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Classical Mechanics

Ch. 10 Hamilton-Jacobi And Action-Angle Variables Lecture Notes

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① Hamiltonian Formulation of dynamics (ch. 8) $\Rightarrow$ Insight

② Canonical transformation $\Rightarrow$ variables satisfy Hamilton's principle, $\underset{H}{(q,p)} \rightarrow \underset{K}{(Q,P)}$

Final Step : $q(t) = q(Q(t), P(t), t)$, $p(t) = \ldots\ldots$

Machinery : generating function, $F(q,p,Q,P,t) \longrightarrow$ Transformation equations
$\longrightarrow$ Kamiltonian, $K = H + \frac{\partial F}{\partial t}$

Example Q3 of A9.

$$Q = \sqrt{\frac{m\omega}{2}}\left(q + \frac{ip}{m\omega}\right), \quad P = i\sqrt{\frac{m\omega}{2}}\left(q - \frac{ip}{m\omega}\right.$$

a.) ✓

b.) $H = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2$, $K = -i\omega QP$: $\dot{Q} = -i\omega Q$, $\dot{P} = i\omega P$

Hamilton - Jacobi Theory (ch. 10)

* Previously : Map mechanical problem to "simple" Kamiltonian [e.g. Q are cyclic]
  Still : Solving for $Q(t)$ & $P(t)$

* Alternative approach : seek a canonical transformation that maps $(\vec{q},\vec{p})$ to some constant quantities at $t = t_0$ (e.g. $\vec{q}_0, \vec{p}_0$)

Would have obtained / used transformation equations

$$\vec{q} = \vec{q}(\vec{Q},\vec{P},t) = \vec{q}(\vec{q}_0,\vec{p}_0,t), \quad \vec{p} = \vec{p}(\vec{Q},\vec{P},t) = \vec{p}(\vec{Q}_0,\vec{P}_0,t)$$

$\Rightarrow$ Formally equivalent to directly solving Hamilton's equation of motion

Task : Find the transformation. (e.g., alternatively "solve" equations of motion)

Hamilton - Jacobi equation & Hamilton's Principle function

Kamiltonian : $K = 0 \Rightarrow \frac{\partial K}{\partial Q} = \frac{\partial K}{\partial P} = 0 \Rightarrow \dot{Q} = 0$, $\dot{P} = 0 \Rightarrow Q(t) = Q(0)$, $P(t) = P(0)$

Also,

$$0 = \frac{\partial F}{\partial t} + H(\vec{q},\vec{p},t) \quad (*)$$

By convention : Assume generating function of type 2, $F_2(\vec{q},\vec{P},t)$

Transformation equations : $p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$

Plug p_i into $(*)$:

(*) $\Rightarrow H(\vec{q}, \frac{\partial F_2}{\partial q_i} \dots \frac{\partial F_2}{\partial q_n}, t) + \frac{\partial F_2}{\partial t} = 0 \longrightarrow$ Hamilton - Jacobi Equation

* Partial differential equation in $(n+1)$ variables $(\vec{q}, t)$

* call generating function : S, Hamilton's principle function

Know : $\dot{P}_i = 0$ by definition $\longrightarrow \vec{P} = \vec{P}(\alpha_1, \dots\dots, \alpha_n)$ n integration constants

In principle, $S = S(\vec{q}, \vec{P}, t) = S(\vec{q}, \vec{\alpha}, t)$, any solution $S \Rightarrow S + \alpha_{n+1}$ also a solution

With a solution :

① $p_i = \frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial q_i}$

② $Q_i = \frac{\partial S}{\partial \alpha_i} = \vec{P}_i$

From ① + initial conditions $(\vec{q} + \vec{p}_0)$ get α_i Similar from ② $\Rightarrow$ get P_i.

These equations are inverted to obtain, $q = q(\vec{\alpha}, \vec{B}, t)$ & $p = p(\vec{\alpha}, \vec{B}, t)$

Quick Summary :

i) Principal function S in the generator of a canonical transformation to constant canonical "position" "momenta"

ii) Solving for S is formally equivalent to obtaining a solution to the mechanical problem.

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Hamilton - Jacobi Theory

* Find a transformation between $(\vec{q}, \vec{p})$ @ time $t \longrightarrow (\vec{q}_0, \vec{p}_0)$ @ $t = t_0$
Canonical Transformation

$q = q(q_0, p_0, t)$, $p = p(q_0, p_0, t) \Rightarrow$ Solution (i.e. solve Hamilton's EOM)

$\dot{Q} = \dot{P} = 0 \Rightarrow$ i.e. $K = 0$

By convention : Type-2 generating function , $S = S(q, P, t)$

$0 = H + \frac{\partial S}{\partial t}$

① $P_i = \alpha_i = const \Rightarrow S(q, \alpha, t)$

② By transformation equations , $p_i = \frac{\partial S}{\partial q_i}$

(i) $0 = H(\vec{q}, \partial S/\partial p, t) + \frac{\partial S}{\partial t}$ Hamilton Jacobi Equation

(ii) $Q_i = \frac{\partial S}{\partial P_i} = \beta_i$

(i) & (ii) w/ S $\Rightarrow$ obtain $q(q_0, p_0, t)$, $p(q_0, p_0, t)$

Reminder: S "Hamilton's principal function"

<u>Physical Significance of S</u>:

Without motivation, $S(q, P, t)$

$$\frac{dS}{dt} = \sum_i \frac{\partial S}{\partial q_i}\dot{q}_i + \cancel{\frac{\partial S}{\partial P_i}}\overset{0}{\dot{P}_i} + \frac{\partial S}{\partial t} = \sum_i p_i\dot{q}_i - H = L$$

Then,

$S = \int L\, dt \longrightarrow$ Indefinite integral

Previously, Hamilton's principle,

$$\delta I = \delta\int_{t_1}^{t_2} L\, dt = 0$$

A final comment: If the Hamiltonian is conserved, $H = H(q,p)$, $\frac{\partial H}{\partial t} = 0$, $H \Rightarrow$ const.

then, the principal function can be written as,

$$S(q, \alpha, t) = \underbrace{W(q, \alpha)}_{\text{Hamilton's characteristic function}} - \alpha t \qquad \frac{\partial S}{\partial t} = 0 - \alpha, \quad \alpha = E$$

Significance of the characteristic function:

$$\frac{dW}{dt} = \sum_i \frac{\partial W}{\partial q_j}\dot{q}_i = \sum_i P_i\dot{q}_i, \quad W = \sum_i \int dt\, p_i\dot{q}_i = \sum_i \int p_i\, dq_i \Rightarrow \text{"Action-angle variables"}$$

<u>Worked Example</u> : <u>1D Harmonic Oscillator</u>

$n+1$ integration constants for PDE w/ $n+1$ variables

$\Rightarrow$ Only 1 needed, $\alpha = \alpha_1$, $\alpha_2 = 0$

(Step 1)

<u>Hamiltonian:</u> $H = \frac{P^2}{2m} + \frac{m\omega^2}{2}q^2$, $\frac{\partial H}{\partial t} = 0$, H is conserved

(Step 2)

Hamilton-Jacobi Equation

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \quad \Rightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2 w^2 q^2\right] - E = 0$$

Try, $S = W - Et$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2 w^2 q^2\right] = E \quad (*)$$

By definition: $p = \frac{\partial W}{\partial t}$, From $(*)$: $p = \sqrt{2mE - m^2 w^2 q^2}$

Re-arrange $(*)$ & integrating

$$W(q,E) = \int dq \sqrt{2mE - m^2 w^2 q^2} \quad \& \quad S(q,E,t) = W(q,E) - Et$$

want:

$$Q_i \equiv \beta = \frac{\partial S}{\partial \alpha} = \frac{\partial}{\partial E}\int dq \sqrt{2mE - m^2 w^2 q^2}$$

$$\beta = \int dq \frac{m}{\sqrt{2mE - m^2 w^2 q^2}} - t = \sqrt{\frac{m}{2E}}\int dq \frac{1}{\sqrt{1 - \frac{mw^2}{2E} q^2}} - t$$

$\beta = \frac{1}{w} \arcsin\left[q \cdot \sqrt{\frac{mw^2}{2E}}\right] - t$, $q = \sqrt{\frac{2E}{mw^2}} \sin(wt + \beta w) = q(\alpha, \beta, t)$: From before,

$p = \sqrt{2mE} \cos(wt + \beta w) = p(\alpha, \beta, t)$

$H = \frac{p^2}{2m} + \frac{mw^2}{2} q^2 = E$ (check yourself)

α & β are functions of initial q_0 & p_0 : [set $q(\alpha, \beta, 0) = q_0$ etc]

i) $\alpha = \frac{p_0^2}{2m} + \frac{mw^2 q_0^2}{2}$

ii) $\tan(\beta w) = m w q_0 / p_0$

Recall by definition:

$P = \alpha = E$ energy, $Q = \beta$ "Initial phase"

Example: Particle In A Linear Potential

* mass m

* $V(x) = kx$

$L = \frac{1}{2} m \dot{x}^2 - kx$

$$p = \frac{\partial s}{\partial q} \quad , \quad Q = \frac{\partial s}{\partial p} = \beta$$

i) write down the Hamilton-Jacobi equation

$$H = \frac{p^2}{2m} + kx \quad , \quad 0 = \frac{\partial s}{\partial t} + H\left(x, \frac{\partial s}{\partial x}, t\right) \quad , \quad 0 = \frac{\partial s}{\partial t} + \frac{1}{2m}\left(\frac{\partial s}{\partial x}\right)^2 + kx$$

ii) using that $H = H(x,p)$. write a Solution for $S(x,\alpha,t)$

$$S(x,\alpha,t) = w(x,\alpha) - \alpha t$$

$$\frac{\partial s}{\partial x} = \sqrt{2m(E - kx)} - Et \quad \therefore \quad S = \int \sqrt{2m(E-kx)}\, dx - Et$$

iii) Use (ii) to obtain solutions of $q \,\&\, p$ as functions of $t\,(t,\alpha,\beta)$

$$Q = \beta = \frac{\partial s}{\partial \alpha} = \int dx \frac{m}{\sqrt{2m(\alpha - kx)}} - t = \dots \quad x = -\frac{1}{k}\frac{1}{2x}(k(\beta+t))^2 + \frac{\alpha}{k}$$

$$p = \frac{\partial s}{\partial x} = \sqrt{2m(\alpha - kx)} = -k(\beta + t)$$

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$$H\left(q, \frac{\partial s}{\partial q}, t\right) + \frac{\partial s}{\partial t} = 0 \quad , \quad p = \frac{\partial s}{\partial q} \quad , \quad Q = \frac{\partial s}{\partial \alpha} = \beta \quad : \quad S(q,\alpha,t)$$

$\frac{\partial H}{\partial t} = 0$? $\Rightarrow$ Hamiltonian is conserved

Separation of Variables: $S(q,\alpha,t) = w(q,\alpha) - \alpha t \; : \; E = \alpha$

The restricted H-J equation (10.3)

Assume that the Hamiltonian is conserved

$F_2(q,P) = w(q,\alpha) \Rightarrow$ Type 2 generating function , $S \neq w$, $K \neq 0$

$P = \frac{\partial w}{\partial q}$, $Q = \frac{\partial w}{\partial P}$ w/ $K = H = \alpha$

Imagine $\vec{p} = (p_1, \dots\dots, p_n)$, $\vec{q} = (q_1, \dots\dots, q_n)$

For $i=1 \Rightarrow \dot{Q}_1 = \frac{\partial K}{\partial P_1} = \frac{\partial K}{\partial \alpha_1} = 1 \Rightarrow Q(t) = t + \beta_1 \quad \dot{P}_1 = 0 \Rightarrow P_1 = \alpha_1$

For $i \neq 1$, $\dot{Q}_i = \frac{\partial K}{\partial \alpha_1} = 0 \quad Q_i(t) = \beta_1$, $\dot{P}_i = -\frac{\partial K}{\partial Q_1} = 0 \quad P(t) = \alpha_1$

All but 1 co-ordinates are cyclic

$P_1 \Rightarrow$ energy = Canonical momenta , $Q_1 \Rightarrow$ Time = Canonical position

time <-> energy , conjugate variable

Restricted H-J equation, $H(q, \frac{\partial w}{\partial q}) = \alpha_1$

① $H(q,P,t) \Rightarrow$ find $S(q,\alpha,t)$ w/ H-J equation (& $K=0$), P_i, Q_i are cyclic

② $H(q,p)$, Find $w(a,\alpha)$ & restricted H-J equation ($k=\alpha_1$)

<u>Separation of variables & cyclic co-ordinates In Higher Dimensions</u>

Definition : A co-ordinate q_j is separable if,

$$S(\vec{q},\vec{\alpha},t) = S_j(q_j,\vec{\alpha},t) + S'(\vec{q} \neq q_j, \vec{\alpha}, t) \quad \text{(Principle function is separable)}$$

and we can break apart the Hamilton-Jacobi equation into two pieces for S_j & S'.

A problem is completely separable if :

$$S = \sum_j S_j(q_j,\vec{\alpha},t) = \sum_j W_j(q_j,\vec{\alpha}) - Et \quad \& \quad H_j(q_j, \partial S_i/\partial q_i, t) + \partial S_j/\partial t = 0$$

Example : 2D H.O

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{m\omega^2 y^2}{2}$$

Example : Particle in 3D subject to linear potential (e.g. gravity)

$$H = \sum_j \frac{P_j^2}{2m} + Kz$$

For <u>cyclic</u> co-ordinates :

$$P_j = \text{const} = \gamma_j$$

If we use a seperation ansatz,

$$W(\vec{q},\vec{\alpha}) = W_j(q_j,\vec{\alpha}) + W'(\vec{q} \neq q_j, \vec{\alpha}) \quad , \quad p_j = \frac{\partial w}{\partial q_j} = \frac{\partial W_j}{\partial q_j} = \gamma_j \Rightarrow W_j(q_j,\vec{\alpha}) = \gamma_j q_j + \ldots$$

$$W(\vec{q},\vec{\alpha}) = q_j\gamma_j + W'(\vec{q} \neq a_j, \vec{\alpha})$$

Generically, for s cyclic co-ordinates, then,

$$W(\vec{q},\vec{\alpha}) = \sum_{j=1}^{s} q_j P_j + \sum_{j=m}^{n} W_j(q_j,\vec{\alpha})$$

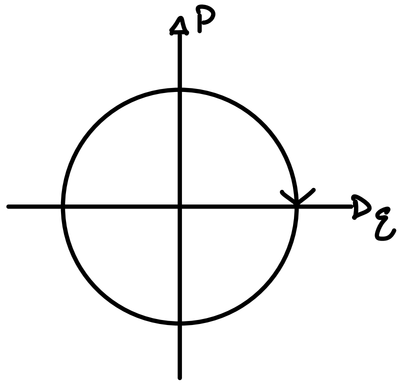
<u>Action Angle variables in 1D (10.6)</u>

Periodic Motion

* Can also use canonical transformation to obtain characteristics of dynamics

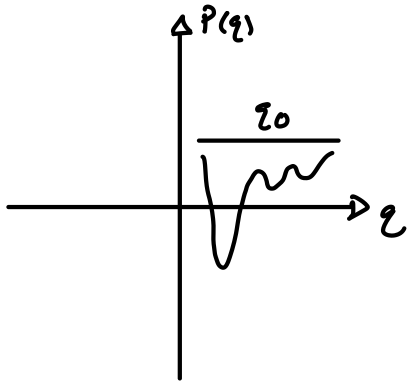
Periodic Motion:

i) Libration :



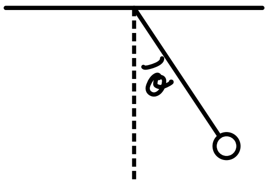
* Closed orbits in phase-Space
* Both q & p are periodic functions in time
* Trajectory deduces to steps

ii) Rotation

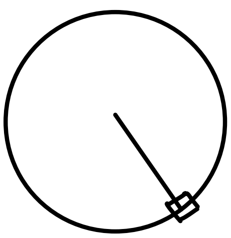


* $P = p(q)$ is a periodic function
* period q_0 . freq : $\frac{2\pi}{q_0}$
* q is unbounded

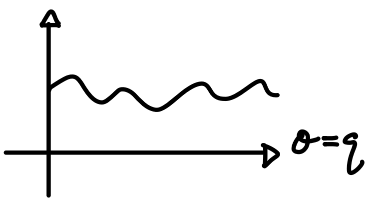
Pendulum:



→ Small angle → H.O



rotation



Previously:

$P = \alpha$

Now:

$\vec{P} = \vec{J}(\vec{\alpha}) \longrightarrow J_i$: action-variable

Formally: $J_i = \oint p_i \, dq_i$

In 1D : $H = \alpha \Rightarrow H = H(J)$

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Previously

* Hamilton's principal (s) characteristic function $S = W - \alpha t$

* Hamilton Jacobi equation $\Rightarrow$ solution for dynamical problem

* Action-angle variables

Purpose: Use a canonical transform to describe periodic motion

Two types: i) Libration

ii) Rotation

i) Periodic motion in time: Ex: Harmonic oscillator

* constantly re-tracing trajectory

ii) Rotation: $p(q)$ is periodic in q, period q_0, freq $2\pi/q_0$

<u>Basic Idea</u>: <u>Action-Angle Variable</u>

$$(p,q) \xrightarrow[S(q,\alpha,t)]{} \underset{q_0,\, p_0}{(Q,P)}$$

Identified: $\vec{P} = \vec{\alpha}$, $P = E$, $\alpha = E$

Now: $\vec{P} = \vec{J}(\alpha) \longrightarrow$ Action variable function $\alpha_1, \dots, \alpha_n$

$$J_i = \oint p_i \, dq_i = J_i(\alpha) \Rightarrow p_i(\vec{q}, \alpha)$$

$\hookrightarrow$ Integrate over a complete period, τ or q_0

* $H = \alpha \rightarrow J(\alpha) \longrightarrow H = H(J)$

* $W = W(q,\alpha) \longrightarrow W(q,J) \rightarrow P$

Just as before:

$$Q = \frac{\partial W(q,J)}{\partial J} \rightarrow \left[\frac{\partial F_2}{\partial P}\right] \rightarrow \text{Angle variable}, \quad Q \Rightarrow w = \frac{\partial w}{\partial J}$$

Equation of Motion for angle variable, $\dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu(J)$, $w(t) = \nu t + \beta$

Overall, so far:

$(q,p) \longrightarrow (Q,P) \Rightarrow (w,J) \rightarrow$ New conjugate variables

$\nu \rightarrow \dot{w}$: rate of change of angle variable

After one period : Change in angle variable :

$$\Delta w = \oint \frac{\partial w}{\partial q}\, dq = \oint \frac{\partial}{\partial q}\left(\frac{\partial w}{\partial J}\right) dq = \oint \frac{\partial}{\partial J}\left(\frac{\partial w}{\partial q}\right) dq = \frac{\partial}{\partial J}\oint p\, dq = \frac{\partial J}{\partial J} = 1$$

Going back, $w = \nu t + \beta$. Define period as τ, then $\Delta w = 1 = \nu t \rightarrow \nu = \frac{1}{t}$

$\therefore$ ν is the frequency of periodic motion

For a periodic 1D system, obtaining $H = H(J)$ then we immediately can get the frequency,

$$\nu = \dot{w} = \frac{\partial H}{\partial J}$$

Example 1 : Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 \Rightarrow p = \sqrt{2m\alpha - m^2\omega^2x^2} \quad w/ \ H = \alpha$$

Step 1 : Obtain your action variable $J = \oint p\, dq = \oint \sqrt{2m\alpha - m^2\omega^2x^2}\, dx$

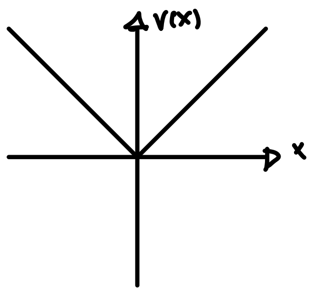
Cheat : Substitute $x = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(a)$

$$J = \int_0^{2\pi} \frac{2\pi}{\alpha}\cos^2(a)\, da = \frac{2\pi\alpha}{\omega}$$

Recall $H = \alpha$, so invert result, $H(J) = \frac{J\omega}{2\pi}$

Step 2 : $\nu = \frac{\partial H}{\partial J} = \frac{\omega}{2\pi}$

Example 2: Linear potential



$V = k|x|$

From energy (H), $p(x) = \pm\sqrt{2m(E - k|x|)}$ → very important

Turning points defined by $E = V(x_\pm)$

This is an example of Libration

Next obtain action variable J, $J = \oint p(x)\, dx$

$$J = \int_0^{x_+} \sqrt{2m(E-k|x|)}\, dx + \int_{x_+}^0 -\sqrt{2m(E-k|x|)}\, dx + \int_0^{x_-} -\sqrt{2m(E-k|x|)}\, dx + \int_{x_-}^0 \sqrt{2m(E-k|x|)}\, dx$$

* $x_\pm = \pm E/k \Rightarrow x_+ = -x_-$

$$J = 4\int_0^{x_+} \sqrt{2m(E-kx)}\,dx = \frac{8}{3k}\sqrt{2m}\,E^{3/2} = J(E)$$

↳ Invert our result, $H = \left(\frac{3kJ}{8\sqrt{2m}}\right)^{2/3} = H(J)$

↳ compute frequency $\nu = \frac{\partial H}{\partial J}$

$$\nu(E) = \frac{k}{4\sqrt{2mE}} \propto \frac{1}{E} \qquad \tau \sim \frac{1}{\nu} = \sqrt{E}$$

<u>Extra Example</u>: Consider a tennis ball bouncing elastically. Calculate the period.

Action - Angle variables in higher <u>dimensions</u>

* Action - Angle formalism works if: Hamilton - Jacobi equation is completely separable

$$W(\vec{q},\vec{J}) = \sum_{i=1}^{n} W_i(q_i,\vec{J})$$

Then, action variables:

$$J_i = \oint p_i\, dq_i$$ ⟶ over complete period in (q_i, p_i) plane.

In 1D ⇒ Periodic
- → Libration
- → Rotation

In higher dimension, periodic in individual $(q_i, p_i) \Rightarrow$ Periodic in full $(\vec{q},\vec{p})$
↳ Not necessarily true

<u>Example</u>: 2D Harmonic oscillator

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_x^2}{2}x^2 + \frac{m\omega_y^2}{2}y^2$$

Solve independently:

$$x(t) = x_0\cos(\omega_x t + \beta_x) \qquad y(t) = y_0\cos(\omega_y t + \beta_y) \quad \text{w/} \quad \omega_{x,y} = 2\pi\nu_{x,y}$$

<u>Case 1</u>: If ν_x & ν_y are commensurate (i.e. $\nu_x/\nu_y \in$ rational) ⇒ periodic in $(\vec{q},\vec{p})$

<u>Case 2</u>: If ν_x & ν_y are incommensurate (i.e. $\nu_x/\nu_y \in$ irrational)

<u>Definition</u>: i) ν_x & ν_y are commensurate motion is periodic w/ one period

ii) ν_x & ν_y incommensurate, multiple periods

In both cases we can define Action-Angle variables

<u>Cyclic</u> <u>co-ordinates</u>: If q_i is cyclic in H, then p_i is constant

$\Rightarrow$ "orbit" which is a straight line in (q_i, p_i) - plane

$\Rightarrow$ Define a "natural" period of 2π associated action, $J_i = 2\pi p_i$

Again: * $H = H(J_1, \dots, J_n)$

* $w_i = \dfrac{\partial W}{\partial J_i}$