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Math Methods in Physics

PHYS 5013 HOMEWORK ASSIGNMENT #5

PROBLEMS: $\{1, 2, 3, 4\}$

Due: October 6, 2021 By 11:59 PM

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Problem 1:

Consider the parameterization of a position vector in 3D:

$$\vec{r}(\eta, \phi, z) = \cosh\eta\cos\phi\hat{i} + \sinh\eta\sin\phi\hat{j} + z\hat{k}$$

(a) Demonstrate that η, ϕ, z constitute a set of orthogonal co-ordinates.

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial \varphi} = (\sinh(\eta)\cos(\varphi)\hat{\imath} + \cosh(\eta)\sin(\varphi)\hat{\jmath}) \cdot (-\cosh(\eta)\sin(\varphi)\hat{\imath} + \sinh(\eta)\cos(\varphi)\hat{\jmath})$$

$$= -\sinh(\eta)\cosh(\eta)\cos(\varphi)\sin(\varphi) + \sinh(\eta)\cosh(\eta)\cos(\varphi)\sin(\varphi) = 0 \checkmark$$

$$\frac{\partial \vec{r}}{\partial \eta} \cdot \frac{\partial \vec{r}}{\partial z} = (\sinh(\eta)\cos(\varphi)\hat{\imath} + \cosh(\eta)\sin(\varphi)\hat{\jmath} + 0\hat{k}) \cdot (0 \cdot \hat{\imath} + 0 \cdot \hat{\jmath} + 1 \cdot \hat{k}) = 0 + 0 + 0 = 0 \checkmark$$

$$\frac{\partial \vec{r}}{\partial \theta} \cdot \frac{\partial \vec{r}}{\partial z} = (-\cosh(\eta)\sin(\theta)\hat{\imath} + \sinh(\eta)\cos(\theta)\hat{\jmath}) + (\hat{k}) = 0 + 0 + 0 = \checkmark$$

These are orthogonal co-ordinates

(b) What is the metric tensor?

$$g = \begin{pmatrix} d\eta & d\varphi & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d\eta \\ d\varphi \\ dz \end{pmatrix} = \begin{pmatrix} d\eta & 0 & 0 \\ 0 & d\varphi & 0 \\ 0 & 0 & dz \end{pmatrix} \begin{pmatrix} d\eta \\ d\varphi \\ dz \end{pmatrix} = d\eta^2 + d\varphi^2 + dz^2$$

$$dx = \frac{dx}{d\eta} d\eta + \frac{dx}{d\varphi} d\varphi \ , \ dx^2 = \left(\frac{dx}{d\eta}\right)^2 d\eta^2 + 2\left(\frac{dx}{d\eta}\right)\left(\frac{dx}{d\varphi}\right) d\eta d\varphi + \left(\frac{dx}{d\varphi}\right)^2 d\varphi^2$$

$$\frac{dx}{d\eta} = \sinh(\eta)\cos(\varphi) \ , \ \left(\frac{dx}{d\eta}\right)^2 = \sinh^2(\eta)\cos^2(\varphi) : \frac{dx}{d\varphi} = -\cosh(\eta)\sin(\varphi) \ , \ \left(\frac{dx}{d\varphi}\right)^2 = \cosh^2(\eta)\sin^2(\varphi)$$

$$\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dx}{d\varphi}\right)^2 = \sinh^2(\eta)\cos^2(\varphi) + \cosh^2(\eta)\sin^2(\varphi) = \sinh^2(\eta)\cos^2(\varphi) + \cosh^2(\eta)(1 - \cos^2(\varphi))$$

$$\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dx}{d\varphi}\right)^2 = \sinh^2(\eta)\cos^2(\varphi) + \cosh^2(\eta) - \cosh^2(\eta)\cos^2(\varphi) = \cosh^2(\eta) - \cos^2(\varphi)$$

$$2\left(\frac{dx}{d\eta}\right)\left(\frac{dx}{d\varphi}\right) = -2\sinh(\eta)\cos(\varphi)\cosh(\eta)\sin(\varphi) = -\sinh(2\eta)\frac{\sin(2\varphi)}{2}$$

$$dx^2 = \cosh^2(\eta)d\varphi^2 - \sinh(2\eta)\frac{\sin(2\varphi)}{2}d\eta d\varphi - \cos^2(\varphi)d\eta^2$$

$$dy = \frac{dy}{d\eta} d\eta + \frac{dy}{d\varphi} d\varphi \ , \ dy^2 = \left(\frac{dy}{d\eta}\right)^2 d\eta^2 + 2\left(\frac{dy}{d\eta}\right)\left(\frac{dy}{d\varphi}\right) d\eta d\varphi + \left(\frac{dy}{d\varphi}\right)^2 d\varphi^2$$

$$\frac{dy}{d\eta} = \cosh(\eta)\sin(\varphi) \ , \ \left(\frac{dy}{d\eta}\right)^2 = \cosh^2(\eta)\sin^2(\varphi) : \left(\frac{dy}{d\varphi}\right) = \sinh(\eta)\cos(\varphi) \ , \ \left(\frac{dy}{d\varphi}\right)^2 = \sinh^2(\eta)\cos^2(\varphi)$$

$$\left(\frac{dy}{d\eta}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = \cosh^2(\eta)\sin^2(\varphi) + \sinh^2(\eta)\cos^2(\varphi) = \cosh^2(\eta)(1 - \cos^2(\varphi)) + \sinh^2(\eta)\cos^2(\varphi)$$

Problem 1: Continued

$$\left(\frac{dy}{d\eta}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2 = \cosh^2(\eta) - \cosh^2(\eta)\cos^2(\varphi) + \sinh^2(\eta)\cos^2(\varphi) = \cosh^2(\eta) - \cos^2(\varphi)$$

$$2\left(\frac{dy}{d\eta}\right)\left(\frac{dy}{d\varphi}\right) = 2\sinh(\eta)\cos(\varphi)\cosh(\eta)\sin(\varphi) = \sinh(2\eta)\frac{\sin(2\varphi)}{2}$$

$$dy^2 = -\cos^2(\varphi)d\varphi^2 + \sinh(2\eta)\frac{\sin(2\varphi)}{2}d\eta d\varphi + \cosh^2(\eta)d\varphi^2$$

$$dx^2 + dy^2 = (\cosh^2(\eta) - \cos^2(\varphi))d\varphi^2 + (\cosh^2(\eta) - \cos^2(\varphi))d\varphi^2$$

$$dz^2 = 1$$

$$dx^2 + dy^2 + dz^2 = (\cosh^2(\eta) - \cos^2(\varphi))d\eta^2 + (\cosh^2(\eta) - \cos^2(\varphi))d\varphi^2 + dz^2$$

$$\boxed{g = \begin{pmatrix} \cosh^2(\eta) - \cos^2(\varphi) & 0 & 0 \\ 0 & \cosh^2(\eta) - \cos^2(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

(c) What is $\vec{\nabla} f$ in these co-ordinates?

$$\vec{\nabla}\varphi = \frac{1}{h_i}\partial_i \varphi \hat{e}_i \ , \ h_\eta^2 = \cosh^2(\eta) - \cos^2(\varphi) \ , \ h_\varphi^2 = \cosh^2(\eta) - \cos^2(\varphi) \ , \ h_z^2 = 1$$

$$\boxed{\vec{\nabla} f = \frac{1}{\sqrt{\cosh^2(\eta) - \cos^2(\varphi)}}\left(\frac{\partial f}{\partial \eta}\hat{e}_\eta + \frac{\partial f}{\partial \varphi}\hat{e}_\varphi\right) + \frac{\partial f}{\partial z}\hat{e}_z}$$

(d) What is $\vec{\nabla} \cdot \mathcal{E}$ in this co-ordinates, where $\vec{\mathcal{E}} = \mathcal{E}_\eta \hat{\eta} + \mathcal{E}_\phi \hat{\phi} + \mathcal{E}_k \hat{k}$?

$$\boxed{\vec{\nabla}\cdot\vec{\mathcal{E}} = \frac{1}{\cosh^2(\eta) - \cos^2(\varphi)}\left(\frac{\partial}{\partial \eta}\sqrt{\cosh^2(\eta) - \cos^2(\varphi)}\,\mathcal{E}_\eta + \frac{\partial}{\partial \varphi}\sqrt{\cosh^2(\eta) - \cos^2(\varphi)}\,\mathcal{E}_\varphi\right) + \frac{\partial \mathcal{E}_z}{\partial z}}$$

(e) What is $\nabla^2 f$ in this co-ordinate system? Where possible, express your answer only in terms of cosines and hyperbolic cosines.

In parts (b)-(e) express you answer in terms of cosines and hyperbolic cosines, where possible.

$$\vec{\nabla}^2 f = \frac{1}{h_1 h_2 h_3}\left[\partial_1\left(\frac{h_2 h_3}{h_1}\partial_1 f\right) + \partial_2\left(\frac{h_3 h_1}{h_2}\partial_2 f\right) + \partial_3\left(\frac{h_1 h_2}{h_3}\partial_3 f\right)\right]$$

$$\boxed{\nabla^2 f = \frac{1}{\cosh^2(\eta) - \cos^2(\varphi)}\left[\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \varphi^2}\right] + \frac{\partial^2 f}{\partial z^2}}$$

Problem 1: Review

Procedure:

- To show that co-ordinates are orthogonal, with co-ordinates labeled as q_1, q_2, q_3, do

$$\frac{\partial f}{\partial q_1}\cdot\frac{\partial f}{\partial q_2} = 0 \ , \ \frac{\partial f}{\partial q_1}\cdot\frac{\partial f}{\partial q_3} = 0 \ , \ \frac{\partial f}{\partial q_2}\cdot\frac{\partial f}{\partial q_3} = 0.$$

- To find the metric tensor for orthogonal co-ordinates, use

$$\hat{g} = \begin{bmatrix} d\eta^2 & 0 & 0 \\ 0 & d\phi^2 & 0 \\ 0 & 0 & dz^2 \end{bmatrix}.$$

- Where to find $d\eta^2, d\phi^2, dz^2$ we must do the following: Differentiate first component of $\vec{f}$ with respect to all variables that show up in the equation. In this case it is ϕ and η, then find dq_i^2 by doing the following. (In this case we will examine dx^2).

$$dx^2 = \left(\frac{\partial f}{\partial \eta}\right)^2 d\eta^2 + \left(\frac{\partial f}{\partial \eta}\right)\left(\frac{\partial f}{\partial \phi}\right) d\eta\, d\phi + \left(\frac{\partial f}{\partial \phi}\right)^2 d\phi^2.$$

- Repeat the above procedure for all components, then calculate $ds^2 = dx^2 + dy^2 + dz^2$, group the terms together for $d\eta^2, d\phi^2, dz^2$. The terms next to the components belong in the diagonals of the metric tensor.
- Use the Gradient and Laplacian equations to compute the rest of the problem.

Key Concepts:

- For co-ordinates to be orthogonal, they must follow the first rule outlined in the procedure section.
- For orthogonal co-ordinates the metric tensor will be diagonal.
- The Gradient and Laplacian can be calculated easily once we know the values in the diagonal.

Variations:

- The function in the problem statement can change.
  - Thus leading to a different metric.

Problem 2:

Consider the function

$$\phi(\mathbf{r}) = \frac{r}{r^2+\epsilon^2}$$

in three dimensions. Calculate $\nabla^2\phi$.

(a) Show that for any fixed value of $r = r_0 \neq 0$,

$$\lim_{r_0\to 0}\left[\lim_{\epsilon\to 0}\nabla^2\phi(r_0)\right] = 0.$$

$$\vec{\nabla}^2\varphi = \frac{1}{h_1h_2h_3}\left(\partial_1\left(\frac{h_2h_3}{h_1}\right)\partial_1\varphi\right) + \partial_2\left(\frac{h_3h_1}{h_2}\partial_2\varphi\right) + \partial_3\left(\frac{h_1h_2}{h_3}\partial_3\varphi\right)$$

$$\vec{\nabla}^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\varphi^2}$$

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(r^2+\epsilon^2)^{-1} - 2r^4(r^2+\epsilon^2)^{-2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2(\epsilon^2-r^2)}{(r^2+\epsilon^2)^2}\right)$$

$$\nabla^2\varphi = \frac{1}{r^2}\left(\frac{2\epsilon^2 r(\epsilon^2-3r^2)}{(r^2+\epsilon^2)^3}\right) = \frac{2\epsilon^2(\epsilon^2-3r^2)}{r(r^2+\epsilon^2)^3} \quad : \quad \nabla^2\varphi = \frac{2\epsilon^2(\epsilon^2-3r^2)}{r(r^2+\epsilon^2)^3}$$

$$\lim_{\epsilon\to 0}\left[\frac{2\epsilon^2(\epsilon^2-3r_0^2)}{r_0(r_0^2+\epsilon^2)^3}\right] = \frac{2\cdot 0\cdot(\epsilon^2-3r_0^2)}{r_0(r_0^2+0^2)^3} = \frac{0}{r_0^6} = 0 \; : \; \lim_{r_0\to 0} 0 = 0 \checkmark$$

$$\boxed{\lim_{r_0\to 0}\left[\lim_{\epsilon\to 0}\frac{2\epsilon^2(\epsilon^2-3r_0^2)}{r_0(r_0^2+\epsilon^2)^3}\right] = 0}$$

(b) Show that for a fixed value of $\epsilon \neq 0$,

$$\lim_{\epsilon\to 0}\left[\lim_{r_0\to 0}\nabla^2\phi(r_0)\right] = \infty.$$

$$\lim_{r_0\to 0}\left[\frac{2\epsilon^2(\epsilon^2-3r_0^2)}{r_0(r_0^2+\epsilon^2)^3}\right] = \frac{2\epsilon^2(\epsilon^2-3(0))}{0(0^2+\epsilon^2)^3} = \frac{2\epsilon^4}{0} = \infty \; : \; \lim_{\epsilon\to 0} \infty = \infty \checkmark$$

$$\boxed{\lim_{\epsilon\to 0}\left[\lim_{r_0\to 0}\frac{2\epsilon^2(\epsilon^2-3r_0^2)}{r_0(r_0^2+\epsilon^2)^3}\right] = \infty}$$

(c) Using (a) and (b), construct an argument that $\nabla^2\frac{1}{r}$ is zero at all points except at the origin, where it diverges. If $f(r)$ is a smooth, well behaved function, calculate an estimate for

$$\int f(r)\,\nabla^2\left(\frac{1}{r}\right)d^3r$$

where integration is over all space.

We can Taylor series expand $F(r) \to \frac{F^0(0)x^0}{1} + \frac{F'(0)x}{1} + \frac{F''(0)x^2}{2} + \cdots\cdots$

We can keep the first term only, $F(0)$, and disregard all of the other terms. Our integral then becomes

Problem 2: Continued

$$F(0)\int \nabla^2\left(\frac{1}{r}\right) d^3r$$

From here we have $\nabla^2\left(\frac{1}{r}\right) d^3r = -4\pi\,\delta(r)$, where our integral now becomes

$$-F(0)\cdot 4\pi \int \delta(r)\, dr$$

Arken Pg. (81) – (1.161)

Since $\delta(r)$ is a dirac delta Function, $\int \delta(r)\, dr = 1$ $\therefore$

$$\boxed{\int f(r)\,\nabla^2\left(\frac{1}{r}\right) dr = -4\pi f(0)}$$

Problem 2: Review

Procedure:

- Calculate the Laplacian in spherical co-ordinates.
- Evaluate the limits for the Laplacian.
- Expand the function with a Taylor series, keeping only the first order term.
- Approximate the Laplacian of $1/r$ and substitute the result.

Key Concepts:

- The Laplacian of this function will go to 0 or ∞ depending on how the extremum is evaluated.
- We can approximate $f(r)$ since it is a smooth function.
- We can then use the common integral for the end result.

Variations:

- The original function can be changed.
  - This in turn would create a different problem. Possibly having to calculate the Laplacian in cylindrical co-ordinates or something similar.

Problem 3:

Your textbook (and many others) state that the curl of the gradient of a scalar function is zero. Consider the function $f(r, \phi) = \phi$ where r and ϕ are the standard polar co-ordinates in two dimensions.

(a) Calculate $\vec{g} = \vec{\nabla} f$ in polar co-ordinates.

$$\vec{\nabla} f = \frac{1}{h_i} \partial_i f \, \hat{e}_i \quad : \quad d\vec{s} = d\vec{r} + r\, d\vec{\theta} + r \sin\theta \, d\vec{\varphi}$$

$$\text{w/ } \theta = \pi/2 \quad : \quad \vec{\nabla} f = \cancelto{0}{\frac{\partial f}{\partial r}} \vec{r} + \frac{1}{r} \cancelto{0}{\frac{\partial f}{\partial \theta}} \vec{\theta} + \frac{1}{r} \frac{\partial f}{\partial \varphi} \vec{\varphi}$$

$$\boxed{\vec{\nabla} f = \frac{1}{r} \vec{\varphi} = \vec{g}}$$

(b) Calculate $\vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{g}$ in polar co-ordinates.

$$\vec{\nabla} \times \vec{\varphi} = \frac{1}{h_2 h_3} \left(\partial_2 (h_3 \varphi_3) - \partial_3 (h_2 \varphi_2) \right) \hat{e}_1 + \frac{1}{h_3 h_1} \left(\partial_3 (h_1 \varphi_1) - \partial_1 (h_3 \varphi_3) \right) \hat{e}_2 + \frac{1}{h_1 h_2} \left(\partial_1 (h_2 \varphi_2) - \partial_2 (h_1 \varphi_1) \right) \hat{e}_3$$

$$\vec{\nabla} \times \vec{g} = \frac{1}{r^2 \sin\theta} \cancelto{0}{\left(\partial_\theta (r \sin\theta \cdot 1/r) - \partial_\varphi (r \cdot 0) \right)} \hat{e}_r + \frac{1}{r \sin\theta} \cancelto{0}{\left(\partial_\varphi (r \cdot 0) - \partial_r (r \sin\theta \cdot 1/r) \right)} \hat{e}_\theta + \frac{1}{r} \cancelto{0}{\left(\partial_r (r \cdot 0) - \partial_\theta (0) \right)} \hat{e}_\varphi$$

$$\boxed{\vec{\nabla} \times \vec{g} = 0}$$

(c) From Stoke's Theorem, we know

$$\int \left(\vec{\nabla} \times \vec{g} \right) \cdot \vec{n} \, dA = \oint \vec{g} \cdot d\vec{l}.$$

Calculate the last line integral in polar co-ordinates.

$$\oint \frac{1}{r} \hat{\varphi} \cdot r \, d\hat{\varphi} = 2\pi$$

$$\boxed{\oint \vec{g} \cdot d\vec{l} = 2\pi}$$

(d) Do your answers agree? Can you explain?

They are not going to agree since there is a discontinuity at $\varphi = 0$.

Problem 3: Review

Procedure:

- Calculate the Gradient of the function in spherical polar co-ordinates.
- Calculate the curl of the Gradient.
- Calculate the line integral of Stokes' theorem.
- Explain why they aren't equal.

Key Concepts:

- Stokes' theorem is not valid for functions that have discontinuities in them (This is why the line integral is different than the LHS in Stokes' theorem).

Variations:

- The function that is given to us can change.
  - The curl of the gradient will still be equal to zero but it is possible the line integral will also be zero as long as there are no discontinuities.