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# Math Methods in Physics

CH. 5 HILBERT SPACE LECTURE NOTES


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Differential Equations $\rightarrow$ Discretize $\rightarrow$ Matrices

One must be a bit careful in infinite vector spaces - Consider the infinite vector

$\vec{x} = (1, 2, 3, 4, \ldots\ldots)$

$\rightsquigarrow$ can never normalize

$\rightsquigarrow$ Can never approximate

Linear Algebra problems can be unsolvable, $A\vec{x} = \vec{b}$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots\ldots \\ 0 & 1 & 2 & 3 & \ldots\ldots \\ 0 & 0 & 1 & 2 & \ldots\ldots \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

With some care we can restrict values to vector spaces without these problems.

we would like to be able to say some function $h(x)$, $h(x) = \sum_{n=0} a_n f_n(x)$

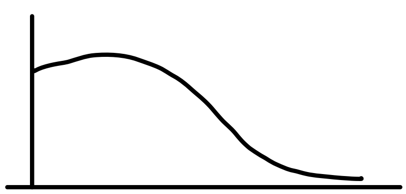
where $h_n(x) = \sum_{j=1} a_n f_n(x)$

(a) Pointwise convergence For all $x$ $\exists$ $N(\epsilon, x)$ such that of $n > N(\epsilon, x)$

$|h(x) - h_n(x)| < \epsilon$

(b) Uniform convergence $\forall x$ $\exists$ $N(\epsilon)$ such that for $N > N(\epsilon)$ $|h(x) - h_n(x)| < \epsilon$

(c) Convergence in the mean $\forall x$ $\exists$ $N(\epsilon)$ such that $\int_a^b |h(x) - h_n(x)|^2 \, dx < \epsilon$



$h(x)$ approaches $h_n(x)$ "Except on a set of measure zero"

11-19-21

Orthogonal Functions

Let's start with polynomials, our basis is $\{x^n\}$, $1, x, x^2, x^3, \ldots\ldots$

What about orthonormality?

$(f(x), g(x)) = \int_a^b w(x) f^* g \, dx$

Choose for simplicity $(f,g) = \int_{-1}^{1} f(x)g(x)\,dx$ , $\int_a^b f(z)g(z)\,dz$

$w = \frac{b-a}{2}$ , $x = \frac{z-2w}{w}$

Now do Gram-Schmidt normalization

$\tilde{P}_0(x) = 1$ , $(\tilde{P}_0, \tilde{P}_0) = \int_{-1}^{1} (1)^2 dx = 2 \quad \therefore \quad P_0(x) = \frac{1}{\sqrt{2}}\tilde{P}_0 = \frac{1}{\sqrt{2}}$

$\tilde{P}_1(x) = x - (P_0, x)P_0(x) = x - \overset{0}{\cancel{\int_{-1}^{1} \frac{1}{\sqrt{2}} x\,dx}} = x$

$(\tilde{P}_1(x), \tilde{P}_1(x)) = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \quad \therefore \quad P_1(x) = \sqrt{\frac{3}{2}}\,x$

$\tilde{P}_2(x) = x^2 - \overset{0}{\cancel{(P_1(x), x^2)}}P_1(x) - (P_0(x), x^2)P_0(x) = x^2 - \frac{1}{2}\int x^2\,dx = x^2 - \frac{1}{3}$

$(\tilde{P}_2(x), \tilde{P}_2(x)) = \int_{-1}^{1} (x^2 - 1/3)^2\,dx = \frac{8}{45} \quad \therefore \quad P_2(x) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$ , $P_3(x) = \sqrt{\frac{7}{5}}\left(\frac{5}{2}x^5 - \frac{3}{2}x\right)$

These are the Legendre Polynomials

Examples: Expand a "vector" $f(x) = e^{-2x^2}$ , $f_n(x) = \sum_{y=0}^{n} a_n P_n(x)$ , $a_n(x) = (P_n(x), f(x))$

$\int_{-1}^{1} P_n(x) e^{-2x^2} dx$

Just like there are many choices for orthonormal vector basis, we can have many orthonormal functions.

Physics $\rightsquigarrow$ Extremiz Quantities $\rightarrow$ Euler-Lagrange equations (often linear differential equations). In principle $\rightarrow$ discretize eigenvectors. So we generalize to look for orthonormal eigen functions of these linear operators.

In general we can get orthonormal functions.

Bessel Functions, Special Subset of orthonormal functions that are orthogonal polynomials, with a specific weight function in the inner product.

$(f(x), g(x)) = \int_a^b w(x) f(x) g(x)\,dx$ , $w(x) \longrightarrow$ weight function

$(f,g) = \int_0^{\infty} e^{-x} f(x) g(x)\,dx \rightsquigarrow$ Laguire polynomials

For $2^{nd}$ order linear DE there are only a finite set of orthogonal polynomials.

What about convergence?

Byron & Fuller prove we can expand any finite function on the interval $[-1,1]$ and the proof is as follows:

(A) Weiserstrauss Theorem

(i) There exists something called a $\delta$-function such that $\int_{-\infty}^{+\infty} f(x)\delta(x-x')\,dx' = f(x)$

Some examples you should know

$$\delta(x) = \lim_{c \to 0} \delta_c(x) = \begin{cases} 1/a & \text{for } |x| < a/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(x) = \lim_{a \to 0} \delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-x^2/a}$$

$$\delta(x) = \lim_{n \to \infty} \delta_n(x) = \frac{(2n+1)!}{2^{n+1}(n!)^2} (1-x^2)^n \quad \text{for } |x| < 1$$

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$$\lim_{n \to \infty} \frac{(2n+1)!}{2^{2n+1}(n!)^2} (1-x^2)^n$$

B & F prove that this is a $\delta$ function. Solve by definition

$$f(x) = \int_{-\infty}^{+\infty} f(x')\,\delta(x-x')\,dx'$$

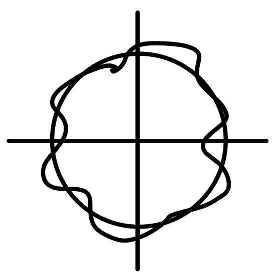
$\delta_n$ is a polynomial. We can always represent a sum of Legendre polynomials.

Is the convergence uniform? As we add terms do the earlier co-efficients decay?

$$f_{n_1}(x) = \sum_{n=0}^{n_1} a_n P_n(x) \quad , \quad f_{n_2}(x) = \sum_{n=0}^{n_2} b_n P_n(x) \quad \text{w/ } n_2 > n_1$$

The trick to the Legendre polynomials of order $(n+1)$ are orthogonal.

Any product can be drawn on circles



$g(\alpha) \to g(x,y)$

Then map $x \to \cos\theta$, $y \to \sin\theta$ & so I can represent $g(\alpha)$ as a sum of powers of $\sin\theta$ & $\cos\theta$ ? $\to$ re-write $\sin\theta$ & $\cos\theta$

Similarily for functions on a sphere $g(x,y,z) \to$ expand $P_n(x)P_{n'}(y)P_n(z)$ & then replace

$$x \to \cos\varphi\sin\theta \quad , \quad y \to \sin\varphi\sin\theta \quad , \quad z \to \cos\theta$$

## Legendre Polynomials

Consider the polynomial given by $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

Note that $P_n(x)$ is $n^{th}$ order polynomial. $P_n(x)$ has only odd powers of $n$ is odd
" even " " even

Also $(P_n(x), P_m(x)) = \int_{-1}^{1} P_n(x) P_m(x)\, dx = \delta_{mn}$

$P_0(x) = 1$ , $P_1(x) = x$

$$(P_n(x), P_m(x)) = \frac{1}{2^n n!} \frac{1}{2^m m!} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^m}{dx^m}(x^2-1)^m\, dx$$

Assume $n > m$ : Integrate by parts $n$ times, each time we do we generate a boundary term with a factor of $(x^2-1)$ on the boundary. $\rightsquigarrow (x^2-1)\big|_{-1}^{1} \to 0$

Eventually because $n > m$ I will take so many derivatives of $(x^2-1)^m \to 0$ $\therefore$ if $m \neq n$ we get zero

But if $n = m$

$$(P_n(x), P_n(x)) = \frac{1}{2^{2n}} \frac{1}{(n!)^2} \int_{-1}^{1} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

We integrate by parts $n$ times

$$= \frac{1}{2^n} \frac{1}{(n!)^2} (-1)^n \int_{-1}^{1} (x^2-1)^n \frac{d^{2n}}{dx^{2n}}(x^2-1)^n\, dx = (2n)!\, 2^{n+1} (n!)^2 (-1)^n/(2n+1)$$

We replace $x = \sin\theta$ , $(1-x^2) \rightsquigarrow \cos^2\theta$ & then lots of integrate by parts

This sort of formula for polynomials is called a Rodriguez formula and is common for all major orthogonal polynomials

Where do they come from? $\longrightarrow$ orthonormalizing $x^n$ $\rightsquigarrow$ odd deriv. formula
$\rightsquigarrow$ Satisfy Legendre Equation

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x} P_n(x)\right] + n(n+1) P_n(x) = 0$$

$\nabla^2$ in spherical co-ordinates

Not done. $\rightsquigarrow$ Show the ODE holds , $\rightsquigarrow$ Connection to $\nabla^2$

11-29-21

Why do you care? Often in physics we solve equations of the form

$$\nabla^2 f(\vec{r}) + V(\vec{r}) f(\vec{r}) = 0$$

And often

$$V(\vec{r}) = V(|\vec{r}|) = V(r)$$

Our PDE in Spherical co-ordinates is

$$\frac{\partial^2}{\partial r^2} f + \frac{1}{r^2 \sin^2\theta} \frac{\partial}{\partial \theta}\left( \sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \varphi^2} f + V(r) f = 0 \quad , \quad f(\vec{r}) = S(r) T(\theta) U(\varphi)$$

$$TU \frac{\partial^2}{\partial r^2} S + \frac{SU}{r^2} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left( \sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{ST}{r^2 \sin^2\theta} \frac{\partial^2 U}{\partial \varphi^2} + V(r) STU = 0$$

Multiply by $\frac{r^2 \sin^2\theta}{STU}$

$$\sin^2\theta \left\{ \frac{r^2}{S(r)} \left[ \frac{\partial^2}{\partial r^2} S + V(r) S(r) \right]\right\} + \frac{\sin\theta}{T} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial T}{\partial \theta} = \frac{-1}{U} \frac{\partial^2 U}{\partial \varphi^2}$$

Since RHS has only $\varphi$ dependence then LHS & RHS must be constants

$$\frac{1}{U} \frac{\partial^2 U}{\partial \varphi^2} = \text{constant} \quad , \quad \frac{\partial^2}{\partial \varphi^2} U = \text{constant } U$$

Since the constant is arbitrary, Choose $= -m^2 \quad \therefore \quad U = e^{im\varphi}$ w/ $m = 0, \pm 1, \pm 2$

Similarily,

$$\sin^2\theta \left\{ \frac{r^2}{S(r)} \left[ \frac{\partial^2}{\partial r^2} S(r) + V(r) S \right] + \frac{\sin\theta}{T} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial T}{\partial \theta} = m^2\right.$$

We can choose

$$\frac{r^2}{S(r)} \left[ \frac{\partial^2}{\partial r^2} S + V(r) S \right] = C_0 \quad , \quad \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left( \sin\theta \frac{\partial}{\partial \theta} T\right) + \left( C_0 - \frac{m^2}{\sin^2\theta}\right) T = 0$$

Set $x = \cos\theta$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} = -\sin\theta \frac{\partial}{\partial x} \quad , \quad \frac{\partial}{\partial x} = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta} \quad , \quad \frac{\partial}{\partial x}\left( (1-x^2) \frac{\partial}{\partial x} T\right) + \left( C_0 - \frac{m^2}{1-x^2}\right) T = 0$$

What if our problem has cylindrical Symmetry $\rightsquigarrow m = 0$

$$\frac{\partial}{\partial x}\left( (1-x^2) \frac{\partial}{\partial x} T\right) + C_0 T = 0 \longrightarrow C_0 = n(n+1) \longrightarrow T(\theta) \text{ is a Legendre polynomial}$$

More generally we set $C_0 = \ell(\ell+1)$

$$\frac{\partial}{\partial x}\left( (1-x^2) \frac{\partial}{\partial x} T\right) + \left( \ell(\ell+1) - \frac{m^2}{1-x^2}\right) T = 0$$

The general solution is

$$T = P_{\ell,m}(x) = (1-x^2)^{m/2} \frac{\partial^m}{\partial x^m} P_\ell(x)$$

More commonly we write

$T\mathcal{U} = Y_{l,m}(\theta,\varphi) \Rightarrow$ Spherical Harmonics

we see this in E&M. If we are in a charge free region then $V(r) = \nabla^2 f = 0$

$S(r) = Ar^{l} + Br^{-(l+1)}$

In spherical co-ordinates the solution is $\nabla^2 f = 0$

$$f = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (A_l r^{l} + B_l r^{-(l+1)}) + (C_{lm} Y_{lm}(\theta,\varphi) + D_{lm} G_{lm}(\theta,\varphi))$$

Example problem: A metal sphere of radius R is set to have a potential $V(R,\theta) = V_0\cos^2\theta$
What is the potential for $|\vec{r}| > R$

"Any solution is a solution", $R < r < \infty$ : So $A_l = 0 \Rightarrow S(r) = Br^{-(l+1)}$

There is no $\varphi$ dependence: Spherical symmetry $\rightarrow m=0$ $\therefore$

$$f = \sum B_l C_{l,0} \frac{1}{r^{l+1}} Y_{l,0} = \sum B_l C_l \frac{1}{r^{l+1}} P_l(\cos\theta) \, , \quad x = \cos\theta$$

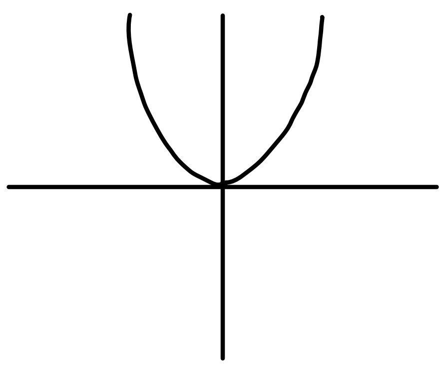
$$f(R,x) = V_0 x^2 = \sum B_l C_l P_l(x) \frac{1}{R^{l+1}} \, , \quad P_2(x) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) . \quad P_0(x) = \frac{1}{\sqrt{2}}$$

we can solve for $B_l C_{l,0}$ : $B_2 C_{2,0} = R^3 \sqrt{\frac{2}{5}} \frac{2}{3} V_0$ , $B_0 C_0 = R \frac{\sqrt{2}}{3} V_0$

12-1-21

When last we met - we showed that Legendre polynomials obey a PDE generated by $\nabla^2$ in Spherical co-ordinates. These lead to $Y_{lm}(\theta,\varphi)$. These then serve as a basis for all Spherically symmetric $\nabla^2$ problems.

Example: Quantum Harmonic Oscillator , $\frac{1}{2}kx^2$



$$\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2 \right] \psi_n = E_n \psi_n$$

Solutions - Hermite polynomials set of orthogonal polynomials orthogonalized under

$$\int H_m(x) H_n(x) e^{-x^2} dx = \delta_{m,n} \, , \quad H_n = -c\,|x|$$

$E_n^{(1)} = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle$ , $E_n^{(2)} = \sum_{j \neq n} \frac{|\langle \psi_n^{(0)} | H_1 | \psi_j^{(0)} \rangle|^2}{E_n^{(0)} - E_j^{(0)}}$

## Stern Louiville Theory

There are seven sets of orthogonal polynomials. Why are they important? Why not more?

In physics many energies or quantitative of interest w.r.t $(\nabla f)^2$ on the calculus of variations $\rightsquigarrow \nabla^2 f$

We get $2^{nd}$ order PDE. These often are eigenvalue problems

- $\rightsquigarrow$ QM
- $\rightsquigarrow$ Normal modes

These generate a discrete set of eigenvalues

If we can factor our $2^{nd}$ order equation.

$$Lf = \left(\frac{\partial}{\partial x} + y(x)\right)\left(\frac{\partial}{\partial x} + g(x)\right) f(x)$$

Then we can look for raising and lowering operators. The lowering operator on the lowest state $\rightsquigarrow 0$

This is hard unless lowest function is a constant & higher ones are polynomials. This gives a discrete spectrum with lowest bound.

Then B & F show that if we start with

$$\left[\alpha(x) \frac{\partial^2}{\partial x^2} + \beta(x) \frac{\partial}{\partial x} + \gamma(x)\right] Q(x) = \lambda Q(x)$$

This is very restricted. There are only seven independent choices....

$\rightsquigarrow$ Each has their own inner product $\int_a^b G_n G_m w(x) = \delta_{mn}$

$\rightsquigarrow$ Each has its own Rodriguez Function

$\rightsquigarrow$ Each form orthogonal & complete bases

## Fourier Series

B & F are a useful source for conventions. If we work in the interval $-\pi \leq x \leq \pi$, Then $f(x) = \frac{a_0}{2} + \sum_{n=0} a_n \cos(nx) + b_n \sin(nx)$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$ , $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$

However, other conventions are common.

Ex: $0 \leq x \leq L$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx$ , $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n x}{L}\right)dx$

or $-\frac{L}{2} \leq x \leq \frac{L}{2}$ & then shifts our function .... But it is more common to work with complex exponentials.

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \;,\; \cos(x) = \frac{1}{2i}(e^{ix} + e^{ix})$$

$$f(x) = \frac{1}{L}\sum g_n e^{2i\pi n x/L} = \frac{1}{k}\sum e^{ikx} \quad k = \frac{2\pi n}{L} \;,\; g_n = \int_{-L/2}^{L/2} f(x) e^{-ikx} dx$$

Q: what happens when $L$ is very long:

A: The spectrum between different values of $k$ becomes small

If $g_k$ is slowly varying then in an interval $dk$ there is $n = \frac{dk}{\Delta k}$ terms for values of $g$

$$f(x) = \frac{L}{2\pi}\int_{-\infty}^{\infty} g(k) e^{-ikx} dk \;,\; g(x) = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-ikx} dx \Rightarrow [f] = [g] \;,\; [\;] \rightarrow \text{units of}$$

12-3-21

## Fourier Transforms

The fourier transform of a function is much like an orthogonal function expression except that for mathematicians they work in the infinite interval.

$$f(x) = \int_{-\infty}^{+\infty} g(x) e^{ikx} dx = \int_{-\infty}^{+\infty} f(k) e^{ikx} dk$$

when the basis functions are on the normalized interval. It is assumed that $f(x)$ is well behaved and the integral exists of $g(x)$ is written as $F(x) \rightsquigarrow$ It is the same thing as a Fourier series.

$$\int_{-\infty}^{+\infty} f(k) e^{ikx} dk = \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-ik'x'} f(x') dx' = \int_{-\infty}^{+\infty} dx' \left\{ \frac{1}{2\pi} e^{ik(x-x')} dk \right\} f(x') = f(x)$$

In a sloppy fashion, if $(x - x') \neq 0$ the integral is oscillating and we get zero "on average" unless $(x - x') = 0$.

Why are Fourier Transforms so common in physics?

* They transform derivatives in powers of $k$ $\quad \int e^{ikx} \frac{d^n}{dx^n} f(x)\, dk = (ik)^n \hat{f}(x)$.

  Every linear differential equation with constant coefficients, become an algebraic equation!

For example the damped harmonic oscillator

$$mx'' + \gamma x' + kx = f(t)$$

$$\left\{ m \frac{d^2}{dt^2} \eta + \gamma \frac{d}{dt} + k \right\} \int_{-\infty}^{+\infty} e^{i\omega t} \hat{x}(\omega)\, d\omega \longrightarrow \int e^{i\omega t} \hat{F}(\omega)\, d\omega$$

$$\int (-m\omega^2 + \gamma\omega + k)\, \hat{x}\, e^{i\omega t} d\omega = \int e^{i\omega t} \hat{F}(\omega)\, d\omega \quad , \quad \hat{x}(\omega) = \frac{\hat{F}(\omega)}{-m\omega^2 + \gamma\omega + k}$$

* They simplify calculations

$$f(x) = \int_{-\infty}^{+\infty} g(x-x')\, h(x')\, dx'$$

Fourier transforms yields

$$\hat{f}(k) = \hat{g}(k)\, \hat{h}(k)$$