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Electrodynamics 1

CH. 16 MAGNETOSTATICS IN MATTER LECTURE NOTES

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As a quick review

$$\vec{D} = \epsilon\vec{E} + \epsilon_0\vec{E} + \vec{P} \quad , \quad \vec{P} = \epsilon_0 \chi_e \vec{E} \quad , \quad \chi_e = \frac{\epsilon - \epsilon_0}{\epsilon_0}$$

Where we have the relationship

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad , \quad \vec{\nabla}\times\vec{E} = 0$$

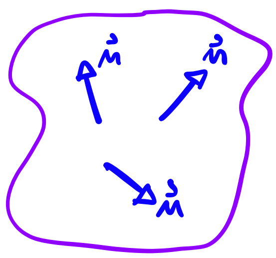
If we then look at how this relates with magnetic field we have

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}_{free} + \mu_0\epsilon_0 \frac{\partial E}{\partial t} \quad , \quad \rho_b = \vec{\nabla}\cdot\vec{P}$$

We can further say

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0 \implies \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P}) + \vec{\nabla}\cdot\vec{J} = 0 \implies \vec{J} = -\frac{\partial}{\partial t}\vec{P}$$

when our charges separate and become a dipole we will have something like



We have a substance that contains a bunch of magnetic dipoles. This means our vector potential is

$$\vec{A} = \int \frac{\vec{m}(\vec{r}) \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d^3r' = \int \vec{m} \times \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|} d^3r$$

Where we say

$$\vec{\nabla}' \times \frac{\vec{m}(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{\vec{\nabla}' \times \vec{m}}{|\vec{r} - \vec{r}'|} - \vec{m} \times \vec{\nabla}' \frac{1}{|\vec{r} - \vec{r}'|}$$

This means our vector potential becomes

$$\vec{A} = \int \frac{\vec{\nabla}' \times \vec{m}}{|\vec{r} - \vec{r}'|} d^3r - \int \vec{\nabla}' \times \left(\frac{\vec{m}}{|\vec{r} - \vec{r}'|}\right) d^3r = \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

We now have our expression for vector potential.

We now have a "new" Maxwell equation

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_{Free} + \vec{J}_b + \vec{J}_m\right) + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

where we introduce a new Field

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{m}$$

we then take the curl of this field to find

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{P}}{\partial t} + \cancel{\vec{\nabla}\times\vec{m}} + \epsilon_0\frac{\partial\vec{E}}{\partial t} - \cancel{\vec{\nabla}\times\vec{m}}$$
$$= \vec{J}_{Free} + \frac{\partial}{\partial t}\left(\epsilon_0\vec{E} + \vec{P}\right) = \vec{J}_{Free} + \frac{\partial}{\partial t}\vec{D}$$

We can further say $\vec{\nabla}\cdot\vec{B} = 0$ is still true

In Magnetostatics we have

$$\vec{\nabla}\times\vec{H} = \vec{J}_{free} + \frac{\partial}{\partial t}\vec{D} \quad , \quad \vec{\nabla}\cdot\vec{D} = \rho_{Free}$$

Where in materials we can Further Say

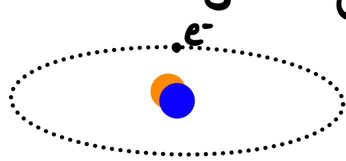
$$\vec{B} = \mu\vec{H} \quad , \quad \vec{m} = \chi_m\vec{H} \quad , \quad \chi_H = \frac{\mu}{\mu_0} - 1 \quad , \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{m} \quad , \quad \vec{B} = \mu_0(\vec{H} + \vec{m})$$

We define something called a pointing vector

$$-\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) = \vec{J}\cdot\vec{E} + \left(\frac{\partial}{\partial t}\vec{B}\right)\cdot\vec{H} + \left(\frac{\partial}{\partial t}\vec{D}\right)\cdot\vec{E}$$

Where the pointing vector tells us how energy is being transferred in materials

We now look at the magnetic susceptibility χ by looking at an atom

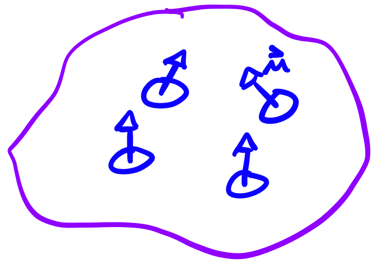


Where our electron going around the atom. If we increase $\vec{H}$ we will inadvertently decrease $\vec{m}$

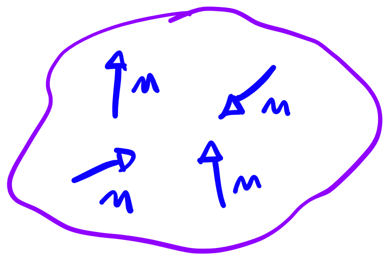
Looking at Diamagnetics we know

$$u = -\vec{m}\cdot\vec{H}$$

which tells us $\chi_m > 0$ for paramagnetics

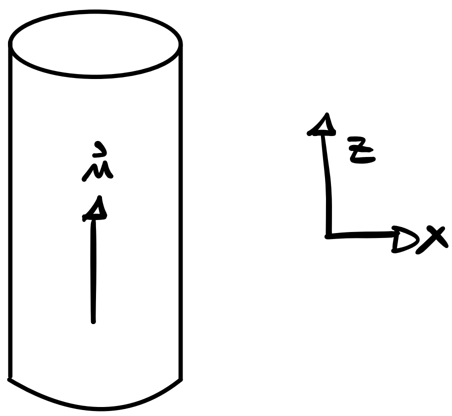


A Ferromagnet is



Magnetic moments align with materials in Ferromagnetism

Looking at a uniformly magnetized cylinder



we can go on to say

$$\vec{\nabla}\cdot\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\cdot\vec{B} - \vec{\nabla}\cdot\vec{M} = 0 \Rightarrow \vec{B} = \mu_0\vec{M}$$

5-2-22

we know in Magnetostatics that

$$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$

When we have time dependence we then have

$$\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \quad , \quad \vec{\nabla}\times\left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = 0$$

Using some other Maxwell equations we have

$$\vec{E} + \frac{\partial}{\partial t}\vec{A} = -\vec{\nabla}\varphi \Rightarrow \vec{E} = -\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}$$

Using the above result we can then say

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad -\nabla^2\varphi - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \frac{\rho}{\epsilon_0}$$

We can then go on to re-writing some of the Maxwell Equations as

$$\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{E} = \mu_0\vec{J}$$

Now taking the Laplacian and curl of the curl of $\vec{A}$, we then have

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\vec{\nabla}\varphi - \frac{\partial}{\partial t}\vec{A}\right) = \mu_0\vec{J} \quad , \quad -\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\varphi = \mu_0\vec{J}$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi\right) = \mu_0\vec{J}$$

We can go on to calculating a Gauge Transformation for this by defining our magnetic field to be

$$\vec{B} = \vec{\nabla}\times\vec{A} \rightarrow \vec{B} = \vec{\nabla}\times\vec{A}'$$

$$\vec{A}' = \vec{A} + \nabla\Lambda \quad , \quad \varphi' = \varphi + \frac{\partial}{\partial t}\Lambda$$

We can then go on to say that our $\vec{E}'$ field is

$$\vec{E}' = -\vec{\nabla}\varphi' - \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\Lambda) = -\vec{\nabla}(\varphi' - \partial/\partial t\,\Lambda) - \frac{\partial\vec{A}}{\partial t}$$

Where we have made the choice

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi = 0$$

The above is then used to define

$$-\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi = \frac{\rho}{\epsilon_0} \quad , \quad -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \mu_0\vec{J}$$

Where we must define a four vector $\vec{A}$ and $\vec{J}$ along with an operator $\Box^2$

These four vectors are then

$$\vec{A} = \begin{pmatrix} \varphi \\ -A_x \\ -A_y \\ -A_z \end{pmatrix}, \quad \vec{J} = \begin{pmatrix} \rho/\epsilon_0 \\ -J_x \\ -J_y \\ -J_z \end{pmatrix}, \quad \Box^2 = \begin{pmatrix} 1/c^2\, \partial^2/\partial t^2 \\ -\partial_x^2 \\ -\partial_y^2 \\ -\partial_z^2 \end{pmatrix}$$

Where we know $\Box^2 \vec{A} = \vec{J}$. We now move on to solving the relativistic Laplacian equation. Defining φ we have

$$\varphi = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2\vec{r}' = \frac{1}{\epsilon_0} \int G(\vec{r}, \vec{r}') \rho(\vec{r}') d^3\vec{r}'$$

We then have a Greens Function of the form

$$-\nabla^2 G(\vec{r}, \vec{r}') = \delta^3(\vec{r} - \vec{r}')$$

This means Laplace's Equation becomes

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi = f(\vec{r}, t) \quad \therefore \quad \psi(\vec{r}, t) = \int G(\vec{r}, \vec{r}', t, t') f(\vec{r}', t')\, d^3\vec{r}' dt'$$

We then have a Greens Function of the form

$$G = G(\vec{r} - \vec{r}', t - t') \Rightarrow R = \vec{r} - \vec{r}', \quad \tau = t - t'$$

We then have an equation of the form

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{r} - \vec{r}', t - t') = \delta^3(\vec{r} - \vec{r}') \delta(t - t') \qquad (*)$$

This means we can write the Greens Function as

$$G(\vec{R}, \tau) = \frac{1}{2\pi} \int G(\vec{R}, \omega) e^{-i\omega\tau} d\omega, \quad G(\vec{R}, \omega) = \int G(\vec{R}, \tau) e^{i\omega\tau} d\tau$$

$$\psi(\vec{r}, t) = \int G(\vec{R}, \tau) f(\vec{r}', t')$$

This changes $(*)$ to

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \frac{1}{2\pi} \int G(\vec{R}, \omega) e^{-i\omega\tau} d\omega = \delta^3(\vec{R}) \frac{1}{2\pi} \int e^{-i\omega\tau} d\omega$$

We then can further conclude

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{R}, \tau) = \delta^3(\vec{R}) \delta(\tau)$$

This means the solution to our Greens Function is then

$$G(\vec{R},\tau) = \frac{1}{2\pi}\int G(\vec{R},\omega)e^{-i\omega\tau}\,d\omega \quad , \quad G^{\pm}(\vec{R},\tau) = \frac{1}{2\pi}\int \frac{e^{-i\omega\tau}e^{\pm i\omega R/c}}{R}\,d\omega$$

$$\Rightarrow \qquad G^{\pm}(\vec{R},\tau) = \frac{1}{2\pi}\int \frac{e^{-i\omega(\tau\mp R/c)}}{R}\,d\omega = \frac{\delta(t-t'\mp R/c)}{|\vec{r}-\vec{r}'|}$$

This means our Wave Function is then

$$\psi^{(r=t)}(\vec{t},t) = \int \frac{\delta((t\mp R/c)-t')}{|\vec{r}-\vec{r}'|} f(\vec{r}',t')\,d^3\vec{r}'\,dt' = \int \frac{f(\vec{r}',(t-\frac{|\vec{r}-\vec{r}'|}{c}))}{|\vec{r}-\vec{r}'|}\,d^3\vec{r}'$$