Solutions to Homework 10
Physics 5393

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P-3.10 Consider a sequence of Euler rotations represented by

$$\tilde{\boldsymbol{\mathcal{D}}}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a single rotation about some axis by an angle θ. Find θ.

The solution to this problem amounts to equating the matrix given above to that given in terms of $\hat{\mathbf{n}}$ and the angle of rotation

$$\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) & (-in_x - n_y)\sin\left(\frac{\theta}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + in_z\sin\left(\frac{\theta}{2}\right) \end{pmatrix} = \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}.$$

The easiest method for equate the angle θ to the Euler angles is by equating the traces of the matrices

$$2\cos\left(\frac{\theta}{2}\right) = \left[e^{-i(\alpha+\gamma)/2} + e^{i(\alpha+\gamma)/2}\right]\cos\left(\frac{\beta}{2}\right)$$
$$= 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)$$
$$\Rightarrow \quad \theta = 2\cos^{-1}\left[\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right]$$

P-3.17 An angular momentum eigenstate $|j, m = m_{\text{max}} = j\rangle$ is rotated by an infintesmal angle ϵ about the y axis. Without using the explicit form of the $d^{(j)}_{m'm}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ϵ^2.

We start by giving the expression for the rotated state to order ϵ^2

$$\tilde{\boldsymbol{\mathcal{D}}}_y(\epsilon)\,|j,j\rangle = \exp\left(\frac{-i\tilde{\mathbf{J}}_y\epsilon}{\hbar}\right)|j,j\rangle \approx \left[\tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{J}}_y\epsilon}{\hbar} - \frac{\tilde{\mathbf{J}}_y^2\epsilon^2}{2\hbar^2}\right]|j,j\rangle\,.$$

We next calculate the amplitude to remain in the original state

$$\left\langle j,j\left|\tilde{\mathbf{1}} - \frac{i\tilde{\mathbf{J}}_y\epsilon}{\hbar} - \frac{\tilde{\mathbf{J}}_y^2\epsilon^2}{2\hbar^2}\right|j,j\right\rangle,$$

which we split into three calculations, one for each term in the sum.

The first term:

$$\left\langle j,j \left| \tilde{\mathbf{1}} \right| j,j \right\rangle = 1$$

The second term:

$$\left\langle j,j \left| \frac{i\tilde{\mathbf{J}}_y \epsilon}{\hbar} \right| j,j \right\rangle = \left\langle j,j \left| \frac{i(\tilde{\mathbf{J}}_+ - \tilde{\mathbf{J}}_-)\epsilon}{2i\hbar} \right| j,j \right\rangle$$
$$= \left\langle j,j \left| \frac{i\tilde{\mathbf{J}}_+ \epsilon}{2i\hbar} \right| j,j \right\rangle - \left\langle j,j \left| \frac{i\tilde{\mathbf{J}}_- \epsilon}{2i\hbar} \right| j,j \right\rangle = 0,$$

since $\left\langle j,j \left| \tilde{\mathbf{J}}_\pm \right| j,j \right\rangle = 0$.

The third term:

$$\left\langle j,j \left| \frac{\tilde{\mathbf{J}}_y^2 \epsilon^2}{2\hbar^2} \right| j,j \right\rangle = \left\langle j,j \left| \frac{(\tilde{\mathbf{J}}_+ - \tilde{\mathbf{J}}_-)^2 \epsilon^2}{-8\hbar^2} \right| j,j \right\rangle$$
$$= \left\langle j,j \left| \frac{(\tilde{\mathbf{J}}_+^2 + \tilde{\mathbf{J}}_-^2) \epsilon^2}{-8\hbar^2} \right| j,j \right\rangle + \left\langle j,j \left| \frac{(\tilde{\mathbf{J}}_- \tilde{\mathbf{J}}_+) \epsilon^2}{8\hbar^2} \right| j,j \right\rangle + \left\langle j,j \left| \frac{(\tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_-) \epsilon^2}{8\hbar^2} \right| j,j \right\rangle$$
$$= \left\langle j,j \left| \frac{(\tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_-) \epsilon^2}{8\hbar^2} \right| j,j \right\rangle$$
$$= \left(\sqrt{2j}\,\hbar\right)^2 \frac{\epsilon^2}{8\hbar^2} = \frac{\epsilon^2 j}{4}$$

Combining the three pieces, and calculating the probability (magnitude squared), we arrive at the solution

$$\left| \left\langle j,j \left| \frac{i\tilde{\mathbf{J}}_y \epsilon}{\hbar} \right| j,j \right\rangle \right|^2 = \left| 1 - \frac{\epsilon^2 j}{4} \right|^2 = 1 - \frac{\epsilon^2 j}{2},$$

where we keep terms to second order in ϵ only.

P-3.15 Ladder operator.

a) Let $\tilde{\mathbf{J}}$ be angular momentum. (It may stand for orbital $\tilde{\mathbf{L}}$, spin $\tilde{\mathbf{S}}$, or $\tilde{\mathbf{J}}_{\text{total}}$.) Using the fact that $\tilde{\mathbf{J}}_x$, $\tilde{\mathbf{J}}_y$, $\tilde{\mathbf{J}}_z$ ($\tilde{\mathbf{J}}_\pm = \tilde{\mathbf{J}}_x \pm \tilde{\mathbf{J}}_y$) satisfy the usual angular momentum commutation relations, prove

$$\tilde{\mathbf{J}}^2 = \tilde{\mathbf{J}}_z^2 + \tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_- - \hbar \tilde{\mathbf{J}}_z.$$

The most straightforward method for solving the problem is to expand the product of the ladder operators

$$\tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_- = \tilde{\mathbf{J}}_x^2 + \tilde{\mathbf{J}}_y^2 - i\left(\tilde{\mathbf{J}}_x \tilde{\mathbf{J}}_y - \tilde{\mathbf{J}}_y \tilde{\mathbf{J}}_x\right) = \tilde{\mathbf{J}}^2 - \tilde{\mathbf{J}}_z^2 - i\left[\tilde{\mathbf{J}}_x, \tilde{\mathbf{J}}_y\right] = \tilde{\mathbf{J}}^2 - \tilde{\mathbf{J}}_z^2 + \tilde{\mathbf{J}}_z \hbar$$
$$\Rightarrow \quad \tilde{\mathbf{J}}^2 = \tilde{\mathbf{J}}_z^2 + \tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_- - \hbar \tilde{\mathbf{J}}_z.$$

b) Using (a), derive the famous expression for the coefficient c_- that appears in

$$\tilde{\mathbf{J}}_- \psi_{jm} = c_- \psi_{j,m-1}.$$

Using the bra-ket notation for simplicity, we take the inner product of the ket with its dual

$$
\begin{aligned}
|c_-|^2 &= \left\langle j,m \middle| \tilde{\mathbf{J}}_-^\dagger \tilde{\mathbf{J}}_- \middle| j,m \right\rangle = \left\langle j,m \middle| \tilde{\mathbf{J}}_+ \tilde{\mathbf{J}}_- \middle| j,m \right\rangle \\
&= \left\langle j,m \middle| \tilde{\mathbf{J}}^2 - \tilde{\mathbf{J}}_z^2 + \hbar \tilde{\mathbf{J}}_z \middle| j,m \right\rangle = \left[j(j+1) - m^2 + m \right] \hbar^2 \\
\Rightarrow \quad c_- &= \sqrt{(j+m)(j-m+1)}\, \hbar,
\end{aligned}
$$

where we take c_- to be real and positive.

P-3.20 Construct the matrix representations of the operators $\tilde{\mathbf{J}}_x$ and $\tilde{\mathbf{J}}_y$ for a spin 1 system, in the $\tilde{\mathbf{J}}_z$ basis, spanned by the kets $|+\rangle \equiv |1,1\rangle$, $|0\rangle \equiv |1,0\rangle$, and $|-\rangle \equiv |1,-1\rangle$. Use these matrices to find the three analogous eigenstates for each of the two operators $\tilde{\mathbf{J}}_x$ and $\tilde{\mathbf{J}}_y$ in terms of $|+\rangle$, $|0\rangle$, and $|-\rangle$.

Using the $\tilde{\mathbf{J}}_\pm$ marix representations in the textbook and the relations $2\tilde{\mathbf{J}}_x = \tilde{\mathbf{J}}_+ + \tilde{\mathbf{J}}_-$ and $2i\tilde{\mathbf{J}}_y = \tilde{\mathbf{J}}_+ - \tilde{\mathbf{J}}_-$, the matrices are

$$
\tilde{\mathbf{J}}_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \tilde{\mathbf{J}}_y \doteq -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
$$

One of many methods of calculating the eigenstates is to use the eigenvalue equation, for example the $\tilde{\mathbf{J}}_x$ problem would have the following form

$$
\tilde{\mathbf{J}}_x \, |1,m\rangle = m \, |1,m\rangle \quad \Rightarrow \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = m \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ with } a^2 + b^2 + c^2 = 1.
$$

The eigenvectors are therefore

$$
\begin{aligned}
|1,1\rangle_{J_x} &= \frac{1}{2}\,|+\rangle + \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{2}\,|-\rangle & |1,1\rangle_{J_y} &= -\frac{1}{2}\,|+\rangle - \frac{i}{\sqrt{2}}\,|0\rangle + \frac{1}{2}\,|-\rangle \\
|1,0\rangle_{J_x} &= -\frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle & |1,0\rangle_{J_y} &= \frac{1}{\sqrt{2}}\,|+\rangle + \frac{1}{\sqrt{2}}\,|-\rangle \\
|1,-1\rangle_{J_x} &= \frac{1}{2}\,|+\rangle - \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{2}\,|-\rangle & |1,-1\rangle_{J_y} &= -\frac{1}{2}\,|+\rangle + \frac{i}{\sqrt{2}}\,|0\rangle + \frac{1}{2}\,|-\rangle
\end{aligned}
$$