# A single qubit

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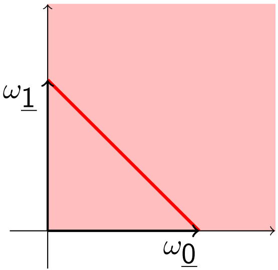
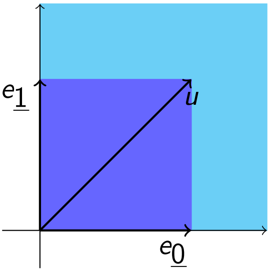
# A single bit

Let $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

- Set of states $= \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}$.

- Set of binary measurements $=$ $\left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}$.



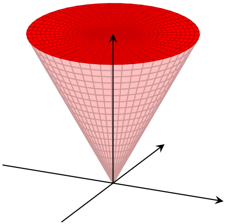



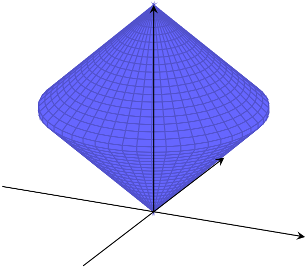
# A single qubit

Let $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$.

- Set of states $= \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}$.

- Set of binary measurements $=$
  $\{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}$.





# A single qubit

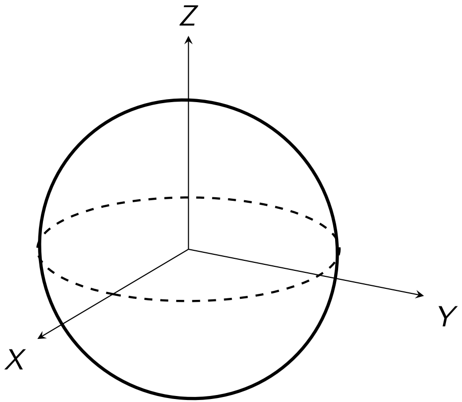
- A qubit can be represented by

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

for $[r_X \ r_Y \ r_Z] \in \mathbb{R}^3$ satisfying $r_X^2 + r_Y^2 + r_Z^2 \leq 1$.

- A qubit can be represented by a point $[r_X \ r_Y \ r_Z]$ in a three-dimensional sphere of radius 1.

# The Bloch sphere



## Complex space and Hermitian operator

- $\mathcal{X}$: A finite-dimensional inner product space on $\mathbb{C}$.
- $\mathcal{L}(\mathcal{X})$: A set of linear operators on $\mathcal{X}$.

For $A \in \mathcal{L}(\mathcal{X})$, an adjoint map $A^\dagger$ of $A$ is a unique operator satisfying

$$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$$

for any $v$, $w \in \mathcal{X}$. $H \in \mathcal{L}(\mathcal{X})$ is Hermitian if and only if $H^\dagger = H$.

- $\mathcal{H}(\mathcal{X})$: A set of Hermitian operators on $\mathcal{X}$.

$\mathcal{L}(\mathcal{X})$ and $\mathcal{H}(\mathcal{X})$ are often regarded as inner product space on $\mathbb{C}$ and $\mathbb{R}$, respectively for the Hilbert–Schmidt inner product $\langle A, B \rangle = \mathrm{Tr}(A^\dagger B)$.

## Spectral decomposition theorem

### Definition (Normal operator)

$A \in \mathcal{L}(\mathcal{X})$ is said to be normal if $AA^\dagger = A^\dagger A$.

Hermitian matrix ($H^\dagger = H$) and unitary matrix ($UU^\dagger = I$) are normal.

### Theorem (Spectral decomposition theorem)

*$A \in \mathcal{L}(\mathbb{C}^n)$ is normal if and only if there exist orthonormal basis $\{|\psi_j\rangle\}$ of $\mathbb{C}^n$ and complex numbers $\{\lambda_j\}$ such that*

$$A = \sum_j \lambda_j \left|\psi_j\right\rangle \left\langle\psi_j\right| .$$

## Pauli matrices

- $$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- $$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

- $$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

# Braket notation

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} |+\rangle &:= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), & |-\rangle &:= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for $|\alpha|^2 + |\beta|^2 = 1$.

$$\langle\psi| = |\psi\rangle^\dagger = \alpha^*\,\langle 0| + \beta^*\,\langle 1| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}$$

# Pauli matrices in braket notation

- $$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \langle 0| - |1\rangle \langle 1|$$

- $$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle \langle +| - |-\rangle \langle -|$$

- $$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

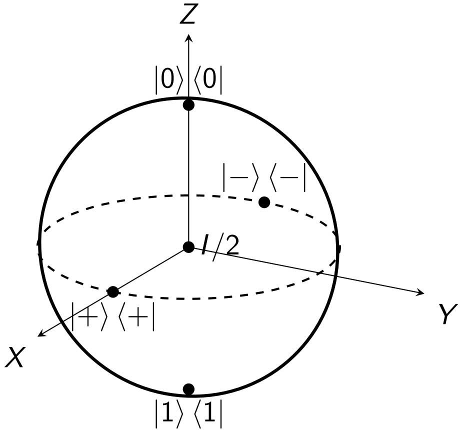
# Special states

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$r_X^2 + r_Y^2 + r_Z^2 \leq 1.$$

| Coordinate | State |
|---|---|
| $[0\ 0\ 0]$ | $\frac{1}{2}I$ |
| $[1\ 0\ 0]$ | $\frac{1}{2}(I + X) = \vert+\rangle\langle+\vert$ |
| $[-1\ 0\ 0]$ | $\frac{1}{2}(I - X) = \vert-\rangle\langle-\vert$ |
| $[0\ 0\ 1]$ | $\frac{1}{2}(I + Z) = \vert0\rangle\langle0\vert$ |
| $[0\ 0\ -1]$ | $\frac{1}{2}(I - Z) = \vert1\rangle\langle1\vert$ |

# Special states in the Bloch sphere



# Pure states are rank-1 density operators

$\rho$ is a pure state

$$\stackrel{\text{def}}{\iff} \rho \neq p\rho_1 + (1-p)\rho_2 \quad \forall p \in (0,1) \text{ and states } \rho_1 \neq \rho_2.$$

## Lemma

*A quantum state $\rho$ is a pure state if and only if $\rho$ is rank-1.*

## Proof.

Let the spectral decomposition of $\rho$ be

$$\rho = \sum_j \lambda_j |\psi_j\rangle \langle\psi_j|$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$. If $\rho$ is not rank-1, $\rho$ is a convex combination of quantum states $(|\psi_j\rangle \langle\psi_j|)_j$.
Assume $\rho = |\varphi\rangle \langle\varphi|$ and $\rho = p_1\rho_1 + p_2\rho_2$. $\mathrm{Tr}(\sigma |\varphi\rangle \langle\varphi|) = 1$ if and only if $\sigma = |\varphi\rangle \langle\varphi|$ since $\mathrm{Tr}(\sigma |\varphi\rangle \langle\varphi|) = \langle\varphi| \sigma |\varphi\rangle = \sum_j \lambda_j |\langle\psi_j|\varphi\rangle|^2$. Then, $\mathrm{Tr}((p_1\rho_1 + p_2\rho_2) |\varphi\rangle \langle\varphi|) = 1$ implies that $\mathrm{Tr}(\rho_1 |\varphi\rangle \langle\varphi|) = \mathrm{Tr}(\rho_2 |\varphi\rangle \langle\varphi|) = 1$, and hence $\rho_1 = \rho_2 = \rho$. $\square$

## Pure states and state vector

Pure state $|\psi\rangle\langle\psi|$ can be represented by a state vector $|\psi\rangle \in \mathbb{C}^n$ with $\langle\psi|\psi\rangle = 1$.

$|\psi\rangle$ and $|\varphi\rangle := e^{i\theta}|\psi\rangle$ represent the same state since $|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$.

A state that is not pure state is called a mixed state.

$\rho$ is called a density matrix.

# Inner product of pure states

- $\rho$ is a qubit pure state with a coordinate $[r_X \ r_Y \ r_Z]$.
- $\sigma$ is a qubit pure state with a coordinate $[-r_X \ -r_Y \ -r_Z]$.

$$\mathrm{Tr}(\rho\sigma) = \mathrm{Tr}(\rho(I-\rho)) = \mathrm{Tr}(\rho) - \mathrm{Tr}(\rho^2) = 1 - 1 = 0$$

- $\rho = |\psi\rangle\langle\psi|$.
- $\sigma = |\varphi\rangle\langle\varphi|$.

$$\begin{aligned}\mathrm{Tr}(\rho\sigma) &= \mathrm{Tr}(|\psi\rangle\langle\psi|\,|\varphi\rangle\langle\varphi|) = \langle\psi|\varphi\rangle\,\mathrm{Tr}(|\psi\rangle\langle\varphi|) \\ &= \langle\psi|\varphi\rangle\langle\varphi|\psi\rangle = |\langle\psi|\varphi\rangle|^2\end{aligned}$$

# Single qubit measurement

$$\text{Set of measurements} = \{(e_1, \ldots, e_k) \mid e_1 + \cdots + e_k = I, e_j \in C_{\succeq 0}$$
$$i = 1, 2, \ldots, k,\ k = 1, 2, \ldots\}$$

If $e_i e_j = \delta_{i,j} e_i$, the measurement is called an orthogonal measurement.

If $|0\rangle\langle 0|$ is measured by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$, the output is 0 with probability $\mathrm{Tr}(|0\rangle\langle 0| |0\rangle\langle 0|) = |\langle 0|0\rangle|^2 = 1$.

If $|+\rangle\langle +|$ is measured by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$, the output is 0 with probability $\mathrm{Tr}(|0\rangle\langle 0| |+\rangle\langle +|) = |\langle 0|+\rangle|^2 = 1/2$.

If $|\psi\rangle\langle\psi|$ is measured by $(|\varphi_0\rangle\langle\varphi_0|, |\varphi_1\rangle\langle\varphi_1|)$, the output is 0 with probability $\mathrm{Tr}(|\varphi_0\rangle\langle\varphi_0| |\psi\rangle\langle\psi|) = |\langle\varphi_0|\psi\rangle|^2$.

## Unitary operation

For unitary operation $U$, let us consider

$$\rho \mapsto U\rho U^\dagger.$$

It is easy to see that

- $\mathrm{Tr}(U\rho U^\dagger) = 1$ (Trace-preserving)
- $U\rho U^\dagger \succeq 0$ (Positive)

A pure state $|\psi\rangle$ is mapped to a pure state $U|\psi\rangle$.

$U$ and $e^{i\theta}U$ are physically equivalent.

# Single qubit quantum channel

Since real vector space spanned by 2x2 Hermitian matrices is 4-dimensional, any linear map $\Phi$ on the linear space is represented by 4x4 real matrix. Let $\sigma_0 = I$, $\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$, and $\mathbf{a} \cdot \sigma_0^3 := \sum_{i=0}^{3} a_i \sigma_i$.

$$\Phi\left(\mathbf{a} \cdot \sigma_0^3\right) = (T\mathbf{a}) \cdot \sigma_0^3 \qquad \text{where}$$

$$T_{i,j} = \frac{1}{2}\mathrm{Tr}(\sigma_i \Phi(\sigma_j)).$$

From the trace-preserving property, i.e., $\mathrm{Tr}(\Phi(\sigma)) = \mathrm{Tr}(\sigma)$,

$$T_{0,j} = \frac{1}{2}\mathrm{Tr}(\sigma_0 \Phi(\sigma_j)) = \frac{1}{2}\mathrm{Tr}(\sigma_j) = \delta_{0,j}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_1 & M_{11} & M_{12} & M_{13} \\ t_2 & M_{21} & M_{22} & M_{23} \\ t_3 & M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\frac{I + \mathbf{r} \cdot \sigma_1^3}{2} \xmapsto{\Phi} \frac{I + (M\mathbf{r} + \mathbf{t}) \cdot \sigma_1^3}{2}$$

# Matrix representation of unitary channel

Unitary channel is unital, i.e., $\Phi(I) = UIU^\dagger = I$.

$$T_{i,0} = \frac{1}{2}\mathrm{Tr}(\sigma_i \Phi(I)) = \frac{1}{2}\mathrm{Tr}(\sigma_i) = \delta_{i,0}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & M_{13} \\ 0 & M_{21} & M_{22} & M_{23} \\ 0 & M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Trace-preserving unital channel is represented by the $3 \times 3$ real matrix $M$.

$$\frac{I + \mathbf{r} \cdot \sigma_1^3}{2} \longmapsto \frac{I + (M\mathbf{r}) \cdot \sigma_1^3}{2}$$

# Examples of unitary operations

- The identity matrix $I$.
- Pauli matrices $X$, $Y$ and $Z$.
- Hadamard matrix $H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- Product $UV$ of unitary operators $U$ and $V$.

# Multiplications of Pauli matrices

For any unitary matrices $U$ and $V$, $UV$ is also unitary matrix.

- $XY = iZ$
- $YZ = iX$
- $ZX = iY$

# Pauli matrices $X$ on the Bloch sphere

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$\begin{aligned} X\rho X^\dagger = X\rho X &= \frac{1}{2}\left(X^2 + r_X X^3 + r_Y XYX + r_Z XZX\right) \\ &= \frac{1}{2}\left(I + r_X X - r_Y Y - r_Z Z\right) \end{aligned}$$

$$[r_X \;\; r_Y \;\; r_Z] \overset{X}{\longmapsto} [r_X \;\; -r_Y \;\; -r_Z]$$

$\pi$-rotation with respect to $X$ axis. $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Similarly, $Y$ and $Z$ corresponds to $\pi$-rotation with respect to $Y$ and $Z$ axes, respectively.

## Hadamard matrix

Hadamard matrix $H$ is unitary and Hermitian.

$$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1|$$
$$= |0\rangle \langle +| + |1\rangle \langle -|$$

$$|0\rangle, |1\rangle \xleftrightarrow{H} |+\rangle, |-\rangle$$

$$HXH = H(|+\rangle \langle +| - |-\rangle \langle -|)H$$
$$= |0\rangle \langle 0| - |1\rangle \langle 1| = Z$$

Similarly, $HZH = X$.
$HYH = H(iXZ)H = iHXHHZH = iZX = -Y$

# Hadamard matrix on the Bloch sphere

$$\rho = \frac{1}{2}(I + r_X X + r_Y Y + r_Z Z)$$

$$\begin{aligned} H\rho H^\dagger = H\rho H &= \frac{1}{2}\left(H^2 + r_X HXH + r_Y HYH + r_Z HZH\right) \\ &= \frac{1}{2}(I + r_X Z - r_Y Y + r_Z X) \end{aligned}$$

$$[r_X \ r_Y \ r_Z] \xmapsto{H} [r_Z \ -r_Y \ r_X]$$

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

# Rotation matrice for the $Z$ axis

$$R_Z(\theta) := \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z$$

$$R_Z(\theta)^\dagger = \cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} Z = R_Z(-\theta)$$

$$\begin{aligned} R_Z(\theta)R_Z(\tau) &= \left(\cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z\right)\left(\cos\frac{\tau}{2} I - i \sin\frac{\tau}{2} Z\right) \\ &= \left(\cos\frac{\theta}{2}\cos\frac{\tau}{2} - \sin\frac{\theta}{2}\sin\frac{\tau}{2}\right) I - i\left(\cos\frac{\theta}{2}\sin\frac{\tau}{2} + \sin\frac{\theta}{2}\cos\frac{\tau}{2}\right) Z \\ &= \cos\frac{\theta+\tau}{2} I - i \sin\frac{\theta+\tau}{2} Z = R_Z(\theta+\tau) \end{aligned}$$

$$R_Z(\theta)R_Z(\theta)^\dagger = R_Z(\theta)R_Z(-\theta) = R_Z(\theta-\theta) = R_Z(0) = I$$

$$R_Z(\theta)X = XR_Z(-\theta), \quad R_Z(\theta)Y = YR_Z(-\theta), \quad R_Z(\theta)Z = ZR_Z(\theta)$$

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## General unitary matrix

$\langle UAU^\dagger, UBU^\dagger \rangle = Tr(UAU^\dagger UBU^\dagger) = \text{Tr}(AB) = \langle A, B \rangle.$

$$\langle UAU^\dagger, UBU^\dagger \rangle = \left\langle \frac{I + (M\mathbf{a}) \cdot \sigma_1^3}{2}, \frac{I + (M\mathbf{b}) \cdot \sigma_1^3}{2} \right\rangle = \frac{1}{2}(1 + \langle M\mathbf{a}, M\mathbf{b} \rangle)$$

$$\langle A, B \rangle = \left\langle \frac{I + \mathbf{a} \cdot \sigma_1^3}{2}, \frac{I + \mathbf{b} \cdot \sigma_1^3}{2} \right\rangle = \frac{1}{2}(1 + \langle \mathbf{a}, \mathbf{b} \rangle)$$

$\langle M\mathbf{a}, M\mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$ implies that $M$ is an orthogonal matrix.

Since for a unitary matrix $U$ with $\det(U) = 1$,

$$U = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j| = VR_Z(\theta)V^\dagger.$$

$$M_U = M_V M_{R_Z(\theta)} M_V^{-1}$$

$\det(M_U) = \det(M_V) \det(M_{R_Z(\theta)}) \det(M_V^{-1}) = \det(M_{R_Z(\theta)}) = 1.$

Conversely, any $3 \times 3$ orthogonal matrix $M$ with $\det(M) = 1$ corresponds to some unitary channel $\rho \mapsto U\rho U^\dagger$. Such matrix is a rotation matrix.

## Assignments

1. Express $\left(a_X X + a_Y Y + a_Z Z\right)^2$ as a linear combination of $I, X, Y, Z$ for $a_X, a_Y, a_Z \in \mathbb{C}$.

2. For a real unit vector $\mathbf{v} := \begin{bmatrix} v_X & v_Y & v_Z \end{bmatrix}$, let
$$R_{\mathbf{v}}(\theta) := \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}\left(v_X X + v_Y Y + v_Z Z\right).$$
Show that $R_{\mathbf{v}}(\theta) R_{\mathbf{v}}(\tau) = R_{\mathbf{v}}(\theta + \tau)$ for any $\theta, \tau \in \mathbb{R}$. Show also that $R_{\mathbf{v}}(\theta)$ is a unitary matrix.

3. Show two pure states $\rho_0$, $\rho_1$ that are invariant by $R_{\mathbf{v}}(\theta)$ for any $\theta \in \mathbb{R}$, i.e., $R_{\mathbf{v}}(\theta) \rho_i R_{\mathbf{v}}(\theta)^\dagger = \rho_i$, as linear combinations of $I$, $X$, $Y$ and $Z$.

4. [Advanced] Show that for any unitary matrix $U$ with $\det(U) = 1$, there exists a real three-dimensional unit vector $\mathbf{v}$ and $\theta \in \mathbb{R}$, such that $U = R_{\mathbf{v}}(\theta)$.