PHYS5153 Assignment 10

Due: 11:59pm on 11/28/2021
Marking: Total of 10 marks (weighting of each question is indicated).
Fine print: Solutions should be presented legibly (handwritten or LaTeX is equally acceptable) so that the grader can follow your line of thinking and any mathematical working should be appropriately explained/described. If you provide only equations you will be marked zero. If you provide equations that are completely wrong but can demonstrate some accompanying logical reasoning then you increase your chances of receiving more than zero. If any of your solution has relied on a reference or material other than the textbook or lectures, please note this and provide details.

Question 1 (4 marks)

A particle of mass m is subject to a force,

$$\mathbf{F} = -\frac{2a}{r^3}\hat{\mathbf{r}}, \tag{1}$$

in three dimensions (3D) with $a > 0$. Here, $\hat{\mathbf{r}} = \mathbf{r}/r$ where $\mathbf{r}$ is the position vector of the particle with respect to the origin and $r = |\mathbf{r}|$.

(a) Explain why you expect the motion of the particle to be confined to a 2D plane. You are not expected to give a mathematical proof, a concise qualitative sentence is sufficient.

(b) Starting from a Lagrangian describing the 2D motion in polar co-ordinates φ and r, show that the motion of the particle is governed by an *effective one-dimensional potential* $V_{\text{eff}}(r)$.

(c) Discuss the particles motion as a function of the initial energy, the value of the constant a and any other relevant factors.

(d) Show that when the 2D motion of the particle [see (a)] is described by polar co-ordinates r and φ, the motion can be parameterized by the integral equation,

$$\varphi - \varphi_0 = \int_{r_0}^{r} \frac{l}{mr'^2} \frac{1}{\sqrt{\frac{2}{m}[E - V_{\text{eff}}(r')]}} dr' \tag{2}$$

where φ_0 and r_0 relate to the particle's initial conditions, E is the total mechanical energy and l is the magnitude of the total angular momentum.

(e) **(Optional) bonus question:** Explicitly solve the integral above to obtain the orbit of the particle and discuss the resulting expression. (This question is not marked/carries no points).

Question 2 (2 marks)

(a) Define and/or briefly discuss the terms:

(i) Limit cycle
(ii) Chaos
(iii) Poincare section/map

Your answer can involve an example illustrative sketch that highlights an important feature(s) if this is useful.

(b) Consider the coupled equations of motion with $b > 0$,

$$\begin{aligned} \dot{x} &= ax - by - C(x^2 + y^2)x, \\ \dot{y} &= bx + ay - C(x^2 + y^2)y. \end{aligned} \tag{3}$$

Introducing polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ the system can be equivalently described via the equations of motion,

$$\begin{aligned} \dot{r} &= ar - Cr^3, \\ \dot{\theta} &= b. \end{aligned} \tag{4}$$

Sketch a useful phase portrait of the system in the $x - y$ plane for:

(i) $a > 0$ and $C > 0$

(ii) $a < 0$ and $C > 0$

(iii) $a < 0$ and $C < 0$

and indicate the important features including, e.g., limit cycles and fixed points. You should also clearly state the radius or position of any limit cycles and fixed points, and comment on their stability (provide evidence!).

Question 3 (4 marks)

A thin uniform disk of radius R and mass m is rigidly fixed to an axle as in Fig. 1. The axle is free to rotate about its own central axis. The disk is oriented such that a normal vector from the disk's surface makes an angle θ with the axle. For simplicity, assume in the following that the axle is massless and the the disk is infinitely thin (i.e., it's thickness is ignorable). For parts (a)-(c) you may assume the angular frequency of the axle's rotation is not fixed (i.e., the axle rotates freely).

(a) By identifying an appropriate set of body fixed axes, compute the principal moments of inertia of the disk.

(b) Give an expression for the instantaneous angular velocity $\vec{\omega}$ with respect to your set of body-fixed axes.

(c) Using your answers to (a) and (b) give an expression for the kinetic energy of the rotating disk.

(d) Assume that the axle is now driven to rotate at a *fixed* angular frequency Ω about its central axis. Compute the magnitude of the applied torque that is required to preserve this motion.

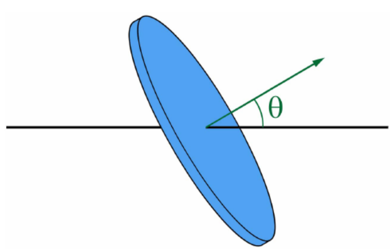

Figure 1: A disk rigidly attached to an axle that can rotate.