11 Fermi systems

$$\log \mathcal{Q} = \frac{PV}{kT} = \sum_{\vec{k}'} \log\left(1 + z e^{-\beta \varepsilon_{\vec{k}'}}\right)$$

Recall : $$\langle n_{\vec{k}} \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \varepsilon_{\vec{k}}} \log \mathcal{Q}$$

the derivative is taken with respect to one particular $\varepsilon_{\vec{k}}$

$$= \frac{z e^{-\beta \varepsilon_{\vec{k}}}}{1 + z e^{-\beta \varepsilon_{\vec{k}}}}$$

$$= \frac{1}{z^{-1} e^{\beta \varepsilon_{\vec{k}}} + 1}$$

non-interacting non-relativistic Fermi gas (spin less)

$$N = z \frac{\partial}{\partial z} \log \mathcal{Q}(z, V, T)$$

$$= \sum_{\vec{k}} \frac{1}{z^{-1} e^{\beta \varepsilon_{\vec{k}}} + 1} = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

see p. 233a for details

$$z \geq 0$$

(occupation numbers must be ≥ 0; ≤ 1)

from angular integration

$$\frac{P}{kT} = \frac{4\pi}{h^3} \int_0^\infty \log\left(1 + z e^{-\beta p^2/(2m)}\right) p^2 \, dp$$

from $\frac{V}{\left(\frac{2\pi}{L}\right)^3}$ factor (sum → integral) and $\vec{k} = \hbar^{-1} \vec{p}$

$$\frac{1}{v} = \frac{4\pi}{h^3} \int_0^\infty \frac{1}{z^{-1} e^{\beta p^2/(2m)} + 1} p^2 \, dp$$

$$v = \frac{V}{N}$$

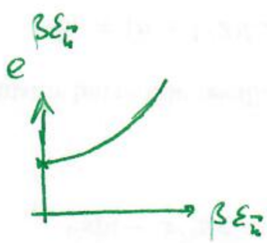
Add-on:

$$\langle n_{\vec{k}} \rangle = \left(z^{-1} e^{\beta \varepsilon_{\vec{k}}} + 1 \right)^{-1}$$

$$\Rightarrow \quad z^{-1} e^{\beta \varepsilon_{\vec{k}}} + 1 = \frac{1}{\langle n_{\vec{k}} \rangle}$$

$$\Rightarrow \quad z^{-1} e^{\beta \varepsilon_{\vec{k}}} = \underbrace{\underbrace{\frac{1}{\langle n_{\vec{k}} \rangle} - 1}_{\dfrac{-\langle n_{\vec{k}} \rangle + 1}{\langle n_{\vec{k}} \rangle}}}_{\geq 0 \text{ since } \langle n_{\vec{k}} \rangle \text{ between } 0 \text{ and } 1}$$

$$\Rightarrow \quad z^{-1} \underbrace{e^{\beta \varepsilon_{\vec{k}}}}_{\text{this is positive}} \geq 0$$



$$\Rightarrow \quad z^{-1} \text{ must be } \geq 0$$

In short: $\frac{P}{kT} = \frac{1}{\lambda^3} f_{5/2}(z)$

where $f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty \log\left(1 + z e^{-x^2}\right) x^2 dx$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} z^\ell}{\ell^{5/2}}$$

f_m: Fermi-Dirac function

$$\lambda = \left(\frac{2\pi \hbar^2}{mkT}\right)^{1/2}$$

$$\frac{1}{v} = \frac{1}{\lambda^3} f_{3/2}(z)$$

where $f_{3/2}(z) = z \frac{\partial}{\partial z} f_{5/2}(z)$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} z^\ell}{\ell^{3/2}}$$

We can also write:

$$f_{3/2}(z) = \int_0^\infty \frac{x^2}{z^{-1} e^{x^2} + 1} dx$$

use: $f_{3/2}(z) = z \frac{\partial}{\partial z} f_{5/2}(z) = z \frac{\partial}{\partial z} \left(\frac{4}{\sqrt{\pi}} \int_0^\infty \log\left(1 + z e^{-x^2}\right) x^2 dx \right)$

$= \dots$

In general:

$$\boxed{f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{y^{m-1}}{z^{-1} e^y + 1}\, dy}$$

Fermi-Dirac function

$$= \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{z^\ell}{\ell^m} \quad (*)$$

As written, the Fermi-Dirac functions are not overly intuitive

$\rightsquigarrow$ look at small z and large z expansions.

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{\lambda^3}{v} \ll 1 \qquad\qquad \frac{\lambda^3}{v} \gg 1$$

n: density $\longrightarrow n\lambda^3 \ll 1$; $\left(\frac{\lambda}{\bar{r}}\right)^3 \ll 1$

$\bar{r}$: average spacing

Small z: Use power series (*) and keep as many terms as desired.

Large z: After some work...

Riemann zeta fct.

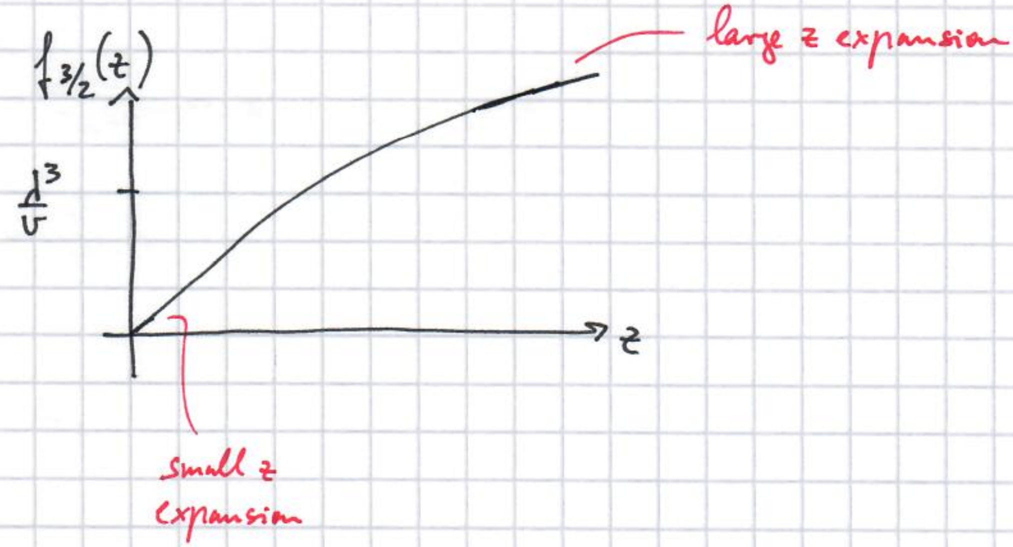
$$f_m(z) = \frac{(\log z)^m}{\Gamma(m+1)} \left[1 + 2m \sum_{j=1,3,5} \left\{ (m-1) \cdot \ldots \cdot (m-j) \left(1 - \frac{1}{2^j}\right) \frac{\zeta(j+1)}{(\log z)^{j+1}} \right\} \right]$$

For example: $$\frac{\lambda^3}{v} = z - \frac{z^2}{2^{3/2}} + \ldots \quad (*)$$

and

$$\frac{\lambda^3}{v} = \frac{(\log z)^{3/2}}{\Gamma(\frac{5}{2})}\left[1 + \frac{3}{2}\cdot\frac{1}{2}\,\frac{\pi^2}{6}\,\frac{1}{(\log z)^2} + \frac{3}{2}\,\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{7\pi^4}{360}\,\frac{1}{(\log z)^4} + \ldots\right]$$

$$= \frac{4}{3\sqrt{\pi}}\left[(\log z)^{3/2} + \frac{\pi^2}{8}(\log z)^{-1/2} + \ldots\right]$$



Solution to (*): $z = \underbrace{\frac{\lambda^3}{v}}_{\text{Boltzmann gas}} + \underbrace{\frac{1}{2^{3/2}}\left(\frac{\lambda^3}{v}\right)^2}_{\text{correction}} + \dots$

(small z regime → high T, low density)

$$\Rightarrow \langle n_{\vec{k}} \rangle \approx \frac{\lambda^3}{v} e^{-\beta \varepsilon_{\vec{k}}}$$

just leading order term

$$\text{and} \quad \frac{Pv}{kT} = \frac{v}{\lambda^3}\left(f_{5/2}(z)\right) \approx \frac{v}{\lambda^3}\Big(\overbrace{z - \frac{z^2}{2^{5/2}} + \dots}^{f_{5/2}}\Big)$$

$$\approx 1 + \frac{z}{2^{5/2}} + \dots$$

$\frac{\lambda^3}{v} \approx z - \frac{z^2}{2^{3/2}} + \dots$

$$= 1 + \frac{1}{2^{5/2}}\frac{\lambda^3}{v} + \dots$$

virial coefficient

$z = \frac{\lambda^3}{v}$

looks like a virial expansion. However, correction to classical ideal gas law are not due to interactions!

Corrections are due to quantum effects!

A bit more discussion of the large z (large $\frac{\lambda^3}{v}$) regime.

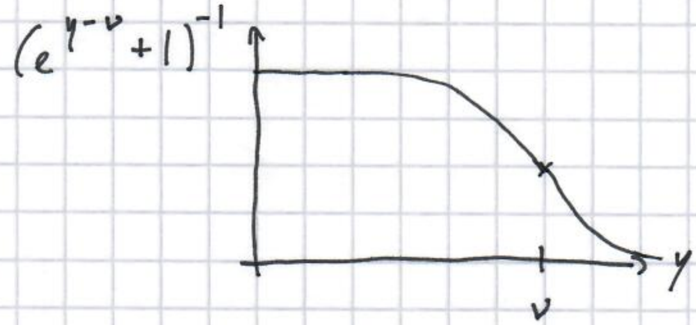
Let $\nu = \log z = \mu/kT$

$$\Rightarrow f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{y^{m-1}}{e^{y-\nu}+1}\, dy$$

Large z implies large ν

$\quad \llcorner \Rightarrow$ denominator dominates

Let's plot $\frac{1}{e^{y-\nu}+1}$:

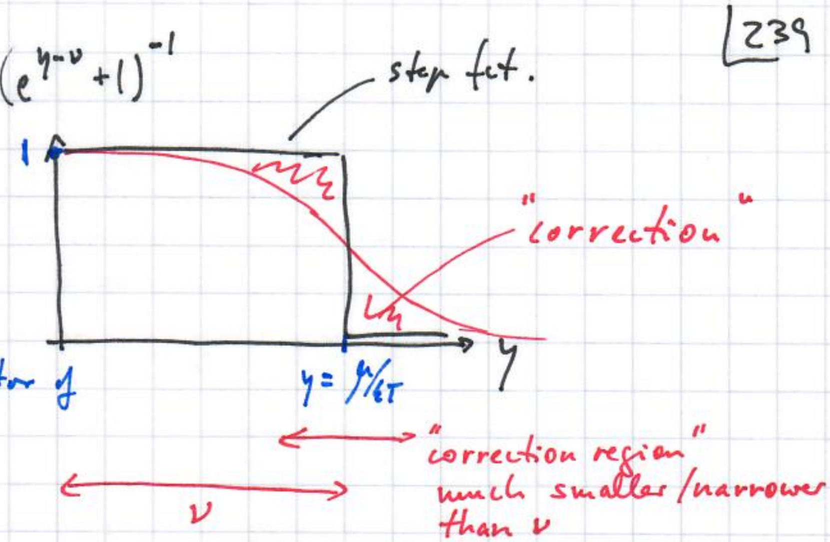


for $y \to 0$:
$(e^{y-\nu}+1)^{-1} \xrightarrow{\nu \text{ large}} 1$

for $y \to \infty$:
$(e^{y-\nu}+1)^{-1} \xrightarrow{\nu \text{ large}} 0$

for $y = \nu$: $(e^{y-\nu}+1)^{-1} = \frac{1}{2}$

Leading-order :



$$\Rightarrow f_m(z) \approx \frac{1}{\Gamma(m)} \int_0^{\nu} y^{m-1}\, dy$$

$$= \frac{1}{\Gamma(m)} \frac{1}{m} \nu^m$$

$$= \frac{1}{\Gamma(m+1)} (\log z)^m$$

Thus: near absolute zero ($T=0$), we have

$$\frac{\lambda^3}{v} \approx \frac{4}{3\sqrt{\pi}} (\log z)^{3/2}$$

$$\Rightarrow z \approx e^{\beta \varepsilon_F}$$

$$\text{with } \boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{2/3}}$$

Fermi energy

mass (not index)

It follows:

$$\boxed{\langle n_{\vec{k}} \rangle \xrightarrow{T \to 0} \begin{cases} 1 & \text{for } \varepsilon_{\vec{k}} < \varepsilon_F \\ 0 & \text{for } \varepsilon_{\vec{k}} > \varepsilon_F \end{cases}}$$

this is "just" the Pauli exclusion principle

In momentum space: $\varepsilon_F = \frac{p_F^2}{2m}$

$$\Rightarrow p_F = \sqrt{2m\varepsilon_F}$$

Fermi momentum

In momentum space, the particles fill a sphere of radius p_F, the surface of which is called Fermi surface.

$$p_F = \hbar k_F \Rightarrow k_F = \frac{p_F}{\hbar}$$

So far: Non-relativistic ideal spinless fermions!

Now: Each single-particle level is g-fold degenerate due to spin degree of freedom:

$$\boxed{g = (2s+1)}$$

s: spin of particle
(s must be positive or zero)
↳ non-negative

degeneracy of 2s+1 arises due to projection quantum number:

$$m_s = \underbrace{-s, -s+1, \ldots, s-1, s}_{2s+1 \text{ possibilities}}$$

Note: degeneracy may be broken by external magnetic field!

$$\boxed{\text{With degeneracy factor: } g \sum_{\vec{k}} \langle n_{\vec{k}} \rangle_{T=0} = N}$$

Using $g \sum_{\vec{k}} \langle n_{\vec{k}} \rangle_{T=0} = N$

together with $\langle n_{\vec{k}} \rangle_{T=0} = \begin{cases} 1 & \text{for } \varepsilon_{\vec{k}} < \varepsilon_F \\ 0 & \text{for } \varepsilon_{\vec{k}} > \varepsilon_F \end{cases}$

we find:

$$\frac{g}{\left(\frac{2\pi}{L}\right)^3} \int d^3k = N$$

$$\Rightarrow \frac{g}{(2\pi)^3} \, 4\pi \underbrace{\int_0^{k_F} k^2 \, dk}_{\frac{k_F^3}{3}} = \frac{N}{V}$$

$$\Rightarrow \frac{g}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{N}{V}$$

$$\overset{\hbar k_F = p_F}{\Longrightarrow} \boxed{\frac{g}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3 = \frac{N}{V}}$$

Solving for p_F and inserting into $\varepsilon_F = \frac{p_F^2}{2m}$,

we have $\boxed{\varepsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{g v}\right)^{2/3}}$ where $v = \frac{V}{N}$

Explicit expressions in $\frac{\lambda^3}{v} \gg 1$ regime:

Recall $\frac{\lambda^3}{v} = f_{3/2}(z)$

$$\approx \frac{4}{3\sqrt{\pi}} \left[(\log z)^{3/2} + \frac{\pi^2}{8} (\log z)^{-1/2} + \ldots \right]$$

$$\Rightarrow kT \log z = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \ldots \right]$$

expansion parameter

define $kT_F = \varepsilon_F$

$\Rightarrow \frac{T}{T_F} \ll 1$

So: $\frac{\lambda^3}{v} \gg 1 \longleftrightarrow \frac{T}{T_F} \ll 1$

$\lambda^3 n \gg 1$

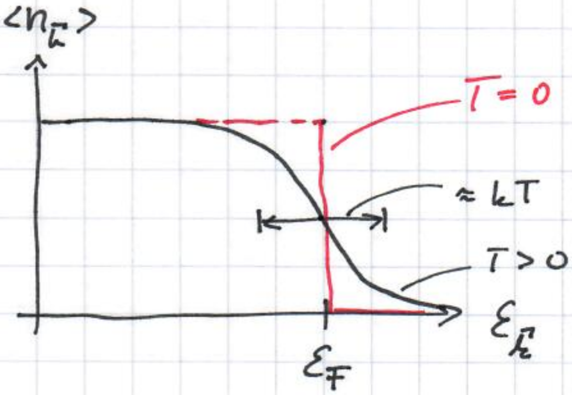
$\frac{\lambda^3}{\bar{r}^3} \gg 1$

de Broglie wave length larger than average interparticle spacing

gas is degenerate: all particles tend to go to the lowest energy levels possible

$\leadsto T_F$ is called degeneracy temperature

In low T regime:



average occupation number in ideal Fermi gas

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\varepsilon_{\vec{k}} - \mu)} + 1}$$

$$\Rightarrow \text{ internal energy } U = \sum_{\vec{k}} \underbrace{\varepsilon_{\vec{k}}}_{\frac{\hbar^2 k^2}{2m}} \langle n_{\vec{k}} \rangle$$

$$= \frac{V}{h^3} \frac{4\pi}{2m} \int_0^\infty p^4 \langle n_{\vec{k}} \rangle \, dp$$

4π comes from angular integral

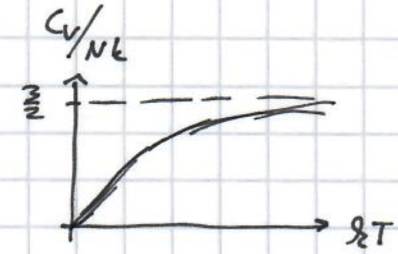
$\rightsquigarrow$ p^2 comes from going to spherical coordinates (volume element)

It follows (using the expansion for $kT \log z = \dots$):

$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{kT}{\varepsilon_F} \right)^2 + \dots \right]$$

$$\sum_{|\vec{k}| < k_F} \frac{\hbar^2 \vec{k}^2}{2m} = \frac{3}{5} N \varepsilon_F$$

$$\Rightarrow C_V = Nk \, \frac{\pi^2}{2} \, \frac{kT}{\varepsilon_F} + \dots$$



exact relation
$$\boxed{p = \frac{2}{3} \frac{U}{V}} = \dots$$

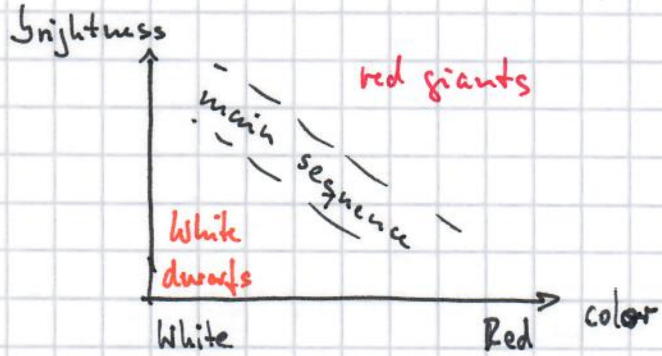
this relation holds for ideal Fermi, Bose and Boltzmann gas (see Chapter 8)

11.2 The theory of White Dwarf Stars

White dwarf : final phase in the life of a star

⇝ has exhausted its usable nuclear fuel

Russel-Hertzsprung diagram



most stars fall into main sequence

Exceptions:
- red giants (huge)
- White dwarfs (small)

White dwarfs : unusually / abnormally faint for their white color

↳ hydrogen has been used up (main energy source of star)

↳ mainly composed of helium

Want to determine mass M_0 (Chandrasekhar mass):

if mass of white dwarf $> M_0$ ⟹ white dwarf will collapse under the

influence of its own
gravitational force

So: We have a "bunch of mass" $\longrightarrow$
Which physical mechanism prevents the collapse due to gravitational force?

In white dwarf: Fermi pressure of the electrons

In an active star like the sun: gravitational collapse is opposed and balanced by the pressure of thermonuclear reactions.

these reactions have been extinguished in white dwarfs, which are in the final phase in the life of a star

Some data for white dwarfs:

mass density of sun

$$\text{mass density} \approx 10^7\ g/cm^3 \approx 10^7\ \rho_\odot$$

$$\text{mass} \approx 10^{33}\ g \approx M_\odot\ ;\ \text{radius} \approx 5 \cdot 10^6\ m$$

$$\text{temperature at center} \approx 10^7 K \approx T_\odot$$

→ mass close to that of Sun and radius close to that of the Earth !

Density roughly 10 Million times that of water ...

Some simplifying assumptions:

- White dwarf is made up of $\underbrace{{}^4He^{++}}$

10^7 K corresponds to thermal energy of ~ 1000 eV
→ He is completely ionized

$\boxed{\text{room temperature } \approx \frac{1}{40} eV}$

and electrons

- homogeneous density

$10^7 \, g/cm^3 \;\hat{\approx}\; 10^{30}$ electrons$/cm^3$

$$\left.\begin{aligned} \Rightarrow \; \varepsilon_F &\sim 20\,MeV \\ T_F &\sim 10^{11}\,K \end{aligned}\right\}$$

these are rough estimates based on non-relativistic non-interacting electron gas

$T/T_F \ll 1$

So: • Electron gas is essentially degenerate

• Gravitational attraction essentially due ~~to~~ helium nuclei

nuclei are heavy

• kinetic motion of nuclei to be neglected

• assume $T=0$ and treat electron gas relativistically

N electrons ; $\frac{N}{2}$ helium nuclei

$$\hookrightarrow \varepsilon_{\vec{k},s} = \sqrt{(\vec{p}c)^2 + (m_e c^2)^2}$$

Ground state energy E_0:

$$E_0 = 2 \sum_{|\vec{k}| < k_F} \sqrt{(\vec{p}c)^2 + (m_e c^2)^2}$$

degeneracy factor

$$= 2 \frac{1}{\left(\frac{2\pi}{L}\right)^3} 4\pi \int_0^{k_F} \sqrt{(\hbar c k)^2 + (m_e c^2)^2}\, k^2 dk$$

where, as before;

$$\frac{g}{(2\pi\hbar)^3}\frac{4\pi}{3}\underbrace{p_F^3}_{(\hbar k_F)^3} = \frac{N}{V}$$

here $g = (2s+1)$

$= (2\tfrac{1}{2}+1) = 2$

$$\Rightarrow k_F = \left(\frac{3\pi^2}{v}\right)^{1/3}$$

To evaluate the integral, we define $x = \dfrac{\hbar c k}{m_e c^2}$

$$\Rightarrow dx = \frac{\hbar}{m_e c}\, dk$$

$$\Rightarrow E_0 = 2\frac{1}{\left(\frac{2\pi}{L}\right)^3} 4\pi \underbrace{m_e c^2 \left(\frac{m_e c}{\hbar}\right)^3}_{\frac{m_e^4 c^5}{\hbar^3}} \underbrace{\int_0^{x_F} \sqrt{x^2+1}\, x^2\, dx}_{f(x_F)}$$

$$\Rightarrow \frac{E_0}{N} = \frac{m_e^4 c^5}{\hbar^3}\frac{1}{\pi^2}\, v\, f(x_F)$$

$$f(x_F) = \frac{1}{8}\left[x_F \sqrt{1+x_F^2}\,(1+2x_F^2) - \log\left(x_F + \sqrt{1+x_F^2}\right)\right]$$

table of integrals

$$= \begin{cases} \frac{1}{3} x_F^3 \left(1 + \frac{3}{10} x_F^2 + \dots\right) & \text{for } x_F \ll 1 \\ \frac{1}{4} x_F^4 \left(1 + \frac{1}{x_F^2} + \dots\right) & \text{for } x_F \gg 1 \end{cases}$$

$$x_F = \frac{\hbar k}{m_e c} = \frac{\hbar}{m_e c}\left(\frac{3\pi^2}{v}\right)^{1/3}$$

$x_F \ll 1$ corresponds to non-relativistic case

$x_F \gg 1$ corresponds to extremely relativistic case

So far, we have only accounted for kinetic energy of the electrons.

Total energy $E_{tot} = E_0 + E_g$

we just worked out...

$$E_g = -\alpha \frac{\gamma M^2}{R}$$

some numerical factor

M: total mass of star

$$M = (m_e + 2m_p)\, N$$

proton mass

($N/2$ He^{++} and mass of $He^{++} \approx 4 m_p$)

R: radius of white dwarf (equilibrium radius)

γ: gravitational constant

this is a bit handwaving: see text for more rigorous approach

Now: Minimizing E_{tot} gives R (this is equivalent to minimizing free energy since we are at $T=0$)

Let's look at ultra relativistic case:

$$E_0 \approx \frac{m_e^4 c^5}{\hbar^3} \frac{1}{\pi^2} V \left(\frac{1}{4} x_F^4 + \frac{1}{4} x_F^2 \right)$$

$$\approx \frac{m_e^4 c^5}{\hbar^3} \frac{1}{\pi^2} \frac{1}{4} \left[\left(\frac{\hbar}{m_e c}\right)^4 \frac{(3\pi^2)^{4/3}}{v^{4/3}} V + \left(\frac{\hbar}{m_e c}\right)^2 \frac{(3\pi^2)^{2/3}}{v^{2/3}} V \right]$$

$$\frac{V}{v^{4/3}} = V^{-1/3} N^{4/3} \sim \frac{1}{R}$$

and $$\frac{V}{v^{2/3}} = V^{1/3} N^{2/3} \sim R$$

So: $$E_0 = c_1 M^{4/3} \frac{1}{R} \left(1 + c_2 R^2 \right)$$

$$\frac{\partial E_{tot}}{\partial R} = - c_1 M^{4/3} \frac{1}{R^2} + c_1 M^{4/3} c_2 + \frac{\alpha \gamma M^2}{R^2} = 0$$

$$\Rightarrow \quad \frac{c_1 M^{4/3} - \alpha \gamma M^2}{c_1 M^{4/3} c_2} = R^2$$

or $$\boxed{R^2 = \frac{1}{c_2} \left(1 - \left(\frac{M}{\tilde{M}} \right)^{2/3} \right)}$$

where $\tilde{M} = \left(\frac{c_1}{\alpha \gamma} \right)^{3/2}$

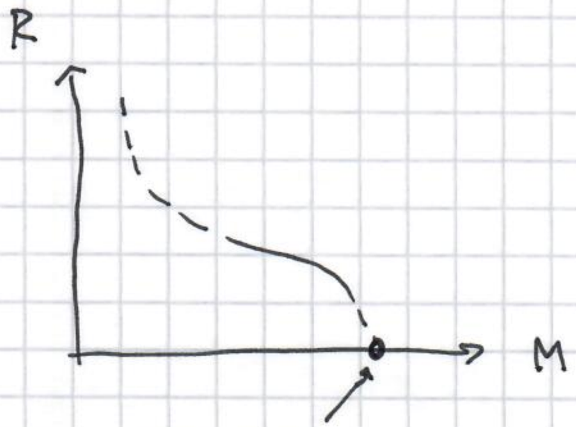
R must be positive

$\rightsquigarrow$ collapse when $\frac{M}{\tilde{M}} = 1$

i.e., $M = \left(\frac{c_1}{\alpha \gamma} \right)^{3/2} = \# \, m_p \left(\frac{\hbar c}{\alpha \gamma m_p^2} \right)^{3/2}$

taking α of order 1 $\longrightarrow \approx M_\odot$

Note: we worked in the ultrarelativistic limit,
near the collapse point



Radius R – mass M relationship of white dwarf

collapse point: $M \approx M_\odot$

$\downarrow$

A more realistic calculation yields $1.4\ M_\odot$