Physics 5403 Homework #4
Spring 2022

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1 Perfect gas of fermions (spin 1/2)

Consider a free gas of N non-relativistic electrons, whose single particle wavefunctions are described by plane-waves normalized by the volume of the space. The kinetic energy of the system given by:

$$K = -\frac{\hbar^2}{2m}\int \mathrm{d}^d x\, \Psi^\dagger(\mathbf{x})\nabla^2\Psi(\mathbf{x}),$$

where

$$\Psi(\mathbf{x}) = \begin{pmatrix} \psi_\uparrow(\mathbf{x}) \\ \psi_\downarrow(\mathbf{x}) \end{pmatrix}$$

is a two component spinor, with ψ_σ ($\sigma =\uparrow,\downarrow$) representing the annihilation field operator for up/down electrons, and d is the dimensionality of the space.

a) Write the Kinetic energy in terms of creation and annihilation operators in the momentum space, $a^\dagger_{\sigma\mathbf{k}}$ and $a_{\sigma\mathbf{k}}$.

b) Write down the momentum operator $\mathbf{P}$ in terms of $\Psi^\dagger(\mathbf{x})$ and $\Psi(\mathbf{x})$ field operators and then rewrite it in terms of the creation and annihilation operators in the momentum space $a^\dagger_{\sigma\mathbf{k}}$ and $a_{\sigma\mathbf{k}}$.

c) The ground state of the system is described by the state:

$$|FS\rangle = \prod_{\alpha\le N} c^\dagger_\alpha|0\rangle,$$

where $c_\alpha \equiv a_{\sigma,\mathbf{k}}$, with the notation $\alpha \le N$ meaning $|\mathbf{k}| \le k_F$, with k_F the radius of the Fermi Surface (FS) and $\sigma =\uparrow,\downarrow$ for momentum states inside the FS. This state has the property that

$$a_{\sigma,\mathbf{k}}|FS\rangle = 0 \qquad \text{for } |\mathbf{k}| > k_F$$

$$a^\dagger_{\sigma,\mathbf{k}}|FS\rangle = 0 \qquad \text{for } |\mathbf{k}| < k_F$$

Assuming $d = 2$ (2D electron gas), compute the total energy of the system at the ground state, $\langle K\rangle_{FS}$ and the total momentum, $\langle \mathbf{P}\rangle_{FS}$. Interpret your result.

d) Suppose now that a uniform magnetic field $\mathbf{B}$ is turned on. The total Hamiltonian becomes $K + \mathcal{H}_B$, where

$$\mathcal{H}_B = -\mu_B \mathbf{S}\cdot\mathbf{B},$$

where μ_B is the Zeman coupling of the electronic spin to the magnetic field and

$$\mathbf{S} = \frac{\hbar}{2} \int \mathrm{d}^d x \Psi^\dagger(\mathbf{x}) \vec{\sigma} \Psi(\mathbf{x})$$

is the spin operator written in terms of field operators, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ as Pauli matrices. Write $\mathcal{H}_B$ in terms of $a^\dagger_{\sigma\mathbf{k}}$ and $a_{\sigma\mathbf{k}}$ operators and then compute the total energy of the system (assume $d = 2$) for each spin in separate. Explain what happens with the ground state (Hint: assume $\mathbf{B}$ along the z-axis).

2 Hydrogen atom

Assume that an electron occupies a p level, which is degenerate among the three states $|n, j = 1, m\rangle$, with $m = \pm 1, 0$. The electron is subjected to a perturbation $V(\mathbf{r}) = \alpha(x^2 - y^2)$.

a) Write the perturbation $V(\mathbf{r})$ in terms of spherical tensors of rank 2.

b) Using the Wigner-Eckart theorem, write the perturbation matrix in the $|n, j = 1, m\rangle$ basis.

c) Find the splitting of the p energy levels in first order in perturbation theory. You don't have to solve the integrals.

d) Assume now that the degeneracy of the p levels is lifted by a *strong* magnetic field $\mathbf{B}$ pointing along the z axis, with the Hamiltonian (ignore spin effects)

$$\mathcal{H}_Z = \mu_B B_z m.$$

If $\mu_B B_z \gg \alpha$, calculate the energy correction to the $|n, j = 1, m\rangle$ states (in the presence of the strong field) due to the perturbation $V(\mathbf{r})$ in lowest order in perturbation theory where the result is non-zero.

3 1D Harmonic oscillator

A particle with mass m is subjected to an harmonic potential

$$V(x) = \frac{1}{2} m \omega^2 x^2.$$

The potential is then perturbed by an anharmonic force with potential

$$\delta V(x) = \lambda \sin \kappa x.$$

a) Find the corrected ground state ket in leading order in perturbation theory.

b) Using your result in a), calculate the expectation value of the position operator in the corrected ground state. Hint: Use the identity:

$$\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B \mathrm{e}^{-[A,B]/2}.$$

c) Now assume that $\kappa x \ll 1$, such that $\delta V(x) \approx \lambda \kappa x$. Use second order perturbation theory to calculate the corrected energy to the ground state.