E & M I

Workshop 10 – Dielectrics, 4/13/2022

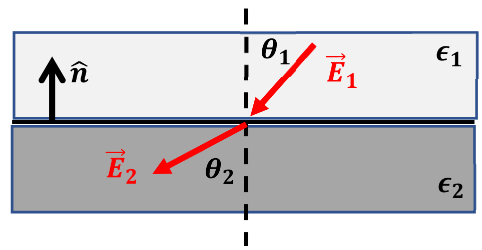
1) Boundary Conditions

In class on Monday, we introduced the concept of the polarization field of a material, the electric displacement, the dielectric, and the polarizability:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}, \qquad \vec{P} = \epsilon_0 \chi_e \vec{E}, \qquad \epsilon = \epsilon_0 (1 + \chi_e)$$

The simplest model for ϵ and χ_e is that they are "Linear, Isotropic, and Homogeneous". Explain, briefly, what these terms mean in the context of dielectrics.

Consider the interface between two dielectric materials, ϵ_1 and ϵ_2. $\hat{n}$ is the normal to the interface and there is no free charge on this surface. There are electric fields $\vec{E}_1$ and$\vec{E}_2$ in the dielectric materials. From Maxwell's equations and the definition of $\vec{D}$ the boundary conditions on the fields can be written:



$$\vec{\nabla} \cdot \vec{D} = 0 \Leftrightarrow \hat{n} \cdot (\vec{D}_1 - \vec{D}_2) = \sigma_{free} = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \Leftrightarrow \hat{n} \times (\vec{E}_1 - \vec{E}_2) = 0$$

A) Define the normal and parallel component fields as, for example:

$$\vec{E} = \vec{E}_n + \vec{E}_p, \qquad \hat{n} \cdot \vec{E}_p = 0, \qquad \hat{n} \times \vec{E}_n = 0$$

Using the boundary conditions above, determine the boundary condition for the parallel fields $\vec{E}_{1p}$ and $\vec{E}_{2p}$.

$$\hat{n} \times (\vec{E}_{n1} - \vec{E}_{n2}) + \hat{n} \times (\vec{E}_{p1} - \vec{E}_{p2}) = 0$$

The $\vec{E}_n$ term vanishes but $\hat{n} \times \vec{E}_p \neq 0$ giving:

$$\vec{E}_{p1} = \vec{E}_{p2}$$

C) Using the boundary conditions above, determine the boundary condition for the normal fields $\vec{E}_{1n}$ and $\vec{E}_{2n}$. (Consider $\vec{D}_{1n}$ and $\vec{D}_{2n}$.)

$$\hat{n} \cdot (\vec{D}_{n1} - \vec{D}_{n2}) + \hat{n} \cdot (\vec{D}_{p1} - \vec{D}_{p2}) = 0$$

But $\hat{n} \cdot \vec{D}_p = \hat{n} \cdot \epsilon \vec{E}_p = 0$ giving

$$\hat{n} \cdot (\vec{D}_{n1} - \vec{D}_{n2}) = 0$$

$$\vec{D}_{n1} = \vec{D}_{n2} \Rightarrow \epsilon_1\, \vec{E}_{n1} = \epsilon_2\, \vec{E}_{n2}$$

D) Defining θ_1 and θ_2 as the angles of the field relative to the normal, derive a relationship between $\tan\theta_1$ and $\tan\theta_2$, giving the change in direction of the electric field in the two dielectrics.

Considering the picture above, the angles are given by:

$$\tan\theta_1 = \frac{E_{1p}}{E_{1n}}, \tan\theta_2 = \frac{E_{2p}}{E_{2n}}$$

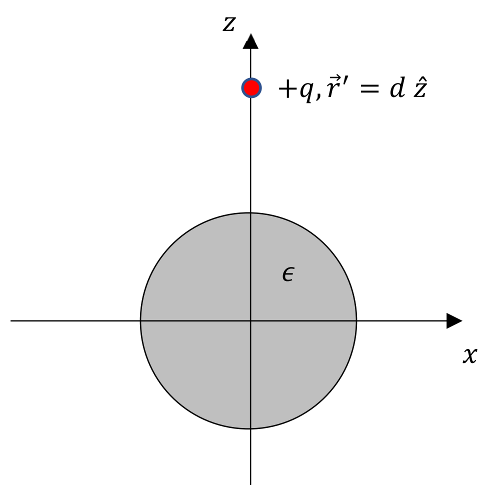
$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{E_{1p}}{E_{1n}}\frac{E_{2n}}{E_{2p}} = \frac{E_{2n}}{E_{1n}} = \frac{\epsilon_1}{\epsilon_2}$$

For the picture above, which is larger, ϵ_1 or ϵ_2?

From above we see that $\theta_2 > \theta_1$ so

$$\frac{\epsilon_1}{\epsilon_2} = \frac{\tan\theta_1}{\tan\theta_2} < 1, \qquad \epsilon_2 > \epsilon_1$$

2) Dielectric Sphere and a charge



Consider an insulating sphere of radius R and dielectric constant ϵ and a point charge $+q$ on the z-axis with a position $\vec{r}\,' = d\,\hat{z}$.

Due to the spherical boundary conditions and the azimuthal symmetry of the problem, we know we can use an expansion in Legendre Polynomials. For any azimuthally symmetric function satisfying Laplace's Equation:

$$F(r,\theta) = \sum_l \left(a_l\, r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos\theta)$$

And for the point charge:

$$\frac{1}{|\vec{r} - \vec{r}\,'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\, P_l(\cos\theta)$$

Where $r_< = r, \text{for } r < r', r_< = r', \text{for } r > r', r_> = r, \text{for } r > r'$, and $r_> = r', \text{for } r < r'$.

NOTE: Only 1 group made significant progress on this problem. It is a useful enough problem that it will show up on the last homework.