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Statistical Mechanics

Ch. 12 Bose Systems Lecture Notes

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In this chapter we begin our examination of Bosons

We first start discussing photons, which are masless light particles that travel at the speed of light.

Our Hamiltonian is then

$$\hat{\mathcal{H}} = \sum_{\vec{k},\varepsilon} \hbar c k \, \hat{a}^{\dagger}_{\vec{k},\varepsilon} \hat{a}_{\vec{k},\varepsilon}$$

Where we call ε our "polarization". The quantity

$$\hat{a}^{\dagger}_n \hat{a}_n = n|n\rangle$$

is essentially the number operator and tells us how many photons we have in a given state.

We can then say the number of particles is an internal variable to be determined by minimizing the Helmholtz free energy. This is of course

$$\left.\frac{\partial A(N,T,V)}{\partial N}\right|_{T,V} = 0$$

Which is when our chemical potential energy is zero ($\mu=0$). We then can write our partition function as

$$Q = \sum_{\{n_{\vec{k},\varepsilon}\}} e^{-\beta E\{n_{\vec{k},\varepsilon}\}}$$

If we take a log of the above function we find

$$\log(Q) = -2\sum_{\vec{k}} \log(1-e^{-\beta\hbar c k})$$

We can then say $\langle n_{\vec{k}} \rangle$ is then

$$\langle n_{\vec{k}} \rangle = -\frac{1}{\beta}\frac{\partial}{\partial(\hbar c k)}\log(Q) = \frac{2}{e^{\beta\hbar c k}-1}$$

We now want to find an equation of state that involves our potential energy. This will look something like

$$PV = \# U$$

Where the number out front turns out to be $1/3$. We then recall some common definitions

$$U = -\frac{\partial}{\partial \beta}\log(Q) \quad , \quad Q = e^{-\beta A} \quad , \quad P = -\left(\frac{\partial A}{\partial V}\right)_T = \frac{1}{\beta}\frac{\partial}{\partial V}\log(Q)$$

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From last time we found that our internal energy is

$$U = 2\sum_{\vec{n}} \frac{\hbar c k}{e^{\beta \hbar c k}-1} = 2\int_0^\infty \mathcal{D}(\varepsilon)\frac{\varepsilon}{e^{\beta\varepsilon}-1}\,d\varepsilon$$

We can also go on to say

$$PV = \frac{1}{3}U$$

For our photonic gas. From Stefan's Law we know

$$\frac{U}{V} \propto T^4$$

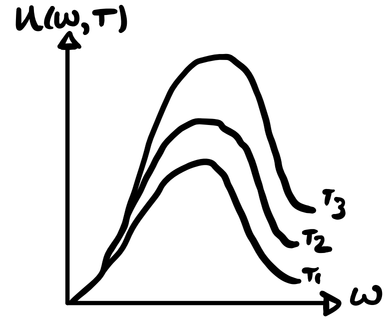
which tells us our internal energy per volume will scale as T^4. Conversely for specific heat we can say

$$C_V = \frac{\partial(U/V)}{\partial T} \propto T^3$$

our specific heat per volume will scale as T^3. With this knowledge we can go on to say the internal energy can now be written as

$$U = \frac{\hbar}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\,d\omega$$

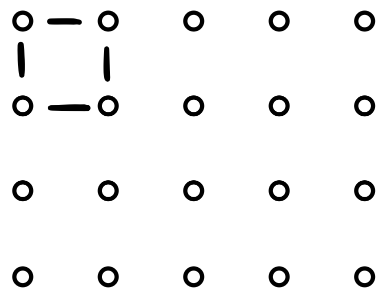
If we then graph this internal energy we will have



where Wien's displacement law tells us

$$\omega_{max} = 2821\,\frac{k_B T}{\hbar}$$

Looking at a solid material



we can say for N atoms $\to$ $3N$ normal modes. We can then say for phonons

$$PV = \frac{2}{3} U$$

From here we have two different types of distributions

Einstein : N_E distinct normal modes with frequency ω_E

Debye : Not all frequencies are equal

If we define our partition function as

$$Q = \left(\frac{1}{1 - e^{-\beta \hbar \omega_E}} \right)^{N_E}$$

For our Einstein distribution, our internal energy is then

$$U = N_E \hbar \omega_E \frac{1}{e^{\beta \hbar \omega_E} - 1}$$

Our specific heat for this distribution is then

$$C_V(T) = \left(\frac{\partial U}{\partial T} \right)_V = k_B N_E \left(\frac{x(T)}{\sinh(x(T))} \right)^2 \quad \text{w/} \quad x(T) = \frac{\hbar \omega_E}{2 k_B T}$$

For the Debye distribution we can determine the number of particles with

$$N = \frac{1}{3} \int_0^{\omega_m} f(\omega)\, d\omega$$

where ω_m is our cutoff frequency. In k-space we can write $f(\omega)\, d\omega$ as

$$f(\omega)\, d\omega = 3 \frac{1}{\left(\frac{2\pi}{L} \right)^3} 4\pi k^2\, dk$$

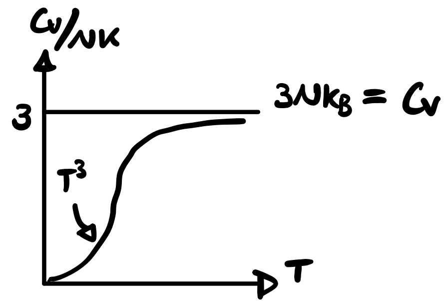
we can finally say

$$\frac{U}{N} = 3 k_B T \cdot D(T_D/T)$$

where we define

$$D(T_D/T) = \frac{3}{x^3}\int_0^x \frac{t^3}{e^t - 1}\,dt$$

Is the Debye Function that is dependent upon the Debye temperature. Plotting the Specific heat vs. temperature



Where the above result is called the Dulong Petit-Law.