E & M I
Homework 4: Faraday, Waves, and Electrostatics

Problem 1: Faraday Generator

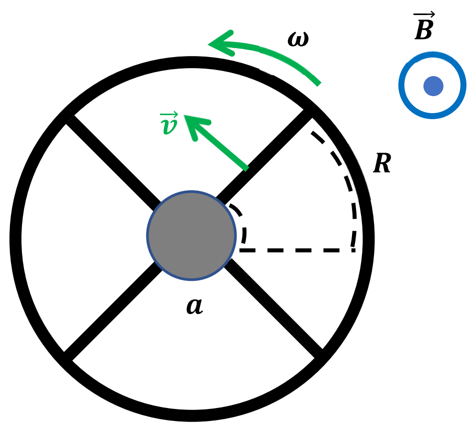
The picture illustrates the concept of a generator based on Faraday's law (as most of them are). It consists of an inner axle of radius a, outer metal wheel of radius R, and metal spokes that connect the two. The wheel is rotating around the axle with angular velocity ω, and the whole thing is in an external magnetic field $\vec{B}$ that is constant and pointing out of the page towards you. Due to this motion, an EMF (voltage) is generated between the axle and the wheel.

A) A common description of this EMF is to use the concept of a "Motional EMF". This is the result of a conductor moving through a magnetic field, which causes the electrons in the conductor to move. For a conductor of length dx moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$:

$$EMF = \mathcal{E} = "E\ dx" = dx\ \vec{v} \times \vec{B}$$

Note: The "$E\ dx$" corresponds to the $\vec{E} \cdot d\vec{l}$ in Faraday's Law:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot \hat{n}\, dS$$



For the situation shown, calculate an expression for the total EMF for each spoke in the wheel.

The velocity of the spoke a distance r from the center is $\vec{v}(r) = r\,\omega\,\hat{\phi}$ so the EMF is:

$$EMF = \int_a^R \vec{v}(r) \times \vec{B}\ dr$$

$$EMF = \int_a^R r\,\omega\,B\,\hat{\phi} \times \hat{z} = \frac{1}{2}\,\omega\,B\,(R^2 - a^2)\,\hat{r}$$

B) As shown in class, the "Motional EMF" is included in Faraday's law by taking the total time derivative (rather than partial derivative) of the magnetic flux.

Consider the loop shown in the picture consisting of the dashed lines and one of the spokes. In this loop, the horizontal dashed line is fixed while the spoke moves. The magnetic flux is changing.

Use Faraday's law to calculate the EMF around the loop. Show that this EMF is the same as found in Part A for a spoke, both in magnitude and direction.

The area of the "loop" shown is:

$$A = \int_a^R \int_0^\theta r\, dr\, d\phi \ = \frac{1}{2}(R^2 - a^2)\, \theta$$

The Flux is:

$$\iint \vec{B} \cdot \hat{n}\, dS = B\, A = \frac{1}{2}\,(R^2 - a^2)\, B\, \theta$$

$$-\frac{d}{dt}\iint \vec{B} \cdot \hat{n}\, dS = -\frac{1}{2}\,(R^2 - a^2)\, B\frac{d\theta}{dt} = -\frac{1}{2}\,(R^2 - a^2)\, B\, \omega$$

$$EMF = \oint \vec{E} \cdot d\vec{l} = \ -\frac{1}{2}\,(R^2 - a^2)\, B\, \omega$$

The minus sign comes from the right-handed coordinate system. A counterclockwise loop is positive, so the EMF is clockwise, or in the +$\hat{r}$ direction as in Part A.

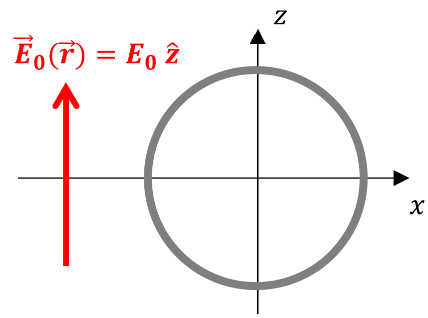
C) If you wish to create an EMF $\mathcal{E} = 5\ V$ using a magnetic field of $B = 0.1\ T$ and a wheel with outer radius $R = 0.4\ m$ and an axle of radius $a = 0.02\ m$, how fast must the wheel turn, ω?

First, check the units: $F = q\ v\ B \Rightarrow B = \frac{F}{qv} = \frac{N\ s}{C\ m}$

So $B\ R^2\ \omega = \frac{Ns}{Cm}\ \frac{m^2}{s} = \frac{Nm}{C} = \frac{J}{C} = Volt$

$$\omega = \frac{2\ EMF}{(R^2 - a^2)B} = 627\frac{rad}{s} = 99.7\ Hz$$

Problem 2) Conductor in a field



A recent Workshop was about the properties of a conducting sphere in a constant electric field, several different approaches to this problem. Here you'll use a general solution to Laplace's equation.

For a problem that has a spherical boundary and is independent of the azimuthal angle ϕ, Laplace's equation is solved by:

$$\phi(\vec{r}) = \sum_l \left(a_l\, r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

a) Consider the boundary condition on $\phi(\vec{r})$ in the limit as $|\vec{r}| \to \infty$. Show that, considering this limit, only one term in the sum will be non-zero.

For $|\vec{r}| \to \infty$, the sphere will have vanishing influence so in this limit the total potential must go to the potential of the constant electric field:

$$\phi(r) \to \phi_0(r) = -E_0\, z = -E_0\, r\cos\theta$$

Thus in this limit, the only term in the sum is $P_1(\cos\theta)$. Because of the orthogonality of the Legendre polynomials, that means the only possible term is $l = 1$.

$$\phi(\vec{r}) = \left(a\, r + \frac{b}{r^2}\right)\cos\theta$$

b) Using the $|\vec{r}| \to \infty$ limit, solve for one of the remaining coefficients (a_l or b_l).

In this limit we have:

$$\phi(\vec{r}) = a\, r\cos\theta = -E_0\, r\cos\theta \;\Rightarrow a = -E_0$$

c) Consider the boundary condition for $\phi(R)$ from the workshop, $\phi(R) = 0$. Use this result to calculate $\phi(\vec{r})$ for all $\vec{r}$. Show that your result is the same as what was found in the workshop (and the text).

Considering $r = R$:

$$\phi(R,\theta) = \left(-E_0\, R + \frac{b}{R^2}\right)\cos\theta = 0$$

$$b = E_0\, R^3$$

$$\phi(\vec{r}) = E_0\left(\frac{R^3}{r^2} - r\right)\cos\theta$$

As found in the book, equation 90.9.

Problem 3) Cartesian Boundary Conditions

The following is a standard problem in electrostatics, so you can probably find the solution if you search. Try to do this problem without looking up the solution.

Consider a conducting cube with walls at $x = 0, x = L, y = 0, y = L, z = 0, z = L$.

The wall at $z = L$ has $\phi(x, y, z = L) = V_0$. All other walls of the cube are grounded, $\phi = 0$.

a) Show that a separable solution of the form:

$$\phi(\vec{r}) = X(x)\, Y(y)\, Z(z)$$

Is a solution to the Laplace equations. What are the most general functional forms for $X(x), Y(y), Z(z)$ that will solve Laplace? What are the relationships between the solutions for $X(x), Y(y), \text{and } Z(z)$?

In Cartesian coordinates, we have:

$$\nabla^2 \phi = \partial_x^2\, X(x)Y(y)Z(z) + \partial_y^2\, X(x)\, Y(y)\, Z(z) + \partial_z^2\, X(x)\, Y(y)\, Z(z) = 0$$

Dividing this by ϕ gives:

$$\frac{1}{X(x)} \partial_x^2\, X(x) + \frac{1}{Y(y)}\, \partial_y^2\, Y(y) + \frac{1}{Z(z)}\, \partial_z^2\, Z(z) = 0$$

Each term depends on only one coordinate, $x, y, or\ z$ so each must be equal to a constant:

$$\partial_x^2\, X(x) = -a^2\, X(x), \qquad \partial_y^2\, Y(y) = -b^2\, Y(y), \qquad \partial_z^2\, Z(z) = c^2\, Z(z)$$

Where $c^2 = a^2 + b^2$. Obviously, we could have any of the coefficients either positive or negative, we just can't have either all three positive or negative.

b) Using the boundary conditions on the cube for the walls $x = 0, x = L, y = 0, y = L$, what are the possible functions $X_m(x)$ and $Y_n(y)$? Explain why these can be indexed by the integers: $m, n = 1, 2, 3, \ldots$

From above, the solution to this differential equation is:

$$X(x) = A \sin(a\, x) + B \cos(a\, x)$$

$$X(0) = 0 \Rightarrow B = 0$$

$$X(L) = 0 \Rightarrow \sin(a\, L) = 0, \qquad a = m\frac{\pi}{L}, m = 1, 2, 3, \ldots$$

$$X_M(x) = A_m \sin\left(m\frac{\pi}{L}\, x\right)$$

Similarly:

$$Y_n(y) = B_n \sin\left(n\frac{\pi}{L}\, y\right)$$

c) Using the boundary condition for $z = 0$ and the results from (b), what are the solutions to the function $Z_{mn}(z)$? Be sure to show and explain the dependence of $Z(z)$ on m, n, the integers indexing the functions $X_m(x), Y_n(y)$.

Using $c^2 = a^2 + b^2$ define:

$$\gamma_{mn} = \sqrt{m^2 + n^2}\frac{\pi}{L}$$

We have:

$$\partial_z^2\, Z_{mn}(z) = \gamma_{mn}^2\, Z_{mn}(z)$$

The solutions will be:

$$Z_{mn}(z) = C\sinh(\gamma_{mn}\, z) + D\cosh(\gamma_{mn}\, z)$$

The boundary condition $Z(0) = 0$ gives

$$Z_{mn}(z) = C_{mn}\sinh(\gamma_{mn}\, z)$$

d) Write down the general solution to this problem:

$$\phi(\vec{r}) = \sum_{m,n} a_{mn}\, X_m(x)\, Y_n(y)\, Z_{mn}(z)$$

Using the boundary condition for $z = L$ and Fourier analysis, determine the coefficients in this sum, a_{mn}. Show your work.

$$\phi(\vec{r}) = \sum_{m,n} a_{mn}\sin\left(\frac{m\pi}{L}\, x\right)\sin\left(\frac{n\pi}{L}\, y\right)\sinh(\gamma_{mn}\, z)\ , \qquad \gamma_{mn} = \sqrt{m^2 + n^2}\frac{\pi}{L}$$

At $z = L$ the potential is:

$$V_0 = \sum_{m,n} a_{mn}\sin\left(\frac{m\pi}{L}\, x\right)\sin\left(\frac{n\pi}{L}\, y\right)\sinh(\gamma_{mn}\, L)$$

Using the relation:

$$\int_0^L \sin\left(\frac{m\pi}{L}\, x\right)\sin\left(\frac{m'\pi}{L}\, x\right)\, dx = \frac{L}{2}\,\delta_{mm'}$$

$$\iint_0^L V_0 \sin\left(\frac{m'\pi}{L}\ x\right) \sin\left(\frac{n'\pi}{L}\ y\ \right) dxdy$$
$$= \sum_{m,n} a_{mn} \sinh(\gamma_{mn}\ L) \iint_0^L \sin\left(\frac{m'\pi}{L}\ x\right) \sin\left(\frac{n'\pi}{L}\ y\ \right) \sin\left(\frac{m\pi}{L}\ x\right) \sin\left(\frac{n\pi}{L}\ y\right)\ dxdy$$

$$V_0 \frac{L^2}{m'n'\ \pi^2}\ (\cos(m'\pi) - 1)\ (\cos(n'\pi) - 1) = \frac{L^2}{4}\ \ a_{m'n'} \sinh(\gamma_{m'n'}\ L)$$

Removing the "Primes" on m and n:

$$a_{mn} = V_0 \frac{4}{m\ n\ \pi^2} \frac{((-1)^m - 1)((-1)^n - 1)}{\sinh(\sqrt{m^2 + n^2}\ \pi)}$$

$$a_{mn} = V_0 \frac{16}{m\ n\ \pi^2} \frac{1}{\sinh(\sqrt{m^2 + n^2}\ \pi)}\ , \qquad m, n\ even$$

$$a_{mn} = 0, \qquad m \text{ or } n\ odd$$

e) Write a sum that gives the potential at the center of the cube, $\phi\left(\frac{L}{2}, \frac{L}{2}, \frac{L}{2}\right)$. You might be able to simplify this result using the relation:

$$\sinh(x) = 2 \sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right)$$

Show that this sum converges quickly by calculating the value of the first few terms. Your answers will be V_0 times a number. You might also determine the ratio of successive terms in the sum for large m and n.

This is a matter of just pugging in the results for the expansion coefficients from above, and $x = y = z = \frac{L}{2}$.

$$\phi(\vec{r}_c) = \frac{16\ V_0}{\pi^2} \sum_{m,n\ odd} \frac{1}{m\ n \sinh\left(\sqrt{m^2 + n^2}\ \pi\right)} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \sinh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right)$$

For m odd:

$$\sin\left(\frac{m\pi}{2}\right) = (-1)^{\frac{m-1}{2}}$$

Using the hint:

$$\frac{\sinh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right)}{\sinh\left(\sqrt{m^2 + n^2}\ \pi\right)} = \frac{\sinh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right)}{2\ \sinh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right) \cosh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right)}$$
$$= \frac{1}{2\ \cosh\left(\sqrt{m^2 + n^2}\frac{\pi}{2}\right)}$$

$$\phi(\vec{r}_c) = \frac{8\,V_0}{\pi^2} \sum_{m,n\ odd} \frac{(-1)^{\frac{m-1}{2}}\,(-1)^{\frac{n-1}{2}}}{m\,n\cosh\left(\sqrt{m^2+n^2}\,\frac{\pi}{2}\right)}$$

Considering the first few terms: $m = n = 1, m = 3\ \&\ n = 1, m = 1\ \&\ n = 3, m = n = 3$

$$\phi(\vec{r}_c) = \frac{8\,V_0}{\pi^2}\left(\frac{1}{\cosh\left(\frac{\pi}{\sqrt{2}}\right)} - \frac{2}{3\ \cosh\left(\sqrt{10}\,\frac{\pi}{2}\right)} + \frac{1}{9\ \cosh\left(\sqrt{18}\,\frac{\pi}{2}\right)} + \cdots\right)$$

$$\phi(\vec{r}_c) = \frac{8\,V_0}{\pi^2}\,(0.2144 - 9.282\times 10^{-3} + 2.835\times 10^{-4} + \cdots)$$

So the terms quickly get smaller so the first few terms will give a good approximation to the solution.

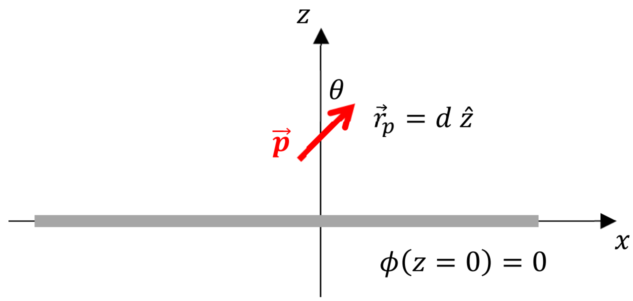
Considering the magnitude of each term (neglecting signs), we have for large m, n:

$$a_{mn} = \frac{1}{m\,n\cosh\left(\sqrt{m^2+n^2}\,\frac{\pi}{2}\right)} \approx \frac{e^{-\sqrt{m^2+n^2}\frac{\pi}{2}}}{m\,n}$$

We see an exponential decrease in these terms as m, n get larger.

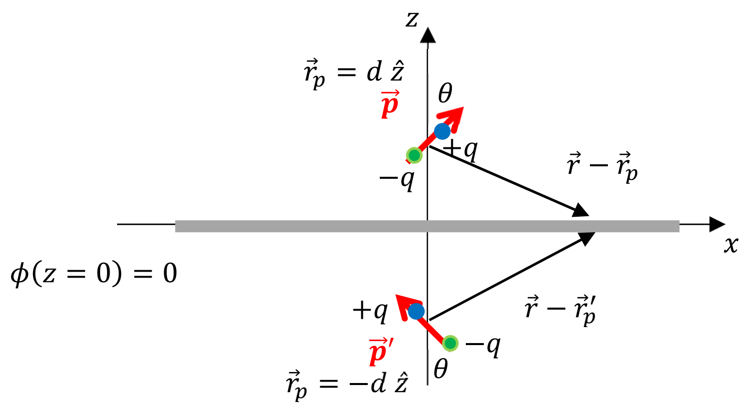
Problem 4) Dipole Image

Consider a very large (assume infinite) grounded, conducting sheet lying in the $z = 0$ plane. A polar molecule is on the z-axis at $\vec{r}_p = d\,\hat{z}$. You can model the molecule as a point dipole, $\vec{p}$, with the dipole moment in the x-z plane at an angle θ as shown.



a) Determine an image that can be used to solve for the electric potential and field for this situation for all $\vec{r}$ with $z \geq 0$. Show that your image gives the correct boundary condition for this problem, $\phi(x, z = 0) = 0$ for any x.

We can picture the dipole as separated $\pm q$ charges. The image should be given by images of these charges, giving the "image dipole" $\vec{p}'$ as shown.



The relation between the source dipole and image dipole is:

$$\vec{p} = p\,(\cos\theta\ \hat{z} + \sin\theta\ \hat{x})$$

$$\vec{p}' = p(\cos\theta\ \hat{z} - \sin\theta\ \hat{x})$$

Checking that this gives the correct boundary conditions, the total potential of the two dipoles (for $z \geq 0$) is:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{\vec{p}\cdot(\vec{r} - \vec{r}_p)}{|\vec{r} - \vec{r}_p|^3} + \frac{\vec{p}'\cdot(\vec{r} - \vec{r}_p')}{|\vec{r} - \vec{r}_p'|^3}\right)$$

On the grounded sheet, $z = 0$:

$$\vec{r} = x\,\hat{x}, \qquad |\vec{r} - \vec{r}_p|^3 = |\vec{r} - \vec{r}_p'|^3 = (x^2 + d^2)^{3/2}$$

$$\phi(x, z = 0) = \frac{1}{4\pi\epsilon_0}\left(\frac{p\,(\cos\theta\ \hat{z} + \sin\theta\ \hat{x})\cdot(x\,\hat{x} - d\,\hat{z}) + p\,(\cos\theta\ \hat{z} - \sin\theta\ \hat{x})\cdot\ (x\,\hat{x} + d\,\hat{z})}{(x^2 + d^2)^{3/2}}\right)$$

$$\phi(x, z = 0) = \frac{p}{4\pi\epsilon_0}\left(\frac{(-d\cos\theta\ + x\ \sin\theta\ \hat{x}) +\ (d\cos\theta - x\sin\theta\ \hat{x})}{(x^2 + d^2)^{3/2}}\right) = 0$$

b) Determine an expression for the potential $\phi(x,z)$ for any x and $z > 0$.

The result above gives the potential everywhere by letting $\vec{r} = x\,\hat{x} + z\,\hat{z}$:

$$\phi(x,z) = \frac{p}{4\pi\epsilon_0}\left(\frac{(\cos\theta\ \hat{z} + \sin\theta\ \hat{x})\cdot(x\,\hat{x} + (z-d)\hat{z})}{(x^2+(z-d)^2)^{3/2}} + \frac{(\cos\theta\ \hat{z} - \sin\theta\ \hat{x})\cdot\ (x\,\hat{x} + (z+d)\,\hat{z})}{(x^2+(z+d)^2)^{3/2}}\right)$$

$$\phi(x,z) = \frac{p}{4\pi\epsilon_0}\left(\frac{(z-d)\cos\theta + x\sin\theta}{(x^2+(z-d)^2)^{3/2}} + \frac{(z+d)\cos\theta\ - x\sin\theta}{(x^2+(z+d)^2)^{3/2}}\right)$$

c) Determine the electric field everywhere on the conducting sheet (meaning for points $z \to 0, z > 0$). You can do this by either taking a derivative of your result from (b) or summing the fields due to the molecule (dipole) and the image (or both, of course).

The field at the surface of the conducting sheet will be in the $\hat{z}$ direction, or

$$E_z(x,0) = \ -\partial_z\,\phi(x,z)|_{z=0}$$

$$E_z(x,0) = -\frac{p}{4\pi\epsilon_0}\left(\frac{\cos\theta}{(x^2+(z-d)^2)^{\frac{3}{2}}} - 3\frac{\big((z-d)\cos\theta + x\sin\theta\big)(z-d)}{(x^2+(z-d)^2)^{\frac{5}{2}}} + \frac{\cos\theta}{(x^2+(z+d)^2)^{\frac{3}{2}}} - 3\,\frac{\big((z+d)\cos\theta\ - x\sin\theta\big)(z+d)}{(x^2+(z+d)^2)^{\frac{5}{2}}}\right)$$

Setting $z = 0$:

$$E_z(x,0) = -\frac{p}{4\pi\epsilon_0}\left(\frac{\cos\theta}{(x^2+d^2)^{\frac{3}{2}}} - 3\frac{(d^2\cos\theta - d\,x\sin\theta)}{(x^2+d^2)^{\frac{5}{2}}} + \frac{\cos\theta}{(x^2+d^2)^{\frac{3}{2}}} - 3\,\frac{(d^2\cos\theta\ - d\,x\sin\theta)}{(x^2+d^2)^{\frac{5}{2}}}\right)$$

$$E_z(x,0) = \frac{p}{2\pi\epsilon_0}\left(3\frac{(d^2\cos\theta - d\,x\sin\theta)}{(x^2+d^2)^{\frac{5}{2}}} - \frac{\cos\theta}{(x^2+d^2)^{\frac{3}{2}}}\right)$$

Compare this to the fields in the z-direction due to the dipoles for $\vec{r} = x\,\hat{x}, z = 0$.

$$E_z = \frac{1}{4\pi\epsilon_0}\left(3\frac{\vec{p}\cdot(\vec{r}-\vec{r}_p)\,(\vec{r}-\vec{r}_p)}{|\vec{r}-\vec{r}_p|^5} - \frac{\vec{p}}{|\vec{r}-\vec{r}_p|^3}\right)\cdot\hat{z}$$

Where for $z = 0$, again:

$$|\vec{r}-\vec{r}_p| = |\vec{r}-\vec{r}_p'| = \sqrt{x^2+d^2}$$

$$\vec{p}\cdot\hat{z} = \vec{p}'\cdot\hat{z} = p\cos\theta$$

$$\vec{p}\cdot(\vec{r}-\vec{r}_p) = p\,x\sin\theta - p\,d\cos\theta\,, \qquad \vec{p}'\cdot(\vec{r}-\vec{r}_p{}') = -p\,x\sin\theta + p\,d\cos\theta$$

$$(\vec{r}-\vec{r}_p)\cdot\hat{z} = -d, \qquad (\vec{r}-\vec{r}_p')\cdot\hat{z} = d$$

So we see that the z-component of the electric field due to $\vec{p}$ and $\vec{p}'$ are equal:

$$E_z = E_z' = \frac{p}{4\pi\epsilon_0}\left(3\frac{d^2\cos\theta - d\,x\sin\theta}{|\vec{r}-\vec{r}_p|^5} - \frac{\cos\theta}{|\vec{r}-\vec{r}_p|^3}\right)$$

Giving the same result as above.

d) Consider the two cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. For each of these angles, calculate:

i) The surface charge density on the conducting sheet for any x.

ii) The force on the molecule (dipole) due to the conducting sheet.

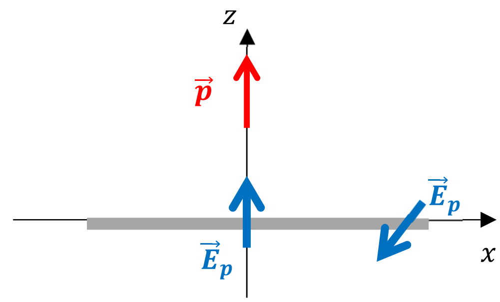
i) Using Gauss at the conducting surface, we know that:

$$E_z(x) = \frac{\sigma(x)}{\epsilon_0} \Rightarrow \sigma(x) = \epsilon_0\,E_z(x)$$

$$\sigma(x) = \frac{p}{2\pi}\left(3\frac{(d^2\cos\theta - d\,x\sin\theta)}{(x^2+d^2)^{\frac{5}{2}}} - \frac{\cos\theta}{(x^2+d^2)^{\frac{3}{2}}}\right)$$

For $\theta = 0$ and $\theta = \frac{\pi}{2}$ this becomes:

$$\sigma_0(x) = \frac{p}{2\pi}\left(3\frac{d^2}{(x^2+d^2)^{\frac{5}{2}}} - \frac{1}{(x^2+d^2)^{\frac{3}{2}}}\right)$$

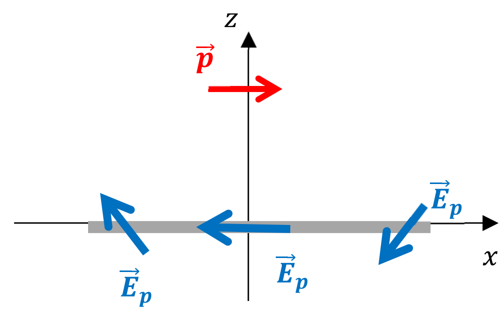


Note that this has the interesting property that:

$\sigma_0 > 0$ for $x < \sqrt{2}\,d$, $\qquad \sigma_0 < 0$ for $x > \sqrt{2}\,d$

This is because the dipole field due to $\vec{p}$ goes from being in the $+\hat{z}$ direction near is axis to being in the $-\hat{z}$ direction farther away, as we've seen before.

$$\sigma_{\frac{\pi}{2}}(x) = -3\frac{p}{2\pi}\left(\frac{d\,x}{(x^2+d^2)^{\frac{5}{2}}}\right)$$



This gives:

$\sigma_{\frac{\pi}{2}} < 0, x > 0;\ \sigma_{\frac{\pi}{2}} > 0, x < 0;\ \sigma_{\frac{\pi}{2}} = 0, x = 0$

Again, plotting the electric field due to $\vec{p}$, this result makes sense.

ii) The force on $\vec{p}$ due to the grounded conductor is equal to the force on $\vec{p}$ due to $\vec{p}'$.

The force between dipoles is derived several places, but let's consider doing this starting with the potential energy. The interaction energy is:

$$U = -\vec{p}\cdot\vec{E}' = \frac{\vec{p}\cdot\vec{p}' - 3\left(\vec{p}\cdot\hat{R}\right)\left(\vec{p}'\cdot\hat{R}\right)}{R^3}$$

(Eq. 20.9 in Garg, for example) where $\vec{R}$ is the vector from one dipole to the other. In this case $\vec{R} = 2\,d\,\hat{z}$ giving:

$$U = \frac{\vec{p}\cdot\vec{p}' - 3\,(\vec{p}\cdot\hat{z})\,(\vec{p}'\cdot\hat{z})}{8\,d^3}$$

And the force is:

$$F = -\partial_d\,U = \frac{3}{8}\,\frac{\vec{p}\cdot\vec{p}' - 3\,(\vec{p}\cdot\hat{z})\,(\vec{p}'\cdot\hat{z})}{d^4}\,\hat{z}$$

For $\theta = 0, \vec{p} = \vec{p}' = p\,\hat{z}$ and the force is:

$$F = \frac{3}{8}\frac{p^2 - 3\,p^2}{d^4}\ \hat{z} = -\frac{3}{4}\ \frac{p^2}{d^4}$$

For $\theta = \frac{\pi}{2}, \vec{p} = p\ \hat{x}, \vec{p}' = \ -p\ \hat{x}$ and the force is:

$$F = \frac{3}{8}\frac{-p^2 - 0}{d^4}\ \hat{z} = \ -\frac{3}{8}\frac{p^2}{d^4}$$

Note that for both cases (and in general) there will be an attractive force on $\vec{p}$ due to the conductor. You might understand this by thinking about the image charges making up the image dipole. The image charges making up $\vec{p}'$ will always be closer to the opposite sign charges on $\vec{p}$.