E & M I

Workshop 2 – Multipole Expansions, 2/7/2022

Last week in class (zoom) and the reading we looked at the concept of multipole expansions of the electric potential in regions outside of a charge distribution. This is, basically, expanding the potential in powers of r, the distance from the charges to the point where the potential is being calculated.

We can write the potential as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_q \frac{q}{\left|\vec{r}-\vec{r}_q\right|}\,, \qquad \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r_q\,\frac{\rho(\vec{r}_q)}{\left|\vec{r}-\vec{r}_q\right|}$$

The first being for point charges, the second being for continuous charge distributions.

The important result here is that:

$$\nabla^2\,\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = 4\,\pi\,\delta^3(\vec{r}-\vec{r}_q)$$

Doing the series expansion for $r \gg r_q$ gives:

$$\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = \sum_n (-1)^n\left(\vec{r}_q\cdot\nabla\right)^n\frac{1}{r}$$

The first three terms are:

$$\frac{1}{\left|\vec{r}-\vec{r}_q\right|} = \frac{1}{r} + \frac{x_{qi}\,x_i}{r^3} + \frac{1}{2}\left(3\,\frac{\left(x_{qi}\,x_i\right)\left(x_{qj}\,x_j\right)}{r^5} - \frac{r_q^2}{r^3}\right) + O\left(\frac{1}{r^4}\right)$$

In writing this, x_i are the Cartesian coordinates of $\vec{r}$, x_{qi} are the Cartesian Coordinates of $\vec{r}_q$, and as always we sum over repeated indices $i, j, k, \ldots$

We also considered Laplace's equation in spherical coordinates and expanded in powers of $\frac{1}{r}$.

Using: $\phi(\vec{r}) = R(r)\,F(\theta,\phi) = R(r)F(\Omega)$ Laplace's equation becomes:

$$\frac{1}{R(r)}\,\partial_r\,r^2\,\partial_r\,R(r) + \frac{1}{F(\Omega)}\,\nabla_\Omega^2\,F(\Omega) = 0$$

$$\nabla_\Omega^2 = \frac{1}{\sin(\theta)}\,\partial_\theta\,\sin(\theta)\,\,\partial_\theta + \frac{1}{\sin^2(\theta)}\,\partial_\phi^2$$

Writing $R(r)$ and $F(\Omega)$ as polynomial expansions in r and $\hat{n} = \frac{\vec{r}}{r}$ (which depends on θ and ϕ) we have:

$$\phi(\vec{r}) = \sum_\ell \left(A_\ell\,r^\ell + \frac{B_\ell}{r^{-(\ell+1)}}\right)\,F_\ell(\Omega)$$

The functions $F_\ell(\Omega)$ will be related to either the Legendre Polynomials (for problems with no ϕ dependence or the Spherical Harmonics in more general cases.

1) Legendre Polynomials:

One result from the expansion in spherical coordinates is that for $\vec{r} > \vec{r}_q$:

$$\frac{1}{|\vec{r} - \vec{r}_q|} = \sum_{\ell} \frac{r_q^{\ell}}{r^{-\ell+1}} \; P_\ell(\cos\theta), \;\; \cos\theta = \frac{\vec{r} \cdot \vec{r}_q}{r \, r_q}$$

Compare the first three terms of this expansion to the Taylor's Series expansion shown above. Using this comparison, determine the polynomials $P_0(x), P_1(x), \text{and } P_2(x)$.

(Of course, in this case the functions will be in terms of $x = \cos(\theta)$.

Check to see that your results for the Legendre Polynomials is correct. (Give the reference you used to check your results.)

2) Dipole Field:

Consider the second, "dipole", ($\ell = 1$) term in the expansion of $\frac{1}{|\vec{r}-\vec{r}_q|}$ when used in the expressions for the potential:

$$\phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\sum_q q\, x_{qi}\right)\frac{x_i}{r^3}, \qquad \phi^1(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\int d^3r_q \, \rho(\vec{r}_q)\, x_{qi}\right)\frac{x_i}{r^3}$$

The term in the parentheses in the equations above is the "Dipole Moment", a vector that depends on the charges and their positions (or the charge density).

In the book, the author labels these $\vec{d}$, but we'll use the more common notation, $\vec{p}$.

i) Write the dipole potential in terms of $\vec{p}$ and $\vec{r}$ (instead of in terms of the components shown above).

ii) For a dipole $\vec{p} = p\,\hat{z}$, for what directions in space does the potential equal zero?

iii) Using the dipole potential, calculate the electric field due to just the dipole moment of a charge distribution, $\vec{p}$.
This was left as an exercise in the book, but it's a good idea to do this.

iv) Using your result from above, draw a sketch showing the electric field due to a dipole $\vec{p} = p\,\hat{z}$. Draw this in the x-z plane. Your sketch should show the E-field vectors at a few representative places with the vector lengths giving the approximate magnitudes.

Repeat for a dipole $\vec{p} = p\,\hat{x}$.

3) Calculating Dipoles:

i) Consider the charge distribution with four charges:

q at (x = a, y = 0, z = 0), -q at (x = -a, y = 0, z = 0), q at (x = 0, y = a, z = 0), -q at (x = 0, y = -a, z = 0)

Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer. A picture might be a good idea.

ii) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

iii) Consider the charge distribution with 3 charges:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

Without doing any calculations, predict what the dipole $\vec{p}$ will be for this charge distribution. Explain your answer. A picture might be a good idea.

iv) Using the definition above for the dipole, calculate the dipole for this charge distribution. If your answer is different from your prediction, explain the difference.

4) Non-Uniformly Charged Line:

Consider a charged line of length $2L$ on the z-axis that extends from $z = -L$ to $z = L$. The linear charge density on the line varies as a function of z as:

$$\lambda(z) = \lambda_0 \sin\left(\frac{\pi}{2L}\, z\right)$$

i) Write down an expression for the charge density $\rho(\vec{r}_q)$ for this charged line.

ii) Describe what you would expect the dipole moment $\vec{p}$ to be for this charged line. Include the direction of the dipole and an estimate of the magnitude of the dipole.

ii) Integrate over the charge density to solve for the dipole moment, $\vec{p}$. Compare your result to your expected result from above and explain any differences.

5) Multipole expansion:

Your results from Question 3, Parts (iii) and (iv) might not be very satisfying. So, let's look a little more at the charge distribution:

q at (x = 0, y = 0, z = a), q at (x = 0, y = 0, z = -a), -2q at (x = 0, y = 0, z = 0)

i) Using the expansions from above and the expression for the potential $\phi(\vec{r})$, calculate the third (quadrupole) term in the expression for the potential.

ii) Write your result in spherical coordinates, r, θ, ϕ. Explain why your result doesn't depend on ϕ. In this, quadrupole, approximation, for what directions in space is the potential equal to zero? (A picture might be nice.)

iii) Using spherical coordinates, calculate the electric field of this charge distribution. Sketch the field in the x-z plane.