Physics 5393
Solutions to Exam I

1. Probability interpretation and the completeness relation.

   (a) Starting with a properly normalized state vector $|\alpha\rangle$ and a set of properly normalized eigenkets $\{|a_i\rangle\}$, expand the state vector in the eigenkets and prove that the sum of the magnitude squared of the expansion coefficients is equal to one

   $$|\alpha\rangle = \sum_i c_i\,|a_i\rangle \quad \Rightarrow \quad \sum |c_i|^2 = 1.$$

   Make sure that you give an explicit form of the expansion coefficients c_i.
   Expand the state ket $|\alpha\rangle$ in the eigenkets $|a_i\rangle$ then apply the completeness relation and identify the coefficients

   $$|\alpha\rangle = \sum_i |a_i\rangle\langle a_i|\alpha\rangle \quad \Rightarrow \quad c_i = \langle a_i|\alpha\rangle\,.$$

   To show that the sum of the magnitude squared of the coefficients is one apply the dual of the state ket on the sum above

   $$\langle\alpha|\alpha\rangle = 1 = \sum_i \langle\alpha|a_i\rangle\langle a_i|\alpha\rangle = \sum_i |\langle\alpha|a_i\rangle|^2\,,$$

   where the duality relation $\langle a_i|\alpha\rangle = \langle\alpha|a_i\rangle^*$ is used.

   (b) Starting from the definition of the expectation value $\left\langle\alpha\left|\tilde{\mathbf{A}}\right|\alpha\right\rangle$, show that this is the average value of the observable $\tilde{\mathbf{A}}$ when the system is in the state $|\alpha\rangle$. *Be sure to explain your result.*
   Again, apply the completeness relation except this time apply it twice in order to isolate the three terms in the inner product

   $$\left\langle\alpha\left|\tilde{\mathbf{A}}\right|\alpha\right\rangle = \sum_{i,j}\langle\alpha|a_i\rangle\left\langle a_i\left|\tilde{\mathbf{A}}\right|a_j\right\rangle\langle a_j|\alpha\rangle = \sum_i a_i\,|c_i|^2\,,$$

   where the result of part (a) is used to identify the c_i and the orthogonality of the eigenkets is also used

   $$\left\langle a_i\left|\tilde{\mathbf{A}}\right|a_j\right\rangle = a_j\,\langle a_i|a_j\rangle = a_j\delta_{ij}.$$

   The expectation value is therefore the sum of the eigenvalues of $\tilde{\mathbf{A}}$ each multiplied by the probability that it occurs, which is the mathematical definition of an average value.

2. Consider a Stern-Gerlach experiment with the magnetic field along the z-axis and the direction of the incident spin 1/2 particles perpendicular to the field. This is followed by a second Stern-Gerlach experiment with the field along the η-axis and again the incident particles are perpendicular to the direction of the field.*Probability interpretation and change of basis.*

   (a) Measurements show that if the first Stern-Gerlach experiment selects particles with only spin up $|+\rangle \equiv |S_z;+\rangle$ then the probability that the second Stern-Gerlach experiment selects particles with spin in the direction of the field is 2/3, where the associated state is defined as $|S_\eta;+\rangle$. Using this information and your knowledge of quantum mechanics, deduce the eigenstates $|S_\eta;\pm\rangle$ in the $|\pm\rangle \equiv |S_z,\pm\rangle$ basis. *Be sure to justify all steps needed to arrive at your solution.*
   Using the probabilistic interpretation of quantum mechanics, the probability to be in the $|S_\eta;-\rangle$ state is $1/3$. Hence, the states $|\pm\rangle$ in the $|S_\eta;\pm\rangle$ basis are

   $$\begin{aligned}|+\rangle &= \sqrt{\frac{2}{3}}\,|S_\eta;+\rangle + \sqrt{\frac{1}{3}}\,|S_\eta;-\rangle\\ |-\rangle &= \sqrt{\frac{1}{3}}\,|S_\eta;+\rangle - \sqrt{\frac{2}{3}}\,|S_\eta;-\rangle\,,\end{aligned}$$

where the orthogonality of the two states is used to arrive at the lower equation. In addition, the coefficients have been selected as real, but in general a phase can exist between the two. Inverting the equations, we arrive at the requested result

$$\begin{aligned} |S_\eta, +\rangle &= \sqrt{\frac{2}{3}}\,|+\rangle + \sqrt{\frac{1}{3}}\,|-\rangle \\ |S_\eta; -\rangle &= \sqrt{\frac{1}{3}}\,|+\rangle - \sqrt{\frac{2}{3}}\,|-\rangle\,. \end{aligned}$$

(b) From the information in part (a), derive the matrix elements for the unitary operator that transforms the eigenstates from the $\tilde{\mathbf{S}}_z$ basis to the $\tilde{\mathbf{S}}_\eta$ basis. *Make sure that you confirm that the operator is indeed unitary.*

The transformation between basis set is given in general by

$$\sum_i |b_i\rangle\langle a_i| \quad \Rightarrow \quad |S_\eta;+\rangle\langle +| + |S_\eta;-\rangle\langle -|\,.$$

Using the result of part (a), the $|S_\eta;\pm\rangle$ can be expressed in the $|\pm\rangle$ basis and the matrix elements extracted. Start by expanding in a complete set

$$\begin{aligned} |S_\eta;+\rangle\langle +| + |S_\eta;-\rangle\langle -| = &\Big[\,|+\rangle\,\langle +\,|S_\eta;+\rangle\,\langle +|\,\Big] + \Big[\,|-\rangle\,\langle -\,|S_\eta;+\rangle\,\langle +|\,\Big] \\ &+ \Big[\,|+\rangle\,\langle +\,|S_\eta;-\rangle\,\langle -|\,\Big] + \Big[\,|-\rangle\,\langle -\,|S_\eta;-\rangle\,\langle -|\,\Big]. \end{aligned}$$

Comparing with part (a), the matrix elements are

$$\tilde{\mathbf{U}} \doteq \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix},$$

which can easily be shown to be unitary.

(c) From the information in part (a), derive the matrix elements for the operator $\tilde{\mathbf{S}}_\eta$ in the $\tilde{\mathbf{S}}_z$ basis set. *Confirm that the operator is Hermitian..*

The matrix representation is derived by expanding in a complete set

$$\tilde{\mathbf{S}}_\eta = \sum_{i,j} |S_z;i\rangle\langle S_z;i|\,\tilde{\mathbf{S}}_\eta\,|S_z;j\rangle\langle S_z;j|\,,$$

where the sum is over the $+$ and $-$ states and the matrix elements are the coefficients $\left\langle S_z;i\left|\tilde{\mathbf{S}}_\eta\right|S_z;j\right\rangle$

$$\tilde{\mathbf{S}}_\eta \doteq \frac{\hbar}{6}\begin{pmatrix} 1 & 2\sqrt{2} \\ 2\sqrt{2} & -1 \end{pmatrix},$$

using the results of part (a) and $\tilde{\mathbf{S}}_\eta\,|S_\eta;\pm\rangle = \pm\frac{1}{2}\,|S_\eta;\pm\rangle$

3. Suppose that a linear operator $\tilde{\mathbf{A}}$, though not Hermitian, satisfies the condition that it commutes with its Hermitian adjoint. Furthermore, assume that the eigenvalues of $\tilde{\mathbf{A}}$ are non-degenerate. The items listed below state what I am looking for not the order that they should be calculated. This problem will be graded on the complete solution not on the individual parts.

Understanding of the duality relations and the associative multiplication axiom.

(a) What is the relation between the eigenstates of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^\dagger$.

The eigenvalue equation of $\tilde{\mathbf{A}}$ is given by

$$\tilde{\mathbf{A}}\,|a_i\rangle = a_i\,|a_i\rangle$$

where a_i and $|a_i\rangle$ are the eigenvlues and eigenkets, repectively. The commutation relation imposes the following condition on the combined operator $\tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger$

$$\left[\tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}\right] = 0 \quad \Rightarrow \quad \tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger = \tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}} = \left(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger\right)^\dagger.$$

Hence the combined operator is Hermitian. Since the combined operator is Hermitian

$$\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}}\,|a_i\rangle = a_i\tilde{\mathbf{A}}^\dagger\,|a_i\rangle = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger\,|a_i\rangle\,.$$

From the condition given above the ket $\tilde{\mathbf{A}}^\dagger\,|a_i\rangle$ is an eigenket of $\tilde{\mathbf{A}}$. Hence $|a_i\rangle$ must also be an eigenket of $\tilde{\mathbf{A}}^\dagger$.

(b) What can be said about the relation between the eigenvalues of $\tilde{\mathbf{A}}$ and of $\tilde{\mathbf{A}}^\dagger$. *To simplify the this problem, do not assume that the eigenvalues of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^\dagger$ are equal.*

To determine the relation between the eigenvalues, we start with the dual corresponds relation

$$\begin{aligned}
\tilde{\mathbf{A}}\,|a_i\rangle = a_i\,|a_i\rangle \quad &\Rightarrow \quad \tilde{\mathbf{A}}\,|a_i\rangle \overset{\text{DC}}{\Longleftrightarrow} \langle a_i|\,\tilde{\mathbf{A}}^\dagger \text{ and } a_i\,|a_i\rangle \overset{\text{DC}}{\Longleftrightarrow} \langle a_i|\,a_i^* \\
\tilde{\mathbf{A}}^\dagger\,|a_i\rangle = b_i\,|a_i\rangle \quad &\Rightarrow \quad \tilde{\mathbf{A}}^\dagger\,|a_i\rangle \overset{\text{DC}}{\Longleftrightarrow} \langle a_i|\,\tilde{\mathbf{A}} \text{ and } b_i\,|a_i\rangle \overset{\text{DC}}{\Longleftrightarrow} \langle a_i|\,b_i^*.
\end{aligned}$$

Next, use the commutation relation and the multiplicative associative axiom to relate the the two sets of eigenvalues

$$\left.\begin{aligned}
\langle a_i|\left(\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}}\,|a_i\rangle\right) &= a_i b_i \\
\left(\langle a_i|\,\tilde{\mathbf{A}}^\dagger\right)\left(\tilde{\mathbf{A}}\,|a_i\rangle\right) &= a_i a_i^*.
\end{aligned}\right\} \quad \Rightarrow \quad b_i = a_i^*$$

Now, the reverse order of the operators is calulated

$$\left.\begin{aligned}
\langle a_i|\left(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger\,|a_i\rangle\right) &= a_i b_i \\
\left(\langle a_i|\,\tilde{\mathbf{A}}\right)\left(\tilde{\mathbf{A}}^\dagger\,|a_i\rangle\right) &= b_i b_i^*.
\end{aligned}\right\} \quad \Rightarrow \quad a = b^*.$$

Hence the two sets give consistent results with the eigenvalues of the two operators being complex conjugates of each other.

(c) What can be said about the scalar product of two eigenstates of $\tilde{\mathbf{A}}$ with unequal eigenvalues?

To determine the relation between different eigenstates, start with the duality relation then calulate the matrix elements of the combined operate in the basis set of $\tilde{\mathbf{A}}$

$$\left.\begin{aligned}
\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}}\,|a_i\rangle &= |a_i|^2\,|a_i\rangle \\
\langle a_j|\,\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}} &= \langle a_j|\,|a_j|^2
\end{aligned}\right\}$$

$$\Rightarrow \quad \left\{\begin{aligned}
\left\langle a_j\left|\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}}\right|a_i\right\rangle &= |a_i|^2\,\langle a_j\,|a_i\rangle \\
\left\langle a_j\left|\tilde{\mathbf{A}}^\dagger\tilde{\mathbf{A}}\right|a_i\right\rangle &= |a_j|^2\,\langle a_j\,|a_i\rangle
\end{aligned}\right\}$$

$$\Rightarrow \quad \left(|a_i|^2 - |a_j|^2\right)\langle a_j\,|a_i\rangle = 0.$$

If $i = j$ the equation is satisfied automatically. If $i \neq j$ then $\langle a_j\,|a_i\rangle = 0$ therefore the states are orthogonal.