COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Electrodynamics 1

Ch. 4 Magnetostatics In Vacuum Lecture Notes

Student
Taylor Larrechea

Professor
Dr. Bruce Mason

2-14-22

Magnetostatics

We first examine an experiment between two magnets

The force between these is

$$\vec{F} \propto \frac{m_1 m_2}{r^2}$$

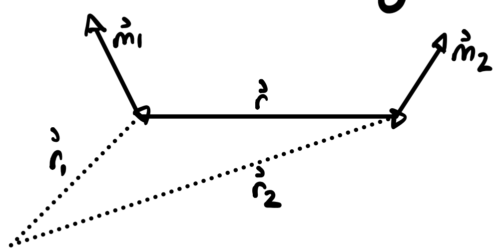
Much like that of $\vec{G}$. We define a magnetic dipole as

$$\vec{m} = m\vec{d}$$

The potential energy between these two magnets is similar to that of dipoles

$$U_{m_1,m_2} = \frac{\mu_0}{4\pi}\left(\frac{(\vec{m}_1\cdot\vec{m}_2)r_{21}^2}{r_{21}^5} - \frac{3(\vec{m}_1\cdot\vec{r}_{21})(\vec{m}_2\cdot\vec{r}_{21})}{r_{21}^5}\right)$$

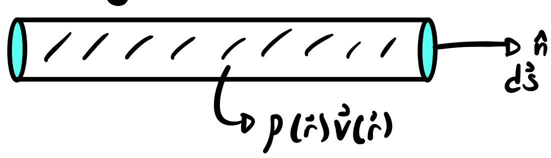
We can visualize the dipole interaction to look something like



In the context above, we can also say that a point magnetic dipole in the presence of an external magnetic field will have the potential energy of

$$U_{21} = -\vec{m}_2\cdot\vec{B}_1 \Rightarrow \vec{B}_1 = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}_1\cdot\vec{r}_{21})\vec{r}}{r^5} - \frac{\vec{m}_1}{r^3}\right)$$

We now move on to to a current carrying wire. For example



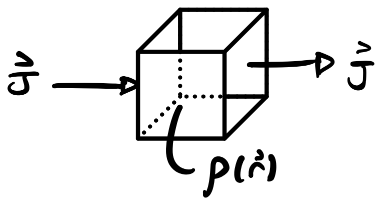
We can calculate the current density with

$$\vec{J} = \int \rho(\vec{r})\,\vec{u}(\vec{r})$$

We can also calculate the current with

$$I = \int \vec{J}\cdot\hat{n}\,d\vec{s}$$

If we then look at charge flowing through some volume



$$Q = \int \rho(\vec{r})\, d^3\vec{r} \Rightarrow \boxed{\frac{\partial Q}{\partial t} + \int \vec{J}\cdot\hat{n}\, d\vec{s} = 0 \quad (*)}$$

We can then take (*) to further show

$$\int \frac{\partial \rho}{\partial t}\, d^3\vec{r} + \int \vec{\nabla}\cdot\vec{J}\, d^3\vec{r} = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$

So, in magnetostatics $\Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla}\cdot\vec{J} = 0.$

Recall from magnetostatics

$$\vec{F} = q(\vec{v}\times\vec{B})$$

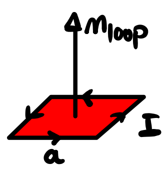
Which, of course is the force due to a charged particle q traveling in a magnetic field B with a velocity $\vec{v}$. We can also calculate

$$\vec{F} = \int (\vec{J}\times\vec{B})\, d^3\vec{r} = \int (\vec{J}\times\vec{B})\, d\vec{A}\cdot d\vec{l} = I\int d\vec{l}\times\vec{B}$$

Which is the force due to a current carrying wire in a magnetic field. We can summarize all of this with

$$u = -\vec{m}\cdot\vec{B}\ ,\quad \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})\ ,\quad \vec{F} = q(\vec{v}\times\vec{B}) = \int \vec{J}\times\vec{B}\, d^3r = I\int d\vec{l}\times\vec{B}$$

Lets now look at a magnetic loop that is carrying current



The magnetic field for this is

$$\vec{B} = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}\cdot\vec{r})}{r^5}\vec{r} - \frac{\vec{m}}{r^3}\right)$$

Where we can calculate the force with

$$\vec{F} = \vec{\nabla}(m_{loop}\cdot\vec{B})$$

If we go off and look at a different loop, say

For an electric dipole the potential is

$$\varphi(\vec{r}) = \frac{1}{c}\int_S \frac{\vec{R}\cdot\hat{n}'}{R^3}\,ds'$$

we then define

$$\varphi_B(\vec{r}) = \frac{ds\cdot R}{R^3}$$

And the magnetic Field is thus

$$\vec{B} = -\vec{\nabla}\varphi_B(\vec{r}) = -\vec{\nabla}\int \frac{d\vec{s}'\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int I(d\vec{s}\times\vec{\nabla})\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\oint I\frac{d\vec{l}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

The results above are used to infer that

$$\frac{\vec{\nabla}\cdot\hat{r}}{r^3} = 0$$

we now examine if we have a current density $\vec{J}$, the $\vec{B}$ field is

$$\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad w/\ \vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \Rightarrow \vec{B} = -\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

Taking the above further we now see

$$\vec{B} = -\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\times\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = \hat{e}_i\epsilon_{ijk}\vec{J}_j\partial_k\left(\frac{1}{R}\right) = \hat{e}_i\epsilon_{ijk}\partial_k\vec{J}_j\left(\frac{1}{R}\right) = -\hat{e}_i\epsilon_{ijk}\partial_k\frac{1}{R}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\vec{\nabla}\times\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'\right) \quad \vec{\nabla}\cdot\vec{B} = 0 \quad \therefore \text{Divergence of a curl is always zero}$$

Continuing we find,

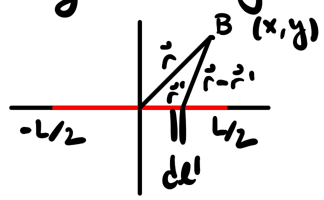
$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\left(\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}(\vec{r})}{|\vec{r}-\vec{r}'|}d^3\vec{r}'\right) = \frac{\mu_0}{4\pi}\left(\int\vec{J}(\vec{r})\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} - \int\vec{\nabla}\cdot\vec{J}\cancelto{0}{\vec{\nabla}}\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= \frac{\mu_0}{4\pi}\int\vec{J}(\vec{r})\,4\pi\delta^3(\vec{r}-\vec{r}') \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})$$

we then arrive at another Maxwell equation

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})}$$

we now shift our focus to calculating the magnetic field due to a wire of length L.



The magnetic field is

$$\vec{B} = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0 I}{4\pi}\int_{-L/2}^{L/2} \frac{dx'\,\hat{x} \times (y\hat{y} + (x-x')\hat{x})}{(y^2+(x-x')^2)^{3/2}} = \frac{\mu_0 I}{4\pi}\hat{z}\, y \int_{-L/2}^{L/2} \frac{dx'}{(y^2+(x-x')^2)^{3/2}}$$

We now make some mathematical definitions

$$\text{Define: } x-x' = y\tan\theta \Rightarrow -dx' = y\frac{d\theta}{\cos^2\theta}$$

$\vec{B}$ then becomes

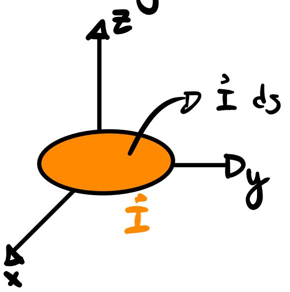
$$\vec{B} = \hat{z}\frac{\mu_0 I}{4\pi}\frac{1}{y}\left[\sin(\theta_+) - \sin(\theta_-)\right] \text{ w/ } \theta_+ = \tan^{-1}\left(\frac{x+\frac{1}{2}}{y}\right),\ \theta_- = \tan^{-1}\left(\frac{x-\frac{1}{2}}{y}\right)$$

2-21-22

We take a look at magnetic moments

$$\vec{m} = I a^2 \hat{n}$$

The magnetic field of a current loop looks something like



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \oint d\vec{l} \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\, d^3r' \qquad (*)$$

We were able to arrive at (*) with the aide of

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3},\quad \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A},\quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r'$$

In conjunction with the Uniqueness Theorem we have

$$\vec{\nabla}\cdot\vec{v} = f(\vec{r}),\quad \vec{\nabla}\times\vec{v} = \vec{\omega}(\vec{r}),\quad \vec{v} = -\vec{\nabla}g(\vec{r}) + \vec{\nabla}\times\vec{u}(\vec{r})$$

Where $g(\vec{r})$ and $u(\vec{r})$ are

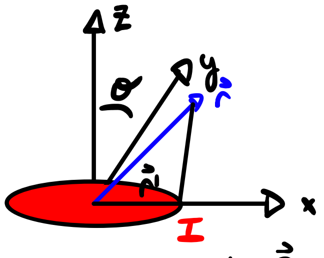
$$g(\vec{r}) = \frac{1}{4\pi}\int \frac{f(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3r',\quad u(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\omega}(\vec{r})}{|\vec{r}-\vec{r}'|}\, d^3r'$$

We can also define $\vec{B}(\vec{r})$ and $\vec{A}(\vec{r})$ as

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}),\quad \vec{A}'(\vec{r}) = \vec{A}(\vec{r}) + \vec{\nabla}\psi(\vec{r}),\quad \vec{\nabla}\times\vec{A} = \vec{B}$$

We now look at solving (*) or something similar.

Example:



we wan to find $\vec{B}(\vec{r})$. By directly integrating we have

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int d\ell' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \qquad (**)$$

we then make the substitutions

$$\vec{r} = r\cos(\theta)\,\hat{z} + r\sin(\theta)\,\hat{x}\ ,\quad \vec{r}' = a\cos(\varphi')\,\hat{x} + a\sin(\varphi')\,\hat{y}\ ,\quad d\ell' = a\,d\varphi'(-\sin(\varphi')\,\hat{x} + \cos(\varphi')\,\hat{y})$$

Putting these into we have

$$|\vec{r}-\vec{r}'| = (r^2+a^2-2\vec{r}\cdot\vec{r}')^{1/2} = (r^2+a^2-2ra\sin(\theta)\cos(\varphi'))^{1/2}$$

$$d\ell' \times (\vec{r}-\vec{r}') = a\,d\varphi'\big(-r\cos(\theta)\sin(\varphi')(-\hat{y}) + r\cos(\theta)\cos(\varphi')(\hat{x}) + r\sin(\varphi')\cos(\theta)(\hat{z})$$
$$+ a\sin^2(\varphi')(\hat{z}) - a\cos^2(\varphi')(-z)\big)$$

$$d\ell' \times (\vec{r}-\vec{r}') = a\,d\varphi'\big(r\cos(\theta)(\cos(\varphi')(\hat{x}) + \sin(\varphi')(\hat{y})) + a - r\cos(\theta)\sin(\varphi'))(\hat{z})\big)$$

we now can integrate $(**)$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I a \oint \frac{(a - r\cos(\theta)\sin(\varphi))\hat{z} + r\cos(\theta)(\cos(\varphi')\,\hat{x} + \sin(\varphi')\,\hat{y})}{(r^2+a^2-2ra\sin(\theta)\cos(\varphi'))^{3/2}}\, d\varphi'$$

Lets first look at $\theta = 0$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I a \int_0^{2\pi} \frac{(a - r\sin(\varphi'))\hat{z} + r(\sin(\varphi'))\,\hat{y}}{(r^2+a^2)^{3/2}}\, d\varphi' = \frac{\mu_0 I}{4\pi}\frac{Ia^2}{(r^2+a^2)} = \frac{\mu_0}{2\pi}\frac{\vec{m}}{(r^2+a^2)^{3/2}}$$

The original integral can be simplified even more

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I a \oint \frac{a(\hat{z}) + r\cos(\theta)\cos(\varphi')(\hat{x})}{(r^2+a^2-2ra\sin(\theta)\cos(\varphi'))^{3/2}}\, d\varphi'$$

This is due to the symmetry of the integral. we now repeat the same example but now we use a different process.

Looking at the vector potential first

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\, d^3r'$$

Where the current density is

$$\vec{J}(\vec{r}') = I\delta(r'-a)\frac{\delta(\cos(\theta))}{a}\hat{\varphi}$$

This changes $\vec{A}$ to

$$\vec{A} = \frac{\mu_0}{4\pi} I \int r'^2 \delta(\vec{r}'-a)\frac{\delta(\cos(\theta)')}{a}\frac{1}{|r-r'|}\, d\cos(\theta)'\, d\varphi' = \frac{\mu_0}{4\pi} Ia \oint \frac{(-\sin(\varphi')\hat{x} + \cos(\varphi')\hat{y})}{(r^2+a^2-2ra\sin\theta\cos\varphi')^{1/2}}$$

$$= \frac{\mu_0}{4\pi} Ia \int \frac{\cos(\varphi')}{(r^2+a^2-2ra\sin\theta\cos\varphi')^{1/2}}\, d\varphi' (\hat{y})$$

We now set $\theta=0 \Rightarrow \vec{A}(z)=0$. Because of difficult integration we look at limits.

$$(r^2+a^2) \Rightarrow \text{Large}\ ,\ (\sin\theta) \Rightarrow \text{Small}$$

Taking these limits A becomes,

$$\vec{A} = \frac{\mu_0}{4\pi} Ia \frac{1}{(r^2+a^2)^{1/2}} \int \frac{\cos(\varphi')\, d\varphi'}{(1-(2ra)/(r^2+a^2)\sin(\theta)\cos(\varphi'))^{1/2}}\hat{y}$$

$$= \frac{\mu_0}{4\pi} Ia \frac{1}{(r^2+a^2)^{1/2}} \int \cos(\varphi')\left(1+\frac{ra}{r^2+a^2}\sin(\theta)\cos(\varphi')+\dots\right)\hat{y}$$

$$\vec{A} = \frac{\mu_0}{4\pi} Ia \frac{1}{(r^2+a^2)^{1/2}}\frac{ra}{(r^2+a^2)}\sin(\theta)\,\pi\cdot\hat{y} = \frac{\mu_0}{4\pi}(I\pi a^2)\frac{\sin(\theta)r}{(r^2+a^2)^{3/2}}\hat{y} \qquad (***)$$

We can now take the curl of $\vec{A}$ to find $\vec{B}$.

$$\vec{B} = \vec{\nabla}\times\vec{A}\ ,\ B_r = \frac{1}{r\sin\theta} d_\theta \sin(\theta) A_\varphi\ ,\ B_\theta = -\frac{1}{r} d_r\, r A_\varphi\ ,\ B_\varphi = 0$$

Finally we have

$$B_r = \frac{\mu_0}{2\pi} m \frac{\cos(\theta)}{(r^2+a^2)^{3/2}}\ ,\ B_\theta = \frac{\mu_0}{4\pi} m \sin(\theta)\left(\frac{3r^2}{(r^2+a^2)^{5/2}} - \frac{2}{(r^2+a^2)^{3/2}}\right)$$