Solutions to Homework 1
Physics 5393

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P-1.1 A beam of silver atoms is created by heating a vapor in an oven to 1000 °C, and selecting atoms with a velocity close to the mean of the thermal distribution. The beam moves through a one-meter long magnetic field with a vertical gradient 10 T/m, and impinges a screen one meter downstream of the end of the magnet. Assuming the silver atom has spin 1/2 with a magnetic moment of one Bohr magneton, find the separation distance in millimeters of the two states on the screen.

We assume that the velociy is at the most likely value of the thermal distribution, which is given by

$$E_{\mathsf{thermal}} = \frac{3}{2}kT = 2.6 \times 10^{-20}\,\mathrm{J} \quad \Rightarrow \quad v_\perp = \sqrt{\frac{2E_{\mathsf{thermal}}}{m}} = 543\,\mathrm{m/s},$$

where $v_\perp$ is perpendicular to the magnetic field. The force is applied while the particle traverses the $1\,\mathrm{m}$ length of the magnetic, which occurs over a total time given by

$$t = \frac{\mathsf{Length}}{v_\perp} = \frac{1\,\mathrm{m}}{543\,\mathrm{m/s}} = 1.8 \times 10^{-3}\,\mathrm{s}.$$

The force and hence the angle at the exit of the magnet are

$$\left.\begin{aligned} &F_\perp = \mu_B \frac{\partial B}{\partial z} = 9.3 \times 10^{-23}\,\mathrm{N} = ma \\ &v_z = at = 0.95\,\mathrm{m/s} \end{aligned}\right\} \quad \Rightarrow \quad \theta = \arctan\left(\frac{v_z}{v_\perp}\right) = 1.76\,\mathrm{mrad}.$$

The seperation at the screen is

$$\Delta z = 2\theta \times \mathsf{distance} = 2 \times 1.76\,\mathrm{mrad} \times 1000\,\mathrm{mm} = 3.5\,\mathrm{mm}.$$

P-1.3 For the spin 1/2 state $|S_x;+\rangle$, evaluate both sides of the inequality

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{4} \left| \left\langle \left[\tilde{\mathbf{A}}, \tilde{\mathbf{B}} \right] \right\rangle \right|^2$$

for the operators $\tilde{\mathbf{A}} = \tilde{\mathbf{S}}_x$ and $\tilde{\mathbf{B}} = \tilde{\mathbf{S}}_y$, and show the inequality is satisfied. Repeat for the operators $\tilde{\mathbf{A}} = \tilde{\mathbf{S}}_z$ and $\tilde{\mathbf{B}} = \tilde{\mathbf{S}}_y$.

The definition of the uncertainty, which is the root mean square deviation, is given by

$$\left\langle (\Delta \tilde{\mathbf{A}})^2 \right\rangle = \left\langle \tilde{\mathbf{A}}^2 \right\rangle - \left\langle \tilde{\mathbf{A}} \right\rangle^2,$$

and $\left\langle \tilde{\mathbf{A}} \right\rangle$ is the average or expectation value of the operator. To simplify some pieces of the calculation, the three spin operators are given in the $|S_z;\pm\rangle$ basis

$$\begin{aligned}
\tilde{\mathbf{S}}_z &= \frac{\hbar}{2}\Big[\, |S_z;+\rangle\langle S_z;+| - |S_z;-\rangle\langle S_z;-|\,\Big] \\
\tilde{\mathbf{S}}_x &= \frac{\hbar}{2}\Big[\, |S_x;+\rangle\langle S_x;+| + |S_x;-\rangle\langle S_x;-|\,\Big] = \frac{\hbar}{2}\Big[\, |S_z;+\rangle\langle S_z;-| + |S_z;-\rangle\langle S_z;+|\,\Big] \\
\tilde{\mathbf{S}}_y &= \frac{\hbar}{2}\Big[\, |S_y;+\rangle\langle S_y;+| + |S_y;-\rangle\langle S_y;-|\,\Big] = -i\frac{\hbar}{2}\Big[\, |S_z;+\rangle\langle S_z;-| - |S_z;-\rangle\langle S_z;+|\,\Big]
\end{aligned}$$

where the kets also in the $|S_z;\pm\rangle$ basis are

$$\begin{aligned}
|S_z;\pm\rangle &= |S_z;\pm\rangle \\
|S_x;\pm\rangle &= \frac{1}{\sqrt{2}}\Big[\, |S_z;+\rangle \pm |S_z;-\rangle\,\Big] \\
|S_y;\pm\rangle &= \frac{1}{\sqrt{2}}\Big[\, |S_z;+\rangle \pm i\,|S_z;-\rangle\,\Big]
\end{aligned}$$

With all the pieces in place, the various expectation values for $|S_x;+\rangle$ are calculated

$$\left.\begin{aligned}
\left\langle \tilde{\mathbf{S}}_x \right\rangle^2 &= \left(\frac{\hbar}{2}\right)^2 \\
\left\langle \tilde{\mathbf{S}}_x^2 \right\rangle &= \left(\frac{\hbar}{2}\right)^2
\end{aligned}\right\} \quad \Rightarrow \qquad \left\langle \left(\Delta\tilde{\mathbf{S}}_x\right)^2 \right\rangle = 0$$

$$\left.\begin{aligned}
\left\langle \tilde{\mathbf{S}}_y \right\rangle^2 &= \left(\frac{\hbar}{2}\right)^2 \\
\left\langle \tilde{\mathbf{S}}_y^2 \right\rangle &= 0
\end{aligned}\right\} \quad \Rightarrow \qquad \left\langle \left(\Delta\tilde{\mathbf{S}}_y\right)^2 \right\rangle = \frac{\hbar^2}{4}$$

$$\left.\begin{aligned}
\left\langle \tilde{\mathbf{S}}_z \right\rangle^2 &= \left(\frac{\hbar}{2}\right)^2 \\
\left\langle \tilde{\mathbf{S}}_z^2 \right\rangle &= 0
\end{aligned}\right\} \quad \Rightarrow \qquad \left\langle \left(\Delta\tilde{\mathbf{S}}_z\right)^2 \right\rangle = \frac{\hbar^2}{4}$$

The last piece required for the calculation are the commutation relations

$$\begin{aligned}
\left[\tilde{\mathbf{S}}_x, \tilde{\mathbf{S}}_y\right] &= i\hbar\tilde{\mathbf{S}}_z \\
\left[\tilde{\mathbf{S}}_z, \tilde{\mathbf{S}}_y\right] &= i\hbar\tilde{\mathbf{S}}_x.
\end{aligned}$$

Finally, substituting into the uncertainty relation

$$\begin{aligned}
\left\langle (\Delta\tilde{\mathbf{S}}_x)^2 \right\rangle \left\langle (\Delta\tilde{\mathbf{S}}_y)^2 \right\rangle &\geq \frac{1}{4}\left|\left\langle \left[\tilde{\mathbf{S}}_x, \tilde{\mathbf{S}}_y\right] \right\rangle\right|^2 \quad \Rightarrow \quad 0 = 0 \\
\left\langle (\Delta\tilde{\mathbf{S}}_z)^2 \right\rangle \left\langle (\Delta\tilde{\mathbf{S}}_y)^2 \right\rangle &\geq \frac{1}{4}\left|\left\langle \left[\tilde{\mathbf{S}}_z, \tilde{\mathbf{S}}_y\right] \right\rangle\right|^2 \quad \Rightarrow \quad \frac{\hbar^4}{16} > 0,
\end{aligned}$$

where the first relation satisfies the equality and the second the inequality. Notice that the uncertainty for the $\tilde{\mathbf{S}}_x$ observable is zero since the S_x state is specified, while it is not zero for the other observable since they can be in either possible state of the observable.

P-1.7 Matrix representations

a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a' |\alpha\rangle$, $\langle a'' |\alpha\rangle$, ... and $\langle a' |\beta\rangle$, $\langle a'' |\beta\rangle$, ... are all known, where $|a'\rangle$, $|a''\rangle$, ... form a complete set of basis kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

Formally, the matrix representation in the basis set $|a_i\rangle$ is

$$X_{ij} = \langle a_i |\alpha\rangle \langle \beta |a_j\rangle = \langle a_i |\alpha\rangle \langle a_j |\beta\rangle^* ,$$

where $|a_i\rangle$ represents a with i primes.

b) We now consider a spin 1/2 system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z = \hbar/2\rangle$ and $|S_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (S_z diagonal) basis.

Start by defining $|S_z; +\rangle \equiv |S_z = \hbar/2\rangle$ and $|S_x; +\rangle \equiv |S_x = \hbar/2\rangle$ in order to simplify the notation. Next, recall that

$$|S_x; +\rangle = \frac{1}{\sqrt{2}} \left[|S_z; +\rangle + |S_z; -\rangle \right]$$

in the $\left|\tilde{\mathbf{S}}_z; \pm\right\rangle$ basis. Therefore, the operator in the S_z basis can be expressed as

$$|S_z; +\rangle\langle S_x; +| = \left[\frac{|S_z; +\rangle\langle S_z; +|}{\sqrt{2}} + \frac{|S_z; +\rangle\langle S_z; -|}{\sqrt{2}} \right] .$$

From this expression, the matrix representation is straightforward to derive

$$\begin{pmatrix} \langle S_z; + |S_z; +\rangle \langle S_x; + |S_z; +\rangle & \langle S_z; + |S_z; +\rangle \langle S_x; + |S_z; -\rangle \\ \langle S_z; - |S_z; +\rangle \langle S_x; + |S_z; +\rangle & \langle S_z; - |S_z; +\rangle \langle S_x; + |S_z; -\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} .$$

P-1.8 Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator $\tilde{\mathbf{A}}$. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of $\tilde{\mathbf{A}}$? Justify your answer.

In general

$$\tilde{\mathbf{A}} \left[|i\rangle + |j\rangle \right] = a_i\, |i\rangle + a_j\, |j\rangle \neq |i\rangle + |j\rangle .$$

On the other hand, if the eigenvalues happen to be degenerate, then $|i\rangle + |j\rangle$ is an eigenket of $\tilde{\mathbf{A}}$

$$\tilde{\mathbf{A}} \left[|i\rangle + |j\rangle \right] = a_i \left[|i\rangle + |j\rangle \right] .$$

Additional Problems

Q-1 Let $\tilde{\mathbf{K}}$ be the operator defined by $\tilde{\mathbf{K}} = |\phi\rangle\langle\psi|$, where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

a) Under what condition is $\tilde{\mathbf{K}}$ Hermitian?
For K to be Hermitian:

$$|\phi\rangle\langle\psi| = |\psi\rangle\langle\phi| \quad \Rightarrow \quad |\phi\rangle = \lambda\,|\psi\rangle \tag{1}$$

where λ is real.

b) Calculate $\tilde{\mathbf{K}}^2$. Under what condition is $\tilde{\mathbf{K}}$ a projection operator?
$K^2 = |\phi\rangle\,\langle\psi\,|\phi\rangle\,\langle\psi|$, For K to be a projection operator $K^2 = K$, this implies that $|\psi\rangle = |\phi\rangle$

c) Show that $\tilde{\mathbf{K}}$ can always be written in the form $\tilde{\mathbf{K}} = \lambda\tilde{\mathbf{P}}_1\tilde{\mathbf{P}}_2$ where λ is a constant to be calculated and $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ are projection operators.
Start with the following:

$$K = \lambda P_1 P_2 = \lambda\,|\phi\rangle\langle\phi|\;|\psi\rangle\langle\psi| \tag{2}$$

The central bra-ket is a number that can be pulled out:

$$K = \lambda\,\langle\phi\,|\psi\rangle\;|\phi\rangle\langle\psi| \tag{3}$$

for this to give back K $\langle\phi\,|\psi\rangle = \lambda^{-1}$ so finally:

$$K = |\phi\rangle\langle\psi|\;. \tag{4}$$