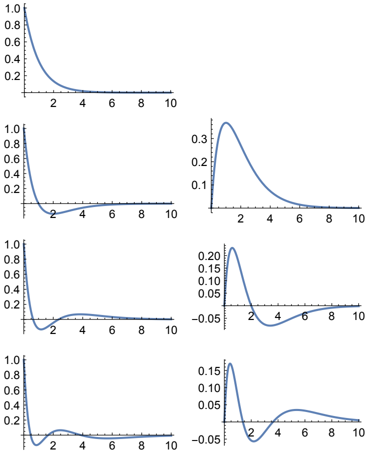
$$
{}_1F_1(a;c;x) = \\
1 + \frac{ax}{c} + \frac{a(a+1)x^2}{2c(c+1)} + \frac{a(a+1)(a+2)x^3}{6c(c+1)(c+2)} \\
+ \frac{a(a+1)(a+2)(a+3)x^4}{24c(c+1)(c+2)(c+3)} \\
+ \frac{a(a+1)(a+2)(a+3)(a+4)x^5}{120c(c+1)(c+2)(c+3)(c+4)} + \cdots
$$

Radial Wavefunction

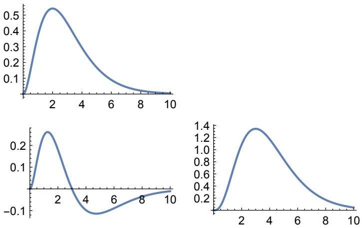
$$R_{n\ell}(r) = \frac{1}{(2\ell+1)!}\left(\frac{2Zr}{na_0}\right)^{\ell} e^{-Zr/na_0}\left[\left(\frac{2Z}{na_0}\right)^3 \frac{(n+1)!}{2n(n-l-1)!}\right]^{1/2}$$

$$\times\, {}_1F_1(-n+\ell+1; 2\ell+2; 2Zr/na_0)$$

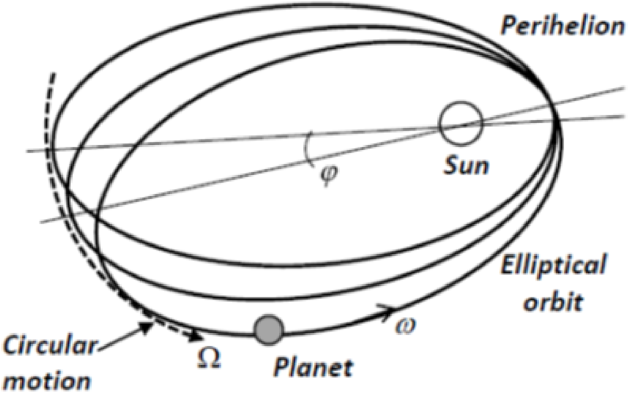
Radial Wavefunction

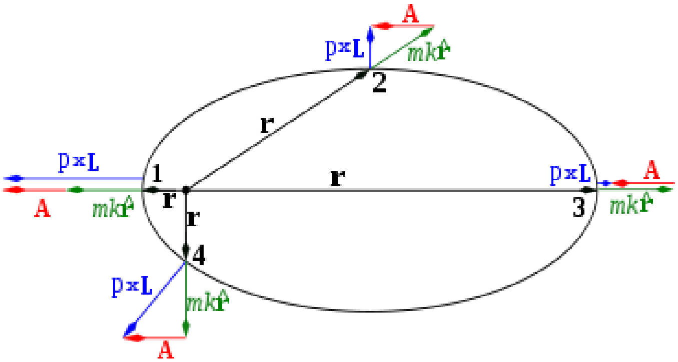


n	N,ℓ	N,ℓ	N,ℓ	N,ℓ
1	0, 0	-	-	-
2	1, 0	0, 1	-	-
3	2, 0	1, 1	0, 2	-
4	3, 0	2, 1	1, 2	0, 3



Orbit Precession




A
p×L
mkr̂
2
r
p×L
1
r
r
p×L
A
A
mkr̂
3
mkr̂
r
4
p×L
mkr̂
A