A single qubit

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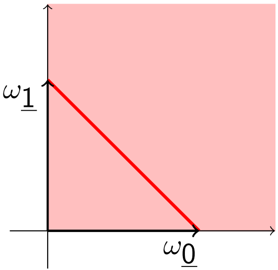
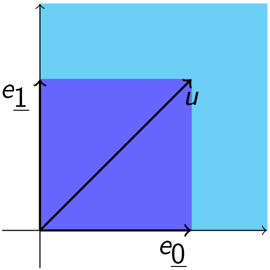
A single bit

Let $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Set of states $= \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}$.

- Set of binary measurements $=$
  $\left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}$.



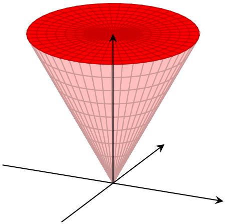


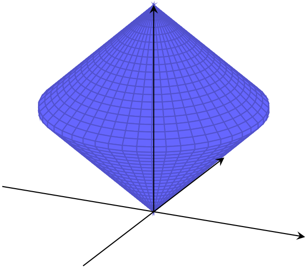
A single qubit

Let $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

- Set of states $= \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}$.

- Set of binary measurements $=$ $\{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}$.





A single qubit

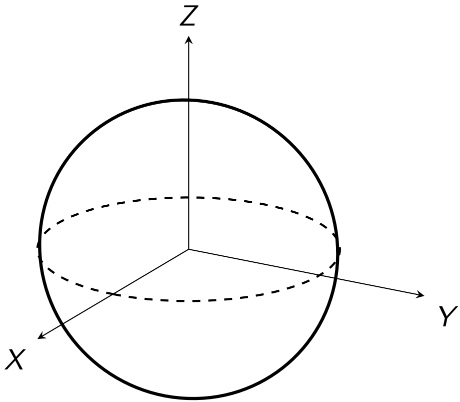
- A qubit can be represented by

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

for $[r_X \; r_Y \; r_Z] \in \mathbb{R}^3$ satisfying $r_X^2 + r_Y^2 + r_Z^2 \leq 1$.

- A qubit can be represented by a point $[r_X \; r_Y \; r_Z]$ in a three-dimensional sphere of radius 1.

The Bloch sphere



Pauli matrices

- $$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- $$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

- $$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

Braket notation

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} |+\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & |-\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for $|\alpha|^2 + |\beta|^2 = 1$.

$$\langle\psi| = |\psi\rangle^\dagger = \alpha^*\,\langle 0| + \beta^*\,\langle 1| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}$$

Pauli matrices in braket notation

- $$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \langle 0| - |1\rangle \langle 1|$$

- $$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle \langle +| - |-\rangle \langle -|$$

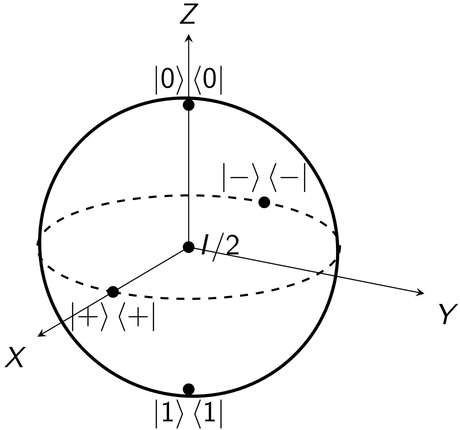
- $$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

Special states

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

Coordinate	State
$[0, 0, 0]$	$\frac{1}{2}I$
$[1, 0, 0]$	$\frac{1}{2}(I + X) = \|+\rangle \langle +\|$
$[-1, 0, 0]$	$\frac{1}{2}(I - X) = \|-\rangle \langle -\|$
$[0, 0, 1]$	$\frac{1}{2}(I + Z) = \|0\rangle \langle 0\|$
$[0, 0, -1]$	$\frac{1}{2}(I - Z) = \|1\rangle \langle 1\|$

Special states in the Bloch sphere



Pure states and state vector

ρ is a pure state

$$\begin{aligned}
&\overset{\text{def}}{\iff} \rho \neq p\rho_1 + (1-p)\rho_2 \quad \forall p \in [0,1] \text{ and states } \rho_1, \rho_2 \neq \rho \\
&\iff \rho \text{ is at surface of the Bloch sphere} \\
&\iff r_X^2 + r_Y^2 + r_Z^2 = 1 \\
&\iff \lambda_1\lambda_2 = 0 \wedge \lambda_1 + \lambda_2 = 1 \\
&\iff \rho \text{ is rank-1 Hermitian with } \mathrm{Tr}(\rho) = 1 \\
&\iff \rho = |\psi\rangle\langle\psi| \text{ for some } |\psi\rangle \in \mathbb{C}^2 \text{ with } \langle\psi|\psi\rangle = 1
\end{aligned}$$

Pure state can be represented by a state vector $|\psi\rangle \in \mathbb{C}^2$ with $\langle\psi|\psi\rangle = 1$.

$|\psi\rangle$ and $|\varphi\rangle := e^{i\theta}|\psi\rangle$ represent the same state since $|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$.

Inner product of pure states

- ρ is a qubit pure state with a coordinate (r_X, r_Y, r_Z).
- σ is a qubit pure state with a coordinate $(-r_X, -r_Y, -r_Z)$.

$$\mathrm{Tr}(\rho\sigma) = \mathrm{Tr}(\rho(I-\rho)) = \mathrm{Tr}(\rho) - \mathrm{Tr}(\rho^2) = 1 - 1 = 0$$

- $\rho = |\psi\rangle\langle\psi|$.
- $\sigma = |\varphi\rangle\langle\varphi|$.

$$\begin{aligned}\mathrm{Tr}(\rho\sigma) &= \mathrm{Tr}(|\psi\rangle\langle\psi|\,|\varphi\rangle\langle\varphi|) = \langle\psi|\varphi\rangle\,\mathrm{Tr}(|\psi\rangle\langle\varphi|) \\ &= \langle\psi|\varphi\rangle\langle\varphi|\psi\rangle = |\langle\psi|\varphi\rangle|^2\end{aligned}$$

Unitary operation

Unitary operation

$$\rho \mapsto U\rho U^\dagger$$

It is easy to see that

- $\mathrm{Tr}(U\rho U^\dagger) = 1$
- $U\rho U^\dagger \succeq 0$

A pure state $|\psi\rangle$ is mapped to a pure state $U|\psi\rangle$.

U and $e^{i\theta}U$ play the same role.

Examples of unitary operation

- The identity matrix I.
- Pauli matrices X, Y and Z.
- Hadamard matrix $H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Pauli matrices X on the Bloch sphere

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$\begin{aligned} X\rho X^\dagger = X\rho X &= \frac{1}{2}\left(X^2 + r_X X^3 + r_Y XYX + r_Z XZX\right) \\ &= \frac{1}{2}\left(I + r_X X - r_Y Y - r_Z Z\right) \end{aligned}$$

$$[r_X\ r_Y\ r_Z] \xmapsto{X} [r_X, -r_Y, -r_Z]$$

π-rotation with respect to X axis.

Similarly, Y and Z corresponds to π-rotation with respect to Y and Z axes, respectively.

Hadamard matrix

Hadamard matrix H is unitary and Hermitian.

$$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1|$$
$$= |0\rangle \langle +| + |1\rangle \langle -|$$

$$|0\rangle, |1\rangle \xleftrightarrow{H} |+\rangle, |-\rangle$$

$$HXH = H(|+\rangle \langle +| - |-\rangle \langle -|)H$$
$$= |0\rangle \langle 0| - |1\rangle \langle 1| = Z$$

Similarly, $HZH = X$.

Hadamard matrix on the Bloch sphere

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$\begin{aligned} H\rho H^{\dagger} = H\rho H &= \frac{1}{2}\left(H^2 + r_X HXH + r_Y HYH + r_Z HZH\right) \\ &= \frac{1}{2}\left(I + r_X Z - r_Y Y + r_Z X\right) \end{aligned}$$

$$[r_X\ r_Y\ r_Z] \xmapsto{H} [r_Z, -r_Y, r_X]$$

Hadamard operation can be decomposed to $\pi/2$-rotation with respect to Y axis

$$[r_X\ r_Y\ r_Z] \overset{R_Y(\pi/2)}{\longmapsto} [-r_Z, r_Y, r_X]$$

and X.

Multiplications of Pauli matrices

For any unitary matrices U and V, UV is also unitary matrix.

- $XY = iZ$
- $YZ = iX$
- $ZX = iY$

Rotation matrices

$$R_X(\theta) := \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X$$

$$R_X(\theta)^\dagger = R_X(-\theta)$$

$$\begin{aligned} R_X(\theta) R_X(\theta)^\dagger &= (\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X)(\cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} X) \\ &= \cos^2\frac{\theta}{2} I + \sin^2\frac{\theta}{2} X^2 = I \end{aligned}$$

$$R_X(\theta) X = X R_X(\theta), \quad R_X(\theta) Y = Y R_X(-\theta), \quad R_X(\theta) Z = Z R_X(-\theta)$$

$$R_X(\theta) R_X(\tau) = R_X(\theta + \tau)$$

$$[1\,0\,0] \overset{R_X(\theta)}{\longmapsto} [1\,0\,0]$$

$$[0\,1\,0] \overset{R_X(\theta)}{\longmapsto} [0\ \cos\theta\ \sin\theta]$$

$$[0\,0\,1] \overset{R_X(\theta)}{\longmapsto} [0\ -\sin\theta\ \cos\theta]$$

Assignments [Deadline is the next of next Friday]

- Show that $R_Y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$ and $R_Z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$.

- Show that for any 2×2 unitary matrix, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha} R_Z(\beta) R_Y(\gamma) R_Z(\delta)$$

- [Advanced] For a real unit vector $\hat{n} = [n_X \; n_Y \; n_Z]$, let

$$R_{\hat{n}}(\theta) := \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}(n_X X + n_Y Y + n_Z Z).$$

Show that for any 2×2 unitary matrix, there exist $\alpha, \theta \in \mathbb{R}$ and a real unit 3d vector $\hat{n}$ such that $U = e^{i\alpha} R_{\hat{n}}(\theta)$.

General 2×2 unitary matrices

$\{I,\ X,\ Y,\ Z\}$ is a orthogonal basis of 2×2 complex matrices.

$$U = \alpha_I I + \alpha_X X + \alpha_Y Y + \alpha_Z Z$$

for some $\alpha_I, \alpha_X, \alpha_Y, \alpha_Z \in \mathbb{C}$.

$$\begin{aligned}
UU^\dagger = &(|\alpha_I|^2 + |\alpha_X|^2 + |\alpha_Y|^2 + |\alpha_Z^2|)I \\
&+ (\alpha_I \alpha_X^* + \alpha_I^* \alpha_X + i(\alpha_Y \alpha_Z^* - \alpha_Z \alpha_Y^*))X \\
&+ (\alpha_I \alpha_Y^* + \alpha_I^* \alpha_Y + i(\alpha_Z \alpha_X^* - \alpha_X \alpha_Z^*))Y \\
&+ (\alpha_I \alpha_Z^* + \alpha_I^* \alpha_Z + i(\alpha_X \alpha_Y^* - \alpha_Y \alpha_X^*))Z
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}(\alpha_I \alpha_X^*) &= \mathrm{Im}(\alpha_Y \alpha_Z^*) \\
\mathrm{Re}(\alpha_I \alpha_Y^*) &= \mathrm{Im}(\alpha_Z \alpha_X^*) \\
\mathrm{Re}(\alpha_I \alpha_Z^*) &= \mathrm{Im}(\alpha_X \alpha_Y^*)
\end{aligned}$$

General rotation matrices

$$|\alpha_I|^2 + |\alpha_X|^2 + |\alpha_Y|^2 + |\alpha_Z^2| = 1$$

Assume α_I is real.

$$\alpha_I \mathrm{Re}(\alpha_X) = \mathrm{Im}(\alpha_Y \alpha_Z^*)$$
$$\alpha_I \mathrm{Re}(\alpha_Y) = \mathrm{Im}(\alpha_Z \alpha_X^*)$$
$$\alpha_I \mathrm{Re}(\alpha_Z) = \mathrm{Im}(\alpha_X \alpha_Y^*)$$

With the direct calculation, we obtain
$\alpha_I^2(\mathrm{Re}(\alpha_X)^2 + \mathrm{Re}(\alpha_Y)^2 + \mathrm{Re}(\alpha_Z)^2) = 0.$