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Classical Mechanics

Ch. 5 The Rigid Body Equations of Motion Lecture Notes

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Rigid Body Motion (Ch.5)

* Orthogonal transformations → rotations
* Euler angles
* Space-Fixed & body-Fixed frames
* Angular velocity

⇒ Example : Heavy symmetries

Angular Momentum & Kinetic Energy (5.1)

Chasles Theorem : The most general displacement of a rigid body can be generated by:

i) A translation along a line

ii) A rotation about an axis collinear w/ screw axis

Natural decomposistion of Kinetic energy:

$$T = \underbrace{\tfrac{1}{2} m \dot{\vec{R}}^2}_{\text{Translational contribution of COM}} + \underbrace{T'(\theta, \varphi, \psi)}_{\text{Rotational contribution}}$$

3 position co-ordinates , 3 Euler angles

Scenario: Rigid body with instantaneous angular velocity $\vec{\omega}$ about fixed point O.

Angular momentum, $\vec{L} = \sum_i m_i (\vec{r}_i \times \dot{\vec{r}}_i)$ $\vec{r}_i$ → relative to O

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i m_i (\vec{\omega} \vec{r}_i^{\,2} - \vec{r}_i(\vec{r}_i \cdot \vec{\omega}))$$

Think of cartesian elements,

$$L_x = \omega_x \sum_i m_i (r_i^2 - x_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$I_{jk} = \sum_i m_i [\delta_{jk} \vec{r}_i^{\,2} - (\vec{r}_i)_j (\vec{r}_i)_k]$$

↳ Elements of moment of inertia tensor, $\underset{\text{matrix}}{I}$: $\vec{L} = I\vec{\omega}$

Returning to K.E: $T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}^2 \rightsquigarrow (\vec{\omega} \times \vec{r}_i) = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i \cdot (\vec{\omega} \times \vec{r}_i)$

$$T = \tfrac{1}{2} \sum_i m_i \vec{\omega} \cdot (\vec{r} \times \dot{\vec{r}}_i) = \vec{\omega} \cdot \tfrac{1}{2} \vec{L} = \tfrac{1}{2} \vec{\omega} \cdot (I \vec{\omega})$$

Simplification: If $\vec{\omega} = \omega \hat{n}$, $T = \frac{1}{2} I \omega^2$, $I = \hat{n} \cdot (I \hat{n}) = \text{scalar}$: moment of inertia about axis of rotation

Principal Axis Transformation (5.4)

I_{jk} depends on :

* Origin & orientation of body-Fixed Frame.

$I_{jk} = I_{kj} \neq 0$

$\Rightarrow$ There is a set of body-Fixed axes For which $I_{jk}=0$!

<u>Consider:</u> * Origin of co-ordinate system is COM

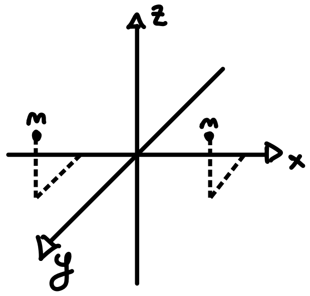
Then, I can be diagonalized, $I_D = RIR^T$, $I_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

* R can be given in terms of euler angles $\rightarrow$ it's a rotation

* $I_{1,2,3} \Rightarrow$ Principle moments of inertia

* Choice of body-fixed x'-y'-z': principle axes :
$\Downarrow$
Preferred co-ordinates System

Symmetries $\longrightarrow$ Identify principle axes



$I_{xy} = I_{xz} = 0$, $I_{xy} = [-Mx_0y_0 - m(-x_0)y_0]$

$x \rightarrow$ Principle axis

Generic Recipe : α-β-γ , symmetric w.r.t α-β plane $\Rightarrow$ γ is principle axis , $L = T - V$

<u>10-25-21</u>

Rate of Change Vector , body-Fixed <=> Space-Fixed

$$\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega} \times \vec{G}$$

$\vec{\omega}$ = angular velocity , $\vec{\omega}$ in body fixed frame : $\vec{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} \rightarrow L = T - V$

$L = T - V$
$\quad \hookrightarrow T_{com} + T_{ref}$

Angular momentum : $\vec{L} = I\vec{\omega}$, Principal axes $\longrightarrow I_D = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$

$T_{rot} = \frac{\vec{\omega}}{2} \cdot (I\vec{\omega})$

<u>Euler's Equations For a Rigid Body (5.5)</u>

2nd Law : $\left(\frac{d\vec{L}}{dt}\right)_{\text{Space}} = \vec{N}$ $\qquad \left(\frac{d\vec{L}}{dt}\right)_{\text{Body}} + \vec{\omega} \times \vec{L} = \vec{N}$

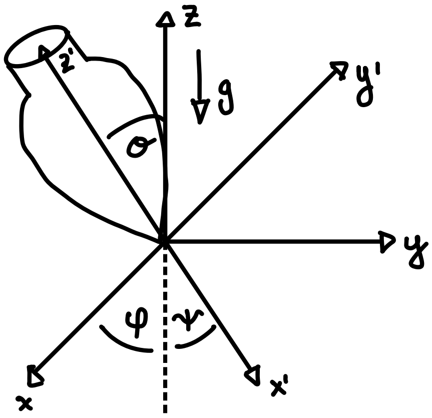
$\hookrightarrow$ Inertial frame

$\dot{L}_j + \sum_{k,l} \varepsilon_{jkl}\, \omega_k L_l = N_j$, Euler's equations : Q3

$I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 = N_1$, $I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_1 \omega_3 = N_2$, $I_3 \dot{\omega}_3 - (I_1 - I_2)\,\omega_1 \omega_2 = N_3$

Example: Heavy Symmetric Top

Diagram



Construct Lagrangian :

$\Rightarrow$ Potential energy : gravity.

$\Rightarrow$ Kinetic energy : rotation

Generalized co-ordinates : Euler angles : φ, θ, ψ

① Gravity : $V = mg z_{com} = mgL\cos(\theta)$

② Kinetic Term : $T = \frac{1}{2}\vec{\omega} \cdot (I\,\vec{\omega})$

So far, z is principal axis, $I_{xz} = I_{yz} = 0$: $I_{zz} \neq 0$, $I_{x'y'} = I_{y'x'} = 0$: $I_{xx} = I_{yy}$

$$T = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right), \quad \vec{\omega} = \begin{pmatrix} \dot{\phi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\ \dot{\phi}\sin(\theta)\cos(\psi) + \dot{\theta}\sin(\psi) \\ \dot{\phi}\cos(\theta) + \dot{\psi} \end{pmatrix}$$

$$T = \frac{I_1}{2}\left(\dot{\varphi}^2 \sin^2(\theta) + \dot{\theta}^2\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos(\theta)\right)^2$$

$\dot{\psi} \rightarrow$ Spin of top , $\dot{\theta} \longrightarrow$ nutation , $\dot{\varphi} \longrightarrow$ Precession

So far : $\Rightarrow$ Have $V, T \Rightarrow L = L(\theta, \varphi, \psi, \dot{\theta}, \dot{\varphi}, \dot{\psi})$

Identify : φ, ψ are cyclic co-ordinates

$\hookrightarrow$ Generalized momenta p_φ, p_ψ are conserved

$P_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos(\theta)) = I_3\omega_3$, $P_\varphi = \frac{\partial L}{\partial \dot\varphi} = (I_1\sin^2(\theta) + I_3\cos^2(\theta))\dot\varphi + I_3\dot\psi\cos(\theta)$

P_ψ & P_φ $\rightarrow$ can use to "eliminate" $\dot\varphi$ & $\dot\psi$

i) $\dot\varphi = \frac{P_\varphi - P_\psi\cos(\theta)}{I_1\sin^2(\theta)} = \dot\varphi(\theta)$

ii) $\dot\psi = \frac{P_\psi}{I_3} - \cos(\theta)\left[\frac{P_\varphi - P_\psi\cos(\theta)}{I_1\sin^2(\theta)}\right]$

In principle, we have :

i) $L = L(\theta, \dot\theta) \rightarrow$ EOM $\rightarrow \theta(t) \rightarrow$ obtain $\varphi(t)$ & $\psi(t)$

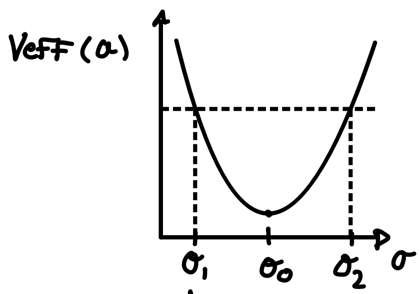
ii) $\dot\varphi$ & $\dot\psi$ as function of θ

Alternative ?

Conserved : $E = \frac{I_1}{2}(\dot\theta + \dot\varphi^2\sin^2(\theta)) + \frac{I_3}{2}\omega_3^2 + mgl\cos(\theta)$

$E' = \frac{I_1}{2}\dot\theta^2 + \frac{1}{2I_1}\left(\frac{P_\varphi - P_\psi\cos(\theta)}{\sin(\theta)}\right)^2 + mgl\cos(\theta)$

$\frac{I_1}{2}\dot\theta^2 \rightarrow$ K.E ; $\frac{1}{2I_1}\left(\frac{P_\varphi - P_\psi\cos(\theta)}{\sin(\theta)}\right)^2 + mgl\cos(\theta) \rightarrow$ P.E , $V_{eff}(\theta)$



$\theta_0 \rightarrow$ minima

θ_1, $\theta_2 \rightarrow$ Turning points

$\theta \in [\theta_1, \theta_2]$

$\dot\varphi \rightarrow$ Two cases

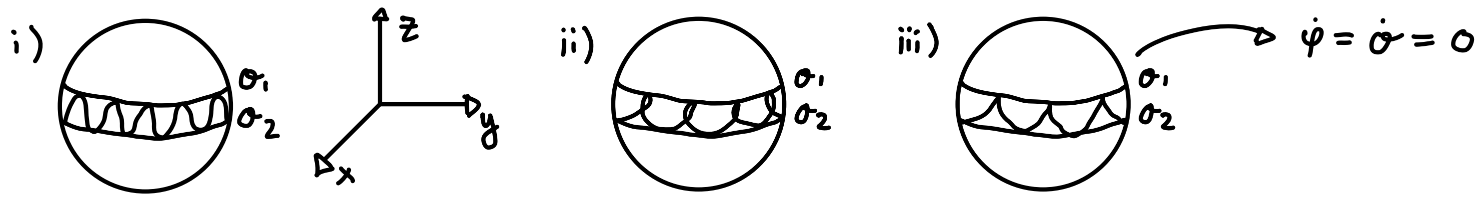
i) $P_\phi - P_\psi\cos(\theta) > 0 \rightarrow \dot\phi > 0$, If $[\theta_1, \theta_2]$ such that $\cos(\theta) < \frac{P_\phi}{P_\psi} \Rightarrow \dot\phi > 0$

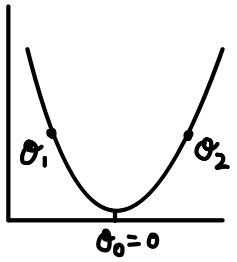
ii) $P_\varphi - P_\psi\cos(\theta) \rightarrow$ changes sign $\Rightarrow \dot\phi$ can change sign

i.e. $\cos(\theta) > \frac{P_\phi}{P_\psi}$ for some $\theta \in [\theta_1, \theta_2]$, Special behavior of ii) : sign

$$\left[\dot\phi\Big|_{\theta=\theta_1}\right] = -\text{sign}\left[\dot\phi\Big|_{\theta=\theta_2}\right]$$

Recall : $\dot\theta \Rightarrow$ Nutation , $\dot\varphi \Rightarrow$ Precession , $\dot\psi \Rightarrow$ Spin (about z')





$\dot\theta = 0$ For all t. $\dot\phi$ & $\dot\psi$ "Pure precession" $(\dot\theta = 0)$

$\left.\dfrac{\partial V_{eff}}{\partial\theta}\right|_{\theta=\theta_0} = 0$, Introduce $\beta = P_\varphi - P_\psi \cos(\theta_0)$

Solution of minima

$\hookrightarrow \beta = \dfrac{P_\psi \sin^2(\theta_0)}{2\cos(\theta_0)}\left(1 \pm \sqrt{1 - \dfrac{4mgLI_1\cos(\theta_0)}{P_\psi^2}}\right)$ $\beta \in \mathbb{R}$, condition on

① $\theta_0 < \pi/2 \longrightarrow \cos(\theta_0) > 0$, $P_\psi^2 \geq 4mglI_1\cos(\theta_0)$

$\hookrightarrow (I_3 w_3) \longrightarrow w_3 \geq \dfrac{2}{I_3}\sqrt{mgLI_1\cos(\theta_0)}$

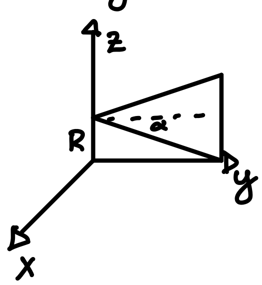
$\dot\phi = \dfrac{P_\varphi - P_\psi\cos(\alpha)}{I_1\sin^2(\alpha)}$, $\dot\phi|_{\theta_0} = \dfrac{\beta}{I_1\sin^2(\theta_0)} \rightsquigarrow \phi_+, \phi_- \longrightarrow$ slow $\dfrac{mgL}{I_3 w_3}$

$\hookrightarrow$ Fast $\dfrac{I_3 w_3}{I_1\cos(\theta_0)}$

② $\theta_0 > \pi/2 \longrightarrow \cos(\theta_0) < 0 \Rightarrow$ no condition for pure precession

11-1-21

Cone rolling on its side : cone rolling on its base



$R = h\tan(\alpha)$

Do everything in principle axis. Put principle axis through COM of cone

$\vec{w} = \Omega\begin{pmatrix} 0 \\ \sin(\alpha) \\ \cos(\alpha)\end{pmatrix} \rightsquigarrow$ In Body-Frame

Moment of Inertia : $I_1 = I_2$ & I_3

$T = \dfrac{I_3}{2}(\Omega\cos(\alpha))^2 + \dfrac{I_{1,2}}{2}(\Omega\sin(\alpha))^2 \Rightarrow$ use no slip condition $\longrightarrow V = Rw = h\dot\varphi$

$R\Omega\cos(\alpha) = h\dot{\varphi}$, $\cancel{h}\tan(\alpha)\Omega\cos(\alpha) = \cancel{h}\dot{\varphi}$, $\sin(\alpha)\Omega = \dot{\varphi}$ $\therefore$ $\Omega = \dot{\varphi}/\sin(\alpha)$

$$T = \frac{3}{40} M h^2 \left(\frac{1}{\cos^2(\alpha)} + 5 \right) \dot{\varphi}^2$$