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# Classical Mechanics

CH. 8 THE HAMILTON EQUATIONS OF MOTION LECTURE NOTES

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## Hamiltonian Dynamics (Ch. 8)

So Far: Lagrangian Mechanics
- Generalized co-ordinates
- Good For constraints
  - Can get Forces of constraint
- Variational approach

Hamiltonian Dynamics is just a reformulation of Lagrangian mechanics

⇒ Provides new physical insight

⇒ calculational tool / basis for other topics

* Canonical transformation
* Hamilton - Jacobi approach
* Action - angle variable

**Other Topics:**

* Quantum mechanics
* statistical mechanics
* Perturbation theory
* Chaos

**Assumptions:**

i) Holonomic constraints $f(r_1, r_2, \dots, t) = 0$

ii) Monogenic system "Forces are conservative", $V(\vec{q}, t)$ or $V(\dot{\vec{q}}, t)$

**The Hamiltonian:**

Recall our Lagrangian, $L = L(\vec{q}, \dot{\vec{q}}, t)$

⇒ $\vec{q}$: generalyzed co-ordinates ($\dot{\vec{q}}$ velocity)

⇒ Defined in configuration space

Lagrangian ⟶ EOM

n generalized co-ordinates, n 2nd order differential equations, 2n initial conditions

Hamiltonian :

$H = (\vec{q}, \vec{p}, t)$ $\to$ $\frac{\partial L}{\partial \dot{\vec{q}}}$ , generalized / conjugate momentum

$\Rightarrow$ $\vec{p}$ is an independent co-ordinate

$\Rightarrow$ $2n$ first-order differential equations

$\Rightarrow$ $2n$ independent variables $\vec{q}, \vec{p}$

$H$ related to $L$ by a Legendre transformation : $L(\vec{q}, \dot{\vec{q}}, t) \longrightarrow H(\vec{q}, \vec{p}, t)$, change of variables $\dot{\vec{q}} \to \vec{p}$

$H(\vec{q}, \vec{p}, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(\vec{q}, \dot{\vec{q}}, t)$ $\longrightarrow$ want EoM for $\vec{q}$ & $\vec{p}$ to describe identical motion.

Toy example of Legendre transformation : consider : $f(x,y)$ , $df = \underbrace{u}_{\frac{\partial f}{\partial x}} dx + \underbrace{v}_{\frac{\partial f}{\partial y}} dy$

$g(u,y)$ , Take : $g = f - ux$

$dg = df - u dx - x du = v dy - x du$

$$dH = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \longrightarrow H(\vec{q}, \vec{p}, x)$$

Alternatively (from $H = \sum_i \dot{q}_i p_i - L$) , $dH = \sum_i \dot{q}_i dp_i + p_i d\dot{q}_i - dL$

$dL = \sum_i \underbrace{\frac{\partial L}{\partial q_i}}_{\to \dot{p}_i} dq_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{\to p_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$ , $dL = \sum_i \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$

Plugging this back in , $dH = \sum_i \dot{q}_i dp_i + \cancel{p_i d\dot{q}_i} - \dot{p}_i dq_i - \cancel{p_i d\dot{q}_i}$

$dH = \sum_i \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$

Comparing both forms & assume $dp_i$ & $dq_i$ are independent ,

i) $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $2n+1$ equations of motion

ii) $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ Canonical equations of motion

iii) $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

A subtle reminder , $H$ looks like $h$ $\Rightarrow$ energy function in Lagrangian formalism

$$h(\vec{q}, \dot{\vec{q}}, t) = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(\vec{q}, \dot{\vec{q}}, t)$$

$h$ " = " $H$

i) Values are the same

ii) Not equal Functions

Fundamental Recipe:

① Obtain $L = T - V$

② Define conjugate momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$

③ Obtain $H = \sum_i \dot{q}_i p_i - L$

④ Obtain $\dot{\vec{q}} = \dot{\vec{q}}(\vec{q}, \vec{p}, t)$

⑤ Eliminate $\dot{\vec{q}}$

Example: Central potential

$$L = T - V \quad \text{w/} \quad T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 \,, \quad V = V(r)$$

Conjugate momenta: $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ , $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$ $\therefore$ $\dot{r} = \frac{p_r}{m}$ , $\dot{\varphi} = \frac{p_\varphi}{mr^2}$

$$H = \dot{r}p_r + \dot{\varphi}p_\varphi - L = \left(\frac{p_r}{m}\right)p_r + \left(\frac{p_\varphi}{m}\right)p_\varphi - \frac{1}{2}m\left(\frac{p_r}{m}\right)^2 - \frac{1}{2}m\left(\frac{p_\varphi}{m}\right)^2 + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r) = T + V$$

Canonical EOM: $\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$ , $\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}$ , $\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3} - \frac{\partial V}{\partial r}$

A shortcut:

Recall: $L(\vec{q}, \dot{\vec{q}}, t) = L_0 + L_1 + L_2$ : Homogeneous function of $\dot{\vec{q}}$ of order $r = 0, 1, 2$

Typically: $L = L_0(\vec{q}, t) + \sum_i a_i(\vec{q}, t)\dot{q}_i + \sum_i T_i(\vec{q}, t)\dot{q}_r^2$

Vector notation,

$$L = L_0 + \dot{\vec{q}}^T \vec{a} + \frac{1}{2}\dot{\vec{q}}^T T \dot{\vec{q}}$$

From this,

$$\vec{p} = T\dot{\vec{q}} + \vec{a} \quad , \quad \dot{\vec{q}} = T^{-1}(\vec{p} - \vec{a})$$

Then:

$$H(\vec{q}, \vec{p}, t) = 1/2\,(\vec{p}^T - \vec{a}^T)T^{-1}(\vec{p} - \vec{a}) - L_0$$

Example: Central Potential , $L = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\varphi}^2 - V(r) \rightarrow L_0 = -V$ , $\vec{a} = 0$

$$T = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix} , \quad \dot{\vec{q}} = \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} , \quad T^{-1} = \frac{1}{m^2r^2}\begin{pmatrix} mr^2 & 0 \\ 0 & m \end{pmatrix} = \frac{1}{m}\begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

$\vec{p} = I\dot{\vec{q}} = \begin{pmatrix} m\dot{r} \\ m^2r^2\dot{\varphi} \end{pmatrix} = \begin{pmatrix} p_r \\ p_\varphi \end{pmatrix}$ , $H = \frac{1}{2}(p_r\ p_\varphi)\begin{pmatrix} 1/m & 0 \\ 0 & 1/mr^2 \end{pmatrix}\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix} + V(r) = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2m} + V(r)$

## Exam Prep

* 1 hr 45 m + 5 m perusal
* 3 questions
* Ch. 3, 4, 5, 6
  * A 4-8

## Central Force Problem:

i) Two bodies interacting

ii) one-body problem , $\vec{F} = f(r)\hat{r}$

  * Symmetries & conserved quantities $L(r, \dot{r}, t)$

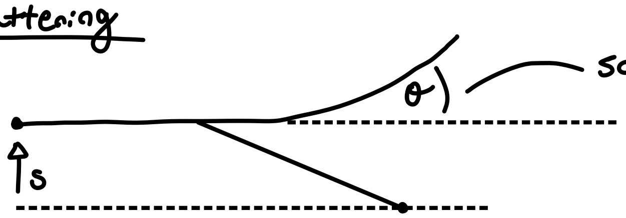
iii) Formal solution of $r(t)$ & $\varphi(t)$ (Kepler Problem)

iv) Parametrized solution : $r(\varphi)$ or $\varphi(r)$

v) Qualitative solutions w/ $V_{eff}(r)$

vi) $\vec{A}$

## Scattering




scattering angle $\theta(s)$

$$\sigma(\theta) = \frac{s}{\sin(\theta)}\left|\frac{ds}{d\theta}\right|$$

## Rigid Body Dynamics

* 6 co-ordinates , 3 translational , 3 rotational $(\theta, \varphi, \psi)$

* space-fixed , body-fixed

$$\left(\frac{d\vec{G}}{dt}\right)_s = \left(\frac{d\vec{G}}{dt}\right)_b + \vec{\omega} \times \vec{G}$$

Angular velocity

$L = T - V$ , $T = T_{com} + T_{ROT}$

* Moment of inertia

* Euler's equations of motion : $\dot{L}_j + \sum \epsilon_{ijk} \omega_k \times L_l = N_j$

## 11-8-21

$L(\vec{q}, \dot{\vec{q}}, t)$ – Defined in configuration space

EOM : $n$ $2^{nd}$ order equations , $n$ co-ords , $2n$ initial conditions

$H(\vec{q},\vec{p},t)$, phase-space : $\vec{P} = \frac{\partial L}{\partial \vec{\dot{q}}}$

EOM : $2n$ 1st order equation, $2n$ co-ordinates, $2n$ initial conditions

## Definition

$$H(\vec{q},\vec{p},t) = \sum_i p_i \dot{q}_i - L(\vec{q},\vec{\dot{q}},t)$$

Canonical EOM : $\dot{q}_i = \frac{\partial H}{\partial p_i}$ , $\dot{p}_i = -\frac{\partial H}{\partial q_j}$ , $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

Shortcut : If:

$$L = L_0(q,t) + \vec{\dot{q}}^T \vec{a} + \frac{1}{2}\vec{\dot{q}}^T I \vec{\dot{q}} \; , \; \vec{P} = I\vec{\dot{q}} + \vec{a}$$

$H(\vec{q},\vec{p},t) = \frac{1}{2}(\vec{P}^T - \vec{a}^T) I^{-1} (\vec{P} - \vec{a}) - L_0(q,t) \rightarrow V(q,t) \rightarrow \boxed{A9}$

$H$ "=" $h \rightarrow$ Energy Function

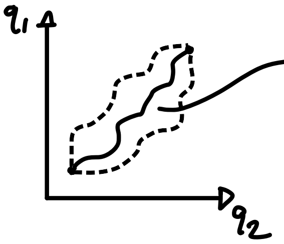
$\rightarrow H = T + V$ "=" Energy , $\frac{\partial L}{\partial q} = 0 \Rightarrow$ Energy conserved

Condition : $\frac{\partial H}{\partial t} = 0$ , $\frac{dH}{dt} = \ldots\ldots = 0 \longrightarrow \frac{\partial H}{\partial t} \rightarrow$ Assume canonical EOM hold

## Variational Approach (8.5)

Lagrangian EOM $\rightarrow$ Hamilton principle $\rightarrow$ Canonical (Hamilton) principle

Principle of least (stationary) action , $\delta I = \delta \int_{t_i}^{t_F} L(q,\dot{q},t)\, dt = 0$




$\rightarrow$ True path $\vec{q}(t,\alpha) = \vec{q}(t,0) + \alpha \vec{\eta}(t)$ , $\vec{\eta}(t_1) = \vec{\eta}(t_0) = 0$

$\delta I = \frac{\partial I}{\partial \alpha}\Big|_{\alpha=0} d\alpha$    * $q_i$ are independent variables $\Rightarrow$ Lagrange's EOM

## Hamiltonian case :

Now : $\vec{q}$ & $\vec{p}$ , Assume variation : $\vec{q}(t,\alpha) = \vec{q}(t,0) + \alpha \vec{\eta}(t)$

$\vec{P}(t,\alpha) = \vec{p}(t,0) + \alpha \vec{\xi}(t)$ w/ $\vec{\xi}(t_1) = \vec{\xi}(t_2) = 0$

Hamilton's Principle:

$$\rightarrow \delta I = \delta \int_{t_0}^{t_1} \Big[ \Big( \sum_{j=1}^{n} \dot{q}_i p_i \Big) - H(q,p,t) \Big] dt = \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} \Big( \sum_{i=1}^{n} \dot{q}_i p_i - H \Big) dt \Big|_{\alpha=0} d\alpha$$

$\frac{\partial q_i}{\partial \alpha}\Big|_{\alpha=0} d\alpha = \delta q_i + \frac{\partial p_i}{\partial \alpha}\Big|_{\alpha=0} d\alpha = \delta p_i$ ...... $\dot{q}_i = \frac{\partial H}{\partial p_i}$ & $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Canonical EOM