Homogeneous Bose gas

bosons of mass m in box with volume L^3

Single-particle ground state energy $\varepsilon_0 = \frac{\pi^2\hbar^2}{2mL^2}(1^2+1^2+1^2)$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$$

N particles: $E_0 = N\varepsilon_0$.

I am shifting the energy scale such that $\varepsilon_0 \to \infty$ becomes $0 \to \infty$.

$$\langle N \rangle = \sum_{\vec{\varepsilon}} \frac{1}{e^{\beta(\varepsilon_{\vec{\varepsilon}}-\mu)}-1} \longrightarrow \int_0^\infty n(\varepsilon)\, \mathcal{D}(\varepsilon)\, d\varepsilon$$

grand canonical ensemble

$$= \frac{V(2m)^{3/2}}{4\pi^2\hbar^3} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}-1}\, d\varepsilon$$

plugging in density of states and distribution fct.

$$\boxed{\mathcal{D}(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} = \frac{(2mL^2)^{3/2}}{4\pi^2\hbar^3}\sqrt{\varepsilon}}$$

see p.147 of notes

Mean density ρ: $\rho = \frac{\langle N \rangle}{V}$

$$\rho = \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}-1}\, d\varepsilon$$

let $\beta\varepsilon = x \quad \Rightarrow \quad dx = \beta\, d\varepsilon$

$$\varepsilon^{1/2} = \beta^{-1/2} x^{1/2}$$

$$\Rightarrow \rho = \underbrace{\frac{(2m\, k_B T)^{3/2}}{4\pi \hbar^3}}_{\text{prefactor}} \int_0^\infty \underbrace{\frac{x^{1/2}}{e^x e^{-\beta\mu} - 1}}_{\text{integrand}} dx$$

we are in the homogeneous phase

⇝ density is "just" a constant

We "want" that the density is constant as the temperature changes.

For bosons : μ is negative

idea: look at expression for ρ to see if we can derive some "general arguments"

as T changes, prefactor changes.

to keep the same ρ, μ also needs to change such that the change of the integral compensates for the change of prefactor.

- integral is maximal for $\mu = 0$

→ $|\mu|$ decreases as T decreases

→ once $\mu = 0$, we have a problem since it cannot go negative

→ Thus, T cannot become smaller since μ "cannot adjust" to keep the density fixed.

→ As $\mu \to 0$, s.th. must happen ...

$$\underbrace{\rho\Big|_{\mu=0}}_{\text{but this is just our density}} = \frac{(2mk_BT)^{3/2}}{4\pi^2\hbar^3} \underbrace{\int_0^\infty \frac{x^{1/2}}{e^x - 1}\,dx}_{2.612\,\frac{\pi^{1/2}}{2}}$$

$$\Rightarrow \rho = \frac{(2m k_B T)^{3/2}}{4\pi^2\hbar^3}\, 2.612\, \frac{\pi^{1/2}}{2}$$

$$\Rightarrow k_BT\Big|_{\mu=0} = \frac{1}{2.612^{2/3}}\, \frac{2\pi\hbar^2}{m}\, \rho^{2/3} \quad (**)$$

we call this k_BT_c

What is going on below $T < T_c$?

Intuitively, there should be no problem with going to very low temperatures.

It turns out that the expression for $\langle N \rangle$ has a problem at very low T:

$$\langle N \rangle = \frac{V(2m)^{3/2}}{4\pi^2 \hbar^3} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}-1}\, d\varepsilon \quad (*)$$

as $T \to 0$, we expect that the small ε portion of the integral is important

The problem is that the integrand contains a factor of $\sqrt{\varepsilon}$, i.e., as $\varepsilon \to 0$, we don't get a contribution (but we should).

So, the expression for $\langle N \rangle$ must be wrong as $T \to 0$.

We know: $\langle N_0 \rangle = \dfrac{1}{e^{-\beta\mu}-1}$ (this uses $\varepsilon_0 = 0$)

number of atoms in ground state

$$\approx \frac{1}{1-\beta\mu-1} = -\frac{kT}{\mu} = \frac{kT}{|\mu|} \gg 1$$

Taylor expand $|\beta\mu|$ small

μ negative

So: for $T < T_c$, we have $\mu = 0$

$$\Rightarrow \langle N_{exc} \rangle = \frac{V(2m)^{3/2}}{4\pi^2\hbar^3} \underbrace{\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta\varepsilon}-1}\, d\varepsilon}_{\beta^{-3/2}\int_0^\infty \frac{x^{1/2}}{e^x-1}dx \,=\, \beta^{-3/2}\, 2.612 \frac{\sqrt{\pi}}{2}}$$

number of atoms not in the ground state (number of atoms in excited states)

Eq. (*) with $\mu = 0$

$$= \langle N \rangle \left(\frac{T}{T_c}\right)^{3/2}$$

the integral can be evaluated (T_c defined in (**))

$$\boxed{\left(\frac{T}{T_c}\right)^{3/2} = \frac{\beta^{-3/2}}{\left(\frac{2\pi\hbar^2\rho^{2/3}}{2.612^{2/3}\, m}\right)^{3/2}}}$$

$$\Rightarrow \langle N \rangle = \langle N_0 \rangle + \langle N_{exc} \rangle$$

$$\text{or} \quad \boxed{\langle N_0 \rangle = \langle N \rangle \left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right) \quad \text{for } T < T_c}$$