Problem 1:

particles are confined to this region

Single-particle Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = \varepsilon\,\psi(x)$$

$$\psi(x) = \psi(x+L)$$

Solution: $\psi(x) = \frac{1}{L^{1/2}} e^{ikx}$, $k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

enforce BC: $\underbrace{e^{ikL}}_{\cos(kL) + i\sin(kL)} = 1$

$\cos(kL) + i\sin(kL) = 1$

$\Rightarrow kL = 0, \pm 2\pi, \pm 4\pi, \ldots$

$k = n\frac{2\pi}{L}$ where $n = 0, \pm 1, \ldots$

$$\Rightarrow \varepsilon = \frac{\hbar^2 k^2}{2m} \quad \text{with } k = n\frac{2\pi}{L}$$

We want to calculate the number of single-particle states with energy less than $E = \frac{\hbar^2 \bar{k}^2}{2m}$

$$\bar{k} = \sqrt{\frac{2mE}{\hbar^2}}$$

we have pos. and negative k

So: integrate over all k: $\int_{-\bar{k}}^{\bar{k}} dk = 2 \int_0^{\bar{k}} dk = 2\bar{k}$

$$= 2\sqrt{\frac{2mE}{\hbar^2}}$$

But we need to divide by $\frac{2\pi}{L}$

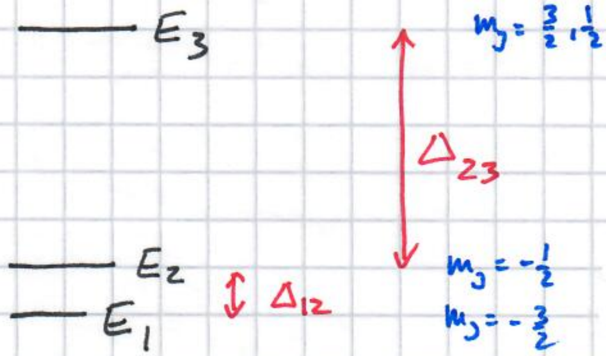
Number of states w/ energy $\leq E$: $N(E) = \frac{2\sqrt{\frac{2mE}{\hbar^2}}}{\frac{2\pi}{L}}$

$$= \frac{1}{\pi}\sqrt{\frac{2mL^2}{\hbar^2}}\sqrt{E}$$

$$\Rightarrow \boxed{D(E) = \frac{\partial N(E)}{\partial E} = \frac{1}{2\pi}\sqrt{\frac{2mL^2}{\hbar^2}}\frac{1}{\sqrt{E}}}$$

I am treating the particles as distinguishable throughout.

Problem 2:



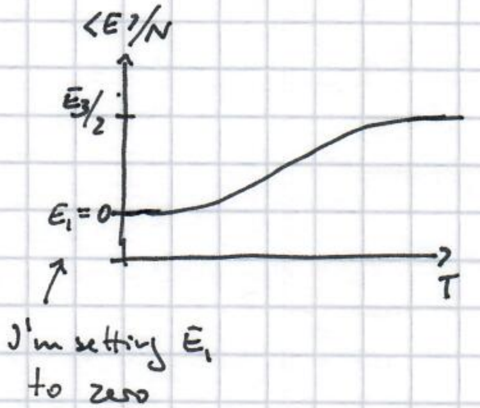
(a) $T = 0$: All particles occupy the energy state E_1

$$\Rightarrow \langle E \rangle = N E_1$$

$\Delta_{12} \ll kT \ll \Delta_{23}$: In this case, the energy states E_1 and E_2 are occupied with roughly the same probability

$$\Rightarrow \langle E \rangle = \frac{N}{2} E_1 + \frac{N}{2} E_2$$

$$= \frac{N}{2}(E_1 + E_2) = N \frac{E_1 + E_2}{2}$$



$\Delta_{23} \ll kT$: all states are occupied with the same probability

$$\Rightarrow \langle E \rangle \approx N \frac{E_1 + E_2 + 2E_3}{4}$$

(b) As $T \to \infty$, all four states are occupied roughly equally $\Rightarrow$ occupation of all four states is $\frac{N}{4}$.

By increasing the temperature, no more energy can be stored in the system.

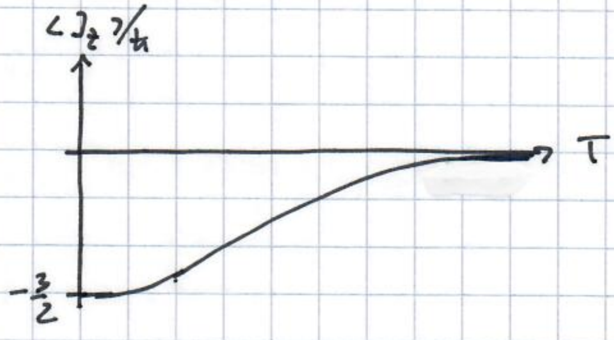
$\Rightarrow$ specific heat $\to 0$ as $T \to \infty$

(c) $T = 0$: $\langle \hat{J}_z \rangle = -\frac{3}{2}\hbar N$

$\Delta_{12} \ll kT \ll \Delta_{23}$: $\langle \hat{J}_z \rangle = \frac{-\frac{1}{2} + (-\frac{3}{2})}{2}\hbar = -1\hbar N$

$\Delta_{23} \ll kT$: $\langle \hat{J}_z \rangle = \frac{-\frac{3}{2} + (-\frac{1}{2}) + \frac{1}{2} + \frac{3}{2}}{4} N = 0$

the reasoning is the same as in (a)



(d) $$\frac{\langle \hat{J}_z \rangle}{\hbar N} = \frac{-\frac{3}{2} e^{-E_1/kT} - \frac{1}{2} e^{-E_2/kT} + \frac{1}{2} e^{-E_3/kT} + \frac{3}{2} e^{-E_3/kT}}{e^{-E_1/kT} + e^{-E_2/kT} + e^{-E_3/kT} + e^{-E_3/kT}}$$

In the limit that $kT \gg E_j$ $\quad (j = 1, 2, 3)$,

we can approximate $e^{-E_j/kT} \approx 1$

$$\Rightarrow \quad \underbrace{\langle \hat{J}_z \rangle / \hbar N \longrightarrow 0}_{\text{in agreement w/ the result in (c)}} \quad \text{as} \quad T \to \infty$$

Homework 6, Problem 3:

(a) We have periodic boundary conditions

SP: single particle

$\Rightarrow$ The SP energies are $\varepsilon_i = \dfrac{\vec{p}_i^{\,2}}{2m} = \dfrac{\hbar^2 \vec{k}_i^{\,2}}{2m}$

We have no degeneracy factor since we are dealing with spin-0 bosons.

Number of states $N(\varepsilon)$ with energy less than $\varepsilon = \dfrac{\hbar^2 \vec{k}^2}{2m}$:

$$N(\varepsilon) = \frac{\frac{4\pi k^3}{3}}{\left(\frac{2\pi}{L}\right)^3}$$

volume in k-space

size of "unit cell"

$$= \frac{L^3 k^3}{6\pi^2}$$

$k = \left(\dfrac{2m\varepsilon}{\hbar^2}\right)^{1/2}$

$$\rightarrow = \frac{(2m)^{3/2} L^3}{6\pi^2 \hbar^3} \varepsilon^{3/2}$$

Then: $$D(\varepsilon) = \frac{\partial N(\varepsilon)}{\partial \varepsilon} = \frac{(2m)^{3/2} L^3}{4\pi^2 \hbar^3} \varepsilon^{1/2}$$

(b)

Let's look at the single-particle Schrö-dinger eq.:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi_{\vec{n}} = \mathcal{E}_{\vec{n}}\psi_{\vec{n}}$$

$\vec{n} = (n_x, n_y, n_z)$ with n_x, n_y, n_z quantum numbers whose values we need to determine

Our solutions are $e^{i\vec{k}\cdot\vec{r}}$ where

$\vec{k} = (k_x, k_y, k_z)$

$\vec{r} = (x, y, z)$

But: We need to enforce BC:

$$\psi(0, y, z) = \psi(x, 0, z) = \psi(x, y, 0)$$
$$= \psi(L, y, z) = \psi(x, L, z) = \psi(x, y, L) = 0$$

Thus, it is easier to work with sin and cos functions instead of $e^{i\vec{k}\cdot\vec{r}}$.

The cos function does not vanish at

zero : $\cos(0) \neq 0$

$\Rightarrow$ drop cos and work with sin only.

$$\psi(x,y,z) = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

normalization factor

we want $\sin\left(\frac{n_x \pi x}{L}\right) = 0$ for $x = L$

$\Rightarrow n_x = \pm 1, \pm 2, \ldots$

note : we need to eliminate (not allow for) $n_x = 0$ since the wave fct. would vanish in its entirety in this case and we wouldn't be able to normalize the fct.

$\longrightarrow$ or said differently, the particle has to be somewhere!

Moreover : The $n_x = 1$ and $n_x = -1$ wave fcts are the same except for a phase factor

$\Rightarrow n_x = 1, 2, 3, \ldots$

Same for n_y and n_z: $n_y = 1, 2, \ldots$

$n_z = 1, 2, \ldots$

The eigenenergies are then:

$$\varepsilon_{\vec{n}} = \frac{\hbar^2 k_{\vec{n}}^2}{2m} = \frac{\hbar^2}{2m}\left(k_{x,n_x}^2 + k_{y,n_y}^2 + k_{z,n_z}^2\right)$$

$$= \frac{\hbar^2}{2m}\left[\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{L}\right)^2\right]$$

$$= \frac{\pi^2 \hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

We want $\varepsilon_{\vec{n}} < \varepsilon$

$$\frac{\pi^2 \hbar^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) < \varepsilon$$

or $$\left(n_x^2 + n_y^2 + n_z^2\right) < \underbrace{\left(\frac{2m\varepsilon}{\pi^2 \hbar^2}\right)}_{K^2} L^2$$

(by dimensional analysis, the circled quantity has units of $\frac{1}{L^2}$)

The number of eigenstates that satisfy the condition is equal to 1/8 of the volume of a sphere of radius K.

Why $\frac{1}{8}$? Recall n_x, n_y, n_z are positive, i.e., we are working in one quadrant; and there are eight quadrants total.

$\Downarrow$

$$N(\mathcal{E}) = \frac{\frac{1}{8}\,\frac{4\pi K^3}{3}}{\left(\frac{\pi}{L}\right)^3}$$

inserting value of K

$$= \frac{1}{8}\;\frac{4\pi}{3}\left(\frac{2m\,\mathcal{E}}{\pi^2\hbar^2}\right)^{3/2} L^3$$

$$= \frac{(2m)^{3/2}\,L^3}{6\pi^2\hbar^3}\;\mathcal{E}^{3/2}$$

Since the expression for $N(\mathcal{E})$ for the case with hard wall BC is the same as that for periodic BC, we find the same expression for $D(\mathcal{E})$:

$$D(\mathcal{E}) = \frac{(2m)^{3/2}\,L^3}{4\pi^2\hbar^3}\;\mathcal{E}^{1/2}$$

Problem 4:

(a) three spinless Boltzmann particles:

We have x_1, x_2, x_3.

Particle 1 can sit in state 1 or 2 : $\psi_1(x_1)$ or $\psi_2(x_2)$.

Same for particles 2 and 3. For gr. st.: particles are in state ψ_1

$$\Rightarrow \quad \Psi(x_1, x_2, x_3) = \underbrace{\psi_1(x_1)\,\psi_1(x_2)\,\psi_1(x_3)}_{\text{product of three single-particle wave functions}}$$

single-particle state label

particle label

$\rightarrow$ Degeneracy $= 1$

$\rightarrow$ energy $= 3E_1$

Note:

$\psi_1(x_1)\,\psi_1(x_2)\,\psi_1(x_3)$

$= \psi_1(x_2)\,\psi_1(x_1)\,\psi_1(x_3)$

(b) $\Psi(x_1; x_2, x_3) = \psi_1(x_1)\,\psi_1(x_2)\,\psi_1(x_3)$

$\rightarrow$ Degeneracy $= 1$

$\rightarrow$ energy $= 3E_1$

(c) Identical fermions with spin $\frac{1}{2}$.

Particle 1 has $m_{s_1} = \pm\frac{1}{2}$ $\rightarrow$ $\psi_1(x_1)\,|m_s\rangle$ or

$\psi_2(x_1)\,|m_s\rangle$

We can put 2 fermions in level 1 $\rightsquigarrow$ one must go into level 2

$\rightarrow$ energy $= 2E_1 + E_2$

$\rightarrow$ degeneracy 2

see next page

Possible wave fcts.: $\overbrace{(1,+\tfrac{1}{2})}^{\psi_1\,|m_s=\frac{1}{2}\rangle}$ & $(1,-\tfrac{1}{2})$ & $\overbrace{(2,+\tfrac{1}{2})}^{\psi_2\,|m_s=\frac{1}{2}\rangle}$

or $(1,+\tfrac{1}{2})$ & $(1,-\tfrac{1}{2})$ & $(2,-\tfrac{1}{2})$

Call $|a\rangle \hat{=} (1,+\tfrac{1}{2})$, $|b\rangle \hat{=} (1,-\tfrac{1}{2})$, $|c\rangle \hat{=} (2,+\tfrac{1}{2})$

$|a\rangle_1$ means particle 1 is in state $\varphi_1(x_1)\,|m_{s1}=+\tfrac{1}{2}\rangle$

The three-particle wave fct. then reads:

$$\frac{1}{\sqrt{3!}}\det\begin{pmatrix} |a\rangle_1 & |a\rangle_2 & |a\rangle_3 \\ |b\rangle_1 & |b\rangle_2 & |b\rangle_3 \\ |c\rangle_1 & |c\rangle_2 & |c\rangle_3 \end{pmatrix} \longrightarrow 6 \text{ terms}$$

$\frac{1}{\sqrt{6}}$

for the second state: $\underbrace{|a\rangle \hat{=} (1,+\tfrac{1}{2}),\ |b\rangle \hat{=} (1,-\tfrac{1}{2})}_{\text{same as above}}$, $|c\rangle \hat{=} (2,-\tfrac{1}{2})$

different!

(d) Same approach: now we have $m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

i.e., we can place three particles in lowest state.

We have $E = 3E_1$.

Degeneracy is 4. We can pick:

case 1: $m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$

case 2: $m_s = -\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}$

case 3: $m_s = -\frac{3}{2}, \frac{1}{2}, \frac{3}{2}$

case 4: $m_s = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

Wave fct. is again determinant. For first case:

$|a\rangle \hat{=} (1,-\tfrac{3}{2})$, $|b\rangle \hat{=} (1,-\tfrac{1}{2})$, $|c\rangle \hat{=} (1,+\tfrac{1}{2})$. Use these to construct determinant of 3×3 matrix.

Same for other 3 cases.