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Electrodynamics 1

PHYS 5573 Homework Assignment 4

Problems: {1, 2, 3, 4}

Due: April 15, 2022 at 5:00 PM

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Problem 1:

The picture illustrates the concept of a generator based on Faraday's law (as most of them are). It consists of an inner axle of radius a, outer metal wheel of radius R, and metal spokes that connect the two. The wheel is rotating around the axle with angular velocity ω, and the whole thing is in an external magnetic field $\vec{B}$ that is constant and pointing out of the page towards you. Due to this motion, an EMF (voltage) is generated between the axle and the wheel.

(a) A common description of this EMF is to use the concept of a "Motional EMF". This is the result of a conductor moving through a magnetic field, which causes the electrons in the conductor to move. For a conductor of length dx moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$:

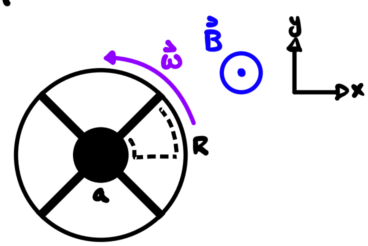
$$EMF \equiv \mathcal{E} \equiv \text{“}E\,dx\text{”} = dx\,\vec{v}\times\vec{B}$$

Note: The "$E\,dx$" corresponds to the $\vec{E}\cdot d\vec{l}$ in Faraday's Law:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int\int \vec{B}\cdot\hat{n}dS$$

For the situation shown, calculate an expression for the total EMF for each spoke in the wheel.

For our situation, we have our wheel



The velocity of this wheel can be found with $\vec{v} = \vec{\omega}\times\vec{r}$. This then means we can find the new expression for EMF with

$$\mathcal{E} = dx\,\vec{v}\times\vec{B} = dx\,(\vec{\omega}\times\vec{r})\times\vec{B}$$

Where we then use

$$(\vec{\omega}\times\vec{r})\times\vec{B} = -\vec{\omega}\times(\vec{r}\times\vec{B})$$

In conjunction with

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

To then write our EMF as

$$\mathcal{E} = ((\overset{0}{\cancel{\vec{\omega}\cdot\vec{r}}})\vec{B} - (\vec{\omega}\cdot\vec{B})\vec{r})\,dx = -(\vec{\omega}\cdot\vec{B})\vec{r}\,dx$$

Where we then use the rules of dot products to say

$$\mathcal{E} = -\omega B\overset{1}{\cancel{\cos(0)}}\vec{r}\,dx$$

Problem 1: Continued

Finally we have an EMF of

$$\mathcal{E} = -\omega B \int_a^R r\,dr\ \hat{r} = -\omega B \frac{r^2}{2}\Big|_a^R \hat{r} = \frac{\omega B}{2}(a^2 - R^2)\,\hat{r}$$

This then means we have an EMF of

$$\boxed{\mathcal{E} = \frac{\omega B}{2}(a^2 - R^2)\,\hat{r}}$$

(b) As shown in class, the "Motional EMF" is included in Faraday's law by taking the total time derivative (rather than partial derivative) of the magnetic flux.

Consider the loop shown in the picture consisting of the dashed lines and one of the spokes. In this loop, the horizontal dashed line is fixed while the spoke moves. The magnetic flux is changing.

Use Faraday's law to calculate the EMF around the loop. Show that this EMF is the same as found in Part A for a spoke, both in magnitude and direction.

Using Faraday's Law, the EMF can be calculated via

$$\mathcal{E} = -\frac{d}{dt}\iint \vec{B}\cdot\hat{n}\,ds$$

We have intentionally set $\vec{B} \parallel \hat{n}$, the ds element for this is

$$ds = r\,dr\,d\varphi$$

This then means EMF is

$$\mathcal{E} = -\frac{d}{dt}\iint \vec{B}\cdot\hat{n}\,ds = -\frac{d}{dt} B\overset{1}{\cancel{\cos}}(\omega)\int_0^{\varphi'}\int_a^R r\,dr\,d\varphi = -\frac{d}{dt} B\cdot\varphi'\cdot\frac{r^2}{2}\Big|_a^R$$

$$= \frac{d\varphi'}{dt}\cdot B\,\frac{r^2}{2}\Big|_R^a = \frac{\omega B}{2}(a^2 - R^2) \quad \Rightarrow \quad \omega \equiv \frac{d\varphi'}{dt}$$

If we then take into account direction the EMF is then

$$\boxed{\mathcal{E} = \frac{\omega B}{2}(a^2 - R^2)\,\hat{r}}$$

Problem 1: Continued

(c) If you wish to create EMF $\mathcal{E} = 5\,V$ using a magnetic field of $B = 0.1\,T$ and a wheel with outer radius $R = 0.4\,m$ and an axle of radius $a = 0.02\,m$, how fast must the wheel turn, ω?

The angular velocity with $\sigma = 0$ is

$$\omega = \frac{2\mathcal{E}}{B}\frac{1}{a^2 - R^2} = \frac{2(5V)}{0.1\,T}\frac{1}{((0.02m)^2 - (0.4m)^2)} = -626.56\ \frac{rad}{s}$$

This then means the magnitude of this angular velocity is

$$\omega = 627\ \frac{rad}{s}$$

For one spoke it is just a quarter of the amount above so

$$\boxed{\omega = 156.75\ \frac{rad}{s}}$$

Problem 1: Review

Procedure:

- – Starting with
  $$\mathcal{E} = dx\,\vec{v} \times \vec{B} \implies \vec{r} = \vec{\omega} \times \vec{r}$$
  find an expression for $\mathcal{E}$ in terms of ω, B, and dx that can be integrated
  – Proceed to change dx to dr and perform the integration
- – Using Faraday's law
  $$\mathcal{E} = -\frac{d}{dt}\int\int \vec{B} \cdot \hat{n}\,dS$$
  set $\vec{B} \parallel \hat{n}$ such that the dS element becomes
  $$dS = r\,dr\,d\phi$$
  – Perform the above integration and show that the same result in (a) will be obtained
- – Solve for ω and plug in the values given to us

Key Concepts:

- – Here we use the relationship $\vec{v} = \vec{\omega} \times \vec{r}$ along with $\vec{v} = r\omega\hat{\phi}$ which tells us that our EMF will point in $\hat{r}$
- – Here we use Faraday's Law to find our EMF along with the fact
  $$\frac{d\theta}{dt} \equiv \omega$$
  to show that we get the same EMF as part (a)
- – This part does not really have a conceptual application

Variations:

- – We could be given a different shape other than a wheel
    * This would most likely affect the co-ordinate system that we use to evaluate our integrals and other aspects of the geometry
  – We could be asked for a different quantity
    * We then would have to use this new equation to find what it is we are looking for
- – This part is essentially the same as part (a)
    * Changing the shape would likely require a change in co-ordinate system to evaluate our quantities in an easier manner
- – We could be given different values of we could be asked to find a different quantity
    * We then just solve for the other quantity and repeat the same process and vice versa

Problem 2:

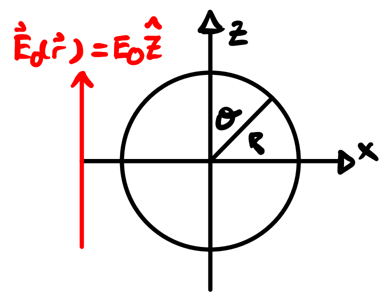
A recent Workshop was about the properties of a conducting sphere in a constant electric field, several different approaches to this problem. Here you'll use a general solution to Laplace's equation.

For a problem that has a spherical boundary and is independent of the azimuthal angle ϕ, Laplace's equation is solved by:

$$\phi(\vec{r}) = \sum_l \left(a_l r^l + \frac{b_l}{r^{l+1}}\right) P_l(\cos(\theta))$$

(a) Consider the boundary condition on $\phi(\vec{r})$ in the limit as $|\vec{r}| \to \infty$. Show that, considering this limit, only one term in the sum will be non-zero.

For our situation we have an object with spherical symmetry in the x-z plane with a constant E-Field pointing in the z-direction. Graphically this looks like



In this Scenario, if we let $\vec{r} \to \infty$

$$\lim_{r\to\infty} \vec{E}(\vec{r}) = E_0\hat{z}$$

which tells us that even at far distances our Electric Field will be a constant. This means if we use

$$\vec{E} = -\vec{\nabla}\varphi \Rightarrow \varphi(\vec{r}) = -\int E(\vec{r})\, d^3\vec{r}$$

Which for us becomes

$$\varphi(\vec{r}) = -E_0\int_0^z dz' = -E_0 z + K = -E_0(r\cos(\theta)) + K$$

So our potential at large $\vec{r}$ will be

$$\varphi(\vec{r}) = -E_0 r\cos(\theta) + K$$

Now, we know that since this object is a conductor we can say

$$\varphi = 0 \to r = R \quad \therefore K = 0$$

$$\varphi = -E_0 r\cos(\theta) \to r >> R$$

Problem 2: Continued

Where θ is the angle off the Z-axis and r is the distance from the center of this ring. Employing these boundary conditions our solution to Laplaces equation will become

$$\lim_{r\to\infty} \varphi(\vec{r}) = \sum_{\ell} \left(a_\ell r^\ell + \cancel{\frac{b_\ell}{r^{\ell+1}}}^{\,0}\right) P_\ell \cos(\alpha) = \sum_{\ell} a_\ell r^\ell P_\ell(\cos(\alpha))$$

This means our potential at large r will become

$$\boxed{\varphi(r \gg R) = \sum_{\ell} a_\ell r^\ell P_\ell(\cos(\alpha))}$$

If we then only take one term from the above expression we will have ($\ell = 0$)

$$\varphi(r \gg R)\Big|_{\ell=0} = a_0 r^0 P_0(z) = -E_0 z = -E_0 r\cos(\alpha)$$

We can then say for very large r we have a potential of

$$\boxed{\varphi(r,\theta) = -E_0 r\cos(\theta)}$$

(b) Using the $|\vec{r}| \to \infty$ limit, solve for one of the remaining coefficients (a_l or b_l).

From the previous part we know that at large r our potential is

$$\varphi(r \gg R) = -E_0 R\cos(\alpha)$$

Conversely we also know at large r our potential is

$$\varphi(r \gg R) = \sum_{\ell} a_\ell R^\ell P_\ell(\cos(\alpha))$$

Where all other values of a_ℓ are zero so therefore a_ℓ is

$$\boxed{a_\ell = -E_0}$$

(c) Consider the boundary condition for $\phi(R)$ from the workshop, $\phi(R) = 0$. Use this result to calculate $\phi(\vec{r})$ for all $\vec{r}$. Show that your result is the same as what was found in the workshop (and the text).

Employing this boundary condition for our solution to Laplaces equation tells us that

$$\varphi(r=R) = a_\ell R^\ell + \frac{b_\ell}{R^{\ell+1}} = 0 \Rightarrow b_\ell = -a_\ell R^{2\ell+1}$$

Problem 2: Continued

From part (b) we know $a_\ell = -E_0$ so this tells us that b_ℓ is

$$b_\ell = E_0 R^{2\ell+1}$$

And so finally we can say our potential everywhere is

$$\boxed{\varphi(r) = E_0 \sum_\ell \left(\frac{R^{2\ell+1}}{r^{\ell+1}} - r^\ell \right) P_\ell(\cos(\theta))}$$

Problem 2: Review

Procedure:

-
  - Begin by showing that the limit as $\vec{E}$ goes to ∞ it will approach a constant value
  - Proceed to calculate the potential with using the relationship of

    $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

    and apply boundary conditions for the different values of r compared to R
  - Take the limit as $r \to \infty$ in

    $$\lim_{r\to\infty}\sum_{l}(a_l r^l + \frac{b_l}{r^{l+1}})P_l\cos{(\theta)}$$

    and compare with the early result
-
  - Compare both results for the potential $\phi(\vec{r})$ in part (a) with one another and solve for a_l from that relationship
-
  - Apply boundary conditions and determine what b_l is in terms of other variables
  - Put this, along with the result in part (b), into the expression for the potential given to us to have a final expression without a a_l or b_l in it

Key Concepts:

-
  - In the limit that $r \to \infty$ we see that the sphere will have less and less of an influence and thus the potential must be that of a constant electric field
  - This also means in this limit we only have $l = 1$ due to the orthogonality of the Legendre polynomials
-
  - Applying the limit in part (a) it is easy to see that the term a_l will be $-E_0$
-
  - Applying this boundary condition allows us to solve for b_l in terms of allowed variables. We then put this into our original expression for $\phi(\vec{r})$ to find a final simplified expression

Variations:

-
  - We can be given a different potential / limit for $\vec{r}$
    * This would require us to enforce our new limit on the same potential or apply the same limit on the new potential. Regardless, the same mathematical formalism will be used
-
  - We could be given a different limit for our potential
    * This then changes what our potential will simplify to, but the same broad procedure is used as before
-
  - We could be given different boundary conditions
    * This would change what our expressions simplify to but it would, again, be the same procedure

Problem 3:

The following is a standard problem in electrostatics, so you can probably find the solution if you search. Try to do this problem without looking up the solution.

Consider a conducting cube with walls at $x = 0, x = L, y = 0, y = L, z = 0, z = L$.

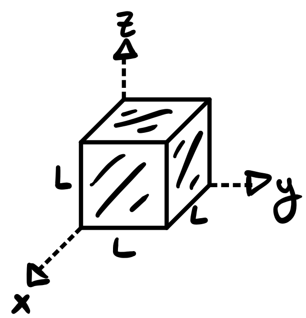
The wall at $z = L$ has $\phi(x, y, z = L) = V_0$. All other walls of the cube are grounded, $\phi = 0$.

(a) Show that a separable solution of the form:

$$\phi(\vec{r}) = X(x)\,Y(y)\,Z(z)$$

Is a solution to the Laplace equations. What are the most general functional forms for $X(x), Y(y), Z(z)$ that will solve Laplace? What are the relationships between the solutions for $X(x), Y(y)$, and $Z(z)$?

Here we have a conducting cube that looks like



With the boundary conditions of

$$V = 0 \begin{cases} z \neq L \end{cases} \quad , \quad V = V_0 \begin{cases} z = L \end{cases}$$

Laplaces Equation is of course

$$\nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) V = 0 \Rightarrow \text{Separable}$$

We now have 3 separable partial differential equations. Each solution for each direction will follow the form of

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = A_1 \ , \ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = A_2 \ , \ \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = A_3$$

Where we also constitute $A_1 + A_2 + A_3 = 0$ from our boundary conditions. This means our solution to Laplaces Equation will follow the form of

$$\boxed{\varphi(\vec{r}) = X(x)\,Y(y)\,Z(z)}$$

Problem 3: Continued

This then means our solution in each direction will follow the form

$$\frac{\partial^2 X}{\partial x^2} = X \cdot A_1 \quad , \quad \frac{\partial^2 Y}{\partial y^2} = Y \cdot A_2 \quad , \quad \frac{\partial^2 Z}{\partial z^2} = Z \cdot A_3$$

Where earlier we said

$$A_1 + A_2 + A_3 = 0 \Rightarrow A_i = -A_j - A_k \Rightarrow A_3 = -A_1 - A_2$$

Since these are second order differential equations we know that A_1, A_2 will be of the form

$$A_1 = \alpha^2 \; , \; A_2 = \beta^2 \Rightarrow A_3 = -\alpha^2 - \beta^2$$

This means for each direction we will have general solution of

$$\boxed{X(x) = A\sin(\alpha x) + B\cos(\alpha x) \; , \; Y(y) = C\sin(\beta y) + D\cos(\beta y) \; , \; Z(z) = Ee^{\sqrt{\alpha^2+\beta^2}\,z} + Fe^{-\sqrt{\alpha^2+\beta^2}\,z}}$$

The solutions to Laplaces Equation in the x & y direction are sinusoidal waves due to the boundary conditions. The solution in the z-direction must be an exponential due to the potential not being zero at $z = L$.

(b) Using the boundary conditions on the cube for the walls $x = 0, x = L, y = 0, y = L$, what are the possible functions $X_m(x)$ and $Y_n(y)$? Explain why these can be indexed by the integers: $m, n = 1, 2, 3, ...$

If we first employ the boundary conditions on $X(x)$ we have

$$X(x=0) = A \cdot \cancel{\sin(0)}^{\,0} + B \cdot \cos(0) = B = 0$$

$$X(x=L) = A \cdot \sin(\alpha \cdot L) = 0 \Rightarrow \alpha = \frac{m\pi}{L}$$

Therefore our solution in the x direction is

$$\boxed{X_m(x) = A \cdot \sin\left(\frac{m\pi x}{L}\right)}$$

We now do the same for the y-direction

Problem 3: Continued

Now, for the Y direction we have

$$Y(y=0) = C\cdot \cancel{\sin(0)}^{0} + D\cdot\cos(0) = D = 0$$

$$Y(y=L) = C\cdot\sin(\beta L) = 0 \Rightarrow \beta = \frac{n\pi}{L}$$

The solution in Y is then

$$\boxed{Y_n(y) = C\cdot\sin\left(\frac{n\pi}{L}y\right)}$$

Where $m,n = 0,1,2,\ldots\infty$ due to the boundary conditions. We cannot have our constants (A,C) being zero so we must find a condition in which the sine function is zero. This occurs for sine at any integer value of π in the argument of sine.

(c) Using the boundary condition for $z = 0$ and the results from (b), what are the solutions to the function $Z_{mn}(z)$? Be sure to show and explain the dependence of $Z(z)$ on m, n, the integers indexing the functions $X_{m(x)}$, $Y_n(y)$.

If we now employ our boundary conditions on Z we will have

$$Z(z=0) = Ee^0 + Fe^0 = E + F = 0 \Rightarrow E = -F$$

$$Z(z=L) = -Fe^{\sqrt{\alpha^2+\beta^2}L} + Fe^{-\sqrt{\alpha^2+\beta^2}L} = F\left(e^{-\sqrt{\alpha^2+\beta^2}L} - e^{\sqrt{\alpha^2+\beta^2}L}\right) = V_0$$

$$= F\cdot 2\sinh\left(\sqrt{\alpha^2+\beta^2}L\right) = V_0$$

We can then go on to say that the solution in the Z-direction is

$$\boxed{Z_{m,n}(z) = 2F\sinh\left(\frac{\pi\sqrt{m^2+n^2}z}{L}\right)}$$

(d) Write down the general solution to this problem:

$$\phi(\vec{r}) = \sum_{m,n} a_{m,n} X_m(x) Y_n(y) Z_{mn}(z)$$

Using the boundary condition for $z = L$ and Fourier analysis, determine the coefficients in this sum, a_{mn}. Show your work.

The general solution for our potential follows the form

$$\varphi(\vec{r}) = \sum_{m,n} a_{m,n} X_m(x) Y(y) Z(z)$$

Problem 3: Continued

The potential then becomes

$$\varphi(\vec{r}) = \sum_m \sum_n a_{mn} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right) \sinh\left(\frac{\pi}{L}\sqrt{m^2+n^2}\, z\right)$$

At $z=L$ our potential becomes

$$\varphi(\vec{r}) = \sum_m \sum_n a_{mn} \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}y\right) \sinh\left(\pi\sqrt{m^2+n^2}\right) \equiv V_0(x,y)$$

We then use

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{m'\pi}{L}x\right) dx = \frac{L}{2}\delta_{m,m'}$$

To evaluate let $a' \doteq \frac{m'\pi}{L}$, $a \doteq \frac{m\pi}{L}$ and $b' \doteq \frac{n'\pi}{L}$, $b \doteq \frac{n\pi}{L}$

$$\int_0^L \int_0^L V_0(x,y) \sin(a'x) \sin(b'y)\, dx dy$$

Where the above becomes

$$\sum_m \sum_n a_{mn} \sinh\left(\pi\sqrt{m^2+n^2}\right) \int_0^L \int_0^L \sin(a'x)\sin(ax)\sin(b'y)\sin(by)\, dx dy$$

We then equate the results of the previous two expressions to get

$$V_0 \frac{L^2}{mn\pi^2} = \frac{L^2}{4} a_{mn} \sinh\left(\pi\sqrt{m^2+n^2}\right) \Rightarrow a_{mn} = \frac{4V_0}{mn\pi^2} \frac{1}{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)}$$

Where the above is only valid for m,n even. a_{mn} is 0 for m,n odd. This means Finally we have

$$\boxed{a_{mn} = \begin{cases} \dfrac{4V_0}{mn\pi^2} \dfrac{1}{\sinh\left(\sqrt{m^2+n^2}\,\pi\right)}, & m,n \text{ even} \\ 0, & m,n \text{ odd} \end{cases}}$$

Problem 3: Continued

(e) Write a sum that gives the potential at the center of the cube, $\phi\left(\frac{L}{2}, \frac{L}{2}, \frac{L}{2}\right)$. You might be able to simplify this result using the relation:

$$\sinh(x) = 2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)$$

Show that this sum converges quickly by calculating the value of the first few terms. Your answers will be V_0 times a number. You might also determine the ratio of successive terms in the sum for large m and n.

The potential at the center of this cube is

$$\varphi\left(\frac{L}{2}, \frac{L}{2}, \frac{L}{2}\right) = \frac{16V_0}{\pi^2} \sum_{m,n \text{ odd}} \frac{1}{mn} \sinh(\pi\sqrt{m^2+n^2}/2) \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right)$$

Where in the Limit that m,n go to infinity

$$\sinh(\pi\sqrt{m^2+n^2}/2) = 2\cdot\sinh(\pi\sqrt{m^2+n^2})\cosh(\pi\sqrt{m^2+n^2})$$

The limit for large m and n is

$$\lim_{m,n\to\infty} 2\cdot\sinh(\pi\sqrt{m^2+n^2})\cosh(\pi\sqrt{m^2+n^2}) = 0$$

So we should only keep the first few terms.

Problem 3: Review

Procedure:

- - Begin by applying Laplaces equation
  $$\nabla^2\phi = 0$$
  in 3D
  - Divide by ϕ and show that there are three separable partial differential equations in x, y, and z
  - Proceed to solve the partial differential equations in each direction
- - Enforce the boundary conditions for the solutions to the three differential equations found in part (a)
- - Same as part (b)
- - Write out the general solution
  $$\phi(\vec{r}) = \sum_{m,n} a_{m,n} X_m(x) Y_n(y) Z_{m,n}(z)$$
  in terms of our new solutions with the applied boundary conditions in parts (b) and (c)
  - Use Fouriers trick to solve $a_{m,n}$ by using the orthogonality of sine
- - Apply a limit to this potential with the relation given in the problem statement to show that only the first few terms are necessary

Key Concepts:

- - The potential given to us is indeed a solution to Laplaces equation
  - Solutions to partial differential equations end up being multiplied by one another to form a general solution
  - Two of our solutions have to be sines and cosines where as one of the directions must be in terms of exponentials. The terms that involves exponentials comes from the boundary conditions that is non zero in that direction. In our case this just so happens to be the z direction
- - Applying these boundary conditions we can find very easily that each direction will only have one term in it
  - m and n must be integers due to the boundary conditions given to us
- - Applying the boundary conditions in the z direction we find that this solution also simplifies to one term
- - Taking the general solution given to us we can use the boundary conditions of our problem, along with Fouriers trick `https://en.wikipedia.org/wiki/Fourier_transform` to find what the constant $a_{m,n}$ will be
  - The constant $a_{m,n}$ will only be non zero is both integers are even
- - It is only necessary to keep the first couple of terms for this potential due to the higher order terms being less and less significant

Variations:

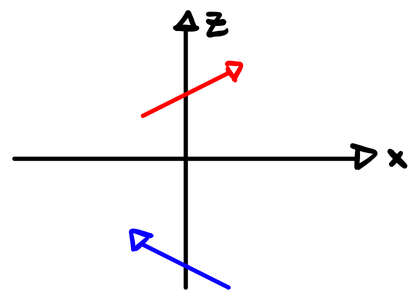
- - We could be given different boundary conditions
    * This would require us to use the same mathematical procedure with different values for our solutions
- - Same as part (a)
- - Same as part (b)
- - Our final solutions could be in 2D instead of 3D
    * This would require us to use the same method (Fouriers trick) but only in two dimensions instead of three to find this other constant
- - We could be given a different limit or potential
    * We then carry out the same math / procedure and determine what this new result would be

Problem 4:

Consider a very long (assume infinite) grounded, conducting sheet lying in the $z = 0$ plane. A polar molecule is on the z-axis at $\vec{r}_p = d\hat{z}$. You can model the molecule as a point dipole, $\vec{p}$, with the dipole moment in the $x - z$ plane at an angle θ as shown.

(a) Determine an image that can be used to solve for the electric potential and field for this situation for all $\vec{r}$ with $z \geq 0$. Show that your image gives the correct boundary condition for this problem, $\phi(x, z = 0) = 0$ for any x.

For the Situation given to us we know



For our potential to be zero we place our image charge equidistant in the -z direction and then orient the dipole to be pointing in the opposite x direction.

Our potential then becomes

$$\varphi(x,z) = \varphi_{Re} + \varphi_{Im}$$

The potential of a dipole is calculated with,

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}$$

where $\vec{r}$ is the location in space and $\vec{r}'$ is the location of the dipole. For our scenario we have

$$\vec{r} = x\hat{x} + z\hat{z} \quad , \quad \vec{r}_p = d\hat{z} \quad , \quad \vec{r}_{p'} = -d\hat{z}$$

The potential due to both dipoles is now

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\vec{p}\cdot(\vec{r}-\vec{r}_p)}{|\vec{r}-\vec{r}_p|^3} + \frac{\vec{p}\,'\cdot(\vec{r}-\vec{r}_{p'})}{|\vec{r}-\vec{r}_{p'}|^3}\right)$$

where $\vec{p}$ and $\vec{p}\,'$ are

$$\vec{p} = p\sin\theta\,\hat{x} + p\cos\theta\,\hat{z} \quad , \quad \vec{p}\,' = -p\sin\theta\,\hat{x} + p\cos\theta\,\hat{z}$$

working this out we get

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{p(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})\cdot(x\hat{x} + (z-d)\hat{z})}{(x^2+(z-d)^2)^{3/2}} + \frac{p(-\sin\theta\,\hat{x} + \cos\theta\,\hat{z})\cdot(x\hat{x} + (z+d)\hat{z})}{(x^2+(z+d)^2)^{3/2}}\right)$$

Problem 4: Continued

Which when simplified becomes

$$\varphi(x,z) = \frac{P}{4\pi\varepsilon_0}\left(\frac{(z-d)\cos\theta + x\sin\theta}{(x^2+(z-d)^2)^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{(x^2+(z+d)^2)^{3/2}}\right)$$

If $z=0$ we then have

$$\varphi(x,0) = \frac{P}{4\pi\varepsilon_0}\left(\frac{-d\cos\theta + \cancel{x\sin\theta}}{(x^2+d^2)^{3/2}} + \frac{d\cos\theta - \cancel{x\sin\theta}}{(x^2+d^2)^{3/2}}\right) = 0$$

As expected

$\square$

(b) Determine an expression for the potential $\phi(x, z)$ for any x and $z > 0$.

The potential here was actually found in part (a) and is

$$\boxed{\varphi(x,z) = \frac{P}{4\pi\varepsilon_0}\left(\frac{(z-d)\cos\theta + x\sin\theta}{(x^2+(z-d)^2)^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{(x^2+(z+d)^2)^{3/2}}\right)}$$

(c) Determine the electric field everywhere on the conducting sheet (meaning for points $z \to 0, z > 0$. You can do this by either taking a derivative of your result from (b) or summing the fields due to the molecule (dipole) and the image (or both, of course).

The relationship between the Electric field and Electric potential is

$$\vec{E} = -\vec{\nabla}\varphi(x,z)$$

The Electric Field is then

$$\vec{E}(x,z) = -\frac{P}{4\pi\varepsilon_0}\left(\frac{\partial}{\partial z}\left(\frac{(z-d)\cos\theta + x\sin\theta}{(x^2+(z-d)^2)^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{(x^2+(z+d)^2)^{3/2}}\right)\hat{z}\right)$$

Putting this in mathematica we find

$$\vec{E}(x,z) = \frac{-P}{4\pi\varepsilon_0}\left(\frac{\cos\theta}{(x^2+(z-d)^2)^{3/2}} - 3\frac{((z-d)\cos\theta + x\sin\theta)(z-d)}{(x^2+(z-d)^2)^{3/2}} + \frac{\cos\theta}{(x^2+(z+d)^2)^{3/2}} - 3\frac{((z+d)\cos\theta - x\sin\theta)(z+d)}{(x^2+(z+d)^2)^{3/2}}\right)\hat{z}$$

Looking at the magnitude of the field and evaluating $z \to 0$ we have

$$E(x, z\to 0) = \frac{P}{2\pi\varepsilon_0}\left(\frac{3(d^2\cos\theta - d x\sin\theta)}{(x^2+d^2)^{5/2}} - \frac{\cos\theta}{(x^2+d^2)^{3/2}}\right)$$

Problem 4: Continued

We can then finally say

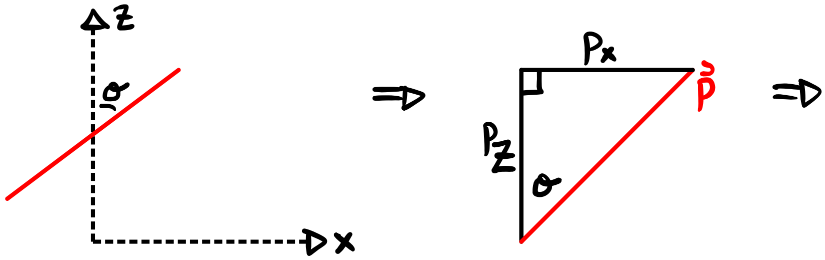
$$\boxed{\vec{E}(x, z\to 0) = \frac{P}{2\pi\varepsilon_0}\left(\frac{3(d^2\cos\theta - dx\sin\alpha)}{(x^2+d^2)^{5/2}} - \frac{\cos\theta}{(x^2+d^2)^{3/2}}\right)\hat{z}}$$

(d) Consider the two cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. For each of these angles, calculate:

(i) The surface charge density on the conducting sheet for any x.

(ii) The force on the molecule (dipole) due to the conducting sheet.

First we examine our dipole to determine what our components are



$$P_x = P\sin(\alpha)$$

$$P_z = P\cos(\alpha)$$

We calculate surface charge density with

$$\sigma'(x) = \varepsilon_0 E_z(x)$$

Which for us is

$$\sigma(x) = \frac{P}{2\pi}\left(\frac{3(d^2\cos\theta - dx\sin\alpha)}{(x^2+d^2)^{5/2}} - \frac{\cos\theta}{(x^2+d^2)^{3/2}}\right)$$

For $\theta = 0$ and $\theta = \pi/2$ we then have

$$\boxed{\sigma(x) = \frac{P}{2\pi}\left(\frac{3d^2}{(x+d^2)^{5/2}} - \frac{1}{(x^2+d^2)^{3/2}}\right), \quad \sigma'(x) = -\frac{P}{2\pi}\frac{3dx}{(x^2+d^2)^{5/2}}}$$

The force is then calculated with

$$F = -\frac{\partial}{\partial d}U(d)$$

Where

$$U = -\vec{p}\cdot\vec{E} = \frac{\vec{P}\cdot\vec{P}' - 3(\vec{P}\cdot\hat{R})(\vec{P}'\cdot\hat{R})}{R^3}$$

Problem 4: Continued

Due to the distance between dipoles we know $\hat{R} = 2d\,\hat{z}$ and thus

$$u = \frac{\vec{p}\cdot\vec{p}' - 3(\vec{p}\cdot\hat{z})(\vec{p}'\cdot\hat{z})}{8d^3}$$

So our Force is then,

$$F = \frac{3}{8}\,\frac{\vec{p}\cdot\vec{p}' - 3(\vec{p}\cdot\hat{z})(\vec{p}'\cdot\hat{z})}{d^4}\,\hat{z}$$

When $\theta = 0$, $\vec{p} = \vec{p}' = pz$ $\therefore$

$$F = \frac{3}{8}\,\frac{p^2 - 3p^2}{d^4} = -\frac{3}{4}\frac{p^2}{d^4}$$

When $\theta = \pi/2$, $\vec{p} = p\hat{x}$ $\vec{p}' = -p\hat{x}$ $\therefore$

$$F = \frac{3}{8}\,\frac{-p^2 - 0^2}{d^4} = -\frac{3}{8}\frac{p^2}{d^4}$$

Our Forces are then

$$\boxed{F(\theta=0) = -\frac{3}{4}\frac{p^2}{d^4}\,,\quad F(\theta=\pi/2) = -\frac{3}{8}\frac{p^2}{d^4}}$$

Problem 4: Review

Procedure:

- – Begin with defining the potential of the real electric dipole and the image as
  $$\phi(x,z) = \phi_{Re} + \phi_{Im}$$
  where the potential of a electric dipole is
  $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}\cdot(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|}$$
  – Proceed to define the quantities above as
  $$\vec{r} = x\hat{x} + z\hat{z} \qquad \vec{r}_p = d\hat{z} \qquad \vec{r'}_p = -d\hat{z}$$
  where the dipole of the real and image are
  $$\vec{P} = P\sin\theta\hat{x} + P\cos\theta\hat{z} \qquad \vec{P'} = -P\sin\theta\hat{x} + P\cos\theta\hat{z}$$
  – Using the above definitions proceed to show that the potential with this image charge at $z=0$ is zero
- – Recycle the same result from part (a) but do not set $z=0$
- – Use the relationship
  $$\vec{E} = -\vec{\nabla}\phi(x,z)$$
  to find what the electric field is
  – Evaluate the electric field for $z \to 0$
- – To calculate surface charge density we use
  $$\sigma(x) = \epsilon_0 E_z(x)$$
  where $E_z(x)$ is that of the result found in part (c). Do this for both $\theta = 0$ and $\theta = \pi/2$
  – Use the standard definition for force with potential energy where the potential energy is defined as
  $$U = -\vec{P}\cdot\vec{E} = \frac{\vec{P}\cdot\vec{P'} - 3(\vec{P}\cdot\hat{r})(\vec{P'}\cdot\hat{r})}{r^3}$$
  where in our case $\hat{r} = 2d\,\hat{z}$

Key Concepts:

- – The correct orientation and placement of a dipole on the $-z$ axis will allow us to create a potential that is zero everywhere along the $z=0$ axis
- – We can use the method of images to find the potential of our system anywhere $z \geq 0$
- – Using the standard relationship between electric field and potential we can find our Electric field along the $z=0$ axis
- – We can find the surface charge density and force of these two dipoles by using the mathematical definitions given to us above

Variations:

- – We could be given a different geometry for our dipoles
  - ∗ This would change the small details in the math but not the same procedure
- – We could be asked to look at a different limit for z
  - ∗ This just requires evaluating a new limit
- – If the potential in (a) and (b) is different this will affect this part
  - ∗ But only what the math evaluates to, not the broad procedure
- – We could be asked to find different quantities
  - ∗ We then would need to use different equations with potentially new geometry