COLLEGE OF ARTS AND SCIENCES
HOMER L. DODGE
DEPARTMENT OF PHYSICS AND ASTRONOMY
The UNIVERSITY of OKLAHOMA

Quantum Mechanics 2

PHYS 5403 Homework Assignment 6

Problems: {1, 2, 3, 4}

Due: April 18, 2022 at 5:00 PM

Student
Taylor Larrechea

Professor
Dr. Bruno Uchoa

Problem 1:

Suppose a plane wave $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$ coming from the left is scattered by a finite range potential in 1D.

(a) Compute the Green's function for a free particle in 1D, and show that it gives

$$G^{(+)}(x,x') = \frac{1}{2ik}e^{ik|x-x'|}.$$

The unperturbed Hamiltonian of a free particle is

$$\mathcal{H} = \frac{\tilde{P}^2}{2m}.$$

If we anticipate elastic scattering our Lipmann-Schwinger equation becomes

$$|\psi^{(\pm)}\rangle = |i\rangle + \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} V|\psi^{(\pm)}\rangle$$

We then find that $\langle x|\psi^{(\pm)}\rangle$ is

$$\langle x|\psi^{(\pm)}\rangle = \langle x|i\rangle + \int d^3x' \left\langle x \middle| \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} \middle| x' \right\rangle \langle x'|V|\psi^{(\pm)}\rangle$$

Where the solution to our term in the integral is a Greens Function

$$G_\pm(x,x') \equiv \frac{\hbar^2}{2m}\left\langle x \middle| \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} \middle| x' \right\rangle \qquad (*)$$

We then expand in a complete set for P, P'

$$G_\pm(x,x') = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\, dp' \langle x|p\rangle\langle p| \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} |p'\rangle\langle p'|x\rangle$$

This of course becomes

$$G_\pm(x,x') = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\, dp' \frac{e^{ipx/\hbar}}{\sqrt{2}} \langle p| \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} |p'\rangle \frac{e^{-ip'x/\hbar}}{\sqrt{2}} \qquad (**)$$

Where the term in the middle is of course

$$\langle p| \frac{1}{E-\mathcal{H}_0 \pm i\varepsilon} |p'\rangle = \frac{\delta(p-p')}{E-\mathcal{H}_0 \pm i\varepsilon}$$

We are also changing the p' in the $\mathcal{H}_0$ of our denominator

Problem 1: Continued

This then means (**) becomes

$$G_{\pm}(x,x') = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dp\, dp' \,\frac{e^{ipx/\hbar}}{\sqrt{2}}\,\frac{e^{-ip'x'/\hbar}}{\sqrt{2}}\,\frac{\delta(p-p')}{E-\mathcal{H}_0 \pm i\varepsilon}$$

And then we have

$$G_{\pm}(x,x') = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\frac{e^{ip'(x-x')/\hbar}}{E-\mathcal{H}_0 \pm i\varepsilon}\,dp'$$

we are defining $\mathcal{H}_0$ as

$$\mathcal{H}_0 = \frac{\hbar^2 p'^2}{2m}$$

We as well know our energy is

$$E = \frac{\hbar^2 k^2}{2m}$$

This allows to re-write our Greens Function as

$$G_{\pm}(x,x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ip'(x-x')/\hbar}}{k^2-p'^2 \pm i\varepsilon}\,dp'$$

We know our function will have a pole at

$$k^2 - p'^2 \pm i\varepsilon = 0 \Rightarrow \bar{p}' = \sqrt{k^2 \pm i\varepsilon} \approx \pm(k^2 \pm i\varepsilon)$$

Looking for $G_+(x,x')$ we then say

$$k_+ = k + i\varepsilon \,,\, -k_+ = -k - i\varepsilon$$

We then look for closure in the upper complex plane with k_+ and we have

$$G_+(x,x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ip'(x-x')/\hbar}}{k^2-p'^2+i\varepsilon}\,dp' = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} dp'\,\frac{e^{ip'(x-x')/\hbar}}{(p+p')(p-p')}$$

using $k = p\hbar$ we then have

$$G_+(x,x') = -\frac{1}{2\pi}\cdot\frac{2\pi i}{1}\,\frac{1}{2k}\,e^{ik(x-x')} = \frac{1}{2ik}\,e^{ik(x-x')}$$

Problem 1: Continued

The closure in the lower complex plane is then the negative of above, so

$$\boxed{G_+(x,x') = \frac{1}{2ik} e^{ik|x-x'|}}$$

(b) In the case of an attractive potential

$$V(x) = -\gamma \frac{\hbar^2}{2m}\delta(x)$$

where $\gamma > 0$, solve the Lipmann-Schwinger equation and compute the reflection and transmission amplitudes of the scattered wave.

Our Lipmann Schwinger equation goes from

$$\langle x|\psi^{(\pm)}\rangle = \langle x|i\rangle + \int d^3x' \, G_\pm(x,x') \langle x'|V|\psi^{(\pm)}\rangle$$

To

$$\langle x|\psi^{(+)}\rangle = \langle x|i\rangle + \frac{2m}{\hbar^2}\int_{-\infty}^{+\infty} dx' \, G_+(x,x') \langle x'| -\frac{\gamma\hbar^2}{2m} |\psi^+\rangle$$

$$= \langle x|i\rangle - \gamma \int_{-\infty}^{+\infty} dx' \, G_+(x,x') \langle x'|\psi^+\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} - \frac{\gamma e^{ikx}}{2ik}\langle x|\psi^+\rangle$$

We then impose $x=0$ to find $\langle 0|\psi^+\rangle$, therefore we then have

$$\langle 0|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}} - \frac{\gamma}{2ik}\langle 0|\psi^{(+)}\rangle \Rightarrow \langle 0|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}\left(1+\frac{\gamma}{2ik}\right)^{-1}$$

We can then say our solution to the Lipmann Schwinger equation is

$$\boxed{\langle x|\psi^{(+)}\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} - \frac{\gamma e^{ikx}}{2ik}\frac{1}{\sqrt{2\pi}}\left(1+\frac{\gamma}{2ik}\right)^{-1}}$$

Where we can then find the reflection and transmission amplitudes with examining when $x>0$ for transmission and $x<0$ for reflection. The transmission and reflection amplitudes are then

$$\boxed{T(k) = \frac{2k}{2k-i\gamma}, \quad R(k) = 1 - \frac{2k}{2k-i\gamma}}$$

Problem 1: Continued

(c) Compute the energy of the bound state in the well. Show that it corresponds to a resonance in the transmission and reflection amplitudes.

Our transmission and reflection amplitudes have a pole at

$$k = \frac{i\gamma}{2}$$

If we take our Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} - \gamma\frac{\hbar^2}{2m}\delta(x)\psi(x) = -E\psi(x)$$

Then it is clear to see the energy of the bound state in the well is

$$\boxed{E = \frac{\hbar^2 \bar{k}^2}{2m}}$$

This then means if we plug in $\bar{k}$ (our pole) we have

$$E = -\frac{\hbar^2\gamma^2}{8m}$$

Which corresponds to a resonance.

Problem 1: Review

Procedure:

- – Begin by expanding the Greens Function $G_{\pm}(x, x')$

  $$G_{\pm}(x, x') \equiv \frac{\hbar^2}{2m}\left\langle x\middle|\frac{1}{E - \mathcal{H}_0 \pm i\mathcal{E}}\middle|x'\right\rangle$$

  in momentum space
  – Begin evaluating the integral and collapse one of the integrals by employing the Dirac delta function
  – Determine where the function has a pole and then look for closure in the upper complex plane and the lower complex plane
  – Sum the two answers to find a final expression
- – Take the Lipmann Schwinger equation

  $$|\psi^{\pm}\rangle = \langle x|i\rangle + \int d^3x'\, G_{\pm}(x, x')\, \langle x'|V|\psi^{\pm}\rangle$$

  and put in the attractive potential
  – Solve for $\langle x|\psi^{\pm}\rangle$ by setting $x = 0$ in the L.S equation
  – Determine when $x > 0$ to find the transmission amplitude and then use $R = 1 - T$ to find the reflection amplitude
- – Determine where the transmission and reflection amplitudes have a pole and use that with the definition of the energy to find the energy in this well

Key Concepts:

- – By using the Lipmann Schwinger equation we can find the Greens function inside of it by expanding in a complete set and finding closure in both complex planes after find the poles of our function
- – We use initial conditions for our system to find a solution to the Lipmann Schwinger equation
  – The transmission and reflection amplitudes are then found by determining what our solution to the Lipmann Schwinger equation is for certain values of x
- – We find the energy of this bound state by determining the poles of our transmission and reflection coefficients

Variations:

- – We could be asked to find the Greens Function for a different type of situation
  - ∗ We then use the same formalism as before but with our new function
- – We could be given a different potential
  - ∗ We then use this new potential in the same way as the previous potential
- – This part can't really change without asking a brand new question

Problem 2:

The scattering potential of an incoming electron due to a hydrogen atom at the origin is

$$V(\mathbf{x}, \mathbf{x}') = -\frac{e^2}{|\mathbf{x}|} + \frac{e^2}{\mathbf{x} - \mathbf{x}'},$$

where e is the electron charge, $\mathbf{x}$ is the coordinate of the incoming electron, and $\mathbf{x}'$ the co-ordinate of the electron in the orbital $\langle \mathbf{x}'|n, l, m\rangle = R_{n,l}(r')Y^l_m(\theta', \phi')$ centered at the origin. Show that in the first Born approximation, the elastic differential scattering cross section of the incoming electron for a hydrogen atom in the *ground state* $|n, l, m\rangle = |1, 0, 0\rangle$ is

$$\sigma(\mathbf{q}) = \frac{4m^2e^4}{\hbar^4}\frac{1}{q^4}\left[1 - \frac{16}{(4 + q^2a_0^2)^2}\right]^2,$$

where $\hbar\mathbf{q} = \hbar(\mathbf{k} - \mathbf{k}')$ s the momentum transferred and a_0 is the Bohr radius. Hint: Calculate the scattering amplitude for the two particle state $|\mathbf{k}\rangle\,|n, l, m\rangle$.

we first start by calculating the scattering amplitude for our potential with the two particle state

$$f(k', k) = \langle k', 1 | V | 1, k \rangle$$

Which for us becomes

$$f(k', k) = \langle k', 1 | \frac{-e^2}{|x|} + \frac{e^2}{|x - x'|} | 1, k \rangle$$

$$= e^2 \Big(\underbrace{\langle k', 1 | \frac{1}{|x - x'|} | 1, k \rangle}_{A} - \underbrace{\langle k', 1 | \frac{1}{|x|} | 1, k \rangle}_{B} \Big)$$

Let us first calculate the matrix element B,

$$\langle k', 1 | \frac{1}{|x|} | 1, k \rangle = \iint d^3x \, d^3x' \, \langle k', 1 | \frac{1}{|x|} | x, x' \rangle \langle x, x' | k', 1 \rangle$$

Using sakurai we know that $\langle x, x' | k', 1 \rangle$ is

$$\langle x, x' | k', 1 \rangle = \frac{e^{(i k \cdot x)}}{\sqrt{\pi} a_0^{3/2} L^{3/2}} e^{(-r'/a_0)}$$

Using the above we can re-write matrix element B as

$$\langle k', 1 | \frac{1}{|x|} | 1, k \rangle = \frac{1}{\pi a_0^3 L^3} \int d^3x \, \frac{e^{(i(k - k') \cdot x)}}{|x|} \int d^3x' \, e^{(-2r/a_0)}$$

$$= \frac{1}{\cancel{\pi a_0^3} L^3} \left(\frac{4\pi}{|k - k'|^2} \right) (\cancel{\pi a_0^3}) = \frac{4\pi}{L^3 |k - k'|^2} = \frac{4\pi}{L^3 q^2}$$

Problem 2: Continued

Where we used common Gaussian integrals to evaluate the above. We made the notational change

$$q \equiv |k - k'|.$$

Calculating matrix element A we have

$$\langle k',1| \frac{1}{|x-x'|} |k,1\rangle = \frac{1}{\pi a_0^3 L^3} \int d^3x\, e^{(iq\cdot x)} \int d^3x' \frac{e^{(-2r'/a_0)}}{|x-x'|}$$

$$= \frac{1}{\pi a_0^3 L^3} \int d^3x\, e^{(iq\cdot(x-x'))} \int d^3x'\, e^{iq\cdot x'} \frac{e^{(-2r'/a_0)}}{|x-x'|}$$

we then make the substitution : $x'' = x - x'$, we then have

$$\langle k',1| \frac{1}{|x-x'|} |k,1\rangle = \frac{1}{\pi a_0^3 L^3} \int d^3x'' \frac{e^{iq\cdot x''}}{|x''|} \int d^3x'\, e^{iq\cdot x'} e^{-2r'/a_0}$$

$$= \frac{1}{\pi a_0^3 L^3} \left(\frac{4\pi}{q^2}\right) \int d^3x'\, e^{iqr'\cos\theta} e^{-2r'/a_0}$$

$$= \frac{4}{q^2 a_0^3 L^3} \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos(\alpha)) \int_0^\infty r'^2 e^{r'(i\cos(\alpha) - 2/a_0)} dr'$$

$$= \frac{4}{q^2 a_0^3 L^3} \cdot (2\pi) \cdot \frac{2}{q} \int_0^\infty \sin(qr')\, r' e^{-2r'/a_0} dr'$$

$$= \frac{16\pi}{q^3 a_0^3 L^3} \int_0^\infty r' \sin(qr') e^{-2r'/a_0} dr' = \frac{16\pi}{q^3 a_0^3 L^3} \left(\frac{4a_0^3 q}{(a_0^2 q^2 + 4)^2}\right)$$

We then can say our scattering amplitude is

$$f(k,k') = -\frac{4\pi e^2}{L^3 q^2} + \frac{64\pi e^2}{L^3 q^2} \frac{1}{(a_0^2 q^2 + 4)^2} = -\frac{4\pi e^2}{L^3 q^2}\left(1 - \frac{16}{(a_0^2 q^2 + 4)^2}\right)$$

Which means the differential cross section scattering is

$$\sigma(q) = \cancel{L^6} \left| \frac{1}{\cancel{4\pi}} \frac{2m}{\hbar^2} \frac{\cancel{4\pi} e^2}{\cancel{L^3} q^2}\left(1 - \frac{16}{(a_0^2 q^2 + 4)^2}\right)\right|^2 = \frac{4m^2 e^4}{\hbar^4 q^4}\left(1 - \frac{16}{(a_0^2 q^2 + 4)^2}\right)^2$$

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Problem 2: Review

Procedure:

- - Calculate the scattering amplitude with
    $$f(k',k) = \langle k',1|V|1,k\rangle$$
    by splitting the calculation into two parts
  - Use the function $\langle x,x'|k',1\rangle$ with the above equation and potential to determine the scattering amplitude
  - Calculate the differential cross section scattering with
    $$\sigma(q) = |f(k',k)|^2 \equiv \frac{d\sigma}{d\Omega}$$
    It should be noted that the extra term in the evaluation is from the fact that we are doing this for the first Born approximation

Key Concepts:

- - We first calculate the scattering amplitude using the wave function in Sakurai
  - We then proceed to calculate the differential cross section by knowing the scattering amplitude of our particle

Variations:

- - The biggest way this problem can change is if we are given a different potential
    * We then use the same procedure but with different values

Problem 3:

Define the Green's function operators

$$G_0^{\pm}(E) \equiv (E - \mathcal{H}_0 \pm i0_+)^{-1},$$

and

$$G^{\pm}(E) \equiv (E - \mathcal{H} \pm i0_+)^{-1},$$

where $\mathcal{H} = \mathcal{H}_0 + V$, where V is the perturbation, and $\mathcal{H}_0$ the unperturbed Hamiltonian.

(a) Show that:

$$G^{\pm}(E) = G_0^{\pm}(E)\Big[1 + VG^{\pm}(E)\Big].$$

To prove the relationship

$$G^{\pm}(E) = G_0^{\pm}(E)[1 + VG^{\pm}(E)]$$

we substitute in the relationships and show that

$$G^{\pm}(E) = \frac{1}{E - \mathcal{H}_0 \pm i0_+}\left(1 + (\mathcal{H} - \mathcal{H}_0)G^{\pm}(E)\right) = \frac{1}{E - \mathcal{H}_0 \pm i0_+}\left(1 + \frac{\mathcal{H} - \mathcal{H}_0}{E - \mathcal{H} \pm i0_+}\right)$$

$$= \frac{1}{E - \mathcal{H}_0 \pm i0_+}\frac{(E - \cancel{\mathcal{H}} \pm i0_+ + \cancel{\mathcal{H}} - \mathcal{H}_0)}{E - \mathcal{H} \pm i0_+} = \frac{1}{E - \cancel{\mathcal{H}_0 \pm i0_+}}\frac{(E - \cancel{\mathcal{H}_0 \pm i0_+})}{E - \mathcal{H} \pm i0_+}$$

$$= \frac{1}{E - \mathcal{H} \pm i0_+} = G^{\pm}(E) \checkmark$$

$\blacksquare$

(b) Using the relation above, show that the equation

$$|\psi_{\mathbf{k}}^{\pm}\rangle = |\mathbf{k}\rangle + G^{\pm}(E)V\,|\mathbf{k}\rangle$$

is equivalent to the Lipmann-Schwinger equation

$$|\psi_{\mathbf{k}}^{\pm}\rangle = |\mathbf{k}\rangle + G_0^{\pm}(E)V\,|\psi_{\mathbf{k}}^{\pm}\rangle\,.$$

Starting with the ket

$$|\psi_k^{\pm}\rangle = |k\rangle + G^{\pm}(E)V|k\rangle$$

we first examine $G^{\pm}(E)V|k\rangle$,

$$G^{\pm}(E)V|k\rangle = G_0^{\pm}(E)(1 + VG^{\pm}(E))V|k\rangle$$

$$= G_0^{\pm}(E)V\underbrace{(|k\rangle + G^{\pm}(E)V|k\rangle)}_{|\psi_k^{\pm}\rangle} = G_0^{\pm}(E)V|\psi_k^{\pm}\rangle$$

Problem 3: Continued

Using the above we can then say

$$|\psi_k^{\pm}\rangle = |k\rangle + G^{\pm}(E)V|k\rangle = |k\rangle + G_0^{\pm}(E)V|\psi_k^{\pm}\rangle$$

$\blacksquare$

(c) Show that the scattered modes follow the orthogonality property:

$$\langle\psi_{\mathbf{k}'}^{\pm}|\psi_{\mathbf{k}}^{\pm}\rangle = \delta(\mathbf{k}-\mathbf{k}').$$

We first define each bra and ket to prove this relationship

$$|\psi_k^{\pm}\rangle = (\mathbb{1} + G^{\pm}(E)V)|k\rangle \quad , \quad \langle\psi_{k'}^{\pm}| = \langle k'| + \langle\psi_{k'}^{\pm}|G_0^{\pm}(E)^{\dagger}V^{\dagger}$$

We then take the inner product of these two to show

$$\langle\psi_{k'}^{\pm}|\psi_k^{\pm}\rangle = \langle\psi_{k'}^{\pm}|k\rangle + G^{\pm}(E)\langle\psi_{k'}^{\pm}|V|k\rangle \qquad (*)$$

We then calculate each component of $(*)$ first with $\langle\psi_{k'}^{\pm}|k\rangle$

$$\langle\psi_{k'}^{\pm}|k\rangle = \langle k'|k\rangle + \langle\psi_{k'}^{\pm}|G_0^{\pm}(E)^{\dagger}V^{\dagger}|k\rangle \qquad (**)$$

We then can say

$$G_0^{\pm}(E)^{\dagger}V^{\dagger} = VG_0^{\mp}(E) \Rightarrow G^{\pm}(E)^{\dagger}V^{\dagger} = VG^{\mp}(E)$$

This then means $(**)$ becomes

$$\langle\psi_{k'}^{\pm}|k\rangle = \delta^3(k-k') + G_0^{\mp}(E)\langle\psi_{k'}^{\pm}|V|k\rangle = \delta^3(k-k') - G^{\pm}(E)\langle\psi_{k'}^{\pm}|V|k\rangle$$

Where we have taken advantage of the eigenvalue relationship for H and H_0 to make a relationship for $G_0^{\mp}(E)$ and $G^{\pm}(E)$. $(*)$ then becomes

$$\langle\psi_{k'}^{\pm}|\psi_k^{\pm}\rangle = \delta^3(k-k') - G^{\pm}(E)\langle\cancel{\psi_{k'}^{\pm}}|V|k\rangle + G^{\pm}(E)\langle\cancel{\psi_{k'}^{\pm}}|V|k\rangle = \delta^3(k-k')$$

$\blacksquare$

(d) If $k' \neq k$, show that

$$\langle\psi_{\mathbf{k}'}^{-}|V|\mathbf{k}\rangle = \langle\mathbf{k}'|V|\psi_{\mathbf{k}}^{+}\rangle\,.$$

We first start with the LHS

$$\langle\psi_{k'}^{-}| = \langle k'| + \langle k'|VG^{+}(E)$$

Problem 3: Continued

Where we now evaluate

$$\langle \psi^-_{k'}|V|k\rangle = \left(\langle k'| + \langle k'|VG^+(E)\right)V|k\rangle = \langle k'|V|k\rangle + \langle k'|VG^+(E)V|k\rangle$$

Looking at the RHS we have

$$|\psi^+_k\rangle = |k\rangle + G^\pm(E)V|k\rangle$$

That then means

$$\langle k'|V|\psi^+_k\rangle = \langle k'|V\left(|k\rangle + G^+(E)V|k\rangle\right) = \langle k'|V|k\rangle + \langle k'|VG^+(E)V|k\rangle$$

Which then means for our relationship

$$\langle \psi^-_{k'}|V|k\rangle = \langle k'|V|\psi^+_k\rangle$$

$$\left(\langle k'| + \langle k'|VG^+(E)\right)V|k\rangle = \langle k'|V\left(|k\rangle + G^+(E)V|k\rangle\right)$$

$$\langle k'|\cancel{V|k}\rangle + \langle k'|V\cancel{G^+(E)}V|k\rangle = \langle k'|\cancel{V|k}\rangle + \langle k'|V\cancel{G^+}(E)V|k\rangle$$

$$\square$$

Problem 3: Review

Procedure:

- – Use the relationships given to us to show that the expression is valid
- – This is the same procedure as part (a) but with different equations
- – Same as (a) and (b)
- – Same as (a), (b), and (c)

Key Concepts:

- – We use other definitions given to us to show that the expression we are asked to prove is valid
- – Because we have previously defined what $|\psi_k^{\pm}\rangle$ is we can show that the expression we are given is valid
- – After taking the inner product of these wave functions we can create relationships between $\mathcal{H}$ and $\mathcal{H}_0$ that allow us to then create relationships for $G_0^{\pm}(x, x')$ and $G^{\pm}(x, x')$
- – We simply show that with specific wave functions and potentials the matrix elements of these will be equal to a different combination

Variations:

- – The only way these parts change is if we are asked to prove different quantities
  - * This is contingent upon the definitions of our functions, where we would use the same procedure but with these new functions

Problem 4:

(a) Using the first Born approximation, compute the differential cross section for the scattering of two identical particles with mass m. Assume that the spin part of the two body wave function is symmetric under the exchange of the particles. The two particles interact through the potential

$$V(r) = V_0 \exp\left[-(|\mathbf{x}_1 - \mathbf{x}_2|/a)^2\right]$$

where x_i, $i = 1, 2$ label their position.

Using the First order Born Approximation we have our Scattering amplitude defined as

$$f(k',k) \approx -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3r \exp(i\vec{q}\cdot\vec{r})\, V_0 \exp(-(r/a)^2)$$

Where we of course define $q \equiv k - k'$ and $r \equiv x - x'$. If we start to evaluate the above expression we find

$$f(k,k') = -\frac{2m}{\hbar^2}\frac{V_0}{q}\int_0^\infty r\cdot\sin(qr)\, e^{-r^2/a^2}\, dr$$

Due to spherical symmetry. The scattering amplitude is then

$$f(k,k') = -\frac{2m}{\hbar^2}\frac{V_0}{\cancel{q}}\cdot\sqrt{\frac{\pi}{16}}\,\frac{\cancel{q}\cdot\exp(-\tfrac{1}{4}a^2q^2)}{\left(\frac{1}{b^2}\right)^{3/2}} = -\frac{mV_0\sqrt{\pi}}{2\hbar^2}\, a^3 e^{-\frac{a^2q^2}{4}}$$

The differential cross section is then calculated with

$$\frac{d\sigma}{d\Omega} = |f(\theta) \pm f(\varphi)|^2$$

Where for the scattered wave we denote φ with $q' \equiv 2k\sin\varphi/2$. This then means

$$\frac{d\sigma}{d\Omega} = \left|-\frac{mV_0a^3\sqrt{\pi}}{2\hbar^2}\left(e^{-\overbrace{\frac{a^2q^2}{4}}^{x}} \pm e^{-\overbrace{\frac{a^2q'^2}{4}}^{y}}\right)\right|^2 = \frac{m^2V_0^2a^6\pi}{4\hbar^4}(e^{-x}+e^{-y})^2$$

$$= \frac{m^2V_0^2a^6\pi}{4\hbar^4}(e^{-2x}+e^{-x-y}+e^{-x-y}+e^{-2y}) = \frac{m^2V_0^2a^6\pi}{4\hbar^4}(e^{-2x}+2e^{-x-y}+e^{-2y})$$

$$= \frac{m^2V_0^2a^6\pi}{4\hbar^4}\left(e^{-a^2q^2/2}+2e^{-a^2(q^2+q'^2)/4}+e^{-a^2q'^2/2}\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2V_0^2a^6\pi}{4\hbar^4}\left(e^{-\frac{a^2q^2}{2}}+2e^{\frac{-a^2(q^2+q'^2)}{4}}+e^{-\frac{a^2q'^2}{2}}\right)}$$

Problem 4: Continued

(b) Using the definition of the partial wave amplitude:

$$f_l = -\frac{\pi}{k} T_l(E),$$

where

$$T_l(E) = \langle E, l, m | T | E, l, m \rangle ,$$

is the matrix element of the transmission matrix, show that the phase shift in the first Born approximation is given by

$$\delta_l(k) = -\frac{2}{\hbar^2} mk \int_0^\infty dr\, r^2 [j_l(kr)]^2 U(r),$$

for an arbitrary (but small) potential $U(r)$, where $j_l(x)$ is a spherical Bessel function.

In the First Order Born Approximation we know $T \approx U$ so

$$T_l(E) \approx -\frac{\pi}{k} \langle E, l, m | U | E, l, m \rangle$$

We then expand in a complete set for x and x'

$$f_l(E) = \frac{-\pi}{k} \iint d^3x\, d^3x' \langle E, l, m | x \rangle \langle x | U | x' \rangle \langle x' | E, l, m \rangle$$

We know that

$$\langle x' | E, l, m \rangle = \frac{i^l}{\hbar} \sqrt{\frac{2mk}{\pi}}\, j_l(kr') Y_l^m(r') \Rightarrow \langle E, l, m | x \rangle = \frac{(-i)^l}{\hbar} \sqrt{\frac{2mk}{\pi}}\, j_l^*(kr') Y_l^{m*}(r')$$

This then means $f_l(E)$ becomes

$$f_l(E) = \frac{-\pi}{k} \iint d^3x\, d^3x' \frac{i^l}{\hbar} \sqrt{\frac{2mk}{\pi}}\, j_l(kr') Y_l^m(r') \langle x | U | x' \rangle \frac{(-i)^l}{\hbar} \sqrt{\frac{2mk}{\pi}}\, j_l^*(kr') Y_l^{m*}(r')$$

$$= -\frac{\cancel{\pi}}{\cancel{k}} \cdot \frac{2mk}{\cancel{\pi}} \cdot \frac{1}{\hbar^2} \iint d^3x\, d^3x' \, |j_l(kr') j_l^*(kr')| \, |Y_l^m(r') Y_l^{m*}(r')| \underbrace{\langle x | U | x' \rangle}_{\delta^3(x-x')}$$

$$= -\frac{2m}{\hbar^2} \int d^3x\, |j_l(kr)|^2 |Y_l^m(r)|^2 U(r)$$

$$= -\frac{2mk}{\hbar^2} \int dr\, r^2 |j_l(kr)|^2 U(r)$$

$\blacksquare$

Where we used $\int |Y_l^m(r)|^2 d\Omega = 1.$

Problem 4: Review

Procedure:

- – Begin with the equation for the scattering amplitude from the first order Born Approximation

  $$f(k',k) \approx -\frac{2m}{\hbar^2}\frac{1}{4\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}V_0 e^{-(r/a)^2}$$

  and proceed to calculate

  – Calculate the differential cross section scattering with

  $$\frac{d\sigma}{d\Omega} = |f(\theta) \pm f(\phi)|^2$$

  where $f(\theta)$ is the incoming particle and $f(\phi)$ is the scattered particle

- – Start with the expression

  $$T_l(E) = \langle E,l,m|T|E,l,m\rangle$$

  and expand in a complete set for x and x'

  – Use the relationship

  $$\langle x'|E,l,m\rangle = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}}j_l(kr')Y_l^m(r')$$

  and show that the expression that is given to us can be found

Key Concepts:

- – We have to take into account the particle that is incoming and the particle that is outgoing when calculating the differential cross section due to our particles being identical

- – Because we can express $\langle x'|E,l,m\rangle$ in terms of spherical Bessel functions we are able to show that the expression we are given is true

  – It is always true that

  $$|Y_l^m(r)|^2\, d\Omega = 1$$

Variations:

- – We can see that the only real way to change this problem is to change the potential
  - ∗ This then changes the functions we do the math with but not the procedure of what math to use