Homework Assignment #3
Math Methods

Homework Due: Monday, September 20th, 4:30pm

Instructions:

Reading: Please review the end of Chapter 2. Then read Chapter 1.

Reading Quiz # 3 covers Chapter 1 and is due before class on Friday, September 17th.

Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.
Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

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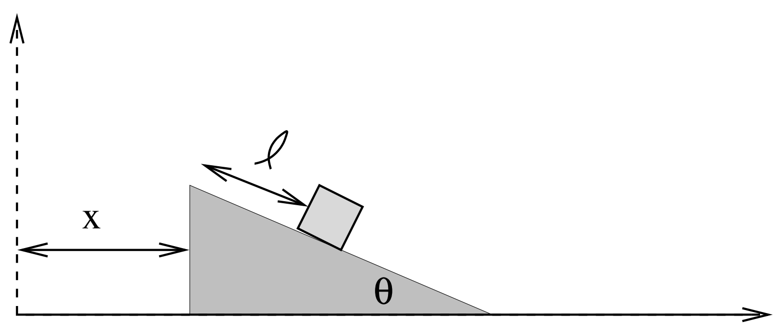
1. *Taken from the 2017 Classical and Statistical Mechanics Qualifier:*

   A wedge of mass M and angle θ moves frictionlessly along the x axis. A small mass, m a distance ℓ from the top of the wedge moves frictionlessly along the wedge. Other than gravity and the normal force on the wedge from the ground, there are no external forces on the system.



   (a) (2 pts) Find the kinetic energy of the system in terms of the generalized coordinates x and ℓ.

   (b) (1 pts) Find the Lagrangian.

   (c) (2 pts) Find the equations of motion for the generalized coordinates and the ratio $\mu = m/(m + M)$.

   (d) (1 pt) Is there a speed of the wedge in which the acceleration of the mass is up the wedge? If so, find it. If not, prove mathematically or explain why.

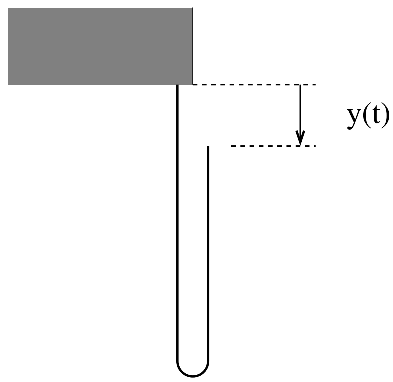
   (e) (2 pts) Integrate the equations of motion and find how long it takes for the particle to slide off if it starts at a height, h above the ground with both wedge and particle at rest.

   (f) (1 pt) Show that in the limit of $M \to \infty$ this agrees with the expected result.

   (g) (1 pt) How far did the wedge move during this time?

2. *Taken from the Spring 2020 Classical and Statistical Mechanics Qualifier:*

   A heavy rope of mass m, and length L has uniform density and is attached at one end to the edge of a (fixed) balcony. It is initially at rest with its free end at a height equal to the point of support so that $y(t = 0) = 0$. The end of the rope is released and it falls downward due to the force of gravity. You should treat the rope as ideal, so that the portion at the left hand side is at rest, and all motion is elastic.



   (a) Determine the position of the center of mass of the rope as a function of y. (2 points)

   (b) Using the above expression for the center of mass, write down the Lagrangian for the rope. (2 points)

   (c) Determine the speed at which the end of the rope falls as a function of y. (2 points)

   (d) Determine the acceleration of the end of the rope as a function of height. (2 points)

   (e) How does your answer compare to g? Explain your result. (2 points)

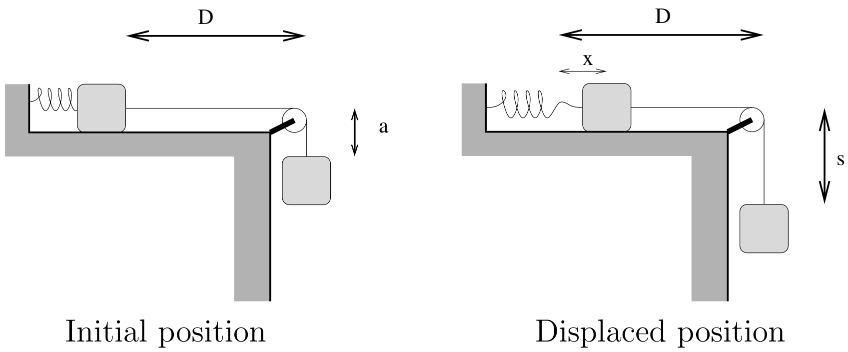
3. *Taken from the 2018 Classical and Statistical Mechanics Qualifier*

   Two blocks each of mass M are connected by a rope of length ℓ and mass $m = \lambda\ell$, where λ is the linear density of the rope. One block slides frictionlessly on a horizontal surface attached to the wall by spring with spring constant, k. The rope, attached to the other side, goes over an ideal, massless pulley and is attached to the second block which hangs freely.

   The figure on the left is the initial condition, where D is the distance between the sliding mass and the edge of the surface when the spring exerts no force (it is neither compressed nor stretched) and the entire system is at rest. In this initial arrangement the length of the rope supporting the suspended block is $a = \ell - D$. (Note that this is *not* an equilibrium position.) The figure on the right shows the system a short time after the blocks are released from rest, so that the spring is now stretched and the blocks are moving. We define x as the displacement of the block on the horizontal surface from the neutral position of the spring and $s = x + a$ is length of the rope supporting the suspended block . (Initially, $s = a$.)

   This problem should be worked using the generalized coordinate s and the variables M, m, k, λ, a and g, the acceleration due to gravity. You should assume that the radius of the pulley is so small that it can be neglected.

   (a) Find the potential energy. (2 Points)



(b) Find the kinetic energy and Lagrangian. (2 Points)

(c) Find the equation of motion for s using the Lagrangian. (2 Points)

(d) Find the equilibrium position for the system. (2 Points)

(e) Find the frequency of oscillation for the system about the equilibrium position. (2 Points)

4. *Taken from the 1996 CSM Qualifier:*

   A point particle of mass m moves under the influence of gravity ($V(x, y, z) = mgz$). The particle itself is constrained to stay on a frictionless surface given by

   $$z = \alpha\left(x^2 + y^2\right)$$

   where $\alpha > 0$.

   (a) Derive the equations of motion for x, y, and z by minimizing the appropriately constrained action, using the method of Lagrange multipliers. **2pt**

   (b) Consider the class of trajectories (i.e. solutions) for which $z = \text{constant} \equiv z_0$. **2pt** Calculate the force required to maintain the constraint for these trajectories.

   (c) Consider the class of trajectories for which $y = 0$. (These are different trajectories than those in part (b).) Assume that the particle reaches a maximum height of z_1

      i. Using conservation of energy derive an expression for $\dot{z} \equiv \frac{dz}{dt}$ as a function of z, and z_1. (That is, remove the dependence on any other spatial coordinate.) **2pt**

      ii. From the expression derived in (i) and the equations of motion you calculated above, determine the force in the z-direction required to maintain the constraint. **2pt**

      iii. Calculate the total force required to maintain the constraint for this class of trajectory. **2pt**

5. Many solid state systems are crystals, and the problems involving them have a rotational symmetry where the system is rotated by $\theta = \pi/2$ (four-fold symmetry) or $\theta = \pi/3$ (six-fold rotational symmetry). Is there a conserved quantity associated with this symmetry by Noether's Theorem? If so, what is it? If not, why not?

6. Consider a two particle interacting system in 3D with the Lagrangian:

   $$L = \frac{1}{2}\left(m_1\,\dot{\vec{r}}_1^{\,2} + m_2\,\dot{\vec{r}}_2^{\,2}\right) - V(\vec{r}_1 - \vec{r}_2)$$

(a) Show that the transformation:

$$\vec{r}_i \rightarrow \vec{r}_i + \epsilon\, \vec{v}_0 t$$

where $\vec{v}_0$ is a constant vector, leaves the Lagrangian unchanged to order ϵ up to a total derivative.

(b) Calculate the conserved quantity associated with with this symmetry of the action.

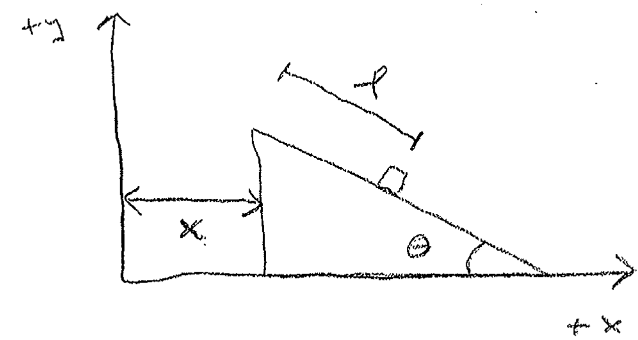
(c) What is this new conserved quantity? Is it useful?

While the above problem is posed in three dimensions, you can work in one dimension and generalize.

A wedge of mass M moves frictionlessly along the ground.

A small mass, m slides frictionlessly on the wedge.

Other than gravity and the normal force on the wedge, there are no external forces on the system.

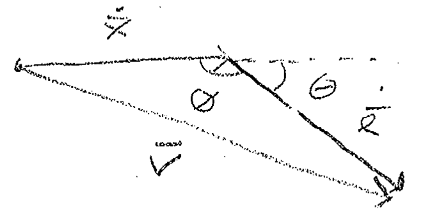


(1) Find the kinetic energy of the system in terms of generalized coordinates, x and ℓ.

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m v^2$$

The velocity of mass relative to wedge : $\dot{\ell}$

velocity relative to ground $\vec{v} = \dot{\vec{x}} + \dot{\vec{\ell}}$



law of cosines => $v^2 = \dot{x}^2 + \dot{\ell}^2 - 2\dot{x}\dot{\ell}\cos\phi$

$$= \dot{x}^2 + \dot{\ell}^2 + 2\dot{x}\dot{\ell}\cos\theta$$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{\ell}^2 + 2\dot{x}\dot{\ell}\cos\theta \right)$$

(2) Find the Lagrangian (1 pt)

$L = T - V$, $V = -mg\ell\sin\theta$

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{\ell}^2 + 2\dot{x}\dot{\ell}\cos\theta \right) + mg\ell\sin$$

(3). Find equations of motion for the generalized coordinates and the ratio $\mu = m/M+m$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + 2m\dot{l}\cos\theta \qquad \frac{\partial L}{\partial x} = 0$$

$$(M+m)\ddot{x} + m\ddot{l}\cos\theta = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \dot{l}} = m\dot{l} + m\dot{x}\cos\theta \qquad \frac{\partial L}{\partial l} = mg\sin\theta$$

$$\cancel{m}\ddot{l} + \cancel{m}\ddot{x}\cos\theta - \cancel{m}g\sin\theta = 0 \qquad (2)$$

substitute and solve for $\ddot{x}$, $\ddot{l}$

using (1) $\ddot{l} = -\frac{1}{\mu\cos\theta}\ddot{x} \Rightarrow -\frac{1}{\mu\cos\theta}\ddot{x} + \ddot{x}\cos\theta - g\sin\theta = 0$

$$\ddot{x}\left(\cos\theta - \frac{1}{\mu\cos\theta}\right) = g\sin\theta \Rightarrow \boxed{\ddot{x} = \frac{g\sin\theta\cos\theta}{\cos^2\theta - 1/\mu}}$$

$$\ddot{l} = -\frac{1}{\mu\cos\theta}\left(\frac{g\sin\theta\cos\theta}{\cos^2\theta - 1/\mu}\right) = \boxed{\frac{+g\sin\theta}{1-\mu\cos^2\theta} = \ddot{l}}$$

Note, w/ no external forces, $\underbrace{\ddot{x} < 0}_{\text{conservation of momentum}}$, $\ddot{l} > 0$, both constants

$$\ddot{x} = -A, \qquad A = \frac{g\sin\theta\cos\theta}{1/\mu - \cos^2\theta}$$

$$\ddot{l} = B, \qquad B = \frac{g\sin\theta}{1-\mu\cos^2\theta}$$

(4) $\mu < 1$, $\cos^2\theta < 1$, thus $\ddot{l} > 0$

(5) integrate the equations of motion and find how long it takes for the particle to slide off if it starts at a height, h, with both at rest.

$$\dot{x} = V_{x_0} - At \quad , \quad x = x_0 + \cancel{V_{x_0}}^{0} t - \frac{1}{2} A t^2$$

$$\dot{l} = V_{l_0} + Bt \quad , \quad l = l_0 + \cancel{V_{l_0}}^{0} t + \frac{1}{2} B t^2$$

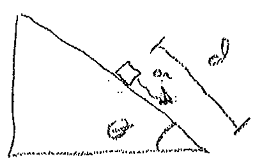
let $x_0 = 0$ and we know $l - l_0 = \frac{h}{\sin\theta}$

$$x = -\frac{1}{2} A t^2 \qquad l - l_0 = \frac{h}{\sin\theta} = \frac{1}{2} B t^2$$

$$t = \sqrt{\frac{2h}{B\sin\theta}} = \sqrt{\frac{2h}{\sin\theta}\left(\frac{1-\mu\cos^2\theta}{g\sin\theta}\right)} = \sqrt{\frac{2h(1-\mu\cos^2\theta)}{g\sin^2\theta}}$$

(don't need x)

(6) In limit $M \to \infty$, $\mu \to 0$, $\ddot{x} \to 0$ and the wedge doesn't move. A particle on a stationary wedge has acceleration $a = g\sin\theta$ and covers a



distance d in $t = \sqrt{\frac{2d}{a}}$ seconds.

$$t = \sqrt{\frac{2d}{g\sin\theta}}$$

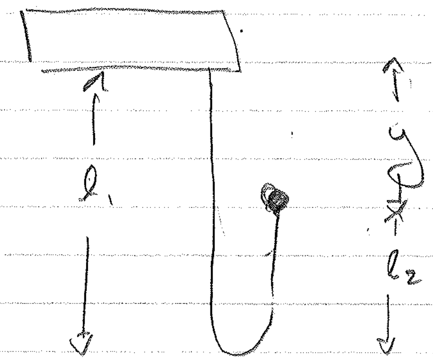
From part (4), $\mu \to 0$ $\quad t = \sqrt{\frac{2h}{g\sin^2\theta}} = \sqrt{\frac{2d}{g\sin\theta}}$

(7)

$$x = -\frac{1}{2}\left(\frac{g\sin\theta\cos\theta}{1/\mu - \cos^2\theta}\right)\left(\frac{2h(1-\mu\cos^2\theta)}{g\sin^2\theta}\right)$$

$$x = -h\cot\theta\left(\frac{1-\mu\cos^2\theta}{1/\mu - \cos^2\theta}\right)$$

PROBLEM 2



Define $\lambda = \frac{M}{L}$

$$l_1 = \left(\frac{L+y}{2}\right)$$

$$l_2 = \frac{L+y}{2} - y = \frac{L-y}{2}.$$

(a) For the rope, m_1 = mass on left, m_2 = mass right.

$$y_{cm} = \frac{1}{M}\left(m_1 \frac{l_1}{2} + m_2\left(y + \frac{l_2}{2}\right)\right)$$

$$= \frac{\lambda}{M}\left(\frac{l_1^2}{2} + l_2\left(y + \frac{l_2}{2}\right)\right)$$

$$\frac{1}{4L}\left(\frac{(L+y)^2}{2} + (L-y)\left(y + \frac{(L-y)}{2}\right)\right)$$

$$= \frac{1}{4L}\left(\frac{(L+y)^2}{2} + (L-y)\left(\frac{L+3y}{2}\right)\right)$$

$$= \frac{\lambda}{8}\left(L^2 + 2Ly + y^2 + L^2 + 2Ly - 3y^2\right)$$

$$= \frac{1}{8L}\left(2L^2 + 4Ly - 2y^2\right)$$

$$= \frac{1}{4L}\left(L^2 + 2Ly - y^2\right)$$

(b) The kinetic energy of the chain on the RHS is -

$$T = \frac{1}{2} m_2 \dot{y}^2$$

$$= \frac{1}{2} \lambda \frac{(L-y)}{2} \dot{y}^2$$

$$= \frac{1}{4} \lambda (L-y) \dot{y}^2$$

So $L = T - V$

$$= \frac{1}{4} \lambda (L-y) \dot{y}^2 + mg \cdot \frac{1}{4L} (L^2 + 2Ly - y^2)$$

$$= \frac{1}{4} \lambda (L-y) \dot{y}^2 + \frac{\lambda g}{4} (L^2 + 2Ly - y^2)$$

(c) Since L is not explicitly time dependent, the energy is a constant of the motion. Setting the initial energy equal to the final -

$$-\frac{\lambda g}{4} L^2 = \frac{\lambda}{4} (L-y) \dot{y}^2 - \frac{\lambda g}{4} (L^2 + 2Ly - y^2)$$

$$(L-y) \dot{y}^2 = g (2Ly - y^2)$$

$$\dot{y}^2 = g \frac{(2Ly - y^2)}{(L-y)}$$

$$\dot{y} = \sqrt{g\frac{(2Ly-y^2)}{L-y}}$$

(d) The acceleration can be found by differentiating $\dot{y}^2$

$$2\dot{y}\ddot{y} = g\left\{\frac{2L-2y}{(L-y)} + \frac{(2Ly-y^2)}{(L-y)^2}(-1)(-1)\right\}\dot{y}$$

$$= g\left\{2 + \frac{(2Ly-y^2)}{(L-y)^2}\right\}\dot{y}$$

$$\ddot{y} = g\left\{1 + \frac{(2Ly-y^2)}{2(L-y)^2}\right\}$$

As $y \to L$ we see that $\ddot{y}$ diverges.

(e) Each segment of the chain that is at the bottom of the curve must come to rest. It does so because it feels a force on either end of that segment that connects it to the rest of the rope. By Newton's third law, it must exert a force on those chains. The balcony supports the portion to the left, while the portion on the right feels an unbalanced force. It therefore accelerates faster than g.

Problem 3

Problem 2.

(a) Potential energy

$$U = U_{spring} + U_{block} + U_{rope}$$

"Obviously"

$$U_{spring} = \frac{1}{2} k x^2 = \frac{1}{2} k (s-a)^2$$

$$U_{block} = -mgs$$

$$U_{rope} = -(\lambda s) g \frac{s}{2} = -\frac{\lambda g s^2}{2}.$$

where the center of mass of the rope is $\frac{s}{2}$ below the edge of the table, & the amount over the edge is λs. So

$$U = \frac{1}{2} k (s-a)^2 - \frac{\lambda g s^2}{2} - mgs$$

n.b. if $\frac{\lambda g}{2} > \frac{k}{2}$ the system is unstable!

(b) Kinetic energy

$$K = \frac{1}{2}(2M)\dot{s}^2 + \frac{1}{2} m \dot{s}^2$$

$$= \frac{1}{2}(m+2M)\dot{s}^2$$

$$L = T - V$$

$$= \frac{1}{2}(m + 2M)\dot{s}^2 - \frac{1}{2}k(s-a)^2 + g\left(ms + \frac{\lambda s^2}{2}\right)$$

(c) Equation of motion from L -

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

$$= (m + 2M)\ddot{s} + k(s-a)$$

$$- mg - \lambda sg = 0.$$

(d) What is the equilibrium position?

In equilibrium $\ddot{s} = 0$

$$ks - ka - mg - \lambda gs = 0$$

$$(k - \lambda g)s = ka + mg.$$

$$S_0 = \frac{ka + mg}{k - \lambda g}.$$

Check: If $\lambda = 0$ then $S = a + \frac{mg}{k}$.

$g = 0$ then $s = a$.

(e) Frequency of small oscillations –

Re-arranging the equation of motion:

$$(m + 2M)\ddot{s} + (k - \lambda g)s - ka - mg = 0$$

Set

$$s = u + s_0$$

$$(m + 2M)\ddot{u} + (k - \lambda g)\left(u + \frac{ka + mg}{k - \lambda g}\right) - ka - mg$$

$$= (m + 2M)\ddot{u} + (k - \lambda g)u = 0$$

$$\omega = \sqrt{\frac{k - \lambda g}{m + 2M}}$$

Problem 4

#3)

Point particle of mass m constrained to the surface:

$$z = \alpha(x^2 + y^2)$$

(a) Find the equations of motion using Lagrange multipliers

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz + \lambda(t)\left(x^2 + y^2 - \frac{z}{\alpha}\right)$$

$\underline{x}$ | $\quad \frac{d}{dt} m\dot{x} - 2\lambda x = 0$

$$m\ddot{x} = 2\lambda x$$

y | $\quad \frac{d}{dt} m\dot{y} - 2\lambda y = 0$

$$m\ddot{y} = 2\lambda y$$

$\underline{z}$ | $\quad \frac{d}{dt} m\dot{z} + mg + \frac{\lambda}{\alpha} = 0$

$$m\ddot{z} = -\left(mg + \frac{\lambda}{\alpha}\right)$$

(b) Consider case where $\ddot{z} = \dot{z} = 0$, $z = z_0$.

What force is required?

From the z-equation:

$$m\ddot{z} = 0 = -\left(mg + \frac{\lambda}{\alpha}\right)$$

$$\Rightarrow \lambda = -\alpha mg.$$

$$\left.\begin{aligned}\ddot{x} &= -2\alpha g x\\ \ddot{y} &= -2\alpha g y\end{aligned}\right\}\ \text{circular motion}$$

$$m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = -2\alpha g m(x\hat{i} + y\hat{j})$$

Thus there is a force in the $-\hat{r}$ direction. The magnitude of ~~the~~ this force is

$$|m\vec{a}| = 2\alpha m g\sqrt{x^2+y^2} = 2\alpha g m\sqrt{\frac{z_0}{\alpha}}$$

$$= 2mg\sqrt{\alpha z_0}$$

This is the force pointing towards the z-axis. The total force is

$$\vec{F} = -2mg\sqrt{\alpha z_0}\,\hat{\rho} + mg\,\hat{k}$$

where $\hat{\rho}$ is the unit vector of cylindrical radius in cylindrical co-ordinates.

So this quantity is constant. Evaluating it at the maximum position, $\dot{x} = \dot{z} = 0$

$$m g z_1 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{z}^2 + m g z .$$

We need to write $\dot{x}$ in terms of z and $\dot{z}$.

$$\alpha x^2(t) = z(t) \rightarrow x = \sqrt{\frac{z}{\alpha}}$$

$$2 \alpha x \dot{x} = \dot{z}$$

These together imply

$$2 \alpha \sqrt{\frac{z}{\alpha}} \dot{x} = \dot{z}$$

$$\dot{x} = \frac{1}{2\alpha} \sqrt{\frac{\alpha}{z}} \dot{z} = \frac{1}{2} \frac{1}{\sqrt{\alpha z}} \dot{z}$$

$$\dot{x}^2 = \frac{1}{4} \frac{\dot{z}^2}{\alpha z}$$

So our energy equation is

$$m g z_1 = \frac{1}{2} m \left\{ \frac{1}{4} \frac{\dot{z}^2}{\alpha z} + \dot{z}^2 \right\} + m g z$$

Solving for $\dot{z}$

$$\dot{z}^2 \left\{ 1 + \frac{1}{4 \alpha z} \right\} = 2 g (z_1 - z)$$

$$\dot{z} = \sqrt{2g} \sqrt{\frac{z_1 - z}{1 + \frac{1}{4 \alpha z}}}$$

(c) Now set $y = \dot{y} = \ddot{y} = 0$, and assume $z_{max} = z_1$.

(i) Using conservation of energy derive an expression for $\dot{z}(z, z_1)$.

The question is unclear if we have to prove conservation of energy or not. To prove it, multiply the x-equation by $\dot{x}$ and the z equation by $\dot{z}$ and add.

$$\dot{x}\frac{d}{dt} m\dot{x} - 2\lambda x(t)\dot{x} + \dot{z}\frac{d}{dt} m\dot{z} + mg\dot{z} + \frac{\lambda(t)}{a}\dot{z} = 0$$

$$= \frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{z}^2 + mgz\right) + \lambda(t)\frac{d}{dt}\left(\frac{z}{a} - x^2\right) = 0.$$

The last term is zero on the constrained surface so integrating we have

$$\frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{z}^2 + mgz \Big|_f^i = 0.$$

As a check, as $\alpha \to \infty$, the particle moves vertically, and we recover the correct 1D result.

(ii) From this result & the equations of motion derive the force in the z-direction required to maintain the constraint.

From the equation of motion -

$$m\ddot{z} = -mg + \frac{\lambda}{\alpha}$$

But

$$m\ddot{z} = F_{z,tot} = -mg + N_z$$

where N_z is the z-component of the normal force. Therefore,

$$N_z = m\ddot{z} + mg$$

This gets a bit messy then -

$$\ddot{z} = \frac{d}{dt}\sqrt{2g}\sqrt{\frac{z_1 - z}{1 + \frac{1}{4\alpha z}}} = \frac{d}{dt} f(z(t))$$

$$= \frac{d}{dz} f(z)\,\dot{z} = \frac{df}{dt} \cdot f$$

$$\ddot{z} = 2g \frac{d}{dz}\left(\frac{z_1 - z}{1+\frac{1}{4\alpha z}}\right)^{1/2} \cdot \left(\frac{z_1 - z}{1+\frac{1}{4\alpha z}}\right)^{1/2}$$

$$= 2g \left\{ \frac{d}{dz}(z_1 - z)^{1/2} \cdot \frac{(z_1 - z)^{1/2}}{1+\frac{1}{4\alpha z}} \right.$$

$$\left. + \frac{d}{dz}\frac{1}{\left(1+\frac{1}{4\alpha z}\right)^{1/2}} \frac{(z_1 - z)}{\left(1+\frac{1}{4\alpha z}\right)^{1/2}} \right\}$$

$$= 2g \left\{ -\frac{1}{2}\frac{1}{(z_1 - z)^{1/2}} \frac{(z - z_1)^{1/2}}{1+\frac{1}{4\alpha z}} \right.$$

$$\left. -\frac{1}{2}\frac{\frac{-1}{4\alpha z^2}}{\left(1+\frac{1}{4\alpha z}\right)^{3/2}} \frac{z_1 - z}{\left(1+\frac{1}{4\alpha z}\right)^{1/2}} \right\}$$

$$= g\left\{ \frac{-1}{1+\frac{1}{4\alpha z}} + \frac{1}{4\alpha z^2}\frac{z_1 - z}{\left(1+\frac{1}{4\alpha z}\right)^2} \right\}$$

$$= g\left\{ -4\alpha z^2\left(1+\frac{1}{4\alpha z}\right) + (z_1 - z)\right\} \times \frac{1}{4\alpha z^2}\frac{1}{\left(1+\frac{1}{4\alpha z}\right)^2}$$

$$= g\left\{ -4\alpha z^2 - z + z_1 - z \right\} 4\alpha \frac{1}{(4\alpha z + 1)^2}$$

$$= 4g\alpha \frac{(z_1 - 2z - 4\alpha z^2)}{(1+4\alpha z)^2}$$

$$N_z = mg\left\{ \frac{4\alpha\,(z_1 - 2z - 4\alpha z^2)}{(1+4\alpha z)^2} + 1 \right\}.$$

N.B. if $\alpha \to \infty$

$$N_z = \left(\frac{4\alpha\,(-4\alpha z^2)}{(1+4\alpha z)^2} + 1 \right) mg.$$

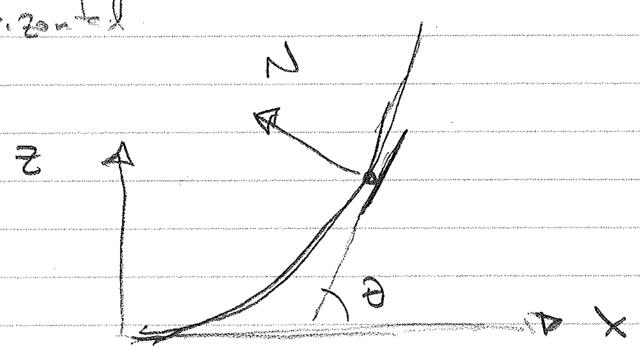
$$= \left(-\frac{(4\alpha z)^2}{(4\alpha z)^2} + 1 \right) mg = 0$$

(iii) What is the total force of contact?

This is the z-component. It is

$$N_z = N\cos\theta$$

where θ is the angle the tangent makes with the horizontal



Since $\tan\theta = \frac{dz}{dx} = 2\alpha x = 2\alpha\sqrt{\frac{z}{\alpha}}$

$$= 2\sqrt{\alpha z}$$

Then

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{4\alpha z+1}}$$

So $N = \frac{N_z}{\cos\theta} = 4\alpha g \frac{(z_1 - \partial z - 4\alpha z^2)}{(1+4\alpha z)^{3/2}}$

The x-component is:

$$N\sin\theta = N\left(1-\cos^2\theta\right)^{1/2}$$

$$= N\left(1 - \frac{1}{1+4\alpha z}\right)^{1/2}$$

$$= N\sqrt{\frac{4\alpha z}{1+4\alpha z}}$$

which I leave you to write out

Problem 5

#4) Is there a conserved quantity associated with discrete rotational symmetry using Noether's Theorem?

<u>No</u>! Noether's theorem is based on infinitesimal variations and <u>continuous</u> symmetries. It does not work with a finite symmetry.

Problem 6

#5) Given the Lagrangian

$$L = \frac{1}{2}\left(m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 \right) - V(\vec{r}_1 - \vec{r}_2)$$

(a) Show that the transformation

$$\vec{r}_i \rightarrow \vec{r}_i + \epsilon \vec{v}_0 t$$

leaves the Lagrangian unchanged up to a total differential.

$$\dot{\vec{r}}_i \rightarrow \dot{\vec{r}}_i + \epsilon \vec{V}_0$$

$$L' = \frac{1}{2}\left(m_1 (\dot{\vec{r}}_1 + \epsilon \vec{v}_0)^2 + m_2 (\dot{\vec{r}}_2 + \epsilon \vec{V}_0)^2 \right) + V(\vec{r}_1 + \cancel{\epsilon \vec{V}_0 t} - \vec{r}_2 - \cancel{\epsilon \vec{V}_0 t})$$

$$L' - L = \frac{1}{2} m_1 \left(\dot{\vec{r}}_1^2 + 2\epsilon \vec{V}_0 \cdot \dot{\vec{r}}_1 + \epsilon^2 V_0^2 \right) - \frac{1}{2} m_1 \dot{\vec{r}}_1^2$$
$$+ \frac{1}{2} m_2 \left(\dot{\vec{r}}_2^2 + 2\epsilon \vec{V}_0 \dot{\vec{r}}_2 + \epsilon^2 V_0^2 \right) - \frac{1}{2} m_1 \dot{\vec{r}}_2^2$$
$$+ V(\vec{r}_2 - \vec{r}_1) - V(\vec{r}_2 - \vec{r}_1)$$
$$= \epsilon \left\{ m_1 \vec{V}_0 \cdot \dot{\vec{r}}_1 + m_2 \vec{V}_0 \cdot \dot{\vec{r}}_2 \right\}$$
$$+ \frac{\epsilon^2}{2} V_0^2 (m_1 + m_2)$$

Neglecting terms of $O(\epsilon^2)$.

$$\frac{L' - L}{\epsilon} = \vec{V}_0 \cdot \frac{d}{dt}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2 \right).$$

The action

$$\frac{I(\epsilon) - I(0)}{\epsilon} = \cdot\, \vec{v}_0 \cdot \int dt \; \frac{d}{dt} \left(m_1 \vec{r}_1 + m_2 \vec{r}_2 \right)$$

which is the integral of a total differential.

(b) In this case we get:

$$\sum_j \eta_j \frac{\partial f}{\partial y_j'} + \xi \left(f - y' \frac{\partial f}{\partial y'} \right) - \Lambda$$

where for us.

$$\eta_j = v_{0j}\, t$$

$$\xi = 0$$

$$\Lambda = m \vec{r}_1 + m_2 \vec{r}_2$$

Our conserved quantity is

$$Q = \sum_i \left(m_i \dot{\vec{r}}_i t - m_i \vec{r}_i \right) \cdot \vec{v}_0$$

where $\vec{v}_0$ is arbitrary. Dividing through by $\sum_i m_i$

$$\vec{v}_{cm} = \frac{\sum_i m_i \dot{\vec{r}}_i}{\sum_i m_i} \qquad \vec{r}_{cm} = \frac{\sum m_i \vec{r}}{\sum m_i}$$

Our conserved charge states that

$$\vec{v}_{cm} t - \vec{r}_{cm}(t) = \vec{v}_{cm}(0)\, 0 - \vec{r}_{cm}(0)$$

$$\vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{v}_{cm}(t)\, t$$

So the center of mass moves at constant velocity.