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Quantum Mechanics 2

PHYS 5403 Homework Assignment 5

Problems: {1, 2, 3}

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Problem 1:

A one dimensional quantum oscillator with frequency ω has the unperturbed Hamiltonian

$$\mathcal{H} = \hbar\omega\left(\tilde{\mathbf{a}}^\dagger\tilde{\mathbf{a}} + \frac{1}{2}\right)$$

where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}^\dagger$ are creation and annihilation operators. At $t = 0$, we turn on a time dependent

$$V(t) = \lambda\Big[f(t)\tilde{\mathbf{a}} + f^*(t)\tilde{\mathbf{a}}^\dagger\Big],$$

where $f(t)$ is some integrable function, such that $f(t \to \infty) = 0$.

(a) Find the time dependence of the creation and annihilation operators in the interaction picture. Use the fact that

$$e^{-\tilde{\mathbf{B}}}\tilde{\mathbf{A}}e^{B} = \sum_{n=0}^{\infty}\frac{1}{n!}[\tilde{\mathbf{A}},\tilde{\mathbf{B}}]_n = \tilde{\mathbf{A}} + [\tilde{\mathbf{A}}.\tilde{\mathbf{B}}] + \frac{1}{2!}[[\tilde{\mathbf{A}},\tilde{\mathbf{B}}],\tilde{\mathbf{B}}] + ...$$

where $[\tilde{\mathbf{A}},\tilde{\mathbf{B}}]_{n+1} \equiv [[\tilde{\mathbf{A}},\tilde{\mathbf{B}}]_n,\tilde{\mathbf{B}}]$ and $[\tilde{\mathbf{A}},\tilde{\mathbf{B}}]_0 \equiv \tilde{\mathbf{A}}$. This relation is known as the Baker-Hausdorff identity.

To calculate the time dependence of these operators we use

$$\tilde{a}^+(t) = \exp(-i\tilde{\mathcal{H}}_0 t/\hbar)\,\tilde{a}^+(t=0)\exp(i\tilde{\mathcal{H}}_0 t/\hbar)$$

$$\tilde{a}(t) = \exp(-i\tilde{\mathcal{H}}_0 t/\hbar)\,\tilde{a}(t=0)\exp(i\tilde{\mathcal{H}}_0 t/\hbar)$$

Where $\mathcal{H}_0$ is our unperturbed Hamiltonian that is given to us. If we then use the Baker Haus-dorff identity for each operator we have

$$\tilde{a}^+(t) = \sum_{n=0}\frac{1}{n!}[\tilde{a}^+, i\tilde{\mathcal{H}}_0 t/\hbar]_n\,,\quad \tilde{a}(t) = \sum_{n=0}\frac{1}{n!}[\tilde{a}, i\tilde{\mathcal{H}}_0 t/\hbar]_n\,. \qquad (*)$$

Before we start taking commutation relations recall that

$$[\tilde{a},\tilde{a}^+] = \tilde{a}\tilde{a}^+ - \tilde{a}^+\tilde{a} = \tilde{\mathbb{1}}.$$

Expanding $(*)$ we then have

$$\tilde{a}^+(t) = [\tilde{a}^+, i\tilde{\mathcal{H}}_0 t/\hbar] = \frac{it}{\hbar}(\tilde{a}^+\tilde{\mathcal{H}}_0 - \tilde{\mathcal{H}}_0\tilde{a}^+)$$

$$= i\omega t\left(\tilde{a}^+\left(\tilde{a}^+\tilde{a} + \frac{\tilde{\mathbb{1}}}{2}\right) - \left(\tilde{a}^+\tilde{a} + \frac{\tilde{\mathbb{1}}}{2}\right)\tilde{a}^+\right)$$

$$= i\omega t\left(\tilde{a}^+\tilde{a}^+\tilde{a} + \cancel{\frac{\tilde{a}^+}{2}} - \tilde{a}^+\tilde{a}\tilde{a}^+ - \cancel{\frac{\tilde{a}^+}{2}}\right) = i\omega t(\tilde{a}^+\tilde{a}^+\tilde{a} - \tilde{a}^+\tilde{a}\tilde{a}^+)$$

$$= i\omega t(\tilde{a}^+(\tilde{a}\tilde{a}^+ - \tilde{\mathbb{1}}) - \tilde{a}^+\tilde{a}\tilde{a}^+) = i\omega t(\tilde{a}^+\cancel{\tilde{a}\tilde{a}^+} - \tilde{a}^+ - \tilde{a}^+\cancel{\tilde{a}\tilde{a}^+}) = -i\omega t\,\tilde{a}^+$$

Problem 1: Continued

We can then say that if want to extend this to n powers we would have

$$\tilde{a}^{+}(t) = \sum_{n=0} (-i\omega t)^n \, \tilde{a}^{+} = \tilde{a}^{+} \sum_{n=0} (-i\omega t)^n = \tilde{a}^{+} \exp(-i\omega t)$$

This then means our time dependent creation operator is

$$\tilde{a}^{+}(t) = \tilde{a}^{+} \exp(-i\omega t).$$

Since the annihilation operator is the conjugate of the creation operator, we can say that the annihilation operator will be

$$\tilde{a}(t) = \tilde{a} \exp(i\omega t).$$

This then means the representations of both will be

$$\boxed{\tilde{a}^{+}(t) = \tilde{a}^{+} \exp(-i\omega t) \;,\; \tilde{a}(t) = \tilde{a} \exp(i\omega t)}$$

(b) At $t = 0$ the quantum oscillator is in the ground state $|0\rangle$. Using leading order of perturbation theory, find the probability for the transition $|0\rangle \to |n\rangle$ at $t \to \infty$ for $n = 1$ and 2.

The transition probability for going from $|0\rangle \to |n\rangle$ is

$$P_n(t) = |\langle n | \tilde{u}(t\to\infty, 0) | 0 \rangle|^2 = |\langle n | \tilde{u}(t\to\infty,0) | 0 \rangle \langle 0 | \tilde{u}^{+}(t\to\infty,0) | n \rangle|^2$$

$$= |c_n(t) \overset{1}{\cancel{\langle n | n \rangle}} c_n^{*}(t)|^2 = |c_n(t)|^2$$

We then finally have

$$P_n(t) = |c_n(t)|^2. \qquad (*)$$

$\tilde{u}(t\to\infty, 0)$ that is used to derive $(*)$ is calculated via

$$u(t\to\infty,0) = \sum_{n=0} \left(\frac{-i}{\hbar}\right)^n \int_0^{\infty} dt_1 \int_0^{t_1} dt_2 \int_0^{t_{n-1}} dt_n \, u(t_1) u(t_2) \dots u(t_n).$$

$c_n(t)$ from $(*)$ is calculated with

$$c_n(t) = \left(\frac{-i}{\hbar}\right)^n \int_0^{\infty} \langle n | u(t') | 0 \rangle \, dt' \qquad (**)$$

Problem 1: Continued

where $u(t')$ in $(**)$ is $\exp(i\mathcal{H}_0 t'/\hbar)\,u(t)\exp(-i\mathcal{H}_0 t'/\hbar)$. $u(t)$ is conversely defined as

$$u(t) = \lambda\left(f(t)\tilde{\alpha} + f^*(t)\tilde{\alpha}^+\right).$$

We then go on to calculate $C_1(t)$ with this

$$C_1(t) = \left(\frac{-i}{\hbar}\right)^1 \int_0^\infty \langle 1|\exp(i\mathcal{H}_0 t'/\hbar)\,\lambda\left(f(t)\tilde{\alpha} + f^*(t)\tilde{\alpha}^+\right)\exp(-i\mathcal{H}_0 t'/\hbar)|0\rangle\, dt'$$

$$= \frac{-i}{\hbar}(\lambda)\int_0^\infty \langle 1|\exp(iE_1 t'/\hbar)\left(f(t)\tilde{\alpha} + f^*(t)\tilde{\alpha}^+\right)\exp(-iE_0 t'/\hbar)|0\rangle\, dt'$$

$$= \frac{-i\lambda}{\hbar}\int_0^\infty \exp(it'(E_1-E_0)/\hbar)\langle 1|f(t)\tilde{\alpha} + f^*(t)\tilde{\alpha}^+|0\rangle\, dt' \qquad (***)$$

we now just look at the bra-ket portion of $(***)$

$$\langle 1|f(t)\tilde{\alpha} + f^*(t)\tilde{\alpha}^+|0\rangle = f(t')\langle 1|\overbrace{\cancel{\tilde{\alpha}}|0\rangle}^{0} + f^*(t')\langle 1|\tilde{\alpha}^+|0\rangle = f^*(t')$$

where we then have a first order probability of

$$\boxed{P_1'(t) = \left(\frac{\lambda}{\hbar}\right)^2\left|\int_0^\infty f^*(t')\exp(it'(E_1-E_0)/\hbar)\, dt'\right|^2}$$

Now looking at when $n=2$, we have the relationship

$$C_{(j)}^{(i)} = \delta_{ij}.$$

This then means we look for $C_2^{(2)}(t)$ and this is of course

$$C_2^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2\int_0^\infty dt_1\int_0^{t_1} dt_2\,\langle 2|u_{21}(t_1)u_{10}(t_2)|0\rangle$$

$$= \frac{1}{\hbar^2}\int_0^\infty \langle 2|u_{21}(t_1)|1\rangle\, dt_1\int_0^{t_1}\langle 1|u_{10}(t_2)|0\rangle\, dt_2 \qquad (****)$$

with our potentials

$$u_{21}(t_1) = \lambda\left(f(t_1)\tilde{\alpha} + f^*(t_1)\tilde{\alpha}^+\right), \quad u_{10}(t_2) = \lambda\left(f(t_2)\tilde{\alpha} + f^*(t_2)\tilde{\alpha}^+\right)$$

Problem 1: Continued

Where when we put these into (****) we find

$$\langle 2|U_{21}(t_1)|1\rangle = \langle 2|\exp(i\mathcal{H}_0 t_1/\hbar)\lambda(f(t_1)\tilde{a} + f^*(t_1)\tilde{a}^\dagger)\exp(-i\mathcal{H}_0 t_1/\hbar)|1\rangle$$
$$= \lambda\exp(it_1(E_2-E_1)/\hbar)(\underbrace{\langle 2|\cancel{f(t_1)}\tilde{a}|1\rangle}_{0} + \langle 2|f^*(t_1)\tilde{a}^\dagger|1\rangle)$$
$$= \lambda\exp(it_1(E_2-E_1)/\hbar)\cdot\sqrt{2}\,f^*(t_1)$$

Conversely we then have

$$\langle 1|U_{10}(t_2)|0\rangle = \langle 1|\exp(i\mathcal{H}_0 t_2/\hbar)\lambda(f(t_2)\tilde{a} + f^*(t_2)\tilde{a}^\dagger)\exp(-i\mathcal{H}_0 t_2/\hbar)|0\rangle$$
$$= \lambda\exp(it_2(E_1-E_0)/\hbar)(\underbrace{\langle 1|\cancel{f(t_2)}\tilde{a}|0\rangle}_{0} + \langle 1|f^*(t_2)\tilde{a}^\dagger|0\rangle)$$
$$= \lambda\exp(it_2(E_1-E_0)/\hbar)\,f^*(t_2)$$

This then means we have a probability of

$$P_2^{(2)}(t) = \frac{1}{\hbar^4}\left|\sqrt{2}\lambda\int_0^\infty f^*(t_1)\exp(it_1(E_2-E_1)/\hbar)\,dt_1\cdot\lambda\int_0^{t_1} f^*(t_2)\exp(it_2(E_1-E_0)/\hbar)\,dt_2\right|^2$$
$$= 2\left(\frac{\lambda}{\hbar}\right)^4\left|\int_0^\infty f^*(t_1)\exp(it_1(E_2-E_1)/\hbar)\,dt_1\int_0^{t_1} f^*(t_2)\exp(it_2(E_1-E_0)/\hbar)\,dt_2\right|^2$$

This means our time dependent probability to leading order is

$$\boxed{P_2^{(2)}(t) = 2\left(\frac{\lambda}{\hbar}\right)^4\left|\int_0^\infty f^*(t_1)\exp(it_1(E_2-E_1)/\hbar)\,dt_1\int_0^{t_1} f^*(t_2)\exp(it_2(E_1-E_0)/\hbar)\,dt_2\right|^2}$$

(c) Suppose now that instead of the perturbed potential (1), we turn on a potential of the form:

$$V(t) = \lambda x^3 e^{-rt}$$

at $t = 0$ ($\tau > 0$). Find the transition probability to the third excited state, $|0\rangle \rightarrow |3\rangle$ in perturbation theory at $t \rightarrow \infty$.

The transition probability to the third excited state is calculated with

$$P_3^{(1)}(t) = \left|\left(\frac{-i}{\hbar}\right)\int_0^\infty dt\,\langle 3|U_{30}(t)|0\rangle\right|^2 \qquad (*)$$

Where we of course have

$$c_3^{(1)}(t) = \left(\frac{-i}{\hbar}\right)\int_0^\infty dt\,\langle 3|U_{30}(t)|0\rangle \qquad (**)$$

Problem 1: Continued

Where we have potentials of the form

$$u_{30}(t) = \lambda \tilde{x}^3 e^{-rt}$$

The position operator in terms of creation and annihilation operators is

$$\tilde{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\tilde{a} + \tilde{a}^{+})$$

Then $\tilde{x}^3$ is

$$\tilde{x}^3 = \left(\frac{\hbar}{2m\omega}\right)^{3/2} (\tilde{a}^3 + \tilde{a}\tilde{a}^{+}\tilde{a} + \tilde{a}^{+}\tilde{a}^2 + \tilde{a}^{+2}\tilde{a} + \tilde{a}^2\tilde{a}^{+} + \tilde{a}\tilde{a}^{+2} + \tilde{a}^{+}\tilde{a}\tilde{a}^{+} + \tilde{a}^{+3}).$$

We can see that when $\tilde{x}^3$ is applied on our state, the only non-zero term is $(\tilde{a}^{+})^3$. This means

$$C_3^{(1)}(t) = \left(\frac{-i}{\hbar}\right) \int_0^\infty dt \left(\frac{\hbar}{2m\omega}\right)^{3/2} \langle 3 | \tilde{a}^{+3} | 0 \rangle \lambda e^{-rt}$$

$$= \frac{-i}{\hbar} \left(\frac{\hbar}{2m\omega}\right)^{3/2} \sqrt{6}\, \lambda \int_0^t \exp(-3i\omega t' - rt')\, dt'$$

$$= \frac{-i\sqrt{6}\lambda}{\hbar} \left(\frac{\hbar}{2m\omega}\right)^{3/2} \left(\frac{1 - \exp(-(3i\omega + r)t)}{3\omega - ir}\right)$$

The probability is then

$$|C_3^{(1)}(t)|^2 = \frac{6\lambda^2}{\hbar^2} \cdot \frac{\hbar^3}{8m^3\omega^3} \left| \left(\frac{1 - \exp(-(3i\omega + r)t)}{3\omega - ir}\right) \right|^2 \longrightarrow \lim_{t \to \infty}$$

$$= \frac{3}{4} \frac{\lambda^2 \hbar}{m^3 \omega^3} \left| \frac{1}{3\omega - ir} \right|^2 = \frac{3}{4} \frac{\lambda^2 \hbar}{m^3 \omega^3} \frac{1}{9\omega^2 + r^2}$$

Our final time dependent transmission probability from $0 \to 3$ to first order is $(t \to \infty)$

$$\boxed{P_3^{(1)}(t \to \infty) = \frac{3}{4} \frac{\lambda^2 \hbar}{m^3 \omega^3} \frac{1}{9\omega^2 + r^2}}$$

Problem 1: Review

Procedure:

- – Begin by finding the time dependent creation and annihilation operators
  – Proceed to use the Baker Haus-dorff identity to find a series representation of these operators and then take a commutator of the two
  – Evaluate the commutator and find a time dependent expression for the operators
- – To calculate the transition probability we use
  $$P_n(t) = |\langle 0|U(t\to\infty,0)|n\rangle|^2 = |c_n(t)|^2$$
  where
  $$U(t\to\infty,0) = \sum_{n=0}\Big(-\frac{i}{\hbar}\Big)^n\int_0^\infty dt_1\int_0^{t_1}dt_2\int_0^{t_n-1}dt_n\,U(t_1)U(t_2)...U(t_n)$$
  and
  $$C_n^1(t) = \Big(-\frac{i}{\hbar}\Big)^n\int_0^\infty dt'\ \langle 0|U(t')|n\rangle$$
  – Write $U(t)$ in terms of creation and annihilation operators as
  $$U(t) = \lambda(f(t)\tilde{\mathbf{a}} + f^*(t)\tilde{\mathbf{a}}^\dagger)$$
  and calculate the transition probability from 0 to 1
  – Repeat the above for 0 to 2
- – Calculate the transition probability from 0 to 3 with this new time dependent potential
  – After evaluating $|C_3^{(1)}(t)|^2$ take the limit as $t\to\infty$

Key Concepts:

- – Using the standard time evolution formalism
  $$A(t) = e^{-i\mathcal{H}t/\hbar}A(t=0)e^{i\mathcal{H}t/\hbar}$$
  to find the time dependent creation and annihilation operators
- – For certain values C_j^i the quantity will be zero
  – We must make certain that we are integrating over the correct variables with the correct perturbed potential
- – We use this new perturbative potential with our formalism to find the transition probability from 0 to 3

Variations:

- – If the Hamiltonian changes this impacts what our perturbative potential is thus what our perturbation evaluates to
  * This would change what our operators would be but we would use the same formalism to find the time evolved state
- – We could be asked to look for a different transition between states
  * This would change what some of our integrals would evaluate to but it would be the same broad procedure
- – Again we could be asked to examine for a different transition
  * We then use the same procedure for this new transition

Problem 2:

Consider a system of three levels with the Hamiltonian

$$\begin{pmatrix} \epsilon_1 & 0 & \triangle(t) \\ 0 & \epsilon_2 & \triangle(t) \\ \triangle^*(t) & \triangle^*(t) & \epsilon_3 \end{pmatrix}$$

where

$$\triangle(t) = \triangle e^{i\omega t},$$

with $\triangle$ real. Find the transition probability between levels ϵ_1 and ϵ_2 in leading order of perturbation theory where the result is non-trivial, when $|\triangle(t)| << |\epsilon_i - \epsilon_j|$, with $i, j = 1, 2, 3$, and $i \neq j$. Interpret your result.

We first begin by finding the eigenvalues of our unperturbed Hamiltonian,

$$\det \begin{pmatrix} \epsilon_1-\lambda & 0 & 0 \\ 0 & \epsilon_2-\lambda & 0 \\ 0 & 0 & \epsilon_3-\lambda \end{pmatrix} = (\epsilon_1-\lambda)(\epsilon_2-\lambda)(\epsilon_3-\lambda) = 0 \Rightarrow \lambda = \epsilon_1, \epsilon_2, \epsilon_3$$

This then means we have eigenstates

$$\epsilon_1 \Rightarrow |1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_2 \Rightarrow |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \epsilon_3 \Rightarrow |3\rangle \doteq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can then show that going from energy level ϵ_1 to ϵ_2 to the first order is trivial with

$$c_n(t) = \left(\frac{-i}{\hbar}\right)^n \int_0^\infty \langle n | u(t') | 0 \rangle \, dt'$$

The transmission probability between energy level $1 \to 2$ to the first order is

$$c_2^{(1)}(t) = \left(\frac{-i}{\hbar}\right)^1 \int_0^t \langle 2 | \exp(iH t/\hbar) u(t) \exp(-iH t/\hbar) | 1 \rangle \, dt$$

$$= \frac{-i}{\hbar} \int_0^t \exp(it(\epsilon_2-\epsilon_1)/\hbar) \langle 2 | u(t) | 1 \rangle \, dt$$

$$= \frac{-i}{\hbar} \int_0^t \exp(it(\epsilon_2-\epsilon_1)/\hbar) \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dt$$

$$= \frac{-i}{\hbar} \int_0^t \exp(it(\epsilon_2-\epsilon_1)/\hbar) \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} dt = 0$$

Problem 2: Continued

Therefore we know $c_2^{(1)}(t) = 0$. We then move on to calculating the second order correction with

$$c_2^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \langle 2| \mathcal{U}'(t_1)\mathcal{U}'(t_2)|1\rangle$$

Where we then expand in the eigenstates of our potentials

$$c_2^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_{\xi=1}^{3} \int_0^t dt_1 \int_0^{t_1} dt_2 \langle 2| \mathcal{U}'(t_1)|\xi\rangle\langle \xi| \mathcal{U}'(t_2)|1\rangle$$

We then write our time-dependent perturbed potentials as

$$\mathcal{U}'(t_1) = \exp(i\mathcal{H}t_1/\hbar)\,\mathcal{U}(t_1)\exp(-i\mathcal{H}t_1/\hbar) \quad , \quad \mathcal{U}'(t_2) = \exp(i\mathcal{H}t_2/\hbar)\,\mathcal{U}(t_2)\exp(-i\mathcal{H}t_2/\hbar)$$

This then means $c_2^{(2)}(t)$ becomes

$$c_2^{(2)}(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_{\xi=1}^{3} \int_0^t dt_1 \int_0^{t_1} dt_2 \langle 2| e^{i\mathcal{H}t_1/\hbar}\mathcal{U}(t_1)e^{-i\mathcal{H}t_1/\hbar}|\xi\rangle\langle \xi| e^{i\mathcal{H}t_2/\hbar}\mathcal{U}(t_2)e^{-i\mathcal{H}t_2/\hbar}|1\rangle$$

$$= \left(\frac{-i}{\hbar}\right)^2 \sum_{\xi=1}^{3} \int_0^t dt_1 \int_0^{t_1} dt_2\, e^{it_1(\epsilon_2-\epsilon_3)/\hbar} \langle 2|\mathcal{U}(t_1)|\xi\rangle\langle \xi|\mathcal{U}(t_2)|1\rangle\, e^{-it_2(\epsilon_3-\epsilon_1)/\hbar}$$

Where we first calculate the matrix elements in the integral

$$\xi = 1 \Rightarrow \langle 2|\mathcal{U}(t_1)|1\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle 1|\mathcal{U}(t_2)|1\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\xi = 2 \Rightarrow \langle 2|\mathcal{U}(t_1)|2\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\langle 2|\mathcal{U}(t_2)|1\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

Problem 2: Continued

$$\xi = 3 \Rightarrow \langle 2|U(t_1)|3\rangle = (0\;1\;0)\begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \Delta(t_1)$$

$$\langle 3|U(t_2)|1\rangle = (1\;0\;0)\begin{pmatrix} 0 & 0 & \Delta(t) \\ 0 & 0 & \Delta(t) \\ \Delta^\dagger(t) & \Delta^\dagger(t) & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \Delta^\dagger(t_2)$$

This then means $C_2^{(2)}(t)$ is

$$C_2^{(2)}(t) = \left(\frac{-i\lambda}{\hbar}\right)^2 \Delta^2 \int_0^t e^{i\omega t_1} e^{it_1(\epsilon_2-\epsilon_3)/\hbar}\, dt_1 \int_0^{t_1} e^{-i\omega t_2} e^{-it_2(\epsilon_3-\epsilon_1)/\hbar}\, dt_2$$

Let us fist make a notational change

$$\epsilon_{31} \doteq (\epsilon_3-\epsilon_1)/\hbar \quad , \quad \epsilon_{23} \doteq (\epsilon_2-\epsilon_3)/\hbar$$

$C_2^{(2)}(t)$ is then

$$C_2^{(2)}(t) = \left(\frac{-i\lambda}{\hbar}\right)^2 \Delta^2 \int_0^t e^{it_1(\omega+\epsilon_{23})} dt_1 \int_0^{t_1} e^{-it_2(\omega+\epsilon_{31})} dt_2$$

$$= \left(\frac{-i\lambda}{\hbar}\right)^2 \Delta^2 \int_0^t e^{it_1(\omega+\epsilon_{23})} dt_1 \left(\frac{e^{-it_2(\omega+\epsilon_{31})}}{-i(\omega+\epsilon_{31})} \Big|_0^{t_1} \right)$$

$$= \left(\frac{-i\lambda}{\hbar}\right)^2 \Delta^2 \frac{i}{(\omega+\epsilon_{31})} \int_0^t e^{it_1(\omega+\epsilon_{23})} \left(e^{-it_1(\omega+\epsilon_{31})} - 1 \right) dt$$

$$= \left(\frac{-i\lambda}{\hbar}\right)^2 \Delta^2 \frac{i}{(\omega+\epsilon_{31})} \int_0^t e^{-it_1(\epsilon_{23}-\epsilon_{31})} - e^{it_1(\omega+\epsilon_{23})}\, dt$$

$$= \left(\frac{-i\lambda}{\hbar}\right)^2 \cancelto{0}{\Delta^2} \frac{i}{(\omega+\epsilon_{31})} \left(\cancel{\frac{ie^{t_1(\epsilon_{23}-\epsilon_{31})}}{\epsilon_{23}-\epsilon_{31}}} + \frac{ie^{it_1(\omega+\epsilon_{23})}}{(\omega+\epsilon_{23})} \right)\Big|_0^t$$

$$= \left(\frac{\lambda}{\hbar}\right)^2 \frac{1}{\omega+\epsilon_{31}} \frac{(e^{it(\omega+\epsilon_{23})} - 1)}{(\omega+\epsilon_{23})}$$

The transmission probability is then

$$P_2^{(2)}(t) = \left(\left(\frac{\lambda}{\hbar}\right)^2 \frac{1}{(\omega+\epsilon_{31})(\omega+\epsilon_{23})} \right)^2 \left(e^{it(\omega+\epsilon_{23})} - 1 \right)\left(e^{-it(\omega+\epsilon_{23})} - 1 \right)$$

Problem 2: Continued

$$= \left(\left(\frac{\lambda}{\hbar} \right)^2 \frac{1}{(\omega + \epsilon_{31})(\omega + \epsilon_{23})} \right)^2 \sqrt{2} \left(1 - \cos^2 (t(\omega + \epsilon_{23})) \right)$$

$$= \left(\left(\frac{\lambda}{\hbar} \right)^2 \frac{2 \cdot \sin (t(\omega + \epsilon_{23}))}{(\omega + \epsilon_{31})(\omega + \epsilon_{23})} \right)^2 \quad \text{w/ } \epsilon_{ij} = \frac{\epsilon_i - \epsilon_j}{\hbar}$$

This means our time dependent transmission probability from 1 → 2 is

$$\boxed{P_2^{(2)}(t) = \left(\left(\frac{\lambda}{\hbar} \right)^2 \frac{2 \cdot \sin ((\omega + \epsilon_{23}) t)}{(\omega + \epsilon_{31})(\omega + \epsilon_{23})} \right)^2}$$

Problem 2: Review

Procedure:

-
  - Calculate the eigenvalues of the unperturbed Hamiltonian
  - Proceed to calculate the transition probability between states 1 to 2 by using

    $$C_2^1(t) = \left(\frac{-i}{\hbar}\right)\int_0^t \langle 1|U(t')|2\rangle \; dt'$$

    and then $C_2^2(t)$ with

    $$C_2^2(t) = \left(\frac{-i}{\hbar}\right)^2 \sum_{\xi=1}^{3} \int_0^t dt_1 \int_0^{t_1} dt_2 \; \langle 1|U'(t_2)\,|\xi\rangle\,\langle\xi|\,U'(t_1)|2\rangle$$

  - Proceed to calculate the matrix elements of the above transition to simplify the above expression
  - Proceed to finish calculating $C_2^2(t)$ and then the probability with

    $$P(t) = |C_2^2(t)|^2$$

Key Concepts:

-
  - Using the potential given to us we can find the transition probability by calculating each separate order of the correction
  - The leading order of this correction is the second order because the first order correction is zero
  - In time dependent perturbation theory, we have to find a time dependent potential in order to correctly find our perturbation
  - Leading order indirectly means find the first non zero correction to the state or energy

Variations:

-
  - With a different Hamiltonian we then have different eigenvalues and eigenstates
    * We then use the same broad procedure but with our new Hamiltonian

Problem 3:

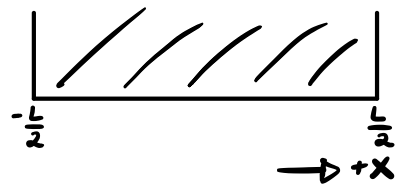
A non-relativistic electron with energy dispersion

$$E_k = \frac{k^2}{2m}$$

is confined to a 1-dimensional square cavity of size L centered at $x = 0$.

(a) Write the wavefunctions of the particle in the box and their corresponding energy levels.

$$V = \begin{cases} 0, & |x| \leq L/2 \\ \infty, & \text{else} \end{cases}$$



With the given boundary conditions we know our wave function is

$$\psi_n(x) = \frac{1}{\sqrt{2L}}\left(e^{in\pi x/L} + (-1)^{n+1} e^{-in\pi x/L}\right)$$

Solving Schrödingers equation we then have

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

We then have,

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{2L}}\left(e^{in\pi x/L} + (-1)^{n+1} e^{-in\pi x/L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

(b) If the system is perturbed by a weak electric field $\mathcal{E}_0$ with potential $V(x) = -\mathcal{E}_0 x$, calculate the first non zero correction to the energy of the ground state. Hint: use the fact that:

$$\sum_{n=1}^{\infty}\left[\frac{1}{(4n^2-1)^3} + \frac{4}{(4n^2-1)^4} + \frac{4}{(4n^2-1)^5}\right] = \frac{1}{2} - \frac{\pi^2}{64}\left(\frac{7}{4} + \frac{\pi^2}{12}\right).$$

We calculate the first order correction with

$$E_m^{(1)} = \langle\psi_m|V|\psi_m\rangle$$

Which for us is

$$E_m^{(1)} = -e\mathcal{E}_0\int_{-L/2}^{L/2} x|\psi_m(x)|^2 dx = 0$$

Due to symmetry. The second order correction to the energy is then

$$E_m^{(2)} = \sum_{l \neq m} \frac{|\langle\psi_m|V|\psi_l\rangle|^2}{E_m^0 - E_l^0}$$

Problem 3: Continued

This then becomes for us

$$E_m^{(2)} = \frac{2m}{\hbar^2}\left(\frac{L}{\pi}\right)^2 e^2 \mathcal{E}^2 \sum_{\ell \neq m} \frac{|\langle \psi_m | x | \psi_\ell \rangle|^2}{m^2 - \ell^2}$$

we know that when $m=1$ $E_1^{(2)} = 0$ for $\ell = 1$ $\therefore$ for odd ℓ other than 1 we have

$$\langle \psi_1 | x | \psi_{2\nu} \rangle = \frac{4iL}{\pi^2}(-1)^\nu \left[\frac{1}{4\nu^2-1} + \frac{2}{(4\nu^2-1)^2}\right]$$

we then know

$$E_1^{(2)} = \frac{32m}{\hbar^2}\left(\frac{L}{\pi}\right)^4 \frac{e^2\mathcal{E}^2}{\pi^2} \sum_{\ell=1}^{\infty} \frac{1}{(4\nu^2-1)^2} + \frac{4}{(4\nu^2-1)^3} + \frac{4}{(4\nu^2-1)^4} \qquad w/\ \ell = 2\nu+1$$

$$= -\frac{32m}{\hbar^2}\left(\frac{L}{\pi}\right)^4 \frac{e^2\mathcal{E}^2}{\pi^2}\left(\frac{1}{2} - \frac{21\pi^2}{768} - \frac{\pi^4}{768}\right)$$

So the second order correction is then

$$\boxed{E_1^{(2)} = -\frac{32m}{\hbar^2}\left(\frac{L}{\pi}\right)^4 \frac{e^2\mathcal{E}^2}{\pi^2}\left(\frac{1}{2} - \frac{21\pi^2}{768} - \frac{\pi^4}{768}\right)}$$

(c) Using your result in b), find the corresponding correction to the ground state ket. Assume now that the particle is prepared in that state. Find the probability of measuring the particle in the first excited state of the unperturbed system.

To calculate the leading correction to $|\psi_1\rangle$ we use

$$|\psi_1^{(1)}\rangle = \sum_{\nu=1} \frac{\langle \psi_{2\nu} | V | \psi_1 \rangle}{E_{2\nu} - E_1} |\psi_{2\nu}\rangle$$

For us this becomes

$$|\psi_1^{(1)}\rangle = -\frac{2m}{\hbar^2}\left(\frac{L}{\pi}\right)^2 \cdot e\mathcal{E} \sum_{\nu=1}^{\infty} \frac{\langle \psi_{2\nu} | x | \psi_1 \rangle}{1 - 4\nu^2} |\psi_{2\nu}\rangle$$

$$= \frac{8im}{\hbar^2}\frac{L^3}{\pi^4} e\mathcal{E} \times \sum_{\nu=1}^{\infty} (-1)^\nu \left[\frac{1}{(4\nu^2-1)^2} + \frac{1}{(4\nu^2-1)^3}\right] |\psi_{2\nu}\rangle$$

We then find the probability of finding our particle in the first $\ell \neq 1$ but odd value $(\nu = 1)$ we find

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$$|\langle \psi_2|\tilde{\psi}_1\rangle|^2 = |\langle \psi_2|\psi_1^{(1)}\rangle|^2 = \frac{25}{9^3}\cdot 8^2\left(\frac{mL^3}{\hbar^2\pi^4}\right)^2$$

Our probability is then

$$\boxed{P = \frac{25}{9^3}\cdot 8^2\left(\frac{mL^3}{\hbar^2\pi^4}\right)^2}$$

(d) Suppose now the electric field is time dependent,

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-t/r},$$

and is turned on at $t = 0$ ($\tau > 0$). If the particle is in the ground state $t < 0$, find the probability of a transition to the first excited level at times $t >> \tau$.

Our time dependent potential will look like

$$u(x,t) = \mathcal{E}_0 e^{-t/r} x$$

We now calculate $C_n(t)$ with

$$C_2^1(t) = \left(\frac{-i}{\hbar}\right)^1 \int_0^\infty \langle 2|u(t')|1\rangle \, dt'$$

From states $1 \to 2$ this

$$C_2^1(t) = \frac{-i}{\hbar}\int_0^t \langle 2|\exp(iH^0t'/\hbar)\cdot e\mathcal{E}_0 e^{-t'/r} x \exp(-iH^0t'/\hbar)|1\rangle \, dt'$$

$$= \frac{-ie\mathcal{E}_0}{\hbar}\int_0^t e^{it'(\omega_2-\omega_1)} e^{-t'/r}\langle 2|x|1\rangle \, dt'$$

$$= \frac{-ie\mathcal{E}_0}{\hbar}\int_0^t e^{it'(\omega_2-\omega_1)} e^{-t'/r}\langle 2|x|1\rangle \, dt'$$

$$= \frac{-ie\mathcal{E}_0}{\hbar}\int_0^t e^{t'(i\omega_{21}-1/r)}\langle 2|x|1\rangle \, dt' = \frac{-ie\mathcal{E}_0}{\hbar}\frac{1}{i\omega_{21}-1/r}\langle 2|x|1\rangle$$

This means the probability is

$$P_2^1(t) = |C_2(t)|^2 = \frac{e^2\mathcal{E}_0^2}{\hbar^2}\frac{r^2}{(\omega_{21}r)^2+1}|\langle 2|x|1\rangle|^2$$

Problem 3: Continued

Finally we have

$$\boxed{P = \frac{16(11)^2}{9^2} \times \frac{L^2}{\pi^4} \frac{\tau^2}{(3\hbar\pi^2/2mL^2)\tau^2+1}}$$

Problem 3: Review

Procedure:

- – Simply look up the wave function for this particle in a box and then write the corresponding energies
- – Calculate the first non zero correction (Ends up being second order in this case) to the eigenenergy
  – Use the identity given to us to simplify our expression
- – Calculate the first order correction to the groundstate and then the probability of the first excited state being found in the aforementioned state with
$$P = |\langle \psi_2^0 | \psi_1^1 \rangle|^2$$
- – Proceed to calculate the probability of the transition with this new time dependent electric field

Key Concepts:

- – This is a simple reference to a particle in a box problem
- – We are required to calculate the second order correction due to the first order correction being zero
- – We only take into account odd numbers of l due to the even ones being zero
  – We are looking at the probability of finding the first excited state in our first order correction to our ground state
- – We use our time dependent potential to find the transition probability of going from 1 to 2

Variations:

- – We could examine this problem in more dimensions
  * This severely complicates the problem but it would be the same procedure with new ground state wave functions and energies
- – For parts (b) and (c) the only real change that can happen here is having a different potential
  * We then use the same procedure as before with a new potential
- – We could be given a different time dependent electric field
  * This changes what our math evaluates to but not the broad procedure we use