Assignment 4
Solutions

Question 1

a) The kinetic & potential energies can be first written in terms of the cartesian co-ordinates (x_1, y_1) & (x_2, y_2) of masses m_1 and m_2, respectively.

Kinetic: $$T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

Potential: $$V = m_1 g y_1 + m_2 g y_2$$

Our task is to rewrite these in terms of the generalized co-ordinates φ_1 & φ_2, as defined in Fig. 1. Note that we only have two generalized co-ordinates as the rods of length l_1 & l_2 imposed a pair of holonomic constraints (see lectures).

Then, transforming to polar co-ordinates:

$$x_1 = l_1 \sin\varphi_1$$
$$y_1 = -l_1 \cos\varphi_1$$

[Note our unusual convention here, trading sin & cos] and,

$$x_2 = x_1 + l_2 \sin \varphi_2$$

$$y_2 = y_1 - l_2 \cos \varphi_2$$

Plugging these into our prior expressions for T & V:

Potential, $V = -(m_1+m_2) g l_1 \cos \varphi_1 - m_2 g l_2 \cos \varphi_2$

Kinetic, $T = \frac{m_1}{2}\left(l_1^2 \dot{\varphi}_1^2\right) + \frac{m_2}{2}\Big[l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2$

$+ 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \big(\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2 \big)\Big]$

Computed $\dot{x}_1$ & $\dot{x}_2$ ... etc from prior definitions w/ only $\varphi_i = \varphi_i(t)$

$= \frac{m_1}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2}\Big[l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)\Big]$

b)

From a) $L = T - V = \text{- - - -}$ details (w/ $l_1 = l_2 = l$ & $m_1 = m_2 = m$)

From L we want to compute,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_j}\right) - \frac{\partial L}{\partial \varphi_j} = 0$$

First, φ_1:

$$\frac{\partial L}{\partial \varphi_1} = -2mgl \sin\varphi_1 + ml^2 \dot\varphi_1 \dot\varphi_2 \left(-\sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2\right)$$

$$= -2mgl\sin\varphi_1 + ml^2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_2 - \varphi_1)$$

$$\frac{\partial L}{\partial \dot\varphi_1} = 2\cancel{3}ml^2 \dot\varphi_1 + ml^2 \dot\varphi_2 \cos(\varphi_1 - \varphi_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right) = 2\cancel{3}ml^2 \ddot\varphi_1 + ml^2 \ddot\varphi_2 \cos(\varphi_1 - \varphi_2) - ml^2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1 - \varphi_2) + ml^2 \dot\varphi_2^2 \sin(\varphi_1 - \varphi_2)$$

$$\therefore \quad ① : \quad 0 = 2\cancel{3}ml^2 \ddot\varphi_1 + ml^2 \ddot\varphi_2 \cos(\varphi_1 - \varphi_2) + ml^2 \dot\varphi_2^2 \sin(\varphi_1 - \varphi_2) \cancel{-2ml^2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1 - \varphi_2)} + 2mgl\sin\varphi_1$$

Next, φ_2:

$$\frac{\partial L}{\partial \varphi_2} = -mlg \sin\varphi_2 + ml^2 \dot\varphi_1 \dot\varphi_2 \left[-\sin\varphi_2 \cos\varphi_1 + \cos\varphi_2 \sin\varphi_1\right]$$

$$= -mgl\sin\varphi_2 + ml^2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1 - \varphi_2)$$

$$\frac{\partial L}{\partial \dot{\varphi}_2} = ml^2\dot{\varphi}_2 + ml\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = ml^2\ddot{\varphi}_2 + ml^2\ddot{\varphi}_1\cos(\varphi_1-\varphi_2)$$
$$+ml^2\dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1-\varphi_2) - ml^2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)$$

$\therefore$ ②: $$0 = ml^2\ddot{\varphi}_2 + ml^2\ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - ml^2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)$$
$$+ mgl\sin\varphi_2$$

From ① & ② we can in principle rearrange into other forms, e.g., $\ddot{\varphi}_1 = \ldots$ & $\ddot{\varphi}_2 = \ldots$, but we won't worry about this here.

c) Look at Eqs ~~B~~ B3 & B4 → they are the equivalent of our ① & ②. ~~Important things to recognize are that:~~

~~⇒ B3 & B4 have divided out a common factor of ml [~~

~~⇒ $\Delta\varphi = \varphi_1 - \varphi_2$ is used for shorthand~~

pto.

To obtain them, observe that:

→ The authors have divided through by m_2 & defined $\mu = 1 + m_1/m_2$.

⇒ We have $\mu = 2$ by $m_1 = m_2$.

⇒ Similarly, they divide through by $l_2 l_1$, but then $l_2/l_2 = 1$

Together: Equivalent to us dividing through by ml^2

Note: Eqs B5 & B6 are what we would obtain if we had completed the last step in (6).

d) Let's take $\dot{\varphi}_1 = 0$ (& $\ddot{\varphi}_1 = 0$) as the top pendulum is pinned in place.

Then, (2) yields:

$$0 = ml^2 \ddot{\varphi}_2 + mgl \sin\varphi_2$$

or $$\ddot{\varphi}_2 = -g/l \sin\varphi_2$$

⇒ Describes a single pendulum, as expected!

e) Invoking a small angle approximation reduces Eqs ① + ② to:

$$① \rightarrow \quad 2ml^2 \ddot{\varphi}_1 + ml^2 \ddot{\varphi}_2 + 2mgl\,\varphi_1 = 0 \qquad Ⓐ$$

$$② \rightarrow \quad ml^2 \ddot{\varphi}_2 + ml^2 \ddot{\varphi}_1 + mgl\,\varphi_2 = 0 \qquad Ⓑ$$

Ⓐ + Ⓑ resemble a system of coupled oscillators (consistent w/ the small angle approximation). So, for a solution let's consider the ansatz,

$$\varphi_1(t) = A_1 e^{i\omega t} \quad + \quad \varphi_2(t) = A_2 e^{i\omega t}$$

Plugging the ansatz into Ⓐ + Ⓑ, we get a matrix eqn (pair of coupled linear eqns):

$$\underbrace{\begin{pmatrix} -2\omega^2 + 2g/l & -\omega^2 \\ -\omega^2 & -\omega^2 + g/l \end{pmatrix}}_{\underline{M}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Solving for,

$$\det(\underline{M}) = 0$$

yields the eigenfrequencies,

$$\omega_\pm^2 = \frac{g}{l}\left(2 \pm \sqrt{2}\right)$$

f) As (d)|(e) demonstrated ~~they~~ the system can be reduced to a pendulum or oscillator → dynamics is integrable in this limit. Nevertheless, the point of the paper is to demonstrate that generically this is *not* the case → the system can be chaotic!

This is exemplified by Fig 9, which shows results of a calculation of the *separation* of the trajectories (in terms of a distance between the generalized co-ords/velocities – see p494 discussion). The plot uses a log-scale for the vertical axis, demonstrating that the separation between slightly perturbed initial conditions grows approximately *exponentially*. This means the system appears to show signs of chaos!

Question 2

This question illustrates a paradigmatic problem of variational calculus, the solution of which yields what is known as the Brachistochrone curve.

a) As the particle starts at rest, this means the initial kinetic energy is 0. Similarly, as we can choose the zero of the potential energy arbitrarily, we can choose the initial total energy of particle to be zero, $E=0$.

Then, using energy conservation we have that at an arbitrary point ~~of~~ of the unknown path,

$$(v^2 = \dot{x}^2 + \dot{y}^2)$$

$$E = 0 = \frac{mv^2}{2} - mgy \quad \text{or} \quad v = \sqrt{2gy} \quad (i)$$

We can use (i) ~~to~~ to write an expression for the time taken to traverse an arbitrary infinitesimal segment of the curve:

distance.

$$dt = \frac{ds}{v} = \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}}\sqrt{\frac{dx^2 + dy^2}{y}}$$

This can be rewritten as,

$$dt = \frac{dy}{\sqrt{2g}} \sqrt{\frac{1 + (dx/dy)^2}{y}}$$

which gives the total time as:

$$t = \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} dy \sqrt{\frac{1+x'^2}{y}} \quad . \text{(ii)}$$

b) We have an integral of the form

$$t = \int_{y_1}^{y_2} dy \; F(x(y), x'(y), y)$$

with $$F = \frac{1}{\sqrt{2g}} \sqrt{\frac{1+x'^2}{y}}$$

Notice the similarity to our expression for the action in terms of the line integral of the Lagrangian. Looking for the stationary value then equivalently gives the requirement,

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$$

similiar to our application of Hamilton's principle.

Now,

$$\frac{\partial F}{\partial x} = 0 \implies \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0 \implies \frac{\partial F}{\partial x'} = \text{const} = 1/c.$$

Explicitly,

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{2g}}\sqrt{\frac{x'^2}{y(1+x'^2)}} = \frac{1}{c} \implies \frac{x'^2}{y(1+x'^2)} = \frac{2g}{c^2} \quad \text{(iii)}$$

(iii) can be rewritten into the form

$$\left(\frac{dx}{dy}\right)^2 = \frac{y^2}{\frac{c^2}{2g}y - y^2}$$

Taking the $+\sqrt{}$ & separating the differentials,

$$x = \int_{y_1}^{y_2} dy \, \frac{y}{\sqrt{\left(\frac{c^2}{2g}\right)y - y^2}}$$

To simplify this integral assume:

i) $(x_1, y_1) = (0,0)$

ii) make the variable change,

$$y = \frac{c^2}{4g}(1 - \cos\varphi)$$

w/ $dy = \frac{c^2}{4g} \sin\varphi \, d\varphi$

Then,

$$x(\theta) = \int_0^\theta \frac{c^2}{4g}(1 - \cos\varphi)\, d\varphi$$

$$= \frac{c^2}{4g}(\theta - \sin\theta)$$

Hence, the path which minimizes (or at least makes $\delta t = 0$), is:

$$x = \frac{c^2}{4g}(\theta - \sin\theta) \quad + \quad y = \frac{c^2}{4g}(1 - \cos\theta)$$

where θ defines our endpoint (ie. parametrizes the curve.)

Question 3

We start by defining an action, (identical to lectures)

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)$$

[We'll do this in 1D for simplicity, but the result is general].

By applying Hamilton's principle we want to find the stationary value of the action, e.g.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t) = 0$$

We obtain the solution by expanding the Lagrangian, ~~as a a di~~

$$\underbrace{\frac{\partial I}{\partial \alpha} d\alpha}_{\delta I} = \int_{t_1}^{t_2} \Bigg(\underset{①}{\frac{\partial L}{\partial q} \underbrace{\frac{\partial q}{\partial \alpha} d\alpha}_{\delta q}} + \underset{②}{\frac{\partial L}{\partial \dot{q}} \underbrace{\frac{\partial \dot{q}}{\partial \alpha} d\alpha}_{\delta \dot{q}}} + \underset{③}{\frac{\partial L}{\partial \ddot{q}} \underbrace{\frac{\partial \ddot{q}}{\partial \alpha} d\alpha}_{\delta \ddot{q}}} \Bigg) dt$$

The integral of ② ~~yield~~ yields: (integrate by parts)

$$\int_{t_1}^{t_2} d\alpha \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \alpha} dt = \underbrace{\frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial \alpha} d\alpha \Big|_{t_1}^{t_2}}_{= 0 \text{ as variation vanishes at endpoints!}} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \frac{\partial q}{\partial \alpha} d\alpha \, dt$$

And for 3:

$\to 0$ at endpoints

$$\int_{t_1}^{t_2} d\alpha \, \frac{\partial L}{\partial \ddot{q}} \frac{\partial \ddot{q}}{\partial \alpha} \, dt = \frac{\partial L}{\partial \ddot{q}} \frac{\partial \dot{q}}{\partial \alpha} \, d\alpha \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) \frac{\partial \dot{q}}{\partial \alpha} \, d\alpha \, dt$$

$\to 0$ again

$$= - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) \frac{\partial q}{\partial \alpha} \, d\alpha \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) \frac{\partial q}{\partial \alpha} \, d\alpha \, dt$$

So together,

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right] \underbrace{\frac{\partial q}{\partial \alpha} \, d\alpha}_{\delta q} \, dt$$

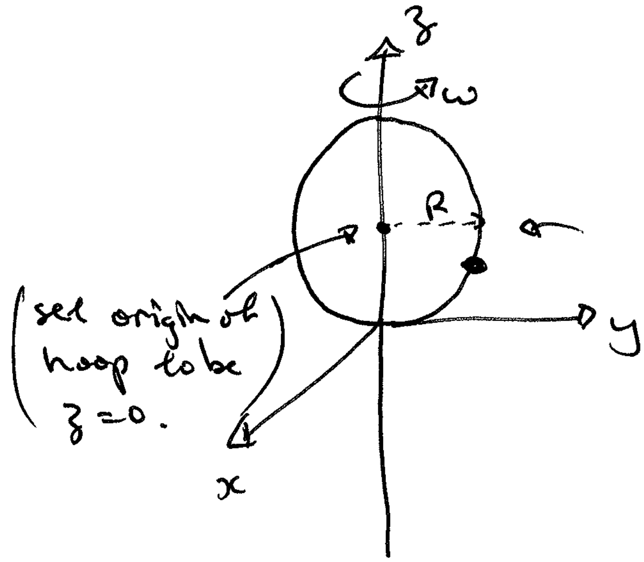
Then, as δq are arbitrary,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0 \quad .$$

Question 4

a) This problem is best suited to be described by spherical co-ordinates,

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta.$$



The constraint of the bead sliding on the hoop enforces two constraints:

$$r = R$$
$$\varphi = \omega t \quad \text{(assume start w/ } \varphi = 0 \text{ @ } t = 0)$$

Thus θ is our remaining generalized co-ordinate.

Next we should construct our Lagrangian, $L = L(\theta, \dot{\theta}, t)$.

Potential energy : $V = mgz = mgR\cos\theta$

Kinetic : $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ ↓ trig rules ...

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right)$$

$$= \underbrace{\frac{m}{2}\left(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta\right)}_{\text{using } \dot{r}=0 \text{ \& } \dot{\varphi}=\omega.}$$

From $L = T - V$ we then obtain eqns of motion:

$$\frac{\partial L}{\partial \theta} = mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$$

Then,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \ddot{\theta} = \omega^2\sin\theta\cos\theta + g/R\,\sin\theta.$$

b) For the bead to be stationary,

$$\ddot{\theta} = \dot{\theta} = 0.$$

From (a), this implies,

$$\omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0$$

or,

$$\sin\theta\left[\omega^2\cos\theta + g/R\right] = 0$$

Then, either:

i) $\sin\theta = 0 \rightarrow \theta = 0$ or π (top or base of hoop)

ii) $\omega^2\cos\theta + g/R = 0 \rightarrow \theta = \mathrm{acos}\left(-g/R\omega^2\right)$

For θ to exist (e.g. $\theta \in \mathbb{R}$) in ii), we have the condition,

$$g/R\omega^2 \leq 1$$

critical frequency, $\omega = \pm\sqrt{g/R}$

Thus:

$g/R\omega^2 \leq 1$	$g/R\omega^2 \geq 1$
• 3 possible stationary states	• 2 possible stationary states.

* $\theta = 0$ (unstable) → corresponds to bead on top of hoop (not rotating & only g acts on bead.)
* $\theta = \pi$ (stable) → bead on base
* $\theta = \mathrm{acos}(-g/R\omega^2)$ → bead is balancing gravity w/ forces due to rotation of hoop. → needs $\omega > \sqrt{g/R}$