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Classical Mechanics

CH. 6 OSCILLATIONS LECTURE NOTES

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Oscillations & Normal Modes (ch. 6)

Generic Problem :

* System's w/ $n >> 1$ degrees of freedom
* Complicated potential that is a function of degrees of freedom

$\{q_i\} \longrightarrow L(\{q_i\},\{\dot{q}_i\}) = T - V$

Question : If $V(\{q\})$ is "complicated" are there simple regimes we can solve / treat ?

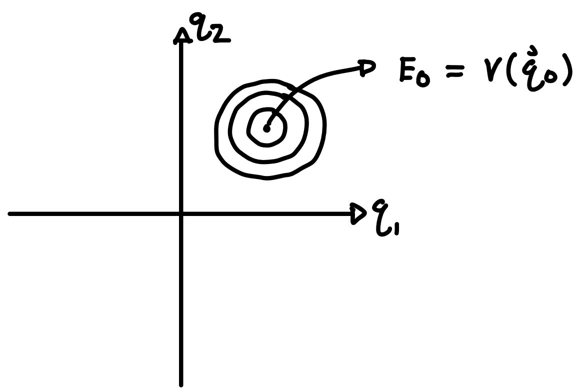
Answer : Possible near equilibrium

Equilibrium : Generalized Forces Vanish

$$Q_j = \left(\frac{-\partial V}{\partial q_j}\right)\Big|_{\vec{q}=\vec{q}_j} = 0$$

Could be : * minimum $\longrightarrow$ Stable

* maximum $\longrightarrow$ unstable



$\Rightarrow$ Want to approximate Lagrangian near equilibrium

$\Rightarrow$ Define new co-ordinates $q_j = q_j + \eta_j \longleftarrow$ Small or displaced co-ordinates $q_j = q_j - q_{j,0}$

Next, Taylor expand potential about equilibrium :

$$V(\vec{q}) \approx V(\vec{q}_i) + \sum_j \partial V/\partial q_j \big|_{\vec{q}=\vec{q}_j} \eta_j + \frac{1}{2}\sum_{jk} \partial^2 V/\partial q_j \partial q_k \big|_{\vec{q}=\vec{q}_i} \eta_j \eta_k + O(n^3)$$

Note :

① $\frac{\partial V}{\partial q_j}\Big|_{\vec{q}=\vec{q}_0} = 0 \longrightarrow$ Terms $O(\eta_j)$ in V vanish

② η_j is small so that we truncate the expansion after $O(\eta^2)$

$$V(\vec{q}) = \tfrac{1}{2} \sum_{jk} V_{jk}\, \eta_j \eta_k \;\longrightarrow\; \left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_{\vec{q}=\vec{q}_0} = V_{jk} = V_{kj}$$

$\Rightarrow$ Similar for kinetic energy ..... $T = \frac{1}{2} \sum_{jk} m_{jk} \dot{q}_j \dot{q}_k$ (homogeneous quadratic function of $\dot{\vec{q}}$), $T = \frac{1}{2} \sum_{jk} T_{jk} \dot{\eta}_j \dot{\eta}_k$ $\longrightarrow$ Somehow depends on m_{jk}

Together, approximate Lagrangian, $L \approx \tfrac{1}{2} \sum_{jk} T_{jk} \dot{\eta}_j \dot{\eta}_k - \tfrac{1}{2} \sum_{jk} V_{jk} \eta_j \eta_k$

$\hookrightarrow$ quadratic in η & $\dot{\eta}$

Lagrangian $\longrightarrow$ E.O.M : $\left\{ \sum_k T_{jk} \ddot{\eta}_k + V_{jk} \eta_k = 0 \right\}$ (*) (n) coupled EOM for n d.o.f

<u>Eigenvalue</u> <u>Equation</u> (6.2)

$\Rightarrow$ Have system of <u>coupled</u> <u>oscillators</u>, Guess : $\eta_j = a_j e^{-i\omega t}$ $\longrightarrow$ Angular frequency of oscillation

$\hookrightarrow$ Amplitude of oscillation

$\ddot{\eta}_j = -\omega^2 \eta_j \Rightarrow$ Plug into (*)

$\hookrightarrow \left\{ \sum_k -\omega^2 T_{jk} \eta_k + V_{jk} \eta_k = 0 \right\} \longrightarrow \left\{ \sum_k -\omega^2 T_{jk} a_k + V_{jk} a_k = 0 \right\}$

$(-\omega^2 I + V)\vec{a} = 0$, $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{pmatrix}$: Immediate solution, $\vec{a} = \vec{0}$

$\hookrightarrow$ No motion, boring

Look for case where determinant of $(-\omega^2 I + V)$ vanishes

i) Solution of vanishing det supplies ω

ii) Obtain $\{a_k\}$ by plugging back into original equation

Generalized eigenvalues problem, $VA = \lambda T A$ $\quad \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ Eigenvalues related to Solution of ω

$A \doteq (\vec{a}^{(1)}, \vec{a}^{(2)}, \vec{a}^{(3)})$

Eigenvalues from,

$$\det \begin{bmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & \cdots & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Keep in mind $V_{jk} = V_{kj}$ & $T_{jk} = T_{kj}$ & real

$n_j = a_j e^{-i\omega t} \longrightarrow$ General solution : $n_j = \sum_\alpha C_\alpha a_{j\alpha} e^{-i\omega_\alpha t}$ $\longrightarrow$ j^{th} component

$\hookrightarrow$ Sum over Solution

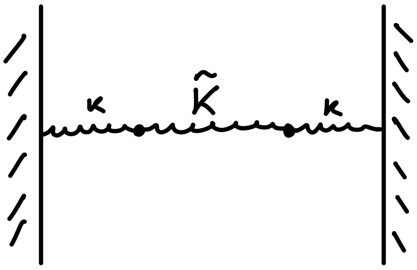
$\longrightarrow$ Sum over oscillators w/ Frequency ω_α

$\eta_j = \sum_\alpha a_{j\alpha} \xi_\alpha$ ⟹ uncoupled oscillator co-ordinates ⟹ Normal mode co-ordinates

Or, $\vec{\eta} = A\vec{\xi}$, Returning to potential: $V \equiv \frac{1}{2}\vec{\eta}\, V \vec{\eta} = \frac{1}{2}\xi^T \underbrace{A^T V A}_{\text{relative to } \omega_\alpha^2} \vec{\eta}$

$V = \frac{1}{2}\sum_\alpha \omega_\alpha^2 \xi_\alpha^2$, ω_α ⟹ Normal mode oscillators

<u>Worked Example</u>: Pair of masses coupled by three springs to walls.



Define: $\eta_{1,2}$ ⟹ Displacement of masses from equilibrium position

Lagrangian: $L = \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 - \frac{k}{2}\eta_1^2 - \frac{k}{2}\eta_2^2 - \frac{\tilde{k}}{2}(\eta_1 - \eta_2)^2$

$T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $V = -\begin{pmatrix} k+\tilde{k} & 2k \\ 2k & k+\tilde{k} \end{pmatrix}$

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<u>Normal Mode Analysis</u>: Complex model w/ many degrees of freedom → Intractable ⟹ Simplify? <u>Near equilibrium</u> effective <u>quadratic</u> lagrangian.

$\left.\frac{\partial V}{\partial q_j}\right|_{q_{0j}} = 0$, $\eta_j = q_j - q_{0,j}$ displaced co-ordinates

i) $L \approx \frac{1}{2}\sum T_{jk}\dot{\eta}_j\dot{\eta}_k - V_{jk}\eta_j\eta_k$ ⟶ System of coupled oscillators

ii) Generate E.O.M $\{\sum_k T_{jk}\ddot{\eta}_k + V_{jk}\eta_k = 0\}$

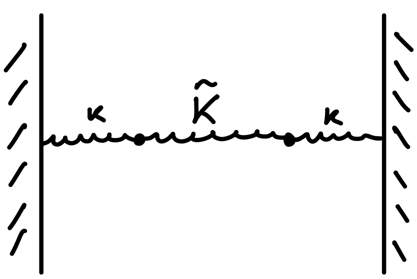
iii) Guess: $\eta_j = a_j e^{-i\omega t}$ ⟶ Matrix eqn: $(-\omega^2 I + V)\vec{a} = 0$

⟹ Generalized evalue problem

iv) Solve for evalues: $\lambda \sim \omega^2$, evector ⟶ Supplied a_j

v) $\eta_j = \sum_k c_k \vec{a}^{(k)} e^{-i\omega_k t}$ ⟶ $\vec{\eta} = A\vec{\xi}$, $A = (\vec{a}^{(1)}, \vec{a}^{(2)}, \dots)$ $L \to \sum_k \frac{1}{2}\dot{\xi}_k^2 + \omega_k^2/2\, \xi_k^2$

<u>Worked Example</u>



$\eta_{1,2}$: displacement of masses from equilibrium configuration

Lagrangian: $L = \frac{m}{2}\dot{\eta}_1^2 + \frac{m}{2}\dot{\eta}_2 - \frac{k}{2}\eta_1^2 - \frac{k}{2}\eta_2^2$, $T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$, $V = \begin{pmatrix} k+\tilde{k} & -k \\ -k & k+\tilde{k} \end{pmatrix}$

<u>EOM</u>

$m\ddot{\eta}_1 = -k\eta_1 - \tilde{k}(\eta_1 - \eta_2)$, $m\ddot{\eta}_2 = -k\eta_2 - \tilde{k}(\eta_2 - \eta_1)$, guess: $\eta_j = a_j e^{-i\omega t}$

$\ddot{\eta}_j = -\omega^2 \eta_j = -\omega^2 a_j e^{-i\omega t}$, generate matrix equation :

$$\begin{pmatrix} -m\omega^2 + k + \tilde{k} & -\tilde{k} \\ -\tilde{k} & -m\omega^2 + k + \tilde{k} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \ , \quad \det(\dots) = 0 \rightarrow (-m\omega^2 + k + \tilde{k})^2 - \tilde{k}^2 = 0$$

$\omega^2 = k/m \rightsquigarrow \lambda_1$ or $(k + 2\tilde{k})/m \rightsquigarrow \lambda_2$, $\omega = \pm\sqrt{\frac{k}{m}}$ or $\omega = \pm\sqrt{\frac{k+2\tilde{k}}{m}}$

For $\omega^2 = k/m$:

$$k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ , \quad \text{eigenvector} : \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightsquigarrow a_1 - a_2 = 0 \rightarrow a_1 = a_2$$

For $\omega^2 = (2k + \tilde{k})/m$

$$k\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a_1 + a_2 = 0 \ \therefore \ a_1 = -a_2 \ , \quad \text{eigenvector} : \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution,

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{i\omega_s t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-i\omega_s t} + c_3\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{i\omega_s t} + c_4\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-i\omega_s t}$$

$\omega_s = \sqrt{\frac{k}{m}} \longrightarrow$ slow , $\omega_f = \sqrt{\frac{2k + \tilde{k}}{m}} \longrightarrow$ Fast

$\eta_1, \eta_2 \in \mathbb{R}$: $c_1 = c_2^* = A_s/2\, e^{i\varphi_s}$: $c_3 = c_4^* = A_f/2\, e^{i\varphi_f}$

<u>Guess</u> : $\eta_1 = A_s\cos(\omega_s t + \varphi_s) + A_f\cos(\omega_f t + \varphi_f)$

$\eta_2 = A_s\cos(\omega_s t + \varphi_s) - A_f\cos(\omega_f t + \varphi_f)$

<u>Interpretation</u> :

1) If $A_f = 0$: $\eta_1(t) = \eta_2(t) = A_s\cos(\omega_s t + \varphi_s) \longrightarrow$ Periodic cos $\Rightarrow$ SHO

$\Rightarrow$ This describes one normal mode

2) If $A_s = 0$: $\eta_1(t) = -\eta_2(t) = A_f\cos(\omega_f t + \varphi_f) \longrightarrow$ Periodic cos $\Rightarrow$ SHO

$\Rightarrow$ This describes the second normal mode

Eigenvectors : $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Recall : $\vec{\xi} = A^T\vec{\eta}$, $\xi_1 = \eta_1 + \eta_2 = 2A_s\cos(\omega_s t + \varphi_s)$ & $\xi_2 = \eta_1 - \eta_2 = 2A_s\cos(\omega_f t + \varphi_f)$

Assignment 8 :

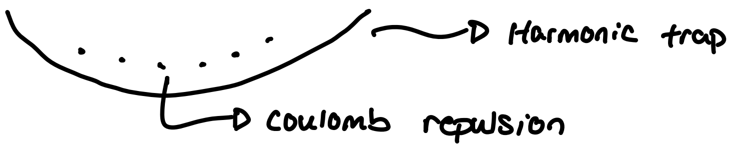
Q1 : Rigid Body Motion problem

Q_2: Normal mode analysis

Q_3: Normal mode analysis

qubit $\to$ Two-level system $\to$ Described by $\hat{\sigma}_\alpha \to \sigma_\xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Trapped Ions



Normal modes of a crystal $\Rightarrow$ Vibrations $\hat{x}_i \to \hat{\xi}_i$ phonons $\to$ Two-particle gates

$\hat{\xi}_i \longleftrightarrow \hat{\sigma}_\alpha$