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Quantum Mechanics 2

PHYS 5403 Homework Assignment 1

Problems: {1, 2, 3}

Due: February 2, 2022 at 5:00 PM

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Problem 1:

Consider two particles with spin 3/2 and spin 1/2.

(a) Compute the total angular momentum states $|j_1, j_2; J, M\rangle$ in terms of the single particle product states $|j_1, m_1\rangle |j_2, m_2\rangle$.

From the above, we can deduce the following

$$\text{Spin-}3/2: j_1 = 3/2, \quad \text{Spin-}1/2: j_2 = 1/2 \quad \therefore \; -j_1 \leq m_1 \leq j_1, \quad -j_2 \leq m_2 \leq j_2$$

$$\Rightarrow \quad |j_1 - j_2| \leq J \leq j_1 + j_2$$

The total angular momentum ranges from,

$$J_1 = 1, 2, \quad \text{with} \quad -J_1 \leq M_1 \leq J_1$$

we then define

$$|3/2, 1/2; J, M\rangle \equiv |J, M\rangle, \quad |3/2, 1/2; M_1, M_2\rangle \equiv |M_1, M_2\rangle$$

Then finally

$$|2,2\rangle = |3/2, 1/2\rangle \tag{1}$$

Which is the max polarized state. we now wish to find the other states for $J=2$. we do this by applying J_-

$$J_-|J, M\rangle = \hbar\sqrt{(J+M)(J-M+1)}\,|J, M-1\rangle$$

Applying this to $|2,2\rangle$ we see

$$J_-|2,2\rangle = 2\hbar|2,1\rangle = (J_{1-} + J_{2-})|3/2, 1/2\rangle = \sqrt{3}\hbar|1/2, 1/2\rangle + \hbar|3/2, -1/2\rangle$$

$$\Rightarrow \quad |2,1\rangle = \frac{\sqrt{3}}{2}|1/2, 1/2\rangle + \frac{1}{2}|3/2, -1/2\rangle \tag{2}$$

we now apply J_- on $|2,1\rangle$

$$J_1|2,1\rangle = \hbar\sqrt{6}\,|2,0\rangle = (J_{1-} + J_{2-})\left(\frac{\sqrt{3}}{2}|1/2, 1/2\rangle + \frac{1}{2}|3/2, -1/2\rangle\right)$$

$$= \hbar\sqrt{3}|-1/2, 1/2\rangle + \hbar\frac{\sqrt{3}}{2}|1/2, -1/2\rangle + \hbar\frac{\sqrt{3}}{2}|1/2, -1/2\rangle$$

we then have $|2,0\rangle$ as

Problem 1: Continued

$$|2,0\rangle = \frac{1}{\sqrt{2}}|-1/2, 1/2\rangle + \frac{1}{\sqrt{2}}|1/2, -1/2\rangle \tag{3}$$

Where we now apply J_- on $|2,0\rangle$

$$J_-|2,0\rangle = \hbar\sqrt{6}\,|2,-1\rangle = (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\Big(|-1/2, 1/2\rangle + |-1/2, -1/2\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}(J_{1-} + J_{2-})(|-1/2, 1/2\rangle + |-1/2, -1/2\rangle)$$

$$= \sqrt{\frac{3}{2}}\hbar|-3/2, 1/2\rangle + \frac{1}{\sqrt{2}}\hbar|-1/2, -1/2\rangle + \hbar\sqrt{2}|-1/2, -1/2\rangle$$

This then means $|2,-1\rangle$

$$|2,-1\rangle = \frac{1}{2}|-3/2, 1/2\rangle + \frac{\sqrt{3}}{2}|-1/2, -1/2\rangle \tag{4}$$

And lastly the $|2,-2\rangle$ state is

$$|2,-2\rangle = |-3/2, -1/2\rangle \tag{5}$$

For $J=1$, we start with $|1,1\rangle$ and impose the condition,

$$\langle 1,1|2,1\rangle = 0$$

Therefore,

$$|1,1\rangle = \frac{1}{2}|1/2, 1/2\rangle - \frac{\sqrt{3}}{2}|3/2, -1/2\rangle \tag{6}$$

We then apply J_- on $|1,1\rangle$

$$J_-|1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle = (J_{1-} + J_{2-})(1/2|1/2, 1/2\rangle - \sqrt{3}/2|3/2, -1/2\rangle)$$

$$= \hbar|-1/2, 1/2\rangle + \frac{\hbar}{2}|1/2, -1/2\rangle - \frac{3\hbar}{2}|1/2, -1/2\rangle$$

Where finally,

$$|1,0\rangle = \frac{1}{\sqrt{2}}|-1/2, 1/2\rangle - \frac{1}{\sqrt{2}}|-1/2, 1/2\rangle \tag{7}$$

Problem 1: Continued

And then we Finally have $|1,-1\rangle$

$$|1,-1\rangle = \frac{\sqrt{3}}{2}|-3/2,1/2\rangle - \frac{1}{2}|-1/2,-1/2\rangle \tag{8}$$

All of these states are

$$\boxed{\begin{aligned} &|2,2\rangle = |3/2,1/2\rangle \\ &|2,1\rangle = \frac{\sqrt{3}}{2}|1/2,1/2\rangle + \frac{1}{2}|3/2,-1/2\rangle,\ |1,1\rangle = \frac{1}{2}|1/2,1/2\rangle - \frac{\sqrt{3}}{2}|3/2,-1/2\rangle \\ J=2,1\ ;\ &|2,0\rangle = \frac{1}{\sqrt{2}}|-1/2,1/2\rangle + \frac{1}{\sqrt{2}}|1/2,-1/2\rangle,\ |1,0\rangle = \frac{1}{\sqrt{2}}|-1/2,1/2\rangle - \frac{1}{\sqrt{2}}|-1/2,1/2\rangle \\ &|2,-1\rangle = \frac{1}{2}|-3/2,1/2\rangle + \frac{\sqrt{3}}{2}|-1/2,-1/2\rangle,\ |1,-1\rangle = \frac{\sqrt{3}}{2}|-3/2,1/2\rangle - \frac{1}{2}|-1/2,-1/2\rangle \\ &|2,-2\rangle = |-3/2,-1/2\rangle \end{aligned}}$$

(b) Suppose the Hamiltonian of the two particles has the form

$$\mathcal{H} = \alpha \mathbf{S}_1 \cdot \mathbf{S}_2,$$

with α a constant. If the system is initially ($t=0$) in the following eigenstate of $\mathbf{S}_1^2$, $\mathbf{S}_2^2$, S_{1z}, S_{2z},

$$|j_1, j_2; m_1, m_2\rangle = |\frac{3}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle,$$

what is the probability of finding the system in state $|\frac{3}{2}, \frac{1}{2}; \frac{3}{2}, -\frac{1}{2}\rangle$ at time $t > 0$?

We First start with the relationship

$$\mathcal{H}|s,m\rangle = \frac{1}{2}\alpha \vec{S}_1 \cdot \vec{S}_2 |s,m\rangle = \frac{\alpha\hbar^2}{2}\left[s(s+1) - \frac{9}{2}\right]|s,m\rangle$$

For $s=1,2$

$$E_1 = -\frac{5\alpha\hbar^2}{4},\quad E_2 = \frac{3\alpha\hbar^2}{4}$$

From the C.G Coefficients

$$|2,1\rangle = \frac{\sqrt{3}}{2}|1/2,1/2\rangle + \frac{1}{2}|3/2,-1/2\rangle,\quad |1,1\rangle = \frac{1}{2}|1/2,1/2\rangle - \frac{\sqrt{3}}{2}|3/2,-1/2\rangle$$

Problem 1: Continued

Inverting the above equations we have

$$|1/2, 1/2\rangle = \frac{\sqrt{3}}{2}|2,1\rangle + \frac{1}{2}|1,1\rangle \quad , \quad |3/2, -1/2\rangle = \frac{1}{2}|2,1\rangle - \frac{\sqrt{3}}{2}|1,1\rangle$$

If our initial state at $t=0$ $\Psi(0) = |1/2, 1/2\rangle$ then

$$\Psi(t) = \frac{\sqrt{3}}{2} e^{-iE_2 t/\hbar}|2,1\rangle + \frac{1}{2} e^{-iE_1 t/\hbar}|1,1\rangle$$

We then calculate the probability for finding $\Psi(t)$ in the state $|3/2, -1/2\rangle$

$$P = |\langle 3/2, -1/2|\Psi(t)\rangle|^2 = \frac{3}{16}|e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar}|^2 = \frac{3}{8}(1 - \cos((E_1 - E_2)t/\hbar))$$

We can then further simplify to say

$$\boxed{P = \frac{3}{4}\sin^2(\alpha t)}$$

Problem 1: Review

Procedure

- – Assign the value of the spin to both j_1 and j_2
  – Find the total angular momentum with
  $$|j_1 - j_2| \leq J \leq j_1 + j_2 \quad \text{with} \quad -J_{1,2} \leq M_{1,2} \leq J_{1,2}$$
  – Define a new ket
  $$|J, M\rangle = |M_1, M_2\rangle$$
  that encapsulates the total angular momentum and the individual angular momentum
  – To find the total angular momentum states in terms of the individual momenta we use
  $$\tilde{\mathbf{J}}_- |J, M\rangle = \hbar\sqrt{(J+M)(J-M+1)}\,|J, M-1\rangle \quad \text{with} \quad \tilde{\mathbf{J}}_- = (\tilde{\mathbf{J}}_{1-} + \tilde{\mathbf{J}}_{2-})$$
  – Repeat the above for each state until there are no more states
- – Use the relationship for the Hamiltonian acting on a spin eigenstate
  $$\tilde{\mathbf{H}}|S, M\rangle = \frac{1}{2}\alpha\mathbf{S}_1 \cdot \mathbf{S}_2 |S, M\rangle = \frac{\alpha\hbar^2}{2}\left[s(s+1) - \frac{9}{2}\right]|S, M\rangle$$
  – Find the energies for $S = 1, 2$ with the above
  – Use the C.G coefficients $|2, 1\rangle$ and $|1, 1\rangle$ and solve for $|1/2, 1/2\rangle$ and $|3/2, -1/2\rangle$ respectively
  – Find the time evolved state for $|1/2, 1/2\rangle$ and proceed to take the probability of it with $|3/2, -1/2\rangle$. Namely
  $$\mathcal{P} = |\langle 3/2, -1/2|\psi(t)\rangle|^2$$
  where $|\psi(t)\rangle$ is the time evolved state for $|1/2, 1/2\rangle$

Key Concepts:

- – The total angular momentum can be found simply by adding the individual momenta
  – J is the total angular momentum, M is the magnetic quantum number. The lower cases of each are for individual particles
  – The magnetic quantum number ranges between $\pm$ the value of the total angular momentum
  – Computing the total angular momentum states in terms of single particle product states are what we call Clebsh Gordan coefficients
  – We find these new states by applying the lowering operator each total angular momentum state
- – Using the C.G coefficients found in part (a) we can rearrange them to have state in a form that we can use
  – Using the eigenvalue equation for this Hamiltonian we then know what values to use in our time evolved state
  – We then proceed to calculate the probability of finding the system in state $|3/2, -1/2\rangle$

Variations:

- – We of course can be given different particles with different spin
    - ∗ We would follow the same procedure but would use different values for J and M for everything that follows
- – We could be asked to find the probability for a different state
    - ∗ We then would have to use different C.G coefficients to solve for the desired state and then follow the same procedure from there

  – We could be given a different Hamiltonian
    - ∗ We would use the same procedure to find the probability but would have different eigenvalues for our states

Problem 2:

Using recursion relations, verify the special case of the Clebsh-Gordan coefficient

$$\langle j_1; j0 | j_1; jj\rangle = \sqrt{\frac{j}{j+1}}.$$

Hint: Write down the relevant Clebsh-Gordan completeness relation that includes this coefficient. That will give you one equation with two unknown coefficients. Find then a convenient recursion relation to obtain the second equation for those two same coefficients and solve them.

We first begin by writing the completeness relation for the C.G coefficients

$$\sum_{m_1, m_2} |\langle m_1, m_2 | j, j\rangle|^2 = \mathbb{1}.$$

We now have the selection rule that

$$m_1 + m_2 = j.$$

For our state

$$\langle j, 1; j, 0 | j, 1; j, j\rangle$$

We can see that $m_1 = 0, \pm 1$ and $m_2 = 0, \pm 1$. We are confined to having $j > 0$ so $m_2 \neq -1$. We use this and the completeness relation to deduce

$$|\langle j, 0 | j, j\rangle|^2 + |\langle j-1, 1 | j, j\rangle|^2 = \mathbb{1}. \qquad (*)$$

Recall the recursion relation for C.G. Coefficients

$$\sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\,\langle m_1 \mp 1, m_2 | J, M\rangle + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\,\langle m_1, m_2 \mp 1 | J, M\rangle$$

If we proceed to use J_+ on our state, $j + 1 = M + 1$, it is forbidden since $J = j$. Because of this we choose $m_1 = j_1 = J = M = j$, $m_2 = j_2 = 1$. We now have

$$\cancelto{\sqrt{2j}}{\sqrt{(j-j+1)(j+j)}}\,\langle j-1, 1 | j, j\rangle + \cancelto{\sqrt{2}}{\sqrt{(1-1+1)(1+1)}}\,\langle j, \cancelto{0}{1-1} | j, j\rangle = 0$$

$$\sqrt{2}\sqrt{j}\,\langle j-1, 1 | j, j\rangle + \sqrt{2}\,\langle j, 0 | j, j\rangle = 0, \quad -\sqrt{j}\,\langle j-1, 1 | j, j\rangle = \langle j, 0 | j, j\rangle \qquad (**)$$

Taking (**) and putting it into (*) we now have

$$|\langle j, 0 | j, j\rangle|^2 + \frac{1}{j}|\langle j, 0 | j, j\rangle|^2 = \left(\frac{j+1}{j}\right)|\langle j, 0 | j, j\rangle|^2 = 1$$

We can then say

$$\langle j, 0 | j, j\rangle = \sqrt{\frac{j}{j+1}} \Rightarrow \langle j, 0 | j, j\rangle \doteq \langle j, 1; j, 0 | j, 1; j, j\rangle = \sqrt{\frac{j}{j+1}} \checkmark$$

Problem 2: Review

Procedure:

- – Begin by writing the completeness relation in the C.G coefficients

  $$\sum_{m_1}\sum_{m_2} |\langle m_1, m_2|j,j\rangle|^2 = \mathbb{I}$$

  where we are given our selection rule $m_1 + m_2 = j$

  – Use the recursion relation for the C.G coefficients

  $$\mathcal{R}_{C.G} = \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\langle m_1 \mp 1, m_2|J,M\rangle + \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}\langle m_1, m_2 \mp 1|J,M\rangle$$

  – Choose the values of m_1, j_1, m_2, j_2 from the recursion relation and use these with the completeness relation of our state and simplify

Key Concepts:

- – We use the completeness relation with our selection rules to help us determine values of m and j for both particles

  – Once these values are obtained we can then use the recursion relation and the completeness relation of our specific state to obtain the desired expression

Variations:

- – We could be asked to prove a different identity
    - ∗ This would then mean that we would have to use different relationships to prove whatever is asked of us

Problem 3:

Consider a system formed by two spinless particles with angular momentum $j_1 = 1$ and $j_2 = 1$.

(a) Assuming that the system is subjected to a spherically symmetric potential, using the Wigner-Eckart theorem, find the selection rules for the matrix elements of the momentum operator components P_x, P_y and P_z

$$\langle \alpha, j', m'|P_i|\alpha, j, m\rangle \,,$$

where α is a quantum number which is independent of the magnetic quantum numbers m and m'. Compute the matrix elements explicitly for $j' = 2$ and $j = 1$ (Use the Clebsh-Gordan table for that).

The Wigner-Eckart Theorem (3.474) mathematically is

$$\langle \alpha', j' m' | T_q^{(k)} | \alpha, j m \rangle = \langle j k ; m q | j k ; j' m' \rangle \underbrace{\frac{\langle \alpha' j' || T^{(k)} || \alpha j \rangle}{\sqrt{2j+1}}}_{\to C^*}$$

For spherical tensors of rank (k). We now wish to express our cartesian tensor spherically starting with P,

$$P_x = \cos(\varphi)\sin(\theta)\,,\quad P_y = \sin(\varphi)\sin(\theta)\,,\quad P_z = \cos(\theta).$$

Taking the above with $l=1$, the spherically symmetric tensor will be

$$Y_1^{\pm 1}(\theta) = Y_1^{-1}(\theta) + Y_1^0(\theta) + Y_1^1(\theta)$$

Expressing the matrix elements with the use of spherical harmonics we see that,

$$Y_1^{-1}(\theta) = \sqrt{\frac{3}{8\pi}}\sin(\theta)e^{-i\varphi} = \sqrt{\frac{3}{8\pi}}\sin(\theta)(\cos(\varphi) - i\sin(\varphi)) = \sqrt{\frac{3}{8\pi}}(\cos(\varphi)\sin(\theta) - i\sin(\varphi)\sin(\theta))$$

$$Y_1^{1}(\theta) = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\varphi} = -\sqrt{\frac{3}{8\pi}}\sin(\theta)(\cos(\varphi) + i\sin(\varphi)) = -\sqrt{\frac{3}{8\pi}}(\cos(\varphi)\sin(\theta) + i\sin(\varphi)\sin(\theta))$$

$$Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$

They can be re-written simply as

$$Y_1^{-1}(\theta) = \sqrt{\frac{3}{8\pi}}(P_x - P_y)\,,\quad Y_1^1(\theta) = -\sqrt{\frac{3}{8\pi}}(P_x + P_y)\,,\quad Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}}P_z$$

We can then go ahead and write P_x, P_y in terms of Tensors

The relationship's for these are

$$P_x = \frac{1}{\sqrt{2}}(\tilde{T}_{-1}^{(1)} - \tilde{T}_1^{(1)})\,,\quad P_y = \frac{i}{\sqrt{2}}(\tilde{T}_1^{(1)} + \tilde{T}_{-1}^{(1)})$$

Problem 3: Continued

We then follow the Wigner Eckhart Theorem to give us our selection rule

$$m' = m + q \quad w/ \quad |j-1| \leq j \leq j+1$$

We can also say $T^{(1)}$ is odd under parity, $j + j' = 2m+1 \therefore j' = |j \pm 1|$

We now proceed to calculate the matrix elements for $j' = 2$, $j = 1$. This then becomes

$$\langle \alpha, 2; m | T_0^{(1)} | \alpha, 1, m \rangle = \langle 11; m0 | 11; 2m \rangle = c \begin{cases} 1/\sqrt{2} & , m = 1 \\ \sqrt{2}/\sqrt{3} & , m = 0 \\ 1/\sqrt{2} & , m = -1 \end{cases}$$

Where we can also say for $m+1$

$$\langle \alpha, 2; m+1 | T_0^{(1)} | \alpha, 1, m \rangle = \langle 11 : m1 | 11 ; 2m+1 \rangle = c \begin{cases} 1 & , m = 1 \\ 1/\sqrt{2} & , m = 0 \\ 1/\sqrt{6} & , m = -1 \end{cases}$$

And for $m-1$

$$\langle \alpha, 2 : m-1 | T_0^{(1)} | \alpha, 1, m \rangle = \langle 11 : m | 11 ; 2m-1 \rangle = c \begin{cases} 1/\sqrt{6} & , m = 1 \\ 1/\sqrt{2} & , m = 0 \\ 1 & , m = -1 \end{cases}$$

We can then say for P_z

$$\langle \alpha, 2; m' | P_z | \alpha, 1; m \rangle = \begin{cases} 1/\sqrt{2} & , m = 1 \\ \sqrt{2}/\sqrt{3} & , m = 0 \\ 1/\sqrt{2} & , m = -1 \end{cases} \times \underbrace{\frac{\langle \alpha, 2 | P_z | \alpha, 1 \rangle}{\sqrt{3}}}_{} \Rightarrow c$$

And then for P_x

$$\langle \alpha, 2; m' | P_x | \alpha, 1; m \rangle = - \langle \alpha, 2, m' | \frac{T_1^{(1)} - T_{-1}^{(1)}}{\sqrt{2}} | \alpha, 1, m \rangle$$

$$= \frac{c}{\sqrt{2}} \times \begin{cases} 1/\sqrt{6} & , m = 1 & , m' = 0 \\ 1/\sqrt{2} & , m = 0 & , m' = -1 \\ 1 & , m = -1 & , m' = -2 \\ -1 & , m = 1 & , m' = 2 \\ -1/\sqrt{2} & , m = 0 & , m' = 1 \\ -1/\sqrt{6} & , m = -1 & , m' = 0 \end{cases}$$

And finally for y

$$\langle \alpha, 2; m' | P_y | \alpha, 1; m \rangle = - \langle \alpha, 2; m' | \frac{T_1^{(1)} + T_{-1}^{(1)}}{\sqrt{2} i} | \alpha, 1; m \rangle$$

Problem 3: Continued

$$= -\frac{c}{\sqrt{2}i} \times \begin{cases} 1/\sqrt{6} & , \mu=1 & , \mu'=0 \\ 1/\sqrt{2} & , \mu=0 & , \mu'=-1 \\ 1 & , \mu=-1 & , \mu'=-2 \\ 1 & , \mu=1 & , \mu'=2 \\ 1/\sqrt{2} & , \mu=0 & , \mu'=1 \\ 1/\sqrt{6} & , \mu=-1 & , \mu'=0 \end{cases}$$

All of the directions finally are

$$\boxed{\begin{gathered}\langle \alpha,2;\mu'|P_x,P_y|\alpha,1;\mu\rangle = \frac{c}{\sqrt{2}} \times \begin{cases} 1/\sqrt{6} & , \mu=1 & , \mu'=0 \\ 1/\sqrt{2} & , \mu=0 & , \mu'=-1 \\ 1 & , \mu=-1 & , \mu'=-2 \\ -1 & , \mu=-1 & , \mu'=2 \\ -1/\sqrt{2} & , \mu=0 & , \mu'=1 \\ -1/\sqrt{6} & , \mu=1 & , \mu'=0 \end{cases} , \quad \frac{-c}{\sqrt{2}i} \times \begin{cases} 1/\sqrt{6} & , \mu=1 & , \mu'=0 \\ 1/\sqrt{2} & , \mu=0 & , \mu'=-1 \\ 1 & , \mu=-1 & , \mu'=-2 \\ 1 & , \mu=-1 & , \mu'=2 \\ 1/\sqrt{2} & , \mu=0 & , \mu'=1 \\ 1/\sqrt{6} & , \mu=1 & , \mu'=0 \end{cases} \\ \langle \alpha,2;\mu'|P_z|\alpha,1;\mu\rangle = \begin{cases} 1/\sqrt{2} & , \mu=1 \\ \sqrt{2}/\sqrt{3} & , \mu=0 \\ 1/\sqrt{2} & , \mu=-1 \end{cases}\end{gathered}}$$

(b) Using the definition for the product of spherical tensors,

$$T_q^{(k)} = \sum_{q_1 q_2} T_{q_1}^{(k_1)} T_{q_2}^{(k_2)} \langle k_1, k_2; q_1, q_2 | k_1, k_2; k, q \rangle ,$$

compute the generic form for $T_{\mathbf{q}}^{(2)}$ in terms of the operator components of $\mathbf{P}$.

We first start out by defining some values

$$k = k_1 + k_2 \Rightarrow T_{q_1}^{k_1} T_{q_2}^{k_2} \text{ are of lower rank than } T_q^k$$

We can also say $q = q_1 + q_2$. W/ $k=2$, $k_1 = k_2 = 1$ this tells us that q will range between ± 2 and $q_{1,2}$ will range between ± 1.

The generic equation looks like

$$T_q^{(k)} = \sum_{q_1} \sum_{q_2} T_{q_1}^{(k_1)} T_{q_2}^{(k_2)} \langle k_1, k_2; q_1, q_2 | k_1, k_2; k, q \rangle$$

With $k=2, k_1=k_2=1$, $q = \pm 2$, $q_1 = q_2 = 0, \pm 1$, these will look like

$$T_{\pm 2}^{(2)} = \sum_{q_1} \sum_{q_2} \underbrace{\langle 1,1;\pm 1, \pm 1 | 1,1;2,\pm 2\rangle}_{=1} T_{\pm 1}^{(1)} T_{\pm 1}^{(1)} \qquad q = \pm 2$$

$$T_{\pm 2}^{(2)} = (1) T_{\pm 1}^{(1)} T_{\pm 1}^{(1)}$$

Problem 3: Continued

We can then write these in their respective momenta components

$$T_2^{(2)} = \frac{1}{2}(P_x + iP_y)^2 \quad , \quad T_{-2}^{(2)} = \frac{1}{2}(P_x - iP_y)^2 \qquad (*)$$

We now do this for $k=2$, $q=\pm 1$, $q_{1,2}=\pm 1$, $q_{2,1}=0$

$$T_{\pm 1}^{(2)} = T_{\pm 1}^{(1)} T_0^{(1)} \underbrace{\langle 1,1;\pm 1,0|1,1;2,\pm 1\rangle}_{=\frac{1}{\sqrt{2}}} + T_0^{(1)} T_{\pm 1}^{(1)} \underbrace{\langle 1,1;0,\pm 1|1,1;2,\pm 1\rangle}_{=\frac{1}{\sqrt{2}}}$$

Where again we write these in momentum operators

$$T_1^{(2)} = -P_z(P_x + iP_y) \quad , \quad T_{-1}^{(2)} = P_z(P_x - iP_y) \qquad (**)$$

And lastly we do this for $k=2$, $q=0$, $\pm q_{1,2} = \mp q_{2,1}$

$$T_0^{(2)} = T_0^{(1)} T_0^{(1)} \underbrace{\langle 1,1;0,0|1,1;2,0\rangle}_{=\sqrt{\frac{2}{3}}} + T_1^{(1)} T_{-1}^{(1)} \underbrace{\langle 1,1;1,-1|1,1;2,0\rangle}_{=\frac{1}{\sqrt{6}}}$$

$$+ T_{-1}^{(1)} T_1^{(1)} \underbrace{\langle 1,1;-1,1|1,1;2,0\rangle}_{=\frac{1}{\sqrt{6}}}$$

Where we of course can write these in terms of P_x, P_y, and P_z

$$T_0^{(2)} = \frac{2}{\sqrt{6}} P_z^2 - \frac{P_x^2 + P_y^2}{\sqrt{6}} \qquad (***)$$

We can then finally say our spherical tensors in components of P is

$$\boxed{\begin{gathered} T_2^{(2)} = \frac{1}{2}(P_x + iP_y)^2 \quad , \quad T_{-2}^{(2)} = \frac{1}{2}(P_x - iP_y)^2 \\ T_1^{(2)} = -P_z(P_x + iP_y) \quad , \quad T_{-1}^{(2)} = P_z(P_x - iP_y) \\ T_0^{(2)} = \frac{2}{\sqrt{6}} P_z^2 - \frac{P_x^2 + P_y^2}{\sqrt{6}} \end{gathered}}$$

Problem 3: Review

Procedure:

-
  - Begin by writing out the components of the momentum operator in terms of spherical tensors. Namely,

$$P_x = \frac{1}{\sqrt{2}}(\tilde{\mathbf{T}}_{-1}^{(1)} - \tilde{\mathbf{T}}_1^{(1)}) \quad \text{and} \quad P_y = \frac{i}{\sqrt{2}}(\tilde{\mathbf{T}}_1^{(1)} + \tilde{\mathbf{T}}_{-1}^{(1)})$$

  - We then use the Wigner Eckart Theorem

$$\langle \alpha', j'm' | \tilde{\mathbf{T}}_q^{(k)} | \alpha, jm \rangle = \langle jk; mq | jk; j'm' \rangle \, \frac{\langle \alpha' j' || \tilde{\mathbf{T}}^k || \alpha j \rangle}{\sqrt{2j+1}}$$

    where the right most term is just a constant
  - The Wigner Eckart Theorem tells us our selection rule $m' = m + q$ and $|j-1| \leq j \leq j+1$
  - Proceed to use the Wigner Eckart Theorem for P_x, P_y, P_z for $j' = 2$ and $j = 1$
-
  - Use the definition for spherical tensors that is given to us write each spherical tensor of $k = 2$ in terms of lower k values
  - Proceed to write the spherical tensors of $k = 1$ in terms of P_x, P_y, P_z

Key Concepts:

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  - We use the Wigner Eckart Theorem to calculate matrix elements for spherical tensors of specific values of j' and j
  - The Wigner Eckart Theorem uses Clebsh Gordan coefficients to calculate matrix elements of spherical tensors
  - To calculate the matrix elements for the components of $\mathbf{P}$ we write them in terms of spherical tensors and use the Wigner Eckart Theorem
-
  - We can write spherical tensors of rank 2 in lower ranks where they can be later expressed in terms of $\mathbf{P}$ by using the spherical tensor product definition
  - We also use Clebsh Gordan coefficients to express these tensors in terms of other components

Variations:

-
  - We could be given different values of j' and j
    * This would just affect the value of the Clebsh Gordan coefficients that we look up
  - We could be asked to use the Wigner Eckart Theorem for a different quantity other than the momentum given to us
    * We then would have to use different relationships to write these new operators in terms of spherical tensors to be used in the Wigner Eckart Theorem
-
  - Like the previous part, we could be given a different initial starting point (Different initial tensor)
    * We would use the same broad procedure in this part but for the new system that we are working with