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Electrodynamics 1

PHYS 5573 Homework Assignment 1

Problems: {1, 2, 3, 4}

Due: February 11, 2022 at 5:00 PM

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Problem 1:

In the first workshop we used Gauss' Law and symmetry considerations to calculate the electric field due to an infinite, uniform, thin charged slab lying in the x-y plane.

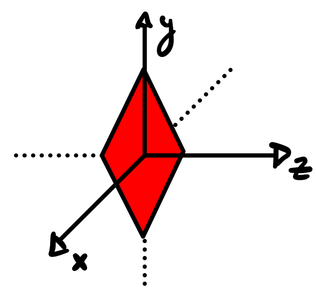
Consider a similar infinite slab, but with a spread-out charge distribution perpendicular to the slap

$$\rho(\vec{r}) = \rho(z) = \rho_0 \, e^{-\alpha|z|}.$$

This might be used as a model for a nano-scale metal surface where the charge distribution is related to the "skin depth" or finite extension of the surface states of the electrons in the metal.

(a) Explain why and how you can use Gauss' Law to determine the electric field due to this slab. What is the general form for the electric field? Justify your answer.

Our slab looks something like



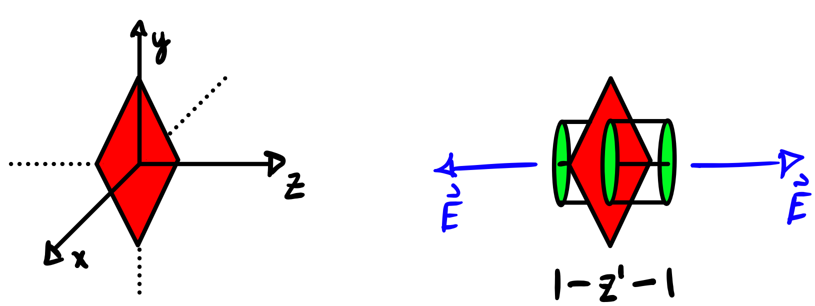
Gauss' Law tells us that the amount of Electric Field that penetrates a surface is equal to the charge distribution of that surface divided by ϵ_0, namely this is

$$\vec{\nabla} \cdot \vec{E}(r) = \frac{\rho(\hat{r})}{\epsilon_0}$$

We use this because we can use symmetries in our field along with Gaussian surfaces to determine $\vec{E}$.

(b) Describe carefully, and draw a picture, showing the volume and surface you will use to solve the integrals in Gauss' Law

Because our slab is perpendicular to the one in the workshop, this slab will have it's normal vector pointing in $\pm y$ direction :



We will use a Gaussian surface of a pill box that looks like the above to solve the integrals.

Problem 1: Continued

(c) Solve the two integrals (volume integral and surface integral) in Gauss' Law to determine the electric field everywhere.

we use the divergence theorem

$$\iiint_V \vec{\nabla}\cdot\vec{F}(\vec{r})\, d^3x = \oint\!\!\oint_{S=\partial V} \vec{F}(\vec{r})\cdot\hat{n}\, ds$$

To show that the electric field can be solved for with,

$$\oint\!\!\oint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds = \oint\!\!\oint_R \vec{E}(\vec{r})\cdot\hat{z}\, ds + \oint\!\!\oint_L \vec{E}(\vec{r})\cdot(-\hat{z})\, ds + \oint\!\!\oint_T \vec{E}(\vec{r})\cdot\hat{r}\, ds$$

$$\iiint_V \vec{\nabla}\cdot\vec{E}(\vec{r})\, d^3x = \iiint_V \rho(\vec{r})/\epsilon\, d^3x$$

First solving the surface integral we have $[\text{w/ } E(\vec{r}) = E_0(\pm\hat{z})]$

$$\oint\!\!\oint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds = \underbrace{\oint\!\!\oint_L E_0\cancel{(-\hat{z})\cdot(-\hat{z})}\, ds}_{A} + \underbrace{\oint\!\!\oint_R E_0\cancel{(\hat{z})\cdot(\hat{z})}\, ds}_{A} + \underbrace{\oint\!\!\oint_S E_0\cancel{(\hat{r})\cdot(\hat{r})\, ds}}_{0} = 2AE_0$$

This means the surface integral evaluates to,

$$\oint\!\!\oint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds = 2AE_0$$

We then turn to solving the volume integral

$$Q_{ENC} = 2\iiint_V \rho(\vec{r})\, d^3x = 2\int_0^{y'}\int_0^{2\pi}\int_0^{r'} \rho_0 e^{-\alpha|z|}\, r\, dr\, d\phi\, dz = r'^2\cdot 2\pi\cdot\rho_0\cdot\left(\frac{e^{-\alpha|z'|}}{-\alpha}+\frac{1}{\alpha}\right)$$

$$= 2\pi r'^2\rho_0(e^{-\alpha|z'|}-1) = -2A\frac{\rho_0}{\alpha}(e^{-\alpha|z'|}-1)$$

This means that if we continue to use Gauss' Law we will have

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{Q_{ENC}}{\epsilon} \Rightarrow \cancel{2}\cancel{A}\vec{E} = \frac{-\cancel{2A}\rho_0(e^{-\alpha|z'|}-1)}{\epsilon\alpha}\hat{z} \quad \therefore \quad \vec{E} = \frac{-\rho_0}{\epsilon\alpha}(e^{-\alpha|z'|}-1)\hat{z}$$

And therefore the electric field is

$$\boxed{\vec{E} = \frac{-\rho_0}{\epsilon\alpha}(e^{-\alpha|z'|}-1)\hat{z}}$$

Problem 1: Continued

(d) Determine the electrostatic potential, $\phi(\vec{r})$, that corresponds to this electric field. (Hint: you can either note that the potential is the integral of the field or guess and check a function for the potential with the property that the negative of the gradient of the function gives the field.

The electric potential of this field can be calculated as

$$\vec{E} = -\vec{\nabla}\varphi(\vec{r})$$

This then turns into

$$\varphi(z) = -\int E_z \, dz = -\frac{\rho_0}{\epsilon\alpha}\left(C_1|z| + C_2\frac{e^{-\alpha|z|}}{\alpha}\right)$$

In this case the potential will be,

$$\boxed{\varphi(z) = \frac{-\rho_0}{\epsilon\alpha}\left(C_1|z| + \frac{C_2}{\alpha}e^{-\alpha|z|}\right)}$$

(e) Consider your results for the field and the potential in the limits as $z \to \infty$ and $z \to 0$. Do these limits make physical sense, comparing them to the thin slab? Explain.

As we let $z \to 0$

$$\boxed{\varphi(z) = \frac{\rho_0^2}{2\epsilon\alpha}} \qquad \boxed{\vec{E}(z) = 0}$$

As we let $z \to \infty$

$$\boxed{\varphi(z) = \infty} \qquad \boxed{\vec{E}(z) = \frac{\rho_0}{\epsilon\alpha}(\hat{z})}$$

Since our charge distribution is dependent upon z, it makes sense for $\vec{E}$ to go to 0 if we get close it as well as it makes sense for us to approach a value at ∞. These Results make sense

(f) Note that your result disagrees with our symmetry arguments from Questions 1a, 1b, and 1c of Workshop 1. Which of the symmetry arguments doesn't work for this case and why?

It breaks scaling because as $(\hat{z})$ grows $\vec{E}$ approachs a fixed value. This is due to the non uniform charge density of our slab.

Problem 1: Review

Procedure:

- – Write out the translational and reflection variance principles and explain how that can be taken advantage of
- – Draw out a picture of our slab and how we plan to use Gauss' Law to approach the problem
- – Gauss' Law mathematically is
$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{Q_{\text{ENC}}}{\epsilon_0}$$
where we can then use the divergence theorem to say
$$\int\int\int_V \vec{\nabla} \cdot \vec{E}(\vec{r})\, d^3x = \oint\oint_{S=\delta V} \vec{E}(\vec{r}) \cdot \hat{n}\, dS \quad , \quad Q_{\text{ENC}} = \int\int\int_V \rho(\vec{r})\, d^3r$$
and we can then solve for $\vec{E}(\vec{r})$
- – Determine the electrostatic potential by using
$$\vec{E}(z) = -\vec{\nabla}\phi(z)$$
- – Apply boundary conditions to our potential and our field to determine what happens to them as we range z
- – Discuss how it breaks scaling symmetry

Key Concepts:

- – There are two types of invariance, Translational ($\vec{E}(x+a) = \vec{E}(x)$) and Reflection ($E_x(z) = -E_x(z)$)
- – We are able to use Gauss' Law for this problem due to the symmetries that are provided to us
- – Thanks to the divergence theorem we can use Gauss' Law to determine the Electric Field
  – When integrating Q_{ENC}, always make sure to encapsulate all of the surface that we are working with
- – We are able to solve for our potential by integrating our Electric field
- – As $z \to 0$ our Electric Field disappears and as $z \to \infty$ our Electric Field approaches a constant value
- – This breaks scaling because as z increases the Electric Field approaches a constant value

Variations:

- – We can be given a different shape
  - ∗ We would then use different symmetries but the same procedure to find the rest of the desired quantities
- – Anything that changes in part (a) will propagate through the rest of the problem
  - ∗ This will affect only small details of the problem, not the overall method of determining the Electric Field and Potential

Problem 2:

If we consider the electron's probability density in a hydrogen atom as a static, classical charge density (I know, quite a stretch) we can make some classical predictions about the atom.

Consider a charge distribution of the form:

$$\rho(r) = \rho_0 e^{-\alpha r}.$$

The charge distribution has a total charge Q and $\alpha = \frac{2}{a_0}$ where a_0 is the length scale in the problem. For a hydrogen atom, $Q = -e$ and a_0 is the Bohr radius.

(a) Determine what ρ_0 needs to be in terms of Q and a_0. Be sure to explain how you determined your answer.

Hint: There is a standard trick for doing these sorts of integrals:

$$\int x^n e^{-\alpha x}\, dx = (-1)^n \frac{\partial^n}{\partial \alpha^n} \int e^{-\alpha x}\, dx$$

Of course, if you're not in a test (or qualifier) you can always pull up Mathematica, Wolfram Alpha, or whatever python library you might use.

We first calculate Q from the charge density

$$Q = \iiint_V \rho(\vec{r})\, d^3r$$

For our problem this becomes

$$Q = \rho_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 e^{-\alpha r} \sin\theta \, dr\, d\theta\, d\phi = 4\pi\rho_0 \int_0^{\infty} r^2 e^{-\alpha r}\, dr \;, \quad w/ \int_0^{\infty} r^n e^{-\alpha r}\, dr = \frac{n!}{\alpha^{n+1}}$$

This then becomes

$$Q = 4\pi\rho_0 \left(\frac{2}{\alpha^3}\right) = \frac{8\pi\rho_0}{\alpha^3}$$

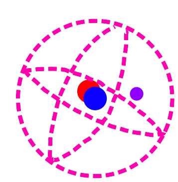
Therefore ρ_0 will become,

$$\boxed{\rho_0 = \frac{Q}{\pi a_0^3}}$$

(b) Explain how you will use Gauss' Law to determine the electric field everywhere for the atom. Draw a picture to illustrate your approach.

We will create a Gaussian surface, determine the charge enclosed and then solve for $\vec{E}$ mathematically.

$$Q_{enc} = \int_V \rho(\vec{r})\, d^3r \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{Q_{enc}}{\epsilon} \Rightarrow$$



By creating a Gaussian surface we can solve for Q_{enc} and then with the Divergence Theorem we can solve for $\vec{E}$.

Problem 2: Continued

(c) Calculate both the volume and the surface integral in Gauss' Law, showing your work. Use these results to determine the electric field everywhere.

Gauss' Law with the Divergence Theorem is

$$\iiint_V \vec{\nabla}\cdot\vec{E}(\vec{r})\,d^3r = \oint\!\!\oint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\,ds \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{Q_{ENC}}{\epsilon} \text{ , } Q_{ENC} = \iiint_V \rho(\vec{r})\,d^3r$$

First let's solve the surface integral,

$$\oint\!\!\oint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\,ds = \int_0^{2\pi}\int_0^{\pi} E\cdot R^2\sin(\theta)\,d\theta d\varphi = 4\pi R^2 E$$

And then the volume integral for Q_{ENC},

$$Q_{ENC} = \iiint_V \rho(\vec{r})\,d^3r = \frac{-e}{\pi a_0^3}\int_0^{2\pi}\int_0^{\pi}\int_0^R r^2 e^{-\alpha r}\sin(\theta)\,dr d\theta d\varphi = \frac{-e}{\pi a_0^3}\cdot 4\pi\int_0^R r^2 e^{-\alpha r}\,dr$$

$$= -\frac{4e}{a_0^3}\int_0^R r^2 e^{-\alpha r}\,dr = \frac{4e}{a_0^3}\frac{1}{\alpha^3}\left((\alpha^2R^2+2\alpha R+2)e^{-\alpha R}-2\right) = e\left(\left(\frac{2R^2}{a_0^2}+\frac{2R}{a_0}+1\right)e^{-2R/a_0}-1\right)$$

We now can solve for $\vec{E}$

$$4\pi R^2 E = \frac{e}{\epsilon}\left(\left(\frac{2R^2}{a_0^2}+\frac{2R}{a_0}+1\right)e^{-2R/a_0}-1\right) \Rightarrow \vec{E} = \frac{e}{4\pi\epsilon}\left(\left(\frac{2}{a_0^2}+\frac{2}{Ra_0}+\frac{1}{R^2}\right)e^{-2R/a_0}-\frac{1}{R^2}\right)\hat{r}$$

The $\vec{E}$ field is then,

$$\boxed{\vec{E} = \frac{e}{4\pi\epsilon}\left(\left(\frac{2}{a_0^2}+\frac{2}{Ra_0}+\frac{1}{R^2}\right)e^{-2R/a_0}-\frac{1}{R^2}\right)\hat{r}}$$

(d) As always, we need to check our results. Show that your result for the field makes physical sense in the limit that r gets large.

NOTE: This means more than finding what happens if $r = \infty$. Check the functional behavior in the limit as $r \to \infty$.

If we let R get large

$$\vec{E}(R\to\infty) = \lim_{R\to\infty}\frac{e}{4\pi\epsilon}\left(\left(\frac{2}{a_0^2}+\frac{2}{Ra_0}+\frac{1}{R^2}\right)e^{-2R/a_0}-\frac{1}{R^2}\right)\hat{r} = 0$$

If we get far enough away $\vec{E}\to 0$. If we get far enough away but not too far away it will be a point charge.

Problem 2: Continued

(e) Check that your result also is well behaved in the limit as $r \to 0$. Remember that this charge distribution is NOT due to a point charge, just the charged cloud.

If we let $r \to 0$

$$\vec{E}(R \to 0) = \lim_{R \to \infty} \frac{e}{4\pi\epsilon}\left(\left(\frac{2}{a_0^2} + \frac{2}{Ra_0} + \frac{1}{R^2}\right)e^{-2R/a_0} - \frac{1}{R^2}\right)\hat{r} = 0$$

The electric field will go to zero.

Problem 2: Review

Procedure:

- – Use the equation for Q_{ENC}
  $$Q_{\text{ENC}} = \int\int\int_V \rho(\vec{r})\, d^3r$$
  and solve for ρ_0
- – Explain how we are able to use Gauss' Law for our system
- – Use Gauss' Law and the Divergence theorem to solve for the Electric Field
  $$\int\int\int_V \vec{\nabla} \cdot \vec{E}(\vec{r})\, d^3x = \oint\oint_{S=\delta V} \vec{E}(\vec{r}) \cdot \hat{n}\, dS$$
  by calculating the new Q_{ENC} with our new charge density
- – Evaluate as the radius goes to infinity
- – Evaluate as the radius goes to zero

Key Concepts:

- – We are able to solve for ρ_0 by using the integral for Q_{ENC}
- – Because of our symmetries in our system we are able to create a Gaussian surface and solve for our Electric field later on
- – Utilizing the Divergence Theorem we are able to also use Gauss' Law to solve for the Electric Field
- – As R goes to infinity we see our Electric Field go that of a point charge
- – As R goes to zero, the Electric Field goes to zero as well

Variations:

- – Since Gauss' Law problems are completely consistent upon the shape that we are observing, there would have to be a new shape
  - * We would use the same broad procedure, possibly with different charge densities etc but the main procedure would not change
- – As always, any change in (a) will propagate through the rest of the problem
  - * We may be asked other conceptual questions about our field or potential

Problem 3:

The equation 16.27 in the textbook gives the classical hydrogen atom electric potential. This is the potential of both the electron cloud and the proton of the hydrogen atom, modeled as a point charge.

(Note, this is in Gaussian Units while we've been using SI units in class. You should be able to handle this difference.)

(a) Consider your results to Problem 2. What is the total electric field of the atom, including both the electron cloud (Problem 2) and the proton? Explain your work.

From problem 2, the Electric Field of the electron is

$$\vec{E} = \frac{e}{4\pi\epsilon}\left(\left(\frac{2}{a_0^2} + \frac{2}{Ra_0} + \frac{1}{R^2}\right)e^{-2R/a_0} - \frac{1}{R^2}\right)\hat{r}$$

The proton has the electric field of a point charge

$$\vec{E} = \frac{e}{4\pi\epsilon}\frac{1}{R^2}\hat{r}$$

using the law of super position the electric field is then

$$\boxed{\vec{E} = \frac{e\,e^{-2R/a_0}}{4\pi\epsilon}\frac{1}{R^2}\left(\frac{2}{a_0^2} + \frac{2}{Ra_0} + 1\right)\hat{r}}$$

(b) Show that your total electric field agrees with the potential given in 16.27.

The electric potential for a hydrogen atom is

$$\varphi(r) = \frac{q}{r}\left(1 + \frac{r}{a_0}\right)e^{-2r/a_0}$$

we will calculate the electric field with,

$$-\vec{\nabla}\varphi = \vec{E} \qquad (*)$$

we can now see that $(*)$ evaluates to

$$-\vec{\nabla}\varphi = -\frac{\partial}{\partial r}\left(\frac{q}{r}e^{-2r/a_0} + \frac{q}{a_0}e^{-2r/a_0}\right) = -\left(-\frac{2}{a_0}\frac{q}{r}e^{-2r/a_0} - \frac{q}{r^2}e^{-2r/a_0} - \frac{2}{a_0}\frac{q}{a_0}e^{-2r/a_0}\right)$$

This then simplifies to

$$-\vec{\nabla}\varphi = \left(\frac{2q}{Ra_0}e^{-2R/a_0} + \frac{q}{R^2}e^{-2R/a_0} + \frac{2q}{a_0^2}e^{-2R/a_0}\right) = \vec{E} \checkmark$$

Problem 3: Continued

(c) Determine the electrostatic potential energy of the electron cloud of the hydrogen atom. Use the total electrostatic potential but don't include the energy due to the proton.

We calculate the electric potential energy with,

$$U = \frac{1}{2}\iiint_V \varphi(\vec{r})\rho(\vec{r})\, d^3r$$

The electric potential and charge density are

$$\varphi(r) = \frac{q}{r}\left(1+\frac{r}{a_0}\right)e^{-2r/a_0} \Rightarrow \text{Full potential}, \quad \rho(\vec{r}) = \rho_0 e^{-2r/a_0} \Rightarrow \text{Just electron}$$

Using this we now have the energy to be

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{q}{r}\left(1+\frac{r}{a_0}\right)e^{-2r/a_0}\rho_0 e^{-2r/a_0} r^2 \sin(\theta)\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{2}\cdot 4\pi\rho_0 q\int_0^{\infty} r\left(1+\frac{r}{a_0}\right)e^{-4r/a_0}dr = 2\pi\rho_0 q\int_0^{\infty} re^{-4r/a_0} + \frac{r^2}{a_0}e^{-4r/a_0}dr$$

$$= 2\pi\rho_0 q\left[\int_0^{\infty} re^{-4r/a_0}dr + \int_0^{\infty}\frac{r^2}{a_0}e^{-4r/a_0}dr\right] = 2\pi\rho_0 q\left[\frac{a_0^2}{16} + \frac{a_0^2}{32}\right]$$

where the common Gaussian integral was used,

$$\int_0^{\infty} x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$$

The electric potential energy is then,

$$\boxed{U = \frac{-3}{8}\frac{\mu_0}{4\pi}\frac{e^2}{a_0} \approx -11\text{ eV}}$$

Problem 3: Review

Procedure:

- – Use the Law of Superposition to add the Electric field of the electron to that of a point charge
- – Take the Electric Potential given to us and use

  $$\vec{E}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

  and show that this is the same as part (a)
- – Calculate the Electric Potential Energy of this electron cloud with

  $$U = \frac{1}{2}\int\int\int_V \phi(\vec{r})\rho(\vec{r})\, d^3r$$

Key Concepts:

- – Using the Law of Superposition allows us to add the two Electric Fields together to create a total Electric Field
- – Here we are showing that if we take the negative gradient of our total potential it will give us the total Electric Field of our cloud
- – We use the full electrostatic potential and just the charge density of the electron to find the potential energy of our electron cloud

Variations:

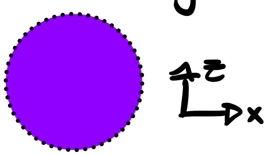
- – We of course can start with a different Electric Field
  - ∗ This would change what our total Electric Field evaluates to as well as what our potential is
- – Just like part (a), we could have a different potential
  - ∗ We would use the same procedure for finding the Electric Field but the math would evaluate slightly differently
- – Just like parts (a) and (b), we can be given a different initial starting point to evaluating this integral
  - ∗ This would change the final answer but not the overall procedure in finding the answer

Problem 4:

The electric potential and electric field of a sphere with a uniform charge on its surface is a common, simple example done in introductory physics classes. Let's change this up a bit for some more useful practice.

(a) As a reminder, write down the electric potential and electric field for a sphere of radius R centered at the origin, $\vec{r} = 0$. Include results for both $r > R$ and $r < R$. You don't have to actually solve this, but it would be good practice just to be sure you can do it.

We first begin by calculating the Electric Field using Gauss' Law



$$\vec{\nabla}\cdot\vec{E} = \frac{Q_{ENC}}{\epsilon} \Rightarrow \iiint_V \vec{\nabla}\cdot\vec{E}(\vec{r})\, d^3r = \oiint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds$$

Inside the sphere : $r < R$

First solving the surface area integral we have

$$\oiint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds = E\int_0^{2\pi}\int_0^{\pi} r^2 \sin(\theta)\, d\theta d\varphi = E\cdot 4\pi r^2$$

Solving the volume integral we have

$$Q_{ENC} = \iiint_V \rho(\vec{r})\, d^3r \Rightarrow \rho(\vec{r}) = 0 \quad \therefore \quad Q_{ENC} = 0 \quad \therefore \quad \vec{E} = 0 \quad \therefore \quad \varphi = 0$$

Outside the sphere : $r > R$, $r = R + a \Rightarrow a \doteq$ distance off sphere

First solving the surface area integral we have

$$\oiint_{S=\partial V} \vec{E}(\vec{r})\cdot\hat{n}\, ds = E\int_0^{2\pi}\int_0^{\pi} r^2 \sin(\theta)\, d\theta d\varphi = E\cdot 4\pi r^2$$

Solving the volume integral we have

$$Q_{ENC} = \iiint_V \rho(\vec{r})\, d^3r = \rho\int_0^{2\pi}\int_0^{\pi}\int_0^{r} R^2 \sin(\theta)\, dR\, d\theta\, d\varphi = \frac{4}{3}\pi r^3 \rho = Q$$

The Electric field and the electric potential is

$$E\cdot 4\pi r^2 = \frac{Q}{\epsilon} \Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon r^2} \ , \ -\nabla\varphi = E \quad \therefore \quad \varphi = -\int E = -\int_0^{r'} \frac{Q}{4\pi\epsilon r^2} = \frac{Q}{4\pi\epsilon r}$$

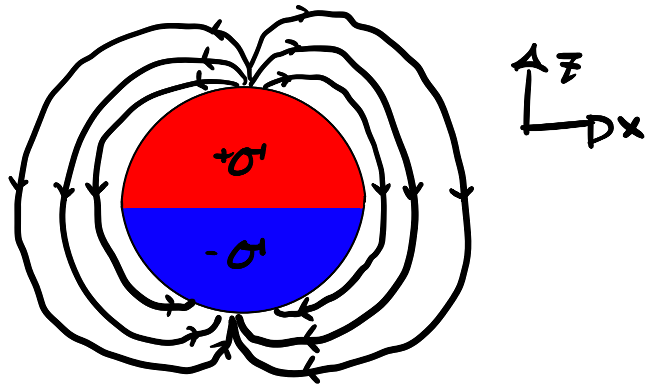
where finally we have

$$\boxed{r < R,\ \vec{E} = 0,\ \varphi = \frac{Q}{4\pi\epsilon r} \ : \ r > R,\ \vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r},\ \varphi = \frac{Q}{4\pi\epsilon r}}$$

Problem 4: Continued

Next, consider the same sphere of radius R but now the surface charge is split between the upper and lower hemispheres. The upper hemisphere ($z > 0$ in Cartesian Co-ordinates, $0 \leq \theta \leq \frac{\pi}{2}$ in Spherical Co-ordinates) has a constant positive surface charge density, $+\sigma$. The lower hemisphere ($z < 0$, $\frac{\pi}{2} \leq \theta \leq \pi$) has a constant negative surface charge density, $-\sigma$. The magnitudes of the charge densities are the same.

(b) Sketch what you think the electric field will look like in the x-z plane shown.



(c) Explain why you can't directly use Gauss' Law to solve for the field of this charge distribution.

We cannot use Gauss' Law to solve for the $\vec{E}$ Field in this distribution because the electric field lines are not perpendicular to the surface of the proposed Gaussian surface and cannot be used.

(d) Using a direct integration, solve for the electric potential everywhere on the z-axis. This includes $R, z < -R$ and $-R < z < R$. Of course, show your work.

The electric potential can be calculated with

$$\varphi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

This then turns into

$$\varphi(\vec{r}) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho(\vec{r}\,')\frac{1}{\sqrt{(\vec{r}-\vec{r}\,')^2}}\, r'^2 \sin(\alpha)\, dr'\, d\alpha\, d\varphi$$

Making some substitutions, we have

$$\varphi(\vec{r}) = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\infty} \frac{\sigma\,\delta(r-R)}{\sqrt{(\vec{r}-\vec{r}\,')^2}}\, r^2\sin(\alpha)\, dr\, d\alpha\, d\varphi - \int_0^{2\pi}\int_{\pi/2}^{\pi}\int_0^{\infty} \frac{\sigma\,\delta(r-R)}{\sqrt{(\vec{r}-\vec{r}\,')^2}}\, r^2\sin(\alpha)\, dr\, d\alpha\, d\varphi$$

$$= R^2\int_0^{2\pi}\int_0^{\pi/2} \underbrace{\frac{\sigma\sin(\alpha)}{\sqrt{R^2+r'^2-2Rr'\cos(\alpha)}}}_{\triangle}\, d\alpha\, d\varphi - R^2\int_0^{2\pi}\int_{\pi/2}^{\pi} \underbrace{\frac{\sigma\sin(\alpha)}{\sqrt{R^2+r'^2-2Rr'\cos(\alpha)}}}_{\square}\, d\alpha\, d\varphi$$

We now make a substitution, $u = \cos(\alpha)$

Problem 4: Continued

$$du = -\sin(\theta)\,d\theta \ , \quad \theta = \pi/2 \to u = 0 \ , \quad \begin{matrix}\theta = 0 \to u = 1 \\ \theta = \pi \to u = -1\end{matrix} \quad \therefore \ d\theta = \frac{-du}{\sin\theta}$$

so now we have

$$\varphi(\vec{r}) = 2\pi R^2 \int_0^1 \sigma (R^2 + r'^2 - 2Rr'u)^{-1/2}\,du - 2\pi R^2 \int_{-1}^0 \sigma (R^2 + r'^2 - 2Rr'u)^{-1/2}\,du$$

now we make another substitution, $V = R^2 + r'^2 - 2Rr'u$

$$dv = -2Rr'\,du \ , \quad u = 0 \to V = R^2 + r'^2 \ , \quad \begin{matrix} u = 1 \to V = R^2 + r'^2 - 2Rr' \\ u = -1 \to V = R^2 + r'^2 + 2Rr' \end{matrix} \ , \quad du = \frac{-dv}{2Rr'}$$

we now have

$$\varphi(r') = 2\pi R^2 \sigma \int_{R^2+r'^2}^{R^2+r'^2-2Rr'} \frac{-1}{2Rr'} V^{-1/2}\,dv + 2\pi R^2 \sigma \int_{R^2+r'^2}^{R^2+r'^2+2Rr'} -\frac{1}{2Rr'} V^{-1/2}\,dv$$

$$= \frac{\pi R \sigma}{r'}\left(-2V^{1/2} \Big|_{R^2+r'^2}^{R^2+r'^2-2Rr'} - 2V^{1/2} \Big|_{R^2+r'^2}^{R^2+r'^2+2Rr'} \right)$$

$$= \frac{-2\pi R \sigma}{r'}\left(\left(\sqrt{R^2+r'^2-2Rr'} - \sqrt{R^2+r'^2}\right) + \left(\sqrt{R^2+r'^2+2Rr'} - \sqrt{R^2+r'^2}\right) \right)$$

$$= \frac{2\pi R \sigma}{r'}\left(2\sqrt{R^2+r'^2} - \sqrt{R^2+r'^2-2Rr'} - \sqrt{R^2+r'^2+2Rr'} \right)$$

The electric potential is Finally, $r' \equiv z$

$$\boxed{\varphi(r') = \frac{2\pi R \sigma}{r'}\left(2\sqrt{R^2+r'^2} - \sqrt{R^2+r'^2-2Rr'} - \sqrt{R^2+r'^2+2Rr'} \right)}$$

(e) Calculate the electric field along the z-axis, $\vec{E}(z) = E(z)\,\hat{e}_z$ using your result for the potential.

we will calculate the electric field with

$$-\vec{\nabla}\varphi = \vec{E}$$

In our case, $r' \equiv z$ so we find $\vec{E}(\hat{z})$ to be

$$\vec{E}(\hat{z}) = -\left(\frac{\partial}{\partial z}\right)\left(\frac{2\pi R \sigma}{z}\left(2\sqrt{R^2+z^2} - \sqrt{R^2+z^2-2Rz} - \sqrt{R^2+z^2+2Rz} \right)\right)\hat{z}$$

Problem 4: Continued

$$\vec{E}(z) = -\left(-\frac{2\pi R\sigma'}{z^2}\left(2\sqrt{R^2+z^2} - \sqrt{R^2+z^2-2Rz} - \sqrt{R^2+z^2+2Rz}\right) \Rightarrow\right.$$

$$\Rightarrow \left. + \frac{2\pi R\sigma'}{z}\left(\frac{2z}{(R^2+z^2)^{1/2}} - \frac{z-R}{(R^2+z^2-2Rz)^{1/2}} - \frac{z+R}{(R^2+z^2+2Rz)^{1/2}}\right)\right)\hat{z}$$

Simplifying this expression we have

$$\vec{E}(z) = -\frac{2\pi R\sigma'}{z}\left(\left(\frac{2z}{(R^2+z^2)^{1/2}} - (\pm 1) - (\pm 1)\right) - \frac{1}{z}\left(2\sqrt{R^2+z^2} - (\pm(R-z)) - (\pm(R+z))\right)\right)\hat{z}$$

And Finally we have

$$\boxed{\vec{E}(z) = \frac{2\pi R\sigma'}{z}\left(\frac{1}{z}\left(2\sqrt{R^2+z^2} \mp (R-z) \mp (R+z)\right) - \left(\frac{2z}{(R^2+z^2)^{1/2}} \mp (2)\right)\right)\hat{z}}$$

(f) The properties of the potential and field of this "Dipole Sphere" are quite different from those of the uniformly charge sphere. What are some of the differences? Explain the physics of these differences.

For example, you might consider what happens for $|z| \to \infty$, $|z| \to R$, and the field and potential in the interior of the spheres.

If we let $z \to \infty$

$$\vec{E}(z\to\infty) = \lim_{z\to\infty} \cancelto{0}{\frac{2\pi R\sigma'}{z}}\left(\frac{1}{z}\left(2\sqrt{R^2+z^2} \mp (R-z) \mp (R+z)\right) - \left(\frac{2z}{(R^2+z^2)^{1/2}} \mp (2)\right)\right)\hat{z} = 0$$

$\vec{E}$ will go to 0. If we let $z \to R$

$$\vec{E}(z\to R) = \lim_{z\to R} \frac{2\pi R\sigma'}{z}\left(\frac{1}{z}\left(2\sqrt{R^2+z^2} \mp (R-z) \mp (R+z)\right) - \left(\frac{2z}{(R^2+z^2)^{1/2}} \mp (2)\right)\right)\hat{z} = 2\sqrt{2}\pi\sigma'\hat{z}$$

$\vec{E}$ will go to $2\sqrt{2}\pi\sigma'\hat{z}$

In the limiting cases as $\hat{z} \to \infty$, $\vec{E} \to 0$ and as $\hat{z} \to 0$, $\vec{E} \to 2\sqrt{2}\pi\sigma'\hat{z}$. This makes sense because $\vec{E}$ should vanish as we get far away and should approach a constant value

Problem 4: Review

Procedure:

- – Use Gauss' Law to determine the Electric Field of our sphere

$$\int\int\int_V \vec{\nabla}\cdot\vec{E}(\vec{r})\,d^3x = \oint\oint_{S=\delta V}\vec{E}(\vec{r})\cdot\hat{n}\,dS \quad , \quad Q_{\text{ENC}} = \int\int\int_V \rho(\vec{r})\,d^3r$$

- – Draw a picture of this sphere
- – Discuss why we cannot use Gauss' Law for this problem
- – Use direct integration to solve for the potential with

$$\phi(\vec{r}) = \int\int\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d^3\vec{r}'$$

- – Calculate the Electric Field by taking the negative gradient of the potential
- – Evaluate what the field will go to with the ranging values

Key Concepts:

- – Due to the symmetry in our system we are able to solve for the Electric Field by using Gauss' Law
- – With this sphere being cut in half it is essentially acting like a dipole
- – Due to the charge distribution of our sphere we cannot use Gauss' Law to solve for the Electric Field
- – When Gauss' Law fails, we can always use direct integration to solve for the Potential and then eventually the Electric Field
- – Once the potential for our system has been found we can take a negative gradient of it and solve for the Electric Field
- – As our distance from our sphere increases the Electric Field goes to zero and as our distance decreases our Electric Field approaches a constant value

Variations:

- – We could be given a different shape
    - ∗ This would cause everything after it to follow this new shape with the same procedure