Universality of quantum circuit

Ryuhei Mori

Tokyo Institute of Technology

Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in L(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, T, CNOT gates computing $\widetilde{U}$ satisfying $D(U, \widetilde{U}) < \epsilon$.

Proof.

1. Any unitary matrix can be decomposed to a product of two-level unitary matrices. Done
2. Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates. Done
3. Any controlled-untary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.
4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Special unitary group

- $\mathrm{U}(n) :=$ the set of $n \times n$ unitary matrices.
- $\mathrm{SU}(n) :=$
  the set of $n \times n$ unitary matrices U with $\det(U) = 1$.
- $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are groups.
- For $U \in \mathrm{SU}(n)$ and $V \in \mathrm{U}(n)$, $VUV^\dagger \in \mathrm{SU}(n)$.
- For $V \in \mathrm{U}(n)$ and $W \in \mathrm{U}(n)$, $VWV^\dagger W^\dagger \in \mathrm{SU}(n)$.
- For $U \in \mathrm{U}(n)$, there exists $V \in \mathrm{SU}(n)$ and $\theta \in \mathbb{R}$ such that $U = \mathrm{e}^{i\theta} V$.

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

Proof.

1. Controlled-U(2) with single controlled qubit.
2. Controlled-SU(2) with n controlled qubits.
3. Controlled-U(2) with n controlled qubits.

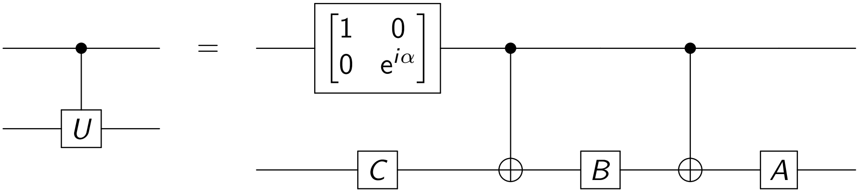
□

Decomposition of single qubit unitary

Lemma

Any single qubit unitary $U \in \mathrm{U}(2)$, there is single qubit unitary matrices A, B, C such that $ABC = I$ and $\mathrm{e}^{i\alpha} AXBXC = U$.

From this lemma,



Decomposition of single qubit unitary

Lemma

Any single qubit unitary $U \in \mathrm{U}(2)$, there is single qubit unitary matrices A, B, C and $\alpha \in \mathbb{R}$ such that $ABC = I$ and $\mathrm{e}^{i\alpha} AXBXC = U$.

Proof.

For any $U \in \mathrm{U}(2)$, there exists $\alpha \in [0, 2\pi)$ and $V \in \mathrm{SU}(2)$ such that $U = e^{i\alpha} V$.

For $R_Z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$, $XR_Z(\theta)XR_Z(-\theta) = R_Z(-2\theta)$.

For any $V \in \mathrm{SU}(2)$, there exists $\theta \in [0, 2\pi)$ and $P \in \mathrm{SU}(2)$ such that

$$V = PR_Z(-2\theta)P^\dagger = PXR_Z(\theta)XR_Z(-\theta)P^\dagger.$$

$A = P$, $B = R_Z(\theta)$, $C = R_Z(-\theta)P^\dagger$ satisfy the conditions. $\square$

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

Proof.

1. Controlled-U(2) with single controlled qubit. Done
2. Controlled-SU(2) with n controlled qubits.
3. Controlled-U(2) with n controlled qubits.

□

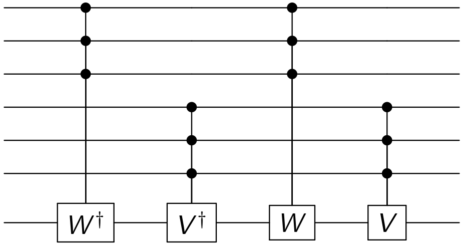
Group commutator and controlled-unitary

Theorem

For any $U \in \mathrm{SU}(2)$, controlled-U gate with n controlled qubits can be realized by $O(n^2)$ CNOT and arbitrary single-qubit gates without ancillas (working qubits).

Proof.

Induction on n. For the group commutator decomposition $U = VWV^\dagger W^\dagger$ using $V = PiXP^\dagger$, $W = PR_Z(\theta)P^\dagger \in \mathrm{SU}(2)$ for some $\theta \in [0, 2\pi)$ and $P \in \mathrm{SU}(2)$.



$S_n = 4S_{n/2} = 4^{\log n} S_1 = O(n^2)$. □

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

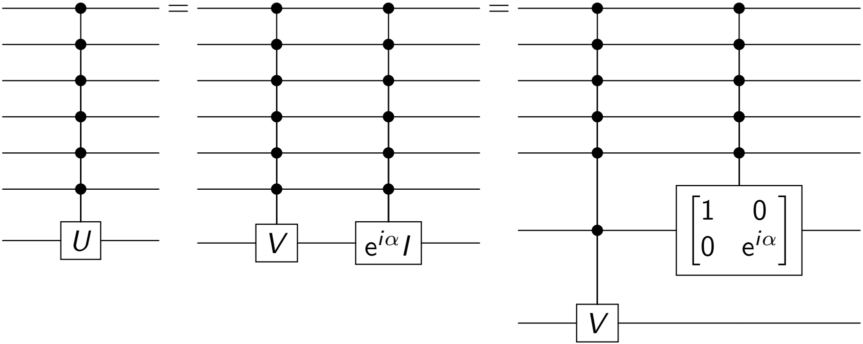
Proof.

1. Controlled-U(2) with single controlled qubit. Done
2. Controlled-SU(2) with n controlled qubits. Done
3. Controlled-U(2) with n controlled qubits.

$\square$

Controlled-$U(2)$ with n controlled qubits

For any $U \in \mathrm{U}(2)$, there exists $V \in \mathrm{SU}(2)$ and $\alpha \in \mathbb{R}$ such that $U = \mathrm{e}^{i\alpha} V$.



$A_n = S_n + A_{n-1} = O(n^3)$

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

Proof.

1. Controlled-U(2) with single controlled qubit. Done
2. Controlled-SU(2) with n controlled qubits. Done
3. Controlled-U(2) with n controlled qubits. Done

□

Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in L(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, T, CNOT gates computing $\widetilde{U}$ satisfying $D(U, \widetilde{U}) < \epsilon$.

Proof.

1. Any unitary matrix can be decomposed to a product of two-level unitary matrices. Done
2. Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates. Done
3. Any controlled-untary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates. Done
4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Approximation of a single-qubit gate is sufficient

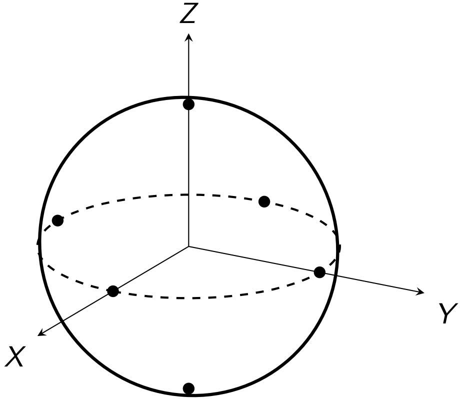
Theorem

Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

This theorem shows the universality of the gate set with CNOT. Assume $D(U_i, V_i) \le \epsilon$ for $i = 1, \ldots, m$.

$$
\begin{aligned}
& D(U_m U_{m-1} \cdots U_1, V_m V_{m-1} \cdots V_1) \\
& \le \sum_{i=1}^{m} D\left(U_m \cdots U_i V_{i-1} \cdots V_1, U_m \cdots U_{i+1} V_i \cdots V_1\right) \quad \text{(triangle inequality)} \\
& = \sum_{i=1}^{m} D\left(U_i, V_i\right) \quad \text{(unitary invariance)} \\
& \le m\epsilon.
\end{aligned}
$$

Universality of X, Y, Z, H, S, T



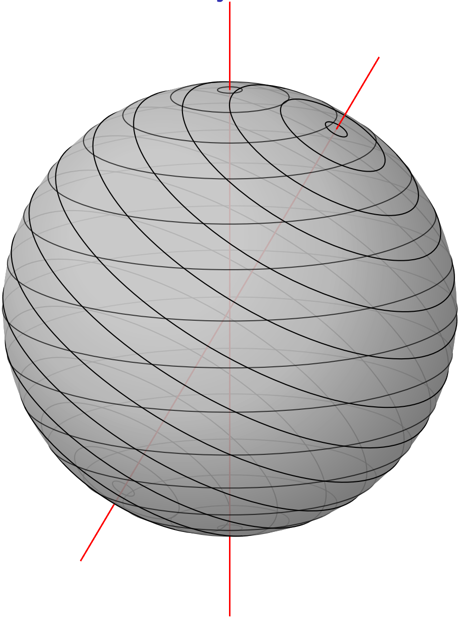
Universality of X, Y, Z, H, S, T

$T \cong R_Z(\pi/4)$. $HTH \cong R_X(\pi/4)$.

$$\begin{aligned}
R_Z(\pi/4)R_X(\pi/4) &= \left[\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}Z\right]\left[\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}X\right] \\
&= \cos^2\frac{\pi}{8}I - i\sin\frac{\pi}{8}\left[\cos\frac{\pi}{8}(X+Z) + \sin\frac{\pi}{8}Y\right] \\
&=: \cos\frac{\eta}{2}I - i\sin\frac{\eta}{2}\left(n_x X + n_Y Y + n_Z Z\right) \\
&= R_{\widehat{n}}(\eta)
\end{aligned}$$

where η satisfying $\cos(\eta/2) = \cos^2(\pi/8)$ and $\widehat{n}$ is a unit vector along with $(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}, \cos\frac{\pi}{8})$. Here, η is an irrational multiple of π. $HR_{\widehat{n}}(\eta)H = R_{\widehat{m}}(\eta)$ where $\widehat{m}$ is a unit vector along with $(\cos\frac{\pi}{8}, -\sin\frac{\pi}{8}, \cos\frac{\pi}{8})$.

Universality of two rotations 1/2



Universality of two rotations 2/2

Theorem

For any $U \in \mathrm{SU}(2)$, there exists $n \in \mathbb{Z}_{\geq 0}$ and $\alpha_1, \ldots, \alpha_n \in (0, 2\pi)$ such that $R_{\widehat{n}}(\alpha_1)R_{\widehat{m}}(\alpha_2)R_{\widehat{n}}(\alpha_3)\cdots R_{\widehat{n}}(\alpha_n)$ is equal to U or $-U$.

Proof.

Let $|\psi\rangle$ and $|\psi^\perp\rangle$ be the eigenvectors of $R_{\widehat{n}}(\theta)$.

Let $|\varphi\rangle := U\,|\psi\rangle$, $|\varphi^\perp\rangle := U\,|\psi^\perp\rangle$.

There exists $n \in \mathbb{Z}_{\geq 0}$ and $\theta_0, \theta_1, \alpha_1, \ldots, \alpha_n \in (0, 2\pi)$ such that

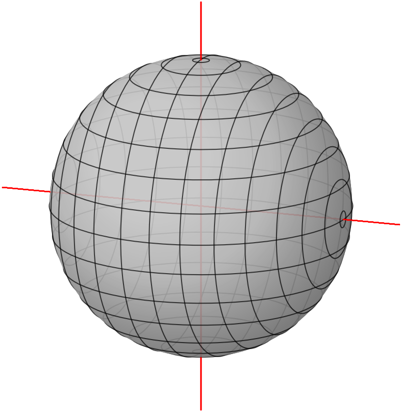
$$
\begin{aligned}
|\varphi\rangle &= e^{i\theta_0} R_{\widehat{n}}(\alpha_1)R_{\widehat{m}}(\alpha_2)R_{\widehat{n}}(\alpha_3)\cdots R_{\widehat{m}}(\alpha_{n-1})\,|\psi\rangle \\
&= e^{i(\theta_0 + \frac{\alpha_n}{2})} R_{\widehat{n}}(\alpha_1)R_{\widehat{m}}(\alpha_2)R_{\widehat{n}}(\alpha_3)\cdots R_{\widehat{m}}(\alpha_{n-1})R_{\widehat{n}}(\alpha_n)\,|\psi\rangle \\
|\varphi^\perp\rangle &= e^{i\theta_1} R_{\widehat{n}}(\alpha_1)R_{\widehat{m}}(\alpha_2)R_{\widehat{n}}(\alpha_3)\cdots R_{\widehat{m}}(\alpha_{n-1})\,|\psi^\perp\rangle \\
&= e^{i(\theta_1 - \frac{\alpha_n}{2})} R_{\widehat{n}}(\alpha_1)R_{\widehat{m}}(\alpha_2)R_{\widehat{n}}(\alpha_3)\cdots R_{\widehat{m}}(\alpha_{n-1})R_{\widehat{n}}(\alpha_n)\,|\psi^\perp\rangle\,.
\end{aligned}
$$

By choosing $\alpha_n = \theta_1 - \theta_0$, then $\theta_0 + \frac{\alpha_n}{2} = \theta_1 - \frac{\alpha_n}{2}$. Hence, $R_{\widehat{n}}(\alpha_1)\cdots R_{\widehat{n}}(\alpha_n)$ maps $|\psi\rangle \mapsto e^{i\theta}\,|\varphi\rangle$, $|\psi^\perp\rangle \mapsto e^{i\theta}\,|\varphi^\perp\rangle$. Since $U \in \mathrm{SU}(2)$, θ must be 0 or π. $\square$

Corollary

For any $U \in \mathrm{U}(2)$*, there exists* $\alpha, \beta, \gamma, \delta \in (0, 2\pi)$ *such that* $U = e^{i\alpha} R_Z(\beta) R_Y(\gamma) R_Z(\delta)$.

Proof.



Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in L(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, T, CNOT gates computing $\widetilde{U}$ satisfying $D(U, \widetilde{U}) < \epsilon$.

Proof.

1. Any unitary matrix can be decomposed to a product of two-level unitary matrices. Done
2. Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates. Done
3. Any controlled-untary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates. Done
4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T. Done

Solovay–Kitaev theorem

Theorem

Assume $\{U_1, \dots, U_k\}$ *generates a dense subset of* $\mathrm{SU}(2)$*. Then, any* $U \in \mathrm{SU}(2)$ *can be approxmiated with error* ϵ *by* $[\log(1/\epsilon)]^c$ *multiplications of* $\{U_1, \dots, U_k\}$ *for some constant* c*.*

Assignments

1. Show a quantum circuit for controlled-$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$ gate with two controlled qubits using the CNOT gates and arbitrary single-qubit gates.

2. [Advanced] Show a quantum circuit for controlled-$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ gate with two controlled qubits using six CNOT gates and seven T and $T^\dagger$ gates.