Homework Assignment #1

Math Methods

Reading Quiz #1 Due: Wednesday, August 25th, 10::30am

Homework Due: Wednesday, September 1st

Reading Quiz #2 Due: Friday, September 3rd, 10:30am

Reading: Please read Chapter 2, sections 1-5. Reading Quiz # 1 on this material is due by the start of class on Wednesday. Reading Quiz #2 covers Chapter 2, sections 6-7.

Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

---

1. You are in a rocket ship, in outer space. You have a nuclear reactor that supplies a *constant* power, P_0, and a large supply of iron pellets. The iron pellets comprise 99/100 of your ship's mass, m. You can use the power to eject the tiny iron beads out the back of your ship with an electromagnetic "gun". You can control the *rate* at which you fire them and their velocity, but are limited by your power plant. (You can't fire an arbitrarily large mass at an arbitrarily large velocity.) As you fire off the beads, your ship moves in the opposite direction to conserve momentum. In addition, the mass of your ship decreases. (You can solve this using a local constraint, but that's the hard way.)

   (a) If you use the energy of your reactor over a time Δt to launch a packet of mass Δm out the rear of your ship, what is the momentum of this "exhaust" packet relative to your ship?

   (b) Now assume that you fire pellets continuously at a constant rate during the interval $0 < t < t_f$. If you start from rest, what is your final velocity?

   (c) However, you do not have to fire pellets at a constant rate. Find the optimal firing rate dm/dt in the interval $0 < t < t_f$ so that your final velocity is a maximum after a time t_f, assuming that you started from rest.

   (d) What is your final velocity in part (c) ? How does it compare to the answer in part (b)?

   You may find it helpful to review rockets in your favorite Freshman physics book.

2. Consider the functional

$$\mathcal{I}[y(x), y'(x)] = \int_0^{x_f} \left\{ \left(\frac{\partial y}{\partial x} \right)^2 + \alpha\, y \frac{\partial y}{\partial x} \right\} dx$$

   (a) Find the function $y(x)$ that extremizes I subject to the boundary conditions that $y(0) = 0$ and $y(x_f) = y_f$.

   (b) How does your answer depend upon α? Why?

3. Byron and Fuller, chapter 2, problem 6.

4. Byron and Fuller, chapter 2, problem 7.

HW #1 problem 1

(a) The power plant can only provide an energy.

$$dE = P_0\, dt$$

in a time interval dt. This goes into the kinetic energy of the pellet.

$$P_0\, dt = -\frac{1}{2}\, dm\, v^2$$

$$P_0 = -\frac{1}{2} \frac{dm}{dt} v^2$$

Note that since $m(t)$ = mass of ship at time t, $\frac{dm}{dt} < 0$ as the ship ejects pellets.

Strictly speaking, some of the energy goes into the KE of the rocket. However, if $dm \ll m$, this is a decent approximation: The exhaust velocity of the pellets relative to the ship is

$$v_{ex} = \sqrt{\frac{2P_0}{-\dot{m}}}$$

The momentum is

$$dp = dm\, v_{ex} = dm \sqrt{\frac{2P_0}{-\dot{m}}}$$

(b) The force on the ship is:

$$F = \frac{dp}{dt} = -\dot{m}\, v_{ex}$$

The change in the ship velocity over time is:

$$v_f = \int a\, dt = \int \frac{F(t)}{m(t)}\, dt = \int -\frac{\dot{m}(t)\, v_{ex}}{m}\, dt$$

N.B. v_f is final velocity of the ship

$v_{ex}(t)$ is exhaust velocity of pellets at time t, which is

$$v_{ex}(t) = \sqrt{\frac{-2P_0}{\dot{m}(t)}} \qquad \dot{m} < 0$$

So

$$v_f = \int -\frac{\dot{m}}{m} \sqrt{\frac{-2P}{\dot{m}}}\, dt$$

If we choose $\dot{m} = \text{constant} = -\mu\, m_0$

$$m(t) = m_0 (1 - \mu t)$$

where $\mu \equiv \frac{99}{100} \frac{1}{t_f}$

$$v_f = \sqrt{2P_0}\int_0^{t_f} \frac{\mu m_0}{m_0(1-\mu t)} \frac{dt}{\sqrt{\mu m_0}}$$

$$= \sqrt{\frac{2P_0}{\mu m_0}} \; - \ln(1-\mu t)\Big|_0^{t_f}$$

$$= -\sqrt{\frac{2P_0}{m_0}\frac{100 t_f}{99}} \ln\left(\frac{1}{100}\right)$$

$$= \sqrt{\frac{2P_0 t_f}{m_0}} \sqrt{\frac{100}{99}} \ln 100$$

(c) Extremizing we first set $u = -m$. Then

$$v_f = \int -\sqrt{2P_0}\,\frac{\sqrt{\dot u}}{u}\,dt$$

So that

$$\dot u \frac{\partial f}{\partial \dot u} - f = -\sqrt{2P}\left\{\dot u \,\frac{1}{2}\,\frac{1}{\sqrt{\dot u}}\,\frac{1}{u} - \frac{\sqrt{\dot u}}{u}\right\} = C_0$$

or $\qquad \sqrt{2P}\,\dfrac{\sqrt{\dot u}}{u} = C_0$

or $\qquad \dot u = C_1 u^2 \qquad \left(C_1 = \dfrac{C_0^2}{2P_0}\right)$

Or $$\dot{m} = -c_1 m^2$$

$$dt = -\frac{1}{c_1} \frac{dm}{m^2}$$

So that $$t = -\frac{1}{c_1} \int \frac{dm}{m^2} = -\frac{1}{c_1} \frac{1}{m} + c_2$$

or $$t - c_2 = -\frac{1}{m c_1}$$

$$m = \frac{1}{c_1} \frac{1}{c_2 - t} = \frac{c_3}{1 - c_4 t}$$ (still two unknown coefficients).

$$m(0) = m_0 \Rightarrow c_3 = m_0 \text{ or}$$

$$m(t_f) = \frac{m_0}{100} \Rightarrow \frac{m_0}{1 - c_4 t_f} = \frac{m_0}{100}$$

So $$1 - c_4 t_f = 100$$

$$c_4 = -\frac{99}{t_f}$$

So $$m(t) = \frac{m_0}{1 + \frac{99}{100} \frac{t}{t_f}}$$

d) Compare the final velocities –

If $m = \dfrac{m_0}{1 + 99\frac{t}{t_f}}$ then $\dot{m} = -\dfrac{99}{t_f} \dfrac{m_0}{\left(1 + \frac{99t}{t_f}\right)^2}$

$$v_f = \sqrt{2P_0}\sqrt{\frac{99}{t_f}} \int \sqrt{\frac{m_0}{\left(1 + \frac{99t}{t_f}\right)^2}} \; \frac{\left(1 + \frac{99t}{t_f}\right)}{m_0} \, dt$$

$$= \sqrt{\frac{P_0}{m_0 t_f}} \sqrt{198} \int_0^{t_f} dt$$

$$= \sqrt{\frac{2 P_0 t_f}{m_0}} \sqrt{99}$$

The ratio of the optimum final velocity to that of firing at a constant rate is –

$$\frac{v_{opt}}{v_{const}} = \frac{\sqrt{99}}{\sqrt{\frac{100}{99}} \ln(100)} = 2.15$$

HW # 1 Problem 2.

Given the functional -

$$I = \int_0^{x_f} \left\{ \left(\frac{\partial y}{\partial x}\right)^2 + \alpha y \frac{\partial y}{\partial x} \right\} dx$$

(a) Extremize $I[y(x)]$ -

Since there is no explicit x - dependence we may immediately write

$$y' \frac{\partial f}{\partial y'} - f = \text{constant} = C_0$$

$$= y'(2y')$$

Therefore

$$y' = \sqrt{\frac{C_0}{2}} \equiv C_1$$

which has the solution

$$y = C_1 x + y_0$$

Using the boundary conditions

$$y(0) = 0 \quad y(x_f) = y_f$$

We have

$$y = \frac{y_f}{x_f} x + \phi .$$

(b) How does your answer depend upon α & why?

The answer does not depend on α at all!

To see why note that the second term in our integral is

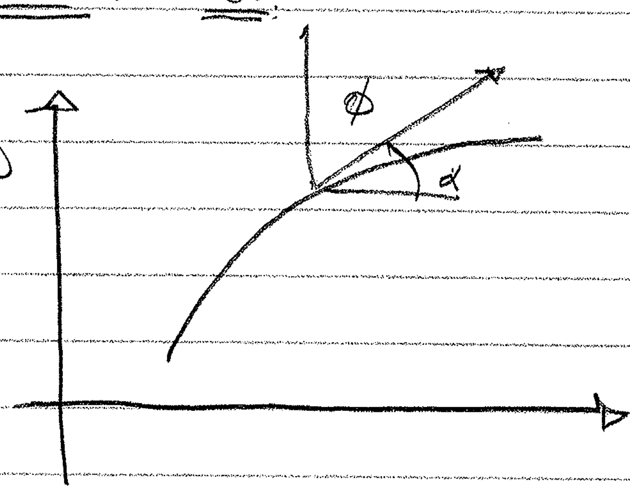
$$\int_0^{x_f} \alpha \cdot y \cdot \frac{dy}{dx} \, dx = \int \alpha \frac{d}{dx}\left(\frac{1}{2} y^2\right)$$

$$= \frac{\alpha}{2}\left(y(x_f)^2 - y(0)^2\right) = \frac{\alpha}{2} y_f^2 .$$

This is independent of the function $y(x)$ - it only depends on it's value at the endpoints. The jargon for this is "An action is invariant to the addition of a total differential.

(3) Byron & Fuller, Chapter 2, problem 7

(a) **Snell's Law**



We minimize the time -

$$t = \int \frac{ds}{v(y)} = \int \frac{\sqrt{1+(y')^2}}{v(y)}\, dx$$

Since $v = v(y)$ we have no x-dependence in the functional. Therefore

$$y'\frac{\partial f}{\partial y'} - f = y'(2y')\frac{1}{2}\frac{1}{\sqrt{1+(y')^2}}\frac{1}{v}$$

$$- \frac{\sqrt{1+(y')^2}}{v} = C$$

for some constant C

$$(y')^2 - \frac{(1+(y')^2)}{x} = c\sqrt{1+(y')^2}$$

$$-1 = c\,x(y)\sqrt{1+(y')^2}$$

But $y' = \tan\alpha$.

$$\frac{1}{(c\,x)^2} = 1+\tan^2\alpha$$

But $\dfrac{\sin^2\alpha}{\cos^2\alpha} + \dfrac{\cos^2\alpha}{\cos^2\alpha} = \dfrac{1}{\cos^2\alpha}$

Or

$$\tan^2\alpha + 1 = \sec^2\alpha$$

$$c\,x = \cos\alpha = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi$$

Or $\sin\phi = c\,x$.

(b) What is trajectory if $u(y) = \beta y$?

Our E-L equation is then

$$\frac{-1}{c\beta y} = \sqrt{1+(y')^2}$$

Solve for y'

$$y' = \sqrt{\frac{1}{(\beta c y)^2} - 1} = \frac{dy}{dx}$$

$$dx = \frac{dy}{\sqrt{\frac{1}{(\beta c y)} - 1}}$$

Define $s = \beta c y$

$v = \beta c x$

Then $dv = \dfrac{ds}{\sqrt{\frac{1}{s^2} - 1}} =$

or $dv = \dfrac{s\,ds}{\sqrt{1 - s^2}}$

$$v - v_0 = -\sqrt{1 - s^2}$$

$$(v - v_0)^2 = 1 - s^2$$

$$(v - v_0)^2 + s^2 = 1$$

or $$\beta c \, (x - x_0)^2 + \beta^2 c^2 y^2 = 1$$

$$(x - x_0)^2 + y^2 = \frac{1}{\beta c}.$$

$$R = \frac{1}{\sqrt{\beta c}}.$$

(4) Byron & Fuller Chapter 2 p 6.

What if $f(y, y', y'', x)$? As before

$$y(x) = \bar{y}(x) + \epsilon \eta(x)$$

where $\bar{y}$ is the optimizing solution and

$$\eta(x_A) = \eta(x_B) = 0$$

$$\eta'(x_A) = \eta'(x_B) = 0$$

Our functional is

$$I(\epsilon) = \int dx \; f(y, y', y'', x)$$

$$\frac{\partial I}{\partial \epsilon} = \int dx \left\{ \frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y'} \eta' + \frac{\partial f}{\partial y''} \eta'' \right\}$$

We wish to pull out a common factor of $\eta(x)$.

We need to integrate by parts — twice. We require that our variations be twice differentiable at all points and

Then $\frac{\partial \eta}{\partial x} = 0$ at $x = x_a$ and $x = x_b$.

Then

$$\frac{\partial I}{\partial \epsilon} = \int dx \left\{ \frac{\partial f}{\partial y} \eta - \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \eta' \right\}$$

$$+ \frac{\partial f}{\partial y'} \eta \Big|_{x_a}^{x_b} + \frac{\partial f}{\partial y''} \eta' \Big|_{x_a}^{x_b}$$

The last two terms are zero by our assumptions

Integrating by parts again

$$\frac{\partial I}{\partial \epsilon} = \int dx \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} \right\} \eta(x)$$

$$- \frac{d}{dx}\frac{\partial f}{\partial y''} \eta \Big|_{x_a}^{x_b}$$

Again our last term $= 0$ by our assumption so

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$