E & M I

Homework 1, Solutions

1) Charged Slab with width:

In the first workshop we used Gauss' Law and symmetry considerations to calculate the electric field due to an infinite, uniform, thin charged slab lying in the x-y plane.

Consider a similar infinite slab, but with a spread-out charge distribution perpendicular to the slap.

$$\rho(\vec{r}) = \rho(z) = \rho_0 \, e^{-\alpha|z|}$$

This might be used as a model for a nano-scale metal surface where the charge distribution is related to the "skin depth" or finite extension of the surface states of the electrons in the metal.

a) Explain why and how you can use Gauss' Law to determine the electric field due to this slab. What is the general form for the electric field? Justify your answer.

Just as we did in class, we can use the symmetry of this infinite slab to argue that the electric field must be in the $\pm\,\hat{z}$ direction and cannot depend on x or y.

Translational Invariance: $\vec{E}(x+a,y,z) = \vec{E}(x,y,z)$ and $\vec{E}(x,y+a,z) = \vec{E}(x,y,z)$ means that

$$\vec{E}(\vec{r}) = \vec{E}(z)$$

Reflection invariance: $E_x(z) = -E_x(z)$ and $E_y(z) = -E_y(z)$ or

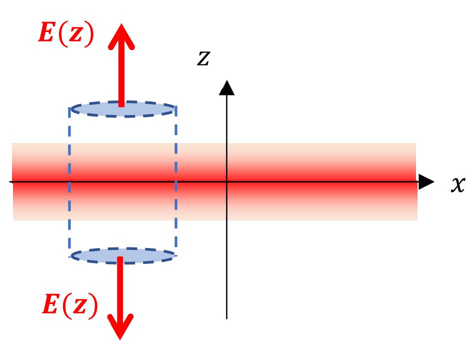
$$\vec{E}(\vec{r}) = E_z(z)\,\hat{z}$$

Note that Reflection invariance in z implies that

$$E_z(z)\,\hat{z} = E_z(-z)(-\hat{z}) \Rightarrow E_z(z) = -E_z(-z)$$

For example, for a positive charge density, the electric field is in the $+\hat{z}$ direction for $z > 0$ and in the $-\hat{z}$ direction for $z < 0$.

b) Describe carefully, and draw a picture, showing the volume and surface you will use to solve the integrals in Gauss' Law



Using a standard "pill box" Gaussian surface, where the ends are at $\pm z$, equal distance from the center of the charge:

Calculate the surface integral of the electric field normal to the surface, which is the integral over the ends where the field is a constant, depending only on z.

Calculate the total charge within the pill box.

Set these two integrals equal to solve for the electric field.

Note: $\frac{1}{\alpha}$ is a measure of the width of the slab. It is the distance from the center of the slab where the charge density decreases by a factor of e^{-1} relative to the center of the slab.

c) Solve the two integrals (volume integral and surface integral) in Gauss' Law to determine the electric field everywhere.

Let A be the cross-sectional area of the pillbox. The surface integral of the field is:

$$\oiint \vec{E}(\vec{r}) \cdot \hat{n}\, dS = E(z)\left(\iint dS_{top} \;+ \iint dS_{bottom}\right) = 2\,A\,E(z)$$

Doing the volume integral, take advantage of the symmetry between $\pm z$:

$$Q(z) = \iiint \rho(\vec{r}')\; d^3r' = 2\,A \int_0^z \rho(z')\; dz'$$

$$Q(z) = 2\,A \int_0^z \rho_0\, e^{-\alpha|z'|}\; dz' = 2\frac{A}{\alpha}\,\rho_0\left(1 - e^{-\alpha\,|z|}\right)$$

Gauss' Law gives:

$$2\,A\,E(z) = 2\frac{A\,\rho_0}{\alpha\,\epsilon_0}\left(1 - e^{-\alpha|z|}\right)$$

$$E(z) = \frac{\rho_0}{\alpha\,\epsilon_0}\left(1 - e^{-\alpha\,|z|}\right)$$

$$\vec{E}(z) = \frac{\rho_0}{\alpha\,\epsilon_0}\left(1 - e^{-\alpha\,|z|}\right)\hat{z}\,sgn(z)$$

To compare with the standard infinite slab, define a "surface density" by considering the total charge from $z = \infty$ to $z = -\infty$.

$$\sigma_0 = \frac{Q(\infty)}{A} = \frac{2\rho_0}{\alpha}$$

$$\vec{E}(z) = \frac{\sigma_0}{2\epsilon_0}\left(1 - e^{-\alpha\,|z|}\right)\hat{z}\,sgn(z)$$

d) Determine the electrostatic potential, $\phi(\vec{r})$, that corresponds to this electric field. (Hint: you can either note that the potential is the integral of the field or guess and check a function for the potential with the property that the negative of the gradient of the function gives the field.

We need a potential:

$$\vec{E}(z) = -\vec{\nabla}\,\phi(\vec{r}) = -\hat{z}\,\partial_z\,\phi(z)$$

Trying a potential:

$$\phi(z) = C_1|z| + C_2\,e^{-\alpha|z|}$$

$$\partial_z\,\phi(z) = C_1\partial_z\,|z| - \alpha\,C_2\,e^{-\alpha|z|}\,\partial_z\,|z| = \left(C_1 - \alpha\,C_2\,e^{-\alpha|z|}\right) sgn(z)$$

This gives $\vec{E}$ if:

$$C_1 = -\frac{\sigma_0}{2\epsilon_0}\,, C_2 = -\frac{1}{\alpha}\frac{\sigma_0}{2\epsilon_0} \Rightarrow \phi(z) = -\frac{\sigma_0}{2\epsilon_0}\left(|z| + \frac{1}{\alpha}\,e^{-\alpha|z|}\right)$$

Note: This result is somewhat pathological because the potential goes to $-\infty$ for as $|z| \to \infty$. This is due to the unphysical model that the charge extends outwards forever. The results of electrostatics assume that the charge density goes to zero for large $\vec{r}$.

If we're really interested in large $\vec{r}$ we need to use a different model.

e) Consider your results for the field and the potential in the limits as $z \to \infty$ and $z \to 0$. Do these limits make physical sense, comparing them to the thin slab? Explain.

The (infinitely) thin slab results are:

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\, \hat{z}\, sgn(z), \qquad \phi(z) = -\frac{\sigma}{2\epsilon_0}\, |z|$$

Our results agree with this for large $|z|$ because $e^{-\alpha|z|} \to 0$.

With the charge smeared out, we get a different, and perhaps more physical, result as $|z| \to 0$.

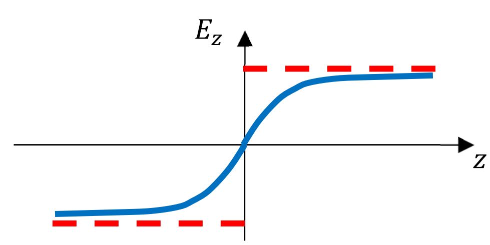
For the infinitely thin slab, the electric field is a constant as $z \to 0$ from above or below, but has a discontinuity of:

$$\Delta E_{slab} = \frac{\sigma}{\epsilon_0}$$

This discontinuity at pure 2D surface charges will be used when we consider the fields around conductors.

The "spread out" charged slab, however, does not have a discontinuity as the electric field goes smoothly through zero. This makes this a somewhat better model if we're interested in the fields near and inside surfaces.

The graph is a sketch of the electric field of the "fuzzy" slab (solid blue line) and the 2D charged sheet (dashed red line).

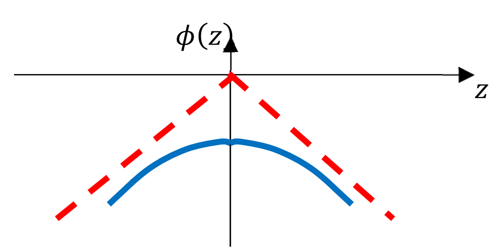


Spreading out the charge removes the discontinuity in the electric field.

For the electric potential as $|z| \to 0$, the potential of the 2D slab goes to zero linearly. It is negative for all $z, |z| \neq 0$ because the potential decreases as one moves away from a positive charge ($\sigma > 0$). For the spread-out charge:

$$\phi(z) \to -\frac{\sigma_0}{2\epsilon_0}\left(|z| + \frac{1}{\alpha}\left(1 - \alpha\,|z| + \frac{1}{2}\,\alpha^2\,|z|^2 + \cdots\right)\right) = -\frac{\sigma_0}{2\epsilon_0}\left(\frac{1}{\alpha} + \frac{\alpha}{2}\,z^2 + \cdots\right)$$

The graph is a sketch of the electric potential of the "fuzzy" slab (solid blue line) and the 2D charged sheet (dashed red line).



f) Note that your result disagrees with our symmetry arguments from Questions 1a, 1b, and 1c of Workshop 1. Which of the symmetry arguments doesn't work for this case and why?

In part (a), the "Scaling" invariance was not used as it was in the Workshop for a 2D slab. Because the charge density depends on z, the problem is not invariant under scaling:

$$\vec{r} \rightarrow a\,\vec{r}$$

The Scaling invariance is what indicated that the electric field of a 2D slab will be constant.

2) Classical "Hydrogen Atom" Field:

If we consider the electron's probability density in a hydrogen atom as a static, classical charge density (I know, quite a stretch) we can make some classical predictions about the atom.

Consider a charge distribution of the form:

$$\rho(r) = \rho_0\, e^{-\alpha\, r}$$

The charge distribution has a total charge Q and $\alpha = \frac{2}{a_0}$ where a_0 is the length scale in the problem. For a hydrogen atom, $Q = -e$ and a_0 is the Bohr radius.

a) Determine what ρ_0 needs to be in terms of Q and a_0. Be sure to explain how you determined your answer.

$$Q = \int \rho(r) d^3r = \rho_0 \iiint e^{-\alpha\, r}\, r^2\, dr \sin\theta\; d\theta\; d\phi$$

$$Q = 4\,\pi\,\rho_0 \int_0^\infty e^{-\alpha\, r}\, r^2\, dr$$

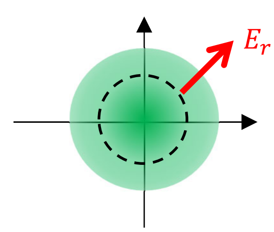
Hint: There is a standard trick for doing these sorts of integrals:

$$\int x^n\, e^{-\alpha x}\, dx = (-1)^n \frac{\partial^n}{\partial \alpha^n} \int e^{-\alpha x}\, dx$$

$$Q = 4\,\pi\,\rho_0\, \partial_\alpha^2 \frac{1}{\alpha} = 8\,\pi \frac{\rho_0}{\alpha^3}$$

$$\rho_0 = Q \frac{\alpha^3}{8\,\pi} = \frac{Q}{\pi\, a_0^3}$$

b) Explain how you will use Gauss' Law to determine the electric field everywhere for the atom. Draw a picture to illustrate your approach.



The spherical symmetry implies that the electric field must be radial and dependent only on r

$$\vec{E}(\vec{r}) = E(r)\,\hat{r}$$

We can use a spherical Gaussian surface.

c) Calculate both the volume and the surface integral in Gauss' Law, showing your work. Use these results to determine the electric field everywhere.

$$\oiint \vec{E} \cdot \hat{n}\, dS = E_r(r) \oiint dS = 4\,\pi\, r^2\, E_r(r)$$

$$Q_{enc}(r) = \iiint \rho(r')\, d^3r' = 4\,\pi\,\rho_0 \int_0^r e^{-\alpha\, r'}\; r'^2\, dr'$$

$$Q_{enc} = 4\,\pi\,\rho_0\, \partial_\alpha^2 \left(\frac{1-e^{-\alpha r}}{\alpha}\right)$$

$$Q_{enc}(r) = 4\,\pi\,\rho_0 \left(2\frac{1-e^{-\alpha r}}{\alpha^3} - 2\frac{r}{\alpha^2}\, e^{-\alpha r} - \frac{r^2}{\alpha}\, e^{-\alpha r}\right)$$

$$Q_{enc}(r) = 8\,\pi\frac{\rho_0}{\alpha^3}\left(1 - e^{-\alpha r}\left(1 + \alpha\, r + \frac{1}{2}\,\alpha^2\, r^2\right)\right)$$

Using Gauss:

$$4\,\pi\, r^2\, E(r) = \frac{8\pi\,\rho_0}{\epsilon_0\,\alpha^3}\left(1 - e^{-\alpha r}\left(1 + \alpha\, r + \frac{1}{2}\,\alpha^2\, r^2\right)\right)$$

$$E(r) = \frac{2}{\epsilon_0}\frac{\rho_0}{\alpha^3\, r^2}\left(1 - e^{-\alpha r}\left(1 + \alpha\, r + \frac{1}{2}\,\alpha^2\, r^2\right)\right)$$

But from above we have $\frac{\rho_0}{\alpha^3} = \frac{Q}{8\,\pi}$ so

$$E(r) = \frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r^2}\left(1 - e^{-\alpha r}\left(1 + \alpha\, r + \frac{1}{2}\,\alpha^2\, r^2\right)\right)$$

d) As always, we need to check our results. Show that your result for the field makes physical sense in the limit that r gets large.

It's obvious that for large r the exponential term goes to zero giving, as one would expect, the field due to a point charge.

$$E(r \to \infty) = \frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r^2}$$

e) Check that your result also is well behaved in the limit as $r \to 0$. Remember that this charge distribution is NOT due to a point charge, just the charged cloud.

Considering terms up to r^3 we have:

$$E(r \to 0) = \frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r^2}\left(1 - \left(1 - \alpha r + \frac{1}{2}\,\alpha^2 r^2 - \frac{1}{6}\,\alpha^3 r^3\right)\left(1 + \alpha\, r + \frac{1}{2}\,\alpha^2\, r^2\right)\right)$$

$$E(r \to 0) =$$
$$\frac{1}{4\,\pi\,\epsilon_0}\frac{Q}{r^2}\left(1 - \left(1 + \alpha r + \frac{1}{2}\alpha^2 r^2 - \alpha r - \alpha^2 r^2 - \frac{1}{2}\alpha^3 r^3 + \frac{1}{2}\,\alpha^2 r^2 + \frac{1}{2}\alpha^3 r^3 - \frac{1}{6}\,\alpha^3 r^3\right)\right)$$

$$E(r \to 0) = \frac{1}{4\,\pi\,\epsilon_0} \frac{Q}{r^2} \left(\frac{1}{6}\,\alpha^3\,r^3\right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{6}\,\alpha^3\,r \to 0$$

Because the electron charge distribution is spread out, without a point charge at $r = 0$, the electric field goes to zero.

3) Classical Hydrogen Atom potential:

The equation 16.27 in the textbook gives the classical hydrogen atom electric potential. This is the potential of both the electron cloud and the proton of the hydrogen atom, modeled as a point charge.

(Note, this is in Gaussian Units while we've been using SI units in class. You should be able to handle this difference.)

a) Consider your results to Problem 2. What is the total electric field of the atom, including both the electron cloud (Problem 2) and the proton? Explain your work.

The total electric field due to the electron cloud and the proton would be the superposition of the field found above and the field due to a point charge. In this case we'll let the $Q = -e$ and add the field due to the proton, $Q = +e$:

$$E_{Tot}(r) = \frac{1}{4\pi\epsilon_0} \frac{e}{r^2} + \frac{1}{4\,\pi\,\epsilon_0} \frac{-e}{r^2} \left(1 - e^{-\alpha r} \left(1 + \alpha\,r + \frac{1}{2}\,\alpha^2\,r^2\right)\right)$$

$$E_{Tot}(r) = \frac{1}{4\,\pi\,\epsilon_0} \frac{e}{r^2} \left(e^{-\alpha r} \left(1 + \alpha\,r + \frac{1}{2}\,\alpha^2\,r^2\right)\right)$$

b) Show that your total electric field agrees with the potential given in 16.27.

The total potential in the textbook is (noting that $\alpha = \frac{2}{a_0}$):

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{e}{r} \left(1 + \frac{1}{2}\,\alpha\,r\right) e^{-\alpha r}$$

The corresponding field is:

$$\vec{E}(r) = -\hat{r}\,\partial_r\,\phi(r)$$

$$E(r) = -\frac{e}{4\pi\epsilon_0} \left(-\frac{1}{r^2} \left(1 + \frac{1}{2}\,\alpha r\right) e^{-\alpha r} + \frac{1}{r} \left(\frac{\alpha}{2}\right) e^{-\alpha r} - \frac{\alpha}{r} \left(1 + \frac{1}{2}\,\alpha r\right) e^{-\alpha r}\right)$$

$$E(r) = -\frac{1}{4\pi\epsilon_0} \frac{e}{r^2} \left(e^{-\alpha r} \left(-\left(1 + \frac{1}{2}\,\alpha r\right) + \frac{1}{2}\,\alpha\,r - \alpha\,r \left(1 + \frac{1}{2}\,\alpha r\right)\right)\right)$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{e}{r^2} \left(e^{-\alpha r} \left(1 + \alpha r + \frac{1}{2}\,\alpha^2\,r^2\right)\right)$$

As above.

c) Determine the electrostatic potential energy of the electron cloud of the hydrogen atom. Use the total electrostatic potential but don't include the energy due to the proton.

The potential energy will be:

$$U = \iiint \rho(r)\, \phi(r)\, d^3r$$

$$U = \iiint \rho_0\, e^{-\alpha\, r}\, \frac{1}{4\pi\epsilon_0} \frac{e}{r} \left(1 + \frac{1}{2}\,\alpha\, r\right) e^{-\alpha r}\, d^3r$$

$$U = \frac{e\,\rho_0}{4\pi\epsilon_0}\, 4\,\pi \int_0^\infty r^2\, dr \frac{1}{r} \left(1 + \frac{1}{2}\,\alpha\, r\right)\, e^{-2\alpha r}$$

$$U = \frac{1}{\pi\epsilon_0} \frac{-e^2}{a_0^3} \int_0^\infty dr\, \left(r + \frac{1}{2}\,\alpha\, r^2\right)\, e^{-2\alpha r}$$

$$U = -\frac{1}{\pi\epsilon_0} \frac{e^2}{a_0^3} \left(\frac{1}{4\alpha^2} + \frac{1}{8\alpha^2}\right)$$

$$U = -\frac{1}{\pi\epsilon_0} \frac{e^2}{a_0^3} \frac{3}{8} \left(\frac{a_0}{2}\right)^2$$

$$U = -\frac{3}{8} \frac{1}{4\,\pi\epsilon_0} \frac{e^2}{a_0} \approx\; -11\; eV$$

This is rather (surprisingly?) close to the binding energy of the ground state of the hydrogen atom.

Hw 1 #4

Sunday, February 20, 2022 8:11 PM

4) The "Dipole Sphere":

The electric potential and electric field of a sphere with a uniform charge on its surface is a common, simple example done in introductory physics classes. Let's change this up a bit for some more useful practice.

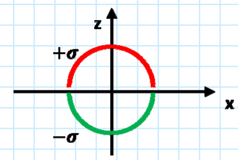
a) As a reminder, write down the electric potential and electric field for a uniform-surface charged sphere of radius R centered at the origin, $\vec{r} = 0$. Include results for both $r > R$ and $r < R$. You don't have to actually solve this, but it would be good practice just to be sure you can do it.

This is basically done using Gauss. For a uniformly charged sphere, the electric field and potential outside are:

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{r}\,, \qquad \phi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Inside the hollow sphere:

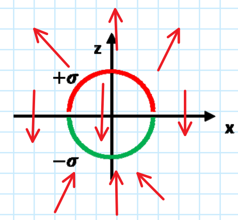
$$\vec{E}(r) = 0, \qquad \phi(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$$



Next, consider the same sphere of radius R but now the surface charge is split between the upper and lower hemispheres. The upper hemisphere ($z > 0$ in Cartesian Coordinates, $0 \le \theta \le \frac{\pi}{2}$ in Spherical Coordinates) has a constant positive surface charge density, $+\sigma$. The lower hemisphere ($z < 0, \frac{\pi}{2} \le \theta \le \pi$) has a constant negative surface charge density, $-\sigma$. The magnitudes of the charge densities are the same.

b) Sketch what you think the electric field will look like in the x-z plane shown.

This will basically look like a dipole, with a positive charge on $+z$ and negative charge on $-z$.

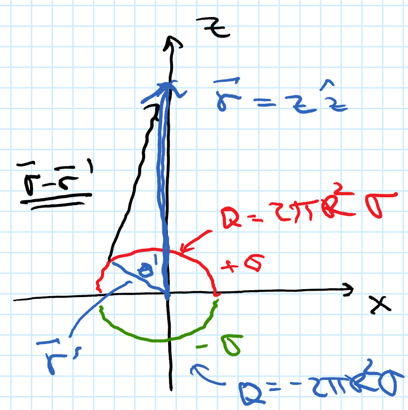


c) Explain why you can't directly use Gauss' Law to solve for the field of this charge distribution.

This problem does not have the symmetry necessary to use Gauss. There isn't an easy-to-integrate surface where $\vec{E}$ will be everywhere normal to the surface and the field is constant (or known).

d) Using a direct integration, solve for the electric potential everywhere on the z-axis. This includes $z > R, z < -R,$ and $-R < z < R$. Of course, show your work.

e) Calculate the electric field along the z-axis, $\vec{E}(z) = E(z)\,\hat{e}_z$ using your result for the potential.



$$\phi(\vec r) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec r')}{|\vec r - \vec r'|}$$

$$\rho(\vec r') = \sigma\,\delta(r' - R) \qquad \theta < \frac{\pi}{2}$$

$$\rho(\vec r') = -\sigma\,\delta(\vec r' - R) \qquad \theta > \frac{\pi}{2}$$

$$\phi(\vec r) = \frac{\sigma}{4\pi\epsilon_0}\left[\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta' \int_0^\infty r'^2 \frac{\delta(r'-R)}{|\vec r - \vec r'|} - \int_0^{2\pi} d\phi' \int_{-1}^0 d\cos\theta' \int_0^\infty r'^2 \frac{\delta(\vec r' - R)}{|\vec r - \vec r'|}\right]$$

$$\phi(\vec r) = \frac{\sigma}{4\pi\epsilon_0} 2\pi R^2 \left[\int_0^1 d\cos\theta' \frac{1}{|\vec r - \vec r'|} - \int_{-1}^0 d\cos\theta' \frac{1}{|\vec r - \vec r'|}\right]$$

as done in class:

$$\frac{1}{|\vec r - \vec r'|} = \frac{1}{(r^2 + r'^2 - 2rr'\cos\theta')^{1/2}}$$

In this case $r = z$, $r' = R$

Also note: $2\pi R^2\sigma = Q$ the total charge on each hemisphere

$$\phi(z) = \frac{Q}{4\pi\epsilon_0}\left[\int_0^1 \frac{d\cos\theta'}{(z^2+R^2-2zR\cos\theta')^{1/2}} - \int_{-1}^0 \frac{d\cos\theta'}{(z^2+R^2-2zR\cos\theta')^{1/2}}\right]$$

Let $z^2+R^2-2zR\cos\theta' = u$

$$-2zR\, d\cos\theta' = du$$

Limits: $\cos\theta' = 1 \Rightarrow u = (z-R)^2$

$\cos\theta' = 0 \Longrightarrow u = z^2+R^2$

$\cos\theta' = -1 \Longrightarrow u = (z+R)^2$

$$\phi(z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{2zR}\int_{(z-R)^2}^{z^2+R^2}\frac{du}{\sqrt{u}} - \frac{1}{2zR}\int_{z^2+R^2}^{(z+R)^2}\frac{du}{\sqrt{u}}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{1}{zR}\left[\sqrt{z^2+R^2} - \sqrt{(z-R)^2} - \sqrt{(z+R)^2} + \sqrt{z^2+R^2}\right]$$

$$\phi(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{zR}\left[2\sqrt{z^2+R^2} - (|z-R| + |z+R|)\right]$$

If $z > R$, $|z-R| + |z+R| = 2z$

If $z < R$, $|z-R| + |z+R| = 2R$

$z > R$: $$\phi_>(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{zR}\left[2\sqrt{z^2+R^2} - 2z\right]$$

$$= \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left[\frac{\sqrt{z^2+R^2}}{z} - 1\right]$$

$$= \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left[\sqrt{1+\frac{R^2}{z^2}} - 1\right]$$

$z < R$: $$\phi_<(z) = \frac{Q}{4\pi\epsilon_0}\frac{1}{zR}\left[2\sqrt{z^2+R^2} - 2R\right]$$

$$= \frac{Q}{2\pi\epsilon_0}\frac{1}{z}\left[\sqrt{1+\frac{z^2}{R^2}} - 1\right]$$

The electric Field for $\bar{r} = z\hat{z}$

$$E_z(z) = -\partial_z\phi(z)$$

For $z > R$: $$E_>(z) = \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\partial_z\sqrt{1+\frac{R^2}{z^2}}$$

For $z > R$: $$E_>(z) = \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\partial_z\sqrt{1+\frac{R^2}{z^2}}$$

$$= -\frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left(\frac{1}{2}\frac{1}{\sqrt{1+\frac{R^2}{z^2}}}\left(-2\frac{R^2}{z^3}\right)\right)$$

$$= \frac{QR}{2\pi\epsilon_0}\frac{1}{z^3\sqrt{1+\frac{R^2}{z^2}}}$$

$$E_>(z) = \frac{QR}{2\pi\epsilon_0}\frac{1}{z^2\sqrt{z^2+R^2}}$$

For $z < R$: $$E_< = -\frac{Q}{2\pi\epsilon_0}\partial_z\left[\frac{\sqrt{1+\frac{z^2}{R^2}}}{z} - \frac{1}{z}\right]$$

$$= -\frac{Q}{2\pi\epsilon_0}\left[\frac{z\frac{1}{\sqrt{1+\frac{z^2}{R^2}}}\cdot\frac{z}{R^2} - \sqrt{1+\frac{z^2}{R^2}}}{z^2} + \frac{1}{z^2}\right]$$

$$= -\frac{Q}{2\pi\epsilon_0}\frac{1}{z^2}\left[\frac{z^2}{R^2}\cdot\frac{1}{\sqrt{1+\frac{z^2}{R^2}}} - \sqrt{1+\frac{z^2}{R^2}} + 1\right]$$

$$= -\frac{Q}{2\pi\epsilon_0}\frac{1}{z^2}\left[\frac{\frac{z^2}{R^2} - \left(1+\frac{z^2}{R^2}\right)}{\sqrt{1+\frac{z^2}{R^2}}} + 1\right]$$

$$= -\frac{Q}{2\pi\epsilon_0}\frac{1}{z^2}\left[1 - \frac{1}{\sqrt{1+\frac{z^2}{R^2}}}\right]$$

f) The properties of the potential and field of this "Dipole Sphere" are quite different from those of the uniformly charge sphere. What are some of the differences? Explain the physics of these differences.

For example, you might consider what happens for $|z| \to \infty$, $|z| \to R$, and the field and potential in the interior of the spheres.

Consider the large z limit:

$$\phi_>(z) = \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left[\sqrt{1+\frac{R^2}{z^2}} - 1\right]$$

$$\xrightarrow[z\to\infty]{} \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left[1+\frac{1}{2}\frac{R^2}{z^2}+\cdots - 1\right]$$

$$\frac{QR}{4\pi\epsilon_0}\frac{1}{z^2}$$

$$E_> = \frac{QR}{2\pi\epsilon_0}\frac{1}{z^2\sqrt{z^2+R^2}} \xrightarrow[z\to\infty]{} \frac{QR}{2\pi\epsilon_0}\frac{1}{z^3}$$

As expected, these are dipole potential $\left(\frac{QR}{r^2}\right)$ and field $\left(\frac{QR}{r^3}\right)$

For $z \to 0$ $$\phi_<(z) = \frac{Q}{2\pi\epsilon_0}\frac{1}{z}\left[\sqrt{1+\frac{z^2}{R^2}} - 1\right]$$

For $z \to 0$

$$\Phi_<(z) = \frac{Q}{2\pi\epsilon_0}\frac{1}{z}\left[\sqrt{1+\frac{z^2}{R^2}} - 1\right]$$

$$\xrightarrow[z\to 0]{} \frac{Q}{2\pi\epsilon_0}\frac{1}{z}\left[1+\frac{1}{2}\frac{z^2}{R^2} - 1\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\frac{z}{R^2}$$

$\phi \to 0$ as $z \to 0$

because there is equal $+Q$ & $-Q$ equidistant from $z=0$

$z \to 0$

$$E_<(z) = -\frac{Q}{2\pi\epsilon_0}\frac{1}{z^2}\left[1 - \frac{1}{\sqrt{1+\frac{z^2}{R^2}}}\right]$$

$$\xrightarrow[z\to 0]{} -\frac{Q}{2\pi\epsilon_0}\frac{1}{z^2}\left[1-\left(1-\frac{1}{2}\frac{z^2}{R^2}\right)\right]$$

$$-\frac{Q}{4\pi\epsilon_0}\frac{1}{R^2}$$

Inside the sphere, the field is in the $-\hat{z}$ direction, rather than zero as for the uniform charged sphere.

Also note that for $z = R$,

$$\phi_>(z=R) = \Phi_<(z=R) = \frac{Q}{2\pi\epsilon_0}\frac{1}{R}\left(\sqrt{2}-1\right)$$

and for the electric field

$$E_>(z=R) = \frac{Q\,R}{2\pi\epsilon_0}\frac{1}{R^3\sqrt{2}} = \frac{Q}{2\pi\epsilon_0}\frac{1}{\sqrt{2}R^2}$$

$$E_<(z=R) = -\frac{Q}{2\pi\epsilon_0}\frac{1}{R^2}\left(1-\frac{1}{\sqrt{2}}\right)$$

$$E_> - E_< = \frac{Q}{2\pi\epsilon_0}\frac{1}{R^2}\left(\frac{1}{\sqrt{2}} + \left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{Q}{2\pi R^2\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

This is the standard discontinuity
in the electric field