Homework #7

1

a) $\beta^2 = 1$, $\alpha_i^2 = 1 \Rightarrow \beta$ and α_i have eigenvalues ± 1. Since

$$\{\beta, \alpha_i\} = 0$$

$\therefore$ $Tr[\beta] = Tr[\alpha_i] = 0$, as shown in class. Since the trace is invariant under unitary transformations, the trace of any matrix is the sum of its eigenvalues $(\pm 1) \Rightarrow$ a zero trace requires an equal number of $+1$ and -1 eigenvalues. $\therefore$ β and α_i ~~are~~ have even dimension.

b)

A generic 2×2 traceless matrix

can be written as:

$$\beta \equiv \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\sigma_z + \frac{b}{2}(\sigma_x + i\sigma_y)$$

$$+ \frac{c}{2}(\sigma_x - i\sigma_y)$$

If $\alpha_i = \sigma_i$ $(i = x,y,z)$ then there is

no nontrivial $\beta \neq 0$ such that

$$\{\beta, \sigma_i\} = 0 \quad , \text{ for all } i,$$

Since:

$$\{\beta, \sigma_x\} = b + c$$

$$\{\beta, \sigma_y\} = i(b-c)$$

$$\{\beta, \sigma_z\} = 2a .$$

e)

If

$$\alpha^3 = \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} = 1 \otimes \sigma_z$$

satisfies the Dirac Algebra:

$$\begin{cases} \{\alpha_i, \alpha_j\} = 2\delta_{ij} \\ \{\alpha_i, \beta\} = 0 \\ \beta^2 = 1 \end{cases}$$

then one can build the matrices

$$\alpha^2 = \sigma_x \otimes \sigma_y$$

$$\alpha^1 = \sigma_x \otimes \sigma_y$$

$$\beta = \sigma_x \otimes \sigma_z$$

which anti-commute with each other

and satisfy $\beta^2 = (\alpha^i)^2 = 1$. The set of matrices that satisfy the Dirac algebra is not unique and can be constructed in different representations.

(A)

(2)

Defining $\bar{\psi}\psi = \psi^{+}\gamma^{0}\psi = \pm 1$

with $\pm$ for positive $(E>0)$ and $(E<0)$ negative energy states, the spinor solution for $E>0$, is:

$$u(k) = A \begin{pmatrix} (k_0 + \frac{mc}{\hbar})\, e(m,0) \\ \hline |\vec{k}|\,(\vec{\sigma}\cdot\hat{k})\, e(m,0) \end{pmatrix}$$

with

$$\psi^{(+)}(x) = e^{-ik\cdot x}\, u(k)$$

$$\Rightarrow \quad u^{+}(k)\,\gamma^{0}\, u(k) = +1.$$

$$\therefore \quad |A|^2 \left[\left(k_0 + \frac{mc}{\hbar}\right)^2 |e(m,0)|^2 - k^2\, e^{+}(m,0)\,(\vec{\sigma}\cdot\hat{k})^2\, e(m,0) \right] = 1$$

$\therefore$

$$1 = |A|^2 \left[\left(k_0 + \frac{mc}{\hbar} \right)^2 - \vec{k}^2 \right]$$

Since:

$$\vec{k}^2 = \frac{1}{\hbar^2} \left(\frac{E^2}{c^2} - m^2 c^2 \right)$$

$$k_0 = \frac{E}{c\hbar}$$

$$\Longrightarrow \quad 1 = \frac{|A|^2}{\hbar^2 c^2} \left[(E + mc^2)^2 - E^2 + m^2 c^4 \right]$$

$$= \frac{2|A|^2}{\hbar^2 \cancel{c^2}} m \cancel{c^2} (E + mc^2)$$

$$\therefore \quad |A| = \frac{\hbar c}{\sqrt{2mc^2(E + mc^2)}}$$

Hence:

$$\psi^{(+)}(x) = e^{-ik\cdot x} \begin{pmatrix} \sqrt{\dfrac{E+mc^2}{2mc^2}}\, e(m,s) \\ \dfrac{\hbar c|\vec{k}|(\vec{\sigma}\cdot\hat{k})\, e(m,s)}{\sqrt{2mc^2(E+mc^2)}} \end{pmatrix}$$

For $E<0$,

$$\bar{\psi}\psi = -1,$$

where

$$\psi^{(-)}(x) = e^{ik\cdot x}\, v(k)$$

with

$$v(k) = -B \begin{pmatrix} |\vec{k}|(\vec{\sigma}\cdot\hat{k})\,\chi(m,s) \\ \left(k_0 - \dfrac{mc}{\hbar}\right)\chi(m,s) \end{pmatrix}$$

and

$$v^{+}(k)\,\gamma_0\, v(k) = -1.$$

Repeating the same steps,

$$|B| = \frac{\hbar c}{\sqrt{2mc^2(|E| + mc^2)}}$$

$$\therefore \quad \psi^{(-)}(x) = e^{ik\cdot x} \begin{pmatrix} \dfrac{\hbar c |\vec{k}| (\vec{\sigma}\cdot\hat{k})}{\sqrt{2mc^2(|E| + mc^2)}} \, \chi_{(m,s)} \\ \sqrt{\dfrac{|E| + mc^2}{2mc^2}} \, \chi_{(m,s)} \end{pmatrix}$$

b)

$$\frac{dL_i}{dt} = \dot{L}_i = \frac{i}{\hbar}[H, L_i],$$

assuming that $\frac{\partial L_i}{\partial t} = 0$.

Since:

$$i\hbar \partial_t \psi = H_0 \psi = E\psi = [c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2]\psi$$

$\therefore$

$$H_0 = c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2$$

$$\Rightarrow [H_0, L_i] = -ic\hbar\, \epsilon_{ijk}\, p^k \alpha^j$$

$\therefore$ $\vec{L}$ is <u>not</u> a constant of the motion.

On the other hand,

d)

$$[H_0, \vec{p}\cdot\vec{\Sigma}] = [H, p^i \Sigma_i]$$

$$= i\hbar c\, \epsilon_{ijk}\, \alpha^j p^k p^i$$

$$= i\hbar c\, \epsilon_{kji}\, \alpha^j p^k p^i \quad (k \leftrightarrow i)$$

$$= -i\hbar c\, \epsilon_{ijk}\, \alpha^j p^k p^i$$

Since:

$$\epsilon_{kji} = -\epsilon_{ijk}$$

$$\Rightarrow [H_0, \vec{p}\cdot\vec{\Sigma}] = -[H_0, \vec{p}\cdot\vec{\Sigma}] = 0.$$

$$\left[H_0, \frac{\hbar}{2}\Sigma_i\right] = \left[c(\alpha_j p^j + \beta mc^2, \frac{\hbar}{2}\mathbb{1}\otimes\sigma^i\right]$$

Since:

$$\left[\beta, \mathbb{1}\otimes\sigma^i\right] = \left[\sigma^z\otimes\mathbb{1}, \mathbb{1}\otimes\sigma^i\right] = 0$$

$$\left[\underbrace{\alpha_j}_{\sigma_x\otimes\sigma_y} p^j, \frac{\hbar}{2}\mathbb{1}\otimes\sigma^i\right] = \sigma_x\otimes\sigma_y\, p^j \left(\frac{\hbar}{2}\mathbb{1}\otimes\sigma_i\right)$$

$$-\left(\frac{\hbar}{2}\mathbb{1}\otimes\sigma_i\right) p^j\, \sigma_x\otimes\sigma_y$$

$$= \hbar\, \sigma_x \otimes \underbrace{(\sigma_i\sigma_j)}_{-i\epsilon_{ijk}\sigma^k} p^j$$

$$= -i\hbar\epsilon_{ijk} \underbrace{(\sigma_x\otimes\sigma^k)}_{\alpha^k} p^j$$

$$= i\hbar\epsilon_{ijk}\, p^k \alpha^j$$

$$\therefore \quad \left[H, L_i + \frac{\hbar}{2}\Sigma_i\right] = 0.$$

d)

$$\vec{V} \equiv \frac{d\vec{x}}{dt} = \dot{\vec{x}} = [H_0, \vec{x}]$$

$\therefore$

$$\vec{v} = [c(\vec{\alpha}\cdot\vec{p}) + \beta mc^2, \vec{x}]$$

or

$$v^i = [c\alpha_j p^j + mc^2\beta, x^i]$$

$$= c\alpha_j [p^j, x^i]$$

$$= -c\alpha_j i\hbar \delta^{ij} = -i\hbar c\alpha^j$$

$\therefore$

$$\frac{dv^i}{dt} = [H_0, \hat{v}^i] =$$

$$= [c\alpha_j p^j + mc^2\beta, -i\hbar c\alpha^i]$$

$$= -i\hbar c^2 p_j [\alpha^j, \alpha^i]$$

$$- i\hbar m c^3 [\beta, \alpha^i]$$

Since

$$[\alpha^j, \alpha^i] = [\sigma_x \otimes \sigma^j, \sigma_x \otimes \sigma^i]$$

$$= \sigma_x \otimes [\sigma^j, \sigma^i]$$

$$= \sigma_x \otimes \underbrace{(\sigma^j \sigma^i - \sigma^i \sigma^j)}_{-2i\epsilon^{ijk}\sigma_k}$$

$$= -2i\epsilon^{ijk}\, \sigma_x \otimes \sigma_k$$

$$= -2i\epsilon^{ijk}\, \alpha_k$$

$$[\beta, \alpha^i] = [\sigma_z \otimes \mathbb{1}, \sigma_x \otimes \sigma^i]$$

$$= [\sigma_z, \sigma_x] \otimes \sigma^i$$

$$= i\sigma_y \otimes \sigma^i$$

$$\to \quad \dot{v}^i = -2\hbar c^2 \epsilon^{ijk} p_j \alpha_k + \hbar m c^3 \sigma_y \otimes \sigma^i$$

$$\neq 0.$$

On the ~~other~~ hand,

$$[H_0, p^i] = [c\alpha_j p^j + mc^2 \beta, p^i] = 0$$

$$\therefore \quad \frac{dp^i}{dt} = 0.$$

5

Since:

a) $H_0 \psi = E\psi = \left[c(\vec{\alpha}\cdot\vec{p} + \beta mc^2 + V(r)) \right]\psi$

$\therefore$

$$H_0 = \left(\begin{array}{c|c} 0 & c(\vec{\sigma}\cdot\vec{p}) \\ \hline c(\vec{\sigma}\cdot\vec{p}) & 0 \end{array}\right) + \left(\begin{array}{c|c} mc^2 + V(r) & 0 \\ \hline 0 & -mc^2 + V(r) \end{array}\right)$$

for

$$\psi = \left(\begin{array}{c} \varphi \\ \hline \chi \end{array}\right)$$

$$\Rightarrow \left.\begin{array}{l} c(\vec{\sigma}\cdot\vec{p})\chi + mc^2\varphi + V(r)\varphi = E\varphi \\ c(\vec{\sigma}\cdot\vec{p})\varphi - mc^2\chi + V(r)\chi = E\chi \end{array}\right\}$$

$$\therefore \quad \chi = \frac{c}{E - V(r) + mc^2}(\vec{\sigma}\cdot\vec{p})\varphi$$

b)

It:

$$\psi(\vec{r},t) = R(r)\, e^{-i\frac{Et}{\hbar}} \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|\downarrow\rangle}$$

we write

$$\vec{\sigma}\cdot\vec{p} = (p_x - i p_y)\frac{1}{2}\sigma_+ + (p_x + i p_y)\frac{1}{2}\sigma_-$$
$$+ p_z \sigma_z$$
$$= -i\hbar(\partial_x - i\partial_y)\frac{1}{2}\sigma_+$$
$$-i\hbar(\partial_x + i\partial_y)\frac{1}{2}\sigma_- - i\hbar\partial_z\sigma_z$$

Since:

$$\frac{1}{2}\sigma_+|\downarrow\rangle = |\uparrow\rangle$$

$$\frac{1}{2}\sigma_-|\downarrow\rangle = 0, \quad \sigma_z|\downarrow\rangle = -|\downarrow\rangle$$

(c)

we calculate:

$$-i\hbar(\partial_x - i\partial_y)R(r) = -i\hbar\left(\frac{x-iy}{r}\right)R'(r)$$

$$-i\hbar(\partial_z)R(r) = -i\hbar\frac{z}{r}R'(r)$$

$\therefore$

$$(\vec{\sigma}\cdot\vec{p})\psi = \Big[-i\hbar\left(\frac{x-iy}{r}\right)R'(r)|\uparrow\rangle$$

$$+ i\hbar\frac{z}{r}R'(r)|\downarrow\rangle\Big]e^{-\frac{i}{\hbar}Et}$$

$$= \left\{-i\hbar\sqrt{\frac{8\pi}{3}}R'(r)\left[Y_1^{-1}|\downarrow\rangle - \frac{1}{\sqrt{2}}Y_1^0|\uparrow\rangle\right]\right.$$

$$e^{-\frac{iEt}{\hbar}}$$

where

$$Y_l^m(x,y,z) \equiv Y_l^m(\theta,\phi) \text{ is a}$$

spherical harmonic.

Hence,

$$\chi = \frac{i\hbar\, k'(r)\sqrt{4\pi}\, c}{E - V(r) + mc^2}\; e^{-i\frac{Et}{\hbar}} \times$$

$$\left\{ \sqrt{\tfrac{1}{3}}\, Y_1^0 \,|\uparrow\rangle - \sqrt{\tfrac{2}{3}}\, Y_1^{-1}\,|\downarrow\rangle \right\}$$

$\therefore$ χ corresponds to a state with

$\underline{l = 1}$ (p-wave).

E

c)

Since:

$$\sqrt{\frac{1}{3}}\, Y_1^0 \,|\downarrow\rangle - \sqrt{\frac{2}{3}}\, Y_1^{-1} \,|\uparrow\rangle$$

$$= \langle \hat{n} | \left(\frac{1}{\sqrt{3}} |1\ 0\rangle |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |1, -1\rangle |\tfrac{1}{2}\ \tfrac{1}{2}\rangle \right)$$

$$= \langle \hat{n} | \underset{\substack{\uparrow \\ l}}{1}\ \underset{\substack{\uparrow \\ s}}{\tfrac{1}{2}} ; \underset{\substack{\uparrow \\ j}}{\tfrac{1}{2}} , \underset{\substack{\uparrow \\ m_j}}{-\tfrac{1}{2}} \rangle$$

according to the table of Clebsh. Gordan coefficients.