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Math Methods in Physics

PHYS 5013 HOMEWORK ASSIGNMENT #1

PROBLEMS: $\{1, 2, 2.6, 2.7\}$

Due: September 1, 2021

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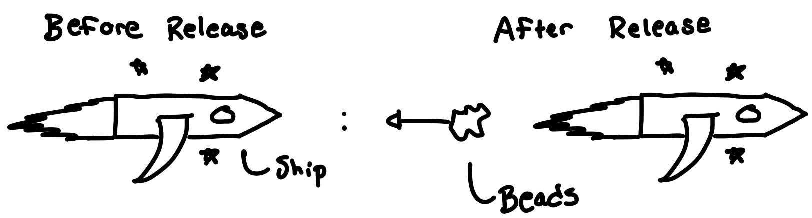
Problem 1:

You are in a rocket ship, in outer space. You have a nuclear reactor that supplies a *constant* power, P_0, and a large supply of iron pellets. The iron pellets comprise $99/100$ of your ship's mass, m. You can use the power to eject the tiny iron beads out the back of your ship with and electromagnetic "gun". You can control the *rate* at which you fire them and their velocity, but are limited by your power plant. (You can't fire an arbitrarily large mass at an arbitrarily large velocity.) As you fire off the beads, your ship moves in the opposite direction to conserve momentum. In addition, the mass of your ship decreases. (You can solve this using a local constraint, but that's the hard way.)

(a) If you use the energy of your reactor over a time $\triangle t$ to launch a packet of mass $\triangle m$ out the rear of your ship, what is the momentum of this "exhaust" packet relative to your ship?



Equation for Power : $P = E/t$: Equation for energy : $E = \bar{p}^2/2m$

$$E = P\cdot\Delta t = \frac{\bar{p}^2}{2m} \quad \therefore \quad \bar{P} = \sqrt{2P_0\Delta m\Delta t}$$

$$\boxed{\bar{P} = \sqrt{2P_0\Delta m\Delta t}}$$

(b) Now assume that you fire pellets continuously at a constant rate during the interval $0 < t < t_f$. If you start from rest, what is your final velocity?

$$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt} \ : \ dv = -\frac{u}{m}dm \ : \ \int dv = -u\int_{m_i}^{m_f}\frac{1}{m}dm \ : \ V_f = -u\ln(m)\Big|_{m_i}^{m_f} = -u\ln(m_f/m_i)$$

$$u = \frac{P}{m} = \frac{\sqrt{2P_0\Delta m\Delta t}}{\Delta m} = \sqrt{2P_0\Delta t/\Delta m} = \sqrt{2P_0/\Delta m'} \ : \ V_f = -\sqrt{2P_0/\Delta m'}\ln(m_f/m_i)\Big|_{m_i = m}^{m_f = 1/100\, m}$$

$$V_f = -\sqrt{2P_0/\Delta m'}\ln(1/100\, m/m) = \sqrt{2P_0/\Delta m'}\ln(100) \ : \ V_f = \sqrt{2P_0/\Delta m'}\ln(100)$$

$$\boxed{V_f = \sqrt{2P_0/\Delta m'}\ln(100)}$$

(c) However, you do not have to fire pellets at a constant rate. Find the optimal firing rate dm/dt in the interval $0 < t < t_f$ so that your final velocity is a maximum after a time t_f, assuming that you started from rest.

$$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt} \ : \ E = P\Delta t \ : \ \frac{1}{2}\Delta m u^2 = P_0\Delta t \ \therefore \ u = \sqrt{\frac{2P_0\Delta t}{\Delta m}} \ : \ \frac{dv}{dt} = -\frac{\sqrt{2P_0/\dot{m}}}{m(t)}\frac{dm}{dt}$$

$$\frac{kg\cdot m^2}{s^3}\cdot\frac{s}{kg} = \frac{m^2}{s^2}$$

$$\frac{dv}{dt} = -\frac{\sqrt{2P_0/\dot{m}}}{m(t)}\dot{m} = -\frac{\sqrt{2P_0\dot{m}}}{m(t)} \ : \ \int dv = -\int\frac{\sqrt{2P_0\dot{m}}}{m(t)}dt \ : \ V_f = -\int\frac{\sqrt{2P_0\dot{m}}}{m(t)}dt$$

$$f = \frac{\sqrt{2P_0\dot{m}}}{m} \ : \ \frac{\partial f}{\partial m} - \frac{d}{dt}\frac{\partial f}{\partial \dot{m}} = 0 \ : \ \dot{m}\frac{\partial f}{\partial \dot{m}} - f = const. \ : \ \frac{\partial f}{\partial \dot{m}} = \frac{1}{2}\sqrt{\frac{2P_0}{\dot{m}}}\frac{1}{m}$$

Problem 1: Continued

$$\dot{m}\frac{\partial f}{\partial \dot{m}} - f = \dot{m}\cdot\frac{1}{2}\sqrt{\frac{2P_0}{\dot{m}}}\,\frac{1}{m} - \frac{\sqrt{2P_0\dot{m}}}{m} = \frac{1}{2}\frac{\sqrt{2P_0\dot{m}}}{m} - \frac{\sqrt{2P_0\dot{m}}}{m} = -\frac{1}{2}\frac{\sqrt{2P_0\dot{m}}}{m}$$

$$K = -\frac{1}{2}\frac{\sqrt{2P_0\dot{m}}}{m} \quad : \quad -2Km = \sqrt{2P_0\dot{m}} \quad : \quad 4K^2m^2 = 2P_0\dot{m} \quad : \quad \dot{m} = \frac{2K^2m^2}{P_0} \quad : \quad \frac{dm}{dt} = \frac{2K^2m^2}{P_0}$$

$$\boxed{\frac{dm}{dt} = \frac{2K^2m^2}{P_0}}$$

(d) What is your final velocity in part (c)? How does it compare to the answer it part (b)?

$$\frac{dm}{dt} = \frac{2K^2m^2}{P_0} \quad : \quad \int \frac{dm}{m^2} = \int \frac{2K^2}{P_0}\,dt \quad : \quad \frac{1}{m}\Big|_{M}^{1/100\,M} = \frac{2K^2}{P_0}\Big|_0^{t_f} \quad : \quad -\frac{99}{M} = \frac{2K^2}{P_0}t_f$$

$$\frac{99P_0}{2Mt_f} = K^2 \quad : \quad K = \sqrt{\frac{99P_0}{2Mt_f}} \quad : \quad V_f = \int_0^{t_f} K\,dt = K\cdot t_f = \sqrt{\frac{99P_0}{2Mt_f}}\,t_f$$

$$\boxed{V_f = \sqrt{\frac{99P_0}{2Mt_f}}\,t_f}$$

Problem 1: Review

Procedure:

- Use the power equation: $P = E/t$ and $E = p^2/(2m)$, to solve for momentum.
- Use the differential equation $dv/dt = -u/m \ dm/dt$ to solve for v with $u = p/\triangle m$.
- Use the same differential equation in (b) to create a functional that can be maximized, then solve for the differential equation: dm/dt.
- Solve for velocity using the differential equation found in part (c).

Key Concepts:

- Extremizing functionals with the use of Euler Lagrange equation.
- Since the functional in part (c) is not explicitly dependent upon the independent variable, the first derivative Euler Lagrange equation can be used.

Variations

- The mass of pellets can be changed.
  - This would change integration limits.
- Pullets can be fired at a non-constant rate.
  - This would change the problem drastically, but the same procedure would more or less be used.
- The functional could end up being explicitly dependent upon the independent variable.
  - Thus making us use the full Euler Lagrange equation.

Problem 2:

Consider the functional

$$I[y(x), y'(x)] = \int_0^{x_f} \left\{ \left(\frac{\partial y}{\partial x}\right)^2 + \alpha y \frac{\partial y}{\partial x} \right\} dx.$$

(a) Find the function $y(x)$ that extremizes I subject to the boundary conditions that $y(0) = 0$ and $y(x_f) = y_f$.

Euler Lagrange equation : $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$: $f = \left(\frac{\partial y}{\partial x}\right)^2 + \alpha y \frac{\partial y}{\partial x}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\left(\frac{\partial y}{\partial x}\right)^2 + \alpha y \frac{\partial y}{\partial x} \right) = \alpha \cdot \frac{\partial y}{\partial x} \quad : \quad \frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(\left(\frac{\partial y}{\partial x}\right)^2 + \alpha y \frac{\partial y}{\partial x} \right) = 2\left(\frac{\partial y}{\partial x}\right) + \alpha y$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(2\left(\frac{\partial y}{\partial x}\right) + \alpha y \right) = 2\left(\frac{\partial^2 y}{\partial x^2}\right) + \alpha \cdot \left(\frac{\partial y}{\partial x}\right)$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad : \quad \cancel{\alpha \cdot \frac{\partial y}{\partial x}} - 2\left(\frac{\partial^2 y}{\partial x^2}\right) - \cancel{\alpha\left(\frac{\partial y}{\partial x}\right)} = 0 \quad \therefore \quad \frac{\partial^2 y}{\partial x^2} = 0$$

$y'' = 0 \quad \therefore \quad y(x) = mx + b \quad : \quad y(0) = 0 \quad \therefore \quad b = 0 \quad : \quad y(x_f) = y_f \quad \therefore \quad m = y_f / x_f$

$$\boxed{y(x) = \frac{y_f}{x_f}(x)}$$

(b) How does your answer depend upon α? Why?

$y(x)$ does not depend on α. This is because $y(x)$ is a strait line that is only dependent upon the boundary conditions and nothing else.

Problem 2: Review

Procedure:

- Begin by applying the Euler Lagrange equation to the defined functional.
- Solve for $y(x)$ with the differential equation given after using the Euler Lagrange equation.
- Apply boundary conditions and solve.
- Answer the qualitative questions with the above equations.

Key Concepts:

- Functionals with explicit independent variable dependence require the full Euler Lagrange equation.
- Use the Euler Lagrange equation to maximize the functional.
- The shortest path is not dependent upon the parameter α.

Variations:

- We can have a different functional that is defined.
  - We would then run through the same process but with a different functional.

Problem 3: 2.6

In all of our discussions so far on finding the function f for which $I = \int_{x_1}^{x_2} f\,dx$ is an extremum, it has been assumed that f depends only on x, y, and y':

$$f = f(x, y, y').$$

Show that if f also involves the second derivative of y with respect to x, $f = f(x, y, y', y'')$, then for fixed endpoints and prescribed y' at the endpoints, the Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

What continuity assumptions must be made about derivatives of y?

$$I(\epsilon) = \int_{x_A}^{x_B} f(x, y, y', y'')\,dx \quad : \text{ we must have } \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = 0$$

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B}\left[\frac{\partial f}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial f}{\partial y'}\frac{dy'}{d\epsilon} + \frac{\partial f}{\partial y''}\frac{dy''}{d\epsilon}\right]dx = \int_{x_A}^{x_B}\left[\frac{\partial f}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{dy}{d\epsilon}\right) + \frac{\partial f}{\partial y''}\frac{d^2}{dx^2}\left(\frac{dy}{d\epsilon}\right)\right]dx$$

$$\frac{dy'}{d\epsilon} = \frac{d}{dx}\left(\frac{dy}{d\epsilon}\right) \quad \therefore \quad \frac{dy''}{d\epsilon} = \frac{d^2}{dx^2}\left(\frac{dy}{d\epsilon}\right)$$

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B}\frac{\partial f}{\partial y}\frac{dy}{d\epsilon}\,dx + \left[\frac{dy}{d\epsilon}\frac{\partial f}{\partial y'}\right]_{x_A}^{x_B} - \int_{x_A}^{x_B}\frac{dy}{d\epsilon}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx + \left[\frac{dy}{d\epsilon}\frac{\partial f}{\partial y''}\right]_{x_A}^{x_B} + \int_{x_A}^{x_B}\frac{dy}{d\epsilon}\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx$$

$$y(x_A, \epsilon) = y_A \text{ and } y(x_B, \epsilon) = y_B \quad \therefore \quad \left.\frac{dy}{d\epsilon}\right|_{x=x_A} = \left.\frac{dy}{d\epsilon}\right|_{x=x_B} = 0$$

$$\frac{dI}{d\epsilon} = \int_{x_A}^{x_B}\frac{\partial f}{\partial y}\frac{dy}{d\epsilon}\,dx + \overset{0}{\cancel{\left[\frac{dy}{d\epsilon}\frac{\partial f}{\partial y'}\right]_{x_A}^{x_B}}} - \int_{x_A}^{x_B}\frac{dy}{d\epsilon}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx + \overset{0}{\cancel{\left[\frac{dy}{d\epsilon}\frac{\partial f}{\partial y''}\right]_{x_A}^{x_B}}} + \int_{x_A}^{x_B}\frac{dy}{d\epsilon}\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)dx$$

$$\frac{dI}{d\epsilon} = \left[\int_{x_A}^{x_B}\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\frac{dy}{d\epsilon} \quad \text{w/} \quad \left.\frac{dy}{d\epsilon}\right|_{\epsilon=0} = \eta(x)$$

$$\frac{dI}{d\epsilon} = \left[\int_{x_A}^{x_B}\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\eta(x) = 0$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0}$$

Problem 3: 2.6 Review

Procedure:

- Expand the first order Euler Lagrange equation in terms of second order differentials.
- Integrate the new functional by parts.
- Define the end points of integration with $y_a = y(x_a, \epsilon)$, $y_b = y(x_b, \epsilon)$.
- Deduce that dy/dt at $x = x_a$ and $x = x_b$ to be equal to zero.
- Integrate by parts again.
- Reduce the equation and show the final form.

Key Concepts:

- First order Euler Lagrange equations can be expanded to second order Euler Lagrange equations.
- End points of integration will be equal to zero in integration by parts.

Variations:

- This problem cannot really change other than creating a whole new problem.

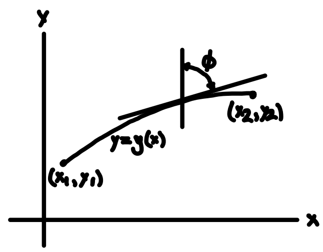
Problem 4: 2.7

Fermat's principle states: If the velocity of light is given by the continuous function $u = u(y)$, the actual light path connecting the points (x_1, y_1) and (x_2, y_2) in a plane is one which extremizes the time integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{u}$$

(Actually, refinements are needed to make this formulation of Fermat's principle hold for all cases).

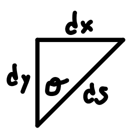
(a) Derive Snell's law from Fermat's principle; that is, prove that $\sin(\phi)/u = \text{const}$, where ϕ is the angle shown in Fig. 2.6.



$$y'\frac{\partial f}{\partial y'} - f = \text{const.} \; : \; f = \frac{\sqrt{1+y'^2}}{u} \; : \; \frac{\partial f}{\partial y'} = \frac{1}{2}\frac{(1+y'^2)^{-1/2}\cdot 2y'}{u} = \frac{y'}{u\sqrt{1+y'^2}}$$

$$y'\cdot\frac{\partial f}{\partial y'} - f = \frac{y'^2}{u\sqrt{1+y^2}} - \frac{\sqrt{1+y'^2}}{u} = \frac{u\cdot y'^2 - u(1+y^2)}{u^2\sqrt{1+y^2}} = \frac{-1}{u\sqrt{1+y'^2}} = \text{Const.}$$

$$\sin\phi = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}} \Big/ \frac{\sqrt{dx^2}}{\sqrt{dx^2}} = \frac{1}{\sqrt{1+(y')^2}} \; : \; \frac{-1}{u\sqrt{1+y'^2}} = \text{const} \, , \; \frac{1}{u\sqrt{1+y'^2}} = \text{const.}$$



$$\boxed{\frac{\sin(\phi)}{u} = \text{const.}}$$

(b) Suppose that light travels in the xy-plane in such a way that its speed is y; then prove that the light rays emitted from any point are circles with their centers on the x-axis.

$$u \propto c_1 \cdot y \; : \; \frac{x'}{c_1 y\sqrt{1+x'^2}} = c \; : \; \frac{x'}{y\sqrt{1+x'^2}} = k \; : \; \frac{x'^2}{y^2(1+x'^2)} = k^2 \; : \; x'^2 = k^2y^2(1+x'^2)$$

$$x'^2(1-k^2y^2) = k^2y^2 \; : \; x'^2 = \frac{k^2y^2}{(1-k^2y^2)} \; : \; \left(\frac{dx}{dy}\right)^2 = \frac{k^2y^2}{(1-k^2y^2)} \; : \; \frac{dx}{dy} = \frac{ky}{(1-k^2y^2)^{1/2}}$$

$$dx = \frac{ky}{(1-k^2y^2)^{1/2}}dy \; : \; \int dx = k\int\frac{y}{(1-k^2y^2)^{1/2}}dy \; : \; u = 1-k^2y^2 \; : \; du = -2k^2y\,dy \; : \; dy = \frac{-1}{2k^2y}$$

$$x - x_0 = \frac{-1}{2k}\int\frac{1}{\sqrt{u}}du = -\frac{1}{2k}\int u^{-1/2}du = \frac{-1}{2k}\cdot 2\cdot u^{1/2} = \frac{\sqrt{1-k^2y^2}}{k} \; : \; (x-x_0) = \frac{\sqrt{1-k^2y^2}}{k}$$

$$k(x-x_0) = \sqrt{1-k^2y^2} \; : \; k^2(x-x_0)^2 = 1-k^2y^2 \; : \; k^2y^2 + k^2(x-x_0)^2 = 1 \; : \; (x-x_0)^2 + y^2 = 1/k^2$$

$$\boxed{(x-x_0)^2 + y^2 = 1/k^2}$$

↳ A circle

Problem 4: 2.7 Review

Procedure:

- Use $ds = \sqrt{1+y'^2}\,dx'$.
- Define f to be ds/u.
- Use implicit Euler Lagrange equation to extremize the functional.
- Define $\sin\theta$ to be dx/ds.
- Solve for the ratio.
- Use the relationship of $u \, \alpha \, c \cdot y$ to deduce $k = x'/y\sqrt{1+x'^2}$.
- Solve for x'.
- Solve for $\triangle x$ and integrate both sides.
- Show the solution is in the form of a circle.

Key Concepts:

- Infinitesimal distances can be defined as : $ds = \sqrt{1+y'^2}\,dx'$.
- We use the implicit Euler Lagrange equation when there is no explicit independent variable dependence.
- $\sin\theta$ can be defined with infinitesimal changes.
- The solution to the differential equation can be written as the form of a circle.

Variations:

- Points of integration can be different.
  - We would use the same procedure but would get a different final answer.