Homework #2

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a)

$$\Pi \psi(\vec{p}) = \langle \vec{p} | \Pi | \psi \rangle$$

$$= \int d\vec{x}\, \underbrace{\langle \vec{p} | \vec{x} \rangle}_{\frac{e^{+i\vec{p}\cdot\vec{x}}}{(2\pi)^3}} \underbrace{\langle \vec{x} | \Pi | \psi \rangle}_{\psi(-\vec{x})}$$

$$= \int d\vec{x}' \, \frac{e^{-i\vec{p}\cdot\vec{x}}}{(2\pi)^3} \cdot \psi(\vec{x})$$

$$= \psi(-\vec{p})$$

b) The eigenstates for a particle in a box $|n\rangle$, with $n = 1 \ldots \infty$ are such that

$$\pi|n\rangle = (-1)^{n+1}|n\rangle .$$

If $U(x) = \alpha x^m$, where $\pi \hat{x} \pi^{-1} = -x$

$$\therefore \quad \pi x^m \pi^{-1} = (\pi x \pi^{-1})^m = (-1)^m x^m .$$

Hence,

$$\langle n'|U(x)|n\rangle$$

$$= \left(\langle n'|\pi^\dagger\right)\left(\pi U(x) \pi^{-1}\right)\left(\pi|n\rangle\right)$$

$$= (-1)^m (-1)^{n+n'+2} \langle n'|U(x)|n\rangle$$

$\therefore$ $m + n + n'$ is even, otherwise,

$$\langle n'|U(x)|n\rangle = 0 .$$

(A)

(2)

a)

If $|\pm\rangle$ are the eigenstates of S_z then

$$|\hat{n},\pm\rangle = e^{i\beta S_z/\hbar}\, e^{i\alpha S_y/\hbar}|\pm\rangle$$

where $\hat{n}$ is parametrized by the Euler angles α and β. For a spin ½ particle,

$$\vec{S}\cdot\hat{n} = \frac{1}{2}\vec{\sigma}\cdot\hat{n} \quad \therefore$$

$$|\hat{n},\pm\rangle = \left[\mathbb{1}\cos\left(\tfrac{\beta}{2}\right) - i\sigma_z \sin\left(\tfrac{\beta}{2}\right)\right]$$

$$\times \left[\mathbb{1}\cos\left(\tfrac{\alpha}{2}\right) - i\sigma_y \sin\tfrac{\alpha}{2}\right] |\pm\rangle$$

Hence, since

$$-i\sigma_y K(i\sigma_z) = (i\sigma_z)(-i\sigma_y K)$$

$$-i\sigma_y K(i\sigma_y) = i\sigma_y(-i\sigma_y K)$$

Then

$$T|\hat{n},\pm\rangle = -i\sigma_y K \overbrace{\left[\mathbb{1}\cos(\beta/2) - i\sigma_z \sin\left(\frac{\beta}{2}\right)\right]}^{e^{i\beta S_z/\hbar}}$$

$$\times \underbrace{\left[\mathbb{1}\cos\left(\alpha/2 - i\sigma_y \sin\frac{\alpha}{2}\right)\right]}_{e^{i\alpha S_y/\hbar}} |\pm\rangle$$

$$= e^{i\beta S_z/\hbar}\, e^{i\alpha S_y/\hbar} \underbrace{(-i\sigma_y K)|\pm\rangle}_{\pm|\mp\rangle}$$

$$= \pm|\hat{n},\mp\rangle.$$

b)

$$H = \alpha S_z^2 + \beta (S_x^2 - S_y^2)$$

In general, since:

$$T\vec{S}T^{-1} = -\vec{S}, \implies T S_i^2 T^{-1} = (T S_i T^{-1})^2$$
$$= S_i^2.$$

then

$$T H T^{-1} = H.$$

c

b)

$$\langle 1m|S_z|1m'\rangle = m\hbar\,\delta_{m,m'}$$

$$\langle 1m|S_x|1m'\rangle = \langle 1m|\frac{S_+ + S_-}{2}|1m'\rangle$$

$$= \frac{\hbar}{2}\sqrt{(1-m')(2+m')}\,\delta_{m,m'+1}$$

$$+\frac{\hbar}{2}\sqrt{(1+m')(2-m')}\,\delta_{m,m'-1} \quad \therefore$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\langle 1m|S_y|1m'\rangle = \langle 1m|\frac{S_+ - S_-}{2i}|1m'\rangle$$

$$= \frac{\hbar}{2i}\sqrt{(1-m')(2+m')}\,\delta_{m,m'+1}$$

$$-\frac{\hbar}{2i}\sqrt{(1+m')(2-m')}\,\delta_{m,m'-1}$$

$\therefore$

$$S_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

That way,

$$S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$S_y^2 = \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Hence:

$$H = \alpha S_z^2 + \beta (S_x^2 - S_y^2)$$

$$= \hbar^2 \begin{pmatrix} \alpha & 0 & \beta \\ 0 & 0 & 0 \\ \beta & 0 & \alpha \end{pmatrix}$$

E

Finding the eigenvalues,

$$(\alpha - E)(-E)(\alpha - E) - \beta^2(-E) = 0$$

$$\therefore \quad E\left[(\alpha - E)^2 - \beta^2\right] = 0 \quad \therefore \quad E = 0, \pm\beta + \alpha$$

For $E = 0$, the eingenket is

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1, 0\rangle$$

For $E = \alpha \pm \beta$,

$$|\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|11\rangle \pm |1, -1\rangle\right)$$

where

$$T|1\rangle = T|1, 0\rangle = |1, 0\rangle$$

$$T|\pm\rangle = T\frac{1}{\sqrt{2}}\left(|11\rangle \pm |1 -1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(-|1,-1\rangle \pm |1\,1\rangle\right)$$

$$= \pm\,|\mp\rangle$$

Since, $\quad T\,|j,m\rangle = (-1)^m\,|j,-m\rangle.$

③

a)

i) $\mathcal{U} = TK$

$\therefore \quad \mathcal{U}J_z = TKJ_z = TJ_zK = -J_zTK$

$= -J_z\mathcal{U}$

$\Rightarrow \quad \mathcal{U}J_z\mathcal{U}^{-1} = -J_z$. (anti. commute)

ii) $\mathcal{U}J_\pm = TKJ_\pm = TK(J_x \pm iJ_y)$

Since $K(iJ_y) = iJ_yK \quad \therefore$

$\mathcal{U}J_\pm = T(J_x \pm iJ_y)K$

$= -(J_x \mp J_y)TK$

$= -J_\mp \mathcal{U}$

$\therefore \quad \mathcal{U}J_\pm\mathcal{U}^{-1} = -J_\mp$.

iii)

$$\mathcal{U} J^2 = TKJ^2 = TJ^2K = J^2TK$$
$$= J^2 \mathcal{U}.$$

$\therefore \quad \mathcal{U} J^2 \mathcal{U}^{-1} = J^2$ (commute)

b)

Since $[\mathcal{U}, J_z] = 0$,

$$0 = \langle j m' | (\mathcal{U} J_z + J_z \mathcal{U}) | j m \rangle$$
$$= \hbar (m + m') \langle j m' | \mathcal{U} | j m \rangle$$

$\therefore \quad \langle j m' | \mathcal{U} | j m \rangle = 0$ unless $m = -m'$.

c)

Since $\mathcal{U} J_\pm = J_\mp \mathcal{U}$,

then

$$0 = \langle j m' | (u J_+ + J_- u) | j m \rangle$$

$$= \hbar \sqrt{(j-m)(j+m+1)} \, \langle j m' | u | j \, m+1 \rangle$$

$$+ \hbar \sqrt{(j-m')(j+m'+1)} \, \langle j \, m'+1 | u | j m \rangle .$$

Since:

$$\langle j m' | u | j \, m+1 \rangle = \langle j m' | u | j \, -m' \rangle \, \delta_{m', -m-1}$$

$\therefore$

$$0 = \hbar \cancel{\sqrt{(j+m'+1)(j-m')}} \, \langle j m' | u | j \, -m' \rangle$$

$$+ \hbar \cancel{\sqrt{(j-m')(j+m'+1)}} \, \langle j, m'+1 | u | j, -m'-1 \rangle$$

$$\therefore \quad \langle j m' | u | j \, -m' \rangle = \underbrace{(-1)}_{i^2} \langle j \, m'+1 | u | j, -m'-1 \rangle$$

$$\Longrightarrow \quad \frac{\langle j \, m'+1 | u | j \, -m'-1 \rangle}{\langle j m' | u | j m' \rangle} = i^2$$

By recursion,

$$\frac{\langle j\, m' | u | j\, -m' \rangle}{\langle j\, m | u | j\, -m \rangle} = i^{2(m'-m)} .$$

Choosing: $\langle j, m | u | j, -m \rangle = (i)^{2m}$

$\therefore \quad \langle j, m' | u | j\, -m' \rangle = (-1)^{m'}$.

$\Rightarrow \quad u | j\, m \rangle = (-1)^{m} | j\, -m \rangle$.

~~Since~~ If $|j\, m\rangle$ is the natural basis of u, then $K |j\, m\rangle = |j\, m\rangle$ $\therefore$

$$u K | j\, m \rangle = T | j\, m \rangle = (-1)^{m} | j, -m \rangle .$$

d)

$$T\,D(k)\,|j\,m\rangle = T\,e^{i\phi\vec{J}\cdot\hat{n}/\hbar}\,|j\,m\rangle$$

$$= e^{i\phi/\hbar\,\vec{J}\cdot\hat{n}}\,T\,|j\,m\rangle$$

$$= (-1)^m\,D(k)\,|j,-m\rangle,$$

where $T\,D(k) = D(k)\,T$.

e)

$$D(k)\,T\,|j\,m\rangle = (-1)^m\,D(k)\,|j,-m\rangle$$

$$\Rightarrow \langle j,-m'|\,D(k)\,T\,|j\,m\rangle = (-1)^m\,D^{(j)}_{-m',-m}$$

$$= \langle j,-m'|\,T\,D(k)\,|j\,m\rangle$$

$$= \langle j,-m'|\,T\sum_{m''}|j\,m''\rangle\underbrace{\langle j\,m''|\,D(k)\,|j\,m\rangle}_{D^{(j)}_{m'',m}}$$

$$= \langle j,-m'|\left(T\,|j,m''\rangle\right)D^{*(j)}_{m'',m}$$

$$= (-1)^{+m'} D^{*(j)}_{+m', m}$$

$$\therefore \quad D^{*(j)}_{m', m} = (-1)^{m-m'} D^{(j)}_{-m', -m}.$$

f)

$$H T|E\rangle = T H |E\rangle = E\,(T|E\rangle)$$

since $[H, T] = 0$. Hence $T|E\rangle$ is an eigenstate of H with energy E. Due to the lack of degeneracy, $\Longrightarrow T|E\rangle = |E\rangle$.

$$\therefore \quad \langle E|\vec{L}|E\rangle = + \langle E|T^{-1}\underbrace{(T\vec{L}T^{-1})}_{-\vec{L}}T|E\rangle$$

$$= -\langle E|\vec{L}|E\rangle \Longrightarrow \langle E|\vec{L}|E\rangle = 0$$