Importantly: The results are independent of how exactly the energy levels of the particles are being grouped into cells.

In fact, the results can be derived without this "grouping into cells". (This is the reason why the grouping is legitimate.)

plural since we will be looking at different statistics

8.6 The ideal gases: grand canonical ensemble

Our starting point is the partition function $Q_N(T,V)$ in the canonical ensemble:

$$Q_N(T,V) = \sum_{\{n_{\vec{k}}\}} g\{n_{\vec{k}}\}\, e^{-\beta E\{n_{\vec{k}}\}}$$

Constraint on occupation numbers: $\sum_{\vec{k}} n_{\vec{k}} = N$

Also: $E\{n_{\vec{k}}\} = \sum_{\vec{k}} n_{\vec{k}} \varepsilon_{\vec{k}}$

$g\{n_{\vec{k}}\}$: statistical weight factor associated with the distribution set $\{n_{\vec{k}}\}$

$\sum_{\{n_{\vec{k}}\}} \ldots$ goes over all distribution sets consistent with the particle number constraint equation

For identical bosons: $g\{n_{\vec{k}}\} = 1$

For identical fermions: $g\{n_{\vec{k}}\} = 1$ provided all $n_{\vec{k}}$ are zero or one

$= 0$ if $n_{\vec{k}}$ are not zero or one

For Boltzmann particles: $g\{n_{\vec{k}}\} = \prod_{\vec{k}} \frac{1}{n_{\vec{k}}!}$

After a bit of work, one finds:

$$\boxed{\langle n_{\vec{k}} \rangle = \frac{1}{z^{-1} e^{\beta \varepsilon_{\vec{k}}} + a}}$$

mean or average occupation number of single-particle energy level $\varepsilon_{\vec{k}}$

$$a = \begin{cases} -1 & \text{identical bosons} \\ +1 & \text{identical fermions} \\ 0 & \text{Boltzmann} \end{cases}$$

$$N = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle$$

$$U = \sum_{\vec{k}} \varepsilon_{\vec{k}} \langle n_{\vec{k}} \rangle$$

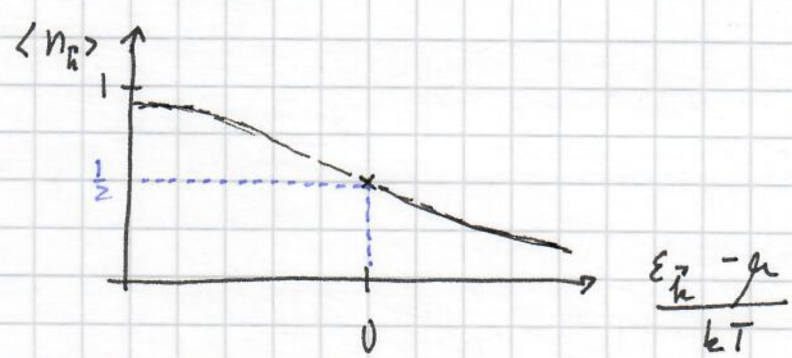
Let's look at the average or mean occupation numbers:

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{(\varepsilon_{\vec{k}} - \mu)/kT} + a}$$

a = 1 (Fermi-Dirac distribution): $n_{\vec{k}}$ can only take the values 0 or 1 $\Rightarrow \langle n_{\vec{k}} \rangle \leq 1$

$\varepsilon_{\vec{k}} < \mu \Rightarrow e^{(\varepsilon_{\vec{k}} - \mu)/kT}$ is exponentially decaying.

If $|\varepsilon_{\vec{k}} - \mu| \gg kT \Rightarrow \langle n_{\vec{k}} \rangle \longrightarrow 1$



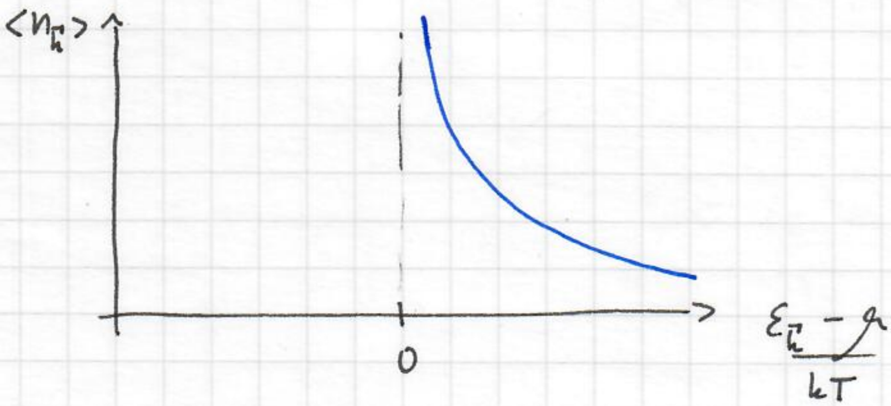
$a = -1$ (Bose-Einstein distribution):

- requires $\underbrace{z e^{-\beta \varepsilon_{\vec{k}}}}_{e^{\beta(\mu - \varepsilon_{\vec{k}})}} < 1$

$$e^{\beta(\mu - \varepsilon_{\vec{k}})} < 1$$

$$\Rightarrow \mu < \varepsilon_{\vec{k}} \quad \text{for all } \varepsilon_{\vec{k}}$$

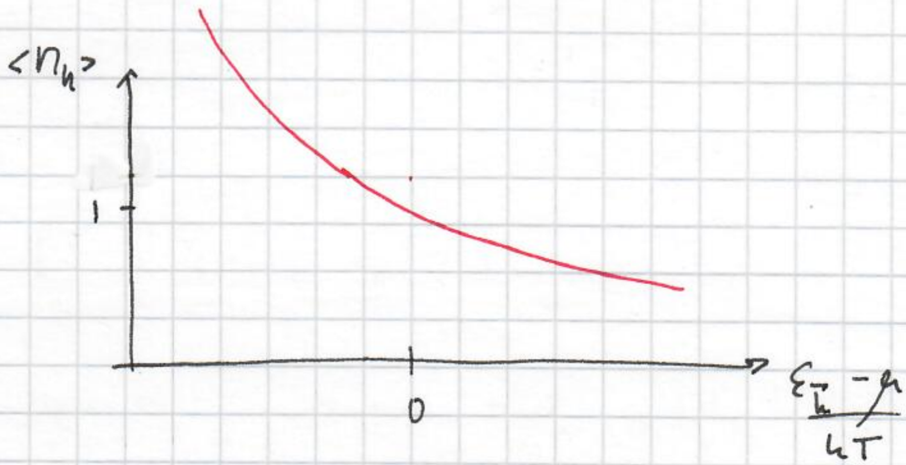
say, the lowest possible $\varepsilon_{\vec{k}}$ is ε_0:
as $\mu \rightarrow \varepsilon_0$, the occupation of that level becomes macroscopically large



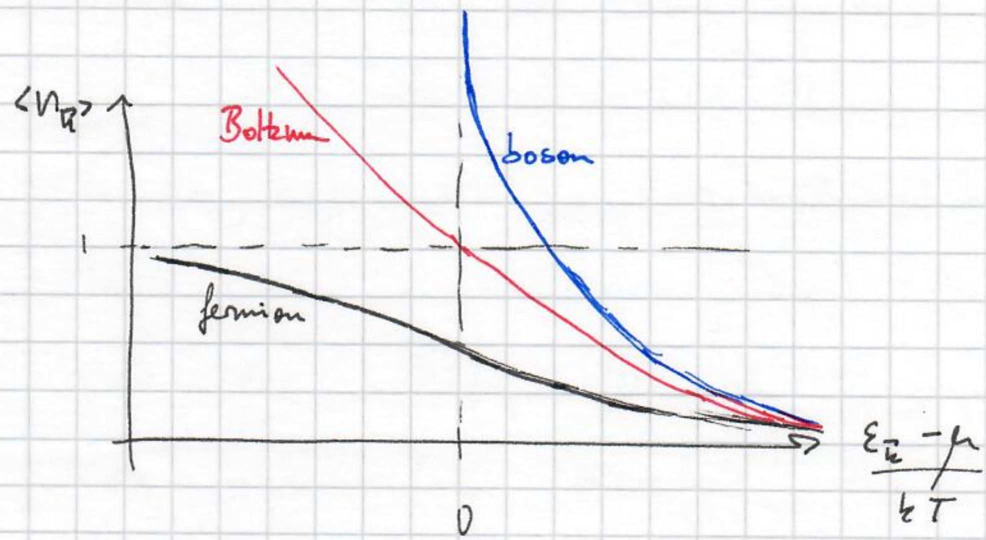
$a = 0$ (Maxwell-Boltzmann distribution):

$$\langle n_{\vec{k}} \rangle = e^{\mu/kT} \, e^{-\varepsilon_{\vec{k}}/kT}$$

recall: this is a quantum result
but without the quantum statistics



Let's put the three plots together:



Large $e^{(\varepsilon_{\vec{k}} - \mu)/kT}$ limit:

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{(\varepsilon_{\vec{k}} - \mu)/kT} + a} \xrightarrow[\text{(small compared to } e^{\cdots})]{\text{neglect } a = \pm 1} \frac{1}{e^{(\varepsilon_{\vec{k}} - \mu)/kT}}$$

$$= e^{\mu/kT} \, e^{-\varepsilon_{\vec{k}}/kT} \quad \text{(Boltzmann)}$$

Physical picture:

large $\exp[(\varepsilon_{\vec{k}} - \mu)/kT]$ implies $\langle n_{\vec{k}} \rangle \ll 1$

If $\langle n_{\vec{k}} \rangle \ll 1$, then the $g\{n_{\vec{k}}\}$ factors for the Boltzmann gas approach 1, i.e., $g\{n_{\vec{k}}\} = \prod_{\vec{k}} \frac{1}{n_{\vec{k}}!} \to 1$.

Alternatively, we can think about what happens to the Bose and Fermi occupations or the associated exchange statistics: if we have very small occupations, then the particles hardly feel the existence of other particles. Therefore, the exchange statistics kicks in less and less as the average or mean occupation numbers go down.

It is also instructive to look at the relative mean-square fluctuations:

$$\frac{\langle n_{\vec{k}}^2 \rangle - \langle n_{\vec{k}} \rangle^2}{\langle n_{\vec{k}} \rangle^2} = \frac{1}{\langle n_{\vec{k}} \rangle} - a$$

without derivation (see exercise 8.4 of text)

Boltzmann gas: $a = 0$ $\rightsquigarrow$ we refer to the $\frac{1}{\langle n_{\vec{k}} \rangle}$ behavior as "normal"

$$\sqrt{\frac{\langle n_{\vec{k}}^2 \rangle - \langle n_{\vec{k}} \rangle^2}{\langle n_{\vec{k}} \rangle^2}} = \frac{1}{\sqrt{\langle n_{\vec{k}} \rangle}}$$

Fermi gas: $a = +1$ $\rightsquigarrow$ "below normal"

relative mean square fluctuations tend to vanish as $\langle n_{\vec{k}} \rangle \to 1$

Bose gas: $a = -1$ $\rightsquigarrow$ "above normal"

$$\frac{1}{\langle n_{\vec{k}} \rangle} + 1$$

"particle" "wave"

Let's look at statistics of occupation numbers:

$p_{\varepsilon_{\vec{k}}}(n)$: probability that there are n particles in a state with energy $\varepsilon_{\vec{k}}$

One finds:

Bose-Einstein : $p_{\varepsilon_{\vec{k}}}(n) = \dfrac{\left(\langle n_{\vec{k}} \rangle\right)^n}{\left(\langle n_{\vec{k}} \rangle + 1\right)^{n+1}}$

Fermi-Dirac : $p_{\varepsilon_{\vec{k}}}(n) = \begin{cases} 1 - \langle n_{\vec{k}} \rangle & \text{for } n = 0 \\ \langle n_{\vec{k}} \rangle & \text{for } n = 1 \end{cases}$

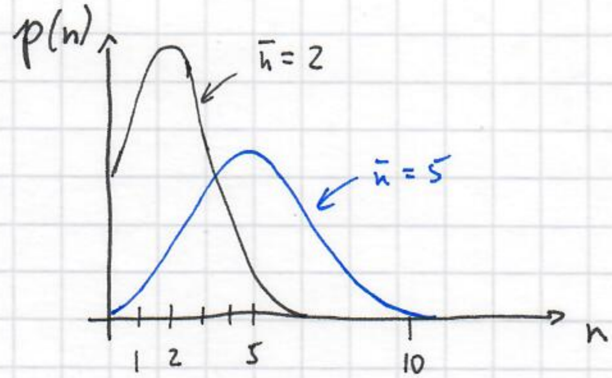
Maxwell-Boltzmann: $p_{\varepsilon_{\vec{k}}}(n) = \dfrac{\left(\langle n_{\vec{k}} \rangle\right)^n}{n!} e^{-\langle n_{\vec{k}} \rangle}$

this is a Poisson distribution

Boltzmann gas:

Let's look at $p_{\varepsilon_{\vec{k}}}(n) = \dfrac{(\langle n_{\vec{k}} \rangle)^n}{n!} e^{-\langle n_{\vec{k}} \rangle}$

Let $\langle n_{\vec{k}} \rangle = \bar{n}$ $\rightsquigarrow$ simplifies notation



$$\langle n \rangle = \langle n_{\vec{k}} \rangle$$ of course, this has to be true by definition

$$\langle n^2 \rangle = \langle n_{\vec{k}} \rangle^2 + \langle n_{\vec{k}} \rangle$$

$$\Rightarrow \Delta n^2 = \langle n^2 \rangle - (\langle n \rangle)^2 = \langle n_{\vec{k}} \rangle$$

$$\Rightarrow \sqrt{\frac{\Delta n^2}{\langle n \rangle^2}} = \sqrt{\frac{\langle n_{\vec{k}}^2 \rangle - \langle n_{\vec{k}} \rangle^2}{\langle n_{\vec{k}} \rangle^2}}$$

$$= \frac{1}{\sqrt{\langle n_{\vec{k}} \rangle}}$$

this is just saying that the statistics of the $n_{\vec{k}}$ (Poisson distr.) is consistent with our result from p. 201

of course, this is what we had before

We can also look at

$$\frac{p_{\varepsilon_{\vec{k}}}(n)}{p_{\varepsilon_{\vec{k}}}(n-1)} = \frac{\langle n_{\vec{k}} \rangle}{n}$$

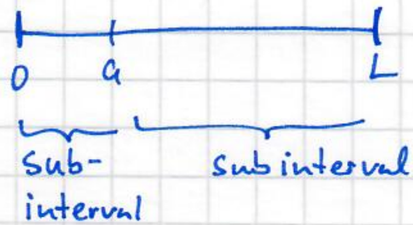
Scaling with $\frac{1}{n}$ is behavior typical of uncorrelated events

recall: all these comments refer to distribution for fixed $\langle n_{\vec{k}} \rangle$

on average, we have 5 particles in state with energy $\varepsilon_{\vec{k}}$: but there is
a probability to have 0,
a probability to have 1,
" " " " 2,
" " " " 3,
⋮ 4
⋮ 5
⋮ 6
⋮ ⋮

Detour on binomial and Poisson distribution:

Let's consider interval $[0, L]$



Want to distribute N particles completely randomly so that the probability that a particle be found in first subinterval is $\frac{a}{L}$ and that a particle be found in second subinterval is $\frac{L-a}{L}$.

Probability that n particles are in interval $[0, a]$ is

$$p_n = \underbrace{\left(\frac{a}{L}\right)^n \left(\frac{L-a}{L}\right)^{N-n}} \binom{N}{n}$$

probability to have n particles in $[0, a]$
$\rightsquigarrow$ probability for one particular arrangement

number of ways of choosing n objects from set of N

Want to rewrite p_n ...

$$\binom{N}{n} = \frac{N(N-1)(N-2)\cdot\ldots\cdot(N-n+1)}{n!}$$

$$= N^n \; \frac{1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{N}\right)}{n!}$$

n terms
in numerator

$$\Rightarrow p_n = \left(\frac{aN}{L}\right)^N \left(1-\frac{a}{L}\right)^{N-n} \frac{1}{n!} \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{N}\right)$$

$$= \bar{n}^n \frac{1}{n!} \left(1-\frac{\bar{n}}{N}\right)^N \frac{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdot\ldots\left(1-\frac{n-1}{N}\right)}{\left(1-\frac{a}{L}\right)^n}$$

using $\sum_{n=0}^{N} p_n \cdot n = \underbrace{\langle n \rangle}_{\bar{n}} = \frac{a}{L} N$

$$\sum_{n=0}^{N} p_n = 1$$

$$\underbrace{\underbrace{\sum_{n=0}^{N} p_n^2}_{\langle n^2 \rangle} - \langle n \rangle^2}_{(\Delta n)^2} = \frac{a}{L}\left(1-\frac{a}{L}\right)N$$

$$p_n \xrightarrow[\substack{\frac{a}{L}\to 0 \\ N\to\infty}]{} \bar{n}^n \frac{1}{n!} e^{-\bar{n}} \cdot 1 = \underbrace{\frac{\bar{n}^n}{n!} e^{-\bar{n}}}_{\text{Poissonian distribution}}$$

Boltzmann particles $\rightsquigarrow$ distinguishable particles

$\rightsquigarrow$ no correlations

(Poissonian distribution for randomly selected "events")

Compare to Bose-Einstein:

$$p_{\varepsilon_{\vec{k}}}(n) = \frac{(\langle n_{\vec{k}} \rangle)^n}{(\langle n_{\vec{k}} \rangle + 1)^{n+1}}$$

$$\Rightarrow \frac{p_{\varepsilon_{\vec{k}}}(n)}{p_{\varepsilon_{\vec{k}}}(n-1)} = \frac{\langle n_{\vec{k}} \rangle}{\langle n_{\vec{k}} \rangle + 1}$$

independent of n, i.e., independent of the number of particles already in the state

$\Rightarrow$ by comparison to Boltzmann case, where we had $\langle n_{\vec{k}} \rangle / n$, bosons have a tendency to bunch together (i.e., we have positive statistical correlations among bosons)

in contrast, fermions exhibit negative statistical correlations