PHYS5153 Assignment 6

Due: 6:00pm on 10/15/2021. *Any changes to the due date will be announced on Canvas.*
Marking: Total of 10 marks (weighting of each question is indicated).
Fine print: Solutions should be presented legibly (handwritten or LaTeX is equally acceptable) so that the grader can follow your line of thinking and any mathematical working should be appropriately explained/described. If you provide only equations you will be marked zero. If you provide equations that are completely wrong but can demonstrate some accompanying logical reasoning then you increase your chances of receiving more than zero. If any of your solution has relied on a reference or material other than the textbook or lectures, please note this and provide details.

Question 1 (3 marks)

A particle of mass m moves in a spherically symmetric potential,

$$V(r) = -\mathcal{C}\frac{e^{-\alpha r}}{r}, \tag{1}$$

with $\mathcal{C}, \alpha > 0$.

(a) Derive an *effective* one-dimensional potential that governs the qualitative motion of the particle. Sketch your potential.

(b) Using your solution to (a), discuss and classify the expected motion of the particle as a function of the initial energy, E_0.

Question 2 (3 marks)

A typical potential describing the interaction between particles is very complicated and the analytic computation of scattering cross sections is difficult. However, we can often gain qualitative insight by approximating the precise form of the potential.

One common treatment of intermolecular scattering is the Lennard-Jones potential (which is in itself already an approximation):

$$V_{\mathrm{LJ}} = \mathcal{V}_0\left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right]. \tag{2}$$

It features a divergent repulsive contribution at short ranges ($r \ll a$) and a long attractive tail ($r \gg a$), with the minimum of the potential occurring at $r = 2^{1/6}a$.

(a) When the short-range repulsive contribution is expected to dominate the physics we can adopt a so-called "hard-core" interaction,

$$V_{\mathrm{hc}}(r) = \begin{cases} \infty & \text{if } r \le a_0 \\ 0 & \text{if } r > a_0 \end{cases}, \tag{3}$$

where a_0 characterizes the length scale of the interaction ($a_0 \neq a$). This potential is equivalent to the scattering of a particle of an impenetrable sphere. Compute the scattering cross section $\sigma(\Theta)$ and total cross section σ_T for this potential. Discuss and interpret both results.

(b) When the attractive part of the interaction is important we instead adopt a so-called "soft-core" interaction,

$$V_{\text{sc}}(r) = \begin{cases} -V_0 & \text{if } r \leq a_0 \\ 0 & \text{if } r > a_0 \end{cases}. \tag{4}$$

Show that the impact parameter associated with this potential can be written as,

$$s = \frac{a \sin(\Theta/2)}{\sqrt{1 + \frac{1}{n^2} - \frac{2}{n}\cos(\Theta/2)}} \tag{5}$$

where $n = \sqrt{1 + V_0/E}$ and E is the energy of the incident particle. Hint: You might find it useful to consider energy and momentum conservation, and consider the associated expressions for $r \leq a_0$ and $r > a_0$ separately.

Question 3 (3 marks)

Consider central force field of the form $F(r) = k/r^3$.

(a) Using the formula,

$$\Theta(s, E) = \pi - 2\int_0^{u_{\max}} \frac{s du}{\sqrt{1 - \frac{V(u)}{E} - s^2u^2}}, \tag{6}$$

with $u = 1/r$, show that,

$$\Theta(s, E) = \pi\left[1 - \frac{s\sqrt{2E}}{\sqrt{k + 2Es^2}}\right]. \tag{7}$$

In principle, one can obtain the scattering cross section from Eq. (7). However, we shall pursue a different route for this example. The motion of a particle in the central force can be written in terms of $u = 1/r$ as,

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dV(u)}{du}. \tag{8}$$

(b) Show that a parametrization of the motion is,

$$\begin{aligned} u(\theta) &= \alpha\cos(\gamma\theta) + \beta\sin(\gamma\theta), \\ \gamma &= \sqrt{1 + \frac{mk}{l^2}}. \end{aligned} \tag{9}$$

(c) Use that the particle approaches from an initial angle $\theta_i = \pi$ to show that: i) $\alpha = -\beta\tan(\gamma\pi)$ and ii) $\gamma = \pi/(\Theta - \pi)$.

(d) Finally, defining $x = \Theta/\pi$, show that the cross section can be obtained as,

$$\sigma(\Theta)d\Theta = \frac{k}{E}\frac{(1-x)dx}{x^2(2-x)^2\sin(\pi x)}. \tag{10}$$

Question 4 (1 marks)

Read Secs. I and II of the article "Elastic scattering by a paraboloid of revolution" by Evan James at https://doi.org/10.1119/1.15593 [Am. J. Phys. **56**, 423 (1988)]. Write a brief summary of the article, discussing the main conclusions and results of the work.