Homework Assignment #5

Math Methods

Homework Due: Monday, September 30th, 10:30am

Instructions:

Reading: Please re-read the rest of Chapter 1. There is no reading quiz this week. **Problems:** Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

---

1. Consider a two dimensional system where the vector $\vec{x}$ is given by

$$\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2$$

   and the x-coordinate is transformed into a different, non-orthogonal coordinates as:

$$\begin{aligned} x'_1 &= \frac{1}{\sqrt{2}}(x_1 + x_2) \\ x'_2 &= x_2 \end{aligned}$$

   The gradient of the scalar function is:

$$\vec{\nabla}\phi(\vec{x}) = \partial_1\phi\,\hat{e}_1 + \partial_2\phi\,\hat{e}_2$$

   Find the components of the gradient in the primed co-ordinates and show that it transforms as a covariant vector.

2. Show that for an orthogonal transformation, there is no distinction between a contravariant and a covariant vector.

3. Consider the parameterization of a position vector in 3D:

$$\vec{r}(\eta, \phi, z) = \cosh\eta\cos\phi\,\hat{\imath} + \sinh\eta\sin\phi\,\hat{\jmath} + z\,\hat{k}$$

   (a) Demonstrate that η, ϕ, z constitute a set of orthogonal coordinates.
   (b) What is the metric tensor?
   (c) What is $\vec{\nabla} f$ in these coordinates?
   (d) What is $\vec{\nabla}\cdot\mathcal{E}$ in this coordinates, where $\vec{\mathcal{E}} = \mathcal{E}_\eta\,\hat{\eta} + \mathcal{E}_\phi\,\hat{\phi} + \mathcal{E}_z\,\hat{k}$?
   (e) What is $\nabla^2 f$ in this coordinate system? Where possible, express your answer only in terms of cosines and hyperbolic cosines.

In parts (b)-(e) Express your answer in terms of cosines and hyperbolic cosines, where possible.

4. Consider the function
$$\phi(\mathbf{r}) = \frac{r}{r^2 + \epsilon^2}$$
in three dimensions. Calculate $\nabla^2 \phi$.

   (a) Show that for any fixed value of $r = r_0 \neq 0$,
   $$\lim_{r_0 \to 0} \left[\lim_{\epsilon \to 0} \nabla^2 \phi(r_0) \right] = 0.$$

   (b) Show that for a fixed value of $\epsilon \neq 0$,
   $$\lim_{\epsilon \to 0} \left[\lim_{r_0 \to 0} \nabla^2 \phi(r_0) \right] = \infty$$

   (c) Using (a) and (b), construct an argument that $\nabla^2 \frac{1}{r}$ is zero at all points save at the origin, where it diverges. If $f(r)$ is a smooth, well behaved function, calculate an estimate for
   $$\int f(r) \, \nabla^2 \left(\frac{1}{r} \right) d^3 r$$
   where the integration range is over all space.

5. Your textbook (and many others) state that the curl of the gradient of a scalar function is zero. Consider the function $f(r, \varphi) = \varphi$ where r and φ are the standard polar coordinates in two dimensions.

   (a) Calculate $\vec{g} = \vec{\nabla} f$ in polar coordinates.

   (b) Calculate $\vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{g}$ in polar coordinates.

   (c) From Stoke's Theorem, We know
   $$\int \left(\vec{\nabla} \times \vec{g} \right) \cdot \hat{n} \; dA = \oint \vec{g} \cdot d\vec{\ell}$$
   Calculate the last line integral in polar coordinates.

   (d) Do your answers agree? Can you explain?

6. This is one long problem, designed to give you experience in working with tensors, and exposure to one of the fundamental tensors in field theory, that of the electromagnetic field. **You have two weeks to do this problem.** The questions below basically encompass questions 11-13 in the text, with some extensions. Those of you interested in other approaches might look at *The Feynman Lectures*, II-27.

   The questions are a bit tedious, but should not be too difficult. If you are having a lot of trouble, then I have probably made a mistake with the question. In that case come see me for clarification.

Maxwell's equations are one of the crowning achievements of classical physics. If we ignore the presence of any polarizing medium, they are:

$$\begin{aligned}
\nabla \cdot \mathcal{E} &= 4\pi\rho \\
\nabla \cdot \mathcal{B} &= 0 \\
\nabla \times \mathcal{E} &= -\frac{1}{c}\frac{\partial \mathcal{B}}{\partial t} \\
\nabla \times \mathcal{B} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t}
\end{aligned}$$

We will develop a more compact notation for expressing these equations.

(a) Derive the law of conservation of charge from Maxwell's equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

(b) Express the electric field in terms of the scalar potential ϕ and the vector potential $\mathcal{A}$ where $\mathcal{B} = \nabla \times \mathcal{A}$. (Note, that in the general time dependent case the answer is *not* simply $\mathcal{E} = -\nabla\phi$).

(c) A second-rank antisymmetric tensor in 3 dimensions has only 3 independent elements. We can make a link between it and a vector in 3 dimensions via:

$$F_{ij} = \epsilon_{ijk} B_k$$

Prove that if this is the case, then $B_k = \frac{1}{2}\epsilon_{ijk} F_{ij}$.

(d) Prove that for F_{ij} as defined above,

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \equiv \partial_i A_j - \partial_j A_i$$

where $\mathcal{A}$ is the vector potential mentioned above.

This tensor is obviously antisymmetric - that is, $F_{ij} = -F_{ji}$. It has $3^2 = 9$ components. We will now extend it to four dimensions, so that it has $4^2 = 16$ components. (Note that it is still a second *rank* tensor. The number of dimensions is not the same as the rank of the tensor.) We will do so by setting $x_4 = ict$, and $A_4 = i\phi$, where $i = \sqrt{-1}$ and ϕ is the electrostatic scalar potential. The indices will now in general run from 1 to 4, rather from 1 to 3.

(e) Prove $F_{4j} = -F_{j4} = iE_j$ where F_{ij} defined as in part (d), and $j = 1, 2, 3$. (F_{44} is zero.) Don't forget that the electric field is *not* simply $-\nabla\phi$.

There are sixteen elements of F_{ij}. The four diagonal ones must be zero, since antisymmetry requires $F_{ii} = -F_{ii}$. Of the 12 left, only half of them are independent, because of the antisymmetry requirement. Of those six, we have just shown that three are the magnetic field, and three are the electric field. If we write out the results of parts (d) and (e) we find:

$$F_{ij} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

Define the current as a four dimensional vector: $\mathcal{J} = (J_x, J_y, J_z, ic\rho)$. Our goal will be to show that all of Maxwell's equations are subsumed in:

[1] $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$

[2] $\partial_k F_{\ell k} = (4\pi/c) J_\ell$.

Eq.(1) is in fact $4^3 = 64$ equations, so it will help to break this down

(f) Show that eq.(1) holds from the definition of F_{ij} given in (d), that is $F_{ij} = \partial_i A_j - \partial_j A_i$. (While this is nice, we haven't really made a link yet to Maxwell's equations!)

(g) Show that eq.(1) trivially holds when any pair of indices is the same (i.e. if $i = j$) or if all are the same.

(h) We have only cases where $i \neq j \neq k$, or $4 \times 3 \times 2 = 24$ cases left. Prove (1) holds when $i = 4$, when j and k are different spatial indices (1,2,3). This covers $3 \times 2 = 6$ cases. However, our choice of starting with $i = 4$ was arbitrary - we could have started with j. This gives us a factor of 3, so we have covered 18 cases.

(i) Prove (1) holds when $i \neq j \neq k \neq 4$. (Note that these all involve just the magnetic field). There are $3 \times 2 \times 1 = 6$ equations. This covers the remaining cases.

(j) Prove eq.(2) gives the rest of Maxwell's equations.

Thus we have shown that eqs (1) and (2) reproduce Maxwell's equations and give nothing more.

For your edification, it is interesting to note that the Lorentz force law is:

$$f_i = \frac{1}{c} F_{ij} J_k,$$

for $i = 1, 2, 3$. (Convince yourself of this, but you don't have to calculate it for the homework.) The quantity icf_4 gives the work done by the electric field.

What have we gained by writing $\mathcal{E}$ and $\mathcal{B}$ in terms of F_{ij}? Well, we've saved a bit of typing. It is very suggestive that we can express everything so simply. But recall that F_{ij} is not some arbitrary matrix, it is a *tensor*. That means if our transformation matrix for coordinates is L_{ij}, then when we change to a different coordinate system,

$$F'_{ij} = L_{im} L_{jn} F_{mn}$$

Let's see if this shows up in our new formulation.

Consider an infinite line of charge along the z-axis. You will recall that it produces an electric field: $\mathcal{E} = \lambda \log(r)\,\hat{r} = \lambda \log(r)\,(\cos\theta\hat{x} + \sin\theta\hat{y})$ where λ is a constant involving the charge per unit length. The charge distribution is $\mathcal{J} = (0, 0, 0, ic\rho_0(\vec{r}))$, where $\rho_0(\vec{r})$ is zero everywhere but on the z-axis, where it is a constant. The transformation matrix we will use will put us in a moving frame, which is sometimes called a "boost" transformation. In this case we will "boost" along the z-axis. The matrix is

$$L_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\alpha & i\sinh\alpha \\ 0 & 0 & -i\sinh\alpha & \cosh\alpha \end{pmatrix}$$

It can be thought of as a rotation, by an angle $i\alpha$, where $\cosh\alpha = 1/\sqrt{1-(v^2/c^2)}$. This transformation puts us in a frame moving at velocity v in the positive $\hat{z}$ direction. The quantity α is sometimes called the "rapidity". It is useful because if we make two successive boosts in the same direction, the final velocity may not be the sum of the two boost velocities, but the final rapidity is the sum of the two boost rapidities. (That is, rapidities are additive in relativistic mechanics.)

(k) Calculate $\mathcal{J}'$. (It transforms as a vector.) Approximate it for $v << c$. In this frame we should now see a current. Can you give a meaning to the change in $\mathcal{J}_4$?

(l) Calculate B'_x and B'_y, by calculating F'_{23} and F'_{31}. Can you explain why in this frame we see a magnetic field?

Thus when we make a transform to a new coordinate system, we see different electric and magnetic fields. The value of the fields is consistent with what we would have gotten if we had started our calculations in the moving frame. All of this information is built in to the tensor F_{ij}, suggesting that this is more than just a convenient notation. Those of you who have a lot of time on your hands might wonder how all of the above changes if we allow for *magnetic* charges (i.e. monopoles).

#4

$$x_1' = \frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2$$

$$x_2' = x_2$$

The above gives our transformation matrix,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Therefor our trasformation a_{ij} is

$$a_{ij} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix}$$

What are the unit vectors, $\hat{e}_1'$ & $\hat{e}_2'$?

$$\vec{x} = \vec{x}'$$

$$x_j \hat{e}_j = x_i' \hat{e}_i'$$

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)\hat{e}_1' + x_2 \hat{e}_2'$$

$$x_1 \hat{e}_1 + x_2 \hat{e}_2 = x_1\left(\frac{1}{\sqrt{2}} \hat{e}_1'\right) + x_2\left(\frac{\hat{e}_1'}{\sqrt{2}} + \hat{e}_2'\right)$$

So
$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix}$$

So our basis vectors transform as

$$\begin{pmatrix} \hat{e}_1' \\ \hat{e}_2' \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^{-1} \begin{pmatrix} 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

Check:

$$x_i' \hat{e}_i' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2\right) \\ x_2 \end{bmatrix} \begin{bmatrix} \sqrt{2}\, \hat{e}_1 \\ -\hat{e}_1 + \hat{e}_2 \end{bmatrix}$$

$$= (x_1 + x_2)\hat{e}_1 + x_2(-\hat{e}_1 + \hat{e}_2)$$

$$= x_1 \hat{e}_1 + \cancel{x_2 \hat{e}_1} - \cancel{x_2 \hat{e}_1} - x_2 \hat{e}_2$$

$$= x_1 \hat{e}_1 + x_2 \hat{e}_2 = x_j \hat{e}_j$$

which checks!

How do the components of $\vec{\nabla}\phi$ change?

$$\vec{\nabla}\phi \cdot \vec{ds} = \frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2$$

$$= \frac{\partial\phi}{\partial x_1'} dx_1' + \frac{\partial\phi}{\partial x_2'} dx_2'$$

but if

$$\begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$$

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx_1' \\ dx_2' \end{pmatrix}$$

So $$\vec{\nabla}\phi \cdot \vec{ds} = \frac{\partial\phi}{\partial x_1}\left(\sqrt{2}\, dx_1' - dx_2'\right) + \frac{\partial\phi}{\partial x_2} dx_2'$$

$$= \left(\sqrt{2}\frac{\partial\phi}{\partial x_1}\right) dx_1' + \left(-\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2}\right) dx_2'$$

$$= \left(\frac{\partial\phi}{\partial x_1'}\right) dx_1' + \left(\frac{\partial\phi}{\partial x_2'}\right) dx_2'$$

So $$\begin{pmatrix} \frac{\partial\phi}{\partial x_1'} \\ \frac{\partial\phi}{\partial x_2'} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\frac{\partial\phi}{\partial x_1} \\ -\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial x_2} \end{pmatrix}$$

which is how $\hat{e}_i$ transforms

#5 If we use the notation of Byron & Fuller then all indices are lower indices. A contravariant vector transforms as

$$x_j' = a_{jk} x_k$$

Then the change of a scalar function ϕ when we move $d\vec{s}$ is the same in all co-ordinate systems. If we write $\frac{\partial \phi}{\partial x_i}$ as $\partial_i \phi$ then

$$d\phi = \partial_j \phi \, dx_j = \partial_j' \phi' \, dx_j'$$

Assume $\nabla\phi$ transforms as $\mathbb{b}$, so that

$$\partial_j' \phi' = b_{jl} \partial_l \phi$$

Then

$$\partial_j \phi \, dx_j = \partial_j' \phi \, \partial x_j'$$

$$= b_{jl} \partial_l \phi \quad a_{jk} \, dx_k$$

$$= b_{jl} a_{jk} \partial_l \phi \, dx_k .$$

This can only hold true if $b_{jl} a_{jk} = \delta_{lk}$.

Then

$$\mathbb{b}^T \mathbb{a} = \underline{\mathbb{1}}.$$

$$\mathbb{b}^T = \mathbb{a}^{-1}$$

But for an orthogonal matrix

$$\mathbb{U}^T = \mathbb{U}^{-1}$$

So for orthogonal transformations there is no distinction between contra- & co-variant vectors.

It is common to write contravariant indices with superscripts & co-variant indices as subscripts. For example

$$d\phi = \partial^\mu \phi \; dx_\mu.$$

Only a expressions that have summed pairs of super & subscripts are true scalars for that set of transformations —

(2) Given

$$\vec{r} = \cosh\eta \cos\phi \,\hat{i} + \sinh\eta \sin\phi \,\hat{j} + z\hat{k}$$

(a) $$d\vec{r} = (\sinh\eta \cos\phi \, d\eta - \cosh\eta \sin\phi \, d\phi)\hat{i}$$

$$+ (\cosh\eta \sin\phi \, d\eta + \sinh\eta \cos\phi \, d\phi)\hat{j}$$

$$+ dz\,\hat{k}$$

$$= (\sinh\eta \cos\phi \,\hat{i} + \cosh\eta \sin\phi \,\hat{j}) \, d\eta$$

$$+ (-\cosh\eta \sin\phi \,\hat{i} + \sinh\eta \cos\phi \,\hat{j}) \, d\phi$$

$$+ dz\,\hat{k}$$

Therefore the un-normalized basis vectors are

$$\vec{e}_\eta = \sinh\eta \cos\phi \,\hat{i} + \cosh\eta \sin\phi \,\hat{j}$$

$$\vec{e}_\phi = -\cosh\eta \sin\phi \,\hat{i} + \sinh\eta \cos\phi \,\hat{j}$$

$$\hat{e}_z = \hat{k}$$

Obviously $\hat{e}_z$ is perpendicular to the other two vectors — What is $\vec{e}_\eta \cdot \vec{e}_\phi$?

$$\vec{e}_\eta \cdot \vec{e}_\phi = -\sinh\eta \cosh\eta \cos\phi \sin\phi + \cosh\eta \sinh\eta \sin\phi \cos\phi$$

$$= 0$$

(b) $d\vec{r}^2 = \left(\sinh^2\eta \cos^2\phi + \cosh^2\eta \sin^2\phi \right) d\eta^2$

$+ \left(\cosh^2\eta \sin^2\phi + \sinh^2\eta \cos^2\phi \right) d\phi^2$

$+ dz^2$

To put this in standard form we write.

$\sinh^2\eta = \cosh\eta^2 - 1$

$\sin^2\phi = 1 - \cos^2\phi$

$d\vec{r}^2 = \left((\cosh^2\eta - 1)\cos^2\phi + \cosh^2\eta\,(1-\cos^2\phi) \right) d\eta^2$

$+ \left(\cosh^2\eta\,(1-\cos^2\phi) + (\cosh^2\eta - 1)\cos^2\phi \right) d\phi^2$

$+ dz^2$

$= \left(\cosh^2\eta - \cos^2\phi \right) d\eta$

$+ \left(\cosh^2\eta - \cos^2\phi \right) d\phi^2 + dz^2$

<u>So</u>:

$$g_{\mu\nu} = \begin{pmatrix} \cosh^2\eta - \cos^2\phi & 0 & 0 \\ 0 & \cosh^2\eta - \cos^2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$h_\eta^2 = h_\phi^2 = \cosh^2\eta - \cos^2\phi$$

$$h_z^2 = 1$$

(c) What is $\vec{\nabla} f$?

$$\vec{\nabla} = \frac{1}{\sqrt{\cosh^2 - \cos^2\phi}} \left\{ \frac{\partial f}{\partial \eta} \hat{e}_\eta + \frac{\partial f}{\partial \phi} \hat{e}_\phi \right\} + \frac{\partial f}{\partial z} \hat{z}$$

note that $\hat{e}_\eta$ and $\hat{e}_\phi$ are the normalized basis vectors —

(d) What is $\vec{\nabla} \cdot \vec{\mathcal{E}}$?

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \frac{1}{(\cosh^2\eta - \cos^2\phi)} \left\{ \frac{\partial}{\partial \eta} \sqrt{\cosh^2\eta - \cos^2\phi}\, \mathcal{E}_\eta + \frac{\partial}{\partial \phi} \sqrt{\cosh^2\eta - \cos\phi^2}\, \mathcal{E}_\phi \right\} + \frac{\partial \mathcal{E}_z}{\partial z}$$

(e) What is $\nabla^2 f$?

$$\nabla^2 f = \frac{1}{\cosh^2\eta - \cos^2\phi} \left\{ \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \phi^2} \right\} + \frac{\partial^2 f}{\partial z^2}$$

(4) If

$$\phi(r,\epsilon) = \frac{r}{r^2+\epsilon^2}$$

Then $\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \phi}{\partial r}$

$$= \frac{2\epsilon^4 - 6r^2\epsilon^2}{(r^2+\epsilon^2)^3}$$

(a) $\lim_{r\to 0} \lim_{\epsilon \to 0} \nabla^2\phi = \lim_{r\to 0} 0 = 0$

(b) $\lim_{\epsilon\to 0} \lim_{r\to 0} = \lim_{\epsilon\to 0} 2\frac{\epsilon^4}{\epsilon^6} = \infty$

(c) Let's assume that $f(r)$ is well behaved.

We want to calculate

$$\lim_{\epsilon\to 0} \int d^3r\, f(r)\, \nabla^2\phi(r).$$

$$= \lim_{\epsilon\to 0} \int d\Omega\, r^2 dr \cdot \frac{2\epsilon^4 - 6r^2\epsilon^2}{(r^2+\epsilon^2)^3}$$

$$\times \left\{ f(0) + r f'(0) + \cdots \right\}.$$

$$= 4\pi \int_0^\infty r^2 \left(\frac{2\epsilon^4 - 6r^2\epsilon^2}{(r^2+\epsilon^2)^3}\right) \times \left\{ f(0) + r f'(0) \ldots \right\}$$

Integrating we obtain -

$$\lim_{\epsilon \to 0} 4\pi \left\{ \frac{\epsilon^2 r^2 \left(3 f(0) + 4 f'(0) r\right) + \epsilon^4 \left(f(0) + 2 r f'(0)\right)}{(r^2 + \epsilon^2)^2} - 2\epsilon f'(0) \arctan \frac{r}{\epsilon} \right\} \Bigg|_0^{\infty}$$

$$= 4\pi f(0)$$

Thus $\nabla^2 \left(\frac{1}{r}\right) = 4\pi \, \delta(\vec{r})$.

(5) Given that $f(\vec{r}) = \varphi$.

(a) $\vec{g} = \vec{\nabla} f = \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} = \frac{1}{r}$

(b) $\vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{g}$

$$= \hat{k}\,\frac{1}{r}\frac{\partial}{\partial r}(r\, g_\phi)$$

$$= \hat{k}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{1}{r}\right) =$$

$$\hat{k}\,\frac{1}{r}\frac{\partial}{\partial r}\,1 = 0.$$

(c) $\int \vec{\nabla} \times \bar{g} \cdot \hat{n}\, dA = \oint \bar{g} \cdot d\bar{l}$

$$= \int \frac{1}{r}\hat{\phi}\,(r\, d\phi\,\hat{\phi})$$

$$= \int d\phi = 2\pi$$

(d) They don't agree. The function $f(r,\phi)$ is multiple valued, & has a discontinuity at $\phi = 0$.