Quantum Mechanics
Qualifying Exam - January 2017

Notes and Instructions

- There are 6 problems. Attempt them all as partial credit will be given.
- Write on only one side of the paper for your solutions.
- Write your alias on the top of every page of your solutions.
- Number each page of your solution with the problem number and page number (e.g. Problem 3, p. 2/4 is the second of four pages for the solution to problem 3.)
- You must show your work to receive full credit.

Possibly useful formulas:

Spin Operator

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

In spherical coordinates,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}r\psi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi. \tag{2}$$

Harmonic oscillator wave functions

$u_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$

$u_1(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega x^2}{2\hbar}}$

Spherical Harmonics:

$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$

$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$

$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\cos\theta\sin\theta$

$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta$

Problem 1: Harmonic Oscillator (10 Points)

Consider the quantum mechanical simple harmonic oscillator.

a. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$ find the average value of X and P for the state $|n>$. **(1 Points)**

b. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$ find the average value of X^2 and P^2 for the state $|n>$. **(2 Points)**

c. Using the raising and lower operators, $\hat{a}$ and $\hat{a}^\dagger$ find the root mean square deviations of X and P for the state $|n>$. **(2 Points)**

d. Find the uncertainty product for the state $|n>$ **(2 Points)**

e. Fine the average potential energy and average kinetic energy for the oscillator when it is in state $|n>$ **(3 Points)**

Problem 2: Variational Method (10 Points)

The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{P^2}{2m} + \frac{m\omega^2 X^2}{2} \, .$$

The ground state energy is $E_0 = \hbar\omega/2$.

Let us employ the variational method with the following trial function as the ground-state wave function

$$\langle x|\psi\rangle = \psi(x) = Ne^{-\beta|x|} \, .$$

a. Determine the constant N by applying the normalization condition. (2 points)

b. Find the value of β that minimizes $\langle\psi|H|\psi\rangle$. (2 points)

c. What is the ground-state energy calculated with the variational method? (5 points)
N.B. *The derivative of the trial function has a discontinuity.*

d. How close do you get to the true ground-state energy? (1 points)

Problem 3: Angular Momentum Hamiltonian (10 points)

Consider the following Hamiltonian for a spinless particle with orbital angular momentum ℓ=2.

$$\hat{H} = \frac{3a}{2\hbar}\hat{L}_z - \frac{a}{\hbar^2}(\hat{L}_x^2 + \hat{L}_y^2)$$

where a is a constant greater than 0 and $\hat{L}_i$ denotes the i^{th} component of the angular momentum operator.

a) Calculate the energy spectrum of this Hamiltonian (2 pts)

b) Suppose a particle with this Hamiltonian has the wavefunction

$$\Psi(\theta, \phi) = A(\sin\theta\cos\theta\cos\phi + \sin^2\theta\sin\phi\cos\phi)$$

where θ is the polar angle, ϕ is the azimuthal angle, and A is a normalization constant. What is the average energy obtained in energy measurements on an ensemble of particles described by the wavefunction above? (3 pts)

c) Assume the particle is in the lowest energy state (with ℓ=2) for $t < 0$. Starting at t=0, an external magnetic field is applied with

$$\hat{V}(t) = \frac{\lambda}{\hbar}\hat{L}_x e^{-t/\tau}$$

where τ is the decay constant and λ is a constant. Calculate the transition probabilities to possible excited states after a very long time ($\tau \ll t \to \infty$) using first order time-dependent perturbation theory. (5 pts)

Problem 4: Hydrogen Atom (10 points)

Schrodinger's equation in spherical coordinates where the potential is only a function of r can be solved by using separation of variables: $\Psi(r,\theta,\phi)=R(r)Y(\theta,\phi)$.

a) In units $2m = 1$ and $\hbar$=1, show that using the change of variables $u(r) \equiv rR(r)$, one can obtain the radial Schrodinger's equation for the hydrogen atom. (1 pt)

$$[-\frac{d^2}{dr^2} - \frac{g^2}{r} + \frac{\ell(\ell+1)}{r^2}]u(r) = \epsilon u(r)$$

where g^2 is the Coulomb strength and ϵ is the energy.

b) The lowest eigenstate of a given ℓ is known to have the form

$$u_\ell^0 = C_\ell r^{\ell+1} exp(-r/a_\ell)$$

For a given ℓ, determine the eigenvalue ϵ_ℓ^0 and the size parameter a_ℓ, in terms of g^2 (2 pts).

Consider that the initial 3-dimensional wave function at time t=0 is a superposition of the above states

$$\Psi(r,0) = D(e^{-g^2\frac{r}{2}} + g^2 r e^{-g^2\frac{r}{4}}\cos\theta)$$

c) Determine $\Psi(r,t)$ (1 pt)

d) Determine $\langle\cos\theta\rangle$ as a function of time (3 pts).

e) Consider the hydrogen atom. Determine the most probable value of r for the ground state. (1 pt)

f) Consider a hydrogen atom placed in a weak constant uniform external electric field. Determine how the energy levels shift for the n=2 state of hydrogen due to the electric field. (2 pts)

Problem 5: 1/x potential (10 points)

An electron moves in one dimension and is confined to the right half space (x > 0) where is has potential energy

$$V(x) = -\frac{e^2}{4x}$$

where e is the charge on an electron.

a) What is the solution of Schrodinder's equation at large x? (2 pts)

b) What are the necessary boundary conditions (1 pt)

c) Using the results of part a) and b), determine the ground state solution of the equation. (3 pts)

d) Determine the ground state energy (2 pts)

e) Find the expectation value $\langle x \rangle$ in the ground state (2 pts)

Problem 6: Measurements and Probability (10 points)

A three-level quantum system has a non-degenerate ground state and a two-fold degenerate excited state, defined by:

$$H|0\rangle = 0 \;, \qquad H|a\rangle = \epsilon|a\rangle \;, \qquad H|b\rangle = \epsilon|b\rangle$$

where ϵ is a positive constant energy.

(a) (1 pt) Write down the matrix representation of H in the basis $|0\rangle$, $|a\rangle$, $|b\rangle$.

(b) (2 pts.) Define the observable C by it's operation on the eigenstates of H.

$$C|0\rangle = \gamma|a\rangle \;, \quad C|a\rangle = \gamma|0\rangle \;, \quad C|b\rangle = -\gamma|b\rangle \tag{3}$$

$\gamma > 0$. What are all the possible outcomes of a measurement of C?

(c) (2 pts.) For each of the eigenstates of H, calculate the probability of measuring the different possible values for C is the system is in that eigenstate.

(d) (1 pts.) Do H and C have common eigenstates? Are H and C compatible observables? Explain.

(e) (2 pts.) At time $t = 0$, the system is in the eigenstate of C with the largest eigenvalue. Calculate the probabilities, as functions of time, of obtaining the different possible results of a measurement of C.

(f) (2 pt.) At time $t = 0$, the system is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |b\rangle\right)$. Calculate the probabilities, as functions of time, of obtaining the different possible results of a measurement of C. Explain the differences in this result and what was found in part (e).