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Electrodynamics 1

PHYS 5573 Homework Assignment 5

Problems: {1, 2, 3, 4}

Due: May 6, 2022 at 11:59 PM

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Problem 1:

This is a workshop problem the only one group got very far in solving. It's worthwhile to do this, as it's a good example of how to deal with the boundary conditions for a dielectric.

Consider an insulating sphere of radius R and dielectric constant ϵ and a point charge $+q$ on the z-axis with a position $\vec{r'} = \hat{z}$.

Due to the spherical boundary conditions and the azimuthal symmetry of the problem, we know we can use an expansion in Legendre Polynomials. For any azimuthally symmetric function satisfying Laplace's Equation:

$$F(r,\theta) = \sum_l \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos\theta)$$

And for the point charge:

$$\frac{1}{|\vec{r} - \vec{r'}|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$

Where $r_< = r$, for $r < r'$, $r_< = r'$, for $r > r'$, $r_> = r$ for $r > r'$, and $r >= r'$, for $r < r'$.

(a) Define the potentials

$\phi_{SI}(r,\theta) \rightarrow$ The potential due to the sphere for $r < R$
$\phi_{SO}(r,\theta) \rightarrow$ The potential due to the sphere for $r > R$
$\phi_{qI}(r,\theta) \rightarrow$ The potential due to the charge for $r < d$
$\phi_{qII}(r,\theta) \rightarrow$ The potential due to the charge for $r > d$

Write down Legendre polynomial expansions for the total potential $\phi_T = \phi_S + \phi_q$ (sum of the sphere potential and potential of q) both inside and outside the sphere, for $r < d$. Do not include terms that must be zero for ϕ_T to remain finite everywhere.

For our sphere, as $r \to 0$ we require $b_l \to 0$. This means mathematically

$$\varphi_{SI}(r,\theta) = \sum_l r^l a_l P_l(\cos\theta)$$

As $r \to \infty$ we require $a_l \to 0$. This is of course

$$\varphi_{SO}(r,\theta) = \sum_l \frac{b_l}{r^{l+1}} P_l(\cos\theta)$$

This means the potential due to our sphere is

$$\varphi_S(r,\theta) = \sum_l (r^l a_l + b_l/r^{l+1}) P_l(\cos\theta)$$

Using the potential for a point charge, when $r<d$ φ is

$$\varphi_{qI}(r,\theta) = \sum_l \frac{r^l}{d^{l+1}} P_l(\cos\theta)$$

We do the same for $r > d$ and we find (we do not include this in φ_T)

Problem 1: Continued

$$\varphi_{q_{II}}(r,\alpha) = \sum_{l} \frac{d^l}{r^{l+1}} P_l(\cos\alpha)$$

The total potential inside and outside the sphere for $r<d$ is

$$\boxed{\varphi_T(r,\alpha) = \sum_l \left(r^l a_l + \frac{b_l}{r^{l+1}} + \frac{r^l}{d^{l+1}} \right) P_l(\cos\alpha)}$$

(b) Using the fact that ϕ_T is continuous at the surface of the sphere, find a relation between the coefficients a_l and b_l in the Legendre expansions ϕ_{SI} and ϕ_{SO}.

φ_T being continuous at the surface tells us

$$\varphi_{SI}(R,\theta) = \varphi_{SO}(R,\theta)$$

This allows us to then say

$$\sum_l \frac{b_l}{R^{l+1}} \cancel{P_l(\cos\alpha)} = \sum_l a_l R^l \cancel{P_l(\cos\alpha)}$$

We can form our relationship now

$$\boxed{b_l = a_l R^{2l+1}}$$

(c) Using the boundary condition that $\hat{r} \cdot \vec{D}$ is continuous at the surface of the sphere (and that $\vec{D} = \epsilon\vec{E} = -\epsilon\vec{\nabla}\phi$), solve for the coefficients a_l and b_l in ϕ_{SI} and ϕ_{SO}.

To perform this calculation, using the boundary condition given to us we can say

$$-\epsilon\vec{\nabla}\varphi_{SI,q_{II}} = -\epsilon_0\vec{\nabla}\varphi_{SO,q_{II}} \Rightarrow -\epsilon\vec{\nabla}(\varphi_{SI} + \varphi_{q_{II}}) = -\epsilon_0\vec{\nabla}(\varphi_{SO} + \varphi_{q_{II}})$$

The LHS of our Boundary Condition equation tells us

$$-\epsilon\vec{\nabla}\left(\sum_l r^l a_l P_l(\cos\alpha) + \sum_l \frac{r^l}{d^{l+1}} P_l(\cos\alpha) \right) = -\epsilon\vec{\nabla}\sum_l r^l a_l P_l(\cos\alpha) - \epsilon\vec{\nabla}\sum_l \frac{r^l}{d^{l+1}} P_l(\cos\alpha)$$

$$\Rightarrow \quad -\epsilon\left(\sum_l l r^{l-1} a_l P_l(\cos\alpha) + \sum_l \frac{l r^{l-1}}{d^{l+1}} P_l(\cos\alpha) \right)$$

Problem 1: Continued

The RHS of our Boundary condition equation is

$$-\epsilon_0 \vec{\nabla}\left(\sum_l \frac{b_l}{r^{l+1}} P_l(\cos a) + \sum_l \frac{d^l}{r^{l+1}} P_l(\cos a)\right) = -\epsilon_0 \vec{\nabla} \sum_l \frac{b_l}{r^{l+1}} P_l(\cos a) - \epsilon_0 \vec{\nabla} \sum_l \frac{d^l}{r^{l+1}} P_l(\cos a)$$

$$\Rightarrow \quad -\epsilon_0 \left(\sum_l -(l+1)\frac{b_l}{r^{l+2}} P_l(\cos a) + \sum_l -(l+1)\frac{d^l}{r^{l+2}} P_l(\cos a)\right)$$

Re-writing the LHS and RHS of our Boundary condition equation gives

$$-\sum_l \epsilon l R^{l-1}\left(a_l + \frac{1}{d^{l+1}}\right) P_l(\cos a) = -\sum_l \epsilon_0 (-l-1)\frac{1}{R^{l+2}}\left(b_l + d^l\right) P_l(\cos a)$$

To determine a_l and b_l we equate the values inside the sum to say

$$\epsilon l R^{l-1}\left(a_l + \frac{1}{d^{l+1}}\right) = \epsilon_0(-l-1)\frac{1}{R^{l+2}}\left(b_l + d^l\right)$$

$$-\frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)}\left(a_l + \frac{1}{d^{l+1}}\right) = \left(b_l + d^l\right) = \left(a_l R^{2l+1} + d^l\right)$$

$$-\frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)} a_l - \frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)}\frac{1}{d^{l+1}} = a_l R^{2l+1} + d^l$$

$$a_l R^{2l+1}\left(1 + \frac{\epsilon}{\epsilon_0}\frac{l}{(l+1)}\right) = -d^l - \frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)}$$

This means a_l is

$$\boxed{a_l = -\left(d^l + \frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)}\right)\left(1 + \frac{\epsilon}{\epsilon_0}\frac{l}{(l+1)}\right)^{-1} R^{-2l-1}}$$

Conversely b_l is then

$$\boxed{b_l = -\left(d^l + \frac{\epsilon}{\epsilon_0}\frac{l R^{2l+1}}{(l+1)}\right)\left(1 + \frac{\epsilon}{\epsilon_0}\frac{l}{(l+1)}\right)^{-1}}$$

Problem 1: Continued

(d) The force on the point charge due to the dipole is:

$$F_q = -q\,\partial_r\,\phi_{SO}(r,0)|_{r=d}$$

Solve for the force due to the dipole, $l = 1$, term in the expansion for ϕ_{SO}.

Taking the $l=1$ expansion for $\varphi_{SO}(r,\theta=0)|_{r=d}$ is

$$\varphi_{SO}(r,0) = \sum_{l} \frac{b_l}{r^{l+1}} P_l(\cos 0) = \frac{b_1}{r^2} P_1(\cos(0))$$

We then take the derivative of our potential

$$\frac{\partial}{\partial r}\varphi_{SO}(r,0) = \frac{\partial}{\partial r}\frac{b_1}{r^2}P_1(\cos(0)) = \frac{-2b_1}{r^3}P_1(\cos(0))\bigg|_{r=d} = \frac{-2b_1}{d^3}$$

We have the constant b_1

$$b_1 = -\left(d + \frac{\epsilon}{\epsilon_0}\frac{R^3}{2}\right)\left(1 + \frac{\epsilon}{\epsilon_0}\frac{1}{2}\right)^{-1}$$

So our Force is then

$$\boxed{F_q = -\frac{2q}{d^3}\left(d + \frac{\epsilon}{\epsilon_0}\frac{R^3}{2}\right)\left(1 + \frac{\epsilon}{\epsilon_0}\frac{1}{2}\right)^{-1}}$$

Problem 2:

A "ferroelectric" sphere of radius R has a constant polarization field $\vec{P}(\vec{r}) = P_0\hat{z}$. In this problem, we'll find the fields inside and outside of the sphere.

The properties of electric fields and electric materials are given by:

$$\vec{\nabla}\times\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{D} = \rho_{free}, \quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}(\rho_{free}+\rho_{bound}), \quad \vec{D} = \epsilon_0\vec{E}+\vec{P},$$

$$\vec{D} = \epsilon\vec{E}, \quad \rho_b = -\vec{\nabla}\cdot\vec{P}, \quad \sigma_b = \hat{n}\cdot\vec{P}, \quad \vec{D}_{\perp,in} - \vec{D}_{\perp,out} = \sigma_{free}, \quad \vec{E}_{\parallel,in} = \vec{E}_{\parallel,out}$$

$\vec{D}$ is created by free charges, $\vec{E}$ by free and "bound" charges from macroscopic dipole moments, and the fields must also satisfy the boundary conditions. We'll solve for $\vec{E}$ using the bound charges, as there are no free charges in the problem.

(a) Using the definitions above, determine the bound charges:

$$\rho_b = \vec{\nabla}\cdot\vec{P}, \quad \sigma_b = \hat{n}\cdot\vec{P}$$

Show that $\rho_b = 0$ while the bound surface charge has a distribution very similar to what we found on a conducting sphere in a uniform electric field (Workshop 9 & HW 4).

Taking the expression for ρ_b we can use

$$\vec{D} = \epsilon_0\vec{E}+\vec{P} \Rightarrow \vec{P} = \vec{D}-\epsilon_0\vec{E}$$

To say that ρ_b is

$$\rho_b = \vec{\nabla}\cdot(\vec{D}-\epsilon_0\vec{E}) = \vec{\nabla}\cdot\vec{D} - \epsilon_0\vec{\nabla}\cdot\vec{E}$$

$$= \rho_f - \cancel{\epsilon_0}\cdot\frac{1}{\cancel{\epsilon_0}}(\rho_f+\rho_b) = \cancel{\rho_f} - \cancel{\rho_f} - \rho_b = -\rho_b$$

$$\rho_b = -\rho_b \rightarrow 2\rho_b = 0 \Rightarrow \rho_b = 0$$

The above expression gives $\rho_b = 0$

Taking the bound surface charge we have

$$\sigma_b = \hat{n}\cdot\vec{P} = \hat{n}\cdot(\vec{D}-\epsilon_0\vec{E}) = \hat{n}\cdot\vec{D} - \epsilon_0\hat{n}\cdot\vec{E}$$

$$= \hat{n}\cdot\epsilon\vec{E} - \epsilon_0\hat{n}\cdot\vec{E} = \hat{n}\cdot\vec{E}(\epsilon-\epsilon_0)$$

Substituting the relationship, $\vec{P} = \cancel{\vec{D}}^{\,0} - \epsilon_0\vec{E}$, we can further say

$$\sigma_b = \hat{n}\cdot\frac{\vec{P}}{\epsilon_0}(\epsilon_0-\epsilon) = \frac{P_0(\epsilon_0-\epsilon)\cos(\alpha)}{\epsilon_0} \qquad (*)$$

where α is the angle between P and $\hat{r}$. When $\alpha = \pi/2, 3\pi/2$ $\sigma_b = 0$.

Problem 2: Continued

The surface charge density is then

$$\boxed{\sigma_b = \frac{P_0(\epsilon_0 - \epsilon)\cos(\alpha)}{\epsilon_0}}$$

Which is similar to previous HW and workshops

(b) Using the bound surface charge, write an integral that can be solved for the potential $\phi(\vec{r})$. This should be an application of Coulomb's law.

The relationship between E and φ is

$$\vec{E} = -\vec{\nabla}\varphi \Rightarrow \varphi(r) = -\int E\, dv \rightarrow \varphi(r) = \int \frac{\sigma'}{r} ds$$

We can then further say for us $(r \doteq |\vec{r} - \vec{r}'|)$

$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\int \frac{P_0(\epsilon_0 - \epsilon)\cos(\alpha)}{\epsilon_0 |\vec{r} - \vec{r}'|} ds = \frac{P_0(\epsilon_0 - \epsilon)}{\epsilon_0}\frac{1}{4\pi\epsilon_0}\int \frac{\cos(\alpha)}{|\vec{r} - \vec{r}'|} ds$$

Taking $ds \equiv d\Omega'$ and letting $\alpha \rightarrow \theta$ we then say our potential is

$$\boxed{\varphi(r) = \frac{P_0(\epsilon_0 - \epsilon)}{\epsilon_0}\frac{R^2}{4\pi\epsilon_0}\iint \frac{\cos(\theta)\sin(\theta')}{|\vec{r} - \vec{r}'|} d\theta' d\varphi'}$$

(c) Solve your integral from (B). A couple of hints that will do most of the work:

$$\frac{\cos(\theta)}{|\vec{r} - \vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\hat{r}' \cdot \hat{r}) \cos(\hat{r}' \cdot \vec{r})$$

$$\int\int d\Omega'\, P_l(\hat{r}' \cdot \hat{r}_1) P_m(\hat{r}' \cdot \hat{r}_2) = \frac{4\pi}{2l+1} P_l(\hat{r}_1 \cdot \hat{r}_2)\delta_{lm}$$

Where: $r_<$ is the smaller of r and r', $r_>$ is the smaller of r and r', and the 2D integral over $d\Omega'$ is over the solid angle of $\vec{r}'$, or $d\Omega' = \sin\theta' d\theta' d\phi'$.

Taking the integrand for the potential, we can say

$$\frac{\cos(\alpha)}{|\vec{r} - \vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\hat{r}' \cdot \hat{r}) \cos(\hat{r}' \cdot \hat{r})$$

Problem 2: Continued

This means our integral for potential becomes

$$\varphi(r) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0}\frac{R^2}{4\pi\epsilon_0}\iint\sum_{l}\frac{r_<^l}{r_>^{l+1}}P_l(\hat{r}'\cdot\hat{r})\cos(\hat{r}'\cdot\hat{r})\,d\Omega'$$

Which turns in to

$$\varphi(r) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0}\frac{R^2}{4\pi\epsilon_0}\sum_{l}\frac{r_<^l}{r_>^{l+1}}\iint P_l(\hat{r}'\cdot\hat{r}_1)P_m(\hat{r}'\cdot\hat{r}_2)\,d\Omega'$$

$$= \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0}\frac{R^2}{4\pi\epsilon_0}\sum_{l}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}P_l(\hat{r}_1\cdot\hat{r}_2)$$

We then have to worry about the potential in and outside the sphere, for inside $(r<R)$ and $l=1$

$$\boxed{\varphi(r<R) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{r}{3}P_1(\cos(\theta))}$$

Conversely, outside the sphere we have

$$\boxed{\varphi(r>R) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{R^3}{3r^3}P_1(\cos(\theta))}$$

(d) Solve for the fields $\vec{E}$ and $\vec{D}$ both outside and inside the sphere. You'll get something very much like the results from Problem 1. Show that, again, the boundary conditions are met.

The Electric Field outside the sphere is

$$\vec{E} = -\vec{\nabla}\varphi = -\left(\frac{\partial\varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta}\right)$$

$$\Rightarrow \quad \frac{\partial\varphi}{\partial r} = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\cdot -2\frac{R^3}{3r^3}\cos(\theta)\hat{r} \;, \quad \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\cdot\frac{-R^3}{3r^3}\sin(\theta)\hat{\theta}$$

$$\Rightarrow \quad \vec{E}(r>R) = \frac{P_0(\epsilon-\epsilon_0)}{\epsilon_0\epsilon_0}\frac{R^3}{3r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right)$$

Problem 2: Continued

On the contrary the $\vec{E}$ Field on the inside is

$$\frac{\partial\varphi}{\partial r} = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\cdot\cos(\theta)\,\hat{r}\ , \qquad \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\cdot -\sin(\theta)\,\hat{\theta}$$

So our Electric Field in the sphere is

$$\vec{E}(r<R) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\left(\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}\right)$$

Converting these to Cartesian we have

$$\vec{E}(r<R) = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\left(\cos(\theta)(\cancel{\sin(\theta)\hat{x}} + \cos(\theta)\hat{z}) - \sin(\theta)(\cancel{\cos(\theta)\hat{x}} - \sin(\theta)\hat{z})\right)$$

$$= \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\left(\cos^2(\theta)+\sin^2(\theta)\right)\hat{z} = \frac{P_0(\epsilon_0-\epsilon)}{\epsilon_0\epsilon_0}\frac{1}{3}\hat{z}$$

$$\vec{E}(r>R) = \frac{P_0(\epsilon-\epsilon_0)}{\epsilon_0\epsilon_0}\frac{R^3}{3r^3}\left(2\cos(\theta)(\sin(\theta)\hat{x} + \cos(\theta)\hat{z}) + \sin(\theta)(\cos(\theta)\hat{x} - \sin(\theta)\hat{z})\right)$$

$$= \frac{P_0(\epsilon-\epsilon_0)}{\epsilon_0\epsilon_0}\frac{R^3}{3r^3}\left(3\cos(\theta)\sin(\theta)\,\hat{x} + (3\cos^2(\theta)-1)\hat{z}\right)$$

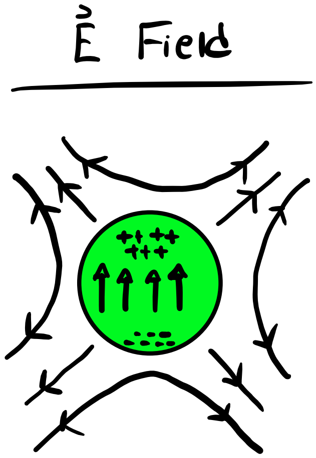
Taking the polarization as $\vec{P} = P_0\hat{z}$ we can say the Electric Displacement Field $\vec{D}$ is, $\vec{D} = \epsilon_0\vec{E} + \vec{P}$

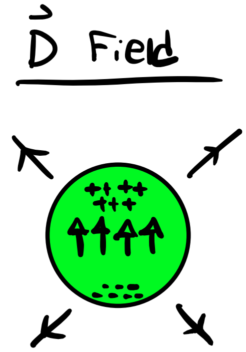
$$\boxed{\begin{aligned} \vec{D}(r>R) &= \frac{P_0(\epsilon-\epsilon_0)}{3\epsilon_0}\frac{R^3}{r^3}\left(3\cos(\theta)\sin(\theta)\,\hat{x} + (3\cos^2(\theta)-1)\hat{z}\right) + P_0\hat{z} \\ \vec{D}(r<R) &= \frac{P_0(\epsilon_0-\epsilon+1)}{3\epsilon_0}\hat{z} \end{aligned}}$$

(e) Draw pictures of the $\vec{E}$ and $\vec{D}$ fields. How do these compare with the fields from Problem 1?

The Electric Field outside the sphere is going to be zero in the x-direction when $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. The Field will be at its greatest when $\theta = \pi/4$. This can be seen in the following illustration

Problem 2: Continued





This is essentially the same results as problem 1 but with a different method

Problem 3:

Interestingly, the methods used in Problem 2 can also solve for the fields due to a magnetized sphere.

The properties of magnetic fields and magnetic materials are given by:

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}, \quad \vec{\nabla} \times \vec{B} = \mu_0(\vec{J}_{free} + \vec{J}_m), \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{B} = \mu_0(\vec{H} + \vec{M}),$$
$$\vec{B} = \mu\vec{H}, \quad \vec{J}_m = \vec{\nabla} \times \vec{M}, \quad \vec{K}_m = \hat{n} \times \vec{M}$$
$$\vec{B}_{\perp,in} = \vec{B}_{\perp,out}, \quad \vec{H}_{\parallel,in} - \vec{H}_{\parallel,out} = I_{free}$$

$\vec{H}$ is created by free currents, $\vec{B}$ by free and "bound" currents from macroscopic magnetic moments, and the fields must also satisfy the boundary conditions at the surface of the materials.

To do this, we'll use the concepts of the "Magnetic Scalar Potential" and "Magnetic Bound Charges":

If there are no free currents, the curl of $\vec{H}$ is zero, so we can define a potential ψ_m:

$$\nabla \times \vec{H} = 0 \implies \vec{H} = -\vec{\nabla}\psi_m$$

Considering the divergence equation:

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0 \implies \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M}$$

This means the solution for $\vec{H}$ is the same as for and electrostatics problem with the analogy of a scalar potential and "magnetic charges" (a problem-solving device, not real magnetic charges):

$$\vec{\nabla} \times \vec{H} = 0, \quad \vec{\nabla} \cdot \vec{H} = \rho_m, \quad \nabla^2\psi_m = \rho_m, \quad \rho_m = -\vec{\nabla} \cdot \vec{M}, \quad \sigma_m = \hat{n} \cdot \vec{M}$$

Considering this analogy and your solution to Problem 2 above, solve for the field $\vec{H}$ and $\vec{B}$ for the magnetized sphere.

Note: The solution to Workshop 12, Problem 1 will be (is) posted for comparison.

The magnetic surface charge density is

$$\sigma_m = \hat{n} \cdot \vec{M} = M\cos(\alpha)$$

Using the results from the previous problem, we can say our magnetic potential ψ can be calculated via

$$\psi(r) = \frac{MR^2}{4\pi} \sum_l \frac{r_<^l}{r_>^{l+1}} \frac{4\pi}{2l+1} P_l(\hat{r}_1 \cdot \hat{r}_2)$$

Inside our sphere $(r<R)$ $\psi(r)$ is found to be $(l=1)$

$$\psi(r<R) = \frac{M\cancel{R^2}}{\cancel{4\pi}} \cdot \frac{r}{\cancel{R^2}} \frac{\cancel{4\pi}}{3} P_1(\cos(\alpha)) = \frac{Mr}{3} P_1(\cos(\alpha))$$

We then say for the outside of our sphere $(r>R)$

Problem 3: Continued

$$\Psi(r>R) = \frac{MR^2}{\cancel{4\pi}}\frac{R}{r^2}\frac{\cancel{4\pi}}{3}P_1(\cos(\alpha)) = \frac{MR^3}{3r^2}P_1(\cos(\alpha))$$

We can then use our magnetic scalar potential to say that the magnetic field can be found with

$$\vec{B} = -\mu_0\vec{\nabla}\Psi$$

So, inside our sphere $\vec{B}$ is

$$\vec{B} = -\mu_0\vec{\nabla}\Psi = -\left(\frac{\partial\Psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\theta}\right)$$

This means for us inside it becomes

$$\frac{\partial\Psi}{\partial r} = \frac{M}{3}\cos(\alpha)\hat{r} \;,\quad \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{M}{3}\sin(\alpha)\hat{\theta}$$

$$\Rightarrow \quad \vec{B}(r<R) = -\mu_0\frac{M}{3}\left(\cos(\alpha)\hat{r} - \sin(\alpha)\hat{\theta}\right)$$

On the outside of the sphere we find it to be

$$\frac{\partial\Psi}{\partial r} = -\frac{2MR^3}{3r^3}\cos(\alpha)\hat{r} \;,\quad \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{MR^3}{3r^3}\sin(\alpha)\hat{\theta}$$

$$\Rightarrow \quad \vec{B}(r>R) = \mu_0\frac{MR^3}{3r^3}\left(2\cos(\alpha)\hat{r} + \sin(\alpha)\hat{\theta}\right)$$

Converting these to Cartesian we find

$$\vec{B}(r<R) = -\mu_0\frac{M}{3}\Big(\cos(\alpha)\big(\sin\cancel{(\alpha)}\hat{x} + \cos(\alpha)\hat{z}\big) - \sin(\alpha)\big(\cancel{\cos(\alpha)}\hat{x} - \sin(\alpha)\hat{z}\big)\Big)$$

$$= -\mu_0\frac{M}{3}\hat{z}$$

$$\vec{B}(r>R) = \mu_0\frac{MR^3}{3r^3}\Big(2\cos(\alpha)\big(\sin(\alpha)\hat{x} + \cos(\alpha)\hat{z}\big) + \sin(\alpha)\big((\cos(\alpha)\hat{x} - \sin(\alpha)\hat{z}\big)\Big)$$

$$= \mu_0\frac{MR^3}{3r^3}\left(3\cos(\alpha)\sin(\alpha)\hat{x} + (3\cos^2(\alpha)-1)\hat{z}\right)$$

Problem 3: Continued

We can then find $\vec{H}(r)$ inside and outside the sphere with

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

With $\vec{M} = M_0 \hat{z}$ $(M_0 \equiv M)$ we can say $\vec{H}$ inside is then

$$\vec{H}(r<R) = \frac{1}{\cancel{\mu_0}} \cancel{\mu_0} \frac{M}{3} \hat{z} - M\hat{z} = -\frac{4}{3} M \hat{z}$$

Where on the outside this is

$$\vec{H}(r>R) = \frac{1}{\cancel{\mu_0}} \cancel{\mu_0} \frac{MR^3}{3r^3} \left(3\cos(\alpha)\sin(\alpha)\hat{x} + (3\cos^2(\alpha)-1)\hat{z}\right) - M\hat{z}$$

And finally we have

$$\boxed{\begin{aligned} \vec{H}(r>R) &= \frac{MR^3}{3r^3}\left(3\cos(\alpha)\sin(\alpha)\hat{x} + (3\cos^2(\alpha)-1)\hat{z}\right) - M\hat{z} \\ \vec{H}(r<R) &= -\frac{4}{3} M \hat{z} \end{aligned}}$$

Problem 4:

Write and solve a problem that you think could be on the E&M Qualifier.

See Following Pages ; Question 2020, From Lim-Yung Kuo

Dielectric Cylinder: (10 Points)

A long, solid dielectric cylinder of radius a is permanently polarized so that the polarization is everywhere radially outward, with a magnitude proportional to the distance from the axis of the cylinder, i.e., $\mathbf{P} = \frac{1}{2}P_0 r\,\hat{r}$

(a) Find the charge density in the cylinder and state the direction that the cylinder is rotating. **(4 Points)**

To begin, we start with using cylindrical coordinates (r, ϕ, z). We were told that $P = P_0 r/2$. The bound charged density is found with

$$\rho = -\vec{\nabla}\cdot\mathbf{P}$$

where the above becomes

$$\boxed{\rho = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{P_0 r}{2}\right) = -P_0.}$$

We can then deduce that the angular velocity ω will point in the **z-direction**.

(b) What is the magnetic field on the axis of the cylinder at points not too near its ends? **(6 Points)**

We know that $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$, the volume current density at a point $\boldsymbol{r} = r\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}}$ in the cylinder is

$$\boldsymbol{j}(\boldsymbol{r}) = \rho(\boldsymbol{\omega}\times\boldsymbol{r}) = -P_0\omega r\hat{\boldsymbol{\phi}}.$$

The cylinder also has a surface charge distribution, of density

$$\sigma = \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{P}} = \hat{\boldsymbol{r}}\cdot\frac{P_0\boldsymbol{r}}{2}|_{r=a} = \frac{P_0 a}{2}.$$

The current density is then

$$\boldsymbol{\alpha} = \sigma\boldsymbol{v} = \frac{P_0}{2}\omega a^2\hat{\boldsymbol{\phi}}.$$

To find the magnetic fiend at a point on the axis of the cylinder not too near its ends, as the cylinder is very long we can take this point as the origin and regard the cylinder as infinitely long. Then the magnetic induction at the origin is given by

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\left(\int_V\frac{\boldsymbol{j}(\boldsymbol{r})\times\boldsymbol{r'}}{\boldsymbol{r'}^3}dV' + \int_S\frac{\boldsymbol{\alpha}(\boldsymbol{r'})\times\boldsymbol{r'}}{\boldsymbol{r'}^3}dS'\right)$$

where V and S are respectively the volume and curved surface area of the cylinder and $\boldsymbol{r'} = (r, \phi, z)$ is a source point. Note the minus sign arises because $\boldsymbol{r'}$ directs from the field point to a source point, rather than the other way around. Through substitution of unit vectors we can then say

$$\int_V\frac{\boldsymbol{j}(\boldsymbol{r})\times\boldsymbol{r'}}{\boldsymbol{r'}^3}dV' = P_0\omega\left[\int_V\frac{r^3 dr d\phi dz}{(r^2+z^2)^{3/2}}\hat{\boldsymbol{z}} - \int_V\frac{r^2 dr d\phi dz}{(r^2+z^2)^{3/2}}\hat{\boldsymbol{r}}\right].$$

As the cylinder can be considered infinitely long, by symmetry the second integral in the above expression vanishes. For the first integral we then make the variable transformation $z = r\tan\beta$. We then can say

$$\int_V\frac{\boldsymbol{j}(\boldsymbol{r})\times\boldsymbol{r'}}{\boldsymbol{r'}^3}dV' = P_0\omega\int_0^{\pi}d\phi\int_0^a r dr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\beta d\beta\hat{\boldsymbol{z}} = 2\pi P_0\omega a^2\hat{\boldsymbol{z}}$$

We then say second integral in $\mathbf{B}$, the surface integral will give

$$\begin{aligned}\int_S \frac{\boldsymbol{\alpha}(\boldsymbol{r'}) \times \boldsymbol{r'}}{\boldsymbol{r'}^3} dS' &= \int_S \frac{\frac{P_0}{2}\omega a^2 \hat{\boldsymbol{\phi}} \times (a\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}})}{(a^2+z^2)^{3/2}} dS' \\ &= -\frac{P_0}{2}\omega a^4 \int_S \frac{d\phi dz}{(a^2+z^2)^{3/2}} \hat{\boldsymbol{z}} = -2\pi P_0 \omega a^2 \hat{\boldsymbol{z}}.\end{aligned}$$

Putting this all together we find

$$\boxed{\mathbf{B} = -\frac{\mu_0}{4\pi}\left(\int_V \frac{\boldsymbol{j}(\boldsymbol{r}) \times \boldsymbol{r'}}{\boldsymbol{r'}^3} dV' + \int_S \frac{\boldsymbol{\alpha}(\boldsymbol{r'}) \times \boldsymbol{r'}}{\boldsymbol{r'}^3} dS'\right) = 2\pi P_0 \omega a^2 \hat{\boldsymbol{z}} - 2\pi P_0 \omega a^2 \hat{\boldsymbol{z}} = 0.}$$