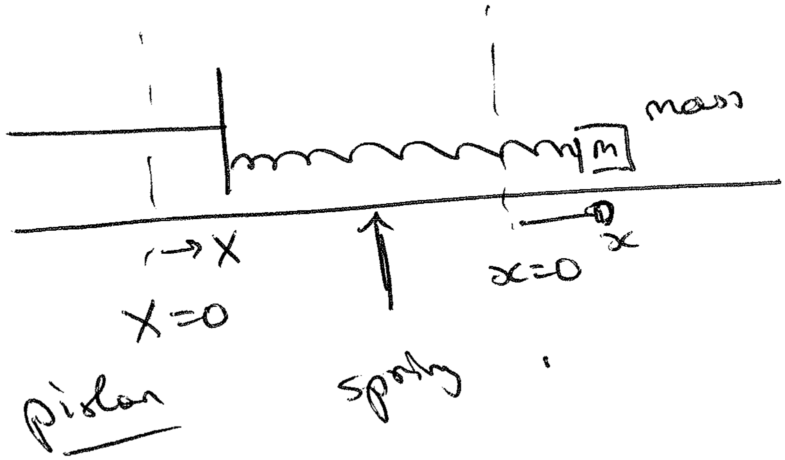
Assignment 1 solutions

Question 1:



a)

The spring will exert a force on the mass,

$$F = -k \Delta l$$

deviation of spring length from equilibrium length.

We define $X = 0$ + $x = 0$ to correspond to the scenario when the spring exerts no restoring force:

$$\therefore F_s = -k(x - X)$$

Sanity check: ✓ $x = 0$ $X > 0$ → spring should push mass to right ($x > 0$) $F > 0$

✓ $X = 0$ $x > 0$ → spring should pull mass to left $F < 0$

The system is also subject damping described by,

$$F_d = -m\nu\dot{x}$$

Hence the total forces in the system (exerted on the mass) are

$$F = F_s + F_d = -k(x - X) - m\nu\dot{x}$$

Newton's laws give us our eqn. of motion,

$$F = m\ddot{x}$$

$\downarrow$ plug in F from above & rearrange

$$m\ddot{x} + m\nu\dot{x} + kx = kX$$

$$\ddot{x} + \nu\dot{x} + \underset{(=k/m)}{\omega_0^2} x = \omega_0^2 X$$

Now, let's assume the piston is driven according to ~~X(t)~~ some function $X(t)$,

$$\ddot{x} + \nu\dot{x} + \omega_0^2 x = \overbrace{\omega_0^2 X(t)}^{\text{define as } F_0(t)}$$

$$\hookrightarrow \ddot{x} + \nu\dot{x} + \omega_0^2 x = F_0(t) \quad \square$$

b) Take: $\quad X(0) = 0 \quad$ & $\quad X(t) = X_o e^{\alpha t} \cos(\omega t)$

$$\Rightarrow \quad \ddot{x} + v\dot{x} + \omega_o^2 x = F_o e^{\alpha t} \cos(\omega t)$$

We can guess the particular solution as following the form of the modulated drive,

$$x_p(t) = D \cos(\omega t - \delta)\, e^{\alpha t}$$

w/ D & δ constants to be determined by solving,

$$\ddot{x}_p + v\dot{x}_p + \omega_o^2 x_p = F_o e^{\alpha t} \cos(\omega t) \quad (*)$$

Now, we need:

$$\ddot{x}_p = -D\omega^2 \cos(\underset{\omega t-\delta}{\cancel{\omega t}})\, e^{\alpha t} - 2D\alpha\omega \sin(\omega t - \delta)\, e^{\alpha t}$$

$$+ D\alpha^2 \cos(\omega t - \delta)\, e^{\alpha t}$$

$$\dot{x}_p = -D\omega \sin(\omega t - \delta)\, e^{\alpha t} + D\alpha \cos(\omega t - \delta)\, e^{\alpha t}$$

To proceed we plug our expressions for $x_p, \dot{x}_p$ & $\ddot{x}_p$ into $*$, express the trig terms as,

$$\cos(\omega t - \delta) = \cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)$$

$$\sin(\omega t - \delta) = \sin(\omega t)\cos(\delta) * - \cos(\omega t)\sin(\delta)$$

& then equate the coefficients of the resulting terms $\propto \cos(\omega t)$ & $\propto \sin(\omega t)$, e.g.

$$\{L_1\}\cos(\omega t) + \{L_2\}\sin(\omega t)$$

$$= \{R_1\}\cos(\omega t) + \{R_2\}\sin(\omega t)$$

st. $L_1 = R_1$ & $L_2 = R_2$ as sin & cos are linearly independent.

Solving $L_2 = R_2$ yields,

$$\tan(\delta) = \frac{2\omega\alpha + r}{\alpha^2 - \omega^2 + \omega_0^2 + \alpha r}$$

&

$$D = \frac{F_0}{[\alpha^2 - \omega^2 + \omega_0^2 + \alpha r]\cos(\delta) + (2\omega\alpha + \omega r)\sin(\delta)}$$

c) We are looking for the frequency (ies) at which the amplitude, D, of the particular solution is maximal:

→ need to solve $$\frac{dD}{d\omega} = 0 \quad \text{for} \quad \omega_R.$$

First we need to remove the δ dependence from our expression for D (as $\delta = \delta(\omega)$). We use:

$$\sin(\delta) = \frac{2\omega\alpha + \nu}{\sqrt{(\alpha^2 - \omega^2 + \omega_0^2 + \alpha\nu)^2 + 4\omega^2(\alpha + \nu/2)^2}}$$

& $$\cos(\delta) = \frac{\alpha^2 - \omega^2 + \omega_0^2 + \alpha\nu}{\sqrt{(\alpha^2 - \omega^2 + \omega_0^2 + \alpha\nu)^2 + 4\omega^2(\alpha + \nu/2)^2}}$$

[we obtained these using trig identities & $\delta = \text{atan}(\ldots)$]

Then,

$$D \equiv \frac{F_0}{\left[(\alpha^2 - \omega^2 + \omega_0^2 + \alpha\nu)^2 + 4\omega^2(\alpha^2 + \nu/2)^2\right]^{1/2}}$$

We then compute $\frac{dD}{d\omega} = 0$ & w/ a bit of work (or using mathematica or similar) obtain that this condition is satisfied for,

$$\omega = \pm\sqrt{\omega_0^2 - \nu^2/2 - \alpha(\alpha+\nu)} = \omega_R$$

This resonance frequency will only be <u>real</u> if

$$\omega_0^2 - \nu^2/2 - \alpha(\alpha+\nu) \geq 0$$

Solving for the critical case where the above is an equality we find,

$$\alpha = -\nu/2 \pm \sqrt{\omega_0^2 - \nu^2/4}$$

which leads to the requirement,

$$-\frac{\nu}{2} - \sqrt{\omega_0^2 - \nu^2/4} \leq \alpha \leq -\frac{\nu}{2} + \sqrt{\omega_0^2 - \nu^2/4}$$

for the resonance frequency to exist.

Question 2:

The numbers in this problem are arbitrary, so let's work through a generic derivation first.

a) We will first consider a model where each spring of the car's suspension is treated as an (undamped) driven oscillator,

$$\ddot{x} + \omega_0^2 x = F_0 \cos(\omega t)$$

x : position / length of spring relative to equilibrium

ω_0 : natural oscillation frequency of spring

F_0 : Force due to ripples in road.

ω : frequency of effective force due to ripples.

Our model is very crude, for a range of reasons. One example is that the ripples in the road may not be well described as a sinusoidal modulation [recall $F_0 \sim \omega_0^2 X_0$ from 1a].

In fact that might be better described as a regular but instantaneous impulse, e.g.,

$$F_0(t) \sim \sum_{n=1}^{\infty} \delta(t - nT)$$

where T is the "time" between ripples. But, let us proceed w/ our model anyway...

To proceed we need to compute/find expressions for some of our inputs. First, let's compute the spring constant.

To estimate k → Use that when 4 adults of mass m_p hop into a car, it lowers (springs are compressed) by some amount Δx_{drop}.

Then: $$m_p g = k \, \Delta x_{drop} \rightarrow k = \frac{mg}{\Delta x_{drop}}.$$

4 adults + 4 springs ~ mg force/per spring.

Next, let us work out the force due to ripples on the road. We can guess these using 1a):

$$F_0 \sim \omega_0^2 X_0 \quad \text{w/} \quad \omega_0 \sim \sqrt{\frac{k}{m_{car}/4}}$$

+ $X_0 \sim$ height of ripples

Note in the expression for ω_0 we used that the relevant mass for each spring is $\sim \frac{m_{car}}{4}$ [4 springs]. This answer will help us later in b).

Last → what is the drive frequency ω? This will depend on how fast the car is moving:

$$\omega = \frac{2\pi}{\lambda_{ripple}} v_{car}$$

velocity of car

"wavelength" of ripples = 2m spacing given in question.

Now, in the absence of damping the resonance frequency is equal to the oscillators natural frequency;

$$\omega_R = \omega_0 = \sqrt{\frac{4k}{m_{car}}}$$

Hence the speed at which the car meets this condition is found via,

$$\underbrace{\frac{2\pi}{\lambda_{ripple}} v_{car}}_{\omega} = \sqrt{\frac{4k}{m_{car}}}$$

$$\downarrow$$

$$v_{car} = \frac{\lambda_{ripple}}{2\pi}\sqrt{\frac{4k}{m_{car}}} = \frac{\lambda_{ripple}}{2\pi}\sqrt{\frac{4m_p g}{m_{car}\,\Delta x_{drop}}}$$

Let's plug in some numbers for an estimate:

$m_{car} \sim 800$ kg

$m_p \sim 80$ kg $\Rightarrow$ $v_{car} \simeq 4.7$ m/s or 16.1 km/h.

$\Delta x_{drop} \sim 2$cm

[right order of magnitude to be relevant...?]

b) A damped oscillator has a resonance which is shifted, relative to the ideal case, and broadened (divergence becomes finite). From our lecture notes,

$$D \sim \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$$

where $A = F_0 = \omega_0^2 X_0$ following our convention here.

The resonance occurs @ $\omega_R = \pm\sqrt{\omega_0^2 - 2\beta^2}$

where,

$$\max D = \frac{X_0 \omega_0^2}{2\beta} \frac{1}{\sqrt{\omega_0^2 - \beta^2}}$$

Let us estimate that the suspension can travel a range of $\sim 10\,\text{cm}$ without causing any issue.
Then, this is equivalent to requiring,

$$\frac{X_0 \omega_0^2}{2\beta} \frac{1}{\sqrt{\omega_0^2-\beta^2}} \lesssim 10\,\text{cm} = D_{max}$$

Re-arranging & solving for the equality we obtain:

$$\beta = \frac{w_0}{\sqrt{2}} \sqrt{1 \pm \sqrt{D_{max}^2 - X_0^2}}$$

It turns out we can eliminate the +ve solution as this would lead to $\omega_R \notin \mathbb{R}$.

For an estimate, plug in all same parameters from a) & guess $X_0 \sim 5cm$. Then,

$$\beta \sim 3.6 \, s^{-1}$$

Question 3:

a) For $m=\omega=1$, we can write the total energy of the oscillator as:

$$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$$

Then the time derivative is written as,

$$\dot{E} = \dot{x}\ddot{x} + x\dot{x}$$
$$= \dot{x}(\ddot{x} + x)$$

From the original EOM we can identify, $\ddot{x} + x = -\dot{x}(x^2 + \dot{x}^2 \cancel{\pm} 1)$

$$\dot{E} = -\dot{x}^2(x^2 + \dot{x}^2 + 1)$$

b) By the definition of the polar co-ordinates, we can find use:

$$r = \sqrt{x^2 + \dot{x}^2} \longrightarrow \dot{r} = \frac{1}{r}(x\dot{x} + \dot{x}\ddot{x})$$

[Note then from a) that,
$$\dot{E} = -\dot{x}^2(r^2 - 1)$$]

To simplify further, use that:

$$\ddot{x} = -x - \dot{x}(x^2 + \dot{x}^2 \cancel{\pm} 1)$$

So,

$$\dot{r} = \frac{1}{r}\left(-\dot{x}^2 x^2 - \dot{x}^4 + \dot{x}^2\right)$$

& plugging in $x = r\cos\theta$, $\dot{x} = r\sin\theta$ yields, (after some work)

$$\dot{r} = r\sin^2\theta\,(1-r^2)$$

To find the EOM for $\dot{\theta}$ → start from $\dot{x} = r\sin\theta$.

Then,

$$\ddot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

original EOM

$$-x - \dot{x}\left(x^2+\dot{x}^2-1\right) = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

↓ plug in $x = r\cos\theta$, $\dot{x} = r\sin\theta$, ...

$$-r\cos\theta - r^3\sin\theta\cos^2\theta - r^3\sin^3\theta + r\sin\theta$$
$$= \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

↓ plug in $\dot{r} = ..$ into RHS, rearranging etc...

$$\dot{\theta} = \sin\theta\cos\theta\,(1-r^2) - 1$$

c) Two points:

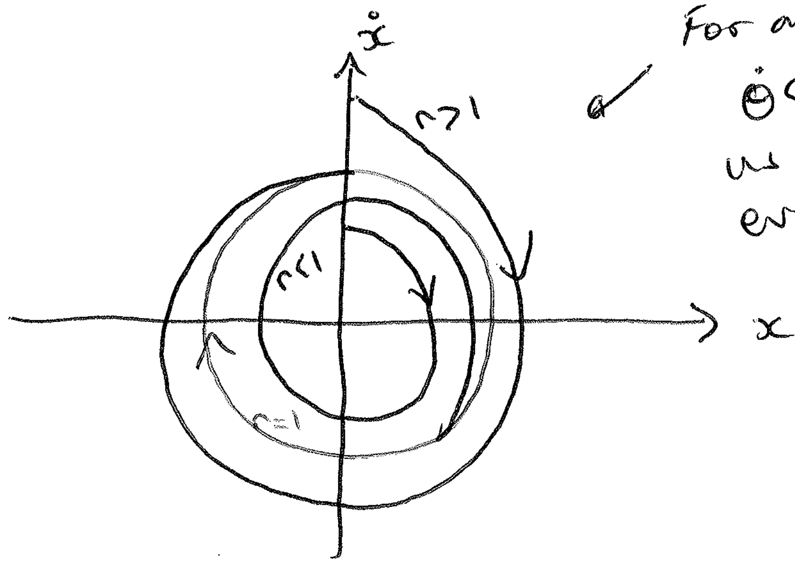
① $r = 1$ is a limit cycle: $\dot{r} = 0$, $\dot{\theta} = -1$

② If $r > 1 \rightarrow \dot{r} < 0$ (until $r = 1$)

If $r < 1 \rightarrow \dot{r} > 0$ (until $r = 1$)

$\rightarrow r = 1$ is a stable attractor.

[Note also $\dot{E} = 0$ for $r = 1$ ✓]



For all three cases $\dot{\theta} < 0$, which tells us how the trajectories evolve.

Solution makes sense from a):

$r < 1$ $\quad \dot{E} > 0$ (exclude $\dot{x} = 0$)

$r > 1$ $\quad \dot{E} < 0$

} energy ~ amplitude, grows or shrinks...