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Quantum Mechanics 1

PHYS 5393 Homework Assignment #9

Problems: {1.8, 3.4, 3.5, 3.9, 3.38}

Due: November 9, 2021 By: 5 PM

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Problem 1: 1.8

Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator $\tilde{\mathbf{A}}$. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of $\tilde{\mathbf{A}}$? Justify your answer.

The only way this will be true is if the two eigenkets are degenerate.

$$\tilde{A}(|i\rangle + |j\rangle) = a_{ij}(|i\rangle + |j\rangle)$$

Problem 1: 1.8 Review

Procedure:

- Show that the only case where this is possible is where the eigenkets are degenerate.

Key Concepts:

- The only way $|i\rangle$ and $|j\rangle$ can be eigenkets of the same operator $\tilde{\mathbf{A}}$ is if the they are degenerate.
- Degeneracy refers to having the same eigenvalue for different eigenkets.

Variations:

- We can be asked what condition can we conclude if this sum of states is not an eigenket of $\tilde{\mathbf{A}}$.
  - This however would alter the problem and would require us to answer a completely different question.

Problem 2: 3.4

Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\sigma \cdot \mathbf{a}}{a_0 - i\sigma \cdot \mathbf{a}}$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

(a) Prove that U is unitary and unimodular.

Unitary: $UU^\dagger = \mathbb{1}$, $A = a_0 + i\vec{\sigma}\cdot\vec{a}$, $A^\dagger = a_0 - i\vec{\sigma}\cdot\vec{a}$

$$U = A(A^\dagger)^{-1} \, , \, UU^\dagger = A(A^\dagger)^{-1}(A(A^\dagger)^{-1})^\dagger = A(A^\dagger)^{-1}A^{-1}A^\dagger = A(AA^\dagger)^{-1}A^\dagger$$

$$AA^\dagger = (a_0 + i\vec{\sigma}\cdot\vec{a})(a_0 - i\vec{\sigma}\cdot\vec{a}) = (a_0^2 + a^2)\mathbb{1} \equiv \alpha\mathbb{1} \;\therefore\; (AA^\dagger)^{-1} = (1/\alpha)\mathbb{1} \;\therefore\; (1/\alpha)AA^\dagger = \mathbb{1} \checkmark$$

Unimodular: $\det(U) = 1$

$$A = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} , \det(A) = \alpha \;:\; A^\dagger = \begin{pmatrix} a_0 - ia_3 & -ia_1 + a_2 \\ -ia_1 - a_2 & a_0 + ia_3 \end{pmatrix} , \det(A^\dagger) = \alpha$$

$$\det(U) = \det(A)/\det(A^\dagger) = \alpha/\alpha = 1 \checkmark$$

$$\boxed{U \text{ is Unitary \& Unimodular } \checkmark}$$

(b) In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

We can write U as: $U = AAA^{-1}(A^\dagger)^{-1} = A^2(AA^\dagger)^{-1}$

$$U = \frac{1}{\alpha^2}\begin{pmatrix} a_0^2 - a^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_1 + 2ia_0a_1 & a_0 - a^2 - 2ia_0a_3 \end{pmatrix}$$

Using (3.80), we can deduce the following for n_x, n_y, n_z. $\quad U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$$n_x = \frac{-\mathrm{Im}(b)}{\sin(\varphi/2)} , \; n_y = \frac{-\mathrm{Re}(b)}{\sin(\varphi/2)} , \; n_z = \frac{-\mathrm{Im}(a)}{\sin(\varphi/2)}$$

with $\sin(\varphi/2) = \frac{2a_0|a|}{\alpha^2}$ and $\cos(\varphi/2) = \frac{a_0^2 - a^2}{\alpha^2}$

$$\boxed{n_x = -\frac{a_1}{|a|} , \; n_y = -\frac{a_2}{|a|} , \; n_z = -\frac{a_3}{|a|}}$$

Problem 2: 3.4 Review

Procedure:

- To show that $\tilde{\mathbf{U}}$ is unitary, begin by expressing $\tilde{\mathbf{U}}$ as $\tilde{\mathbf{U}} = \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^\dagger)^{-1}$.
- Use the above relationship to arrive at $\tilde{\mathbf{U}} = \tilde{\mathbf{A}}(\tilde{\mathbf{A}}\tilde{\mathbf{A}}^\dagger)^{-1}\tilde{\mathbf{A}}^\dagger$.
- Use the above relationship with $\tilde{\mathbf{A}} = a_0 + i\hat{\sigma}\cdot\hat{\mathbf{a}}$ and $\tilde{\mathbf{A}}^\dagger = a_0 - i\hat{\sigma}\cdot\hat{\mathbf{a}}$ to show that $\tilde{\mathbf{U}}$ is unitary.
- To prove that $\tilde{\mathbf{U}}$ is unimodular, express $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^\dagger$ as:

$$\tilde{\mathbf{A}} = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{A}}^\dagger \begin{pmatrix} a_0 - ia_3 & a_2 - ia_1 \\ -ia_1 - a_2 & a_0 + ia_3 \end{pmatrix}$$

  and show that $\det(\tilde{\mathbf{U}}) = 1$.
- To find the components for $\mathbf{n}$ write $\tilde{\mathbf{U}}$ as $\tilde{\mathbf{U}} = \tilde{\mathbf{A}}\tilde{\mathbf{A}}\tilde{\mathbf{A}}^{-1}(\tilde{\mathbf{A}}^\dagger)^{-1}$ and calculate the matrix representation.
- Using equation (3.80) in the third edition of Sakurai, we can deduce

$$n_x = \frac{-\text{Im}(b)}{\sin(\phi/2)} \quad , \quad n_y = \frac{-\text{Re}(b)}{\sin(\phi/2)} \quad , \quad n_z = \frac{-\text{Im}(a)}{\sin(\phi/2)}.$$

Key Concepts:

- For an operator to be unitary, it must follow that $\tilde{\mathbf{X}}\tilde{\mathbf{X}}^\dagger = \mathbb{I}$.
- For an operator to be unimodular, the determinant of that operator must be equal to 1.
- We can calculate the components of $\mathbf{n}$ by using the equations for n_x, n_y, n_z as long as we have a matrix representation for $\tilde{\mathbf{U}}$.

Variations:

- We can be given a different operator $\tilde{\mathbf{U}}$.
  - This would change the matrix representations for $\tilde{\mathbf{A}}$ but it would not change the overall process with calculating the components and showing the properties of $\tilde{\mathbf{U}}$.

Problem 3: 3.5

The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

$$\tilde{\mathbf{H}} = A\tilde{\mathbf{S}}^{(e^-)} \cdot \tilde{\mathbf{S}}^{(e^+)} + \left(\frac{eB}{mc}\right)(S_z^{(e^-)} - S_z^{(e^+)}).$$

Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.

(a) Is this an eigenfunction of $\tilde{\mathbf{H}}$ in the limit $A \to 0, eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of $\tilde{\mathbf{H}}$?

$$\tilde{H} = A\tilde{S}^{(e^-)} \cdot \tilde{S}^{(e^+)} + \left(\frac{eB}{mc}\right)(\tilde{S}_z^{(e^-)} - \tilde{S}_z^{(e^+)}) \quad \text{w/ } A = 0, \quad \tilde{H} = \left(\frac{eB}{mc}\right)(\tilde{S}_z^{(e^-)} - \tilde{S}_z^{(e^+)})$$

$$\tilde{H}|\alpha\rangle = \left(\frac{eB}{mc}\right)(\tilde{S}_z^{(e^-)}|\alpha\rangle - \tilde{S}_z^{(e^+)}|\alpha\rangle) = \left(\frac{eB}{mc}\right)\left(\frac{\hbar}{2} - \left(\frac{-\hbar}{2}\right)\right)|\alpha\rangle = \frac{\hbar eB}{mc} = \hbar\omega|\alpha\rangle$$

$$\boxed{E = \hbar\omega}$$

(b) Same problem when $eB/mc \to 0, A \neq 0$.

$$\tilde{H} = A\tilde{S}^{(e^-)} \cdot \tilde{S}^{(e^+)} \quad \text{w/ } S^2 = S_1^2 + S_2^2 + 2S_1S_2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$\tilde{H}|\alpha\rangle = A\tilde{S}^{(e^-)} \cdot \tilde{S}^{(e^+)}|\alpha\rangle, \quad \tilde{S}^{(e^-)}\tilde{S}^{(e^+)} = S_z^{(e^-)}S_z^{(e^+)} + \frac{1}{2}S_+^{(e^-)}S_-^{(e^+)} + \frac{1}{2}S_-^{(e^-)}S_+^{(e^+)}$$

$$\tilde{H}|\alpha\rangle = A\left[S_z^{(e^-)}S_z^{(e^+)} + \frac{1}{2}S_+^{(e^-)}S_-^{(e^+)} + \frac{1}{2}S_-^{(e^-)}S_+^{(e^+)}\right]|\alpha\rangle$$

$$\tilde{H}|\alpha\rangle = A\left[S_z^{(e^-)}S_z^{(e^+)}|+-\rangle + \frac{1}{2}S_+^{(e^-)}S_-^{(e^+)}|+-\rangle + \frac{1}{2}S_-^{(e^-)}S_+^{(e^+)}|-+\rangle\right]$$

$$S_z^{(e^-)}S_z^{(e^+)}|\alpha\rangle = \frac{\hbar}{2}\left(-\frac{\hbar}{2}\right)|\alpha\rangle, \quad \frac{1}{2}S_+^{(e^-)}S_-^{(e^+)}|\alpha\rangle = 0, \quad \frac{1}{2}S_-^{(e^-)}S_+^{(e^+)}|\alpha\rangle = \frac{\hbar^2}{2}|\alpha\rangle$$

$$\tilde{H}|\alpha\rangle = A\left[-\frac{\hbar^2}{4}|+-\rangle + \frac{\hbar^2}{2}|-+\rangle\right] = \frac{A\hbar^2}{4}\left[-|+-\rangle + 2|-+\rangle\right]$$

Because $|+-\rangle$ is not the same as $|-+\rangle$, this cannot be an energy eigenstate

$$\text{So,} \quad \langle H\rangle = \langle +-|H|+-\rangle, \quad H|+-\rangle = \frac{A\hbar^2}{4}\left[-|+-\rangle + 2|-+\rangle\right]$$

$$\langle H\rangle = \frac{A\hbar^2}{4}\langle +-|(-|+-\rangle + 2|-+\rangle) = -\frac{A\hbar^2}{4}\langle +-|+-\rangle + \frac{A\hbar^2}{2}\underbrace{\langle +-|-+\rangle}_{0} = -\frac{A\hbar^2}{4}$$

$$\boxed{\langle H\rangle = -\frac{A\hbar^2}{4}}$$

Problem 3: 3.5 Review

Procedure

- Begin by applying the limits defined in the problem statement for the Hamiltonian that is defined in the problem statement.
- Apply it to the spin functions $\chi_{+}^{(e_-)}$ and $\chi_{-}^{(e^+)}$ with the eigenstate $\alpha \equiv |+-\rangle$.
- Use the eigenvalues for $\chi_{+}^{(e_-)}$ and $\chi_{-}^{(e^+)}$ to show that this is indeed and eigenfunction of the Hamiltonian.
- To then show the result when $A \neq 0$, we use equation (3.339) in the third edition of Sakurai

$$\tilde{\mathbf{S}}^{(e^-)} \cdot \tilde{\mathbf{S}}^{(e^+)} = \tilde{\mathbf{S}}_z^{(e^-)} \tilde{\mathbf{S}}_z^{(e^+)} + \frac{1}{2} \tilde{\mathbf{S}}_+^{(e^-)} \tilde{\mathbf{S}}_-^{(e^+)} + \frac{1}{2} \tilde{\mathbf{S}}_-^{(e^-)} \tilde{\mathbf{S}}_+^{(e^+)}.$$

- Using the above equation we can show that when this is applied on the spin function it is clearly not an eigenfunction of the Hamiltonian.
- Proceed to calculate the expectation value of $\tilde{\mathbf{H}}$.

Key Concepts:

- If we apply $\tilde{\mathbf{H}}$ on a spin function and the same eigenstate is returned this tells us that the spin function is an eigenfunction of the Hamiltonian.

Variations:

- We can be given a different Hamiltonian.
  - This would change the math of the problem but not the overall process.
- We could be given a different set of limiting conditions.
  - This would change the Hamiltonian that we apply on the spin function.
- We could be given a different function to see if it were an eigenfunction of the Hamiltonian.
  - This would change what the eigenvalues and eigenstates of that function would end up being.

Problem 4: 3.9

What is the meaning of the following equation:

$$U^{-1}A_kU = \sum R_{kl}A_l,$$

where the three components of $\tilde{\mathbf{A}}$ are matrices? From this equation show that matrix elements $\langle m|\tilde{\mathbf{A}}_k|n\rangle$ transform like vectors.

$\tilde{\mathcal{U}}^{-1}\tilde{A}_K\tilde{\mathcal{U}} \longrightarrow$ This is an operator "$\tilde{A}_K$" that is under rotation.

This means, $\tilde{\mathcal{U}}^{-1}\tilde{A}_K\tilde{\mathcal{U}} = \sum R_{K\ell}A_\ell$ represents a matrix under rotation on the LHS and a matrix that has been rotated on the RHS. "$R_{K\ell}$" are the matrix locations. we can Find the elements by doing:

$$\tilde{\mathcal{U}}^{-1}\tilde{A}_K\tilde{\mathcal{U}} = \sum_{K\ell} R_{K\ell}\langle m|A_K|n\rangle$$

where we can see that $\langle m|A_\ell|n\rangle$ are the matrix elements. From this we can see that this transforms as a vector.

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Problem 4: 3.9 Review

Procedure:

- Show that the LHS of the given equation is clearly a linear combination of unrotated operators.
- Calculate the matrix elements of these unrotated operators with the standard convention.
- Proceed to show that these elements will transform like components of a vector.

Key Concepts:

- When something is of the form $\tilde{\mathbf{U}}^{-1} A_k \tilde{\mathbf{U}}$ it is of the form for a an operator A_k that is undergoing a rotation.
- We can proceed to calculate the matrix elements of this unknown operator with applying a complete set on each side, and then disregarding the location of each component.
- After this we can show that the matrix elements are analogous to components of a vector and they must transform like a vector.

Variations:

- This cannot change that much other than maybe the position of $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{U}}^{-1}$.
  - This however could represent something different than an operator that is undergoing a transformation.

Problem 5: 3.38

(b) Show that for $j = 1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta).$$

Using equation (3.207) in the book we know that the Taylor expansion looks like:

$$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \dots\dots \quad w/\ x = \frac{J_y\beta}{\hbar}$$

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - i\left(\frac{J_y}{\hbar}\right)\beta - \left(\frac{J_y}{\hbar}\right)^2\frac{\beta^2}{2!} + i\left(\frac{J_y}{\hbar}\right)^3\frac{\beta^3}{3!} + \dots\dots,$$ using (3.207) we can write

$$e^{\frac{-iJ_y\beta}{\hbar}} = 1 - \left(\frac{J_y}{\hbar}\right)^2\underbrace{\left(\frac{\beta^2}{2!} - \frac{\beta^4}{4!} + \dots\right)}_{\to\, 1-\cos(\beta)} - i\left(\frac{J_y}{\hbar}\right)\underbrace{\left(\beta - \frac{i\beta^3}{3!} + \frac{i\beta^5}{5!} + \dots\right)}_{\to\, \sin(\beta)} = 1 - i\left(\frac{J_y}{\hbar}\right)\sin(\beta) - \left(\frac{J_y}{\hbar}\right)^2(1-\cos(\beta))$$

$\blacksquare$

(c) Using (b), prove

$$d^{(j=1)}(\beta) = \begin{pmatrix} (\frac{1}{2})(1+\cos\beta) & -(\frac{1}{\sqrt{2}})\sin\beta & (\frac{1}{2})(1-\cos\beta) \\ (\frac{1}{\sqrt{2}})\sin\beta & \cos\beta & -(\frac{1}{\sqrt{2}})\sin\beta \\ (\frac{1}{2})(1-\cos\beta) & (\frac{1}{\sqrt{2}})\sin\beta & (\frac{1}{2})(1+\cos\beta) \end{pmatrix}.$$

Utilize equations (3.203) and (3.206)

$$d^{(j=1)}(\beta) = \langle 1,m'|e^{(-iJ_y\beta/\hbar)}|1,m\rangle = \mathbb{1} - \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\sin(\beta) - \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}(1-\cos(\beta))$$

$$d^{(j=1)}(\beta) = \mathbb{1} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sin(\beta) & 0 \\ -\sin(\beta) & 0 & \sin(\beta) \\ 0 & -\sin(\beta) & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1-\cos(\beta) & 0 & \cos(\beta)-1 \\ 0 & 2-2\cos(\beta) & 0 \\ \cos(\beta)-1 & 0 & 1-\cos(\beta) \end{pmatrix}$$

$$d^{(j=1)}(\beta) = \mathbb{1} - \begin{pmatrix} 0 & \sin(\beta)/\sqrt{2} & 0 \\ -\sin(\beta)/\sqrt{2} & 0 & \sin(\beta)/\sqrt{2} \\ 0 & -\sin(\beta)/\sqrt{2} & 0 \end{pmatrix} - \begin{pmatrix} (1-\cos(\beta))/2 & 0 & (\cos(\beta)-1)/2 \\ 0 & 1-\cos(\beta) & 0 \\ (\cos(\beta)-1)/2 & 0 & (1-\cos(\beta))/2 \end{pmatrix}$$

$$d^{(j=1)}(\beta) = \begin{pmatrix} 1 & \sin(\beta)/\sqrt{2} & 0 \\ \sin(\beta)/\sqrt{2} & 1 & \sin(\beta)/\sqrt{2} \\ 0 & \sin(\beta)/\sqrt{2} & 1 \end{pmatrix} - \begin{pmatrix} (1-\cos(\beta))/2 & 0 & (\cos(\beta)-1)/2 \\ 0 & 1-\cos(\beta) & 0 \\ (\cos(\beta)-1)/2 & 0 & (1-\cos(\beta))/2 \end{pmatrix}$$

$$d^{(j=1)}(\beta) = \begin{pmatrix} 1/2(1+\cos(\beta)) & -\sin(\beta)/\sqrt{2} & 1/2(1-\cos(\beta)) \\ \sin(\beta)/\sqrt{2} & \cos(\beta) & -\sin(\beta)/\sqrt{2} \\ 1/2(1-\cos(\beta)) & \sin(\beta)/\sqrt{2} & 1/2(1+\cos(\beta)) \end{pmatrix}$$

$\blacksquare$

Problem 5: 3.38 Review

Procedure:

- We begin first by expanding the operator with equation (3.207) of the third edition of Sakurai

$$\left(\frac{\tilde{\mathbf{J}}_y^{(j=1)}}{\hbar}\right)^3 = \frac{\tilde{\mathbf{J}}_y^{(j=1)}}{\hbar}.$$

- Expand the operator with a Taylor Series.
- Once the operator has been expanded, we collect the even terms to show the representation with cos and the odd terms for sin.
- Use equation (3.203)

$$d^{(j)}{}_{m'm}(\beta) \equiv \langle j, m'|\exp\left(\frac{-i\tilde{\mathbf{J}}_y\beta}{\hbar}\right)|j, m\rangle$$

  and (3.206) from the third edition of Sakurai to show that the desired result can be obtained.

Key Concepts:

- Angular momentum operators can be expanded in the form of a Taylor Series and can be shown to exhibit sinusoidal behavior.
- Using the result obtained in (b) we can show that the result in (c) can be obtained while using equation (3.203).

Variations:

- We can be given an operator that is in the x or z direction instead.
  - This would change the matrix representation that is found in (c).
  - It would also change how the operator can be written in something like part (b).