Solutions to Homework 7
Physics 5393

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P-2.1 Consider the spin-precession problem discussed in the text. It can also be solved in the Heisenberg picture. Using the Hamiltonian

$$\tilde{\mathbf{H}} = -\frac{eB}{mc}\tilde{\mathbf{S}}_z = \omega\tilde{\mathbf{S}}_z$$

write the Heisenberg equations of motion for the time-dependent operators $\tilde{\mathbf{S}}_x(t)$, $\tilde{\mathbf{S}}_y(t)$, and $\tilde{\mathbf{S}}_z(t)$. Solve them to obtain $\tilde{\mathbf{S}}_{x,y,z}$ as functions of time.

The solution to this problem is a straightforward application of the Heisenberg equations of motion

$$\frac{d\tilde{\vec{\mathbf{S}}}}{dt} = \frac{1}{i\hbar}\left[\tilde{\vec{\mathbf{S}}}, \tilde{\mathbf{H}}\right] \quad \Rightarrow \quad \left\{\begin{aligned} \frac{d\tilde{\mathbf{S}}_x}{dt} &= \frac{eB}{mc}\tilde{\mathbf{S}}_y \\ \frac{d\tilde{\mathbf{S}}_y}{dt} &= -\frac{eB}{mc}\tilde{\mathbf{S}}_x \\ \frac{d\tilde{\mathbf{S}}_z}{dt} &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} \tilde{\mathbf{S}}_x(t) = \tilde{\mathbf{S}}_x(0)\left[a\sin\omega t + b\cos\omega t\right] \\ \tilde{\mathbf{S}}_y(t) = \tilde{\mathbf{S}}_y(0)\left[c\sin\omega t + d\cos\omega t\right] , \\ \tilde{\mathbf{S}}_z(t) = \text{constant}, \end{cases}$$

where the $\tilde{\mathbf{S}}_x$ and $\tilde{\mathbf{S}}_y$ equations are decoupled by taking appropriate derivatives to derive two second order differential equations and the angular momentum commutation relation $\left[\tilde{\mathbf{S}}_i, \tilde{\mathbf{S}}_j\right] = i\hbar\epsilon_{ijk}\tilde{\mathbf{S}}_k$; recall that the Levi-Civita symbol ϵ_{ijk} is defined as

$$\begin{aligned} &\epsilon_{123} = +1 \\ &\text{Even permutations of 1,2,3} = +1 \\ &\text{Odd permutations of 1,2,3} = -1 \\ &\text{If any indices are equal} = 0. \end{aligned}$$

P-2.3 An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$, the electron is known to be in an eigenstate of $\tilde{\vec{\mathbf{S}}} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector, lying in the x-y plane, that makes angle β with the z-axis.

a) Obtain the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.

Using the previously solved problem (P-1.7), the eigenstate at $t = 0$ and at a laeter time t in the $|S_z; \pm\rangle$ eigenkets are

$$\begin{aligned} |\alpha, 0\rangle &= \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle \\ |\alpha, 0; t\rangle &= e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|-\rangle . \end{aligned}$$

The $S_x = (1/2)\hbar$ state is given as

$$|S_x; +\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right).$$

Hence, the probability of being in the state $S_x = \hbar/2$ is

$$|\langle S_x;+|\alpha,0;t\rangle|^2 = \frac{1}{2}\left(1+\sin\beta\cos\omega t\right).$$

b) Find the expectation value of $\tilde{\mathbf{S}}_x$ as a function of time.

The expectation value is

$$\left\langle \alpha;t\left|\tilde{\mathbf{S}}_x\right|\alpha;t\right\rangle = \frac{\hbar}{2}\sin\beta\cos\omega t.$$

c) For your own peace of mind, show that your answers make good sense in the extreme cases (i) $\beta \to 0$ (ii) $\beta \to \pi/2$.

For $\beta = 0$, the spin at $t = 0$ is in the $+z$ direction so the probability of being in the $S_x = +$ direction is $1/2$ and the expectation value is zero.

If $\beta = \pi/2$, then the initial state is $S_x = +$. Therefore, the spin precesses about the z axis so the probability oscillates between one and zero, while the expectation value oscillates between $\pm\hbar/2$.

P-2.4 Derive the neutrino oscillation probability (2.1.65) and use it, along with the data in Fig. 2.2, to estimate the values of Δm^2c^4 (in units of eV^2) and θ.

To derive the time dependence of the probability that an electron neutrino is still an electron neutrino at a later time, we start by writing the most general form of the flavor eigenstates in the mass basis

$$\begin{aligned}
|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle \\
|\nu_\mu\rangle &= \cos\theta\,|\nu_2\rangle + \sin\theta\,|\nu_1\rangle.
\end{aligned}$$

The statement of the problem states that the system is in the $|\nu_e\rangle$ state at $t = 0$. The system is then evolved in time by applying the time evolution operator keeping in mind that the $|\nu_e\rangle$ is not an eigenstate of the Hamiltonian, but $|\nu_{1,2}\rangle$ are

$$\begin{aligned}
\boldsymbol{\mathcal{U}}(t)\,|\nu_e\rangle &= e^{-iE_1t/\hbar}\cos\theta\,|\nu_1\rangle - e^{-iE_2t/\hbar}\sin\theta\,|\nu_2\rangle \\
&= e^{-ipct/\hbar}\left[e^{-im_1^2c^3t/2p\hbar}\cos\theta\,|\nu_1\rangle - e^{-im_2^2c^3t/2p\hbar}\sin\theta\,|\nu_2\rangle\right].
\end{aligned}$$

The probability that the state at $t > 0$ is still $|\nu_e\rangle$ is

$$\begin{aligned}
\mathcal{P}(\nu_e\to\nu_e) &= |\langle\nu_e;0\,|\nu_e;t\rangle|^2 \\
&= \left|e^{-im_1^2c^3t/2p\hbar}\cos^2\theta + e^{-im_2^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \left|\cos^2\theta + e^{-i\Delta m^2c^3t/2p\hbar}\sin^2\theta\right|^2 \\
&= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\left[e^{i\Delta m^2c^3t/2p\hbar} + e^{-i\Delta m^2c^3t/2p\hbar}\right] \\
&= 1 - \frac{4}{2}\cos^2\theta\sin^2\theta\left[1-\cos\left(\frac{\Delta m^2c^3t}{2p\hbar}\right)\right] \\
&= 1 - \sin^2 2\theta\sin^2\left(\frac{\Delta m^2c^3t}{4p\hbar}\right),
\end{aligned}$$

where the following relation was used

$$\cos^4\theta + \sin^4\theta = \cos^2\theta(1-\sin^2\theta) + \sin^2\theta(1-\cos^2\theta) = 1 - 2\cos^2\theta\sin^2\theta.$$

The final steps are to convert the momentum to energy and time to a length

$$\left.\begin{array}{r} E = pc \\ L = ct \end{array}\right\} \quad\Rightarrow\quad \mathcal{P}(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 c^4 L}{4E\hbar}\right).$$

To calculate Δm^2, use the difference between the first maximum and first minimum, which allows the extraction of L/E over half a wavelength

$$\left.\begin{array}{l} \dfrac{L}{E} \approx 20 \text{ km/MeV} \\ \dfrac{\Delta m^2 c^4 L}{4E\hbar} = \pi \end{array}\right\} \quad\Rightarrow\quad \Delta m^2 c^4 = 4\pi\hbar c\frac{E}{L} \approx 1.2\times 10^{-4}\ \text{eV}^2.$$

The angle θ can be derived at the first minimum where the time dependent $\sin$ function is a maximum

$$\sin^2\left(\frac{\Delta m^2 c^3 t}{4p\hbar}\right) = 1 \quad\Rightarrow\quad 1 - \sin^2(2\theta) \approx 0.4 \quad\Rightarrow\quad \theta \approx 25^\circ.$$

P-2.9 Let $|a'\rangle$ and $|a''\rangle$ be eigenstate of a Hermitian operator $\tilde{\mathbf{A}}$ with eigenvalues a' and a'', respectively ($a' \neq a''$). The Hermitian operator is given by

$$\tilde{\mathbf{H}} = |a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|\,,$$

where δ is a real number.

a) Clearly, $|a'\rangle$ and $|a''\rangle$ are not eigenstates of the Hamiltonian. Write down the eigenstates of the Hamiltonian. What are their energy eigenvalues?

There are two approaches to solving this problem: The first is to express the Hamiltonian in a matrix representation, use the characteristic equation to derive the eigenvalues, and finally calculate the eigenvectors. An alternative approach is by solving the problem using a purely algebraic approach. Start with an assumed form for the eigenvectors

$$|E\rangle = \cos\theta\,|a'\rangle + \sin\theta\,|a''\rangle\,,$$

where the obvious constraint $\sin^2\theta + \cos^2\theta = 1$ will be used. The eigenvalue equation is then

$$\begin{aligned}\left[|a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|\right]&\left[\sqrt{1-\sin^2\theta}\,|a'\rangle + \sin\theta\,|a''\rangle\right] \\ &= E\left[\sqrt{1-\sin^2\theta}\,|a'\rangle + \sin\theta\,|a''\rangle\right].\end{aligned}$$

This equation can be expressed, using the orthogonality of the eigenkets, as two equations

$$\left.\begin{array}{r} \delta\sin\theta = E\sqrt{1-\sin^2\theta} \\ \delta\sqrt{1-\sin^2\theta} = E\sin\theta \end{array}\right\} \quad\Rightarrow\quad \left\{\begin{array}{l} E_\pm = \pm\delta \\ |E_\pm\rangle = \dfrac{1}{\sqrt{2}}\left[|a'\rangle \pm |a''\rangle\right] \end{array}\right.$$

b) Suppose the system is known to be in state $|a'\rangle$ at $t = 0$. Write down the state vector in the Schrödinger picture for $t > 0$.

At $t = 0$, the state in the energy representation is

$$|\alpha, 0\rangle = \left|a'\right\rangle = \frac{1}{\sqrt{2}}\left[|E_+\rangle + |E_-\rangle\right].$$

For time $t > 0$, the state is given by

$$|\alpha, 0; t\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\delta t/\hbar}\,|E_+\rangle + e^{+i\delta t/\hbar}\,|E_-\rangle\right]$$

c) What is the probability for finding the system in $|a''\rangle$ for $t > 0$ if the system is known to be in state $|a'\rangle$ at $t = 0$?

The probability of being in the state $|a''\rangle$ is

$$\left|\left\langle a''\,|\alpha, 0; t\right\rangle\right|^2 = \frac{1}{4}\left|\left[\langle E_+| - \langle E_-|\right]\left[e^{-i\delta t/\hbar}\,|E_+\rangle + e^{+i\delta t/\hbar}\,|E_-\rangle\right]\right|^2 = \sin^2\left(\frac{\delta t}{\hbar}\right)$$

d) Can you think of a physical situation corresponding to this problem?

This represents a classic two state problem.

Additional Problem

P-1 Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$ and $|u_3\rangle$. In this basis, the Hamiltonian operator $\tilde{\mathbf{H}}$ of the system and the two observable $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are written:

$$\tilde{\mathbf{H}} = \hbar\omega\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} \qquad \tilde{\mathbf{A}} = a\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad \tilde{\mathbf{B}} = b\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

The physical system at time $t = 0$ is in the state

$$|\alpha, 0\rangle = \frac{1}{\sqrt{2}}\,|u_1\rangle + \frac{1}{2}\,|u_2\rangle + \frac{1}{2}\,|u_3\rangle$$

a) At time $t = 0$, the energy of the system is measured. What values can be found and with what probabilities? Calculate, for the system in the state $|\alpha, 0\rangle$, the mean value $\left\langle\tilde{\mathbf{H}}\right\rangle$ and the RMS deviation ΔH.

The following energies can be measured:

$$E = \hbar\omega \qquad \mathcal{P} = \frac{1}{2}$$
$$E = 2\hbar\omega \qquad \mathcal{P} = \frac{1}{2}.$$

The expectation value of the energy is given by:

$$\left\langle\alpha, 0\left|\tilde{\mathbf{H}}\right|\alpha, 0\right\rangle = \left[\frac{1}{2} + 2\frac{1}{4} + 2\frac{1}{4}\right]\hbar\omega = \frac{3}{2}\hbar\omega$$

The RMS is given by $(\Delta\tilde{\mathbf{H}})^2 = \left\langle \tilde{\mathbf{H}}^2 \right\rangle - \left\langle \tilde{\mathbf{H}} \right\rangle^2$, therefore calculate $\left\langle \tilde{\mathbf{H}}^2 \right\rangle$;

$$\left\langle \alpha, 0 \left| \tilde{\mathbf{H}}^2 \right| \alpha, 0 \right\rangle = \left[\frac{1}{2} + 4\frac{1}{4} + 4\frac{1}{4} \right] \hbar\omega = 2.5\hbar\omega$$

giving:

$$\Delta\tilde{\mathbf{H}} = \sqrt{2.5 - 1.5}\hbar\omega = \hbar\omega$$

b) Instead of measuring $\tilde{\mathbf{H}}$ at time $t = 0$, one measures $\tilde{\mathbf{A}}$; what results can be found and with what probabilities? What is the state vector immediately after the measurement?
First of all, the eigenvalues of $\tilde{\mathbf{A}}$ are $+a$, $+a$ and $-a$. Next the eigenvectors of $\tilde{\mathbf{A}}$ are:

$$\begin{aligned} -a: \quad & |a_1\rangle = \frac{1}{\sqrt{2}} [|u_2\rangle - |u_3\rangle] \\ a: \quad & |a_2\rangle = \frac{1}{\sqrt{2}} [|u_2\rangle + |u_3\rangle] \\ a: \quad & |a_3\rangle = |u_1\rangle \end{aligned}$$

Therefore, the statevector at $t = 0$ is:

$$|\alpha, 0\rangle = \frac{1}{\sqrt{2}} |a_2\rangle + \frac{1}{\sqrt{2}} |a_3\rangle$$

A measurement of the observable $\tilde{\mathbf{A}}$, gives a with probability one.

c) Calculate the state vector $|\alpha, 0; t\rangle$ of the system at time t.
The time evolution of the statevector is given by:

$$|\alpha, 0\rangle = \frac{1}{\sqrt{2}} |u_1\rangle e^{-i\omega t} + \left[\frac{1}{2} |u_2\rangle + \frac{1}{2} |u_3\rangle \right] e^{-i2\omega t}$$

d) Calculate the mean values $\left\langle \tilde{\mathbf{A}}(t) \right\rangle$ and $\left\langle \tilde{\mathbf{B}}(t) \right\rangle$ of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ at time t. What comments can be made?
Start with the form of the statevector in terms of the eigenstates of $\tilde{\mathbf{A}}$. The expectation value is;

$$\left\langle \alpha; t \left| \tilde{\mathbf{A}} \right| \alpha, t \right\rangle = a$$

It is time independent since $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{A}}$ commute
We start by calculating the eigenvalues and vectors for operator $\tilde{\mathbf{B}}$. Using the standard technique of using the characteristic equation to extract the eigenvalue and then substituting them back in to calculate the eigenvectors, we find

$$\begin{aligned} -b: \quad & |b_1\rangle = \frac{1}{\sqrt{2}} [|u_1\rangle - |u_2\rangle] \\ b: \quad & |b_2\rangle = \frac{1}{\sqrt{2}} [|u_1\rangle + |u_2\rangle] \\ b: \quad & |b_3\rangle = |u_3\rangle . \end{aligned}$$

To include the time dependence, we also need the energy eigenvectors expressed in the $|b_i\rangle$ basis

$$\begin{aligned} |u_1\rangle &= \frac{1}{\sqrt{2}}\left[|b_1\rangle + |b_2\rangle\right] \\ |u_2\rangle &= \frac{1}{\sqrt{2}}\left[|b_2\rangle - |b_1\rangle\right] \\ |u_3\rangle &= |b_3\rangle\,. \end{aligned}$$

Therefore, the time dependent state vector in the b representation is

$$\begin{aligned} |\alpha;t\rangle &= \frac{e^{-i\omega t}}{2}\left(|b_1\rangle + |b_2\rangle\right) + \frac{e^{-i2\omega t}}{2\sqrt{2}}\left(|b_2\rangle - |b_1\rangle\right) + \frac{e^{-i2\omega t}}{2}\,|b_3\rangle \\ &= \frac{e^{-i\omega t}}{2}\left[\left(|b_1\rangle + |b_2\rangle\right) + \frac{e^{-i\omega t}}{\sqrt{2}}\left(|b_2\rangle - |b_1\rangle\right) + e^{-i\omega t}\,|b_3\rangle\right] \\ &= \frac{e^{-i\omega t}}{2}\left[\left(1 - \frac{e^{-i\omega t}}{\sqrt{2}}\right)|b_1\rangle + \left(1 + \frac{e^{-i\omega t}}{\sqrt{2}}\right)|b_2\rangle + e^{-i\omega t}\,|b_3\rangle\right] \end{aligned}$$

Using the orthogonality of the $|b_i\rangle$ and the eigenvalues of $\tilde{\mathbf{B}}$, the expectation value of $\tilde{\mathbf{B}}$ is:

$$\left\langle \alpha;t \left|\tilde{\mathbf{B}}\right| \alpha;t\right\rangle = \frac{b}{4}\left[-\left(\frac{3}{2} - \sqrt{2}\cos\omega t\right) + \left(\frac{3}{2} + \sqrt{2}\cos\omega t\right) + 1\right] = \frac{b}{4} + \frac{b}{\sqrt{2}}\cos\omega t$$

e) What results are obtained if the observable $\tilde{\mathbf{A}}$ is measured at time t? Same question for the observable $\tilde{\mathbf{B}}$. Interpret.

The observable $\tilde{\mathbf{A}}$ is independent of time, so remains the same. The observable $\tilde{\mathbf{B}}$ depends on time as given in the previous part.