COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Classical Mechanics

CH. 3 THE CENTRAL FORCE PROBLEM LECTURE NOTES

STUDENT
Taylor Larrechea

PROFESSOR
Dr. Robert Lewis-Swan

10-4-21

The central Force Problem

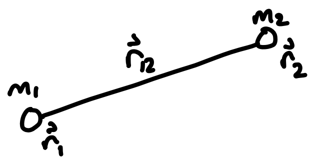
Force : $\vec{F} = f(r)\,\hat{r}$

$f(r)$ ← Characterizing type and strength of force

Conceptually : Non-static force

Example : Centripetal Force $\quad F(r) = \dfrac{mv^2}{r}$

Key-Application : Two-body problem



$\vec{F}_{21} \propto f(r_{12})\,\hat{r}_{12}$: Strong and weak law of action & re-action : $F_{12} = -F_{21}$

We will consider a monogenic system : $U(r_1, r_2) = U(r_{12})$ also $U(\dot{r}_{12})$

Reduce two body problem to one-body problem

$\Rightarrow$ Introduce center of mass & relative position vectors :

$\vec{r}_1, \vec{r}_2$: position vectors for particles 1 & 2

$\vec{r} = \vec{r}_2 - \vec{r}_1$: relative position vector : $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$: center of mass position vector

Then, in the "original frame" :

$\vec{r}_1 = \vec{R} + \vec{r}_1{}' \quad , \quad \vec{r}_2 = \vec{R} + \vec{r}_2{}'$

$\vec{r}_2{}'$ ← Position in center of mass / rel frame

$\vec{R}$ ← Position center of mass

Lagrangian : $L(\vec{r}, \vec{R}) = T(\dot{\vec{r}}, \dot{\vec{R}}) - U(\vec{r}, \dot{\vec{r}})$

Kinetic Term : $T(\dot{\vec{R}}, \dot{\vec{r}}) = \dfrac{m_1+m_2}{2}\dot{\vec{R}}^2 + T' \quad , \quad T' = \dfrac{m_1}{2}\dot{\vec{r}}_1{}'^2 + \dfrac{m_2}{2}\dot{\vec{r}}_2{}'^2 = \dfrac{1}{2}\dfrac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^2$

$L = \dfrac{1}{2}\mu\dot{\vec{r}}^2 + \dfrac{1}{2}M\dot{\vec{R}}^2 - U(\vec{r}, \ldots)$

$\vec{R} \rightarrow$ cyclic co-ordinate(s) , Associated conserved quantity : $\dfrac{\partial \vec{L}}{\partial \dot{\vec{R}}} = \vec{P}$, Total linear com momentum

$M\dot{R}(0) = \vec{P}(0) \longrightarrow R(t) = \dot{R}(0)t + R(0)$

$L = \dfrac{1}{2}\mu\dot{\vec{r}}^2 - U(\vec{r}) \longrightarrow$ one-body problem

In general (for us) : $U = U(r) \quad , \quad r = |\vec{r}^{\,2}|$

$\Rightarrow$ Spherical Symmetry : Lagrangian is invariant under rotations

$\Rightarrow$ Conserved quantit(ies) ?

$\Rightarrow$ Total angular momentum is conserved !

$\vec{L} = \vec{r} \times \vec{p}$ $\quad\longrightarrow$ 3 conserved quantities

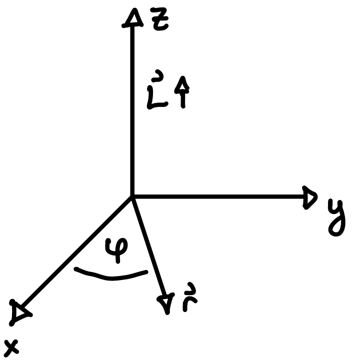
$\vec{r}$ —— defined w.r.t relative co-ordinates

$\Rightarrow$ Use conserved quantities to solve dynamics

<u>First Consequence</u>: Motion of the system is restricted to a 2D plane to which $\vec{L}$ is normal

$\vec{L} = \mu \vec{r} \times \dot{\vec{r}} = \text{const.} \Rightarrow \vec{r}\cdot\vec{L} = 0 \;,\quad \vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times\dot{\vec{r}}) = \dot{\vec{r}}\cdot(\vec{r}\times\vec{r}) = 0$

Concretely choose $\vec{L} = L\hat{z}$, Plane of motion , x-y plane



In spherical co-ordinate,

$$\vec{r} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \longrightarrow \vec{r} \equiv \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} \text{ w/ } \theta = \pi/2$$

In 'polar' co-ordinates,

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) \;,\quad \varphi \text{ is cyclic} \;,\quad P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \text{const.} \;,\quad P_\varphi = \mu r^2\dot{\varphi} \rightarrow \text{ang. mom.}$$

$$|\vec{L}| = L = \mu|\vec{r}\times\dot{\vec{r}}| = \mu r^2(\dot{\theta} + \dot{\varphi}^2\sin^2\theta)^{1/2} = \mu r^2\dot{\varphi}$$

$$L = L(r,\dot{r}) = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - U(r) \xrightarrow[\text{EOM}]{} \frac{d}{dt}(\mu\dot{r}) + \frac{L^2}{\mu r^3} + \frac{\partial U}{\partial r} = 0$$

Energy $\rightarrow$ New conserved quantity (obtain from energy function h)

$$E = \frac{\mu\dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r) = \text{const.}$$

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}}\sqrt{E - U(r) - \frac{L^2}{2\mu r^2}} \quad (*) \qquad dt = \pm\sqrt{\frac{\mu}{2}}\frac{dr}{\sqrt{E - U(r) - \frac{\ell^2}{2\mu r^2}}}$$

$$\int_{t_0}^{t} dt' = \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r} dr' \frac{1}{\sqrt{E - U(r) - \frac{\ell^2}{2\mu r^2}}} \quad : \quad t - t_0 \Rightarrow t(r) \longrightarrow r(t)$$

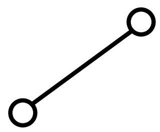
r & φ : typically want to obtain $r(\varphi)$

$$\hookrightarrow \frac{d\varphi}{dr} = \frac{d\varphi}{dt}\frac{dt}{dr} = \left(\frac{\dot{\varphi}}{\dot{r}}\right) \;:\; d\varphi = \frac{\dot{\varphi}}{\dot{r}}dr \;:\; \varphi - \varphi_0 = \pm\int_{r_0}^{r} \frac{\ell}{\mu r^2}\frac{dr'}{\sqrt{E - U(r') - \ell^2/2\mu r^2}}$$

10-6-21

Central Force Problem

$\vec{F} = -f(r)\,\hat{r}$



$\vec{F}_{21} = -\vec{F}_{12} \propto \hat{r}_{12}$

Two body problem

Lagrangian: $L(x,y,z)$, $L(r,\varphi,\theta) \longrightarrow$ 2D problem (dir of $\vec{L}$)

$L = \frac{\mu \dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} - U(r)$, $E = \frac{\mu \dot{r}^2}{2} + \frac{\ell^2}{2\mu r^2} + U(r)$, w/ $h = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} - L$

$\dot{r} \Rightarrow t = t(r)$: $t - t_0 = \pm \int_{r_0}^{r} dr' \sqrt{\frac{\mu}{2}} \frac{1}{\sqrt{E - U(r') - \ell^2/2\mu r'^2}}$, $t(r) \rightarrow r(t)$, $r(\varphi) \rightarrow \varphi(r)$

Classification of Orbits (3.3)

* Use analysis of particle's energy to make comments about r

$E = \frac{\mu \dot{r}^2}{2} + \frac{L^2}{2\mu r^2} + U(r)$

$\frac{L^2}{2\mu r^2} + U(r) \rightarrow$ Effective potential energy

$\frac{\mu \dot{r}^2}{2} \rightarrow$ kinetic energy

$F_{eff} = -\frac{\partial V_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} - \frac{\partial V}{\partial r}$: $\frac{\ell^2}{\mu r^3} \equiv \frac{(\mu r^2 \dot{\varphi})^2}{\mu r^3} = \frac{\mu (r^2 \dot{\varphi}^2)^2}{r} \equiv \frac{\mu v_\varphi^2}{r}$

$\frac{\ell^2}{\mu r^3} \rightarrow$ Centrifugal force $\longrightarrow$ Angular momentum barrier

From now on: $V(r) = -k/r^n$, $n \geq 1 \longrightarrow$ Integer

Example: $n = 1$

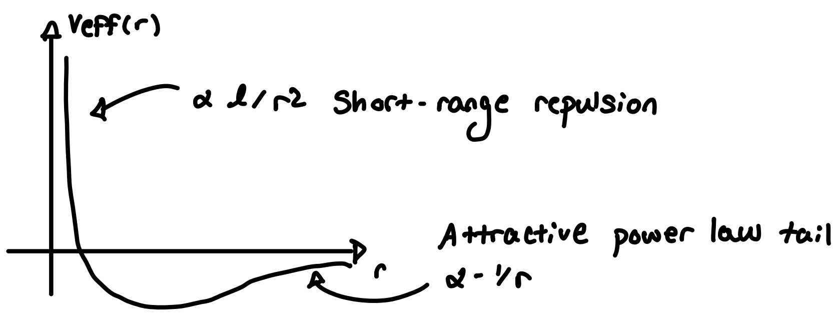
$\nearrow$ gravitational potential $\rightarrow k > 0$

$\rightarrow$ Coulomb interaction $\rightarrow k > 0$ or $k < 0$

$\Rightarrow$ n controls behavior at large & small distances
($r \to \infty$) ($r \to 0$)

$n = 1$ ($n > 0$) $\longrightarrow V_{eff} = -\frac{k}{r} + \frac{L^2}{2\mu r^2}$, $r \to 0$ V_{eff} diverges & positive (repulsive short range)
$r \to \infty$ V_{eff} vanishes ($\to 0$) weakly negative $\propto -\frac{k}{r}$

Sketch V_{eff}:



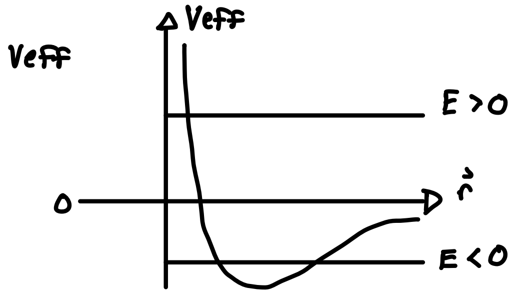
$n=3$:

$V_{eff} = -\frac{k}{r^3} + \frac{l^2}{2\mu r^2}$ $(k>0)$, $r \to 0$: attractive (negative) Veff diverges to $-\infty$ as $-1/r^3$
$r \to 0$: Veff $\to 0$ as $1/r^2$

* Use energy conservation & effective potential to classify behavior

* "Particle in a potential approach"

* Initial value $E_0 \Rightarrow$ describes allowable motion



① $E>0$: particle can probe $r \in [r_{min}, \infty) \longrightarrow V_{eff}(r_{min}) = E$

② $E<0$: particle can probe between two turning points $r \in [r_{min}, r_{max}]$
$V_{eff}(r_{min}) = V_{eff}(r_{max}) = E$

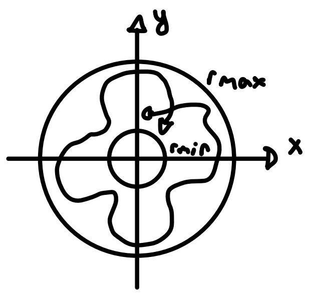
③ $E=0$: $V_{eff}(r_{min}) = 0$, $r \in [r_{min}, \infty]$

Physically:

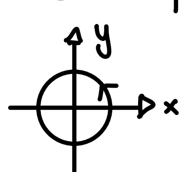
① $E>0$: * 1 turning point
* "Scattering motion" (Start at large r, bounce off short range barrier and go back)
* At $r \to \infty$: Free motion

② $E<0$: * 2 turning points
* "bound motion" $\leftarrow$ due to attractive potential (never escape V)

Note ①: $E<0$ (bound case) is periodic in r. We have said nothing about φ.
$\Rightarrow$ Don't assume we have a closed orbit.



Note ②: A particle w/ $E = E_{min}$ is not stationary $(\min(V_{eff}) = E_{min}) \Rightarrow$ Orbit is circular



$\dot{\varphi}(t) = \dot{\varphi}(0) \longleftarrow l = \mu r^2 \dot{\varphi} = \text{const}$

Q: Can we deduce general conditions for which bound motion / particles exhibit closed orbits?

Closed orbits: A particle repeatedly returns to its initial condition

Math: $r(\varphi) = r(\varphi + 2\pi j)$ $j \in \mathbb{Z}$ $\dot{r}(\varphi) = \dot{r}(\varphi + 2\pi_j)$

Bertrand's Theorem: The only forces that allow closed orbits for all bound particles are:
i) Inverse square law
ii) Hooke's law

Kepler's Problem (3.7)

* Problems of two bodies connected by a central force

* Want to obtain position and velocity of bodies as functions of time

Recall: $$\varphi - \varphi_0 = \pm \int_{r_0}^{r(\varphi)} \frac{dr'}{r'^2 \sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu V(r')}{\ell^2} - \frac{1}{r'^2}}}$$

Plug in $F = K/r^2$ or $V = -K/r$

Then,

i) Define $u = 1/r$

ii) $$\frac{-2\mu V(r')}{\ell^2} = \frac{2\mu K}{\ell^2 r'^2} = \frac{2\mu K}{\ell^2} u$$

$$\varphi - \varphi_0 = \pm \int_{u(\varphi_0)}^{u(\varphi)} \frac{du'}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu K}{\ell^2} u' - u'}}$$

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = (-\gamma)^{1/2} \arccos\left[\frac{-\beta + 2\gamma x}{\sqrt{q}}\right], \quad q = \beta^2 - 4\alpha\gamma \quad \gamma < 0$$

Abrahomowitz & Stegum

$$\alpha = \frac{2\mu E}{\ell^2}, \quad \beta = \frac{2\mu k}{\ell^2}, \quad \gamma = -1, \quad q = \frac{4\mu^2 k^2}{\ell^4} + \frac{8\mu E}{\ell^2} \geq 0, \quad E \geq \frac{-\mu k^2}{2\ell^2}$$

$$E_{min} \Rightarrow \frac{\partial V_{eff}}{\partial r} = 0 \Rightarrow r_0 = \frac{\ell^2}{\mu k}, \quad V_{eff}(r_0) = \frac{-\mu k^2}{2\ell^2} = E_{min}$$

$$\varphi - \varphi_0 = \pm \arccos\left[\frac{\ell^2 u/\mu k - 1}{\sqrt{1 + 2E\ell^2/\mu k^2}}\right], \quad r = \frac{1}{C}\left[1 + e\cos(\varphi - \varphi_0)\right]^{-1}$$

10-11-21

QQ: a.) Mathematically or geometrically
b.) Geometrically, $\rightarrow$ Light refracting through medium

Kepler Problem

$\varphi(r) = \int_{r_0}^{r(\varphi)} \ldots\ldots \quad V(r) \sim -k/r$

$\varphi - \varphi_0 = \pm \arccos\left(\frac{l^2 u/\mu k - 1}{\sqrt{1 + 2El^2/\mu k^2}}\right), \quad u = \frac{1}{r} \; : \; r(\varphi) = 1/C\left[1 + e\cos(\varphi - \varphi_0)\right]^{-1}, \quad C = \frac{\mu k}{l^2}$

$1/r = C\left[1 + e\cos(\varphi - \varphi_0)\right]$

$e = \sqrt{1 + \frac{2El^2}{\mu k^2}} \longrightarrow$ eccentricity

$e = 0$: circle $\longrightarrow$ (φ drops out), $e < 0$: ellipse $\longrightarrow$ (closed orbit)
$E = E_{min}$ $\qquad E < 0$

$e > 1$: Hyperbola, $e = 1$: Parabola $\longleftrightarrow$ Scattering $E \geq 0$

Want to get $r(t)$ and $\varphi(t)$ at the very end

$t - t_0 = \pm \frac{\mu}{2}\int_{r_0}^{r} \frac{dr'}{\sqrt{E - V(r') - l^2/2\mu r'^2}}, \quad t(r) = \ldots\ldots \Rightarrow r(t)$ "messy", $L = \mu r^2 \frac{d\varphi}{dt} = C$

$dt = \frac{\mu r^2}{L} d\varphi \; : \; t - t_0 = \frac{\mu}{L}\int_{\varphi_0}^{\varphi} r(\varphi)^2 d\varphi \Rightarrow \varphi(t) \Rightarrow r(\varphi(t))$

$t - t_0 = \frac{L^3}{\mu k^2}\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\left[1 + e\cos(\varphi - \varphi_0)\right]^2} \longrightarrow$ Goldstein: $e = 1 \rightarrow e$ general

Laplace-Runge-Lenz Vector

① Start = Newton's 2nd law, $\vec{F} = \dot{\vec{p}}$: $\vec{F} \rightarrow f(r)\hat{r} = \frac{f(r)}{r}\vec{r}$

② $\dot{\vec{p}} \times \vec{L} = \left(\frac{f(r)}{r}\vec{r}\right) \times \vec{L} = \left(\frac{f(r)}{r}\vec{r}\right) \times \left(\mu \vec{r} \times \dot{\vec{r}}\right)$: $\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$\left(\frac{f(r)}{r}\vec{r}\right) \times \left(\mu \vec{r} \times \dot{\vec{r}}\right) = \frac{\mu f(r)}{r}\left[\vec{r}(\vec{r}\cdot\dot{\vec{r}}) - \dot{\vec{r}}(\vec{r}\cdot\vec{r})\right]$: use : $\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) = r\dot{r}$

$\frac{\mu f(r)}{r}\left[\vec{r}(\vec{r}\cdot\dot{\vec{r}}) - \dot{\vec{r}}(\vec{r}\cdot\vec{r})\right] = \frac{\mu f(r)}{r}\left[(r\dot{r})\vec{r} - r^2\dot{\vec{r}}\right] = \mu r^2 f(r)\left[\frac{\dot{r}}{r^2}\vec{r} - \frac{1}{r}\dot{\vec{r}}\right]$

$\mu r^2 f(r)\left[\frac{\dot{r}}{r^2}\vec{r} - \frac{1}{r}\dot{\vec{r}}\right] = -\mu f(r) r^2 \frac{d}{dt}(\hat{r}) = \mu k \frac{d}{dt}(\hat{r}) \longrightarrow$ Assumption

$\frac{d}{dt}(\vec{p} \times \vec{L}) = \dot{\vec{p}} \times \vec{L} = \mu k \frac{d}{dt}(\hat{r}) \longrightarrow \frac{d}{dt}(\vec{p} \times \vec{L} - \mu k \hat{r}) = 0 \longrightarrow$ conserved quantity

$\vec{A} = \vec{p} \times \vec{L} - \mu k \hat{r}$: Laplace-Runge-Lenz vector w/ $\frac{d\vec{A}}{dt} = 0$ (Fig 3.18)

Features :

① $\vec{A}$ lives in the same plane as the motion of the system.

$$\vec{A}\cdot\vec{L} = 0 \longrightarrow \underbrace{\vec{p}\times\vec{L}\cdot\vec{L}}_{\perp\vec{L}=0} - \mu k \underbrace{\hat{r}\cdot\vec{L}}_{\hat{r}\perp\vec{L}=0}$$

② $\vec{A}\cdot\vec{r} = Ar\cos(\varphi)$, seperately $\vec{A}\cdot\vec{r} = (\vec{p}\times\vec{L})\cdot\vec{r} - \frac{\mu k}{r}\cancel{\vec{r}\cdot\vec{r}}^{\,r^2} = \vec{L}\cdot(\vec{r}\times\vec{p}) - \mu k r = L^2 - \mu k r$

we can get $r(\varphi)$!, $\ell^2 - \mu k r = Ar\cos(\varphi) \longrightarrow r = \frac{\ell^2}{\mu k}\,\frac{1}{1 + A/\mu k \cos(\varphi)}$

$e \longleftrightarrow \frac{A}{\mu k}$, $c \longleftrightarrow \frac{\mu k}{l^2}$ $\quad \varphi \to \varphi - \varphi_0$

Conserved quantities :

1) $\dot{\vec{R}} = \text{const}$

2) $E = \text{const}$ $\quad 1$

3) $\vec{L} = \text{const}$ $\quad 3$

4) $\vec{A} = \text{const}$ $\quad 3$

} 7 consered quantities

Recall

i) $\frac{A}{\mu k} = e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$, $A = A(E, L)$ $\quad -1$

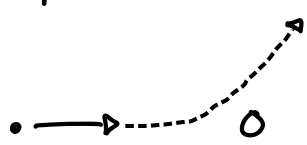
ii) $\vec{A}\cdot\vec{L} = 0$ $\quad -1$

$\vec{A} \longrightarrow 1$ cons. quantity $= 5$ overall

$\underline{\vec{A} = \vec{p}\times\vec{L} - \mu k\hat{r}}$ unique to $F(r) \sim -\frac{k}{r^2}$, "$\vec{A}$" exists $\longrightarrow$ closed orbits

<u>Scattering in a central force field</u>

Some particle traveling w/ $\vec{v}$:

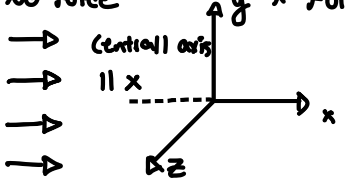


Rutherford experiment : α- particles scattering off a nucleus
(light) (heavy)

⇒ Consider a <u>beam</u> of particles incoming to a <u>force center</u>

⇒ Assume : * Particles in the beam have equal mass
* All have some (incoming) <u>energy</u>
* Force vanishes as $r \to \infty$

no force



⇒ Beam is characterized by a flux intensity I,

I = # of free particles crossing a unit area normal to beam per unit time

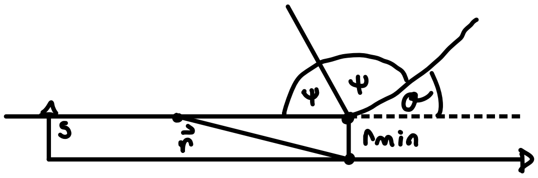
⇒ Beam has particles of mass μ, energy E

Impact parameter s: s = perpendicular distance between central path of beam (towards force center) & particle in beam.

Assume we have radial symmetry

Care about asymptotic behavior after deflection:

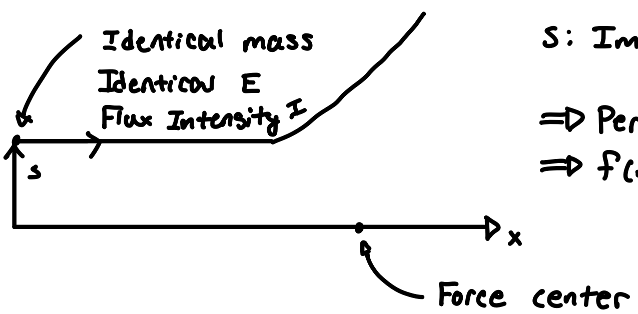
Goal: Determine final path of particle as $t \to \infty$ (interaction @ $t \sim 0$)



θ: Scattering angle. The angle between the incoming trajectory & outgoing asymptotic trajectory

10-13-21

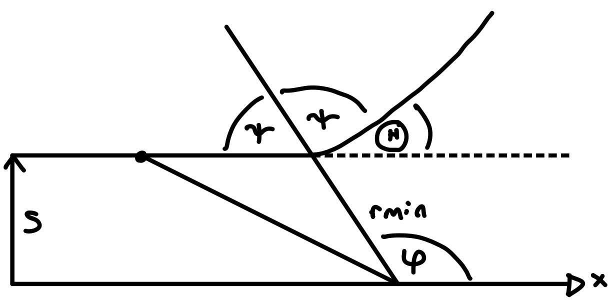
Scattering Problem



s: Impact parameter

⇒ Perpendicular height from central axis (x-axis) of beam

⇒ $f(r) \longrightarrow 0$ as $r \to \infty$



r_{min}: distance of closest approach to force center

Θ: Scattering angle "angle between incoming & outgoing trajectories"

Clearly,

① $\Theta + 2\Psi = \pi$, $\varphi_{min} = \Theta + \Psi$ when $\vec{r} = \vec{r}_{min}$

↳ Polar angle

Differential cross section, $\sigma(\vec{\Omega})\, d\vec{\Omega}$ = # of particles scattered into some solid angle $d\Omega$ / unit time : Incident intensity (I)

Solid angle $\longrightarrow d\Omega = 2\pi \sin(\alpha)\, d\alpha$

Assume : different s lead to different θ

particles w/ impact parameter $s \to s+ds$ = # particles scattered between $\alpha \to \alpha + d\alpha$

$2\pi I s|ds| = 2\pi \sigma(\alpha) I \sin(\alpha) |d\alpha|$

$\hookrightarrow$ Rearrange $\sigma(\alpha) = \dfrac{s}{\sin(\alpha)}\left|\dfrac{ds}{d\alpha}\right|$

Want : $s = s(\theta, E)$ or $\theta = \theta(s, E)$

Recall: $\displaystyle\int_{r_0}^{r(t)} \frac{dr'}{r'^2\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu V(r')}{\ell^2} - \frac{1}{r'^2}}}$

$\varphi_0 \Rightarrow$ polar angle @ $t = -\infty$

i) $\varphi_0 = \pi$ as $r \to \infty$ @ $t = -\infty$

ii) For $r = r_{min} \to \varphi = \varphi_{min} = \pi - \Psi$

iii) $\Theta + 2\Psi = \pi \to \dfrac{\Theta}{2} - \dfrac{\pi}{2} = -\Psi$

ii) $\to \varphi_{min} - \pi = -\Psi = \dfrac{\Theta}{2} - \dfrac{\pi}{2}$: $\varphi_{min} - \varphi_0 = \dfrac{\Theta}{2} - \dfrac{\pi}{2}$: $\varphi_{min} - \varphi_0 = \displaystyle\int_{r_0}^{r_{min}} \cdots\cdots$

$$\Theta = \pi - 2\int_{r_{min}}^{\infty} \frac{dr'}{r'^2\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu V(r')}{\ell^2} - \frac{1}{r'^2}}}$$

impact parameter $\longleftrightarrow$ angular momentum

* as $r \to \infty$ & $t = -\infty$, $\dot{x}_{incoming} = v_0$
* $s^2 = y^2_{incoming} + z^2_{incoming}$
* $E_{incoming} = E = \frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu \dot{x}^2_{incoming}$
* $\ell = -\mu |\vec{r} \times \dot{\vec{r}}|$ $\quad \dot{z} = \dot{y} = 0$ initially

$= \mu \dot{x}\sqrt{z^2 + y^2} = \mu \dot{x} s$: re-expressing using E : $s = \dfrac{L}{\sqrt{2\mu E}}$

Finally,

$$\Theta(s,E) = \pi - 2\int_{r_{min}}^{\infty} \frac{dr'}{r'\sqrt{\frac{1}{s^2} - \frac{V(r')}{Es^2} - \frac{1}{r'^2}}}, \quad u = \frac{1}{r'} : \Theta = \pi - 2\int_0^{u_{max}} \frac{s\,du}{\sqrt{1 - \frac{V(1/u)}{E} - s^2u^2}}$$

$\Theta \longrightarrow \sigma(\alpha) = \dfrac{s}{\sin(\alpha)}\left|\dfrac{ds}{d\alpha}\right|$

Example: Coulomb interaction

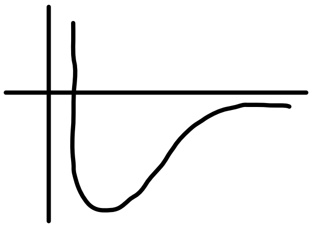
$\Rightarrow$ Consider two particles w/ charge q & q' $[q = 2e, \ q' = 79e]$

$\Rightarrow$ Coulomb potential, $V(r) = \frac{qq'}{r}$, $[f(r) = qq'/r^2]$

$\Rightarrow$ Effective potential, $V_{eff} = \frac{qq'}{r} + \frac{l^2}{2\mu r^2}$

Step 1: Find r_{min}

$$E_i = V_{eff}(r_{min}) = \frac{qq'}{r} + \frac{l^2}{2\mu r_{min}^2} \quad \text{math} \Rightarrow \quad r_{min}^2 - \frac{qq'}{E} r_{min} - \frac{l^2}{2\mu E} = 0 \quad l^2 = 2\mu E s^2$$



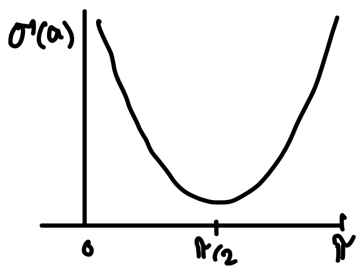
$$r_{min} = \frac{qq'^2}{2E} + \sqrt{\left(\frac{qq'}{2E}\right)^2 + s^2}$$, $r_{min} \geq 0$, using old solution, $s = \frac{qq'}{2E} \cot\left(\frac{\theta}{2}\right)$

case 1: $\theta = 0 \longrightarrow s \rightarrow \infty$

Case 2: $\theta = \pi \longrightarrow s = 0$

Plug into cross-section formula, $\sigma(\theta) = \frac{s}{\sin(\theta)}\left|\frac{ds}{d\theta}\right| = \frac{1}{4}\left(\frac{qq'}{2E}\right)\sin^{-4}\left(\frac{\theta}{2}\right)$

quantum <-> Classical



Total scattering cross section: $\sigma = \int \sigma(\Omega)\, d\Omega$

For a coulomb interaction,

$$\sigma_T = \int_0^{\pi} \sin^{-4}(\theta/2)\sin(\theta)\, d\theta \longrightarrow \text{diverges w/ Experiment}$$

$$V_{real}(r) = V(r)e^{-\alpha r}$$