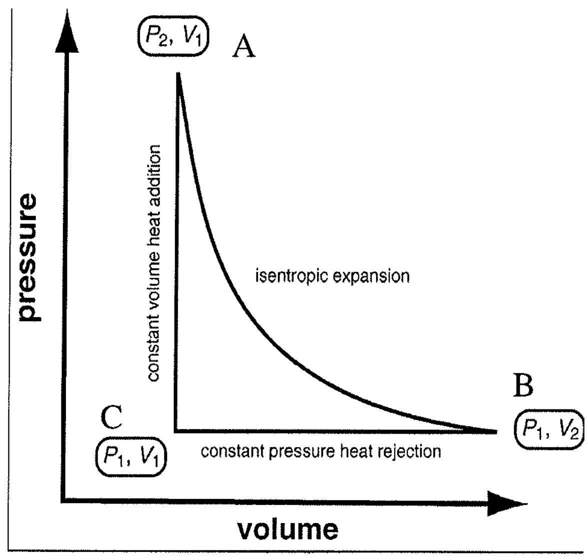
Statistical Mechanics

4. **Heat Engines:** A pulse jet operates under a Lenoir cycle. This consists of an adiabat, an isobar, and an isochore, as shown.



Assuming that the working fluid is an ideal 3D monoatomic gas of N particles:

(a) Find the work done in one complete cycle. (3 points)

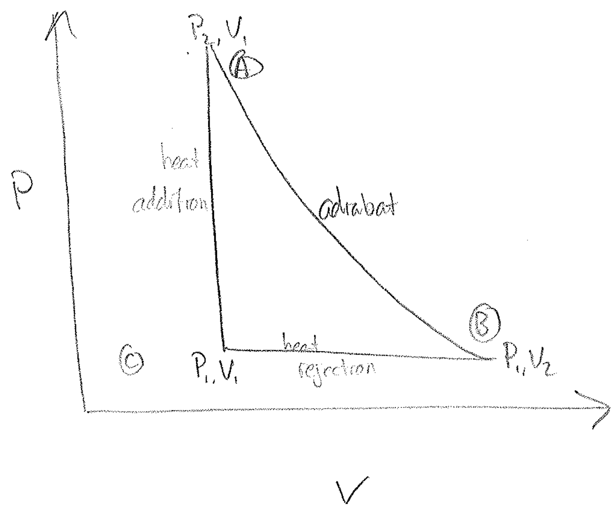
(b) Find the heat exchanged in each step in the cycle. (3 points)

(c) Find the efficiency of the engine. Express your answer in terms of pressures and volutmes. (3 points)

(d) To produce work, should the engine cycle operate clockwise ($A \to B \to C \to A$) or counterclockwise ($A \to C \to B \to A$) ? (1 point)

Jan 2008

Stat Mech #1



* Assume 3-D monatomic gas of N particles

$\Rightarrow PV = Nk_BT$

a) Find work done in one complete cycle

$P_A = P_2$ $\quad$ $P_B = P_1$ $\quad$ $P_c = P_1$

$V_A = V_1$ $\quad$ $V_B = V_2$ $\quad$ $V_c = V_1$

$T_A = \frac{Nk_B}{P_2 V_1}$ $\quad$ $T_B = \frac{Nk_B}{P_1 V_2} = \frac{Nk_B}{P_2 V_1}$ $\quad$ $T_c = \frac{Nk_B}{V_2 P_2}$

$\frac{V_B}{T_B} = \frac{V_c}{T_c}$

$\Rightarrow T_c = \frac{V_c T_B}{V_B}$

$T_c = \frac{V_1}{V_2} \frac{Nk_B}{P_2 V_1}$

$= \frac{Nk_B}{V_2 P_2}$

$W = \int P dV$

$W_{A\to C} = 0, \quad dV = 0$

$W_{C\to B} = P_1(V_2 - V_1)$

$W_{B\to A} = \frac{P_A V_A - P_B V_B}{1-\gamma}$

$= \frac{P_2 V_1 - P_1 V_2}{1 - 5/3}$

$= -\frac{3}{2}(P_2 V_1 - P_1 V_2)$

$W_{TOT} = 0 + P_1 V_2 - P_1 V_1 - \frac{3}{2}(P_2 V_1 - P_1 V_2)$

$= \frac{5}{2} P_1 V_2 - P_1 V_1 - \frac{3}{2} P_2 V_1$

$= \frac{5}{2} P_1 V_2 - V_1 (P_1 + \frac{3}{2} P_2)$

b) Find the heat exchanged in each step of the cycle.

$$Q_{A\to C} = nC_V \Delta T$$
$$= \frac{N}{6.02\cdot 10^{23}} \cdot \frac{3}{2}R \cdot \left(\frac{nR}{V_2P_2} - \frac{nR}{P_2V_1}\right)$$
$$= \frac{3nR}{2} \cdot \frac{nR}{P_2}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$
$$= \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

$$Q_{C\to B} = nC_P \Delta T$$
$$= \frac{5nR}{2}\left(\frac{nR}{P_2V_1} - \frac{nR}{V_2P_2}\right)$$
$$= \frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

$Q_{B\to A} = 0$ b/c adiabatic

$$Q_{TOT} = \frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2} - \frac{1}{V_1}\right) + \frac{5n^2R^2}{2P_2}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$
$$= \frac{n^2R^2}{P_2}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)$$

c) Find the efficiency of the engine

$$\eta = \frac{W_{out}}{Q_{in}}$$

*Note: Q_{in} occurs in $A\to C$

$$\eta = \frac{\frac{5}{2}P_1V_2 - V_1\left(P_1 + \frac{3}{2}P_2\right)}{\frac{3n^2R^2}{2P_2}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

$$= \frac{5P_1V_2 - 2V_1P_1 - 3V_1P_2}{\frac{3n^2R^2}{P_2}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

$$= \frac{5P_1P_2V_2 - 2P_1P_2V_1 - 3P_2^2V_1}{3n^2R^2\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

d) To produce work, does engine operate clockwise or counter-clockwise?

CW

5. Consider a classical ideal gas in 3D that feels a linear gravitational potential,

$$V(z) = mgz$$

where m is the mass of a single gas atom and $0 < z < \infty$. This is not an interaction between gas atoms, it is simply their gravitational potential energy near the surface of the Earth.

The gas is in a box of dimensions L_x, L_y, and L_z, so that:

$$\begin{aligned} 0 < \ & z & < L_z \\ 0 < \ & x & < L_x \\ 0 < \ & y & < L_y \end{aligned}$$

(a) Calculate the partition function in the canonical ensemble. (3 points)

(b) Determine the internal energy of the gas. (3 points)

(c) Calculate the specific heat c_v. (3 points)

(d) Explain the behavior of the specific heat when $\beta mgL_z >> 1$ and when $\beta mgL_z << 1$. (The approximation for the gravitational potential may or may not be valid for large L_z. Don't worry about that.) (1 point)

Stat Mech #2

*Consider a classical ideal gas in 3-D w/ linear gravitational potential $V(z) = mgz$. Note: m is mass of single atom, $0 < z < \infty$. Dimensions of box are: $0 < z < L_z$, $0 < x < L_x$, $0 < y < L_y$

a) Calculate the partition function in the classical ensemble

$$Z = \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta E}\right]^N$$

$$* \text{ let } E = \frac{P_x^2 + P_y^2 + P_z^2}{2m} + mgz$$

$$= \left[\frac{1}{h^3}\int dp^3 dq^3 e^{-\beta\left(\frac{1}{2m}[p_x^2 + p_y^2 + p_z^2] + mgz\right)}\right]^N$$

$$= \left[\frac{1}{h^3}\int dp_x dp_y dp_z\, dx dy dz\, e^{-\beta[\frac{1}{2m}[p_x^2+p_y^2+p_z^2] + mgz]}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\int_0^\infty dp\, e^{-\beta p^2/2m}\right)^3 \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, e^{-\beta mgz}\right]^N$$

$$= \left[\frac{1}{h^3}\left(\frac{1}{2}\sqrt{\frac{\pi}{\beta/2m}}\right)^3 L_x L_y \left(\frac{1}{-\beta mg} e^{-\beta mgz}\Big|_0^{L_z}\right)\right]^N$$

$$= \left[\left(\frac{1}{2h}\sqrt{\frac{2\pi m}{\beta}}\right)^3 L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]^N$$

b) Determine the internal energy of the gas

$$U = -\frac{\partial}{\partial\beta}\ln(Z)$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2\beta}\right)^{3/2} L_x L_y \frac{1}{-\beta mg}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N \ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg} \frac{1}{-\beta^{5/2}}\left(e^{-\beta mgL_z} - 1\right)\right]$$

$$= \frac{-\partial}{\partial\beta} N\left(\ln\left[\left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{L_x L_y}{mg}\right] + \ln\left[1 - e^{-\beta mgL_z}\right] - \ln\left(\beta^{5/2}\right)\right]$$

$$= -N\left[\left(1 - e^{-\beta mgL_z}\right)^{-1} + mgL_z e^{-\beta mgL_z} - \frac{5}{2\beta}\right]$$

c) Calculate the specific heat C_V

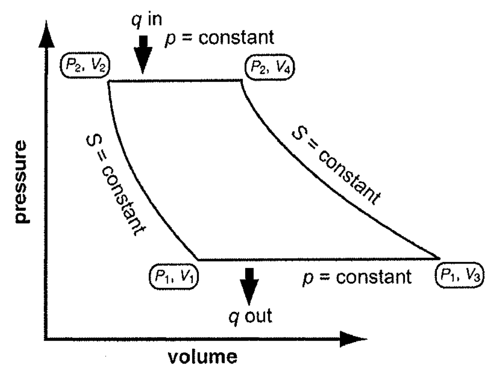
$$
\begin{aligned}
C_V &= \frac{\partial U}{\partial T} \\
&= \frac{\partial}{\partial T}\left(-N\left(\frac{mgL_z e^{-\beta mgL_z}}{1-e^{-\beta mgL_z}} - \frac{5}{2\beta}\right)\right) \\
&= \frac{\partial}{\partial T}\left(\frac{5}{2}Nk_BT - \frac{mgL_z}{e^{-mgL_z/k_BT}-1}\right) \\
&= \frac{5}{2}Nk_B + mgL_z\left(e^{-mgL_z/k_BT}-1\right)^{-2} \cdot -mgL_z/k_BT^2\, e^{-mgL_z/k_BT} \\
&= \frac{5}{2}Nk_B - \frac{(mgL_z)^2 e^{-mgL_z/k_BT}}{T^2k_B\left(e^{-mgL_z/k_BT}-1\right)^2}
\end{aligned}
$$

Classical Mechanics and Statistical/Thermodynamics

January 2009

Statistical Mechanics

4. The gas turbine (jet engine) can be modeled as a Brayton cycle. Below is the P-V diagram for this process.



Assume that the working fluid is an ideal monatomic gas.

(a) Calculate the work done by the gas on each step in the cycle. (3 pts.)

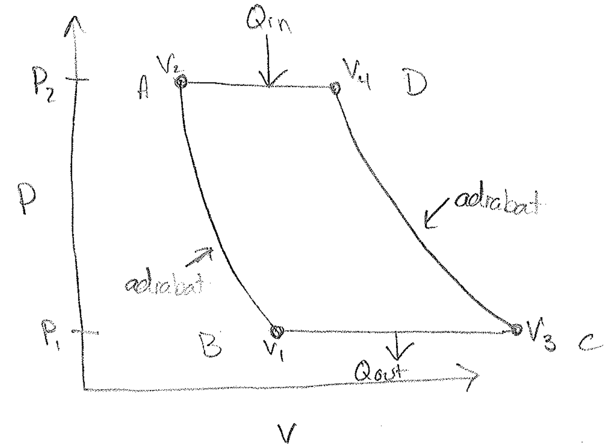
(b) Find the heat for each step in the cycle. (3 pts.)

(c) Find the efficiency of this engine. Your answer should be in terms of the pressures (P_1 and P_2) and the volumes (V_1, V_2, V_3, and V_4). (3 pts.)

(d) To produce work, which way does the cycle operate? Clockwise or counter clockwise? (1 pt.)

Jan 2009

Stat Mech #1



* Ideal monatomic gas

$$\Rightarrow \begin{matrix} C_V = \frac{3}{2}R \\ C_P = \frac{5}{2}R \end{matrix} \Big\} \ \gamma = \frac{5}{3}$$

$P_A = P_2$ $\quad$ $P_B = P_1$ $\quad$ $P_C = P_1$ $\quad$ $P_D = P_2$

$V_A = V_2$ $\quad$ $V_B = V_1$ $\quad$ $V_C = V_3$ $\quad$ $V_D = V_4$

$T_A = \frac{nR}{P_2V_2}$ $\quad$ $T_B = \frac{nR}{P_1V_1}$ $\quad$ $T_C = \frac{nR}{P_1V_3}$ $\quad$ $T_D = \frac{nR}{P_2V_4}$

a) Calculate the work done in each step.

$$W_{A\to D} = P\Delta V = P(V_4 - V_2)$$

$$W_{D\to C} = \frac{P_C V_C - P_D V_D}{1-\gamma} = \frac{P_1V_3 - P_2V_4}{1 - 5/3} = -\frac{3}{2}(P_1V_3 - P_2V_4)$$

$$W_{C\to B} = P\Delta V = P_1(V_1 - V_3)$$

$$W_{B\to A} = \frac{P_A V_A - P_B V_B}{1-\gamma} = \frac{P_2V_2 - P_1V_1}{1-5/3} = -\frac{3}{2}(P_2V_2 - P_1V_1)$$

b) Find the heat for each step

$$Q_{A\to D} = nC_p\Delta T = n\frac{5}{2}R\left(\frac{P_2V_4}{nR} - \frac{P_2V_2}{nR}\right) = \frac{5}{2}P_2(V_4 - V_2)$$

$Q_{D\to C} = 0$ b/c adiabatic

$$Q_{C\to B} = nC_p\Delta T = n\frac{5}{2}R\left(\frac{P_1V_3}{nR} - \frac{P_1V_1}{nR}\right) = \frac{5}{2}P_1(V_3 - V_1)$$

$Q_{B\to A} = 0$ b/c adiabatic

c) Find the efficiency of the engine

$$\eta = 1 - \frac{Q_{in}}{Q_{out}}$$

$$= 1 - \left(\frac{P_2 (V_4 - V_2)}{P_1 (V_3 - V_1)} \right)^{-1}$$

d) Which way does the cycle operate?

CW

6. Consider a free, non-interacting spin zero Bose gas in two dimensions. The energy of each particle is given by:

$$\mathcal{E}(\vec{k}) = \hbar^2 k^2 / 2m$$

where m is the mass of the boson. Assume your system is confined to a square region of length L on a side.

(a) Write down an expression for the grand canonical free energy $\mathcal{G}(T, V, \mu)$ as a sum over $\vec{k}$ states. Do not evaluate the sum. (1 pt.)

(b) Calculate the number of particles in the system as a function of T, V and μ. (3 pts.)

(c) Analyze your expression for $N(T, V, \mu)$ in the limit $T \to 0$. What does it imply about the possibility of a Bose-Einstein transition in this system? (3 pts.)

(d) Prove that the pressure is equal to the energy density, so that $PV = U$. (Hint: you do not have to do any sums over states - you need only prove that this holds using analytic expressions for P and U in this particular system). (3 pts.)

Jan 2009

Stat Mech #3

* Consider a free, non-interacting spin 0 Bose gas in 2-D, where the energy of each particle is: $E(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

  - Assume m is mass of boson and the system is confined to a square region of side length L.

a) Write down the expression for the grand canonical free energy Ω

$$\Omega = -PV$$
$$= -kT \ln(\mathcal{Z}_1)$$

*but $\mathcal{Z}_1 = \prod_j [1 - \exp[\beta(\mu - \epsilon_j)]]^{-1}$

$$= -kT \sum_j \ln(1 - \exp[\beta(\mu - \epsilon_j)])$$
$$= -kT \sum_k \ln(1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})])$$

b) Calculate the # of particles in the system as a function of T, V, and μ

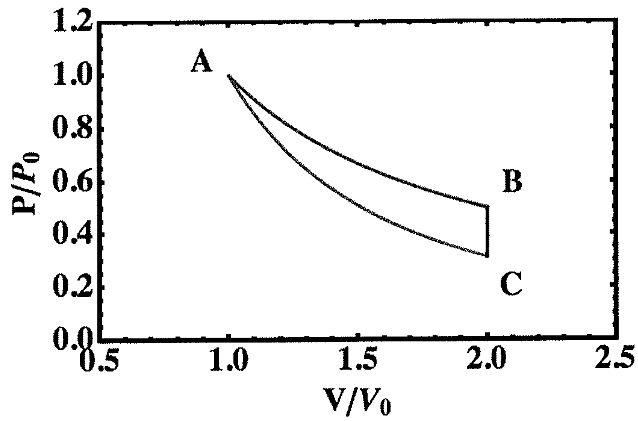
$$\bar{N} = \left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V}$$
$$= \frac{\partial}{\partial \mu} kT \sum$$
$$= -kT \sum_k \frac{-\beta \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]}{1 - \exp[\beta(\mu - \frac{\hbar^2 k^2}{2m})]}$$
$$= \sum_k [\exp[-\beta(\mu - \frac{\hbar^2 k^2}{2m})] - 1]$$
$$\approx \int$$

Classical Mechanics and Statistical/Thermodynamics

August 2011

Statistical Mechanics

4. Consider an ideal monatomic gas used as the working fluid in a thermodynamic cycle. The number of particles is n_0. It follows a cycle consisting of one adiabat, one isochore and one isotherm, as shown below.



(a) Calculate the pressure, temperature, and volume at each corner of the cycle, A, B, and C, expressing your answer in terms of P_0, V_0, n_0 and perhaps R, the ideal gas constant. Note that point A the pressure is P_0 and the volume is V_0. (3pts)

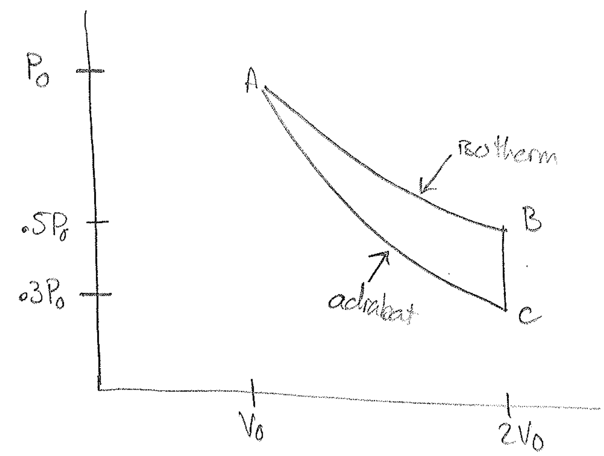
(b) Calculate the work done on the system, the heat into the system and the change in the internal energy of the system for each process step. (4.5pts)

(c) What direction around the ycle must the system follow to be used as a functional heat engine? (1/2pt)

(d) What is the efficiency of the cycle, run as an engine? (1pt)

(e) What is the efficiency of an ideal Carnot engine run between reservoirs B and C? (1pt)

Aug 2011 Stat Mech # 1



* cycle consists of adiabat, isochore, and isotherm

* gas is ideal + monatomic

$\Rightarrow C_V = \frac{3}{2}R$, $C_P = \frac{5}{2}R$ $\Big\rangle\ \gamma = 5/3$

a) Find P, V, and T at each corner of the cycle in terms of P_0, V_0, n_0, and R

$P_A = P_0$ $\qquad P_B = .5P_0$ $\qquad P_C = .3P_0$

$V_A = V_0$ $\qquad V_B = 2V_0$ $\qquad V_C = 2V_0$

$T_A = \frac{P_0V_0}{nk_B}$ $\qquad T_B = \frac{P_0V_0}{nk_B}$ $\qquad T_C = \frac{.6P_0V_0}{nk_B}$

$PV = nk_BT$

$\Rightarrow T = \frac{PV}{nk_B}$

* For A → C (adiabat)

$P_AV_A^\gamma = P_CV_C^\gamma$

$T_AV_A^{\gamma-1} = T_CV_C^{\gamma-1}$

$\frac{P_AV_A}{T_A} = \frac{P_CV_C}{T_C}$

$\frac{\cancel{P_A}V_A}{\cancel{T_A}} = \frac{\frac{\cancel{P_A}V_A^\gamma}{V_C^\gamma}V_C}{\frac{\cancel{T_A}V_A^{\gamma-1}}{V_C^{\gamma-1}}}$

$V_A = \frac{V_A^\gamma V_C^{1-\gamma}}{V_A^{\gamma-1}V_C^{1-\gamma}}$

b) Find the work done on the system, the heat into the system, and the change in internal energy during each step of the cycle

$W_{B\to C} = 0$

$$Q_{B\to C} = nC_V\Delta T = \frac{\cancel{n}}{n_{av}}\left(\frac{3}{2}R\right)\left(\frac{2}{5}\frac{P_0V_0}{\cancel{n}k_B}\right) = \frac{3}{5}P_0V_0$$

$\Delta E = Q = \frac{3}{5}P_0V_0$

$$W_{C\to A} = \frac{P_fV_f - P_iV_i}{1-\gamma} = \frac{.6P_0V_0 - P_0V_0}{1 - 5/3} = \frac{2/5}{-2/3}P_0V_0 = -3/5\,P_0V_0$$

$Q_{C\to A} = 0$

$\Delta E = -W = 3/5\,P_0V_0$

b) $W_{A\to B} = nk_BT\ln\left(\frac{V_B}{V_A}\right)$

$$= n\,k_B \frac{P_0V_0}{n\,k_B}\ln\left(\frac{2V_0}{V_0}\right)$$

$$= P_0V_0\ln(2)$$

$Q = W = P_0V_0\ln(2)$

$\Delta E = 0$ b/c isotherm

c) Which direction does the heat engine flow?

CW

d) What is the efficiency of the heat engine?

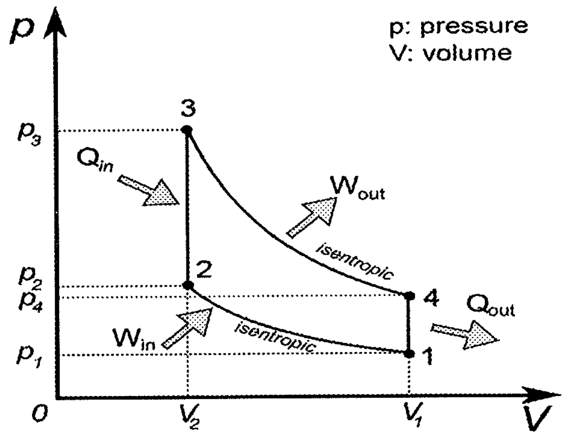
$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$

$$= 1 - \frac{.6P_0V_0}{P_0V_0\ln(2)}$$

$$= 1 - \frac{.6}{\ln(2)}$$

Problem 4 (10 Points):

The diesel engine uses the Otto cycle. Below is the P-V diagram for this process. Assume a monatomic ideal gas.



a. Find the work done during each cycle. **(3 Points)**

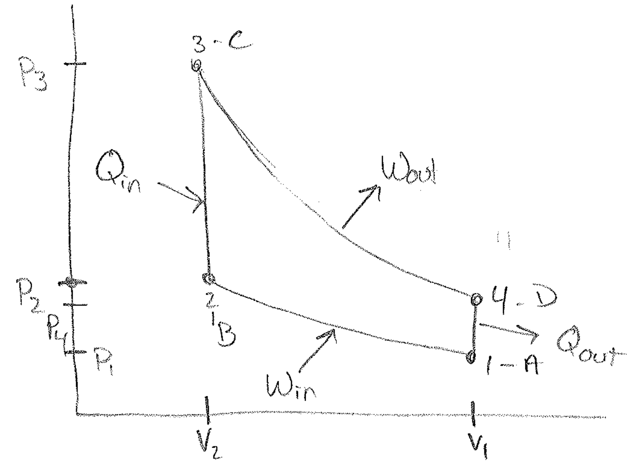
b. Find the heat exchanged each cycle. **(3 Points)**

c. What is the efficiency of this engine? **(3 Points)**

d. To produce work, which way does the cycle operate? Clockwise or counter clockwise in the diagram. **(1 Points)**

Aug 2013

Stat Mech #1



* Assume ideal monatomic gas
$\rightarrow C_V = \frac{3}{2}R$
$C_P = \frac{5}{2}R \quad \gamma = 5/3$

$P_A = P_1$ $\quad P_B = P_2$ $\quad P_C = P_3$ $\quad P_D = P_4$

$V_A = V_1$ $\quad V_B = V_2$ $\quad V_C = V_2$ $\quad V_D = V_1$

$T_A = \frac{P_1 V_1}{nR}$ $\quad T_B = \frac{P_1 V_1^{5/3}}{nR V_2^{2/3}}$ $\quad T_C = \frac{P_1 P_3 V_1^{5/3}}{nR P_2 V_2^{2/3}}$ $\quad T_D = \frac{P_4 V_1}{nR}$

$$T_B = T_A\left(\frac{V_A}{V_B}\right)^{\gamma-1} = \frac{P_1V_1}{nR}\left(\frac{V_1}{V_2}\right)^{2/3} = \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}$$

$$T_C = T_B\left(\frac{P_C}{P_B}\right) = \frac{P_1V_1^{5/3}}{nRV_2^{2/3}}\left(\frac{P_3}{P_2}\right)$$

$$T_D = T_A\left(\frac{P_D}{P_A}\right) = \frac{P_1V_1}{nR}\left(\frac{P_4}{P_1}\right)$$

a) Find the work done during the cycle.

$$W_{A\to B} = \frac{P_BV_B - P_AV_A}{1-\gamma} = \frac{P_2V_2 - P_1V_1}{1-5/3} = \frac{-3}{2}(P_2V_2 - P_1V_1)$$

$$W_{C\to D} = \frac{P_DV_D - P_CV_C}{1-\gamma} = \frac{P_4V_1 - P_3V_2}{1-5/3} = \frac{-3}{2}(P_4V_1 - P_3V_2)$$

$W_{B\to C} = 0$ b/c isochoric

$W_{D\to A} = 0$ b/c isochoric

$$\Rightarrow W_{TOT} = \frac{3}{2}(P_1V_1 - P_2V_2) + \frac{3}{2}(P_3V_2 - P_4V_1)$$
$$= \frac{3}{2}\left(V_1[P_1 - P_4] + V_2[P_3 - P_2]\right)$$

b) Find the heat exchanged each cycle

$Q_{A\to B} = 0$ b/c adiabatic

$$Q_{B\to C} = nC_V\Delta T$$
$$= n\left(\tfrac{3}{2}R\right)\left(\frac{P_1 P_3 V_1^{5/3}}{nR\,P_2 V_2^{2/3}} - \frac{P_1 V_1^{5/3}}{nR\,V_2^{2/3}}\right)$$
$$= \frac{3}{2}\,\frac{P_1 V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2} - 1\right)$$

$Q_{C\to D} = 0$ b/c adiabatic

$$Q_{D\to A} = nC_V\Delta T$$
$$= n\left(\tfrac{3}{2}R\right)\left(\frac{P_1 V_1}{nR} - \frac{P_4 V_1}{nR}\right)$$
$$= \frac{3}{2}\left(P_1 - P_4\right)$$

$$Q_{TOT} = \frac{3}{2}\left[\frac{P_1 V_1^{5/3}}{V_2^{2/3}}\left(P_3/P_2 - 1\right) + P_1 - P_4\right]$$

c) What is the efficiency of the engine?

$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$
$$= 1 - \left|\frac{\frac{3}{2}(P_1 - P_4)}{\frac{3}{2}\frac{P_1 V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2} - 1\right)}\right|$$
$$= 1 - \left|\frac{P_1 - P_4}{\frac{P_1 V_1^{5/3}}{V_2^{2/3}}\left(\frac{P_3}{P_2} - 1\right)}\right|$$

d) Which direction does the engine operate?

CW

Classical Mechanics and Statistical/Thermodynamics

August 2015

Statistical Mechanics

4. Consider a thermally insulated vessel, divided into two parts by a partition. One side contains n_1 moles of nitrogen gas that occupies a volume V_1 at temperature T_1 and pressure P_1 and the other contains n_2 moles of argon gas that occupies a volume V_2 at T_2 and P_2. Assume nitrogen to be an ideal gas with $c_v = (5/2)R$ and argon to be an ideal gas with $c_v = (3/2)R$. The goal of this problem is to calculate the change in entropy of the system when the partition is removed and each gas expands freely through the container.

   Since entropy is a function of state, the change in entropy between an initial and final state of a system is independent of the path taken to get from one state to another. That means we can break this problem into separate segments of a path connecting the initial and final states such that the entropy change for each segment is more easily calculated.

   (a) First let the two parts of the system equilibrate thermally at constant volumes. Find the final temperature, T_f, and the entropy change of the system. (3 points)

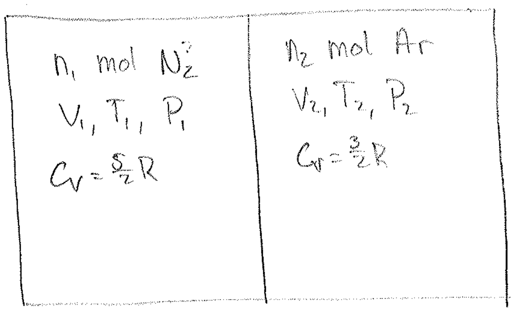
   (b) Second let the pressure of the two parts of the system equilibrate at this constant temperature (i.e., letting the partition between the chambers move). Find the entropy change of the system for this step. (3 points)

   (c) Finally, remove the partition and let the molecules of the gas mix. Find the entropy change for this step. (3 points)

   (d) What is the total entropy change in this process? (1 point)

Aug 2015

Stat Mech #1



* System is thermally isolated

$$S = \int \frac{dQ}{T}$$

a) Let the two parts of the system equilibrate thermally at constant V. Find the final temperature and the entropy change of the system

* for N_2

$$\frac{P_1}{T_1} = \frac{P_{f,1}}{T_f}$$

$$Q = mC_V \Delta T$$

* for Ar

$$\frac{P_2}{T_2} = \frac{P_{f,2}}{T_f}$$

$$Q = mC_V \Delta T$$

$$S = \int \frac{mC_V dT}{T}$$

$$\Delta S = mC_V \ln\left(\frac{T_f}{T_i}\right)$$

$$\Delta S_{N_2} = m_{N_2}$$

$$m_{N_2} \tfrac{5}{2} R (T_f - T_1) = m_{Ar} \tfrac{3}{2} R (T_f - T_2)$$

$$5m_{N_2}(T_f - T_1) = 3m_{Ar}(T_f - T_2)$$

$$5m_{N_2}T_f - 3m_{Ar}T_f = -3m_{Ar}T_2 + 5m_{N_2}T_i$$

$$T_f = \frac{5m_{N_2}T_1 - 3m_{Ar}T_2}{5m_{N_2} - 3m_{Ar}}$$

Classical Mechanics and Statistical/Thermodynamics

January 2016

Statistical Mechanics

4. A heat engine is made from N atoms of an ideal mono-atomic gas starting at an initial temperature T_1, and volume V_1. Call this state "1." It is initially heated isochorically (at constant volume) to a state "2" with a temperature $T_2 = 4T_1$. It then undergoes an adiabatic expansion to state "3" where it has returned to its original pressure. Finally it is then cooled isobarically (at constant pressure) until it returns to its original condition.

   (a) Draw the thermodynamic cycle in the PV plane. (1 point).

   (b) Calculate the volume and temperature at states 2 and 3 in terms of V_1, T_1 and N. (1 point).

   (c) Calculate the work done by the gas in each step of the cycle. (3 points).

   (d) Calculate the heat in (or out) of the gas during each step. (3 points)

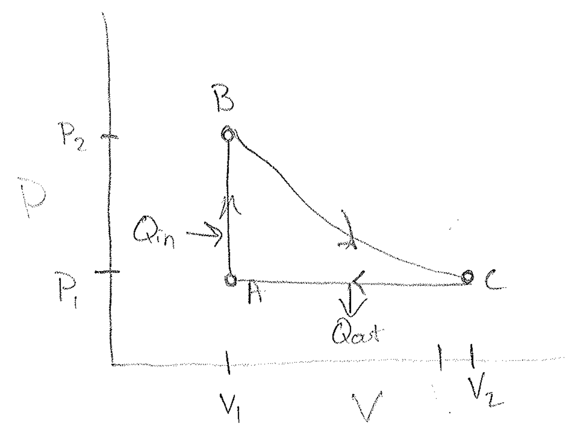
   (e) What is the efficiency of this engine? (2 points)

Jan 2016

Stat Mech #1

A heat engine made of N atoms of an ideal monatomic gas starting at initial temp T_1 and volume V_1. It is heated isochorically to $T_2 = 4T_1$. It then expands adiabatically to state C where it has returned to its initial pressure. It is then isobarically cooled to its initial state.

a) Draw the cycle in the P-V plane.



b) Calculate the temperature at states B and C in terms of V_1, T_1, and N

$P_A = \frac{Nk_BT_1}{V_1}$ $\quad$ $P_B = \frac{4Nk_BT_1}{V_1}$ $\quad$ $P_C = \frac{Nk_BT_1}{V_1}$

$4^3 = 64$

$V_A = V_1$ $\quad$ $V_B = V_1$ $\quad$ $V_C = 4^{3/5}V_1$

$T_A = T_1$ $\quad$ $\boxed{T_B = 4T_1}$ $\quad$ $T_C = 4^{3/5}T_1$

$$P_B V_B^{\gamma} = P_C V_C^{\gamma}$$

$$\frac{4N\cancel{k_B}T_1}{\cancel{V_1}} V_1^{5/3} = \frac{N\cancel{k_B}T_1}{\cancel{V_1}} V_C^{5/3}$$

$$4^{3/5}V_1 = V_C$$

$$\frac{V_A}{T_A} = \frac{V_C}{T_C}$$

$$\Rightarrow T_C = \frac{V_C}{V_A}T_A = \frac{4^{3/5}V_1}{V_1}T_1 = 4^{3/5}T_1$$

c) Calculate the work done by the gas in each step.

$$W_{A\to B} = 0 \quad b/c \text{ isochoric}$$

$$W_{B\to C} = \frac{P_C V_C - P_B V_C}{1-\gamma}$$

$$= \frac{3}{2}\left(4^{3/5} N k_B T - 4 N k_B T\right)$$

$$= -\frac{3}{2} N k_B T \left(4^{3/5} - 4\right)$$

$$W_{C\to A} = P\Delta V$$

$$= \frac{N k_B T}{\cancel{V_1}}\left(\cancel{V_1} - 4^{3/5}\cancel{V_1}\right)$$

$$= N k_B T \left(1 - 4^{3/5}\right)$$

d) Calculate the heat during each step

$$Q_{A\to B} = n C_V \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{3}{2}R\right)\left(4T_1 - T_1\right)$$

$$= \frac{9RNT_1}{2(6.02\cdot 10^{23})}$$

$$Q_{B\to C} = 0 \quad b/c \text{ adiabatic}$$

$$Q_{C\to A} = n C_P \Delta T$$

$$= \frac{N}{n_{av}}\left(\frac{5}{2}R\right)\left(T_1 - 4^{3/5}T_1\right)$$

$$= \frac{5NRT_1\left(1-4^{3/5}\right)}{2\cdot(6.02\cdot 10^{23})}$$

e) What is the efficiency of the engine?

$$\eta = 1 - \left|\frac{Q_{out}}{Q_{in}}\right|$$

$$= 1 - \left|\frac{\frac{5NRT_1\left(1-4^{3/5}\right)}{2\cdot(6.02\cdot 10^{23})}}{\frac{9NRT_1}{2\cdot(6.02\cdot 10^{23})}}\right| = 1 - \left|\frac{5\left(1-4^{3/5}\right)}{9}\right|$$

5. Consider a system of N distinguishable particles with only 3 possible energy levels: 0, ϵ and 2ϵ. The system occupies a fixed volume V and is in thermal equilibrium with a reservoir at temperature T. Ignore interactions between particles and assume that Boltzmann statistics applies.

   (a) What is the partition function for a single particle in the system? (1 point).

   (b) What is the average energy per particle? (1 points).

   (c) What is probability that the 2ϵ level is occupied in the high temperature limit, $k_B T \gg \epsilon$? Explain your answer on physical grounds. (1 point).

   (d) What is the average energy per particle in the high temperature limit, $k_B T \gg \epsilon$? (1 point).

   (e) At what approximate temperature is the ground state 1.1 times as likely to be occupied as the 2ϵ level? (1 point).

   (f) Find the heat capacity of the system, c_v, analyze the low-T (when $k_B T \ll \epsilon$) and high-T ($k_B T \gg \epsilon$) limits, and sketch c_v as a function of T. Explain your answer on physical grounds. (5 points).

Jan 2016

Stat Mech #2

System: N distinguishable particles

3 possible energy levels $(0, \epsilon, 2\epsilon)$

V is a fixed volume

T is temperature of heat resevoir, in thermal equilibrium

* Ignore particle interactions, assume Boltzmann statistics

a) What is the partition function for a single particle?

$$Z = \sum_i e^{-\beta E_i}$$

$$= e^{-\beta 0} + e^{-\beta\epsilon} + e^{-\beta 2\epsilon}$$

$$= 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}$$

b) What is the avg. energy per particle

$$\langle E \rangle = \frac{-\partial}{\partial\beta} \ln(z)$$

$$= \frac{-\partial}{\partial\beta} \ln(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon})$$

$$= \frac{-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

c) What is the probability that the 2ϵ energy level is occupied in the high T limit ($k_BT \gg \epsilon$)? Explain answer on physical grounds

$$P = \frac{1}{z} e^{-\beta E}$$

$$= \frac{e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$= \frac{1}{e^{2\beta\epsilon} + e^{\beta\epsilon} + 1}$$

$$= \frac{1}{3}$$

d) What is the avg energy per particle in the high T limit?

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln(z)$$

$$= \frac{-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$= \frac{-\epsilon - 2\epsilon}{3}$$

$$= -\epsilon$$

e) At what approximate T is the ground state 1.1 times as likely to be occupied as the 2ϵ level?

$$\frac{P_0}{P_{2\epsilon}} = 1.1 = \frac{\frac{1}{z} e^{-\beta 0}}{\frac{1}{z} e^{-\beta 2\epsilon}}$$

$$1.1 = \frac{e^{-\beta 0}}{e^{-2\beta\epsilon}}$$

$$1.1 = \frac{1}{e^{-2\beta\epsilon}}$$

$$e^{2\beta\epsilon} = \frac{1}{1.1}$$

$$2\beta\epsilon = \ln\left(\frac{1}{1.1}\right)$$

$$\frac{1}{k_B T} = \frac{1}{2\epsilon} \ln\left(\frac{1}{1.1}\right)$$

$$T = \frac{2\epsilon}{k_B \ln\left(\frac{1}{1.1}\right)}$$

f) Find the heat capacity of the system in both the high and low T limits. Sketch C_V as a function of T. Explain your answer on physical grounds.

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$$U = -\frac{\partial}{\partial \beta} \ln(Z)$$

$$= -\frac{\partial}{\partial \beta} N \ln(1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon})$$

$$= \frac{N(-\epsilon e^{-\beta\epsilon} - 2\epsilon e^{-2\beta\epsilon})}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}$$

$$C_V = \frac{\partial U}{\partial T}$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{-\beta\epsilon} + 2e^{-2\beta\epsilon})}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}}\right]$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{\beta\epsilon} + 2)}{e^{2\beta\epsilon} + e^{\beta\epsilon} + 1}\right]$$

$$= \frac{\partial}{\partial T}\left[\frac{-N\epsilon(e^{\epsilon/kT} + 2)}{e^{2\epsilon/kT} + e^{\epsilon/kT} + 1}\right]$$

$$= \frac{-N\epsilon \frac{-\epsilon}{kT^2} e^{\epsilon/kT}}{e^{2\epsilon/kT} + e^{\epsilon/kT} + 1} + -N\epsilon(e^{\epsilon/kT} + 2)(-)(e^{2\epsilon/kT} + e^{\epsilon/kT} + 1)^{-2} \cdot \left(-\frac{2\epsilon}{kT^2} e^{2\epsilon/kT} + -\frac{\epsilon}{kT^2} e^{\epsilon/kT}\right)$$

$$=$$

f) $C_V = +N\epsilon \frac{\epsilon}{kT^2} e^{\epsilon/kT}\left(e^{2\epsilon/kT}+e^{\epsilon/kT}+1\right)^{-1} - N\epsilon\left(e^{\epsilon/kT}+2\right)\left(\frac{2\epsilon}{kT^2}e^{2\epsilon/kT}+\frac{\epsilon}{kT^2}e^{\epsilon/kT}\right)\left(e^{2\epsilon/kT}+e^{\epsilon/kT}+1\right)^{-2}$

$= \frac{N\epsilon^2}{kT^2}\left[e^{\epsilon/kT}\left(e^{2\epsilon/kT}+e^{\epsilon/kT}+1\right)^{-1} - \left(e^{\epsilon/kT}+2\right)\left(e^{2\epsilon/kT}+e^{\epsilon/kT}\right)\left(e^{2\epsilon/kT}+e^{\epsilon/kT}+1\right)^{-2}\right]$

* in the high T limit

$C_V \to 0$

* in the low T limit

$C_V \to \infty$