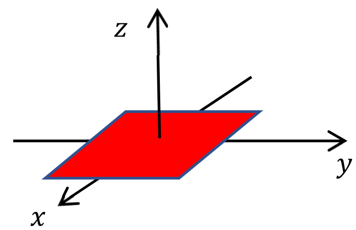
E & M I

Workshop 1 – Fun with Gauss, 1/26/2022

In class on Monday, we solved for the electric potential and field of a uniformly charged sphere the "hard way". We can, of course, do this problem using Gauss' Law without much thought at all. However, we need to be very careful about problems that we *think* we can do without much thought.

In this workshop, we'll take a close look at using Gauss' Law as a calculational tool.

1) Using Symmetry properties:



Consider a large, thin, uniform slab of positive charge lying in the x-y plane. We'll model this as having an infinite area, zero thickness, and a charge per unit area σ. The electric field can be written in the most general form as:

$$\vec{E}(x,y,z) = E_x(x,y,z)\hat{x} + E_y(x,y,z)\hat{y} + E_z(x,y,z)\hat{z}$$

We can use the symmetry properties of the physical problem (the slab) to greatly simplify this expression.

a) Translations and Rotations:

Consider the coordinate translations: $(x,y,z) \rightarrow (x+a,y,z)$ and $(x,y,z) \rightarrow (x,y+a,z)$; And the rotations: $\vec{r} \rightarrow R_{\hat{z},\phi}\,\vec{r}$ where $R_{\hat{z},\phi}$ is a rotation about the z-axis by an angle ϕ.

i) How will these transformations change the problem?

Because the charged slab is uniform and infinite, the plate looks the same under the translation and rotation transformations. Therefore, the electric field must not change under these transformations.

ii) What does this imply about the electric field, $\vec{E}(x,y,z)$? Use these symmetries to rewrite a somewhat simplified expression for the electric field. Explain your result.

$$\vec{E}(x+a,y,z) = \vec{E}(x,y,z)$$

$$\vec{E}(x,y+a,z) = \vec{E}(x,y,z)$$

$$\vec{E}(x - r\cos\theta\,, y + r\sin\theta\,, z) = \vec{E}(x,y,z)$$

This implies that the field can't depend on either x or y:

$$\vec{E}(\vec{r}) = E_x(z)\,\hat{x} + E_y(z)\,\hat{y} + E_z(z)\,\hat{z}$$

b) Inversions:

Consider the inversions in the x and y coordinates: $\hat{x} \to -\hat{x}$ and $\hat{y} \to -\hat{y}$

i) How will these inversions change the problem?

Again, because of the uniformity of the infinite plate, the plate looks the same when the coordinates are inverted. This means the field must be the same on inversion.

ii) Show that this results in a very simple, but expected, form for the electric field, $\vec{E}(\vec{r})$. Explain your results clearly. Using a picture to visualize the inversions is highly encouraged.

Considering the inversion on the x-inversion ($\hat{x}$ – unit vector before inversion):

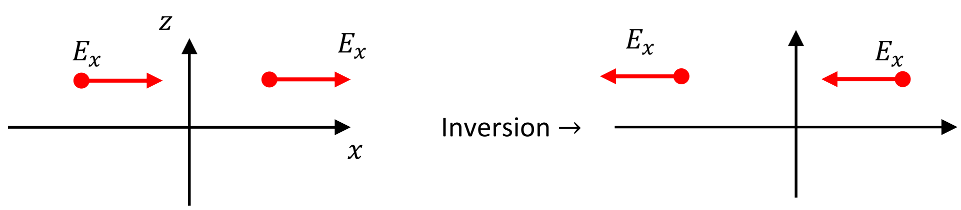
$$E_x(z)\hat{x} = E_x(z)(-\hat{x}) = -E_x(z)\,\hat{x} \to E_x(z) = 0$$

$$E_y(z)\hat{y} = E_y(z)(-\hat{y}) = -E_y(z)\,\hat{y} \to E_y(z) = 0$$

This means:

$$\vec{E}(\vec{r}) = E_z(z)\,\hat{z}$$

Picturing this, considering what a field would look like in the x-direction, considering it can only depend on z:



c) Scaling:

Consider the scaling of the coordinates: $\vec{r} \to \alpha\,\vec{r}$ where α is any number.

How will scaling the coordinate change the problem?

Again, for an infinite slab it will look the same as all the dimensions are scaled. Another way of thinking about this is that an infinite slab will continue to look infinite regardless of how far away it is.

Show that this simplifies $\vec{E}(\vec{r})$ to something *really* simple.

$$E_z(\alpha\, z) = E_z(z) \Rightarrow E_z(z) = E_0 = \text{Constant}$$

This implies a constant field:

$$\vec{E}(\vec{r}) = E_0\,\hat{z}$$

2) The "Gaussian Slab":

I'm sure you have all done this many times, but let's solve for the infinite slab's electric field using Gauss' Law and the Divergence Theorem:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

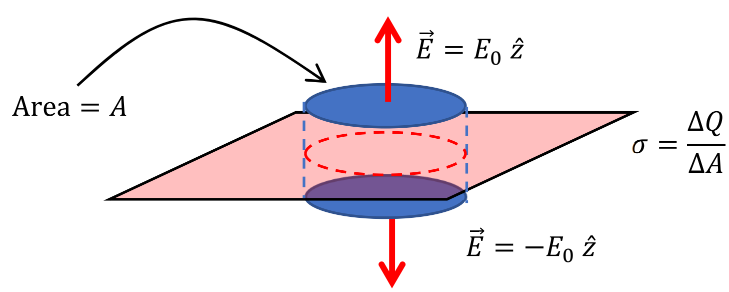
$$\iiint_V \vec{\nabla} \cdot \vec{F}(\vec{r}) d^3x = \oiint_{S=\delta V} \vec{F}(\vec{r}) \cdot \hat{n}\, dS \quad \text{for a vector field } \vec{F}(\vec{r}) \text{ in a volume } V$$

a) Draw a picture(s) and explain how you will apply the two expressions above to solve for the electric field of the slab. Be sure to describe the volume V and the surface S you will use in the integrals above.

Doing the surface integral generally requires a field that is has symmetries that have the field either parallel or perpendicular to the surface.

For a slab, the usual approach is to consider a "pill-box", a volume with sides parallel to $\hat{z}$ where $\vec{E} \cdot \hat{n} = 0$ and two surfaces perpendicular to $\hat{z}$ where $\vec{E} \cdot \hat{n} = E_z$

Consider a side-on view of the charged slab with charge density σ.



b) Write out the integrals needed to determine the electric field. Be clear how you are doing the surface integral.

$$\iiint_V \vec{\nabla} \cdot E(\vec{r}) d^3x = \iiint_V \frac{\rho(\vec{r})}{\epsilon_0}\, d^3x = \frac{Q_{Tot}}{\epsilon_0} = \sigma \frac{A}{\epsilon_0}$$

$$\oiint_{S=\delta V} \vec{E}(\vec{r}) \cdot \hat{n}\, dS = \oiint_{Top} \vec{E}(\vec{r}) \cdot \hat{z}\, dS + \oiint_{Bottom} \vec{E}(\vec{r}) \cdot (-\hat{z}\,) dS + \oiint_{Sides} \vec{E}(\vec{r}) \cdot \hat{r}\, dS = 2\, A\, E_0$$

where $\hat{\rho}$ is the usual radial unit vector in cylindrical (polar) coordinates.

c) Solve for the electric field due to the slab. Explain why this model is an approximation and not physical.

From above: $2\, A\, E_0 = \sigma \frac{A}{\epsilon_0}$ or $\vec{E}(\vec{r}) = sgn(z) \frac{\sigma}{2\epsilon_0}\, \hat{z}$ where $sgn(z) = \pm 1$ is the sign of z.

3) Spherical symmetry and Gauss:

The general form for the electric field in spherical coordinates is, of course:

$$\vec{E}(r,\theta,\phi) = E_r(r,\theta,\phi)\,\hat{r} + E_\theta(r,\theta,\phi)\,\hat{\theta} + E_\phi(r,\theta,\phi)\,\hat{\phi}$$

a) Using symmetry arguments like those in Question 1 above for a spherically symmetric charge density:

$$\rho(\vec{r}) = \rho(r), \qquad r = \vec{r}\cdot\hat{r}$$

provide a convincing argument to write down a much simpler form for the electric field. Remember, you need to be convincing and complete!

For a spherically symmetric charge, such as a sphere, the charge distribution looks the same as the angles θ and ϕ change.

$$\vec{E}(r,\theta,\phi) = \vec{E}(r,\theta+\alpha,\phi)$$

$$\vec{E}(r,\theta,\phi) = \vec{E}(r,\theta,\phi+\alpha)$$

This means:

$$\vec{E}(r) = E_r\,(r)\,\hat{r} + E_\theta(r)\hat{\theta} + E_\phi(r)\,\hat{\phi}$$

The sphere has mirror symmetry so reflection can't change the field.

Consider a point on the x-axis. $\theta = \frac{\pi}{2}$ and $\hat{\theta} = \hat{z}$. A reflection in the $z = 0$ plane will take $\hat{\theta} \to -\hat{\theta}$. This requires for the field

$$E_\theta(r)\hat{\theta} = E_\theta(r)\big(-\hat{\theta}\big) \Rightarrow E_\theta(r) = 0$$

Similarly, for a point on the x-axis, $\hat{\phi} = \hat{x}$ and a reflection in the $x = 0$ plane takes

$$\hat{\phi} \to -\hat{\phi} \;\Rightarrow E_\phi(r) = 0$$

Just as for θ. So

$$\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$$

b) Using Gauss' Law and the Divergence Theorem, determine an integral (a 1D integral) that can be solved for the electric field, $E_r(r)$, for any symmetric charge distribution, $\rho(r)$.

Consider a spherical surface of radius R centered at the origin. The normal to this surface is $\hat{r}$ and $\vec{E}(\vec{r})$ will be constant on the surface and pointing radially.

$$\oiint_{S=\delta V} \vec{E}\cdot\hat{r}\;dS = \iiint_V \vec{\nabla}\cdot\vec{E}(\vec{r})\;d^3\,r = \int r^2\,dr \int d\Omega\frac{\rho(r)}{\epsilon_0}$$

$$E_r\,(R)\;4\,\pi\,R^2 = \frac{4\pi}{\epsilon_0}\int_0^R r^2\,\rho(r)\;dr$$

$$E_r(R) = \frac{1}{\epsilon_0} \frac{\int_0^R r^2 \, \rho(r) \, dr}{R^2}$$