Physics 5573, Spring 2022
Test 1, Solutions

1) Consider a sphere of radius R and total charge Q that has been embedded with an r-dependent charge density:

$$\rho(\vec{r}) = C\, r$$

a) Write and solve an integral to determine C in terms of the properties of the sphere.

The value of C depends on the charge and radius of the sphere:

$$Q = \iiint \rho(\vec{r})\, d^3r$$

$$Q = C \int r^2\, dr \int d\Omega\, r = 4\,\pi\, C \int_0^R r^3\, dr = \pi\, C\, R^4$$

$$C = \frac{Q}{\pi\, R^4}$$

b) Explain and justify your approach to solving for the electric field and potential due to this sphere everywhere. This should, of course, be the simplest (correct) approach to the problem.

Because of the spherical symmetry of the problem, the easiest approach to solving for the field and potential is to use Gauss' Law to solve for the electric field. The potential can be determined from the integral of the field.

The symmetry of the sphere means the radial coordinate and direction is the only possible dependence of the field. This gives: $\vec{E}(\vec{r}) = E(r)\,\hat{r}$

c) Set up your solution method and determine $\vec{E}(\vec{r})$ for all r.

Gauss' Law can used with a spherical surface of radius r. This can be done for $r < R$ and $r > R$:

For $r < R$

$$\oiint \vec{E}(\vec{r}) \cdot \hat{r}\, dS = \frac{1}{\epsilon_0} \iiint \rho(r)\, d^3r$$

$$E(r)\, 4\,\pi\, r^2 = \frac{C}{\epsilon_0} \iint d\Omega \int_0^r r'^3\, dr'$$

$$E(r)\, 4\,\pi\, r^2 = 4\,\pi \frac{C}{\epsilon_0} \frac{r^4}{4}$$

$$E(r) = \frac{C}{4\epsilon_0}\, r^2 = \frac{Q}{4\pi\epsilon_0} \frac{r^2}{R^4}$$

For $r > R$

$$\oiint \vec{E}(\vec{r}) \cdot \hat{r}\, dS = \frac{1}{\epsilon_0} \iiint \rho(r)\, d^3r = \frac{Q}{\epsilon_0}$$

$$E(r) = \frac{Q}{4\pi\epsilon_0\, r^2}$$

Note that in this case, the electric field is continuous at the surface of the sphere. This is true because there is not a 2D surface charge on the sphere.

d) Set up your solution method and solve for $\phi(\vec{r})$ for all r.

The potential is given by the integral of the electric field. Setting the potential equal to 0 at

$r = \infty$ as usual, for $r > R$:

$$\phi(\vec{r}) = -\int_{\infty}^{r} \vec{E}(\vec{r}) \cdot d\vec{l} = -\int_{\infty}^{r} \frac{Q}{4\pi\epsilon_0} \frac{dr'}{r'^2} = \frac{Q}{4\pi\epsilon_0\ r}$$

For $r < R$:

$$\phi(\vec{r}) - \phi(R) = -\int_{R}^{r} \vec{E}(\vec{r}) \cdot d\vec{l} = -\int_{R}^{r} \frac{Q}{4\pi\epsilon_0} \frac{r'^2}{R^4}\, dr'$$

$$\phi(r) - \frac{Q}{4\pi\epsilon_0\ R} = -\frac{1}{3} \frac{Q}{4\pi\epsilon_0\ R^4}(r^3 - R^3) = \frac{1}{3} \frac{Q}{4\pi\epsilon_0\ R}\left(1 - \frac{r^3}{R^3}\right)$$

$$\phi(r) = \frac{4}{3} \frac{Q}{4\pi\epsilon_0\ R}\left(1 - \frac{1}{4}\frac{r^3}{R^3}\right) = \frac{Q}{3\pi\epsilon_0 R}\left(1 - \frac{r^3}{4R^3}\right)$$

This result shows that the potential increases towards the interior of the sphere, from $r = R$ to $r = 0$.

e) Solve for the electric potential energy of the sphere.

There are two approaches possible for this problem using either the potential found in part d or the field found in part c.

Using the potential, it is only necessary to integrate over the sphere where $\rho(\vec{r}) \neq 0$:

$$U = \frac{1}{2} \iiint \rho(r)\, \phi(r)\, d^3r$$

$$U = \frac{1}{2} \iint d\Omega \int_0^R r^2\, dr \left(\frac{Q}{\pi R^4}\, r\right)\left(\frac{Q}{3\pi\epsilon_0 R}\right)\left(1 - \frac{r^3}{4R^3}\right)$$

$$U = 2\pi \frac{Q^2}{3\pi^2\epsilon_0 R^5} \int_0^R dr \left(r^3 - \frac{r^6}{4R^3}\right)$$

$$U = \frac{2Q^2}{3\pi\epsilon_0 R^5}\ R^4 \left(\frac{1}{4} - \frac{1}{28}\right) = \frac{Q^2}{7\,\pi\,\epsilon_0\, R}$$

Calculating the energy from the magnitude of the electric field:

$$U = \frac{\epsilon_0}{2} \iiint \left|\vec{E}(\vec{r})\right|^2 d^3r = 4\pi \frac{\epsilon_0}{2} \left(\int_R^\infty E^2\, r^2\, dr + \int_0^R E^2\, r^2 dr \right)$$

$$U = \frac{4\pi\epsilon_0}{2} \left(\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4}\, r^2\, dr + \int_0^R \left(\frac{Q}{4\pi\epsilon_0}\right)^2 \frac{r^4}{R^8}\, r^2\, dr \right)$$

$$U = \frac{Q^2}{8\pi\epsilon_0} \left(\frac{1}{R} + \frac{R^7}{7R^8} \right)$$

$$U = \frac{Q^2}{8\pi\epsilon_0\, R} \left(1 + \frac{1}{7} \right) = \frac{Q^2}{7\pi\epsilon_0\ R}$$

The two results are equal, as expected.

f) Describe your approach to solving this problem if the charge density is: $\rho(\vec{r}) = C\, r \cos^2\theta$
You do NOT have to solve this problem; just explain how you would approach it.

Although a solution through direct integration is always possible (numerically), this problem lends itself well to a multipole expansion. Using the result:

$$, \qquad \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

And the expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ (for $r > R$, for example)

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \iiint d^3r'\, r'^l\, \rho(\vec{r}')\, Y_{lm}^*(\theta', \phi')$$

But using:

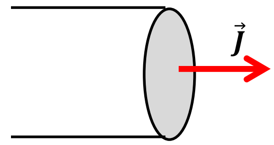
$$Y_{20}(\theta', \phi') = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\,(3\cos^2\theta' - 1), \qquad Y_{00}(\theta', \phi') = \frac{1}{\sqrt{4\pi}}$$

We have:

$$\cos^2\theta' = \frac{2}{3}\sqrt{\frac{4\pi}{5}}\, Y_{20} + \sqrt{4\pi}\, Y_{00}$$

The integral over the angles $d\Omega' = \sin\theta'\ d\theta' d\phi'$ will result in just the $l = 0$ and $l = 2, m = 0$ terms remaining. We would just be left with simple radial integrals to get the potential. The electric field would be calculated just from the potential.

2) The Magnetic analog to the problem above is a long, current-carrying wire with a current density that varies across the radius of the wire.



Use polar coordinates: $\hat{z}$ along the wire, $\hat{r}$ the radial direction perpendicular to $\hat{z}$, and $\hat{\phi}$ the azimuthal angle around $\hat{z}$.

The wire has a radius R and total current I in the $\hat{z}$ direction. The current density is:

$$\vec{J}(\vec{r}) = C\,r\,\hat{z}$$

a) Derive an expression for the constant C in terms of properties of the wire.

The current will be given by an integral over the cross-section of the wire:

$$I = \iint \vec{J}(\vec{r}) \cdot \hat{z}\, dS = C \int_0^{2\pi} d\phi \int_0^R r\, dr\; r = \frac{2\pi}{3}\, C\, R^3$$

$$C = \frac{3}{2}\frac{I}{\pi R^3}$$

b) Calculate $\vec{B}(\vec{r})$ everywhere. Be sure to describe and justify your approach.

The symmetry of the infinite wire indicates that the field can only depend on the distance from the center of the wire, r, not on either ϕ or z. The field at any point $\vec{r}$ is in the direction of the cross product $\vec{J} \times \vec{r}$, or the $\hat{\phi}$ direction. This allows us to use Ampere's law for a loop centered on the z-axis:

For $r < R$:

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_0 \iint \vec{J}(\vec{r}) \cdot \hat{n}\, dS$$

$$B_\phi(r)\, 2\,\pi\, r = \mu_0 C \int_0^{2\pi} d\phi' \int_0^r r'\, dr'\; r'$$

$$B_\phi(r)\, 2\,\pi\, r = 2\pi\, \mu_0\, C \frac{r^3}{3}$$

$$B_\phi(r) = \mu_0\, C \frac{r^2}{3} = \frac{\mu_0\, I}{2\pi} \frac{r^2}{R^3}$$

For $r > R$:

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_0 \iint \vec{J}(\vec{r}) \cdot \hat{n}\, dS = \frac{I}{\mu_0}$$

$$B_\phi(r)\, 2\,\pi\, r = \mu_0 I$$

$$B_\phi(r) = \frac{\mu_0\, I}{2\pi\, r}$$

c) What direction is the vector potential, $\vec{A}(\vec{r})$, for the wire? Explain/justify your answer.

The vector potential must be in the $\hat{z}$ direction:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint \vec{J}(\vec{r}\,') \frac{1}{|\vec{r} - \vec{r}\,'|} \, d^3r'$$

And $\vec{J}(\vec{r}) = J(r)\,\hat{z}$

d) Solve for the magnetic potential energy of the wire per unit length of the wire.

As for the problem above, the potential energy can be calculated either with the current density and the vector potential or with the magnitude of the field. As we have solved for the magnetic field, we'll use that approach:

$$U = \frac{1}{2\mu_0} \iiint \left|\vec{B}(\vec{r})\right|^2 \, d^3r$$

As the B-field is independent of z, we can do the integral over z for a length L to get the energy per length:

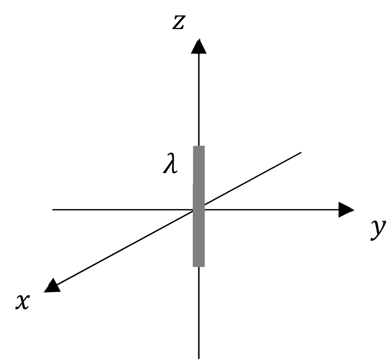
$$\frac{U}{L} = \frac{1}{2\mu_0} \left(\int_0^{2\pi} d\phi \int_0^R r\,dr \left(\frac{\mu_0\, I}{2\pi} \frac{r^2}{R^3} \right)^2 + \int_0^{2\pi} d\phi \int_R^{\infty} r\,dr \left(\frac{\mu_0\, I}{2\pi\, r} \right)^2 \right)$$

It turns out that the second term here diverges as the $\ln(r)$ so that the answer is, in fact, infinite. It shouldn't be too much of a surprise that creating an infinite current takes an infinite energy.

Note on Grading: Everyone with a reasonable setup for this problem will get credit.

3) Consider a uniform charged rod of charge Q and length L on the z-axis, with its center at the origin. The linear charge density is

$$\lambda = \frac{Q}{L}$$



a) Calculate the electric potential due to the rod on the z-axis for all $z > \frac{L}{2}$.

The potential on the z-axis can be determined by an integral over the rod:

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int_{-\frac{L}{2}}^{\frac{L}{2}} \lambda\, dz' \frac{1}{z - z'} = \frac{\lambda}{4\pi\epsilon_0} \int_{z-\frac{L}{2}}^{z+\frac{L}{2}} \frac{du}{u}$$

$$\phi(z) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{z + \frac{L}{2}}{z - \frac{L}{2}}\right) = \frac{\lambda}{4\pi\epsilon_0} \ln\left(\frac{1 + \frac{L}{2z}}{1 - \frac{L}{2z}}\right)$$

b) Expand your result for $\phi(z)$ in powers of $L/(2\,z)$.
Perhaps useful:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \;,\;\; \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

This gives:

$$\phi(z) = \frac{\lambda}{4\pi\epsilon_0} \left(\ln\left(1 + \frac{L}{2z}\right) - \ln\left(1 - \frac{\mathrm{L}}{2z}\right)\right)$$

$$\phi(z) = \frac{\lambda}{4\pi\epsilon_0} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{L}{2z}\right)^n ((-1)^{n-1} + 1)$$

Define a new index:

$$l = n - 1$$

Giving:

$$\phi(z) = \frac{\lambda}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(\frac{1}{l+1}\right) \left(\frac{L}{2}\right)^{l+1} \frac{1}{z^{l+1}} ((-1)^l + 1)$$

All the terms where l is odd will be zero, giving:

$$\phi(z) = \frac{\lambda}{4\pi\epsilon_0} \sum_{l=even}^{\infty} \left(\frac{1}{l+1}\right) \left(\frac{L}{2}\right)^{l+1} \frac{2}{z^{l+1}} = \frac{Q}{4\pi\epsilon_0} \sum_{l=even}^{\infty} \left(\frac{1}{l+1}\right) \left(\frac{L}{2}\right)^{l} \frac{1}{z^{l+1}}$$

c) Using this result, determine an expression (a sum) for the potential $\phi(r,\theta)$ anywhere. Why is the potential independent of ϕ? Remember that any such function can be written as:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \frac{a_l}{r^{l+1}} P_l(\cos\theta)$$

The charged rod is symmetric in the ϕ direction, so the potential must be independent of the azimuthal angle. The expression above is an expansion for $z = r, \theta = 0$. So we can write this expansion as:

$$\phi(r,\theta = 0) = \frac{Q}{4\pi\epsilon_0} \sum_{l=even}^{\infty} \left(\frac{1}{l+1}\right)\left(\frac{L}{2}\right)^l \frac{1}{r^{l+1}}$$

But this is exactly the form as the expansion in the Legendre Polynomials and each term in the sum, powers of $\frac{1}{r}$, must be equal. This gives:

$$a_l = 0, \qquad l = 1, 3, 5, \dots$$

$$a_l = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{l+1}\right)\left(\frac{L}{2}\right)^l \frac{1}{P_l(1)} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{l+1}\right)\left(\frac{L}{2}\right)^l, \qquad l = 0, 2, 4, \dots$$

This last result comes from the fact that $P_l(1) = 1$, although that isn't something that you needed to know.

d) Calculate the monopole, dipole, and quadrupole terms for the line charge and show that these agree with the first three terms in (c).

The Monopole for the line charge is, of course, Q, so the monopole term in the potential should be

$$\phi^{(0)}(r) = \frac{Q}{4\pi\epsilon_0\, r}$$

The result above gives $a_0 = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$ exactly as expected.

The Dipole is:

$$\vec{p} = \hat{z}\,\lambda \int_{-\frac{L}{2}}^{\frac{L}{2}} z\ dz = 0$$

The dipole term is zero, just as $a_1 = 0$ in the expansion.

The quadrupole term will be:

$$\phi^{(2)}(\vec{r}) = \frac{\lambda}{8\pi\epsilon_0 r^3} \int_{-\frac{L}{2}}^{\frac{L}{2}} dz\ \left(3\,(z\,\hat{z}\cdot\hat{r})^2 - z^2\right)$$

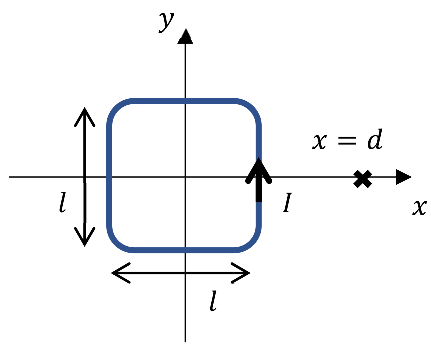
$$\phi^{(2)}(\vec{r}) = \frac{\lambda}{8\pi\epsilon_0 r^3}\,(3\cos^2\theta - 1)\int_{-\frac{L}{2}}^{\frac{L}{2}} z^2\,dz$$

$$\phi^{(2)}(\vec{r}) = \frac{\lambda}{8\pi\epsilon_0 r^3}\,(3\cos^2\theta - 1)\frac{2}{3}\left(\frac{L}{2}\right)^3$$

$$\phi^{(2)}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^3}\,P_2(\cos\theta)\,\frac{1}{3}\left(\frac{L}{2}\right)^2 = \frac{Q}{4\pi\epsilon_0}\frac{L^2}{12}\,P_2(\cos\theta)$$

Exactly the result found above for the $l = 2$ term.

4) Consider the current loop shown. It is a square with sides of length l and counter-clockwise current I, lying in the x-y plane and centered at the origin. Your task is to calculate the magnetic field at the point $x = d, y = 0, z = 0$.



a) To get a first approximation to the field, what is the magnetic field at $x = d$ if the loop is considered a point magnetic dipole at the origin?

In this case, the magnetic moment of the current is

$$\vec{m} = I\,l^2\,\hat{z}$$

And the field at $r = d\,\hat{x}$ is:

$$\vec{B}(x = d) = \frac{\mu_0}{4\pi\,d^3}\,(3\,(\vec{m}\cdot\hat{x})\,\hat{x} - \vec{m}) = -\frac{\mu_0\,I\,l^2}{4\pi\,d^3}\,\hat{z}$$

The approximation to the magnetic field will be in the $-\hat{z}$ direction and decrease as d^{-3}.

b) Write down integrals to calculate the magnetic field at $x = d$ due to each side of the current loop. Simplify these integrals as much as possible, without actually doing the integrals. What is the direction of the magnetic field due to each side?

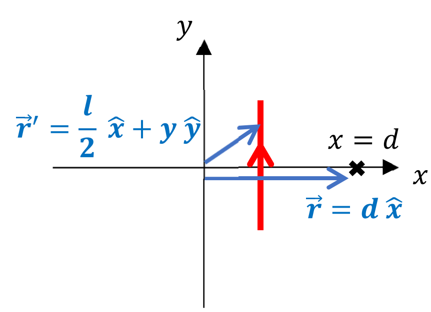
The field due to each side of the loop would be of the form:

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Reminder: $\vec{r}$ is the "field point", in this case $\vec{r} = d\,\hat{x}$.

$\vec{r}'$ is the "source point", a point on the current-carrying loop.

Let's draw a couple of pictures to define things. For the right side of the loop:



$$d\vec{l}' = dy\,\hat{y}, \qquad \vec{r} = d\,\hat{x}, \qquad \vec{r}' = \frac{l}{2}\,\hat{x} + y\,\hat{y}, \qquad \vec{r} - \vec{r}' = \left(d - \frac{l}{2}\right)\hat{x} - y\,\hat{y}$$

This means the direction of the magnetic field due to the right side of the loop would be:

$$\hat{y} \times \left(d - \frac{l}{2}\right)\hat{x} = -\hat{z}$$

Writing the integral:

$$\vec{B}_r = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dy\,\hat{y} \times \left(d\,\hat{x} - \left(\frac{l}{2}\,\hat{x} + y\,\hat{y}\right)\right)}{\left(\left(d - \frac{l}{2}\right)^2 + y^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dy\,\left(d - \frac{l}{2}\right)(-\hat{z})}{\left(\left(d - \frac{l}{2}\right)^2 + y^2\right)^{3/2}}$$

For the left side of the loop, this would be similar except that the current is in the $-\hat{y}$ direction giving a field in the $+\hat{z}$ direction:

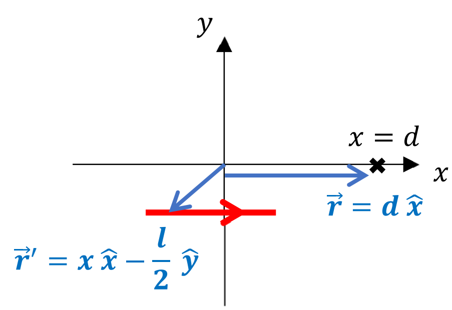
$$\vec{B}_l = \frac{\mu_0 I}{4\pi}\int_{\frac{l}{2}}^{-\frac{l}{2}} \frac{dy\,\hat{y} \times \left(d\,\hat{x} - \left(-\frac{l}{2}\,\hat{x} + y\,\hat{y}\right)\right)}{\left(\left(d + \frac{l}{2}\right)^2 + y^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dy\,\left(d + \frac{l}{2}\right)(\hat{z})}{\left(\left(d + \frac{l}{2}\right)^2 + y^2\right)^{3/2}}$$

Note: We need to be careful about the directions in the integral here. The direction of the integral, in the $-\hat{y}$ direction comes from the limits on the integral. The integral is on dy and the minus sign arises from the limits going from positive to negative. In the last term the limits of the integral is switched giving a negative sign.

For the bottom side of the loop:

$$d\vec{l}' = dx\,\hat{x}, \qquad \vec{r} = d\,\hat{x}, \qquad \vec{r}' = x\,\hat{x} - \frac{l}{2}\,\hat{y},$$

$$\vec{r} - \vec{r}' = (d - x)\hat{x} + \frac{l}{2}\,\hat{y}$$



This means the direction of the magnetic field due to the right side of the loop would be:

$$\hat{x} \times \frac{l}{2}\hat{y} = +\hat{z}$$

Writing the integral:

$$\vec{B}_b = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx\,\hat{x} \times \left(d\,\hat{x} - \left(-\frac{l}{2}\,\hat{y} + x\,\hat{x}\right)\right)}{\left((d - x)^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx\,\left(\frac{l}{2}\right)(\hat{z})}{\left((d - x)^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}}$$

For the top of the loop, the field will also be in the $+\hat{z}$ direction because the direction of the current and the y-component of $\vec{r}'$ have both changed sign:

$$\vec{B}_t = \frac{\mu_0 I}{4\pi}\int_{\frac{l}{2}}^{-\frac{l}{2}} \frac{dx\,\hat{x}\times\left(d\,\hat{x} - \left(\frac{l}{2}\,\hat{y} + x\,\hat{x}\right)\right)}{\left((d-x)^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}} = \frac{\mu_0 I}{4\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx\,\left(\frac{l}{2}\right)(\hat{z})}{\left((d-x)^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}}$$

Considering the symmetry of the problem, not surprisingly $\vec{B}_b = \vec{B}_t$.

c) Solve for the magnetic field due to the loop at $x = d$. If you use results found in class, be sure to explain and justify them. If you do the integrals, it might be useful to know that:

$$\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$$

Doing the integrals:

$$\vec{B}_r = (-\hat{z})\frac{\mu_0 I}{4\pi}\frac{d - \frac{l}{2}}{\left(d - \frac{l}{2}\right)^2}\left.\frac{y}{\sqrt{\left(d - \frac{l}{2}\right)^2 + y^2}}\right|_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$\vec{B}_r = (-\hat{z})\frac{\mu_0 I}{4\pi}\frac{1}{d - \frac{l}{2}}\frac{l}{\sqrt{\left(d - \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}$$

Similarly

$$\vec{B}_l = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{1}{d + \frac{l}{2}}\frac{l}{\sqrt{\left(d + \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}$$

For the top and bottom:

$$\vec{B}_t = \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{l}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dx}{\left((d-x)^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}} = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{l}{2}\int_{d-\frac{l}{2}}^{d+\frac{l}{2}} \frac{du}{\left(u^2 + \left(\frac{l}{2}\right)^2\right)^{3/2}}$$

$$\vec{B}_t = \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{2}{l}\left(\frac{d + \frac{l}{2}}{\sqrt{\left(d + \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}} - \frac{d - \frac{l}{2}}{\sqrt{\left(d - \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}\right)$$

All this doesn't really simplify much, but the result is:

$$\vec{B} = -\hat{z}\frac{\mu_0 I}{4\pi}\left(\left(\frac{l}{d - \frac{l}{2}} + 4\,\frac{d - \frac{l}{2}}{l}\right)\frac{1}{\sqrt{\left(d - \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}} - \left(\frac{l}{d + \frac{l}{2}} + 4\frac{d + \frac{l}{2}}{l}\right)\frac{1}{\sqrt{\left(d + \frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}\right)$$

d) Compare your result to the dipole approximation show that they agree in the limit $d \gg l$.

This expansion is more difficult than expected (I made a mistake on my initial solutions). This part of the test has only been looked a briefly in grading.

Consider the left and right wires:

$$\vec{B}_r + \vec{B}_l = (-\hat{z})\frac{\mu_0 I}{4\pi}\left(\frac{1}{d-\frac{l}{2}}\frac{l}{\sqrt{\left(d-\frac{l}{2}\right)^2+\left(\frac{l}{2}\right)^2}} - \frac{1}{d+\frac{l}{2}}\frac{l}{\sqrt{\left(d+\frac{l}{2}\right)^2+\left(\frac{l}{2}\right)^2}}\right)$$

Expanding the terms in the square roots and pulling out factors of d gives:

$$\vec{B}_r + \vec{B}_l = (-\hat{z})\frac{\mu_0 I}{4\pi}\frac{l}{d^2}\left(\frac{1}{1-\frac{l}{2d}}\frac{l}{\sqrt{1-\frac{l}{d}+\frac{l^2}{2d^2}}} - \frac{1}{1+\frac{l}{2d}}\frac{l}{\sqrt{1+\frac{l}{d}+\frac{l^2}{2d^2}}}\right)$$

To first order is $\frac{l}{d}$ this becomes:

$$\vec{B}_r + \vec{B}_l = (-\hat{z})\frac{\mu_0 I}{4\pi}\frac{l}{d^2}\left(\left(1+\frac{l}{2d}\right)\left(1+\frac{l}{2d}\right) - \left(1-\frac{l}{2d}\right)\left(1-\frac{l}{2d}\right)\right)$$

$$\vec{B}_r + \vec{B}_l = (-\hat{z})\frac{\mu_0 I}{4\pi}\frac{l}{d^2}\left(\left(1+\frac{l}{d}\right) - \left(1-\frac{l}{d}\right)\right) = 2\frac{(-\hat{z})\mu_0\, I l^2}{4\pi\ d^3}$$

This is a factor of 2 larger than the dipole approximation from part a.

Considering the top and bottom wires together:

$$\vec{B}_t + \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{4}{l}\left(\frac{d+\frac{l}{2}}{\sqrt{\left(d+\frac{l}{2}\right)^2+\left(\frac{l}{2}\right)^2}} - \frac{d-\frac{l}{2}}{\sqrt{\left(d-\frac{l}{2}\right)^2+\left(\frac{l}{2}\right)^2}}\right)$$

Factoring out the $d \pm \frac{l}{2}$ terms:

$$\vec{B}_t + \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{4}{l}\left(\frac{1}{\sqrt{1+\left(\frac{l}{2d+l}\right)^2}} - \frac{1}{\sqrt{1+\left(\frac{l}{2d-l}\right)^2}}\right)$$

Expanding this for $\frac{l}{d} \ll 1$ or $\left(\frac{l}{2d \pm l}\right)^2 \ll 1$

$$\vec{B}_t + \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{4}{l}\left(\left(1-\frac{1}{2}\left(\frac{l}{2d+l}\right)^2\right) - \left(1-\frac{1}{2}\left(\frac{l}{2d-l}\right)^2\right)\right)$$

$$\vec{B}_t + \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{2}{l}\left(\left(\frac{l}{2d-l}\right)^2 - \left(\frac{l}{2d+l}\right)^2\right) = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{2}{l}\left(\frac{l}{2d}\right)^2\left(\left(1-\frac{l}{2d}\right)^{-2} - \left(1+\frac{l}{2d}\right)^2\right)$$

Doing a final expansion of these terms gives:

$$\vec{B}_t + \vec{B}_b = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{2}{l}\left(\frac{l}{2d}\right)^2\left(\left(1+\frac{l}{d}\right) - \left(1-\frac{l}{d}\right)\right) = (\hat{z})\frac{\mu_0 I}{4\pi}\frac{2}{l}\left(\frac{l}{2d}\right)^2\left(2\frac{l}{d}\right)$$

This gives:

$$\vec{B}_t + \vec{B}_b = \frac{(\hat{z})\mu_0\, Il^2}{4\pi\ d^3}$$

So, the total field in the large d expansion is equal to the dipole field:

$$\vec{B}_r + \vec{B}_l + \vec{B}_t + \vec{B}_b = 2\frac{(-\hat{z})\,\mu_0\, Il^2}{4\pi\ d^3} + \frac{(\hat{z})\,\mu_0\, Il^2}{4\pi\ d^3} = -\frac{\mu_0}{4\pi}\frac{I\,l^2}{d^3}\ \hat{z}$$