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Quantum Mechanics 2

CH. 8 RELATIVISTIC QUANTUM MECHANICS LECTURE NOTES

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Relativity and Quantum Mechanics

The Schrödinger Equation needs to be modified when particles move close to the Speed of Light, where Quantum Mechanics needs to be reconcile with special relativity.

In relativity the quantity

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \vec{x}^2$$

is a fundamental invariant. Defining the quadravector

$$x^\mu = (ct, x^1, x^2, x^3) = (x^0, \vec{x})$$

with $\mu = 0,1,2,3$, one can introduce the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Which raises and lowers indices,

$$x_\mu = g_{\mu\nu} x^\mu = (ct, -\vec{x})$$

where repeated indices are to be summed,

$$a_\mu b^\mu \longrightarrow \sum_\mu a_\mu b^\mu$$

Therefore our metric becomes

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^u = g^{\mu\nu} \Delta x_\mu \Delta x_u = \Delta x_\mu \Delta x^\mu$$

and

$$g^\mu_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

All allowed transformations that preserve the invariant Δs^2, are called Lorentz transformations,

$$x'^\mu = f^\mu(x^\nu) \Rightarrow dx'^\mu = \frac{\partial f^\mu}{\partial x^\lambda} dx^\lambda$$

The metric becomes

$$ds^2 = dx'^{\mu} dx'_{\mu} = g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = g'_{\mu\nu} \frac{\partial f^{\mu}}{\partial x^{\lambda}} \frac{\partial f^{\nu}}{\partial x^{\sigma}} dx^{\lambda} dx^{\sigma} = g_{\lambda\sigma}\, dx^{\lambda} dx^{\sigma}$$

we can then say

$$g'_{\mu\nu} \frac{\partial f^{\mu}}{\partial x^{\lambda}} \frac{\partial f^{\nu}}{\partial x^{\sigma}} = g_{\lambda\sigma} \qquad \text{(covariance)}$$

Besides covariance, Lorentz Transformations leave the metric tensor invariant ∴ they must be linear transformations that combines translations with homogeneous transformations

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

where

$$\Lambda^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \text{const.}$$

we can finally say

$$g'_{\mu\nu} \times \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} = g_{\lambda\sigma}$$

In QM we adopt the substitution

$$E \to i\hbar \frac{\partial}{\partial t} \ , \quad \vec{p} = -i\hbar \vec{\nabla}$$

In special relativity the momentum quadrivector is

$$p^{\mu} = m u^{\mu} = (E/c, \vec{p})$$

whereas the relativistic energy dispersion is :

$$E^2 = m^2 c^4 + p^2 c^2$$

Then

$$\nabla_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right) \ , \quad \nabla^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

we then know

$$p^{\mu} = \left(\frac{E}{c}, \vec{p}\right) \to i\hbar \nabla^{\mu} \ , \quad p_{\mu} = \left(\frac{E}{c}, -\vec{p}\right) \to i\hbar \nabla_{\mu}$$

4-25-22

Previous class

We have the tensor

$$X^{\mu} = (ct, \vec{x}) = g^{\mu\nu} X_{\nu} = g^{\mu\nu}(x_0, -\vec{x})$$

where

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \Delta S^2 = X^{\mu} X_{\mu} \quad \text{(Scalar)}$$

Lorentz transformations keep scalars ΔS^2 invariant.

In Quantum Mechanics

$$p^{\mu} = \left(\frac{E}{c}, \vec{p}\right) \longrightarrow i\hbar \nabla^{\mu}$$

We can go on to say

$$\nabla^{\mu} \nabla_{\mu} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box \quad \text{(Dalambartian operator)}$$

The simplest relativistic equation, that is invariant under Lorentz transformations is:

$$\left(\Box + \frac{m^2 c^2}{\hbar^2}\right) \varphi(x) = 0 \tag{1}$$

Where (1) is the Klein Gordon Equation.

For a probabilistic interpretation of this equation, one would need a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = \partial_{\mu} j^{\mu} = 0$$

With $j^{\mu} = (c\rho, \vec{j})$. We have:

$$\varphi^* \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi = \varphi^* \left(\nabla^2 \varphi - \frac{m^2 c^2}{\hbar^2}\right) \tag{2}$$

$$\varphi \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \varphi^* = \varphi \left(\nabla^2 \varphi^* - \frac{m^2 c^2}{\hbar^2} \varphi^*\right) \tag{3}$$

Subtracting (2) and (3),

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varphi^*\frac{\partial \varphi}{\partial t} - \varphi\frac{\partial \varphi^*}{\partial t}\right) = \varphi^*\nabla^2\varphi - \varphi\nabla^2\varphi^*$$

Since,

$$\varphi^*\nabla^2\varphi = \vec{\nabla}\cdot(\varphi^*\vec{\nabla}\varphi) - \vec{\nabla}\varphi^*\cdot\vec{\nabla}\varphi$$

We can then say

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\varphi^*\frac{\partial \varphi}{\partial t} - \varphi\frac{\partial \varphi^*}{\partial t}\right) = \vec{\nabla}\cdot(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*)$$

We then go on to say

$$\vec{j} \equiv \frac{i\hbar}{2m}(\varphi\vec{\nabla}\varphi^* - \varphi^*\vec{\nabla}\varphi) \quad , \quad \rho \equiv \frac{i\hbar}{2mc^2}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right)$$

While $\vec{j}$ is the standard definition in QM, ρ however is not positively defined (lack of probability conservation) with negative energy states for a free particle,

$$\varphi(\vec{x},t) = A e^{\frac{-i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$$

Satisfying $E^2 = m^2c^4 + p^2c^2 \Rightarrow$

$$E = \pm\sqrt{(pc)^2 + (mc^2)^2} \tag{5a}$$

Those issues motivated Dirac to write a dynamical equation in the form of a linear Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \mathcal{H}_D\psi \tag{6}$$

Where the components of ψ (a spinor) satisfy the Klein Gordon eq., with plane wave solutions that obey (5a).

They also have a density $j^0 = c\rho$ that is positive defined. Also, equation (6) must also be Lorentz invariant.

Since:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \quad , \quad \vec{p} = -i\hbar\vec{\nabla}$$

the Dirac Hamiltonian has the form

$$\mathcal{H}_D = \vec{A}\cdot\vec{\nabla} + B$$

where the matrices $(\vec{A}, B)$ do not depend on time or co-ordinates $\Rightarrow \mathcal{H}_D$ is a linear operator.

Hence,

$$[\partial/\partial t, \mathcal{H}_D] = 0$$

Taking the square of (6),

$$i\hbar \frac{\partial}{\partial t}\left(i\hbar \frac{\partial \psi}{\partial t}\right) = i\hbar \frac{\partial}{\partial t}\mathcal{H}_D = i\hbar \mathcal{H}_D \frac{\partial}{\partial t}\psi = \mathcal{H}_D^2 \psi = -\hbar^2 \frac{\partial^2 \psi}{\partial t^2}$$

or equivalently

$$\begin{aligned}
-\hbar^2 \frac{\partial^2}{\partial t^2} = \mathcal{H}_D^2 &= m^2c^4 - c^2\hbar^2\nabla^2 \\
&= (\vec{A}\cdot\vec{\nabla} + B)(\vec{A}\vec{\nabla} + B) \\
&= (A^i\nabla_i + B)(A^j\nabla_j + B) \\
&= A^iA^j\nabla_i\nabla_j + (A^jB + BA^j)\nabla_j + B^2
\end{aligned}$$

For $(i,j = 1,2,3)$

$$\begin{aligned}
&= \frac{1}{2}(A^iA^j + A^jA^i)\nabla_i\nabla_j + (A^jB + BA^j)\nabla_j + B^2 \\
&= m^2c^4 - c^2\hbar^2\delta^{ij}\nabla_i\nabla_j
\end{aligned}$$

Comparing the last two lines we find

$$A^iA^j + A^jA^i = \{A^i, A^j\} = -2c^2\hbar^2\delta^{ij}$$

We can then conclude

$$A^jB + BA^j = \{A^j, B\} = 0 \quad \Rightarrow \quad B^2 = m^2c^4$$

Those relations define the <u>Dirac algebra</u>. Normalizing the matrices,

$$-i\hbar c\alpha^i \equiv A^i$$

$$mc^2\beta \equiv B$$

We then have

$$\alpha^i\alpha^j + \alpha^j\alpha^i = \{\alpha^i, \alpha^j\} = 2\delta^{ij} \quad , \quad \{\alpha^i, \beta\} = 0 \quad , \quad \beta^2 = 1 \qquad (7)$$

The Dirac equation becomes

$$i\hbar \frac{\partial}{\partial t}\psi = -ic\hbar\, \vec{\alpha}\cdot\vec{\nabla}\psi + mc^2\beta\psi$$

with

$$\mathcal{H}_D = \frac{c\hbar}{i}\vec{\alpha}\cdot\vec{\nabla} + mc^2\beta = c\,\vec{\alpha}\cdot\vec{p} + mc^2\beta$$

Since $\mathcal{H}_D$ is hermitian, then

$$(\alpha^i)^\dagger = \alpha^i \Rightarrow \beta^\dagger = \beta$$

Also, from (7),

$$(\alpha^i)^2 = 1 \;,\quad \beta^2 = 1 \;,\quad i = 1,2,3$$

We then say

i) All eigenvalues of α's and β are ± 1

ii) $Tr(\alpha^i) = Tr(\beta) = 0$, $i = 1,2,3$

The proof of this is

$$\alpha^i\beta = -\beta\alpha^i \Rightarrow \alpha^i\beta(\alpha^i)^{-1} = -\beta$$

Taking the trace,

$$Tr[\alpha^i\beta(\alpha^i)^{-1}] = Tr(\beta) = -Tr(\beta) = 0$$

In the same way,

$$\alpha^i\beta = -\beta\alpha^i \Rightarrow \beta^{-1}\alpha^i\beta = -\alpha^i$$

Therefore

$$Tr[\beta^{-1}\alpha^i\beta] = Tr[\alpha^i] = -Tr[\alpha^i] = 0$$

iii) Since α^i and β have eigenvalues ± 1, then β and α^i must have even dimension

<u>First Attempt $N=2$</u>

The algebra of α^i is identical to Pauli matrices, if we set

$$\alpha^1 = \sigma_x \;,\quad \alpha^2 = \sigma_y \;,\quad \alpha^3 = \sigma_z$$

Since

$$\sigma^1_i\sigma^1_j + \sigma^1_j\sigma^1_i = 2\delta_{ij} \Rightarrow \mathrm{Tr}(\sigma^1_i) = 0 \quad , \mathrm{Tr}(\sigma^{1\,2}_i) = \mathbb{1}$$

However it is <u>impossible</u> to find a fourth 2×2 matrix (β) which anti-commutes with the other three.

<u>Second Attempt $N=4$</u>

Let's choose the representation,

$$\alpha^i = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline \sigma^i & 0 \end{array}\right) \qquad i = 1,2,3$$

$$\beta = \left(\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

These matrices can be multiplied in Blocks,

$$\alpha^i\alpha^j = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline \sigma^i & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & \sigma^j \\ \hline \sigma^j & 0 \end{array}\right) = \left(\begin{array}{c|c} \sigma^i\sigma^j & 0 \\ \hline 0 & \sigma^i\sigma^j \end{array}\right)$$

$\Rightarrow$ $$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}$$

<u>Gamma Matrices</u> γ^μ

$$\gamma^0 \equiv \beta \quad , \quad \gamma^i \equiv \beta\alpha^i \quad , \quad i = 1,2,3$$

We then

$$\{\gamma^i,\gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i = -\beta^2\alpha^i\alpha^j - \beta^2\alpha^j\alpha^i = -2\delta^{ij}$$

We also have

$$\{\beta,\gamma^i\} = \beta^2\alpha^i + \beta\alpha^i\beta = \alpha^i - \alpha^i = 0$$

Therefore

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad \mu,\nu = 0,1,2,3$$

4-27-22

<u>Definition</u>: <u>Gamma</u> <u>Matrices</u>, γ^μ

$$\gamma^0 \equiv \beta \quad , \quad \gamma^i = \beta\alpha^i$$

Where

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

With

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

<u>Definition</u>: γ^5

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

In the chosen representation, we have

$$\gamma^5 = \left(\begin{array}{c:c} 0 & \mathbb{1} \\ \hdashline \mathbb{1} & 0 \end{array}\right)$$

Hence, ψ is a 4 component spinor,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi^\dagger = (\psi_1^* \; \psi_2^* \; \psi_3^* \; \psi_4^*)$$

<u>Definition</u>: Feynman "slash"

$$\not{a} \equiv a_\mu \gamma^\mu$$

We can write the Dirac equation in the form

$$i\hbar \frac{\partial \psi}{\partial(ct)} + i\hbar\alpha^i \nabla_i \psi = \mu c \beta \psi$$

Multiplying by β on the left,

$$i\hbar\left[\beta \frac{\partial}{\partial(ct)} + \beta\alpha^i \nabla_i\right]\psi = \mu c \psi$$

a

$$(i\hbar\gamma^\mu\partial_\mu - \mu c)\psi = (i\hbar\not{\partial} - \mu c)\psi = 0$$

Which is the most compact form

$$(i\hbar\not{\partial} - \mu c)\psi = 0$$

To show that the Dirac equation has a continuity equation with a positively defined probability density $j^0 = c\rho$, we write

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla}\psi + mc^2\beta\psi \qquad \text{(A)}$$

and the adjoint form:

$$-i\hbar \frac{\partial \psi^+}{\partial t} = i\hbar c\,\vec{\nabla}\psi^+\cdot\vec{\alpha} + mc^2\psi^+\beta \qquad \text{(B)}$$

Since β and $\vec{\alpha}$ are hermitian. Multiplying (A) on the left by ψ^+ and (B) on the right by

$$\psi^+(A) - (B)\psi = i\hbar\psi^+\frac{\partial\psi}{\partial t} + i\hbar\left(\frac{\partial\psi^+}{\partial t}\right)\psi$$

$$= i\hbar\left[\psi^+\frac{\partial\psi}{\partial t} + \frac{\partial\psi^+}{\partial t}\psi\right] = i\hbar\frac{\partial(\psi^+\psi)}{\partial t}$$

$$= -i\hbar c\,\psi^+(\vec{\alpha}\cdot\vec{\nabla}\psi) + mc^2\psi^+\beta\psi - i\hbar c(\vec{\nabla}\psi^+\cdot\vec{\alpha})\psi - mc^2\psi^+\beta\psi$$

$$= -i\hbar\,\vec{\nabla}\cdot(\psi^+ c\vec{\alpha}\psi)$$

Where finally we have

$$\frac{\partial}{\partial t}(\psi^+\psi) + \vec{\nabla}\cdot(\psi^+ c\vec{\alpha}\psi) = 0$$

That gives the current density

$$J^0 = c\rho = c\psi^+\psi \Rightarrow \vec{J} = \psi^+ c\vec{\alpha}\psi$$

with density $\rho = \psi^+\psi$

$$\rho = (\psi_1^* \ldots\ldots \psi_4^*)\begin{pmatrix}\psi_1\\ \vdots\\ \psi_4\end{pmatrix} = |\psi_1|^2 + |\psi_2|^2 + \ldots + |\psi_4|^2 \geq 0$$

is positive defined. In the bi spinors representation,

$$\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} \Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = -i\hbar c\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\nabla\varphi\\ \nabla\chi\end{pmatrix} + mc^2\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$$

Separating in components

$$i\hbar\frac{\partial\varphi}{\partial t} = -i\hbar c(\vec{\sigma}\cdot\vec{\nabla}\chi) + mc^2\varphi \quad , \quad i\hbar\frac{\partial\chi}{\partial t} = -i\hbar c(\vec{\sigma}\cdot\vec{\nabla}\varphi) - mc^2\chi$$

Where

$$J^0 = c\rho = \psi^\dagger \psi c = (\varphi^\dagger, \chi^\dagger)\, c \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = (\varphi^\dagger\varphi + \chi^\dagger\chi)\, c$$

$$\vec{J} = \psi^\dagger c \vec{\alpha} \psi = (\varphi^\dagger, \chi^\dagger)\, c \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c(\varphi^\dagger \vec{\sigma} \chi + \chi^\dagger \vec{\sigma} \varphi)$$

<u>Dirac Equation Free Particle Solution</u>

The general solution of the Dirac equation for a free particle has the general form

$\psi^{(+)}_{(x)} = e^{-ik\cdot x} u(k) \rightarrow$ (Positive energy) , $\psi^{(-)}_{(x)} = e^{ik\cdot x} v(k) \rightarrow$ (Negative energy)

Where

$$k\cdot x \equiv k^\mu x_\mu = k_\mu x^\mu = k^0 ct - \vec{k}\cdot\vec{x}$$

with $k^0 > 0$, and $p^\mu = \hbar k^\mu$

From the Klein Gordon equation

$$p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = \hbar^2 k^\mu k_\mu = m^2 c^2$$

$\Rightarrow$

$$k^\mu k_\mu = \frac{m^2c^2}{\hbar^2} = (k^0)^2 - \vec{k}^2$$

For $m \neq 0$, we choose the rest reference frame of the particle,

$$k^\mu = \left(\frac{mc}{\hbar}, 0\right)$$

The Dirac equation becomes:

$$\left(i\gamma^\mu \frac{\partial}{\partial \mu} - \frac{mc}{\hbar}\right)\psi = 0 \qquad (1)$$

Therefore

$$i\gamma^\mu \frac{\partial \psi^{(+)}}{\partial \mu} = i\gamma^0 \left(\frac{1}{c}\frac{\partial}{\partial t} e^{-ik^0 ct}\right) u(k) = \gamma^0 k_0 \gamma^0 u(k) = \frac{mc}{\hbar}\gamma^0 u(k)$$

Hence, from (1)

$$\frac{mc}{\hbar}(\gamma^0 - \mathbb{1})u(k) = 0 \quad E > 0 \quad , \quad -\frac{mc}{\hbar}(\gamma^0 + \mathbb{1})v(k) = 0 \quad E < 0$$

Since:

$$\gamma^0 - \mathbb{1} = \left(\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & -2\mathbb{1} \end{array}\right) \quad , \quad \gamma^0 + \mathbb{1} = \left(\begin{array}{c:c} 2\mathbb{1} & 0 \\ \hdashline 0 & 0 \end{array}\right)$$

then we have four Linearly independent solutions,

$$u^{(1)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 1 \\ 0 \\ \hdashline 0 \\ 0 \end{pmatrix} \quad , \quad u^{(2)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 1 \\ \hdashline 0 \\ 0 \end{pmatrix}$$

and

$$v^{(1)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ \hdashline 1 \\ 0 \end{pmatrix} \quad , \quad v^{(2)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ \hdashline 0 \\ 1 \end{pmatrix}$$

For $\vec{k} \neq 0$, we need to solve the equation

$$(\not{K} - \frac{mc}{\hbar}) u(k) = 0 \qquad (\not{K} + \frac{mc}{\hbar}) v(k) = 0$$

Those solutions can be found by applying a Lorentz transformation with speed

$$v = \frac{\hbar |\vec{k}|}{m} = \frac{|\vec{k}| c}{k_0}$$

Equivalently, we notice that since

$$\not{K}^2 = K_\mu \gamma^\mu K^\mu \gamma_\mu = K_\mu K^\mu \qquad \text{(check!)}$$

$$(\not{K} - \frac{mc}{\hbar})(\not{K} + \frac{mc}{\hbar}) = K^\mu K_\mu - \frac{m^2 c^2}{\hbar^2} = 0$$

then one can build the solutions in the form,

$$u^{(\alpha)}_{(k)} = A^{(\alpha)} (\not{K} + \frac{mc}{\hbar}) u^{(\alpha)}(m, 0) \quad , \quad v^{(\alpha)}_{(k)} = -B^{(\alpha)} (\not{K} - mc/\hbar) v^{(\alpha)}_{(m,0)}$$

5-2-22

From the previous class we have

$$\psi^{(+)}(x) = e^{-iK\cdot x} u(k) \qquad \text{(positive energy)}$$

$$\psi^{(-)}(x) = e^{iK\cdot x} v(x) \qquad \text{(negative energy)}$$

We then have the relationships

$$u^{(1)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u^{(2)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(2)}(mc/\hbar, \vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

In general,

$$u^{(\alpha)}(k) = A^{(\alpha)}(\not{K} + mc/\hbar)\, u^{(\alpha)}(m,0)$$

$$v^{(\alpha)}(k) = -B^{(\alpha)}(\not{K} - mc/\hbar)\, v^{(\alpha)}(m,0)$$

In a more explicit form,

$$\not{K} = k_\mu \gamma^\mu = k_0\beta - \beta(\vec{k}\cdot\vec{\alpha}) = k_0\beta - |\vec{k}|\beta(\hat{k}\cdot\vec{\alpha}) = k_0 \left(\begin{array}{c:c} \mathbb{1} & 0 \\ \hdashline 0 & -\mathbb{1} \end{array}\right) - |\vec{k}| \left(\begin{array}{c:c} 0 & \hat{k}\cdot\vec{\sigma} \\ \hdashline -\hat{k}\cdot\vec{\sigma} & 0 \end{array}\right)$$

For $E > 0$,

$$\not{K}\, u(m,0) = \not{K}\left(\begin{array}{c} \varphi(m,0) \\ \hdashline 0 \end{array}\right) = k_0 \left(\begin{array}{c} \varphi(m,0) \\ \hdashline 0 \end{array}\right) + |\vec{k}| \left(\begin{array}{c} 0 \\ \hdashline \hat{k}\cdot\vec{\sigma}\varphi(m,0) \end{array}\right)$$

This then means $u(k)$ is

$$u(k) = A \left(\begin{array}{c} (k_0 + \frac{mc}{\hbar})\varphi(m,0) \\ \hdashline |\vec{k}|(\vec{\sigma}\cdot\hat{k})\varphi(m,0) \end{array}\right)$$

Where $\varphi(m,0)$ is some linear superposition of the two component spinors

$$\varphi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{s.t} \quad \varphi^{\dagger}\cdot\varphi = 1$$

The normalization follows from

$$u^{\dagger}(k) = A^{*}\Big((k_0 + mc/\hbar)\varphi^{\dagger}(m,0),\, |\vec{k}|\varphi^{\dagger}(m,0)(\vec{\sigma}\cdot\hat{k})\Big),$$

Imposing that:

$$1 = u^{\dagger}(k)u(k) = |A|^2 \left\{(k_0 + mc/\hbar)^2 \varphi^{\dagger}(m,0)\varphi(m,0) + \vec{k}^2 \varphi^{\dagger}(m,0)(\vec{\sigma}\cdot\hat{k})^2\varphi(m,0)\right\}$$

We have

$$1 = |A|^2 \left\{(k_0 + mc/\hbar)^2 + \vec{k}^2\right\}$$

where:

$$\vec{k}^2 = \frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - m^2c^2\right) \quad w/ \quad k_0 = \frac{E}{c\hbar}$$

we can then say

$$1 = |A|^2 \frac{1}{\hbar^2}\left\{\left(\frac{E}{c} + mc\right)^2 + \frac{E^2}{c^2} - m^2c^2\right\} = |A|^2 \frac{1}{c^2\hbar^2}\left\{(E+mc^2)^2 + E^2 - m^2c^4\right\}$$

$$= \frac{2|A|^2}{\hbar^2c^2} E(E+mc^2)$$

This tells us

$$|A| = \frac{\hbar c}{\sqrt{2E(E+mc^2)}}$$

Hence, for $E > 0$

$$\psi^{(+)}(x) = \frac{e^{-ikx}}{\sqrt{2E(E+mc^2)}} \times \begin{pmatrix} \hbar c(k_0 + mc/\hbar)\varphi(m,0) \\ \hbar c(\vec{\sigma}\cdot\vec{k})\varphi(m,0) \end{pmatrix}$$

The normalization above is not the most common in the Literature. Re-writing the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \tag{1}$$

In the adjoint Form:

$$-\varphi^\dagger(i\hbar(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu + mc) = 0 \tag{2}$$

We note that:

$$(\gamma^0)^2 = \beta^2 = \mathbb{1} \Rightarrow (\gamma^0 \text{ unitary})$$

$$(\gamma^i)^\dagger = (\beta\alpha^i)^\dagger = \alpha^i\beta = \alpha^i\gamma^0 = \gamma^0(\gamma^0\alpha^i)\gamma^0 = \gamma^0\gamma^i\gamma^0$$

Also,

$$(\gamma^0)^\dagger = \gamma^0 = \gamma^0\gamma^0\gamma^0 \quad \therefore \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Equation (2) becomes

$$\psi^\dagger\gamma_0(i\hbar\gamma^\mu\overleftarrow{\partial}_\mu + mc)\gamma^0 = 0$$

We then define something called the Dirac Adjoint

$$\bar{\psi} \equiv \psi^{+}\gamma^{0}$$

We have

$$\bar{\psi}(i\hbar\overleftarrow{\not{\partial}} + \mu c) = 0 \ (2A) \quad , \quad (i\hbar\overrightarrow{\not{\partial}} - \mu c)\psi = 0 \ (1A)$$

This then means

$$\Rightarrow \qquad (2A)\psi + \bar{\psi}(1A) = \bar{\psi}(i\hbar\overleftarrow{\not{\partial}} + \mu c)\psi + \bar{\psi}(i\hbar\overrightarrow{\not{\partial}} - \mu c)\psi = 0$$

Therefore we then have

$$0 = \bar{\psi}(\overleftarrow{\not{\partial}} + \overrightarrow{\partial})\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi)$$

Which is the covariant form of the continuity equation, with the current density quadrivector

$$J^\mu = \bar{\psi}\gamma^\mu\psi \quad , \quad \partial_\mu J^\mu = 0$$

The normalization is chosen that

$$\bar{\psi}\psi = 1 \qquad (E>0)$$

$$\bar{\psi}\psi = -1 \qquad (E<0)$$

In that case, the normalized spinors satisfy

$$(\varphi^{+}, \chi^{+})\gamma^{0}\begin{pmatrix}\varphi \\ \chi\end{pmatrix} = \pm 1 = \varphi^{+}\varphi - \chi^{+}\chi$$

Electromagnetic Coupling

In order to include the coupling of a charged particle with classic electromagnetic fields described by the quadrivector

$$A^\mu = (\varphi, \vec{A}) \ , \ A_\mu = (\varphi, -\vec{A})$$

We use the substitution

$$i\hbar\partial_\mu \longrightarrow i\hbar\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)$$

or

$$i\hbar\gamma^\mu\partial_\mu \longrightarrow i\hbar\left(\gamma^\mu\partial_\mu + \frac{ie}{\hbar c}\gamma^\mu A_\mu\right)$$

The Dirac equation becomes

$$\left(i\hbar\not{\partial} - \frac{e}{c}\not{A} - mc\right)\psi = 0 \qquad (3)$$

The Dirac equation is invariant under gauge transformation

$$A_\mu \longrightarrow A_\mu - \partial_\mu \Lambda(x) \equiv A'_\mu \quad , \quad \psi(x) \longrightarrow e^{ie/\hbar c \Lambda(x)}\psi(x) \equiv \psi'(x)$$

where $\Lambda(x)$ is some arbitrary function. Indeed

$$\partial_\mu (e^{ie/\hbar c \Lambda}\psi) = e^{ie/\hbar c \Lambda}\psi + \frac{ie}{\hbar c}(\partial_\mu \Lambda)\psi'$$

Hence,

$$\partial_\mu \psi' + \frac{ie}{\hbar c}A'_\mu \psi' = e^{ie/\hbar c \Lambda}\partial_\mu \psi + \frac{ie}{\hbar c}(\cancel{\partial_\mu \Lambda})\psi' + \frac{ie}{\hbar c}(A_\mu - \cancel{\partial_\mu \Lambda})\psi'$$

$$= e^{ie/\hbar c \Lambda}(\partial_\mu + ie/\hbar c\, D_\mu)\psi$$

From (3), the transformed Dirac equation becomes

$$e^{ie/\hbar c \Lambda}\left[i\hbar\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right) - mc\right]\psi = 0$$

or equivalently

$$(i\hbar\not{D} - mc)\psi = 0$$

Where

$$D_\mu = \partial_\mu + \frac{ie}{\hbar c}A_\mu$$

Is the Gauge Invariant derivative, such that

$$D'_\mu \psi' = e^{ie/\hbar c \Lambda} D_\mu \psi$$

5-4-22

<u>Summary of Dirac Algebra</u>

i) $\gamma^0 = \beta \;,\; \gamma^i = \beta\alpha^i \;,\; i = 1,2,3$

ii) $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \;,\; (\gamma^0)^2 = 1 \;,\; (\gamma^i)^2 = -1$

iii) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \;,\; (\gamma^5)^\dagger = \gamma^5 = (\gamma^5)^* = (\gamma^5)^T$

iv) $\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \;,\; \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \;,\; \vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \;,\; \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

In explicit form

$$\gamma^0 = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & -1 \end{array}\right) \Rightarrow (\gamma^0)^* = \gamma^0 = (\gamma^0)^T = \gamma^{0\dagger}$$

$$\gamma^1 = \left(\begin{array}{cc|cc} & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \\ \hline 0 & -1 & & \\ -1 & 0 & \multicolumn{2}{c}{0} \end{array}\right) \Rightarrow (\gamma^1)^* = \gamma^1 \;,\; (\gamma^1)^\dagger = (\gamma^1)^T = -\gamma^1$$

$$\gamma^2 = \left(\begin{array}{cc|cc} & & 0 & -i \\ \multicolumn{2}{c|}{0} & i & 0 \\ \hline 0 & i & & \\ -i & 0 & \multicolumn{2}{c}{0} \end{array}\right) \Rightarrow (\gamma^2)^* = -\gamma^2 \;,\; (\gamma^2)^T = \gamma^2 \;,\; (\gamma^2)^\dagger = -\gamma^2$$

$$\gamma^3 = \left(\begin{array}{cc|cc} & & 1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & -1 \\ \hline -1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \end{array}\right) \Rightarrow (\gamma^3)^* = \gamma^3 \;,\; (\gamma^3)^\dagger = (\gamma^3)^T = -\gamma^3$$

v) $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $(\gamma^\mu)^* = [(\gamma^\mu)^\dagger]^T = \gamma^0(\gamma^\mu)^T\gamma^0$, $\gamma^0(\gamma^\mu)^*\gamma^0 = (\gamma^\mu)^T$

$\Rightarrow \gamma^0(\gamma^\mu)^* = (\gamma^\mu)^T\gamma^0$

Charge Conjugation

The Symmetry of the Dirac equation imply that the absence of a position with charge $+|e|$ and $E<0$ in the Dirac sea is equivalent to the presence of an electron with opposite charge $(-|e|)$ and energy $(E>0)$, and vice-versa.

That symmetry operation is called charge conjugation,

$$\psi \rightarrow \psi^c \quad , \quad \text{such that}$$

This leads to

$$(i\hbar\not{\partial} - \frac{e}{c}\not{A} - mc)\psi = 0 \tag{1}$$

$$(i\hbar\not{\partial} + \frac{e}{c}\not{A} - mc)\psi^c = 0 \tag{2}$$

with $e \rightarrow -e$. In order to express ψ^c in terms of ψ, Some A_μ is real, we take the complex conjugation

$$\left[-(i\hbar\partial_\mu + \frac{e}{c}A_\mu)(\gamma^\mu)^* - mc\right]\psi^* = 0 \tag{3}$$

To express (3) in the same form as (2) we need a transformation such that:

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu$$

or equivalently

$$C\gamma^0(\gamma^\mu)^*\gamma^0 C^{-1} = -\gamma^\mu$$

This tells us we can deduce

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \quad , \quad C(\gamma^\mu)^T = -\gamma^\mu C \tag{4}$$

or conversely

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$$

Since:

$$(\gamma^0)^T = \gamma^0 \quad , \quad (\gamma^2)^T = \gamma^2$$

But

$$(\gamma^1)^T = -\gamma^1 \quad , \quad (\gamma^3)^T = -\gamma^3$$

From (4), we have that C commutes with γ^1, γ^3 and anti-commutes with γ^0 and γ^2.
We can choose the following representations for C,

$$C = i\gamma^2\gamma^0$$

With

$$C^{-1} = C^\dagger = C^T = C$$

The Dirac equation for a conjugated charge becomes:

$$0 = C^0\gamma^0\left[-(i\hbar\partial_\mu + \frac{e}{c}A_\mu)\gamma^{\mu *} - mc\right]((\gamma^0)^{-1}((\gamma^0)\psi^*$$

$$= \left[(i\hbar\partial_\mu + \frac{e}{c}A_\mu)\gamma^\mu - mc\right]\psi^c = 0$$

Where

$$\psi^c = C\gamma^0\psi^*$$

For a negative energy state $(E<0)$ with spin <u>down</u> in the rest reference frame,

$$\psi(x) = e^{\frac{i}{\hbar}mc^2 t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\psi^c(x) = C\gamma^0\psi^* = e^{\frac{-i}{\hbar}mc^2 t}\, i\gamma^2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e^{-\frac{imc^2 t}{\hbar}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

This then means

$$\psi^c(x) = e^{\frac{i}{\hbar}mc^2t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$\therefore$ The conjugate particle has opposite energy, charge, and spin.