Quantum teleportation

Ryuhei Mori

Tokyo Institute of Technology

15, Oct., 2021

Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

$$T : \text{The set of states} \to \text{the set of states.}$$

$$pT(\rho_1) + (1-p)T(\rho_2) = T(p\rho_1 + (1-p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0, 1]$.

$T : \mathcal{H}(V) \to \mathcal{H}(W)$ must be linear (a proof is needed).

Schrödinger picture and Heisenberg picture

$T^\dagger$: The set of binary measurements $\to$ the set of binary measurements.

$$\langle T(\rho), P\rangle = \langle \rho, T^\dagger(P)\rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. $T^\dagger$ is an adjoint map of T.

$$\langle T_3(T_2(T_1(\rho))), P\rangle = \langle T_2(T_1(\rho)), T_3^\dagger(P)\rangle$$
$$= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P))\rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P)))\rangle$$

No-cloning theorem

$$\begin{aligned} |0\rangle\langle 0| &\longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ |1\rangle\langle 1| &\longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1| \end{aligned}$$

From the linearlity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Axioms for quantum channel

$T: \mathcal{H}(V) \to \mathcal{H}(W)$.

1. Trace-preserving: $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$.

2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.

3. Completely positive: $\mathrm{id} \otimes T$ is positive.

Axioms for quantum channel

$T \colon \mathcal{H}(V) \to \mathcal{H}(W)$.

1. Trace-preserving: $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$.
2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.
3. Completely positive: $\mathrm{id} \otimes T$ is positive.

Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$: transposition according to $\{|0\rangle, |1\rangle\}$.

The transposition is obviously trace-preserving.

The transposition is positive.

Proof.

For any $A \succeq 0$ and $|\psi\rangle \in \mathbb{C}^2$,

$$\langle\psi| T(A) |\psi\rangle = \langle\psi| A^T |\psi\rangle = \langle\psi| A^* |\psi\rangle = (\langle\psi|^* A |\psi\rangle^*)^* \geq 0 \qquad \square$$

Positive but not completely positive 2/2

But, the transposition is not completely positive.

Proof.

For $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$
\begin{aligned}
&(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|) \\
&= (\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)\Bigg(\frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\
&\qquad + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big)\Bigg) \\
&= \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
&\qquad + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big)
\end{aligned}
$$

$$
|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |10\rangle \qquad |10\rangle \mapsto |01\rangle \qquad |11\rangle \mapsto |11\rangle
$$

Hence, $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$. $(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$ is not positive semidefinite. $\square$

Unitary operations

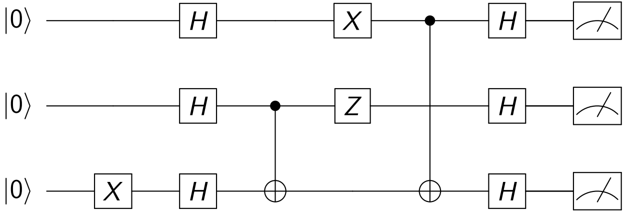
$$\rho \longmapsto U\rho U^\dagger.$$

1. Trace-preserving: $\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}(\rho)$.

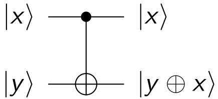
2. Completely positive: $(\mathrm{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only pure states and unitary operations are used.

Quantum circuit



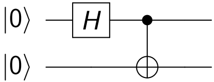
Controlled not



$$\mathrm{CNOT}\,|x\rangle\,|y\rangle \longmapsto |x\rangle\,|y \oplus x\rangle$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit



$$
\begin{aligned}
|0\rangle\,|0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|x\rangle\,|y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle)\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|y\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|\bar{y}\rangle).
\end{aligned}
$$

Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \mathrm{Tr}(\rho_{V\otimes W}(P_a \otimes Q_b)).$$

$$P(a \mid b) = \frac{1}{P(b)}\mathrm{Tr}(\rho_{V\otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)}\mathrm{Tr}(\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b))P_a).$$

$\rho_{V|Q_b} := \frac{1}{P(b)}\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b)).$

For $\rho_{V\otimes W} = |\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}$ and $Q_b = |\psi_b\rangle_W\langle\psi_b|_W$,

$$\mathrm{Tr}_W(\rho_{V\otimes W}(I \otimes Q_b)) = \mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W))$$

Trick

$$A_{W\to V} = \sum_{i,j} A_{i,j} \left|i\right\rangle_V \left\langle j\right|_W$$
$$\left|A_{W\to V}\right\rangle\!\rangle := \mathcal{M}^{-1}(A_{W\to V}) = \sum_{i,j} A_{i,j} \left|i\right\rangle_V \left|j\right\rangle_W$$

Examples

$$\left|\frac{1}{\sqrt{2}} I\right\rangle\!\!\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|1\right\rangle\right)$$
$$\left|\frac{1}{\sqrt{2}} X\right\rangle\!\!\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|1\right\rangle + \left|1\right\rangle\left|0\right\rangle\right)$$
$$\left|\frac{1}{\sqrt{2}} Y\right\rangle\!\!\right\rangle = \frac{1}{\sqrt{2}}\left(-i\left|0\right\rangle\left|1\right\rangle + i\left|1\right\rangle\left|0\right\rangle\right)$$
$$\left|\frac{1}{\sqrt{2}} Z\right\rangle\!\!\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle - \left|1\right\rangle\left|1\right\rangle\right)$$

Trick

$$
\begin{aligned}
(B_V \otimes C_W)|A_{W\to V}\rangle\!\rangle &= (B_V \otimes C_W)\sum_{i,j} A_{i,j}\,|i\rangle_V\,|j\rangle_W \\
&= \sum_{i,j} A_{i,j}(B_V\,|i\rangle_V) \otimes (C_W\,|j\rangle_W) \\
&\xmapsto{\mathcal{M}} \sum_{i,j} A_{i,j}(B_V\,|i\rangle_V)(\langle j|_W\, C_W^\dagger)^* \\
&= B_V \sum_{i,j} A_{i,j}\,|i\rangle_V\,\langle j|_W\, C_W^T \\
&= B_V A_{W\to V} C_W^T \\
&\xmapsto{\mathcal{M}^{-1}} \left|B_V A_{W\to V} C_W^T\right\rangle\!\!\right\rangle
\end{aligned}
$$

Trick

$$\langle\langle A_{W\to V} || B_{S\to W}\rangle\rangle = \sum_{i,j} A^*_{i,j} \langle i|_V \langle j|_W \sum_{k,\ell} B_{k,\ell} |k\rangle_W |\ell\rangle_S$$
$$= \sum_{i,j,\ell} A^*_{i,j} B_{j,\ell} \langle i|_V |\ell\rangle_S = (A^*_{W\to V} B_{S\to W})^T = B^T_{S\to W} A^\dagger_{W\to V}$$

$$\langle\langle A_{V\to W} || B_{W\to S}\rangle\rangle = \sum_{i,j} A^*_{i,j} \langle i|_W \langle j|_V \sum_{k,\ell} B_{k,\ell} |k\rangle_S |\ell\rangle_W$$
$$= \sum_{i,j,k} A^*_{i,j} B_{k,i} \langle j|_V |k\rangle_S = B_{W\to S} A^*_{V\to W}$$

Quantum teleportation

Let $A = B = C = \mathbb{C}^d$. For $L_{B\to A}$ satisfying $\mathrm{Tr}(L^\dagger_{B\to A} L_{B\to A}) = 1$,

$$\begin{aligned}
&\mathrm{Tr}_{AB}\left(\rho_A \otimes \left|\frac{1}{\sqrt{d}} I_{C\to B}\right\rangle\!\!\right\rangle\left\langle\!\!\left\langle\frac{1}{\sqrt{d}} I_{C\to B}\right| \quad \left|L_{B\to A}\right\rangle\!\!\right\rangle\left\langle\!\!\left\langle L_{B\to A}\right|\right) \\
&= \mathrm{Tr}_{AB}\left(I_A \otimes \left|\frac{1}{\sqrt{d}} I_{C\to B}\right\rangle\!\!\right\rangle\left\langle\!\!\left\langle\frac{1}{\sqrt{d}} I_{C\to B}\right| \quad \left|\rho_A L_{B\to A}\right\rangle\!\!\right\rangle\left\langle\!\!\left\langle L_{B\to A}\right|\right) \\
&= \mathrm{Tr}_{AB}\left(I_A \otimes \left|\frac{1}{\sqrt{d}} I_{C\to B}\right\rangle\!\!\right\rangle \frac{1}{\sqrt{d}} \rho_A L_{B\to A} I^*_{C\to B} \left\langle\!\!\left\langle L_{B\to A}\right|\right) \\
&= \left\langle\!\!\left\langle L_{B\to A}\right|\right|\left.\frac{1}{\sqrt{d}} I_{C\to B}\right\rangle\!\!\right\rangle \frac{1}{\sqrt{d}} \rho_A L_{C\to A} \\
&= \frac{1}{\sqrt{d}} (I_{C\to B})^T L^\dagger_{B\to A} \frac{1}{\sqrt{d}} \rho_A L_{C\to A} = \frac{1}{d} L^\dagger_{C\to A} \rho_A L_{C\to A}
\end{aligned}$$

When $L_{B\to A} = \frac{1}{\sqrt{d}} U_{B\to A}$ for some unitary matrix (isometry) $U_{B\to A}$,

$$\frac{1}{d} L^\dagger_{C\to A} \rho_A L_{C\to A} = \frac{1}{d^2} U^\dagger_{C\to A} \rho_A U_{C\to A}.$$

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V\otimes W} \langle\varphi|_{V\otimes W}$ and $Q_b = |\psi_b\rangle_W \langle\psi_b|_W$,

$$\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b)) = \mathrm{Tr}_W(|\varphi\rangle_{V\otimes W} \langle\varphi|_{V\otimes W} (I \otimes |\psi_b\rangle_W \langle\psi_b|_W))$$

From an expression $|\varphi\rangle_{V\otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W$,

$$\begin{aligned}
&\mathrm{Tr}_W(|\varphi\rangle_{V\otimes W} \langle\varphi|_{V\otimes W} (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W)) \\
&= \mathrm{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j}\varphi^*_{k,l} |i\rangle_V |\psi_j\rangle_W \langle k|_V \langle\psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W) \right) \\
&= \mathrm{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j}\varphi^*_{k,l} |i\rangle_V \langle k|_V \otimes |\psi_j\rangle_W \langle\psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W) \right) \\
&= \sum_{i,j,k,l} \varphi_{i,j}\varphi^*_{k,l} |i\rangle_V \langle k|_V \,\mathrm{Tr}\left(|\psi_j\rangle_W \langle\psi_l|_W |\psi_b\rangle_W \langle\psi_b|_W \right) \\
&= \sum_{i,k} \varphi_{i,b}\varphi^*_{k,b} |i\rangle_V \langle k|_V = \left(\sum_i \varphi_{i,b} |i\rangle_V \right) \left(\sum_k \varphi^*_{k,b} \langle k|_V \right)
\end{aligned}$$

$$|\varphi\rangle_{V\otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W \longmapsto \frac{1}{\sqrt{P(b)}} \sum_i \varphi_{i,b} |i\rangle_V$$

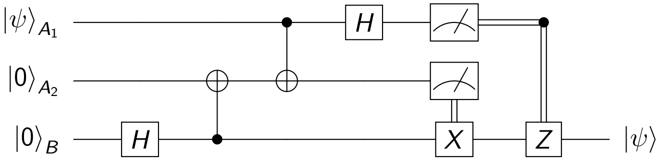
Examples of conditional density operator

For $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^{1} \alpha_{i,j} |i\rangle_V |j\rangle_W$, we measure the system W by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$.

if the outcome is 0, the state $\frac{1}{\sqrt{|\alpha_{0,0}|^2+|\alpha_{1,0}|^2}} \sum_{i=0}^{1} \alpha_{i,0} |i\rangle_V$.

if the outcome is 1, the state $\frac{1}{\sqrt{|\alpha_{0,1}|^2+|\alpha_{1,1}|^2}} \sum_{i=0}^{1} \alpha_{i,1} |i\rangle_V$.

Quantum teleportation



Quantum teleportation

$$|\Phi\rangle_{A_2\otimes B} = \frac{1}{\sqrt{2}}\left(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B\right)$$

$$(H_{A_1}\otimes I_{A_2}\otimes I_B)(\mathrm{CNOT}_{A_1A_2}\otimes I_B)\,|\psi\rangle_{A_1}|\Phi\rangle_{A_2\otimes B}$$

For $|\psi\rangle_{A_1} = \alpha|0\rangle_{A_1} + \beta|1\rangle_{A_1}$

$$\begin{aligned}
|\psi\rangle_{A_1}|\Phi\rangle_{A_2\otimes B} &= \left(\alpha|0\rangle_{A_1} + \beta|1\rangle_{A_1}\right)\frac{1}{\sqrt{2}}\left(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B\right)\\
&\xmapsto{\mathrm{CNOT}_{A_1A_2}} \frac{\alpha}{\sqrt{2}}\left(|0\rangle_{A_1}|0\rangle_{A_2}|0\rangle_B + |0\rangle_{A_1}|1\rangle_{A_2}|1\rangle_B\right)\\
&\quad + \frac{\beta}{\sqrt{2}}\left(|1\rangle_{A_1}|1\rangle_{A_2}|0\rangle_B + |1\rangle_{A_1}|0\rangle_{A_2}|1\rangle_B\right)\\
&\xmapsto{H_{A_1}} \frac{\alpha}{2}\left(|000\rangle_{A_1A_2B} + |100\rangle_{A_1A_2B} + |011\rangle_{A_1A_2B} + |111\rangle_{A_1A_2B}\right)\\
&\quad + \frac{\beta}{2}\left(|010\rangle_{A_1A_2B} - |110\rangle_{A_1A_2B} + |001\rangle_{A_1A_2B} - |101\rangle_{A_1A_2B}\right)
\end{aligned}$$

Quantum teleportation

$$
\begin{aligned}
&\frac{\alpha}{2}\left(|000\rangle_{A_1A_2B} + |100\rangle_{A_1A_2B} + |011\rangle_{A_1A_2B} + |111\rangle_{A_1A_2B}\right) \\
&\quad + \frac{\beta}{2}\left(|010\rangle_{A_1A_2B} - |110\rangle_{A_1A_2B} + |001\rangle_{A_1A_2B} - |101\rangle_{A_1A_2B}\right) \\
&= \frac{1}{2}\Big[|00\rangle_{A_1A_2}(\alpha\,|0\rangle_B + \beta\,|1\rangle_B) + |01\rangle_{A_1A_2}(\alpha\,|1\rangle_B + \beta\,|0\rangle_B) \\
&\qquad + |10\rangle_{A_1A_2}(\alpha\,|0\rangle_B - \beta\,|1\rangle_B) + |11\rangle_{A_1A_2}(\alpha\,|1\rangle_B - \beta\,|0\rangle_B)\Big]
\end{aligned}
$$

According to the measurement outcome of A_1A_2

$$
\begin{aligned}
00 &\Rightarrow I \\
01 &\Rightarrow X \\
10 &\Rightarrow Z \\
11 &\Rightarrow ZX
\end{aligned}
$$

Assignments

1. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

   when $|0\rangle\langle 0|$ is measured at the second system.

2. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|+\rangle\langle +|$ is measured at the second system.

3. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|\varphi\rangle\langle\varphi|$ is measured at the second system where $|\varphi\rangle := \alpha|0\rangle + \beta|1\rangle$.