COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Math Methods in Physics

PHYS 5013 HOMEWORK ASSIGNMENT #6

PROBLEMS: $\{1, 2, 3, 4, 5, 6\}$

Due: October 19, 2021 By 11:59 PM

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Kieran Mullen

Problem 1

This is one long problem, designed to give you experience in working with tensors, and exposure to one of the fundamental tensors in field theory, that of the electromagnetic field. The questions below basically encompass questions 11-13 in the text, with some extensions. Those of you interested in other approaches might look at *The Feynman Lectures*, II-27.

The questions are a bit tedious, but should not be too difficult. If you are having a lot of trouble, then I have probably made a mistake with the question. In that case come see me for clarification.

Maxwell's equations are one of the crowning achievements of classical physics. If we ignore the presence of any polarizing medium, they are:

$$\nabla \cdot \mathcal{E} = 4\pi\rho$$
$$\nabla \cdot \mathcal{B} = 0$$
$$\nabla \times \mathcal{E} = -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t}$$
$$\nabla \times \mathcal{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}$$

We will develop a more conceptual compact notation for expressing these equations.

(a) Derive the law of conservation of charge from Maxwell's equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}, \quad \nabla\cdot(\nabla \times \tilde{x}) = 0 : \quad \nabla\cdot(\nabla\times B) = \frac{4\pi}{c}\nabla J + \frac{1}{c}\nabla\frac{\partial \mathcal{E}}{\partial t}$$

$$0 = 4\pi \nabla J + \nabla \frac{\partial \mathcal{E}}{\partial t} = 4\pi\nabla J + \frac{\partial}{\partial t}\nabla \mathcal{E} = 4\pi \nabla J + \frac{\partial}{\partial t}(4\pi\rho) = \nabla J + \frac{\partial}{\partial t}\rho \therefore$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla J = 0}$$

(b) Express the electric field in terms of the scalar potential ϕ and the vector potential $\mathcal{A}$ where $\mathcal{B} = \nabla \times \mathcal{A}$. (Note, that in the general time dependent case the answer is *not* simply $\mathcal{E} = -\nabla\phi$).

$$B = \nabla \times A, \quad \nabla \times \mathcal{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times A) : \quad \nabla\times\mathcal{E} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times A) = 0$$

$$\nabla\times\mathcal{E} + \frac{1}{c}\nabla\times\left(\frac{\partial A}{\partial t}\right) = \nabla\times\left(\mathcal{E} + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0 \quad \text{using} \quad \nabla\times(\nabla\varphi) = 0$$

This means,

$$-\nabla\varphi = \mathcal{E} + \frac{1}{c}\frac{\partial A}{\partial t}$$

$$\boxed{-\nabla\varphi = \mathcal{E} + \frac{1}{c}\frac{\partial A}{\partial t}}$$

(c) A second-rank antisymmetric tensor in 3 dimensions has only 3 independent elements. We can make a link between it and a vector in 3 dimensions via:

$$F_{ij} = \epsilon_{ijk}B_k$$

Prove that if this is the case, then $B_k = \frac{1}{2}\epsilon_{ijk}F_{ij}$.

Problem 1 Continued

$\epsilon_{ijk} F_{ij} = \epsilon_{ijk}\epsilon_{ijk} B_k$: we have to change the index on B_k to B_n :

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$$

$$i = l,\ j = m : \epsilon_{ijn}\epsilon_{ijk} = (\delta_{jj}\delta_{nk} - \delta_{jk}\delta_{nj})B_n = 3\delta_{nk}B_n - \delta_{jk}\delta_{nj}B_n = 3\delta_{jk}B_j - \delta_{jk}B_j$$

$$\epsilon_{ijn}\epsilon_{ijk} = 3\cancel{\delta_{kk}}B_k - \cancel{\delta_{kk}}B_k = 3B_k - B_k = 2B_k$$

$$F_{ij} = \epsilon_{ijk}B_k = \frac{1}{2}\epsilon_{ijk}\epsilon_{ijk}F_{ij} = \frac{1}{2}\epsilon_{ijn}\epsilon_{ijk}F_{ij} = \frac{1}{2}(2)F_{ij} = F_{ij} \quad \checkmark$$

■

(d) Prove that for F_{ij} as defined above,

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \equiv \partial_i A_j - \partial_j A_i$$

where $\mathcal{A}$ is the vector potential mentioned above.

This tensor is obviously antisymmetric - that is, $F_{ij} = -F_{ji}$. It has $3^2 = 9$ components. We will now extend it to four dimensions, so that it has $4^2 = 16$ components. (Note that it is still a second *rank* tensor. The number of dimensions is not the same as the rank of the tensor.) We will do so by setting $x_4 = ict$, and $A_4 = i\phi$, where $i = \sqrt{-1}$ and *phi* is the electrostatic scalar potential. The indices will now in general run from 1 to 4, rather than 1 to 3.

$$F_{ij} = \epsilon_{ijk}B_k \ ,\ B = \nabla \times A \ ,\ B_k = \epsilon_{kxy}\partial_x A_y \ \therefore\ F_{ij} = \epsilon_{ijk}\epsilon_{kxy}\partial_x A_y$$

$$\epsilon_{ijk}\epsilon_{kxy} = \delta_{ix}\delta_{jy} - \delta_{iy}\delta_{jx} \ \therefore\ F_{ij} = (\delta_{ix}\delta_{jy} - \delta_{iy}\delta_{jx})\partial_x A_y = \delta_{ix}\delta_{jy}\partial_x A_y - \delta_{iy}\delta_{jx}\partial_x A_y$$

$$F_{ij} = \delta_{ix}\partial_x\delta_{jy}A_y - \delta_{jx}\partial_x\delta_{iy}A_y = \partial_i A_j - \partial_j A_i \doteq \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \quad \checkmark$$

■

(e) Prove $F_{4j} = -F_{j4} = iE_j$ where F_{ij} defined as in part (d). and $j = 1, 2, 3$. (F_{44} is zero.) Don't forget that the electric field is *not* simply $-\bigtriangledown\phi$.

There are sixteen elements of F_{ij}. The four diagonal ones must be zero, since antisymmetry requires $F_{ii} = -F_{ii}$. Of the 12 left, only half of them are independent, because of the antisymmetry requirement. Of those six, we have just shown that three are the magnetic field, and three are the electric field. If we write out the results of parts (d) and (e) we find:

$$F_{ij} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_z & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

$$F_{4j} = \partial_4 A_j - \partial_j A_4 \ :\ F_{j4} = \partial_j A_4 - \partial_4 A_j \ ,\ F_{j4} = \partial_4 A_j - \partial_j A_4 \ \rightarrow\ F_{4j} = -F_{j4} \ \checkmark$$

From part b.) : $\mathcal{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\tilde{A}}{\partial t}$, $F_{4j} = iE_j$, $\partial_4 A_j - \partial_j A_4 = i(-\nabla\varphi - \frac{1}{c}\partial\tilde{A}/\partial t)$

Problem 1 Continued

$$i(-\nabla\varphi - \tfrac{1}{c}\partial\tilde{A}/\partial t) = -\nabla(i\varphi) - \frac{i}{c}\frac{\partial \tilde{A}}{\partial t} = -\nabla(i\varphi) - \frac{i}{c}\partial t A_j \quad \text{where } j = 1,2,3$$

$$-\nabla(i\varphi) - \frac{i}{c}\partial t \partial A_j = -\partial_j\varphi - \frac{i}{c}\partial t A_j = i\left(-\partial_j\varphi - \frac{1}{c}\partial t A_j\right) = iE_j = F_{4j} = -F_{j4} \ \checkmark$$

Define the current as a four dimensional vector $\mathcal{J} = (J_x, J_y, J_z, icp)$. Our goal will be to show that all of Maxwell's equations are subsumed in:
[1] $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$
[2] $\partial_k F_{lk} = (4\pi/c)J_l$.
Eq. (1) is in fact $4^3 = 64$ equations, so it will help to break this down.

(f) Show that eq.(1) holds from the definition of F_{ij} given in (d), that is $F_{ij} = \partial_i A_j - \partial_j A_i$. (While this is nice, we haven't really made a link yet to Maxwell's equations!)

$$\text{eq (1)}: \ \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

$$F_{jk} = \partial_j A_k - \partial_k A_j, \quad F_{ki} = \partial_k A_i - \partial_i A_k, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

$$\partial_i(\partial_j A_k - \partial_k A_j) + \partial_j(\partial_k A_i - \partial_i A_k) + \partial_k(\partial_i A_j - \partial_j A_i)$$

$$\partial_i\partial_j A_k - \partial_i\partial_k A_j + \partial_j\partial_k A_i - \partial_j\partial_i A_k + \partial_k\partial_i A_j - \partial_k\partial_j A_i$$

$$\Delta + \tilde{\Delta} = \partial_i\partial_j A_k - \partial_j\partial_i A_k = 0, \quad \square + \tilde{\square} = \partial_k\partial_i A_j - \partial_i\partial_k A_j = 0, \quad \circ + \tilde{\circ} = \partial_j\partial_k A_i - \partial_k\partial_j A_i = 0$$

$\therefore$ Eq(1) Holds ■

(g) Show that eq.(1) trivially holds when any pair of indices is the same (i.e. if $i = j$) or if all are the same.

$$\text{eq (1)}: \ \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i, \quad i = j: \ \partial_i F_{ik} + \partial_i F_{ki} + \partial_k F_{ii} = 0$$

$$\partial_i(\partial_i A_k - \partial_k A_i) + \partial_i(\partial_k A_i - \partial_i A_k) + \partial_k(\overset{0}{\cancel{\partial_i A_i - \partial_i A_i}}) = 0$$

$$\partial_i\partial_i A_k - \partial_i\partial_k A_i + \partial_i\partial_k A_i - \partial_i\partial_i A_k = A_k(\overset{0}{\cancel{\partial_i\partial_i - \partial_i\partial_i}}) + A_i(\overset{0}{\cancel{\partial_i\partial_k - \partial_i\partial_k}}) = 0 \ \checkmark$$

■

(h) We have only cases where $i \neq j \neq k$, or $4 \times 3 \times 2 = 24$ cases left. Prove (1) holds when $i = 4$, when j and k are different spatial indices (1,2,3). This covers $3 \times 2 = 6$ cases. However, our choice of starting with $i = 4$ was arbitrary - we could have started with j. This gives us a factor of 3, so we have covered 18 cases.

$$\text{eq (1)}: \ \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i: \quad i = 4, \ \partial_4 F_{jk} + \partial_j F_{k4} + \partial_k F_{4j} = 0$$

$$\partial_4(\partial_j A_k - \partial_k A_j) + \partial_j(\partial_k A_4 - \partial_4 A_k) + \partial_k(\partial_4 A_j - \partial_j A_4) = 0$$

$$\partial_4\partial_j A_k - \partial_4\partial_k A_j + \partial_j\partial_k A_4 - \partial_j\partial_4 A_k + \partial_k\partial_4 A_j - \partial_k\partial_j A_4 = 0$$

Problem 1 Continued

$$A_k(\cancel{\partial_4\partial_j - \partial_j\partial_4}^{\,0}) + A_4(\cancel{\partial_j\partial_k - \partial_k\partial_j}^{\,0}) + A_j(\cancel{\partial_k\partial_4 - \partial_4\partial_k}^{\,0}) = \nabla \times \mathcal{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0$$

$\blacksquare$

(i) Prove (1) holds when $i \neq j \neq k \neq 4$. (Note that these all involve just the magnetic field). There are $3 \times 2 \times 1 = 6$ equations. This covers the remaining cases.

$$\text{eq (1)} : \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \quad F_{ij} = \partial_i A_j - \partial_j A_i \quad : \quad i \neq j \neq k = 4 \quad : \quad i = 1, \; j = 2, \; k = 3$$

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \partial_1(\partial_2 A_3 - \partial_3 A_2) + \partial_2(\partial_3 A_1 - \partial_1 A_3) + \partial_3(\partial_1 A_2 - \partial_2 A_1) = 0$$

$$\partial_1\partial_2 A_3 - \partial_1\partial_3 A_2 + \partial_2\partial_3 A_1 - \partial_2\partial_1 A_3 + \partial_3\partial_1 A_2 - \partial_3\partial_2 A_1 = 0$$

$$A_1(\cancel{\partial_2\partial_3 - \partial_3\partial_2}^{\,0}) + A_2(\cancel{\partial_3\partial_1 - \partial_1\partial_3}^{\,0}) + A_3(\cancel{\partial_1\partial_2 - \partial_2\partial_1}^{\,0}) = \vec{\nabla}\cdot\vec{B} = 0 \; \checkmark$$

$\blacksquare$

(j) Prove eq.(2) gives the rest of Maxwell's equations. Thus we have shown that eqs (1) and (2) respectively reproduce Maxwell's equations and nothing more.

$$\partial_k F_{4k} = (4\pi/c) J_4 \quad : \quad \partial_k F_{4k} = \partial_1 F_{41} + \partial_2 F_{42} + \partial_3 F_{43} + \partial_4 F_{44} = (4\pi/c) J_4$$

$$\partial_4 F_{44} = -\partial_4 i E_4 = -\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} \quad : \quad \partial_1 F_{41} + \partial_2 F_{42} + \partial_3 F_{43} = \partial_i \epsilon_{ijk} B_k = \vec{\nabla}\times\vec{B} \quad \therefore$$

$$\partial_1 F_{41} + \partial_2 F_{42} + \partial_3 F_{43} + \partial_4 F_{44} = -\frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + (\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c} J_4 \rightarrow \vec{\nabla}\times\vec{B} = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}$$

$\blacksquare$

$$\vec{\nabla}\cdot\mathcal{E} = 4\pi\rho \quad : \quad \partial_k F_{lk} = (4\pi/c) J_l \quad : \quad l = 4 \quad \therefore \quad \partial_k F_{4k} = (4\pi/c) J_4 = (4\pi/c)\cdot ic\rho = i4\pi\rho$$

$$\partial_k F_{4k} = \partial_k i E_k = i(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) = i(\nabla\cdot\mathcal{E}) \quad \therefore \quad i(\vec{\nabla}\cdot\mathcal{E}) = i4\pi\rho \rightarrow \vec{\nabla}\cdot\mathcal{E} = 4\pi\rho$$

$\blacksquare$

For your edification, it is interesting to note that the Lorentz force law is:

$$f_i = \frac{1}{c} F_{ij} J_k,$$

for $i = 1, 2, 3$. (Convince yourself of this, but you don't have to calculate it for the homework.) The quantity of icf_4 gives the work done by the electric field.

What have we gained by writing $\mathcal{E}$ and $\mathcal{B}$ in terms of F_{ij}? Well, we've saved a bit of typing. It is very suggestive that we can express everything so simply. But recall that F_{ij} is not some arbitrary matrix, it is a *tensor*. That means if our transformation matrix for coordinates is L_{ij}, then when we change to a different coordinate system,

$$F'_{ij} = L_{im} L_{jn} F_{mn}$$

Let's see if this shows up in our new formulation.

Problem 1 Continued

Consider an infinite line of charge along the z-axis. you will recall that it produces an electric field: $\mathcal{E} = \lambda \log(r)\hat{r} = \lambda \log(r)(\cos\theta\hat{x} + \sin\theta\hat{y})$ where λ is a constant involving the charge per unit length. The charge distribution is $\mathcal{J} = (0, 0, 0, ic\rho_0(\vec{r}))$, where $\rho_0(\vec{r})$ is zero everywhere but on the z-axis, where it is a constant. The transformation matrix we will use will put us in a moving frame, which is sometimes called a "boost" transformation. In this case we will "boost" along the z-axis. The matrix is

$$L_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha & i\sinh\alpha \\ 0 & 0 & -i\sinh\alpha & \cosh\alpha \end{pmatrix}$$

It can be thought of as a rotation, by an angle $i\alpha$, where $\cosh\alpha = 1/\sqrt{1-(v^2/c^2)}$. This transformation puts us in a frame moving at velocity v in the positive $\hat{z}$ direction. The quantity α is sometimes called the "rapidity". It is useful because if we make two successive boosts in the same direction, the final velocity may not be the sum of the two boost velocities, but the final rapidity is the sum of the two boost rapidities. (That is, rapidities are additive in relativistic mechanics.)

(k) Calculate $\mathcal{J}'$. (It transforms as a vector.) Approximate it for $v << c$. In this frame we should now see a current. Can you give a meaning to the change in $\mathcal{J}_4$?

$$J' = L_{ij} J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\alpha) & i\sinh(\alpha) \\ 0 & 0 & -i\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ ic\rho_0(\vec{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -c\rho_0(\vec{r})\sinh(\alpha) \\ ic\rho_0(\vec{r})\cosh(\alpha) \end{pmatrix}$$

$$\boxed{J' = \begin{pmatrix} 0 \\ 0 \\ -c\rho_0(\vec{r})\sinh(\alpha) \\ ic\rho_0(\vec{r})\cosh(\alpha) \end{pmatrix}}$$

$\cosh(\alpha) \rightarrow \dfrac{1}{\sqrt{1-(v/c)^2}}$ if $v << c$ then $v/c \rightarrow 0$ $\therefore$ $\cos(\alpha) \rightarrow 1$ $\sinh(\alpha) \rightarrow 0$

when J_4' transforms, it is essentially a boosted vector that has extra terms

(l) Calculate B_x' and B_y', by calculating F_{23}' and F_{31}'. Can you explain why in this frame we see a magnetic field?

Thus when we transform we make a transform to a new coordinate system, we see different electric and magnetic fields. The value of the fields is consistent with what we would have gotten if we had started our calculations in the moving frame.

$$F_{23}' = L_{2m}L_{3n}F_{mn}$$

$$L_{2m}L_{3n}F_{mn} = L_{21}\overset{0}{\cancel{L_{31}}}F_{11} + L_{21}\overset{0}{\cancel{L_{32}}}F_{12} + L_{21}\overset{0}{\cancel{L_{33}}}F_{13} + L_{21}\overset{0}{\cancel{L_{33}}}F_{14} + L_{22}L_{31}F_{21} + L_{22}L_{32}F_{22}$$
$$+ L_{22}L_{33}F_{23} + L_{22}L_{34}F_{24} + L_{23}\overset{0}{\cancel{L_{31}}}F_{31} + L_{23}\overset{0}{\cancel{L_{32}}}F_{32} + L_{23}\overset{0}{\cancel{L_{33}}}F_{33} + L_{23}\overset{0}{\cancel{L_{34}}}F_{34}$$
$$+ L_{24}\overset{0}{\cancel{L_{31}}}F_{41} + L_{24}\overset{0}{\cancel{L_{32}}}F_{42} + L_{24}\overset{0}{\cancel{L_{33}}}F_{43} + L_{24}\overset{0}{\cancel{L_{34}}}F_{44}$$

Problem 1 Continued

$$L_{2m}L_{3n}F_{mn} = L_{22}\overset{0}{\cancel{L_{31}}}F_{21} + L_{22}\overset{0}{\cancel{L_{32}}}F_{22} + L_{22}L_{33}F_{23} + L_{22}L_{34}F_{24}$$

$$L_{2m}L_{3n}F_{mn} = L_{22}L_{33}F_{23} + L_{22}L_{34}F_{24} = -\cosh(\alpha)B_x + \sinh(\alpha)E_y$$

$$\boxed{L_{2m}L_{3n}F_{mn} = \sinh(\alpha)E_y - \cosh(\alpha)B_x}$$

$$F_{31}' = L_{3m}L_{1n}F_{mn}$$

$$L_{3m}L_{1n}F_{mn} = L_{31}\overset{0}{\cancel{L_{11}}}F_{31} + L_{31}\overset{0}{\cancel{L_{12}}}F_{12} + L_{31}\overset{0}{\cancel{L_{13}}}F_{13} + L_{31}\overset{0}{\cancel{L_{14}}}F_{14} + L_{32}\overset{0}{\cancel{L_{11}}}F_{21} + L_{32}\overset{0}{\cancel{L_{12}}}F_{22}$$
$$L_{32}\overset{0}{\cancel{L_{13}}}F_{23} + L_{32}\overset{0}{\cancel{L_{14}}}F_{24} + L_{33}L_{11}F_{31} + L_{33}L_{12}F_{32} + L_{33}L_{13}F_{33} + L_{33}L_{14}F_{34} + L_{34}L_{11}F_{41}$$
$$+ L_{34}L_{12}F_{42} + L_{34}L_{13}F_{43} + L_{34}L_{14}F_{44}$$

$$L_{3m}L_{1n}F_{mn} = L_{33}L_{11}F_{31} + L_{33}\overset{0}{\cancel{L_{12}}}F_{32} + L_{33}\overset{0}{\cancel{L_{13}}}F_{33} + L_{33}\overset{0}{\cancel{L_{14}}}F_{34} + L_{34}L_{11}F_{41}$$
$$+ L_{34}\overset{0}{\cancel{L_{12}}}F_{42} + L_{34}\overset{0}{\cancel{L_{13}}}F_{43} + L_{34}\overset{0}{\cancel{L_{14}}}F_{44} = L_{33}L_{11}F_{31} + L_{34}L_{11}F_{41}$$

$$L_{3m}L_{1n}F_{mn} = L_{33}L_{11}F_{31} + L_{34}L_{11}F_{41} = \cosh(\alpha)B_y - \sinh(\alpha)E_x$$

$$\boxed{L_{3m}L_{1n}F_{mn} = \cosh(\alpha)B_y - \sinh(\alpha)E_x}$$

We have boosted along the z-axis, thus changing our position and this is why we will see a magnetic field.

All of this information is built in to the tensor F_{ij}, suggesting that this is more than just a convenient notation. Those of you who have a lot of time on your hands might wonder how all of the above changes if we allow for *magnetic* charges (i.e. monopoles).

Problem 1: Review

Procedure:

- Use tensor rules and procedures to do parts (a) through (l).
- Use the information given in each question to determine what type of procedure that is necessary.

Key Concepts:

- We can use tensor rules and procedures to derive the Maxwell equations.
- We can have anytisymmetric tensors represent physical quantities.

Variations:

- Since this problem is asking us to derive Maxwell's equations, there isn't very many different ways this problem can be changed into a new one.

Problem 2

Consider the function

$$f(x) = 4x^3 - 32x^2 + 66x - 18$$

(a) Solve $f(x) = 0$ for x, obtaining both symbolic and numeric answers.

```
Solve[{4*x^3 - 32*x^2 + 66*x - 18 == 0}, {x}]
```

$$\left\{\{x \to 3\}, \left\{x \to \frac{1}{2}\left(5 - \sqrt{19}\right)\right\}, \left\{x \to \frac{1}{2}\left(5 + \sqrt{19}\right)\right\}\right\}$$

```
NSolve[{4*x^3 - 32*x^2 + 66*x - 18 == 0}, {x}]

{{x → 0.32055052822966323`}, {x → 2.99999999999999`}, {x → 4.6794494717703365`}}
```

(b) Solve $f'(x) = 0$ for x, obtaining both symbolic and numeric answers.

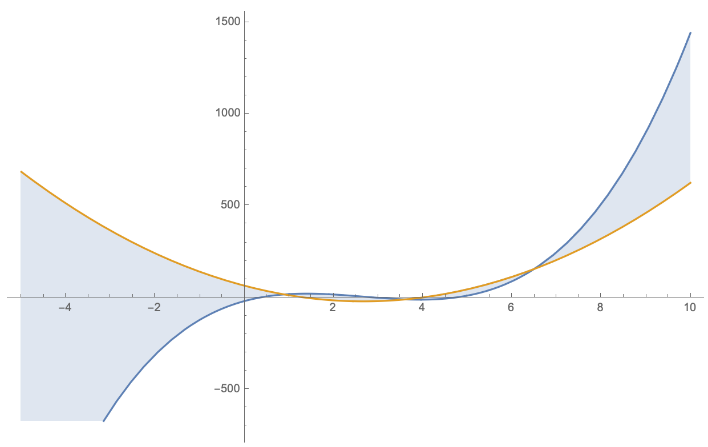
```
Solve[{D[4*x^3 - 32*x^2 + 66*x - 18, x] == 0}, {x}]
```

$$\left\{\left\{x \to \frac{1}{6}\left(16 - \sqrt{58}\right)\right\}, \left\{x \to \frac{1}{6}\left(16 + \sqrt{58}\right)\right\}\right\}$$

```
NSolve[{D[4*x^3 - 32*x^2 + 66*x - 18, x] == 0}, {x}]

{{x → 1.397371149022682`}, {x → 3.9359621843106516`}}
```

(c) Plot the function and verify that your roots and extrema are correct.



a_1	a_2		
`H[3.9359621843106516]`	`H[1.397371149022682]`	`F[3.9359621843106516]`	`F[1.397371149022682]`
30.4631	-30.4631	-10.0635	22.6561

2nd D Test

$a_2 = \text{max}$, $a_1 = \text{min}$

Smallest

Largest

Shows these are maxes

Problem 2: Review

Procedure:

- Use the ***Solve*** command to obtain symbolic answers and the ***NSolve*** command to obtain numerical ones.
- To take a derivative of the function use the ***D***[***f(x)***,**x**] syntax, and then solve that derivative for the roots of x.
- Use the ***Plot*** command to plot the functions to compare the extrema in them.

Key Concepts:

- The ***Solve*** command will give us symbolic solutions to an equation.
- The ***NSolve*** command will give us numerical solutions to an equation.
- We can take a derivative of a function using ***D***[***f(x)***,**x**].
- We can plot functions with the ***Plot*** command and the correct corresponding syntax.

Variations:

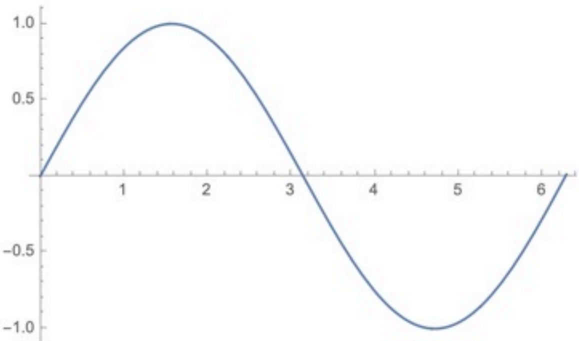
- We could be give a different equation for $f(x)$.
  - Thus changing the function we would input but not the syntax used to solve our equation.

Problem 3

Plot the Taylor Series expansion of the sine function over the interval $0 < x < 2\pi$ and determine how many terms you need in order to get reasonable accuracy. (*Hint*: You can use the **Table** command to generate a list of the terms and the **Total** command to sum up the list.)

```
F[x_] = Sin[x];
H[x_] = D[F[x], {x, n}];
Total[Table[H[0] / (n!) * x^n, {n, 20}]]
Plot[Total[Table[H[0] / (n!) * x^n, {n, 17}]], {x, 0, 2 Pi}]
```

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800} - \frac{x^{15}}{1307674368000} + \frac{x^{17}}{355687428096000} - \frac{x^{19}}{121645100408832000}$$



we need 17 terms for this to be reasonable for $0 \leq x \leq 2\pi$

Problem 3: Review

Procedure:

- Define the expression $\sin(x)$ as a function.
- Define the n^{th} derivative of the above expression as a separate function.
- Pick a number for how many derivatives you wish to take on the expression and proceed use the ***Table*** and ***Total*** commands to add these derivatives together.
- Now use the ***Plot*** command to plot these derivatives on a range from 0 to 2π.
- Play with the value of n to see what value will best approximate the $\sin(x)$ function on the interval from 0 to 2π.

Key Concepts:

- We can use the summation
$$\sum_{n=0}^{N} \frac{f^n(a)(x-a)^n}{n!}$$
to a approximate a function with polynomials. We set $a = 0$ here.
- We choose an arbitrary value for n to see how many terms are necessary to appropriately approximate $\sin(x)$ on the interval of 0 to 2π.
- With the method that we have chosen, we only need 17 terms to approximate $\sin(x)$ on our specified interval.

Variations:

- We could be given a different function, like $\cos(x)$.
  - This would only change what we define as $f(x)$ in our MATHEMATICA code.
- Our interval could be changed.
  - This would affect the value of n to see how many terms are needed to approximate the function.

Problem 4

Consider the 10×10 matrix $A_{ij} = \sin(ij)$ and vector 10×1 column vector $b_i = i$. Solve the numerical problem

$$A\vec{x} = \vec{b}$$

for the unknown vector $\vec{x}$ using the command **LinearSolve**. (*Hint*: If this takes a while to calculate on your computer, then you have gotten caught in one of the traps I warned you about in lecture.)

In[125]:=
```
matrixA = Table[N[Sin[i*j]], {i, 1, 10}, {j, 1, 10}];
MatrixForm[matrixA]
```

Out[126]//MatrixForm=

$$\begin{pmatrix}
0.841471 & 0.909297 & 0.14112 & -0.756802 & -0.958924 & -0.279415 & 0.656987 & 0.989358 & 0.412118 & -0.544021 \\
0.909297 & -0.756802 & -0.279415 & 0.989358 & -0.544021 & -0.536573 & 0.990607 & -0.287903 & -0.750987 & 0.912945 \\
0.14112 & -0.279415 & 0.412118 & -0.536573 & 0.650288 & -0.750987 & 0.836656 & -0.905578 & 0.956376 & -0.988032 \\
-0.756802 & 0.989358 & -0.536573 & -0.287903 & 0.912945 & -0.905578 & 0.270906 & 0.551427 & -0.991779 & 0.745113 \\
-0.958924 & -0.544021 & 0.650288 & 0.912945 & -0.132352 & -0.988032 & -0.428183 & 0.745113 & 0.850904 & -0.262375 \\
-0.279415 & -0.536573 & -0.750987 & -0.905578 & -0.988032 & -0.991779 & -0.916522 & -0.768255 & -0.558789 & -0.304811 \\
0.656987 & 0.990607 & 0.836656 & 0.270906 & -0.428183 & -0.916522 & -0.953753 & -0.521551 & 0.167356 & 0.773891 \\
0.989358 & -0.287903 & -0.905578 & 0.551427 & 0.745113 & -0.768255 & -0.521551 & 0.920026 & 0.253823 & -0.993889 \\
0.412118 & -0.750987 & 0.956376 & -0.991779 & 0.850904 & -0.558789 & 0.167356 & 0.253823 & -0.629888 & 0.893997 \\
-0.544021 & 0.912945 & -0.988032 & 0.745113 & -0.262375 & -0.304811 & 0.773891 & -0.993889 & 0.893997 & -0.506366
\end{pmatrix}$$

In[130]:=
```
vectorB = Table[i, {i, 1, 10}];
MatrixForm[vectorB]
```

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

In[132]:=
```
LinearSolve[matrixA, vectorB];
MatrixForm[LinearSolve[matrixA, vectorB]]
```

Out[132]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Out[134]//MatrixForm=

$$\begin{pmatrix} 2.83492 \\ -5.65071 \\ -16.77 \\ -9.82246 \\ 2.24527 \\ -5.75988 \\ -2.63877 \\ 2.96037 \\ 25.6627 \\ 23.0544 \end{pmatrix}$$

Problem 4: Review

Procedure:

- Begin by defining the matrix $\hat{A}$ by using the ***Table*** and ***N*** commands.
- Define column vector $\vec{b}$.
- Use the ***LinearSolve*** command to solve for the vector $\vec{x}$.

Key Concepts:

- We use the ***LinearSolve*** command to solve for a system of equations where we have a matrix (In this case $\hat{A}$) and known column vector (In this case $\vec{b}$).

Variations:

- We can be given a different matrix for $\hat{A}$ or column vector $\vec{b}$.
  - In this case it would be the same procedure but with a different matrix and column vector.

Problem 5

Euler Method Breakdown: Consider a simple harmonic oscillator governed by

$$\ddot{x} = -x$$

with $x(0) = 1$, and $\dot{x}(0) = 0$. (We have set the spring constant and mass to one). Define $v \equiv \dot{x}$.

(a) Consider the curve $(x(t), v(t))$. What should it look like for the exact solution?

The exact solution of this D.E is

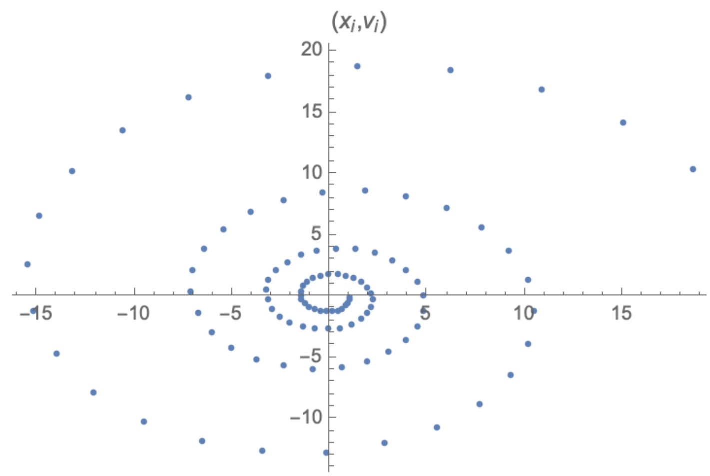
$$\boxed{x(t) = \cos(t)}$$

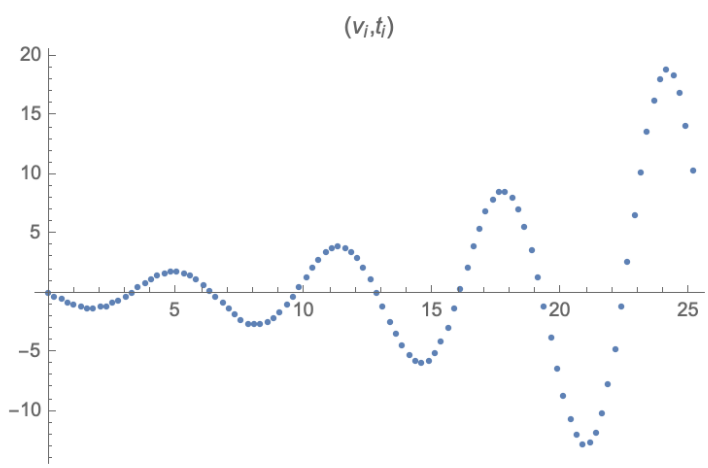
(b) Solve this with a standard Euler approach:

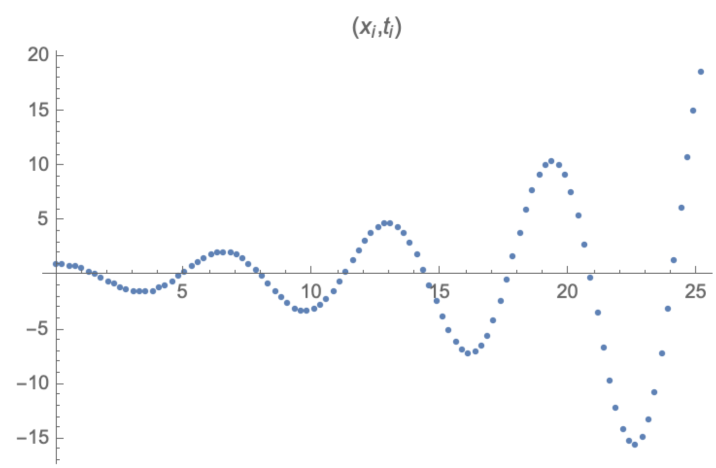
$$v_{i+1} = v_i - x_i \triangle t$$
$$x_{i+1} = x_i + v_i \triangle t$$

(Convince yourself that this is a reasonable discretization scheme and is consistent with the equation of motion.) Plot the orbit (x_i, v_i) for $0 \leq t \leq 8\pi$ for some reasonable choice of step size $\triangle t$. What type of orbit do you get?

```
Clear["Global`*"];
pts = 100.0;
dt = (8 Pi) / pts;
xfxn = Table[0, {i, 1, pts + 1}];
vfxn = Table[0, {i, 1, pts + 1}];
time = Table[0, {i, 1, pts + 1}];
xfxn[[1]] = 1;
vfxn[[1]] = 0;
Do[
 xfxn[[i + 1]] = xfxn[[i]] + vfxn[[i]] * dt;
 vfxn[[i + 1]] = vfxn[[i]] - xfxn[[i]] * dt;
 time[[i + 1]] = time[[i]] + dt,
 {i, 1, pts}
]
Transpose[{xfxn, vfxn}];
ListPlot[Transpose[{xfxn, vfxn}], PlotLabel → "(xi,vi)"]
ListPlot[Transpose[{time, xfxn}], PlotLabel → "(xi,ti)"]
ListPlot[Transpose[{time, vfxn}], PlotLabel → "(vi,ti)"]
```







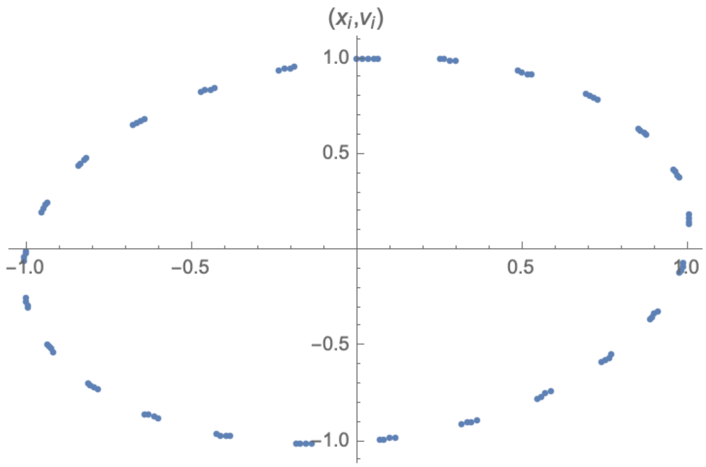
Problem 5 Continued

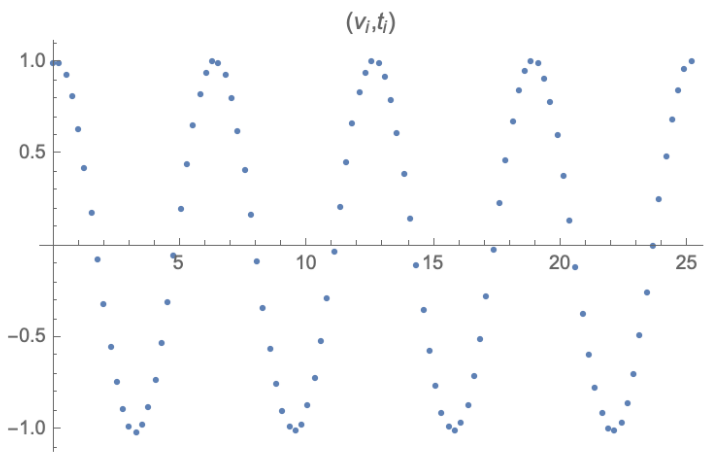
(c) Alter the algorithm so that you instead use:

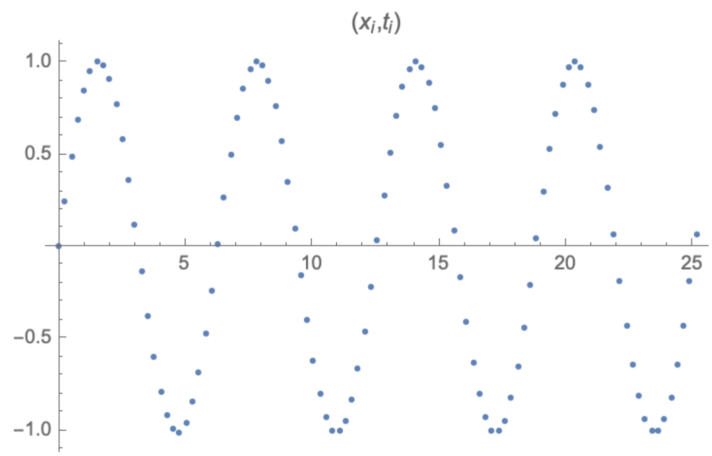
$$v_{i+1} = v_i - x_i \bigtriangleup t$$
$$x_{i+1} = x_i + v_{i+1} \bigtriangleup t$$

Thus the new position depends upon the *new* velocity. Plot the orbits again. does this give a better result? Note that this is partially an *implicity scheme*, in that we use some information at time $i+1$ to calculate the solution at time $i+1$.

```
Clear["Global`*"];
pts = 100.0;
dt = (8 Pi) / pts;
xfxn = Table[0, {i, 1, pts + 1}];
vfxn = Table[0, {i, 1, pts + 1}];
time = Table[0, {i, 1, pts + 1}];
xfxn[[1]] = 0;
vfxn[[1]] = 1;
Do[
 vfxn[[i + 1]] = vfxn[[i]] - xfxn[[i]] * dt;
 xfxn[[i + 1]] = xfxn[[i]] + vfxn[[i + 1]] * dt;
 time[[i + 1]] = time[[i]] + dt,
 {i, 1, pts}
]
Transpose[{xfxn, vfxn}];
ListPlot[Transpose[{xfxn, vfxn}], PlotLabel → "(xi,vi)"]
ListPlot[Transpose[{time, xfxn}], PlotLabel → "(xi,ti)"]
ListPlot[Transpose[{time, vfxn}], PlotLabel → "(vi,ti)"]
```







(d) The energy E_i at time i is given by $(x_i^2 + v_i^2)/2$. Calculate analytically the energy at time $i+1$ in the algorithm of part (b), in terms of x_i and v_i. What do you get for $(E_{i+1} - E_i)/\bigtriangleup t \sim \dot{E}$? Is this consistent with your graph? Do the same for the algorithm of part (c). Why is it better?

$E_{i+1} = ((x_{i+1})^2 + (v_{i+1})^2)/2$, $(x_{i+1})^2 = x_i^2 + 2x_iv_i\Delta t + v_i^2\Delta t^2$, $(v_{i+1})^2 = v_i^2 - 2x_iv_i\Delta t + x_i^2\Delta t^2$

$$E_{i+1} = (x_i^2 + 2x_i\cancel{v_i}\Delta t + v_i^2\Delta t^2 + v_i^2 - 2\cancel{x_iv_i}\Delta t + x_i^2\Delta t^2)/2 = (1+\Delta t^2)(x_i^2+v_i^2)$$

$$E_{i+1} - E_i = \frac{1}{\Delta t}\left(\frac{x_i^2+v_i^2+x_i^2\Delta t^2+v_i^2\Delta t^2}{2} - \frac{x_i^2+v_i^2}{2}\right) = \frac{(x_i^2+v_i^2)\Delta t}{2} \sim \dot{E}$$

consistent!

$E_{i+1} = ((x_{i+1})^2 + (v_{i+1})^2)/2$, $(x_{i+1})^2 = x_i^2 + 2x_iv_{i+1}\Delta t + v_{i+1}^2\Delta t^2$, $(v_{i+1})^2 = v_i^2 - 2x_iv_i\Delta t + x_i^2\Delta t$

$E_{i+1} = (x_i^2+v_i^2+(x_i^2+v_{i+1}^2)\Delta t + 2x_i\Delta t(v_{i+1}-v_i))/2$, $(E_{i+1}-E_i)/\Delta t = \dfrac{(x_i^2+v_{i+1}^2)+2x_i(v_{i+1}-v_i)}{2}$

This is better because it is looking forward in time as we as present so it will give a more precise plot than that of b.).

Problem 5: Review

Procedure:

- Begin by finding the exact solution to the differential equation in MATHEMATICA.
- Proceed to use a Standard Euler approach in solving the differential equation numerically. (Follow along with the provided code to see how this is done exactly)
- Alter the algorithm in (b) to do what is asked of us in part (c). (Again follow along with the code to see how this is carried out)
- Use equations that are defined in the problem statement to analytically solve for the energy.
- Compare and contrast the results and talk about why this new method is better than the previous.

Key Concepts:

- We can use MATHEMATICA to solve differential equations analytically (When applicable) and numerical methods for all differential equations.
- The standard Euler approach does not approximate the solution to the differential equation very accurately.
- The orbits we get with the standard Euler approach are phase portraits and more specifically attractors.
- The modified Euler approach does a much better job at approximating a solution to the differential equation.
- The orbits we get with the modified Euler approach are circular (Elliptical) with a lot more accurate solution.
- The algorithm from (c) is better than (b) because the algorithm in (c) is looking forward in time at the same time to approximate the next values.

Variations:

- We can be given a different differential equation to solve for.
  - This would include the same procedure but with different functions for the differential equation that is to be solved.
- We could be asked different qualitative questions about our solutions.
  - This would cause us to use our results and interpret them to answer these questions.

Problem 6

Assume you wish to solve for the eigenstates of an electron in an infinite 1D quantum well with an electric field, $\mathcal{E}$ applied across the well. Your Hamiltonian is given by:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - e\mathcal{E}x\psi = E\psi$$

where $0 \leq x \leq L$. You are interested in finding out how the energy of the first two eigenstates changes as a function of the strength of the electric field.

(a) What is the unit in which you will measure distance?

We need to make sure that all our units in this D.E are dimensionless. For x, we will first use the following transformation

$$x = \ell s$$

Where "s" is dimensionless, thus for distance we will use : $s = x/\ell$

(b) What is the unit with which you will measure the energy? What is its physical meaning?

If we rearrange our D.E, it will now look like :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - e\mathcal{E}\ell s\psi - E\psi = 0 \; : \; \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + (e\mathcal{E}\ell s + E)\psi = 0$$

We again want E to be dimensionless, so we will use

$$E = \epsilon E_0. \quad w/ \; E_0 = \hbar^2/2m\ell^2$$

This value is essentially our maximum kinetic energy of our system.

(c) What is the dimensionless parameter that is the control "control knob" that corresponds to increasing the field? What is its meaning?

Our new D.E will look something like this:

$$\frac{\hbar^2}{2m}\frac{1}{\ell^2}\frac{\partial^2}{\partial s^2} + (e\mathcal{E}s\ell + \epsilon E_0)\psi = 0$$

To make sure we continue to have dimensionless units, we will use:

$$\mathcal{E} = \xi\frac{\mathcal{E}_0}{e}$$

This value is essentially our max potential energy of our system.

(d) At what value of this parameter would you expect to see deviations from the infinite square well solutions?

If we rearrange our D.E, it will look something like this

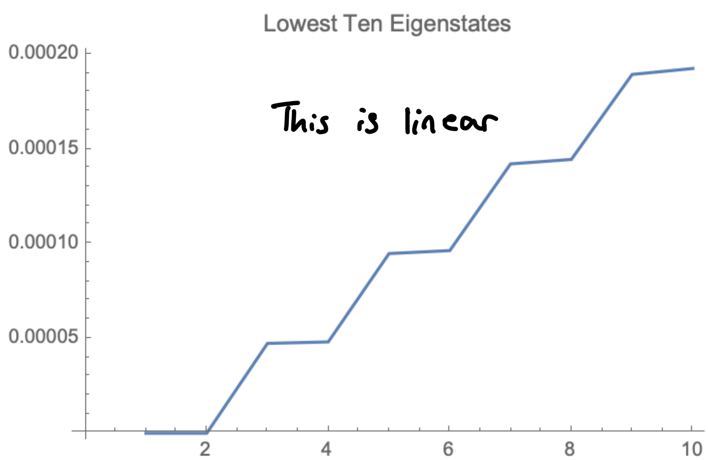
$$-\frac{\hbar^2}{2m\ell^2}\frac{\partial^2}{\partial s^2} - \xi\mathcal{E}_0 s\ell\psi = \epsilon\cdot\frac{\hbar^2}{2m\ell^2}\psi \; : \; \frac{2m\ell^3\xi\mathcal{E}_0}{\hbar^2} = \delta \quad \therefore \quad -\frac{\partial^2}{\partial s^2} - \delta s\psi = \epsilon\psi$$

Problem 6 Continued

where ξ is in units of "e" and δ is dimensionless. If this value "δ" is increased, the Harmonic oscillator approximation will be a good one. If it is small it will not be a good approximation.

(e) Solve the problem numerically. Plot the energy of the lowest ten eigenstates for a "small" value of the applied field. How do they vary with n, the number of the eignenstate?

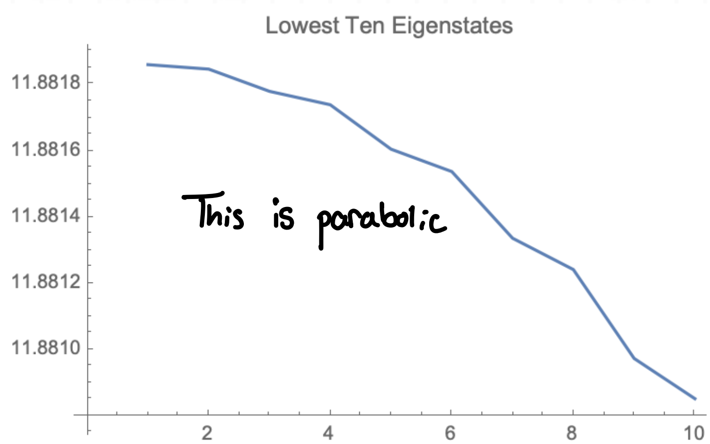
```
Clear["Global`*"];
ndiv = 1000;
ds = N[1 / ndiv];
npts = ndiv - 1;
delta = 1;
matrix = Table[If[i == j, (-2/ds^2 - (delta ds j)), 0] + If[Abs[i - j] == 1, 1/ds^2, 0], {i, 1, npts}, {j, 1, npts}];
MatrixForm[matrix];
epsilon = 0.1;
vector = Table[N[-epsilon], {j, 1, npts}];
vector[[1]] = 0;
vector[[npts]] = 0;
MatrixForm[vector];
phi = LinearSolve[matrix, vector];
phiTot = Join[{0}, phi, {0}];
phidat = Table[{(j - 1) * ds, phiTot[[j]]}, {j, 1, ndiv + 1}];
ListPlot[TakeSmallest[phidat[[All, 2]], 10], PlotLabel → "Lowest Ten Eigenstates", Joined → True]
```



Problem 6 Continued

(f) Solve the problem numerically. Plot the energy of the lowest ten eigenstates for a "large" value of the applied field. How do they vary with n, the number of the eigenstate?

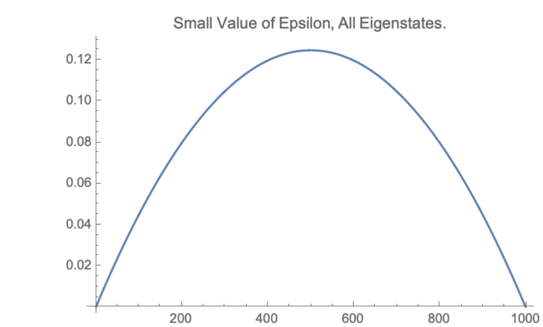
```
Clear["Global`*"];
ndiv = 1000;
ds = N[1 / ndiv];
npts = ndiv - 1;
delta = 1;
matrix = Table[If[i == j, (-2/ds^2 - (delta ds j)), 0] + If[Abs[i - j] == 1, 1/ds^2, 0], {i, 1, npts}, {j, 1, npts}];
MatrixForm[matrix];
epsilon = 100;
vector = Table[N[-epsilon], {j, 1, npts}];
vector[[1]] = 0;
vector[[npts]] = 0;
MatrixForm[vector];
phi = LinearSolve[matrix, vector];
phiTot = Join[{0}, phi, {0}];
phidat = Table[{(j - 1) * ds, phiTot[[j]]}, {j, 1, ndiv + 1}];
ListPlot[TakeLargest[phidat[[All, 2]], 10], PlotLabel → "Lowest Ten Eigenstates", Joined → True]
```

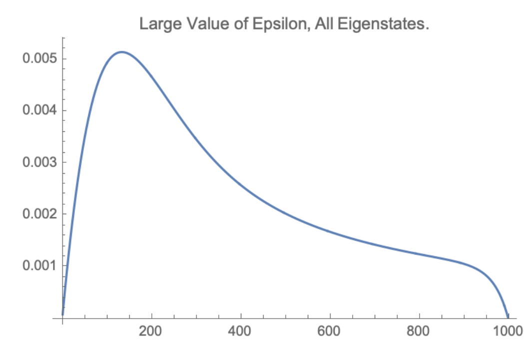


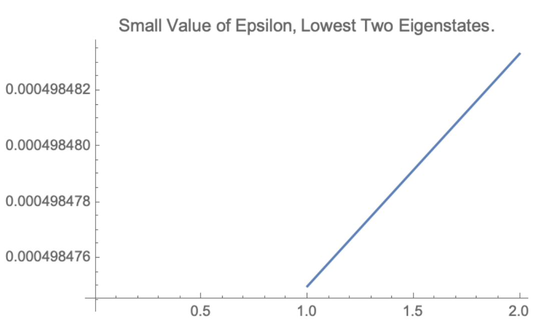
Problem 6 Continued

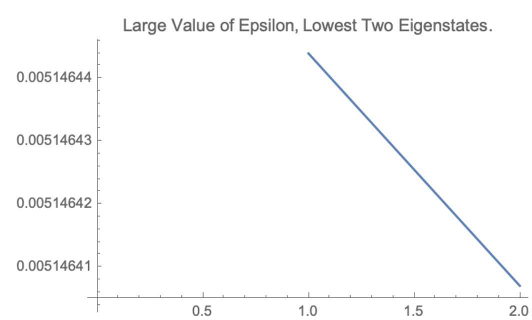
(g) Plot the energy of the lowest two eigenstates as a function of the applied field sweeping from small to large values of the applied field. Is your estimate above for the critical values of the field correct?

```
Clear["Global`*"];
eigenstates[delta_, epsilon_, ndiv_] :=
Module[{ds, npts, matrix, vector, phi},
ds = N[1 / ndiv];
npts = ndiv - 1;
matrix = Table[If[i == j, (-2/ds^2 - (delta ds j)), 0] + If[Abs[i - j] == 1, 1/ds^2, 0], {i, 1, npts}, {j, 1, npts}];
vector = Table[-epsilon, {j, 1, npts}];
vector[[1]] = 0;
vector[[Length[vector]]] = 0;
phi = LinearSolve[matrix, vector]
];
delta = 1;
Subscript[epsilon, large] = 1000;
Subscript[epsilon, small] = 0.001;
ndiv = 1000;
small = eigenstates[Subscript[epsilon, small], delta, ndiv];
large = eigenstates[Subscript[epsilon, large], delta, ndiv];
ListPlot[TakeSmallest[small, 2], Joined -> True, PlotLabel -> "Small Value of Epsilon, Lowest Two Eigenstates."]
ListPlot[small, Joined -> True, PlotLabel -> "Small Value of Epsilon, All Eigenstates."]
ListPlot[TakeLargest[large, 2], Joined -> True, PlotLabel -> "Large Value of Epsilon, Largest Two Eigenstates."]
ListPlot[large, Joined -> True, PlotLabel -> "Large Value of Epsilon, All Eigenstates."]
```









This is a correct estimate

Problem 6: Review

Procedure:

- Begin by creating a dimensionless variable for distance with the equation

$$s = x/l.$$

- Create dimensionless variable for energy with the equation

$$E = \epsilon E_0 \quad \text{with} \quad E_0 = \frac{\hbar^2}{2ml^2}.$$

  We solve for E_0 by setting it equal to the first terms in the differential equation.

- We re-arrange our differential equation to get it to look something like

$$-\frac{\partial^2}{\partial s^2} - \delta s\psi = \epsilon\psi \quad \text{with} \quad \delta = \frac{2ml^3\xi\mathcal{E}_0}{\hbar^2}.$$

  We now have to "control knobs" that allow us to tweak our system.

- We then proceed to solve the differential equation with the numerical techniques that are defined in parts (e) - (g).

Key Concepts:

- When we make our differential equation dimensionless, we can create control knobs to alter values of parameters that are specific to our problem.
- We have to make our equations dimensionless so that they can be solved in the correct manner.

Variations:

- We can be given a different differential equation.
  - The same procedure would be required to solve the problem.
- We could have a different set of eigenvalues that we wished to plot.
  - This would change some of the MATHEMATICA code to reflect those new eigenvalues.