Physics 5403 Homework #6
Spring 2022

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1 Scattering in 1D

Suppose a plane wave $\langle x|\phi\rangle = \mathrm{e}^{ikx}/\sqrt{2\pi}$ coming from the left is scattered by a finite range potential in 1D.

a) Compute the Green's function for a free particle in 1D, and show that it gives

$$G^{(+)}(x, x') = \frac{1}{2ik}\mathrm{e}^{ik|x-x'|}.$$

b) In the case of an attractive potential

$$V(x) = -\gamma\frac{\hbar^2}{2m}\delta(x)$$

with $\gamma > 0$, solve the Lipmann-Schwinger equation and compute the reflection and transmission amplitudes of the scattered wave.

c) Compute the energy of the bound state in the well. Show that it corresponds to a resonance in the transmission and reflection amplitudes.

2 Hydrogen atom

The scattering potential of an incoming electron due to a hydrogen atom at the origin is

$$V(\mathbf{x}, \mathbf{x}') = -\frac{e^2}{|\mathbf{x}|} + \frac{e^2}{|\mathbf{x} - \mathbf{x}'|},$$

where e is the electron charge, $\mathbf{x}$ is the coordinate of the incoming electron, and $\mathbf{x}'$ the coordinate of the electron in the orbital $\langle \mathbf{x}'|n, \ell, m\rangle = R_{n,l}(r')Y_m^\ell(\theta', \varphi')$ centered at the origin. Show that in the first Born approximation, the elastic differential scattering cross section of the incoming electron for a hydrogen atom in the *ground state* $|n, \ell, m\rangle = |1, 0, 0\rangle$ is

$$\sigma(\mathbf{q}) = \frac{4m^2e^4}{\hbar^4}\frac{1}{q^4}\left[1 - \frac{16}{(4 + q^2a_0^2)^2}\right]^2,$$

where $\hbar\mathbf{q} = \hbar(\mathbf{k} - \mathbf{k}')$ is the momentum transfered and a_0 is the Bohr radius. Hint: Calculate the scattering aplitude for the two particle state $|\mathbf{k}\rangle|n, \ell, m\rangle$.

3 Lipmann-Schwinger equation

Define the Green's function operators

$$G_0^{\pm}(E) \equiv (E - \mathcal{H}_0 \pm i0_+)^{-1},$$

and

$$G^{\pm}(E) \equiv (E - \mathcal{H} \pm i0_+)^{-1},$$

where $\mathcal{H} = \mathcal{H}_0 + V$, where V is the perturbation, and $\mathcal{H}_0$ the unperturbed Hamiltonian.

a) Show that:

$$G^{\pm}(E) = G_0^{\pm}(E)\left[1 + VG^{\pm}(E)\right].$$

b) Using the relation above, show that the equation

$$|\psi_{\mathbf{k}}^{\pm}\rangle = |\mathbf{k}\rangle + G^{\pm}(E)V|\mathbf{k}\rangle$$

is equivalent to the Lipmann-Schwinger equation

$$|\psi_{\mathbf{k}}^{\pm}\rangle = |\mathbf{k}\rangle + G_0^{\pm}(E)V|\psi_{\mathbf{k}}^{\pm}\rangle.$$

c) Show that the scattered modes follow the orthogonality property:

$$\langle\psi_{\mathbf{k}'}^{\pm}|\psi_{\mathbf{k}}^{\pm}\rangle = \delta(\mathbf{k} - \mathbf{k}').$$

d) If $k' \neq k$, show that

$$\langle\psi_{\mathbf{k}'}^{-}|V|\mathbf{k}\rangle = \langle\mathbf{k}'|V|\psi_{\mathbf{k}}^{+}\rangle.$$

4 Identical particles

a) Using the first Born approximation, compute the differential cross section for the scattering of two identical particles with mass m. Assume that the spin part of the two body wave function is symmetric under the exchange of the particles. The two particles interact through the potential

$$V(r) = V_0 \exp[-(|\mathbf{x}_1 - \mathbf{x}_2|/a)^2]$$

where $\mathbf{x}_i$, $i = 1, 2$ label their position.

b) Using the definition of the partial wave amplitude:

$$f_\ell = -\frac{\pi}{k} T_\ell(E),$$

where

$$T_\ell(E) = \langle E, \ell, m|T|E, \ell, m\rangle,$$

is the matrix element of the transmission matrix, show that the phase shift in the first Born approximation is given by

$$\delta_\ell(k) = -\frac{2}{\hbar^2} mk \int_0^\infty \mathrm{d}r\, r^2 \left[j_\ell(kr)\right]^2 U(r),$$

for an arbitrary (but small) potential $U(r)$, where $j_\ell(x)$ is a spherical Bessel function.