Classical Mechanics and Statistical/Thermodynamics

January 2022

1. Write your answers only on the answer sheets provided, only on **one** side of the page.
2. Write your alias (not your name) at the top of every page of your answers.
3. At the top of each answer page write:
   (a) The problem number,
   (b) The page number *for that problem*,
   (c) The total number of pages of your answer *for that problem.*

   For example if your answer to problem 3 was two pages long, you would label them "Problem 3, page 1 of 2" and "Problem 3, page 2 of 2".
4. If the answer to your problem involves units, such as SI or Gaussian units, state which ones you are using.
5. Use only the math reference provided (*Schaum's Guide*). No other references are allowed.
6. Do not staple your exam when done.

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\, e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\, e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x - b\,x^2}\, dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^{n+1}\, \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(-1) = -\zeta(p)$$

$$\zeta(1) = \infty \qquad \zeta(-1) = -\tfrac{1}{12} = 0.0833333$$
$$\zeta(2) = \tfrac{\pi^2}{6} = 1.64493 \qquad \zeta(-2) = 0$$
$$\zeta(3) = 1.20206 \qquad \zeta(-3) = \tfrac{1}{120} = 0.0083333$$
$$\zeta(4) = \tfrac{\pi^4}{90} = 1.08232 \qquad \zeta(-4) = 0$$

Physical Constants:

Coulomb constant K $= 8.998 \times 10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85 \times 10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi \times 10^{-7}$T m/A	electronic charge $e = 1.60 \times 10^{-19}$C
electronic mass $m_e = 9.11 \times 10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38 \times 10^{-23}$J/K	Planck's constant: $\hbar = 6.63 \times 10^{-34}\mathrm{m}^2\mathrm{kg/s}$
speed of light: $c = 3.00 \times 10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. A uniform rod of length L and mass M has a bead of mass m glued a distance $L/4$ from its end as shown. The rod is supported by a thin horizontal, frictionless pin acting as an axle, and passing through the rod's center of mass. The system starts at rest, with the the rod placed as shown, and gravity pointing down (down the page). The radius of the bead is small enough that you can consider it to be a point mass located a distance $L/2$ from the axle.

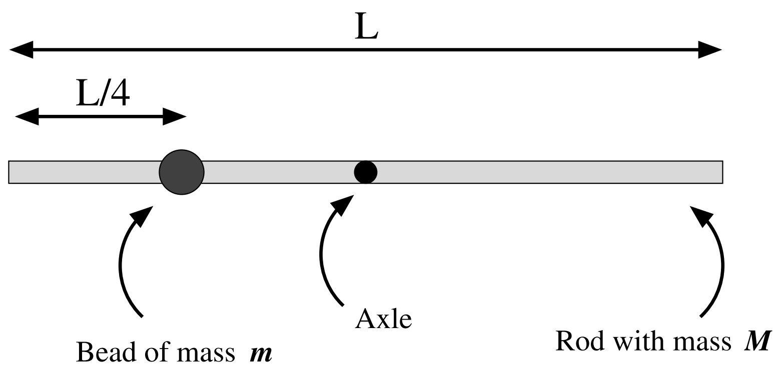

Figure 1: The rod rotates without friction.

   (a) Calculate the moment of inertia of the rod (without the bead) around its center of mass. (1 point)

   (b) If the system is released from rest so that it may rotate freely about the axle, what is its initial angular acceleration? (2 points)

   (c) Calculate the angular velocity of the rod when the bead reaches its lowest point. (2 points)

   (d) What is the magnitude of the total acceleration of the bead (as a function of either time or angle) between its initial release and its lowest point on the arc? At what time or angle is it maximum? (3 points)

   (e) When the bead is at its lowest point on the arc the glue fails and the bead slides to the end of the rod in a very short time, which we will treat as instantaneous. The bead does not fall off the end, but stops at a distance $L/2$ from the axle. When the bead reaches the end of the rod, will the angular velocity of the rod increase, decrease, or remain the same? Explain your answer. (1 point)

   (f) In the situation described in part (e) above, will the total kinetic energy of the system increase, decrease, or remain the same? Explain your answer. (1 point)

2. A particle of mass, m moves without friction, confined to a surface given by $z = \frac{1}{2}a\rho^2$, where $a > 0$ is a constant, and the cylindrical coordinates are (ρ, θ, z). A gravitational field acts on the particle with a uniform acceleration g in the $-z$ direction.

   (a) Find the Lagrangian for the system. (1 point)

   (b) Find the Euler-Lagrange equations. (1 point)

   (c) Construct the Hamiltonian for the system. (1 point)

   (d) Find Hamilton's equations for the system. (1 point)

   (e) Assume the particle moves on a trajectory such that z is a constant, so that $z = h$. Find the energy and angular momentum in terms of m, h, a, and g. (2 points)

   (f) A small perturbation is applied in the z direction to the particle moving on the trajectory in part (e). Find the frequency of the oscillation of ρ about its unperturbed value, assuming that the amplitude of oscillation is very small. You may use either the Lagrangian or Hamiltonian formulation of the problem. (4 points)

3. Consider a small bead of mass m that is constrained to lie on a rigid wire that is wound into a helix of radius R centered on the z-axis. The pitch of the helix is z_0, so that circling the z-axis exactly once counter-clockwise the bead would increase its vertical co-ordinate by z_0 . The bead is assumed to move upon the wire without friction but is subject to gravity, oriented in the z-directon (along the axis of the helix).

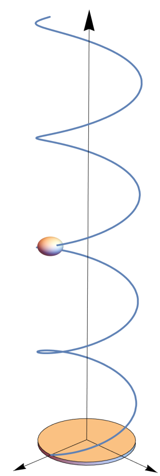

Figure 2: The bead slides without friction.

(a) Write the general form for the kinetic energy of a particle of mass m in cylindrical coordinates, (ρ, θ, z). (1 point)

(b) Construct a Lagrangian in terms of (ρ, θ, z) that includes the constraints required to confine the bead to move on the helix. Assuming the bead is released from rest, from a height h above the base of the helix, use this Lagrangian to calculate the motion of the bead as a function of time, and the generalized forces of constraint that act on the bead. For this part of the problem, the helix itself does not move. (3 points)

(c) Show that total mechanical energy of the bead is conserved. (1 point)

(d) Compute the time required for the bead to reach the base, and discuss how it depends upon the pitch. (2 points)

(e) The base of the helical wire is now affixed to a motor that generates a rotation of the wire about the z-axis at a fixed angular frequency ω_0. Discuss the behavior of the bead if it is started at a height $z(0) = h$ above the base, and the bead starts with $dz/dt = 0$. Compare this to your results above(d) (3 points)

Statistical Mechanics

4. A cylindrical container is initially separated by a clamped, thermally conductive piston into two compartments of equal volume. The left compartment is filled with one mole of neon gas at a pressure of four atmospheres and the right with argon gas at one atmosphere. The gases may be considered as ideal. The whole system is initially at temperature $T = 300\,\mathrm{K}$, and is thermally insulated from the outside world. The heat capacity of the cylinder-piston system is C (a constant). The piston is now unclamped and released to move freely without friction. Eventually, due to slight dissipation, it comes to rest in an equilibrium position.

   (a) Find the new temperature of the system. (2 points)

   (b) Find the ratio of final neon to argon volumes. (2 points)

   (c) Find the total entropy change of the system. (2 points)

   (d) Find the additional entropy change which would be produced if the piston were removed. (2 points)

   (e) If, in the initial state, the gas in the left compartment were a mole of argon instead of a mole of neon, which, if any, of the answers to (a), (b) and (c) would be different? (2 points)

5. A statistical system is characterized by N distinguishable and non-interacting atoms in thermal equilibrium with a reservoir at temperature T. Each atom can occupy the energy levels $E_n = (n+1)\epsilon$, with $\epsilon > 0$ and $n = 0, 1, 2, ..., +\infty$. The degeneracy of the n-th level is equal to $g_n = \lambda^n$, with $\lambda > 1$.

   (a) Find the canonical partition function $Z(T, N)$. (3 points)

   (b) Find the average energy $U(T, N)$ for this system. (2 points)

   (c) Find the specific heat $C(T, N)$ for this system. (2 points)

   (d) What happens to the specific heat at low temperatures? (1 point)

   (e) Is there a temperature above which the canonical description becomes invalid? If yes, what is this temperature, expressed as a function of λ and ϵ? (2 points)

6. An ideal paramagnet composed of magnetic moments pointing in arbitrary directions is described by the Hamiltonian

$$\mathcal{H} = -\sum_{i-1}^{N} \vec{m}_i \cdot \vec{H} = -\sum_{i=1}^{N} mH \cos\theta_i$$

where the magnetic field $\vec{H}$ is non-zero only in the z-direction. The magnetic moments can be represented in terms of the spherical coordinates as:

$$\vec{m}_i(\theta_i, \phi_i) = m \left(\sin\theta_i \cos\phi_i \, \hat{\imath} + \sin\theta_i \sin\phi_i \, \hat{\jmath} + \cos\theta_i \, \hat{k} \right).$$

The phase-space of each moment consists of a unit sphere with volume element $d\Omega_i = \sin\theta_i \, d\theta_i \, d\phi_i$

(a) Using the canonical ensemble, calculate the Helmholtz free energy and the internal energy of the paramagnet. (3 points)

(b) Find the heat capacity

$$c_H = \left. \frac{\partial U}{\partial T} \right|_H .$$

Does it vanish as $T \to 0$? (2 points)

(c) Calculate the average magnetization $\langle \vec{m} \rangle$. (2 points)

(d) Calculate the magnetic susceptibility

$$\chi_{zz} = - \left. \frac{\partial^2 F}{\partial H_z^2} \right|_{T,N}$$

(2 points)

(e) The magnetic susceptibility is related to a measurable quantity of this model via the fluctuation-dissipation theorem. To what quantity is it proportional and why? (1 point)