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Electrodynamics 1

CH. 14 ELECTROSTATICS AROUND CONDUCTORS LECTURE NOTES

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Chapter 14 Conductors

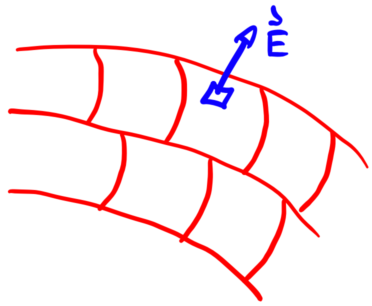
We have the current density defined as

$$\vec{J} = \sigma' \vec{E}$$

We know that inside the conductor

$$J = 0 \ , \ \vec{E} = 0 \ , \ \vec{\nabla} \cdot \vec{E} = 0 \ \Rightarrow \ \varphi \equiv \text{const.}$$

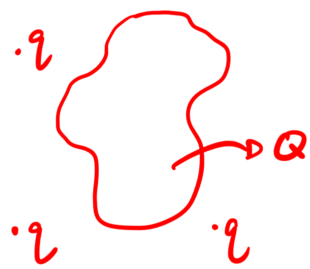
Outside of a conductor we know that $\vec{E} \perp$ to the surface of the conductor



We then know outside this conductor

$$\vec{E} \cdot \hat{n} = \frac{\sigma'}{\varepsilon_0} \Rightarrow -\vec{\nabla}\varphi \cdot \hat{n} = \frac{\sigma'}{\varepsilon_0} \quad \therefore \quad \frac{\partial \varphi}{\partial n} = -\frac{\sigma'}{\varepsilon_0}$$

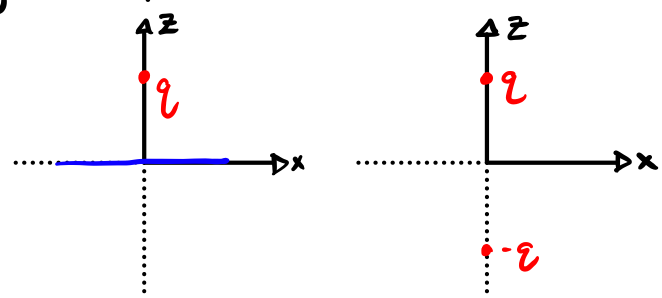
If we then have a source with surrounding charged point particles



We can then say

$$\nabla^2 \varphi(r) = -\frac{\rho}{\varepsilon_0}$$

If we have two charge set ups like



Where when we solve for φ where $z>0$, we will have the same solution for q with a conductor and with another charge $-q$. This principal is called The Uniqueness Theorem.

Looking at the conductor example with potential

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r}-\vec{d}|} - \frac{1}{|\vec{r}+\vec{d}|}\right) \quad \text{with } \vec{d} = d\hat{z}$$

The potential then becomes

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+(z+d)^2}}\right)$$

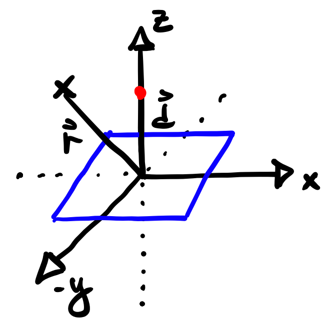
Where we can of course solve for $\vec{E}$ with

$$\vec{E} = -\vec{\nabla}\varphi \ , \quad \vec{E}_x = -\partial_x \varphi\, \hat{x} \ , \quad \vec{E}_z = -\partial_z \varphi\, \hat{z}$$

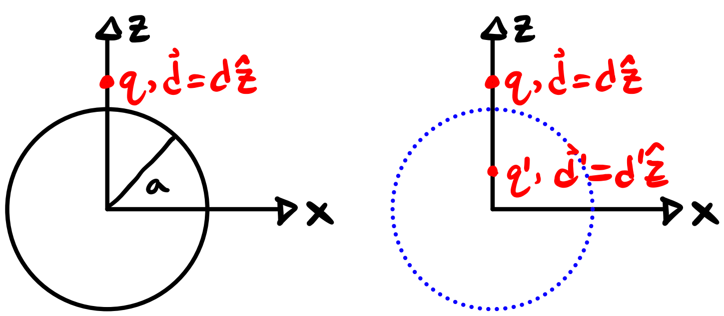
This then means the E field is

$$\vec{E}_z = \frac{q}{4\pi\varepsilon_0}\left(\frac{z-d}{(x^2+(z-d)^2)^{3/2}} + \frac{z+d}{(x^2+(z+d)^2)^{3/2}}\right)\hat{z}$$

Graphically this looks like



We now look at an example where we use images. Looking at a sphere with radius a



The potential at a is

$$\varphi(z=a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d-a} + \frac{q'}{a-d'}\right) = 0, \quad \varphi(z=-a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d+a} + \frac{q'}{d'+a}\right) = 0$$

Going through the math we find

$$d' = \frac{a^2}{d} \quad , \quad q' = q\frac{a}{d}$$

Where we constructed our equations for φ at the edges of the sphere on the z-axis. This is because we are doing it at known points.

The method of images is used to take a real life scenario and find a first order approximactio to a problem we always kno

Looking at Laplaces Equation

$$\nabla^2 \varphi = 0$$

This expanded is of cours

$$\partial_x^2 \varphi + \partial_y^2 \varphi + \partial_z^2 \varphi = 0 \implies \varphi = X(x)\,Y(y)\,Z(z)$$

where we then have

$$\left(YZ\,\partial_x^2 X + XZ\,\partial_y^2 Y + XY\,\partial_z^2 Z = 0\right)\frac{1}{\varphi}$$

$$\frac{1}{X(x)}\partial_x^2 X(x) + \frac{1}{Y(y)}\partial_y^2 Y(y) + \frac{1}{Z(z)}\partial_z^2 Z(z) = 0$$

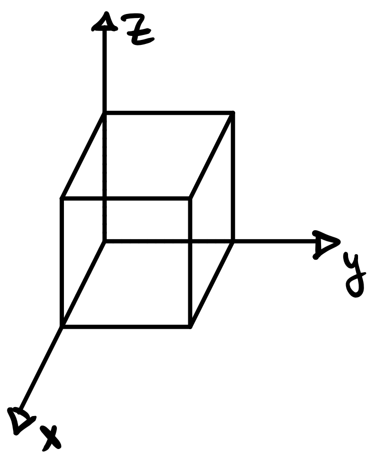
We can further say

$$\partial_x^2 X = a^2 x \quad , \quad \partial_y^2 Y = b^2 y \quad , \quad \partial_z^2 Z = c^2 Z$$

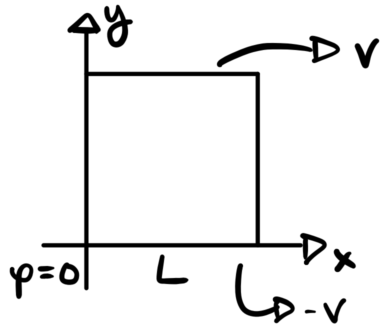
The general solution is then

$$X = A\sin(ax) + B\cos(ax) \quad , \quad Y = C\sin(bx) + D\cos(bx) \quad , \quad Z = Ee^{cz} + Fe^{-cz}$$

Which graphically looks like



Looking at a 2D case of this we have



Our solutions then become

$$\partial_x^2 X - a^2 X = 0 \quad , \quad \partial_y^2 Y + a^2 Y = 0 \Rightarrow \frac{1}{x}\partial_x^2 X + \frac{1}{y}\partial_y^2 Y = 0$$

The general solutions are now

$$X(x) = A\sin(ax) + B\cos(ax)$$

Applying Boundary conditions $X(0)=0$, $X(L)=0$ we have

$$X(x) = A\sin\left(\frac{n\pi}{L}x\right)$$

For the y-direction we have

$$\partial_y^2 Y + a^2 Y = 0$$

we then have after applying boundary conditions

$$Y(y) = C_n \sinh\left(\frac{n\pi}{L}y\right)$$

The potential is then

$$\varphi = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right)$$