Two-particle partition function for quantum particles

We want to compare Q_2 for the 3D isotropic harmonic oscillator for

(i) Boltzmann particles

(ii) identical bosons

(iii) identical fermions

Without symmetry:

"normal" way of writing the energy

$$E_{n_{x1} n_{y1} n_{z1} n_{x2} n_{y2} n_{z2}} = (n_{x1} + n_{y1} + n_{z1} + n_{x2} + n_{y2} + n_{z2} + 3) \times \hbar\omega$$

$$\Rightarrow Q_2^{\text{Boltzmann}} = \underbrace{Q_1 \cdot Q_1}_{\substack{\text{we have two independent} \\ \text{single particle problems}}} = Q_1^2$$

For identical particles, it's most convenient to switch to relative and center-of-mass coordinates:

$$E_{NLM\,n\ell m} = \Big(\underbrace{(2N + L)}_{\text{CM}} + \underbrace{(2n + \ell)}_{\text{rel}} + 3\Big)\hbar\omega = E_{rel} + E_{CM}$$

$n = 0, 1, 2, \ldots$
$\ell = 0, 1, 2, \ldots$
$N = 0, 1, 2, \ldots$
$L = 0, 1, 2, \ldots$

each energy level has a $2L+1$ or $2\ell+1$ degeneracy

For the center-of-mass degrees, we have no restriction

$$\Rightarrow \quad Q^{CM} = Q_1 = \Bigg(\underbrace{\frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{\beta\hbar\omega}}}_{\text{1D result}} \Bigg)^3$$

using our result from class

$$= \frac{e^{\frac{3}{2}\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^3}$$

We can calculate this in many ways:

$$Q_{CM} = \left(\sum_{N_x = 0,1,\ldots} e^{-\beta(N_x + \frac{1}{2})\hbar\omega} \right)^3$$

$$\text{or} \quad Q_{CM} = \sum_{\substack{N \\ N=0,1,2,\ldots}} \; \sum_{\substack{L \\ L=0,1,2,\ldots}} (2L+1)\, e^{-\beta(2N + L + \frac{3}{2})\hbar\omega}$$

For the relative degrees of freedom, we want to restrict the sum over l to even l for identical bosons and to odd l for identical fermions.

$$Q_{rel,\,bosons} = \sum_{n=0,1,\ldots} \sum_{\text{even } l} (2l+1)\, e^{-\beta(2n+l+\frac{3}{2})\hbar\omega}$$

$$= \frac{e^{\frac{5}{2}\beta\hbar\omega}\left(3+e^{2\beta\hbar\omega}\right)}{\left(e^{2\beta\hbar\omega}-1\right)^3}$$

$$Q_{rel,\,fermions} = \sum_{n=0,1,\ldots} \sum_{l \text{ odd}} (2l+1)\, e^{-\beta(2n+l+\frac{3}{2})\hbar\omega}$$

$$= \frac{e^{\frac{3}{2}\beta\hbar\omega}\left(1+3e^{2\beta\hbar\omega}\right)}{\left(e^{2\beta\hbar\omega}-1\right)^3}$$

$$\Rightarrow \quad Q_2^{bosons} = Q_{CM}\; Q_{rel,\,bosons}$$

$$Q_2^{fermions} = Q_{CM}\; Q_{rel,\,fermions}$$