

Quantum Mechanics
Qualifying Exam - January 2016

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1)$$

In spherical coordinates,

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \psi + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \psi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi. \quad (2)$$

Harmonic oscillator wave functions

$$u_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$u_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}} x e^{-\frac{m\omega x^2}{2\hbar}}$$

Problem 1: Clebsh-Gordon coefficients (10 pts)

A system of two particles with spins $s_1 = \frac{3}{2}$ and $s_2 = \frac{1}{2}$ is described by the Hamiltonian

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2$$

with α a constant and \mathbf{S}_i ($i = 1, 2$) is the spin operator of the i -th particle.

a) What are the allowed values for the quantum numbers of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$? (2 Points)

b) Calculate the energy levels of the Hamiltonian. (2 Points)

c) Let us define the basis of eigenstates of the \mathbf{S}_1^2 , \mathbf{S}_2^2 , S_{1z} , S_{2z} operators, $|s_1 s_2; m_1 m_2\rangle$, where m_1 and m_2 are the quantum numbers of the projection operators S_{1z} and S_{2z} respectively. The system at time $t = 0$ is initially in the state

$$\left| s_1 s_2; \frac{1}{2}, \frac{1}{2} \right\rangle.$$

Find the state of the system at times $t > 0$. (4 Points)

d) Assuming the initial state above, what is the probability of finding the system in the state

$$\left| s_1 s_2; \frac{3}{2}, -\frac{1}{2} \right\rangle$$

at $t > 0$? (2 Points)

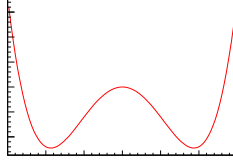


Figure 1: $U(x)$

Problem 2: Perturbation to a Harmonic Oscillator (10 pts)

Consider a particle of mass, m , moving in a 1-dimensional potential (see Figure 1)

$$U(x) = \lambda x^4 - kx^2.$$

λ and k are positive, and $\lambda \ll \frac{(k^{3/2}m^{1/2})}{4\hbar}$. Approximate the potential near the minima by a simple harmonic oscillator. Here are some useful integrals:

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} dx = \frac{1}{4A^{5/2}} (3 + 4a^2 A (3 + a^2 A)) \sqrt{\pi}, \text{ for } A > 0$$

$$\int_{-\infty}^{\infty} x^4 e^{-A(x-a)^2} e^{-A(x+a)^2} dx = \frac{3}{16A^{5/2}} e^{-2a^2 A} \sqrt{\frac{\pi}{2}}, \text{ for } A > 0$$

- Sketch the wavefunctions of the state $|\psi_R\rangle$ which is defined as the state when the particle is found at $x > 0$ and the state $|\psi_L\rangle$ which is the state when the particle is found at $x < 0$. Only consider the lowest energy states near the minima. **(2 Points)**
- Since the potential is invariant under reflection about the origin, the stationary states must be eigenstates of the parity operator. Express the ground-state and first excited state wavefunctions in terms of $|\psi_R\rangle$ and $|\psi_L\rangle$. **(2 Points)**
- Estimate the energies of the 2 lowest states using the approximations already described. Hint: use the space representation of the harmonic oscillator wavefunctions and carry out the integrals to find the perturbed energies. **(6 Points)**

Problem 3: Identical particles (10 pts)

Two non-interacting particles of mass m are trapped in a 1-dimensional infinite box of length L situated between $x = 0$ and $x = L$. (In the cases you are considering fermions, assume them to all be spin up.)

- (a) [1 points] Write down the single particle energy eigenvalues and wavefunctions.
- (b) [1 points] Write down the energy eigenvalues and wavefunctions for two distinguishable particles. Label the states by n_1 for particle 1 and n_2 for particle 2.
- (c) [2 points] An energy measurement of the *two identical particle* system yields $E = \hbar^2\pi^2/mL^2$. Write down the state vector/wave function of the system.
- (d) [2 points] Suppose instead the energy of the two identical particle system is measured to be $E = 5\hbar^2\pi^2/mL^2$. What is the wave function?
Hint: there are two possibilities.
- (e) [2 points] Show that the fermion state you found in part (d) is an eigenfunction of the Hamiltonian, with the appropriate eigenvalue.
- (f) [1 points] Write down the wavefunction for two identical spin-up fermions in the $n_1 = 2$ and $n_2 = 2$ state.
- (g) [1 points] If instead you had three particles in the orthonormal states Ψ_1, Ψ_2 , and Ψ_3 , construct the three particle state for identical fermions.

Problem 4: Matrix Mechanics (10 pts)

Consider a system governed by a Hamiltonian H , with an observable C . The Hamiltonian is represented in the $|e_i\rangle$ basis as:

$$H = \hbar\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Where } |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |e_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of H are

$$|E_1 = -\hbar\omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, |E_2 = \hbar\omega, 1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, |E_2 = \hbar\omega, 2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let C be represented in the $|e_i\rangle$ basis as

$$C = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

At $t=0$, the system is in the state: $|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$

- At time $t=0$, the observable C is measured. What results are possible and with what probabilities? (2 pts)
- Determine the representation of the time evolution operator $U(t, t_0 = 0)$ in the $|e_i\rangle$ representation. (2 pts)
- Determine $|\Psi(t)\rangle$ in the $|e_i\rangle$ basis. (2 pts)
- If C is measured at some later time t , what results are possible and with what probabilities? (2 pts)
- Are your probabilities time dependent or time independent? Explain (2 pts)

Problem 5: Magnetic Moments and Spin (10 pts)

Consider a spin 1/2 particle with a magnetic moment. We can write the interaction between the spin and an external magnetic field using the Hamiltonian:

$$H = -\gamma \vec{B} \cdot \vec{S} \quad (1)$$

where \vec{B} is the external field, \vec{S} is the spin operator for the particle, and γ is a real positive constant. In this problem, use the usual basis states that are eigenstates of S_z

$$S_z \chi_{\pm} = \pm \frac{\hbar}{2} \chi_{\pm}, \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2)$$

For this problem, assume the magnetic field lies in the x-z plane:

$$\vec{B} = B_x \hat{e}_x + B_z \hat{e}_z \quad (3)$$

- (a) [1 pt] Solve for the eigenenergies for the Hamiltonian, showing your work. Explain the physics of your results.
- (b) [2 pts] Any state of the spin can be written in the χ_{\pm} basis as:

$$\Psi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} \quad (4)$$

Using the Hamiltonian, derive the first-order coupled differential equations that give the time dependence for $\alpha(t)$ and $\beta(t)$. In other words, derive the equations for $\dot{\alpha}(t)$ and $\dot{\beta}(t)$.

- (c) [2 pts] Show that you can re-write your results from part (b) as two uncoupled second-order differential equations:

$$\begin{aligned} \ddot{\alpha}(t) &= -\frac{\gamma^2 B_T^2}{4} \alpha(t) \\ \ddot{\beta}(t) &= -\frac{\gamma^2 B_T^2}{4} \beta(t) \end{aligned} \quad (5)$$

where $B_T = \sqrt{B_x^2 + B_z^2}$ is the magnitude of the total magnetic field. How is this result related to what you found in part (a)?

Of course, the solutions to these equations are:

$$\begin{aligned} \alpha(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ \beta(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{aligned} \quad (6)$$

with $\omega = \frac{\gamma B_T}{2}$.

- (d) [3 pts] Consider the situation where the spin is in the spin-up S_z state χ_+ at time $t = 0$. Using the boundary conditions at time $t = 0$, determine the values for the constants C_1 , C_2 , C_3 , C_4 that will solve for the time-dependence of the state. Remember that the equations in part (c) are second-order, so you need two boundary conditions at $t = 0$ for each.
- (e) [2 pt] Write down the time-dependent probabilities, P_{\pm} of the spin being in the spin-up and spin-down S_z states. Show that your results are correct in the two cases where $B_x = 0$ and $B_z = 0$.

Problem 6: Electron in a Finite Square Well (10 pts)

Consider an electron of energy E incident from $x=-\infty$ on a symmetric one-dimensional square well of depth V_0 and width L .

$$V(x) = \begin{cases} 0, & x < -L/2 \\ -V_0, & -L/2 < x < L/2 \\ 0, & x > L/2 \end{cases}$$

- a) Write down the solutions to the time-independent Schrodinger Equation for this situation. There should be five integration constants (2 points)
- b) Apply boundary conditions to find the probability that the electron is transmitted past the finite well (4 points)
- c) For what values of E is there a 100% probability for transmission past the well? (2 points)
- d) Consider a potential well with V_0 large enough for there to be two bound states. For this well, what is the smallest electron energy ($E > 0$) for which there is a 100% probability for transmission? Your answer will depend on V_0 and other parameters in the problem. (2 points)