Physics 5403 Homework #2
Spring 2022

Instructor Bruno Uchoa

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1 Parity operator

A wavefunction is written in the momentum representation $\psi(\mathbf{p}) \equiv \langle \mathbf{p}|\psi\rangle$.

a) Using the Wigner definition of parity, $\pi\psi(\mathbf{p}) = \langle \mathbf{p}|\pi|\psi\rangle$, calculate the inverted wavefunction in momentum space.

b) A charged particle with charge q sits in a 1D quantum well with eigenstates $|n\rangle$, $n = 1, 2, \ldots$. A small potential $U(x) = -qE_0x^m$ due to a weak electric field is then introduced, where m is a positive integer and $x = 0$ at the center of the well. Using the properties of the parity operator, derive the parity selection rule for the matrix element

$$\langle n'|U(x)|n\rangle.$$

2 Time reversal symmetry I

a) If $|\hat{n}, -\rangle$ is a two component eigenstate of the spin projection $\mathbf{S}\cdot\hat{n} = \frac{1}{2}\vec{\sigma}\cdot\hat{n}$, with eigenstate $-\hbar/2$, show that application of the time reversal operator on this state, namely $-i\sigma_y K|\hat{n}, -\rangle$, results in a state with the spin reversed.

b) A spin 1 particle has the Hamiltonian

$$\mathcal{H} = \alpha S_z^2 + \beta(S_x^2 - S_y^2).$$

Is the Hamiltonian invariant under time reversal symmetry? Prove your answer using the properties of the time reversal symmetry operator $\mathcal{T}$.

c) Calculate the exact eigenstates of the Hamiltonian in part b, and show that those states obbey the same symmetry you found for the Hamiltonian under time reversal symmetry.

3 Time reversal symmetry II

Consider the time reversal symmetry operator $\mathcal{T} = \mathcal{U}K$ acting in angular momentum states $|j, m\rangle$, where $\mathcal{U}$ is a unitary operator and K the conjugation.

a) Using the properties of the time reversal symmetry operator $\mathcal{T}$ and of the conjugation operator K, calculate:

i) $\mathcal{U}J_z\mathcal{U}$
ii) $\mathcal{U}J_\pm\mathcal{U}$

iii) $\mathcal{U}J^2\mathcal{U}$

Express your answer in angular momentum operators only. Find whether each of those angular momentum operators commute or anticommute with $\mathcal{U}$.

b) Using your result in a), calculate the selection rule for the matrix elements $\langle j, m'|\mathcal{U}|j, m\rangle$.

c) Now show that

$$\frac{\langle j, m'|\mathcal{U}|j, -m'\rangle}{\langle j, m|\mathcal{U}|j, -m\rangle} = i^{2(m'-m)}.$$

Choosing $\langle j, m|\mathcal{U}|j, -m\rangle = (i)^{2m}$, then find that

$$\mathcal{T}|j, m\rangle = (-1)^m|j, -m\rangle.$$

d) Find the time reversed state of the rotated ket $\mathcal{D}(R)|jm\rangle$.

e) Starting from the ket $\mathcal{D}(R)\mathcal{T}|j, m\rangle$, show that

$$\mathcal{D}^{(j)*}_{m',m}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R).$$

Hint: Use the commutator $[\mathcal{D}(R), \mathcal{T}]$ in your derivation.

f) If a system is time reversal invariant and has no degeneracy in the energy spectrum $\mathcal{H}|E\rangle = E|E\rangle$, show that

$$\langle E|\mathbf{L}|E\rangle = 0.$$