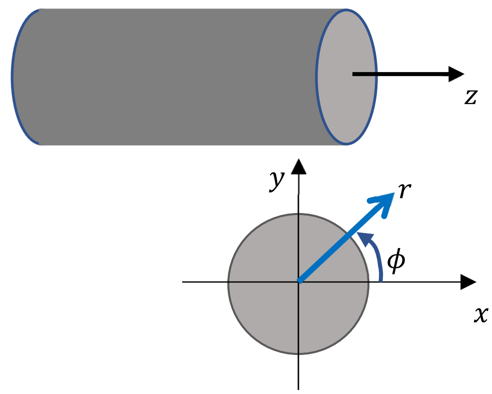
E & M I

Workshop 11 – 2D Cylindrical Coordinates, 4/20/2022

1) General Solutions

We'll be getting to Cylindrical Coordinates and Bessel Functions in a bit, but let's start with a similar problem but without the z-dependence.

Consider a very long cylindrical object, say a long dielectric cylinder. The natural coordinates to describe this are 2D polar coordinates.



Here I'll use r as the radial coordinate and ϕ as the azimuthal coordinate. (These are the same as spherical coordinates in the x-y plane.)

Because of the large (infinite) extent of the cylinder, symmetry arguments state that solutions to Laplace's equation are independent of z.

If $\Phi(\vec{r})$ is the electric potential, everywhere there is no charge it must satisfy:

$$\nabla^2\,\Phi(\vec{r}) = \frac{1}{r}\partial_r\, r\,\partial_r\,\Phi(\vec{r}) + \frac{1}{r^2}\,\partial_\phi^2\,\Phi(\vec{r}) = 0$$

A) Show that you can use separation of variables to find solutions to $\Phi(\vec{r})$:

$$\Phi(\vec{r}) = R(r)\,\chi(\phi)$$

Derive the differential equations for $R(r)$ and $\chi(\phi)$.

Using the separable form for the potential, Laplace's equation become:

$$\nabla^2\,\Phi(\vec{r}) = \frac{\chi(\phi)}{r}\partial_r\, r\,\partial_r R(r) + \frac{R(r)}{r^2}\,\partial_\phi^2\chi(\phi) = 0$$

Multiplying by $\frac{r^2}{\Phi}$ gives

$$\frac{r}{R(r)}\partial_r\, r\,\partial_r R(r) + \frac{1}{\chi(\phi)}\,\partial_\phi^2\chi(\phi) = 0$$

As each term depends on only one coordinate, r or ϕ, they must both be equal to a constant, and the sum of those constants equal to zero. This gives:

$$\partial_\phi^2\,\chi_\nu(\phi) = -\nu^2\,\chi_\nu(\phi)$$

$$r\,\partial_r\, r\,\partial_r\,R(r) = +\nu^2\,R(r)$$

B) The differential equation for $\chi(\phi)$ is of the form:

$$\partial_\phi^2 \, \chi_\nu(\phi) = -\nu^2 \, \chi_\nu(\phi)$$

Solve for the solutions to this equation and explain why you must have $\nu = n$, an integer.

This being a second order differential equation, you'll have two solutions for each value of n. Show, however, that there is just one physical solution for $n = 0$.

The solution is:

$$\chi_\nu(\phi) = A \sin(\nu \, \phi) + B \cos(\nu \, \phi)$$

To be single valued, we must have:

$$\chi(\phi + 2\,\pi) = \chi(\phi)$$

$$A \sin(\nu \, \phi + 2\,\pi\,\nu) + B \cos(\nu \, \phi + 2\,\pi\,\nu) = A \sin(\nu \, \phi) + B \cos(\nu \, \phi)$$

This means that $\nu = n$, an integer. Note that n can be limited to a positive integer or zero as negative n gives the same solutions (modulo a minus sign for the $\sin(n\,\phi)$ term.

For $n = 0$, the differential equation is:

$$\partial_\phi^2 \, \chi_0(\phi) = 0, \;\; \chi_0(\phi) = B_0 + A_0 \, \phi$$

The second term does not obey the boundary conditions of χ being single valued.

C) Using the results from B, solve for the functions $R_n(r)$ corresponding the values of $n \neq 0$ from the solution to the angular equation. Again, for a second-order differential equation, there will be two solutions for each n.

Hint: You might want to start with the simplest possible solutions you can think of.

The radial equation becomes:

$$r \, \partial_r \, r \, \partial_r \, R(r) = n^2 \, R(r)$$

Looking at this equation, it seems to indicate that when you take the derivative and then multiply by r you get the same function plus a constant. This is satisfied by a power:

$$R(r) = r^n, \qquad r \, \partial_r(r \, \partial_r r^n) = n \, (r \, \partial_r \, r^n) = n^2 \, R(r)$$

$$R(r) = r^{-n}, \qquad r \, \partial_r(r \, \partial_r r^{-n}) = -n \, (r \, \partial_r \, r^{-n}) = n^2 \, R(r)$$

D) Determine the two solutions for the $n = 0$ radial function $R_0(r)$. Explain why one of these solutions is not physical while the second one is a potential you should recognize.

Hint: Consider the geometry of the problem.

For $n = 0$ this gives:

$$r\,\partial_r(\,r\,\partial_r\,R(r)) = 0$$

The trivial solution is, of course $R(r)$ is a constant. This solution is not physical for systems where $r \to \infty$ and there are no charges at infinity. It should be considered for confined systems where there is a finite boundary on r.

For the second solution, we would need:

$$r\,\partial_r\,R(r) = C$$

Where C is a constant. This gives:

$$\partial_r\,R(r) = \frac{C}{r}$$

Or

$$R(r) = C\ln\left(\frac{r}{r_0}\right)$$

Where r_0 is some constant of integration. This is the electrostatic potential of a uniform line charge, something we should expect.

E) Pull together your results from parts B, C, and D to write the most general solution to Laplace's equation for a 2D cylindrical problem.

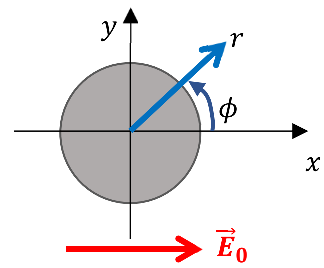
The most general form for the potential is a sum over these solutions:

$$\begin{aligned}\Phi(r,\phi) = A_0 + B_0 \ln\left(\frac{r}{r_0}\right) \\ + \sum_{n=1}^{\infty}\bigg(A_n\,r^n\sin(n\phi) + B_n\,r^n\cos(n\phi) + \frac{C_n}{r^n}\sin(n\phi) \\ + \frac{D_n}{r^n}\cos(n\phi)\bigg)\end{aligned}$$

Or

$$\Phi(r,\phi) = A_0 + B_0 \ln\left(\frac{r}{r_0}\right) + \sum_{n=-\infty,\neq 0}^{\infty} \left(A_n r^n\ \sin(n\phi) + B_n\,r^n\cos(n\phi)\right)$$

2) Cylinder in a Constant Field



Next, considering putting the dielectric cylinder in a constant electric field perpendicular to the axis of the cylinder:

$$\vec{E}_0 = E_0\,\hat{x}$$

The cylinder has a radius R and no free charge.

You will use the general solution you found in Question 1 to determine the potential and electric field everywhere.

This is like homework 4 where you considered a conducting sphere in a constant field. See if you can apply the same reasoning as the homework problem to solve the problem in this case.

A) What is the functional form for the total potential $\Phi(\vec{r})$ for $x \to \infty$?

For a constant field:

$$\Phi_0(x) = -E_0\,x = -E_0\,r\cos\phi$$

As usual, we'll use two separate potentials inside and outside of the sphere $\Phi_{in}(\vec{r})$ and $\Phi_{out}(\vec{r})$

B) Using part A, explain why there will only be two terms in the expansion for $\Phi_{out}(\vec{r})$. Determine the coefficient (constant) for one of these terms.

Far away from the cylinder, the potential needs to reduce to $\Phi_0 \propto \cos\phi$. This means that in the general expansion above, there can only be the $n = 1$ terms (or the $n = \pm 1$ terms if you some over positive and negative n.)

$$\Phi_{out}(r,\phi) = A_1\,r\sin(\phi) + B_1\,r\cos(\phi) + \frac{C_1}{r}\sin(\phi) + \frac{D_1}{r}\cos(\phi)$$

We can't have the $\sin\phi$ terms leaving:

$$\Phi_{out}(r,\phi) = B_1\,r\cos(\phi) + \frac{D_1}{r}\cos(\phi)$$

To match the potential as $r \to \infty$, $B_1 = -E_0$.

C) Considering the limit $r \to 0$, write down a general expansion (sum) for $\Phi_{in}(\vec{r})$.

For $r \to 0$, all the r^{-n} terms must go away, along with the $\ln(r)$ term, so the potential inside the cylinder is (relabeling coefficients to keep them distinct):

$$\Phi_{\mathrm{In}}(r,\phi) = F_0 + \sum_{n=1}^{\infty}\left(F_n\,r^n\sin(n\phi) + G_n\,r^n\cos(n\phi)\right)$$

D) What are the boundary conditions on Φ at $r = R$? Remember that this is a dielectric.

The potential must be continuous at the boundary, and the perpendicular component of $\vec{D} = \epsilon \vec{E}$ must be continuous.

$$\Phi_{Out}(R, \phi) = \Phi_{In}(R, \phi)$$

$$\epsilon_0 \, \partial_r \, \Phi_{Out}(R, \phi) = \epsilon \, \partial_r \, \Phi_{In}(R, \phi)$$

E) Use the boundary conditions to solve for both $\Phi_{in}(\vec{r})$ and $\Phi_{out}(\vec{r})$ everywhere.

For the potential to be continuous at $r = R$:

$$B_1 \, R \cos(\phi) + \frac{D_1}{R} \cos(\phi) = F_0 + \sum_{n=1}^{\infty} (F_n \, R^n \sin(n\phi) + G_n \, R^n \cos(n\phi))$$

This implies that:

$$F_n = 0, G_{n \neq 1} = 0$$

$$B_1 \, R + \frac{D_1}{R} = G_1 \, R, \qquad B_1 = -E_0$$

The continuity of $\vec{D} \cdot \hat{r}$ gives:

$$\epsilon_0 \, \partial_r \left(B_1 \, r \cos(\phi) + \frac{D_1}{r} \cos(\phi) \right) \Big|_{r=R} = \epsilon \, \partial_r \, G_1 \, r \cos(\phi) |_{r=R}$$

$$B_1 - \frac{D_1}{R^2} = \frac{\epsilon}{\epsilon_0} \, G_1$$

Solving:

$$G_1 = B_1 + \frac{D_1}{R^2}$$

$$B_1 - \frac{D_1}{R^2} = \frac{\epsilon}{\epsilon_0} \left(B_1 + \frac{D_1}{R^2} \right)$$

$$D_1 = R^2 \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} \, B_1 = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \, E_0 \, R^2$$

$$G_1 = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} \, E_0$$

$$\Phi_{In}(\vec{r}) = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} \, E_0 \, r \cos \phi$$

$$\Phi_{out}(\vec{r}) = E_0 \left(-r + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r} \right) \cos \phi$$

F) Solve for the electric field $\vec{E}_{in}(\vec{r})$ and the Polarization $\vec{P}(\vec{r})$ inside the cylinder, and the bound surface charge, σ_b. Remember the definitions:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}, \qquad \vec{D} = \epsilon\, \vec{E}, \qquad \sigma_b = \vec{P} \cdot \hat{n}$$

Inside the sphere:

$$\Phi_{In}(\vec{r}) = -\frac{2\epsilon_0}{\epsilon + \epsilon_0}\, E_0\, r \cos\phi = -\frac{2\epsilon_0}{\epsilon + \epsilon_0}\, E_0\, x$$

$$\vec{E}_{In} = -\vec{\nabla}\, \Phi_{In} = \frac{2\epsilon_0}{\epsilon + \epsilon_0}\, E_0\, \hat{x}$$

The field is constant in the $\hat{x}$ direction.

$$\vec{P}(\vec{r}) = \vec{D} - \epsilon_0\, \vec{E} = (\epsilon - \epsilon_0)\, \vec{E} = 2\, \epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\, E_0\, \hat{x}$$

G) Solve the total electric field outside of the cylinder, $\vec{E}_{Tot}(\vec{r})$.

Show that $\vec{E}_{Tot}$ is the sum of three terms: $\vec{E}_0$, an electric field due to the cylinder that is in the same direction as $\vec{E}_0$ ($\hat{x}$), and a radial electric field due to the cylinder in the $\hat{r}$ direction. Hint: On this last part, it may be useful to write

$$E_0 \cos\phi = \frac{\vec{E}_0 \cdot \vec{r}}{r}$$

Using the hint, we can write the external potential:

$$\Phi_{out}(\vec{r}) = E_0 \cos\phi \left(-r + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r} \right) = \vec{E}_0 \cdot \vec{r} \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} - 1 \right)$$

$$\vec{E}_{out}(\vec{r}) = \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} - 1 \right) \left(-\vec{\nabla} \left(\vec{E}_0 \cdot \vec{r} \right) \right) + \left(\vec{E}_0 \cdot \vec{r} \right) \left(-\partial_r \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} - 1 \right) \hat{r} \right)$$

$$\vec{E}_{out}(\vec{r}) = \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} - 1 \right) \left(-\vec{E}_0 \right) + \left(\vec{E}_0 \cdot \vec{r} \right) \left(2\, \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^3} \right) \hat{r}$$

$$\vec{E}_{out}(\vec{r}) = \vec{E}_0 - \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} \right) E_0\, \hat{x} + 2E_0 \cos\phi \left(\frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} \frac{R^2}{r^2} \right) \hat{r}$$

As advertised, the first term is the external field, the second term is due to the cylinder and is in the $-\hat{x}$ direction, and the third term is radial.

To do this without using the hint, the electric field is:

$$\vec{E}_{out}(\vec{r}) = -\vec{\nabla}\left(E_0\left(-r + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r}\right)\cos\phi\right)$$

$$\vec{E}_{out}(\vec{r}) = \hat{r}E_0\cos\phi\left(-\partial_r\left(-r + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r}\right)\right) - \hat{\phi}\frac{E_0}{r}\left(-r + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r}\right)\partial_\phi\cos\phi$$

$$\vec{E}_{out}(\vec{r}) = E_0\cos\phi\left(1 + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r^2}\right)\hat{r} + E_0\left(-1 + \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r^2}\right)\sin\phi\ \hat{\phi}$$

$$\vec{E}_{out}(\vec{r}) = E_0\left(\cos\phi\ \hat{r} - \sin\phi\ \hat{\phi}\right) + E_0\ \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r^2}\left(\cos\phi\ \hat{r} + \sin\phi\ \hat{\phi}\right)$$

The first term is just the external field:

$$\hat{r} = \cos\phi\ \hat{x} + \sin\phi\ \hat{y}, \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}, \cos\phi\ \hat{r} - \sin\phi\ \hat{\phi} = \hat{x}$$

$$E_0\left(\cos\phi\ \hat{r} - \sin\phi\ \hat{\phi}\right) = E_0\,\hat{x}$$

Considering the second term:

$$\cos\phi\ \hat{r} + \sin\phi\ \hat{\phi} = \cos\phi\ \hat{r} + \sin\phi\left(\cos\phi\ \hat{y} - \sin\phi\ \hat{x}\right)$$

$$\cos\phi\ \hat{r} + \sin\phi\ \hat{\phi} = \cos\phi\ \hat{r} + \sin\phi\cos\phi\ \hat{y} - (1 - \cos^2\phi)\hat{x}$$

$$\cos\phi\ \hat{r} + \sin\phi\ \hat{\phi} = \cos\phi\ \hat{r} + \cos\phi\left(\sin\phi\ \hat{y} + \cos\phi\,\hat{x}\right) - \hat{x}$$

$$\cos\phi\ \hat{r} + \sin\phi\ \hat{\phi} = 2\cos\phi\ \hat{r} - \hat{x}$$

The second term becomes:

$$-E_0\ \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r^2}\ \hat{x} + 2E_0\ \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{R^2}{r^2}\cos\phi\ \hat{r}$$

Giving a field in the direction of the external field and radial field, just as found above.