Homework #3

(1)

a) Rotations $\theta_n = \frac{2\pi}{3} n$, $n = 1, 2, 3$

$$R(\theta_n) = \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \text{ for } \theta = 4\pi/3$$

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \text{ for } \theta = 2\pi/3$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ for } \theta = 0.$$

Reflections

$$\sigma_1 : x \to -x \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2 : R(4\pi/3)\,\sigma_1 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$\sigma_3 : R(2\pi/3)\,\sigma_1 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$R(\sigma_1) : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \Rightarrow \delta_0 = +1$$

$$\mathrm{Det}\left[R(\sigma_1)\right] = 1$$

③

$$R(\theta_2): \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Rightarrow \delta_\sigma = +1$$

$$\mathrm{Det}\left[R(\theta_2)\right] = 1$$

$$R(\theta_3): \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \delta_\sigma = +1$$

$$\mathrm{Det}\left[R(\theta_3)\right] = 1$$

$$\sigma_1: \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \Rightarrow \delta_\sigma = -1$$

$$\sigma_2: \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \Rightarrow \delta_\sigma = -1$$

$$\sigma_3: \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow \delta_\sigma = -1$$

$$\mathrm{Det}\left[\sigma_1\right] = \mathrm{Det}\left[\sigma_2\right] = \mathrm{Det}\left[\sigma_3\right] = -1$$

b)

$$|\alpha\rangle = \sum_{\sigma \in P_3} |\sigma(1)\rangle |\sigma(2)\rangle |\sigma(3)\rangle$$

which is a symmetric state (spinless bosons)

$$\therefore \quad D(\theta = \pi/3)\,|\alpha\rangle = e^{\frac{i L_z}{\hbar} 2\pi/3}\,|\alpha\rangle$$

If:

$$|\alpha\rangle = \sum_{\sigma \in P_3} |j_{\sigma(1)}, m_{\sigma(1)}\rangle \cdots |j_{\sigma(3)}, m_{\sigma(3)}\rangle$$

$$D(\pi/3)\,|\alpha\rangle = e^{\frac{2i}{\hbar}(m_1 + m_2 + m_3)\pi/3}\,|\alpha\rangle$$

$$= |\alpha\rangle$$

Given that $|\alpha\rangle$ is invariant under $2\pi/3$ rotations.

$$\frac{2\pi M}{3} = \pi\left(\frac{m_1 + m_2 + m_3}{3}\right) = 2\pi \cdot m, \quad m \in \mathbb{Z}$$

$$\therefore M = 0, \pm 3, \pm 6 \ldots$$

c) The orbital part is ANTI-SYMMETRIC

$\therefore$ the spin part is ALSO ANTISYMMETRIC

under the exchange of two particles.

Defining the states: $\{|+\rangle, |0\rangle, |-\rangle\}$ AS

the single particle states with spin 1,

$$|S\rangle = \frac{1}{\sqrt{6}}\Big[|+\rangle|0\rangle|-\rangle + |0\rangle|-\rangle|+\rangle + |-\rangle|+\rangle|0\rangle - |0\rangle|+\rangle|-\rangle - |-\rangle|0\rangle|+\rangle - |+\rangle|-\rangle|0\rangle \Big].$$

(2) $\boxed{A}$

The energy spectrum of the Harmonic oscillator is:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

For spin 5/2 particles, there are $g = 2(5/2) + 1 = 6$ states per energy level. For N identical fermions, the ground state energy is:

$$E_0 = N \bmod (6) \quad \text{for } N < 6$$

and

$$E_0 = 6 \sum_{n=0}^{\bar{n}-1} \hbar\omega\left(n + \frac{1}{2}\right)$$

$$+ \; N \bmod (6) \times \hbar\omega\left(\bar{n} + \frac{1}{2}\right)$$

$(N \geq 6)$

$$= 6\hbar\omega \frac{\bar{n}^2}{2} + N\,\mathrm{mod}(6) \times \hbar\omega\left(\bar{n} + \tfrac{1}{2}\right)$$

$$(N \geq 6)$$

where

$$\bar{n} = \frac{N - N\,\mathrm{mod}(6)}{6}$$

is the highest occupied level.

③ $\boxed{A}$

$$a_m = S_x(m) - iS_y(m)$$

$$a_m^+ = S_x(m) + iS_y(m)$$

with $a_m^+ a_m = S_z(m) + 1/2$

a)

i) $[a_i^+, a_j] = [S_x(i) + iS_y(i), S_x(j) - iS_y(j)]$

$= 0$

since $[S_k(i), S_\ell(j)] = 0$ for $i \neq j$, $(k, \ell = x, y)$

In the same way,

$[a_i^+, a_j^+] = [a_i, a_j] = 0.$

ii) $\{a_i^+, a_i\} = \{S_x(i) + iS_y(i), S_x(i) - iS_y\}$

$$= \overbrace{i\{S_y, S_x\} - i\{S_x, S_y\}}^{0}$$

$$+ \overbrace{\{S_x, S_x\}}^{1/2} + \overbrace{\{S_y, S_y\}}^{1/2}$$

$$= 1$$

In the same way,

$$\{a_i, a_i\} = \{S_x, S_x\} - \{S_y, S_y\} = 0$$

$$= \{a_i^+, a_i^+\}.$$

$$\therefore a_i^2 = a_i^{+2} = 0$$

b)

$$\{c_i, c_j^+\} = \Big\{ \exp\Big[i\pi \sum_{k=1}^{i-1} a_k^+ a_k\Big] a_i ,$$

$$a_j^+ \exp\Big[-i\pi \sum_{k'=1}^{j-1} a_{k'}^+ a_{k'}\Big] \Big\}$$

Defining $U_{i-1} \equiv e^{i\pi \sum_k^{i-1} a_k^+ a_k}$

$\boxed{c}$

$$\{c_i, c_j^\dagger\} = U_{i-1} a_i a_j^\dagger U_{j-1}^\dagger$$
$$+ a_j^\dagger U_{j-1}^\dagger U_{i-1} a_i$$

for $i=j$,

$$= U_{i-1} a_i a_i^\dagger U_{i-1}$$
$$+ a_i^\dagger a_i$$
$$= \{a_i, a_i^\dagger\} = 1$$

Since:

$$\cancel{e^{-B} A e^{B} = A + [A,B]}$$

$$\left[a_i a_i^\dagger, \sum_k^{i-1} a_k^\dagger a_k\right] = 0$$

for $i \neq j$ $(i > j)$

$$\{c_i, c_j^\dagger\} = a_i U_{i-1} U_{j-1}^\dagger a_j^\dagger$$
$$+ a_j^\dagger U_{j-1}^\dagger U_{i-1} a_i$$

Since $[a_i, U_{i-1}] = 0$

$$U_{i-1} U_{j-1}^{\dagger} = \bar{U} = \exp\left(+i\pi \sum_{k=j}^{i-1} a_k^{\dagger} a_k\right) \quad \text{D}$$

$$\therefore \quad \{c_i, c_j^{\dagger}\} = a_i \bar{U} a_j^{\dagger} + a_j^{\dagger} \bar{U} a_i$$

$$= a_i e^{i\pi a_j^{\dagger} a_j} a_j^{\dagger} e^{i\pi \sum_{k=j+1}^{i-1} a_k^{\dagger} a_k}$$

$$+ a_j^{\dagger} e^{i\pi a_j^{\dagger} a_j} a_i e^{i\pi \sum_{k=j+1}^{i-1} a_k^{\dagger} a_k}$$

$$= \left[a_i e^{i\pi(1 - \overset{0}{\cancel{a_j a_j^{\dagger}}})} a_j^{\dagger} \right] e^{i\pi \sum_{k=j+1}^{i-1} a_k^{\dagger} a_k}$$

$$+ \left[a_j^{\dagger} e^{i\pi \overset{0}{\cancel{a_j^{\dagger} a_j}}} a_i \right] e^{i\pi \sum_{k=j+1}^{i-1} a_k^{\dagger} a_k}$$

$$= [a_j^{\dagger}, a_i] \, e^{i\pi \sum_{k=j+1}^{i-1} a_k^{\dagger} a_k}$$

$$= 0.$$

$\boxed{E}$

Also:

$$c_i^\dagger c_i = a_i^\dagger \underbrace{U^\dagger U}_{1} a_i$$

$$= a_i^\dagger a_i \qquad (1)$$

$$c_i^\dagger c_{i+1} = a_i^\dagger e^{i\pi \cancel{a_i^\dagger a_i}} a_{i+1}$$

$$= a_i^\dagger a_{i+1}$$

From (1),

$$\therefore \begin{cases} a_i = e^{-i\pi \sum_k^{i-1} c_k^\dagger c_k} \cdot c_i \\ a_i^\dagger = c_i^\dagger e^{i\pi \sum_k^{i-1} c_k^\dagger c_k} \end{cases}$$

Is the inverse transformation.

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c)

$$H = J \sum_{i=1}^{N} \vec{S}_i \vec{S}_{i+1}$$

$$= J \sum_i \left[S_{z_i} S_{z_{i+1}} + S^x_i S^x_{i+1} + S^y_i S^y_{i+1} \right]$$

$$= J \sum_i \left[\left(a^+_i a_i - \tfrac{1}{2} \right) \left(a^+_{i+1} a_{i+1} - \tfrac{1}{2} \right) \right.$$

$$\left. + \frac{1}{2} a^+_i a_{i+1} + \frac{1}{2} a^+_{i+1} a_i \right]$$

Since

$$a^+_i a_i = c^+_i c_i$$

$$c^+_i c_{i+1} = a^+_i a_{i+1}$$

$$a^+_{i+1} a_i = c^+_{i+1} c_i ,$$

$\therefore$

$$H = J \sum_i \left\{ \frac{1}{2} \left[c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right] + \left(c_i^\dagger c_i - \tfrac{1}{2} \right) \left(c_{i+1}^\dagger c_{i+1} - \tfrac{1}{2} \right) \right\}.$$

4

$$a|z\rangle = z|z\rangle$$

where $\langle z|z\rangle = 1$.

$$|z\rangle = \sum_{n=0}^{\infty} \phi_n |n\rangle$$

$$= \sum_{n=0}^{\infty} \phi_n \frac{(a^+)^n}{\sqrt{n!}} |0\rangle$$

then:

$$a|z\rangle = \sum_{n=1}^{\infty} \phi_n \sqrt{n} \, |n-1\rangle$$

$$= \sum_{k=0}^{\infty} \phi_{k+1} \sqrt{k+1} \, |k\rangle$$

$$\therefore \; z\,\phi_n = \phi_{n+1} \sqrt{n+1} \quad \Longrightarrow$$

setting $\phi_0 = A$,

$$\phi_m = \frac{z^m}{\sqrt{m!}} \cdot A$$

$$\therefore \quad |z\rangle = A \sum_{m=0}^{\infty} \frac{z^m}{m!} (a^+)^m |0\rangle$$

$$= A \exp\left(z a^+ \right) |0\rangle .$$

Now finding the normalization condition,

$\langle z | z \rangle = 1 \ \therefore$

$$\langle 0 | e^{z^* a} e^{z a^+} |0\rangle \cdot |A|^2$$

$$= \langle 0 | \sum_{m m'} \frac{1}{m!} \frac{1}{m'!} \cdot (z^* a)^m (z a^+) |0\rangle |A|^2$$

$$= \langle 0 | \sum_{m m'} \frac{1}{\sqrt{m!}} \frac{1}{\sqrt{m'!}} \cdot (z^* \cancel{a})^m z^{m'} |m' - m\rangle |A|^2$$

$$= |A|^2 \langle 0 | \sum_m \frac{1}{m!} |z|^2 |0\rangle = \langle 0 | e^{|z|^2} |0\rangle |A|^2$$

$$= e^{|z|^2}\,|A|^2 = 1$$

$$\therefore |A| = e^{-\frac{|z|^2}{2}} \;\therefore$$

$$|z\rangle = \exp\left(\frac{-|z|^2}{2}\right)\cdot \exp(z a^+)|0\rangle,$$

b) In general, If $C = [A, B]$ where $[A, C] = [B, C] = 0$,

then:

$$e^A e^B = e^{A+B}\, e^{C/2}$$

$$\therefore\; e^{+z a^+}\, e^{-z^* a} = e^{z a^+ - z^* a}\cdot e^{+|z|^2/2},$$

since $[-z^* a, z a^+] = -|z|^2$.

$$\therefore \quad e^{za^\dagger - z^* a} |0\rangle$$

$$= e^{-|z|^2/2} e^{za^\dagger} \underbrace{e^{-z^* a} |0\rangle}_{|0\rangle}$$

$$= |z\rangle$$

c) $$\langle z | z' \rangle = \langle 0 | e^{-\frac{|z|^2}{2} - \frac{|z'|^2}{2}} e^{z^* a} e^{z' a^\dagger} |0\rangle$$

$$= \langle 0 | \sum_{m m'} \frac{1}{\sqrt{m!}\sqrt{m'!}} (z^*)^m (z')^{m'} |m' - m\rangle$$

$$\times e^{-\left(\frac{|z|^2 + |z'|^2}{2}\right)}$$

$$= e^{z^* z'} \cdot e^{-\frac{|z|^2 + |z'|^2}{2}}$$

d)

$$\langle m|z\rangle = \langle m| e^{-|z|^2/2} e^{za^\dagger}|0\rangle$$

$$= e^{-|z|^2/2} \langle m| \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} |m\rangle$$

$$= e^{-|z|^2/2} \frac{z^m}{\sqrt{m!}}.$$

$$P_m = |\langle m|z\rangle|^2$$

$$= e^{-|z|^2} \frac{z^{2m}}{m!}$$