E & M I

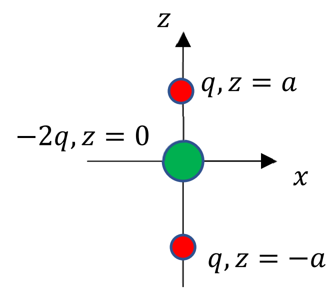
Homework 2, Solutions

1) Multipole expansion:

Consider the second charge distribution considered in the Multipole Expansion workshop:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

a) Using the multipole expansions for the potential $\phi(\vec{r})$, calculate the third (quadrupole) term in for $\phi(\vec{r})$ for this charge distribution.



We saw in the workshop that the monopole and dipole term for this charge distribution is zero. This looks like two dipoles in opposite directions, $\pm\hat{z}$. The quadrupole term is:

$$\phi^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\sum_{q_a} q_a\,(3\,(\vec{r}_a\cdot\hat{r})(\vec{r}_a\cdot\hat{r}) - r_a^2\,)$$

Where the sum is over the charges, q_a, with positions $\vec{r}_a$, and $\hat{r} = \frac{\vec{r}}{r}$ the unit vector from the origin to the position where the potential is being calculated.

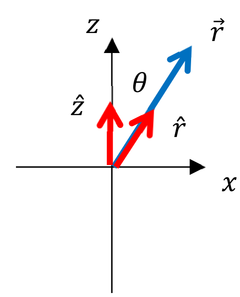
The negative charge will not contribute to the sum, being at the origin, giving:

$$\phi^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{2r^3}\left((3\,(a\,\hat{z}\cdot\hat{r})(a\,\hat{z}\cdot\hat{r}) - a^2) + (3\,(-a\,\hat{z}\cdot\hat{r})(-a\,\hat{z}\cdot\hat{r}) - a^2)\right)$$

$$\phi^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}(3\,(a\,\hat{z}\cdot\hat{r})^2 - a^2)$$

$$\phi^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,a^2}{r^3}(3\,(\hat{z}\cdot\hat{r})^2 - 1)$$

b) Write your result in spherical coordinates, r, θ, ϕ. Explain why your result doesn't depend on ϕ. In this, quadrupole, approximation, for what directions in space is the potential equal to zero? (Draw a picture.)



Considering the vectors shown, we have:

$$\hat{z}\cdot\hat{r} = \cos\theta$$

Note: For spherical coordinates:

$$\hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$

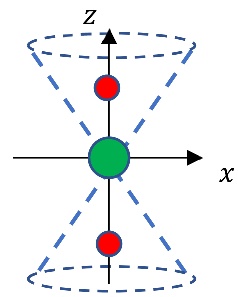
$$\phi^{(2)}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q\,a^2}{r^3}(3\cos^2\theta - 1)$$

The charge distribution is symmetric for rotations about the z-axis. This means the potential cannot depend on the azimuthal angle ϕ.

The potential is zero for:

$$\cos^2\theta = \frac{1}{3} \rightarrow \cos\theta = \frac{1}{\sqrt{3}} \rightarrow \theta = 54.7^o$$

This means that the potential is zero on cones about the z-axis:



c) Using spherical coordinates, calculate the electric field of this charge distribution in the quadrupole approximation. Sketch the field.

The electric field can be determined from:

$$\vec{E}(\vec{r}) = -\vec{\nabla}\,\phi(\vec{r})$$

$$\vec{E}(\vec{r}) = -\frac{q\,a^2}{4\pi\epsilon_0}\left(\partial_r\,\frac{1}{r^3}(3\cos^2\theta - 1)\,\hat{r} + \frac{1}{r}\,\partial_\theta\,\frac{1}{r^3}(3\cos^2\theta - 1)\,\hat{\theta}\right)$$

$$\vec{E}(\vec{r}) = -\frac{q\,a^2}{4\pi\epsilon_0}\left(\frac{-3}{r^4}(3\cos^2\theta - 1)\,\hat{r} + \frac{1}{r^4}(-6\cos\theta\sin\theta)\,\hat{\theta}\right)$$

$$\vec{E}(\vec{r}) = 3\frac{q\,a^2}{4\pi\epsilon_0}\frac{1}{r^4}\,\left((3\cos^2\theta - 1)\,\hat{r} + \;2\cos\theta\sin\theta\;\;\hat{\theta}\right)$$

$$\vec{E}(\vec{r}) = 3\frac{q\,a^2}{4\pi\epsilon_0}\frac{1}{r^4}\,\left((3\cos^2\theta - 1)\,\hat{r} + \;\sin 2\theta\;\;\hat{\theta}\right)$$

We see $\vec{E} \propto \frac{1}{r^4}$ as expected for a quadrupole.

On the $\pm\, z$ axis, $\theta = 0, \theta = \pi$ the field is in the $\hat{r} = \;\pm\hat{z}$ direction.

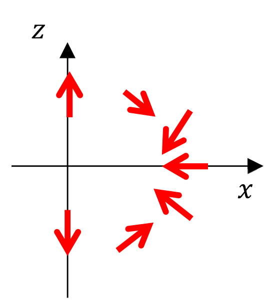
$$\vec{E}(z) = 6\frac{q\,a^2}{4\pi\epsilon_0}\frac{1}{z^4}\,sgn(z)\,\hat{z}$$

On the x-axis, $\theta = \frac{\pi}{2}$, the field is in the $-\hat{r} = -\hat{x}$ direction:

$$\vec{E}(x) = -\,3\frac{q\,a^2}{4\pi\epsilon_0}\frac{1}{x^4}\,\hat{x}$$

We also see that the radial field is zero at the angles found above, $\cos\theta_0 = \pm\frac{1}{\sqrt{3}}$. In fact, the field is radially outward for $\cos^2\theta > \frac{1}{3}$ and radially inward for $\cos^2\theta < \frac{1}{3}$

$$E_r\,(r,\theta) > 0$$



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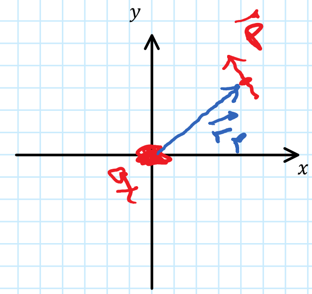

2) Charge-Dipole interaction:

Consider a point charge Q at the origin $\vec{r} = 0$ and an electric point dipole $\vec{p}$ at a position $\vec{r}_p$ (not at the origin). Calculate:

a) The potential energy of the Q & $\vec{p}$ system, by:
   i. Calculate the potential energy of $\vec{p}$ in the electric potential of Q
   ii. Calculate the potential energy of Q in the electric potential of $\vec{p}$.
   iii. Show these are equal.

i) $\vec{E}_q(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r}$

$U_p = -\vec{p} \cdot \vec{E}_q(\vec{r}_p) = -\frac{q}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}_p}{r_p^3}$ dipole potential energy

ii) $\phi_p(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot (\vec{r} - \vec{r}_p)}{|\vec{r} - \vec{r}_p|^3}$

$U_q = q\,\phi_p(\vec{r} = 0) = \frac{q}{4\pi\epsilon_0} \frac{\vec{p} \cdot (-\vec{r}_p)}{r_p^3}$ charge potential energy

iii) $U_p = U_q = -\frac{q}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}_p}{r_p^3}$

b) The forces on $\vec{p}$ and Q, by:
   i. Calculate the force on $\vec{p}$ due to the field of Q
   ii. Calculate the force on Q due to the field of $\vec{p}$
   iii. Compare the results.

i) $\vec{F}_p = \vec{\nabla}(\vec{p} \cdot \vec{E}) = \frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{\vec{p} \cdot \vec{r}}{r^3}$

$= \frac{q}{4\pi\epsilon_0} \hat{e}_i \partial_i \frac{p_j r_j}{(r_k r_k)^{3/2}}$

$= \frac{q}{4\pi\epsilon_0} \hat{e}_i p_j \left(\frac{\delta_{ij}}{(r_k r_k)^{3/2}} - r_j \frac{3}{2} \frac{2 r_k \delta_{ik}}{(r_k r_k)^{5/2}} \right)$

$= \frac{q}{4\pi\epsilon_0} \left(\frac{\vec{p}}{r_p^3} - 3 \frac{(\vec{p} \cdot \vec{r}_p)\vec{r}}{r_p^5} \right)$

ii) $\vec{F} = q\,\vec{E}_p(\vec{r} = 0)$

The dipole field is

$\vec{E}_p(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left\{ 3 \frac{(\vec{p} \cdot (\vec{r} - \vec{r}_p))(\vec{r} - \vec{r}_p)}{|\vec{r} - \vec{r}_p|^5} - \frac{\vec{p}}{|\vec{r} - \vec{r}_p|^3} \right)$

$\vec{E}_p(0) = \frac{1}{4\pi\epsilon_0} \left(3 \frac{(\vec{p} \cdot (-\vec{r}_p))(-\vec{r}_p)}{r_p^5} - \frac{\vec{p}}{r_p^3} \right)$

$$E_r(0) = \frac{1}{4\pi\epsilon_0}\left(3\frac{(\bar{p}\cdot\bar{r}_p)\,\bar{r}_p}{r_p^5} - \frac{\bar{p}}{r_p^3}\right)$$

$$\bar{F}_q = \frac{q}{4\pi\epsilon_0}\left(3\frac{(\bar{p}\cdot\bar{r}_r)\,\bar{r}_r}{r_p^5} - \frac{\bar{p}}{r_p^3}\right)$$

iii) $\bar{F}_p = -\bar{F}_q \Rightarrow$ Newton 3^{rd}

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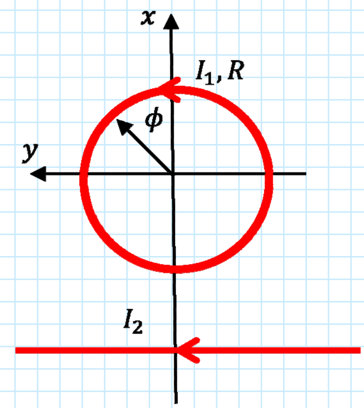


3) Magnetic Force on a Loop:
Consider the two currents shown: (i) a circular loop of radius R with counterclockwise current I_1 centered at the origin in the x-y plane

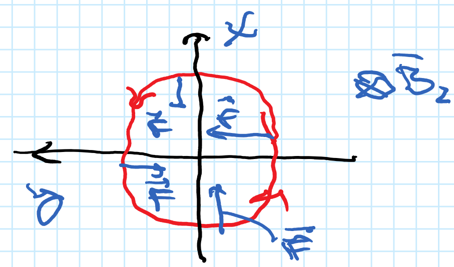
and (ii) a very long straight wire with current I_2 parallel to the y-direction at $z = 0, x = -d$.

We're going to find solutions for the force on the loop, first by integration and then by considering the potential energy.

a) Predict the direction of the total force on the loop due to the magnetic field of the long-straight wire. Explain your prediction.

The B-field, $\bar{B}_2$, is in the $-\hat{z}$ direction through the loop.

$$(\vec{B} \sim d\vec{l} \times \vec{r} = \hat{y} \times \hat{x} = -\hat{z})$$



The force on the loop is everywhere towards the center of the loop

The forces in the y-direction cancel

The forces in the x-direction are smaller at the top of the loop, larger at the bottom because the field is smaller farther from the wire

Total force is in the +x direction

b) What is the magnetic field $\vec{B}_2(\vec{r})$ (magnitude and direction) due to the long wire everywhere in the x-y plane?

Ampere's Law for a circle of radius r:

$$\oint \bar{B} \cdot d\vec{l} = \mu_0 I_2$$

$$B \cdot 2\pi r = \mu_0 I_2$$

$$B(r) = \frac{\mu_0 I_2}{2\pi r}$$

Using the geometry given

$$r = d + x, \text{ so}$$

$$\vec{B}(x) = -\frac{\mu_0 I_2}{\quad} \hat{z}$$

$2\pi(d+x)$

c) The force on the loop due to the magnetic field of the wire is:

$$\vec{F}_{21} = I_1 \oint d\vec{l} \times \vec{B}_2$$

It should be clear that we want to do this integral over the circle by integrating over $d\phi$.

i)Derive expressions for $\vec{B}_2(\vec{r})$ and$d\vec{l}$ in terms of R, ϕ, d, and $d\phi$ (for points on the current loop). Write these in terms of the $\hat{x}$ and $\hat{y}$ components.

Check your answer at a few simple points (such as for the angles $\phi = 0, \phi = \frac{\pi}{2}, \dots$)

iii) Write out an integral that gives the force on the loop. You should simplify this as much as possible, but you don't need to solve it, as it's somewhat messy.

Does the direction agree with your prediction?

Using the coordinates given

$$d\vec{l} = R\,d\phi\,\hat{\phi} = R\,d\phi\,(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y})$$

$$\vec{B}(\vec{r}) = -\frac{\mu_0 I_2}{2\pi(d + R\cos\phi)}\hat{z}$$

$$\vec{F} = \frac{\mu_0 I_1 I_2}{2\pi} R \oint d\phi \frac{(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) \times (-\hat{z})}{d + R\cos\phi}$$

$$= -\frac{\mu_0 I_1 I_2}{2\pi} R \oint d\phi \frac{\sin\phi\,\hat{y} + \cos\phi\,\hat{x}}{d + R\cos\phi}$$

Considering the $\hat{y}$ integral

$$\int_{-\pi}^{\pi} d\phi \frac{\sin\phi}{d + R\cos\phi} = 0$$ by symmetry

$\sin(-\phi) = -\sin\phi$

$\cos(-\phi) = \cos\phi$

For the $\hat{x}$ integral

$$\int_{-\pi}^{\pi} d\phi \frac{\cos\phi}{d + R\cos\phi} = 2\int_0^{\pi} d\phi \frac{\cos\phi}{d + R\cos\phi}$$

This integral is negative because the magnitude of the integrand is larger for $\frac{\pi}{2} < \phi < \pi$ where $\cos\phi < 0$

$$\left|\int_0^{\frac{\pi}{2}} d\phi \frac{\cos\phi}{d + R\cos\phi}\right| < \left|\int_{\frac{\pi}{2}}^{\pi} d\phi \frac{\cos\phi}{d + R\cos\phi}\right|$$

because of the $d + R\cos\phi$ in the denominator.

$$\vec{F} = \frac{\mu_0 I_1 I_2}{\quad} \int^{2\pi} \frac{d\phi}{\quad} = +|F|\hat{x}$$

$$\vec{F} = -\frac{\mu_0 I_1 I_2}{2\pi} R\,\hat{x} \int_0^{2\pi} d\phi \frac{\cos\phi}{R + d\cos\phi} = +|F|\hat{x}$$

d) Another approach to this problem is to determine the potential energy of the loop due to the magnetic field of the wire, U_{Loop},and then use that the force is $F = -\vec{\nabla} U_{Loop}$. In this case you're interested in the change in energy as the loop moves relative to the wire

$F_x = -\partial_d U_{Loop}$

i) A current loop is equivalent to a sheet of small current loops, each a magnetic dipole (Fig. 4.7 in the textbook). Using small loops, $d\vec{S} = dA\,\hat{n}$, gives magnetic dipoles

$d\vec{m} = I\,dA\,\hat{n}$

The potential energy of the magnetic dipoles

$dU(\vec{r}) = -d\vec{m} \cdot \vec{B}_2(\vec{r})$

Write a surface integral for the potential energy of the loop. Use polar coordinates.

ii) Write the field $\vec{B}_2(\vec{r})$ everywhere inside the loop using polar coordinates. (This is a simple extension to part c above.)

iii) Solve your integral to determine U_{Loop} and take the derivative to get ~~F_y~~ F_x. Does your result make sense? It might be useful to know that:

$$\int_0^{2\pi} \frac{d\phi}{d + r\cos\phi} = \frac{2\pi}{\sqrt{d^2 - r^2}}$$

$$d\vec{m} = I_1 r\,dr\,d\phi\,\hat{z}$$

$$\vec{B}_2(\vec{r}) = -\frac{\mu_0 I_2}{2\pi} \frac{\hat{z}}{(d+x)} = -\frac{\mu_0 I_2}{2\pi} \frac{\hat{z}}{(d + r\cos\phi)}$$

$$dU = \frac{\mu_0 I_1 I_2}{2\pi} \frac{r\,dr\,d\phi}{d + r\cos\phi}$$

$$U = \frac{\mu_0 I_1 I_2}{2\pi} \int_0^R r\,dr \int_0^{2\pi} \frac{d\phi}{d + r\cos\phi}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi} \int_0^R r\,dr \frac{2\pi}{\sqrt{d^2 - r^2}}$$

$u = d^2 - r^2$
$du = -2r\,dr$

$$= \frac{\mu_0 I_1 I_2}{2} \int_{d^2-R^2}^{d^2} \frac{du}{\sqrt{u}}$$

$$= \frac{\mu_0 I_1 I_2}{\cancel{2}} \cancel{2}\left(d - \sqrt{d^2 - R^2}\right)$$

$$F_x = -\frac{\partial u}{\partial d} = -\mu_0 I_1 I_2 \left(1 - \frac{d}{\sqrt{d^2 - R^2}}\right)$$

$$F_x = \mu_0 I_1 I_2 \left(\frac{d}{\sqrt{d^2 - R^2}} - 1\right)$$

Because $\frac{d}{\sqrt{d^2 - R^2}} > 1$ F_x is in the $+\hat{x}$ direction

$F_x \to 0$ as $d \to \infty$

F_x gets large as $d \to R$

iv) Consider your result for the force on the loop in the limit that $d \gg R$. Show that this is equivalent to the force on a magnetic point dipole due to the magnetic field of the wire. Remember that the force on a dipole is

$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$

For $d \gg R$

$$F_x = \mu_0 I_1 I_2 \left(\frac{1}{\sqrt{1 - \frac{R^2}{d^2}}} - 1 \right)$$

$$\approx \mu_0 I_1 I_2 \left(1 + \frac{1}{2}\frac{R^2}{d^2} + \cdots - 1 \right)$$

$$F_x \approx \frac{\mu_0}{2} I_2 \frac{I_1 R^2}{d^2}$$

$$= \frac{\mu_0 I_2}{2\pi d^2} (I_1 \pi R^2)$$

$$= \frac{\mu_0 I_2 m_1}{2\pi d^2}$$

But the dipole approximation gives

$$F = \nabla(\vec{m}_1 \cdot \vec{B}) = \vec{\nabla}\left(m_1 \hat{z} \cdot \left(\frac{\mu_0 I_2}{2\pi d} (-\hat{z}) \right) \right)$$

$$= -\vec{\nabla} \frac{\mu_0 I_2}{2\pi d} m_1$$

$$= -\partial_d \frac{\mu_0 I_2}{2\pi d^2} m_1 = \frac{\mu_0 I_2 m_1}{2\pi d^2}$$

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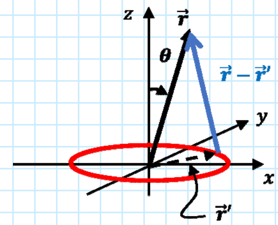
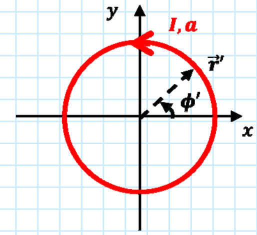




4) Back to the Ring:

Consider one more time the current-carrying ring considered in class. The ring is in the x-y plane with radius a and a counter-clockwise current I. (Bottom picture)

As before, we want to calculate the magnetic field at a point

$\vec{r} = r\cos(\theta)\ \hat{z} + r\sin(\theta)\ \hat{x}$

In this case we'll look at this problem using a multipole expansion in spherical harmonics. We'll use the result:

$$\frac{1}{|\vec r - \vec r'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{a^l}{r^{l+1}} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi)$$

To be exact, this should be r'^l

a) Write down (or look up from class) and expression for the current density of the ring, $\vec{J}(\vec{r}')$ in terms of the spherical coordinates r', θ', ϕ'.

Remember that for a 1D distribution you'll need two δ-functions.

The current is at $r' = a$, so it must have a term $\delta(r'-a)$

It is also at $\theta' = \frac{\pi}{2}$. This gives a term $\frac{1}{r'}\delta(\theta' - \frac{\pi}{2})$

The $\frac{1}{r'}$ is to give the units of $\frac{1}{r}$ and so the integral over $r\,d\theta$ works out

$$\vec J(\vec r') = I\,\delta(r'-a)\,\frac{1}{r'}\,\delta(\theta' - \tfrac{\pi}{2})\,\hat\phi'$$

b) Using the definition for the vector potential:

$$\vec{A}(\vec r) = \frac{\mu_0}{4\pi}\iiint \vec{J}(\vec r')\frac{1}{|\vec r - \vec r'|}\,d^3r'$$

Write down the multipole (spherical harmonics) expansion for the vector potential. This should still include sums over l and m and the volume integral over $\vec r'$. Remember to include the vector direction(s) of the current density.

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\sum_l \sum_m \frac{4\pi}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\iiint r'^2 dr' \sin\theta' d\theta' d\phi'$$

$$\times\, r'^l\, I\,\delta(r'-a)\,\frac{1}{r'}\,\delta(\theta' - \tfrac{\pi}{2})\,\hat\phi'\, Y^*_{lm}(\theta', \phi')$$

c) Perform the integrals over dr' and $d\theta'$ to determine an expression for $\vec{A}(\vec r)$ just in terms of an integral on $d\phi'$. Of course, you'll still have the sums over l and m.

$$\vec A(\vec r) = \frac{\mu_0 I}{4\pi}\sum_l \sum_m \frac{4\pi}{2l+1}\frac{a^{l+1}}{r^{l+1}} Y_{lm}(\theta,\phi)\int_0^{2\pi} d\phi'\, Y^*_{lm}(\tfrac{\pi}{2}, \phi')\,\hat\phi'$$

The integrals we'll need to do are:

$$\int_0^{2\pi} d\phi'\, Y^*_{lm}(\tfrac{\pi}{2}, \phi')\,\hat\phi'$$

$$= \int_0^{2\pi} d\phi'\, Y^*_{lm}(\tfrac{\pi}{2}, \phi')(-\sin\phi'\,\hat x + \cos\phi'\,\hat y)$$

$$= \int_0^{2\pi} d\phi' \, Y^*_{\ell m}\left(\frac{\pi}{2}, \phi'\right)\left(-\sin\phi' \hat{x} + \cos\phi' \hat{y}\right)$$

d) Show that the $l = 0$ term in the expansion is zero.

$$Y^*_{00}\left(\frac{\pi}{2}, \phi'\right) = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

giving the integral $\frac{1}{2}\frac{1}{\sqrt{\pi}} \int_0^{2\pi} d\phi' \left(-\sin\phi' \hat{x} + \cos\phi' \hat{y}\right)$

$$= 0$$

e) Calculate the $l = 1$ term in the multipole expansion. This will include the sum over $m = 0, \pm 1$.

For $\ell = 1$, $Y^*_{10}\left(\frac{\pi}{2}, \phi'\right) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\left(\frac{\pi}{2}\right) = 0$

$$Y^*_{11}\left(\frac{\pi}{2}, \phi'\right) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\left(\frac{\pi}{2}\right) e^{-i\phi'} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{-i\phi'}$$

$$Y^*_{1-1}\left(\frac{\pi}{2}, \phi'\right) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{i\phi'}$$

Doing the integrals we have integrals like

$$\int_0^{2\pi} e^{\pm i\phi} \sin\phi = \mp \frac{\pi}{i}$$

$$\int_0^{2\pi} e^{\pm i\phi} \cos\phi = \pi$$

The $m=1$ term is

$$C_{+1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \int_0^{2\pi} d\phi' e^{-i\phi'} \left(-\sin\phi' \hat{x} + \cos\phi' \hat{y}\right)$$

$$= -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \left(-\frac{\pi}{i}\hat{x} + \pi\hat{y}\right)$$

The $m=-1$ term is

$$C_{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \int_0^{2\pi} d\phi' e^{i\phi'} \left(-\sin\phi' \hat{x} + \cos\phi' \hat{y}\right)$$

$$= \frac{1}{2}\sqrt{\frac{3}{2\pi}} \left(\frac{\pi}{i}\hat{x} + \pi\hat{y}\right)$$

The $\ell=1$ term will be:

$$A^{(1)}(\vec{r}) = \frac{\mu_0 I}{3}\frac{a^2}{r^2}\left(Y_{11}(\theta,\phi)\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(\frac{\pi}{i}\hat{x} - \pi\hat{y}\right) + Y_{1-1}(\theta,\phi)\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(\frac{\pi}{i}\hat{x} + \pi\hat{y}\right)\right)$$

$$= \frac{\mu_0 I}{3}\frac{a^2}{r^2}\Big[-\frac{3}{8\pi}\sin\theta\, e^{i\phi}\left(\frac{\pi}{i}\hat{x}-\pi\hat{y}\right)$$
$$+\frac{3}{8\pi}\sin\theta\, e^{-i\phi}\left(\frac{\pi}{i}\hat{x}+\pi\hat{y}\right)\Big]$$
$$= \frac{\mu_0 I}{8}\frac{a^2}{r^2}\sin\theta\left(\frac{1}{i}\left(-e^{i\phi}+e^{-i\phi}\right)\hat{x}+\left(e^{i\phi}+e^{-i\phi}\right)\hat{y}\right)$$
$$= \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin\theta\left(-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}\right)$$
$$= \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin\theta\,\hat{\phi}$$

So the $l=1$ term will only have an $A_\phi(r,\theta)$

f) Calculate the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$. Compare your results to those found in class, including the results for $\vec{B}(\vec{r} = z\,\hat{z})$ and for $r \gg a$.

Because $\vec{A}(\vec{r}) = A_\phi(\vec{r})\,\hat{\phi}$, many terms in the curl will be zero. Doing the curl

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$B_r = \frac{1}{r\sin\theta}\partial_\theta \sin\theta\, A_\phi$$
$$= \frac{1}{r\sin\theta}\partial_\theta \frac{\mu_0 I}{4}\frac{a^2}{r^2}\sin^2\theta$$
$$= \frac{\mu_0 I}{2}\frac{a^2}{r^3}\cos\theta$$

$$B_\theta = -\frac{1}{r}\partial_r\, r A_\phi$$
$$= -\frac{1}{r}\partial_r \frac{\mu_0 I}{4}\frac{a^2}{r}\sin\theta$$
$$= \frac{\mu_0 I}{4}\frac{a^2}{r^3}\sin\theta$$

Letting $m = I\pi a^2$

$$\vec{B}(\vec{r},\theta) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right)$$

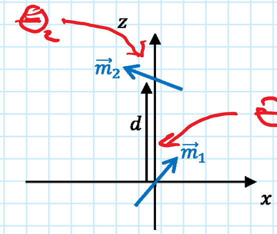
For $\vec{r} = z\hat{z}$, $\theta = 0$, $\hat{r} = \hat{z}$

$$B = \frac{\mu_0 m}{2\pi z^3}\hat{z}$$

as found in class

Hw2-5

Tuesday, March 8, 2022 3:30 PM



5) Magnetic Dipole Interactions in a Magnetic Field:
NOTE: In this problem, you can assume for all your answers that $\vec{m}_1$ and $\vec{m}_2$ are either parallel or anti-parallel. I'm fairly sure that this has to be true, but there are still a couple of cases where I need a more complete proof. If you want to prove this, please do. If not, you can assume it.

Two identical magnetic dipoles are shown, $\vec{m}_1$ at the origin and $\vec{m}_2$ at $\vec{r} = d\,\hat{z}$. The magnetic dipoles are free to rotate in the x-z plane (they don't have components in $\hat{y}$). To simplify the solution to this problem, define a quantity related to the magnetic field due to the dipoles:

$$B_d = \frac{\mu_0}{4\pi}\frac{m}{d^3}, |\vec{m}_1| = |\vec{m}_2| = m$$

There is a uniform, constant magnetic field $\vec{B}$ in the x-z plane.

a) Write down an expression for the total potential energy of the two dipoles interacting with each other and the magnetic field.

The interaction between dipoles is

$$U_{mm} = -\vec{m}_2 \cdot \vec{B}_1(\vec{r}_2)$$

$$= -\vec{m}_2 \cdot \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1\cdot\hat{r}_2)\hat{r}_2}{r_2^3} - \frac{\vec{m}_1}{r_2^3}\right)$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r_2^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r}_2)(\vec{m}_2\cdot\hat{r}_2)\right)$$

but $|r_2| = d$ $\hat{r}_2 = \hat{z}$ giving

$$U_{mm} = \frac{\mu_0}{4\pi d^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3m_{1z}m_{2z}\right)$$

The total potential Energy is, therefore

$$U = \frac{\mu_0}{4\pi d^3}\left(\vec{m}_1\cdot\vec{m}_2 - 3m_{1z}m_{2z}\right) - (\vec{m}_1+\vec{m}_2)\cdot\vec{B}$$

$$= \frac{\mu_0}{4\pi d^3}\left(m_{1x}m_{2x} - 2m_{1z}m_{2z}\right) - (\vec{m}_1+\vec{m}_2)\cdot\vec{B}$$

b) First, consider the case where $\vec{B} = 0$. What are the are the configurations of the dipoles that give the lowest potential energy (there are two)? Show that adding a magnetic field $\vec{B} = B_{ext}\,\hat{z}$ will result in one configuration having the lowest energy.

For $\vec{B} = 0$ $\quad U = \frac{\mu_0}{4\pi d^3}\left(m_{1x}m_{2x} - 2m_{1z}m_{2z}\right)$

The components have limits

$$-m \le \begin{matrix} M_{1x} \\ M_{1z} \end{matrix} \le m$$

The lowest energy is when the negative term is largest and the positive term is zero - the dipoles are parallel and in the $\pm\hat{z}$ direction

$$\vec{m}_1 = \vec{m}_2 = m(\pm\hat{z})$$

If $\vec{B} = B_{ext}\hat{z}$, there is an extra term:

$$U = -2\frac{\mu_0 m^2}{4\pi d^3} - (m_{1z} + m_{2z})B_{ext}$$

$$= -2B_d m - (m_{1z} + m_{2z})B_{ext}$$

Lowest energy is for $\vec{m}_1 = \vec{m}_2 = m\hat{z}$

c) If instead the magnetic field is $\vec{B} = B_{ext}\,\hat{x}$, the lowest energy configuration of the dipoles will depend on the magnitude of the field.

i) If $B_{ext} \ll B_d$ what do you expect the lowest-energy configuration of the dipoles to be? If $B_{ext} \gg B_d$ what do you expect the lowest-energy configuration of the dipoles to be? Explain.

ii) Assuming $\vec{m}_1$ and $\vec{m}_2$ remain parallel, determine the lowest energy configuration of the two dipoles as a function of the magnitude B_{ext}.

Show that there is a "critical" value, B_c, for the external field where the lowest-energy configuration changes abruptly to the dipoles being aligned with $\vec{B}$.

If $\vec{B} = B_{ext}\hat{x}$ we have

$$U = \frac{\mu_0}{4\pi d^3}(m_{1x}m_{2x} - 2m_{1z}m_{2z}) - (m_{1x} + m_{2x})B_{ext}$$

If $B_{ext} << B_d$ the first term is large so the minimum energy is again $\vec{m}_1 = \vec{m}_2 = m(\pm\hat{z})$

If $B_{ext} >> B_d$ the second term is large so the minimum energy is $\vec{m}_1 = \vec{m}_2 = m\hat{x}$

Assuming $\vec{m}_1$ & $\vec{m}_2$ are parallel we have

$$\vec{m}_1 = (\cos\theta_1\hat{z} + \sin\theta_1\hat{x})m$$

$$\vec{m}_2 = (\cos\theta_2\hat{z} + \sin\theta_2\hat{x})m$$

$$\theta_1 = \theta_2 = \theta$$

$$U = \frac{\mu_0 m^2}{4\pi d^3}(\sin^2\theta - 2\cos^2\theta) - 2\sin\theta\, m B_{ext}$$

$$= m\left(B_d(\sin^2\theta - 2\cos^2\theta) - 2B_{ext}\sin\theta\right)$$

$$= m\left(B_d(3\sin^2\theta - 2) - 2B_{ext}\sin\theta\right)$$

Find the minimum $\quad \frac{\partial U}{\partial\theta} = m\left(B_d(6\sin\theta\cos\theta) - 2B_{ext}\cos\theta\right)$

$$\frac{\partial U}{\partial \theta} = 0 \Rightarrow B_d(6 \sin\theta \cos\theta) - 2B_{ext}\cos\theta = 0$$

Two solutions

$\cos\theta = 0$ $\quad \vec{m}_1 + \vec{m}_2$ in $\hat{x}$

Large B_{ext}

or $3 B_d \sin\theta = B_{ext}$

$$\sin\theta = \frac{B_{ext}}{3 B_d}$$

$B_{ext} \to 0 \Rightarrow \theta = 0$ as expected

We see that the second solution is not possible if $B_{ext} > 3 B_d$ after which the only solution is $\theta = \frac{\pi}{2}$

Putting in our result for $\sin\theta$:

$$U_\theta = m\left(B_d(3\sin^2\theta - 2) - 2B_{ext}\sin\theta\right)$$

$$= m\left(B_d\left(3\frac{B_{ext}^2}{9B_d^2} - 2\right) - 2\frac{B_{ext}^2}{3B_d}\right)$$

$$= m\left(\frac{B_{ext}^2}{3B_d} - 2B_d - \frac{2}{3}\frac{B_{ext}^2}{B_d}\right)$$

$$= -m\left(\frac{B_{ext}^2}{3B_d} + 2B_d\right)$$

The energy for $\theta = \frac{\pi}{2}$

$$U_{\frac{\pi}{2}} = m(B_d - 2B_{ext})$$

These are equal if

$$B_d - 2B_{ext} = -\frac{B_{ext}^2}{3B_d} - 2B_d$$

$$3B_d^2 - 6B_{ext}B_d = -B_{ext}^2 - 6B_d^2$$

$$9B_d^2 - 6B_{ext}B_d + B_{ext}^2 = 0$$

$$(3B_d - B_{ext})^2 = 0$$

or as found above as our "critical" field

$$B_{ext} = 3B_d$$

For $0 \leq B_{ext} \leq 3B_d$

the ground state direction of the dipoles goes from $\theta = 0$ to $\theta = \frac{\pi}{2}$

For $B_{ext} > 3B_d$

the dipoles are fixed along the x-axis.