E & M Qualifier

January 11, 2017

To insure that the your work is graded correctly you MUST:

1. use only the blank answer paper provided,
2. use only the reference material supplied (Schaum's Guides),
3. write only on one side of the page,
4. start each problem by stating your units e.g., SI or Gaussian,
5. put your alias **(NOT YOUR REAL NAME)** on every page,
6. when you complete a problem put 3 numbers on **every** page used for **that** problem as follows:
   (a) the first number is the problem number,
   (b) the second number is the page number for **that** problem (start each problem with page number 1),
   (c) the third number is the total number of pages you used to answer **that** problem,
7. **DO NOT** staple your exam when done.

Problem 1: Electrostatics

Consider a capacitor composed of two concentric spherical metal shells, the inner one with radius a and the outer one with radius b. The region between the spherical metal shells is filled with a linear dielectric with permittivity $\epsilon = \frac{k}{r^2}$. Place charge $+Q$ on the inner metallic shell and $-Q$ on the outer metallic shell.

1. Find the electric displacement $\vec{D}$ everywhere in space. [3 points]

2. Find the capacitance of the configuration. [3 points]

3. Calculate the bound charge densities in the linear dielectric and verify that the total net bound charge is zero. [4 points]

Problem 2: Magnetostatics

Consider a sphere of radius R composed of magnetic material with a magnetization given by $\vec{M} = M_0 \hat{z}$.

1. Starting with the Maxwell equations for a static magnetic field $\vec{H}$ and a static magnetic induction $\vec{B}$, prove that the magnetic field $\vec{H}$ can be written in terms of a scalar magnetic potential. From this derive the Poisson equation that solves the potential. In addition, derive expressions for the magnetic volume charge density and the bound current density from the Maxwell equations. [2 points]

2. Derive the boundary conditions on $\vec{H}$ and $\vec{B}$. Be sure to clearly define the effective surface magnetic charge density and the surface magnetic current density. [2 points]

3. Derive the fields inside and outside the sphere. *You can assume that $\vec{B}$ and $\vec{M}$ are parallel.* [6 points]

Problem 3: Maxwell equations

Consider a medium with nonzero scalar conductivity σ ($\mathbf{J}_f = \sigma\mathbf{E}$ is the current density), permeability μ, permittivity ϵ, and with no free charge ($\rho_f = 0$).

1. Write down the set of four differential Maxwell's equations appropriate for this medium. [2points]

2. Derive the wave equation for $\mathbf{E}$ in this medium. Highlight the additional term arising from the non-zero $\mathbf{J}_f$. [3 points]

3. Consider a monochromatic wave moving in the $+x$ direction with E_y given by

$$E_y = A\, e^{i(kx-\omega t)}$$

Show that this wave has an amplitude A which decreases exponentially. Find the attenuation length Δx, the distance after which the amplitude has decayed by a factor of $1/e$ from its initial value, as a function of σ. Show that your solution correctly predicts $\Delta x = 0$ if $\sigma = 0$. [5 points]

Problem 4: EM radiation

A particle of mass m and charge q is attached to a spring with force constant k, which is hanging from the ceiling. The particle's equilibrium position is a distance h above the floor. Suppose the particle is pulled down a distance d below its equilibrium position and released at time $t = 0$. Useful information: When the wavelength is much greater than the spatial amplitude, the electric field from an oscillating dipole is

$$E = -\frac{\mu_o p_o \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos(\omega t - [\omega r/c]) \tag{1}$$

1. Calculate the intensity of the radiation hitting the floor, as a function of the distance R from the point directly below the point particle. Assume $d \ll \lambda \ll h$ and neglect radiative damping of the oscillator. [3 points]

2. At what R is the radiation most intense? [2 points]

3. Assume the floor is of infinite extent. Calculate the average energy per unit time striking the entire floor. How does this compare to the total power radiated by the oscillating charge? [3 points]

4. Because energy is lost in the form of radiation, the amplitude of the oscillation will gradually decrease. At what time τ has the oscillator's energy been reduced to $1/e$ of its initial value? Assume the fraction of the total energy lost in one cycle is very small. [2 points]

Problem 5: Special relativity

Consider two frames K and K' with a uniform relative velocity. Observers at rest in K' are moving along the positive x axis of K with a velocity v. $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$, and c is the speed of light.

1. Let x^ν be the four-dimensional space-time vector in the K frame with the components: $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, and x'^μ be the corresponding vector in the K' frame with the Lorentz transformation of $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu$, where Einstein's summation rule is implied. What are the components of $\Lambda^\mu_{\ \nu}$? [2 points]

2. An object is moving with a three-dimensional velocity $\vec{u}$ in K, and the velocity is measured to be $\vec{u'}$ in K'. What are the components of the object's four-velocity in K' in terms of u_x, u_y and u_z? [4 points]

3. Let θ be the angle between $\vec{u}$ and x in K, and θ' be the angle between $\vec{u'}$ and x' in K'. Show that
$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}. \tag{2}$$
[2 point]

4. A source is emitting isotropically in its rest-frame and moves with an ultra-relativistic velocity in K with $\gamma \gg 1$. Show that in K half of the radiation power is concentrated in a cone with a half open angle of $1/\gamma$. [2 points]

Problem 6: Relativistic electrodynamics

The Lagrangian for the EM field generated from a 4-current j_μ is given (in SI units) by

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$, $A_\mu = (\phi/c, -\vec{A})$ and $j_\mu = (c\rho, -\vec{j})$ and where $c^2 = 1/\epsilon_0\mu_0$.

1. Show that $\mathcal{L}$ is invariant under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(t, \vec{x})$. [2 points]
2. Derive the covariant form of Maxwell's equations from the Euler-Lagrange equations using $\mathcal{L}(A_\mu, \partial_\nu A_\mu)$. [4 points]
3. Show that these reduce to the usual form of Maxwell's equations in 3-vector notation. [4 points]