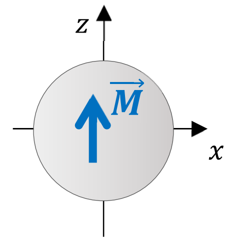
E & M I
Workshop 12 – Polarized Spheres, 4/27/2022



1) Magnetic Sphere

A ferromagnetic sphere of radius R has a constant magnet moment $\vec{M}(\vec{r}) = M_0\,\hat{z}$. In this problem, we'll find the fields inside and outside of the sphere.

The properties of magnetic fields and magnetic materials are given by:

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}, \qquad \vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_{free} + \vec{J}_m), \quad \vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{B} = \mu_0\,(\vec{H} + \vec{M}),$$

$$\vec{B} = \mu\,\vec{H}, \qquad \vec{J}_m = \vec{\nabla} \times \vec{M}, \qquad \vec{K}_m = \hat{n} \times \vec{M}$$

$$\vec{B}_{\perp,in} = \vec{B}_{\perp,out}, \qquad \vec{H}_{\parallel,in} - \vec{H}_{\parallel,out} = I_{free}$$

$\vec{H}$ is created by free currents, $\vec{B}$ by free and "bound" currents from macroscopic magnetic moments, and the fields must also satisfy the boundary conditions at the surface of the materials.

Today we'll approach this problem by solving for $\vec{B}$.

A) There are no free currents, but there are magnetic "bound currents". Using the definitions above, determine the currents:

$$\vec{J}_m = \vec{\nabla} \times \vec{M}, \qquad \vec{K}_m = \hat{n} \times \vec{M}$$

Show that $\vec{J}_m = 0$ and that $\vec{K}_m$, the magnetic current on the surface of the sphere has a simple form.

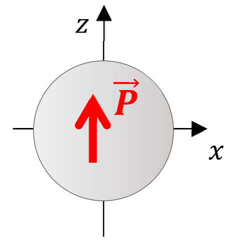
B) In fact, considering $\vec{K}_m$ as a surface current, we have already solved this problem. Where have we done this before? (If you're not sure, ask.)

C) Considering our previous solutions, determine the fields $\vec{B}$ and $\vec{H}$ both inside and outside of the sphere. You don't need to do much to solve the fields except make comparisons with previous solutions. Explain the physics of the magnetic field $\vec{B}$ outside the sphere considering that $\vec{M}$ is a macroscopic magnetic dipole field.

D) To illustrate your results, draw two spheres, one with a sketch of $\vec{B}$ the other with a sketch of $\vec{H}$.

E) As another check on the answer, make sure your results give the correct boundary conditions for $\vec{B}$ and $\vec{H}$ everywhere on the surface of the sphere.

2) Electric Dipole Sphere



A "ferroelectric" sphere of radius R has a constant polarization field $\vec{P}(\vec{r}) = P_0\,\hat{z}$. In this problem, we'll find the fields inside and outside of the sphere.

The properties of electric fields and electric materials are given by:

$$\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{D} = \rho_{free}, \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\left(\rho_{free} + \rho_{bound}\right), \qquad \vec{D} = \epsilon_0 \vec{E} + \vec{P},$$

$$\vec{D} = \epsilon\,\vec{E}, \qquad \rho_b = -\vec{\nabla} \cdot \vec{P}, \qquad \sigma_b = \hat{n} \cdot \vec{P}\,, \qquad \vec{D}_{\perp,in} - \vec{D}_{\perp,out} = \sigma_{free}, \qquad \vec{E}_{\parallel,in} = \vec{E}_{\parallel,out}$$

$\vec{D}$ is created by free charges, $\vec{E}$ by free and "bound" charges from macroscopic dipole moments, and the fields must also satisfy the boundary conditions. We'll solve for $\vec{E}$ using the bound charges. (This is another case of a problem very similar to one already done.)

A) There are no free charges, but there are "bound charges". Using the definitions above, determine the bound charges:

$$\rho_b = \vec{\nabla} \cdot \vec{P}, \qquad \sigma_b = \hat{n} \cdot \vec{P}$$

Show the bound volume and surface charge densities are very similar to the results above.

B) Using the bound surface charge, write an integral that can be solved for the potential $\phi(\vec{r})$. This should be an application of Coulomb's law.

C) Solve your integral from (B). A couple of hints that will do most of the work:

$$\frac{\cos(\theta')}{|\vec{r} - \vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\, P_l(\hat{r}' \cdot \hat{r}) \cos(\hat{r}' \cdot \hat{r})$$

$$\iint d\Omega'\; P_l(\hat{r}' \cdot \hat{r}_1)\, P_m(\hat{r}' \cdot \hat{r}_2) = \frac{4\pi}{2\,l + 1}\, P_l(\hat{r}_1 \cdot \hat{r}_2)\, \delta_{lm}$$

Where: $r_<$ is the smaller of r and r', $r_>$ is the smaller of r and r', and the 2D integral over $d\Omega'$ is over the solid angle of $\vec{r}'$, or $d\Omega' = \sin\theta'\ d\theta'\ d\phi'$.

D) Solve for the fields $\vec{E}$ and $\vec{D}$ both outside and inside the sphere. You'll get something very much like the results from Problem 1. Show that, again, the boundary conditions are met.

E) Draw pictures of the $\vec{E}$ and $\vec{D}$ fields. How do these compare with the fields from Problem 1?

3) Magnetic Sphere Again:

Interestingly, Problem 1 has exactly the same solution as Problem 2 using the concepts of the "Magnetic Scalar Potential" and "Magnetic Bound Charges" given by $\rho_m = -\vec{\nabla} \cdot \vec{M}, \sigma_m = \hat{n} \cdot \vec{M}$. These arise from using the relations for a region with no free currents:

$$\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\vec{\nabla}\,\psi_m, \qquad \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{H} + \vec{M}\right) = 0 \Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$