Physics 5573, Spring 2022
Test 1, Statics

1) Consider a sphere of radius R and total charge Q that has been embedded with an r-dependent charge density:

$$\rho(\vec{r}) = C\, r$$

a) Write and solve an integral to determine C in terms of the properties of the sphere.

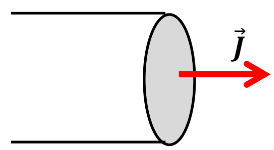
b) Explain and justify your approach to solving for the electric field and potential due to this sphere everywhere. This should, of course, be the simplest (correct) approach to the problem.

c) Set up your solution method and determine $\vec{E}(\vec{r})$ for all r.

d) Set up your solution method and solve for $\phi(\vec{r})$ for all r.

e) Solve for the electric potential energy of the sphere.

f) Describe your approach to solving this problem if the charge density is: $\rho(\vec{r}) = C\, r \cos^2\theta$
You do NOT have to solve this problem; just explain how you would approach it.



2) The Magnetic analog to the problem above is a long, current-carrying wire with a current density that varies across the radius of the wire.

Use polar coordinates: $\hat{z}$ along the wire, $\hat{r}$ the radial direction perpendicular to $\hat{z}$, and $\hat{\phi}$ the azimuthal angle around $\hat{z}$.

The wire has a radius R and total current I in the $\hat{z}$ direction. The current density is:

$$\vec{J}(\vec{r}) = C\, r\, \hat{z}$$

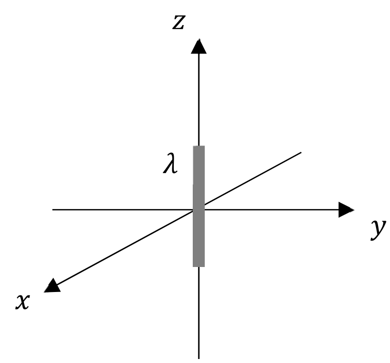
a) Derive an expression for the constant C in terms of properties of the wire.

b) Calculate $\vec{B}(\vec{r})$ everywhere. Be sure to describe and justify your approach.

c) What direction is the vector potential, $\vec{A}(\vec{r})$, for the wire? Explain/justify your answer.

d) Solve for the magnetic potential energy of the wire per unit length of the wire.

3) Consider a uniform charged rod of charge Q and length L on the z-axis, with its center at the origin. The linear charge density is



$$\lambda = \frac{Q}{L}$$

a) Calculate the electric potential due to the rod on the z-axis for all $z > \frac{L}{2}$.

b) Expand your result for $\phi(z)$ in powers of $L/(2\,z)$. Perhaps useful:

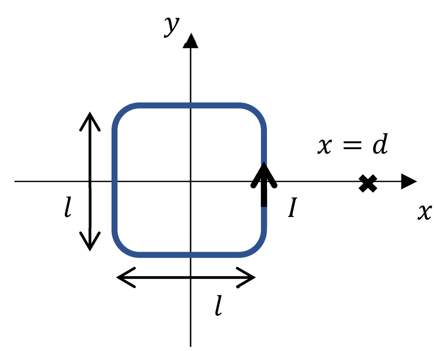
$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\ , \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

c) Using this result, determine an expression (a sum) for the potential $\phi(r,\theta)$ anywhere. Why is the potential is independent of ϕ. Remember that any such function can be written as:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \frac{a_l}{r^{l+1}}\, P_l(\cos\theta)$$

d) Calculate the monopole, dipole, and quadrupole terms for the line charge and show that these agree with the first three terms in (c).

4) Consider the current loop shown. It is a square with sides of length l and counter-clockwise current I, lying in the x-y plane and centered at the origin. Your task is to calculate the magnetic field at the point $x = d, y = 0, z = 0$.



a) To get a first approximation to the field, what is the magnetic field at $x = d$ if the loop is considered a point magnetic dipole at the origin?

b) Write down integrals to calculate the magnetic field at $x = d$ due to each side of the current loop. Simplify these integrals as much as possible, without actually doing the integrals. What is the direction of the magnetic field due to each side?

c) Solve for the magnetic field due to the loop at $x = d$. If you use results found in class, be sure to explain and justify them. If you do the integrals, it might be useful to know that:

$$\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \frac{x}{a^2\,\sqrt{a^2+x^2}}$$

d) Compare your result to the dipole approximation show that they agree in the limit $d \gg l$.

Possibly useful information:

Spherical Coordinates:

$$\vec{r} = r\sin\theta\cos\phi\ \hat{x} + r\sin\theta\sin\phi\ \hat{y} + r\cos\theta\ \hat{z}$$

$$\vec{\nabla} f(\vec{r}) = \hat{r}\partial_r f + \hat{\theta}\frac{1}{r}\ \partial_\theta f + \hat{\phi}\frac{1}{r\sin\theta}\ \partial_\phi f$$

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \frac{1}{r^2}\ \partial_r\, r^2\ A_r + \frac{1}{r\sin\theta}\ \partial_\theta \sin\theta\ A_\theta + \frac{1}{r\sin\theta}\ \partial_\phi\ A_\phi$$

$$\vec{\nabla}\times\vec{A}(\vec{r}) = \frac{\hat{r}}{r\sin\theta}\ \left(\partial_\theta\ sin\theta\ A_\phi - \partial_\phi\ A_\theta\right) + \frac{\hat{\theta}}{r}\ \left(\frac{1}{\sin\theta}\ \partial_\phi A_r - \partial_r\ r\ A_\phi\right) + \frac{\hat{\phi}}{r}(\partial_r\ r\ A_\theta - \partial_\theta\ A_r)$$

Polar (Cylindrical) Coordinates:

$$\vec{r} = r\cos\phi\ \hat{x} + r\sin\phi\ \hat{y} + z\ \hat{z}$$

$$\vec{\nabla} f(\vec{r}) = \hat{r}\partial_r f + \hat{\phi}\frac{1}{r}\ \partial_\phi f + \hat{z}\ \partial_z f$$

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \frac{1}{r}\ \partial_r\ r\ A_r + \frac{1}{r}\ \partial_\phi\ A_\phi +\ \partial_z\ A_z$$

$$\vec{\nabla}\times\vec{A}(\vec{r}) = \hat{r}\left(\frac{1}{r}\partial_\phi\ sin\theta\ A_z - \partial_z\ A_\phi\right) +\ \hat{\phi}(\partial_z A_r - \partial_r\ A_z) + \frac{\hat{\phi}}{r}\left(\partial_r\ A_\phi - \partial_\phi\ A_r\right)$$

Legendre Polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\ (3\ x^2 - 1)$$

$$P_3(x) = \frac{1}{2}\ (5\ x^3 - 3\ x)$$

Spherical Harmonics: $\left(Y_{l,-m}(\theta,\phi) = (-1)^m\ Y^*_{lm}(\theta,\phi)\right)$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\,\pi}},\qquad Y_{1,0}(\theta,\phi) = \frac{1}{2}\ \sqrt{\frac{3}{\pi}}\cdot\cos\theta$$

$$Y_{1,1}\ (\theta,\phi) =\ -\frac{1}{2}\ \sqrt{\frac{3}{2\,\pi}}\cdot\sin\theta\cdot e^{i\phi},\ \ Y_{1,-1}(\theta,\phi) = \frac{1}{2}\ \sqrt{\frac{3}{2\,\pi}}\cdot\sin\theta\cdot e^{-i\phi}$$

$$Y_{20} = \frac{1}{2}\sqrt{\frac{5}{4\,\pi}}\ (3\cos^2\theta - 1), Y_{21} = -\sqrt{\frac{15}{8\,\pi}}\sin\theta\cos\theta\ \ e^{i\phi}, Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\,\pi}}\sin^2\theta\ \ e^{2i\phi},$$

Multipole Expansion:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{r}\,Q_0 + \frac{1}{r^2}\,\hat{r}\cdot\vec{p} + \frac{1}{2r^3}\,\hat{r}\cdot\vec{\vec{D}}\cdot\hat{r} + \cdots\right)$$

Where

$$Q_0 = \sum_{q_\alpha} q_\alpha\,, \qquad \vec{p} = \sum_{q_\alpha} q_\alpha\,\vec{r}_\alpha\,, \qquad \left(\vec{\vec{D}}\right)_{ij} = \sum_{q_\alpha} q_\alpha\left(3\,r_{\alpha,i}\,r_{\alpha,j} - r_\alpha^2\,\delta_{ij}\right)$$

And

$$\hat{r}\cdot\vec{\vec{D}}\cdot\hat{r} = \hat{r}_i\,D_{ij}\,\hat{r}_j, \qquad \text{Summed over i and j}$$