Exam #2

(1)

a)

$$E_n^0 = \hbar\omega(n + 1/2)$$

$$\hat{V} = A m \omega^2 x^2$$

$$= A m \omega^2 \frac{\hbar}{2m\omega} (a + a^\dagger)^2$$

$$= \frac{A}{2} \hbar\omega (a + a^\dagger)^2 .$$

$$\Delta E_n = \underbrace{\langle n|\hat{V}|n\rangle}_{\Delta E_n^{(1)}} + \underbrace{\sum_{m\neq n} \frac{|\langle n|\hat{V}|m\rangle|^2}{E_n^0 - E_m^0}}_{\Delta E_n^{(2)}} + \dots$$

$$\Delta E_n^{(1)} = \frac{A\hbar\omega}{2}\left[\langle n|a a^\dagger|n\rangle + \langle n|a^\dagger a|n\rangle\right]$$

$$= (2n+1)\frac{A\hbar\omega}{2} . = A\hbar\omega(n + \tfrac{1}{2})$$

$$\Delta E_n^{(2)} = \frac{A^2(\hbar\omega)^2}{4}\left[\frac{|\langle n|aa|n+2\rangle|^2}{E_n^0 - E_{n+2}^0} + \frac{|\langle n|a^\dagger a^\dagger|n-2\rangle|^2}{E_n^0 - E_{n-2}^0}\right]$$

$$= \left[-\frac{(n+2)(n+1)}{2\hbar\omega} + \frac{n(n-1)}{2\hbar\omega}\right]\frac{A^2(\hbar\omega)^2}{4}$$

$$= -\frac{A^2\hbar\omega}{8}\left[\cancel{n^2} + 3n + 2 - \cancel{n^2} + n\right]$$

$$= -\frac{A^2\hbar\omega}{2}\cdot\left(n + \frac{1}{2}\right).$$

Solving the problem exactly,

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \bar{\omega}^2 x^2$$

where

$$\bar{\omega} = \omega (1 + 2A)^{1/2}$$

$$\Longrightarrow E_n = \hbar \bar{\omega} (n + 1/2)$$

$$\cong \hbar\omega (n + 1/2) \times \left[1 + A - \frac{A^2}{2} + \dots \right]$$

In agreement with the perturbative result.

b)

$$c^{(1)}_{(t)} = -\frac{i}{\hbar}\int_0^t dt' \langle m| e^{-iH_0 t'/\hbar} V(t') e^{iH_0 t'/\hbar}|0\rangle$$

$$= -\frac{i}{\hbar}\int_0^t dt' e^{i\omega_{m0} t'} e^{(i\omega-\lambda)t'} \times$$

$$\frac{A}{2}\hbar\omega \underbrace{\langle m|(a+a^{+})^2|0\rangle}_{\sqrt{2}\,\delta_{m,2}+\delta_{m,0}}$$

for $m=2$,

$$c^{(1)}_{(t)} = \frac{-i}{\sqrt{2}\hbar}\int_0^t e^{(2\omega i+\omega i-\lambda)t'} \times A\hbar\omega$$

$$= \frac{+i}{\sqrt{2}\hbar} \frac{1-e^{(3i\omega-\lambda)t}}{3\omega i-\lambda} A\hbar\omega.$$

$$\therefore |c^{(1)}_{(\omega)}|^2 = \frac{(A\omega)^2}{2}\frac{|1-e^{3i\omega t}|^2}{9\omega^2+\lambda^2}, \quad \lambda\to 0.$$

(1)

(2)

a) $$Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\, z\cdot(x \pm iy)$$

$$\therefore \quad T^2_{\pm 1} = \mp \underline{\text{Const}}\,.\, z(x \pm iy)$$

$$\Longrightarrow \; -\frac{T^2_{+1} - T^2_{-1}}{2i} = zy$$

b)

For $l = 1$,

$$\langle 1, m' | \frac{(T^2_{+1} - T^2_{-1})}{2i} | 1, m \rangle$$

$$= \underbrace{\langle 1, 2; m, 1 | 1\, 2; l' = 1\, m' \rangle}_{\cancel{m' = m+1}\ \delta_{m', m+1}} \times \frac{\langle 1 \| T^{(2)} \| 1 \rangle}{2i\sqrt{3}}$$

$$- \underbrace{\langle 1,2; m-1 | 12; 1m' \rangle}_{\delta_{m',m-1}} \frac{\langle 1 \| T^{(2)} \| 1 \rangle}{2i\sqrt{3}} .$$

~~Since m' =~~

therefore the matrix elements of $\hat{V}$ in the $|l=1, m\rangle$ basis are:

$$\hat{V} = V_0 \begin{pmatrix} 0 & \Delta & 0 \\ \Delta^* & 0 & \Delta \\ 0 & \Delta^* & 0 \end{pmatrix}$$

where Δ is a constant, $(\Delta = i|\Delta|)$

$$\Rightarrow \det \begin{vmatrix} \Delta E & i|\Delta| & 0 \\ -i|\Delta| & \Delta E & i|\Delta| \\ 0 & -i|\Delta| & \Delta E \end{vmatrix} = 0$$

$$\Rightarrow \Delta E^3 - 2\Delta E |\Delta|^2 = 0 \Rightarrow \Delta E = 0, \pm |\Delta| \sqrt{2}$$

with eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for } \Delta E = 0.$$

and

$$\frac{1}{2}\begin{pmatrix} -1 \\ \pm\sqrt{2}i \\ 1 \end{pmatrix}, \quad \text{for } \Delta E = \pm|\Delta|\sqrt{2}$$

3

a) $$|\psi^+\rangle = |k\rangle + \frac{1}{E - H_0 + i0_+}\hat{V}|\psi^+\rangle$$

$$\langle \vec{p}|\psi^+\rangle \equiv \psi^+(\vec{p}) = \langle \vec{p}|\vec{k}\rangle$$

$$+ \langle \vec{p}|\frac{1}{E - H_0 + i0_+}\hat{V}|\psi^+\rangle$$

$$= \delta(\vec{p}-\vec{k}) + \frac{1}{E - \frac{\hbar^2 p^2}{2m} + i0_+} \times$$

$$\int d\vec{k}' \underbrace{\langle \vec{p}|\hat{V}|\vec{k}'\rangle}_{V(\vec{p}-\vec{k}')} \underbrace{\langle \vec{k}'|\psi^+\rangle}_{\psi^+(\vec{k}')}$$

b)

$$f^{+}(\vec{k},\vec{k}') = -\frac{2\pi^2}{\hbar^2} 2m \langle \vec{k}' | V | \psi_{k}^{+} \rangle$$

$$= -2\pi^2 \frac{2m}{\hbar^2} \langle \vec{k}' | V | \vec{k}^{0} \rangle$$

$$- 2\pi^2 \frac{2m}{\hbar^2} \langle \vec{k}' | \hat{V} | \frac{1}{E - H_0 + i0_+} \hat{V} | \psi_{k}^{+} \rangle$$

$$= -2\pi \frac{2m}{\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle$$

$$+ \frac{2m}{\hbar^2} \int d^3\vec{q} \langle \vec{k}' | V | \vec{q} \rangle \frac{1}{E - \frac{\hbar^2}{2m} q^2 + i0_+}$$

$$(-2\pi)^2 \times \langle \vec{q} | \hat{V} | \psi_{k}^{+} \rangle$$

$$= -2\pi \frac{2m}{\hbar^2} \langle \vec{k}' | V | \vec{k} \rangle$$

$$+ \frac{2m}{\hbar^2} \int d^3q \, V(\vec{k}' - \vec{q}) \frac{1}{k^2 - q^2 + i0_+} f^{(+)}(\vec{q}, \vec{k}$$

where

$$f^{(+)}_{(\vec{q},\vec{k})} = -2\pi^2 \frac{2m}{\hbar^2} \langle \vec{q}' | V | \psi^+_k \rangle$$

c)

In the first Born approximation,

$$\sigma_T = \int d\Omega_{k'} \, | f^{(+)}(\vec{k},\vec{k}') |^2$$

$$= 4\pi^4 \left(\frac{2m}{\hbar^2}\right)^2 \int d\Omega_{k'} \, | V(\vec{k}' - \vec{k}) |^2 .$$

In the second Born approximation,

$$\mathrm{Im}\, f^{(+)}(\vec{k},\vec{k}) = -2\pi^2 \frac{2m}{\hbar^2} \times$$

$$\mathrm{Im} \langle \vec{k} | \hat{V} + \hat{V} G \hat{V} | \vec{k} \rangle$$

$$= -2\pi^2 \left(\frac{2m}{\hbar^2}\right)^2 \int d^3k' \, |V(\vec{k}-\vec{k}')|^2$$

$$\times \underbrace{\operatorname{Im} \frac{1}{k^2 - p^2 + i0_+}}_{-\pi\,\delta(k^2-p^2) \,=\, -\frac{\pi}{2k}\,\delta(k-p)}$$

$$= \frac{\pi^3}{k} \left(\frac{2m}{\hbar^2}\right)^2 k^2 \int d\Omega_{k'} \, |V(\vec{k}-\vec{k}')|^2$$

$$\Longrightarrow \quad \operatorname{Im} f^{(+)}(\vec{k},\vec{k}) = \frac{k}{4\pi}\,\sigma_+ \,.$$

⑤

d)

$$f^{(+)}(\vec{k}^0,\vec{k}') = -2\pi^2 \frac{2m}{\hbar^2} \langle \vec{k} | V | \vec{k}' \rangle$$

$$= -2\pi^2 \frac{2m}{\hbar^2} \underbrace{\int d\vec{x}\, V(x)\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{I}$$

$$I = \int_0^\infty dr\, r^2 V(r) \int_0^{2\pi} d\phi \int_0^\pi d\theta\, e^{i|\vec{k}-\vec{k}'| r\cos\theta}$$

$$= 2\pi \int_0^\infty dr\, r^2 V(r) \int_{-1}^{1} du\, e^{i|\vec{k}-\vec{k}'| r u}$$

$$= 2\pi \int_0^\infty dr\, r^2 V(r)\, \frac{e^{i|\vec{k}-\vec{k}'| r} - e^{-i|\vec{k}-\vec{k}'| r}}{i|\vec{k}-\vec{k}'| r}$$

$$= \frac{2\pi}{i|\vec{k}-\vec{k}'|} \int_0^\infty dr\, r V(r) \left[e^{i|\vec{k}-\vec{k}'| r} - e^{-i|\vec{k}-\vec{k}'| r} \right.$$

$$\therefore\ f^{(+)}(\vec{k},\vec{k}') = -\frac{2\pi^2\, 2m}{\hbar^2} \frac{2\pi}{i|\vec{k}-\vec{k}'|} \times$$

⑥

$$\times \left[\frac{1}{(\mu + i|\vec{k}-\vec{k}'|)^2} - \frac{1}{(\mu - i|\vec{k}-\vec{k}'|)^2} \right]$$

$$= + \frac{2\pi^2}{\hbar^2} \frac{2\pi}{i\cancel{|\vec{k}-\vec{k}'|}} \times \frac{2i\mu\cancel{|\vec{k}-\vec{k}'|}}{\left[\mu^2 + |\vec{k}-\vec{k}'|^2\right]^2} \cdot 2m$$

$$= \frac{(2\pi)^3 2m}{\hbar^2} \frac{\mu}{\left[\mu^2 + |\vec{k}-\vec{k}'|^2\right]^2}.$$

The scattering cross section is:

$$|f^{(+)}(\vec{k}.\vec{k}')|^2.$$