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Quantum Mechanics 1

PHYS 5393 HOMEWORK ASSIGNMENT #5

PROBLEMS: {1.33, 1.34, 1.35, 1.36, Q-1}

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Problem 1: 1.33

In the main text we discussed the effect of $\mathscr{J}(\mathbf{dx'})$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle \tilde{\mathbf{x}} \rangle$ and $\langle \tilde{\mathbf{p}} \rangle$ under infinitesimal translation. Using (1.207), (1.227), and $|\alpha\rangle \to \mathscr{J}(\mathbf{dx'})\,|\alpha\rangle$ only, prove $\langle \tilde{\mathbf{x}} \rangle \to \langle \tilde{\mathbf{x}} \rangle + \mathbf{dx'}$, $\langle \tilde{\mathbf{p}} \rangle \to \langle \tilde{\mathbf{p}} \rangle$ under infinitesimal translation.

$$[x, J(dx')] = dx' \longrightarrow (1.207)$$

$$[P, J(dx')] = 0 \longrightarrow (1.227)$$

$[x, J(dx')] = xJ(dx') - J(dx')x = dx'$, if we use $J^{+}(dx')$ on (1.207)

$$J^{+}(dx')[x, J(dx')] = J^{+}(dx')\,x\,J(dx') - J^{+}(dx')J(dx')x \qquad (*)$$

For $(*)$ to be true, $J^{+}(dx')\,x\,J(dx') = x + dx'$ and $J^{+}(dx')J(dx')x = x$ s.t $1.207 = dx'$

$$\langle\alpha|x|\alpha\rangle = \langle\alpha|J^{+}(dx')|x|J(dx')|\alpha\rangle = \langle\alpha|x + dx'|\alpha\rangle = x + dx'\langle\alpha|\alpha\rangle = x + dx' \checkmark$$

$[P, J(dx')] = PJ(dx') - J(dx')P = 0$, if we use $J^{+}(dx')$ on (1.227)

$$J^{+}(dx')[P, J(dx')] = J^{+}(dx')PJ(dx') - J^{+}(dx')J(dx')P \qquad (**)$$

For $(**)$ to be true, $J^{+}(dx')PJ(dx') = P$ and $J^{+}(dx')J(dx')P = P$ s.t $1.227 = 0$

$$\langle\alpha|P|\alpha\rangle = \langle\alpha|J^{+}(dx')|P|J(dx')|\alpha\rangle = \langle\alpha|P|\alpha\rangle = P\langle\alpha|\alpha\rangle = P \checkmark$$

$$\boxed{\langle x\rangle \rightarrow x + dx', \ \langle P\rangle \rightarrow P}$$

Problem 1: 1.33 Review

Procedure:

- Start with a commutation relation
- Proceed to apply the adjoint infinitesimal translation operator $\mathcal{J}^\dagger$ onto the commutation relationship defined above.
  - Use the mathematical rules of translation operators to deduce how this works out mathematically.
- Repeat the steps above for both the position and momentum relationship.
- Once these steps are completed, put the results into the expectation value equation and calculate the end results.

Key Concepts:

- Because the translation operator is being applied in the x direction, it will slightly change the expectation value of position by a small amount **dx'**
- Because the translation operator is being applied in the x direction, there is no translation that is recorded for the expectation value of momentum.
- This problem can be proved in other methods than stated above, the key result to take away is how the translation operator affects momentum and position. Also, how these the translation operators act arithmetically is another huge part of this problem.

Variations:

- The translation operator could be translating a small amount of momentum (**dp'**) instead of (**dx'**).
  - This would lead to a flipped result. Where the translation operator will no longer have an effect on the x direction. Instead, it would affect momentum only.
- We could be asked to prove these relations through other methods than above.
  - We would use the same mathematical principles, just through different means.

Problem 2: 1.34

Starting with a momentum operator $\tilde{\mathbf{p}}$ having eigenstates $|p'\rangle$, define an infinitesimal boost operator $\tilde{\mathbf{B}}(\mathbf{dp'})$ that changes one momentum eigenstate into another, that is

$$\tilde{\mathbf{B}}(\mathbf{dp'})\,|p'\rangle = |p' + \mathbf{dp'}\rangle$$

Show that the form $\tilde{\mathbf{B}}(\mathbf{dp'}) = 1 + i\tilde{\mathbf{W}} \cdot \mathbf{dp'}$, where $\tilde{\mathbf{W}}$ is Hermitian, satisfies the unitary, associative, and inverse properties that are appropriate for $\tilde{\mathbf{B}}(\mathbf{dp'})$. Use dimensional analysis to express $\tilde{\mathbf{W}}$ in terms of the position operator $\tilde{\mathbf{x}}$, and show that the result satisfies the canonical commutation relations $[\tilde{\mathbf{x}}_i, \tilde{\mathbf{p}}_j] = i\hbar\delta_{ij}$. Derive an expression for the matrix element $\langle p'|\tilde{\mathbf{x}}|\alpha\rangle$ in terms of a derivative with respect to p' of $\langle p'|\alpha\rangle$.

The translation operator is defined as: $J(dx')\,|x'\rangle = |x' + dx'\rangle$

So, our infinitesimal operator must be, $B(dx')\,|x'\rangle = |x' + dx'\rangle$, w/ momentum this is

$$B(dp')\,|p'\rangle = |p' + dp'\rangle$$

The generator function is, $J(dx')\,|x'\rangle = 1 + ik \cdot dx' \quad \therefore \quad B(dp')\,|p'\rangle = 1 + iw \cdot dp'$

Unitary: $\hat{\mathcal{U}}^{+}\hat{\mathcal{U}} = 1$,

$$\tilde{B}^{+}(dp')\,\tilde{B}(dp') = (1 + i\tilde{\omega}^{+} \cdot dp')(1 - i\tilde{\omega} \cdot dp') = 1 + i(\tilde{\omega}^{+} - \tilde{\omega})dp'^{2} : \quad \tilde{\omega}^{+} = \tilde{\omega} \quad \therefore \quad 1 \ \checkmark$$

Associative: $\tilde{B}(dp')\,\tilde{B}(dp'') = \tilde{B}(dp' + dp'')$

$$\tilde{B}(dp')\,\tilde{B}(dp'') = (1 - i\tilde{\omega} \cdot dp')(1 - i\tilde{\omega} \cdot dp'') = 1 - i\tilde{\omega}(dp' + dp'') = \tilde{B}(dp' + dp'') \ \checkmark$$

Inverse:

$$\tilde{B}(-dp')\,\tilde{B}(dp') = (1 + i\tilde{\omega}dp')(1 - i\tilde{\omega}dp') = 1 + i(\tilde{\omega} - \tilde{\omega})dp' : \quad \hat{\omega} = \hat{\omega} \quad \therefore \quad 1 \ \checkmark$$

The operator must have dimensionsions of inverse $\tilde{p}$ $\therefore$ $\tilde{B} = 1 + i\tilde{x} \cdot dp'/\hbar$

$$[\tilde{p}, \tilde{B}(dp')]\,|p'\rangle = (\tilde{p}\tilde{B}(dp') - \tilde{B}(dp')\tilde{p})\,|p'\rangle = (\tilde{p} + dp')\,|p' + dp'\rangle - \tilde{p}\,|p' + dp'\rangle = dp'\,|p'\rangle$$

$$[\tilde{p}, \hat{B}(dp')]\,|p'\rangle = \hat{p}(1 + i\tilde{x} \cdot dp'/\hbar)\,|p'\rangle - (1 + i\tilde{x} \cdot dp'/\hbar)\,\hat{p}\,|p'\rangle = \frac{i}{\hbar}[\tilde{p}, \tilde{x} + dp']\,|p'\rangle$$

We then have $dp' = \dfrac{i}{\hbar}\left(\tilde{p}\sum_j \tilde{x}_j dp'_j - \sum_j \tilde{x}_j dp' p'\right) \Rightarrow [\tilde{x}_j, \tilde{p}_i] = i\hbar\delta_{ij}$

$$\langle p'|\tilde{B}(dp')|\alpha\rangle = \langle p' + dp'|\alpha\rangle = \langle p'|\alpha\rangle + \sum_j \frac{\partial\langle p'|\alpha\rangle}{\partial p'_j}dp'_j \quad (*)$$

$$\langle p'|\tilde{B}(dp')|\alpha\rangle = \langle p'|\alpha\rangle + \frac{i}{\hbar}\sum_j \langle p'|x_j|\alpha\rangle dp'_j \quad (**)$$

Equating the two we get: $\dfrac{i}{\hbar}\langle p'|x_j|\alpha\rangle = \dfrac{\partial\langle p'|\alpha\rangle}{\partial p'_j} \Rightarrow \langle p'|\tilde{x}|\alpha\rangle = i\hbar\tilde{\nabla}_{p'}\langle p'|\alpha\rangle$

Problem 2: 1.34 Review

Procedure:

- Apply the mathematical definitions of Unitary, Associative, and Inverse identities to show that this boost operator follows these rules with the use of the generator function for this operator.
- Define the dimensions of the operator in the generator function.
- Apply a commutation relation between $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{B}}$ acting on an eigenstate of p'. Proceed to use the mathematical definitions of translation operators on this eigenstate.
- The step above can be repeated but with instead of the actual operator $\tilde{\mathbf{B}}$, one can use the generator function.
- Equate the two results above to show the canonical commutation relation holds.
- Calculate the expectation value of the $\tilde{\mathbf{B}}$, do this once for the operator in co-ordinate space and for the explicit form of the momentum translation operator.
- Equate the two results above and show there relation.

Key Concepts:

- Boost operators, like translation operators, must follow the same mathematical rules. These rules say these operators must be Unitary, Associative, and have an Inverse.
- The units of the Hermitian operator in the generator function must make the final quantity dimensionless.
- We can apply commutation relations of these operators acting on eigenstates of either position, energy, or momentum.
  - This can be done either explicitly or with the generator function.
- The translation and boost operators can have a matrix representation.

Variations:

- This operator can translate position space instead of momentum.
  - This would affect the commutation relations and eventually the matrix representation.

Problem 3: 1.35

(a) Verify (1.271 a) and (1.271 b) for the expectation value of **p** and $\mathbf{p}^2$ from the Gaussian wave packet (1.267).

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{(ikx' - x'^2/2d^2)} \quad (1.267), \quad \langle P\rangle = \hbar k \quad (1.271a), \quad \langle P^2\rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2 \quad (1.271b)$$

$$\langle P\rangle = \langle\alpha|P|\alpha\rangle = \iint \langle\alpha|x'\rangle\langle x'|P|x''\rangle\langle x''|\alpha\rangle\, dx'dx'' = \int \langle\alpha|x'\rangle\langle x'|P|\alpha\rangle\, dx'$$

$$\text{w/ } P = -i\hbar\frac{d}{dx'} \qquad \langle P\rangle = \int \langle\alpha|x'\rangle - i\hbar\frac{d}{dx'}\langle x'|\alpha\rangle\, dx'$$

using mathematica : $\langle P\rangle = \frac{-i\hbar}{d\sqrt{\pi}}\int ik - \frac{x'}{d^2}\, e^{(-x'^2/d^2)}dx' = \frac{-i\hbar}{d\sqrt{\pi}}\cdot\frac{ik\sqrt{\pi}}{1/d} = \hbar k$ ✓

$$\langle P^2\rangle = \langle\alpha|P^2|\alpha\rangle = \iint \langle\alpha|x'\rangle\langle x'|P^2|x''\rangle\langle x''|\alpha\rangle\, dx'dx'' = \int \langle\alpha|x'\rangle\langle x'|P^2|\alpha\rangle\, dx'$$

$$\text{w/ } P^2 = -\hbar^2\frac{d^2}{dx'^2} \qquad \langle P^2\rangle = -\hbar^2\int \langle\alpha|x'\rangle\frac{d^2}{dx'^2}\langle x'|\alpha\rangle dx'$$

$\langle P^2\rangle = -\hbar^2\int e^{(-ikx' - x'^2/2d^2)}\frac{d^2}{dx^2}e^{(ikx' - x'^2/2d^2)}dx'$ using mathematica . . . . .

$\langle P^2\rangle = \frac{-\hbar^2}{d\sqrt{\pi}}\int\left[\left(a - \frac{2x}{b}\right)^2 - \frac{2}{b}\right]e^{-x'^2/d^2}dx'$ w/ $a = ik$ and $b = 2d^2$

$$\langle P^2\rangle = \hbar^2\left[a^2 + \frac{2}{b}\right] - \frac{\hbar^2}{b} = \hbar^2\left[k^2 + \frac{1}{d^2}\right] - \frac{\hbar^2}{2d^2} = \hbar^2k^2 + \frac{\hbar^2}{2d^2} \checkmark$$

$$\boxed{\langle P\rangle = \hbar k\ , \ \langle P^2\rangle = \hbar^2k^2 + \frac{\hbar^2}{2d^2}}$$

(b) Evaluate the expectation value of **p** and $\mathbf{p}^2$ using the momentum-space wave function (1.274).

$$\langle P'|\alpha\rangle = \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{-[(P-\hbar k)^2d^2/2\hbar^2]} \quad (1.274)$$

$$\langle P\rangle = \langle\alpha|P|\alpha\rangle = \iint \langle\alpha|P'\rangle\langle P'|P|P''\rangle\langle P''|\alpha\rangle\, dp'dp'' = \int\langle\alpha|P'\rangle\langle P'|P|\alpha\rangle dp'$$

$$\langle P\rangle = \int\langle\alpha|P'\rangle P'\langle P'|\alpha\rangle dp' = \int P'|\langle P|\alpha|\rangle|^2dp' = \frac{d}{\hbar\sqrt{\pi}}\int P' e^{-[(P'-\hbar k)^2d^2/\hbar^2]}dp'$$

using mathematica . . . . . $\langle P\rangle = \frac{\cancel{d}}{\cancel{\hbar\sqrt{\pi}}}\,\frac{\cancel{\hbar}k\cancel{\sqrt{\pi}}}{\cancel{d}/\cancel{\hbar}} = \hbar k$

$$\langle P^2\rangle = \frac{d}{\hbar\sqrt{\pi}}\int P'^2 e^{-[(P'-\hbar k)^2d^2/\hbar^2]}dp' = \frac{d}{\hbar\sqrt{\pi}}\left(\frac{\hbar^3}{d^3}\frac{\sqrt{\pi}}{2} + \frac{\hbar\sqrt{\pi}}{d}(\hbar k)^2\right) = \hbar^2k^2 + \frac{\hbar^2}{2d^2}$$

$$\boxed{\langle P\rangle = \hbar k\ , \ \langle P^2\rangle = \hbar^2k^2 + \frac{\hbar^2}{2d^2}}$$

Problem 3: 1.35 Review

Procedure:

- Apply the expectation value equation for momentum and the square of momentum.
- Apply a complete continuous set to the left of the momentum operator so that when the operator acts on the eigenstate of position, it will create a wave function and a conjugate of the wave function.
- Evaluate the integrals in the equations.

Key Concepts:

- Because we are dealing with position and momentum space, the eigenvalues of these are continuous and thus when we apply the completeness relation we must do it in a continuous set.
- The expectation values will not change for either, regardless of being in position space or momentum space.

Variations:

- These expectation values can be calculated for position instead of momentum.
  - Thus changing the details of this problem, but not the overall procedure.

Problem 4: 1.36

(a) Prove the following:

(i) $\langle p'|\tilde{\mathbf{x}}|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$

$$\langle p'|\tilde{x}|\alpha\rangle = \int \langle p'|\tilde{x}|p''\rangle\langle p''|\alpha\rangle dp'' \quad : \quad \langle p'|\tilde{x}|p''\rangle = \int \langle p'|\tilde{x}|x'\rangle\langle x'|p''\rangle dx'$$

$$\langle p'|\tilde{x}|p''\rangle = \int x'\langle p'|x'\rangle\langle x'|p''\rangle dx' = \int x'\phi_{x'}(p')\phi_{x'}(p'')\,dx' = i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$$

$$\langle p'|\tilde{x}|\alpha\rangle = \int i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\langle p''|\alpha\rangle dp'' \text{ if } p' = p'' = i\hbar\frac{\partial}{\partial p}\langle p'|\alpha\rangle$$

$\blacksquare$

(ii) $\langle\beta|\tilde{\mathbf{x}}|\alpha\rangle = \int dp'\phi_\beta^*(dp')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

$$\langle\beta|\tilde{x}|\alpha\rangle = \int\langle\beta|p'\rangle\langle p'|\tilde{x}|\alpha\rangle dp' = \int\langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle dp' = \int \varphi_\beta^*(p')\,i\hbar\frac{\partial}{\partial p'}\varphi_\alpha(p')\,dp'$$

$\blacksquare$

(b) What is the physical significance of

$$\exp\left(\frac{i\tilde{\mathbf{x}}\Theta}{\hbar}\right)$$

where $\tilde{\mathbf{x}}$ is the position operator and Θ is some number with the dimensions of momentum? Justify your answer.

$$J(p')|p\rangle = (p' + \Theta)|p\rangle \quad \text{w/} \quad J(p') = -i\hbar\frac{\partial}{\partial x}e^{ix\Theta/\hbar} = -i^2\frac{\hbar}{\hbar}\Theta e^{ix\Theta/\hbar} = \Theta + p'$$

This is a translation momentum operator

Problem 4: 1.36 Review

Procedure:

- First apply a complete set in momentum space to the value on the left in i.).
- Expand the first expectation value with another complete set, this time in position space.
- Reorganize the result so that two plane waves are created.
- Evaluate the integral. Insert the result into the original expanded complete set, and then show that the result simplifies to the desired result.
- Use the result in part i.), with expanding the expectation value in momentum space, to simplify the above result.
- For part b.), apply this onto an eigenstate of momentum and show that the result will simplify to a momentum translation operator.

Key Concepts:

- A complete set that is continuous is required when expanding in position or momentum space.
- Multiple complete sets can be used to expand a relationship, either in position or momentum space.

Problem 5: Q-1

Consider the Hamiltonian $\tilde{\mathbf{H}}$ of a particle in one-dimension problem defined by

$$\tilde{\mathbf{H}} = \frac{\tilde{\mathbf{p}}^2}{2m} + V(\tilde{\mathbf{x}})$$

where $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{x}}$ are momentum and position operators respectively. These operators satisfy the commutation relation: $[\tilde{\mathbf{x}}, \tilde{\mathbf{p}}] = i\hbar$. The eigenvectors of $\tilde{\mathbf{H}}$ are denoted by $|\phi_n\rangle$ and satisfy the eigenvalue equation $\tilde{\mathbf{H}}\,|\phi_n\rangle = E_n\,|\phi_n\rangle$, where n is a discrete index.

(a) Show that:

$$\langle\phi_n|\tilde{\mathbf{p}}|\phi_{n'}\rangle = \alpha\,\langle\phi_n|\tilde{\mathbf{x}}|\phi_{n'}\rangle$$

where α is a coefficient that depends on the difference between E_n and $E_{n'}$. Calculate α.

$$[\tilde{x},\tilde{H}] = [\tilde{x}, \tilde{p}^2/2m + V(x)] = \frac{1}{2m}[\tilde{x},\tilde{p}^2] = \frac{i\hbar}{m}\tilde{p} \quad \therefore \quad \tilde{p} = \frac{m}{i\hbar}[\tilde{x},\tilde{H}]$$

$$\frac{m}{i\hbar}\langle\varphi_n|[\tilde{x},\tilde{H}]|\varphi_{n'}\rangle = \frac{m}{i\hbar}\left[\langle\varphi_n|\tilde{x}\tilde{H} - \tilde{H}\tilde{x}|\varphi_{n'}\rangle\right] = \frac{m}{i\hbar}\left[\langle\varphi_n|\tilde{x}\tilde{H}|\varphi_{n'}\rangle - \langle\varphi_n|\tilde{H}\tilde{x}|\varphi_{n'}\rangle\right]$$

$$\frac{m}{i\hbar}\left[E_{n'}\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle - E_n\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle\right] = \frac{im}{\hbar}(E_n - E_{n'})\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle \checkmark$$

$$\boxed{\alpha = \frac{im}{\hbar}(E_n - E_{n'})}$$

(b) Form this, deduce, using the completeness relation, the equation:

$$\sum_{n'}(E_n - E_{n'})^2|\langle\phi_n|\tilde{\mathbf{x}}|\phi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\langle\phi_n|\tilde{\mathbf{p}}^2|\phi_n\rangle\,.$$

$$|\alpha\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle|^2 = (E_n - E_{n'})^2\cdot\frac{m^2}{\hbar^2}|\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle|^2 = |\langle\varphi_n|\tilde{p}|\varphi_{n'}\rangle|^2$$

$$\sum_{n'}(E_n - E_{n'})^2\cdot\frac{m^2}{\hbar^2}|\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle|^2 = \sum_{n'}\langle\varphi_n|\tilde{p}|\varphi_{n'}\rangle\langle\varphi_{n'}|\tilde{p}|\varphi_n\rangle = \langle\varphi_n|\tilde{p}^2|\varphi_n\rangle \therefore$$

$$\boxed{\sum_{n'}(E_n - E_{n'})^2|\langle\varphi_n|\tilde{x}|\varphi_{n'}\rangle|^2 = \frac{\hbar^2}{m^2}\langle\varphi_n|\tilde{p}^2|\varphi_n\rangle}$$

Problem 5: Q-1 Review

Procedure:

- Compute a commutation relation between $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{H}}$ which will give a relationship between $\tilde{\mathbf{p}}$.
- Expand this commutation relation between the two eigenstates of $\tilde{\mathbf{H}}$, and evaluate.
- Proceed to use the result from part a.) in part b.) by first squaring both sides of the first relationship.
- Proceed to expand the side where $\tilde{\mathbf{p}}$ in a complete set.
- Simplify and show the result at the end.

Key Concepts:

- Because it was stated initially we can use a discrete completeness relation. This is because we are dealing with energy that is bounded.
- There is a commutation relation between $\tilde{\mathbf{x}}$, $\tilde{\mathbf{H}}$, and $\tilde{\mathbf{p}}$ that can be used to calculate this relationship.
- Because the Hamiltonian is hermitian, it does not matter which eignestate it operates on.