Joint system

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Joint system

- System = Set of states & set of measurements
- Joint system = "Product" of systems.
- Joint system of a system of a coin (two-dimentional classical system) and a system of a dice (six-dimentional classical system) is twelve-dimentional classical system.
- What is a joint system of quantum systems ?

Tensor product of linear spaces

For linear product spaces V and W over a field F (usually $\mathbb{R}$ or $\mathbb{C}$), a tensor space $V \otimes W$ is a linear space spanned by $v \otimes w$ for all $v \in V$, $w \in W$.

- $\forall c \in F, \forall v \in V, \forall w \in W, c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.
- $\forall u, v \in V, \forall w \in W, (u + v) \otimes w = u \otimes w + v \otimes w$.
- $\forall v \in V, \forall w, y \in W, v \otimes (w + y) = v \otimes w + v \otimes y$.

$\dim(V \otimes W) = \dim(V) \dim(W)$.

If V and W are inner product spaces, $V \otimes W$ is also a inner product space defined by

$$\langle v \otimes w, u \otimes y \rangle = \langle v, u \rangle \langle w, y \rangle.$$

Vector representation in tensor product

Let $V := \mathbb{R}^n$, $W := \mathbb{R}^m$.

$$e_i := \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \\ i-1 \\ i \\ i+1 \\ \\ n \end{array} \in \mathbb{R}^n, \qquad f_j := \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ \\ j-1 \\ j \\ j+1 \\ \\ m \end{array} \in \mathbb{R}^m$$

$$e_i \otimes f_j = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} (1,1) \\ \\ (i,j-1) \\ (i,j) \\ (i,j+1) \\ \\ (n,m) \end{array} \in \mathbb{R}^n \otimes \mathbb{R}^m$$

Vector representation in tensor product

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$$e_i \otimes f_j = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} (1,1) \\ \\ (i,j-1) \\ (i,j) \\ (i,j+1) \\ \\ (n,m) \end{array} \end{array} \in \mathbb{R}^n \otimes \mathbb{R}^m$$

$$v \otimes w = \begin{array}{cc} \begin{bmatrix} \vdots \\ v_i w_j \\ \vdots \end{bmatrix} & \begin{array}{c} \\ (i,j) \\ \\ \end{array} \end{array} = \begin{bmatrix} v_1 w \\ v_2 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{R}^n \otimes \mathbb{R}^m$$

Linear spaces

- $L(V, W)$: A linear space spanned by linear maps from a linear space V to a linear space W.

- $L(V) := L(V, V)$.

- $H(V)$: A real linear space spanned by Hermitian operators acting on a complex linear space V.

- $J(V, W) := L(H(V), H(W))$.

Tensor product of linear maps

$$L(V,X)\otimes L(W,Y)\cong L(V\otimes W,X\otimes Y)$$

since the both the linear spaces have dimension

$$\dim(V)\dim(W)\dim(X)\dim(Y).$$

A natural choice of an isomorphism is

$$\begin{aligned}\Phi : L(V,X)\otimes L(W,Y) &\to L(V\otimes W,X\otimes Y)\\ A\otimes B &\longmapsto (v\otimes w\mapsto A(v)\otimes B(w)).\end{aligned}$$

$$A\otimes B=\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B\\ A_{21}B & A_{22}B & \dots & A_{2m}B\\ \vdots & \vdots & \vdots & \vdots\\ A_{n1}B & A_{n2}B & \dots & A_{nm}B\end{bmatrix}$$

Tensor product of Hermitian maps

$$H(V) \otimes H(W) \cong H(V \otimes W)$$

since the both the linear spaces have dimension

$$\dim(V)^2 \dim(W)^2.$$

A natural choice of an isomorphism is

$$\begin{aligned} \Phi : H(V) \otimes H(W) &\to H(V \otimes W) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

Joint quantum system

A quantum system on a complex linear space V: A states and a measurements are elements of $H(V)$ and ...

For a quantum systems on V and W, a joint system is a quantum system on $V \otimes W$.

Examples: two-qubit system

Examples of states

- $|0\rangle\langle 0| \otimes |1\rangle\langle 1| = |01\rangle\langle 01|$

- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) =$
  $|0\rangle\langle 0| \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2} I$.

- $\frac{1}{2}(|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|)$.

- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| +$
  $|1\rangle\langle 1| \otimes |1\rangle\langle 1|) = |\varphi\rangle\langle\varphi|$ for $|\varphi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$.

Local tomography

For measurements $\{P_a\}_a$ of quantum system on V and $\{Q_b\}_b$ of quantum system on W, a measurement $\{P_a \otimes Q_b\}_{a,b}$ in the joint system is said to be local.

A useful formula.

$$\begin{aligned}
\mathrm{Tr}(A \otimes B) &= \sum_{i,j} \langle i| \otimes \langle j| \, A \otimes B \, |i\rangle \otimes |j\rangle \\
&= \sum_{i,j} \langle i| A |i\rangle \langle j| B |j\rangle \\
&= \mathrm{Tr}(A)\mathrm{Tr}(B)
\end{aligned}$$

Separable states & entangled states

A quantum state ρ in a joint system is said to be separable if

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$$

for some probability distribution p and quantum states $\{\rho_1^i\}$ and $\{\rho_2^i\}$ for subsystems.

If a quantum state is not separable, the state is said to be entangled state.

Partial trace and reduced density matrix

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \mathrm{Tr}(\rho(P_a \otimes Q_b)).$$

$$\begin{aligned}
\sum_b P(a, b) &= \sum_b \mathrm{Tr}(\rho(P_a \otimes Q_b)) \\
&= \mathrm{Tr}\left(\rho\left(P_a \otimes \sum_b Q_b\right)\right) \\
&= \mathrm{Tr}\left(\rho\left(P_a \otimes I\right)\right) \\
&= \mathrm{Tr}(\mathrm{Tr}_W(\rho) P_a).
\end{aligned}$$

Reduced state of a pure state is not necessarily pure

A two-qubit pure state (called Bell state, Bell pair or EPR pair)

$$|\varphi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\begin{aligned}|\varphi\rangle\langle\varphi| = \frac{1}{2}(&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\ &+ |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)\end{aligned}$$

By taking the partial trace for the second qubit, we obtain a reduced density matrix $I/2$.

Scmidt decomposition

Theorem (Schmidt decomposition)

For any pure state $|\psi\rangle \in V \otimes W$, there exist orthonormal basis $\{|v_i\rangle\}$ of V and $\{|w_i\rangle\}$ of W such that

$$|\psi\rangle = \sum_i \lambda_i |v_i\rangle |w_i\rangle .$$

Sketch of a proof.

By a natural isomorphism

$$\begin{aligned} \Phi : & V \otimes W \to L(W, V) \\ & |v\rangle |w\rangle \mapsto |v\rangle \langle w| \end{aligned}$$

the Schmidt decomposition for $V \otimes W$ corresponds to the singular value decomposition for $L(W, V)$. $\square$

Superdense coding

Alice can send two bits to Bob by sending a single qubit and using a shared Bell state.

$$
\begin{aligned}
|\varphi_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) && \\
|\varphi_{01}\rangle &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), && \text{by } X \\
|\varphi_{10}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), && \text{by } Z \\
|\varphi_{11}\rangle &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle), && \text{by } XZ
\end{aligned}
$$

These are orthogonal.

Quantum teleportation

Assignments

- Show reduced density matrices (both sides) of your three favorite quantum states (not necessarily pure state) in any joint system.