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Math Methods in Physics

CH. 4 VECTOR SPACES IN PHYSICS LECTURE NOTES

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The return of vector spaces

⇝ Inner product

⇝ Applications to Functions

⇝ Rayleigh Schrödinger principles

⇝ Pertubation theory

Now that we have vector spaces of finite and possibly infinite dimensional, it would be useful to generalize some concepts from our physical or cartesian vectors.

How do we define length of a vector?

Given $\vec{x}$, the length of $\vec{x}$: $\|\vec{x}\|^2 = \sum_i x_i^2$

We want to generalize this to complex vector spaces

Given $\vec{z}$, the length of $\vec{z}$: $\|\vec{z}\|^2 = \sum_i z_i^* z_i$, $\vec{z} = \vec{x} + i\vec{y}$ w/ $x, y \in \mathbb{R}$ & $i = \sqrt{-1}$

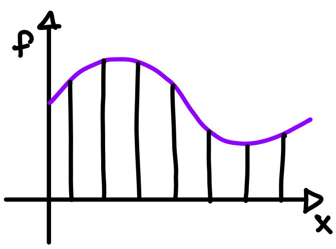
Then $\vec{z}^* = \vec{x} - i\vec{y}$

$\vec{x} \cdot \vec{y} = \sum_i x_i y_i \longrightarrow (u, v) = \sum_i u_i^* v_i \longrightarrow$ Lose commutativity $\neq (v, u)$

$(u, v)^* = (v, u)$

What about functions? How do we take an inner product of functions?

Recall, we can discretize continuous operator & $f(x) \to f(x_i) \to f_i$



Then our discretized function inner product: $(f, g) \longrightarrow \sum_i f_i^* g_i \, \Delta x$

$(f, g) = \int f^*(x) g(x) \, dx$

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Inner Products: $(\vec{x}, \vec{y}) = x_i^* y_i$ & $(f(x), g(x)) = \int_{x_i}^{x_f} f^*(x) g(x) \, dx$ are examples of inner products.

An inner product is a form that produces a scalar from two inputs in the vector space.

It must satisfy:

(1) $(x, y)^* = (y, x)$

(2) Linearity ~ $(\alpha x + \beta y) = \alpha^*(x,z) + \beta^*(y,z)$ For all scalars α, β, & $x, y, z \in V(F)$

(3) $(x,x) \geq 0 \ \forall \ x \in V(F)$ and $= 0$ if and only if $\vec{x} = 0$

Other examples,

$(x,y) = x^{(i)} g_{ij} y^{(j)}$, $(x,x) = x^{(i)} x^{(j)} g_{ij}$ of $(f(x), g(x)) = \int_0^\infty e^{-x^2} f(x) g^*(x)\, dx$ for real fxn's f & g.

$(f(x), g(x)) = \int_0^{2\pi} \sin(x) f^*(x) g(x)\, dx$, this can be negative .....

$(x^{(u)}, x^{(v)}) = L_{uv} x^{(u)} x^{(v)}$ For the Lorentz Matrix $\rightarrow$ time like intervals are negative

We define "orthogonality" in terms of the inner products. A set of vectors is orthonormal if

$(x_i, x_j) = \delta_{ij}$

Given a set of vectors ....

$\rightarrow$ How can we get an orthonormal set of basis from this,

$\rightarrow$ Give as a check on linear independence.

Given a set of vectors $\{y_i\}$ we construct an orthonormal set .....

$x_1 = \dfrac{y_1}{\|y_1\|} = \dfrac{y_1}{(y_1, y_1)^{1/2}}$, $\tilde{x}_2 = y_2 - (x_1, y_2) x_1$

Note $(x_1, \tilde{x}_2) = (x_1, y_2) - (x_1, y_2)(x_1, x_1)$

$x_2 = \dfrac{\tilde{x}_2}{\|\tilde{x}_2\|} = \dfrac{\tilde{x}_2}{(\tilde{x}_2, \tilde{x}_2)^{1/2}}$, $\tilde{x}_3 = y_3 - (x_1, y_3) x_1 - (x_2, y_3) x_2$, $x_3 = \dfrac{\tilde{x}_3}{\|\tilde{x}_3\|}$

$\tilde{x}_n = y_n - \sum_{i=1}^{n-1} (x_i, y_n) x_i$, $x_n = \dfrac{\tilde{x}_n}{\|\tilde{x}_n\|}$ $\longrightarrow$ Gram Schmdt Orthonormalization

what if we get $\tilde{x}_K = 0$? We also would like our basis to be complete so that For all y & $z \in V(F)$ our $\{x_i\}$ are complete if $(y, z) = \sum_{i=1}^{n} (y, x_i)(x_i, z)$

Schwartz Inequality : $|(x,y)| \leq \|x\| \, \|y\|$

We can ask the following question : Given an operator $\mathbb{A}$ does there exist an operator $\mathbb{B}$ such that

$(x, \mathbb{A} y) = (\mathbb{B} x, y)$

Returning to our idea that a is the matrix operator of $\mathbb{A}$ then by assumption

$(x_i, Ax_j) = a_{ij}$, $(x_i, \mathbb{B}x_j) = b_{ij}$

$b_{ij} = (\mathbb{B}x_j, x_i)^* = (x_j, Ax_i)^* = a_{ji}^*$, $b = (a^T)^* \Rightarrow$ adjoint $\Rightarrow a^\dagger$

If $A^\dagger = A$ then A is self adjoint. If A is real and self adjoint then A is symmetric.

Complex & self adjoint is called Hermitian.

While this true and simple for matrices we can do this for any linear operator.

$(f,g) = \int_0^L f,g \, dx$ $\quad f,g \in \mathbb{R}$ then

$(f, \mathbb{D}g) = \int_0^L f(x) \, d/dx \, g(x) \, dx = -\int d/dx \, f(x) + f(x)g(x)\Big|_0^L$ then

$\int ((-d/dx + \delta(x-L) - \delta(x)) f(x)) g(x) \, dx$, $\mathbb{D}^* = -\frac{d}{dx} + \delta(x-L) - \delta(x)$, $(\mathbb{D}^* f, g) = (f, \mathbb{D}g)$

11-1-21

Adjoint operators obey : $(A + B)^* = A^* + B^*$, $(AB)^* = B^* A^*$, $(A^*)^* = A$

If A and B are self-adjoint then $A \cdot B$ are self-adjont. If for a self-adjoint operator $(x, Ax) = 0$ for all x then $A = 0$.

Important Result : A Hermitian linear operator has only real eigenvalues!

$(x, Ax) = (A^* x, x)$, $(x, Ax) = (Ax, x)$, $\lambda(x, x) = \lambda^*(x, x)$ $\therefore$ $\lambda = \lambda^*$

If $U^\dagger U = \mathbb{I}$ then U is an isometric transformation $\rightarrow$ Lengths don't change.

If $UU^\dagger = \mathbb{I} \longrightarrow U$ is unitary , If U is finite dimensional and $U^\dagger U = \mathbb{I}$ then $UU^\dagger = \mathbb{I}$

Another Important Result : If A is self-adjoint then the eigenvectors of distinct (unequal) eigenvalues and orthogonal.

$Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$ then $(x_1, x_2) = 0$

$(x_1, Ax_2) = (Ax_1, x_2), (x_1, \lambda_2 x_2) = (\lambda_1 x_1, x_2), \lambda_2(x_1, x_2) = \lambda_1(x_1, x_2), (\lambda_2 - \lambda_1)(x_1, x_2) = 0$

If $\lambda_1 = \lambda_2$ and $x_1 \neq x_2$ then Gram-Schmidt orthogonalization will give you an orthogonal set.

Then these, with the other eigenvectors will form an orthogonal & complete set.

Another Important Result : Let A be a linear operator on a finite dimensional "vector" space V_0 Iff $Ax = 0$ implies that $x = 0$ then A^{-1} exists. We say x is a null vector for A.

Minimization

Assume that we have a Hermitian operator A, then each normalized vector y that extremizes (y, Ay) is an eigenvector of A!

Informal Proof – Assume that we know all of the eigenvectors & values of A

$$\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots\ldots$$

with eigenvectors $x_1, x_2, x_3 \ldots\ldots$ Then any y can be written as

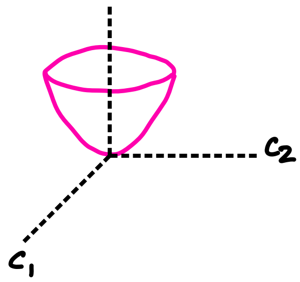
$$y = c_0 x_0 + c_1 x_1 + \ldots\ldots = \sum c_j x_j ,$$

Then $(y, Ay) = (\sum c_j x_j , A \sum c_k x_k) = \sum_{kj} c_j^* c_k (x_j, \lambda_k x_k) = \sum_{jk} c_j^* c_k \lambda_k (x_j, x_k)$

And finally $\sum_j |c_j|^2 A_j$ for any y. $\sum |c_j|^2 = 1$

Pictorially in some multi dimensional space

$$\frac{\langle y, Ay \rangle}{\langle y, y \rangle}$$



In Quantum Mechanics if you have no idea or can not find the groundstate wave function & energy then you can make a guess. If in 3D with one particle

$\psi(\vec{r}; \alpha_1, \alpha_2, \alpha_3) \rightsquigarrow$ Then you minimize $\dfrac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$ & this gives you a boundary on E_0!

11-3-21

If $\mathcal{U}$ is finite dimensional then $\mathcal{U}^\dagger \mathcal{U} = \mathbb{1}$ implies $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$. Infinite dimensional counter-example: Let V be the space of infinite normalizable vectors

$\bar{x} = (x_1, x_2, x_3, \ldots\ldots) \longrightarrow (\vec{x}, \vec{x}) = 1 = \sum x_i^2$. Then define the shift right operator

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \end{pmatrix} = \mathbb{R}\vec{x} , \quad (\vec{x}, \mathbb{R}\vec{x}) = (\mathbb{L}\vec{x}, \vec{x}) , \quad (\mathbb{L}\vec{x}, \mathbb{R}\vec{x}) = (\vec{x}, \vec{x})$$

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \end{pmatrix}$$

Minimization: Given argument, assume we want it minimized $(x, Ax) = \lambda(x,x) = K$

In a sloppy notation: $\frac{\delta K}{\delta x} = Ax - \lambda x = 0 \rightsquigarrow Ax = \lambda x$, x is an eigenvector of A.

More honest proof — $\vec{x} = \Sigma c_i \vec{v}_i$ where $A\vec{v}_i = \lambda_i \vec{v}_i$ & this demands that $\Sigma |c_i|^2 = 1$ and then $\frac{\delta}{\delta c_1}\left((x, Ax) - \lambda(x,x)\right)$

Example: we need to know a little bit about QM & Delta functions Qualifier Q!!

we can get a bound on the ground state energy of a Hamiltonian H by the following:

① Create a guess $\psi(\alpha_1, \alpha_2, \alpha_3, x)$, Make $\langle\psi|\psi\rangle = \int \psi^* \psi \, dx = 1$

② Calculate $\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = E(\alpha_1, \alpha_2, \ldots \alpha_n)$

③ Calculate $\frac{\partial E}{\partial \alpha_i} = 0$ & solve for the $\alpha_i = \tilde{\alpha}_i$ from this equation

④ $E(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots \tilde{\alpha}_n)$ will give you a bound on the ground state energy

Notes:

Ⓐ This assumes that there is a bound energy $> -\infty$

Ⓑ Usually the bound state is unique (non-degenerate)

$$H\psi = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \gamma\delta(x)\right)\psi = E\psi$$ Switch to dimensionless variables

$E = \epsilon E_0$, $x = x_0 s$

$$= \left(\frac{-\hbar^2}{2m}\frac{1}{x_0^2}\frac{\partial^2}{\partial s^2} - \gamma\delta(x_0)\right)\psi = E_0\epsilon\psi = \left(\frac{-\hbar^2}{2m x_0^2}\frac{\partial^2}{\partial s^2} - \frac{\gamma}{x_0}\delta(x)\right)\psi = E_0\epsilon\psi$$

$$\frac{\hbar^2}{2m}\frac{1}{x_0^2} = \frac{\gamma}{x_0} \rightsquigarrow x_0 = \frac{\hbar^2}{2m}\frac{1}{\gamma} \qquad E_0 = \frac{\gamma}{x_0} = \frac{\gamma^2}{\hbar^2/2m} \quad \therefore -\frac{\partial^2}{\partial s^2}\psi - \delta(s)\psi = \epsilon\psi$$

Guess that $\psi = (\pi\alpha)^{-1/4} e^{-s^2/2\alpha} \rightsquigarrow \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*\psi \, ds = 1$

$$(\psi, H\psi): \frac{\partial}{\partial s}\psi = \frac{1}{(\pi\alpha)^{1/4}}\left(-\frac{s}{\alpha}e^{-s^2/2\alpha}\right) = \frac{-1}{(\pi)^{1/4}}\frac{s}{\alpha^{5/4}} e^{-s^2/2\alpha}$$

$$\frac{\partial^2}{\partial s^2}\psi = \frac{-1}{(\pi)^{1/4}}\left(\frac{1}{\alpha^{5/4}} - \frac{s}{\alpha^{5/4}}\left(\frac{s}{\alpha}\right)\right)e^{-s^2/2\alpha} = \frac{1}{\pi^{1/4}}\left(\frac{1}{\alpha^{5/4}} - \frac{s^2}{\alpha^{9/4}}\right)e^{-s^2/2\alpha}$$

$$-(\psi, \partial^2/\partial s^2 \psi) = \frac{1}{\sqrt{\pi}}\frac{1}{\alpha^{1/4}}\int\left(\frac{1}{\alpha^{5/4}} - \frac{s^2}{\alpha^{9/4}}\right)e^{-s^2/\alpha}\,ds = \frac{1}{\sqrt{\pi}}\int\left(\frac{1}{\alpha^{3/2}} - \frac{s^2}{\alpha^{5/2}}\right)e^{-s^2/\alpha}\,ds$$

$$= \frac{1}{\sqrt{\pi}}\left\{\frac{1}{\alpha^{3/2}}\sqrt{\pi\alpha} - \frac{1}{\alpha^{5/2}}\cdot\frac{1}{2}\sqrt{\pi}\,\alpha^{3/2}\right\} = \frac{1}{\alpha} - \frac{1}{2\alpha} = \frac{1}{2\alpha}$$

$$-(\psi, \delta(s)\psi) = -\int \frac{1}{\sqrt{\pi\alpha}} e^{-s^2/\alpha} \delta(s)\, ds = -\frac{1}{\sqrt{\pi\alpha}}$$

$$E(\alpha) = \frac{1}{2\alpha} - \frac{1}{\sqrt{\pi\alpha}}, \quad \frac{\partial E}{\partial\alpha} = -\frac{1}{2\alpha^2} + \frac{1}{2\sqrt{\pi}\alpha^{3/2}} = 0 \quad \frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{\pi}}, \quad \alpha = \pi$$

$$E(\alpha=\pi) = \frac{1}{2\pi} - \frac{1}{\pi} = -\frac{1}{2\pi}, \quad \text{Ans: } \epsilon = -\frac{1}{4}$$

11-5-21

Given: $\left[\frac{\partial^2}{\partial s^2} + \delta(s)\right]\psi(s) = \epsilon\psi(s)$ w/ $\psi = e^{-|s|/2}$ is a solution

Proof: $\frac{\partial}{\partial s}\psi(s) = -\frac{1}{2}\,\mathrm{sgn}(x)e^{-|s|/2}, \quad \frac{\partial^2}{\partial x^2} = \frac{1}{4}\mathrm{sgn}(x)^2 e^{-|s|/2} - \delta(x)e^{-|s|/2}$

so,

$$\left[\frac{\partial^2}{\partial s^2} + \delta(s)\right]e^{-|s|/2} = \frac{-1}{4}e^{-|s|/2}, \quad \epsilon \leq \frac{1}{2\pi}$$

Side comment, Good for energies, not good for ψ.

Solvability

Q: when does on $Mx = y$ (Given M & y) have a solution for the linear operator M?

A: If we know the eigenvectors of M^+, $M^+u_K = \lambda_K u_K$, then we can expand x & y in units of K

$$x = \sum a_i u_i, \quad y = \sum b_i u_i$$

If $Mx = y$, $(u_i, Mx) = (M^+u_i, x) = (u_i, y) = \lambda_i(u_i, x) = (u_i, y)$

$\lambda_i(u_i, \sum_K a_K u_K) = (u_i, \sum_K b_K u_K)$, $\lambda_i \sum_K a_K(u_i, u_K) = \sum b_K(u_i, u_K)$ $\therefore$

$a_i\lambda_i = b_i \longrightarrow a_i = \frac{b_i}{\lambda_i}$ Unless $\lambda_i = 0$

Let $\tilde{u}_i$ be the set of eigenvectors of M^+ such that $M^+\tilde{u}_K = 0 = \text{"}0\text{"}u_K$
$\lambda_K = 0$

Then the $\{\tilde{u}_K\}$ span the "null" space $\longrightarrow$ we can solve this iff $b_i = 0$ for all $\tilde{u}_i$

y has no component in the null space

Example:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ \& so does } M^+ \text{ w/ } y = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$Mx = y \to \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} \rightsquigarrow x = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1&1\\1&1\end{pmatrix}\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

There is a null vector that is $\begin{pmatrix}1\\-1\end{pmatrix}$, $\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

by y is $\perp$ to $\vec{0}$, $\begin{pmatrix}3\\3\end{pmatrix}^T = \begin{pmatrix}1\\-1\end{pmatrix} = 0$, $x = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} + c\begin{pmatrix}1\\-1\end{pmatrix}$

" when does $Mx = y$ have a solution" $\neq$ when does M have an inverse

Perturbation Theory

often in physics we cannot solve a problem of interest, but we can solve one nearby. That is for some operator A_0 we know everything, the eigenvalues and eigenvectors. unfortunately we want to solve for the eigenvalues and vectors of $A = A_0 + \epsilon A_1$ for ϵ small.

we are going to seek a solution in a power series in ϵ and hope that the first few terms are sufficient to approximate the true answer.

we wish to solve $Ax_n = \lambda_n x_n$, $\lambda_n = \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)} + \dots\dots$
$x_n = x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)} + \dots\dots$ where we know all $\lambda_n^{(0)}$ & $x_n^{(0)}$ where
$A = A_0 + \epsilon A_1$, $A_0 x_n^{(0)} = \lambda_n^{(0)} x_n^{(0)}$.

we assume that $\lambda_0 < \lambda_1 < \lambda_2$ so no λ's are equal. "non-degenerate"
Take our forms for A, x_n & plug them in. Then work order by order in ϵ.

$(A_0 + \epsilon A_1)(x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)}) = \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)} \dots (x_n^{(0)} + \epsilon x_n^{(1)} + \epsilon^2 x_n^{(2)} \dots\dots)$

$A_0 x_n^{(0)} = \lambda_n^{(0)} x_n^{(0)}$, $A_0 x_n^{(0)} + A_1 x_n^{(1)} = \lambda_n^{(0)} x_n^{(1)} + \lambda_n^{(1)} x_n^{(0)}$

11-8-21

Non-degenerate PT. Given $A_0 x_j^{(0)} = \lambda_j x_j^{(0)}$ we want to solve $Ax = \lambda x$ where $A = A_0 + \epsilon A$ & ϵ is small

Power Series Assumption

$$\lambda = \lambda_i^{(0)} + \epsilon \lambda_i^{(1)} + \epsilon^2 \lambda_i^{(2)} + \dots \quad x_i = x_i^{(0)} + \epsilon x_i^{(1)} + \epsilon^2 x_i^{(2)}$$

Insert and collect terms of order ϵ

$u = 0 \quad (A_0 - \lambda_u^{(0)} \mathbb{1}) x_n^{(0)} = 0$, $u = 1 \quad (A_0 - \lambda_u^{(0)} \mathbb{1}) x_n^{(1)} = \lambda_n^{(1)} x_n^{(0)} - A_n x_n^{(0)}$

$u = 2 \quad (A_0 - \lambda_u^{(0)} \mathbb{1}) x_n^{(2)} = -(A_1 - \lambda_u^{(1)} \mathbb{1}) x_n^{(1)} + \lambda_u^{(2)} x_n^{(2)}$

$u = 3 \quad (A_0 - \lambda_u^{(0)}) x_n^{(3)} = -(A_1 - \lambda_u^{(1)}) x_n^{(2)} + \lambda_u^{(2)} x_n^{(1)} + \lambda_u^{(3)} x_n^{(2)}$

$M x_n^{(1)} = y$, $M = (A_0 - \lambda_0 \mathbb{1}) x_n^{(0)}$ we require $(x_n^{(0)}, y) = 0$

$(x_n^{(0)}, \lambda_n^{(1)} x_n^{(0)} - A_1 x_n^{(0)}) \longrightarrow \lambda^{(1)} (x_n^{(0)}, x_n^{(0)}) = (x_n^{(0)}, A_1 x_n^{(0)})$, $\lambda_n^{(1)} = (x_n^{(0)}, A_1 x_n^{(0)})$

BTW : A_0, A_1 are self-adjoint

What about $x_n^{(1)}$? We need some convenient orthonormal basis in what to express $x_n^{(1)}$. $\sim \{x_n^{(0)}\}$

$$x_n^{(1)} = \sum_{j\neq n} \alpha_{n,j} x_j^{(0)}$$

$n=1$, $(x_j^{(0)}, (A_0 - \lambda_n^{(0)} \mathbb{1}) x_n^{(0)}) = (x_j^{(0)}, \lambda^{(1)} x_n^{(0)}) - (x_j^{(0)}, A_1 x_n^{(0)})$

$$(\lambda_j^{(0)} - \lambda_n^{(0)})(x_j^{(0)}, x_n^{(0)}) = \lambda^{(1)} \overset{0}{\cancel{(x_j^{(0)}, x_n^{(0)})}} - (x_j^{(0)}, A_1 x_n^{(0)})$$

$$(\lambda_j^{(0)} - \lambda_n^{(0)}) \alpha_{n,j} = -(x_j^{(0)}, A_1 x_n^{(0)}) , \quad \alpha_{n,j} = \frac{(x_j^{(0)}, A_1 x_n^{(0)})}{\lambda_n^{(0)} - \lambda_j^{(0)}}$$

$$x_n^{(1)} = \sum_{j\neq n} \frac{(x_j^{(0)}, A_1 x_n^{(0)})}{\lambda_n^{(0)} - \lambda_j^{(0)}}$$

$n=2$, $(A_0 - \lambda_n^{(0)} \mathbb{1}) x_n^{(2)} = -(A_1 - \lambda_n^{(1)} \mathbb{1}) x_n^{(1)} + \lambda_n^{(2)} x_n^{(0)}$

RHS must be orthogonal to $x_n^{(0)}$ $\therefore$ $-(x_n^{(0)}, (A_1 - \lambda_n^{(1)} \mathbb{1}) x_n^{(1)}) + \lambda_n^{(2)} (x_n^{(0)}, x_n^{(0)}) = 0$

$\lambda_n^{(2)} = (x_n^{(0)}, A_1 x_n^{(1)}) - \lambda_n^{(1)} (x_n^{(0)}, x_n^{(1)})$ $\therefore$ $\lambda_n^{(2)} = (x_n^{(0)}, A_1 x_n^{(1)})$

$$\lambda_n^{(2)} = \sum_{j\neq n} \frac{|(x_n^{(0)}, A_1 x_j^{(0)})|^2}{\lambda_n^{(0)} - \lambda_j^{(0)}} , \quad x_n^{(1)} = \sum_{j=n} \frac{(x_j^{(0)}, A_1 x_n^{(0)})}{(\lambda_n^{(0)} - \lambda_j^{(0)})}$$

11-10-21

$$A = A_0 + \epsilon A_1$$

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,$$ A, A_0, A_1 are Hermitian

$\lambda_1^{(0)} = -1$, $x_1^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: $\lambda_2^{(0)} = 0$, $x_2^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$: $\lambda_3^{(0)} = 1$, $x_3^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\lambda_n^{(1)} = \langle x_n^{(0)} | A | x_n^{(0)} \rangle$$

$$\lambda_1^{(1)} = \langle x_1^{(0)} | A_1 | x_1^{(0)} \rangle = (1\ 0\ 0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1\ 0\ 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\lambda_1 = -1 + \epsilon(0)$$

$$\lambda_2^{(1)} = \langle x_2^{(0)} | A_1 | x_2^{(0)} \rangle = (0\ 1\ 0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0\ 1\ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$\lambda_2 = 0 + \epsilon(1)$$

$$\lambda_3^{(1)} = \langle x_3^{(0)} | A_1 | x_3^{(0)} \rangle = (0\ 0\ 1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (0\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\lambda_3 = 1 + \epsilon(0)$$

$$X_n^{(1)} = \sum_{j \neq n} \frac{(x_j^{(0)}, A_1 x_n^{(0)})}{\lambda_n^{(0)} - \lambda_j^{(0)}} x_j^{(0)}$$

$$x_1^{(1)} = \sum_{j \neq 1} \frac{(x_j^{(0)}, A_1 x_1^{(0)})}{\lambda_1^{(0)} - \lambda_j^{(0)}} x_j^{(0)} = \sum_{j \neq 1} \frac{(x_2^{(0)}, A_1 x_1^{(0)})}{\lambda_1^{(0)} - \lambda_2^{(0)}} x_2^{(0)} + \frac{(x_3^{(0)}, A_1 x_1^{(0)})}{\lambda_1^{(0)} - \lambda_3^{(0)}} x_3^{(0)}$$

$$x_1^{(1)} = \frac{(0\,1\,0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{-1-0} x_2^{(0)} + \frac{(0\,0\,1)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{-1-(+1)} x_3^{(0)}$$

$$= \frac{(0\,1\,0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{-1} x_2^{(0)} + \frac{(0\,0\,1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{-2} x_3^{(0)} = \frac{-1}{2} x_3^{(0)}$$

$$x_1^{(1)} = x_1^{(0)} - \frac{\epsilon}{2} x_3^{(0)}$$

$$\lambda_n^{(2)} = \sum_{j \neq n} \frac{|(x_j^{(0)}, A_1 x_n^{(0)})|^2}{\lambda_n^{(0)} - \lambda_j^{(0)}}, \quad \lambda_1^{(2)} = \frac{|(x_2^{(0)}, A_1 x_1^{(0)})|^2}{\lambda_1^{(0)} - \lambda_2^{(0)}} + \frac{|(x_3^{(0)}, A_1 x_1^{(0)})|^2}{\lambda_1^{(0)} - \lambda_3^{(0)}}$$

$$\lambda_1^{(2)} = \frac{(0\,1\,0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{-1-0} + \frac{(1\,0\,0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{-1-(+1)}$$

$$\lambda_1^{(2)} = -1\left|(0\,1\,0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 - \frac{1}{2}\left|(0\,0\,1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|^2 = -\frac{1}{2} \quad \therefore \quad \lambda_1^{(2)} = -1 + 0\epsilon - \frac{\epsilon^2}{2}$$

$$\lambda_1^{(2)} = -1 + 0\epsilon - \frac{\epsilon^2}{2}, \quad \lambda_2^{(2)} = 0 + \epsilon + \frac{\epsilon^2}{2}, \quad \lambda_3^{(2)} = 1 + 0\epsilon + \frac{\epsilon^2}{2}$$

$$\det\begin{pmatrix} -1 & 0 & \epsilon \\ 0 & \epsilon & 0 \\ \epsilon & 0 & 1 \end{pmatrix} = 0 \longrightarrow \lambda = \epsilon, \quad \lambda = \pm\sqrt{1+\epsilon^2} = \pm\left(1 + \frac{\epsilon^2}{2} + \frac{\epsilon^4}{4}\right)$$

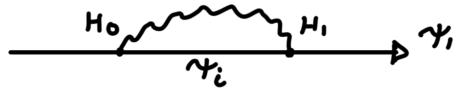
Comments

① Typically one calculates $\lambda^{(1)}$ & $\lambda^{(2)}$ & not $x^{(1)}$ & $x^{(2)}$ as often

② In QM think of PT as first order average value of the perturbation in the unperturbed state

$H = H_0 + H_1$, $E_n^{(1)} = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle \rightsquigarrow$ The second order correction represents fluctuations in $|\langle \psi_f | H | \psi_i \rangle|$

The denominator $\dfrac{1}{E_i - E_f}$ reflects that large fluctuations in energy are suppressed

You will see this represented dyacramactically 

Denerate Perturbation Theory

Again we have Hermitian A_0 & A_1 & $A = A_0 + \epsilon A_1$. But now each eigenvalue $\lambda_K^{(0)}$ has an index associated with it

$$A_0 X_{Ki}^{(0)} = \lambda_K^{(0)} x_{Ki} \quad \text{with} \quad i = 1, 2, \dots \mu_K$$

There are several eigenvectors that all have the same eigenvalues

Recall, $Bx = y$ has a solution when y has no component of the null space.

At first order solvability gave us $(x_n^{(0)}, (A_1 - \lambda_n^{(0)} \mathbb{1}) x_n^{(0)}) = 0$

11-12-21

Degenerate Perturbation Theory:

A_0 & A_1 are Hermitian. We know the $X_{n,i}^{(0)}$ such that $A_0 X_{n,i}^{(0)} = \lambda_n^{(0)} x_{n,i}^{(0)}$ for $i = 1 \dots \mu_n$

Want to solve $A = A_0 + \epsilon A$. Previously solvability required $(x_n^{(0)}, (A_1 - \lambda_n^{(0)}) x_n^{(0)}) = 0$ for $\lambda_n^{(0)}$. Now $(x_{n,j}^{(0)}, A_1 - \lambda_n^{(0)} \mathbb{1}) x_{n,i}^{(0)} = 0$. For all $j \dots$

Remember that the exact set of $x_{n,i}^{(0)}$ are not entirely fixed $\longrightarrow$ Only linear complete combination is also!

$$x_{2,1}^{(0)} \text{ \& } x_{2,2}^{(0)} , \quad \frac{1}{\sqrt{2}} (x_{2,1}^{(0)} \pm x_{2,2}^{(0)})$$

Solution - Find a new set of zeroth order vectors $y_{n,i}^{(0)} = \sum_{j=1}^{\mu} \alpha_{i,j} x_{nj}^{(0)}$, $y_{ni} = y_{ni}^{(0)} + \epsilon y_{ni}^{(0)}$

$$(x_{nj}^{(0)}, (A_1 - \lambda_{n,i}^{(1)} \mathbb{1}) y_{n,i}^{(0)}) = 0 , \quad (x_{n,j}^{(0)}, (A_1 - \lambda_n^{(1)} \mathbb{1}) \sum \alpha_{n,k} x_{n,k}^{(0)}) = 0$$

$$\sum_K (x^{(0)}_{n,j}, A_1 x^{(0)}_{n,k}) = \sum_K \lambda^{(0)}_{n,i} \underbrace{(x^{(0)}_{n,j}, x^{(0)}_{n,k})}_{\delta_{jk}} \alpha_{ik} \quad , \quad \sum_K a_{j,k}\, \alpha_{i,k} = \lambda_{n,i}\, \alpha_{i,k} \quad , \quad a\, \vec{\alpha}_i = \lambda^{(1)}_{n,i}\, \vec{\alpha}_i$$

Example

$$A_0 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad , \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad \det(A_1 - \lambda \mathbb{1}) = (1+\lambda)(1+\lambda)(\lambda-3)$$

$$\lambda_1 = -1 , \; \mu_1 = 2 \; : \; \lambda_2 = 3 , \; \mu_2 = 1$$

$$x^{(0)}_{1,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad x^{(0)}_{1,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad x^{(0)}_{2,1} \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$a_{11} = (x^{(0)}_{1,1}, A_1 x^{(0)}_{1,1}) = \frac{1}{\sqrt{2}} (-1 \; 1 \; 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$a_{12} = (x^{(0)}_{1,1}, A_1 x^{(0)}_{1,2}) = \frac{1}{\sqrt{2}} (-1 \; 1 \; 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} = a_{21}$$

$$a_{22} = (x^{(0)}_{1,2}, A_1 x^{(0)}_{2,1}) = \frac{1}{\sqrt{2}} (1 \; 1 \; 0) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$\lambda^{(0)}_{i,j}$ are the eigenvalues of a. The $y^{(0)}_{i,j}$ are given by eigenvectors

$$\det \begin{pmatrix} \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & \lambda \end{pmatrix} = 0 \quad , \quad \lambda^2 - \frac{1}{2} = 0 \quad \therefore \quad \lambda = \pm \frac{1}{\sqrt{2}}$$

$$\lambda^{(0)}_{1,1} = \frac{-1}{\sqrt{2}} \quad , \quad \vec{\alpha} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \; : \; \lambda^{(0)}_{1,2} = \frac{1}{\sqrt{2}} \quad , \quad \vec{\alpha} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y^{(0)}_{1,1} = -1 \cdot x^{(0)}_{1,1} + 1 \cdot x^{(0)}_{1,2} \; : \; y^{(0)}_{1,2} = 1 \cdot x^{(0)}_{1,1} + 1 \cdot x^{(0)}_{1,2}$$

$$y_{1,1} \approx y^{(0)}_{1,1} + \epsilon\, y^{(0)}_{1,1} \; \ldots\ldots \quad y_{1,1} = \frac{1}{\sqrt{2}} (x^{(0)}_{1,1} + x^{(0)}_{1,2}) + \ldots$$

$$\lambda_{1,1} = \lambda^{(0)}_{1,1} + \epsilon \lambda^{(1)}_{1,1} + \epsilon^2 \lambda^{(1)}_{1,1} = -1 - \frac{\epsilon}{\sqrt{2}} \; \ldots\ldots$$

$$y_{1,2} \approx y^{(0)}_{1,2} + \epsilon\, y^{(0)}_{1,2} \; \ldots\ldots \quad y_{1,2} = \frac{1}{\sqrt{2}} (x^{(0)}_{1,1} + x^{(0)}_{1,2}) + \ldots$$

$$\lambda_{1,2} = \lambda^{(1)}_{1,2} + \epsilon \lambda^{(1)}_{1,2} + \epsilon^2 \lambda^{(1)}_{1,2} = -1 + \frac{\epsilon}{\sqrt{2}} \; \ldots\ldots$$

$$\lambda^{(0)}_2 = -3 \quad , \quad \lambda^{(1)}_2 = (x^{(0)}_2, A_1 x^{(0)}_2) = 0$$

Let's divide and conquer. For all eigenvalues that are not degenerate.

$$x_n^{(1)} = \sum_{kj} \frac{(x_{kj}^{(0)}, A_1 x_n^{(0)})}{\lambda_n^{(0)} - \lambda_k^{(0)}} x_{kj}^{(0)}$$

11-15-21

DPT : The algebra ......

$$A_0 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\lambda_1^{(0)} = -1$, $\mu_1 = 2$, $x_{1,1}^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $x_{1,2}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$: $\lambda_2^{(0)} = 3$, $\mu_2 = 1$, $x_2^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Need now. Zeroth order vectors

$$y_{n,i} = y_{n,i}^{(0)} + \epsilon y_{n,i}^{(1)}, \quad y_{n,i}^{(0)} = \sum \alpha_{ij}^{(1)} x_{n,j}^{(0)}$$

Determined by a_I

$$a_{I\,ij} = (x_{n,i}^{(0)}, A_1 x_{nj}^{(0)}) \longrightarrow \mu_n \times \mu_n \quad \therefore \quad a_I = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_I \vec{d} = \lambda^{(1)} \vec{d}$$

$$\lambda_{1,1}^{(1)} = \frac{1}{\sqrt{2}}, \quad \alpha_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \lambda_{1,2}^{(1)} = -\frac{1}{\sqrt{2}}, \quad \alpha_{1,i} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We say A_1 splits the deneracy

$$\lambda_{1,1} = -1 + \frac{\epsilon}{\sqrt{2}} + \ldots\ldots, \quad \lambda_{1,2} = -1 - \frac{\epsilon}{\sqrt{2}} + \epsilon^2 + \ldots$$

$$y_{(1)} = \frac{1}{\sqrt{2}}(x_{1,1}^{(0)} + x_{1,2}^{(0)}) + \epsilon y_{1,1}^{(1)} + \ldots\ldots$$

Non-denerate states obey <u>NDPT</u> :

$$\lambda_2^{(1)} = (x_2^{(0)}, A_1 x_2^{(0)}), \quad \lambda_2^{(0)} = 3 + 0\epsilon + \ldots\ldots$$

$$x_p^{(1)} = \sum_{ni} \frac{(x_{n,i}^{(0)}, A_1 x_p^{(0)}) x_{n,i}^{(0)}}{\lambda_p^{(0)} - \lambda_n^{(0)}}, \quad x_2^{(1)} = \frac{(x_{1,1}^{(0)}, A_1 x_2^{(0)})}{3-(-1)} x_{1,1}^{(0)} + \frac{(x_{1,2}^{(0)}, A_1 x_2^{(0)})}{3-(-1)} x_{1,2}^{(0)}$$

$$\lambda_2^{(2)} = (x_2^{(0)}, A_1 x_2^{(0)}) = \frac{1}{8} \not\longrightarrow \lambda_2^{(2)} = 3 + 0\epsilon + \frac{\epsilon^2}{8}$$

The First order correction to $y_{n,i}$ helps determine $\lambda_{n,i}^{(2)}$. We want to divide

$$y_{n,i}^{(0)} = \underbrace{f_{n,i}}_{\text{Inside subspace}} + \underbrace{g_{n,i}}_{\text{Outside subspace}}$$

$\lambda_{n,i}^{(2)}$ depends only on $g_{n,i}$, $\lambda_{n,i}^{(2)} = \sum_{m \neq n} \sum_{j=1}^{\mu_n} \frac{|(x_{m,j}^{(0)}, A_1 x_{n,i}^{(0)})|^2}{\lambda_n^{(0)} - \lambda_m^{(0)}}$

$$y_{n,i} = y_{n,i}^{(0)} + y_{n,i}^{(1)}\epsilon + \ldots\ldots, \quad \lambda_{n,i} = \lambda_n^{(0)} + \lambda_{n,i}^{(1)}\epsilon + \lambda_{n,i}^{(2)}\epsilon^2$$

Solvability Requires

$$(x^{(0)}_{p,j}, (A_1 - \lambda^{(1)}_{n,i} \mathbb{1}) y^{(1)}_{n,i}) - \lambda^{(2)}_{n,i} (x^{(0)}_{p,j}, y^{(0)}_{n,i})$$

Our first order equation in ϵ_i (Long eqn.)

$$(A_0 - \lambda_n^{(0)} \mathbb{1}) y^{(0)}_{n,i} = -(A_1 - \lambda^{(0)}_{n,i}) y^{(0)}_{n,i}$$

$(x^{(0)}_{p,j}, (A_0 - \lambda_n^{(0)}\mathbb{1})(f_{n,i}+g_{n,i})) = (x^{(0)}_{p,j}, (A_1 - \lambda^{(0)}_{n,i}\mathbb{1}) y^{(0)}_{n,i})$, If $p \neq n$

$$(\lambda_p^{(0)} - \lambda_n^{(0)})(x^{(0)}_{p,j}, \overset{0}{\cancel{f_{n,i}+}} g_{n,i}) = -(x^{(0)}_{p,j}, A_1 y^{(0)}_{n,i}) - \lambda^{(0)}_{n,i}\overset{0}{\cancel{(x^{(0)}_{p,j}, y^{(0)}_{n,i})}}$$

$$\sum_{p,j} (x^{(0)}_{p,j}, g_{n,i}) x^{(0)}_{p,j} = \sum_{p,j} \frac{(x^{(0)}_{p,j}, A_1 y^{(0)}_{n,i})}{\lambda_n^{(0)} - \lambda_p^{(0)}} x^{(0)}_{p,j} = g_{n,i}$$

Our second order equation is

$$(A_0 - \lambda_n^{(0)}\mathbb{1}) y^{(0)}_{n,i} = -(A_1 - \lambda^{(1)}_{n,i}\mathbb{1}) y^{(0)}_{n,i} + \lambda^{(0)}_{n,i} y^{(0)}_{n,i}$$

we require : $x^{(0)}_{n,j} \longrightarrow 0$, $(x^{(0)}_{n,j}, (A_1 - \lambda_n^{(1)}\mathbb{1}) y^{(0)}_{n,i}) = \lambda^{(2)}_{n,i}(x_{n,j}, y^{(0)}_{n,i}) = 0$

$$(x^{(0)}_{n,j}, (A_1 - \lambda^{(1)}_{n,i}\mathbb{1}) f_{n,i}) + (x^{(0)}_{n,j}, A_1 g_{n,i}) - \lambda^{(1)}_{n,1} \overset{0}{\cancel{(x^{(0)}_{n,j}, g_{n,i})}} = \lambda^{(0)} \overset{\alpha_{i,j}}{\cancel{(x^{(0)}_{n,j}, y^{(0)}_{n,i})}}$$

$$(x^{(0)}_{n,j}, (A_1 - \lambda^{(1)}_{n,i}\mathbb{1}) f_{n,i}) = -(x^{(0)}_{n,j}, A_1 g_{n,i}) + \lambda^{(2)}_{n,i}\alpha_{ij}$$

Expand $f_{ni} = \beta_k^{(i)} x^{(0)}_{n,k}$

11-17-21

$\lambda^{(2)}(\vec\alpha_{n,i}, \vec\alpha_{n,j}) = (\alpha_{n,i}, c_{i,j})$, $\lambda^{(2)}_{n,i} = \sum_j \alpha_{i,j}(x^{(0)}_{n,j}, A_1 g_{ni}) = \sum_{p,n} \frac{|(y_{n,j}^{(0)}, A_1 x^{(0)}_{n,j})|^2}{\lambda_n^{(0)} - \lambda_p^{(0)}}$

$y^{(1)}_{1,1} = f_{1,1} + g_{1,1}$. we know $g_{1,1} = -\frac{1}{8} x_2^{(0)}$

$c_1 = (x^{(0)}_{1,1}, A_1 g_{1,1})$, $c_1 = \frac{1}{\sqrt2}(0\ 1\ -1)\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} -\frac18 \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix} = 0$

$c_2 = (x^{(0)}_{1,2}, A_1 g_{1,1})$, $c_2 = (1\ 0\ 0)\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\frac18 \frac{1}{\sqrt2}\begin{pmatrix}1\\1\\0\end{pmatrix} = -\frac{1}{8\sqrt2}$

$\lambda^{(2)}_{1,1} = -\frac{1}{16}$, $\lambda^{(2)}_{1,1}\vec\alpha - \vec c = -\frac{1}{16}\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{8\sqrt2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{16}\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}$

$a_{11} - \lambda^{(1)}\mathbb{1} = \frac{1}{\sqrt2}\begin{pmatrix}-1&1\\1&-1\end{pmatrix}$, $\frac{1}{\sqrt2}\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \frac{1}{16}\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}$: $\begin{matrix}-\beta_1+\beta_2 = -1/16\\ \beta_1 - \beta_2 = 1/16\end{matrix}$

We know that $y^{(1)}_{1,1} \perp y^{(0)}_{1,1}$, $\vec\beta = \frac{1}{32}\begin{pmatrix}1\\-1\end{pmatrix}$, $y^{(1)}_{1,1} = \frac{1}{32}(x^{(0)}_{1,1} - x^{(0)}_{1,2})$