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Math Methods in Physics

PHYS 5013 Homework Assignment #7

Problems: $\{3.1, 3.9, 3.12, 3.17, 3.26, 3.28\}$

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Problem 1: 3.1

Prove that the n complex nth roots of unity together with multiplication of complex numbers form an abelian group. [*Hint*: From De Moivre's formula, the n complex nth roots of unity are given by

$$e_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2ik\pi/n}, \quad k = 0, 1, \cdots, n-1,$$

where $i^2 = -1$.]

For a group to be Abelian, there must be closure, associativity, commutative, has an inverse, and is an identity.

Closure : $K_3 = K_1 + K_2$, Some addition of elements = that element in set

$$e^{2i(K_1)\pi/n} \cdot e^{2i(K_2)\pi/n} = e^{2i\pi/n\,(K_1+K_2)} = e^{2i\pi(K_3)/n} \checkmark$$

Associativity : $A(BC) = (AB)C$

$$e^{2i(0)\pi/n}\left(e^{2i(1)\pi/n}\, e^{2i(2)\pi/n}\right) = 1 \cdot e^{6i\pi/n} = e^{6i\pi/n}$$

$$\left(e^{2i(0)\pi/n} \cdot e^{2i(1)\pi/n}\right) e^{2i(2)\pi/n} = e^{2i\pi/n} \cdot e^{4i\pi/n} = e^{6i\pi/n} \checkmark$$

Commutative : $AB = BA$

$$e^{2i(1)\pi/n} \cdot e^{2i(2)\pi/n} = e^{2i\pi/n\,(1+2)} = e^{6i\pi/n}$$

$$e^{2i(2)\pi/n} \cdot e^{2i(1)\pi/n} = e^{2i\pi/n(2+1)} = e^{6i\pi/n} \checkmark$$

Inverse : $f(x) \cdot f(-x) = 1$, $K = n - k$

$$e^{i2\pi(n-k)/n}\, e^{i2\pi k/n} = e^{i2\pi} \cdot e^{-i2\pi k/n} \cdot e^{i2\pi k/n} = 1 \cdot e^{i2\pi k/n - i2\pi k/n} = 1 \cdot e^{0} = 1 \checkmark$$

Identity : $f(x) \cdot f(x(\text{value})) = f(x)$

$$\text{w/ } k=0 : \; e^{2ik\pi/n} \cdot e^{2i(0)\pi/n} = e^{2ik\pi/n} \cdot e^{0} = e^{2ik\pi/n} \cdot 1 = e^{2ik\pi/n} \checkmark$$

All conditions are met $\therefore$ this is an abelian group

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Problem 1: 3.1 Review

Procedure:

- Show that for this function has closure, associativity, is commutative, has an inverse, and has an identity.

Key Concepts:

- For a group to be Abelian, it mus have closure, associativity, is commutative, has an inverse, and has an identity.

Variations:

- We can be given a different function and be asked to show that is abelian.
  - This would be the same process but with a different function.

Problem 2: 3.9

Define the commutator $[A, B]$ of two matrices by $[A, B] = AB - BA$, and the anticommutator $\{A, B\}$ by $\{A, B\} = AB + BA$. Show that

(a) $[A, B] = -[B, A]$

$$[A,B] = AB - BA \;,\; [B,A] = BA - AB \;,\; -[B,A] = AB - BA \quad \therefore \quad [A,B] = -[B,A] \;\checkmark$$

$\blacksquare$

(b) $[AB, C] = A[B, C] + [A, C]B$

$$[B,C] = BC - CB \;,\; A[B,C] = ABC - ACB \;:\; [A,C] = AC - CA \;,\; [A,C]B = ACB - CAB$$

$$A[B,C] + [A,C]B = ABC - A\cancel{C}B + A\cancel{C}B - CAB = ABC - CAB$$

$$[AB,C] = ABC - CAB \quad \therefore \quad [AB,C] = A[B,C] + [A,C]B \;\checkmark$$

$\blacksquare$

(c) $[AB, C] = A\{B, C\} - \{A, C\}B$

$$\{B,C\} = BC + CB \;,\; A\{B,C\} = ABC + ACB \;:\; \{A,C\} = AC + CA \;,\; \{A,C\}B = ACB + CAB$$

$$A\{B,C\} - \{A,C\}B = ABC + A\cancel{C}B - A\cancel{C}B - CAB = ABC - CAB$$

$$[AB,C] = ABC - CAB \quad \therefore \quad [AB,C] = A\{B,C\} - \{A,C\}B$$

$\blacksquare$

(d) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

$$[B,C] = BC - CB \;,\; [A,[B,C]] = ABC - ACB - BCA + CBA$$

$$[C,A] = CA - AC \;,\; [B,[C,A]] = BCA - BAC - CAB + ACB$$

$$[A,B] = AB - BA \;,\; [C,[A,B]] = CAB - CBA - ABC + BAC$$

$$\cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA} + \cancel{BCA} - \cancel{BAC} - \cancel{CAB} + \cancel{ACB} + \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC} = 0$$

$$\therefore \; [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

$\blacksquare$

Problem 2: 3.9 Review

Procedure:

- Use the common commutator identities

$$[\tilde{\mathbf{A}}, \tilde{\mathbf{B}}] = \tilde{\mathbf{A}}\tilde{\mathbf{B}} - \tilde{\mathbf{B}}\tilde{\mathbf{A}} \quad \text{and} \quad \{\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} + \tilde{\mathbf{B}}\tilde{\mathbf{A}}$$

  with each individual case presented in parts (a) through (d).

Key Concepts:

- Each commutation relation can be proved by using other simpler commutator relations.
- These commutator relationships can be used in Quantum Mechanics.

Variations:

- These identities are pretty common identities and thus can't really be changed.
  - We could however be asked to prove a similar identity by using the same simple identities.

Problem 3: 3.12

The Pauli spin matrices are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$. Prove that

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \quad \text{and} \quad \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\sum_{k=1}^{3} \epsilon_{ijk}\sigma_k.$$

Thus the Pauli matrices anticommute.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$$

$$\sigma_1\sigma_3 + \sigma_3\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$$

We can see from above that $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, $\sigma_1\sigma_3 = -\sigma_3\sigma_1$, $\sigma_2\sigma_3 = -\sigma_3\sigma_2$

As long as $i \neq j$, $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$: If $i = j$

$$\sigma_1\sigma_1 + \sigma_1\sigma_1 = 2\cdot\sigma_1\sigma_1 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\mathbb{1}, \sigma_2\sigma_2 + \sigma_2\sigma_2 = 2\cdot\sigma_2\sigma_2 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\mathbb{1}$$

$$\sigma_3\sigma_3 + \sigma_3\sigma_3 = 2\sigma_3\sigma_3 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\mathbb{1}, \therefore \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} \text{ with } \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_i\sigma_j = -\sigma_j\sigma_i \quad \therefore \quad \sigma_i\sigma_j - \sigma_j\sigma_i = 2\sigma_i\sigma_j$$

$$i=1, j=2 : 2\sigma_1\sigma_2 = 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 2i\sigma_3, \quad i=2, j=1 : 2\sigma_2\sigma_1 = 2\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -2i\sigma_3$$

$$i=1, j=3 : 2\sigma_1\sigma_3 = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_2, \quad i=3, j=1 : 2\sigma_3\sigma_1 = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -2i\sigma_2$$

$$i=2, j=3 : 2\sigma_2\sigma_3 = 2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 2i\sigma_1, \quad i=3, j=2 : 2\sigma_3\sigma_2 = 2\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -2i\sigma_1$$

When i and j switch, the sign switches, thus $\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\sum \epsilon_{ijk}\sigma_k$

Problem 3: 3.12 Review

Procedure:

- Use different combinations of $\sigma_i\sigma_j$ with matrix multiplication to show the first identity of $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$.
- Do the same procedure to prove the second identity $\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$.

Key Concepts:

- The Pauli matrices commute with one another.
- $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ is a common identity for the Pauli matrices.
- $\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$ is another common identity for the Pauli matrices.

Variations:

- These identities are common and cannot be changed.
  - We could be asked to prove something similar for a different set of matrices.

Problem 4: 3.17

Denote the two-dimensional rotation matrix for a rotation of the coordinate axes through and angle θ by

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A. Find a diagonlizing matrix for A, that is, a matrix P such that $P^{-1}AP$ is a diagonal matrix. Demonstrate that P is such a matrix by inverting P and forming the product $P^{-1}AP$.

$$A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} \cos(\theta) - \lambda & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) - \lambda \end{pmatrix}, \quad (\cos(\theta) - \lambda)^2 + \sin^2(\theta) = 0$$

$$(\cos(\theta) - \lambda)^2 = -\sin^2(\theta), \quad \cos(\theta) - \lambda = \pm i\sin(\theta) \quad \therefore \quad \lambda = \cos(\theta) \mp i\sin(\theta) = e^{\mp i\theta}$$

$$\lambda_1 = e^{-i\theta}, \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{-i\theta}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \quad \begin{matrix} \cos(\theta)\alpha + \sin(\theta)\beta = e^{-i\theta}\alpha \\ \cos(\theta)\beta - \sin(\theta)\alpha = e^{-i\theta}\beta \end{matrix}$$

$$\text{rref}\left(\begin{bmatrix} (\cos(\theta) - e^{-i\theta})\alpha & \sin(\theta)\beta & 0 \\ (-\sin(\theta))\alpha & (\cos(\theta) - e^{-i\theta})\beta & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \alpha - i\beta = 0 \quad \therefore \quad \alpha = i\beta$$

$$\boxed{\lambda_1 = e^{-i\theta}, \quad |\lambda_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

$$\lambda_2 = e^{i\theta}, \quad \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{i\theta}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \quad \begin{matrix} \cos(\theta)\alpha + \sin(\theta)\beta = e^{i\theta}\alpha \\ \cos(\theta)\beta - \sin(\theta)\alpha = e^{i\theta}\beta \end{matrix}$$

$$\text{rref}\left(\begin{bmatrix} (\cos(\theta) - e^{i\theta})\alpha & \sin(\theta)\beta & 0 \\ (-\sin(\theta))\alpha & (\cos(\theta) - e^{i\theta})\beta & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \alpha + i\beta = 0 \quad \therefore \quad \alpha = -i\beta$$

$$\boxed{\lambda_2 = e^{i\theta}, \quad |\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \rightarrow \text{Combination of eigenvectors}, \quad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

$$\tilde{A} = P^{-1}AP = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$\boxed{\tilde{A} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \quad P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}$$

Problem 4: 3.17 Review

Procedure:

- Begin by finding the eigenvalues of the matrix $\hat{A}$.
- Proceed to find the eigenvectors of the matrix $\hat{A}$.
- To find the matrix $\hat{P}$, combine each eigenvector into a 2×2 matrix.
- Proceed to calculate the product $\hat{P}^{-1}\hat{A}\hat{P}$.

Key Concepts:

- The eigenvalues of this rotation matrix are the Euler identities.
- We can find a diagonalizing matrix for a matrix by combining the eigenvectors of the matrix that we are given.

Variations:

- We can be given a different initial matrix $\hat{A}$.
  - This would change the end results but not the process to get to those results.

Problem 5: 3.26

Show that $(C+D)^{-1} = C^{-1} - C^{-1}D(C+D)^{-1}$. Thus for D "small" we can iterate to get $(C+D)^{-1} = C^{-1} - C^{-1}DC^{-1} + C^{-1}DC^{-1}DC^{-1} \cdots$ which is the operator analogue of the familiar expansion $(x+y)^{-1}$ when y is "small".

Because $(\text{something})^{-1} * \text{something} = 1$, if we do $(C^{-1} - C^{-1}D(C+D)^{-1}) * (C+D)$ we can show that

$$(C+D)^{-1} = (C^{-1} - C^{-1}D(C+D)^{-1})$$

$$(C+D)^{-1}(C+D) = (C^{-1} - C^{-1}D(C+D)^{-1})(C+D) = C^{-1}(C+D) - C^{-1}D\cancelto{1}{(C+D)^{-1}(C+D)} = \cancelto{1}{C^{-1}C} + \cancel{C^{-1}D} - \cancel{C^{-1}D} = 1 \checkmark$$

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Problem 5: 3.26 Review

Procedure:

- Use the expanded form of $(C+D)^{-1}$ and show that $(C+D)^{-1}(C+D)=\mathbb{I}$.

Key Concepts:

- Some linear combination of operators' inverse times that same linear combination of operators must equal the identity matrix, or $\mathbb{I}$.

Variations:

- This is essentially another identity that can be proven with the outlined procedure.
  - We could be given a different identity but would use the same process to prove it.

Problem 6: 3.28

Show that the operator

$$T = \mathbb{I} + \frac{xD}{1!} + \frac{(xD)^2}{2!} + \cdots + \frac{(xD)^n}{n!},$$

where $D = d/dt$, acts as a translation operator on the space of polynomials (in the variable t) of degree $\leqq n$, that is,

$$Tf(t) = f(t+x),$$

if $f(t)$ is in the space of polynomials of degree n.

First we will write out the operator acting on the polynomial $f(t)$

$$Tf(t) = f(t) + \frac{xDf(t)^i}{1} + \frac{x^2D^2f(t)^i}{2!} + \ldots + \frac{x^nD^nf(t)^i}{n!} \Longrightarrow$$

we can then go on to write this as a sum of terms

$$\sum_{in} \frac{x^nD^nt^i}{n!}$$

where the sum can be rewritten as

$$\frac{i!}{n!(i-n)!}t^{i-n}x^n$$

where we can re-write this as

$$\binom{i}{n}t^{i-n}x^n = (t+x)^i$$

And thus

$$Tf(t) = (t+x)^i \quad w/ \quad i \in \mathbb{R} \;\&\; i \in \mathbb{Z} > 0$$

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Problem 6: 3.28 Review

Procedure:

- Begin by expanding out the operator in the summation of a power series.
- Show that the series can be written as
$$\sum_i \sum_n = \frac{x^n D^n t^i}{n!}.$$
- The above can then be re-written as
$$\frac{i!}{n!(i-n)!} t^{i-n} x^n.$$
- Where the above can also be re-written as
$$\frac{i!}{n!(i-n)!} \rightarrow \binom{i}{n} \quad \therefore \quad \binom{i}{n} t^{i-n} x^n = (t+x)^i.$$
- The above is then shown to prove that T acts as a translation operator.

Key Concepts:

- The operator T acts as a translation operator.
- We do this by expanding T in a power series and using common identities to show that it can be written in the form of $Tf(t) = f(t+x)$.

Variations:

- This operator that is of the form that we are given essentially cannot change.
  - If the operator were to change it could possibly be completely different and thus wouldn't act as a translation operator.