Solutions to Homework 11
Physics 5393

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P-3.23 The wave function of a particle subjected to a spherically potential $V(r)$ is given by

$$\psi(\vec{\mathbf{x}}) = (x + y + 3z)f(r).$$

a) Is ψ an eigenfunction $\tilde{\mathbf{L}}^2$? If so, what is the l-value? If not, what are the possible values of l that we may obtain when $\tilde{\mathbf{L}}^2$ is measured?

We start by writing the wavefunction in spherical coordinates

$$\psi(\vec{\mathbf{x}}) = r[\cos\phi\sin\theta + \sin\phi\sin\theta + 3\cos\theta]f(r),$$

then in terms of the spherical harmonics

$$\psi(r,\theta,\phi) = \sqrt{\frac{8\pi}{3}}\left[\frac{Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)}{2} + \frac{Y_1^1(\theta,\phi) - Y_1^{-1}(\theta,\phi)}{2i} + \frac{3}{\sqrt{2}}Y_1^0(\theta,\phi)\right] rf(r).$$

This is clearly an eigenfunction of $\tilde{\mathbf{L}}^2$ with eigenvalue $l = 1$, but not an eigenstate of $\tilde{\mathbf{L}}_z$.

b) What are the probabilities for the particle to be found in various m_l states?

The eigenvalues of $\tilde{\mathbf{L}}_z$ and probabilities are

$$\begin{aligned} m &= -1 \quad \Rightarrow \quad \mathcal{P} = \frac{1}{11} \\ m &= 0 \quad\;\; \Rightarrow \quad \mathcal{P} = \frac{9}{11} \\ m &= 1 \quad\;\; \Rightarrow \quad \mathcal{P} = \frac{1}{11}, \end{aligned}$$

where the probabilities are calculated by calculating the magnitude square of the amplitude of each m_l term and dividing by the sum of the three to ensure proper normalization.

c) Suppose it is known somehow that $\psi(\vec{\mathbf{x}})$ is an energy eigenfunction with eigenvalues E. Indicate how we may find $V(r)$.

Since we know the wavefunction is $\psi(\vec{\mathbf{x}}) = F_\ell(\theta,\phi)rf(r)$ with $\ell = 1$ and assume that $f(r)$ is given, we can apply the radial differential equation to it and solve for $V(r)$

$$\begin{aligned} \left[-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)\right] R_{E\ell}(r) &= ER_{E\ell}(r) \\ \Rightarrow \quad V(r)rf(r) = \left[\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{\ell(\ell+1)\hbar^2}{2mr^2} + E\right] rf(r). \end{aligned}$$

P-3.24 A particle in a spherically symmetric potential is known to be in an eigenstate of $\tilde{\mathbf{L}}^3$ and $\tilde{\mathbf{L}}_z$ with eigenvalues $\ell(\ell+1)\hbar$ and $m\hbar$, respectively. Prove that the expectation values between $|\ell, m\rangle$ states satisfy:

$$\left\langle \tilde{\mathbf{L}}_x \right\rangle = \left\langle \tilde{\mathbf{L}}_y \right\rangle = 0, \quad \left\langle \tilde{\mathbf{L}}_x^2 \right\rangle = \left\langle \tilde{\mathbf{L}}_y^2 \right\rangle = \frac{\ell(\ell+1)\hbar - m^2\hbar}{2}.$$

Interpret this result semiclassically.

The most straightforward method of attacking this problem is to express the angular momentum operators in terms of the ladder operators

$$\left.\begin{aligned}\tilde{\mathbf{L}}_x &= \frac{1}{2}(\tilde{\mathbf{L}}_+ + \tilde{\mathbf{L}}_-)\\ \tilde{\mathbf{L}}_y &= \frac{i}{2}(\tilde{\mathbf{L}}_+ - \tilde{\mathbf{L}}_-)\end{aligned}\right\} \quad \Rightarrow \quad \left\{\begin{aligned}\tilde{\mathbf{L}}_x^2 &= \frac{1}{4}(\tilde{\mathbf{L}}_+^2 + \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- + \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ + \tilde{\mathbf{L}}_-^2)\\ \tilde{\mathbf{L}}_y^2 &= -\frac{1}{4}(\tilde{\mathbf{L}}_+^2 - \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- - \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ + \tilde{\mathbf{L}}_-^2).\end{aligned}\right.$$

The expectation value of the linear operators is

$$\begin{aligned}\left\langle \tilde{\mathbf{L}}_x \right\rangle &= \frac{1}{2}\left[\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle + \left\langle \ell, m \middle| \tilde{\mathbf{L}}_- \middle| \ell, m\right\rangle\right] = 0\\ \left\langle \tilde{\mathbf{L}}_- \right\rangle &= \frac{i}{2}\left[\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle - \left\langle \ell, m \middle| \tilde{\mathbf{L}}_- \middle| \ell, m\right\rangle\right] = 0,\end{aligned}$$

where the relation $\tilde{\mathbf{L}}_\pm \left|\ell, m\right\rangle \propto \left|\ell, m \pm 1\right\rangle$ and the orthogonality of the eigenstates are used.

The expectation value of the squared operators is derived as follows

$$\begin{aligned}\left\langle \tilde{\mathbf{L}}_x^2 \right\rangle &= \frac{1}{4}\left[\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+^2 \middle| \ell, m\right\rangle + \left\langle \ell, m \middle| \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- \middle| \ell, m\right\rangle + \left\langle \ell, m \middle| \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle + \left\langle \ell, m \middle| \tilde{\mathbf{L}}_-^2 \middle| \ell, m\right\rangle\right]\\ &= \frac{1}{4}\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- + \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle\\ &= \frac{1}{2}\left\langle \ell, m \middle| \tilde{\mathbf{L}}^2 - \tilde{\mathbf{L}}_z^2 \middle| \ell, m\right\rangle = \frac{1}{2}\left\{\ell(\ell+1) + m^2\right\}\hbar^2\end{aligned}$$

$$\begin{aligned}\left\langle \tilde{\mathbf{L}}_y^2 \right\rangle &= -\frac{1}{4}\left[\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+^2 \middle| \ell, m\right\rangle - \left\langle \ell, m \middle| \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- \middle| \ell, m\right\rangle - \left\langle \ell, m \middle| \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle + \left\langle \ell, m \middle| \tilde{\mathbf{L}}_-^2 \middle| \ell, m\right\rangle\right]\\ &= \frac{1}{4}\left\langle \ell, m \middle| \tilde{\mathbf{L}}_+\tilde{\mathbf{L}}_- + \tilde{\mathbf{L}}_-\tilde{\mathbf{L}}_+ \middle| \ell, m\right\rangle\\ &= \frac{1}{2}\left\langle \ell, m \middle| \tilde{\mathbf{L}}^2 - \tilde{\mathbf{L}}_z^2 \middle| \ell, m\right\rangle = \frac{1}{2}\left\{\ell(\ell+1) + m^2\right\}\hbar^2,\end{aligned}$$

where $\left\langle \tilde{\mathbf{L}}_\pm \right\rangle = 0$ was used.

P-3.25 Suppose a half integer l-value, say $1/2$, were allowed for orbital angular momentum. From

$$\tilde{\mathbf{L}}_+ Y_{1/2}^{1/2}(\theta, \phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2}^{1/2}(\theta, \phi) \propto e^{i\phi/2}\sqrt{\sin\theta}.$$

Now try to construct $Y_{1/2}^{-1/2}(\theta, \phi)$ by (a) applying $\tilde{\mathbf{L}}_-$ to $Y_{1/2}^{1/2}(\theta, \phi)$; and (b) using $\tilde{\mathbf{L}}_- Y_{1/2}^{-1/2}(\theta, \phi) = 0$. Show that the two procedures lead to contradictory results.

We start with

$$\tilde{\mathbf{L}}_- Y_{1/2}^{1/2}(\theta, \phi) = \sqrt{(s + s_z)(s - s_z + 1)}\,\hbar Y_{1/2}^{-1/2}(\theta, \phi) = \hbar Y_{1/2}^{-1/2}(\theta, \phi).$$

We then apply the operator $\tilde{\mathbf{L}}_-$ in the position representation

$$-ie^{-i\phi}\hbar\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)ce^{i\phi/2}\sqrt{\sin\theta}=-c\hbar e^{-i\phi/2}\cot\theta\sqrt{\sin\theta}.$$

If we apply the lowering operator on this result, we find

$$-ie^{-i\phi}\hbar\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)c\hbar e^{-i\phi/2}\cot\theta\sqrt{\sin\theta}=c\hbar^2\frac{e^{-3i\phi/2}}{2\sqrt{\sin^3\theta}}[\cos\theta-2\sin^2\theta-\cos^2\theta]\neq 0$$

which does not equal zero in general. Furthermore, solving the differential equation directly, we find

$$-ie^{-i\phi}\hbar\left(-i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right)Y_{1/2}^{-1/2}(\theta,\phi)=0 \quad\Rightarrow\quad Y_{1/2}^{1/2}(\theta,\phi)=ce^{-i\phi/2}\sqrt{\sin\theta}.$$

Combined, these equations give contradictions.

P-3.26 Consider an orbital angular momentum eigenstate $|l=2,m=0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m=0$ ±1, and ±2.

It is best to use the Euler angles to calculate the rotation operator

$$\tilde{\boldsymbol{\mathcal{D}}}(\beta)\ |2,0\rangle \text{ where } \tilde{\boldsymbol{\mathcal{D}}}(\beta)\equiv\tilde{\boldsymbol{\mathcal{D}}}(\alpha=0,\beta,\gamma=0).$$

Next we expand this in a complete set

$$\tilde{\boldsymbol{\mathcal{D}}}(\beta)\ |2,0\rangle=\sum_{m'}\left|2,m'\right\rangle\left\langle 2,m'\left|\tilde{\boldsymbol{\mathcal{D}}}(\beta)\right|2,0\right\rangle=\sum_{m'}\left|2,m'\right\rangle\tilde{\boldsymbol{\mathcal{D}}}^{(l)}_{m',0}(\beta).$$

Since the rotation is relative to the z-axis, $\tilde{\boldsymbol{\mathcal{D}}}^{(l)}_{m',0}(\beta)$ is the rotation operator derive in Eq. 3.6.52. Therefore,

$$\tilde{\boldsymbol{\mathcal{D}}}(\beta)\ |2,0\rangle=\sum_{m'}|2,m\rangle\sqrt{\frac{4\pi}{5}}\ Y_l^{m'*}(\beta,0),$$

and the probabilities are

$$\begin{aligned}
\left|\left\langle 2,m\left|\tilde{\boldsymbol{\mathcal{D}}}(\beta)\right|2,0\right\rangle\right|^2 &= \frac{4\pi}{5}\left|Y_l^{m'}(\beta,0)\right|^2\\
\left|\left\langle 2,\pm2\left|\tilde{\boldsymbol{\mathcal{D}}}(\beta)\right|2,0\right\rangle\right|^2 &= \frac{3}{8}\sin^4\beta\\
\left|\left\langle 2,\pm1\left|\tilde{\boldsymbol{\mathcal{D}}}(\beta)\right|2,0\right\rangle\right|^2 &= \frac{3}{2}\sin^2\beta\cos^2\beta\\
\left|\left\langle 2,0\left|\tilde{\boldsymbol{\mathcal{D}}}(\beta)\right|2,0\right\rangle\right|^2 &= \frac{1}{4}(3\cos^2\beta-1)^2.
\end{aligned}$$