COLLEGE OF ARTS AND SCIENCES
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The UNIVERSITY of OKLAHOMA

Classical Mechanics

PHYS 5153 HOMEWORK ASSIGNMENT #7

PROBLEMS: $\{1, 2, 3, 4\}$

Due: October 28, 2021 By 6:00 PM

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PROFESSOR
Dr. Robert Lewis-Swan

Problem 1:

In class we expressed components of the angular velocity ω along the body-fixed axes in terms of Euler angles as

$$\omega_{bf} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}. \tag{1}$$

Show that the angular velocity along the space-fixed axes in terms of Euler angles is instead given by,

$$\omega_{sf} = \begin{pmatrix} \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi \\ -\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{pmatrix} \tag{2}$$

$$B = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{Rotation about body fixed z-axis}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \longrightarrow \text{Rotation about body fixed x-axis}$$

$$D = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{Rotation about space fixed z-axis}$$

$$A = BCD = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \cos(\psi)\cos(\varphi) - \cos(\alpha)\sin(\varphi)\sin(\psi) & \cos(\psi)\sin(\varphi) + \cos(\alpha)\cos(\varphi)\sin(\psi) & \sin(\psi)\sin(\alpha) \\ -\sin(\psi)\cos(\varphi) - \cos(\alpha)\sin(\varphi)\cos(\psi) & -\sin(\psi)\sin(\varphi) + \cos(\alpha)\cos(\varphi)\cos(\psi) & \cos(\psi)\sin(\alpha) \\ \sin(\alpha)\sin(\varphi) & -\sin(\alpha)\cos(\varphi) & \cos(\alpha) \end{pmatrix}$$

$\hookrightarrow$ Goldstein p.g. 153 EQ: (4.43) - (4.46)

$$A^{-1} = \begin{pmatrix} \cos(\psi)\cos(\varphi) - \cos(\alpha)\sin(\varphi)\sin(\psi) & -\sin(\psi)\cos(\varphi) - \cos(\alpha)\sin(\varphi)\cos(\psi) & \sin(\alpha)\sin(\varphi) \\ \cos(\psi)\sin(\varphi) + \cos(\alpha)\cos(\varphi)\sin(\psi) & -\sin(\psi)\sin(\varphi) + \cos(\alpha)\cos(\varphi)\cos(\psi) & -\sin(\alpha)\cos(\varphi) \\ \sin(\psi)\sin(\alpha) & \cos(\psi)\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Because A is orthogonal, we can transpose A to find A^{-1}. To find the space axes from the body co-ordinates we do $x' = A^{-1}x$.

$$\omega_{sF} = A^{-1}\omega_{bf}$$

Using mathematica we see this is true

Problem 1: Continued

Problem 1:

In[ ]:=

```
MatrixA =
  {{Cos[ψ] * Cos[φ] - Cos[θ] * Sin[φ] * Sin[ψ], -Sin[ψ] * Cos[φ] - Cos[θ] * Sin[φ] * Cos[ψ],
    Sin[θ] * Sin[φ]}, {Cos[ψ] * Sin [φ] + Cos[θ] * Cos[φ] * Sin[ψ],
    -Sin[ψ] * Sin[φ] + Cos[θ] * Cos[φ] * Cos[ψ], -Sin[θ] * Cos[φ]},
   {Sin[ψ] * Sin[θ], Cos[ψ] * Sin[θ], Cos[θ]}};
MatrixForm[MatrixA]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\varphi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\varphi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\varphi] - \text{Cos}[\varphi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\varphi \\ \text{Cos}[\psi]\,\text{Sin}[\varphi] + \text{Cos}[\theta]\,\text{Cos}[\varphi]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\varphi]\,\text{Cos}[\psi] - \text{Sin}[\varphi]\,\text{Sin}[\psi] & -\text{Cos}[\varphi]\,\text{Sin}[\\ \text{Sin}[\theta]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

$\rightarrow A^{-1}$

In[ ]:=

```
MatrixW =
  {{φ̇ * Sin[θ] * Sin[ψ] + Cos[ψ] * θ̇}, {φ̇ * Sin[θ] * Cos[ψ] - θ̇ * Sin[ψ]}, {φ̇ * Cos[θ] + ψ̇}};
MatrixForm[MatrixW]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\psi]\,\dot{\theta} + \dot{\varphi}\,\text{Sin}[\theta]\,\text{Sin}[\psi] \\ \text{Cos}[\psi]\,\dot{\varphi}\,\text{Sin}[\theta] - \dot{\theta}\,\text{Sin}[\psi] \\ \text{Cos}[\theta]\,\dot{\varphi} + \dot{\psi} \end{pmatrix} \longrightarrow \omega_{bf}$$

```
MatrixForm[FullSimplify[MatrixA.MatrixW]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\varphi]\,\dot{\theta} + \dot{\psi}\,\text{Sin}[\theta]\,\text{Sin}[\varphi] \\ -\text{Cos}[\varphi]\,\dot{\psi}\,\text{Sin}[\theta] + \dot{\theta}\,\text{Sin}[\varphi] \\ \dot{\varphi} + \text{Cos}[\theta]\,\dot{\psi} \end{pmatrix} \longrightarrow \omega_{sf} \quad \checkmark$$

Problem 1: Review

Procedure:

- Calculate the rotation matrix $\mathbf{A}$ by multiplying the matrices $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ together. These equations can be found in Goldstein p.g. 153 EQ: (4.43) - (4.46).
- Find the inverse of $\mathbf{A}$.
- The resulting column matrix is calculated via
$$\omega_{sf} = \mathbf{A}^{-1}\omega_{bf}.$$

Key Concepts:

- We can transform between space fixed to body fixed with the relationship,
$$\omega_{sf} = \mathbf{A}^{-1}\omega_{bf}.$$
- We can transform between body fixed to space fixed with
$$\omega_{bf} = \mathbf{A}\omega_{sf}.$$
- The matrices $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ correspond to rotations about the body fixed z-axis, body fixed x-axis, and space fixed z-axis respectively.

Variations:

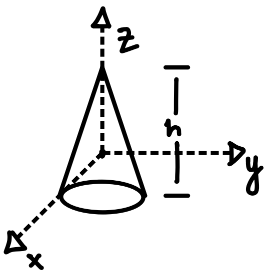
- We can be asked to transform between different co-ordinate systems.
  - This would correspond mathematically to using different orders of rotation matrices defined above.
- We can be asked to transform to the body fixed frame.
  - This corresponds to the equation found in the second bullet point of the **Key Concepts** section.

Problem 2:

Consider a 3D cone with uniform mass density (and total mass M).

(a) Find the moment of inertia tensor assuming the body-fixed co-ordinate system is such that the origin is placed at:

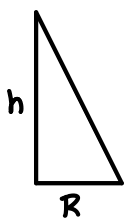
i The center-of-mass (COM) of the cone and the z-axis passes through the sharp tip of the cone.



Because of the placement of this cone, we have symmetries about each axis. This makes our moment of inertia tenser look like:

$$I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

We can calculate the individual components with:



$y = mx + b$

$r = -\frac{R}{h}(z - 3/4 h)$

$$I_{jk} = \int_V \rho(r)(r^2 \delta_{jk} - x_j x_k)\, dv \longleftrightarrow \text{Goldstein (5.8)}$$

Volume of cone : $\int_0^{2\pi}\int_0^h\int_0^{Rz/h} r\, dr\, dz\, d\alpha = \frac{\pi R^2 h}{3} \quad \therefore \quad \rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$

If this rotated about any axis, we will have the same rotation at the end of the day, therefore our matrix tensor will be a diagonal.

For I_{xx}, $I_{xx} = \int_V \rho(r)(r^2 - x^2)\, dv$, $r^2 = x^2+y^2+z^2$, $r^2 - x^2 = y^2 + z^2$ w/ $y = r\sin(\alpha)$

$$I_{xx} = \rho \int_0^{2\pi}\int_{-1/4h}^{3/4h}\int_0^{R/h(3/4h - z)} (r^2 \sin^2(\alpha) + z^2)\, r\, dr\, dz\, d\alpha = \frac{\rho \pi R^2 h (4R^2 + h^2)}{80}, \quad \rho = \frac{3M}{\pi R^2 h}$$

Via mathematica

For I_{yy}, $I_{yy} = \int_V \rho(r)(r^2 - y^2)\, dv$, $r^2 = x^2+y^2+z^2$, $r^2 - y^2 = x^2 + z^2$ w/ $x = r\cos(\alpha)$

$$I_{yy} = \rho \int_0^{2\pi}\int_{-1/4h}^{3/4h}\int_0^{R/h(3/4h - z)} (r^2 \cos^2(\alpha) + z^2)\, r\, dr\, dz\, d\alpha = \frac{\rho \pi R^2 h (4R^2 + h^2)}{80}, \quad \rho = \frac{3M}{\pi R^2 h}$$

Via mathematica

For I_{zz}, $I_{zz} = \int_V \rho(r)(r^2 - z^2)\, dv$, $r^2 = x^2+y^2+z^2$, $r^2 - z^2 = x^2 + y^2$ w/ $x^2 + y^2 = r^2$

$$I_{zz} = \rho \int_0^{2\pi}\int_{-1/4h}^{3/4h}\int_0^{R/h(3/4h - z)} r^3\, dr\, dz\, d\alpha = \frac{\rho \pi R^4 h}{10}, \quad \rho = \frac{3M}{\pi R^2 h} : \rho = \frac{M}{V}$$

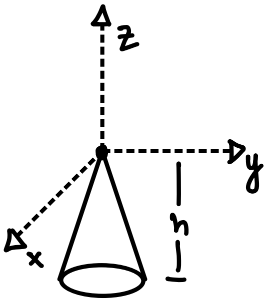
Via mathematica

$$I_{xx} = I_{yy} = \frac{3M(4R^2 + h^2)}{80}, \quad I_{zz} = \frac{3MR^2}{10}$$

$$\boxed{I = \frac{3M}{80}\begin{pmatrix} 4R^2+h^2 & 0 & 0 \\ 0 & 4R^2+h^2 & 0 \\ 0 & 0 & 8R^2 \end{pmatrix}}$$

Problem 2: Continued

ii The sharp tip (apex) of the cone and the z-axis passes through the COM



The difference between part i and ii is the placement of the cone in respect to the origin. We will place the tip of the cone at the origin. This changes the limits of integration for z, but this rotation will still be symmetric and the matrix will be diagonal.

For I_{xx}, $I_{xx} = \int_V \rho(r)(r^2 - x^2)\,dV$, $r^2 = x^2+y^2+z^2$, $r^2 - x^2 = y^2+z^2$ w/ $y = r\sin(\alpha)$

$$I_{xx} = \rho \int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h} (r^2\sin^2(\alpha) + z^2)\, r\,dr\,dz\,d\alpha = \rho\,\frac{h\pi R^2(R^2+4h^2)}{20}, \quad \rho = \frac{M}{V}$$

Via mathematica

For I_{yy}, $I_{yy} = \int_V \rho(r)(r^2 - y^2)\,dV$, $r^2 = x^2+y^2+z^2$, $r^2 - y^2 = x^2+z^2$ w/ $x = r\cos(\alpha)$

$$I_{yy} = \rho \int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h} (r^2\cos^2(\alpha) + z^2)\, r\,dr\,dz\,d\alpha = \rho\,\frac{h\pi R^2(R^2+4h^2)}{20}, \quad \rho = \frac{M}{V}$$

Via mathematica

For I_{zz}, $I_{zz} = \int_V \rho(r)(r^2 - z^2)\,dV$, $r^2 = x^2+y^2+z^2$, $r^2 - z^2 = x^2+y^2$ w/ $r^2 = x^2+y^2$

$$I_{zz} = \rho \int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h} r^3\,dr\,dz\,d\alpha = \rho\,\frac{h\pi R^4}{10}, \quad \rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

Via mathematica

$$I_{xx} = I_{yy} = \frac{3M(R^2+4h^2)}{20}, \quad I_{zz} = \frac{3MR^2}{10}$$

$$\boxed{I = \frac{3M}{20}\begin{pmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}}$$

(b) Check your results to (a) by verifying that they are consistent with the Steiner's parallel axes theorem:

$$I_{jk} = I'_{jk} - M(|\mathbf{R}|^2\delta_{jk} - R_j R_k), \tag{3}$$

where I is the moment of inertia tensor in the COM basis, I' relative to the cone tip and $\mathbf{R}$ is the vector from the origin O of the COM basis to the cone tip.

$$I_{xx} = I_{xx}' - M(|R|^2 - R_x R_x) = \frac{3M}{20}(R^2+4h^2) - M\left(\left(\frac{3}{4}h\right)^2 - 0(0)\right) = \frac{3M}{20}(R^2+4h^2) - \frac{9Mh^2}{16}$$

$$I_{xx} = \frac{12M}{80}(R^2+4h^2) - \frac{45Mh^2}{80} = \frac{12MR^2}{80} + \frac{3Mh^2}{80} \underset{0}{=} \frac{3M}{80}(4R^2+h^2), \quad I_{xx} = I_{yy}$$

$$I_{zz} = I_{zz}' - M(|R|^2 - R_z R_z) = \frac{3MR^2}{10} - M\left(\left(\frac{3}{4}h\right)^2 - \cancel{\left(\frac{3}{4}h\right)}\left(\frac{3}{4}h\right)\right) = \frac{3MR^2}{10}, \quad I_{zz} = I_{zz}'$$

These tensors are the same. ✓

Problem 2: Review

Procedure:

- Begin by calculating the volume of the solid, in this case a cone.
- Calculate the center of mass via
$$q_{COM} = \frac{\rho}{m}\int q_i \, dV.$$
- Place the center of mass of our cone along the principle axis. This axis is arbitrarily chosen, in this case the principle axis is the z axis.
- For the first part of the problem, place the origin of our co-ordinate system where the center of mass of our cone is.
  - Doing this will yield the moment of inertia for each co-ordinate w.r.t the center of mass. Which at the same time is the origin as well.
- Proceed to calculate the moment of inertia with
$$I_{jk} = \int_v \rho(r)(\mathbf{r}^2\delta_{jk} - x_j x_k) \, dV.$$
- Proceed to calculate the moments of inertia for the I_{xx}, I_{yy}, and I_{zz} components.
- Repeat the above steps, but instead of placing the origin at the center of mass, place it at the tip of the cone.
  - Doing this will yield the moments of inertia for each co-ordinate w.r.t the origin. Which in this case is no longer at the center of mass.
- Use Steiner's parallel axes theorem to calculate the moment of inertia with respect to the center of mass using
$$I_{jk} = I'_{jk} - M(|\mathbf{R}|^2\delta_{jk} - R_j R_k). \tag{1}$$
  - $I'_{jk} \rightarrow$ moment of inertia w.r.t origin. $\mathbf{R} \rightarrow$ distance from origin to center of mass. $R_{j,k} \rightarrow$ distance from center of mass to origin w.r.t the co-ordinate j, k.
- Show that the Steiner's parallel axes theorem is correct.

Key Concepts:

- Placing the center of mass at the origin of our co-ordinate system eliminates the need to calculate the moment of inertia via Steiner's parallel axes theorem.
- When the center of mass is placed along the principle axis, we will have a moment of inertia tensor that is diagonal.
- We use Steiner's parallel axis theorem to calculate the moment of inertia w.r.t the center of mass when we have already calculated the moment of inertia with respect to the origin.
- It is always best to place the origin of our solid where the integration will occur along the principle axis, regardless of whether or not that origin is at the center of mass of the solid.

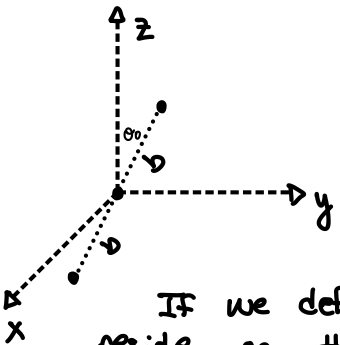
Variations:

- This specific problem cannot be changed that much.
  - We can be asked to find the moments of inertia for other shapes, like other problems in other assignments.

Problem 3:

Consider a toy model of a diatomic molecule where the atoms are taken to be point particles with masses m_1 and m_2 connected by a massless rigid rod of length $2b$. Assume the molecule rotates in such a fashion that the rigid rod makes a constant angle θ_0 with respect to the z-axis of the space-fixed co-ordinate system and the atoms trace out circular orbits in respective xy-planes (i.e., their space-fixed z co-ordinates are fixed). What is the angular momentum of the molecule and what is the magnitude of the torque that must be applied (in the space-fixed frame) for the molecule to continue precessing about a fixed axis? Solve this question by:

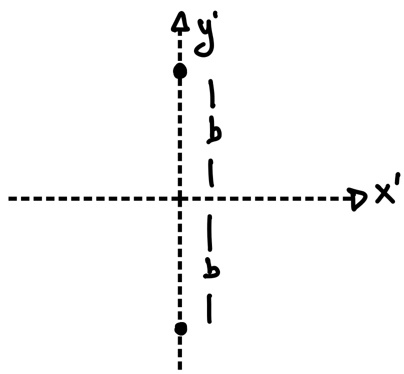
(a) Computing the momentum of inertia tensor using a body fixed co-ordinate system set by the principle axes.



Our diatomic molecule is in a body-fixed co-ordinate system so this means that our moment of inertia tensor will be symmetric. We calculate the moments of inertia via:

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2), \quad I_{yy} = \sum_i m_i(r_i^2 - y_i^2), \quad I_{zz} = \sum_i m_i(r_i^2 - z_i^2)$$

If we define a new set of co-ordinate's such that our molecules only reside on the y-axis, it will look like.



$$I_{xx} = \sum_i m_i(\cancel{x_i^2} + y_i^2 + z_i^2 - \cancel{x_i^2}) = \sum_i m_i(y_i^2 + z_i^2) = m_1(\overset{b^2}{\cancel{y_1^2}} + \overset{0}{\cancel{z_1^2}}) + m_2(\overset{b^2}{\cancel{y_2^2}} + \overset{0}{\cancel{z_2^2}}) = (m_1+m_2)b^2$$

$$I_{yy} = \sum_i m_i(x_i^2 + \cancel{y_i^2} + z_i^2 - \cancel{y_i^2}) = \sum_i m_i(x_i^2 + z_i^2) = m_1(\overset{0}{\cancel{x_1^2}} + \overset{0}{\cancel{z_1^2}}) + m_2(\overset{0}{\cancel{x_2^2}} + \overset{0}{\cancel{z_2^2}}) = 0$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2 + \cancel{z_i^2} - \cancel{z_i^2}) = \sum_i m_i(x_i^2 + y_i^2) = m_1(\overset{0}{\cancel{x_1^2}} + \overset{b^2}{\cancel{y_1^2}}) + m_2(\overset{0}{\cancel{x_2^2}} + \overset{b^2}{\cancel{y_2^2}}) = (m_1+m_2)b^2$$

Therefore $\hat{I}$ will be

$$\hat{I} = \begin{pmatrix} (m_1+m_2)b^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (m_1+m_2)b^2 \end{pmatrix}$$

Therefore the angular momentum would be

$$L_x = I_{xx}w_x + I_{xy}\overset{0}{\cancel{w_y}} + I_{xz}\overset{0}{\cancel{w_z}} = (m_1+m_2)b^2 w_x$$

$$L_y = 0$$

Problem 3: Continued

$$L_z = I_{zz}\omega_z + \cancel{I_{zy}}^{\,0}\omega_y + \cancel{I_{zx}}^{\,0}\omega_x = (m_1+m_2)b^2\omega_z$$

$$\vec{L} = \begin{pmatrix} (m_1+m_2)b^2\omega_x \\ 0 \\ (m_1+m_2)b^2\omega_z \end{pmatrix}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = (m_1+m_2)b^2\dot{\omega}_x\hat{\imath} + (m_1+m_2)b^2\dot{\omega}_z\hat{k}$$

$$\vec{\omega} = \begin{pmatrix} \dot{\varphi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi) \\ \dot{\varphi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi) \\ \dot{\varphi}\cos(\theta) + \dot{\psi} \end{pmatrix} \quad w/ \quad \dot{\theta} = \dot{\psi} = \text{const}, \quad \dot{\vec{\omega}} = \begin{pmatrix} 0 \\ 0 \\ \ddot{\varphi}\cos(\theta) \end{pmatrix} \quad \begin{matrix} \dot{\omega}_x = 0 \\ \dot{\omega}_y = 0 \\ \dot{\omega}_z = \ddot{\phi} \end{matrix}$$

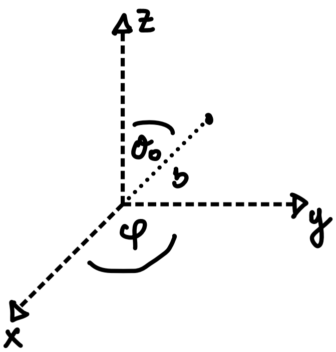
$$\vec{\tau} = \frac{d\vec{L}}{dt} = (m_1+m_2)b^2\dot{\omega}_x\hat{\imath} + (m_1+m_2)b^2\dot{\omega}_z\hat{k}, \quad \dot{\theta} = 0, \quad \psi = \dot{\psi} = 0, \quad \tau_s = \tau_B + \vec{\omega}_B \times \vec{L}_B$$

$$\vec{\omega} = \begin{pmatrix} 0 \\ \dot{\varphi}\sin(\alpha) \\ \dot{\varphi}\cos(\alpha) \end{pmatrix}, \quad \vec{L} = (m_1+m_2)b^2\begin{pmatrix} 0 \\ 0 \\ \dot{\varphi}\cos(\alpha) \end{pmatrix} \; : \; \vec{\omega}_B \times \vec{L}_B = (m_1+m_2)b^2\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & \dot{\varphi}\sin(\alpha) & \dot{\varphi}\cos(\alpha) \\ 0 & 0 & \dot{\varphi}\cos(\alpha) \end{vmatrix}$$

$$\vec{\tau} = (m_1+m_2)b^2\left(\cancel{\dot{\omega}_x}^{\,0} + \dot{\varphi}^2\sin(\alpha)\cos(\alpha)\right)\hat{\imath} + (m_1+m_2)b^2\ddot{\varphi}\cos(\alpha)\hat{k}, \quad \ddot{\varphi} = 0 \quad \therefore$$

$$\boxed{\tau = (m_1+m_2)b^2\dot{\varphi}^2\sin(\alpha)\cos(\alpha)}$$

(b) Computing the rate of change of the angular momentum using a space-fixed co-ordinate system.



$$x = b\cos(\varphi)\sin(\theta_0), \quad y = b\sin(\varphi)\sin(\theta_0), \quad z = b\cos(\theta_0)$$

$$\vec{r} = b\cos(\varphi)\sin(\theta_0)\hat{\imath} + b\sin(\varphi)\sin(\theta_0)\hat{\jmath} + b\cos(\theta_0)\hat{k}$$

$$\dot{\vec{r}} = -b\dot{\varphi}\sin(\varphi)\sin(\theta_0)\hat{\imath} + b\dot{\varphi}\cos(\varphi)\sin(\theta_0)\hat{\jmath} + 0K$$

we can find τ w/ $\dfrac{d\vec{L}}{dt} = \dfrac{d}{dt}(m_1+m_2)(\vec{r} \times \dot{\vec{r}})$

$$\vec{L} = (m_1+m_2)b^2 \cdot \vec{r} \times \dot{\vec{r}} = (m_1+m_2)b^2\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ b\cos(\varphi)\sin(\theta_0) & b\sin(\varphi)\sin(\theta_0) & b\cos(\theta_0) \\ -b\dot{\varphi}\sin(\varphi)\sin(\theta_0) & b\dot{\varphi}\cos(\varphi)\sin(\theta_0) & 0 \end{vmatrix}$$

$$\vec{L} = (m_1+m_2)\left((-b^2\dot{\varphi}\cos(\varphi)\cos(\theta_0)\sin(\theta_0))\hat{\imath} + (-b^2\dot{\varphi}\sin(\varphi)\cos(\theta_0)\sin(\theta_0))\hat{\jmath} + (b^2\dot{\varphi}\sin^2(\theta_0))\hat{k}\right)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} = (m_1+m_2)b^2\dot{\varphi}\left((\dot{\varphi}\sin(\varphi)\cos(\theta_0)\sin(\theta_0))\hat{\imath} + (\dot{\varphi}\cos(\varphi)\cos(\theta_0)\sin(\theta_0))\hat{\jmath} + (0)\hat{k}\right)$$

Problem 3: Continued

$$\Upsilon = (m_1 + m_2) b^2 \dot{\varphi}^2 \sqrt{\sin^2(\varphi)\cos^2(\sigma_0)\sin^2(\sigma_0) + \cos^2(\varphi)\cos^2(\sigma_0)\sin^2(\sigma_0)}$$

$$\Upsilon = (m_1 + m_2) b^2 \dot{\varphi}^2 \sqrt{\sin^2(\sigma_0)\cos^2(\sigma_0)\left(\sin^2(\varphi) + \cos^2(\varphi)\right)}$$

$$\boxed{\Upsilon = (m_1 + m_2) b^2 \dot{\varphi}^2 \sin(\sigma_0)\cos(\sigma_0)}$$

Problem 3: Review

Procedure:

- Begin by placing the rod of the diatomic molecules along a new set of co-ordinates y' and x'. Place this rod solely along the y' axis. This is called body-fixed co-ordinates.
- Calculate the moment of inertia w.r.t the origin of the body-fixed co-ordinates with using
$$I_{ij} = \sum_{ij} m_{i,j}(\mathbf{r}_{i,j}^2 - q_{i,j}^2).$$
- Proceed to calculate the moment of inertia tensor w.r.t the origin of the co-ordinate system.
- Calculate the angular momentum vector with
$$L_i = \sum_i I\omega_i.$$
- Proceed to calculate the torque with
$$\vec{\tau} = \frac{d\vec{L}}{dt}.$$
- Convert this torque into the space-fixed frame by calculating
$$\vec{\tau}_{sf} = \vec{\tau}_{bf} + \dot{\vec{\omega}}_{bf} \times \vec{L}_{bf}$$
where $\dot{\vec{\omega}}_{bf}$ is the time derivative of the $\vec{\omega}_{bf}$ vector.
- Take the above result and find the magnitude of the vector.
- To find the torque of this molecule in the space-fixed frame right away use
$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}I\vec{\omega}_{sf} = \frac{d}{dt}I \cdot \vec{r} \times \dot{\vec{r}}.$$
- Use the above result and compare with the result obtained in (a).

Key Concepts:

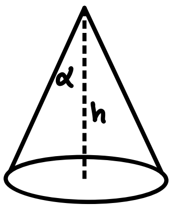
- Since we are looking at two discrete molecules, we do not have to integrate to find the moments of inertia.
- We can calculate the torque in the body-fixed frame and then convert it to the space fixed frame.
- We can calculate the torque in the space-fixed frame directly with the second to last bullet in the **Procedure** section.

Variations:

- We can have the number of particles changed to possibly three.
  - This adds another term to our moment of inertia tensor and also complicates the way we calculate the torque directly in the space-fixed frame.
- The molecules can move in different directions as previously stated.
  - This wouldn't change much of the dynamics of the problem mathematically, but would slightly change some constants like θ and ψ in part (a).

Problem 4:

Consider a cone of mass M, height h and half-angle α rolling on a plane without slipping. The cone is taken to be oriented such that it is rolling on its slanted side (i.e., both the apex of the cone and edge of the base make contact with the surface upon which the cone is rolling). Compute the kinetic energy.

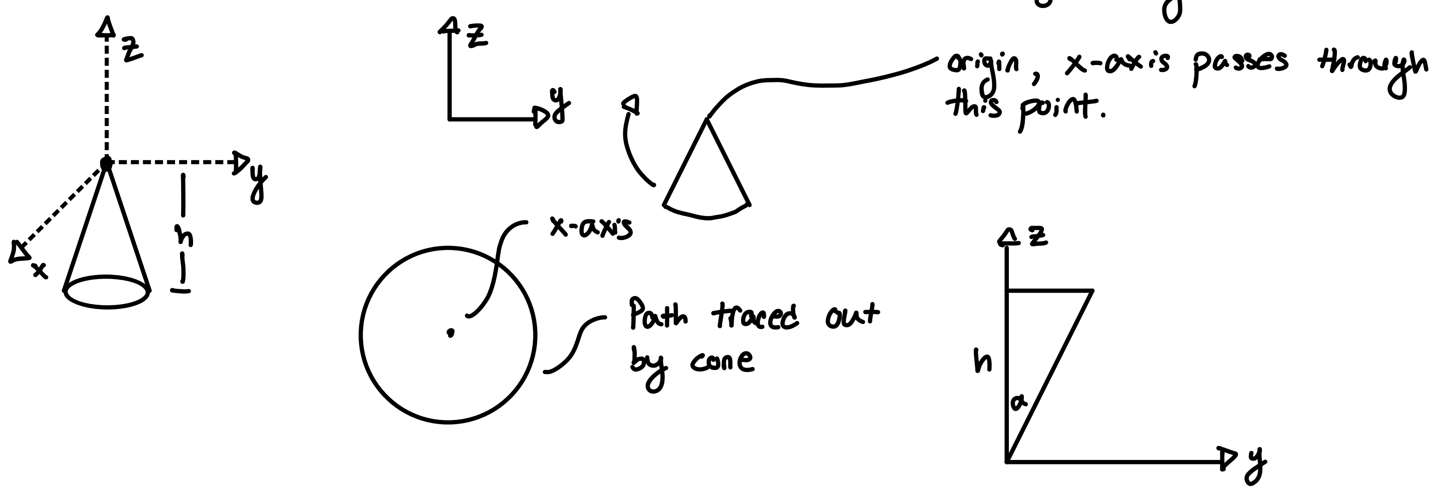


The Kinetic energy of a rotating rigid body is going to be

$$T = \frac{1}{2}\vec{\omega}^{T}\vec{I}\vec{\omega}$$

where I is the moment of inertia and ω is the angular velocity

We first need to find the moment of inertia for this rigid body:



Because this cone has its tip placed at the origin, our moment of inertia will be the same as in problem 2, a.) part ii.). We can re-orient our cone to match the orientation of our cone in problem 2, a.) part ii.). Thus our integrals will become:

$$\text{Volume of cone}: \int_0^{2\pi}\int_0^{h}\int_0^{Rz/h} r\,dr\,dz\,d\alpha = \frac{\pi R^2 h}{3} \quad \therefore \quad \rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

For I_{xx}, $I_{xx} = \int_V \rho(r)(r^2 - x^2)\,dv$, $r^2 = x^2+y^2+z^2$, $r^2 - x^2 = y^2 + z^2$ w/ $y = r\sin(\alpha)$

$$I_{xx} = \rho\int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h}(r^2\sin^2(\alpha)+z^2)\,r\,dr\,dz\,d\alpha = \rho\,\frac{h\pi R^2(R^2+4h^2)}{20}, \quad \rho = \frac{M}{V}$$

Via mathematica

For I_{yy}, $I_{yy} = \int_V \rho(r)(r^2 - y^2)\,dv$, $r^2 = x^2+y^2+z^2$, $r^2 - y^2 = x^2 + z^2$ w/ $x = r\cos(\alpha)$

$$I_{yy} = \rho\int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h}(r^2\cos^2(\alpha)+z^2)\,r\,dr\,dz\,d\alpha = \rho\,\frac{h\pi R^2(R^2+4h^2)}{20}, \quad \rho = \frac{M}{V}$$

Via mathematica

Problem 4: Continued

For I_{zz}, $I_{zz} = \int_V \rho(r)(r^2 - z^2)\,dv$, $r^2 = x^2+y^2+z^2$, $r^2 - z^2 = x^2+y^2$ w/ $r^2 = x^2+y^2$

$$I_{zz} = \rho \int_0^{2\pi}\int_{-h}^{0}\int_0^{-Rz/h} r^3\,dr\,dz\,d\sigma = \rho\,\frac{h\pi R^4}{10}, \quad \rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

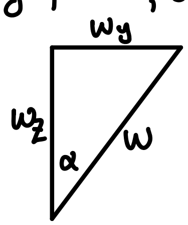
Via mathematica

$$I_{xx} = I_{yy} = \frac{3M(R^2+4h^2)}{20}, \quad I_{zz} = \frac{3MR^2}{10}$$

This means our moment of Inertia tensor is thus:

$$\vec{I} = \frac{3M}{20}\begin{pmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}$$

Now we wish to find the angular velocity tensor. Because our motion is confined to the zy-plane, our angular velocity equation will then be:



$$w_z = w\cos(\alpha)$$

$$w_y = w\sin(\alpha)$$

$$\vec{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \begin{pmatrix} 0 \\ w\sin(\alpha) \\ w\cos(\alpha) \end{pmatrix}, \quad \vec{w}^{\,+} = \begin{pmatrix} 0 & w\sin(\alpha) & w\cos(\alpha) \end{pmatrix}$$

Then our kinetic energy becomes:

$$T = \frac{1}{2}\begin{pmatrix} 0 & w\sin(\alpha) & w\cos(\alpha) \end{pmatrix} \frac{3M}{20}\begin{pmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}\begin{pmatrix} 0 \\ w\sin(\alpha) \\ w\cos(\alpha) \end{pmatrix}$$

$$T = \frac{3M}{40}\begin{pmatrix} 0 & w\sin(\alpha) & w\cos(\alpha) \end{pmatrix}\begin{pmatrix} 0 \\ (R^2+4h^2)w\sin(\alpha) \\ 2R^2 w\cos(\alpha) \end{pmatrix} = \frac{3M}{40}\left((R^2+4h^2)w^2\sin^2(\alpha) + 2R^2 w^2\cos^2(\alpha)\right)$$

Finally, our kinetic energy becomes:

$$\boxed{T = \frac{3Mw^2}{40}\left((R^2+4h^2)\sin^2(\alpha) + 2R^2\cos^2(\alpha)\right)}$$

Problem 4: Review

Procedure:

- Place the cone tip at the origin and recycle the moment of inertia tensor from Problem 2 where we calculated the moment of inertia tensor from an origin that is not the center of mass.
- Use the equation
$$T = \frac{1}{2}\vec{\omega}^{\dagger}\hat{I}\vec{\omega}$$
to calculate the kinetic energy of our cone.
- Use geometrical interpretations to define the vector $\vec{\omega}$.
- Use the former two results to calculate the kinetic energy of this cone.

Key Concepts:

- Since we placed the tip of our cone at the origin of the co-ordinate system we can re-use the moment of inertia tensor found in Problem 2.
- With this recycled tensor all we have to do additionally is find the vector $\vec{\omega}$ and plug it into the kinetic energy equation.

Variations:

- We can be given a different solid to calculate the kinetic energy of.
  - This would make us recalculate the moment of inertia tensor and thus the $\vec{\omega}$ would change as well.