Quantum teleportation

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

T : The set of states $\rightarrow$ the set of states.

$$pT(\rho_1) + (1-p)T(\rho_2) = T(p\rho_1 + (1-p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0, 1]$.

T must be linear (a proof is needed).

Schrödinger picture and Heisenberg picture

$T^\dagger$: The set of binary measurements $\to$ the set of binary measurements.

$$\langle T(\rho), P\rangle = \langle \rho, T^\dagger(P)\rangle$$

for any $\rho \in H(V)$ and $P \in H(W)$. $T^\dagger$ is an adjoint map of T.

$$\begin{aligned} &\langle T_3(T_2(T_1(\rho))), P\rangle = \langle T_2(T_1(\rho)), T_3^\dagger(P)\rangle \\ =&\langle T_1(\rho), T_2^\dagger(T_3^\dagger(P))\rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P)))\rangle \end{aligned}$$

No-cloning theorem

$$\begin{aligned} |0\rangle\langle 0| &\longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ |1\rangle\langle 1| &\longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1| \end{aligned}$$

From the linearlity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Axioms for operations

$T\colon H(V) \to H(W)$.

1. Trace preserving: $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$.

2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.

3. Completely positive: $\mathrm{id} \otimes T$ is positive.

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Unitary operations

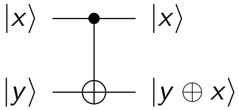
$$\rho \longmapsto U\rho U^\dagger.$$

1. Trace preserving: $\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}(\rho)$.

2. Completely positive: $(\mathrm{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only pure states and unitary operations are used.

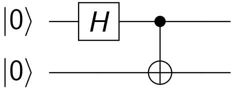
Controlled not



$$\mathrm{CNOT}\,|x\rangle\,|y\rangle \longmapsto |x\rangle\,|y \oplus x\rangle$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit



$$
\begin{aligned}
|0\rangle\,|0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|x\rangle\,|y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle)\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|y\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|\bar{y}\rangle).
\end{aligned}
$$

Conditional probability

A probability of outcome of local measurement in a joint system is

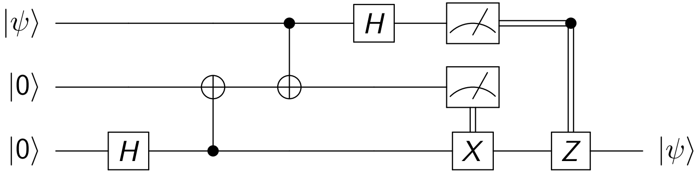
$$P(a, b) = \mathrm{Tr}(\rho(P_a \otimes Q_b)).$$

$$\begin{aligned} P(a \mid b) &= \frac{1}{P(b)}\mathrm{Tr}(\rho(P_a \otimes Q_b)) \\ &= \frac{1}{P(b)}\mathrm{Tr}(\mathrm{Tr}_W(\rho(I \otimes Q_b))P_a). \end{aligned}$$

For $Q_b = |\psi_b\rangle \langle\psi_b|$,

$$\mathrm{Tr}_W(\rho(I \otimes Q_b)) = \mathrm{Tr}_W(\rho(I \otimes |\psi_b\rangle \langle\psi_b|))$$

Quantum teleportation



Assignments [Deadline is the next Friday]

1. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

when $|0\rangle\langle 0|$ is measured at the second system.

2. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|+\rangle\langle +|$ is measured at the second system.

3. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|\psi\rangle\langle \psi|$ is measured at the second system where $|\psi\rangle := \alpha|0\rangle + \beta|1\rangle$.