Problem:

Consider two very dilute classical gases of slightly different densities and temperatures separated by a thin wall with a small hole of area A.
Let the density of the gas on the left be n_L and that of the gas on the right n_R. Let the temperature of the gas on the left be T_L and of the gas on the right T_R. Define: $\Delta n = n_L - n_R$

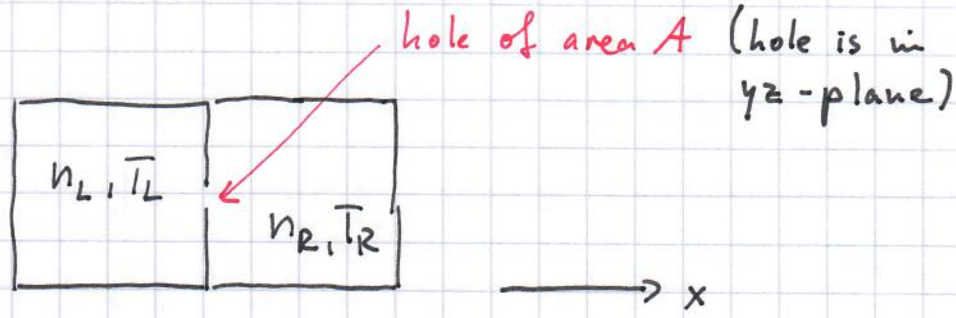
$$\Delta T = T_L - T_R$$

Assume that collisions can be neglected.

(a) Calculate the rate at which particles are transferred through the hole as a fct of Δn and ΔT. Make sure to provide the net change of particles to first order in Δn and ΔT, and make sure that your rate has units of $(\text{time})^{-1}$.

(b) Calculate the rate at which energy is transferred through the hole as a fct of Δn and ΔT. Make sure to provide the net change of the energy to first order in Δn and ΔT, and make sure that your rate has units of energy $(\text{time})^{-1}$.

(a)



Maxwell Boltzmann distribution

$$P(v_x) = n_x \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{m v_x^2}{2kT}}$$

for one degree of freedom

$$P(\vec{v}) = n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m \vec{v}^2}{2kT}}$$

Let's look at the rate $R_{L \to R}$ at which particles pass through the area A from left to right:

$$R_{L \to R} = A \int_{v_x > 0} \underbrace{P(\vec{v}, T_L)}_{\substack{\text{Maxwell Boltzmann} \\ \text{distribution with } T = T_L \\ n = n_L}} v_x \, dv_x \, dv_y \, dv_z$$

we're looking for the particles that are moving toward the +x-direction

$$= n_L A \underbrace{\int_0^\infty \left(\frac{m}{2\pi k T_L}\right)^{1/2} e^{-\frac{m v_x^2}{2kT_L}} v_x \, dv_x}_{\left(\frac{m}{2\pi k T_L}\right)^{1/2} \frac{1}{2} \frac{2kT_L}{m} = \frac{1}{2}\left(\frac{2kT_L}{m\pi}\right)^{1/2}}$$

So: $R_{L\to R} = n_L \, A \left(\frac{kT_L}{2m\pi}\right)^{1/2}$

Similarly: $R_{R\to L} = n_R \, A \left(\frac{kT_R}{2m\pi}\right)^{1/2}$

Now, we want to look at the difference $R_{L\to R} - R_{R\to L}$.

To this end, it is useful to introduce the following change of variables:

$$\left.\begin{aligned} n_L &= n + \frac{\Delta n}{2} \\ n_R &= n - \frac{\Delta n}{2} \end{aligned}\right\} \quad n_L - n_R = \Delta n$$

$$\left.\begin{aligned} T_L &= T + \frac{\Delta T}{2} \\ T_R &= T - \frac{\Delta T}{2} \end{aligned}\right\} \quad T_L - T_R = \Delta T$$

Then: $$R_{L\to R} - R_{R\to L} = A\left(\frac{k}{2\pi m}\right)^{1/2}\left[\left(n + \frac{\Delta n}{2}\right)\left(T + \frac{\Delta T}{2}\right)^{1/2} - \left(n - \frac{\Delta n}{2}\right)\left(T - \frac{\Delta T}{2}\right)^{1/2}\right]$$

use: $\left(T + \frac{\Delta T}{2}\right)^{1/2} = T^{1/2}\left(1 + \frac{\Delta T}{2T}\right)^{1/2} \approx T^{1/2}\left(1 + \frac{\Delta T}{4T}\right)$

$$\Rightarrow \underbrace{R_{L\to R} - R_{R\to L}}_{\frac{dN}{dt}} \approx A\left(\frac{k}{2\pi m}\right)^{1/2}\left[n \frac{\Delta T}{2T^{1/2}} + \Delta n\, T^{1/2}\right]$$

can easily check: units are s^{-1}

net change of # of particles (more particles will move to the right than to the left, provided ΔT and Δn are positive)

(b) The energy carried by a particle is just its kinetic energy $\frac{1}{2} m \vec{v}^2$

We proceed as in (a), except that our integrand needs to include the kinetic energy term:

$$\bar{R}_{L\to R} = A \int_{v_x > 0} P(\vec{v}, T_L)\, \frac{1}{2} m \vec{v}^2\, v_x\, dv_x\, dv_y\, dv_z$$

$$= n_L A \left(\frac{m}{2\pi k T_L}\right)^{3/2} \frac{\pi}{2} \left(\frac{2kT_L}{m}\right)^{3} m$$

So: $$\bar{R}_{L\to R} = n_L\, A\, k T_L \left(\frac{2kT_L}{\pi m}\right)^{1/2}$$

$$= n_L A \left(\frac{2}{\pi m}\right)^{1/2} k^{3/2}\, T_L^{3/2}$$

$$\Rightarrow \quad \frac{dE}{dt} = A\left(\frac{2k^3}{m\pi}\right)^{1/2}\left(n_L\, T_L^{3/2} - n_R\, T_R^{3/2}\right)$$

$$= A\left(\frac{2k^3}{m\pi}\right)^{1/2}\left[\left(n + \frac{\Delta n}{2}\right) T^{3/2}\left(1 + \frac{\Delta T}{2T}\right)^{3/2} - \left(n - \frac{\Delta n}{2}\right) T^{3/2}\left(1 - \frac{\Delta T}{2T}\right)^{3/2}\right]$$

$$\approx A\left(\frac{2k^3}{m\pi}\right)^{1/2}\left(\Delta n\, T^{3/2} + \frac{3}{2} n\, T^{1/2}\, \Delta T\right)$$

$$\left(1 + \frac{x}{2}\right)^{3/2} \approx 1 + \frac{3x}{4} + \ldots$$

checking units:

$$\left(\frac{kT}{m}\right)^{1/2} \underbrace{kT}_{\text{energy}} \; \underbrace{A\, n}_{\frac{1}{\text{length}}}$$

$$\rightsquigarrow \text{energy} \underbrace{\left(\frac{kT}{\text{mass length}^2}\right)^{1/2}}_{\underbrace{\left(\frac{\text{energy}}{\text{mass length}^2}\right)^{1/2}}_{\frac{1}{\text{time}}}}$$

so: units work out : energy/time