COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #11

PROBLEMS: $\{1, 2, 3, 4\}$

Due: December 8, 2021 By 6:00 PM

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Robert Lewis-Swan

Problem 1:

Consider a harmonic oscillator with the Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \tag{1}$$

(a) Use the Hamilton-Jacobi equation to show that Hamilton's principal function for this problem is,

$$S(q,\alpha,t) = \frac{m\omega}{2}(q^2+\alpha^2)\cot(\omega t) - m\omega\alpha q\csc(\omega t). \tag{2}$$

We start off with the Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

Since $\partial H/\partial t = 0$, we can proceed to use,

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \quad \text{w/} \quad S(q,\alpha,t) = W(q,\alpha) - \alpha t$$

We will sub in the principal function to show it is correct,

$$\frac{\partial S}{\partial q} = m\omega q\cot(\omega t) - m\omega\alpha\csc(\omega t) \quad , \quad \left(\frac{\partial S}{\partial q}\right)^2 = m^2\omega^2\left(q^2\cot^2(\omega t) + \alpha^2\csc^2(\omega t) - 2\alpha q\cot(\omega t)\csc(\omega t)\right)$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \frac{1}{2m}\left[m^2\omega^2\left(q^2\cot^2(\omega t) + \alpha^2\csc^2(\omega t) - 2\alpha q\cot(\omega t)\csc(\omega t)\right) + m^2\omega^2 q^2\right]$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \frac{m\omega^2}{2}\left[\left(q^2\cot^2(\omega t) + \alpha^2\csc^2(\omega t) - 2\alpha q\cot(\omega t)\csc(\omega t)\right) + q^2\right]$$

$$\frac{\partial S}{\partial t} = -\frac{m\omega^2}{2}(q^2+\alpha^2)\csc^2(\omega t) + m\omega^2\alpha q\cot(\omega t)\csc(\omega t)$$

Putting these results into our Hamilton Jacobi equation we get:

$$\frac{m\omega^2 q^2}{2}\cot^2(\omega t) + -\frac{m\omega^2 q^2}{2}\csc^2(\omega t) + \frac{m\omega^2 q^2}{2} = \frac{m\omega^2 q^2}{2}\left(\cot^2(\omega t) - \csc^2(\omega t)\right) + \frac{m\omega^2 q^2}{2}$$

Using the common trig identity we can deduce our equation simplifies to,

$$-\frac{m\omega^2 q^2}{2} + \frac{m\omega^2 q^2}{2} = 0 \quad \therefore \quad \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0 \ \checkmark$$

The principal function satisfies the Hamilton-Jacobi equation

$\blacksquare$

Problem 1: Continued

(b) Demonstrate that $S(q, \alpha, t)$ furnishes the expected solution for the dynamics of the oscillator.

we want to find p and q in terms of α and β,

$$\beta = \frac{\partial S}{\partial \alpha} = m\omega\alpha\cot(\omega t) - m\omega q\csc(\omega t) \quad \therefore \quad q = \alpha\cos(\omega t) - \frac{\beta}{m\omega}\sin(\omega t)$$

we then want to find p in terms of α and β,

$$p = \frac{\partial S}{\partial q} = m\omega q\cot(\omega t) - m\omega\alpha\csc(\omega t)$$

Substituting q into the above,

$$P = m\omega\frac{\cos(\omega t)}{\sin(\omega t)}\left(\alpha\cos(\omega t) - \frac{\beta}{m\omega}\sin(\omega t)\right) - m\omega\alpha\frac{1}{\sin(\omega t)}$$

$$P = m\omega\alpha\frac{\cos^2(\omega t)}{\sin(\omega t)} - \beta\cos(\omega t) - m\omega\alpha\frac{1}{\sin(\omega t)} = \frac{m\omega\alpha}{\sin(\omega t)}(\cos^2(\omega t) - 1) - \beta\cos(\omega t)$$

Simplifying we get,

$$P = -m\omega\alpha\sin(\omega t) - \beta\cos(\omega t)$$

Then we Finally have,

$$\boxed{q = \alpha\cos(\omega t) - \frac{\beta}{m\omega}\sin(\omega t), \quad P = -m\omega\alpha\sin(\omega t) - \beta\cos(\omega t)}$$

Problem 1: Review

Procedure:

- Plug the principal function into the Hamilton Jacobi equation

$$\left[\left(\frac{\partial S}{\partial q}\right)^2 + V(q)\right] + \frac{\partial S}{\partial t} = 0$$

  and show that the Hamilton Jacobi equation is satisfied.

- Use the principal function in the below equations

$$\beta = \frac{\partial S}{\partial \alpha} \quad , \quad p = \frac{\partial S}{\partial q}$$

  and solve for p and q in terms of α and β.

Key Concepts:

- We can show that a principal function is valid if it satisfies the Hamilton Jacobi equation.
- Since p and q have sinusoidal terms it demonstrates expected solution for that of an oscillator.

Variations:

- We can be given a different Hamiltonian and principal function.
  - We then would use the some equations and procedures to show the results.

Problem 2:

Consider a projectile whose motion is restricted to a 2D plane (defined by x and y co-ordinates). Use the Hamilton-Jacobi equation to confirm that the path of the projectile under the influence of gravity is:

$$\begin{aligned} x(t) &= v_x t + x(0), \\ y(t) &= -\frac{1}{2}gt^2 + v_y t + y(0). \end{aligned} \tag{3}$$

Make sure that your solution includes an explicit form for the Hamilton principal function.

The Lagrangian for our system is,

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

So the Hamiltonian is then,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad \therefore \quad H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy$$

Since the Hamiltonian is conserved we can use,

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0$$

where we have the characteristic function equal to

$$S = S_x(x) + S_y(y) - Et$$

We then write E in terms of x and y,

$$E = \alpha_x + \alpha_y$$

So the characteristic function is then,

$$S = S_x(x) + S_y(y) - \alpha_x t - \alpha_y t \ , \quad \frac{\partial S}{\partial t} = -\alpha_x - \alpha_y$$

Because our Hamilton-Jacobi equation is separable we get the equations

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 = \alpha_x \ , \quad \frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + mgy = \alpha_y \quad w/ \quad \frac{\partial W}{\partial x,y} = \frac{\partial S}{\partial x,y}$$

We now solve these differential equations

$$\frac{\partial W}{\partial x} = \sqrt{2m\alpha_x} \quad \therefore \quad W_x = \sqrt{2m\alpha_x}\, x$$

Problem 2: Continued

The principal function is then,

$$S = W_x - \alpha_x t$$

we then use,

$$\beta = \frac{\partial S}{\partial \alpha} \quad , \quad \beta = \sqrt{2m}\,\frac{1}{2}\alpha_x^{-1/2} x - t = \frac{1}{2}\sqrt{\frac{2m}{\alpha_x}}\,x - t$$

Solving for x we get,

$$x = \underbrace{2\sqrt{\frac{\alpha_x}{2m}}\,\beta}_{x(0)} + \underbrace{2\sqrt{\frac{\alpha_x}{2m}}}_{V_x}\,t \quad \therefore \quad x(t) = x(0) + V_x t$$

Solving for y we now have,

$$\frac{\partial W}{\partial y} = \sqrt{2m}\sqrt{\alpha_y - mgy} \quad \therefore \quad \frac{\partial W}{\partial y} = \sqrt{2m\alpha_y}\sqrt{1 - mgy/\alpha_y}$$

$$W_y = \sqrt{2m\alpha_y}\int\sqrt{1 - mgy/\alpha_y}\,dy \quad , \quad 1 - mgy/\alpha_y = \gamma \quad \therefore \quad dy = -\frac{\alpha_y}{mg}\,d\gamma$$

$$W_y = -\frac{\alpha_y}{g}\sqrt{\frac{2\alpha_y}{m}}\int \gamma^{1/2}\,d\gamma = -\sqrt{\frac{2\alpha_y^3}{mg^2}}\,\frac{2}{3}\gamma^{3/2} = -\frac{2}{3}\sqrt{\frac{2\alpha_y^3}{mg^2}}\,(1 - mgy/\alpha_y)^{3/2}$$

$$W_y = -\frac{2}{3}\sqrt{\frac{2}{mg^2}}\,(\alpha_y - mgy)^{3/2} \quad , \quad S = W_y - \alpha_y t$$

We now solve for β,

$$\beta = \frac{\partial S}{\partial \alpha_y} = -\frac{1}{g}\sqrt{\frac{2}{m}}\sqrt{\alpha_y - mgy} - t \Rightarrow \sqrt{\frac{m}{2}}\beta g = -\sqrt{\alpha_y - mgy} - \sqrt{\frac{m}{2}}gt$$

$$-\sqrt{\alpha_y - mgy} = \sqrt{\frac{m}{2}}gt + \sqrt{\frac{m}{2}}\beta g = \sqrt{\frac{m}{2}}g(t+\beta) \quad \therefore \quad \alpha_y - mgy = \frac{m}{2}g^2(t^2 + 2\beta t + \beta^2)$$

$$y = \frac{\alpha_y}{mg} - \frac{g}{2}t^2 - g\beta t - \frac{g}{2}\beta^2 \quad \therefore \quad y = \underbrace{\frac{\alpha_y}{mg} - \frac{g}{2}\beta^2}_{y(0)} \underbrace{-g\beta}_{V_y} t - \frac{g}{2}t^2$$

Therefore we can say,

$$\boxed{x(t) = x(0) + V_x t \quad , \quad y(t) = y(0) + V_y t - \frac{g}{2}t^2}$$

Problem 2: Review

Procedure:

- Write out the Lagrangian for this system.
- Calculate the canonical momenta and the Hamiltonian for this system.
- Use the Hamilton Jacobi equation
$$\left[\left(\frac{\partial S}{\partial q}\right)^2 + V(q)\right] + \frac{\partial S}{\partial t} = 0.$$
- Use the equation for β and rearrange the results to come up with equations in (3).

Key Concepts:

- Using the Hamilton Jacobi equation we can solve for the kinematic equations of motion.

Variations:

- Since this problem has us solving for equations of motion it cannot be changed without creating an entirely new problem.

Problem 3:

Consider a particle of mass m in a one-dimensional box potential,

$$V(x) = \begin{cases} V_0 & \text{if } |x| \leq a_0 \\ \infty & \text{if } x > a_0. \end{cases} \tag{4}$$

Use the action-angle formalism to compute the frequency of the particle's periodic motion. Make sure your expression is given in terms of the particle's initial condition. Is your answer consistent with your expectations.

we first start off with writing out our Lagrangian

$$L = \frac{m}{2}\dot{x}^2 - V_0$$

we then find our canonical momenta

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \therefore \quad p_x = m\dot{x}$$

we can then write out our Hamiltonian:

$$H = \frac{p_x^2}{2m} + V_0$$

we know since energ is conserved we can write

$$p_x = \pm\sqrt{2m(E-V_0)}$$

we now integrate p_x to find our action angle variable with

$$J_x = \int p_x \, dx = \int_{-a}^{a} p_x \, dx - \int_{a}^{-a} p_x \, dx = 2\int_{-a}^{a} p_x \, dx = 2p_x(2a) = 4ap_x$$

This means our canonical momenta in terms of our action-angle variable is:

$$p_x = \frac{J_x}{4a}$$

This means our Hamiltonian is then,

$$H = \frac{J_x^2}{32ma^2} + V_0$$

The frequency is calculated by

$$\nu_x = \frac{\partial H}{\partial J_x} = \frac{J_x}{16ma^2} \quad \text{w/} \quad J_x = 4a\sqrt{2m(E-V_0)} \quad \therefore \quad \nu_x = \frac{\sqrt{2m(E-V_0)}}{4ma} = \sqrt{\frac{E-V_0}{8ma^2}} \quad a_0 \equiv a$$

$$\boxed{\nu_x = \sqrt{\frac{E-V_0}{8ma^2}}}$$

Problem 3: Continued

Our particle will constantly oscillate between the two walls $-a$ and a with a constant momentum due to our potential also being constant.

As our potential drops, the frequency increases. As the potential increases, the frequency will decrease as well. If the value of a_0 increases, the frequency will decrease and vice versa.

With the answer I have derived it makes sense physically.

Problem 3: Review

Procedure:

- Calculate the Lagrangian for this system.
- Calculate the canonical momenta and the Hamiltonian for this system.
- Proceed to solve for the canonical momenta in this system from the Hamiltonian.
- Calculate the action angle variable with
$$J_{q_i} = \int P_{q_i}\, dq_i.$$
- Proceed to plug this action angle variable back into the Hamiltonian.
- Calculate the frequency with
$$\nu = \frac{\partial H}{\partial J_{q_i}}.$$

Key Concepts:

- We can calculate the frequency of this motion with the use of action angle formalism.
- The bounds on our integral are because of the motion of our particle from one wall to another back to the same wall.

Variations:

- We can be given a different system to solve for.
  - We then would use the same formalism and solve for the frequency.
- We would be asked to solve for the period of this system.
  - We would then calculate $1/\nu$.

Problem 4:

Consider a particle of mass m that is constrained to move along a wire parameterized by

$$x(\phi) = l[\phi + \sin(\phi)],$$
$$y(\phi) = l[1 - \cos(\phi)], \tag{5}$$

where l is taken to be a constant. The particle is subject to gravity along y (pointing down).

(a) Compute a Lagrangian describing the particle's motion.

Our generalized co-ordinate is φ and the Lagrangian is:

$$L = T - U$$

We first start by writing out the kinetic energy:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \quad w/ \quad \dot{x} = l\dot{\varphi}[1+\cos(\varphi)] \quad \text{and} \quad \dot{y} = l\dot{\varphi}\sin(\varphi)$$

$$\dot{x}^2 = l^2\dot{\varphi}^2(1 + 2\cos(\varphi) + \cos^2(\varphi)), \quad \dot{y}^2 = l^2\dot{\varphi}^2\sin^2(\varphi) \quad \therefore \quad \dot{x}^2 + \dot{y}^2 = 2l^2\dot{\varphi}^2(1+\cos(\varphi))$$

$$T = ml^2\dot{\varphi}^2(1+\cos(\varphi))$$

The potential energy is then:

$$U = mgh \quad w/ \quad h = l[1-\cos(\varphi)] \quad \therefore \quad U = mgl[1-\cos(\varphi)]$$

Therefore our Lagrangian is:

$$\boxed{L = ml^2\dot{\varphi}^2(1+\cos(\varphi)) - mgl(1-\cos(\varphi))}$$

(b) Use your answer to (a) to compute the associated Hamiltonian. Justify that the Hamiltonian is equal to the energy and is conserved (but `do not` assume $H = T + V$ to initially compute the Hamiltonian).

We calculate the Hamiltonian with,

$$H = \sum_i \dot{\varphi}_i P_{\varphi_i} - L$$

We calculate the canonical momenta with,

$$P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = 2ml^2\dot{\varphi}(1+\cos(\varphi)) \quad \therefore \quad P_\varphi = 2ml^2\dot{\varphi}(1+\cos(\varphi))$$

We then calculate the Hamiltonian,

$$H = 2ml^2\dot{\varphi}^2(1+\cos(\varphi)) - ml^2\dot{\varphi}^2(1+\cos(\varphi)) + mgl(1-\cos(\varphi)) = ml^2\dot{\varphi}^2(1+\cos(\varphi)) + mgl(1-\cos(\varphi))$$

Therefore our Hamiltonian is:

$$H = ml^2\dot{\varphi}^2(1+\cos(\varphi)) + mgl(1-\cos(\varphi))$$

Problem 4: Continued

But, we need to eliminate $\dot{\varphi}$ from our Hamiltonian,

$$\dot{\varphi} = \frac{p_\varphi}{2ml^2(1+\cos(\varphi))} \quad , \quad \dot{\varphi}^2 = \frac{p_\varphi^2}{4m^2l^4(1+\cos(\varphi))^2}$$

We now put this into our prior equation for our Hamiltonian,

$$H = \frac{p_\varphi^2}{4ml^2(1+\cos(\varphi))} + mgl(1-\cos(\varphi))$$

Therefore our Hamiltonian is,

$$\boxed{H = \frac{p_\varphi^2}{4ml^2}(1+\cos(\varphi))^{-1} + mgl(1-\cos(\varphi))}$$

Because $\frac{\partial H}{\partial t} = 0$, our Hamiltonian is conserved and is therefore our energy

(c) Obtain an expression for the canonical momentum p_ϕ in terms of ϕ and the energy E.

We now take our energy function and solve for canonical momenta p_φ,

$$E - mgl(1-\cos(\varphi)) = \frac{p_\varphi^2}{4ml^2}(1+\cos(\varphi))^{-1} \quad , \quad 4ml^2(E - mgl(1-\cos(\varphi))) = p_\varphi^2(1+\cos(\varphi))^{-1}$$

$$p_\varphi^2 = 4ml^2(1+\cos(\varphi))(E - mgl(1-\cos(\varphi))) \quad , \quad p_\varphi = 2l\sqrt{m(1+\cos(\varphi))(E-mgl(1-\cos(\varphi)))}$$

Our canonical momenta is thus,

$$\boxed{p_\varphi = 2l\sqrt{m(1+\cos(\varphi))(E-mgl(1-\cos(\varphi)))}}$$

(d) The motion described by (a) and (b) is periodic. Use action-angle variables to show that the frequency of the motion is identical for any initial condition with $\phi \leq \pi$. Hints: You might find the substitution $u = \sqrt{2mgl/E}\sin(\phi/2)$ handy to solve an integral that arises.

We use the trig identity:

$$\cos(\varphi) = 1 - 2\sin^2(\varphi/2) \quad , \quad \cos(\varphi) = 2\cos^2(\varphi/2) - 1$$

Our canonical momenta becomes,

$$p_\varphi = 2l\sqrt{m(2\cos^2(\varphi/2))(E - 2mgl\sin^2(\varphi/2))} = 2l\cos(\varphi/2)\sqrt{2m(E - 2mgl\sin^2(\varphi/2))}$$

$$p_\varphi = 2l\cos(\varphi/2)\sqrt{2mE(1 - 2mgl/E\sin^2(\varphi/2))} \quad w/ \quad u = \sqrt{\frac{2mgl}{E}}\sin(\varphi/2)$$

Problem 4: Continued

$$du = \frac{1}{2}\sqrt{\frac{2mgl}{E}}\cos(\varphi/2)\, d\varphi \quad \therefore \quad d\varphi = \sqrt{\frac{E}{2mgl}}\frac{2}{\cos(\varphi/2)}\, du$$

$$\text{when } \varphi = 0,\ u = 0,\ \varphi = \pi,\ u = \sqrt{\frac{2mgl}{E}}$$

We then proceed to calculate our action angle variable,

$$J = \oint p_\varphi \, d\varphi = \int_{-\sqrt{\frac{2mgl}{E}}}^{\sqrt{\frac{2mgl}{E}}} \sqrt{\frac{E}{2mgl}}\frac{2}{\cancel{\cos(\varphi/2)}} \cdot 2l\cancel{\cos(\varphi/2)}\sqrt{2mE}\sqrt{1-u^2}\, du$$

If we say our particle is initially at rest then $E = 2mgl$ and thus

$$J_\varphi = 4E\sqrt{\frac{l}{g}}\int_{-1}^{1}\sqrt{1-u^2}\, du = 4E\sqrt{\frac{l}{g}}\left(\frac{\arcsin(u)}{2} + \frac{u\sqrt{1-u^2}}{2}\right)\Bigg|_{-1}^{1} = 2E\sqrt{\frac{l}{g}}\pi$$

Because our Hamiltonian is conserved, $H = E$. We find the frequency by,

$$\frac{\partial H}{\partial J} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \nu$$

Therefore we can see that our frequency is identical for any initial condition.

$$\boxed{\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}}}$$

Problem 4: Review

Procedure:

- Calculate the Lagrangian of this system.
- Proceed to calculate the canonical momenta and then the Hamiltonian of this system.
- Proceed to calculate the canonical momenta in terms of the total energy.
- Calculate the action angle variable
$$J_{q_i} = \int P_{q_i} \, dq_i$$
and then the frequency with
$$\nu = \frac{\partial H}{\partial J_{q_i}}.$$

Key Concepts:

- Since the Hamiltonian for this system is conserved, it is our total energy.
- We can use action angle formalism to solve for the frequency of our motion.

Variations:

- We can be given a different system.
  - We then would use the same formalism to solve for the frequency.