

# A single qubit

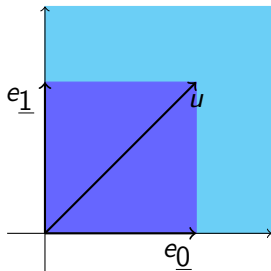
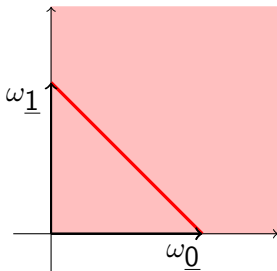
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## A single bit

Let  $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

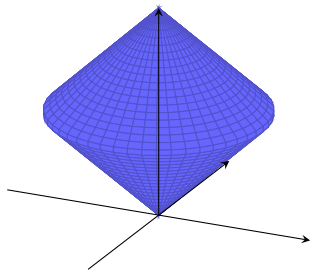
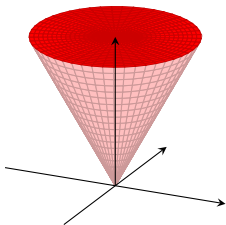
- Set of states =  $\{\omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1\}$ .
- Set of binary measurements =  $\{e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0}\}$ .



## A single qubit

Let  $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $\langle e, \omega \rangle := \text{Tr}(e\omega)$ .

- Set of states =  $\{\omega \in V \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1\}$ .
- Set of binary measurements =  $\{e \in V \mid e \in C_{\geq 0}, u - e \in C_{\geq 0}\}$ .



## A single qubit

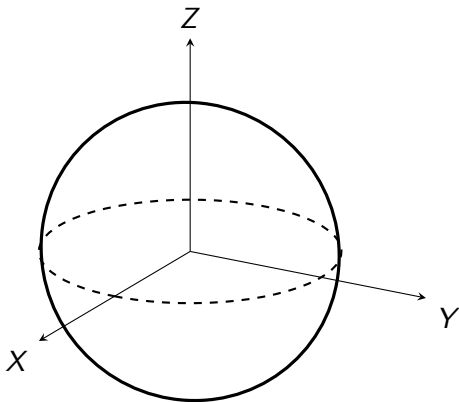
- A qubit can be represented by

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

for  $[r_X \ r_Y \ r_Z] \in \mathbb{R}^3$  satisfying  $r_X^2 + r_Y^2 + r_Z^2 \leq 1$ .

- A qubit can be represented by a point  $[r_X \ r_Y \ r_Z]$  in a three-dimensional sphere of radius 1.

## The Bloch sphere



## Complex space and Hermitian operator

- $\mathcal{X}$ : A finite-dimensional inner product space on  $\mathbb{C}$ .
- $\mathcal{L}(\mathcal{X})$ : A set of linear operators on  $\mathcal{X}$ .

For  $A \in \mathcal{L}(\mathcal{X})$ , an **adjoint** map  $A^\dagger$  of  $A$  is a unique operator satisfying

$$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$$

for any  $v, w \in \mathcal{X}$ .  $H \in \mathcal{L}(\mathcal{X})$  is Hermitian if and only if  $H^\dagger = H$ .

- $\mathcal{H}(\mathcal{X})$ : A set of Hermitian operators on  $\mathcal{X}$ .

$\mathcal{L}(\mathcal{X})$  and  $\mathcal{H}(\mathcal{X})$  are often regarded as inner product space on  $\mathbb{C}$  and  $\mathbb{R}$ , respectively for the Hilbert–Schmidt inner product  $\langle A, B \rangle = \text{Tr}(A^\dagger B)$ .

# Spectral decomposition theorem

## Definition (Normal operator)

$A \in \mathcal{L}(\mathcal{X})$  is said to be **normal** if  $AA^\dagger = A^\dagger A$ .

Hermitian matrix ( $H^\dagger = H$ ) and unitary matrix ( $UU^\dagger = I$ ) are normal.

## Theorem (Spectral decomposition theorem)

$A \in \mathcal{L}(\mathbb{C}^n)$  is **normal** if and only if there exist orthonormal basis  $\{|\psi_j\rangle\}$  of  $\mathbb{C}^n$  and complex numbers  $\{\lambda_j\}$  such that

$$A = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|.$$

## Pauli matrices

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$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- 

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

- 

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$



## Bracket notation

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$|1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} |+\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \end{aligned}$$

$$\begin{aligned} |-\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for  $|\alpha|^2 + |\beta|^2 = 1$ .

$$\langle\psi| = |\psi\rangle^\dagger = \alpha^* \langle 0| + \beta^* \langle 1| = [\alpha^* \quad \beta^*]$$

## Pauli matrices in bracket notation

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$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

- 

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle +| - |-\rangle\langle -|$$

- 

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

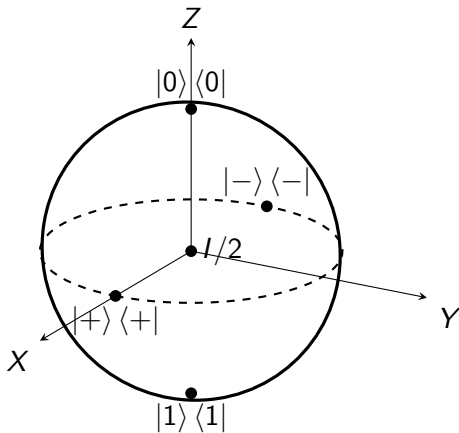
## Special states

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$r_X^2 + r_Y^2 + r_Z^2 \leq 1.$$

Coordinate	State
$[0 \ 0 \ 0]$	$\frac{1}{2}I$
$[1 \ 0 \ 0]$	$\frac{1}{2}(I + X) =  +\rangle \langle + $
$[-1 \ 0 \ 0]$	$\frac{1}{2}(I - X) =  -\rangle \langle - $
$[0 \ 0 \ 1]$	$\frac{1}{2}(I + Z) =  0\rangle \langle 0 $
$[0 \ 0 \ -1]$	$\frac{1}{2}(I - Z) =  1\rangle \langle 1 $

## Special states in the Bloch sphere



## Pure states are rank-1 density operators

$\rho$  is a **pure state**

$$\stackrel{\text{def}}{\iff} \rho \neq p\rho_1 + (1-p)\rho_2 \quad \forall p \in (0,1) \text{ and states } \rho_1 \neq \rho_2.$$

### Lemma

A quantum state  $\rho$  is a pure state if and only if  $\rho$  is **rank-1**.

### Proof.

Let the spectral decomposition of  $\rho$  be

$$\rho = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|$$

where  $\lambda_j \geq 0$  and  $\sum_j \lambda_j = 1$ . If  $\rho$  is not rank-1,  $\rho$  is a convex combination of quantum states  $(|\psi_j\rangle \langle \psi_j|)_j$ .

Assume  $\rho = |\varphi\rangle \langle \varphi|$  and  $\rho = p_1\rho_1 + p_2\rho_2$ .  $\text{Tr}(\sigma |\varphi\rangle \langle \varphi|) = 1$  if and only if  $\sigma = |\varphi\rangle \langle \varphi|$  since  $\text{Tr}(\sigma |\varphi\rangle \langle \varphi|) = \langle \varphi | \sigma | \varphi \rangle = \sum_j \lambda_j |\langle \psi_j | \varphi \rangle|^2$ . Then,  $\text{Tr}((p_1\rho_1 + p_2\rho_2) |\varphi\rangle \langle \varphi|) = 1$  implies that  $\text{Tr}(\rho_1 |\varphi\rangle \langle \varphi|) = \text{Tr}(\rho_2 |\varphi\rangle \langle \varphi|) = 1$ , and hence  $\rho_1 = \rho_2 = \rho$ . □

## Pure states and state vector

Pure state  $|\psi\rangle \langle\psi|$  can be represented by a **state vector**  $|\psi\rangle \in \mathbb{C}^n$  with  $\langle\psi|\psi\rangle = 1$ .

$|\psi\rangle$  and  $|\varphi\rangle := e^{i\theta} |\psi\rangle$  represent the same state since  $|\psi\rangle \langle\psi| = |\varphi\rangle \langle\varphi|$ .

A state that is not pure state is called a **mixed state**.

$\rho$  is called a **density matrix**.

## Inner product of pure states

- $\rho$  is a qubit pure state with a coordinate  $[r_X \ r_Y \ r_Z]$ .
- $\sigma$  is a qubit pure state with a coordinate  $[-r_X \ -r_Y \ -r_Z]$ .

$$\text{Tr}(\rho\sigma) = \text{Tr}(\rho(I - \rho)) = \text{Tr}(\rho) - \text{Tr}(\rho^2) = 1 - 1 = 0$$

- $\rho = |\psi\rangle \langle\psi|$ .
- $\sigma = |\varphi\rangle \langle\varphi|$ .

$$\begin{aligned}\text{Tr}(\rho\sigma) &= \text{Tr}(|\psi\rangle \langle\psi| |\varphi\rangle \langle\varphi|) = \langle\psi|\varphi\rangle \text{Tr}(|\psi\rangle \langle\varphi|) \\ &= \langle\psi|\varphi\rangle \langle\varphi|\psi\rangle = |\langle\psi|\varphi\rangle|^2\end{aligned}$$

## Single qubit measurement

Set of measurements =  $\{(e_1, \dots, e_k) \mid e_1 + \dots + e_k = I, e_j \in C_{\geq 0}$   
 $i = 1, 2, \dots, k, k = 1, 2, \dots\}$

If  $e_i e_j = \delta_{ij} e_i$ , the measurement is called an **orthogonal measurement**.

If  $|0\rangle\langle 0|$  is measured by  $(|0\rangle\langle 0|, |1\rangle\langle 1|)$ , the output is 0 with probability  $\text{Tr}(|0\rangle\langle 0| |0\rangle\langle 0|) = |\langle 0|0\rangle|^2 = 1$ .

If  $|+\rangle\langle +|$  is measured by  $(|0\rangle\langle 0|, |1\rangle\langle 1|)$ , the output is 0 with probability  $\text{Tr}(|0\rangle\langle 0| |+\rangle\langle +|) = |\langle 0|+\rangle|^2 = 1/2$ .

If  $|\psi\rangle\langle \psi|$  is measured by  $(|\varphi_0\rangle\langle \varphi_0|, |\varphi_1\rangle\langle \varphi_1|)$ , the output is 0 with probability  $\text{Tr}(|\varphi_0\rangle\langle \varphi_0| |\psi\rangle\langle \psi|) = |\langle \varphi_0|\psi\rangle|^2$ .



## Unitary operation

For unitary operation  $U$ , let us consider

$$\rho \mapsto U\rho U^\dagger.$$

It is easy to see that

- $\text{Tr}(U\rho U^\dagger) = 1$  (Trace-preserving)
- $U\rho U^\dagger \succeq 0$  (Positive)

A pure state  $|\psi\rangle$  is mapped to a pure state  $U|\psi\rangle$ .

$U$  and  $e^{i\theta}U$  are physically equivalent.

## Single qubit quantum channel

Since real vector space spanned by  $2 \times 2$  Hermitian matrices is 4-dimensional, any linear map  $\Phi$  on the linear space is represented by  $4 \times 4$  real matrix. Let  $\sigma_0 = I$ ,  $\sigma_1 = X$ ,  $\sigma_2 = Y$ ,  $\sigma_3 = Z$ .

$$\Phi(\mathbf{a} \cdot \sigma_0^3) = (T\mathbf{a}) \cdot \sigma_0^3$$

Here,

$$T_{i,j} = \frac{1}{2} \text{Tr}(\sigma_i \Phi(\sigma_j)).$$

From the trace-preserving property

$$T_{0,j} = \frac{1}{2} \text{Tr}(\sigma_0 \Phi(\sigma_j)) = \frac{1}{2} \text{Tr}(\sigma_j) = \delta_{0,j},$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & M_{11} & M_{12} & M_{13} \\ t_2 & M_{21} & M_{22} & M_{23} \\ t_3 & M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\frac{I + \mathbf{r} \cdot \sigma_1^3}{2} \xrightarrow{\Phi} \frac{I + (M\mathbf{r} + \mathbf{t}) \cdot \sigma_1^3}{2}$$

## Matrix representation of unitary channel

Unitary channel is **unital**, i.e.,  $\Phi(I) = UIU^\dagger = I$ .

$$T_{i,0} = \frac{1}{2} \text{Tr}(\sigma_i \Phi(I)) = \frac{1}{2} \text{Tr}(\sigma_i) = \delta_{i,0}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & M_{13} \\ 0 & M_{21} & M_{22} & M_{23} \\ 0 & M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Unitary channel is represented by the  $3 \times 3$  real matrix  $M$ .

$$\frac{I + \mathbf{r} \cdot \boldsymbol{\sigma}_1^3}{2} \longmapsto \frac{I + (M\mathbf{r}) \cdot \boldsymbol{\sigma}_1^3}{2}$$

## Examples of unitary operations

- The identity matrix  $I$ .
- Pauli matrices  $X$ ,  $Y$  and  $Z$ .
- Hadamard matrix  $H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- Product  $UV$  of unitary operators  $U$  and  $V$ .

## Multiplications of Pauli matrices

For any unitary matrices  $U$  and  $V$ ,  $UV$  is also unitary matrix.

- $XY = iZ$
- $YZ = iX$
- $ZX = iY$

## Pauli matrices $X$ on the Bloch sphere

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$\begin{aligned} X\rho X^\dagger &= X\rho X = \frac{1}{2} (X^2 + r_X X^3 + r_Y XYX + r_Z XZX) \\ &= \frac{1}{2} (I + r_X X - r_Y Y - r_Z Z) \end{aligned}$$

$$[r_X \ r_Y \ r_Z] \xrightarrow{X} [r_X \ -r_Y \ -r_Z]$$

$\pi$ -rotation with respect to  $X$  axis.  $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

Similarly,  $Y$  and  $Z$  corresponds to  $\pi$ -rotation with respect to  $Y$  and  $Z$  axes, respectively.

## Hadamard matrix

Hadamard matrix  $H$  is unitary and Hermitian.

$$\begin{aligned} H &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1| \\ &= |0\rangle \langle +| + |1\rangle \langle -| \end{aligned}$$

$$|0\rangle, |1\rangle \xleftrightarrow{H} |+\rangle, |-\rangle$$

$$\begin{aligned} HXH &= H(|+\rangle \langle +| - |-\rangle \langle -|)H \\ &= |0\rangle \langle 0| - |1\rangle \langle 1| = Z \end{aligned}$$

Similarly,  $HZH = X$ .

$$HYH = H(iXZ)H = iHXHRZH = iZX = -Y$$

## Hadamard matrix on the Bloch sphere

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$\begin{aligned} H\rho H^\dagger &= H\rho H = \frac{1}{2} (H^2 + r_X HXH + r_Y HYH + r_Z HZH) \\ &= \frac{1}{2} (I + r_X Z - r_Y Y + r_Z X) \end{aligned}$$

$$[r_X \ r_Y \ r_Z] \xrightarrow{H} [r_Z \ -r_Y \ r_X]$$

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$



## Rotation matrix for the $Z$ axis

$$R_Z(\theta) := \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Z$$

$$R_Z(\theta)^\dagger = \cos \frac{\theta}{2} I + i \sin \frac{\theta}{2} Z = R_Z(-\theta)$$

$$\begin{aligned} R_Z(\theta)R_Z(\tau) &= \left( \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Z \right) \left( \cos \frac{\tau}{2} I - i \sin \frac{\tau}{2} Z \right) \\ &= \left( \cos \frac{\theta}{2} \cos \frac{\tau}{2} - \sin \frac{\theta}{2} \sin \frac{\tau}{2} \right) I - i \left( \cos \frac{\theta}{2} \sin \frac{\tau}{2} + \sin \frac{\theta}{2} \cos \frac{\tau}{2} \right) Z \\ &= \cos \frac{\theta + \tau}{2} I - i \sin \frac{\theta + \tau}{2} Z = R_Z(\theta + \tau) \end{aligned}$$

$$R_Z(\theta)R_Z(\theta)^\dagger = R_Z(\theta)R_Z(-\theta) = R_Z(\theta - \theta) = R_Z(0) = I$$

$$R_Z(\theta)X = XR_Z(-\theta), \quad R_Z(\theta)Y = YR_Z(-\theta), \quad R_Z(\theta)Z = ZR_Z(\theta)$$

$$M = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## General unitary matrix

$$\langle UAU^\dagger, UBU^\dagger \rangle = \text{Tr}(UAU^\dagger UBU^\dagger) = \text{Tr}(AB) = \langle A, B \rangle.$$

$$\langle UAU^\dagger, UBU^\dagger \rangle = \left\langle \frac{I + (M\mathbf{a}) \cdot \sigma_1^3}{2}, \frac{I + (M\mathbf{b}) \cdot \sigma_1^3}{2} \right\rangle = \frac{1}{2}(1 + \langle M\mathbf{a}, M\mathbf{b} \rangle)$$

$$\langle A, B \rangle = \left\langle \frac{I + \mathbf{a} \cdot \sigma_1^3}{2}, \frac{I + \mathbf{b} \cdot \sigma_1^3}{2} \right\rangle = \frac{1}{2}(1 + \langle \mathbf{a}, \mathbf{b} \rangle)$$

$\langle M\mathbf{a}, M\mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$  implies that  $M$  is an orthogonal matrix.

Since for a unitary matrix  $U$  with  $\det(U) = 1$ ,

$$U = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j| = V R_Z(\theta) V^\dagger.$$

$$M_U = M_V M_{R_Z(\theta)} M_V^{-1}$$

$$\det(M_U) = \det(M_V) \det(M_{R_Z(\theta)}) \det(M_V^{-1}) = \det(M_{R_Z(\theta)}) = 1.$$

Conversely, any  $3 \times 3$  **orthogonal** matrix  $M$  with  $\det(M) = 1$  corresponds to some unitary channel  $\rho \mapsto U\rho U^\dagger$ . Such matrix is a **rotation matrix**.

## Assignments

- ① Express  $(a_X X + a_Y Y + a_Z Z)^2$  as a linear combination of  $I, X, Y, Z$  for  $a_X, a_Y, a_Z \in \mathbb{C}$ .

- ② For a real unit vector  $\mathbf{v} := [v_X \ v_Y \ v_Z]$ , let

$$R_{\mathbf{v}}(\theta) := \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} (v_X X + v_Y Y + v_Z Z).$$

Show that  $R_{\mathbf{v}}(\theta)R_{\mathbf{v}}(\tau) = R_{\mathbf{v}}(\theta + \tau)$  for any  $\theta, \tau \in \mathbb{R}$ . Show also that  $R_{\mathbf{v}}(\theta)$  is a unitary matrix.

- ③ Show two pure states  $\rho_0, \rho_1$  that are invariant by  $R_{\mathbf{v}}(\theta)$  for any  $\theta \in \mathbb{R}$ , i.e.,  $R_{\mathbf{v}}(\theta)\rho_i R_{\mathbf{v}}(\theta)^\dagger = \rho_i$ , as linear combinations of  $I, X, Y$  and  $Z$ .
- ④ [Advanced] Show that for any unitary matrix  $U$  with  $\det(U) = 1$ , there exists a real three-dimensional unit vector  $\mathbf{v}$  and  $\theta \in \mathbb{R}$ , such that  $U = R_{\mathbf{v}}(\theta)$ .