Question 1:

Assignment 3 solutions

Let's begin by defining the position of the centerpoint of each wheel (e.g., where the axles connect to the wheels) by the vectors $\vec{r}$ & $\vec{r}\,'$ respectively. Then, the co-ordinates of the midpoint of the axle are:

$$(x, y) = \frac{1}{2}\left(r_x + r_x', \; r_y + r_y'\right) \qquad (i)$$

Now, per the definitions given in the question, when one wheel rotates by $d\phi$ ($d\phi'$) the vector $\vec{r}$ ($\vec{r}\,'$) is shifted by a distance $Rd\phi$ ($Rd\phi'$) in ~~the~~ a direction defined by θ (see Fig. 1). Then, we can equivalently write:

$$d\vec{r} = Rd\phi\left[\sin\theta\,\hat{x} - \cos\theta\,\hat{y}\right]$$

$$d\vec{r}\,' = Rd\phi'\underbrace{\left[\sin\theta\,\hat{x} - \cos\theta\,\hat{y}\right]}_{\text{see Fig. 1 to understand this dependence.}}$$

From the eqns for $d\vec{r}$ & $d\vec{r}'$ & Eq (i) we then have:

$$dx = \frac{R}{2}\sin\theta\,(d\phi + d\phi') \qquad (ii)$$

$$dy = -\frac{R}{2}\cos\theta\,(d\phi + d\phi') \qquad (iii)$$

Multiply Eqs (ii) & (iii) by $\sin\theta$ ~~&~~ or $\cos\theta$ & then adding | subtracting the resulting equations we can then also write:

$$\sin\theta\,dx - \cos\theta\,dy = \frac{R}{2}\underbrace{(\sin^2\theta + \cos^2\theta)}_{1}(d\phi + d\phi')$$

& $$\cos\theta\,dx + \sin\theta\,dy = 0$$

Which is the constraint Eq. 1 ~~for~~ $\frac{1}{2}$

Now to obtain the holonomic constraint (Eq. 2 of Q1), consider the relative vector:

$$\vec{r}_{12} = \vec{r} - \vec{r}'$$

By definition, the angle θ can be written as,

$$\tan\theta = \frac{r_{12,y}}{r_{12,x}}$$

and the differential of this ~~vector~~ result is,

$$\sec^2\theta \, d\theta = -\frac{r_{12,y}}{r_{12,x}^2} dr_{12,x} + \frac{dr_{12,y}}{r_{12,x}}$$

Plugging $d\vec{r}$ & $d\vec{r}'$ into the definition of $\vec{r}_{12}$ & computing differentials then let's us write:

$$\sec^2\theta \, d\theta = R(d\phi - d\phi')\left[-\frac{r_{12,y}}{r_{12,x}^2}\sin\theta - \frac{\cos\theta}{r_{12,x}}\right]$$

$$\equiv -\frac{R(d\phi - d\phi')}{r_{12,x}}\left[\tan\theta\sin\theta + \cos\theta\right]$$

$\downarrow$

$$d\theta = -\frac{R(d\phi - d\phi')}{r_{12,x}}\left[\sin^2\theta\cos\theta + \cos^3\theta\right]$$

trig identities, $\sin^2\theta + \cos^2\theta = 1$

$$= -\frac{R(d\phi - d\phi')}{r_{12,x}}\cos\theta$$

Again, we consider the definition of θ to realize,

$$\cos\theta = \frac{r_{12,x}}{L}$$

So that,

$$d\theta = -\frac{R}{L}(d\phi - d\phi')$$

Integrating both sides then yields the result for Eq. 2 ~~(...)~~ :

$$\theta = \underbrace{C}_{\text{const.}} - \frac{R}{L}(\phi - \phi')$$

$$\hookrightarrow \quad \theta + \frac{R}{L}(\phi - \phi') = \text{const.}$$

So in summary, the condition of no slipping leads to two nonholonomic constraint eqns:

$$\sin\theta\, dx - \cos\theta\, dy = \frac{R}{2}(d\phi + d\phi')$$

&

$$\cos\theta\, dx + \sin\theta\, dy = 0$$

(nonholonomic → they can be equivalently written w/ $\frac{dx}{dt}, \frac{dy}{dt}, \frac{d\phi}{dt}$ & $\frac{d\phi'}{dt}$ → depend on velocities!)

and one holonomic constraint:

$$\theta + \frac{R}{L}(\phi - \phi') = \text{const.}$$ → depends only on co-ordinates θ, ϕ, ϕ'

Question 2:

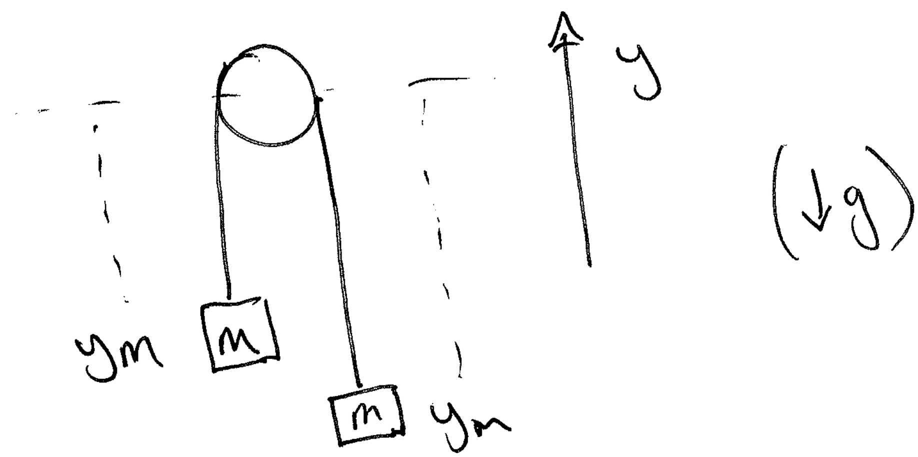
Part 1

a) D'Alembert's principle states that,

$$\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \cdot \delta r_i = 0$$

Let us define the co-ordinates of the system by:

(The $y=0$ point will be irrelevant for us...)



Then for mass m,

$$F_m^{(a)} = -mg \quad \& \quad \dot{p}_m = (m\dot{y}_m)^{\cdot} = m\ddot{y}_m$$

& for mass M,

$$F_M^{(a)} = -Mg \qquad \dot{p}_M = M\ddot{y}_M$$

Then d'Alembert's principle gives:

$$-m(g+\ddot{y}_m)\cdot\delta y_m - M(g+\ddot{y}_M)\cdot\delta y_M = 0 \quad \text{(i)}$$

b) y_m & y_M are not independent, because the masses are connected by the fixed length of rope. In particular,

$$l + y_m + y_M = 0 \quad \text{(assuming } y=0 \text{ at pulley ...)}$$

Then, $\ddot{y}_m = -\ddot{y}_M$ & $\delta y_m = -\delta y_M$

and thus,

$$\text{(i)} \rightarrow -g(m-M) + (m+M)\ddot{y}_M = 0$$

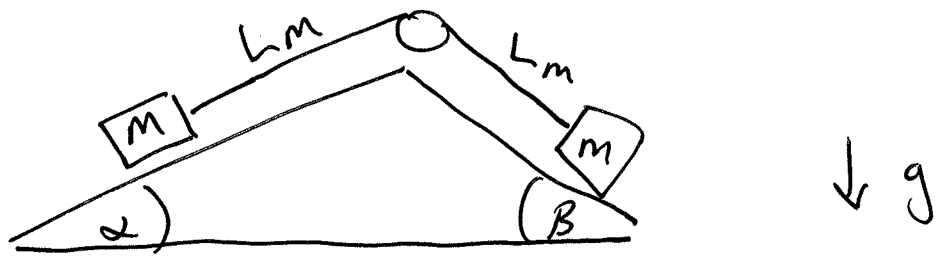
$$\ddot{y}_M = \frac{m-M}{m+M}\, g$$

or $\ddot{y}_m = \dfrac{M-m}{m+M}\, g$.

c) We can again use d'Alembert's principle to compute the relevant equations of motion. Appropriately computing the component of gravity along the surfaces of the wedge:

$$F_m^{(a)} = +mg\sin\beta \qquad\qquad F_M^{(a)} = +Mg\sin\alpha$$

$$\dot{p}_m = +m\ddot{L}_m \qquad\qquad p_M = +M\ddot{L}_M$$



Then, $\sum_i (F_i^{(a)} - \dot{p}_i)\cdot\delta r_i = 0$, yields:

$$\delta L_m(mg\sin\beta - m\ddot{L}_m) + \delta L_M(Mg\sin\alpha - M\ddot{L}_M) = 0$$

As in (b), we have that L_M & L_m are not independent, i.e.,

$$\ddot{L}_m = -\ddot{L}_M \quad \& \quad \delta L_m = -\delta L_M$$

Thus, d'Alembert's principle then implies,

$$+ mg\sin\beta \mp m\ddot{L}_m - Mg\sin\alpha - M\ddot{L}_m = 0$$

$$\downarrow$$

$$g(m\sin\beta - M\sin\alpha) - (m+M)\ddot{L}_m = 0$$

$$\downarrow$$

$$\ddot{L}_m = g\,\frac{m\sin\beta - M\sin\alpha}{m+M}$$

$\boxed{\text{Part 2}}$

d)

In principle we might think at least 3 co-ordinates are needed to describe the positions of the three weights. However, given that the two ropes (length L & l) form a pair of constraints, we will find only 2 generalized co-ordinates are required.

Why 2? In general,

$$n_{dof} = D N - m$$

m constraints (holonomic)

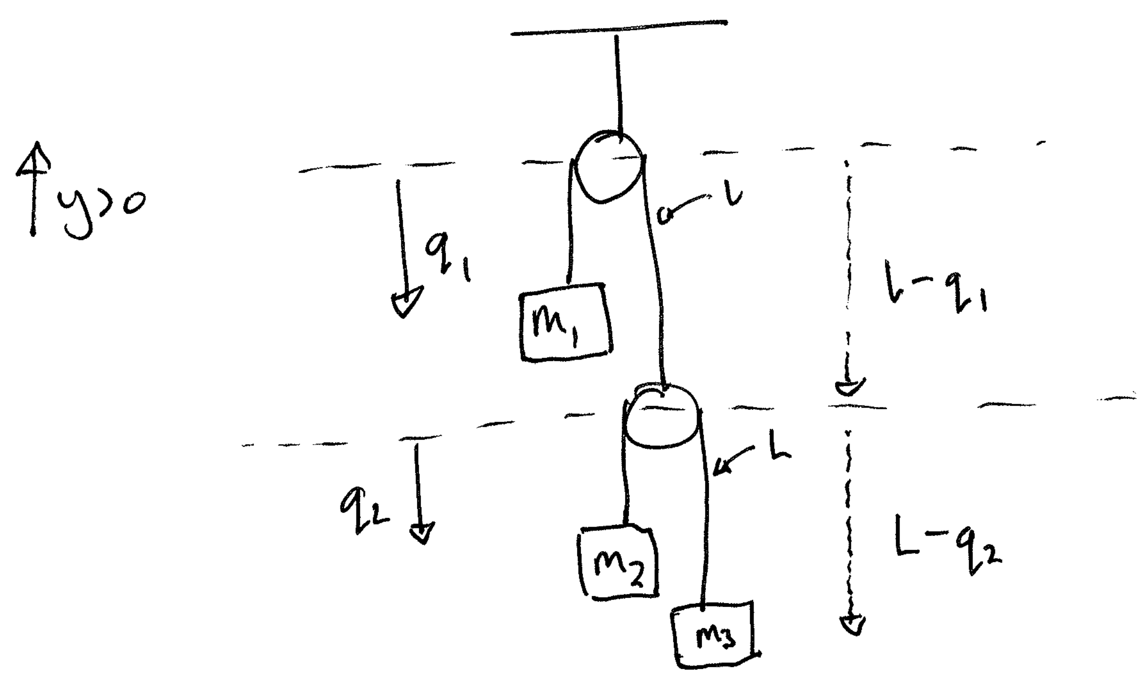
N moving particles

D dimensional motion

# of d.o.f or generalized co-ordinates

We have $N = 4$ moving objects (3 blocks + 1 moving pulley) moving in $D=1$. The two rope lengths give us $m=2$ constraints. Thus $n_{dof} = 2$!

Let's next draw a diagram to set up our relevant co-ordinates:



Defining y_i as the original cartesian co-ordinate of each block w/ mass m_i, we can define everything in terms of the two generalized co-ordinates q_1 + q_2.

[$q_1 \Rightarrow$ distance of m_1 from top pulley height,
$q_2 \Rightarrow$ distance of m_2 from bottom pulley height.]

We have:

$$y_1 = -q_1 \,, \qquad y_2 = -l + q_1 - q_2$$

$$\text{\& } y_3 = -(l+L) + q_1 + q_2 \,.$$

Also, for completeness,

$$\delta y_1 = -\delta q_1 \,, \qquad \delta y_2 = \delta q_1 - \delta q_2$$

$$\text{\& } \delta y_3 = \delta q_1 + \delta q_2 \,.$$

e) We can obtain the relevant equations of motion for q_1 & q_2 from d'Alembert's principle.

$$\sum_i (F_i^{(a)} - \dot{p}_i) \cdot \delta r_i = 0$$

~~$m_1 g = m_1 \ddot{q}_1$~~

plo

$$\Rightarrow \quad 0 = (-m_1 g - m_1 \ddot{y}_1)\,\delta y_1 + (-m_2 g - m_2 \ddot{y}_2)\,\delta y_2 + (-m_3 g - m_3 \ddot{y}_3)\,\delta y_3$$

$$= (m_1 g - m_1 \ddot{q}_1)\,\delta q_1 + (-m_2 g + m_2 \ddot{q}_2 - m_2 \ddot{q}_1)(\delta q_1 - \delta q_2) + (-m_3 g - m_3 \ddot{q}_1 - m_3 \ddot{q}_2)(\delta q_1 + \delta q_2)$$

As both δq_1 & δq_2 are independent variations,

$$\delta q_1 \rightarrow \quad 0 = (m_1 g - m_2 g - m_3 g) + \ddot{q}_1(-m_1 - m_2 - m_3) + \ddot{q}_2(m_2 - m_3) \quad ①$$

$$\delta q_2 \rightarrow \quad 0 = (m_2 g - m_3 g) + \ddot{q}_1(m_2 - m_3) + \ddot{q}_2(-m_2 - m_3) \quad ②$$

Eqns ① + ② are a linear system w/ two unknowns (e.g. $\ddot{q}_1$ + $\ddot{q}_2$) that can be solved. In particular, using the shorthand,

$$① \rightarrow \quad A_1 \ddot{q}_1 + B_1 \ddot{q}_2 + C_1 = 0$$

$$② \rightarrow \quad A_2 \ddot{q}_1 + B_2 \ddot{q}_2 + C_2 = 0$$

then, for example,

$$\ddot{q}_1 = \frac{B_2 C_1 - B_1 C_2}{A_2 B_1 - A_1 B_2}$$

$$\ddot{q}_2 = \frac{A_2 C_1 - A_1 C_2}{A_1 B_2 - A_2 B_1}$$

f) By the definitions established in (d), we have:

$$\ddot{y}_1 = -\ddot{q}_1 = \frac{B_1 C_2 - B_2 C_1}{A_2 B_1 - A_1 B_2}$$

and similarly,

$$\ddot{y}_2 = \ddot{q}_1 - \ddot{q}_2 = \ldots$$

$$\ddot{y}_3 = \ddot{q}_1 + \ddot{q}_2 = \ldots$$

For m_1 to be stationary, we want to establish when:

$$\ddot{y}_1 = \frac{B_1 C_2 - B_2 C_1}{A_2 B_1 - A_1 B_2} = 0$$

ie., want to solve:

$$B_1 C_2 - B_2 C_1 = 0 \qquad (\text{assuming } A_2 B_1 - A_1 B_2 \neq 0)$$

$$\downarrow$$

$$(m_2 - m_3)^2 g + (m_2 + m_3)(m_1 - (m_2 + m_3)) g = 0$$

$$\downarrow$$

$$m_1 = m_2 + m_3 - \frac{(m_2 - m_3)^2}{m_2 + m_3}$$

An example is for $m_1 = m_2 + m_3$ & $m_2 = m_3$. Then m_1 is stationary (which makes sense because the mass of the lower pulley system balances the mass m_1) and it is also straightforward to show that $m_2 = m_3$ implies $\ddot{q}_2 = 0$ & thus $\ddot{y}_2 = 0$, so the lower pulley system is also stationary.

Question 3:

a) First, let us assume the length of the rope is such that $L = h$ (see Fig 3), such that the gravitational potential energy for the block can be written as,

$$V = Mgd$$

height of mass M above floor.

Further, we can use polar co-ordinates to characterize the motion of the block of mass m on the table,

$\theta \rightarrow$ angular co-ordinate

$r \rightarrow$ radial co-ord. (defined relative to hole in table)

With this in hand, the kinetic energy of both masses can be ~~wr~~ written as,

$$T_m = \cancel{\tfrac{m}{2}} \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 \qquad \left(\Rightarrow \text{from } T = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2\right)$$

$$T_M = \frac{M}{2}\dot{d}^2$$

However, note that the fixed length ~~holding the~~ of the rope holding both masses together implies,

$$\dot{r} = \dot{d}$$

& thus:

$$T_m = \frac{m}{2}\dot{d}^2 + \frac{m}{2}d^2\dot{\theta}^2$$

[d & θ are thus our two generalized co-ordinates]

Hence, the Lagrangian of the total system is then,

$$L = \left(\frac{m+M}{2}\right)\dot{d}^2 + \frac{m}{2}d^2\dot{\theta}^2 - Mgd$$

Lagrange's eqns of motion then yield,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{for } q_j = \theta, d$$

So

i) $$\frac{d}{dt}\left(md^2\dot{\theta}\right) = 0$$

ii) $$\frac{d}{dt}\left((m+M)\dot{d}\right) = md\dot{\theta}^2 - Mg$$

The former equation illustrates that angular momentum is conserved,

$$m d^2 \dot{\theta} = \text{const.} = c_1$$

when enables us to rewrite (ii) as,

$$(m+M)\ddot{d} = -Mg + \frac{c_1^2}{md^3} \qquad (**)$$

Formally solving for d from this equation implicitly allows us to subsequently solve for θ from $\dot{\theta}(d)$.

b) Clearly we have identified that angular momentum is conserved:

$$L = \underset{\substack{\uparrow \\ I = md^2 \\ \text{(moment of inertia)}}}{I} \; \underset{\substack{\uparrow \\ \omega = \dot{\theta} \\ \text{(angular frequency)}}}{\omega} = \text{const.}$$

Moreover inspection of (*) indicates that:

$$\ddot{d} = 0 \quad \rightarrow \quad \frac{c_1^2}{md^3} = Mg$$

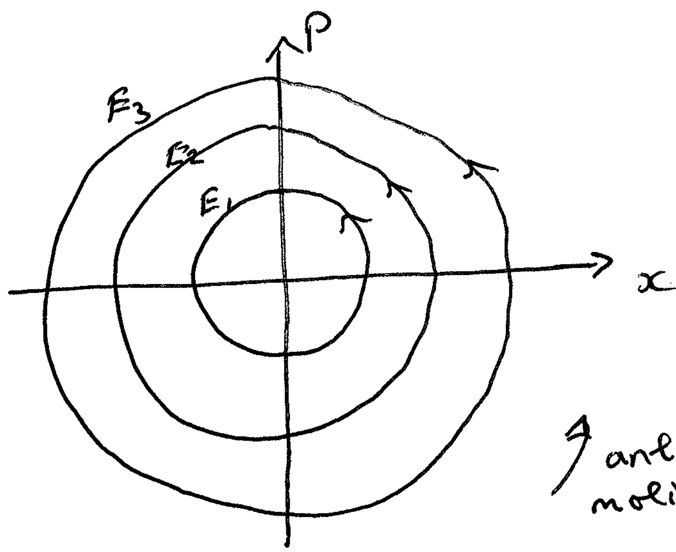
e.g., the lower block is stationary when the force due to rotation of upper block balances gravity.

Question 4

a) The harmonic oscillator in 1D traces out circles in phase space (for appropriately normalized position + momentum co-ordinates).
Here we take $m = \omega = 1$ for simplicity such that for any trajectory we have that,

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{p^2}{2} + \frac{x^2}{2} = \text{constant.}$$

Phase portrait:



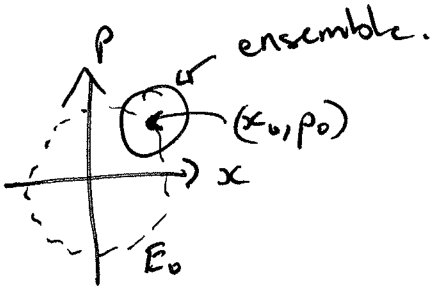
$E_3 > E_2 > E_1$

anti-clockwise motion for $\omega > 0$.

b) We consider a swarm of points w/in an area defined by,

$$(x - x_0)^2 + (p - p_0)^2 = a$$

e.g., a circle centred @ (x_0, p_0) w/ initial energy $E_0 = \frac{x_0^2}{2} + p_0^2/2$



Now, by definition of simple harmonic motion, each point w/in the ensemble will orbit about $(x,p)=(0,0)$ maintaining a fixed radius (due to conservation of energy) and at a fixed angular frequency ($\omega=1$). Hence, the circular area itself orbits w/ freq $\omega=1$ about the origin w/out being deformed. ~~there~~.

$\Rightarrow$ Phase-space area occupied by ensemble is conserved under evolution.

c) See Fig 4 of assignment.

$q \equiv$ spatial co-ordinate.

Area of ensemble $\quad A = \displaystyle\int_{p_1}^{p_2} dp \int_{q(p)} dq \cdot 1$

where $q(p)$ is defined by $E' + E''$ & the relation,

$$p = \pm\sqrt{2m(E - mgq)}$$

$E = T + V = \text{const.}$

$V(q)$ due to gravity

Then,

$$q = \frac{(E - p^2/2m)}{mg} \quad \cancel{(18)} \quad (19)$$

$$A = \int_{p_1}^{p_2} dp \left[\left(E'' - p^2/2m\right) - \left(E' - p^2/2m\right) \right] \times \frac{1}{mg}$$

$$= \frac{(E'' - E')(p_2 - p_1)}{mg} \quad (\sim \Delta E \Delta P)$$

d) Conservation of energy implies that,

$$p(t) = \pm \sqrt{2m\left(E - mgq(t)\right)}$$

or

$$q(t) = \frac{1}{mg}\left(E - \frac{p(t)^2}{2m}\right)$$

Now, for a particle in 1D subject to gravity,

$$\ddot{q} = -g \longrightarrow q(t) = -\frac{gt^2}{2} + \frac{p(0)t}{m} + q(0)$$

and thus, $p(t) = -mgt + p(0)$

The latter solution for $p(t)$ implies:

$$p_2(t) - p_1(t) = p_2(0) - p_1(0) \text{ is conserved.}$$

Thus the boundaries of the integrated region of the ensemble evolve in a well defined way:

$$A(t) = \int_{p_1(t)}^{p_2(t)} \int_{q(p)} dq$$

$$= \int_{p_1(t)}^{p_2(t)} dp \left[\left(E'' - p^2/2m\right) - \left(E' - p^2/2m\right) \right] \frac{1}{mg}$$

$$= \frac{(E'' - E')(p_2(t) - p_1(t))}{mg}$$

$$= \frac{(E'' - E')(p_2(0) - p_1(0))}{mg} = A(0)$$

Thus the area of the ensemble in phase-space is conserved, consistent w/ Liouville's theorem.