

Classical Mechanics and Statistical/Thermodynamics

January 2016

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i a x - b x^2} dx = \sqrt{\frac{\pi}{b}} e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \quad \text{for } |x| < 1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^n}{n^p} \equiv g_p(z)$$

$$\sum_{n=1}^\infty (-1)^n \frac{z^n}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p)$$

$$f_p(1) = \zeta(-p)$$

$$\zeta(1) = \infty$$

$$\zeta(-1) = -\frac{1}{12} = 0.0833333$$

$$\zeta(2) = \frac{\pi^2}{6} = 1.64493$$

$$\zeta(-2) = 0$$

$$\zeta(3) = 1.20206$$

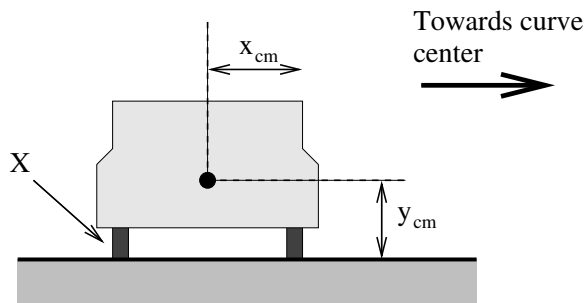
$$\zeta(-3) = \frac{1}{120} = 0.0083333$$

$$\zeta(4) = \frac{\pi^4}{90} = 1.08232$$

$$\zeta(-4) = 0$$

Classical Mechanics

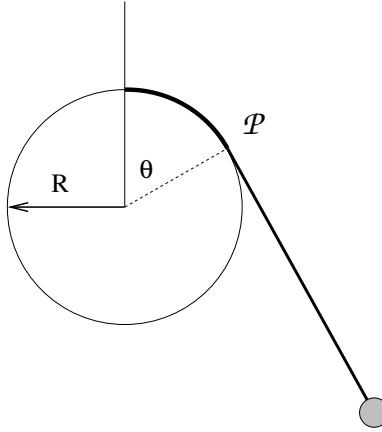
1. A car of mass m travels on a flat road at constant velocity v_0 , and the coefficient of friction between the car tires and the road is μ . It comes to a circular curve of radius R . If the car is moving very fast, it may either slide off the road, (if μ is small) or even flip up on two wheels (if μ is large). In the figure below, the velocity of the car is into the page, and the center of the curve is to the right.



When the car is at rest, the center of mass of the car is located at a point y_{cm} above the road and a distance x_{cm} from either wheel. Treat the wheels as having a negligible width.

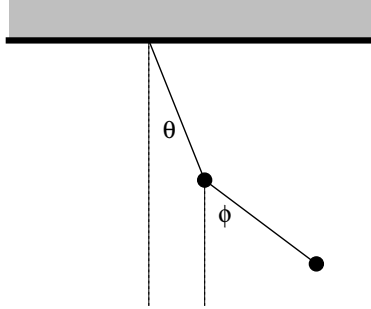
- (a) Draw a free-body diagram for the car showing the forces that act on the car. (1 point).
- (b) Write down Newton's second law for the motion of the car tangential to the curve, perpendicular to the curve (outward) and for rotation of the car about the point marked "X". (2 point).
- (c) If the coefficient of friction is small, the car will skid or slide off the road. At what velocity will that occur for a given value of m ? (3 points)
- (d) Assume that the car never skids or slides sideways. At what velocity will the car start to flip over? (3 points)
- (e) Finally assume that the car does not skid, but exceeds the above speed and starts to flip. If it continues at the same speed, will it continue to flip, or will it achieve equilibrium at an angle θ_{eq} with the road? If the former, prove it, if the latter, calculate the equilibrium angle. (1 point)

2. A stationary disk of radius R is aligned vertically so that its axis is parallel to the ground. The disk is fixed and does not rotate. A string of length ℓ is attached to the top of the disk, and $\ell > \pi R$. A point mass m is attached to the end of the string and can swing in a vertical plane (left to right in the figure below). As the mass m swings, the point \mathcal{P} where the string just contacts the disk will move. Assume that the string is always taut. The angle between \mathcal{P} and the vertical is θ ; it will be the generalized coordinate in this problem.



- Determine the x and y position of the point mass as a function of θ , R and ℓ . Use the center of the disk as the origin of your coordinate system. (Hint: Knowing the value of θ determines both the amount of string wrapped on the disk and the angle the straight length of string makes with the vertical.) (1 point)
- Treating $\theta(t)$ as the generalized coordinate, determine the kinetic energy of the point mass as a function of m , θ , $\dot{\theta}$, R , and ℓ . (2 points)
- What is the Lagrangian for the system in terms of this generalized coordinate? (2 points)
- What are the equations of motion? (1 point)
- Assume that the point mass makes small oscillations about some angle θ_0 (which might not be zero). Determine θ_0 and the angular frequency of these oscillations. (4 points).

3. Consider a double pendulum, consisting of a mass m suspended from a point with a massless cord of length ℓ , with a second mass m suspended from the first with another massless cord of equal length ℓ . At a given instant, the first mass makes an angle θ with respect to the vertical, while the second mass makes an angle ϕ with respect to the vertical. A uniform gravitational field, with gravitational acceleration g , acts in the vertical direction.



- (a) Starting from the description of kinetic and potential energy in Cartesian coordinates, obtain the Lagrangian in terms of the angles θ and ϕ and their time derivatives, $\dot{\theta}$ and $\dot{\phi}$. (2 points)
- (b) Now simplify the Lagrangian to the situation when both angles are small, $\theta \ll 1$, $\phi \ll 1$, and obtain the form of two coupled harmonic oscillators. (1 point)
- (c) For this system, obtain the mass matrix \mathbf{M} and the spring-constant matrix \mathbf{K} , where the Lagrangian is written:

$$L = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \mathbf{M} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \theta & \phi \end{pmatrix} \mathbf{K} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

(2 points)

- (d) Show that the normal modes satisfy

$$(\omega^2 \mathbf{M} - \mathbf{K}) \cdot \mathbf{Q} = 0.$$

(1 point)

- (e) Determine the characteristic frequencies (eigenfrequencies) ω in terms of the quantity $\omega_0^2 = g/l$. (2 points)
- (f) If we write the normal mode vector as

$$\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

determine the ratio q_2/q_1 , which characterize the normal modes. (2 points)

Statistical Mechanics

4. A heat engine is made from N atoms of an ideal mono-atomic gas starting at an initial temperature T_1 , and volume V_1 . Call this state “1.” It is initially heated isochorically (at constant volume) to a state “2” with a temperature $T_2 = 4T_1$. It then undergoes an adiabatic expansion to state “3” where it has returned to its original pressure. Finally it is then cooled isobarically (at constant pressure) until it returns to its original condition.
- (a) Draw the thermodynamic cycle in the PV plane. (1 point).
 - (b) Calculate the volume and temperature at states 2 and 3 in terms of V_1 , T_1 and N . (1 point).
 - (c) Calculate the work done by the gas in each step of the cycle. (3 points).
 - (d) Calculate the heat in (or out) of the gas during each step. (3 points)
 - (e) What is the efficiency of this engine? (2 points)

5. Consider a system of N distinguishable particles with only 3 possible energy levels: 0, ϵ and 2ϵ . The system occupies a fixed volume V and is in thermal equilibrium with a reservoir at temperature T . Ignore interactions between particles and assume that Boltzmann statistics applies.
- (a) What is the partition function for a single particle in the system? (1 point).
 - (b) What is the average energy per particle? (1 points).
 - (c) What is probability that the 2ϵ level is occupied in the high temperature limit, $k_B T \gg \epsilon$? Explain your answer on physical grounds. (1 point).
 - (d) What is the average energy per particle in the high temperature limit, $k_B T \gg \epsilon$? (1 point).
 - (e) At what approximate temperature is the ground state 1.1 times as likely to be occupied as the 2ϵ level? (1 point).
 - (f) Find the heat capacity of the system, c_v , analyze the low- T (when $k_B T \ll \epsilon$) and high- T ($k_B T \gg \epsilon$) limits, and sketch c_v as a function of T . Explain your answer on physical grounds. (5 points).

6. Consider an ideal gas of bosons confined in a three-dimensional box of linear size L on each side. The gas obeys a dispersion relation $\epsilon_{\vec{k}} = \alpha k$, where c is a positive constant and $k \equiv |\vec{k}|$ and where

$$\vec{k} = \frac{2\pi}{L} (n_x \hat{i} + n_y \hat{j} + n_z \hat{k})$$

and where the n_x , n_y , and n_z are integers.

- (a) Find the density of states. (2 points)
- (b) Now assume that the bosons are photons with two possible polarizations. This fixes μ , the chemical potential, so that $\mu = 0$. Why? (1pt).
- (c) Show that the average energy density in the box varies as T^4 . (3 points)
- (d) Calculate the pressure on the walls of the box and show that it is proportional to the average energy density. (4 points).