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Quantum Mechanics 1

PHYS 5393 Homework Assignment #1

Problems: {1.1, 1.3, 1.7, 1.8, Q-1}

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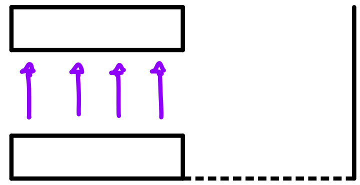
Problem 1: 1.1

A beam of silver atoms is created by heating a vapor in an oven to 1000^o c, and selecting atoms with a velocity close to the mean of the thermal distribution. The beam moves through a one-meter long magnetic field with a vertical gradient $10T/m$, and impinges a screen one meter downstream of the end of the magnet. Assuming the silver atom has spin 1/2 with a magnetic moment of one Bohr magneton, find the separation distance in millimeters of the two states on the screen.



$F_z = \frac{\partial}{\partial z}(\vec{u}\cdot\vec{B}) \simeq \mu_z \frac{\partial B_z}{\partial z}$: 1 Bohr magneton $= 9.27\times10^{-24}\,\frac{J}{T}$ $\therefore$ $\mu_z = 9.27\times10^{-24}\,\frac{J}{T}$

$$F_z = 9.27\times10^{-24}\,\frac{J}{T} \times 10\,\frac{T}{m} = 9.27\times10^{-23}\,\frac{J}{m} = 9.27\times10^{-23}\,N$$

$m_{Ag} = 1.79\times10^{-25}\,kg$: $F_z = m_{Ag}a_z = 9.27\times10^{-23}\,N$: $a_z = \frac{9.27\times10^{-23}\,N}{1.79\times10^{-25}\,kg} = 517.88\,\frac{m}{s^2}$

Thermal velocity of atoms : $V_{th} = \sqrt{\frac{8k_BT}{m\pi}}$: $k_B = 1.38\times10^{-23}\,\frac{m^2 kg}{s^2 K}$: $T = 1273\,K$

$$V_{TH} = \sqrt{\frac{8\cdot 1.38\times10^{-23}\,m^2kg/s^2K\cdot 1273\,K}{1.79\times10^{-25}\,kg\cdot\pi}} = 499.92\,m/s$$

Time to get through magnets : $\Delta x = v_0\Delta t + \overset{a=0}{\cancel{\frac{1}{2}a\Delta t^2}}$ $\therefore$ $\Delta t = \frac{\Delta x}{v_0}$

$$\Delta t = \frac{\Delta x}{v_0} = \frac{1\,m}{499.92\,m/s} = 0.002\,s$$

Velocity in z at end of magnets : $V_1 = \overset{0=V_0}{\cancel{V_0}} + a_z\Delta t$

$$V_1 = (517.88\,m/s^2)(0.002\,s) = 1.036\,m/s$$

Distance traveled in z after through magnets : $\Delta z = \overset{0=V_0}{\cancel{V_0\Delta t}} + \frac{1}{2}a_z\Delta t^2$

$$\Delta z_1 = \frac{1}{2}a_z\Delta t^2 = \frac{1}{2}(517.88\,m/s)(0.002\,s)^2 = 0.001\,m$$

Time to get to screen from end of magnets will be the same as to travel through magnets due to there being no acceleration in the x direction $\therefore$ $\Delta t = 0.002\,s$.

Distance traveled in z after magnets : $\Delta z = V_0\Delta t + \overset{0=a}{\cancel{\frac{1}{2}a\Delta t^2}}$

$$\Delta z_2 = V_0\Delta t = 1.036\,m/s\,(0.002\,s) = 0.002\,m$$

Total distance between slits : $\Delta z = 2(\Delta z_1 + \Delta z_2) = 2(0.001\,m + 0.002\,m) = 0.006\,m$

$$\boxed{\Delta z = 6\,mm}$$

Problem 1: 1.1 Review

Procedure:

- Use the equation
$$F_z = \frac{\partial}{\partial z}(\vec{u} \cdot \vec{B}) = \mu_z \frac{\partial B_z}{\partial z}$$
and the equation for thermal velocity of atoms
$$V_{\text{TH}} = \sqrt{\frac{8k_B T}{m\pi}}$$
to find the initial conditions of this system.
- Proceed to use the initial conditions found above with kinematic equations to determine how far the atoms travel after the magnet until they hit the screen.

Key Concepts:

- We can use Newton's Second Law with kinematic equations to determine how far these atoms will travel.
- We can calculate the thermal velocity of these atoms if we know the mass of the substance we are working with along with the temperature of the oven.
- It is important to realize that the atoms have to travel the length of the magnet as well as a distance after the magnet. These two distances must be added together to find the total distance.

Variations:

- We can be given a different atom.
  - This would change the thermal velocity but not the overall procedure.
- We can be asked to find the time or some other variable instead of distance.
  - We then would use the kinematic equations again but we would look for something different instead of distance.

Problem 2: 1.3

For the spin 1/2 state $|S_x;+\rangle$, evaluate both sides of the inequality (1.146), that is

$$\langle(\triangle A)^2\rangle\langle(\triangle B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

for the operators $A = S_x$ and $B = S_y$, and show that the inequality is satisfied. Repeat for the operators $A = S_z$ and $B = S_y$.

$\Delta A = A - \langle A\rangle$: $\quad \langle(\Delta A)^2\rangle = \langle(A^2 - 2A\langle A\rangle + \langle A\rangle^2)\rangle = \langle A^2\rangle - \langle A\rangle^2$

$$[A,B] = AB - BA$$

$$\langle A\rangle = \langle\alpha|A|\alpha\rangle$$

$$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle, \quad |S_y;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

$$[\hat{S}_x,\hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y,\hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z,\hat{S}_x] = i\hbar\hat{S}_y$$

$$\tilde{S}_z = \frac{\hbar}{2}\left[|S_z:+\rangle\langle S_z:+| - |S_z:-\rangle\langle S_z:-|\right]$$

$$\tilde{S}_x = \frac{\hbar}{2}\left[|S_x:+\rangle\langle S_x:+| + |S_x:-\rangle\langle S_x:-|\right] = \frac{\hbar}{2}\left[|S_z:+\rangle\langle S_z:-| + |S_z:-\rangle\langle S_z:+|\right]$$

$$\tilde{S}_y = \frac{\hbar}{2}\left[|S_y:+\rangle\langle S_y:+| + |S_y:-\rangle\langle S_y:-|\right] = \frac{-i\hbar}{2}\left[|S_z:+\rangle\langle S_z:-| - |S_z:-\rangle\langle S_z:+|\right]$$

$$|S_z:\pm\rangle = |S_z:\pm\rangle, \quad |S_x:\pm\rangle = \frac{1}{\sqrt{2}}\left[|S_z:+\rangle \pm |S_z:-\rangle\right], \quad |S_y:\pm\rangle = \frac{1}{\sqrt{2}}\left[|S_z:+\rangle \pm i|S_z:-\rangle\right]$$

Expectation values For $|S_x:+\rangle$:

$$\langle\tilde{S}_x\rangle^2 = \left(\frac{\hbar}{2}\right)^2, \quad \langle\tilde{S}_x^2\rangle = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \langle(\Delta\tilde{S}_x)^2\rangle = 0,$$

$$\langle\tilde{S}_y\rangle^2 = \left(\frac{\hbar}{2}\right)^2, \quad \langle\tilde{S}_y^2\rangle = 0 \Rightarrow \langle(\Delta\tilde{S}_y)^2\rangle = \frac{\hbar^2}{4}, \quad \langle\tilde{S}_z\rangle^2 = \left(\frac{\hbar}{2}\right)^2, \quad \langle\tilde{S}_z^2\rangle = 0 \Rightarrow \langle(\Delta\tilde{S}_z)^2\rangle = \frac{\hbar^2}{4}$$

$$|\langle[\tilde{S}_x,\tilde{S}_y]\rangle|^2 = 0, \quad |\langle[\tilde{S}_z,\tilde{S}_y]\rangle|^2 = 0$$

$$\langle(\Delta\tilde{S}_x)^2\rangle\langle(\Delta\tilde{S}_y)^2\rangle \geq 1/4\,|\langle[\tilde{S}_x,\tilde{S}_y]\rangle|^2 \Rightarrow 0 = 0$$

$$\langle(\Delta\tilde{S}_z)^2\rangle\langle(\Delta\tilde{S}_y)^2\rangle \geq 1/4\,|\langle[\tilde{S}_z,\tilde{S}_y]\rangle|^2 \Rightarrow \hbar^4/16 > 0$$

Problem 2: 1.3 Review

Procedure:

- Begin by using the equation

$$\triangle\tilde{\mathbf{A}} = \tilde{\mathbf{A}} - \langle\tilde{\mathbf{A}}\rangle \quad \rightarrow \quad \langle(\triangle\tilde{\mathbf{A}})^2\rangle = \langle\tilde{\mathbf{A}}^2\rangle - \langle\tilde{\mathbf{A}}\rangle^2 \quad , \quad \langle\tilde{\mathbf{A}}\rangle = \langle\alpha|\tilde{\mathbf{A}}|\alpha\rangle$$

  to calculate the dispersion of an observable. $\tilde{\mathbf{A}}$ is our observable and $|\alpha\rangle$ is the state of our system.

- Proceed to use the above equation with the equations for commutators and the common identities

$$[\tilde{\mathbf{S}}_i, \tilde{\mathbf{S}}_j] = i\hbar\,\epsilon_{ijk}\tilde{\mathbf{S}}_k$$

  for the RHS of the equation.

- We can use the common rule that if a Spin 1/2 operator acts on a state that is not its own eigenstate, it follows

$$\text{w/ } i \neq j, \quad \tilde{\mathbf{S}}_i\,|\pm\alpha_j\rangle = \pm\frac{\hbar}{2}\,|\mp\alpha_j\rangle \quad \text{e.g.} \quad \tilde{\mathbf{S}}_x\,|S_z;+\rangle = \frac{\hbar}{2}\,|S_z;-\rangle$$

  where $|\alpha\rangle$ is the state of our Spin 1/2 particle.

- Use the above formalism to deduce the answers for each case.

Key Concepts:

- We can use the shortcut method for deducing how the Spin 1/2 operators act on eigenstates instead of expanding in complete sets and using the long way.
- The commutations of the Spin 1/2 operators can be simplified with the above Levi-Civita tensor relationship.
- The only quantity that is not zero in the above scenario is that when $\tilde{\mathbf{A}} = \tilde{\mathbf{S}}_z$ and $\tilde{\mathbf{B}} = \tilde{\mathbf{S}}_y$. This is because $\langle\tilde{\mathbf{S}}_x\rangle^2 = \langle\tilde{\mathbf{S}}_x^2\rangle$ and thus the dispersion of $\tilde{\mathbf{S}}_x$ is zero.
- The dispersion of an observable that is in its own eigenstate will always be zero.

Variations:

- Our Spin 1/2 particle could be in a different state.
  - This would alter our expectation values equations, but it would not change the process that we used.
- The operators $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ could change.
  - Thus changing the math but not the overall procedure of the calculation.

Problem 3: 1.7

(a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$,... and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle$,... are all known, where $|a'\rangle$, $|a''\rangle$,... form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

$$\tilde{X} = |\alpha\rangle\langle\beta| = \sum_i \sum_j |a_i\rangle\langle a_i|\alpha\rangle\langle\beta|a_j\rangle\langle a_j| = \sum_i \sum_j \langle a_i|\alpha\rangle\langle\beta|a_j\rangle \, |a_i\rangle\langle a_j|$$

$$\boxed{\tilde{X} = \begin{pmatrix} \langle a_1|\alpha\rangle\langle\beta|a_1\rangle & \langle a_1|\alpha\rangle\langle\beta|a_2\rangle & \cdots \\ \langle a_2|\alpha\rangle\langle\beta|a_1\rangle & \langle a_2|\alpha\rangle\langle\beta|a_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}$$

(b) We now consider a spin 1/2 system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|S_z;+\rangle$ and $|S_x;+\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (S_z diagonal) basis.

$$|S_z:\pm\rangle = |S_z:\pm\rangle \ , \ |S_x;\pm\rangle = \frac{1}{\sqrt{2}}\left[|S_z:+\rangle \pm |S_z;-\rangle\right]$$

$$|S_z:+\rangle\langle S_x:+| = \frac{1}{\sqrt{2}}\left[|S_z;+\rangle\langle S_z:+| + |S_z:+\rangle\langle S_z;-|\right]$$

$$\boxed{|S_z:+\rangle\langle S_x:+| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}$$

Problem 3: 1.7 Review

Procedure:

- Begin by taking the operator $\tilde{\mathbf{X}}$ and expanding in a complete set twice.
- Proceed to show that
$$\tilde{\mathbf{X}} = |\alpha\rangle \langle\beta| = \langle a_i|\alpha\rangle \langle\beta|a_j\rangle = \langle a_i|\alpha\rangle \langle a_j|\beta\rangle$$
and then expand this to a matrix form.
- To diagnoalize a matrix, do the following
$$|S_i;\pm\rangle \langle S_i;\pm| \, .$$

Key Concepts:

- We expand in a complete set to determine the matrix elements of this matrix.
- Once we expand in a complete set, we rearrange the equation to show what the matrix elements are.
- We use the above equation to diagonalize a matrix of our choosing.

Variations:

- Because this problem is essentially a proof, there cannot be many variations of it without making a brand new problem.

Problem 4: 1.8

Suppose $|i\rangle$ and $|j\rangle$ are eigenvalues of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

The only way this will be true is if the two eigenkets are degenerate.

$$\tilde{A}(|i\rangle + |j\rangle) = a_{ij}(|i\rangle + |j\rangle)$$

Problem 4: 1.8 Review

Procedure:

- Show that the only case where this is possible is where the eigenkets are degenerate.

Key Concepts:

- The only way $|i\rangle$ and $|j\rangle$ can be eigenkets of the same operator $\tilde{\mathbf{A}}$ is if the they are degenerate.
- Degeneracy refers to having the same eigenvalue for different eigenkets.

Variations:

- We can be asked what condition can we conclude if this sum of states is not an eigenket of $\tilde{\mathbf{A}}$.
  - This however would alter the problem and would require us to answer a completely different question.

Problem 5: Q-1

Let $\hat{K}$ be the operator defined by $\hat{K} = |\phi\rangle\langle\psi|$, where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

(a) Under what condition is $\hat{K}$ Hermitian?

In order for $\hat{K}$ to be Hermitian, $\hat{K}$ must be its own Hermitian Conjugate. Namely, $\hat{K}$ must be equal to its conjugate.

Consider two arbitrary kets $|\alpha\rangle$ and $|\beta\rangle$

$$\langle\alpha|\hat{K}^\dagger|\beta\rangle = \langle\beta|\hat{K}|\alpha\rangle^* = (\langle\beta|\phi\rangle\langle\psi|\alpha\rangle)^* = \langle\alpha|\psi\rangle\langle\phi|\beta\rangle = \langle\alpha|\hat{K}^\dagger|\beta\rangle$$

$$\langle\alpha|\hat{K}^\dagger|\beta\rangle = \langle\beta|\hat{K}|\alpha\rangle^* \quad \therefore \quad \hat{K} = \hat{K}^\dagger$$

$$\boxed{\hat{K}^\dagger = \hat{K}}$$

(b) Calculate $\hat{K}^2$. Under what condition is $\hat{K}$ a projection operator?

In order for $\hat{K}$ to be a projection operator, the square of $\hat{K}$ must be equal to $\hat{K}$. Namely,

$$\hat{K}^2 = |\phi\rangle\langle\psi|\phi\rangle\langle\psi| \; : \; \text{If } \langle\psi|\phi\rangle = 1 \text{ Then } \hat{K}^2 = |\phi\rangle\langle\psi| = \hat{K}$$

$$\boxed{\hat{K}^2 = \hat{K}}$$

(c) Show that $\hat{K}$ can always be written in the form $\hat{K} = \lambda\hat{P}_1\hat{P}_2$ where λ is a constant to be calculated and $\hat{P}_1$ and $\hat{P}_2$ are projection operators.

$$\tilde{K} = \lambda\tilde{P}_1\tilde{P}_2 = \lambda\,|\varphi\rangle\langle\varphi|\psi\rangle\langle\psi| = \lambda\langle\varphi|\psi\rangle|\varphi\rangle\langle\psi|$$

$$\text{w/ } K = |\varphi\rangle\langle\psi| \,, \quad \lambda^{-1} = \langle\varphi|\psi\rangle$$

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Problem 5: Q-1 Review

Procedure:

- For an operator to be Hermitian the following must be true

$$\tilde{\mathbf{K}}^\dagger = \tilde{\mathbf{K}}.$$

- Take an expectation value of each operator and show that with the rules of mathematics that the following is true.
- In order for $\tilde{\mathbf{K}}$ to be a projection operator it must be idempotent. Namely,

$$\tilde{\mathbf{K}}^2 = \tilde{\mathbf{K}}.$$

- Prove that $\tilde{\mathbf{K}}$ is idempotent and thus a projection operator.
- To show that $\tilde{\mathbf{K}}$ can be written in this form, write out $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ as

$$\tilde{\mathbf{P}}_i = \sum_i |a_i\rangle \langle a_i|$$

  which is the standard definition of a projection operator.
- Use the above rules and conclude what λ^{-1} must be.

Key Concepts:

- Hermitian operators are self adjoint and thus equal their complex transpose.
- Projection operators are idempotent by definition.
- We can use completeness relations to determine the final question (c).

Variations:

- This problem is proving properties of operators and thus cannot be changed without producing a brand new problem.