Assignment 1, Problem 1:

(a) Let's call the doors A, B, C.

There are three possible arrangements:

A	B	C	
Car	G	G	arrangement 1
G	car	G	arrangement 2
G	G	car	arrangement 3

We will analyze arrangement 1 (the other two arrangements give the same result).

We will look at the three possibilities the contestant has: go to door A, door B, or door C.

Go to A : (i) host goes to B $\xrightarrow{\text{switch}}$ loose ; $\xrightarrow{\text{stay}}$ win

(ii) host goes to C $\xrightarrow{\text{switch}}$ loose ; $\xrightarrow{\text{stay}}$ win

So: for this first possibility, staying guarantees winning (car is better than goat)

Go to B: host has to go to C $\xrightarrow{\text{switch}}$ win

$\xrightarrow{\text{stay}}$ loose

So, for this second possibility, switching guarantees a car.

Go to C: host has to go to B $\xrightarrow{\text{switch}}$ win

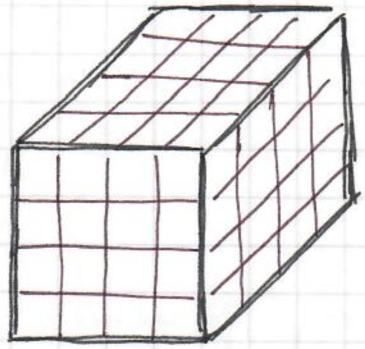
$\xrightarrow{\text{stay}}$ loose

So, for this third possibility, switching guarantees a car.

All together: always Switching wins a car with $\frac{2}{3}$ probability.

Always staying wins a car with $\frac{1}{3}$ probability.

(b) $(64)^{1/3} = 4$



P_0 : only the inside cubes have no painted faces : $(2)^3 = 8$

$$\boxed{P_0 = \frac{8}{64} = \frac{1}{8}}$$

P_1 : we have four on each side and six sides : 24

$$\boxed{P_1 = \frac{24}{64} = \frac{3}{8}}$$

P_2 : looking from the top, we have eight. Same for the bottom. The sides contribute another 8 : 24

$$\boxed{P_2 = \frac{3}{8}}$$

P_3 : just the corners : 8

$$\boxed{P_3 = \frac{8}{64} = \frac{1}{8}}$$

Check: $\boxed{P_0 + P_1 + P_2 + P_3 = 1}$

(c) important, the probabilities are independent...

The first coin can be any of the four.
The second coin can be any of the four.
-- third ...
-- fourth ...

$\Rightarrow$ total number of possibilities $= 4^4 = 256$

With a little bit of try and error, we see that only the combination $10 + 25 + 1 + 1$ gives 37.

Say, we are given a quarter: it could be the first, second, third, or fourth coin (four options).
Now, we're given a dime: Since we already have 1 coin, only three slots are left
For the pennies, no options are left.

So, there are $12 = 4 \cdot 3$ possibilities to be given a quarter, dime, and two pennies.

$\boxed{\text{probability to get } 37c : \frac{12}{256} = \frac{3}{64}}$

Problem 2:

(a) ideal gas ⇒ no interactions

$$\mathcal{H}_{cl} = \sum_{i=1}^{K} \frac{1}{2} m \vec{v}_i^{\,2} = \sum_{i=1}^{K} \frac{\vec{p}_i^{\,2}}{2m}$$

we need $\mathcal{H}$ in terms of generalized momenta

mass of point particles m

of i^{th} particle:

position vector $\vec{r}_i$

velocity vector $\vec{v}_i = \dot{\vec{r}}_i$

superscript "$\cdot$" means $\frac{d}{dt}$

conjugate momentum vector $\vec{p}_i = m\vec{v}_i = m\dot{\vec{r}}_i$

$$\mathcal{H}_{qm} = \sum_{i=1}^{K} \frac{\hat{\vec{p}}_i^{\,2}}{2m} = \sum_{i=1}^{K} -\frac{\hbar^2}{2m} \vec{\nabla}_i^{\,2}$$

$$\hat{\vec{p}}_i = -i\hbar \vec{\nabla}_i$$

$$\vec{\nabla}_i = \begin{pmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial z_i} \end{pmatrix}$$

this is for 3D system
(2D: just x, y)
(1D: just x)

(b) 8 mass m particles & 5 mass M particles

$$\mathcal{H}_{cl} = \sum_{i=1}^{8} \frac{\vec{p}_{im}^{\,2}}{2m} + \sum_{i=1}^{5} \frac{\vec{p}_{iM}^{\,2}}{2M}$$

$$+ \sum_{i<j}^{8} V_{mm}(\vec{r}_{im}, \vec{r}_{jm})$$

$$+ \sum_{i<j}^{5} V_{MM}(\vec{r}_{iM}, \vec{r}_{jM})$$

$$+ \sum_{i=1}^{8} \sum_{j=1}^{5} V_{mM}(\vec{r}_{im}, \vec{r}_{jM})$$

here: $\vec{r}_{im}$ position vector of i^{th} mass m particle

$\vec{r}_{iM}$ position vector of i^{th} mass M particle

$\vec{p}_{im} = m \dot{\vec{r}}_{im}$

$\vec{p}_{iM} = M \dot{\vec{r}}_{iM}$

the interaction potentials depend on the position vectors of the two particles involved

↳ more generally, we could also have a dependence on spin, isospin, ...

The quantum mechanical Hamiltonian is obtained by replacing $\vec{p}_{im}$ and $\vec{p}_{iM}$ by $-i\hbar \vec{\nabla}_{im}$ and $-i\hbar \vec{\nabla}_{iM}$, respectively.

$$\vec{\nabla}_{im} = \frac{\partial}{\partial \vec{r}_{im}} = \begin{pmatrix} \frac{\partial}{\partial x_{im}} \\ \frac{\partial}{\partial y_{im}} \\ \frac{\partial}{\partial z_{im}} \end{pmatrix} \quad \text{with } \vec{r}_{im} = \begin{pmatrix} x_{im} \\ y_{im} \\ z_{im} \end{pmatrix}$$

similarly for $\vec{\nabla}_{iM}$

(c) (i) let the number of electrons be K
let " " " protons " K

Assuming that / treating protons as point-particles with positive charge (i.e., neglecting for simplicity the strong interaction and considering only electro-magnetic forces), we have to account for the Coulomb interaction between each pair of electrons, between each pair of protons, and between each electron-proton pair.

$$V_{ee}(r_{jk}) = \frac{e^2}{4\pi\varepsilon_0 r_{jk}}$$

$$V_{pp}(R_{jk}) = \frac{e^2}{4\pi\varepsilon_0 R_{jk}}$$

$$V_{ep}(|\vec{r}_j - \vec{R}_k|) = \frac{-e^2}{4\pi\varepsilon_0 |\vec{r}_j - \vec{R}_k|}$$

here: $\vec{r}_j$ position vector of j^{th} electron

$\vec{R}_k$ position vector of k^{th} proton

$$r_{jk} = |\vec{r}_j - \vec{r}_k|$$

$$R_{jk} = |\vec{R}_j - \vec{R}_k|$$

$$\Rightarrow V_{int}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_K, \vec{R}_1, \vec{R}_2, \dots, \vec{R}_K)$$

$$= \sum_{j<k}^{K} V_{ee}(r_{jk}) + \sum_{j<k}^{K} V_{pp}(R_{jk})$$

$$+ \sum_{j=1}^{K} \sum_{k=1}^{K} V_{ep}(|\vec{r}_j - \vec{R}_k|)$$

$$\mathcal{H}_{cl} = \sum_{j=1}^{K} \frac{\vec{p}_j^{\,2}}{2m} + \sum_{j=1}^{K} \frac{\vec{P}_j^{\,2}}{2M} + V_{int}(\dots)$$

$\vec{p}_j = m\dot{\vec{r}}_j$ with m electron mass

$\vec{P}_j = M\dot{\vec{R}}_j$ with M proton mass

$\mathcal{H}_{qm}$ is the same as $\mathcal{H}_{cl}$ with $\vec{p}_j$ and $\vec{P}_j$ replaced by $\hat{\vec{p}}_j$ and $\hat{\vec{P}}_j$, respectively.

(ii) Low temperature regime:

We expect hydrogen atoms to form

→ this requires quantum → gas of H atoms

If we lower the temperature even more, we expect H_2 molecules to form → gas of H_2 molecules, with vibrational and rotational degrees of freedom

(iii) If the temperature is large, we expect a gas of electrons to be mixed with a gas of protons

In the extremely high temperature regime, we expect to be able to treat the system as a mixture of ideal gases (two-component gas)

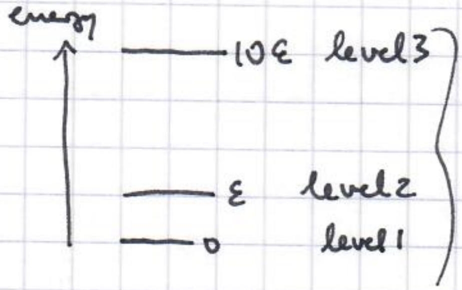
(iv) Room temperature $\hat{\approx} \frac{1}{40}\, eV$

this is small compared to the binding

energy of the H-atom (which is 13.6 eV).
Thus, at room temperature, we don't expect to find individual electrons and protons.
We can also try to compare the energy of $\frac{1}{40}$ eV to typical binding energies of molecules. Generally, $|E_{molecule}| > \frac{1}{40}$ eV
Thus, we might expect the formation of diatomic molecules, i.e., of H_2.

Problem 3:

(a)



"spectrum" of single classical particle

$$\Gamma(E, N) = \frac{N!}{N_1!\, N_2!\, N_3!}$$

N_i : # of particles in level i

constraint: $N_1 = N_2 \quad \Rightarrow \Gamma(E,N) = \dfrac{N!}{(N_1!)^2\, N_3!}$

we need to express N_1 and N_3 in terms of E, N !!!

$$E = N_1 \cdot 0 + N_2\, \varepsilon + N_3 \cdot 10\,\varepsilon \underset{N_1 = N_2}{=} \varepsilon\,(N_1 + 10 N_3) \quad ①$$

$$N = N_1 + N_2 + N_3 \underset{N_1 = N_2}{=} 2N_1 + N_3 \quad ②$$

Solve ① and ② for N_1: $\quad N_1 = \frac{E}{\varepsilon} - 10 N_3$

$$N_1 = \frac{N}{2} - \frac{N_3}{2}$$

Set the r.h.s.'s equal: $\frac{E}{\varepsilon} - 10 N_3 = \frac{N}{2} - \frac{N_3}{2}$

Solve for N_3: $19\, N_3 = \frac{2E}{\varepsilon} - N$

$$\Rightarrow \boxed{N_3 = \frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19}}$$

Plug $N_3 = \frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19}$ into $N_1 = \frac{N}{2} - \frac{N_3}{2}$:

$$\Rightarrow N_1 = \frac{N}{2} - \frac{1}{2}\left(\frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19}\right)$$

$$\boxed{N_1 = \frac{10}{19} N - \frac{1}{19}\frac{E}{\varepsilon}}$$

Dummy check : $2N_1 + N_3 = 2\underbrace{\left(\frac{10}{19} N - \frac{1}{19}\frac{E}{\varepsilon}\right)}_{N_1 \text{ (from above)}} + \underbrace{\left(\frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19}\right)}_{N_3 \text{ (from above)}} = N$

ok, things check out...

Now we have N_1 and N_3 as fcts of E and N!

As an aside: Require $N_3 \geq 0$ & $N_1 \geq 0$

$$\Rightarrow \frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19} \geq 0 \quad \& \quad \frac{10}{19} N - \frac{1}{19}\frac{E}{\varepsilon} \geq 0$$

$$\Rightarrow 2\frac{E}{\varepsilon} - N \geq 0 \quad \& \quad 10N - \frac{E}{\varepsilon} \geq 0$$

$$\Rightarrow \frac{E}{\varepsilon} \geq \frac{N}{2} \quad \& \quad \frac{E}{\varepsilon} \leq 10N$$

So : $\frac{N}{2} \leq \frac{E}{\varepsilon} \leq 10N$

Back to $T(E,N)$:

$$T(E,N) = \frac{N!}{(N_1!)^2 N_3!}$$

$$\log(T(E,N)) = \log(N!) - 2\log(N_1!) - \log(N_3!)$$

$$\approx N \log N - \underset{\sim}{N} - 2N_1 \log N_1 + \underset{\sim}{2N_1}$$

$$- N_3 \log N_3 + \underset{\sim}{N_3}$$

the underlined terms cancel since $N = 2N_1 + N_3$

$$= N\log N - 2N_1 \log N_1 - N_3 \log N_3$$

Now: We want to calculate $S = k \log(T(E,N))$

and then $\left(\frac{\partial S}{\partial E}\right)_N = \left(\frac{\partial}{\partial E}\Big(k \log(T(E,N))\Big)\right)_N$

for this, we need to insert the expressions for N_1 and N_3 into $\log(T(E,N))$:

$$k\log(T(E,N)) \approx k\Big[N\log N - 2\left(\tfrac{10}{19}N - \tfrac{1}{19}\tfrac{E}{\varepsilon}\right)\log\left(\tfrac{10}{19}N - \tfrac{1}{19}\tfrac{E}{\varepsilon}\right)$$

$$-\left(\tfrac{2}{19}\tfrac{E}{\varepsilon} - \tfrac{N}{19}\right)\log\left(\tfrac{2}{19}\tfrac{E}{\varepsilon} - \tfrac{N}{19}\right)\Big]$$

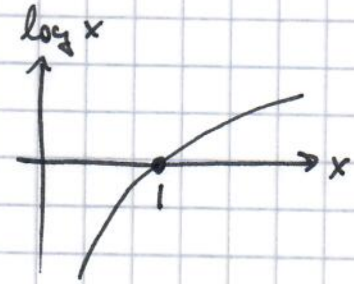
$$\left(\frac{\partial S}{\partial E}\right)_N = k\left[\frac{2}{19}\frac{1}{\varepsilon} - \frac{2}{19}\frac{1}{\varepsilon}\right.$$

$$+ \frac{2}{19}\frac{1}{\varepsilon}\log\left(\frac{10}{19}N - \frac{1}{19}\frac{E}{\varepsilon}\right)$$

$$\left. - \frac{2}{19}\frac{1}{\varepsilon}\log\left(\frac{2}{19}\frac{E}{\varepsilon} - \frac{N}{19}\right)\right]$$

$$\Rightarrow \boxed{\frac{\varepsilon}{kT} = \frac{2}{19}\log\left(\frac{10N - E/\varepsilon}{2\frac{E}{\varepsilon} - N}\right)}$$

now: log ( ) is negative if argument is < 1



$$\Rightarrow \frac{10N - E/\varepsilon}{2\frac{E}{\varepsilon} - N} < 1$$

$$10N - \frac{E}{\varepsilon} < 2\frac{E}{\varepsilon} - N$$

$$11N < 3\frac{E}{\varepsilon}$$

$$\Rightarrow \boxed{\frac{E}{\varepsilon} > \frac{11}{3}N}$$

(b)

How can we think of a negative T?

$$\left(\frac{\partial S}{\partial E}\right)_N = k\left(\frac{\partial}{\partial E} \log(\Gamma(E,N))\right)_N$$

$$= k\left(\frac{\frac{\partial}{\partial E}\Gamma(E,N)}{\Gamma(E,N)}\right)_N$$

$$\approx k \frac{\frac{\Delta\Gamma(E,N)}{\Gamma(E,N)}}{\Delta E}$$

fractional change of # of microstates is negative per energy interval

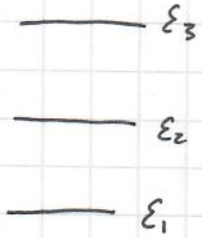
# of available microstates decreases with increasing energy

⇝ this happens for sufficiently large energy, provided we have a finite # of energy levels

(c) need an energy level structure w/ cutoff (or finite # of microstates)

Assignment 1, Problem 4:

(a)



let the # particles in ε_1 be N_1

- - - - . ε_2 be N_2

- - - . ε_3 be N_3

$$\Rightarrow N_{total} = N_1 + N_2 + N_3 = 1200 \quad (*)$$

$$E_{total} = N_1 \varepsilon_1 + N_2 \varepsilon_2 + N_3 \varepsilon_3$$

$\uparrow$

each particle has an energy E_j ($j = 1, \ldots, N_{total}$) and E_j can take the values ε_1, ε_2 or ε_3

Using that $\varepsilon_1 = 1eV$, $\varepsilon_2 = 2eV$ and $\varepsilon_3 = 3eV$,

we obtain

$$2400\,eV = N_1 \cdot 1eV + N_2 \cdot 2eV + N_3 \cdot 3eV$$

$$\text{or} \quad 2400 = N_1 + 2N_2 + 3N_3 \quad (**)$$

From $(*)$: $N_2 = N_{total} - N_1 - N_3$

From (**) : $N_2 = 1200 - \frac{N_1}{2} - \frac{3N_3}{2}$ (**')

Setting the r.h.s.'s of the last two eqs. equal:

$$\cancel{N_{total}} - N_1 - N_3 = \cancel{1200} - \frac{N_1}{2} - \frac{3N_3}{2}$$

$$\Rightarrow \boxed{N_1 = N_3}$$

Using $N_1 = N_3$ in (*), we find : $1200 = 2N_1 + N_2$

$$\Rightarrow \boxed{N_2 = 1200 - 2N_1}$$

The number of microstates is

$$\frac{1200!}{N_1!\,N_2!\,N_3!} = \boxed{\frac{1200!}{(N_1!)^2\,(1200-2N_1)!}}$$

using the constraints set by the total energy and the total # of particles

To find the most probable value of N_1, we can take the derivative w.r.t. N_1 and set the result to zero.

Factorials are not nice for this...
Work w/ logarithm instead and use Stirling's approximation:

$$\log a! \underset{\text{for large } a}{\approx} a \log a - a$$

$$T(N_1) = \frac{1200!}{(N_1!)^2 (1200-2N_1)!}$$

$$\Rightarrow \log T(N_1) = \log(1200!) - 2\log(N_1!) - \log((1200-2N_1)!)$$

will not contribute when taking the derivative

$$\approx -2(N_1 \log N_1 - N_1)$$

$$\approx -(1200-2N_1)\log(1200-2N_1) + (1200-2N_1)$$

$$\Rightarrow \frac{\partial}{\partial N_1}\left(\log T(N_1)\right) \approx -2\log N_1 - 2 + 2$$

$$+ \ 2\log(1200-2N_1) - 1 - 2 \overset{!}{=} 0$$

So: $-2\log N_1 + 2\log(1200-2N_1) - 3 = 0$

$$-2\log\left(\frac{N_1}{1200-2N_1}\right) - \cancel{3} = 0$$

I will drop this (this is consistent with the Stirling approximation)

Thus: look for $\dfrac{N_1}{1200-2N_1} = 1$

$$\Rightarrow N_1 = 1200 - 2N_1 \quad \Rightarrow \boxed{N_1 \approx 400}$$

(b) Before answering part (b), let's think a bit about

$T(N_1)$ from part (a):

$$T(N_1 = 600) = \frac{1200!}{(600!)^2} \rightsquigarrow \text{large \#}$$

$T(N_1 = 0) = 1$ exactly one possibility

(makes sense since the energy level E_2 has energy $2eV$ and placing all particles in that level gives the desired energy:

$2eV \cdot 1200 = 2400\, eV$)

Now, let's consider Bose - Einstein or indistinguishable particles. $\rightsquigarrow$ the particles are quantum!

If we specify N_1, N_2 and N_3, then the quantum state is fully determined.

Let's pick a value for N_1:

$$\text{Then,} \quad \left.\begin{aligned} N_2 &= 1200 - 2N_1 \\ N_3 &= N_1 \end{aligned}\right\} \text{from part (a)}$$

But we also know that N_2 cannot be negative $\Rightarrow N_1 \leq 600$

$\Rightarrow$ For $N_1 = 0, 1, 2, \ldots, 600$, there exists exactly one microstate and all of these microstates are equally probable by assumption $\Rightarrow$ all N_1 are equally probable.