Homework Assignment #7

Math Methods

Reading: Please read sections 1-9 of chapter 4. Reading Quiz #5 on this material will be due Monday before class. Much of the chapter deals with concepts in matrices that I believe you already understand. If this is not the case, be sure to let me know by posting to the discussion board and also providing a detailed answer to the quiz question about what material I should review.

Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Due: Tuesday, October 27th

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1. B & F Chapter 3, problem 1.
2. B & F Chapter 3, problem 9.
3. B & F Chapter 3, problem 12.
4. B & F Chapter 3, problem 17.
5. B & F Chapter 3, problem 26.
6. B & F Chapter 3, problem 28.

1. "∘" Closure: $p_k p_{k'} = e^{\frac{2\pi i k}{N}} e^{\frac{2\pi i k'}{N}}$

$= e^{2\pi i (k+k')/N} = p_{k''}$

If $k+k' < N$, it is clear $p_{k''}$ is in the group. If $k+k' > N$, then

$p_{k''} = e^{2\pi i (k+k')/N} e^{-2\pi i}$

$= e^{2\pi i (k+k'-N)/N}$

Then $k+k'-N < N$ & it is also clear $p_{k''}$ is in the group.

(1) Associativity:

$(p_k p_{k'}) p_{k''} = \left(e^{2\pi i k/N} e^{2\pi i k'/N}\right) e^{2\pi i k''/N}$

$= e^{2\pi i (k+k'+k'')/N}$

$p_k (p_{k'} p_{k''}) = e^{2\pi i k/N} \left(e^{\frac{2\pi i k'}{N}} e^{\frac{2\pi i k''}{N}} \right)$

$= e^{2\pi i (k+k'+k'')N}$

QED

(2) Identity: $e^{2\pi i 0} = 1 = p_0 =$ "e"

$p_k p_0 = p_0 p_k = p_k$ for all p_k

(3) Inverse: $(p_k)^{-1} p_k = p_0 \rightarrow (p_k)^{-1} = e^{-2\pi i k/N}$

$= e^{2\pi i (N-k)/N}$ which is in the group

(4) Abelian: $p_k p_{k'} = p_{k'} p_k$ for exponentials

But then we cannot express x_1 in terms of the other x_i. Thus the set of $\{x_i\}$ without x_1 cannot represent every vector in the space, Thus the set is no longer a basis.

9. a) $[A,B] = AB - BA = -(BA - AB)$

$$= -[B,A].$$

b) $[AB,C] = ABC - CAB$

$$= ABC - ACB + ACB - CAB$$

$$= A[B,C] + [A;C]B.$$

c) $[AB,C] = ABC - CAB$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B,C\} - \{A,C\}B.$$

d) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]]$

$$= A[B,C] - [B,C]A + B[C,A] - [C,A]B$$

$$+ C[A,B] - [A,B]C.$$

$$= ABC - ACB - BCA + CBA + BCA - BAC$$

$$- CAB + ACB + CAB - CBA - ABC + BAC = 0$$

#12

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can do each individual case

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

Then note that $\sigma_i^* = \sigma_i$. Then

$$(\sigma_1\sigma_2)^* = (i\sigma_3)^*$$

$$= \sigma_2^*\sigma_1^* = -i\sigma_3^*$$

$$\sigma_2\sigma_1 = -i\sigma_3$$

Therefore $\sigma_{ij} = -\sigma_{ij}$ if $i \neq j$. For the diagonal cases -

$$\sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore.

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

$$\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\,\epsilon_{ijk}\sigma_k.$$

17 $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\det(A - \lambda \mathbb{1}) = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$\lambda^2 - 2\lambda\cos\theta + 1$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

$\lambda = e^{i\theta}$

$$\begin{pmatrix} \cos\theta - e^{i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{i\theta} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

$$a\left[\tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) - e^{i\theta}\right] + b\,\tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = 0$$

$$a(e^{-i\theta} - e^{i\theta}) + b\,\tfrac{1}{i}(e^{i\theta} - e^{-i\theta}) = 0$$

$$-a + \tfrac{1}{i}b = 0$$

$$b = ia$$

eigen vector is $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ ia \end{pmatrix} = a\begin{pmatrix} 1 \\ i \end{pmatrix}$

choose $a = \frac{1}{\sqrt{2}}$ for normalization:

check: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} e^{i\theta}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\lambda = e^{-i\theta}$

$$\begin{pmatrix} \cos\theta - e^{-i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{-i\theta} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow a\left(e^{i\theta} - e^{i\theta}\right) + b\,\frac{1}{i}\left(e^{i\theta} - e^{-i\theta}\right) = 0.$$

$\therefore a = ib$

Eigenvector: $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ib \\ b \end{pmatrix} = b\begin{pmatrix} i \\ 1 \end{pmatrix}$.

choose $b = \frac{1}{\sqrt{2}}$ for normalization.

check! $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i\cos\theta + \sin\theta \\ -i\sin\theta + \cos\theta \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\, e^{-i\theta}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\mathbb{P}$ is made of eigenvectors —

$\mathbb{P} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ $\qquad \mathbb{P}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

check $\mathbb{P}\mathbb{P}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$$

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \cos\theta - i\sin\theta & i\cos\theta + \sin\theta \\ -\sin\theta + i\cos\theta & -i\sin\theta + \cos\theta \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

26 Show that: $(C+D)^{-1} = C^{-1} - C^{-1}D(C^{-1}+D)^{-1}$

Proof: the definition of $(C+D)^{-1}$ is that

$$(C+D)^{-1}(C+D) = 1$$

Let's see if it works:

$$\left[C^{-1} - C^{-1}D(C+D)^{-1}\right](C+D)$$

$$= \underline{\mathbb{1}} + C^{-1}D - C^{-1}D\,\underline{\mathbb{1}} = \underline{\mathbb{1}}$$

Note we must be careful about left & right inverses in infinite dimensional vector spaces —

28)

$$T = 1 + \frac{xD}{1!} + \frac{(xD)^2}{2!} \ldots$$

where $D = \frac{d}{dt}$

$$T f(t) = f(t) + x f'(t) + \frac{x^2}{2!} f'' \ldots$$

$$= f(t+x)$$

This is just the Taylor series expansion —