Homework # 1

(A)

(1)

a) $j_1 = 3/2 \Longrightarrow -j_1 \leq m_1 \leq j_1$

$j_2 = 1/2 \longrightarrow -j_2 \leq m_2 \leq j_2$

$$|j_1 - j_2| \leq J \leq j_1 + j_2$$

$\therefore \; J_1 = 1, 2$, with $-J \leq n \leq J$.

Defining

$$|3/2, 1/2; Jn\rangle \equiv |Jn\rangle$$

$$|3/2\, 1/2; m_1, m_2\rangle \equiv |m_1, m_2\rangle$$

then

$$|22\rangle = |3/2, 1/2\rangle \qquad (1)$$

which is the maximally polarized ~~state~~.

To find the other states for

$J=2$, we apply J_-

$$J_-|J,m\rangle = \hbar\sqrt{(J+m)(J-m+1)}\,|J,m-1\rangle$$

$\therefore$

$$J_-|2,2\rangle = 2\hbar|2,1\rangle$$

$$= (J_{1-}+J_{2-})|3/2\ 1/2\rangle$$

$$= \sqrt{3}\hbar\,|1/2\ 1/2\rangle + \hbar\,|1/2,-1/2\rangle$$

$$\Rightarrow\quad |2,1\rangle = \frac{\sqrt{3}}{2}|1/2,1/2\rangle + \frac{1}{2}|3/2,-1/2\rangle \qquad (2)$$

Also,

$$J_-|2\,1\rangle = \hbar\sqrt{6}\,|2\,0\rangle$$

$$= (J_{1-} + J_{2-})\left(\frac{\sqrt{3}}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \frac{1}{2}|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle\right)$$

$$= \hbar\sqrt{3}\,|-\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \frac{\sqrt{3}}{2}\hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$+ \hbar\frac{\sqrt{3}}{2}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$\therefore$

$$|20\rangle = \frac{1}{\sqrt{2}}|-\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad (3)$$

$$J_-|20\rangle = \hbar\sqrt{6}\,|2,-1\rangle$$

$$= (J_{1-} + J_{2-})\left(\frac{1}{\sqrt{2}}|-\tfrac{1}{2}\,\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$= \hbar\frac{\sqrt{3}}{\sqrt{2}}|-\tfrac{3}{2}\,\tfrac{1}{2}\rangle + \frac{\hbar}{\sqrt{2}}|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$+ \hbar\sqrt{2}\,|-\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$\therefore$

$$|2,-1\rangle = \frac{1}{2}|-3/2,1/2\rangle + \frac{\sqrt{3}}{2}|-1/2,-1/2\rangle \quad (4)$$

and finally:

$$|2,-2\rangle = |-3/2,-1/2\rangle \quad (5)$$

For the states with $\underline{J=1}$, we start with the state:

$$|1,1\rangle$$

Imposing the condition that:

$$\langle 11|21\rangle = 0$$

$\therefore$

$$|11\rangle = \frac{1}{2}|1/2\ 1/2\rangle - \frac{\sqrt{3}}{2}|3/2,-1/2\rangle \quad (6)$$

which is well defined up to a global phase. Applying J_- on this state,

$$J_-|11\rangle = \hbar\sqrt{2}\,|10\rangle$$

$$= \left(J_{1-} + J_{2-}\right)\left(\frac{1}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \frac{\sqrt{3}}{2}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle\right)$$

$$= \hbar\left|-\tfrac{1}{2},+\tfrac{1}{2}\right\rangle + \frac{\hbar}{2}\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle - \frac{3\hbar}{2}\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

$\Rightarrow$

$$|10\rangle = \frac{1}{\sqrt{2}}\left|-\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle \qquad (7)$$

Finally,

$$|1,-1\rangle = \frac{\sqrt{3}}{2}\left|-\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle - \frac{1}{2}\left|-\tfrac{1}{2}\,-\tfrac{1}{2}\right\rangle$$

by applying J_- on $|10\rangle$.

b)

$$H|s,m\rangle = \frac{1}{2}\alpha \vec{S}_1 \cdot \vec{S}_2 |s,m\rangle$$

$$= \frac{\alpha\hbar^2}{2}\Big[s(s+1) \underbrace{- \tfrac{3}{2}\left(\tfrac{5}{2}\right) - \tfrac{1}{2}\tfrac{3}{2}}_{-9/2}\Big]|s,m\rangle$$

For $s=1$,

$$E_1 = -\frac{5\alpha\hbar^2}{4}$$

For $s=2$,

$$E_2 = \frac{3\alpha\hbar^2}{4}.$$

From the C.G. coefficients,

$$|2\,1\rangle = \frac{\sqrt{3}}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{1}{2}|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

$$|1\,1\rangle = \frac{1}{2}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle - \frac{\sqrt{3}}{2}|\tfrac{3}{2},-\tfrac{1}{2}\rangle$$

Inverting,

$$\begin{cases} |\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{\sqrt{3}}{2}|2,1\rangle + \frac{1}{2}|1,1\rangle \\ |\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{2}|2,1\rangle - \frac{\sqrt{3}}{2}|1,1\rangle \end{cases}$$

If $\psi(0) = |\tfrac{1}{2},\tfrac{1}{2}\rangle$, then:

$$\psi(t) = \frac{\sqrt{3}}{2} e^{-iE_2 t/\hbar}|2,1\rangle + \frac{1}{2} e^{-iE_1 t/\hbar}|1,1\rangle$$

$\therefore$ the probability of finding it in the state $|3/2,-1/2\rangle$ is:

$$P = |\langle 3/2,-1/2|\psi(t)\rangle|^2$$

$$= \frac{3}{16}\left| e^{-iE_2 t/\hbar} - e^{-iE_1 t/\hbar}\right|^2$$

$$= \frac{3}{8} \left\{ 1 - \cos\left[(E_1 - E_2) t/\hbar \right] \right\}$$

$$= \frac{3}{8} \left[1 - \cos(2\alpha\hbar t) \right]$$

$$= \frac{3}{4} \sin^2(\alpha\hbar t) .$$

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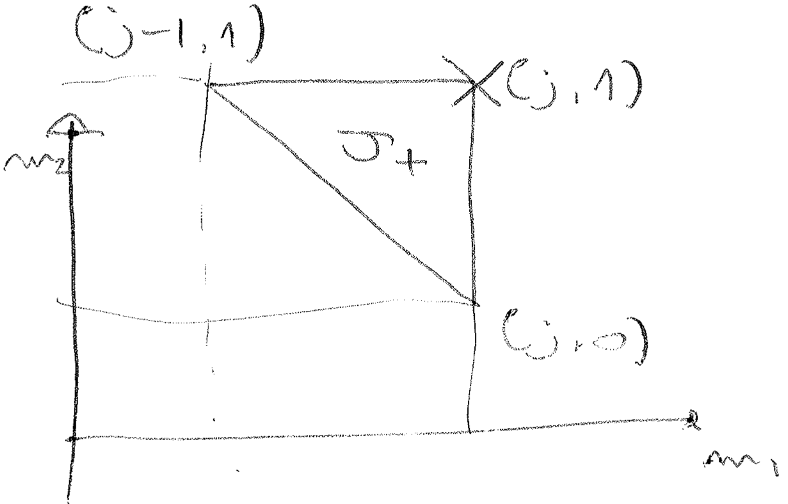
Using the recursion relation J_+



$$\left| \begin{array}{l} j_1 = j \\ j_2 = 1 \end{array} \right.$$

where $\left. \begin{array}{l} m_1 = j \\ m_2 = 1 \end{array} \right\} \quad \begin{array}{l} m_1 + m_2 = m + 1 \\ \therefore\ m = j \end{array}$

$$\sqrt{(j-m)(j+m+1)}\,\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j\, m+1\rangle$$

$$= \sqrt{(j_1-m_1+1)(j_1+m_1)}\,\langle j_1 j_2 ; m_1-1\, m_2 | j\, m\rangle$$

$$+ \sqrt{(j_2-m_2+1)(j_2+m_2)}\,\langle j_1 j_2 ; m_1\, m_2-1 | j\, m\rangle$$

$$0 = \sqrt{2j}\, \langle j1; j-1, 1 | j j \rangle$$

$$+ \sqrt{2}\, \langle j1; j 0 | j j \rangle$$

Using the normalization relation:

$$\sum_{m_1 m_2} |\langle j_1 j_2; m_1 m_2 | j_1 j_2; J M \rangle|^2 = 1$$

where $m_1 + m_2 = M = j$

$$\therefore |\langle j1; j-1, 1 | j1; j j \rangle|^2 + |\langle j1; j 0 | j1; j j \rangle|^2 = 1$$

$$\Longrightarrow \langle j1; j0 | j1; jj \rangle = \sqrt{\frac{j}{j+1}}.$$

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$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, n_z$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}} (n_x \pm i n_y)$$

a)

mapping $\sqrt{\frac{4\pi}{3}}\, Y_1^m \longrightarrow T_m^{(1)}$

$$\hat{n} \longrightarrow \vec{p}$$

then:

$$\begin{cases} p_z = T_0^1 \\ \mp \dfrac{p_x \pm i p_y}{\sqrt{2}} = T_{\pm 1}^1 \end{cases}$$

$$\therefore \quad p_y = -\frac{1}{\sqrt{2}\, i}\left(T_{+1}^1 + T_{-1}^1\right)$$

$$p_x = \frac{1}{\sqrt{2}}\left(-T_{+1}^1 + T_{-1}^1\right)$$

Following the Wigner Eckart theorem,

$$\langle \alpha', j', m' | T_q^{(1)} | \alpha\, j, m \rangle$$

$$= \langle j 1 ; m q | j 1 ; j' m' \rangle \times \underbrace{\frac{\langle \alpha', j' \| T^{(1)} \| \alpha, j \rangle}{\sqrt{2j+1}}}_{\text{Const.}}$$

The selection rules are:

$$m' = m + q$$

$$|j-1| \leq j' \leq j+1$$

Also, since $T^{(1)}$ is odd under parity,

$$j + j' = 2n+1, \; n \in \mathbb{Z}, \; \therefore \; j' = |j \pm 1|$$

For $j' = 2$
$j = 1$

$$\langle \alpha, 2, m \,|\, T^{(1)}_{0} \,|\, \alpha, 1, m \rangle$$

$$= \langle 11; m0 \,|\, 11; 2m \rangle \times e$$

$$= \begin{cases} e/\sqrt{2}, & m = 1 \\ \sqrt{2}\,e/\sqrt{3}, & m = 0 \\ e/\sqrt{2}, & m = -1 \end{cases}$$

Also,

$$\langle \alpha, 2, m+1 \,|\, T^{(1)}_{1} \,|\, \alpha, 1, m \rangle$$

$$= \langle 11; m1 \,|\, 11; 2, m+1 \rangle \times e$$

$$= \begin{cases} e, & m = 1 \\ e/\sqrt{2}, & m = 0 \\ e/\sqrt{6}, & m = -1 \end{cases}$$

$$\langle \alpha, 2, m-1 | T^{(1)}_{-1} | \alpha, 1, m \rangle$$

$$= \langle 11; m, -1 | 11; 2, m-1 \rangle \times c$$

$$= \begin{cases} c/\sqrt{6}, & m = 1 \\ c/\sqrt{2}, & m = 0 \\ c, & m = -1 \end{cases}$$

Hence,

$$\langle \alpha, 2, m | P_z | \alpha, 2, m \rangle$$

$$= \begin{cases} \frac{1}{\sqrt{2}}, & m = 1 \\ \sqrt{2}/\sqrt{3}, & m = 0 \\ \frac{1}{\sqrt{2}}, & m = -1 \end{cases} \times \underbrace{\frac{\langle \alpha, 2 \| P_z | \alpha, 1 \rangle}{\sqrt{3}}}_{c}$$

$$\langle \alpha, 2, m' | P_x | \alpha, 1, m \rangle$$

$$= -\langle \alpha, 2, m' | \frac{T^{(1)}_{+1} - T^{(1)}_{-1}}{\sqrt{2}} | \alpha, 1, m \rangle$$

$$= \frac{1}{\sqrt{2}} \times \begin{cases} c/\sqrt{6}, & m = 1, m' = 0 \\ c/\sqrt{2}, & m = 0, m' = -1 \\ c, & m = -1, m' = -2 \\ -c, & m = 1, m' = 2 \\ -c/\sqrt{2}, & m = 0, m' = 1 \\ -c/\sqrt{6}, & m = -1, m' = 0. \end{cases}$$

Finally,

$$\langle \alpha, 2, m' | P_y | \alpha, 1, m \rangle$$

$$= -\langle \alpha, 2, m' | \frac{T^{(1)}_{+1} + T^{(1)}_{-1}}{\sqrt{2}\, i} | \alpha, 1, m \rangle$$

$$= -\frac{C}{\sqrt{2}\, i} \times \begin{cases} \frac{1}{\sqrt{6}}, & m = 1,\ m' = 0 \\ \frac{1}{\sqrt{2}}, & m = 0,\ m' = -1 \\ 1, & m = -1,\ m' = -2 \\ 1, & m = 1,\ m' = 2 \\ \frac{1}{\sqrt{2}}, & m = 0,\ m' = 1 \\ \frac{1}{\sqrt{6}}, & m = -1,\ m' = 0 \end{cases}$$

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b) $$T^{(2)}_q = \sum_{q_1 q_2} T^{(1)}_{q_1} T^{(1)}_{q_2} \langle 11; q_1 q_2 | 11; 2q \rangle$$

where $\underline{q_1 + q_2 = q}$.

$$T^{(2)}_0 = T^{(1)}_0 T^{(1)}_0 \overbrace{\langle 11; 00 | 11, 20 \rangle}^{2/\sqrt{6}}$$

$$+ 2 T^{(1)}_{-1} T^{(1)}_{+1} \underbrace{\langle 11; 1-1 | 11; 20 \rangle}_{1/\sqrt{6}}$$

$$= p_z^2 \frac{2}{\sqrt{6}} - \frac{p_x^2 + p_y^2}{\sqrt{6}}$$

$$T^{(2)}_1 = 2 T^{(1)}_0 T^{(1)}_1 \underbrace{\langle 11; 01 | 11; 21 \rangle}_{1/\sqrt{2}}$$

$$= - p_z (p_x + i p_y)$$

$$T^{(2)}_{-1} = 2 T^{(1)}_{-1} T^{(1)}_0 \overbrace{\langle 11; -10 | 11; 2-1 \rangle}^{1/\sqrt{2}}$$

$$= p_z (p_x - i p_y)$$

$$T^{(2)}_{2} = T^{(1)}_{1}\, T^{(1)}_{1} \underbrace{\langle 11; 11 | 11, 22 \rangle}_{1}$$

$$= \frac{1}{2} (p_x + i p_y)^2$$

$$T^{(2)}_{-2} = T^{(1)}_{-1}\, T^{(1)}_{-1} \underbrace{\langle 11; -1 -1 | 11, 2, -2 \rangle}_{1}$$

$$= \frac{1}{2} (p_x - i p_y)^2 .$$