Classical Mechanics and Statistical/Thermodynamics

January 2017

Possibly Useful Information

Handy Integrals:

$$\int_0^\infty x^n e^{-\alpha x}\, dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x\; e^{-\alpha x^2}\, dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2\; e^{-\alpha x^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

$$\int_{-\infty}^\infty e^{i\,a\,x-b\,x^2}\, dx = \sqrt{\frac{\pi}{b}}e^{-a^2/4b}$$

Geometric Series:

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x} \qquad \text{for} \quad |x|<1$$

Stirling's approximation:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Levi-Civita tensor:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

Riemann and related functions:

$$\sum_{n=1}^\infty \frac{1}{n^p} \equiv \zeta(p)$$

$$\sum_{n=1}^\infty \frac{z^p}{n^p} \equiv g_p(z) \qquad \sum_{n=1}^\infty (-1)^p\ \frac{z^p}{n^p} \equiv f_p(z)$$

$$g_p(1) = \zeta(p) \qquad f_p(1) = \zeta(-p)$$

$$\begin{array}{cc} \zeta(1) = \infty & \zeta(-1) = -\frac{1}{12} = 0.0833333 \\ \zeta(2) = \frac{\pi^2}{6} = 1.64493 & \zeta(-2) = 0 \\ \zeta(3) = 1.20206 & \zeta(-3) = \frac{1}{120} = 0.0083333 \\ \zeta(4) = \frac{\pi^4}{90} = 1.08232 & \zeta(-4) = 0 \end{array}$$

Physical Constants:

Coulomb constant $K = 8.998\times10^9$ N-m$^2/C^2$	$\epsilon_0 = 8.85\times10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$
$\mu_0 = 4\pi\times10^{-7}$T m/A	electronic charge $e = 1.60\times10^{-19}$C
electronic mass $m_e = 9.11\times10^{-31}$kg	Density of pure water: 1.00gm/cm^3.
Boltzmann's constant: $k_B = 1.38\times10^{-23}$J/K	Planck's constant: $\hbar = 6.63\times10^{-34}\mathrm{m}^2\mathrm{kg/s}$
speed of light: $c = 3.00\times10^8$m/s	Ideal Gas Constant: $R = 0.0820\,\ell\mathrm{atm}\cdot\mathrm{mol}^{-1}\mathrm{K}^{-1}$

Classical Mechanics

1. A solid sphere with mass M and radius R is initially rotating clockwise at an angular velocity of ω_0 about a line through its center and parallel to the x-axis, but with a horizontal velocity of zero. It is set on a horizontal surface with coefficient of kinetic friction given by μ_k. It has no initial horizontal velocity. Once it touches the surface, it will initially roll while slipping. It will accelerate horizontally until the point where it will start to roll without slipping.

   (a) Show that the moment of inertia for a solid sphere with mass M and radius R rotating around its center of mass is given by $(2/5)MR^2$. (2 points)

   (b) What is the initial linear acceleration of the sphere, when it first touches the surface? (1 point)

   (c) What is the initial angular acceleration (about the center of mass) of the sphere? (1 point)

   (d) Initially the sphere is only slipping on the surface, but will eventually roll without slipping. Determine the time it takes and the distance traveled before the sphere begins rolling without slipping. (3 points)

   (e) How much work was done by the frictional force on the sphere during this time? (3 points)

2. A particle is constrained to move on a surface of a cylinder of radius a, and is attracted to the origin by a force proportional to the distance from the origin:

$$\vec{F}(\vec{r}) = -k\,\vec{r}$$

where $\vec{r}$ is the 3D vector from the origin to the location of the particle.

(a) (1 pts) Find the Lagrangian in cylindrical coordinates.

(b) (2 pts) Use Lagrange's equations to find the equations of motion.

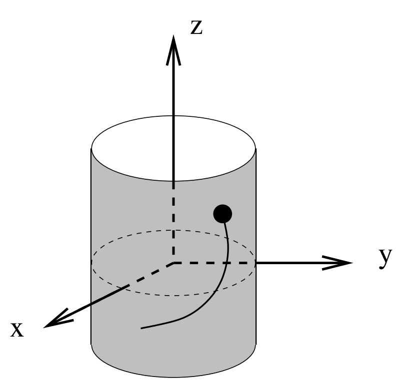
(c) (1 pt) Find the generalized momenta and the kinetic energy in terms of the generalized momenta.

(d) (2 pts) Find the Hamiltonian. Is the Hamiltonian equal to the total energy for this system? Why or why not?

(e) (2 pts) Find Hamilton's equations and show they reproduce the equations of motion from Lagrange's equations.

(f) (1 pt) What are the constants of motion? Explain or show mathematically why.

(g) (1pt) Are the orbits open or closed? If they are closed, in general, prove it. If they are open, find the set of initial conditions that will produce closed orbits.



3. Consider a free particle of mass m moving on a plane.

   (a) Write down the Lagrangian of the system in polar coordinates r and ϕ. (1 point)

   (b) Calculate the Hamiltonian of the particle in the coordinates r and ϕ and in the generalized momenta p_r and p_ϕ. (2 points)

   (c) Set up the time independent Hamilton-Jacobi equation and find the Jacobi complete integral $W(r, \phi, p_r, p_\phi)$ (characteristic function). (3 points)

   (d) Using your solution of $W(r, \phi, p_r, p_\phi)$, calculate r and ϕ as a function of time. (4 points)

Statistical Mechanics

4. Consider two metal samples, one hot and one cold. The cold sample consists of n_1 moles of atoms at temperature T_1, while the hot sample consists of n_2 moles of atoms at temperature T_2. The molar specific heat of the solid metal is c_s, its melting temperature is T_m, and the latent heat of fusion of the metal is L_f. The molar specific heat of the liquid (molten) metal is c_ℓ.

   (a) Consider the case where initially both metal samples are solid bricks that are placed in contact with each other, but are otherwise thermally isolated.

      i. Calculate the final, equilibrium temperature of the system. (1 point)

      ii. Calculate the total entropy change of the system. (3 points)

   (b) Now consider the case where the hot sample is so hot that it is molten, but the cold sample is not. Again the samples are placed in contact with each other, but are otherwise thermally isolated. The initial conditions are such that the final temperature is greater than T_m.

      i. Calculate the final, equilibrium temperature of the system. (2 points)

      ii. Calculate the total entropy change of the system. (4 points)

5. Consider a gas of N distinguishable, free, non-interacting particles moving in a fixed three dimensional volume V and in thermal equilibrium with a reservoir at temperature T. The particles have a spin 1/2 and are in a uniform, constant magnetic field, $\vec{B}_0$. The energy of the i-th particle is

$$E_i = \frac{\vec{p_i}^{\,2}}{2m} + \sigma_i \, \Delta$$

where $\vec{p}$ is the particle momentum, m is its mass, $\sigma \in \{-1, +1\}$, and the interaction of the spin with the field is the constant $\Delta = g\mu_0 B_0/2$. We assume that the temperature is high and the density is low so that we can treat the problem classically.

(a) What is the partition function for the system in the canonical ensemble? (2 points)

(b) What is the average energy of the system? (2 points).

(c) What is the specific heat at constant volume for the system? (3 points)

(d) Calculate the specific heat in the limiting cases below. In each case explain why the presence of the spin degree of freedom affects (or does not affect) the specific heat .

i. $\Delta \gg kT$ (1.5 points).

ii. $\Delta \ll kT$ (1.5 points).

Simply stating a mathematical result is insufficient - you must give a physical explanation of the result.

6. Consider a gas of N free, non-interacting electrons (spin 1/2 fermions) in a three dimensional volume, V.

   (a) What is the fermi energy (the chemical potential at $T = 0$) for the gas as a function of N and V? (2 points)

   (b) Calculate the energy of the electron gas, at $T = 0$, and show that it can be written as
   $$U(N, V, T)\Big|_{T=0} = C\,NE_f$$
   for a constant C, and determine C. (2 points)

   (c) Calculate the pressure the gas exerts on the wall of the container at $T = 0$, as a function of V and N. (2 points)

   (d) The density of conduction electrons in copper is approximately $8.5 \times 10^{28}\text{m}^{-3}$. What is the pressure exerted by the copper atoms? (2 points)

   (e) What is the pressure of an ideal classical gas at room temperature and with the same density? (1 point)

   (f) Both of your results are for non-interacting, ideal gasses, but one pressure is much larger than the other. Why is it larger, even though there are no interactions in either system? (1 point)

   You may find it helpful to refer to the table of constants at the front of the exam in evaluating your expressions.