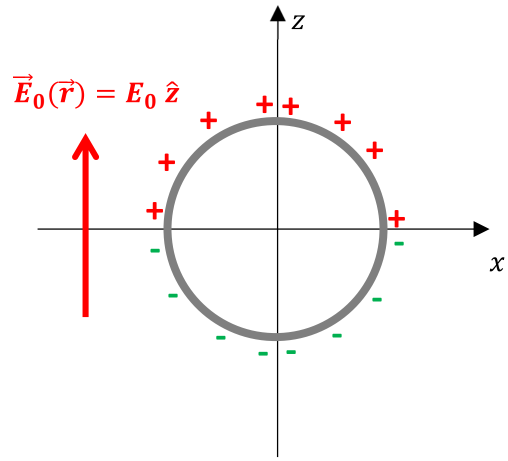
E & M I
Workshop 9 – Spherical Conductor in a Field, Solutions

Today's workshop will consider the fairly standard problem of a spherical conductor placed in a constant, external electric field. We'll consider three different ways to solve this problem to better understand both the physics and the methods for solving this problem.



0) Setup:

The diagram shows the metal sphere, centered at the origin, and the external electric field. Let:

Radius of the sphere = R, Charge on the sphere = 0.
The electric field $\vec{E}_0(\vec{r})$ is the result of charges very far away, so we can approximate them as being at infinity. E_0 is a constant.

a) What is the potential, $\phi_0(\vec{r})$, corresponding to the field $\vec{E}(\vec{r})$? Write this in terms of both Cartesian and Spherical coordinates.

The electric field needs to be given by:

$$\vec{E}(\vec{r}) = E_0\hat{z} = -\nabla\,\phi(\vec{r})$$

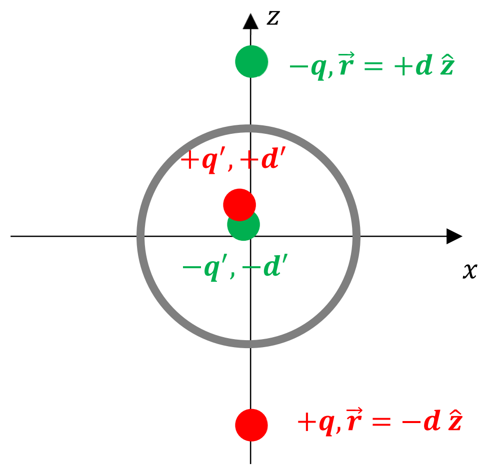
$$\phi(\vec{r}) = -E_0\,z = -E_0\,r\cos\theta$$

b) Predict what will happen to the charge on the sphere. Draw a picture of your prediction and explain your reasoning.

The charge on the conductor will move to cancel $\vec{E}_0$ inside the sphere. The positive and negative charges will separate as shown, giving a dipole. There will be greater charge density near the z-axis to create a field in the $-\hat{z}$ direction.

1) Images:

First, consider a slightly different problem. Instead of an external field, the sphere has two point charges equal distance from the center with charges $\pm q$ as shown.



a) Using results from last week*, what are the values and positions of two image charges inside the sphere that will allow a solution for the potential and field everywhere $r > R$? A picture would be a good idea.

From our previous work, we found that an image charge inside of the conducting sphere will create a sphere with zero potential. In this case, two charges as shown will result in a constant potential sphere.

$$|q'| = |q|\frac{R}{d} \quad d' = \frac{R^2}{d}$$

This also gives the net image charge to be zero, as is necessary because the conducting sphere is not charged.

b) Show (without doing much work) that the potential on the sphere is the same at the points $(x = 0, z = R), (x = 0, z = -R), (x = R, z = 0), (x = -R, z = 0)$.

For $(x = \pm R, z = 0)$ it is obvious that $\phi = 0$ because those points are equal distant from opposite charges.

For $(x = 0, z = \pm R)$ we showed previously that $\phi = 0$ for each pair of charge/image separately. This means the total is zero.

c) What is the potential on the sphere, $\phi(R)$?

The potential on the sphere is $\phi(R) = 0$. It will be left as a future exercise to show this is true for any point on the sphere.

d) This problem approaches the constant-field problem above if we consider $d \to \infty$ but changing q along with d in such a way that the field due to these two charges, $\vec{E}_{2q}(\vec{r})$, at the origin is a constant:

$$\vec{E}_{2q}(\vec{r} = 0) = E_0\,\hat{z}$$

What is the relation between $q, d,$ and E_0 that will keep the field at the origin constant?

If the sphere wasn't there, the total electric field due to the two charges at the origin will be:

$$\vec{E}_{2q}(0) = \frac{2}{4\pi\epsilon_0}\frac{q}{d^2}\ \hat{z}$$

Setting this equal to $\vec{E}_0$:

$$\frac{1}{2\pi\epsilon_0}\frac{q}{d^2} = E_0$$

The field at the origin is a constant if both $d \to \infty, q \to \infty$ in such a way that

$$q = 2\ \pi\ \epsilon_0\ E_0\ d^2$$

e) Considering this limit, $d \to \infty, q \to \infty, E_{2q}(0) = E_0$, show that the images charges found above become a point electric dipole. What is the magnitude of the dipole moment?

The image charges would change as:

$$d' = \frac{R^2}{d} \to 0$$

$$|q'| = |q|\frac{R}{d} = 2\pi\epsilon_0\ E_0\ R\ d$$

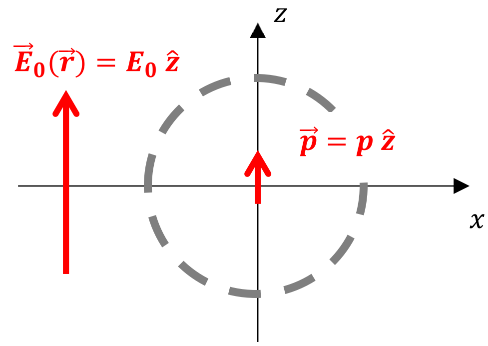
The dipole moment of the image charges will be:

$$\vec{p} = 2\ q'\ d'\ \hat{z}$$

$$\vec{p} = 4\pi\epsilon_0\ R^3\ E_0\ \hat{z}$$

In this limit, the image charge become a point dipole at the origin.

2) Dipole



In the textbook, Garg solved this problem by jumping immediately to the idea that the sphere can be modeled by an "image dipole" at the center. This seems to be true from Part 1, but let's check that this really works.

a) Consider the problem of a point dipole at the origin in a constant electric field. Write down the total potential, due to the sphere and the external field everywhere.

Reminder: For an electric dipole,

$$\phi(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}, \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3}\,(3\,(\vec{p}\cdot\hat{r})\hat{r} - \vec{p})$$

The total potential will be:

$$\phi_T(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} - E_0\,r\cos\theta$$

$$\vec{E}_T(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3}\,(3\,p\cos\theta\,\,\hat{r} - p\,\hat{z}) + E_0\,\hat{z}$$

b) What is the value of the image dipole $\vec{p}$ that will give the same result for $\phi(R)$ as Part 1-c above? How does this compare with your answer to Part 1-e?

$$\phi_T(R) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{R^2} - E_0\,R\cos\theta = 0$$

$$\frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{R^2} = E_0\,R\cos\theta$$

$$p = \,4\pi\epsilon_0\,R^3\,E_0$$

As above.

c) Calculate the total electric field at the surface of the sphere, $\vec{E}_{tot}(|\vec{r}| = R)$ and show that it is perpendicular to the surface of the sphere.

$$\vec{E}_T(R) = \frac{p}{4\pi\epsilon_0 R^3}\,(3\cos\theta\,\,\hat{r} - \,\hat{z}) + E_0\,\hat{z} = E_0(3\cos\theta\,\,\hat{r} - \,\hat{z}) + E_0\,\hat{z}$$

$$\vec{E}_T(R) = 3\,E_0\cos\theta\,\,\hat{r}$$

As needed, the total electric field is radially outward, perpendicular to the surface of the sphere.

d) What is the surface charge density on the sphere (as a function of θ of course)? Does this match your prediction from the start of the problem?

Using Gauss, the discontinuity in the electric field is given by the surface charge density (with the electric field being zero inside the sphere).

$$\vec{E}_T(R) \cdot \hat{n} = \vec{E}_T(R) \cdot \hat{r} = \frac{\sigma}{\epsilon_0}$$

$$\sigma(\theta) = 3\,\epsilon_0\,E_0 \cos\theta$$

This shows the charge density is positive for $\theta < \frac{\pi}{2}$, negative for $\theta > \frac{\pi}{2}$, zero for $\theta = \frac{\pi}{2}$. It is largest in magnitude for $\theta = 0, \pi$ and decreases closer to the equator.