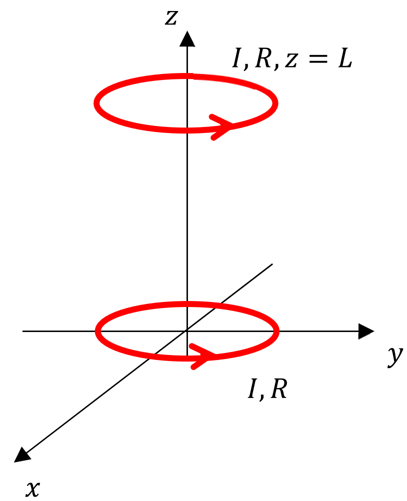
E & M I

Workshop 6 – Helmholtz Coils, 3/7/2022

We have done several calculations of magnetic fields, but we haven't really applied our results. That's the goal of today's workshop.



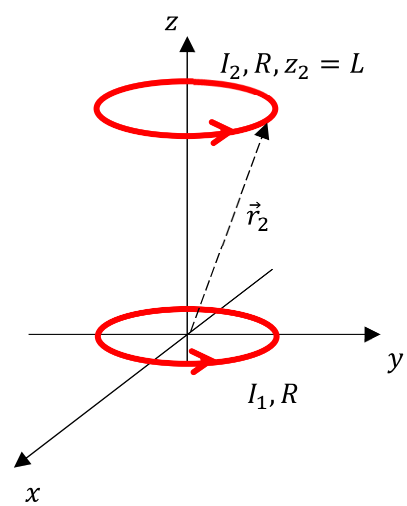
1) Helmholtz coils are used to create reasonably uniform magnetic fields in labs, or sometimes to cancel out external unwanted fields. This consists of two current-carrying loops (or multiple loops), parallel to each other and centered on the same axis.

In this case, consider loop of radius R in the x-y plane, centered at the origin, and carrying a current I, and a second, identical loop centered on the z-axis at $z = L$.

a) Solve for (or write down) an expression for the total magnetic field of the two loops on the z-axis, $B_z(z)$, for $0 \leq z \leq L$.

Note: Qualifier Practice – We have done this (several different ways) so you can look it up, but this is the type of thing you will be expected to do quickly and efficiently on a qualifier. If you look up the answer, you might think about how to prepare for doing this on a test.

b) For experiments a uniform field is sometimes desired. Solve for the value of z where $\partial_z B_z = 0$. The answer should be obvious but show how you got your answer and explain the results.



2) In designing the Helmholtz coil pair, we need to be careful that the structure is mechanically stable. The two coils will exert a magnetic force on each other. Here we'll approximate the magnetic force on the top ring due to the magnetic field of the bottom ring. We'll distinguish the rings by labeling the bottom ring "1" and the top ring "2".

a) Write an integral that will give the force on the top ring, current I_2 and positions of the ring $\vec{r}_2$, due to the magnetic field of the bottom ring, $\vec{B}_1$.

Your answer should include something that looks like $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$

Doing this integral exactly is rather messy, so let's make the approximation that the field due to the bottom ring can be replaced by the magnetic dipole approximation. Remember that the field and vector potential of a magnetic dipole are:

$$\vec{B}_1(\vec{r}_2) = \frac{\mu_0}{4\pi}\left(3\frac{(\vec{m}_1 \cdot \hat{r}_2)\,\hat{r}_2}{r_2^3} - \frac{\vec{m}_1}{r_2^3}\right), \quad \vec{A}_1(\vec{r}_2) = \frac{\mu_0}{4\pi}\frac{\vec{m}_1 \times \hat{r}_2}{r_2^2}, \quad \vec{m}_1 = I_1\,\pi R^2\,\hat{z}$$

b) As we have seen in similar problems, we need to be careful with the vector nature of our integrand when doing integrals such as in part (a). This is due, in part, to the fact that the directions of spherical coordinates $\hat{r}, \hat{\theta}$, and $\hat{\phi}$ all change for different positions $\vec{r}$. There are a couple of ways to handle this.

(i) Write the cross product, $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$ in terms of the Cartesian unit vectors of $\hat{x}, \hat{y}$, and $\hat{z}$ and do the cross product. This should be a function of r_2, θ_2, ϕ_2.

(ii) Write the cross product, $\hat{\phi}_2 \times \vec{B}_1(\vec{r}_2)$ in terms of the spherical unit vectors of $\hat{r}_2, \hat{\theta}_2$, and $\hat{\phi}_2$ and do the cross product. This again should be a function of r_2, θ_2, ϕ_2.

Hint: What are $\hat{r}_2 = \frac{\vec{r}_2}{r_2}$ and $\hat{\phi}_2$ in terms and the angles θ_2 and ϕ_2? What is $\hat{z}$ in terms of the directions $\hat{r}_2$ and $\hat{\theta}_2$?

c) Solve for the force on the top loop and show that it will only have a z-component. You should be able to use either of the expressions for the integrand found in part (b).

d) An even simpler model for this problem is to treat BOTH loops as dipoles. In this case, the potential energy due to the interaction between the dipoles is:

$$U = -\vec{m}_2 \cdot \vec{B}_1(\vec{r}_2)$$

Using this approximation, explain the physics behind the direction of the force found in part (c) makes physical sense.

(Note: After class you might want to calculate the force on the top loop using the dipole-dipole approximation and compare it to the result from part (c).)

3) It is also of interest to understand the energy of the Helmholtz coil pair. We have found a couple of different ways to calculate the magnetic energy, including the interaction between a current density and vector potential:

$$U_{21} = \iiint \vec{J_2}(\vec{r}_2) \cdot \vec{A}_1(\vec{r}_2)\, d^3 r_2$$

Note: This is the energy the top loop current due to the bottom loop, so there isn't a factor of $\frac{1}{2}$ out front which is there to avoid double-counting interactions.

Using the same dipole approximation as above for the bottom loop, calculate the interaction potential energy above. Consider how the steps outlined above in Question 2 can be used in this case.