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Math Methods in Physics

CH. 2 CALCULUS OF VARIATIONS LECTURE NOTES

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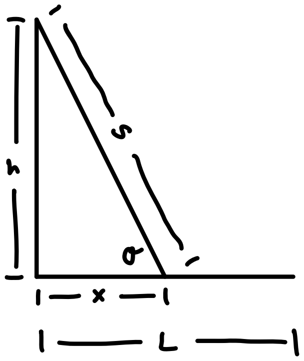
8-23-21



What angle gives the fastest speed?

Total Time : $t_{total} = t_1 + t_2$

$\frac{1}{2}at_1^2 = s$: $\frac{1}{2}g\sin\theta\, t_1^2 = \sqrt{h^2+x^2}$ $\therefore$ $t_1 = \left[\frac{2}{g}\frac{\sqrt{h^2+x^2}}{\sin\theta}\right]^{1/2}$: $\sin\theta = \frac{h}{\sqrt{h^2+x^2}}$

$t_1 = \left[\frac{2}{gh}(h^2+x^2)\right]^{1/2}$: $t_2 = \frac{L-x}{v_f}$: $\frac{1}{2}mv_f^2 = mgh$: $v_f = \sqrt{2gh}$ $\therefore$ $t_2 = \frac{L-x}{\sqrt{2gh}}$

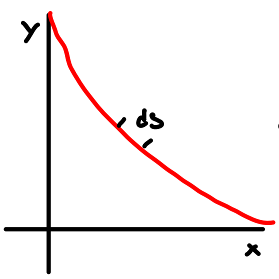
$t_{total} = \left[\frac{2}{g}\frac{\sqrt{h^2+x^2}}{\sin\theta}\right]^{1/2} + \frac{L-x}{\sqrt{2gh}}$: $\frac{dt_{total}}{dx} = \sqrt{\frac{2}{gh}}\frac{x}{\sqrt{x^2+h^2}} - \frac{1}{\sqrt{2gh}}$

$\sqrt{\frac{2}{gh}}\frac{x}{\sqrt{x^2+h^2}} - \frac{1}{\sqrt{2gh}} = 0$: $\sqrt{\frac{2}{gh}}\,x = \frac{\sqrt{x^2+h^2}}{\sqrt{2gh}}$: $\frac{2}{\cancel{gh}}x^2 = \frac{x^2+h^2}{2\cancel{gh}}$: $4x^2 = x^2+h^2$

$3x^2 = h^2$: $\frac{h^2}{x^2} = 3$: $\frac{h}{x} = \sqrt{3}$ $\therefore$ $\tan\theta = \sqrt{3} \rightarrow \theta = 60°$

$\theta = 60°$

Curve of fastest descent = Brachristachrone problem



$ds = \sqrt{dx^2+dy^2}$: $dt = \frac{ds}{v} = \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gy}} = \frac{\sqrt{1+(dy/dx)^2}}{y} \cdot \frac{1}{\sqrt{2g}}$

$t_{tot} = \frac{1}{\sqrt{2g}}\int_0^L \frac{\sqrt{1+(y')^2}}{y}\,dx$

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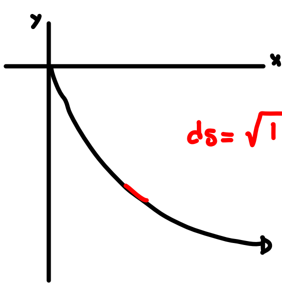


$ds = \sqrt{1+(y')^2}\,dx$

$T[y(x)]$: $\int_{x_0}^{x_1} \frac{1}{\sqrt{2g}}\sqrt{\frac{1+(y')^2}{y}}\,dx$: $T[y(x)]$ is a functional
$y(x) \rightarrow$ Scalar

Outline

1. unconstrained optimization
2. constrained optimization
3. Generalizations
4. Nuethers theorem

Extremization w/ no constraints

Assume we are given a functional

$$I[y(x)] = \int_{x_a}^{x_b} f(x, y(x), y'(x))\, dx$$

we wish to extremize I (optimal function $y(x)$)

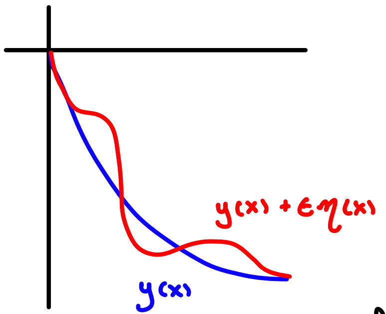
<u>Assumption</u>:

- End points $y(x_a)$ and $y(x_b)$ are fixed
- f is a well behaved function (derivatives, continuous)

<u>Big Idea</u>: If $y(x)$ is not an extreme then we can change it by $\Delta y(x)$ no change in $I(y(x) + \Delta y(x))$

If it is an extremum then small changes $\Delta y(x) \rightsquigarrow \Delta I \propto (\Delta y)^2$

We write our solution as: $\bar{y}(x) = y(x) + \epsilon\, y(x)$ where $y(x)$ is the true solution & $\eta(x)$ is a variation $\eta(x_a) = \eta(x_b) = 0$



$\eta(x)$ is any other well behaved function

If $y(x)$ is an extremum then $I(\epsilon)$ has a maximum where $\epsilon = 0$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_a}^{x_b} \left\{ \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x} \right\} dx = \int_{x_a}^{x_b} \left\{ \frac{\partial f}{\partial y}\frac{\partial \bar{y}}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial}{\partial x}\frac{\partial \bar{y}}{\partial x} \right\} dx$$

$$= \int_{x_a}^{x_b} \left\{ \frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta} \right\} dx = \int_{x_a}^{x_b} \left\{ \frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} \right\} \eta(x)\, dx + \left.\frac{\partial f}{\partial y'}\eta(x)\right|_{x_a}^{x_b}$$

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_a}^{x_b} \left\{ \frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} \right\} \eta(x)\, dx = 0$$

: From our definition : $\frac{\partial \bar{y}'}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}\frac{\partial \bar{y}}{\partial x} = \frac{\partial}{\partial x}\frac{\partial}{\partial \epsilon}\bar{y}$

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\frac{\partial f}{\partial y'} = 0 \longrightarrow \text{Euler Lagrange Equation}$$

Our problem was to extremize $T[y(x)] = \frac{1}{\sqrt{2g}}\int_{x_a}^{x_b} \sqrt{\frac{1+(y')^2}{y}}\, dx$

If $F(x, y(x), y'(x)) = f(y, y')$ that there is no explicit dependence on x on f then our Σ can be integrated once to arrive at :

$$y'\frac{\partial f}{\partial y'} - f = C_0 = \text{constant}$$

: <u>Proof</u> ∴

$$\frac{\partial c_0}{\partial x} = 0 = y''\frac{\partial f}{\partial y'} + y'\frac{\partial}{\partial x}\frac{\partial f}{\partial y'} + y'\frac{\partial}{\partial}\frac{\partial f}{\partial y} - \left\{ \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial x} \right\} : 0 = -y'\left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right\}$$

Back to our problem ... $f = \sqrt{\frac{1+(y')^2}{y}}$: $\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+(y')^2}}$: $\frac{y'}{\sqrt{y}}\frac{1}{\sqrt{1+(y')^2}} - \sqrt{\frac{1+(y')^2}{y}} = c$

$$\frac{(y')^2}{\sqrt{y(1+(y')^2)}} - \frac{(1+(y')^2)}{\sqrt{y(1+(y')^2)}} - \frac{-1}{\sqrt{y(1+(y')^2)}} = C_0 \longrightarrow y(1+(y')^2) = \frac{1}{C_0^2} = 2a$$

Solve for y'

$$(y')^2 = \frac{2a}{y} - 1 = \frac{2a-y}{y} \;:\; \frac{\partial y}{\partial x} = \sqrt{\frac{2a-y}{y}} \;:\; \int_{x_a}^{x} dx = \int \sqrt{\frac{y}{2a-y}}\, dy \;:\; (x-x_a) = \int_{y_a}^{y} \sqrt{\frac{y}{2a-y}}\, dy$$

$$y = a(1-\cos\theta) = 2a\cdot\sin^2\theta/2 \;:\; y = 2a\int \tan\frac{\theta}{2}\cdot 2a\sin\frac{\theta}{2}\, d\theta \rightsquigarrow a(\theta - \sin\theta)$$

$$x - x_a = a(\theta - \sin\theta) \;:\; y = a(1-\cos\theta)$$

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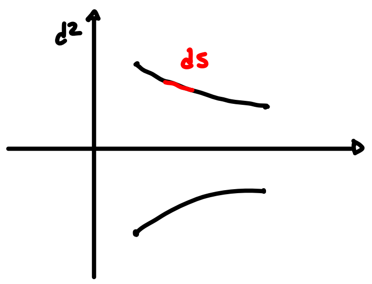
When taking a derivative of: $\frac{d}{dx}\left\{ y'\frac{\partial f}{\partial y'} - f \right\} = y''\frac{\partial f}{\partial y'} + \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{d}{dx}f$

Example: Soap Fins

Soap fins have an energy cost / area that drives the system to minimize the total area (Analog)



$$\text{Area} = 2\pi\int y\, ds = 2\pi\int y\sqrt{1+y'^2}\, dx$$

Then our $f(y,y',x) = y\sqrt{1+(y')^2} \longrightarrow y'\frac{\partial f}{\partial y'} - f = C_0$

$$y'\cdot y\cdot\frac{1}{2}(2y')\frac{1}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = C_0 \;:\; C_0\sqrt{1+(y')^2} = y(y')^2 - y(1+y'^2) = -y$$

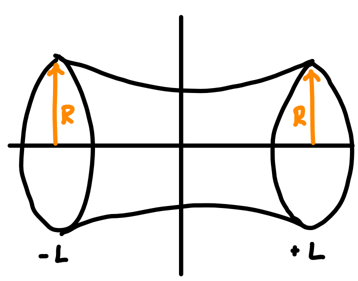
$$C_0\sqrt{1+(y')^2} = -y \;\therefore\; y'^2 = \frac{y^2}{C_0^2} - 1 \;:\; \frac{dy}{dx} = \sqrt{\frac{y^2}{C_0^2} - 1} \;:\; dx = \frac{dy}{\sqrt{y^2/C_0^2 - 1}} \;:\; y = C_0\cosh(t)$$

$$x - x_0 = \int \frac{C_0\sinh(t)}{\sqrt{\cosh^2 t - 1}}\, dt = \int C_0\, dt \;:\; x - x_0 = C_0(t - t_0) \;:\; \frac{x-x_0}{C_0} = \operatorname{arccosh}\left(\frac{y}{C_0}\right)$$

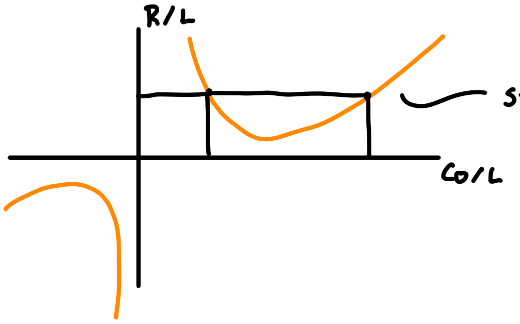
$$y = C_0\cosh\left(\frac{x-x_0}{C_0}\right)$$

In general finding the correct x_0 and C_0 for a given set of points on the curve is hard

Consider a simple geometry where the rings has equal radii



$$\frac{R}{L} = \frac{C_0}{L}\cosh\left(\frac{L}{C_0}\right)$$



Notes :

① we have assumed well behaved functions and at times the extremal solution is not well-behaved

② we looked at fixed endpoint problems. This requirement can be relaxed → this generates boundary equations that must be solved.

③ The E-L equation is a necessary but not sufficient condition → might be at a maxima & not a minima.

Extremization w/ constraints

First - Lets extremize a function subject by a constraint. Imagine a function $f(x_1, x_2)$ that we wish to extremize.

$f(x, x_0) = x_1^2 + x_0^2$, Clearly the extrema is at $x_1 = x_2 = 0$. Now we will impose a constraint.

$$g(x_1, x_2) = g_0$$

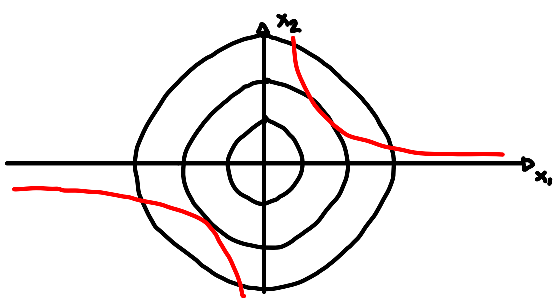
$$g(x_1, x_2) - g_0 = G(x_1, x_2) = 0$$

We could "invent" $g(x_1, x_2)$ to get

$$x_2 = X(g_0, x_1)$$

and substitute back into $F(x_1, x_2)$

$g(x_1, x_2) = x_1 x_2 - 1 = 0$



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$f(x_1, x_2) = x_1^2 + x_2^2 : g(x_1, x_2) = x_2 x_1 - 1 = 0$

An allowable step is one where $\Delta g = 0 = \bar{\nabla} g + \Delta \hat{r} = 0$

Along that step we want

$\Delta f = 0 = \vec{\nabla} \cdot f \cdot \vec{\nabla} r$: $\vec{\nabla} f \parallel \vec{\nabla} g$ They are parallel / anti-parallel $\vec{\nabla} f - \lambda \vec{\nabla} g = 0$

we extremize $K = f - \lambda g = k(x_1, x_2)$: $K = x_1^2 + x_2^2 - \lambda(x_2 x_1 - 1)$

$\frac{\partial k}{\partial x_1} = \partial x_1 - \lambda x_1 = 0$: $\partial x_1 = \lambda x_2$: $\partial x_2 = \lambda x_1$: $\frac{x_1}{x_2} = \frac{x_2}{x_1}$: If $x_1 x_2 - 1 = 0 \rightarrow x_1 = \pm 1$

Constraints on Functionals

There are two types of constraints

Local : These must hold at every point along the solution

Global : These are a property of an integral of the solution - property of the whole & not a point

Global or Local

Curve $y(x)$ total Length $L \rightarrow$ Global

Must stay on the surface of the sphere $\rightarrow$ Local

Must enclose a volume $V \rightarrow$ Local

Time to travel the path is $T \rightarrow$ Global

The time is a minimum $\longrightarrow$ Not a constraint

Global $\neq$ Holds everywhere : Global = property of the whole

Global Constraints

We consider problems of the Form

$I[y(x)] = \int_{x_a}^{x_b} f(x, y, y')\, dx$: $J[y(x)] = \int_{x_a}^{x_b} g(x, y, y')\, dx = J_0$: $\bar{y}(x) = y(x) + \epsilon_1 y_1(x) + \epsilon_2 y(x)$

We need two degrees of freedom because for a given value of ϵ_1 (& curve y_1) we need ϵ_2 (& y_2) to be able to satisfy the constraint

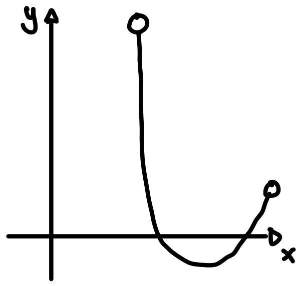
We introduce

$$K[y(x)] = I[y(x)] + \lambda J[y(x)] \quad \text{where} \quad K(\epsilon_1, \epsilon_2) = \int_{x_a}^{x_b} \left\{ f(x,y,y') - \lambda g(x,y,y') \right\} dx$$

We extremize w.r.t ϵ_1 and ϵ_2

$\left.\frac{\partial K}{\partial \epsilon_1}\right| = \frac{\partial K}{\partial \epsilon_2} = 0 \longrightarrow$ we will generate the identical Euler Lagrange equation that we have derived earlier. $-\frac{d}{dx}\frac{\partial h}{\partial y'} + \frac{\partial h}{\partial y} = 0$

Hangin Chain : Caternary



Gravitational potential energy - of a segment ds

$dv = dm\, g y = (\rho ds) g y \;:\; \rho = \frac{m}{L} \;:\; dv = \rho g y \sqrt{1+(y')^2}\, dx$

$u = \int_{x_a}^{x_b} \rho g y \sqrt{1+(y')^2}\, dx$ The length is L

$\int_{x_a}^{x_b} ds = \int \sqrt{1+(y')^2}\, dx = L \;:\; h = f - \lambda g = \rho g y \sqrt{1+(y')^2} - \lambda \sqrt{1+(y')^2}$

Set $\lambda = \rho g \tilde{y} \;:\; h = \rho g (y - \tilde{y}) \sqrt{1+(y')^2} \longrightarrow z = y - \tilde{y} \;:\; z' = y'$

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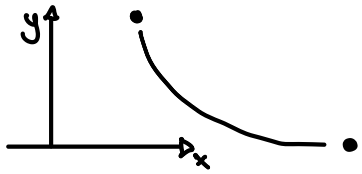
Catenary



Extremize

$K[y(x)] = \int_{x_a}^{x_b} \rho g \sqrt{1+(y')^2}\,(y - y_0)\, dx \;:\; \rho g y_0 = \lambda$

$h - y' \frac{\partial h}{\partial y'} = c_b$ Shift : $z = y - y_0$, $z' = y'$: $K[z(y)] = \int \rho g \sqrt{1+(z')^2}\, z\, dx$

$h - y' \frac{\partial h}{\partial y'} = c_b \Rightarrow h \cdot z' \frac{\partial h}{\partial z'} \;:\; \rho g \left\{ \sqrt{1+(z')^2}\, z - z' \cdot \frac{1/2\, z'(2)}{\sqrt{1+(z')^2}}\, z \right\} = c_0$

$\left\{ \frac{z(1+(z')^2) - (z')^2 z}{\sqrt{1+z'^2}} \right\} = \frac{c_0}{\rho g} \;:\; \frac{z}{\sqrt{1+z'^2}} = c_1 \;:\; \frac{z^2}{c_1^2} = (1+(z')^2) \;:\; z' = \sqrt{\frac{z^2}{c^2} - 1}$

$\frac{dz}{\sqrt{\frac{z^2}{c^2} - 1}} = dx \longrightarrow$ Lagrange Multiplier

Isoperimetric

Maximize the area for fixed perimeter : $A = \frac{1}{2} \int x\dot{y} - y\dot{x}\, dt$

Digression

What if we have more than one dependent variable ?

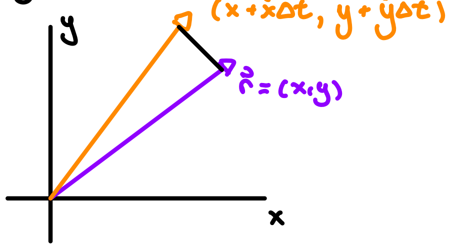
Independent variable $\rightarrow t$, Dependent variable $\rightarrow x, y$

$f(x, y, \dot{x}, \dot{y}, t) \longrightarrow \bar{y} = y + \epsilon_1 y_1$, $\bar{x} = x + \epsilon_2 y_2$

We must allow for two independent variations $\frac{dI(x(t), y(t), \epsilon_1, \epsilon_2)}{d\epsilon_1} = \frac{dI}{d\epsilon_2} = 0$

$I = \int_{t_a}^{t_b} f(x, y, \dot{x}, \dot{y}, t)\, dt$ All the math goes through as before $\frac{d}{dt}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} = 0 \rightarrow$ Same for y

Why is this the area Formula?



$\vec{r}_1 = x\hat{x} + y\hat{y}$, $\vec{r}_2 = (x+\dot{x}\Delta t)\hat{x} + (y+\dot{y}\Delta t)\hat{y}$

$\dot{\vec{r}}\Delta t = \vec{r}_1 + (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})\Delta t$

$A = \vec{r}_1 \times \vec{r}_2 = \vec{r}_1 \times (\vec{r}_1 + \dot{\vec{r}}\Delta t) = \vec{r}_i \times \dot{\vec{r}}$

$A = (x\hat{\imath} + y\hat{\jmath}) \times (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})\, dt = (x\dot{y} - y\dot{x})\,\hat{k}\, dt$: $dA = \frac{1}{2}(x\dot{y} - y\dot{x})dt$

$A = \frac{1}{2}\int (x\dot{y} - y\dot{x})\, dt$: $L = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

$h = \frac{1}{2}(x\dot{y} - y\dot{x}) - \lambda\left(\sqrt{\dot{x}^2 + \dot{y}^2} - L\right)$: $h(x, y, \dot{x}, \dot{y}, t)$

$\frac{d}{dt}\frac{\partial h}{\partial \dot{x}} - \frac{\partial h}{\partial x} = 0$: $\frac{d}{dt}\left(-\frac{1}{2}y - \frac{\lambda}{2}\cdot 2\dot{x}\,\frac{1}{(\dot{x}^2+\dot{y}^2)^{1/2}}\right) - \frac{1}{2}\dot{y} = 0$

$-\dot{y} - \frac{d}{dt}\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = 0$: $\dot{y} = -\lambda\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$

$y - y_0 = -\lambda\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$: $x - x_0 = -\lambda\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$

$(y-y_0)^2 + (x-x_0)^2 = \lambda^2 \cdot \cancel{\frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2+\dot{y}^2}}$: $(y-y_0)^2 + (x-x_0)^2 = \lambda^2$

Probability Distributions : Assume that we want to find the distribution of particle energies $p(t)$.

$\langle \ln p(E) \rangle$ is a maximum

The total energy : $E_{tot} = \int E\, p(E)\, dE$

Total Number : $N = \int p(E)\, dE$

$h[p(E)] = p(E)\ln(p(E)) - \lambda_1 E\, p(E) - \lambda_2\, p(E)$

$\frac{d}{dE}\frac{\partial h}{\partial p'} - \frac{\partial}{\partial p}h = 0$: $\ln(p) + p(E)\frac{1}{p(E)} - \lambda_1 E - \lambda_2 = 0$: $\ln(p) = \lambda_1 E - \lambda_2 - 1$

$p = \exp\{(\lambda_2 - 1)\}\, e^{+\lambda_1 E} = p_0\, e^{-E/kt}$

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Local Constraints

Local constraints are much stronger and more restrictive than global constraints

Example - Staying on a surface

In general we write them as $g(y_1, y_2, y_3) = g_0$

g is not a function of y_1, y_2, y_3 or x

Derivative constraints do not uniqually restrict coordinates. Insufficiently restrictive.

This is an equation not an inequality

How do you handle local constraints?

$g(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 = R_0^2$: Eliminating one of the variables $y_3 = \sqrt{R^2 - y_1^2 - y_2^2}$

Q: Why not always substitute this?

A: This may complicate things. Need symmetries that are helpful.

$g(y_1, y_2, y_3) = y_1 y_3 - (y_1 y_2 + y_2 y_3) = g_0$

$y_1 \rightarrow y_3$

So how can we enforce an infinite number of constraints?

Introduce $\lambda(x)$

$h = f(x_1, y_1, y_1', y_2', y_3' \dots) - \lambda(x) g(y_1, y_2, \dots)$

Extremize $\int_{x_a}^{x_b} h(x, y_1, y_1', y_2, y_2', \dots)\, dx$

As before when we derived our EL equation we will obtain $\longrightarrow$

$$\int \sum_i \left\{ \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i} - \lambda(x) \frac{\partial g}{\partial y_i} \right\} \eta_i(x)\, dx = 0$$

If the η were independent we would be ok

The η_i are not independent! $\bar{y}_i = y_i + \epsilon_i \eta_i$

Imagine that we have q constraints $\longrightarrow \eta_2, \eta_3, \dots \eta_q$

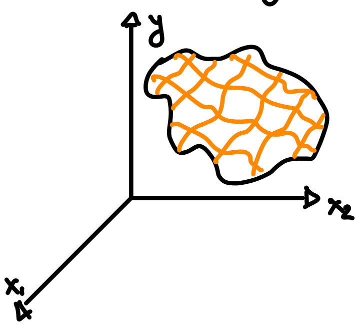
we require η_1 such that the constraint is satisfied

$y_1 + y_2 = \pi$

So now we simply require that $\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y_1} = -\lambda(x) \frac{\partial g}{\partial y_1}$

Several Independent Variables

Now we have $y = y(x_1, x_2)$ with $y(x_1,x_2)\Big|_{\text{Const } C} = y(c)$



we want to extremize $I[y(x_1,x_2)] : \int_{\text{surface bounded by } C} f(x_1 x_1, \partial y/\partial x_1, \partial y/\partial x_2, y)\, dx_1 dx_2$

$\bar{y} = y(x_1,x_2) + \epsilon\, \eta(x_1,x_2)$ require $\eta(x_1,x_2) = 0$

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Anholonomic or Non-Holonomic Constraints

B + f deal only with constraints of this form

$g(y_1, y_2, y_3, \ldots\ldots) = 0$

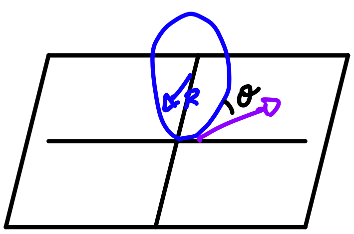
Dependent not y_1' or y_2'

Q: Is it possible to have constraints dependent on y_1'?

A: Yes, but its less clear

A constraint should restrict the path $\longrightarrow$ confined to contain region of the independent variables

Rolls w/o slipping $V = R\dot{\phi}$: $\dot{x} = V\sin\theta = R\sin\theta\,\dot{\phi}$: $\dot{y} = V\cos\theta = R\cos\dot{\phi}$



I can obtain any value of ϕ I want & return to exactly the same x, y, θ

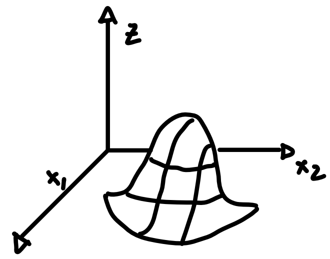
$dx = -R\sin\theta\, d\phi$, $dy = -R\cos\theta\, d\phi$

In a space with coordinates of x, y, θ, ϕ. A holonomic set of equations will restrict motion such that any pair of displacements "commute" $dx + dy \rightarrow dy + dx$

It is possible to have a constraint that involves derivatives $\partial y_1 \dot{y}_1 + \partial y_2 \dot{y}_2 + \partial y_3 \dot{y}_3 = 0$

$\frac{d}{dt}(y_1^2 + y_2^2 + y_3^2)$: Don't forget you can't have inequalities as a constraint

We were discussing multiple independent variables.



$$I[y(x_1,x_2)] = \iint f\left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, x_1, x_2\right) dx_1\, dx_2$$

$$\bar{y}(x_1,x_2) = y(x_1,x_2) + \epsilon\,\eta(x_1,x_2) \quad \text{w/} \quad \eta(x_1,x_2)\Big|_{x_1,x_2} = C$$

$$\frac{dI}{d\epsilon} = \iint \left\{ \frac{\partial f}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial f}{\partial(\partial y/\partial x_1)}\frac{d}{d\epsilon}\frac{\partial y}{\partial x_1} + \frac{\partial f}{\partial(\partial y/\partial x_2)}\frac{d}{d\epsilon}\frac{\partial y}{\partial x_2} \right\} dx_1 dx_2$$

$$\iint \left\{ \frac{\partial f}{\partial y}\eta(x_1,x_2) + \frac{\partial f}{\partial(\partial y/\partial x_1)}\frac{\partial \eta}{\partial x_1} + \frac{\partial f}{\partial(\partial y/\partial x_2)}\frac{\partial \eta}{\partial x_2} \right\} dx_1\, dx_2$$

B + F invok Green's Theorem

$$\iint \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right) dx_1\, dx_2 = \int_C P\,dx_2 - Q\,dx_1$$

<u>Lazy</u> <u>Physics</u> <u>Argument</u>

$$\iint (\vec{\nabla} \times \vec{B}) \cdot \hat{\eta}\, ds = \int_C \vec{B} \cdot d\vec{l} \quad : \quad d\vec{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} \quad : \quad \vec{B} = -Q(x_1,x_2)\,\hat{\imath} - P(x_1,x_2)\,\hat{\jmath} + 0\hat{k}$$

$$\vec{\nabla} \times \vec{B} \cdot \hat{n} = \left\{ \left[\frac{\partial 0}{\partial x_2} - \frac{\partial Q}{\partial x_3}\right]\hat{e}_1 + \left[\frac{\partial P}{\partial x_3} - \frac{\partial 0}{\partial x_1}\right]\hat{e}_2 + \left[\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2}\right]\hat{e}_3 \right\} \cdot \hat{n}$$

$$\iint (\vec{\nabla} \times \vec{B}) \cdot \hat{n}\, ds = \oint_C \vec{B} \cdot d\vec{l} = \oint (-Q\hat{\imath} + P\hat{\jmath}) \cdot (dx\,\hat{\imath} + dy\,\hat{\jmath}) = \int (P(x_1,x_2)\,dx\,\hat{\imath} - Q(x_1,x_2)\,dx\,\hat{\jmath})$$

$$\iint \left\{ \frac{\partial f}{\partial(\partial y/\partial x_1)}\frac{\partial \eta}{\partial x_1} + \frac{\partial f}{\partial(\partial y/\partial x_2)}\frac{\partial \eta}{\partial x_2} \right\} dx_1 dx_2 = \iint \left\{ \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial(\partial y/\partial x_1)}\eta\right) - \frac{\partial}{\partial x_1}\frac{\partial f}{\partial(\partial y/\partial x_1)}\eta \right. \longrightarrow$$

$$\longrightarrow \left. \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial(\partial y/\partial x_2)}\eta\right) - \frac{\partial}{\partial x_2}\frac{\partial f}{\partial(\partial y/\partial x_2)}\eta \right\} dx_1\, dx_2 = \longrightarrow$$

$$\longrightarrow -\iint \left\{ \frac{\partial}{\partial x_1}\frac{\partial f}{\partial(\partial y/\partial x_1)} + \frac{\partial}{\partial x_2}\frac{\partial f}{\partial(\partial y/\partial x_2)} \right\} \eta\, dx_1\, dx_2 + \oint_C \underbrace{\cancel{\eta(x_1,x_2)}}_{0} \left[\frac{\partial f}{\partial(\partial y/\partial x_1)}dx_2 - \frac{\partial f}{\partial(\partial y/\partial x_2)}dx_1 \right]$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx_1}\frac{\partial f}{\partial(\partial y/\partial x_1)} - \frac{d}{dx_2}\frac{\partial f}{\partial(\partial y/\partial x_2)} = 0 \longrightarrow \text{Two independent variables}$$

If we have $x_1, x_2, x_3, \ldots\ldots x_n$ w/ 1 dependent variable

$$\frac{\partial f}{\partial y} - \sum_j \frac{d}{dx_1}\frac{\partial f}{\partial(\partial y/\partial x_2)}$$: Unfortunately we lose the first integral shortcut.

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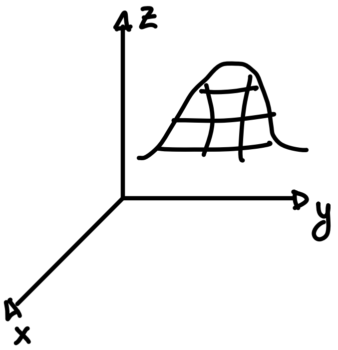
Several Independent Variables

If we have $y(x_1, x_2, \dots x_N)$ & $I[y(x_1,x_2,\dots x_N)] = \int f(y, \partial y/\partial x_1, \partial y/\partial x_2, \dots, x_1, \dots x_N)\, dx_{1,2}$

E.L EQN. $\dfrac{\partial f}{\partial y} - \sum_j \dfrac{d}{dx_j} \dfrac{\partial f}{\partial \, \partial y/\partial_j} = 0$

Minimal Surfaces



$$d\vec{v}_1 = dx\,\hat{\imath} + \frac{\partial z}{\partial x} dx\,\hat{k} \;,\quad d\vec{v}_2 = dy\,\hat{\jmath} + \frac{\partial z}{\partial y} dy\,\hat{k}$$

$$d\vec{A} = d\vec{v}_1 \times d\vec{v}_2 = dx\,dy\,(\hat{\imath}\times\hat{\jmath}) + dx\,dy\,\frac{\partial z}{\partial y}(\hat{\imath}\times\hat{k}) + dx\,dy\,\frac{\partial z}{\partial x}(\hat{k}\times\hat{\jmath})$$

$$d\vec{A} = dx\,dy\left(\hat{k} - \frac{\partial z}{\partial y}\hat{\jmath} - \frac{\partial z}{\partial x}\hat{\imath}\right)$$

$$dA = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy \;:\; A[z(x,y)] = \int \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy$$

$$f = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \;:\; 0 - \frac{d}{dx}\frac{\partial f}{\partial z_x} - \frac{d}{dy}\frac{\partial f}{\partial z_y} = 0$$

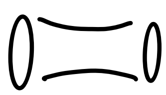
$$-\frac{\partial}{\partial x}\frac{z_x}{\sqrt{1+z_x^2+z_y^2}} - \frac{\partial}{\partial y}\frac{z_y}{\sqrt{1+z_x^2+z_y^2}} = 0$$

We can simplify : $\dfrac{\partial^2 z}{\partial x^2}\left(1 + \left(\dfrac{\partial z}{\partial y}\right)^2\right) + \dfrac{\partial^2 z}{\partial y^2}\left(1 + \left(\dfrac{\partial z}{\partial x}\right)^2\right) - 2\left(\dfrac{\partial z}{\partial x}\right)\left(\dfrac{\partial z}{\partial y}\right)\dfrac{\partial^2 z}{\partial x \partial y} = 0$

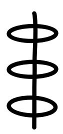
Minimal Surfaces are an area of current research

In physics - non Liquid crystals , non complex spaces

Catenoid



Helicoids



$x = \rho\cos\theta$, $y = \rho\sin\theta$

$z = \arctan(y/x)$: $\dfrac{\partial z}{\partial x} = \dfrac{-y}{x^2+y^2}$, $\dfrac{\partial z}{\partial y} = \dfrac{x}{x^2+y^2}$

Classical Mechanics

The calculus of variations is applied throughout classical mechanics, here the independent Variable is time t

$f(y, y', x) \longrightarrow f(x(t), \dot{x}(t), t)$ where the Lagrangian is $f = L = T - V$

For example : $L = \frac{1}{2} m \dot{x}^2 - V(x)$

The functional : $I[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt$

our Euler Lagrange Equation is

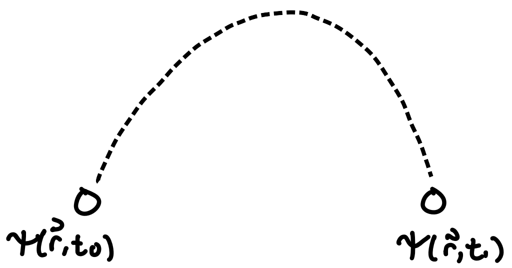
$$\frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0 \longrightarrow m\ddot{x} = -\frac{\partial V}{\partial x} = F \quad \text{(Newton's second Law)}$$

So .... why does this work?

① Because it yields Newton's 2nd Law

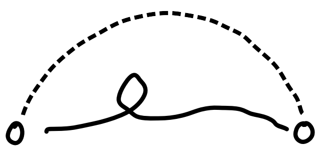
② It's a deep mystery

③ It's Quantum Mechanics



$$\Psi(\vec{r}, t_1) = \int \Psi(\vec{r}\,', t_1)\, G(\vec{r}, t_1, \vec{r}\,', t_0)\, d\vec{r}\,'$$

$$G(\vec{r}, t_1, \vec{r}\,', t_0) = \int \exp\left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} L(\vec{r}, \dot{r}, t)\, dt \right\} \partial[\vec{r}(t)] \longrightarrow \text{Path integral}$$



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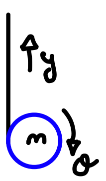
Simple Example

A string is wrapped around a disk of radius R and the disk is released from rest. Determine the motion.



$L = T - V$, $T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$, $V = mgy$: Constraint , $y = -R\theta$, $\dot{y} = -R\dot{\theta}$, $\ddot{y} = -R\ddot{\theta}$

$L = \frac{1}{2} m\dot{y}^2 + \frac{1}{2} I \dot{\theta}^2 - mgy - \lambda(t)(y + R\theta)$

$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \quad -mg - \frac{d}{dt} m\dot{y} - \lambda(t) = 0 \rightarrow mg + \frac{d}{dt} m\dot{y} + \lambda(t) = 0$

$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0 \quad -\frac{d}{dt} I\dot{\theta} - \lambda(t) R = 0 \rightarrow \frac{d}{dt} I\dot{\theta} + \lambda(t) R = 0$

$\ddot{y} = -g - \frac{\lambda(t)}{m}$, $\ddot{\theta} = -\frac{R}{I}\lambda(t)$: $\lambda(t) = -\frac{I}{R}\ddot{\theta} = -\frac{I}{R^2}\ddot{y}$: $\ddot{y} = -g - \frac{I}{R^2}\ddot{y}$: $\ddot{y} = -\frac{g}{1 + I/mR^2}$

$$y(t) = -\frac{1}{2} g \frac{t^2}{1+I/mR^2} + y_0 \quad : \quad m\ddot{y} = mg + \lambda(t) \,, \quad -\frac{mg}{1+I/mR^2} + mg + \lambda(t) = 0$$

$$mg\left(\frac{-1+(1+I/mR^2)}{1+I/mR^2}\right) + \lambda(t) = 0 \quad : \quad \lambda(t) = \frac{-I/mR^2}{1+I/mR^2}\, mg$$

What is $\lambda(t)$? It is the force necessary to keep the disk spinning at the correct rate.

Sometimes said that $\lambda(t)$ is the force necessary to maintain the constraint.

$[L]$ = Energy , constraint $x=y \rightsquigarrow \lambda(t)(x-y)$.

If our constraint is $g(y_1, y_2, y_1', y_2', t) - g_0 = 0$, The generalized force more properly is

$F_i = \lambda(t) \frac{\partial g}{\partial x_i}$.

<u>Complicated</u> <u>Example</u>

Charged particle moving in an Electromagnetic Field

Claim ~ $L = \frac{1}{2} m \dot{\vec{x}}^2 + q(\vec{A}(\vec{x}) \cdot \dot{\vec{x}} - \phi(\vec{x}))$, $\vec{x} = x_1(t)\hat{\imath} + x_2(t)\hat{\jmath} + x_3(t)\hat{k}$

$\vec{A} \to$ Magnetic vector magnitude , $\vec{\phi} \to$ Electric potential

There are three Euler Lagrange equations each of the form.

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

Note that $\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \frac{\partial \dot{x}_i}{\partial \dot{x}_j}$ w/ $\frac{\partial x_i}{\partial \dot{x}_j} = 0$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i(\vec{x},t) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i + q\left\{\frac{\partial A_i}{\partial t} + \sum_i \frac{\partial A_i}{\partial x_j}\dot{x}_j\right\}$$

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$$\frac{\partial L}{\partial x_i} = q\sum_j \frac{\partial A_j}{\partial x_i}\dot{x}_j - \frac{\partial \varphi}{\partial x_i} \quad : \quad m\ddot{x}_i + q\left\{\left[\frac{\partial \varphi}{\partial x_i} + \frac{\partial A_i}{\partial t}\right] + \sum_j\left[\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right]\dot{x}_j\right\}$$

$$m\ddot{x}_i = q[\vec{\mathcal{E}} + \dot{\vec{x}} \times \vec{B}]$$

To show that we get the $\dot{\vec{x}} \times \vec{B}$ term, consider the first component

$$(\dot{\vec{x}} \times \vec{B})_1 = (\dot{\vec{x}} \times \vec{\nabla} \times \vec{A})_1 = \dot{x}_2(\vec{\nabla}\times\vec{A})_3 - \dot{x}_3(\vec{\nabla}\times\vec{A})_2$$

$$= \dot{x}_2\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) - \dot{x}_3\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) - \frac{\partial A_1}{\partial x_2}\dot{x}_2 - \frac{\partial A_1}{\partial x_3}\dot{x}_3 + \frac{\partial A_2}{\partial x_1}\dot{x}_2 + \frac{\partial A_3}{\partial x_1}\dot{x}_3$$

$$= \sum_j \left[\frac{\partial A_1}{\partial x_j} \dot{x}_j - \frac{\partial A_j}{\partial x_1} \dot{x}_j \right]$$

Often the Lagrangian L does not explicitly depend upon one of the coordinates in your classical mechanical problem.

The corresponding E.L equation : $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$

L does not depend explicitly on q. We say "q is a cyclic co-ordinate." Then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \quad , \quad \frac{\partial L}{\partial \dot{q}} = \text{const} \longrightarrow \text{" canonical momentum"}$$

$$L = \frac{1}{2} m\dot{x}^2 + V(x) \; : \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i = p_i$$

For a free particle : $H = \frac{\vec{p}^2}{2m}$, w/ a $\vec{B}$ field $H = \frac{1}{2m}(\vec{P} - q\vec{A})^2$

Noether's Theorem

" If an infinitesimal transformation on the dependent and / an independent variable leaves the action invariant , then there is a conserved quantity associated with the transformation "

So assume we make the most general variation around the classical path $y(x)$

$$\bar{y}(x) = y_i(x) + \epsilon\eta_i(x, y_i, y_i') \quad , \quad \bar{x} = x + \epsilon(x_i, y_i, y_i')$$

Further we will assume that the action is invariant

Our plan is to calculate $I(\epsilon) = \int_{x_a}^{x_b} F(\vec{x}, \vec{y}_i, \vec{y}_i')\, dx$

and expand to lowest order in ϵ. Then what we need to know is that the actual shift w/ the action / Functional is .....

We expand to lowest order in ϵ. We will derive an algorithm (set of values) you give an η and a conserved quantity

$$\bar{y}(x) = y(\bar{x} - \epsilon\xi) + \epsilon\eta(\bar{x}, \bar{y}, \bar{y}') \ldots\ldots = y(\bar{x}) - \epsilon\frac{\partial y}{\partial x}\xi + \epsilon\eta = y(\bar{x}) + \epsilon(\eta - \partial y_i/\partial x)$$

We can plug this into our integral

$$I(\epsilon) = \int_{\bar{x}_a}^{\bar{x}_b} f(x, y_i + \epsilon p_i, y_i' + \epsilon p')\, d\bar{x} \quad , \quad \bar{x}_a = x_a + \epsilon\xi(x, y_i, y_i') \; , \; \bar{x}_b = x_b + \epsilon\xi(x, y_i, y_i')$$

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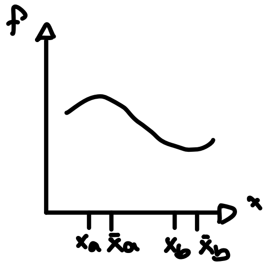
$$I[\epsilon] = \int_{x_a}^{x_b} f(\bar{x}, \bar{y}_i, \bar{y}_i')\, d\bar{x} \longrightarrow \text{change the limits to } x_a \text{ and } x_b$$



$$I[\epsilon] = \int_{x_a}^{x_b} f(\bar{x},\bar{y}_i,\bar{y}_i')\,d\bar{x} = \int_{x_a}^{x_b} f(\bar{x},\bar{y}_i,\bar{y}_i')\,d\bar{x} = \int_{x_b}^{x_b+\delta x_b} f(\bar{x},\ldots\ldots,)\,d\bar{x}$$

$$-\int_{x_a}^{x_a+\delta x_a} f(\bar{x},\bar{y}_i,\bar{y}_i')\,d\bar{x} = \int_{x_b}^{x_b+\delta x_b} f\,d\tilde{x} = \int (\bar{x}_i\bar{y}_i,\bar{y}_i')\Big|_{x=x_b}$$

At the other end point

$$-\epsilon f \xi\Big|_{x=x_b} = -\int_{x_a}^{x_a+\delta x_a}$$

Put this together : $I(\epsilon) = \int_{x_a}^{x_b} f(\bar{x},y_i,y_i')\,d\bar{x} + \epsilon\int\left[\frac{\partial f}{\partial \bar{y}_i}p_i + \frac{\partial f}{\partial \bar{y}_i}p_i'\right]d\bar{x} + \epsilon f\Big|_{x_a}^{x_b}$

$$I(\epsilon) = I(0) + \epsilon\left\{\int_{x_b}^{x_a}\left\{\frac{\partial f}{\partial \bar{y}_i} - \frac{d}{dx}\frac{\partial f}{\partial \bar{y}_i'}\right\}p_i + \epsilon\left\{p_i\frac{\partial f}{\partial \bar{y}_i}\Big|_{x_a}^{x_b} + f\xi\Big|_{x_a}^{x_b}\right\}\right\}$$

$\rightarrow = 0$

$$I(\epsilon) - I(0) = \epsilon\left\{p\frac{\partial f}{\partial y_i'} + f\xi\right\}\Big|_{x_a}^{x_b} = 0 \;:\; p_i\frac{\partial f}{\partial \bar{y}_i'} + f\xi\Big|_{x_a} = p_i\frac{\partial f}{\partial \bar{y}_i'} + f\xi\Big|_{x_b}$$

① Identify a transformation that leaves $f(L)$ invariant* to order ϵ. What is η? what is ξ?

② Thus $p_i\frac{\partial L}{\partial y'} + f\xi = \text{const}$

<u>Example</u> : Is space translation invariance ?

N isolated interacting particles in 1D

$$L = \sum_{i=1}^{N}\frac{1}{2}m\dot{y}_i^2 - \sum_{i\neq j}V(y_1-y_2)$$

My transformation is $\bar{y}_1 = y_1 + (1)\epsilon$, $\bar{x} = x + (0)\epsilon$: $\eta = 1$, $\xi = 0$

$$\eta\frac{\partial F}{\partial y'} + \xi(f - y'\partial f/\partial y') = \text{const.}$$

$$p_i = \eta - \frac{\partial L}{\partial y} + \xi(f - y'\partial f/\partial y') \;:\; p_i\frac{\partial f}{\partial \bar{y}_i'} + f\xi\Big|_{x_a} = p_i\frac{\partial f}{\partial \bar{y}_i'} + f\xi\Big|_{x_b}$$

$$p = \eta - \xi y', \quad \delta x_n = \epsilon\xi|_{x_a}, \quad \delta x_b = \epsilon\xi|_{x_b}$$

$$\eta\frac{\partial f}{\partial \bar{y}} + \xi(f - y'\partial f/\partial y') = \text{const.}$$

$$\sum_i \eta_i\frac{\partial L}{\partial \dot{y}_i} + \overset{0}{\cancel{\xi}}\left(f - \sum_i \dot{y}_i\,\partial f/\partial \dot{y}\right), \quad \sum_i\frac{\partial L}{\partial \dot{y}_i} = \text{const}, \quad \sum_i m\dot{y}_i = \text{total momentum}$$

Spacial translation invariation $\longrightarrow$ momentum conservation

The infinitesimal translation leaves L (or f) invariant up to a total derivative $\frac{d\Lambda}{dt}$

Assume we do this &

$$\int L(t, y, \dot{y})\,dt \quad L \to L + \alpha y\dot{y} = L + \frac{1}{2}\alpha y^2 \Big|_{x_a}^{x_b}$$

Example: Time Translation

① Transformation : $y_i \to y_i + \epsilon(0)$, $t \to t + \epsilon(1)$: $\eta_i = 0$, $\xi = 1$

$$f - y' \frac{\partial f}{\partial y'} = \text{const.} \; : \; L - \dot{y}_i \frac{\partial L}{\partial \dot{y}_i} = \sum_i \frac{1}{2} m \dot{y}_i^2 - \sum_i U(y_i) - \dot{y} \sum m\dot{y}^2 = \text{const.} : -T - U = \text{const}$$

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① Invent an infinitesimal transformation

$$\bar{y}_i = y_i + \epsilon \eta_i \, , \; \bar{x}_i = x_i + \epsilon \xi_i$$

② verify that under this

$$f \to f + \frac{\partial \Lambda}{\partial x} + \mathcal{O}(\epsilon^2)$$

③ Then

$$\sum \eta_i \frac{\partial f}{\partial y_i} + \xi \left(F - \sum y_i' \partial F / \partial y_i'\right) - \Lambda = \text{const.} = \text{Noether charge}$$

Time

$$\bar{y}_i = y + \epsilon(0) \, , \; \bar{x} = x + \epsilon(1) \, , \; \bar{t} = t + \epsilon(1) \; : \; \text{So } \eta = 0 , \; \xi = 1$$

② If L is not a function of time $\to L \to L$

③ $$(0) \sum_i \frac{\partial L}{\partial y_i} + (1)\left[L - \sum_i \dot{y}_i \frac{\partial L}{\partial \dot{y}_i}\right] = \text{const} = -E$$

$$y_i \to y + \epsilon \dot{y}_i \, , \; t \to t + \epsilon(0) \, , \; \dot{y}_1 = \dot{y}_1 + \epsilon \ddot{y}_1 \, , \; \eta_1 = \dot{y}_1$$

$$\frac{\partial L}{\partial \epsilon} = \left(\frac{\partial L}{\partial y_i} y' + \frac{\partial L}{\partial \dot{y}} \ddot{y}\right) + c = \epsilon\left(\frac{\partial L}{\partial y}\dot{y} + \frac{\partial L}{\partial \dot{y}}\ddot{y}\right) = \frac{d}{dt}(L)$$

Noether Charge $= \sum_i \eta_i \frac{\partial L}{\partial \dot{y}_i} - L = E$: So, Noethers theorem when it applies will always generate a constant of the motion but not a new one

r → Generate an old one
→ Generate a function of conserved quantity
→ Generate a Noether
→ New quantity

Noethers Theorem For a Field

Given a field $\psi(\bar{x},t)$ the Lagrangian density might be

$$\mathcal{L} = \frac{\hbar^2}{2m}(\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*) + V(\bar{x})\psi^*\psi - \frac{i\hbar}{2}(\dot{\psi}\psi^* - \dot{\psi}^*\psi) \; : \; I[\psi(x,t)\psi^*(x,t)] = \int d^3x\,dt\,\mathcal{L}$$

$$\frac{\partial^2}{\partial x^2}\frac{\hbar^2}{2m}\psi = \epsilon\psi \; : \; \psi = \sin(kx) \text{ w/ } k = \frac{\pi}{L}n$$

we can view this $\psi = R(x,t) + i\,Im(x,t)$ and vary this parts seperately. We treat ψ and ψ^* as independent variables of each other.

- Our dependent variables are ψ and ψ^*
- Our independent variables are $\bar{x}$ & t w/ $x_1, x_2, x_3 = \bar{x}$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \; : \; \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}} + \sum_i \frac{\partial}{\partial x_j}\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_j)} - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0 \; : \; \text{Total variation w.r.t } \psi^*$$

$$\frac{\partial}{\partial t}\left(\frac{i\hbar}{2}\psi\right) + \sum_j \frac{\partial}{\partial x_j}\left(\frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x_j}\right) - \left[V(x)\psi(x) - \frac{i\hbar}{2}\dot{\psi}\right] = 0$$

$$-i\hbar\dot{\psi} + \sum_j \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_j^2} - V(x)\psi = 0 \; : \; i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + V(\vec{x})\psi(x,t)$$

Do we have a Noether charge? Can we find a nontrivial example?

$$\psi \to \psi(1+i\epsilon), \quad \psi^* \to \psi(1-i\epsilon) \; : \; L \to L + (\;)\epsilon^2$$

How many η_i's do we have? $\eta_1 = \eta_{(\psi)} = i\psi$, $\eta_2 = \eta_{(\psi^*)} = -i\psi^*$, $\xi = 0$

9-22-21

$$\mathcal{L} = \frac{\hbar^2}{2m}\vec{\nabla}\psi\cdot\vec{\nabla}\psi^* + V(x)\psi^*\psi = \frac{i\hbar}{2}(\dot{\psi}\psi^* - \dot{\psi}^*\psi)$$

$$I = \int d^3x\,dt\;\mathcal{L} \longrightarrow \psi \to (1+i\epsilon)\psi, \quad \psi^* \longrightarrow (1-i\epsilon)\psi^*$$

so $\eta_1 = i\psi$, $\eta_2 = -i\psi^*$, $\xi_x = \xi_t = 0$, $\Lambda = 0$

Q: Can use our earlier formula?

A: We have to alter it because we have 4 independent variables. Previously we had

$$\int_{x_a}^{x_b} dx\,\frac{\partial f}{\partial y'}\frac{\partial P}{\partial x}$$

So integration by parts will be more complicated. Now we have

$$\int d^4x \sum_{i=0}^{3} \frac{\partial L}{\partial(\partial\psi/\partial x_i)}\frac{\partial P_1}{\partial x_i} + \frac{\partial L}{\partial(\partial\psi/\partial x_i)}\frac{\partial P_2}{\partial x_i}$$

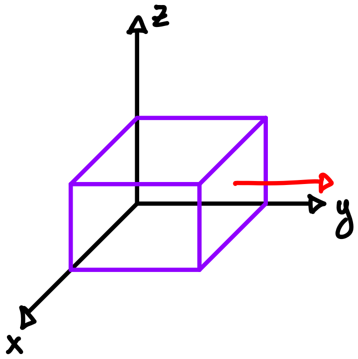
$$= -\int d^4x \sum_{i=0}^{3} \left\{ \frac{\partial}{\partial x_i} \frac{\partial L}{\partial(\partial\psi/\partial x_i)} \rho_1 + \frac{\partial}{\partial x_i} \frac{\partial L}{\partial(\partial\psi/\partial x_i)} \rho_2 \right\}$$

$$+ \sum_i \oint \left\{ \frac{\partial L}{\partial(\partial\psi/\partial x_i)} \rho_1 + \frac{\partial L}{\partial(\partial\psi^*/\partial x_i)} \rho_2 \right\} \hat{\eta}_i \cdot d\xi_i$$

Before with one independent variable the boundary term was of the form

$$\left(\eta - \xi y'\right) \frac{\partial f}{\partial y'} \Big|_{x_a}^{x_b} = 0 \quad \text{Now} \quad \oint \left(\eta - \xi y'\right) \vec{\nabla} f \cdot \hat{n}\, ds = 0$$

In 3D we can draw a picture



For $i = 1,2,3$

$$\frac{\partial L}{\partial\left(\frac{\partial\psi}{\partial x_i}\right)} \eta_1 = \left(\frac{\partial\psi^*}{\partial x_i}\right)(i\psi)\frac{\hbar^2}{2m}, \quad \frac{\partial L}{\partial\left(\frac{\partial\psi^*}{\partial x_i}\right)} \eta_2 = \left(\frac{\partial\psi}{\partial x_i}\right)(-i\psi^*)\frac{\hbar^2}{2m}$$

These terms will yield $\int dt \int \hat{\eta} \cdot \frac{\hbar^2}{2m}\left(\psi \vec{\nabla}\psi^* - \psi^* \vec{\nabla}\psi\right) \cdot d\vec{s}$

The time term yields : $\frac{\partial L}{\partial\dot{\psi}} \eta_1 = -\frac{i\hbar}{2} \psi^*(i\psi)$, $\frac{\partial L}{\partial\dot{\psi}} \eta_2 = \frac{i\hbar}{2} \psi(-i\psi^*)$

$$\hbar \int d^3x \; \psi^*(\vec{x})\psi(\vec{x})\, d^3x \Big|_{t_a}^{t_b}$$