Quantum Mechanics
Qualifying Exam - August 2016

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega x^2}{2\hbar}}$

Problem 1: Time dependent solutions to Schrodinger's Equation (10 pts)

Consider a particle of mass m in an infinite square well.

$$V(x) = \begin{cases} 0, & -\frac{a}{2} \leq x \leq -\frac{a}{2} \\ \infty, & x < -\frac{a}{2} \text{ or } x > +\frac{a}{2} \end{cases}$$

The solutions to the time independent Schrodinger Equation are:
$H|\Psi_n\rangle = E_n|\Psi_n\rangle$ for n=1,2,3, ... where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ and

$$\langle x|\Psi_n\rangle = \Psi_n(x) = \sqrt{\frac{2}{a}}\cos(\frac{n\pi x}{a})\ n = 1,3,5,... \quad \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})\ n = 2,4,6,...$$

Assume at t_o, the particle is in the state:

$$|\Psi(t_o = 0)\rangle = \sqrt{3/10}\,|\Psi_1\rangle - i\sqrt{7/10}\,|\Psi_3\rangle$$

Answer the following questions:

a) Using Dirac notation, write down the expression for the time evolution operator, $U(t, t_o = 0)$ in terms of energy eigenvalues and eigenstates. (1 pt)

b) Find $|\Psi(t)\rangle = U(t, t_o = 0)|\Psi(t_o = 0)\rangle$ (1 pt)

c) Does your $|\Psi(t)\rangle$ in part b) satisfy the time independent Schrodinger Equation? Demonstrate explicitly. (1 pt)

d) Does your $|\Psi(t)\rangle$ in part b) satisfy the time dependent Schrodinger Equation? Demonstrate explicitly. (1 pt)

e) Is the uncertainty in the energy $\Delta E > 0$, < 0 or $= 0$ for $|\Psi(t)\rangle$? Discuss. (1 pt)

f) State whether the following properties are time dependent or time independent for a system in the state $|\Psi(t)\rangle$. (4 pts)
i) ΔE
ii) $\langle x^2\rangle$
iii) $\langle p\rangle$
iv) $\langle P\rangle$, where P is the parity operator

g) How do your answers to part f) change after the energy is measured at time t and the result is $E = \frac{9\pi^2\hbar^2}{2ma^2}$? (1 pt)

Problem 2: Hydrogen Atom (10 pts)

In this problem you will calculate the relativistic correction to the energies of the hydrogen atom. The hydrogen atom Hamiltonian is in terms of its electron in the field of the positively charged nucleus

$$H_0 = \frac{p^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0 r}$$

where p is the electrons momentum, r its position, m_e its mass, and e the charge. This Hamiltonian is nonrelativistic $(p/(mc) \ll 1)$. The correct relativistic expression to use for the kinetic energy is

$$T = \sqrt{p^2c^2 + m_e^2c^4} - m_ec^2$$

recall that

$$\langle r \rangle_{nl} = n^2 a_0 \{1 + \frac{1}{2}[1 - \frac{l(l+1)}{n^2}]\}$$

$$\langle r^2 \rangle_{nl} = n^4 a_0^2 \{1 + \frac{3}{2}[1 - \frac{l(l+1) - 1/3}{n^2}]\}$$

$$\langle \frac{1}{r} \rangle_{nl} = \frac{1}{a_0 n^2}$$

$$\langle \frac{1}{r^2} \rangle_{nl} = \frac{1}{a_0^2 n^3} \frac{1}{l + 1/2}$$

$$\langle \frac{1}{r^3} \rangle_{nl} = \frac{1}{a_0^3 n^3} \frac{1}{l(l+1/2)(l+1)}$$

a. Use this information to find the first non-zero order correction to the Hamiltonian due to the relativistic motion of the electron. **(2 Points)**

b. Show that this correction is diagonal in the $|nlm\rangle$ basis by proving that it commutes with the angular momentum operator $\vec{L}$. Why is it sufficient to prove that the perturbation commutes with $\vec{L}$ to show that the perturbation is diagonal in the $|nlm\rangle$ basis? **(4 Points)**

c. Using the fact that

$$\frac{p^2}{2m_e} = H_0 + \frac{e^2}{4\pi\epsilon_0 r}$$

find the relativistic energy correction to the energy levels of the Hydrogen atom. **(4 Points)**

Problem 3: Angular momentum (10 pts)

One particle has spin j_1 and another particle has spin j_2.

(a) [1 point] What are the good quantum numbers for the two-particle system with $\vec{J} = \vec{J}_1 + \vec{J}_2$ in the direct product basis? Write down the basis vectors labelled according to their eigenvalues.

(b) [1 points] Write down the basis vectors in the total j basis. What are the good quantum numbers in this case?

(c) [2 points] Write down the completeness relation for the direct product basis states.

(d) [2 points] Use the completeness relation to relate the total -j basis to the direct product basis. Identify the Clebsch-Gordon coefficient.

(e) [2 points] Write down the relation between total-j and direct product bases for $j_1 = 1/2$ and $j_2 = 1/2$. Recall

$$J_\pm |j, m >= \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1 >$$

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(f) [2 points] Suppose you have an interaction of the form $H_I = A\vec{J}_1 \cdot \vec{J}_2$ where $\vec{J} = \vec{J}_1 + \vec{J}_2$. Which basis vectors are best to use and why?

Problem 4: 3D Attractive Potential (10 pts)

Consider a particle that moves subjected to a three dimensional attractive potential

$$V(x, y, z) = -\frac{\hbar^2}{2m}[\lambda_1 \delta(x) + \lambda_2 \delta(y) + \lambda_3 \delta(z)],$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$.

a) Find the energy and the wavefunction of the particle in this potential. (4 points)

b) Interpret the meaning of this state. Calculate the probability of finding the particle inside a rectangular volume centered at the origin, with size $\ell_i = 1/\lambda_i$, with $i = 1, 2, 3$ for the x, y, z directions respectively. (2 points)

c) Compute the spatial and momentum uncertainties $(\Delta \mathbf{x})^2$ and $(\Delta \mathbf{p})^2$ for the state of item a) and explicitly check Heisenberg's inequality. (4 points)

Hint:

$$\frac{d|x|}{dx} = \frac{x}{|x|} \equiv \text{sign}(x) \qquad \frac{d}{dx}\text{sign}(x) = 2\delta(x)$$

Problem 5: Expanding Harmonic Oscillator (10 pts)

Consider a particle of mass m confined in a 1D harmonic oscillator potential with frequency ω_0

$$H_a = \frac{P^2}{2m} + \frac{m}{2}\omega_0^2\, X^2 \tag{1}$$

The raising and lowering operators are useful for harmonic oscillator problems:

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} - i\frac{\lambda}{\hbar}P\right) \qquad a = \frac{1}{\sqrt{2}}\left(\frac{X}{\lambda} + i\frac{\lambda}{\hbar}P\right) \tag{2}$$

where $\lambda = \sqrt{\frac{\hbar}{m\omega_0}}$ is the length scale for the harmonic oscillator:

(a) [2 pts] Use the raising and lowering operators to derive the ground state wavefuction, $\psi_0(x)$, and the first excited state wavefuction, $\psi_1(x)$, for the Hamiltonian H_a. Be sure to show your work.

(b) [1 pt] Consider a sudden change in the potential, modeled by a change in the original frequency of the oscillator by some multiplicative value f, to the new Hamiltonian:

$$H_b = \frac{P^2}{2m} + \frac{m}{2}\omega_1^2\, X^2, \quad \omega_1 = f\omega_0, \quad 0 < f < 1 \tag{3}$$

"Sudden" in this case means that one can ignore the time it takes to change the potential.

If $\phi_0(x)$ and $\phi_1(x)$ are the ground and first excited state wavefunctions of H_b, what are the functional forms for these wavefunctions? Explain your answer.

(c) [3 pts] The oscillator is in the ground state $\psi_0(x)$ when the potential suddenly changes. What is the expectation value of the energy of the oscillator after the potential changes? Show your work.

(d) [2 pts] If the oscillator is in the state $\psi_0(x)$ when the potential suddenly changes, what is the probability of the oscillator being in the ground state of H_b after the potential changes? Show your work.

(e) [1 pt] If the oscillator is in the state $\psi_0(x)$ when the potential suddenly changes, what is the probability of the oscillator being in the first excited state of H_b after the potential changes? Explain your answer.

(f) [1 pt] Finally, assume the oscillator is in the first excited state of H_a, $\psi_1(x)$, when the potential suddenly changes. What is the expectation value of the energy of the oscillator after the potential changes? Is the change in the expectation value of the energy, from H_a to H_b, for ψ_1 larger than, smaller than, or the same as ψ_0? Explain.

Remember that the Gaussian integrals have the form:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ax^2} dx &= \sqrt{\frac{\pi}{a}} \\ \int_{-\infty}^{\infty} x^{2n} e^{-ax^n} dx &= \frac{1\cdot 3\cdot 5 ...(2n-1)}{2^n a^2}\sqrt{\frac{\pi}{a}} \end{aligned} \tag{4}$$

Problem 6: Delta function in a 1-D well(10 pts)

A particle of mass m is placed in an attractive 1-D delta function potential

$$V(x) = -\hbar^2 \lambda \delta(x)/m$$

with positive λ. The particle and the potential are located in an infinite box with walls at x=$\pm$ $a/2$ (i.e $V(a/2) = V(-a/2) = \infty$)

a) Determine the condition on the parameters for which the system will have exactly one bound state with negative energy eigenvalue E and give its wave function (4 pts).

b) For the same system, determine the energy eigenvalues and eigenvectors for states with positive E. (3 pts)

c) If the coefficient $\lambda < 0$, explain in detail how your results change for parts a) and b) (3 pts)