Physics 5403 Exam #1
Spring 2022

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Time: 3 hours

March 9, 2022

1 Atomic orbitals

An electron occupies a p-wave orbital, with angular momentum state $|l = 1, m\rangle$.

(a) Write down the total angular momentum states $|J, M\rangle$ of the electron in the basis of product states $|l, m\rangle\,|s, m_s\rangle$ where m_s is the spin projection quantum number of the S_z operator.

(b) Suppose we have two *indistinguishable* particles with spin 1/2 in the same $l = 0$ orbital of an atom. What are the allowed *total* spin quantum numbers of the two-particle state? Justify.

2 Spherical tensors

The expectation value

$$Q \equiv e\,\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

is known as the *quadrupole moment*, where x, y, z are position operators, $r^2 = x^2 + y^2 + z^2$ and e is the electric charge.

(a) Write $(3z^2 - r^2)$, xy and $(x^2 - y^2)$ in terms of irreducible spherical tensors of rank 2.

(b) Using the Wigner-Eckart theorems

$$\langle \alpha', j', m'|T_q^{(k)}|\alpha, j, m\rangle = \langle j, k; mq|jk; j'm'\rangle\,\frac{\langle \alpha' j'||T^{(k)}||\alpha j\rangle}{\sqrt{2j+1}}$$

evaluate

$$e\,\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle$$

in terms of Q and appropriate Clebsh-Gordan coefficients. You don't need to evaluate the Clebsch-Gordan coefficients. Hint: use the Wigner-Eckart theorem to compute Q as well and express it in terms of a Clebsh-Gordan coefficient.

3 Identical particles

The Hamiltonian of a two-level system with identical *bosons* is

$$\mathcal{H} = \omega\Big(a_1^\dagger a_2 + a_2^\dagger a_1\Big),$$

where $a_i(a_i^\dagger)$ is an annihilation (creation) operator in level $i = 1, 2$. Those operators satisfy the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$ and $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$.

(a) Defining the single particle states $\Psi_i = a_i^\dagger |0\rangle$, where $|0\rangle$ is the vacuum of the Fock space, calculate the matrix elements

$$\langle \Psi_i | \mathcal{H} | \Psi_j \rangle$$

and the energy levels of the two-level system. Derive the eigenkets of the Hamiltonian and write them explicitly in terms of the vacuum $|0\rangle$.

(b) Consider now a given *two*-particle state

$$|\Phi\rangle = A \sum_{ij} c_i c_j a_i^\dagger a_j^\dagger |0\rangle \, ,$$

where $c_i (i = 1, 2)$ are *real* coefficients. Compute the normalization coefficient A of this state.