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Math Methods in Physics

CH. 3 VECTORS & MATRICES LECTURE NOTES

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Numerics

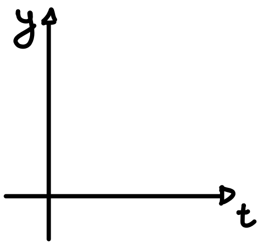
Our Euler Lagrange equations yield ODE & PDE. These are not all easy to solve.

1. Classical Mechanics : $L = \frac{1}{2} m \dot{\vec{x}}^2 - U(\vec{x})$, $\ddot{\vec{x}} = -\vec{\nabla} U(\vec{x})$ Initial value P.
2. Electro Magnetism : $\int (\vec{\nabla}\varphi)\, d^2x$, but $\varphi(x)\Big|_{\vec{x} = \vec{l} = \text{curve}} = f(\vec{x})$ Boundary value P.
3. Field problem : Extremization might yield $-\frac{\hbar^2}{2m}\nabla^2\varphi + V(\vec{x})\Psi = E\Psi(\vec{x})$ Eigenvalue P.

In general 1 IVP is of the form : $\dot{x}_1 = f_1(x_1, x_2, x_3, \ldots)$
$\dot{x}_2 = f_2(x_1, x_2, x_3, \ldots)$

For 2nd order : $\dot{x}_1 = x_2$, $\dot{x}_2 = f(x)$ → Any order ODE can be handled this way

without loss of generality (WLOG) we can focus on a 1D problem



$\dot{y}(t) = f(y,t)$

we discretize time

$t_i = i\Delta t$, i = integer

Then we represent the derivatives : $\frac{d}{dt} y(t) \approx \frac{y(t+\Delta t) - y(t)}{\Delta t}$

Q : How accurate is this ?
A : For a well behaved $y(t)$

$$y(t+\Delta t) = y(t_i) + \dot{y}(t_i)\Delta t + \frac{1}{2}\ddot{y}(t_i)\Delta t^2$$

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} = \dot{y}(t) + \frac{1}{2}\ddot{y}\Delta t$$

Given our ordinary differential equation , $y_{i+1} = y_i + f(y_i)\Delta t$ → we know all of our quantities on the RHS

⇝ Explicit method

$\frac{y_{i+1} - y_i}{\Delta t} = f(y,t)$, we could choose $f(y_{i+1}, t_{i+1})$

⇝ Implicit method

Backwards Euler

We can choose a mid-point method

$f(y_{i+1}, t_{i+1}) \rightarrow y_{i+1} = y_i + \frac{\Delta t}{2}(f_{i+1} + f_i) \rightarrow$ often the best behaved approach

$\frac{y(t+\Delta t) - y'(t)}{\Delta t}$:

$\frac{y_{i+1} - y_{i-1}}{2\Delta t} = \dot{y}(t) + \frac{2\Delta t}{2}\ddot{y} + \frac{2\Delta t^2}{3!}\dddot{y}$, $\frac{y_{i+2} - y_{i-2}}{4\Delta t} = \dot{y}(t) + \frac{4\Delta t}{2} + \frac{(4\Delta t)^2}{3!}\dddot{y}$

↳ A ↳ B

$2A - B = \dot{y}(t) - \frac{4}{3}\Delta t^2 \dddot{y}$

5-pt formula $\rightarrow$ errors in Δt^2

$$\dot{y}_i = \frac{-\frac{1}{4} y_{i+2} + y_{i+1} - y_{i+1} + \frac{1}{4} y_{i-2}}{\Delta t} - \frac{4}{3}\Delta t^2 \dddot{y}$$

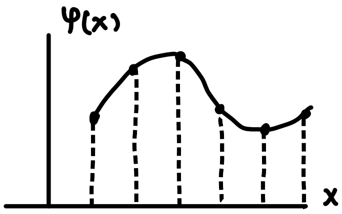
A very common package $\rightsquigarrow$ Predictor or corrector codes that evaluate the derivatives at multiple Δt's

10-11-21

Boundary Value Problems

Typically 2nd order ODE or PDE

$\frac{\partial^2}{\partial x^2}\varphi(x) = 0$, $\frac{\partial^2 \varphi}{\partial x^2} = F(x)$ with specific BV's $\varphi(0) = 1$, $\varphi(\pi) = 0$



$$\varphi(x_{j+1}) = \varphi(x_j + \Delta x) = \varphi(x_j) + \varphi'(x_j)\Delta x + \frac{1}{2}\varphi''(x_j)\Delta x^2 + \frac{1}{3!}\varphi'''(x_j)\Delta x_j^3 + \frac{1}{4!}\varphi^4(x_j)\Delta x_j^4$$

$$\varphi_{j+1} = \varphi_j + \varphi'(x_j)\Delta x + \frac{1}{2}\varphi''(x_j)\Delta x^2 + \frac{1}{3!}\varphi'''(x_j)\Delta x^3 + \frac{1}{4!}\varphi^4(x_j)\Delta x^4$$

$$\varphi_{j-1} = \varphi_j - \varphi'(x_j)\Delta x - \frac{1}{2}\varphi''(x_j)\Delta x^2 - \frac{1}{3!}\varphi'''(x_j)\Delta x^3 - \frac{1}{4!}\varphi^4(x_j)\Delta x^4$$

$$\varphi_{j+1} + \varphi_{j-1} = 2\varphi_j + \varphi''(x_j)\Delta x^2 + \frac{1}{4!}\varphi^4 \Delta x^4$$

$\frac{\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j}{\Delta x^2} = \varphi''(x_j) + \frac{1}{12}\Delta x^2 \varphi^4(x_j)$, How can I use this?

$\varphi(x) \rightarrow \varphi(x_j) \rightarrow \varphi_j = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix}$, $\frac{\partial^2}{\partial x^2} \rightarrow D^2_{ij}$

$$\begin{pmatrix} & & & \\ \frac{1}{\Delta x^2} & -2\frac{1}{\Delta x^2} & +\frac{1}{\Delta x^2} \\ & & & \end{pmatrix}\begin{pmatrix} \vdots \\ \varphi_{j-1} \\ \varphi_j \\ \varphi_{j+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \varphi''_{j-1} \\ \varphi''_j \\ \varphi''_{j+1} \\ \vdots \end{pmatrix} \qquad \mathbb{D}^2 = \frac{1}{\Delta x^2}\begin{pmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \end{pmatrix}$$

Boundary Conditions

Let's be specific, $0 \le x \le L$, $\Delta x = L/N$, $N = 6$

① Zero - Value B.C, $\varphi(0) = \varphi(L) = 0$

Q: How many unknowns are there?

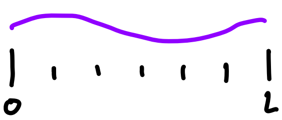
A: For zero value boundary conditions, number of unknowns $= N-1$

$$\begin{pmatrix} \frac{-2}{\Delta x^2} & \frac{1}{\Delta x^2} & & \\ \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} & \frac{1}{\Delta x^2} & \\ & & & \\ & & \frac{1}{\Delta x^2} & -\frac{2}{\Delta x^2} \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{pmatrix}$$

$j=1$

What if $\varphi_0 = c_0$

② Zero ... derivative boundary conditions

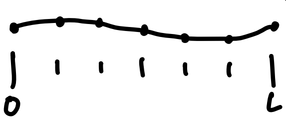


Q: How many unknowns are there?

A: We have $N+1$ unknowns

$$\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 2 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & & & & 2 & -2 & 1 \end{pmatrix} \qquad \varphi'_0 = \frac{\varphi_1 - \varphi_{-1}}{2\Delta x} = 0$$

③ Periodic Boundary Conditions



There are N unknowns

$$\frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & & & & \\ & & & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}\begin{pmatrix} \varphi_2 \\ \vdots \\ \vdots \\ \varphi_{N-1} \end{pmatrix}$$

10-13-21

Discretizing ODE: we can turn an equation

$$\frac{\partial^2\varphi}{\partial x^2} + \cos(x)\varphi(x) = 0 \quad \text{w/ Boundary Conditions } \varphi(0) \dots \varphi(2\pi) = C$$

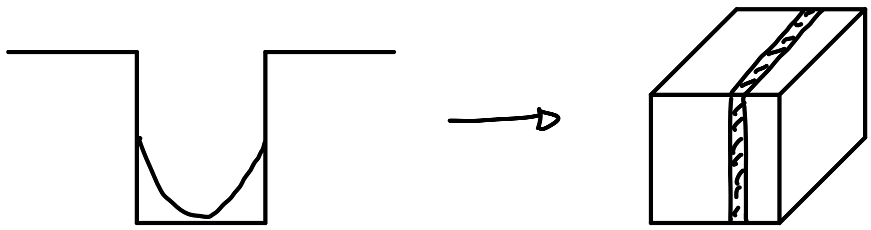
we can turn this into a matrix equation, $M_{ij}\varphi_j = b_i$

Similarly we can turn Eigenvalue DE's → Matrix eigenvalue equations

Turning a Physics problem → numerical problem

Example: $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}kx^2\psi = E\psi$$

Let's say this is an electron in a box with a width of this $V(x)$



computers do not understand units.

→ Using MKS units can lead to accuracy problems

→ Lose intuition on what is going on

You want to go to dimensionless units, change from dimension-full variables to dimensionless.

$$x = \underbrace{\ell}_{\text{Length we choose for convenience}} \underbrace{s}_{\text{dimensionless}}$$

$$-\frac{\hbar^2}{2m}\frac{1}{\ell^2}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}k\ell^2 s^2 = E\psi$$

We want to <u>choose</u> ℓ s.t. our equation becomes simpler. Do not make $\ell < 0$ or imaginary.

If solution $\sim e^{iks\ell}$

$$\frac{\hbar^2}{2m\ell^2} = \frac{1}{2}k\ell^2 \quad \therefore \quad \ell^2 = \frac{\hbar}{\sqrt{km}} \longrightarrow \frac{\hbar^2}{2m}\frac{\sqrt{km}}{\hbar} = \frac{1}{2}\hbar\sqrt{\frac{k}{m}} = \frac{1}{2}\hbar\omega_0$$

$$-\frac{\partial^2}{\partial s^2}\psi + s^2\psi = \frac{2E}{\hbar\omega_0}\psi \quad : \quad E = \underbrace{\epsilon}_{\text{Dimensionless}}\underbrace{E_0}_{\text{Dimension-Full}} \qquad \text{Set } E_0 = \frac{1}{2}\hbar\omega_0$$

$$-\frac{\partial^2}{\partial s^2}\psi + s^2\psi = \epsilon\psi \quad : \quad \psi(0) = \psi(L) = 0$$

Aside: A differential equation is not well posed without Boundary Conditions : $\psi(\pm\infty)=0$

We are now working in a box of finite width and this will change the true answers. $x = L$

$$-\frac{\hbar^2}{2m}\frac{1}{L^2}\frac{\partial^2}{\partial s^2}\psi + \frac{1}{2}kL^2 s^2\psi = \epsilon E_0\psi \text{ , Do either } \frac{\hbar^2}{2mL^2} = E_0 \text{ or } \frac{1}{2}kL^2 = E_0$$

$\frac{\hbar^2}{2mL^2} \longrightarrow$, $k = \frac{n\pi}{L}$: Energy of a particle in a box , KE

$\frac{1}{2}kL^2 \longrightarrow$: Max potential energy

$$-\frac{\partial^2}{\partial s^2}\psi + \underbrace{\frac{\frac{1}{2}kL^2}{\hbar^2/2mL^2}}_{\delta} s^2\psi \quad : \quad -\frac{\partial^2}{\partial s^2}\psi + \delta s^2\psi$$

: $\delta \to$ Large HO approx is good!

$\to$ Small , KE dominates

10-18-21

Groups

A group is a set of elements and binary operations such that the operation only yields other members of the group.

$\forall a,b \in G$ if $a \circ b = c$ then $c \in G \longrightarrow$ Closure

(1) The operation is associative

$$\forall a,b,c \in G \quad (a \circ b) \circ c = a \circ (b \circ c)$$

(2) $\exists$ an identity element e such that

$$a \circ e = e \circ a = a \quad \forall a \in G$$

(3) $\forall a \in G \ \exists \ a^{-1} \in G$ (For all a is G then exists a^{-1} in G)

$$a \circ a^{-1} = a^{-1} a = e$$

If $a \circ b = b \circ a$ for all $a,b \in G$ then we say that this an Abelian Group

Quiz! Is it a Group?

(1) Integers under addition $\therefore$ Abelian Group

(2) Integers under multiplication $\therefore$ Not a Group

(3) Real numbers under multiplication $\therefore$ Not a Group

(4) " excluding zero " ∴ Abelian Group

(5) Rotation about $\hat{k}$ axis ∴ Abelian Group

(6) General 3D rotation ∴ Group, not Abelian Group

(7) Rules of chess ∴ Not a Group

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Groups

Representations: we can have different representations, because it is truly defined by its table of binary operations.

Example: Rotate a 2D vector about the origin

$$\mathbb{R}(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$ Form a group.

The set of complex numbers of unit modulus

$$|z| = 1 \;,\; z = e^{i\theta} \;:\; z_1, z_2 = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

Consider numbers & 2D rotation that have the same binary operation

IF $\mathbb{R}(\theta_1)\mathbb{R}(\theta_2) = \mathbb{R}(\theta_3) \;:\; e^{i\theta_1}e^{i\theta_2} = e^{i\theta_3}$

These are both representations of $U(1)$. The unitary group in 1D.

Field: A field has two operations and a set of elements F $(F, +, \cdot)$ that satisfy

(1) $(F, +, \cdot)$ is an Abelian group with identity "o"

(2) Let F' be the set F without "0" then $\{F', \cdot\}$ Form a group with identity "e"

(3) For $a, b, c \in F$ then $a \circ (b+c) = a \circ b + a \circ c$

Real numbers, with "+" & "x" are a Field. : Need two operations for field

2D vectors? No! 3D vectors? No!

Vector Space is defined over a Field F & consists of a set of elements V(F) such that

(1) There exists an operation "o" such that $\{V(F), \text{"o"}\}$ form an Abelian group

(2) For all $\alpha, \beta \in F$ and $x \in V(F)$ that $\alpha\vec{x} \in V(F)$

(a) $\alpha(\beta\vec{x}) = \alpha\beta\vec{x}$

(b) There is an identity element $\mathbb{1}$. $\mathbb{1}\vec{x} = \vec{x} \quad \forall \vec{x} \in V(F)$

(c) $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x} + \alpha\vec{y}$

(d) $(\alpha+\beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$

Ex: F is the real numbers $\mathbb{R}$, $V(\vec{F})$ = all 3D vectors, F is the complex numbers $\mathbb{C}$
V(F) Span complex 2-vectors

F is a complex number

$V(F) = z_1\mathbb{1} + i z_x\hat{\sigma}_x + i z_y\hat{\sigma}_y + i z_z\hat{\sigma}_z$ where $\mathbb{1}, \hat{\sigma}$ are 2x2 matrices

Is the space of polynomials a vector space? Yes!

10-22-21

Isomorphism: Given groups G with binary operation "o" & H with operation * if there exists a one-to-one mapping $\varphi(g_i) = h_i$ and $\forall g_1, g_2, g_3 \in G$ if $g_1 \circ g_2 = g_3$ then $h_1 * h_2 = h_3$, then the two groups are isomorphic.

G are the + reals .... with operation "x" H the reals mapping $h_1 = \mathrm{Im}(g_1)$
$g_1 \times g_2 = g_3$

Linear Transformations: A linear transformation or linear operation A on a vector space V(F) is a correspondence that exists to every x ~~dy~~ $\in V(F)$ the vectors $A(x)$ & $A(y)$ such that $A(\alpha x + \beta y) = \alpha A x + \beta A y$ for scalors $\alpha, \beta \in F$.

Q: Is it a linear operation?

Rotation of vectors? yes. Differentiation of polynomials? yes.

Transformation of Vectors:

$\Pi \cdot \vec{x} = \vec{x} + \vec{V}_0$, where $\vec{V}_0$ is constant

$\Pi\vec{x} + \Pi\vec{y} = \vec{x} + \vec{V}_0 + \vec{y} + \vec{V}_0 \neq \Pi(\vec{x}+\vec{y})$, $\Pi'(a) f(x) = f(x+a)'$ G

Linear operations look like multiplication but they might not commute

$A_1 A_2 x \stackrel{?}{=} A_2 A_1 x$

Big Question: If $Ax = y$ and we know y and A, can we find x? Does A have an "inverse" operator? If & only if A posseses the following properties.

(A) If $x \neq y$ then we know (or this implies) $Ax \neq Ay$

(B) $\forall\, y \in V(F)\ \exists\, x \in V(F)$ such that $Ax = y$ then A^{-1} exists such that $AA^{-1} = A^{-1}A = \mathbb{1}$

Note, even if $AB = \mathbb{1}$ this does not mean that A^{-1} exists

$\mathbb{D}$ derivatives $\mathbb{D}p(x) = \frac{d}{dx}p(x)$, $\mathbb{S}p(x) = \int_0^x p(t)\,dt$, $\mathbb{D}\mathbb{S} = \mathbb{1}$ but $\mathbb{S}\mathbb{D} \neq \mathbb{1}$

When can we use A? Often = "when can we solve problems":

Matrices & Matrix Representations

Recall that the vector $\vec{x}$ exists independent of our basis – if we write $x = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 = x\hat{i} + y\hat{j} + z\hat{k}$ then $(x, y \,\&\, z) = (x_1, x_2, x_3)$

Similarily if I write $Ax = y$, w/ that of A on a matrix. How do I find the matrix representation of A?

To start we need a basis – Lets call it $\{u_i\}$ (B & F use "x_i" but I find that confusing)

Then $x = \alpha_i u^i$, $y = \beta_i u^i$, we also must define A in terms of this basis – For each u^i in basis we need to know how A acts on u^i.

$Au^{(i)} = a_{ij}u^{(i)}$, then the equation $Ax = y$ $\sum\limits_i \beta_i u^{(i)} = A\sum\limits_j \alpha_j u^{(j)} = \sum\limits_{ij} a_j a_{ij} u^{(i)}$
or subtract $\sum(\beta_i - a_{ij}\alpha_j)u^{(i)} = 0$ $\therefore$ $\beta_i = a_{ij}\alpha_j$

Return to our Question from chapter 1. How will A look in a different basis?
→ What will change? → what doesn't change?

Let $\mathbb{B}$ be a linear operator: It has the representation b_{ij} w.r.t $\{u^{(i)}\}$
b_{ij}' w.r.t $\{v^{(j)}\}$

That is – $\mathbb{B}u^{(j)} = b_{ij}u^{(i)}$, $\mathbb{B}v^{(j)} = b_{ij}'v^{(i)}$

Finally – Let A transform $u^{(i)} \rightarrow v^{(i)}$, $Au^{(j)} = v^{(j)}$

$\mathbb{B}v^{(j)} = \mathbb{B}Au^{(j)} = \mathbb{B}\sum\limits_k a_{kj}u^{(k)} = \sum\limits_k a_{kj}\mathbb{B}u^{(k)} = \sum\limits_k a_{kj}\sum\limits_i b_{ik}u^{(i)}$

$\sum\limits_k b_{kj}'v^{(k)} = \sum\limits_k\sum\limits_i b_{kj}a_{ik}u^{(i)}$, $\sum\limits_k b_{ik}a_{kj} = \sum\limits_k a_{ik}b_{kj}'$

$[\mathbb{B}]_u[A]_{u\to v} = [A]_{u\to v}[\mathbb{B}]_v$, $[\mathbb{B}]_u = [A]_{u\to v}[\mathbb{B}]_v[A]^{-1}_{u\to v}$

↳ Similarity transformation

10-25-21

Given $\mathbb{B}$ in u and v representations and A that maps $u \to v$

$[\mathbb{B}]_u[A]^{-1}_{u\to v} = [\mathbb{B}]_v[A]_{u\to v}$, $\mathbb{b} = a_{ii}'\,\mathbb{b}_a$ ⟶ Similarity transformation why do we care ?

$\rightsquigarrow$ Communication ....
$\rightsquigarrow$ Simplification

$\mathbb{b}'$ might be diagonal, $\mathbb{b}'$ might be block diagonal

Given $\mathbb{b}$ how do we find a_{11} ?

Eigenvectors

IF there is a non-zero vector x such that $Ax = \lambda x$ when λ is a scalar then x is an eigenvector of a_{11} with eigenvalue λ.

To find λ, $(Ax - \lambda \mathbb{1} x) = 0$, $(A - \lambda \mathbb{1})x = 0$. These are linear operators $\rightarrow$ some basis $(a_{11} - \lambda \mathbb{1})x = 0$.

This has a non-trivial solution $(a_{11} - \lambda \mathbb{1})$ is a singular matrix or $\det[a_{11} - \lambda \mathbb{1}] = 0$.

Numerically / computationally, if λ is a variable. $\det(a_{11} - \lambda \mathbb{1})$ is a N^{th} order polynomial if dim a_{11} is $N \times N$

It is easy to find the zeros of this polynomial by searching numerically. To find the end points of the vector. $\rightsquigarrow$ Power method — Assume $y^{(1)}$ is a radial vector.

I can expand it in the eigenvectors of a_{11}

$$y^{(1)} = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots\dots$$

$$= c_i x_i$$

$$a_{11} y = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots\dots$$

$$a_{11}^2 y = c_1 \lambda_1^2 x_1 + c_2 \lambda_2^2 x_2 + \dots\dots$$

$$a_{11}^n y = c_1 \lambda_1^n x_1 + c_2 \lambda_2^n x_2 + \dots\dots$$

Largest eigenvalue grows fastest so $y^{(P)} = \dfrac{a_{11} y^{P-1}}{a_{11} y^{P-1}}$

$\rightsquigarrow$ Flow to largest eigenvalue & its eigenvector, $a_{11}^{-1} y$ $\rightsquigarrow$ Flow to smallest

$$a_{11} y^{(P+1)} = y^{(P)}$$

Byron & Fuller prove $\det(a_{11}\mathbb{b}) = \det(a_{11})\det(\mathbb{b})$, Then

$$\det(\mathbb{1}) = \det(a_{11}^{-1} a_{11}) = \det(a_{11}^{-1})\det(a_{11})$$

have inverse determinants

Similar matrices

$\hookrightarrow \mathbb{b} = \mathcal{S}^{-1} \mathbb{a} \mathcal{S}$

then $\det(\mathbb{b}) = \det(\mathcal{S}^{-1}\mathbb{a}\mathcal{S}) = \det(\mathbb{a})$

Q: Can we find an $\mathcal{S}$ that gives a simple $\mathbb{b}$ given $\mathbb{a}$?

A: Yes! Assume that we know all of the eigenvectors of $\mathbb{a}$

$$\mathbb{A}\, x^{(i)} = \lambda^{(i)} x^{(i)} \quad , \quad \mathbb{a}_{jk}\, x_k^{(i)} = \lambda^{(i)} x^{(i)}$$

Construct a square matrix $\mathbb{X}_{jk}$ for the eigenvectors

$$\mathbb{X}_{jk} = x_j^{(k)} \text{ so that } (\mathbb{a}\, \mathbb{X})_{ik} = \sum_j \mathbb{a}_{ij}\, x_j^{(k)} = \lambda^{(k)} x_i^{(k)} = \sum_j x_i^{(k)} (\lambda^{(j)} \delta_{jk})$$

$$x^{-1} \mathbb{a}\, x = x^{-1} x\, \mathbb{D}(\lambda^{(i)}) \longrightarrow \text{Diagonalizes } \mathbb{a}$$

Since the determinant of $\mathbb{D}$ is the product of its elements, $\det(\mathbb{a}) = \prod \lambda^{(i)}$. This is true for the trace of $\mathbb{a}$ as well

$$\det(\mathbb{a}) = \epsilon_{abc}\, a_{1a} a_{2b} a_{3c}$$

10-27-21

Kroenicker products are useful when you have two separate vector spaces that you want to describe jointly.

If we have a Hamiltonian. As for $\dot{s}$ and H_σ for a σ, how do we write the Hamiltonian.

$$H_{\text{simplified}} = H_s \otimes \mathbb{1}_{\sigma} + H_s \otimes \mathbb{1}$$

Given two linear operators $\mathbb{A}$ & $\mathbb{B}$ with matrix representations $\mathbb{a}$ & $\mathbb{b}$. The kroenecker product

$$\mathbb{a} \otimes \mathbb{b} = \begin{pmatrix} a_{11}\mathbb{b} & a_{12}\mathbb{b} & a_{13}\mathbb{b} \\ a_{21}\mathbb{b} & \cdots & \cdots \\ a_{31}\mathbb{b} & \vdots & \vdots \end{pmatrix}$$

A simple example

$$\vec{x} = x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k} \quad , \quad \vec{y} = y_1 \hat{i} + y_3 \hat{j} + y_3 \hat{k}$$

$$\vec{x} \otimes \vec{y} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix} , \quad H_\sigma = \begin{pmatrix} \epsilon_+ & A \\ A & \epsilon_- \end{pmatrix} , \quad H_s = \begin{pmatrix} E_+ & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_- \end{pmatrix}$$

$$H = H_\sigma \otimes \mathbb{1}_s + \mathbb{1}_\sigma \otimes H_s = \begin{pmatrix} \epsilon_+ \mathbb{1} & \Delta \mathbb{1} \\ \Delta \mathbb{1} & \epsilon_- \mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} H_s & 0 \\ 0 & \mathbb{1} H_s \end{pmatrix}$$