Solutions to Homework 3
Physics 5393

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P-1.6 Using the rules of the bra-ket algebra, prove or evaluate the following:

a) $\mathrm{tr}(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}) = \mathrm{tr}(\tilde{\mathbf{Y}}\tilde{\mathbf{X}})$

The first step is to express the operator product in a specific basis and recall that the trace of an matrix is the sum of its diagonal components

$$\mathrm{tr}(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}) = \sum_i \left\langle a_i \middle| \tilde{\mathbf{X}}\tilde{\mathbf{Y}} \middle| a_i \right\rangle .$$

Next, use the completenesss relation in order to express the matrix elements of each operator independently

$$\sum_i \left\langle a_i \middle| \tilde{\mathbf{X}}\tilde{\mathbf{Y}} \middle| a_i \right\rangle = \sum_i \sum_j \left\langle a_i \middle| \tilde{\mathbf{X}} \middle| a_j \right\rangle \left\langle a_j \middle| \tilde{\mathbf{Y}} \middle| a_i \right\rangle .$$

Finally, since the matrix elements are numbers, they commute leading to the following result

$$\begin{aligned}\sum_i \sum_j \left\langle a_i \middle| \tilde{\mathbf{X}} \middle| a_j \right\rangle \left\langle a_j \middle| \tilde{\mathbf{Y}} \middle| a_i \right\rangle & \\ &= \sum_j \sum_i \left\langle a_j \middle| \tilde{\mathbf{Y}} \middle| a_i \right\rangle \left\langle a_i \middle| \tilde{\mathbf{X}} \middle| a_j \right\rangle \\ &= \sum_j \left\langle a_j \middle| \tilde{\mathbf{Y}}\tilde{\mathbf{X}} \middle| a_j \right\rangle = \mathrm{tr}(\tilde{\mathbf{Y}}\tilde{\mathbf{X}}).\end{aligned}$$

Therefore, the desired result is achieved $\mathrm{tr}(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}) = \mathrm{tr}(\tilde{\mathbf{Y}}\tilde{\mathbf{X}})$.

b) $(\tilde{\mathbf{X}}\tilde{\mathbf{Y}})^\dagger = \tilde{\mathbf{Y}}^\dagger\tilde{\mathbf{X}}^\dagger$

To prove this relation, the duality of the bra and ket vectors is used. Start by applying the operator $\tilde{\mathbf{Y}}$ on a ket $|\alpha\rangle$ and then define $|\beta\rangle = \tilde{\mathbf{Y}}\,|\alpha\rangle$. Duality then yields

$$\left.\begin{aligned}\tilde{\mathbf{Y}}\,|\alpha\rangle &\overset{\mathsf{DC}}{\Longleftrightarrow} \langle\alpha|\,\tilde{\mathbf{Y}}^\dagger \\ \tilde{\mathbf{X}}\,|\beta\rangle &\overset{\mathsf{DC}}{\Longleftrightarrow} \langle\beta|\,\tilde{\mathbf{X}}^\dagger\end{aligned}\right\} \quad \Rightarrow \quad \tilde{\mathbf{X}}\tilde{\mathbf{Y}}\,|\alpha\rangle \overset{\mathsf{DC}}{\Longleftrightarrow} \langle\alpha|\,\tilde{\mathbf{Y}}^\dagger\tilde{\mathbf{X}}^\dagger.$$

But $\tilde{\mathbf{X}}\tilde{\mathbf{Y}} = (\tilde{\mathbf{X}}\tilde{\mathbf{Y}})$ can be treated as a single operator, which leads to

$$\left(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}\right)|\alpha\rangle \overset{\mathsf{DC}}{\Longleftrightarrow} \langle\alpha|\left(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}\right)^\dagger .$$

Comparison of the two forms of the duality relation for $\tilde{\mathbf{X}}\tilde{\mathbf{Y}}\,|\alpha\rangle$, leads immediately to the desired result

$$\left(\tilde{\mathbf{X}}\tilde{\mathbf{Y}}\right)^\dagger = \tilde{\mathbf{Y}}^\dagger\tilde{\mathbf{X}}^\dagger.$$

c) $\exp[if(\tilde{\mathbf{A}})] = ?$ in bra-ket form, where $\tilde{\mathbf{A}}$ is a Hermitian operator whose eigenvalues are known.

An explicit form for this operator can be derived by applying it to the completeness relation

$$\exp[if(\tilde{\mathbf{A}})] \sum_i |a_i\rangle\langle a_i| = \sum_i \exp[if(\tilde{\mathbf{A}})]\, |a_i\rangle\langle a_i| = \sum_i \exp[if(a_i)]\, |a_i\rangle\langle a_i|\ ,$$

where in the last step we apply the operator on the ket; we are assuming the $|a_i\rangle$ is an eigenket of $\tilde{\mathbf{A}}$. This immediately leads to the desired result

$$\exp[if(\tilde{\mathbf{A}})] = \sum_i \exp[if(a_i)]\, |a_i\rangle\langle a_i|\ .$$

P-1.15 A beam of spin 1/2 atoms goes through a series of Stern-Gerlach-type measurements as follows:

a) The first measurement accepts $S_z = \hbar/2$ and rejects $S_z = -\hbar/2$ atoms.
b) The second measurement accepts $S_n = \hbar/2$ atoms and rejects $S_n = -\hbar/2$, where S_n is the eigenvalue of the operator $\tilde{\mathbf{S}} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ making an angle β in the x-y plane with respect to the z-axis.
c) The third measurement accepts $S_z = -\hbar/2$ atoms and rejects $S_z = \hbar/2$ atoms.

What is the intensity of the final $S_z = -\hbar/2$ beam when the $S_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $S_z = -\hbar/2$ beam?

Atoms emerging the first apparatus are all in the $|+\rangle$ state. The second apparatus projects out the $|S_n;+\rangle$ state. Therefore, it behaves like a projection operator

$$|S_n;+\rangle\langle S_n;+| = \left[\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right]\left[\cos\left(\frac{\beta}{2}\right)\langle +| + \sin\left(\frac{\beta}{2}\right)\langle -|\right].$$

The probability of being in the $|S_n;+\rangle$ state is

$$|\langle S_n;+|+\rangle|^2 = \cos^2\left(\frac{\beta}{2}\right).$$

The third apparatus projects out the $|-\rangle$ state from the $|S_n;+\rangle$ state. Therefore, the probability to end in the $|-\rangle$ state is

$$|\langle -|S_n;+\rangle|^2 = \sin^2\left(\frac{\beta}{2}\right).$$

Finally the probability of going from the $|+\rangle$ state to the $|-\rangle$ state through the described apparatus is the product of the probabilities

$$\mathcal{P} = |\langle +|S_n;+\rangle|^2\, |\langle S_n;+|-\rangle|^2 = \cos^2\left(\frac{\beta}{2}\right)\sin^2\left(\frac{\beta}{2}\right) = \frac{1}{4}\sin^2\beta.$$

The maximum probability occurs when $\beta = \pi/2$ with 25% of the atoms being transmitted.

P-1.17 Let $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ be observables. Suppose the simultaneous eigenkets of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ $\{|a', b'\rangle\}$ form a complete orthonormal set of base kets. Can we always conclude that

$$\left[\tilde{\mathbf{A}}, \tilde{\mathbf{B}}\right] = 0? \tag{1}$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

The answer is yes. This can be proved using the completeness relation

$$\tilde{\mathbf{A}}\tilde{\mathbf{B}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} \sum_{i,j} \left|a_i, b_i^{(j)}\right\rangle\left\langle a_i, b_i^{(j)}\right| = \sum_{i,j} a_i b_i^{(j)} \left|a_i, b_i^{(j)}\right\rangle\left\langle a_i, b_i^{(j)}\right| = \tilde{\mathbf{B}}\tilde{\mathbf{A}},$$

where in the last step we note that the order of the operators does not matter in deriving the eigenvalues and the notation i, j denotes the degenerate eigenstates of b_i associated with a_i . Also, it is important that the sum be over both a_i and $b_{i,j}$ to ensure the full space is spanned.

P-1.19 Two observables $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$, which do not involve time explicitly, are known not to commute,

$$\left[\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2\right] \neq 0$$

yet we also know that $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ both commute with the Hamiltonian:

$$\left[\tilde{\mathbf{A}}_1, \tilde{\mathbf{H}}\right] = 0 \qquad \left[\tilde{\mathbf{A}}_2, \tilde{\mathbf{H}}\right] = 0$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $\tilde{\mathbf{H}} = \tilde{\mathbf{p}}^2/2m + V(\tilde{\mathbf{r}})$, with $\tilde{\mathbf{A}} \to \tilde{\mathbf{L}}_z$ and $\tilde{\mathbf{A}}_2 \to \tilde{\mathbf{L}}_x$.

We start with the compatible observables

$$\begin{aligned}
\left[\tilde{\mathbf{A}}_1, \tilde{\mathbf{H}}\right] = 0 \quad &\Rightarrow \quad \tilde{\mathbf{A}}_1\tilde{\mathbf{H}} \left|E_i, a_{1,i}\right\rangle = a_{1,i}E_i \left|E_i, a_{1,i}\right\rangle \\
\left[\tilde{\mathbf{A}}_2, \tilde{\mathbf{H}}\right] = 0 \quad &\Rightarrow \quad \tilde{\mathbf{A}}_2\tilde{\mathbf{H}} \left|E_2, a_{2,i}\right\rangle = a_{2,i}E_i \left|E_i, a_{2,i}\right\rangle
\end{aligned}$$

Since $\tilde{\mathbf{H}} \left|E_i, a_{1,i}\right\rangle$ is not an eigenstate of $\tilde{\mathbf{A}}_2$ and $\tilde{\mathbf{H}} \left|E_i, a_{2i}\right\rangle$ is not an eigenstate of $\tilde{\mathbf{A}}_1$, the only way that the compatabiliy between $\tilde{\mathbf{A}}_i$ and $\tilde{\mathbf{H}}$ can be compatable is if the energy eigenstates are degenerate so that a linear combination of $\tilde{\mathbf{H}} \left|E_i, a_{1,i}\right\rangle$ can be used to give $\tilde{\mathbf{H}} \left|E_i, a_{2,i}\right\rangle$ and vise versa. As an example, consider the $\tilde{\mathbf{L}}^2$ operator, which commutes with $\tilde{\mathbf{L}}_x$ and $\tilde{\mathbf{L}}_z$ but the latter operators don't commute with each other. In this case, the $\tilde{\mathbf{L}}^2$ operator is degenerate with each state of the $\tilde{\mathbf{L}}^2$ operator having $2(\ell + 1)$ eigenvalues of $\tilde{\mathbf{L}}_z$ and $\tilde{\mathbf{L}}_x$.

P-1.24 Not previously used. Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it *in seconds.*

The following is my solution to this problem. It is not the only solution.

I will treat this as an inverted pendulum, for which the Newton's second law gives the following solution

$$I\ddot{\theta} = m\ell^2\ddot{\theta} = mg\ell\sin\theta$$

where the moment of inertia of the ice pick is approximated as $I = m\ell^2$ with ℓ being the length and m the mass of the ice pick. If we assume a small displacement from the unstable equilibrium, the differential equation becomes

$$m\ell^2\ddot{\theta} = mg\ell\theta \quad \Rightarrow \quad \ddot{\theta} - \frac{g}{\ell}\theta = 0$$

with the solution

$$\theta(t) = Ae^{t/\tau} + Be^{-t/\tau} \text{ where } \tau = \sqrt{\frac{\ell}{g}}.$$

Since we will apply the momentum position uncertainty relation $\Delta x\Delta p \approx x_0 p_0 \approx \hbar$ to relate the two quantities, the initial conditions are given in terms of the initial position $x_0 = \ell\theta(0)$ and initial momentum $p_0 = m\ell\dot{\theta}(0)$ and the coefficients are

$$\left.\begin{aligned} &x_0 = \ell\theta(0) = (A+B)\ell \\ &p_0 = m\ell\dot{\theta}(0) = m\tau\ell(A-B) \\ &x_0 p_0 = \hbar \end{aligned}\right\} \quad \Rightarrow \quad x_0 p_0 = \frac{m\ell^2}{\tau}(A^2 - B^2) = \hbar.$$

Since the negative exponential term will be negligible relative to the other term after a short period of time, we therefore set $B = 0$ and

$$A^2 = \frac{\hbar\tau}{m\ell^2}.$$

Selecting $m = 0.1\,\mathrm{kg}$, $\ell = 0.1\,\mathrm{m}$, and a tilt angle of 0.1° the ice pick is stable for

$$t = \tau\ln\left(\Delta\theta\sqrt{\frac{m\ell^2}{\hbar\tau}}\right) = \tau\left[\ln(\Delta\theta) - \frac{1}{2}\ln\left(\frac{\hbar\tau}{m\ell^2}\right)\right] \approx 3\,\mathrm{s}$$

where it should be noted that the second (quantum) term dominates the results.