Assignment 8
solutions

Question 1:

a) To obtain the moment of inertia relative to the com it will in fact be easier to obtain it via Steiner's parallel axis thm.

Step 1: Compute com,

$$\vec{r}_{com} = \frac{1}{m} \int dV \, \rho(\vec{r}) \, \vec{r}$$

If we choose our axes such that z lies along the long axis of the cylinder & y-x lie on the plane of the cylinder, then the x component of the com will vanish and the distance of the com from the flat face of the semi-cylinder will be:

$$y_{com} = \frac{1}{m} \int dV \, y \, \rho(\vec{r})$$

work in quasi 2D as long axis will drop out...

$$\equiv \frac{1}{m} \rho_0 \int_0^{\pi} \int_0^{R} r\,dr\,d\theta \; r\sin\theta$$

in polar co-ords.

Here $\rho_0 \equiv \dfrac{m}{\left(\dfrac{\pi r_0^2}{2}\right)}$ as we are working in 2D effectively.

Solving the integral yields,

$$y_{com} = \frac{4}{3\pi} r_0$$

<u>Step 2</u>: The only relevant moment of inertia is I'_{zz})

As we are dealing w/ a <u>half</u>-cylinder, then

$$I'_{zz} = \frac{1}{2}\left(I_{zz}^{\text{cylinder}}\right) = \frac{m r_0^2}{2}.$$

Next, using Steiner's theorem (see A7):

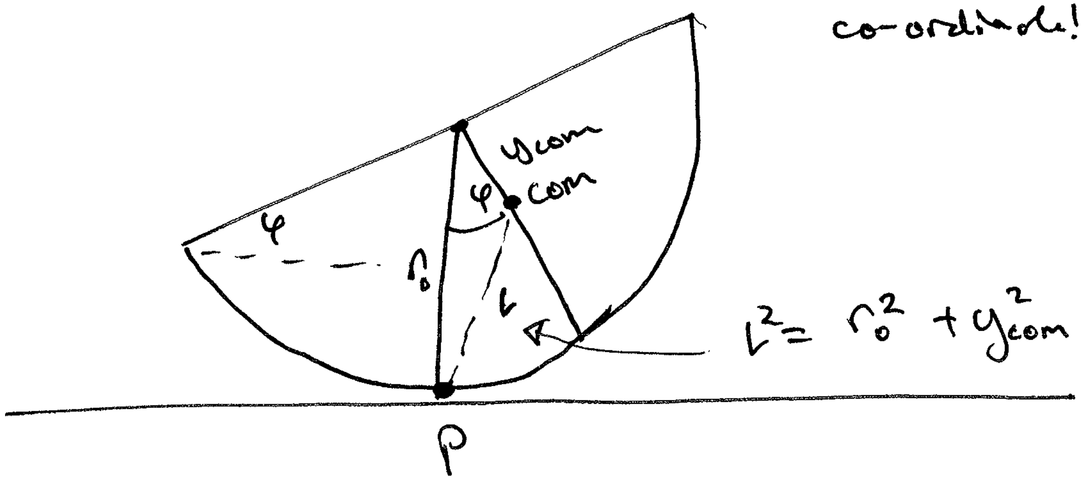
$$I_{zz} = I'_{zz} - m y_{com}^2$$

relative to com

$$= \left(\frac{1}{2} - \frac{16}{9\pi^2}\right) m r_0^2 .$$

b) To obtain the Lagrangian we will need to obtain expressions for T + V. First, let's draw a diagram!

φ is our generalized co-ordinate!



$$L^2 = r_0^2 + y_{com}^2 - 2r_0 y_{com} \cos\varphi$$

Using the com of mass as our reference, the potential contribution due to gravity is identical to that of a pendulum:

just some length, not a co-ordinate!

$$V = -mg\,y_{com}\cos\varphi = -\frac{4mgr_0}{3\pi}\cos\varphi$$

The kinetic term can be computed entirely from the rotational motion of the cylinder, if we consider rotation about the contact point P. This means we need to recompute the moment of inertia relative to this axis!

i.e. $I^{P}_{zz} = I^{com}_{zz} + ml^2$ from Steiner's theorem.

$\downarrow$ plug in l^2

$$I^{P}_{zz} = I = \left(\frac{3}{2} - \frac{8}{3\pi}\right) m r_0^2$$

Then the kinetic energy is given by

$$T = \frac{1}{2} I \omega^2$$ rotation $\omega = \dot{\varphi}$ about P

$$= \frac{1}{2}\left(\frac{3}{2} - \frac{8}{3\pi}\cos\varphi\right) m r_0^2 \, \dot{\varphi}^2$$

& $L = T - V$ as required.

c) Near the equilibrium point ($\varphi = 0$) we can use the small angle approximation to describe small oscillations, so:

$$L \approx \frac{1}{2}\left(\frac{3}{2} - \frac{8}{3\pi}\right) m r_0^2 \dot{\varphi}^2 + \frac{4 m g r_0}{3\pi}\left(1 - \frac{1}{2}\varphi^2\right)$$

when we only keep terms up to quadratic in $\varphi, \dot{\varphi}$.

We can the describe the motion of the system using the EOM:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

$$\Downarrow$$

$$\left(\frac{3}{2} - \frac{8}{3\pi}\right) m r_0^2 \ddot{\varphi} + \frac{4}{3\pi} m g r_0 \varphi = 0$$

or, dividing through:

$$\ddot{\varphi} + \frac{4g/3\pi}{\left(3/2 - 8/3\pi\right) r_0} \varphi = 0$$

defining $l = \frac{3\pi r_0}{4}\left(\frac{3}{2} - 8/3\pi\right) = r_0\left(\frac{9\pi}{8} - 2\right)$

we get:

$$\ddot{\varphi} + g/l \, \varphi = 0$$

which is the EOM for a pendulum w/ length l for small oscillations!

Question 2:

a) To obtain the normal modes we first need to write down a Lagrangian. Let's adopt generalized co-ordinates $\underbrace{\theta_1}_{2m}, \underbrace{\theta_2, \theta_3}_{m}$ to describe the masses. Then, working near equilibrium:

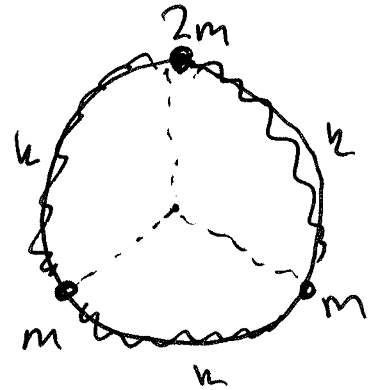
Kinetic term:

$$T = \frac{1}{2}(2m) r_0^2 \dot{\theta}_1^2 + \frac{1}{2} m r_0^2 \dot{\theta}_2^2 + \frac{1}{2} m r_0^2 \dot{\theta}_3^2$$

Potential term:

$$V = \frac{1}{2} k r_0^2 (\theta_1 - \theta_3)^2 + \frac{1}{2} k r_0^2 (\theta_1 - \theta_2)^2 + \frac{1}{2} k r_0^2 (\theta_2 - \theta_3)^2$$

Sketch



($\theta_i = 0$ at equilib)

Our Lagrangian is then $L = T - V$, and we generate the coupled EOM:

$$(*)\quad \begin{cases} 2m\ddot{\theta}_1 + 2k\theta_1 - \cancel{2}k(\theta_2+\theta_3) = 0 \\ m\ddot{\theta}_2 + 2k\theta_2 - k(\theta_1+\theta_3) = 0 \\ m\ddot{\theta}_3 + 2k\theta_3 - k(\theta_1+\theta_2) = 0 \end{cases}$$

Let's guess a solution to these equations of the form,

$$\theta_j = a_j e^{-i\omega t}$$

Plugging into $(*)$ we generate a matrix eqn:

$$\begin{pmatrix} 2k-2m\omega^2 & -k & -k \\ -k & 2k-m\omega^2 & -k \\ -k & -k & 2k-m\omega^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The non-trivial solution is when the determinant of the ~~coefficient~~ matrix vanishes. Some manipulation puts this in the form of an equation

$$\det|\cdots| = 0 = -\frac{2m}{k}\omega^2\left(\frac{m\omega^2}{k}-3\right)\left(\frac{m\omega^2}{k}-2\right)$$

From this we read off the roots:

$$\omega_s^2 = 0 \qquad \omega_m^2 = \frac{2k}{m} \qquad \omega_f^2 = \frac{3k}{m}$$

These solutions are used to obtain the corresponding eigenvectors of the problem:

i) $\omega_s^2 = 0 \Rightarrow \begin{pmatrix} 2k - 2m\omega^2 & -k & -k \\ -k & 2k - m\omega^2 & -k \\ -k & -k & 2k - m\omega^2 \end{pmatrix} \vec{a} = 0$

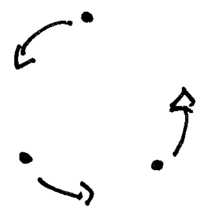
$\therefore \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ after plugging in $\omega^2 = \omega_s^2$

ii) $\omega_m^2 = \dfrac{2k}{m} \Rightarrow \vec{a} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

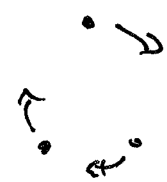
iii) $\omega_f^2 = \dfrac{3k}{m} \Rightarrow \vec{a} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

~~These~~ solutions i) – iii) describe 3 normal modes!

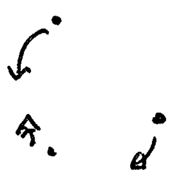
i) Describes a zero-frequency mode where the springs are not compressed → all masses rotate in unison + the springs stay at their equilibrium lengths:



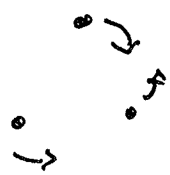
or



ii) Describes a mode w/ frequency $\omega_m = \pm \frac{2k}{m}$ where the heavy mass moves/rotates in an opposite manner to the light masses:



or



iii) Describes a mode where the heavy mass remains pinned in place the ~~[illegible]~~ lighter masses move, in opposite fashion.

or

b) First, a general solution for the motion of the masses will be:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = C_s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos(\omega_s t - \phi_s)$$

$$+ C_m \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cos(\omega_m t - \phi_m)$$

$$+ C_f \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\omega_f t - \phi_f)$$

[we got here by either using $\pm$ freq. solutions & enforcing that $\theta_j \in \mathbb{R}$, or jumping straight to the cosine solutions based on the same condition]

The constants $C_{s,m,f}$ & $\phi_{s,m,f}$ are to be determined from our initial conditions:

$\theta_1(0) = \theta_0$ ← initial displacement

$\theta_2(0) = \theta_3(0) = 0$

& $\dot{\theta}_1(0) = \dot{\theta}_2(0) = \dot{\theta}_3(0) = 0$ [initially stationary]

In principle these provide us w/ 6 equations to solve for the 6 unknowns.

Motivate by $\dot{\theta}_j(0) = 0$ we guess that a solution could include $\delta_{s,m,f} = 0$, leaving us w/ 3 eqns + unknowns:

$$\begin{cases} \theta_0 = C_s + C_m \\ 0 = C_s - C_m + C_f \\ 0 = C_s - C_m - C_f \end{cases}$$

A solution is: $C_f = 0$, $\underbrace{C_s = C_m = \frac{\theta_0}{2}}$

only the modes w/ $\omega_s = 0$ + $\omega_m = \frac{2k}{m}$ participate!

Question 3:

b) Eq 1 is describing the potential seen by each ion (summed over all ions!).

The first term: $\sum_{i=1}^{N} \frac{1}{2} m \nu^2 x_i^2$

describes a harmonic confining potential in which the ions sit (Penning & Paul traps are some of the tools used in many ion setups). The potential has frequency ν (we usually use ω_{-}) & each ion feels the trap independently, so for the total system we just sum over all ions.

The second term: $\sum_{i \neq j} \frac{Z^2 e^2}{8\pi\epsilon_0} \frac{1}{|x_i - x_j|}$

describes the Coulomb repulsion between the ions. The competition of this term w/ the harmonic trap can lead the ions to assemble into crystals or linear arrays!

c) From the paper, we understand that Eq 5 is just the Eq 3 → i.e. finding the point of equilibrium for which $\frac{\partial V}{\partial x_m} = 0$

To get Eq 5 let's start from Eq 1 & introduce the rescaled co-ordinate $\tilde{x}_k = x_k / l$ w/

$l \Rightarrow$ given by Eq 4 in the paper.

From Eq 1:

$$V \equiv Mv^2 \left[\sum_i \frac{1}{2} x_i^2 + \underbrace{\frac{\left(\frac{Z^2 e^2}{4\pi\epsilon_0}\right)}{Mv^2}}_{l^3} \sum_{i \neq j} \frac{1}{2} \frac{1}{|x_i - x_j|} \right]$$

so,

$$V = Mv^2 l^2 \left[\sum_i \frac{1}{2} \tilde{x}_i^2 + \sum_{i \neq j} \frac{1}{2} \frac{1}{|\tilde{x}_i - \tilde{x}_j|} \right]$$

or $$\tilde{V} = \frac{V}{Mv^2 l^2} = \sum_i \frac{1}{2} \tilde{x}_i^2 + \sum_{i \neq j} \frac{1}{2} \frac{1}{|\tilde{x}_i - \tilde{x}_j|}$$

To simplify following computation, let's assume $x_1 > x_2 > \ldots x_N$ & split the double sum:

$$\hat{V} = \sum_i \frac{1}{2} \tilde{x}_i^2 + \frac{1}{2} \sum_i \left[\sum_{j=1}^{i-1} \frac{1}{\tilde{x}_j - \tilde{x}_i} + \sum_{j=i+1}^{N} \frac{1}{\tilde{x}_i - \tilde{x}_j} \right]$$

Then we can proceed to compute the derivative (Eq 3)

as $\frac{\partial V}{\partial x_k}$ & $\frac{\partial \hat{V}}{\partial \tilde{x}_k}$ are interchangeable,

$$\frac{\partial \hat{V}}{\partial \tilde{x}_k} = \tilde{x}_k + \sum_{j=1}^{k-1} \frac{-1}{(\tilde{x}_j - \tilde{x}_i)^2} + \sum_{j=k+1}^{N} \frac{1}{(\tilde{x}_j - \tilde{x}_i)^2}$$

Now, if $\tilde{x}_m = u_m + \hat{q}_m$ & the derivative vanishes at the equilibrium solution defined by $\{u_m\}$,

$$0 = u_k + \sum_{j=1}^{k-1} \frac{-1}{(u_j - u_k)^2} + \sum_{j=k+1}^{N} \frac{1}{(u_j - u_k)^2}$$

which is Eq 5 if $k \to m$. & $j \to n$.

d)

i) As N increases → the distance between the two central ions decreases. This is a consequence of an increasingly large chain probing the quadratic trapping potential

⇒ there is an energy tradeoff & it is favorable to increase the energy from the coulomb interaction to offset the increased energy from ions at the edge of the chain due to the trap.

ii) Nevertheless → the position of the outermost ions will still slowly increase because the energy potential for $x_i - x_j \to 0$ is too great

⇒ the chain must grow larger.