Introduction to quantum thoery: Quantum states and quantum measurements

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What is “Quantum theory”?

Quantum theory

- Physics for microscopic phenomena, e.g., atoms, light.

Why is quantum theory important ?

- Just because it’s reality.

- Because it gives more efficient information processing, e.g., quantum factoring algorithm, quantum secret-key sharing, etc.

On this course

We study mathematical foundation of quantum information.

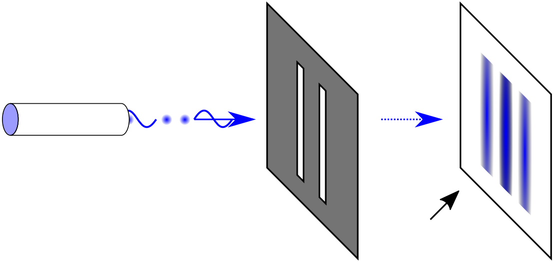
- Mathematical foundation of quantum physics
- Quantum algorithms
- Other quantum information processing, e.g, quantum communication, quantum error-correction.

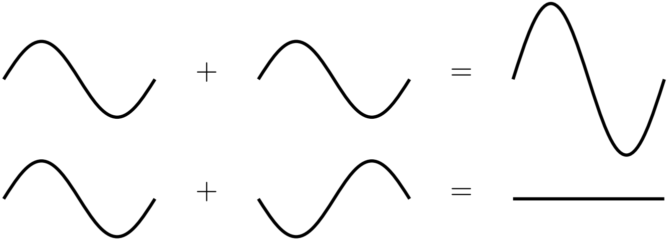
Score

- Assignments: 70%
- Final exam: 30%

Light is wave (1801)

Thomas Young
(1773 – 1829)

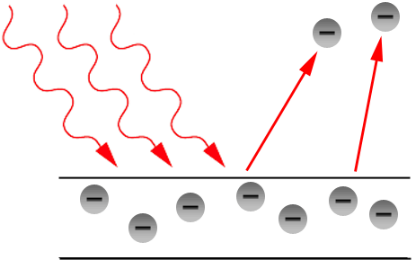




https://en.wikipedia.org/wiki/Young's_interference_experiment

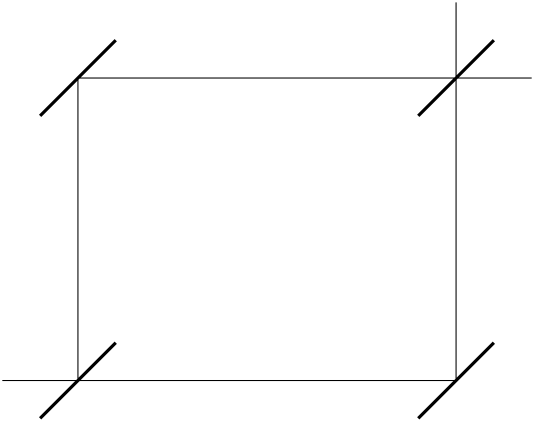
Light is particle (1905)

Albert Einstein
(1879 – 1955)



https://en.wikipedia.org/wiki/File:Photoelectric_effect.png

Mach–Zehnder interferometer



Quantum states and quantum measurements

A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection

State	Measurement
A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2	detection
A single photon $\Rightarrow$ BS1	BS2 $\Rightarrow$ detection
A single photon	BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection

All understandings are valid

Mathematical representations of states and measurements

How "States" and "Measurements" are treated mathematically ?

A table of probabilities of outcome 'YES' for each binary measurment on each state

	Measurement 1	Measurement 2	$\cdots$
State A	p_{A1}	p_{A2}	$\cdots$
State B	p_{B1}	p_{B2}	$\cdots$
$\vdots$			

* The number of states and measurements are not necessarily countable.

Classical theory

States: $\underline{0}$, $\underline{1}$
Binary measurements: $\underline{0}?$, $\underline{1}?$

$$\underline{0}?(\underline{0}) = 1, \qquad \underline{0}?(\underline{1}) = 0,$$
$$\underline{1}?(\underline{0}) = 0, \qquad \underline{1}?(\underline{1}) = 1$$

	$\underline{0}?$	$\underline{1}?$
$\underline{0}$	1	0
$\underline{1}$	0	1

State and measurement

	$\underline{0}?$	$\underline{1}?$
$\underline{0}$	1	0
$\underline{1}$	0	1

$S :=$ $\underline{0}$ with probability p, $\underline{1}$ with probability $1-p$.
S is also regarded as a state.

$$\underline{0}?(S) = p, \qquad \underline{1}?(S) = 1 - p.$$

Similarly,
$E_1 :=$ $\underline{0}?$ with probability p, $\underline{1}?$ with probability $1-p$.
$E_2 := (\underline{0}$ or $\underline{1})?$.
E_1 and E_2 are also regarded as a binary measurement.

Linear space

$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{0}?(\underline{0}) = \langle e_{\underline{0}}, \omega_{\underline{0}} \rangle, \qquad \underline{0}?(\underline{1}) = \langle e_{\underline{0}}, \omega_{\underline{1}} \rangle$$

$$\underline{1}?(\underline{0}) = \langle e_{\underline{1}}, \omega_{\underline{0}} \rangle, \qquad \underline{1}?(\underline{1}) = \langle e_{\underline{1}}, \omega_{\underline{1}} \rangle$$

$S := \underline{0}$ with probability p, $\underline{1}$ with probability $1 - p$

$$\omega_S = p\omega_{\underline{0}} + (1-p)\omega_{\underline{1}} = \begin{bmatrix} p \\ 1-p \end{bmatrix}.$$

$$\underline{0}?(S) = p = \langle e_{\underline{0}}, \omega_S \rangle.$$

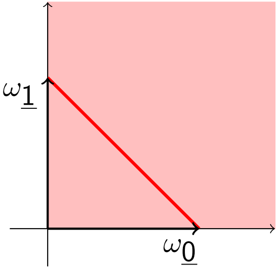
States and measurements in a linear space

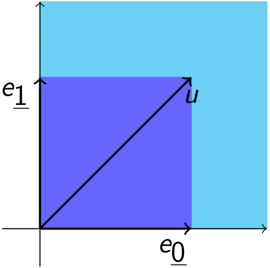
$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$





State and measurement in a linear space

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$

Let $C_{\geq 0}$ be the set of nonnegative vectors and $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}.$$

$$\begin{aligned} \text{Set of measurements} = \{(e_1, \dots, e_k) \mid\ & e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0} \\ & i = 1, 2, \dots, k,\ k = 1, 2, \dots \} \end{aligned}$$

Outcome of the measurement $M = (e_1, \dots, e_k)$ on ω is i with probability $\langle e_i, \omega \rangle$.

Quantum theory

$C_{\geq 0} \subseteq \mathbb{R}^2$: the set of nonnegative vectors, $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}.$$

$$\text{Set of measurements} = \{ (e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0} \\ i = 1, 2, \dots, k,\ k = 1, 2, \dots, \}$$

V : the linear space on $\mathbb{R}$ spanned by 2×2 Hermitian matrices.
$\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$ for $\omega, e \in V$ (Hilbert-Schmidt inner product).
$C_{\succeq 0} \subseteq V$: the set of positive semidefinite matrices, $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{Set of states} = \{ \omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1 \}.$$

$$\text{Set of binary measurements} = \{ e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0} \}.$$

Linear space spanned by 2x2 Hermitian matrices

Basis

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Another choice of basis

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Both are orthogonal basis.
The second basis (I and Pauli matrices X, Y and Z) has nice properties.

1. $\mathrm{Tr}(I) = 2$. $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$.
2. $X^2 = Y^2 = Z^2 = I$ (X, Y and Z have eigenvalues ± 1).
3. $XY = -YX$, $YZ = -ZY$, $ZX = -XZ$.

Positive semidefinite cone

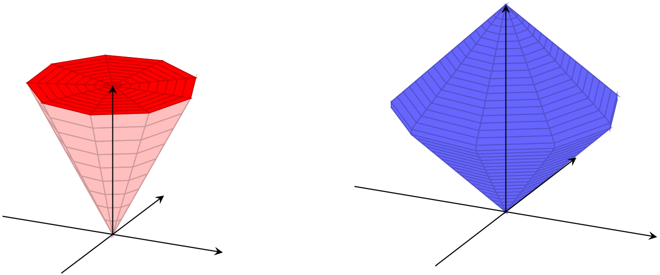
$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rho = \frac{1}{\sqrt{2}} \left(a_I I + a_X X + a_Y Y + a_Z Z \right)$$

$$\begin{aligned}
\lambda_1 \geq 0,\ \lambda_2 \geq 0 \iff & \lambda_1 + \lambda_2 \geq 0,\ \lambda_1 \lambda_2 \geq 0 \\
\iff & \mathrm{Tr}(\rho) \geq 0,\ \mathrm{Tr}(\rho)^2 - \mathrm{Tr}(\rho^2) \geq 0 \\
\iff & a_I \geq 0,\ 2a_I^2 - (a_I^2 + a_X^2 + a_Y^2 + a_Z^2) \geq 0 \\
\iff & a_I \geq 0,\ a_I^2 \geq a_X^2 + a_Y^2 + a_Z^2
\end{aligned}$$

$$\mathrm{Tr}(\rho) = 1 \iff a_I = \frac{1}{\sqrt{2}}$$

Geometry of quantum states and effects



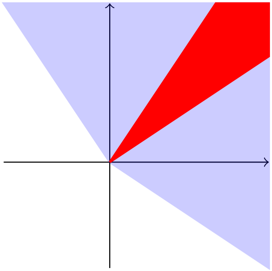
Convex cone and dual cone

$$C \subseteq V \text{ is a convex cone} \iff x + y \in C,\ \lambda x \in C,$$
$$\forall x \in C, y \in C,\ \lambda \geq 0$$

Proper cone: closed, not V, full-dimensional.

$$C^* \subseteq V \text{ is a dual cone of } C$$
$$\iff C^* := \{x \in V \mid \langle x, y\rangle \geq 0,\ \forall y \in C\}$$

$C_{\geq 0}$ and $C_{\succeq 0}$ are self-dual cones.



Generalized probabilistic theories

C : convex cone.
$u \in$ interior of C^*.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of effects} = \{e \in V \mid e \in C^*, u - e \in C^*\}.$$

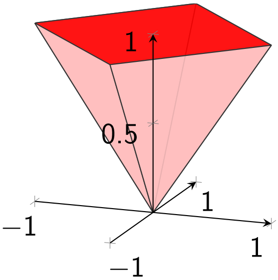
$$\text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u,\ k = 1, 2, 3, \dots\}$$

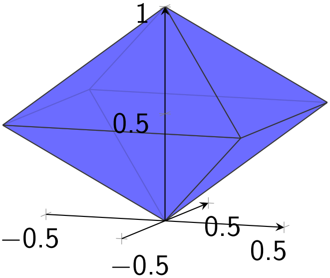
Classical theory
$V = \mathbb{R}^n,\ C = C_{\geq 0},\ u =$ the all-1 vector.

Quantum theory
$V =$ A set of $n \times n$ Hermitian matrices, $C = C_{\succeq 0},\ u = I$.

Other theories ?

Toy theory: Gbit





$$\omega_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \qquad \omega_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix},$$
$$\omega_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, \qquad \omega_3 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}.$$

$$e_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad e_1 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix},$$
$$e_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \qquad e_3 = \frac{1}{2}\begin{bmatrix} -1 & -1 & 1 \end{bmatrix},$$
$$u = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Nonlocality

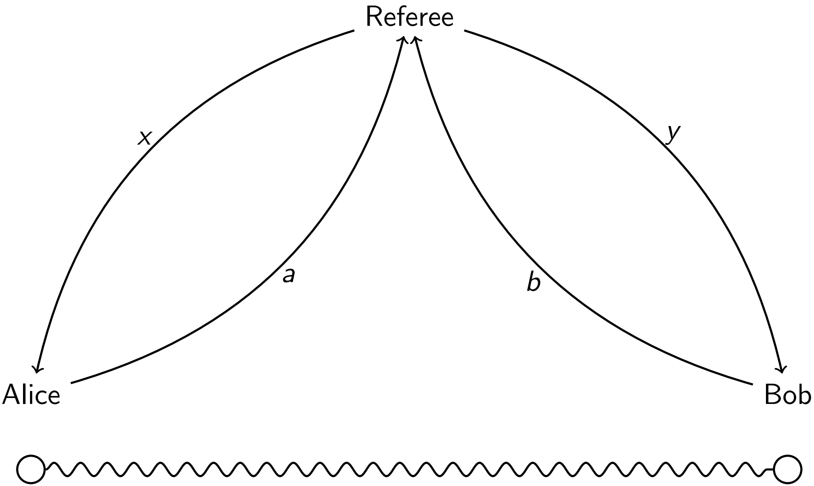
OK, generalized probabilistic theory is quite simple and easy to understand.

But, what is essential difference between classical theory and quantum theory ?

Can quantum theory be “simulated” or “explained” by classical theory ?

Are Hermitian positive-semidefinite matrices really needed for explaining reality ?

Bell test: CHSH game (1964, 1969)



Alice and Bob win iff $a \oplus b = x \wedge y$.

Bell inequality

a_x: Output of Alice for given x.
b_y: Output of Bbob for given y.

$$\begin{aligned} a_0 \oplus b_0 &= 0 \\ a_1 \oplus b_0 &= 0 \\ a_0 \oplus b_1 &= 0 \\ a_1 \oplus b_1 &= 1 \end{aligned}$$

By adding all equations, we get $0 = 1$, which means there is no solution. Hence, the winning probability 1 cannot be achieved.

Three equalities can be satisfied, so that the largest winning probability is $3/4$ (Bell inequality or CHSH inequality).

If Alice and Bob share quantum states, then the largest winning probability is $(2 + \sqrt{2})/4 \approx 0.854$ (Violation of Bell/CHSH inequality)

Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution $P(a, b \mid x, y)$ is said to be **local** if

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Quantum physics allow nonlocal behaviors.

Einstein–Podolsky–Rosen (EPR) paradox (1935)

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

$\iff$ there exists a joint distribution of (a_0, a_1, b_0, b_1).

$$\Downarrow$$

In quantum physics, a_0, a_1, b_0, b_1 cannot *exists* simultaneously.

$$\Downarrow$$

In quantum physics, position and momentum cannot *exists* simultaneously.

Summary

- .
- Classical theory and quantum theory are special cases of generalized probabilistic theories.
- Joint system, e.g., two qubits.

Assignments

1. Show the dimension and one of the basis of real linear space spanned by $n \times n$ Hermitian matrices
2. Show that $XY = -YX$, $YZ = -ZY$ and $ZX = -XZ$.
3. Show that the Hilbert–Schmidt inner product satisfies the axioms of inner product.
4. Show that $C_{\succeq 0}$ is a self-dual cone.