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Statistical Mechanics

PHYS 5163 Homework Assignment 4

Problems: {1,2,3,4}

Due: February 18, 2022 at 6:00 PM

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Problem 1:

A solid is composed of N atoms whose nuclei have angular momentum $\hbar/2$ and thus a nuclear magnetic moment of magnitude μ. When the solid is placed in a homogeneous field along a given axis, the magnetic moment of each of the atoms can have the value μ or $-\mu$. Let us assume that the interaction between the magnetic moments and the external field is much, much stronger than the interaction between neighboring magnetic moments.

(a) Use the canonical ensemble to calculate the internal energy of the solid in the external field at temperature T.

We will calculate the internal energy with

$$U = A - T\left(\frac{\partial A}{\partial T}\right)_V \qquad (*)$$

To do $(*)$, we need to calculate the Helmholtz free energy. We do that with

$$A = -KT\log(Q_N)$$

We are told that the atoms have a magnetic moment of $\pm\mu$. This then means our Hamiltonian will have the form of

$$\mathcal{H} = \pm\mu B$$

The partition function is then calculated via

$$Q_N = \sum_\nu e^{-\beta E_\nu}$$

This then turns into,

$$Q_1 = e^{-\beta(\mu B)} + e^{-\beta(-\mu B)} = e^{-\mu\beta B} + e^{\mu\beta B} = 2\cosh(\mu\beta B) = 2\cosh\left(\frac{\mu B}{KT}\right)$$

Putting this into $(Q_1)^N$ and then the Helmholtz free energy we have

$$A = -KT\log\left(2^N\cosh^N\left(\frac{\mu B}{KT}\right)\right) = -NKT\log\left(2\cosh\left(\frac{\mu B}{KT}\right)\right)$$

We now put the above into $(*)$

$$\left(\frac{\partial A}{\partial T}\right)_V = -NK\log\left(2\cosh\left(\frac{\mu B}{KT}\right)\right) - NK\cancel{T}\cdot\left(\frac{-\mu B}{KT^{\cancel{2}}}\right)\tanh\left(\frac{\mu B}{KT}\right)$$

$$= \frac{\mu BN}{T}\tanh\left(\frac{\mu B}{KT}\right) - NK\log\left(2\cosh\left(\frac{\mu B}{KT}\right)\right)$$

$$A - T\left(\frac{\partial A}{\partial T}\right)_V = -NKT\cancel{\log\left(2\cosh\left(\frac{\mu B}{KT}\right)\right)} - T\left(\frac{\mu BN}{T}\tanh\left(\frac{\mu B}{KT}\right) - NK\cancel{\log\left(2\cosh\left(\frac{\mu B}{KT}\right)\right)}\right)$$

Problem 1: Continued

This then means the internal energy is,

$$\boxed{U = -\mu B N \tanh\left(\frac{\mu B}{kT}\right)}$$

(b) Use the canonical ensemble to calculate the entropy of the solid in the external field at temperature T.

Another way we can write (*) is

$$U = A + TS \qquad (**)$$

Using the result for U in part (a), the prior definition for A, we can solve for S. Doing this we find

$$-\mu B N \tanh\left(\frac{\mu B}{kT}\right) = -NkT\log\left(2\cosh\left(\frac{\mu B}{kT}\right)\right) + T(S)$$

If we re-arrange the above for S we find

$$\boxed{S = Nk\log\left(2\cosh\left(\frac{\mu B}{kT}\right)\right) - \frac{\mu B N}{kT}\tanh\left(\frac{\mu B}{kT}\right)}$$

Problem 1: Review

Procedure:

- – Calculate the internal energy with
  $$U = \mathcal{A} - T\Big(\frac{\partial \mathcal{A}}{\partial T}\Big)_V$$
  – Calculate the Helmholtz free energy with
  $$\mathcal{A} = -KT\log\left(Q_N\right)$$
  – Since our energy levels are discrete, we can calculate the partition function with
  $$Q_N = \sum_{\nu}^{N}\exp\big(-\beta E_\nu\big)$$
  – Proceed to calculate the internal energy with the above information
- – Using the definition for U that is
  $$U = \mathcal{A} + TS,$$
  use the prior result from (a) and solve for S

Key Concepts:

- – Working in the canonical ensemble we will make our bridge from Statistical Mechanics to Thermodynamics by calculating the partition function for our system
  – Since our energy levels are discrete, we can use the summation for our energy levels instead of the integral form
  – Once the partition function is calculated we then proceed to calculate the Helmholtz free energy and eventually the internal energy
- – We can use a separate definition for the internal energy to solve for the entropy of our system

Variations:

- – We can be given a system that has continuous energy levels
  - ∗ We then would use the integral form of the partition function
  – We could be given different values for the energies
  - ∗ This would change what our sum evaluates to
- – Excluding any changes from part (a) the only way this part changes is if we are asked to find a separate quantity

Problem 2:

(a) An experimentalist measures the specific heat C_V of a gas of non-interacting one- dimensional particles of mass m at temperature T. The potential of the j^{th} particle is given by

$$V(x_j) = \epsilon_0 \left| \frac{x_j}{a} \right|^n, \tag{1}$$

where ϵ_0 and a are constants with units "energy" and "length", respectively, and n is an integer, $n = 1, 2, ...$ From the measurements, can the experimentalist determine the value of n?

To calculate the value of n we will calculate C_V in terms of n. C_V is calculated via

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V$$

Where the internal energy for our system is

$$U = A - T \left(\frac{\partial A}{\partial T} \right)_V$$

We of course need to calculate the Helmholtz Free energy

$$A = -kT \log(Q_N)$$

And to do this we need to calculate Q_N

$$Q_N = \frac{1}{N! h^N} \int_\Gamma \int_\Gamma e^{-\beta \mathcal{H}} d\vec{p} d\vec{q}$$

Where our partition function is dependent upon $\mathcal{H}$. For our system $\mathcal{H}$ is

$$\mathcal{H}(p, x) = \frac{p^2}{2m} + \epsilon_0 \left| \frac{x_i}{a} \right|^n$$

Putting this hamiltonian into our partition function we have

$$Q_N = \frac{1}{N! h^N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\beta \left[\frac{\vec{p}^2}{2m} + \epsilon_0 \left| \frac{q_i}{a} \right|^n \right]} d\vec{p} d\vec{q}$$

$$Q_N = \frac{1}{N! h^N} \int_{-\infty}^{+\infty} e^{-\beta \left[\frac{\vec{p}^2}{2m} \right]} d\vec{p} \int_{-\infty}^{+\infty} e^{-\beta \left[\epsilon_0 \left| \frac{q_i}{a} \right|^n \right]} d\vec{q}$$

Using common rules for Gaussians the integral simplifies to

$$Q_N = \frac{1}{N! h^N} \left(\frac{2\pi m}{\beta} \right)^{1/2} \int_{-\infty}^{+\infty} e^{-\beta \left[\epsilon_0 \left| \frac{q_i}{a} \right|^n \right]} d\vec{q}$$

Problem 2: Continued

The above integral is very difficult to solve, therefore we make a change of variables

$$z = \beta \epsilon_0^n q^n a^{-n} \Rightarrow dz = n\beta \left(\frac{\epsilon_0}{a}\right)^n q^{n-1} dq \quad \therefore \quad dq = \frac{1}{n\beta}\left(\frac{\epsilon_0}{a}\right)^{-n} q^{1-n} dz$$

$$\Rightarrow \quad dq = \frac{a^n}{\beta\epsilon_0 n} \frac{a^{1-n}}{(\beta\epsilon_0)^{(1-n)/n}} z^{(1-n)/n} dz = \frac{a}{n(\beta\epsilon_0)^{1/n}} z^{(1-n)/n} dz$$

Our integral is now

$$Q_N = \frac{1}{N!h^N}\left(\frac{2\pi m}{\beta}\right)^{\frac{N}{2}}\left(\frac{2a}{n(\beta\epsilon_0)^{1/n}}\right)^{\frac{N}{2}}\int_0^\infty e^{-z} z^{\frac{1-n}{n}} dz = \frac{1}{N!}\left(\frac{2ma\Gamma(1/n)}{nh\epsilon_0^{1/n}}\right)^{N/2} \beta^{-\frac{N}{2}\left(\frac{n+2}{n}\right)}$$

We have now found the partition function, we proceed to find $\langle \mathcal{H} \rangle$

$$\langle \mathcal{H} \rangle = -\frac{\partial}{\partial\beta}\log(Q_N) = -\frac{\partial}{\partial\beta} N\left[\log\left(\beta^{-\frac{1}{2}\left(\frac{n+2}{n}\right)}\right) + \text{const.}\right] = \frac{NkT}{2}\left(\frac{n+2}{n}\right) = \frac{NkT}{2}\left(\frac{n+2}{n}\right)$$

We then find C_V to be

$$\boxed{C_V = Nk\left(\frac{1}{2} + \frac{1}{n}\right)}$$

(b) The experimentalist repeats the measurement for a gas of non-interacting two-dimensional particles of mass m at temperature T for which the potential of the j^{th} particle is given by

$$V(x_j, y_j) = \epsilon_0\left(\frac{\rho_j}{a}\right)^n, \tag{2}$$

where $\rho_j = \sqrt{x_j^2 + y_j^2}$. What is C_V in this case? The following integral may be helpful:

$$\int_0^\infty \exp(-z)^{(1-n)/n}\, dz = \Gamma(1/n). \tag{3}$$

We will follow a similar procedure for (a) but now in 2D for (b). This immediately means our Hamiltonian will change to

$$\mathcal{H}(\vec{p}, \vec{q}) = \frac{\vec{P}_x^2 + \vec{P}_y^2}{2m} + \epsilon_0\left(\frac{\rho_i}{a}\right)^n$$

This then means our partition function will now be

$$Q_N = \frac{1}{N!h^{2N}}\int_\Gamma\int_\Gamma \exp\left[-\beta\left(\frac{P_x^2 + P_y^2}{2m} + \epsilon_0\left(\frac{\rho_i}{a}\right)^n\right)\right] d^{2N}\vec{p}\, d^{2N}\vec{q}$$

Problem 2: Continued

To simplify this we will make the substitutions $\tilde{p}_i^2 = p_{x_i}^2 + p_{y_i}^2$, $N=1$. This means

$$Q_1 = \frac{1}{h^2}\int_\gamma\int_\gamma \exp\left[-\beta\left(\left(\frac{\tilde{p}_i^2}{2m}\right) + \varepsilon_0\left(\frac{p_i}{a}\right)^n\right)\right] d^2\vec{p}\,d^2\vec{q}$$

Making the transistion from cartesian to polar, Q_1 becomes

$$Q_1 = \frac{1}{h^2}\left[\int_0^{2\pi}\int_0^\infty \tilde{p}\exp\left(\frac{-\beta\tilde{p}_i^2}{2m}\right)d\tilde{p}\,d\theta\right]\left[\int_0^{2\pi}\int_0^\infty \rho\exp\left(-\beta\frac{\varepsilon_0}{a^n}\rho^n\right)d\rho\,d\theta\right]$$

$$= \frac{1}{h^2}\left[(2\pi)\int_0^\infty \tilde{p}\exp\left(\frac{-\beta\tilde{p}_i^2}{2m}\right)d\tilde{p}\right]\left[(2\pi)\int_0^\infty \rho\exp\left(-\beta\frac{\varepsilon_0}{a^n}\rho^n\right)d\rho\right]$$

$$= \frac{(2\pi)^2}{h^2}\left[\frac{1}{2}\left(\frac{2m}{\beta}\right)\right]\left[\frac{1}{n}\left(\frac{a^2}{(\varepsilon_0\beta)^{2/n}}\right)\Gamma(2/n)\right] = \frac{4\pi^2 m a^2}{h^2\beta n}\frac{\Gamma(2/n)}{(\varepsilon_0\beta)^{2/n}} = \frac{ma^2}{n\hbar^2}\frac{KT^{\left(\frac{n+2}{n}\right)}}{\varepsilon_0^{2/n}}\Gamma\left(\frac{n+2}{n}\right)$$

Where we have used the common integral

$$\int_0^\infty x^N e^{-\alpha x^b}dx = \frac{1}{b}\alpha^{-\frac{N+1}{b}}\Gamma\left(\frac{N+1}{b}\right)$$

This then means Q_N is finally

$$Q_N = \frac{(Q_1)^N}{N!} = \left(\frac{ma^2}{n\hbar^2(N!)}\frac{KT^{\left(\frac{n+2}{n}\right)}}{\varepsilon_0^{2/n}}\Gamma\left(\frac{n+2}{n}\right)\right)^N$$

Our Helmholtz free energy is then

$$A = -NKT\left[\log(ma^2) + \log\left(KT^{(n+2/n)}\right) + \log(\Gamma(n+2/n)) - \log(n\hbar^2 N!) - \log(\varepsilon_0^{2/n})\right]$$

This then means our internal energy is

$$U = NKT\left(\frac{n+2}{n}\right) \qquad (*)$$

We now take $(*)$ and solve for $C_V \Rightarrow n$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = NK\left(\frac{n+2}{n}\right) \Rightarrow n = \frac{2}{C_V/NK - 1}$$

Problem 2: Continued

We then finally have a value for n

$$\boxed{n = \frac{2}{C_V/Nk - 1}}$$

The same procedure can be used in part a.)

Problem 2: Review

Procedure:

- – We will calculate C_v with
  $$C_v = \left(\frac{\partial U}{\partial T}\right)_V$$
  – The internal energy is calculated with
  $$U = \mathcal{A} - T\left(\frac{\partial \mathcal{A}}{\partial T}\right)_V$$
  – The internal energy and partition function are calculated via
  $$\mathcal{A} = -KT\log{(Q_N)} \qquad \text{with} \qquad Q_N = \frac{1}{N!h^{\alpha N}}\int_\tau\int_\tau \exp(-\beta\mathcal{H})d^{\alpha N}\vec{p}\,d^{\alpha N}\vec{q}$$
  where α is the dimension that we are working in
  – Proceed to solve for average value of the Hamiltonian $\langle\mathcal{H}\rangle$ with
  $$\langle\mathcal{H}\rangle = -\frac{\partial}{\partial\beta}\log{(Q_N)}$$
  and eventually the specific heat with
  $$C_v = \frac{\partial\langle\mathcal{H}\rangle}{\partial T}$$
- – Use the same process for part (b) as in part (a)
  – Solve for n after calculating C_v

Key Concepts:

- – We are working in the canonical ensemble so we will calculate a partition function to create a connection from Statistical Mechanics to Thermodynamics
  – Because our energy levels are not discrete, we are required to use the integral form for calculating the partition function
  – The internal energy of our system is equal to the average value of our Hamiltonian
- – Since we are working in two dimensions this will slightly change our Hamiltonian and everything that follows after it

Variations:

- – We can be given a different potential
    - ∗ This would change what our partition function evaluates to thus changing the exact answer but not the process
  – We could be given discrete energy levels
    - ∗ We could then use a simplified version for finding the partition function
- – Essentially the same as part (a) but in 2D

Problem 3:

For zero-mass particles, the energy-momentum relation is $E = c|\vec{p}|$. This relationship can also be used to approximate "ordinary" particles when $KT >> mc^2$. A gas described by this relationship is sometimes referred to as an extremely relativistic classical gas. Assuming three-dimensional space, calculate the pressure and energy per particle of such a gas as functions of the density and temperature.

To do this problem we will calculate the pressure and the energy per particle by

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} , \quad U = A - T\left(\frac{\partial A}{\partial T}\right)_{N,V}$$

In order to follow this framework we must have the Helmholtz Free energy. This of course is calculated with

$$A = -kT\log(Q_N)$$

Where since we are in the canonical ensemble we must calculate the partition function with

$$Q_N = \frac{1}{N!h^{3N}}\int_\Gamma\int_\Gamma e^{-\beta c\vec{p}}\, d^{3N}\vec{p}\, d^{3N}\vec{q}$$

It is of course easier to first calculate the partition function for one particle

$$Q_1 = \frac{1}{h^3}\int_\Gamma\int_\Gamma e^{-\beta c\vec{p}}\, d^3\vec{p}\, d^3\vec{q} = \frac{1}{h^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} d\vec{q}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\vec{p}^2 e^{-\beta\vec{p}c}\sin(\theta)\,d\vec{p}\,d\theta\,d\varphi$$

$$= \frac{V}{h^3}(2\pi)(2)\int_0^\infty \vec{p}^2 e^{-\beta\vec{p}c}\,d\vec{p} = \frac{4\pi}{h^3}V(2)\left(\frac{kT}{c}\right)^3 = 8\pi V\left(\frac{kT}{hc}\right)^3$$

We can now raise this to N

$$Q_N = (8\pi V)^N\left(\frac{kT}{hc}\right)^{3N}$$

We then of course now have our Helmholtz Free energy

$$A = -kT\log\left((8\pi V)^N\left(\frac{kT}{hc}\right)^{3N}\right) = -NkT\left[\log(8\pi V) + 3\log(kT) - 3\log(hc)\right]$$

This then means our internal energy and pressure are

$$U = 3NkT \implies \frac{U}{N} = 3kT , \quad P = \frac{NkT}{V} \implies V = \frac{NkT}{P}$$

$$\boxed{U/N = 3kT , \quad V = NkT/P}$$

Problem 3: Review

Procedure:

- – Since we are working in the canonical ensemble we will first calculate our partition function with

$$Q_N = \frac{1}{N!h^{\alpha N}} \int_\tau \int_\tau \exp(-\beta\mathcal{H}) d^{\alpha N}\vec{p}\, d^{\alpha N}\vec{q}$$

  – We then calculate the Helmholtz free energy with

$$A = -KT\log\left(Q_N\right)$$

  – We then calculate the pressure and energy per particle with

$$P = -\left(\frac{\partial \mathcal{A}}{\partial V}\right)_{T,N} \quad , \quad U = \mathcal{A} - T\left(\frac{\partial \mathcal{A}}{\partial T}\right)_{N,V}$$

Key Concepts:

- – Working in the canonical ensemble we once again create the bridge from Statistical Mechanics to Thermodynamics by calculating a partition function and eventually the Helmholtz free energy
  – Our energy levels are not discrete and that is why we have to use the integral form for the partition function
  – We can derive the ideal gas law this way

Procedure:

- – We can be given a different Hamiltonian
    * This would cause the evaluation of the partition function to be different and everything that follows after

Problem 4:

This problem serves as a review of undergraduate quantum knowledge in preparation for quantum statistical mechanics.

(a) A one-dimensional harmonic oscillator potential is a potential of the form

$$V(x) = \frac{1}{2}kx^2. \tag{4}$$

What is the energy and degeneracy of the ground state of a system consisting of five non- interacting particles of mass m that are confined by $V(x)$ in the cases that

(i) The particles are spin-0 bosons,
(ii) The particles are spin-$\frac{1}{2}$ fermions,
(iii) The particles are spin-$\frac{1}{2}$ bosons,
(iv) The particles are spin-0 fermions, and
(v) The particles are spin-$\frac{5}{2}$ fermions?

The quantized energy levels for the Quantum Harmonic Oscillator for n levels in one dimension is

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$$

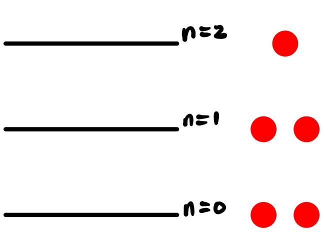
If we are looking at 5 Spin-0 Bosons, then the ground state energy is

$$\boxed{E_0 = \frac{5}{2}(\hbar\omega)}$$

Since our Bosons are spin 0, this means we want to count our number of permutations for (0,0,0,0,0). In this case there is only one and thus the degeneracy is 1.

$$\boxed{\text{Degeneracy of } 1}$$

When dealing with 5 Spin-1/2 Fermions, we know that the lowest energy levels must be occupied first. In our case, two particles at most can be in one level. This means our energy is then



$$\Rightarrow E_0 = \hbar\omega\left(0+\frac{1}{2}\right) + \hbar\omega\left(0+\frac{1}{2}\right) + \hbar\omega\left(1+\frac{1}{2}\right) + \hbar\omega\left(1+\frac{1}{2}\right) + \hbar\omega\left(2+\frac{1}{2}\right) = \frac{13}{2}\hbar\omega$$

This then means our ground state energy is

$$\boxed{E_0 = \frac{13}{2}\hbar\omega}$$

Problem 4: Continued

Since these particles are indistinguishable, there are only two ways this can be possible $\therefore$ This means we have a degeneracy of

$$\boxed{\text{Degeneracy of } 2}$$

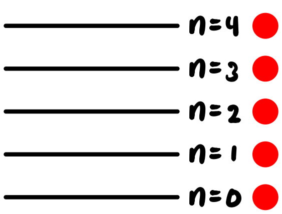
Since Bosons are going to still occupy the $n=0$ state it is the same for Spin-0 Bosons as Spin-½ Bosons. Therefore the ground state energy for 5 Spin-½ Bosons is

$$\boxed{E_0 = \frac{5}{2}(\hbar\omega)}$$

Since these are indistinguishable particles and Bosons can all fit in the same energy level this means there are 6 distinct combinations $\therefore$ the degeneracy is

$$\boxed{\text{Degeneracy of } 6}$$

For Spin-0 Fermions, this means there can only be one particle per energy level (value of n). This means for 5 Spin-0 Fermions the ground state energy is



$$E_0 = \hbar\omega\left(0+\frac{1}{2}\right) + \hbar\omega\left(1+\frac{1}{2}\right) + \hbar\omega\left(2+\frac{1}{2}\right) + \hbar\omega\left(3+\frac{1}{2}\right) + \hbar\omega\left(4+\frac{1}{2}\right) = \frac{25\hbar\omega}{2}$$

This means our ground state energy is

$$\boxed{E_0 = \frac{25}{2}\hbar\omega}$$

Since we have spin 0, there is only one way these can be oriented $\therefore$ the degeneracy is

$$\boxed{\text{Degeneracy of } 1}$$

Lastly, for Spin-5/2 Fermions we can have up to six particles for a given n. This is because $S_z = \pm 5/2, \pm 3/2, \pm 1/2$. So for 5 Spin-5/2 Fermions the ground state energy is

$$\boxed{E_0 = \frac{5}{2}(\hbar\omega)}$$

Problem 4: Continued

The Fermions can all fit inside the same energy level, and since the particles are indistinguishable the degeneracy will be

$$\boxed{\text{Degeneracy of } 6}$$

(b) Repeat part (a) for the isotropic two-dimensional harmonic oscillator potential.

For a 2D isotropic Harmonic Oscillator the potential looks something like

$$V(x,y) = \frac{1}{2}k(x^2+y^2)$$

When we now look for the energy, this means $E = E_x + E_y$, so

$$E_n = \hbar\omega\left(n_x + \frac{1}{2}\right) + \hbar\omega\left(n_y + \frac{1}{2}\right) = \hbar\omega(n_x + n_y + 1)$$

Since spin-0 Bosons can all occupy the same energy level for <u>5 spin-0 Bosons</u>

$$E_0 = 5\hbar\omega(0+0+1) = 5\hbar\omega$$

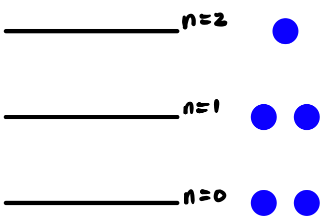
This then means our ground state energy is

$$\boxed{E_0 = 5\hbar\omega}$$

The degeneracy is then

$$\boxed{\text{Degeneracy is } 1}$$

For Spin-½ Fermions, there can be at most 2 particles per energy level. For <u>5 Spin-½ Fermions</u> the ground state energy is



$$E_0 = (0+0+1)\hbar\omega + (0+0+1)\hbar\omega + (1+0+1)\hbar\omega$$
$$+ (0+1+1)\hbar\omega + (1+0+1)\hbar\omega = 8\hbar\omega$$

The ground state energy finally is

$$\boxed{E_0 = 8\hbar\omega}$$

Problem 4: Continued

The degeneracy for this scenario will be twice that of the 1D problem

$$\boxed{\text{Degeneracy is } 4}$$

Since spin-½ Bosons can all occupy the same energy level for 5 spin-½ Bosons

$$E_0 = 5\hbar\omega(0+0+1) = 5\hbar\omega$$

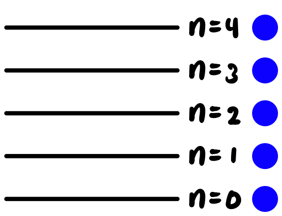
This then means our ground state energy is

$$\boxed{E_0 = 5\hbar\omega}$$

The degeneracy for this in 2D is the same as 1D ∴

$$\boxed{\text{Degeneracy is } 6}$$

For Spin-0 Fermions, this means there can only be one particle per energy level (value of n). This means for 5 Spin-0 Fermions the ground state energy is



$$E_0 = (0+0+1)\hbar\omega + (0+1+1)\hbar\omega + (1+0+1)\hbar\omega + (2+0+1)\hbar\omega + (0+2+1)\hbar\omega$$
$$= 11\hbar\omega$$

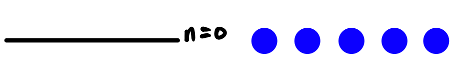
This then means our ground state energy is

$$\boxed{E_0 = 11\hbar\omega}$$

The degeneracy for this will be 3

$$\boxed{\text{Degeneracy is } 3}$$

Lastly, for Spin-5/2 Fermions we can have up to six particles for a given n. This is because $S_z = \pm 5/2, \pm 3/2, \pm 1/2$. So for 5 Spin-5/2 Fermions the ground state energy is



$$E_0 = 5\hbar\omega(0+0+1) = 5\hbar\omega$$

Therefore

$$\boxed{E_0 = 5\hbar\omega}$$

Problem 4: Continued

The degeneracy is then simply the same as the 1D

$\boxed{\text{Degeneracy of 6}}$

(c) Which of the cases (i)-(v) is physically possible/impossible? Explain.

Cases (iii) and (iv) are not possible since Bosons must have full integer spin where as Fermions must have half integer spin.

Problem 4: Review

Procedure:

- - The equation for calculating the energy in the 1D harmonic oscillator is

    $$E_n = \hbar\omega(n + 1/2)$$

  - We calculate the degeneracy of each system by counting the number of arrangements in the system
  - For Fermions, we calculate the occupancy with

    $$\mathcal{O} = 2S + 1$$

    where S is the spin of our particle
- - For a 2D harmonic oscillator the energy is calculated with

    $$E_n = \hbar\omega(n_x + n_y + 1)$$

  - The rest of the procedure from (a) carries over to (b)
- - Answer the question

Key Concepts:

- - There is no limit for the amount of Bosons that can fit in one energy level
  - The limit for the amount of Fermions that can fit in one energy level is calculated with the occupancy rule
  - Degeneracy is calculated by counting the number of arrangements that create the same energy for the system
  - Quantum particles are indistinguishable and therefore we cannot count permutations of particles
- - See part (a) but for 2D
- - Bosons are only allowed integer spin
  - Fermions are only allowed half integer spin

Variations:

- - We can be asked to do this for a different type of oscillator
    * We then would use a different equation for calculating energy and for counting degeneracy
  - We could be given more particles
    * This would just change the total energy and would require us to count more particles for degeneracy
- - Same as part (a) but in 2D
- - We could be asked a completely different question