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Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #10

PROBLEMS: $\{1, 2, 3\}$

Due: November 28, 2021 By 11:59 PM

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PROFESSOR
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Problem 1:

A particle of mass m is subject to a force,

$$\mathbf{F} = -\frac{2a}{r^3}\hat{\mathbf{r}},$$

in three dimensions (3D) with $a > 0$. Here, $\hat{\mathbf{r}} = \mathbf{r}/r$ where $\mathbf{r}$ is the position vector of the particle with respect to the origin and $r = |\mathbf{r}|$.

(a) Explain why you expect the motion of the particle to be confined to a 2D plane. You are not expected to give a mathematical proof, a concise qualitative sentence is sufficient.

we expect this to happen because of spherical symmetry that exists in our Lagrangian. θ is also cyclic $\therefore$ angular momentum is conserved.

(b) Starting from a Lagrangian describing the 2D motion in polar co-ordinates *phi* and r, show that the motion of the particle is governed by an *effective one-dimensional potential* $V_{eff}(r)$.

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{a}{r^2}, \quad \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0, \quad \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \varphi \text{ is cyclic}$$

This means we can write $\dot{\varphi}$ as $\dot{\varphi}^2 = \frac{\ell^2}{m^2 r^4}$, our Lagrangian then becomes...

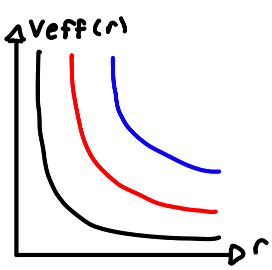
$$L = \underbrace{\frac{1}{2}m\dot{r}^2}_{T} + \underbrace{\frac{1}{2}\frac{\ell^2}{mr^2} + \frac{a}{r^2}}_{V} \quad \therefore \quad V_{eff}(r) = \frac{1}{2}\frac{\ell^2}{mr^2} + \frac{a}{r^2}$$

$$\boxed{V_{eff}(r) = \frac{1}{2}\frac{\ell^2}{mr^2} + \frac{a}{r^2}}$$

(c) Discuss the particles motion as a function of the initial energy, the value of the constant a and any other relevant factors.

$$V_{eff}(r) = \frac{1}{2}\frac{\ell^2}{mr^2} + \frac{a}{r^2} = \frac{1}{r^2}\left(\frac{\ell^2}{2m} + a\right) \Rightarrow \frac{1}{r^2}\alpha \quad \text{where} \quad \alpha \equiv \frac{\ell^2}{2m} + a \;\;\&\;\; a > 0$$

we will fix the values of ℓ and m while we let a vary



Black curve : $V_{eff}(r) = \frac{1}{r^2}$ w/ $\alpha = 1$ $\therefore$ $a = \frac{\ell^2}{2m} + 1$

Red curve : $V_{eff}(r) = \frac{5}{r^2}$ w/ $\alpha = 5$ $\therefore$ $a = \frac{\ell^2}{2m} + 5$

Blue curve : $V_{eff}(r) = \frac{10}{r^2}$ w/ $\alpha = 10$ $\therefore$ $a = \frac{\ell^2}{2m} + 10$

As $r \to 0$, $V_{eff} \to \infty$: $r \to \infty$, $V_{eff} \to 0$. we can see on the graph there is no point where $E < 0$ or where $E = 0$. This plot is only for when $E > 0$

$E > 0$: particle can probe $r \in [r_{min}, \infty) \longrightarrow V_{eff}(r_{min}) = E$

$E > 0$: * 1 turning point
* "Scattering motion" (Start at large r, bounce
* At $r \to \infty$: Free motion

Problem 1: Continued

(d) Show that when the 2D motion of the particle [see (a)] is described by polar co-ordinates r and ϕ, the motion can be parameterized by integral equation,

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{l}{mr'^2} \frac{1}{\sqrt{\frac{2}{m}[E - V_{eff}(r')]}} dr'$$

where ϕ_0 and r_0 relate to the particle's initial conditions, E is the total mechanical energy and l is the magnitude of the total angular momentum.

we can write the energy for our system as :

$$E = T + U = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \frac{\ell^2}{r^2} + U(r)$$

we solve this for $\dot{r}$ and obtain $\therefore$

$$\dot{r} = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(r) - \frac{\ell^2}{2mr^2}}$$

Where this can be solved for t w/ $\dot{r} = dr/dt$ $\therefore$

$$dt = \pm \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{E - U(r) - \ell^2/2mr^2}}$$

But since we want φ in terms of r we use : $\frac{d\varphi}{dr} = \frac{d\varphi}{dt}\frac{dt}{dr} = \frac{\dot{\varphi}}{\dot{r}}$ $\therefore$ $d\varphi = \frac{\dot{\varphi} \cdot dr}{\dot{r}}$

w/ $V_{eff}(r) = U(r) + \frac{\ell^2}{2mr^2}$ we get the following :

$$\int_{\varphi_0}^{\varphi} d\varphi = \int_{r_0}^{r} \frac{\ell}{mr^2} \frac{dr}{\sqrt{2/m(E - V_{eff}(r))}} \checkmark$$

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Problem 1: Review

Procedure:

- Because of spherical symmetry this will leave our Lagrangian unchanged and thus angular momentum is conserved.
- Write out a Lagrangian for this system, show that it is cyclic in $\dot{\phi}$ and then proceed to identify the effective potential.
- Discuss how the effective potential changes by slightly changing values for variables and plotting them.
- Express the energy in terms of kinetic and potential energies, solve for $\dot{r}$ and then rearrange for $d\phi$.

Key Concepts:

- Because of spherical symmetry angular momentum is conserved.
- Because one co-ordinate is cyclic we can solve for an effective potential.
- We can parameterize the motion of our particle by expressing the energy and rearranging it.

Variations:

- We can be given a different force.
  - Where we would use the same formalism and solve for what is asked of us.

Problem 2:

(a) Define and / or briefly discuss the terms:

- (i) Limit cycle
- (ii) Chaos
- (iii) Poincare section / map

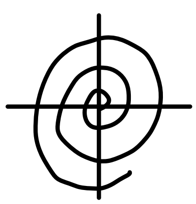
Your answer can involve an example or illustrative sketch that highlights an important feature(s) if this is useful.

Limit cycle : Closed trajectory (orbit) where at least one or another trajectory flow into it as $t \to \infty$ or $t \to -\infty$

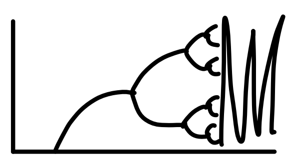
Chaos : A system in which a slight perturbance in initial conditions will produce different dynamics. These can be described by Lyapunov exponents or presented with Poincare maps or bifurcation diagrams.

Poincare Maps : The continuous curves in a poincare map indicate non-chaotic motion where as the dots in each map indicate chaos. This procedure is done by keeping track of when a system crosses a specific plane designated by the creator of the map.

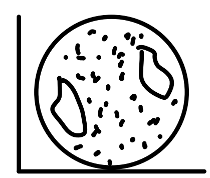
Limit cycle : Phase Spaces



Bifurcation Diagram :



Poincare maps



(b) Consider the coupled equations of motion with $b > 0$,

$$\dot{x} = ax - by - C(x^2 + y^2)x,$$
$$\dot{y} = bx + ay - C(x^2 + y^2)y.$$

Introducing polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ the system can be equivalently described via the equations of motion,

$$\dot{r} = ar - Cr^3,$$
$$\dot{\theta} = b.$$

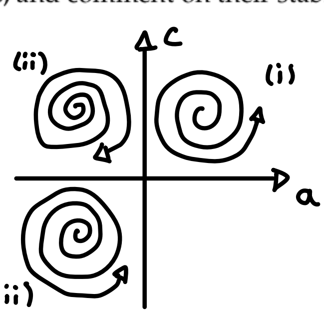
Sketch a useful phase portrait of the system in the $x - y$ plane for:

- (i) $a > 0$ and $C > 0$
- (ii) $a < 0$ and $C > 0$
- (iii) $a < 0$ and $C < 0$

and indicate the important features including, e.g., limit cycles and fixed points. You should also clearly state the radius or position of any limit cycles and fixed points, and comment on their stability (provide evidence!).

$$\dot{r} = r(a - cr^2)$$

Fixed point at : $\dot{r} = 0,\ r = 0,\ r^2 = \frac{a}{c}$



Problem 2: Continued

Stability of fixed points:

$a > 0\ ,\ c > 0$

$r = 0 : \quad \frac{\partial \dot{r}}{\partial r} = a - 3cr^2 \Big|_{r=0} = a \ : \quad \frac{\partial \dot{r}}{\partial r}\Big|_{r=0} > 0 \quad \therefore \quad r = 0$ is $\underline{\underline{\text{unstable}}}$

$r = \pm\sqrt{\frac{a}{c}} : \quad \frac{\partial \dot{r}}{\partial r} = a - 3cr^2 \Big|_{r=+\sqrt{\frac{a}{c}}} = -2a \ : \quad \frac{\partial \dot{r}}{\partial r}\Big|_{r=\sqrt{\frac{a}{c}}} < 0 \quad \therefore \quad r = \pm\sqrt{\frac{a}{c}}$ is $\underline{\underline{\text{stable}}}$

$a < 0\ ,\ c > 0$

$r = 0 : \quad \frac{\partial \dot{r}}{\partial r} = a - 3cr^2 \Big|_{r=0} = a \ : \quad \frac{\partial \dot{r}}{\partial r}\Big|_{r=0} < 0 \quad \therefore \quad r = 0$ is $\underline{\underline{\text{stable}}}$

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$a < 0\ ,\ c < 0$

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Problem 2: Review

Procedure:

- Proceed to discuss the terms.
- Draw the phase portrait for each scenario.

Key Concepts:

- Limit Cycle's are closed trajectory orbits where one trajectory flows into another.
- Chaotic systems are systems where a small perturbance in initial conditions will produce different dynamics.
- Poincare maps describe chaotic behavior.
- Phase portraits can be used to describe the motion of a system over time.

Variations:

- We can be given different coupled equations.
  - This would lead to different phase spaces.

Problem 3:

A thin uniform disk of radius R and mass m is rigidly fixed to an axle as in Fig. 1. The axles is free to rotate about its own central axis. The disk is oriented such that a normal vector from the disk's surface makes an angle θ with the axle. For simplicity, assume in the following that the axle is massless and the disk is infinitely thin (i.e., it's thickness is ignorable). For parts (a)-(c) you may assume the angular frequency of the axle's rotation is not fixed (i.e., the axle rotates freely).

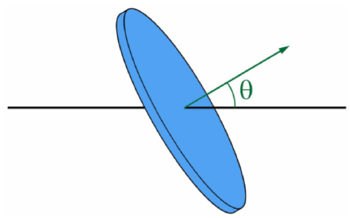
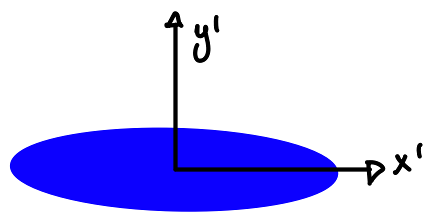


Figure 1: A disk rigidly attached to an axle that can rotate.

(a) By identifying an appropriate set of body fixed axes, compute the principal moments of inertia of the disk.

we will orient our disk such that the normal vector of our disk points straight up.

$$I_{ij} = \int \rho(r)(r^2 - x_i x_j)\, dV$$



$$I_{ij} = \int \rho(r)(r^2 - x_i x_j)\, dA$$

$$I_{xx} = \frac{m}{\pi R^2}\int_0^{2\pi}\int_0^R r^3 \sin^2\varphi \, dr\, d\varphi \, , \quad I_{yy} = \frac{m}{\pi R^2}\int_0^{2\pi}\int_0^R r^3 \cos^2\varphi \, dr\, d\varphi \, , \quad I_{zz} = \frac{m}{\pi R^2}\int_0^{2\pi}\int_0^R r^3 \, dr\, d\varphi$$

$$I_{xx} = \frac{m}{\pi R^2}\cdot\frac{R^4}{4}\cdot\pi = \frac{mR^2}{4} \, , \quad I_{yy} = \frac{m}{\pi R^2}\cdot\frac{R^4}{4}\cdot\pi = \frac{mR^2}{4} \, , \quad I_{zz} = \frac{m}{\pi R^2}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{mR^2}{2}$$

$$\boxed{I = \frac{1}{4}\begin{pmatrix} mR^2 & 0 & 0 \\ 0 & mR^2 & 0 \\ 0 & 0 & 2mR^2 \end{pmatrix}}$$

(b) Give an expression for the instantaneous angular velocity $\vec{\omega}$ with respect to your set of body-fixed axes.

Using the body-fixed co-ordinates the angular velocity vector is

$$\boxed{\vec{\omega}_{bf} = \dot{\varphi}\begin{pmatrix} \sin(\varphi)\sin(\theta) \\ \cos(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}}$$

(c) Using your answers to (a) and (b) give an expression for the kinetic energy of the rotating disk.

The kinetic energy is then calculated by

$$T = \frac{1}{2}\vec{\omega}^{+}\hat{I}\,\vec{\omega}$$

Problem 3: Continued

Which in this case will be:

$$T = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \quad w/ \quad I_1 = I_2 \doteq I_{12} \,, \quad \vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3 \rightarrow \vec{\omega}_{bf}$$

$$T = \frac{I_{12}}{2}\dot{\varphi}^2\left(\sin^2(\theta)\underbrace{(\sin^2(\varphi) \cancel{+} \cos^2(\varphi))}_{1}\right) + \frac{I_3}{2}\dot{\varphi}^2\cos^2(\theta) = \frac{I_{12}}{2}\dot{\varphi}^2\sin^2(\theta) + \frac{I_3}{2}\dot{\varphi}^2\cos^2(\theta)$$

$$\boxed{T = \frac{\dot{\varphi}^2}{2}\left(I_{12}\sin^2(\theta) + I_3\cos^2(\theta)\right)}$$

(d) Assume that the axle is now driven to rotate at a *fixed* angular frequency Ω about its central axis. Compute the magnitude of the applied torque that is required to preserve this motion.

we want $\dot{\vec{\omega}} = 0$ for the motion to be preserved

$$\vec{\omega} \rightarrow \Omega \begin{pmatrix} \sin(\varphi)\sin(\theta) \\ \cos(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

Euler's equations will give us

$$(I_3 - I_2)\omega_2\omega_3 = N_1 \,, \quad (I_1 - I_3)\omega_1\omega_3 = N_2 \,, \quad (I_2 - I_1)\omega_1\omega_2 = N_3$$

With $I_1 = I_2 \Rightarrow N_3 = 0$

$$|\vec{N}|^2 = \sqrt{N_1^2 + N_2^2} = \sqrt{\left(\frac{mR^2}{4}\omega_2\omega_3\right)^2 + \left(\frac{mR^2}{4}\omega_1\omega_3\right)^2} = \frac{mR^2}{4}\sqrt{\omega_1^2 + \omega_2^2}\,|\omega_3| = \frac{mR^2}{4}\Omega^2\cos\theta\sin\theta$$

$$\boxed{\Upsilon = \frac{mR^2}{4}\Omega^2\cos(\theta)\sin(\theta)}$$

Problem 3: Review

Procedure:

- Begin by calculating the moment of inertia tensor with

$$I_{ij} = \int \rho(\mathbf{r})(\mathbf{r}^2 - x_i x_j)\, dV.$$

- The Body Fixed angular momentum is a definition that is given to us in Goldstein.
- We calculate the kinetic energy with

$$\hat{T} = \frac{1}{2}\vec{\omega}^\dagger \hat{I}\vec{\omega}.$$

- We then proceed to calculate the applied torque with the use of Euler's equations

$$|\vec{N}|^2 = \sqrt{N_1^2 + N_2^2 + N_3^2}$$

Key Concepts:

- We can calculate the quantities asked of us with simple equations given to us in Goldstein.

Variations:

- We can be given a different system.
  - Where we would use the same formalism for finding the quantities that are asked of us.