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Quantum Mechanics 1

PHYS 5393 Homework Assignment #3

Problems: $\{1.6, 1.15, 1.17, 1.19, 1.24\}$

Due: September 14, 2021 By: 5 PM

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Problem 1: 1.6

Using the rules of bra-ket algebra, prove or evaluate the following:

(a) $\mathrm{Tr}(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}) = \mathrm{Tr}(\tilde{\mathbf{Y}}\tilde{\mathbf{X}})$, where $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are operators.

$$\mathrm{Tr}(\tilde{X}\tilde{Y}) = \sum_i \langle a_i|\tilde{X}\tilde{Y}|a_i\rangle = \sum_i\sum_j \langle a_i|\tilde{X}|a_j\rangle\langle a_j|\tilde{Y}|a_i\rangle = \sum_j\sum_i \langle a_j|\tilde{Y}|a_i\rangle\langle a_i|\tilde{X}|a_j\rangle$$

$$= \sum_j \langle a_j|\tilde{Y}\tilde{X}|a_j\rangle = \mathrm{Tr}(\tilde{Y}\tilde{X})$$

$$\boxed{\mathrm{Tr}(\tilde{X}\tilde{Y}) = \mathrm{Tr}(\tilde{Y}\tilde{X})}$$

(b) $(\tilde{\mathbf{X}}\tilde{\mathbf{Y}})^\dagger = \tilde{\mathbf{Y}}^\dagger\tilde{\mathbf{X}}^\dagger$, where $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are operators.

$$\tilde{Y}|\alpha\rangle \overset{DC}{\Longleftrightarrow} \langle\alpha|\tilde{Y}^\dagger \quad , \quad \tilde{X}|\alpha\rangle \overset{DC}{\Longleftrightarrow} \langle\alpha|\tilde{X}^\dagger$$

$$(\tilde{X}\tilde{Y})|\alpha\rangle \overset{DC}{\Longleftrightarrow} \langle\alpha|(\tilde{X}\tilde{Y})^\dagger$$

$$\boxed{(\tilde{X}\tilde{Y})^\dagger = \tilde{Y}^\dagger\tilde{X}^\dagger}$$

(c) $\exp[if(\tilde{\mathbf{A}})] =?$ in ket-bra form, where $\tilde{\mathbf{A}}$ is a Hermitian operator whose eigenvalues are known.

$$\exp[if(\tilde{A})] = \exp[if(\tilde{A})]\sum_i |a_i\rangle\langle a_i| \quad \text{w/} \quad \tilde{A}|a_i\rangle = a_i|a_i\rangle$$

$$\boxed{\exp[if(\tilde{A})] = \sum_i \exp[if(a_i)]|a_i\rangle\langle a_i|}$$

(d) $\sum_{a'} \psi^*_{a'}(x')\psi_{a'}(x")$, where $\psi_{a'}(x') = \langle x'|a'\rangle$.

$$\sum_{a'} \psi^*_{a'}(x')\psi_{a'}(x'') = \sum_{a'} \psi^*_{a'}(x')\langle x''|a'\rangle = \sum_{a'} \langle a'|x'\rangle\langle x''|a'\rangle = \sum_{a'} \langle x''|a'\rangle\langle a'|x'\rangle$$

$$\sum_{a'} \psi^*_{a'}(x')\psi_{a'}(x'') = \langle x''|x'\rangle = \delta(x''-x')$$

$$\boxed{\sum_{a'} \psi^*_{a'}(x')\psi_{a'}(x'') = \delta(x''-x')}$$

Problem 1: 1.6 Review

Procedure:

- Begin by applying the equation for a trace
$$\text{Tr} = \sum_i \langle a_i | \tilde{\mathbf{X}}\tilde{\mathbf{Y}} | a_i \rangle$$
apply a complete set between the operators, and carry out the algebra.
- Use the dual compliment equation acting on an arbitrary state to show that the Hermiticity condition is associative.
- Apply a complete set on the quantity $\exp[\text{i}f(\tilde{\mathbf{A}})]$ to show that this is in bra ket form.
- Use the definitions that are presented in the problem statement to show that the product of these wave functions must be equal to the Dirac delta function.

Key Concepts:

- The trace of operators is associative.
- The Hermitcity of two operators being multiplied together is associative.
- We can write the expression in (c) in bra ket form as long as we expand in a complete set.
- The expression in (d) evaluates to be a Dirac Delta function because of the rules of linear algebra.

Variations:

- Because this problem is proving identities the only way this problem can change is if we are asked to prove other identities.

Problem 2: 1.15

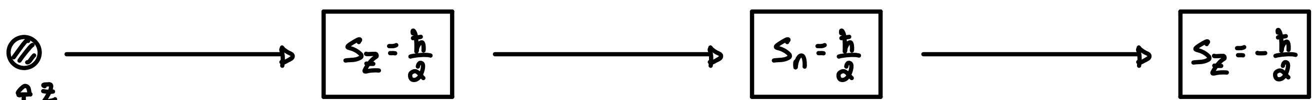
A beam of spin 1/2 atoms goes through a series of Stern-Gerlach type measurements as follows.

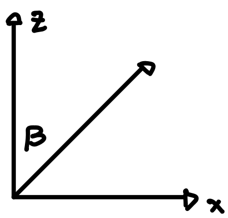
(a) The first measurement accepts $\tilde{\mathbf{S}}_z = \hbar/2$ atoms and rejects $\tilde{\mathbf{S}}_z = -\hbar/2$ atoms.

(b) The second measurement accepts $\tilde{\mathbf{S}}_n = \hbar/2$ atoms and rejects $\tilde{\mathbf{S}}_n = -\hbar/2$ atoms, where S_n is the eigenvalue of the operator $\tilde{\mathbf{S}} \cdot \hat{n}$, with $\hat{n}$ making an angle β in the xz-plane with respect to the z-axis.

(c) The third measurement accepts $\tilde{\mathbf{S}}_z = -\hbar/2$ atoms and rejects $\tilde{\mathbf{S}}_z = \hbar/2$ atoms.

What is the intensity of the final $\tilde{\mathbf{S}}_z = -\hbar/2$ beam when the $\tilde{\mathbf{S}}_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $\tilde{\mathbf{S}}_z = -\hbar/2$ beam?





$|S_n+\rangle = a|+\rangle + b|-\rangle$, $|S_n-\rangle = a|+\rangle + b-\rangle$

Atoms from the first apparatus are in the $|+\rangle$ state. The second apparatus projects out the $|S_n:+\rangle$ state. This acts like a projection operator.

$$|S_n:+\rangle\langle S_n:+| = \left(\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right)\left(\cos\left(\frac{\beta}{2}\right)\langle +| + \sin\left(\frac{\beta}{2}\right)\langle -|\right)$$

Probability of being in state $|S_n:+\rangle$ is:

$$|\langle S_n:+|+\rangle|^2 = \cos^2\left(\frac{\beta}{2}\right)$$

Third apparatus projects out $|-\rangle$ on the $|S_n:+\rangle$ state:

$$|\langle -|S_n:+\rangle|^2 = \sin^2\left(\frac{\beta}{2}\right)$$

Total probability of this becomes:

$$P = |\langle S_n:+|+\rangle|^2|\langle -|S_n:+\rangle|^2 = \cos^2\left(\frac{\beta}{2}\right)\sin^2\left(\frac{\beta}{2}\right) = \frac{1}{4}\sin^2(\beta)$$

Maximized at $\beta = \frac{\pi}{2}$ w/ 25% probability.

Problem 2: 1.15 Review

Procedure:

- Begin by writing out the product
$$|\tilde{\mathbf{S}}_n;+\rangle\,\langle\tilde{\mathbf{S}}_n;+|$$
where we have
$$|\tilde{\mathbf{S}}_n;+\rangle = \cos(\beta/2)\,|+\rangle + \sin(\beta/2)\,|-\rangle$$
where the above product acts like a projection operator.

- Proceed to calculate the probability of a state with
$$\mathcal{P} = |\,\langle\tilde{\mathbf{S}}_n;\pm|\alpha\rangle\,|^2$$
where $|\alpha\rangle$ is the state of our system that we wish to calculate the probability of being in.

- Multiply the probabilities of the two apparatus' together to get the total probability.

- Find the maximum of this total probability for the angle β which happens to be 25%.

Key Concepts:

- We calculate the probabilities of these measurements with the above formalism depending on what the experiment rejects and accepts.

- When the apparatus is configured the correct way, the maximum probability of this measurement after two apparatus' is at most 25/

Variations:

- The apparatus can accept different spins.
  - This would change what state we use to calculate probabilities but not the overall procedure.

Problem 3: 1.17

Let $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ be observables. Suppose the simultaneous eigenkets of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$[\tilde{\mathbf{A}}, \tilde{\mathbf{B}}] = 0?$$

If your answer is yes, prove the assertion. If your answer is no, give a counter-example.

Simultaneous means if an observable acts on one of two states, it will not affect the other state.

$$\tilde{A}|a',b'\rangle = a|a',b'\rangle \;,\; \tilde{B}|a',b'\rangle = b|a',b'\rangle \;:\; [\tilde{A},\tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$$

To show $[\tilde{A},\tilde{B}] = 0$ we will do the following : $[\tilde{A},\tilde{B}]|a',b'\rangle$

$$[\tilde{A},\tilde{B}]|a',b'\rangle = [\tilde{A}\tilde{B} - \tilde{B}\tilde{A}]|a',b'\rangle = \tilde{A}\tilde{B}|a',b'\rangle - \tilde{B}\tilde{A}|a',b'\rangle$$

$$\tilde{A}\tilde{B}|a',b'\rangle = a\tilde{B}|a',b'\rangle = ab|a',b'\rangle \;:\; \tilde{B}\tilde{A}|a',b'\rangle = b\tilde{A}|a',b'\rangle = ba|a',b'\rangle$$

Scalar products are commutative therefore $a \cdot b = b \cdot a$, and thus

$$[\tilde{A},\tilde{B}]|a',b'\rangle = ab|a',b'\rangle - ba|b',a'\rangle = ab|a',b'\rangle - ab|a',b'\rangle = 0$$

From this we can conclude that simultaneous eigenkets of A and B that form a complete orthornormal set of base kets will always have

$$\boxed{[A,B] = 0}$$

Problem 3: 1.17 Review

Procedure:

- Take the commutation of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ and apply it on an arbitrary eigenstate of $|a', b'\rangle$.
- From the above we can deduce that the coefficients ab must be equal to ba.

Key Concepts:

- Simultaneous eigenkets mean if an observable acts on two states, it will not affect the other state. It also means that the order in which they are applied is arbitrary and will produce the same result.
- For the observables to commute this means that $ab = ba$.
- If two observables are simultaneous this means that the their eigenstates form a complete orthonormal set of base kets.

Variations:

- We could be asked to prove this for the anticommutator.
  - We would use the same procedure and deduce what the eigenvalues would have to be relative to one another.

Problem 4: 1.19

Two observables $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$, which do not involve time explicitly, are known not to commute,

$$[\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2] \neq 0$$

yet we also know that $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ both commute with the Hamiltonian:

$$[\tilde{\mathbf{A}}, \tilde{\mathbf{H}}] = 0 \ , \ [\tilde{\mathbf{A}}_2, \tilde{\mathbf{H}}] = 0.$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = p^2/2m + V(r)$, with $\tilde{\mathbf{A}}_1 \rightarrow L_z$, $\tilde{\mathbf{A}}_2 \rightarrow L_x$.

An energy eigenstate is an Eigenstate of the Hamiltonian.

$$\tilde{H}|n\rangle = E_n|n\rangle$$

$$[\tilde{A}_1, \tilde{A}_2] = \tilde{A}_1\tilde{A}_2 - \tilde{A}_2\tilde{A}_1 \ , \ [\tilde{A}_1, \tilde{H}] = \tilde{A}_1\tilde{H} - \tilde{H}\tilde{A}_1 \ , \ [\tilde{A}_2, \tilde{H}] = \tilde{A}_2\tilde{H} - \tilde{H}\tilde{A}_2$$

$$[\tilde{A}_{1,2}, \tilde{H}]|n\rangle = \tilde{A}_{1,2}\tilde{H}|n\rangle - \tilde{H}\tilde{A}_{1,2}|n\rangle \ : \ \tilde{H}\tilde{A}_{1,2}|n\rangle = E_n \cdot a_{1,2}|n\rangle \ : \ \tilde{A}_{1,2}\tilde{H}|n\rangle = E_n \cdot a_{1,2}|n\rangle$$

$$[\tilde{A}_1, \tilde{H}]|n\rangle = \tilde{A}_1\tilde{H}|n\rangle - \tilde{H}\tilde{A}_1|n\rangle = E_n \cdot a_1|\tilde{n}\rangle - E_n \cdot a_1|\tilde{n}\rangle = 0 \ \checkmark$$

$$[\tilde{A}_2, \tilde{H}]|n\rangle = \tilde{A}_2\tilde{H}|n\rangle - \tilde{H}\tilde{A}_2|n\rangle = E_n \cdot a_2|\tilde{n}\rangle - E_n \cdot a_2|\tilde{n}\rangle = 0 \ \checkmark$$

Because $|n\rangle$ is not a simultaneous eigenstate of $\tilde{A}_1$ and $\tilde{A}_2$, it is a requirement for $\tilde{A}_1$ and $\tilde{A}_2$ to be have degeneracy with the Hamiltonian in order for $\tilde{A}_{1,2}$ to commute with $\tilde{H}$.

Problem 4: 1.19 Review

Procedure:

- Begin by calculating the commutation relations of each operator that is defined.
- Apply this to each eigenstate of the Hamiltonian.
- Show that the eigenvalues must be degenerate because the state $|n\rangle$ is not a simultaneous eigenstate of $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$.

Key Concepts:

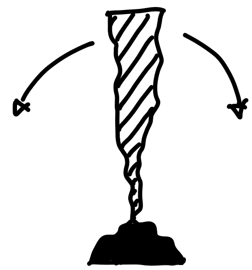
- Because the state $|n\rangle$ is not a simultaneous eigenstate of $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ it is a requirement for the eigenvalues to be degenerate for the above equations to be correct.

Variations:

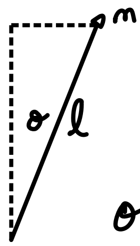
- We can be given different commutation relations.
  - Where we must use the same procedures to end up at the same result.

Problem 5: 1.24

Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it in seconds.



Upside down pendulum ⟶ Assumption :



$U = mgl\sin\theta$, $T = mL^2$ $\sin\theta \approx \theta$

$mgl\theta = mL^2\ddot{\theta}$ ∴ $\ddot{\theta} - \frac{g}{l}\theta = 0$ w/ $\omega = \sqrt{\frac{g}{l}}$

$\theta(t) = Ae^{\omega t} + Be^{-\omega t}$

Initial Point : $\theta(t=0) = A + B$, we want a position so @ $t=0$, $x(t=0) = \theta(t=0)\cdot l$

To Find velocity : $\dot{\theta} = A\cdot\omega e^{\omega t} - B\cdot\omega e^{-\omega t}$, Initial $\dot{\theta} = A\omega - B\omega = \omega(A-B)\cdot l$

Heisenberg uncertainty principle : $\sigma_x\sigma_p = \hbar/2$, $\sigma_x\sigma_p = m(A+B)(A-B)\cdot l^2\omega = \hbar/2$

$m(A^2 - B^2)l^2\omega = \hbar/2$: $\omega = \sqrt{\frac{g}{l}}$, $\omega\cdot l^2\cdot m = \sqrt{m^2 g l^3}$: $\sqrt{m^2 g l^3}\,(A^2 - B^2) = \hbar/2$

we want to assume that the pick will start to fall at $5^\circ = \pi/36$

$\frac{\pi}{36} = Ae^{\omega t}$, $\frac{\pi}{36}\cdot\frac{1}{A} = e^{\omega t}$ ∴ $\ln\left(\frac{\pi}{36}\cdot\frac{1}{A}\right) = \omega t$ ∴ $t = \sqrt{\frac{l}{g}}\ln\left(\frac{\pi}{36}\cdot\frac{1}{A}\right)$

$A^2 = \frac{1}{2}\frac{\hbar}{\sqrt{m^2 g l^3}}$: w/ $m = 1\text{kg}$, $l = 1\text{m}$, $g = 9.8\text{ m/s}^2$, $\hbar = 6.626\times10^{-34}\text{ Js}$

$$t = \sqrt{\frac{1\text{m}}{9.8\text{ m/s}^2}}\cdot\ln\left(\frac{\pi}{36}\cdot\left(\frac{1}{2}\frac{6.626\times10^{-34}\text{ Js}}{\sqrt{(1\text{kg})^2(9.8\text{m/s}^2)(1\text{m})^3}}\right)^{-1/2}\right) = 11.716\text{ s}$$

$$\boxed{t = 11.716\text{ s}}$$

Problem 5: 1.24 Review

Procedure:

- Assume a solution of an inverted pendulum.
- Use the initial conditions to simplify the solution.
- Apply the uncertainty principle.
- Assume values for our ice pick and solve for the time t.

Key Concepts:

- Because our ice pick is going to oscillate we assume a solution of an inverted pendulum.
- We apply initial conditions to simplify our problem.
- The Heisenberg uncertainty principle can be used to determine the time that it would take for this ice pick to fall off the rock.

Variations:

- We can be asked to solve for a different value.
  - Thus requiring us to use the same procedure and solve algebraically for a different result.