5163, Exam 3

10 hours during a 30-hour time period, with the 30-hour time period being between Friday, May 6, 2022, 8am, and Tuesday, May 10, 2022, 2pm.

- The 30-hour time period starts when you first look at/download the Exam (pdf) from Canvas.
- You can work on the exam in several blocks, say, 8am to noon and 4-8pm on one day, followed by two hours on the second day, 10am to noon.
- For example: If you download the exam on Sunday at 2pm, then you must upload your solution by Monday, 8pm.
- Please sort your solutions before uploading: Problem 1 first, Problem 2 second, Problem 3 third, Problem 4 fourth, and Problem 5 last.

The exam consists of five problems. Please select four problems to work on (only four problems will be graded). If you are working on more than four problems, please state clearly which four problems you would like to be counted for your grade. The four problems that will be graded carry equal weight.

If you are unclear about the wording of a problem, please indicate your questions and reasoning in your solutions.

Please upload your solutions and the next page to Canvas (the upload feature has a sharp cut-off; please plan accordingly).

A couple of rules:

- You **are not allowed** to search the internet while taking the exam, except for downloading the exam and uploading your solutions after you are done.
- You **are allowed** to refer to lecture notes and materials, this semester's homework assignments and solutions, and two pre-selected text books.
- You **are allowed** to use a calculator or plotting program.
- You **are not allowed** to discuss, by any means, with others while taking the exam. Also, please do not discuss the exam with anybody before 2pm on Tuesday, 05/10/2022.

After completing the exam, please sign—provided this is true—the following statements:

- I spent no more than 10 hours on the exam, distributed over a 30-hour time period.
- I first looked at/downloaded the exam on/at (please provide date and time):
- I worked on the exam solutions by myself.
- I resorted to the following materials while taking the exam (please provide the resources):

Date, time, and signature

Problem 1:
This problem considers a non-relativistic two-dimensional non-interacting quantum gas (point particles with mass m) contained in a two-dimensional area L^2.
(a) What is the spatial density n ($n = N/L^2$, where N is the average number of particles) for a two-dimensional spinless fermionic gas? Express your answer in terms of the temperature T and the chemical potential μ.
The substitution $x = \exp(-\beta\epsilon)$ might be helpful.
Note:

$$\int_a^b \frac{1}{z^{-1} + x} dx = \log(z^{-1} + x)|_a^b. \tag{1}$$

(b) What is the spatial density n ($n = N/L^2$, where N is the average number of particles) for a two-dimensional Bose gas? Express your answer in terms of the temperature T and the chemical potential μ?
(c) Compare your results from parts (a) and (b). As part of the comparison, sketch your results (you may use a calculator or plotting program). Please explain why you decided to sketch the results in the way you did and provide a discussion.

Problem 2:
(a) Explain the following terms using complete and grammatically sound sentences (the grammar does not have to be perfect but the meaning has to be clear); please do not copy from a textbook—use your own words:
(ai) micro-canonical ensemble.
(aii) canonical ensemble.
(aiii) grand-canonical ensemble.
(b) Provide a concrete example for each of the three above-mentioned ensembles that can be treated conveniently in one ensemble but not in the other two.
(c) Explain the following terms using complete and grammatically sound sentences (the grammar does not have to be perfect but the meaning has to be clear); please do not copy from a textbook—use your own words:
(ci) classical statistical mechanics.
(cii) quantum statistical mechanics.
(d) Consider a gas of H_2-molecules. Without referring to the literature, what is your expectation for the phase diagram? What phases do you expect and why? Are there regimes in the phase diagram where you expect that quantum effects might play a role? If so, what are these regimes and why? Do molecular para- and ortho-hydrogen have anything to do with this? Please explain.

Problem 3:
Consider N massless non-interacting spin-s fermions in a three-dimensional box of volume V.
(a) Find the Fermi energy ϵ_F as a function of N, V, and s.
(b) At zero temperature, find the pressure in terms of N, V, and ϵ_F.
(c) Sketch the occupation of states as a function of the energy at a temperature of $T \approx \epsilon_F/(10k_B)$. You do not have to perform a calculation to do this but the effect of the finite temperature should be indicated clearly and the sketch should be accompanied by a discussion.
(d) Suppose that you start with a gas of these particles in a box of volume V near $T = 0$. Now suppose that a valve is opened such that the gas can undergo a free expansion, allowing it to come to equilibrium at a new volume that is twice as large as the initial volume. Assuming no work or heat transfer occurs during the expansion, does the temperature of the gas go up, down, or stay the same? Please explain your reasoning.
Please note: To address part (d), it will be helpful to perform a $T = 0$ calculation, assuming a volume $2V$.

Problem 4:
Cosmic microwave background (CMB) radiation is an isotropic radiation with a black body spectrum at temperature $T \approx 2.7$ K.
(a) Find the density n of the CMB photons. How many CMB photons are there on average inside a volume $V = 1\ \text{cm}^3$?
(b) Find the rate (i.e., the number of photons per seconds) at which a sphere of radius $R = 1$ cm is struck by CMB photons. You can use a calculator for this.
Hint: The following integral might be useful,

$$\int_0^\infty \frac{x^2}{\exp(x) - 1} dx = 2\zeta(3) \approx 2.404. \tag{2}$$

Problem 5:

A model of a rubber band (a rubber band is a classical system) consists of a one-dimensional chain containing a large number N of linked rigid segments as shown in figure 1. Each segment is independent of the others. A segment occupies one of two possible states: horizontal, which contributes a length a to the chain, or vertical, which contributes nothing to the length. The segments are linked so that they cannot come apart. The chain is in contact with a heat bath that has a temperature T.

(a) If there is no energy difference between the two states, what (expressed in terms of N, a, and T) is the average length of the chain?

(b) Now assume that the chain is fixed at one end to a wall and stretched horizontally at the other end by a weight with mass m hung over a pulley that supplies a horizontal force F (see figure 2). Determine the average length of the chain (again in terms of N, a, and T) at any temperature. Find the average length in the limits that $T \to 0$ and $T \to \infty$.

(c) What are the requirements on the temperature T and the force F such that Hooke's law (namely, that the change in length from equilibrium is proportional to F) applies?

(d) As the temperature is raised, do we need a larger or a smaller force to stretch the rubber band? If you warm the rubber band up (assuming thermal equilibrium throughout) while applying a fixed force, will it expand or contract?

Fig. 1:

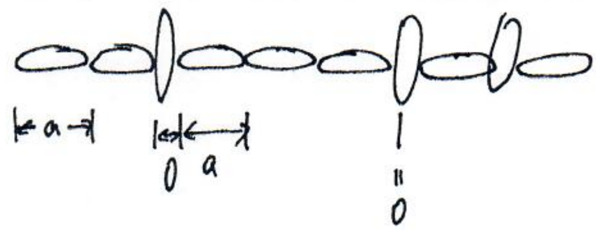

Fig. 2: