Want to calculate $J = \int_{\substack{\text{all} \\ \text{space}}} \theta\left(E - \sum_{i=1}^{N} \vec{u}_i^{\,2}\right) d^{3N}\vec{u}$

How?

Define $\vec{R}_E = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_N)$

$$\Rightarrow |\vec{R}_E| = \sqrt{|\vec{u}_1|^2 + \ldots + |\vec{u}_N|^2} = R_E$$

Define $\vec{R}_E = (\underbrace{R_E^2}_{\text{1 coordinate}}, \underbrace{\hat{R}_E}_{\text{3N-1 coordinates}})$

$$\Rightarrow J = \underbrace{\int_0^{\infty} \theta(E - R_E) R_E^{3N-1} \, dR_E}_{\int_0^{\sqrt{E}} R_E^{3N-1} \, dR_E} \cdot \underbrace{\int_{\substack{3N-1 \\ \text{angles}}} d\hat{R}_E}_{\text{3N-1 coordinates}}$$

$$= 3N \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}$$

through comparison w/ Gaussian integral (next page)

$$= \frac{1}{3N} R_E^{3N} \Big|_0^{\sqrt{E}} = \frac{1}{3N} E^{3N/2}$$

So: $$\boxed{J = E^{3N/2} \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}}$$

How can we calculate $\int_{\substack{3N-1 \\ \text{angles}}} d\hat{R}_E$?

Let us look at :

$$\int\cdots\int\int e^{-\sum \vec{u}_i^2/\varepsilon}\, d^{3N}\vec{u} = (\varepsilon\pi)^{3N/2}$$

straight forward gaussian integral

$$= \int e^{-R_E^2/\varepsilon} R_E^{3N-1}\, dR_E \int_{\substack{3N-1 \\ \text{angles}}} d\hat{R}_E$$

using R_E & $\hat{R}_E$

this is a bit of work

$$= \frac{1}{2}\varepsilon^{3N/2}\,\Gamma\left(\frac{3N}{2}\right)$$

combining l.h.s and r.h.s.

$$\Rightarrow \int d\hat{R}_E = 3N \frac{\pi^{3N/2}}{\frac{3N}{2}\Gamma\left(\frac{3N}{2}\right)} = 3N \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}$$