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Math Methods in Physics

PHYS 5013 Homework Assignment #4

Problems: $\{1, 2, 3, 4, 5, 6\}$

Due: September 29, 2021 By 10:30 AM

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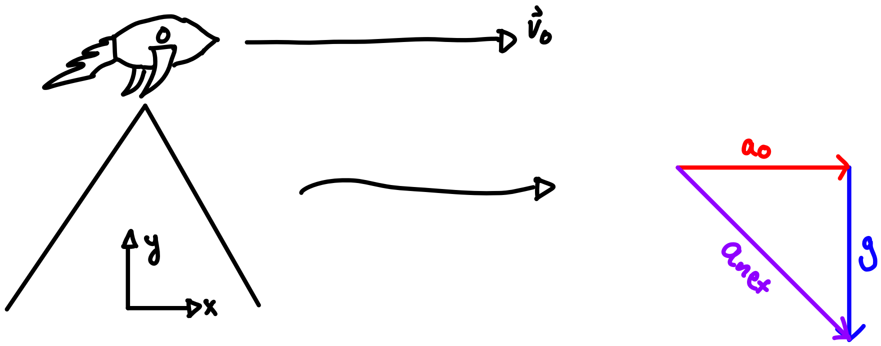
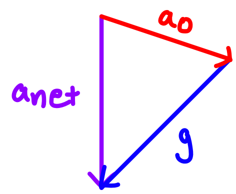
Problem 1:

A rocket is fired horizontally off of a rooftop. As it leaves the rooftop it has an initial horizontal velocity v_0 and a constant horizontal acceleration a_0 in addition to the acceleration, g, downward due to gravity. What is the shape of its trajectory? (Hyperbola? Parabola? Straight line? Or something else?) Hint: This question can be answered without any need for calculation, if you think geometrically.



If someone turns their head (or rotates their co-ordinate system), such that



This motion will be parabolic. In fact, this motion will also be parabolic if we don't rotate our co-ordinate system as well.

Problem 1: Review

Procedure:

- Draw vectors depicting the accelerations.
- Rotate the vectors such that the net vector points straight down.
- Realize that this is symbolic of parabolic motion.

Key Concepts:

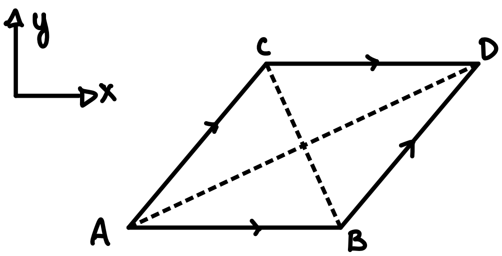
- Even with horizontal acceleration, independent of initial velocity, the motion will be parabolic.

Variations:

- Horizontal acceleration can be removed.
    - Leading to normal trajectory motion.

Problem 2:

Prove by vector methods that the diagonals of a rhombus are orthogonal.



want to show that $C \to B$ and $A \to D$ are orthogonal

$$|\vec{AB}| = |\vec{CD}| = |\vec{AC}| = |\vec{BD}| \; : \; \vec{AB} = \vec{CD} \; , \; \vec{AC} = \vec{BD}$$

$$\vec{AD} \cdot \vec{BC} = (\vec{AB} + \vec{BD}) \cdot (\vec{BD} + \vec{CD}) = (\vec{AB} + \vec{BD}) \cdot (\vec{BD} - \vec{AB})$$

$$\vec{AD} \cdot \vec{BC} = \cancel{\vec{AB} \vec{BD}} - \vec{AB} \cdot \vec{AB} + \vec{BD} \cdot \vec{BD} - \cancel{\vec{AB} \vec{BD}} = \vec{BD} \cdot \vec{BD} - \vec{AB} \cdot \vec{AB}$$

$$\vec{AD} \cdot \vec{BC} = |\vec{BD}|^2 - |\vec{AB}|^2$$

Since $|\vec{BD}| = |\vec{AB}| \quad \therefore \quad \vec{AD} \cdot \vec{BC} = |\vec{AB}|^2 - |\vec{AB}|^2 = 0$

And since when dot products are equal to zero, the vectors that are in those dot products have to be orthogonal or 0. Since $\vec{AD}$ & $\vec{BC} \neq 0$ these vectors are orthogonal.

$\blacksquare$

Problem 2: Review

Procedure:

- Define points of Rhombus as A through D.
- Proceed to define vectors between points.
- Identify that the magnitudes of the vectors are all equal.
- Compute the dot product between the diagonals of the shape.
- Show that the dot product is zero.

Key Concepts:

- The side length's of a rhombus are all equal.
- For the diagonals of the rhombus to be orthogonal, the dot product must be zero.

Variations:

- We could be asked to do this for a different shape.
  - We would use the same procedure but would get a different result potentially.

Problem 3:

Show that $\epsilon_{ijk}\epsilon_{ijk} = 6$.

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \longrightarrow (\text{Eq. } 1.30)$$

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}$$

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad \text{w/ } j = m \quad \epsilon_{ijk}\epsilon_{ljk} = \delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl}$$

$$\delta_{jj} = \overset{1}{\cancel{\delta_{11}}} + \overset{0}{\cancel{\delta_{12}}} + \overset{0}{\cancel{\delta_{13}}} + \overset{0}{\cancel{\delta_{21}}} + \overset{1}{\cancel{\delta_{22}}} + \overset{0}{\cancel{\delta_{23}}} + \overset{0}{\cancel{\delta_{31}}} + \overset{0}{\cancel{\delta_{32}}} + \overset{1}{\cancel{\delta_{33}}} \quad \therefore \quad \delta_{jj} = 3$$

$$\epsilon_{ijk}\epsilon_{ljk} = 3\delta_{il} - \delta_{ij}\delta_{jl} \quad \text{w/ } \delta_{ij}\delta_{jl} = \delta_{il} \quad \epsilon_{ijk}\epsilon_{ljk} = 3\delta_{il} - \delta_{il} = 2\delta_{il}$$

$$\text{w/ } i = l,\ k = m \quad \epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} \quad : \quad \text{w/ } \delta_{ii} = 6 \quad \therefore \quad \epsilon_{ijk}\epsilon_{ijk} = 2\cdot 3 = 6 \ \checkmark$$

$\blacksquare$

Problem 3: Review

Procedure:

- Use common Levi-Cevita relationships : $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.
- Set $j = m$.
- Identify $\delta_{jj} = 3$.
- Identify $\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} \rightarrow 3\delta_{il} - \delta_{ij}\delta_{jl}$.
- Identify $\delta_{ij}\delta_{jl} \rightarrow \delta_{il}$.
- Set $i = l$ and then show the rest.

Key Concepts:

- We can write out Levi-Cevita's in terms of Kroenecker Delta's.
- Setting indices equal to one-another will reduce the equations.
- $\delta_{ii} = 3$ with 3 indices to be summed over.
- Even permutations of $\epsilon_{ijk} = 1$.
- Odd permutations of $\epsilon_{ijk} = 0$.

Variations:

- The indices of the Levi-Cevita can change.
  - Use the same rules as before.

Problem 4:

Demonstrate algebraically whether or not the cross product is associative. That is, verify or falsify the following:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}.$$

If they are not in general equal, are they at least of equal magnitude?

$$\vec{a} = a_i \hat{e}_i \;,\; \vec{b} = b_j \hat{e}_j \;,\; \vec{c} = c_k \hat{e}_k$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$$

From the above we can say

$$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \neq (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \epsilon_{ijk} a_j \epsilon_{klm} b_l c_m = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) a_j b_l c_m = b_i a_j c_j - c_i a_j b_j = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \epsilon_{klm} a_l b_m \epsilon_{ijk} c_j = (\delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}) a_l b_m c_j = a_i b_j c_j - a_j b_i c_j = \vec{a}(\vec{b} \cdot \vec{c}) - \vec{b}(\vec{a} \cdot \vec{c})$$

And thus,

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}}$$

$$\vec{a} \cdot \vec{c} = a_i c_k \delta_{ik} \;:\; (\vec{a} \cdot \vec{c})\vec{b} = a_i c_k \delta_{ik} * b_j \hat{e}_j = a_i b_j c_k \delta_{ik} \hat{e}_j$$

$$\vec{a} \cdot \vec{b} = a_i b_j \delta_{ij} \;:\; (\vec{a} \cdot \vec{b})\vec{c} = a_i b_j \delta_{ij} * c_k \hat{e}_k = a_i b_j c_k \delta_{ij} \hat{e}_k$$

$$D = \vec{a} \times (\vec{b} \times \vec{c}) = a_i b_j c_k \delta_{ik} \hat{e}_j - a_i b_j c_k \delta_{ij} \hat{e}_k$$

$$\vec{c} \cdot \vec{a} = c_k a_i \delta_{ki} \;:\; (\vec{c} \cdot \vec{a})\vec{b} = c_k a_i \delta_{ki} * b_j \hat{e}_j = c_k a_i b_j \delta_{ki} \hat{e}_j$$

$$\vec{c} \cdot \vec{b} = c_k b_j \delta_{kj} \;:\; (\vec{c} \cdot \vec{b})\vec{a} = c_k b_j \delta_{kj} * a_i \hat{e}_i = c_k b_j a_i \delta_{kj} \hat{e}_i$$

$$D^* = (\vec{a} \times \vec{b}) \times \vec{c} = c_k a_i b_j \delta_{ki} \hat{e}_j - c_k b_j a_i \delta_{kj} \hat{e}_i$$

$$|D| = \sqrt{(a_i b_j c_k \delta_{ik})^2 + (a_i b_j c_k \delta_{ij})^2} \;,\; |D^*| = \sqrt{(c_k a_i b_j \delta_{ki})^2 + (c_k b_j a_i \delta_{kj})^2}$$

$\boxed{\text{The magnitudes are also not equal}}$

Problem 4: Review

Procedure:

- Use the fact that a cross product can be written as $(\vec{a} \times \vec{b})_i = \epsilon_{ijk} a_j b_k$.
- Proceed to show the $\vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ rule.
- Show that the vector products are not equal.
- Use the rule for dot products can be written as $(\vec{a} \cdot \vec{b}) = a_i b_j \delta_{ij}$.
- Show that the magnitudes are not equal.

Key Concepts:

- The vector products can be written in terms of Levi-Cevita's.
- $\vec{A} \times (\vec{A} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$ and neither are the magnitudes.
- $(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B})$.
- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} \cdot (\vec{A} \cdot \vec{B})$.

Variations:

- The order of the cross products can be changed.
  - Thus having us prove a different identity with the same procedure.

Problem 5:

Consider a two dimensional system where the vector $\vec{x}$ is given by

$$\vec{x} = x_1\,\hat{e}_1 + x_2\,\hat{e}_2$$

and the x-co-ordinate is transformed into a different, non-orthogonal co-ordinate as:

$$x_1' = \frac{1}{\sqrt{2}}(x_1 + x_2)$$
$$x_2' = x_2.$$

The gradient of the scalar function is:

$$\vec{\nabla}\phi(\vec{x}) = \partial_1\phi\,\hat{e}_1 + \partial_2\phi\,\hat{e}_2.$$

Find the components of the gradient in the primed co-ordinates and show that it transforms as a covariant vector.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In order to transform between two co-ordinates:

$$\begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix} = a_{ij}\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = ((a_{ij})^T)^{-1}\begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix}$$

In our case:

$$(a) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \end{bmatrix} \therefore (a^T) = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 \end{bmatrix} \therefore (a^T)^{-1} = \begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix}$$

The components of the gradient are:

$$\vec{\nabla}\varphi(\vec{x}) = (a^T)^{-1}\,\hat{e}' : \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = (a^T)^{-1}\begin{bmatrix} \partial_1\varphi' \\ \partial_2\varphi' \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \partial_1\varphi' \\ \partial_2\varphi' \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\partial_1\varphi' \\ \partial_2\varphi' - \partial_1\varphi' \end{bmatrix}$$

$$\hat{e}_1 = \sqrt{2}\,\partial_1\varphi'(\hat{e}_1'),\quad \hat{e}_2 = (\partial_2\varphi' - \partial_1\varphi')(\hat{e}_2') \rightarrow \vec{\nabla}\varphi(\vec{x}) = (\sqrt{2}\,\partial_1\varphi')\,\hat{e}_1' + (\partial_2\varphi' - \partial_1\varphi')\,\hat{e}_2'$$

$$\boxed{\vec{\nabla}\varphi(\vec{x}) = (\sqrt{2}\,\partial_1\varphi')\,\hat{e}_1' + (\partial_2\varphi' - \partial\varphi')\,\hat{e}_2'}$$

For this vector to transform as a covariant vector:

$$\begin{bmatrix} \partial_1\varphi' \\ \partial_2\varphi' \end{bmatrix} = (a^T)^{-1}\begin{bmatrix} \partial_1\varphi \\ \partial_2\varphi \end{bmatrix} : \begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix} = a\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}$$

Which has been shown above

Problem 5: Review

Procedure:

- Use the rules
$$\begin{bmatrix}\hat{e}_1' \\ \hat{e}_2'\end{bmatrix} = a_{ij}\begin{bmatrix}\hat{e}_1 \\ \hat{e}_2\end{bmatrix} \quad , \quad \begin{bmatrix}\hat{e}_1 \\ \hat{e}_2\end{bmatrix} = ((a_{ij})^T)^{-1}\begin{bmatrix}\hat{e}_1' \\ \hat{e}_2'\end{bmatrix}$$
where a_{ij} is the matrix defined in the problem.
- The inverse of a matrix can be calculated via
$$(a_{ij})^{-1} = \frac{1}{ab - cd}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$
and the transpose is calculated via
$$(a_{ij})^T = \begin{bmatrix} a & c \\ b & d\end{bmatrix}.$$
- Proceed to use the above equations to transform the co-ordinates.

Key Concepts:

- The above equations must be used to go between the co-ordinates.
- Primed co-ordinates mean $\hat{e}_i = [...]x_i' + [...]x_j'$.
- A covariant transformation is one which follows the first rule outlined in the procedure section.

Variations:

- We could be asked to write the co-ordinates in the un-primed co-ordinates.
  - This would mean that we would have to undo $(a^T)^{-1} \rightarrow a$ and then use the equations above.
- We could be given different primed co-ordinates.
  - This would change some of the values in our calculations, would lead to a different final answer, but would not change the procedure.

Problem 6:

Show that for an orthogonal transformation, there is no distinction between a contravariant and covariant vector.

For an orthogonal transformation

$$(a) = (a^T)^{-1} = \hat{A}$$

$$\underbrace{\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}}_{\hat{E}} = (a^T)^{-1} \begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix} , \quad \underbrace{\begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix}}_{\hat{E}'} = a \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} \quad \therefore \quad \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \hat{A} \begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix} , \quad \begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \end{bmatrix} = \hat{A} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$\hat{a} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} , \quad \hat{a}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} , \quad (\hat{a}^T)^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$a = \frac{d}{ad-bc} , \quad b = \frac{-c}{ad-bc} , \quad c = \frac{-b}{ad-bc} , \quad d = \frac{a}{ad-bc}$$

$$a = \frac{a}{(ad-bc)^2} \quad : \quad (ad-bc)^2 = 1 \quad ad-bc = \pm 1$$

$$a = \pm d , \quad b = \mp c \quad : \text{ if } a = d \text{ then } b = -c , \text{ if } a = -d \text{ then } b = c$$

$$\text{w/ the condition } a^2 + b^2 = 1$$

With the above conditions, we will have a matrix $\hat{A}$ s.t $\hat{a}$ and $(\hat{a}^T)^{-1}$ will be equal and thus this will show that this transformation is orthogonal.

Because the transformation is orthogonal, there will be no difference between contravariant and covariant vectors. This is because when $\hat{A}$ acts on the components and basis' of a vector there will be no difference between them since the transformation is equal.

■

Problem 6: Review

Procedure:

- Use the rule that $(a^T)^{-1} = a$ for an orthogonal transformation.
- Proceed to find a generic matrix that satisfies $(a^T)^{-1} = a$.

Key Concepts:

- For orthogonal transformations: $(a^T)^{-1} = a$.
- If a transformation is indeed orthogonal, then there is no difference between contravariant and covariant vectors.
- When the matrix "a" acts on these basis', there is no difference in the transformation.

Variations:

- Because this problem involves proving something, this problem cannot change without making a new, completely different problem.