COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Electrodynamics 1

PHYS 5573 Homework Assignment 3

Problems: {1, 2, 3, 4}

Due: March 28, 2022 at 5:00 PM

Student
Taylor Larrechea

Professor
Dr. Bruce Mason

Problem 1:

Consider a sphere of radius R and total charge Q that has been embedded with an r-dependent charge density:

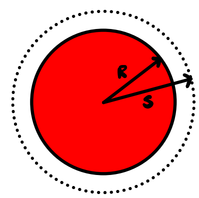
$$\rho(\vec{r}) = Cr$$

(a) Write and solve an integral to determine C in terms of the properties of the sphere, Q and r. For the rest of the problem, use this result for C in $\rho(\vec{r})$.

For our sphere of radius R, we will calculate what the charge enclosed is for this sphere with

$$Q_{ENC} = \int_V \rho(\vec{r})\, d\tau \qquad (*)$$

We are interested in the charge enclosed by our sphere. Pictorally this looks like



Where the red sphere is our actual sphere and the dotted sphere is the arbitrary distance we are looking at. But since there is no charge beyond the sphere, $R=S$ $\therefore$ Q_{ENC}

$$Q_{ENC} = C\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^3 \sin^2(\theta)\, dr\, d\theta\, d\phi = C \cdot 4\pi \cdot \frac{r^4}{4}\Big|_0^R = C\pi R^4$$

If we say the total charge enclosed is Q, then C is

$$\boxed{C = \frac{Q}{\pi R^4}}$$

(b) Explain why and how you can use Gauss' Law to solve for the electric field of the charged sphere.

By taking advantage of our translational and reflexive symmetries we can use the divergence theorem to say

$$\int_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \oint_S \vec{E}\cdot d\vec{A} \qquad (**)$$

We can then say the RHS of $(**)$ due to Gauss' Law, $\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$, then changes

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\epsilon_0} \qquad (***)$$

Where we then solve $(***)$ for $\vec{E}$.

Problem 1: Continued

(c) Set up your solution and determine $\vec{E}(\vec{r})$ for all r. You will have somewhat different results for $0 \leq r \leq R$ and for $r \geq R$. These results should agree for $r = R$.

We then finish solving for $\vec{E}$ inside the sphere

$$\oint \vec{E}\cdot d\vec{A} = E r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta = E \cdot 4\pi r^2$$

$$Q_{ENC} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} C r'^3 \sin(\theta)\, dr' d\theta d\varphi = C 4\pi \frac{r^4}{4}$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\varepsilon_0} \Rightarrow E \cdot 4\pi r^2 = \frac{C r^4 \pi}{\varepsilon_0} \Rightarrow \vec{E}(\vec{r}) = \frac{C r^2}{4\varepsilon_0}(\hat{r}) \quad (*)$$

Outside the sphere $\vec{E}$ is

$$\oint \vec{E}\cdot d\vec{A} = E r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin(\theta)\, d\theta = E \cdot 4\pi r^2 \, , \quad Q_{ENC} = C\pi R^4$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{ENC}}{\varepsilon_0} \Rightarrow E \cdot 4\pi r^2 = C\pi R^4 \quad \therefore \quad \vec{E}(\vec{r}) = \frac{C R^4}{4 r^2 \varepsilon_0}(\hat{r}) \quad (**)$$

If $r=R$ then we see

$$(*) \Rightarrow \vec{E}(\vec{R}) = \frac{C R^2}{4\varepsilon_0}(\hat{R}) \, , \quad (**) \Rightarrow \vec{E}(\vec{R}) = \frac{C R^{\cancel{4}\,2}}{4\cancel{R^2}\varepsilon_0}(\hat{R}) = \frac{C R^2}{4\varepsilon_0}(\hat{R}) \quad \therefore (*) = (**)$$

We can then see that these are indeed equal for $r=R$

$$\boxed{0 < r \leq R \, , \ \vec{E}(\vec{r}) = \frac{C r^2}{4\varepsilon_0}(\hat{r}) \quad : \quad r > R \, , \ \vec{E}(\vec{r}) = \frac{C R^4}{4 r^2 \varepsilon_0}(\hat{r})}$$

(d) Solve for $\phi(\vec{r})$ for all r. As usual, define $\phi(\infty) = 0$.

We use the relationship

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$$

To find our potential. Inside the sphere the potential is

$$\varphi(\vec{r}) = -\int \vec{E}(\vec{r})\cdot d\vec{r} = -\int_{\infty}^{R} \vec{E}_o(\vec{r})\, d\vec{r} - \int_R^r \vec{E}_I(\vec{r})\, d\vec{r}$$

$$= -\int_{\infty}^{R} \frac{C R^4}{4 r^2 \varepsilon_0}\, dr - \int_R^r \frac{C r'^2}{4\varepsilon_0}\, dr' = \frac{C R^3}{4\varepsilon_0} - \frac{C r^3}{12\varepsilon_0} + \frac{C R^3}{12\varepsilon_0}$$

Problem 1: Continued

This means our potential inside the sphere is

$$\varphi(r) = \frac{c}{3\varepsilon_0}\left(R^3 - \frac{r^3}{4}\right)$$

At the sphere the potential is

$$\varphi(r)\Big|_{R=r} = \frac{c}{3\varepsilon_0}\left(R^3 - \frac{R^3}{4}\right) = \frac{cR^3}{4\varepsilon_0}$$

outside the sphere the electric potential is

$$\varphi(r) = -\int \vec{E}(\vec{r})\cdot d\vec{r} = -\int_\infty^r \vec{E}_0(\vec{r})\cdot d\vec{r} = -\frac{c}{4\varepsilon_0}\int_\infty^r \frac{R^4}{r^2}\,dr = \frac{cR^4}{4\varepsilon_0 r}$$

This means Finally

$$\boxed{r<R,\ \varphi(r) = \frac{c}{3\varepsilon_0}\left(R^3 - \frac{r^3}{4}\right) \ :\ r=R,\ \varphi(r) = \frac{cR^3}{4\varepsilon_0} \ :\ r>R,\ \varphi(r) = \frac{cR^4}{4\varepsilon_0 r}}$$

(e) Solve for the total electric potential energy of the charged sphere. This can be considered the total energy (work) necessary to bring all the charges together in the sphere.

The equation for work is

$$W = \frac{1}{2}\int \rho(r)\varphi(r)\,d\tau$$

This means the "Potential Energy" in the sphere is

$$W = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_s^R cr\cdot\frac{c}{3\varepsilon_0}\left(R^3 - \frac{r^3}{4}\right) r^2\sin(\alpha)\,dr\,d\alpha\,d\varphi = \frac{2\pi c^2}{3\varepsilon_0}\int_s^R r^3R^3 - \frac{r^6}{4}\,dr$$

$$= \frac{2\pi c^2}{3\varepsilon_0}\left(\frac{3R^7}{7} - \frac{s^4R^3}{4} + \frac{s^7}{28}\right) = \frac{\pi c^2}{42\varepsilon_0}\left(12R^7 - 7s^4R^3 + s^7\right) = \frac{Q^2}{42\pi R^8}\left(12R^7 - 7s^4R^3 + s^7\right)$$

Outside the sphere the electric potential energy is

$$W = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty} cR\cdot\frac{cR^4}{4\varepsilon_0 r} r^2\sin(\alpha)\,dr\,d\alpha\,d\varphi = \frac{c^2\pi R^5}{2\varepsilon_0}\int_R^\infty r\,dr = \frac{c^2\pi R^5}{4\varepsilon_0}\, r^2\Big|_R^\infty$$

Problem 1: Continued

This then means the electric potential energy is infinite out of the sphere. Finally we have

$$\boxed{r<R\,,\ u=\frac{Q^2}{42\pi R^8}\left(12R^7-7s^4R^3+s^7\right)\ :\ r>R\,,\ u=\infty}$$

(f) Consider a similar sphere with the charge density $\rho(\vec{r}) = Cr\cos^2\theta$. Set up a multipole expansion to solve for $\phi(\vec{r})$ and show (explain) that there will only be two terms in the expansion. If you wish, you can solve this.

The equation for the the multipole expansion for $\varphi(r)$ is

$$\boxed{\varphi(r)=\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{C}{r^{(n+1)}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}(r')^{n+2}P_n(\cos(\alpha))\cos^2(\alpha)\sin(\alpha)\,dr'\,d\alpha\,d\varphi}$$

And due to the orthogonality of the Legendre polynomials there will only be two terms, $n=0,1$.

Problem 2:

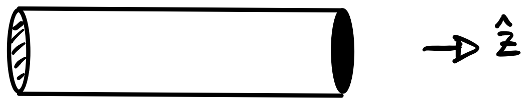
The Magnetic analog to the problem above is a long, current-carrying wire with a current density that varies across the radius of the wire.

Use polar coordinates: $\hat{z}$ along the wire, $\hat{r}$ the radial direction perpendicular to $\hat{z}$, and $\hat{\phi}$ the azimuthal angle around $\hat{z}$.

The wire has a radius R and total current I in the $\hat{z}$ direction. The current density is:

$$\vec{J}(\vec{r}) = Cr\hat{z}$$

(a) Write an equation relating the total current in the wire, I, to the current density, $\vec{J}(\vec{r})$ for the wire. Use this to Derive an expression for the constant C in terms of properties of the wire, I and R.



Here we do the magnetic analog to the last problem. The relationship for current and current density is

$$I = \iint \vec{J}(\vec{r}) \cdot \hat{n}\, ds$$

which for us this is,

$$\hat{n} = \hat{z} \ , \ ds = R^2 d\varphi\, dz$$

Since the wire is uniform in z, we can reduce this to just a integral over $d\varphi$ and thus I is

$$I = CR^3 \int_0^{2\pi} d\varphi = 2\pi R^3 \cdot C \Rightarrow C = \frac{I}{2\pi R^3}$$

This then means C in terms of I and R are,

$$\boxed{C = \frac{I}{2\pi R^3}}$$

(b) Use the symmetry of the problem and the relation between the current (density) and the magnetic field to determine the general form for the magnetic field, $\vec{B}(\vec{r})$. Specifically, what direction is the field and how does it depend on r, ϕ, and z?

Here we take advantage of Stokes' Theorem

$$\iint (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} = \oint \vec{B} \cdot d\vec{l} \qquad (*)$$

Where we can say with Amperes Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{Enc} \qquad (**)$$

Problem 2: Continued

Where the relationship between the current density and the magnetic field is that they are perpendicular to one another. If the current density is in the $\hat{z}$ direction, it is reasonable to say that the magnetic field must point in the $\hat{\varphi}$ direction. Namely

$$\vec{B}(d) \propto \hat{z} \times \hat{r} = \hat{\varphi}$$

Therefore we can say that $\vec{B}$ is roughly

$$\boxed{\vec{B}(d) \propto \hat{\varphi}}$$

(c) Explain why you can use Ampere's Law to solve for $\vec{B}(\vec{r})$.

Where we can then use Amperes Law to say

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{Enc}$$

With equations (*) and (**) we have the magnetic analog of Gauss' Law. Considering our $d\vec{l}$ must point in the $\hat{\varphi}$ direction, it is reasonable to assume that the

$$\underline{\vec{B}} \text{ points in } \underline{\hat{\varphi}} \text{ direction}$$

This is because Amperes Law constitutes we align these two in the same direction.

Once $\vec{B}$ and $d\vec{l}$ point in the same direction we can pull out the magnitude of the dot product from the integral after we define dl.

(d) Calculate $\vec{B}(\vec{r})$ everywhere. Be sure to explain your approach.

This field is then

$$\oint \vec{B} \cdot d\vec{l} = \oint B \cdot \cos(\alpha)\, dl \quad w/\ dl = d \cdot d\varphi \implies Bd\cos(\alpha)\int_0^{2\pi} d\varphi = Bd \cdot (2\pi)\cos(\alpha)$$

Because we aligned $\vec{B}$ with $d\vec{l}$, $\cos(\alpha) = 1$ $\therefore$ our field is

$$\boxed{\vec{B}(d) = \frac{\mu_0 I_{Enc}}{2\pi d}(\hat{\varphi})}$$

Where we have taken advantage of symmetries to find $\vec{B}$

Problem 2: Continued

(e) Show that the magnetic potential energy of the wire is infinite. This shouldn't be surprising, as it is an infinitely long wire.

The magnetic potential energy of this wire is calculated with

$$U = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{R} \vec{J}(\vec{z})\cdot\vec{A}(\vec{z})\, r\, dr\, d\phi\, dz \qquad (***)$$

Where when we integrate (***), the magnetic potential energy will go to infinity due to the wire being infinite.

Problem 3:

Consider a uniform charged rod of charge Q and length L, $\lambda = Q/L$, on the z-axis, centered at the origin, extending from $z = -\frac{L}{2}$ to $z = \frac{L}{2}$.

(a) Write an integral over the charged rod for the electric potential on the z-axis. Calculate the electric potential due to the rod on the z-axis for all $z > \frac{L}{2}$.

The generic equation for determining the electric potential is

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}_q)}{|\vec{r}-\vec{r}_q|}\, d^3r_q$$

In 1D this becomes,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(L)}{|\vec{r}-\vec{r}_q|}\, dr_q$$

The potential for the entire rod is

$$\varphi(z) = \frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2} \frac{Q}{L}\frac{1}{z-z_q}\, dz_q = \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\ln(z-z_q)\Big|_{-L/2}^{L/2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{L}\Big(\ln(z-L/2)-\ln(z+L/2)\Big) = \frac{Q}{4\pi L\varepsilon_0}\ln\left(\frac{z-L/2}{z+L/2}\right)$$

we finally have

$$\boxed{\varphi(z) = \frac{Q}{4\pi L\varepsilon_0}\ln\left(\frac{z-L/2}{z+L/2}\right)}$$

(b) Expand your result for $\phi(z)$ in powers of $L/(2z)$. It will be useful to know that: (If both expressions don't help you, you'll probably want to redo your integral from part (a).)

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \quad \ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Taking the previous expression for $\varphi(z)$ and re-writing it so that we can use these expansions we have

$$\varphi(z) = \frac{Q}{4\pi L\varepsilon_0}\Big(\ln(1-L/2z)-\ln(1+L/2z)\Big) \qquad (*)$$

Where if we call $x = L/2z$, and using this new potential $(*)$, we can then write this expansion as

Problem 3: Continued

$$\varphi(x) = \frac{Q}{4\pi L\varepsilon_0}\left(\ln(1-x) - \ln(1+x)\right) = \frac{Q}{4\pi L\varepsilon_0}\left(-\sum_{n=1}^{\infty}\frac{x^n}{n} - (-1)^{n-1}\frac{x^n}{n}\right)$$

$$= -\frac{Q}{4\pi L\varepsilon_0}\sum_{n=1}^{\infty}\frac{x^n}{n}\left(1+(-1)^{n-1}\right)$$

Substituting in $x = L/2z$ we then have

$$\boxed{\varphi(z) = -\frac{Q}{4\pi L\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{2z}\right)^n\left(1+(-1)^{n-1}\right)}$$

(c) Using this expansion, you can determine a sum for the potential $\phi(r,\theta)$ everywhere. Note that for points along the z-axis:

$$\phi(z) = \phi(r, \theta = 0)$$

The potential everywhere is given by:

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\frac{a_l}{r^{l+1}}P_l(\cos\theta)$$

Using the result from part (b), determine an expression for all the coefficients a_l. Note: $P_l(\cos(0)) = P_l(1) = 1$ for all l.

Knowing that $\theta = 0$, our potential everywhere is then

$$\varphi(r,\theta=0) = \sum_{l=0}^{\infty}\frac{a_l}{r^{l+1}}$$

Comparing this with what we found in part b) we can say

$$\varphi(z) = \varphi(r,\theta=0) \Rightarrow -\frac{Q}{4\pi L\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{2z}\right)^n\left(1+(-1)^{n-1}\right) = \sum_{l=0}^{\infty}\frac{a_l}{r^{l+1}}$$

If we wish to determine an expression for a_l, we simply solve for a_l and say

$$\boxed{a_l = -\frac{Q}{4\pi L\varepsilon_0}\sum_{l=0}^{\infty} r^{l+1}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{2z}\right)^n\left(1+(-1)^{n-1}\right)}$$

(d) Using the multipole expansion for the electric potential, calculate the monopole, dipole, and quadrupole potential terms due to the line charge. Show that these terms agree with part (c).

The multipole expansion is defined as

$$\frac{1}{|\vec{r}-\vec{r}_q|} = \frac{1}{r} + \frac{x_{q_i}x_i}{r^3} + \frac{1}{2}\left(3\frac{(x_{q_i}x_i)(x_{q_j}x_j)}{r^5} - \frac{r_q^2}{r^3}\right)$$

Problem 3: Continued

This then means we can write the equation for the potential as

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{r})\left(\frac{1}{r} + \frac{\vec{r}_q\cdot\vec{r}}{r^3} + \frac{1}{2}\left(3\frac{(\vec{r}_q\cdot\vec{r})^2}{r^5} - \frac{\vec{r}_q^{\,2}}{r^3}\right)\right) d^3\vec{r}$$

This then means the monopole term is

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r}\int_{-L/2}^{L/2}\frac{Q}{L}\,dz = \frac{1}{4\pi\varepsilon_0 r}\frac{Q}{L}\,z\Big|_{-L/2}^{L/2} = \frac{Q}{4\pi\varepsilon_0 r}$$

The dipole term is then

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\int_{-L/2}^{L/2}\frac{Q}{L}(\vec{r}_q\cdot\vec{r})\,dz = \frac{Q}{4\pi\varepsilon_0 r^2} = \int_{-L/2}^{L/2}\frac{1}{L}(\vec{r}_q\cdot\hat{r})\,dz = \frac{Q r_q}{4\pi\varepsilon_0 r^2}$$

The quadrupole term is

$$\varphi(\vec{r}) = \frac{1}{8\pi\varepsilon_0 r^3}\int_{-L/2}^{L/2}\frac{Q}{L}\left(3\cdot(\vec{r}_q\cdot\hat{r})^2 - \vec{r}_q^{\,2}\right)dz = \frac{Q}{4\pi\varepsilon_0 r^3}\,r_q^2$$

Therefore our Full expansion is then

$$\boxed{\varphi(r) = \frac{Q}{4\pi\varepsilon_0 r}\left(1 + \frac{r_q}{r} + \frac{r_q^2}{r^2}\right)}$$

Problem 4:

A square current-carrying loop with sides of length l and counter-clockwise current I is in the $x-y$ plane and centered at the origin. Calculate the magnetic field at the point $\vec{r} = d\hat{x}$.

(a) What is the magnetic field at the point $\vec{r} = d\hat{x}$ if you approximate the loop by a magnetic dipole?

The magnetic field that comes from the dipole approximation is calculated with

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\hat{n}\cdot\vec{m})\hat{n} - \vec{m}}{r^3}$$

Where the magnetic dople moment and $\hat{n}$ are

$$\vec{m} = I l^2 \hat{z} \quad , \quad \hat{n} = \hat{x} \quad \Rightarrow \quad \vec{B}(\vec{r}) = -\frac{\mu_0}{4\pi} \frac{\vec{m}}{r^3}$$

If we look at this a distance d away we find

$$\boxed{\vec{B}(d) = -\frac{\mu_0 I l^2}{4\pi d^3} (\hat{z})}$$

(b) Use cross products to determine the direction of the magnetic field due to each side of the loop at $\vec{r}$.

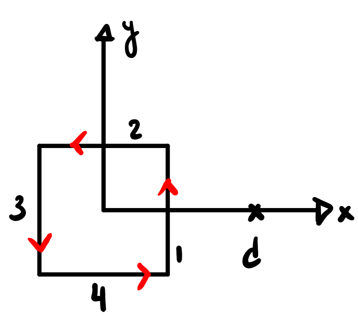
For this loop, we calculate the magnetic field using the Biot-Savart Law. This is of course

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint d\vec{l}' \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|}$$

This means the direction of our field due to each side of the loop is determined with the cross product

$$d\vec{l}' \times (\vec{r} - \vec{r}')$$

To do this, we look at each side of the loop and then add them together. Looking at our loop we have



Problem 4: Continued

We first examine side 1,

$$d\vec{l}' = dy\,\hat{y}\ ,\ \vec{r} = d\,\hat{x}\ ,\ \vec{r}' = l/2\,\hat{x} + y\hat{y}\ \therefore\ (\vec{r}-\vec{r}') = (d-l/2)\hat{x} - y\hat{y}$$

$$d\vec{l}' \times (\vec{r}-\vec{r}') = (dy\,\hat{y}) \times ((d-l/2)\hat{x} - y\hat{y}) = (d-l/2)\,dy\,(-\hat{z})$$

we then examine side 2,

$$d\vec{l}' = -dx\,(\hat{x})\ ,\ \vec{r} = d\,\hat{x}\ ,\ \vec{r}' = x\hat{x} + l/2\hat{y}\ \therefore\ (\vec{r}-\vec{r}') = (d-x)\hat{x} - \frac{l}{2}\hat{y}$$

$$d\vec{l}' \times (\vec{r}-\vec{r}') = (-dx\,\hat{x}) \times ((d-x)\hat{x} - l/2\,\hat{y}) = (l/2)\,dx\,(\hat{z})$$

Next is side 3,

$$d\vec{l}' = -dy\,(\hat{y})\ ,\ \vec{r} = d\,\hat{x}\ ,\ \vec{r}' = -\frac{l}{2}\hat{x} + y\hat{y}\ \therefore\ (\vec{r}-\vec{r}') = (d+l/2)\hat{x} - y\hat{y}$$

$$d\vec{l}' \times (\vec{r}-\vec{r}') = (-dy\,\hat{y}) \times ((d+l/2)\hat{x} - y\hat{y}) = (d+l/2)\,dy\,(\hat{z})$$

Lastly, side 4,

$$d\vec{l}' = dx\,(\hat{x})\ ,\ \vec{r} = d\hat{x}\ ,\ \vec{r}' = x\hat{x} - \frac{l}{2}\hat{y}\ \therefore\ (\vec{r}-\vec{r}') = (d-x)\hat{x} + \frac{l}{2}\hat{y}$$

$$d\vec{l}' \times (\vec{r}-\vec{r}') = (dx\,\hat{x}) \times ((d-x)\hat{x} + l/2\,\hat{y}) = (l/2)\,dx\,(\hat{z})$$

If we add all of the sides together we find that the magnetic field will point in the z-direction.

$$\boxed{\vec{B}(d) \propto (\hat{z})}$$

(c) Write down integrals for the magnetic field at $\vec{r}$ due to each side of the current loop.

For side 1, the integral is

$$\boxed{\vec{B}_1(\vec{r}) = \int_{-l/2}^{l/2} \frac{(d-l/2)}{\sqrt{(d-l/2)^2+y^2}}\,dy\,(-\hat{z})}$$

Problem 4: Continued

For side 2, the integral is

$$\boxed{\vec{B}_2(\vec{r}) = \int_{\ell/2}^{-\ell/2} \frac{\ell/2}{\sqrt{(d-x)^2 + \ell^2/4}}\, dx \; (\hat{z})}$$

Side 3 is then

$$\boxed{\vec{B}_3(\vec{r}) = \int_{\ell/2}^{-\ell/2} \frac{(d+\ell/2)}{\sqrt{(d+\ell/2)^2 + y^2}}\, dy \; (\hat{z})}$$

where lastly side 4 is

$$\boxed{\vec{B}_4(\vec{r}) = \int_{-\ell/2}^{\ell/2} \frac{\ell/2}{\sqrt{(d-x)^2 + \ell^2/4}}\, dx \; (\hat{z})}$$

(d) Solve for the magnetic field due to the loop at $\vec{r}$. It might be useful to know that:

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Note: The results are somewhat messy.

The solution to side 1 is,

$$\boxed{\vec{B}_1(\vec{r}) = 2(\ell/2 - d)\ln\left(\frac{\left|\sqrt{\ell^2/2 - d\ell + d^2} + \ell/2\right|}{|\ell/2 - d|}\right)(-\hat{z})}$$

The solution to side 2 is,

$$\boxed{\vec{B}_2(\vec{r}) = \frac{\ell}{2}\ln\left(\left|\frac{\sqrt{\ell^2/2 + d\ell + d^2} - a - d}{\sqrt{\ell^2/2 - d\ell + d^2} + a - d}\right|\right)(\hat{z})}$$

Problem 4: Continued

The solution to side 3 is,

$$\boxed{\vec{B}_3(\vec{r}) = 2(\ell/2+d)\ln\left(\frac{\left|\sqrt{\ell^2/2 + d\ell + d^2} - \frac{\ell}{2}\right|}{|\ell/2+d|}\right)(\hat{z})}$$

The solution to side 4 is finally

$$\boxed{\vec{B}_4(\vec{r}) = \frac{\ell}{2}\ln\left(\left|\frac{\sqrt{\ell^2/2 + d\ell + d^2} - a - d}{\sqrt{\ell^2/2 - d\ell + d^2} + a - d}\right|\right)(\hat{z})}$$

Where the above integrals were calculated with Mathematica. $\underline{\mu_0 I}$ ... $\underline{\text{omitted}}$

(e) Expand your result above for $d >> l$ $\frac{l}{d} << 1$ to the lowest non-zero power in $1/d$. Show that this gives the dipole approximation from part (a).

Hint: It's a good idea to expand the fields from the right and left wires together, and the top and bottom wires together.

When we apply this expansion our field becomes

$$\vec{B}(\vec{r}) = \frac{3\ell^2 \cdot \mu_0 I}{4\pi}\int_0^d \frac{1}{r^4}\,dr$$

If we then integrate this we find

$$\vec{B}(\vec{r}) = -\frac{3\ell^2\mu_0 I}{4\pi}\cdot\frac{1}{3r^3}\Big|_0^d = -\frac{\mu_0 I \ell^2}{4\pi d^3}(\hat{z})$$

Where we can clearly see that this is equal to the dipole. ✓