Classical Mechanics and Statistical/Thermodynamics
January 2019

Problem 1:

(a) If an asteroid that strikes Earth has a speed v_0 at a very large distance from Earth, what will its impact speed be in terms of v_0 and the mass (M_E) and radius of Earth (R_E)? (Ignore any gravitational effects from any other bodies not mentioned in this problem.) (3 points)

(b) What is the maximum impact parameter this asteroid could have with respect to Earth and still strike the Earth? (3 points)

(c) Earth's escape velocity is $\sim$11 km/s. What would v_0 have to be for Earth's gravitational cross-section to be three times as large as its physical/geometrical cross-section? (4 points)

Problem 2:

A bead with negligible size slides along a frictionless wire bent in the shape of a parabola, $z = Cr^2$ (see schematic on the next page). The wire is rotating about its vertical z-axis with a constant angular velocity ω. Choose r, θ, and z as the generalized cylindrical coordinates for this problem. Gravity is directed along the negative z-axis.

(a) Find the kinetic energy and the potential energy of the bead using the generalized coordinates. (1 point)

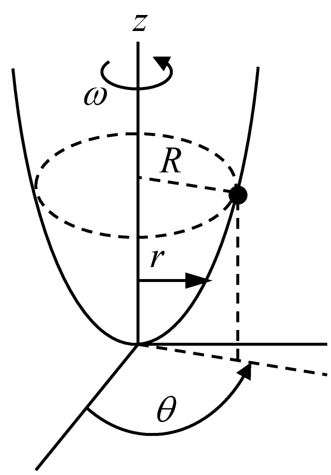
(b) Write the equation(s) of constraint for the system. (1 point)

(c) How many degrees of freedom does the system have? (1 point)

(d) Find Langrage's equations of motion for the bead. (4 points)

(e) Find the value for C that causes the bead to rotate in a circle of fixed radius R. (1 point)

(f) With C set as determined in part (e), is there a radius that is a point of stable equilibrium? If yes, find that radius. If not, provide a detailed physical explanation of the result obtained in part (e). (2 points)


z
ω
R
r
θ

Problem 3:

A single particle moves under the Hamiltonian $H = \frac{1}{2}p^2$.

(a) Find the Hamilton-Jacobi generating function $S(q, \alpha, \beta)$. (2 points)

(b) Find the canonical transformation $q = q(\beta, \alpha)$ and $p = p(\beta, \alpha)$, where β and α are the transformed coordinate and momentum, respectively. Interpret what it means. (2 points)

(c) Add a perturbing Hamiltonian $H_p = \frac{1}{2}q^2$. What is the transformed Hamiltonian using the generating function from part (a)? (2 points)

(d) Find Hamilton's equations for the transformed Hamiltonian. (1 point)

(e) Derive a differential equation for α and interpret what it means. (1 point)

(f) Find equations of motion for $p(t)$ and $q(t)$. (2 points)

Problem 4:
Some substance has the entropy function

$$S = \lambda V^{1/2}(NE)^{1/4}, \tag{1}$$

where N is in moles, λ is a constant with appropriate units, and E and V denote energy and volume, respectively. A cylinder is separated by a partition into two halves, each of volume 1 m^3. One mole of the substance with an energy of 200 J is placed in the left half, while two moles of the substance with an energy of 400 J is placed in the right half.

(a) Assuming that the partition is fixed but conducts heat, what will be the distribution of the energy between the left and right halves at equilibrium? (4 points)

(b) Does your result from part (a) make sense? If so, provide an intuitive explanation of your result. If not, explain why your result does not make sense. (1 point)

(c) Assuming that the partition moves freely and also conducts heat, what will be the volumes and energies of the samples in both sides at equilibrium? (4 points)

(d) Does your result from part (c) make sense? If so, provide an intuitive explanation of your result. If not, explain why your result does not make sense. (1 point)

Problem 5:

(a) A one-dimensional harmonic oscillator potential is a potential of the form

$$V(x) = \frac{1}{2}kx^2. \tag{2}$$

What is the energy and degeneracy of the ground state of a system consisting of five non-interacting particles of mass m that are confined by $V(x)$ in the cases that
(i) the particles are spin-0 bosons, (1 point)
(ii) the particles are spin-$\frac{1}{2}$ fermions, (1 point)
(iii) the particles are spin-$\frac{1}{2}$ bosons, (1 point)
(iv) the particles are spin-0 fermions, and (1 point)
(v) the particles are spin-$\frac{5}{2}$ fermions? (1 point)

(b) Repeat part (a) for the isotropic two-dimensional harmonic oscillator potential. (4 points)

(c) Which of the cases (i)-(v) is physically possible/impossible? Explain. (1 point)

Problem 6:

Generalized Fermi gas.

A non-interacting gas of fermions has an energy spectrum $\varepsilon(\mathbf{p}) = |\mathbf{p}|^{\mathbf{s}}$, with $s > 0$. Assume that the system is in $d = 2$ spatial dimensions and that it occupies an area A in real space (e. g., a square with hard walls).

(a) Calculate the density of states of the system. (2 points)

(b) Calculate the grand potential, $\Omega(\mu, T) = -k_B T \ln \mathcal{Z}$, where μ is the chemical potential, T the temperature, k_B the Boltzmann constant, and $\mathcal{Z}$ the partition function. Express your answer in terms of s and $f_m(z)$, where $z = \mathrm{e}^{\beta\mu}$ is the fugacity ($\beta = (k_B T)^{-1}$) and

$$f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \mathrm{d}x \frac{x^{m-1}}{z^{-1}\mathrm{e}^x + 1}.$$

Hint: Write $\ln \mathcal{Z}$ and integrate by parts. (3 points)

(c) Calculate the density $n = N/A$. Express it in terms of $f_m(z)$. (3 points)

(d) Calculate the ratio PA/E, where P is the pressure and E the energy. (2 points)