Homework Assignment #2

Math Methods

Homework Due: Monday, September 13th

Instructions:

Note that there is no class one Labor Day, Monday September 6th.

Reading: Reading Quiz # 2 is due on Friday by the start of class and covers the rest of Chapter 2.

Please read Chapter 1. Reading Quiz # 3 on this material is due by the start of class on Wednesday, September 15th.

Problems: Below is a list of questions and problems from the texbook due by the time and date above. It is not sufficient to simply obtain the correct answer. You must also explain your calculation, and each step so that it is clear that you understand the material.

Homework should be written legibly, on standard size paper. Do not write your homework up on scrap paper. If your work is illegible, it will be given a zero.

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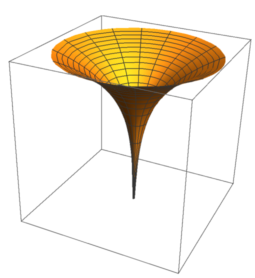
1. Use a Lagrange multiplier approach to find the extremal points (x, y) of the function

$$f(x,y) = 2x^2 + \frac{1}{2}y^2 - xy$$

subject to the constraint

$$g(x,y) = 4x^2 + y^2 - 4 = 0$$

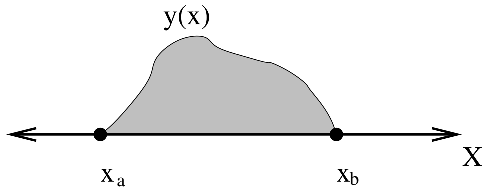
2. A bug moves on a surface with cylindrical symmetry. The surface is given by $z = \log\rho$ where ρ is the distance of the surface from the z-axis. . As an intelligent bug, it wants to move the shortest distance between two points.



(a) Calculate the differential equation of such a curve. between any two points on the surface. (You can solve this using a local constraint, but that's the hard way.)

(b) Solve for either the function $\rho(\theta)$ or $\theta(\rho)$, which ever you find easier. (You may need to consult a good integral table).

3. Consider a curve $y(x)$ where $y(x_a) = y(x_b) = 0$, and the total length of the curve is L. Find the curve that gives you the maximum area enclosed between $y(x)$ and the x-axis. (Assume that $L < \pi(x_b - x_a)/2$. There is a simple reason why the problem changes when L is too large. Can you see why?)



4. Often constrained problems yield difficult integral problems or differential equations to solve. In each case below, apply the calculus of variations and derive a final expression for the curve specified as an integral, that is:

$$x - x_a = \int_{y_a}^{y} F(y)\, dy$$

where $F(y)$ is a known function of y. (This is called *reducing the problem to quadrature.*)

   (a) Determine the curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) on a path of fixed length but along which a frictionless particle would slide with the shortest time. (That is, similar to the brachistochrone but on a path of fixed length).

   (b) Determine the shortest length curve $y(x)$ that connects the points (x_a, y_a) and (x_b, y_b) along which a frictionless particle would slide with a fixed time.

   (c) A solid object can be defined by rotating a curve $\rho(z)$ about the z-axis to determine a surface of revolution. The top and bottom surfaces of the object would be disks parallel to the x-y plane. Determine the curve $\rho(z)$ connecting the points $\rho(z_a) = \rho_a$ to $\rho(z_b) = \rho_b$ so that it minimizes the surface area of the object but has a fixed volume and moment of inertia. Assume that the density of the object has a fixed mass/volume, denoted by the constant α.

5. Consider the variational problem:

$$I = \int \left(f(x, y_1, y_1', y_2, y_2') - \lambda(x) g(y_1, y_2)\right) dx$$

where f is the optimization function and g is a local constraint.

   (a) If f is just a square root, such as $\sqrt{1 + y_1'^2}$, it would be easier to minimize f^2. Does this give you the same curve?

   (b) If g is just a square root, such as $\sqrt{y_1^2 + y_2^2}$, it would be easier to replace g with g^2. Does this give you the same curve?

6. Consider a 3D crystal with a surface defined by a function $z(x, y)$. We seek to minimize the total surface free energy:

$$F[z] = \int dx\, dy\, \left\{\alpha(x, y)\sqrt{1 + z_x^2 + z_y^2}\right\}$$

where $\alpha(z_x, z_y$ is the direction dependent surface tension, and we have introduced the compact but confusing notation, $z_x \equiv \partial z/\partial x$ and $z_y \equiv \partial z/\partial y$. We wish to enforce the constraint of a constant volume to the crystal where the volume is given by:

$$V[z] = \int dx\, dy\, z$$

via a Lagrange multiplier, λ.

(a) What is the Euler-Lagrange equation for the system?

(b) Show that in the isotropic case, where $\alpha(z_x, z_y) = \alpha_0$, that the shape that minimizes the surface energy for fixed volume is a sphere. You may do this either by direct substitution to verify the solution

$$z(x, y) = \sqrt{R^2 - x^2 - y^2}$$

or by changing to polar coordinates $z(\rho, \varphi)$ in the original functional, and invoking cylindrical symmetry, so that your 2D problem becomes effectively a one dimensional problem for $z(\rho)$. In either case, determine the relation between λ and the radius of the sphere.

7. **Variational derivation of Poisson equation:**

The energy in an electrostatic field in the presence of charges is :

$$E = \int \left\{ \frac{1}{8\pi} \left(\nabla\phi(\vec{r})\right)^2 - \rho(\vec{r})\, \phi(\vec{r}) \right\} d\vec{r}$$

Show that the minimum energy configuration of the potential $\varphi(\vec{r})$ satisfies the equation:

$$\vec{\nabla}^2 \phi(\vec{r}) = -4\pi\, \rho(\vec{r})$$

Thus the *minimal energy* configuration for the field is also the one given by the Poisson equation.

The above problem is rather simple. I am assigning it to you for the notes below, so that you see *why* it works. It is a common technique in quantum field theory, so it's worth knowing the background.

Background: The expression for the energy above avoids counting the "self-energy", so that a charge does not feel the force of the electric field it creates. To see this assume that we have a set of point charges q_i each with potential $\phi_i(\vec{r})$. Then the total potential is the sum of the individual contributions:

$$\phi(\vec{r}) = \sum_i \phi_i(\vec{r})$$

The energy of the charges (in some units) can be written as

$$E = \frac{1}{8\pi} \int \left(\vec{\mathcal{E}}(\vec{r})\right)^2 d\vec{r}$$

(including the self-energy (Jackson, *Classical Electrodynamics*, p.46) which we can write as

$$2E = \int \left\{ \frac{1}{8\pi} \sum_{i \neq j} \left(\nabla\phi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r})\right) \right\} d\vec{r}$$

where the factor of 2 comes from double counting in the sum over i and j. The energy can also be written as:

$$E = \int \left\{ \sum_{i \neq j} q_i \, \delta(\vec{r} - \vec{r}_i) \, \phi_j(\vec{r}) \right\} d\vec{r}$$

where $\delta(\vec{r} - \vec{r}_i)$ is a function sharply peaked at $\vec{r}_1$, the location of the point charge, and there is no double counting. If we remove the restriction $i \neq j$, then both expressions pick up a self-energy term. However, if we subtract the two expressions, these cancel out, and we are left with simply the total energy:

$$\int \sum_{i,j} \left\{ \frac{1}{8\pi} \sum_{i,j} \left(\nabla\phi_i(\vec{r}) \cdot \nabla\phi_j(\vec{r}) \right) - q_i \, \delta(\vec{r} - \vec{r}_i) \, \phi_j(\vec{r}) \right\} = 2E + E_S - E - E_S = E$$

If we take the continuum limit and replace the discrete distribution by a continuous one, then we obtain the expression at the top of the page.

This "background" discussion does not contain any questions. I am including it merely because the above formulation is routinely invoked in some field theory courses without explanation.

8. Obtain access to MATHEMACTICA on some computer. As proof of having access, if N is your OU ID number, calculate the prime factors of N^2+1 by typing in the command:

   FactorInteger$[N * N + 1]$

   and then hold down the "shift" key and the "return" key at the same time to execute the command.

HW#2, problem 1

Us a Lagrange multiplier approach to find the extremal points (x,y) of the function:

$$f(x, y) = 2\,x^2 + \tfrac{1}{2}\,y^2 - x\,y$$

subject to the constraint:

$$g(x, y) = 4\,x^2 + y^2 - 4 = 0$$

We construct our functional integrand:

In[∘]:=
```
hfxn = 2 x^2 + 1/2 y^2 -  x y - λ (4 x^2 + 1 y^2 - 4);
```

We extremize the functional for values of x and y, by calculating the partials of our functional.

In[∘]:=
```
eqns = {D[hfxn, x] == 0, D[hfxn, y] == 0, 4 x^2 + 1 y^2 == 4} // FullSimplify
```

Out[∘]= $\{x\,(4 - 8\,\lambda) == y,\ y == x + 2\,y\,\lambda,\ 4\,x^2 + y^2 == 4\}$

We solve our equations.

In[∘]:=
```
solnxy = Solve[eqns, {x, y, λ}];
TableForm[solnxy]
```

Out[∘]//TableForm=

$x \to -\frac{1}{\sqrt{2}}$	$y \to -\sqrt{2}$	$\lambda \to \frac{1}{4}$
$x \to -\frac{1}{\sqrt{2}}$	$y \to \sqrt{2}$	$\lambda \to \frac{3}{4}$
$x \to \frac{1}{\sqrt{2}}$	$y \to -\sqrt{2}$	$\lambda \to \frac{3}{4}$
$x \to \frac{1}{\sqrt{2}}$	$y \to \sqrt{2}$	$\lambda \to \frac{1}{4}$

There are four extremal points. Technically that's all the problem asks you to do. However, let's look at the problem graphically. We can calculate a contour plot of the function *f(x,y)* with a boundary given by the constraint. The extremal points are where the contours are tangent to the boundary.

In[∘]:=
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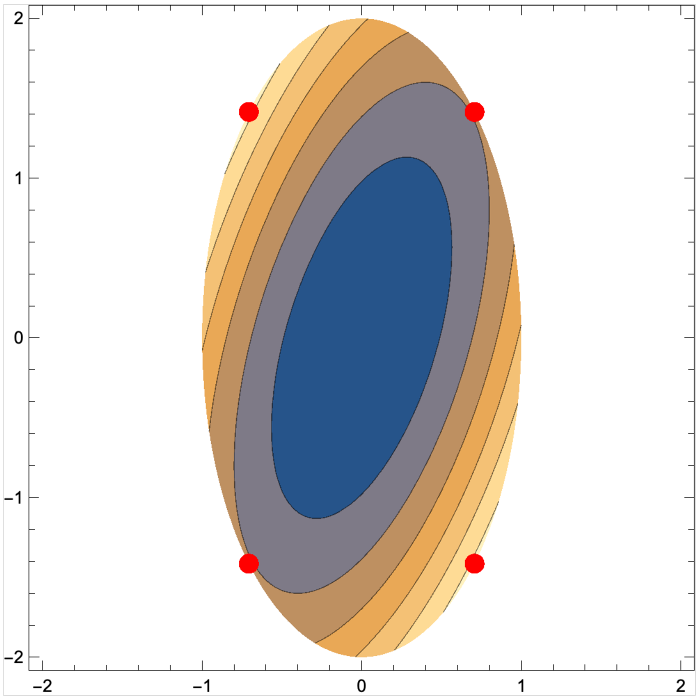
solnPts = {x, y} /. solnxy;
```

In[∘]:=
```
lim = 2.0;
cplot = ContourPlot[2 x^2 + 1/2 y^2 -  x y , {x, -lim, lim}, {y, -lim, lim},
   RegionFunction → Function[{x, y}, 4 x^2 + y^2 ≤ 4], Contours → 28];
extrema = ListPlot[solnPts, PlotStyle → {Red, PointSize[.03]}];
Show[cplot, extrema]
```

Out[∘]=

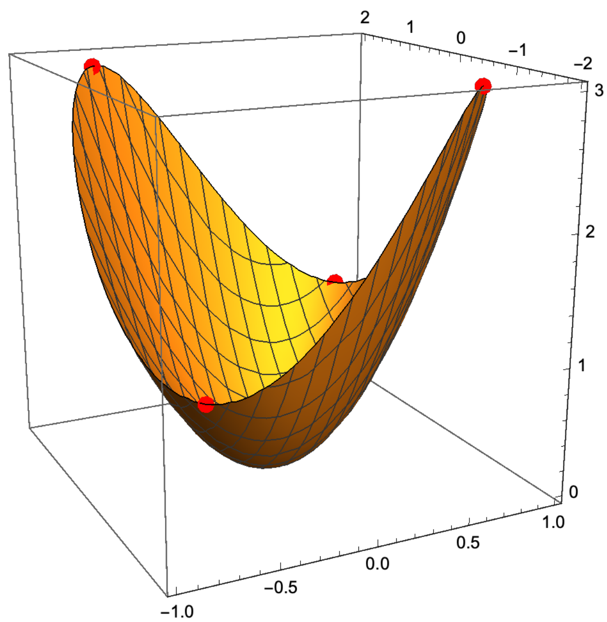


We can also plot the surface *f(x,y)* in 3D. Again we fix the boundary of the plotted region to be the constraint itself. One wrinkle is that we also need to calculate the values of *f* at the extrema.

In[∘]:=
```
surf = Plot3D[2 x^2 + 1/2 y^2 -  x y , {x, -lim, lim},
   {y, -lim, lim}, RegionFunction → Function[{x, y}, 4 x^2 + y^2 ≤ 4]];
pts = Table[{x, y, 2 x^2 + 1/2 y^2 -  x y } /. solnxy[[i]], {i, 1, 4}]
ptsPlot = ListPointPlot3D[pts, PlotStyle → {Red, PointSize[.03]}];
Show[surf, ptsPlot, BoxRatios → {1, 1, 1}]
```

Out[∘]= $\left\{\left\{-\frac{1}{\sqrt{2}}, -\sqrt{2}, 1\right\}, \left\{-\frac{1}{\sqrt{2}}, \sqrt{2}, 3\right\}, \left\{\frac{1}{\sqrt{2}}, -\sqrt{2}, 3\right\}, \left\{\frac{1}{\sqrt{2}}, \sqrt{2}, 1\right\}\right\}$

Out[∘]=



GEODESIC ON LOG-CONE

#1 Let $z = a \log \rho$

In our case $a = 1$, but it is useful to introduce such a factor so that units are right.

$$dz = a \frac{d\rho}{\rho}$$

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$$

$$= d\rho^2 + \rho^2 d\theta^2 + \frac{a^2}{\rho^2} d\rho^2$$

$$= \left(1 + \frac{a^2}{\rho^2}\right) d\rho^2 + \rho^2 d\theta^2$$

$$ds = \sqrt{\rho^2 + \left(\frac{a^2}{\rho^2} + 1\right)\left(\frac{d\rho}{d\theta}\right)^2} \; d\theta$$

We want to minimize.

$$\int \sqrt{\left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2 + \rho^2} \; d\theta$$

Since there is no explicit θ dependence:

$$\frac{\left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2}{\sqrt{\left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2 + \rho^2}} - \sqrt{\left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2 + \rho^2} = c$$

$$\Rightarrow \quad -\rho^2 = C \sqrt{\left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2 + \rho^2}$$

$$\frac{\rho^4}{c^2} = \left(1 + \frac{a^2}{\rho^2}\right)(\rho')^2 + \rho^2$$

$$(\rho')^2 = \left(\frac{\rho^4}{c^2} - \rho^2\right) \Big/ \left(1 + \frac{a^2}{\rho^2}\right)$$

$$\rho' = \frac{d\rho}{d\theta} = \sqrt{\left(\frac{\rho^6}{c^2} - \rho^4\right) \Big/ (\rho^2 + a^2)}$$

$$\theta - \theta_a = \int_{\rho_a}^{\rho} \sqrt{\frac{\rho^2 + a^2}{\frac{\rho^6}{c^2} - \rho^4}}\, d\rho$$

$$= \int_{\rho_0}^{\rho} \sqrt{\frac{\rho^2 + a^2}{\rho^2 - c^2}}\ \frac{c}{\rho^2}\, d\rho$$

Let $u = \frac{1}{\rho} \Rightarrow du = -\frac{d\rho}{\rho^2}$

$$= \int \sqrt{\frac{1/u^2 + a^2}{1/u^2 - c^2}}\ c\, du$$

$$= \int \sqrt{\frac{1 + a^2u^2}{1 - c^2u^2}}\ c\, du.$$

Now set $cu = \sin t$

$$du = \frac{1}{c} \cos t \cdot dt$$

$$\theta - \theta_0 = \int \sqrt{\frac{1 + (a/c \sin t)^2}{1 - \sin^2 t}}\ \frac{1}{c} \cos t\, dt.$$

$$= \int \sqrt{1 + (a/c \sin t)^2}\ \frac{1}{c}\, dt$$

$$\theta - \theta_0 = \frac{1}{c} \mathcal{E}(t, -a) \Big|_{t_{min}}^{t}$$

where $\mathcal{E}(t, -a)$ is the "elliptic integral of the second kind". From above

$$t = \arcsin cu = \arcsin \frac{c}{\rho}$$

$$\theta - \theta_0 = \frac{1}{c} \left\{ \mathcal{E}\left[\arcsin\left(\frac{c}{\rho}\right), -a\right] - \mathcal{E}\left[\arcsin\left(\frac{c}{\rho_0}\right), -a\right] \right\}$$

#2 We want to maximize:

$$\int_{x_A}^{x_B} \left\{ y(x) - \lambda \sqrt{1+(y')^2} \right\} dx$$

where λ is a constant such that

$$L = \int \sqrt{1+(y')^2}\, dx$$

Our function is: $h = y + \lambda \sqrt{1+(y')^2}$

It is independent of x, so we use shortcut:

$$y' \frac{\partial h}{\partial y'} - h = \frac{\lambda (y')^2}{\sqrt{1+(y')^2}} - y - \lambda \sqrt{1+(y')^2}$$

$$= C$$

$$\Rightarrow \lambda (y')^2 - y\sqrt{1+(y')^2} - \lambda(1+(y')^2) = C\sqrt{1+(y')^2}$$

Arranging terms:

$$-\lambda = (C+y)\sqrt{1+(y')^2}$$

$$\Rightarrow \frac{\lambda^2}{(C+y)^2} = 1+(y')^2$$

$$y' = \sqrt{\frac{\lambda^2}{(C+y)^2} - 1} = \sqrt{\lambda^2 - (C+y)^2}\ \frac{1}{(C+y)}$$

$$dx = \frac{(C+y)}{\sqrt{\lambda^2-(C+y)^2}}\, dy$$

Let $u = y + c$

$$x - x_a = \int \frac{u\,du}{\sqrt{\lambda^2 - u^2}} = -(\lambda^2 - u^2)^{1/2}$$

$$= -(\lambda^2 - (y+c)^2)^{1/2} + (\lambda^2 - c^2)^{1/2}$$

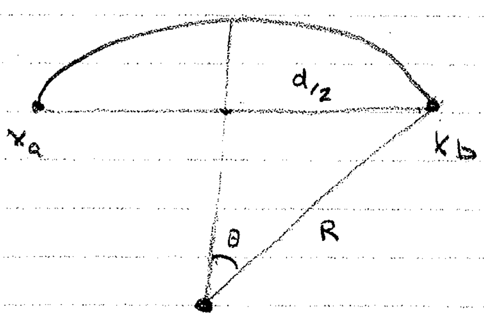
$$\Rightarrow (x - x_0)^2 = \lambda^2 - (y+c)^2$$

$$\Rightarrow (x - x_0)^2 + (y+c)^2 = \lambda^2$$

where $x_0 = x_a - (\lambda^2 - c^2)^{1/2}$.

Clearly this is a circle of radius λ!

By inspection $x_0 = \frac{1}{2}(x_b - x_a)$. Draw a diagram!



Let $d = x_b - x_a$

$$R = \left(\frac{d}{2}\right)\frac{1}{\sin\theta}$$

Also, $R\theta = L/2$

This is two equations with 2 unknowns — Solve:

$$R \sin\left(\frac{L}{2R}\right) = \frac{d}{2} \quad \text{(transcendental equ. for R)}$$

Then $y_0 = -R\cos\theta$. For $L \sim d$

$$R\left(\frac{L}{2R} - \frac{1}{3!}\left(\frac{L}{2R}\right)^3\right) = \frac{d}{2}.$$

$$\frac{L}{2} - \frac{1}{6}\frac{L^3}{2R^2} = \frac{d}{2}$$

$$\frac{1}{6}\frac{L^3}{R^2} = (L-d)$$

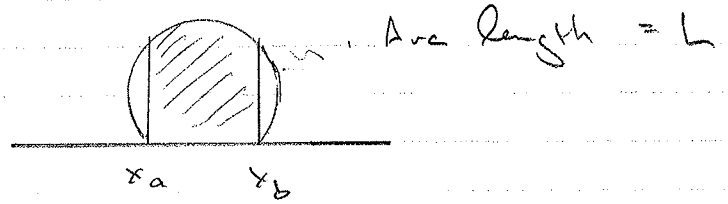
$$R \sim \sqrt{\frac{L^3}{(L-d)}}$$

$$\theta \sim \frac{L}{2R} = \frac{1}{2}\sqrt{\frac{(L-d)}{L}}$$

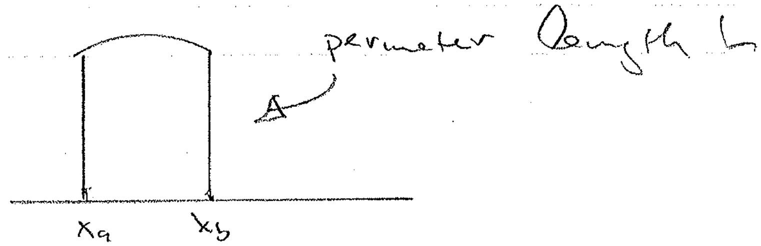
$$y_0 = -R\cos\frac{1}{2}\sqrt{\frac{(L-d)}{L}}$$

$$= -R\left\{1 - \frac{1}{2!}\left(\frac{1}{2}\sqrt{\frac{(L-d)}{L}}\right)^2 + \cdots\right\}$$

Note, if $L = \frac{\pi}{2}d$ then our solution is a half-circle with $R = d/2$ and $\theta = \frac{\pi}{2}$. However for larger L we have a discontinuous solution.



We cannot integrate "backwards"



#3

(a) Find the integral that gives the shortest time path for a curve of fixed length.

The time for the path is from the brachistochrone problem

$$I[y] = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1+(y')^2}{y}}\, dx$$

The fixed length constraint is

$$J[y] = \int \left\{ \sqrt{1+(y')^2} - \frac{L}{(x_b - x_a)} \right\} dx = 0$$

So that

$$h(y, y') = \frac{1}{\sqrt{2g}} \sqrt{\frac{1+(y')^2}{y}} - \lambda \left\{ \sqrt{1+(y')^2} - \frac{L}{x_a - x_b} \right\}.$$

Note that λ is a constant because this is a global constraint & not local. For simplicity let

$$\tilde{\lambda} = \frac{\lambda}{\sqrt{2g}}\ ; \quad \tilde{h} = \sqrt{2g}\, h$$

So that

$$\tilde{h} = \sqrt{\frac{1+(y')^2}{y}} - \tilde{\lambda} \left\{ \sqrt{1+(y')^2} - \frac{L}{x_a - x_b} \right\}.$$

Since $\tilde{h}$ has no explicit x dependence, then

$$y' \frac{\partial \tilde{h}}{\partial y'} - \bar{h} = C_0$$

$$= y' \left\{ \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1+(y')^2}} - \tilde{\lambda} \frac{y'}{\sqrt{1+(y')^2}} \right\}$$

$$- \left\{ \sqrt{\frac{(1+(y')^2)}{y}} - \tilde{\lambda}\left(\sqrt{1+(y')^2} - \frac{L}{x_b - x_a}\right) \right\}$$

We define

$$C_1 \equiv C_0 - \tilde{\lambda} \frac{L}{x_b - x_a}$$

and multiply through by $\sqrt{1+(y')^2}$

$$\frac{y'^2}{\sqrt{y}} - \lambda y'^2 - \frac{(1+(y')^2)}{\sqrt{y}} + \tilde{\lambda}(1+(y')^2)$$

$$= c_1 \sqrt{1+(y')^2}$$

$$-\frac{1}{\sqrt{y}} + \tilde{\lambda} = c_1 \sqrt{1+(y')^2}$$

Solve for y'. Let $\tilde{\tilde{\lambda}} = -\frac{\tilde{\lambda}}{c_1}$

$$\left(\tilde{\tilde{\lambda}} + \frac{1}{c_1\sqrt{y}}\right)^2 = \left(1 + (y')^2\right)$$

$$y' = \sqrt{\left(\tilde{\lambda} + \frac{1}{c_1\sqrt{y}}\right)^2 - 1} = \frac{dy}{dx}$$

$$\int_{x_a}^{x} dx = x - x_a = \int_{y_a}^{y} \frac{dy}{\sqrt{\left(\frac{1}{c_1\sqrt{y}} + \tilde{\lambda}\right)^2 - 1}}$$

(b) Find the shortest curve for fixed time.

From above - minimize

$$I[y(x)] = \int_{x_a}^{x_b} \sqrt{1 + (y')^2}\, dx$$

Subject to the constraint:

$$J[y] = \int_{x_a}^{x_b} \left\{ \frac{1}{\sqrt{2g}} \sqrt{\frac{1 + (y')^2}{y}} - \frac{T}{x_b - x_a} \right\} dx$$

So that h is given by:

$$h = \sqrt{1 + (y')^2} - \lambda \left\{ \frac{1}{\sqrt{2g}} \sqrt{\frac{1 + (y')^2}{y}} - \frac{T}{x_b - x_a} \right\}$$

Define $\tilde{h} = \sqrt{2g}\, h$.

$$\tilde{T} = \frac{T}{\sqrt{2g}}$$

$$\tilde{h} = \sqrt{1+(y')^2} - \lambda \left\{ \sqrt{\frac{1+(y')^2}{y}} - \frac{\tilde{T}}{x_b - x_a} \right\}.$$

Now let $\tilde{\tilde{h}} \equiv -\frac{1}{\lambda}\tilde{h}$

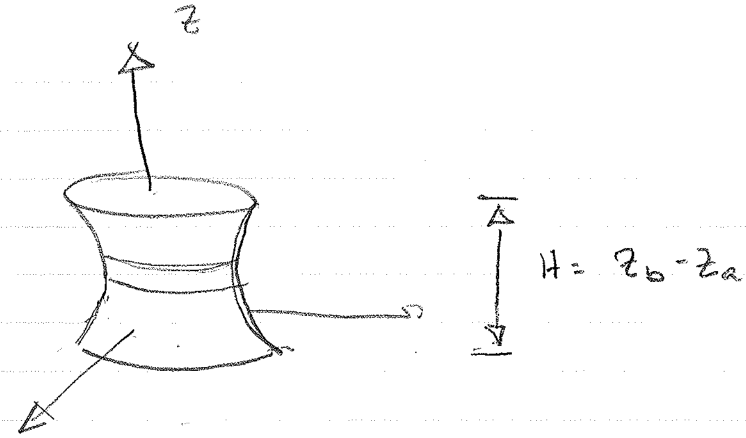
$$\lambda_2 \equiv \frac{1}{\lambda}$$

$$T \lambda = -\lambda_2 L$$

(yes - L is not known here - just treat it as a dummy variable), We have

$$\tilde{\tilde{h}} = \sqrt{\frac{1+(y')^2}{y}} - \lambda_2 \left\{ \sqrt{1+(y')^2} - \frac{L}{x_a - x_b} \right\}.$$

This is the same quantity as in (a). The extremizing curve is therefore the same as above.



We can view the object as made up of a series of thin disks of radius ρ and thickness dz. The volume of the disk is

$$dV = \pi \rho^2 \, dz$$

The moment of inertia is given by

$$dI = \frac{1}{2}(dm)\rho^2$$

$$= \frac{1}{2}\left(\mu \pi \rho^2 \, dz\right)\rho^2$$

$$= \frac{1}{2} \mu \pi \rho^4 \, dz$$

where μ is the mass/volume of the object.

The surface area of the disk is

$$dA = 2\pi \rho \sqrt{1 + (\rho')^2} \, dz$$

Putting these together we have -

$$K = \int_{z_a}^{z_b} dz \left\{ 2\pi\rho\sqrt{1+(\rho')^2} - \lambda_1\left(\pi\rho^2 - \frac{V}{H}\right) - \lambda_2\left(\frac{1}{2}\mu\pi\rho^4 - \frac{I_0}{H}\right)\right\}$$

We optimize

$$h = 2\pi\rho\sqrt{1+(\rho')^2} - \lambda_1\left(\pi\rho^2 - \frac{V}{H}\right) - \lambda_2\left(\frac{1}{2}\mu\pi\rho^4 - \frac{I_0}{H}\right)$$

Again h does not depend explicitly on z so.

$$\rho'\frac{\partial h}{\partial \rho'} - h = \text{constant} = C_0$$

$$\rho'\left(2\pi\rho\frac{\rho'}{\sqrt{1+(\rho')^2}}\right) - \left[2\pi\rho\sqrt{1+(\rho')^2} - \lambda_1\left(\pi\rho^2 - \frac{V}{H}\right) - \lambda_2\left(\frac{1}{2}\mu\pi\rho^4 - \frac{I_0}{H}\right)\right] = C_0$$

To simplify let

$$C_1 = C_0 + \lambda_1\frac{V}{H} + \lambda_2\frac{I_0}{H}$$

$$\frac{2\pi\rho(\rho')^2}{\sqrt{1+(\rho')^2}} - 2\pi\rho\sqrt{1+(\rho')^2} - \lambda_1\pi\rho^2$$

$$- \frac{\lambda_2}{2}\mu\pi\rho^4 = C_1$$

$$\frac{2\pi\rho(\rho')^2}{\sqrt{1+(\rho')^2}} - 2\pi\rho\sqrt{1+(\rho')^2} = C_1 - \lambda_1\pi\rho^2 - \frac{\lambda_2}{2}\mu\pi\rho^4$$

Multiplying through by $\sqrt{1+(\rho')^2}$

$$2\pi\rho\,(\cancel{\rho'})^2 - 2\pi\rho\left(1+\cancel{\rho'^2}\right) =$$

$$\left[C_1 - \lambda_1\pi\rho^2 - \frac{\lambda_2}{2}\mu\pi\rho^4\right]\sqrt{1+(\rho')^2}$$

$$\frac{-2\pi\rho}{\left[C_1 - \lambda_1\pi\rho^2 - \lambda_2\mu\pi\rho^4\right]} = \sqrt{1+(\rho')^2}$$

$$\rho' = \frac{d\rho}{dz} = \left\{\frac{(2\pi\rho)^2}{\left[C_1 - \lambda_1\pi\rho^2 - \lambda_2\mu\pi\rho^4\right]^2} - 1\right\}^{1/2}$$

$$\int_{z_1}^{z} dz = z - z_1 = \int \frac{d\rho}{\sqrt{\left(\frac{2\pi\rho}{C_1 - \lambda_1\pi\rho^2 - \lambda_2\mu\pi\rho^4}\right)^2 - 1}}$$

#6 Assume $y(x)$ minimizes $\int f(y, y', x)$. Does it minimize $\int f^2(y, y'; x)\, dx$? $\equiv I_2$

$$\frac{dI_2}{d\epsilon} = \frac{d}{d\epsilon} \int f^2(y, y', x)\, dx$$

$$= \int 2f \left\{ \frac{\partial f}{\partial y'} \frac{dy'}{d\epsilon} + \frac{\partial f}{\partial y} \frac{dy}{d\epsilon} \right\} dx$$

Integrate by parts:

$$2 \int \left\{ -\frac{d}{dx}\left(f \frac{\partial f}{\partial y'} \right) + \frac{\partial f}{\partial y} \right\} \frac{\partial y}{\partial \epsilon}\, dx$$

Our standard variation $\Rightarrow \frac{\partial y}{\partial \epsilon} = \eta(x)$

$$I_2 = 2 \int \left\{ -\frac{df}{dx} \frac{\partial f}{\partial y'} \cancel{+ \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y}} \right\} \eta(x)\, dx$$

$$= -2 \int \left\{ \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial x} \right\} \frac{\partial f}{\partial y} \eta(x)\, dx$$

This is not in general zero. In fact, we can choose $\eta(x) = -\frac{df}{dx} \frac{\partial f}{\partial y'}$ and get:

$$I_2 = +2 \int \left(\frac{df}{dx} \frac{\partial f}{\partial y'} \right)^2 dx$$

which is non-zero if $\frac{df}{dx} \neq 0$ and $\frac{df}{dy'} \neq 0$.

What about a constraint? Assume

$$h = f(x) + \lambda(x)\, g(y_1, y_2) \quad \text{is extremal.}$$

Let $\tilde{h} = f(x) + \tilde{\lambda}(x)\, g^2(y_1, y_2)$.

$$\tilde{I}(\epsilon) = \int \left\{ f(x) + \tilde{\lambda}(x)\, g^2(y_1, y_2) \right\} dx$$

$$\frac{d\tilde{I}}{d\epsilon} = \int \left\{ \frac{\partial f}{\partial y_1'} \frac{dy'}{d\epsilon} + \frac{\partial f}{\partial y_1} \frac{dy}{d\epsilon} + \tilde{\lambda}(x)\, g(y_1, y_2) \frac{\partial g}{\partial y_1} \frac{dy_1}{d\epsilon} \right\} =$$

where we are only varying y_1.

$$\frac{d\tilde{I}}{d\epsilon} = \int \left\{ -\frac{d}{dx} \frac{\partial f}{\partial y_1'} + \frac{\partial f}{\partial y_1} + \tilde{\lambda}(x)\, g(y_1, y_2) \frac{\partial g}{\partial y_1} \right\} dx$$

If we choose $\tilde{\lambda}(x) = \dfrac{\lambda(x)}{g(\bar{y}_1(x), \bar{y}_2(x))}$

We obtain correct soln again. Thus the Lagrange multiplier changes, but still same soln.

(5) Given.

$$F[z] = \int dx\, dy \left\{ \alpha(x,y) \sqrt{z_x^2 + z_y^2 + 1} \right\}$$

with the volume constant.

(a) What is the E-L equation?

The constant volume constraint is a global constraint. We wish to extremize

$$\Pi = \int dx\, dy \left\{ \alpha(x,y) \sqrt{1 + z_x^2 + z_y^2} - \lambda z \right\}$$

$$\frac{\partial}{\partial q_i} \frac{\partial f}{\partial z_{q_i}} - \frac{\partial f}{\partial z} = 0$$

$$= \frac{\partial}{\partial x} \left\{ \alpha(x,y) \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}} \right\} + \frac{\partial}{\partial y} \left\{ \alpha(x,y) \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right\}$$

$$- \lambda = 0$$

(b) Now assume $\alpha(x,y) = \alpha_0$.

Make the change to cylindrical co-ordinates —

By symmetry we know z is a function of ρ & not φ. Then

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \rho}\frac{\partial \rho}{\partial x} = z_\rho \frac{x}{\sqrt{x^2+y^2}} = z_\rho \cos\varphi$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \rho}\frac{\partial \rho}{\partial y} = z_\rho \frac{y}{\sqrt{x^2+y^2}} = z_\rho \sin\varphi$$

So $z_x^2 + z_y^2 = z_\rho^2$

Therefore we have.

$$\Pi = \int d\rho\, d\varphi \left\{ \rho\sqrt{1+z_\rho^2} - \rho\lambda z \right\}$$

Since Π is independent of φ, we can ignore the degree of freedom. Our E-L equation becomes -

$$\frac{d}{d\rho}\frac{\partial f}{\partial z_\rho} - \frac{\partial f}{\partial z} = 0$$

Note that $f = \rho\left[\sqrt{1+z_\rho^2} - \lambda z\right]$ is <u>not</u> independent of ρ, so we may not assume a first integral.

$$\frac{d}{d\rho} \frac{\rho z_\rho}{\sqrt{1+z_\rho^2}} - \lambda\rho = 0$$

$$= \frac{d}{d\rho}\left\{ \frac{\rho z_\rho}{\sqrt{1+z_\rho^2}} - \frac{1}{2}\lambda\rho^2 \right\} = 0$$

Then $\frac{\rho z_\rho}{\sqrt{1+z_\rho^2}} - \frac{1}{2}\lambda\rho^2 = \text{constant}$

If we keep the constant, the algebra is a bit messy. We will find that we can set it equal to zero. The

$$z_\rho = \frac{\lambda}{2}\rho \cdot \sqrt{1+z_\rho^2}$$

$$z_\rho^2 = \frac{\lambda^2}{4}\rho^2 \left(1 + z_\rho^2\right)$$

$$z_\rho^2\left(1 - \frac{\lambda^2}{4}\rho^2\right) = \frac{\lambda^2}{4}\rho^2$$

$$\frac{dz}{d\rho} = \frac{\lambda}{2}\rho \frac{1}{\sqrt{1-\frac{\lambda^2}{4}\rho^2}}$$

$$dz = \frac{\lambda}{2} \frac{\rho\, d\rho}{\sqrt{1-\frac{\lambda^2}{4}\rho^2}}$$

Let $u = \frac{\lambda}{2}\rho$.

$$\frac{\lambda}{2}\, dz = \frac{u\, du}{\sqrt{1-u^2}}$$

$$\frac{\lambda}{2}(z - z_0) = -\sqrt{1-u^2}$$

$$\frac{\lambda^2}{4}(z-z_0)^2 = 1 - u^2 = 1 - \frac{\lambda^2}{4}\rho^2$$

$$\left(\frac{\lambda}{2}\right)^2 \left\{ (z-z_0)^2 + \rho^2 \right\} = 1$$

$$(z-z_0)^2 + \rho^2 = \left(\frac{2}{\lambda}\right)^2 = R^2$$

So $\lambda = \dfrac{2}{R}$

The constant z_0 is determined by the requirement that the volume by V.

HW4: Problem 5

If you don't want to solve differential equations, you can try substitution to verify the solution:

Substitution in x and y co-ordinates:

$$\mathbf{zfxn} = \sqrt{R^2 - x^2 - y^2}\;;$$

```
zx = D[zfxn, x];
zy = D[zfxn, y];
```

$$\mathbf{f1} = \mathbf{Simplify}\left[\frac{\mathbf{zx}}{\sqrt{1 + \mathbf{zx}^2 + \mathbf{zy}^2}}\right];$$

$$\mathbf{f2} = \mathbf{Simplify}\left[\frac{\mathbf{zy}}{\sqrt{1 + \mathbf{zx}^2 + \mathbf{zy}^2}}\right];$$

```
FullSimplify[D[f1, x] + D[f2, y]]
```

0

Substitution in cylindrical co-ordinates

$$\mathbf{zfxn} = \sqrt{R^2 - \rho^2}$$

$$\sqrt{R^2 - \rho^2}$$

$$\mathbf{zr} = \mathbf{D}[\mathbf{zfxn}, \rho]$$

$$-\frac{\rho}{\sqrt{R^2 - \rho^2}}$$

$$\mathbf{temp} = \mathbf{D}\left[\rho\,\frac{\mathbf{zr}}{\sqrt{1 + \mathbf{zr}^2}}, \rho\right] \;//\; \mathbf{Simplify} \;//\; \mathbf{Together}$$

$$-\frac{2\rho}{\sqrt{R^2 - \rho^2}\,\sqrt{-\frac{R^2}{-R^2+\rho^2}}}$$

$$\rho\,\frac{\mathbf{zr}}{\sqrt{1 + \mathbf{zr}^2}}$$

$$-\frac{\rho^2}{\sqrt{R^2 - \rho^2}\,\sqrt{1 + \frac{\rho^2}{R^2-\rho^2}}}$$

It can be hard to force *Mathematica* to recognize that it can simplify such expressions. We will discuss why this is so in another class. However, if you think the problem has no subtle branch cuts, then you can define a function to simplify it for you.

```
PowerContract[expr_] :=
 expr //. {m_^q_ n_^q_ :> (m n)^q /; ! IntegerQ[m] && ! IntegerQ[n],
   m_^q_ n_^p_ :> (m / n)^q /; q ≥ 0 && p == -q && ! IntegerQ[m] && ! IntegerQ[n]}
```

```
PowerContract[temp] // Simplify // PowerExpand
```

$$-\frac{2\rho}{R}$$

Looking at the Euler-Lagrange equation in the notes, you can see this means $\lambda = 2/R$.

(6) a) $E = \int \left\{ \frac{1}{8\pi} (\vec{\nabla}\varphi)^2 - \rho\varphi \right\} dx\,dy\,dz$

$$= \int \left\{ \frac{1}{8\pi} \left((\partial_x\varphi)^2 + (\partial_y\varphi)^2 + (\partial_z\varphi)^2 \right) - \rho\varphi \right\} d^3x$$

$$U = \int f(\varphi, \partial_x\varphi, \partial_y\varphi, \partial_z\varphi)\, dx\,dy\,dz$$

The extremizing φ satisfies

$$\partial_i \frac{\partial}{\partial(\partial_i\varphi)} f - \frac{\partial f}{\partial\varphi} = 0$$

$$= \frac{1}{8\pi}\left(\partial_x\, 2\partial_x\varphi + \partial_y\, 2\partial_y\varphi + \partial_z\, 2\partial_z\varphi \right) + \rho = 0$$

$$\partial_x^2\varphi + \partial_y^2\varphi + \partial_z^2\varphi = -4\pi\rho$$

$$\vec{\nabla}^2\varphi = -4\pi\rho$$

QED!

b) ~~Our action is:~~

$$\cancel{\int \left\{ \frac{1}{16\pi} F_{ij}F_{ij} + \frac{1}{c} J_i A_i \right\} d^4x}$$

~~We wish to use A_i as our dependent variable. Note that the derivatives of A_i are independent of each other. That is, $\frac{\partial}{\partial(\partial_k A_i)} \partial_j A_i = 0$ if $k \neq j$. Furthermore~~ each of the A_i are independent of each other.

HW#2, problem 8

Here's a random nine digit number:

```
In[1308]:= numMax = 999 999 999;

num = RandomInteger[numMax]

Out[1303]= 839 043 256
```

Here's the factorization of $n^2 + 1$

```
In[1304]:= FactorInteger[num * num + 1]

Out[1304]= {{169 217, 1}, {4 160 300 592 961, 1}}
```

The point was to give you a problem that required access to Mathematica.

What do you think that chance is that your answer is actually a prime number? The gap between primes is typically on the order of the natural log of the number. So

```
In[1309]:= gap = Log[numMax * numMax + 1.]

Out[1309]= 41.4465
```

So it's roughly a 1 in 41 chance. (Acually a bit lower since OU ID numbers seem to start with "11":

```
In[1310]:= numMax = 119 999 999;

In[1311]:= gap = Log[numMax * numMax + 1.]

Out[1311]= 37.206
```