PHYS5153 Assignment 3

Due: 1:30pm on 9/15/2021 (prior to class commencing).
Marking: Total of 10 marks (weighting of each question is indicated).
Fine print: Solutions should be presented legibly (handwritten or LaTeX is equally acceptable) so that the grader can follow your line of thinking and any mathematical working should be appropriately explained/described. If you provide only equations you will be marked zero. If you provide equations that are completely wrong but can demonstrate some accompanying logical reasoning then you increase your chances of receiving more than zero. If any of your solution has relied on a reference or material other than the textbook or lectures, please note this and provide details.

Question 1 (2 marks)

Consider the situation of Fig. 1. Two wheels with radius R are mounted on an axle of length l but can rotate independently. The axle-wheels contraption is allowed to roll (without slippage) on a 2D plane defined by the Cartesian co-ordinates x and y. Taking ϕ and ϕ' to be the angle of rotation of each wheel about the axis defined by the axle, θ to be the angle the axle makes with respect to the x-axis of the 2D plane, show that the system has: i) two *nonholonomic* constraint equations,

$$\begin{aligned} \cos(\theta)dx + \sin(\theta)dy &= 0, \\ \sin(\theta)dx - \cos(\theta)dy &= R(d\phi + d\phi'), \end{aligned} \tag{1}$$

where the co-ordinates (x, y) correspond to the centre of the axle, and ii) one *holonomic* constraint

$$\theta + \frac{R}{l}(\phi - \phi') = \text{const.} \tag{2}$$

Hint: For ii) consider the motion of the relative vector between the two wheels.

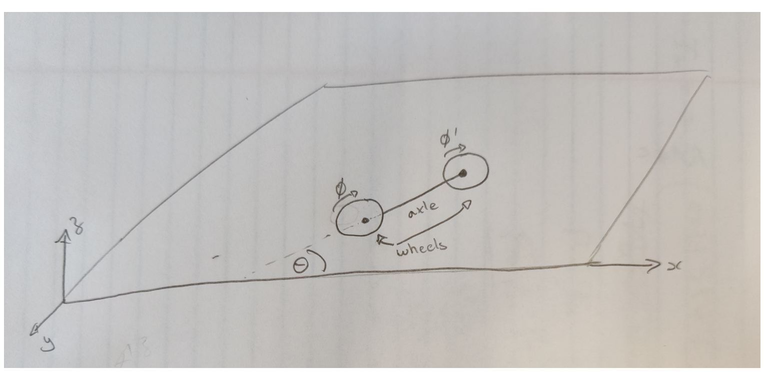


Figure 1: Physical system for Question 1

Question 2 (3 marks)

This question involve two parts, but both make use of d'Alembert's principle explicitly.
Part 1: Consider the Atwood machine illustrated in Fig. 2(a).

(a) Write down d'Alembert's principle for the system.

(b) Use your answer to (a) to show that the motion of the system is entirely governed by

$$\ddot{y}_m = \frac{M - m}{M + m} g, \tag{3}$$

where y_m is the position of the mass block of weight m.

(c) Assume now that the masses are allowed to rest on the sides of a fixed wedge, as per Fig. 2(b). Show that the equation of motion becomes,

$$\ddot{L}_m = \frac{m \sin(\beta) - M \sin(\alpha)}{M + m} g, \tag{4}$$

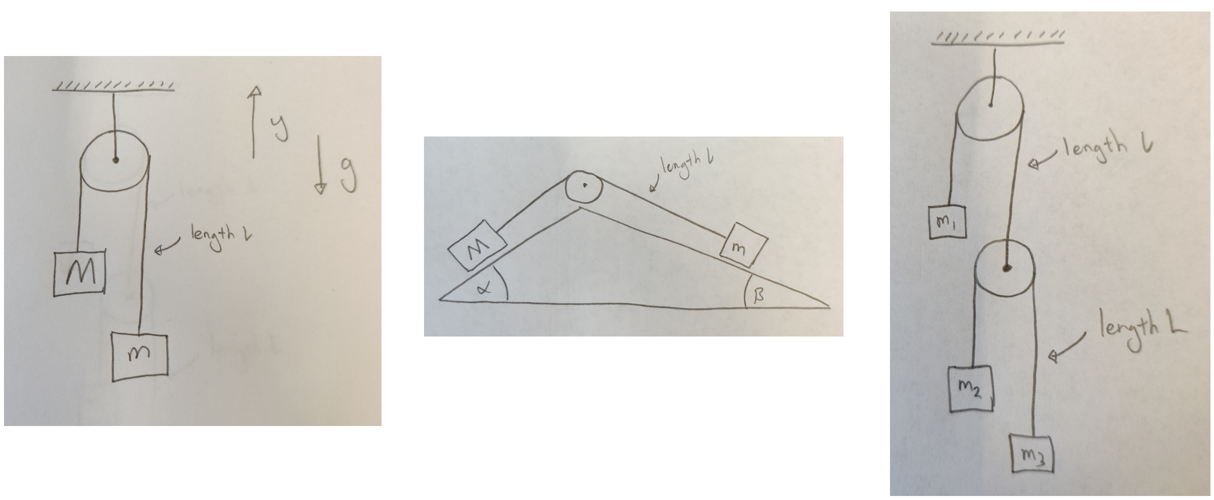
where L_m is the distance of the block parallel along the relevant surface of the wedge.

Part 2: Consider the mass-pulley system illustrated in Fig. 2(c).

(d) Identify a set of generalized co-ordinates and the constraints in the system.

(e) Using the coordinates defined in (c), compute the equations of motion for the system.

(f) What is the acceleration of each mass? From your equations, identify conditions on the masses such that m_1 would be stationary? Does your result make sense?



(a) (b) (c)

Figure 2: Physical systems for Question 2

Question 3 (2 marks)

Consider a pair of blocks of mass M and m connected by a massless string. The former mass lies on top of a table while the latter hangs below, in a configuration shown in Fig. 4. You may assume the motion of the hanging block is restricted to be only in the vertical direction (e.g., along $\hat{z}$ only), while the block on the table is restricted to move in a 2D plane (e.g., the $\hat{x} - \hat{y}$ plane defined by the table's surface). Moreover, assume the string is precisely the same length as the height of the table above the ground.

(a) Use the Lagrange formalism to write down the equations of motion for appropriate generalized coordinates.

(b) Discuss the physical interpretation of the equations derived in (a) and identify any relevant conserved quantities.

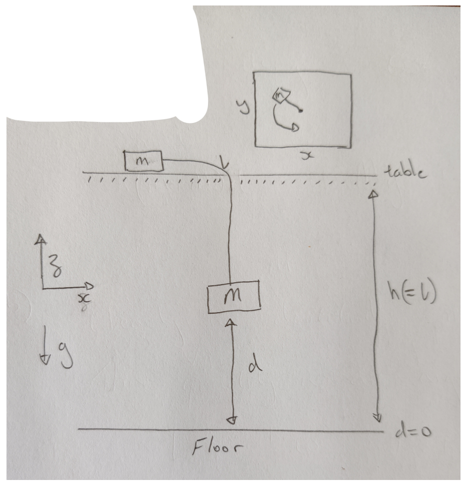


Figure 3: Physical system for Question 3

Question 4 (3 marks)

This questions serves as a useful introduction to some notions of *statistical distributions* in classical phase space. We will investigate two simple examples.

First, consider a massive particle in 1D subject to a harmonic potential with frequency $\omega = m = 1$.

(a) Sketch a phase portrait of the system that gives a sufficient description of the general dynamics.

(b) Consider an ensemble of points (e.g., a set of particles with a variety of initial conditions) that all fall within some region bounded by the circle $(x - x_0)^2 + (p - p_0)^2 = a$ where the radius satisfies $0 < a < \sqrt{x_0^2 + p_0^2}/2$. The area of phase-space (typically referred to as the *phase-space volume*) occupied by the ensemble is that of a circle πa^2. Use physical reasoning (i.e., 'hand-wavy' arguments) based off your solution to (a) to explain why the area of phase-space occupied by the ensemble is conserved in time. Your solution does not need to be quantitative.

Now, instead consider a massive particle in 1D subject to gravity. A phase-portrait for a single-particle in shown in Fig. 4.

(c) Consider an ensemble of points in the phase-space confined to an area defined by the constraints $p_1 \leq p \leq p_2$ and $E' \leq E \leq E''$ where p is the momentum and E the energy of the particle. Compute the area of phase space occupied by the ensemble.

(d) By solving the equations of motion to yield $q(t)$ and $p(t)$, show/argue that the phase space area enclosed by the evolving ensemble is preserved.

The result that the phase-space volume is constant in time is a result of Liouville's theorem, which plays a crucial role in the relating deterministic classical mechanics to the more tractable framework of statistical mechanics. The latter allows us to describe the thermodynamic properties of macroscopic systems composed of *many* microscopic particles. We will revisit Liouville's theorem when we address Hamiltonian mechanics.

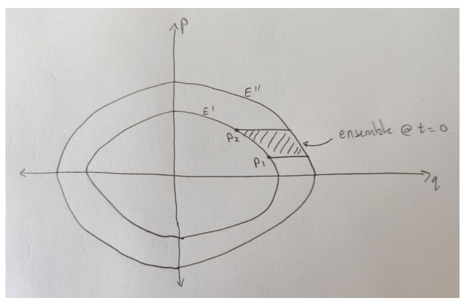


Figure 4: Phase portrait for Question 3 parts (c) and (d).