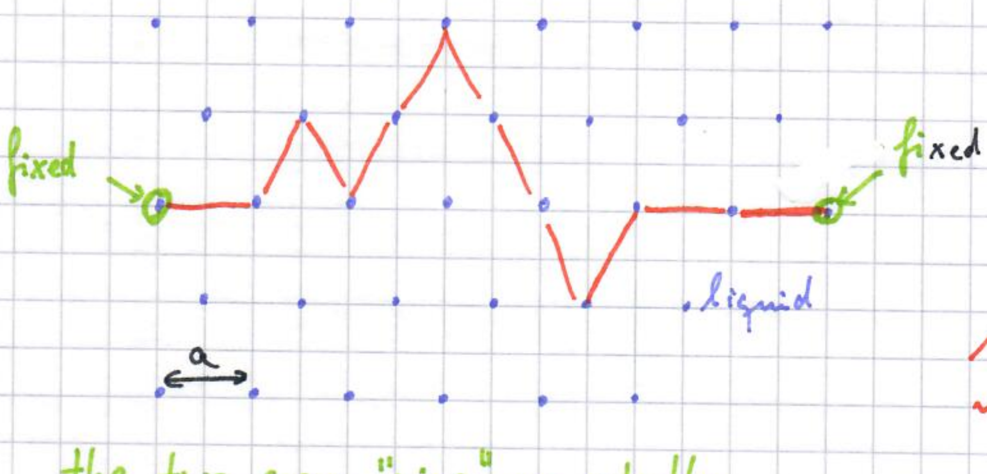
Problem:

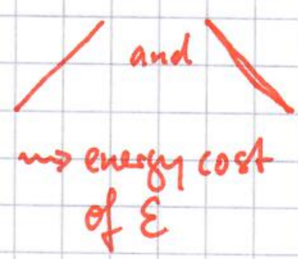
The surface of a 3D liquid is approximately 2D.

The surface of a 2D liquid is approximately 1D.

The surface is not entirely flat. At zero T, flat surface. At high T, deformed surface (deformations cost energy).

Let's assume that our liquid "lives" on regularly distributed positions — let's assume that the structure is made from triangles.





the two green "pins" are at the same height

Calculate the length of the surface as a fct. of T.

To make the calculation a little simpler, let's assume that there is an energy cost ε associated with every line, regardless of whether the line is horizontal, up, or down.

Then $E = \varepsilon\,(N_0 + N_+ + N_-)$

However, for convenience, we can offset E such that $E = 0$ for $N_0 = N$, $N_+ = 0$, $N_- = 0$

Thus: $E = \varepsilon\,(N_0 + N_+ + N_-) - \varepsilon \frac{L}{a}$

N_0 : # of horizontal lines
N_+ : # of up lines
N_- : # of down lines

fixed pins $\Rightarrow N_+ = N_- \Rightarrow E = \varepsilon\left(N_0 + 2N_+\right) - \varepsilon \frac{L}{a}$

Also: $L = \left[N_0 + \frac{1}{2}(N_+ + N_-)\right]a = (N_0 + N_+)a$

So, we know:

$$\boxed{L = (N_0 + N_+)\,a} \quad (2') \qquad \Rightarrow \quad N_0 = \frac{L}{a} - N_+ \quad (2'')$$

$$\boxed{E = \varepsilon\left(N_0 + 2N_+ - \frac{L}{a}\right)} \quad (3')$$

Solve (3') for L: $\quad \frac{aE}{\varepsilon} - aN_0 - 2aN_+ = -L$

$$\Rightarrow L = -\frac{aE}{\varepsilon} + aN_0 + 2aN_+ \quad (3'')$$

Set the r.h.s. of (2') equal to the r.h.s. of (3''):

$$(N_0 + N_+)\,a = a\left(-\frac{E}{\varepsilon} + N_0 + 2N_+\right)$$

$$\Rightarrow N_+ = \frac{E}{\varepsilon}$$

Insert $N_+ = \frac{E}{\varepsilon}$ into (2''): $\quad N_0 = \frac{L}{a} - \frac{E}{\varepsilon}$

So: $\boxed{N_+ = N_- = \frac{E}{\varepsilon} \text{ and } N_0 = \frac{L}{a} - \frac{E}{\varepsilon}}$

$\rightsquigarrow$ We have N_+, N_-, N_0 in terms of L (fixed) and E (macro variable in micro canonical ensemble).

$$\Rightarrow \text{We can get } \Gamma(E) = \frac{(N_0 + N_+ + N_-)!}{N_0!\,N_+!\,N_-!} = \frac{\left(\frac{E}{\varepsilon} + \frac{L}{a}\right)!}{\left(\frac{L}{a} - \frac{E}{\varepsilon}\right)!\left(\left(\frac{E}{\varepsilon}\right)!\right)^2}$$

Why do we want this? We want $\ell(T)$! But

$$\ell/a = N_0 + N_+ + N_- = N_0 + 2N_+ \underset{(3')}{=} \frac{E}{\varepsilon} + \frac{L}{a}$$

So: if we know $E(T)$, we also know $\ell(T)$!!!

$$S = k \log(\Gamma(E))$$

$$\boxed{\begin{array}{l}\log N! \approx N \log N - N \\ \text{Stirling formula}\end{array}}$$

$$\log(\Gamma(E)) = \log\left(\left(\frac{E}{\varepsilon} + \frac{L}{a}\right)!\right) - \log\left(\left(\frac{L}{a} - \frac{E}{\varepsilon}\right)!\right) - 2\log\left(\left(\frac{E}{\varepsilon}\right)!\right)$$

$$\approx \left(\frac{E}{\varepsilon} + \frac{L}{a}\right)\log\left(\frac{E}{\varepsilon} + \frac{L}{a}\right) - \left(\underline{\frac{E}{\varepsilon}} + \underline{\underline{\frac{L}{a}}}\right)$$

$$- \left(\frac{L}{a} - \frac{E}{\varepsilon}\right)\log\left(\frac{L}{a} - \frac{E}{\varepsilon}\right) + \left(\underline{\underline{\frac{L}{a}}} - \underline{\frac{E}{\varepsilon}}\right)$$

$$- 2\frac{E}{\varepsilon}\log\left(\frac{E}{\varepsilon}\right) + \underline{2\frac{E}{\varepsilon}}$$

$$\frac{\partial}{\partial E}\log(\Gamma(E)) = \underline{\frac{1}{\varepsilon}} + \underline{\frac{1}{\varepsilon}} - \underline{\frac{2}{\varepsilon}}$$

$$+ \frac{1}{\varepsilon}\log\left(\frac{E}{\varepsilon} + \frac{L}{a}\right) + \frac{1}{\varepsilon}\log\left(\frac{L}{a} - \frac{E}{\varepsilon}\right)$$

$$\boxed{\text{We want } \frac{\partial S}{\partial E} = \ldots}$$

$$- \frac{2}{\varepsilon}\log\left(\frac{E}{\varepsilon}\right)$$

$$= \frac{1}{\varepsilon}\log\left(\frac{\left(\frac{E}{\varepsilon} + \frac{L}{a}\right)\left(\frac{L}{a} - \frac{E}{\varepsilon}\right)}{\left(\frac{E}{\varepsilon}\right)^2}\right)$$

$$= \frac{1}{\varepsilon}\log\left(\frac{-\left(\frac{E}{\varepsilon}\right)^2 + \left(\frac{L}{a}\right)^2}{\left(\frac{E}{\varepsilon}\right)^2}\right)$$

$$\Rightarrow \frac{1}{T} = k\frac{1}{\varepsilon}\log\left(\frac{\left(\frac{L}{a}\right)^2 - \left(\frac{E}{\varepsilon}\right)^2}{\left(\frac{E}{\varepsilon}\right)^2}\right)$$

$$\Rightarrow \frac{\varepsilon}{kT} = \log\left(\frac{\left(\frac{L}{a}\right)^2 - \left(\frac{E}{\varepsilon}\right)^2}{\left(\frac{E}{\varepsilon}\right)^2}\right)$$

$$\Rightarrow \quad e^{\varepsilon/kT} = \frac{\left(\frac{L}{a}\right)^2 - \left(\frac{E}{\varepsilon}\right)^2}{\left(\frac{E}{\varepsilon}\right)^2}$$

$$\Rightarrow \quad \left(\frac{E}{\varepsilon}\right)^2 \left[e^{\varepsilon/kT} + 1 \right] = \left(\frac{L}{a}\right)^2$$

$$\Rightarrow \quad \frac{E}{\varepsilon} = \frac{\frac{L}{a}}{\left(1 + \exp\left(\frac{\varepsilon}{kT}\right)\right)^{1/2}}$$

Now: $\frac{\ell}{a} = \frac{E}{\varepsilon} + \frac{L}{a}$

$$\Rightarrow \quad \boxed{\frac{\ell}{a} = \frac{L}{a}\left[1 + \left(1 + e^{\varepsilon/kT}\right)^{-1/2} \right]}$$