Solutions to Homework 6
Physics 5393

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P-2.2 Look again at the Hamiltonian of Chapter 1, Problem 1.11. Suppose the typist made an error and wrote $\tilde{\mathbf{H}}$ as

$$\tilde{\mathbf{H}} = H_{11}\,|1\rangle\langle 1| + H_{22}\,|2\rangle\langle 2| + H_{12}\,|1\rangle\langle 2|$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. (You may assume $H_{11} = H_{22} = 0$ for simplicity.)

This is not a Hermitian operator

$$\tilde{\mathbf{H}} \neq \tilde{\mathbf{H}}^\dagger \qquad \text{where} \qquad \tilde{\mathbf{H}}^\dagger = H_{11}\,|1\rangle\langle 1| + H_{22}\,|2\rangle\langle 2| + H_{12}\,|2\rangle\langle 1|\,;$$

notice the difference in the last term. In a matrix representation, the lack of Hermiticity is clearly displayed

$$\tilde{\mathbf{H}} = \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix} \qquad \text{and} \qquad \tilde{\mathbf{H}}^\dagger = \begin{pmatrix} H_{11} & 0 \\ H_{12} & H_{22} \end{pmatrix}.$$

This being the case, the time evolution operator will no longer be unitary and therefore the state vector normalization will no longer be preserved. This can be seen as follows: first applying the simplification suggested in the statement of the problem $H_{11} = H_{22} = 0$. Then expand the time evolution operator

$$\boldsymbol{\mathcal{U}}(t) = \exp\left(-\frac{i\tilde{\mathbf{H}}t}{\hbar}\right) = 1 - \frac{i}{\hbar}\tilde{\mathbf{H}}t,$$

since

$$\tilde{\mathbf{H}}^2 = H_{12}^2\,|1\rangle\langle 2|\,|1\rangle\langle 2| = 0 \quad \Rightarrow \quad \tilde{\mathbf{H}}^n = 0 \quad \text{if} \quad n = \text{integer and} \quad n \geq 2.$$

The normalization of an arbitrary state will be time dependent for this Hamiltonian.

P-2.3 An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$, the electron is known to be in an eigenstate of $\tilde{\vec{\mathbf{S}}} \cdot \hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector, lying in the x-y plane, that makes angle β with the z-axis.

a) Obtain the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.
Using the previously solved problem (P-1.7), the eigenstate at $t = 0$ and at a laeter time t in the $|S_z;\pm\rangle$ eigenkets are

$$\begin{aligned} |\alpha, 0\rangle &= \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle \\ |\alpha, 0; t\rangle &= e^{-i\omega t/2}\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{i\omega t/2}\sin\left(\frac{\beta}{2}\right)|-\rangle\,. \end{aligned}$$

The $S_x = (1/2)\hbar$ state is given as

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right).$$

Hence, the probability of being in the state $S_x = \hbar/2$ is

$$|\langle S_x;+|\alpha,0;t\rangle|^2 = \frac{1}{2}\left(1+\sin\beta\cos\omega t\right).$$

b) Find the expectation value of $\tilde{\mathbf{S}}_x$ as a function of time.
The expectation value is

$$\left\langle \alpha;t\left|\tilde{\mathbf{S}}_x\right|\alpha;t\right\rangle = \frac{\hbar}{2}\sin\beta\cos\omega t.$$

c) For your own peace of mind, show that your answers make good sense in the extreme cases (i) $\beta\to 0$ (ii) $\beta\to\pi/2$.
For $\beta = 0$, the spin at $t=0$ is in the $+z$ direction so the probability of being in the $S_x = +$ direction is $1/2$ and the expectation value is zero.
If $\beta = \pi/2$, then the initial state is $S_x = +$. Therefore, the spin precesses about the z axis so the probability oscillates between one and zero, while the expectation value oscillates between $\pm\hbar/2$.

P-2.4 Derive the neutrino oscillation probability (2.1.65) and use it, along with the data in Fig. 2.2, to estimate the values of $\Delta m^2 c^4$ (in units of eV^2) and θ.

To derive the time dependence of the probability that an electron neutrino is still and electron neutrino at a later time, we start by writing the most general form of the flavor eigenstates in the mass basis

$$\begin{aligned}|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle\\ |\nu_\mu\rangle &= \cos\theta\,|\nu_2\rangle + \sin\theta\,|\nu_1\rangle.\end{aligned}$$

The statement of the problem states that the system is in the $|\nu_e\rangle$ state at $t=0$. The system is then evolved in time by applying the time evolution operator keeping in mind that the $|\nu_e\rangle$ is not an eigenstate of the Hamiltonian, but $|\nu_{1,2}\rangle$ are

$$\begin{aligned}\boldsymbol{\mathcal{U}}(t)\,|\nu_e\rangle &= e^{-iE_1t/\hbar}\cos\theta\,|\nu_1\rangle - e^{-iE_2t/\hbar}\sin\theta\,|\nu_2\rangle\\ &= e^{-ipct/\hbar}\left[e^{-im_1^2c^3t/2p\hbar}\cos\theta\,|\nu_1\rangle - e^{-im_2^2c^3t/2p\hbar}\sin\theta\,|\nu_2\rangle\right]\end{aligned}$$

The probability that the state at $t>0$ is still $|\nu_e\rangle$ is

$$\begin{aligned}\mathcal{P}(\nu_e\to\nu_e) &= |\langle\nu_e;0|\nu_e;t\rangle|^2\\ &= \left|e^{-im_1^2c^3t/2p\hbar}\cos^2\theta + e^{-im_2^2c^3t/2p\hbar}\sin^2\theta\right|^2\\ &= \left|\cos^2\theta + e^{-i\Delta m^2c^3t/2p\hbar}\sin^2\theta\right|^2\\ &= \cos^4\theta+\sin^4\theta+2\cos^2\theta\sin^2\theta\left[e^{i\Delta m^2c^3t/2p\hbar}+e^{-i\Delta m^2c^3t/2p\hbar}\right]\\ &= 1-\frac{4}{2}\cos^2\theta\sin^2\theta\left[1-\cos\left(\frac{\Delta m^2c^3t}{2p\hbar}\right)\right]\\ &= 1-\sin^2 2\theta\sin^2\left(\frac{\Delta m^2c^3t}{4p\hbar}\right),\end{aligned}$$

where the following relation was used

$$\cos^4\theta + \sin^4\theta = \cos^2\theta(1-\sin^2\theta) + \sin^2\theta(1-\cos^2\theta) = 1 - 2\cos^2\theta\sin^2\theta.$$

The final steps are to convert the momentum to energy and time to a length

$$\left.\begin{aligned} E &= pc \\ L &= ct \end{aligned}\right\} \quad\Rightarrow\quad \mathcal{P}(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 c^4 L}{4E\hbar}\right)$$

To calculate Δm^2, use the difference between the first maximum and first minimum, which allows the extraction of L/E over half a wavelength

$$\left.\begin{aligned} &\frac{L}{E} \approx 20 \text{ km/MeV} \\ &\frac{\Delta m^2 c^4 L}{4E\hbar} = \pi \end{aligned}\right\} \quad\Rightarrow\quad \Delta m^2 c^4 = 4\pi\hbar c\frac{E}{L} \approx 1.2\times 10^{-4} \text{ eV}^2$$

The angle θ can be derived at the first minimum where the time dependent $\sin$ function is a maximum

$$\sin^2\left(\frac{\Delta m^2 c^3 t}{4p\hbar}\right) = 1 \quad\Rightarrow\quad 1-\sin^2(2\theta) \approx 0.4 \quad\Rightarrow\quad \theta \approx 25^\circ$$

P-2.6 Consider a particle in one dimensions whose Hamiltonian is given by

$$\tilde{\mathbf{H}} = \frac{\tilde{\mathbf{p}}^2}{2m} + V(\tilde{\mathbf{x}}).$$

By calculating $\left[\left[\tilde{\mathbf{H}},\tilde{\mathbf{x}}\right],\tilde{\mathbf{x}}\right]$, prove

$$\sum_i \left|\langle a_j\left|\tilde{\mathbf{x}}\right| a_i\rangle\right|^2 (E_i - E_j) = \frac{\hbar^2}{2m},$$

where $|a_i\rangle$ is an energy eigenket with eigenvalue E_i.

Using the canonical commutation relation, the commutation relation requested is found by brute force to be

$$\left[\tilde{\mathbf{H}},\tilde{\mathbf{x}}\right] = -i\hbar\frac{\tilde{\mathbf{p}}}{m} \quad\Rightarrow\quad \left[\left[\tilde{\mathbf{H}},\tilde{\mathbf{x}}\right],\tilde{\mathbf{x}}\right] = -\frac{\hbar^2}{m}.$$

In addition, this commutator can be expanded into the following form

$$\left[\left[\tilde{\mathbf{H}},\tilde{\mathbf{x}}\right],\tilde{\mathbf{x}}\right] = \tilde{\mathbf{H}}\tilde{\mathbf{x}}^2 + \tilde{\mathbf{x}}^2\tilde{\mathbf{H}} - 2\tilde{\mathbf{x}}\tilde{\mathbf{H}}\tilde{\mathbf{x}}$$

$$\Rightarrow\quad \left\langle a_j\left|\left[\left[\tilde{\mathbf{H}},\tilde{\mathbf{x}}\right],\tilde{\mathbf{x}}\right]\right|a_j\right\rangle = 2E_j\left\langle a_j\left|\tilde{\mathbf{x}}^2\right|a_j\right\rangle - 2\left\langle a_j\left|\tilde{\mathbf{x}}\tilde{\mathbf{H}}\tilde{\mathbf{x}}\right|a_j\right\rangle = -\frac{\hbar^2}{m}.$$

To separate the operator terms in the equation above, we introduce a complete set into the equation above

$$\sum_i\left[\left\langle a_j\left|\tilde{\mathbf{x}}\tilde{\mathbf{H}}\right|a_i\right\rangle\langle a_i\left|\tilde{\mathbf{x}}\right|a_j\rangle - E_j\langle a_j\left|\tilde{\mathbf{x}}\right|a_i\rangle\langle a_i\left|\tilde{\mathbf{x}}\right|a_j\rangle\right] = \frac{\hbar^2}{2m}.$$

This expression can be written as

$$\sum_i\left[(E_i - E_j)\left|\left\langle a_i\left|\tilde{\mathbf{x}}^2\right|a_j\right\rangle\right|^2\right] = \frac{\hbar^2}{2m}.$$

Additional Problems

Q-1 Suppose the state vectors $|\alpha\rangle$ and $|\beta\rangle$ are eigenvectors of a unitary operator $\boldsymbol{\mathcal{U}}$ with eigenvalues λ and λ', respectively. What relation must λ and λ' satisfy if $|\alpha\rangle$ is not orthogonal to $|\beta\rangle$?

Consider the following three inner products:

$$\begin{aligned}
\langle\alpha\,|\alpha\rangle &= \left\langle\alpha\left|\boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\mathcal{U}}\right|\alpha\right\rangle = \lambda\lambda^* = 1 \qquad \Rightarrow \quad \lambda = e^{i\phi_1}\\
\langle\beta\,|\beta\rangle &= \left\langle\beta\left|\boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\mathcal{U}}\right|\beta\right\rangle = \lambda'\lambda'^* = 1 \qquad \Rightarrow \quad \lambda' = e^{i\phi_2}\\
\langle\beta\,|\alpha\rangle &= \left\langle\beta\left|\boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\mathcal{U}}\right|\alpha\right\rangle = \lambda'^*\lambda\,\langle\beta\,|\alpha\rangle \quad \Rightarrow \quad \lambda'^*\lambda = 1 \quad \Rightarrow \quad \lambda = \lambda' = e^{i\phi_1},
\end{aligned}$$

where use of the non-orthogonality of the eigenkets is used; $\langle\beta\,|\alpha\rangle \neq 0$. Therefore, the eigenvalues of the two eigenkets are equal and in general complex.