Assignment ~~4~~ 5 solutions

Question 1

Although the problem in principle involves a 3D sphere, we can simplify our analysis by restricting motion to a 2D x-y plane (gravity lies along -y). Our solution will be valid for any other plane generated by a rotation ~~through~~ about z.

a) For a point particle (ie., w/ zero radius), we have a constraint of the general form:

$$x^2+y^2 \geqslant R^2$$

where (x,y) denotes the location of the particle & R is the sphere radius.

In part (b) we will be interested in determining how/ the height at which the particle falls off the sphere. Hence for ~~our~~ our solution we can assume we only need a description when the constraint is holonomic,

$$x^2 + y^2 - R^2 = 0$$

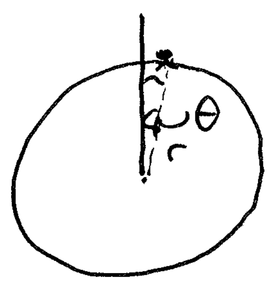
and the inequality will only become relevant once the particle has left the sphere.

Next, given that we are looking at motion on a spherical surface, let's move to polar co-ordinates,

$x = r \sin\theta$

$y = r\cos\theta$

[chosen s.t. $\theta = 0 \rightarrow$ north pole of sphere.



In these co-ordinates, the constraint simplifies to $r - R = 0$. We could just plug this straight into our definitions of T & V (ie. set $r = R$ in $x + y$) but we are going to want the forces of constraint, so let's leave r as a 'free' variable for now.

To get the forces of constraint & EOM, let's start w/ the Lagrangian:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \underbrace{\frac{m}{2}\dot{r}^2}_{\text{radial contribution}} + \underbrace{\frac{m}{2}r^2\dot{\theta}^2}_{\text{angular contribution.}}$$

[see class or do the trig to get this...]

& $V = mgy = mgr\cos\theta$

Then, $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$.

Next, we use the generalized Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) - \dfrac{\partial L}{\partial r} = \lambda \dfrac{\partial f}{\partial r} \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = \lambda \dfrac{\partial f}{\partial \theta} \end{cases}$$

where λ is an unknown lagrange multiplier stemming from our <u>single</u> constraint,

$$f(r) = r - R \quad (=0)$$

From f we the obtain:

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda & ① \quad \left(\frac{\partial f}{\partial r} = 1\right) \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0 & ② \quad \left(\frac{\partial f}{\partial \theta} = 0\right) \end{cases}$$

Now, f supplies the 3rd eqn to enable us to eliminate λ in principle, ie. by setting $r=R$ & $\dot{r}=\ddot{r}=0$.

So,

$$① \Rightarrow \quad -mR\dot{\theta}^2 + mg\cos\theta = \lambda \qquad (1a)$$

$$② \Rightarrow \quad mR^2\ddot{\theta} - mgR\sin\theta = 0 \qquad (2a)$$

(2a) can be rearranged, $\ddot{\theta} = g/R \sin\theta$, solution of which will allow us to then solve for λ in (1a). To solve our equation for $\ddot{\theta}$ lets use a trick,

$$\ddot{\theta}\, d\theta = \dot{\theta}\, d\dot{\theta} \qquad (*)$$

(which is obtained via,

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$
)

We could formally integrate (*),

$$\int_{\theta_1}^{\theta_2} \ddot{\theta}\, d\theta = \int_{\dot{\theta}_1}^{\dot{\theta}_2} \dot{\theta}\, d\dot{\theta}$$

Plugging in $\ddot{\theta} = g/_R \sin\theta$,

$$\int_{\theta_1}^{\theta_2} \frac{g}{R} \sin\theta\, d\theta = \int_{\dot{\theta}_1}^{\dot{\theta}_2} \dot{\theta}\, d\dot{\theta}$$

$$\Downarrow$$

$$-g/_R \cos\theta \Big|_{\theta_1}^{\theta_2} = \frac{\dot{\theta}^2}{2} \Big|_{\dot{\theta}_1}^{\dot{\theta}_2}$$

Our particle is initially at rest, $\dot{\theta}_1 = 0$, & on the north pole, $\theta_1 = 0$, so:

$$-\frac{g}{R}\left(\cos\theta_2 - 1\right) = \frac{1}{2}\dot{\theta}_2^2$$

or $\theta_2 \to \theta$,

$$\dot{\theta} = \frac{2g}{R}\left(1 - \cos\theta\right)$$

Our solution for $\dot{\theta}(\theta)$ can then be used to solve for λ,

$$\dot{\theta} \Rightarrow (1a) \Rightarrow \quad \lambda = -2mg + 3mg\cos\theta$$

The force of constraint associated w/ r is then,

$$Q_r = +\lambda \frac{\partial f}{\partial r} = -2mg + 3mg\cos\theta$$

b) When does the ball/point particle fall off the sphere? Precisely when the force of constraint vanishes (this is when we instead move to $x^2+y^2 \geq R^2$)

$$Q_R = 0 \Rightarrow 2mg = 3mg\cos\theta$$

$$\Downarrow$$

$$\cos\theta = 2/3$$ is the ~~angle~~ at which it falls off.

Converting back to our cartesian co-ordinates,

$$y = R\cos\theta = \frac{2R}{3}$$

$\therefore$ The ~~big~~ height at which the particle falls off the sphere is $y = \frac{2R}{3}$.

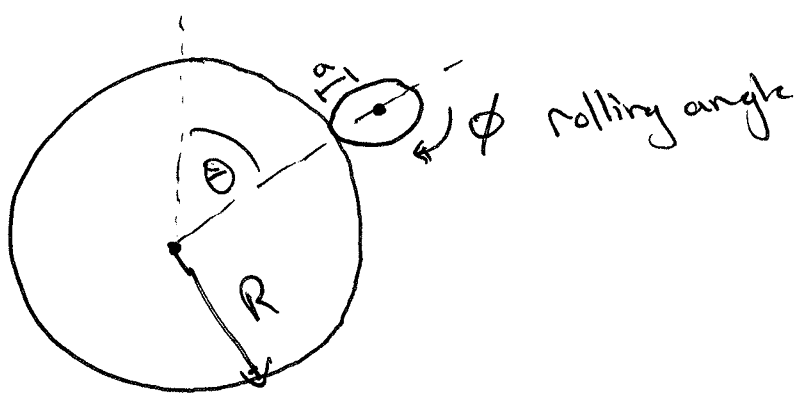
Notice → this is above $R/2$!!

c) Our marble is not a point particle! It has a radius a, which will enforce a new pair of constraints (care of our new no slip condition)),

i) $r = R + a$ [center of marble is a above sphere]

ii) $a(\phi - \theta) = R\theta$ [no slip]

Where does (ii) come from?!? First, ~~setp~~ set up co-ordinates/diagram.



A marble when rolling w/out slipping traces out an arclength,

$$s = a\phi$$

radius of marble

angle marble rolls through.

But also, in terms of the angle θ above the large sphere,

$$s = (R+a)\,\theta$$

Combining both : $a(\phi - \theta) = R\theta \;.\Rightarrow \text{(ii)}$

As holonomic constraints :

$$\text{(i)} \rightarrow \quad f_1(r,\theta,\phi) = r - a - R$$

$$\text{(ii)} \rightarrow \quad f_2(r,\theta,\phi) = R\theta - a(\phi - \theta)$$

Building from a), our Lagrangian is:

r describes com of marble.

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \underbrace{\frac{m}{2}a^2\dot{\phi}^2}_{\text{new rotational contribution.}}$$

$$V = mgr\cos\theta$$

$$\Rightarrow L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + a^2\dot{\phi}^2\right) - mgr\cos\theta$$

Our EOM in this case become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{j=1}^{2} \lambda_j \frac{\partial f_j}{\partial q_k}$$

$q_k = r, \theta, \phi$

$\Downarrow$

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_1 & (1) \\ mr^2\ddot{\theta} + 2mr\dot{\theta}\dot{r} + mgr\sin\theta = \lambda_2 (R+a) & (2) \\ mr^2\ddot{\phi} = -\lambda_2 a & (3) \end{cases}$$

w/ 2 constraint eqns:

$f_1 \rightarrow \quad r = R+a \, , \quad \dot{r} = \ddot{r} = 0$

$f_2 \rightarrow \quad \phi = \frac{a+R}{a}\theta \rightarrow \quad \dot{\phi} = \frac{R+a}{a}\dot{\theta}$

$+ \quad \ddot{\phi} = \frac{R+a}{a}\ddot{\theta}$

The insight from the constraint eqns can be used to solve (3):

$$(3) \rightarrow \quad mr^2\left(\frac{R+a}{a}\ddot{\theta}\right) = -\lambda_2 a$$

$$\Rightarrow \quad \ddot{\theta} = \frac{-\lambda_2}{m(R+a)} \qquad (3a)$$

Similarly, from 2:

$$2 \rightarrow \quad m(R+a)\ddot{\theta} + mg\sin\theta = \lambda_2$$

$$\hookrightarrow \quad \ddot{\theta} = \frac{\lambda_2 - mg\sin\theta}{m(R+a)} \quad 2a$$

Setting 3a = 2a, we obtain:

$$\lambda_2 = \frac{mg\sin\theta}{2}$$

We then ~~plug this~~ plug this back into 3a to solve for,

$$\ddot{\theta} = -\frac{g}{2(R+a)}\sin\theta$$

Using the relation $\ddot{\theta}\,d\theta = \dot{\theta}\,d\dot{\theta}$ from b), and following similar working,

$$\dot{\theta}^2 = \frac{g}{R+a} - \frac{g}{R+a}\cos\theta$$

Finally, this goes into ① to ~~yt~~ yield,

$$\lambda_1 = mg\,(2\cos\theta - 1)$$

From $\lambda_1 \propto Q_r$ we know that the condition for the marble to fall off the sphere is equivalent to,

$$\lambda_1 = 0 \implies 2\cos\theta - 1 = 0$$

$$\downarrow$$

$$\cos\theta = \frac{1}{2}$$

This converts to a height:

$$y = \frac{R+a}{2}$$

Question 2

(Note m + M are interchanged)

a) Let us start by setting $z=0$ to correspond to the table surface ($z<0$ under the table).

We obtain our lagrangian as,

$$L = \frac{M}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{m}{2}\dot{z}^2 \underbrace{- mgz}_{V}$$

The rope (string) provides a constraint:

$$f(r,\varphi,z) = l + z - r = 0 \qquad (z \leq 0)$$

Then we have equations of motion in ~~the~~ the form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \lambda \frac{\partial f}{\partial q_k}$$

for $q_k = r, \varphi, z$.

i.e.,

$$
\begin{cases}
M\ddot{r} - Mr^2\dot{\varphi} = -\lambda & (1) \\
\frac{d}{dt}\left(Mr^2\dot{\varphi}\right) = 0 & (2) \\
m\ddot{z} + mg = \lambda & (3)
\end{cases}
$$

(2) says that $p_\varphi = Mr^2\dot{\varphi}$ is a constant of motion!

Additionally, our constraint let's us set,

$$f \rightarrow \quad \ddot{z} = \ddot{r}$$

and using this w/ $M \times$ (3):

$$mM\ddot{r} + mMg = M\lambda \qquad (3a)$$

Similiarly, $m \times$ (1) yields,

$$mM\ddot{r} - mM\dot{\varphi}^2 r = -m\lambda \qquad (1a)$$

Then taking (1a) − (3a) gives us,

$$\lambda = \frac{mM}{m+M}\dot{\varphi}^2 r + \frac{mM}{m+M}g$$

Defining the reduced mass, $\mu = \frac{mM}{m+M}$, the forces of contraint are then obtained as:

$$Q_r = -\mu g - \mu \dot{\varphi}^2 r$$

$$Q_\varphi = 0$$

$$Q_z = \mu \dot{\varphi}^2 r + \mu g$$

const of motion.

b) To avoid the block being pulled down through the hole, we want

$$Q_z + F_{grav} = 0$$

constraint force

$-mg$ due to gravity

ie.,

$$\mu g + \mu \dot{\varphi}^2 r - mg = 0$$

$$\text{so } r\dot{\varphi}^2 = \frac{mg - \mu g}{\mu}$$

$$= \frac{m - \frac{mM}{m+M}}{\frac{mM}{m+M}} g = \frac{m}{M} g$$

or/ $\dot{\varphi} = \sqrt{\dfrac{mg}{Mr_0}}$ (taking +ve squareroot)

$\uparrow$ set by initial condition $r(0) = r_0$.

c) In principle, we know that,

$$E = T + V = \frac{M}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{m}{2}\dot{z}^2 + mgz = \text{const.}$$

We can use this to obtain an equation for $t(r)$.

First, we manipulate our expression for E by using that our constraint implies $\dot{r} = \dot{z}$,

$$E = \frac{m+M}{2}\dot{r}^2 + \frac{p_\varphi^2}{2Mr^2} + mg(r-l)$$

$\uparrow$ $(l + z - r = 0)$

This can be rearranged to solve for $\dot{r}$,

$$\dot{r} = \pm\sqrt{\frac{2}{m+M}}\sqrt{E - \frac{p_\varphi^2}{2Mr^2} - mg(r-l)}$$

Writing $\dot{r} = \frac{dr}{dt}$ we then get,

$$dt = \pm \sqrt{\frac{m+M}{2}} \; \frac{dr}{\sqrt{E - \frac{p_\varphi^2}{2Mr^2} - mg(r-l)}}$$

& integrating yields,

$$t - t_0 = \pm \sqrt{\frac{m+M}{2}} \int_{r(t_0)}^{r(t)} \frac{dr'}{\sqrt{E - p_\varphi^2/2Mr'^2 - mg(r'-l)}}$$

Question 3

a)

From Lagrange's equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$\hookrightarrow \quad \ddot{q} + \gamma \dot{q} + \frac{k}{m} q = 0$$

This eqn looks just like that of a damped harmonic oscillator.

makes sense because of terms $\propto \dot{q}^2 + q^2$ in brackets

The $e^{\gamma t}$ ~~factor~~ factor in L is thus connected to the exponential damping we expect for a damped SHO.

b)

$$\frac{\partial L}{\partial t} \neq 0 \quad \text{cause of the } e^{\gamma t} \text{ prefactor.}$$

Hence the energy function,

$$h = \dot{q}\frac{\partial L}{\partial \dot{q}} - L \quad \text{is not conserved.}$$

It would thus appear there are no constants of motion...

c) Define $Q = q e^{\gamma t/2}$ or $q = Q e^{-\gamma t/2}$

Then, notice,

$$e^{\gamma t} q^2 = e^{\gamma t} Q^2 e^{-\gamma t} = Q^2$$

and,

$$e^{\gamma t} \dot{q}^2 = e^{\gamma t}\left(\dot{Q} e^{-\gamma t/2} - \frac{\gamma}{2} Q e^{-\gamma t/2}\right)^2$$

$$= \left(\dot{Q} - \frac{\gamma}{2} Q\right)^2$$

Thus,

$$L = \frac{m}{2}\left(\dot{Q} - \frac{\gamma}{2} Q\right)^2 - \frac{k}{2} Q^2$$

~~w/ equation of motion,~~

d) Our new Lagrangian features a conserved energy func^n, $h = \dot{Q}\frac{\partial L}{\partial \dot{Q}} - L$ as $\frac{\partial L}{\partial t} = 0$.

Explicitly,

$$h = m\dot{Q}\left(\dot{Q} - \frac{\gamma}{2}Q\right) - \frac{m}{2}\left(\dot{Q} - \frac{\gamma}{2}Q\right)^2 + \frac{k}{2}Q^2$$

$$= \frac{m}{2}\dot{Q}^2 + \frac{\bar{k}}{2}Q^2$$

where, $\bar{k} = k - m\left(\frac{\gamma}{2}\right)^2$

Clearly, h looks like the energy of a simple harmonic oscillator. We can also rewrite:

$$\bar{k} = m\omega^2 - m\left(\frac{\gamma}{2}\right)^2$$

$$= m\left[\omega^2 - \left(\frac{\gamma}{2}\right)^2\right]$$

This is equivalent to the effective frequency, $\bar{\omega} = \sqrt{\omega^2 - \left(\frac{\gamma}{2}\right)^2}$ we deduced for the damped harmonic oscillator.

In fact our eqn for h ties nicely into our discussion of under/critical/over-damped systems: For our system to be an effective SHO $\rightarrow$ $\bar{\omega}^2 > 0$ required! i.e., $\omega^2 > \left(\frac{\gamma}{2}\right)^2$!

Question 4

a) The problem is best tackled by using cylindrical co-ordinates,

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

The constraint for the particle to remain on the surface of the cylinder can be written via:

$$x^2 + y^2 - R^2 = 0$$

or $\quad r - R = 0$

It is thus a holonomic constraint.

With this in hand, we are left with two generalized co-ordinates: z, θ

$$\left[\begin{array}{l} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{array}\right]$$

b) To compute the Lagrangian we need to first find the potential associate w/ the force. Clearly,

$$\vec{F} = -k\vec{r}$$

exhibits an indepence of θ (ie. strength depends only on $|\vec{r}|$). It is simplest to write the associated potential ~~in~~ terms of r, $\underline{\text{w/ } r^2 = R^2 + z^2}$

$$V(r) = \frac{1}{2}kr^2 \Rightarrow \vec{F}(r) = -\frac{\partial V}{\partial r}\hat{r}$$
$$= -kr\hat{r}$$

$\hat{r} \equiv \vec{r}/r$

Note you could have also done this in cartesian co-ordinates w/ a few more lines of working.

Our kinetic contribution is,

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(R^2\dot{\theta}^2 + \dot{z}^2\right)$$

trig or identity as angular motion

& thus,

$$L = T - V = \frac{m}{2}\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$$

Note: the contribution $\propto \frac{1}{2}kR^2$ could be ignored as a constant shift in the potential

Next, we obtain the EOM, from Lagrange's eqn:

$$\theta \rightarrow \quad \frac{d}{dt}\left(mR^2\dot{\theta}\right) = 0$$

$$z \rightarrow \quad m\ddot{z} - kz = 0$$

The former enables us to identify $mR^2\dot{\theta}$ as a constant of motion. It corresponds to a conserved angular momentum.

c) The equations describe a particle that:

i) rotates at fixed rate about the z-axis ($\dot{\theta} = \text{const}$)

ii) undergoes harmonic motion ~~about~~ in z $\left(\ddot{z} = \frac{k}{m} z\right)$

because the force is independent of θ!