

Quantum phase estimation

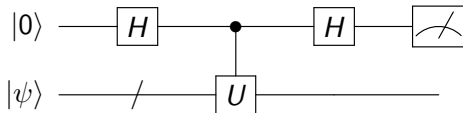
Ryuhei Mori

Tokyo Institute of Technology

Quantum algorithms

- Quantum phase estimation: Integer factoring.
- Grover search, quantum walk: Unstructured search.

Hadamard test

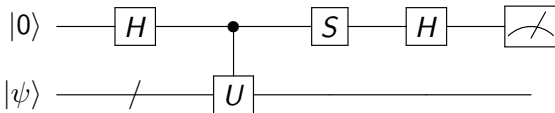


$$\begin{aligned}
 |0\rangle |\psi\rangle &\mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} |\psi\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle |\psi\rangle + |1\rangle U |\psi\rangle) \\
 &\mapsto \frac{1}{\sqrt{2}} (|+\rangle |\psi\rangle + |-\rangle U |\psi\rangle) \\
 &= \frac{1}{2} (|0\rangle (|\psi\rangle + U |\psi\rangle) + |1\rangle (|\psi\rangle - U |\psi\rangle)).
 \end{aligned}$$

0 is measured with probability $\left\| \frac{|\psi\rangle + U|\psi\rangle}{2} \right\|^2 = \frac{1 + \text{Re}(\langle \psi | U | \psi \rangle)}{2}$.

1 is measured with probability $\left\| \frac{|\psi\rangle - U|\psi\rangle}{2} \right\|^2 = \frac{1 - \text{Re}(\langle \psi | U | \psi \rangle)}{2}$.

Hadamard test



$$\begin{aligned}
 |0\rangle |\psi\rangle &\mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} |\psi\rangle \mapsto \frac{1}{\sqrt{2}} (|0\rangle |\psi\rangle + i |1\rangle U|\psi\rangle) \\
 &\mapsto \frac{1}{\sqrt{2}} (|+\rangle |\psi\rangle + i |-\rangle U|\psi\rangle) \\
 &= \frac{1}{2} (|0\rangle (|\psi\rangle + iU|\psi\rangle) + |1\rangle (|\psi\rangle - iU|\psi\rangle)).
 \end{aligned}$$

0 is measured with probability $\left\| \frac{|\psi\rangle + iU|\psi\rangle}{2} \right\|^2 = \frac{1 + \text{Im}(\langle\psi|U|\psi\rangle)}{2}$.

1 is measured with probability $\left\| \frac{|\psi\rangle - iU|\psi\rangle}{2} \right\|^2 = \frac{1 - \text{Im}(\langle\psi|U|\psi\rangle)}{2}$.

Hadamard test for eigenvector

If $|\psi_\theta\rangle$ is an eigenvector of U for eigenvalue $e^{i\theta}$.

$$\operatorname{Re}(\langle\psi_\theta| U |\psi_\theta\rangle) = \operatorname{Re}(e^{i\theta}) = \cos(\theta)$$

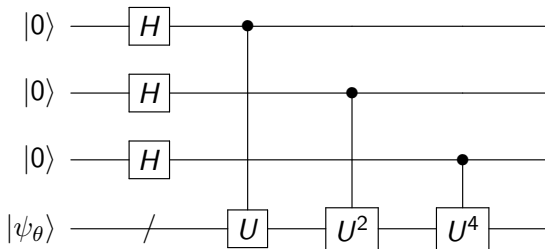
$$\operatorname{Im}(\langle\psi_\theta| U |\psi_\theta\rangle) = \operatorname{Im}(e^{i\theta}) = \sin(\theta)$$

Hence, we can estimate θ . But, this algorithm doesn't work when the eigenvector $|\psi_\theta\rangle$ is not given. For

$$|\psi\rangle := \sum_{j=1}^N \alpha_j |\psi_{\theta_j}\rangle$$

$$\langle\psi| U |\psi\rangle = \sum_{j=1}^N |\alpha_j|^2 e^{i\theta_j}.$$

Quantum phase estimation



$$\begin{aligned}
 |0\rangle^{\otimes n} |\psi_\theta\rangle &\mapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |\psi_\theta\rangle \\
 &\mapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{i\theta(x_1 + 2x_2 + 2^2x_3 + \dots + 2^{n-1}x_n)} |x\rangle |\psi_\theta\rangle \\
 &= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1,2,\dots,2^n-1\}} e^{i\theta x} |x\rangle |\psi_\theta\rangle
 \end{aligned}$$

Quantum Fourier transform

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \longleftrightarrow \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega^9 \end{bmatrix}$$

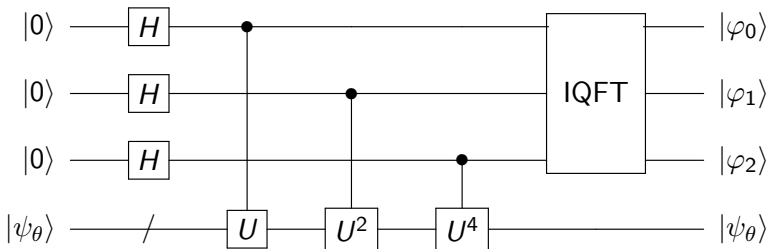
$$|x\rangle \longleftrightarrow |\hat{x}\rangle := \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_N^{xy} |y\rangle$$

where $\omega_N := e^{i\frac{2\pi}{N}}$.

$$U_{\text{QFT}} := \sum_{x=0}^{N-1} |\hat{x}\rangle \langle x|$$

Hadamard operator is the quantum Fourier transform for $N = 2$.

Quantum phase estimation



Assume $\theta = 2\pi \frac{\varphi}{2^n}$ for some integer $\varphi \in \{0, 1, \dots, 2^n - 1\}$.

$$\begin{aligned}
 & \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1, 2, \dots, 2^n - 1\}} e^{i \frac{2\pi}{2^n} \varphi x} |x\rangle |\psi_\theta\rangle \\
 &= |\widehat{\varphi}\rangle |\psi_\theta\rangle \\
 &\mapsto |\varphi\rangle |\psi_\theta\rangle
 \end{aligned}$$

Quantum phase estimation for superposition of eigenvectors

$$|\psi\rangle := \sum_{i=1}^N \alpha_i |\psi_{\theta_i}\rangle$$

$$\begin{aligned} |0\rangle \left(\sum_{i=1}^N \alpha_i |\psi_{\theta_i}\rangle \right) &\mapsto \sum_{i=1}^N \alpha_i |\widehat{\varphi}_i\rangle |\psi_{\theta_i}\rangle \\ &\mapsto \sum_{i=1}^N \alpha_i |\varphi_i\rangle |\psi_{\theta_i}\rangle \end{aligned}$$

Then, φ_i is measured with probability $|\alpha_i|^2$.

Probability of the best approximation

Assume that the eigenvalue is $e^{2\pi i\phi}$. Let $\varphi \in \{0, 1, \dots, 2^n - 1\}$ be the best n -bit approximation of ϕ , i.e., $|\phi - \frac{\varphi}{2^n}| \leq \frac{1}{2^{n+1}}$.

$$\begin{aligned}\Pr(\varphi) &= \frac{1}{2^{2n}} \left| \left(\sum_{x=0}^{2^n-1} e^{-i\frac{2\pi}{2^n}\varphi x} \langle x| \right) \left(\sum_{x=0}^{2^n-1} e^{i2\pi\phi x} |x\rangle \right) \right|^2 \\&= \frac{1}{2^{2n}} \left| \sum_{x=0}^{2^n-1} e^{2\pi i(\phi - \frac{\varphi}{2^n})x} \right|^2 \\&= \frac{1}{2^{2n}} \left| \frac{1 - e^{2\pi i(2^n\phi - \varphi)}}{1 - e^{2\pi i(\phi - \frac{\varphi}{2^n})}} \right|^2 \\&= \frac{1}{2^{2n}} \frac{\sin^2(\pi(2^n\phi - \varphi))}{\sin^2(\pi(\phi - \frac{\varphi}{2^n}))} \\&\geq \frac{1}{2^{2n}} \frac{\sin^2(\pi(2^n\phi - \varphi))}{(\pi(\phi - \frac{\varphi}{2^n}))^2} \\&\geq \frac{1}{2^{2n}} \frac{(2(2^n\phi - \varphi))^2}{(\pi(\phi - \frac{\varphi}{2^n}))^2} = \frac{4}{\pi^2} \approx 0.405\end{aligned}$$

Quantum Fourier transform

$$U_{\text{QFT}(N)} := \sum_{x=0}^{N-1} |\widehat{x}\rangle \langle x|.$$

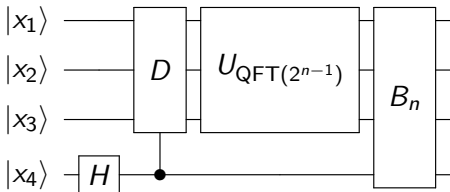
$$\begin{aligned}
 U_{\text{QFT}(2^n)} &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} \omega_{2^n}^{xz} |z\rangle \langle x| \\
 &= \frac{1}{\sqrt{2^n}} \sum_{a,b \in \{0,1\}} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{(x+2^{n-1}a)(2z+b)} |zb\rangle \langle ax| \\
 &= \frac{1}{\sqrt{2^n}} \sum_{a,b \in \{0,1\}} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb+2^{n-1}ab} |zb\rangle \langle ax| \\
 &= \frac{1}{\sqrt{2^n}} \sum_{a,b \in \{0,1\}} (-1)^{ab} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} B_n |bz\rangle \langle ax| \\
 &= B_n \frac{1}{\sqrt{2^n}} \sum_{a,b \in \{0,1\}} (-1)^{ab} |b\rangle \langle a| \otimes \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} |z\rangle \langle x|
 \end{aligned}$$

Quantum Fourier transform

$$\begin{aligned}
 U_{\text{QFT}(2^n)} &= B_n \frac{1}{\sqrt{2^n}} \sum_{a,b \in \{0,1\}} (-1)^{ab} |b\rangle \langle a| \otimes \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} |z\rangle \langle x| \\
 &= B_n \frac{1}{\sqrt{2^{n-1}}} \sum_{b \in \{0,1\}} |b\rangle \langle b| H \otimes \sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{xb} \sum_{z=0}^{2^{n-1}-1} \omega_{2^{n-1}}^{xz} |z\rangle \langle x| \\
 &= B_n \sum_{b \in \{0,1\}} |b\rangle \langle b| \otimes \sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{xb} |\hat{x}\rangle \langle x| (H \otimes I_{2^{n-1}}) \\
 &= B_n \left(\sum_{b \in \{0,1\}} |b\rangle \langle b| \otimes \sum_{x=0}^{2^{n-1}-1} |\hat{x}\rangle \langle x| \right) \\
 &\quad \cdot \left(\sum_{b \in \{0,1\}} |b\rangle \langle b| \otimes \sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{xb} |x\rangle \langle x| \right) (H \otimes I_{2^{n-1}}) \\
 &= B_n (I_2 \otimes U_{\text{QFT}(2^{n-1})}) \Lambda \left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^x |x\rangle \langle x| \right) (H \otimes I_{2^{n-1}})
 \end{aligned}$$

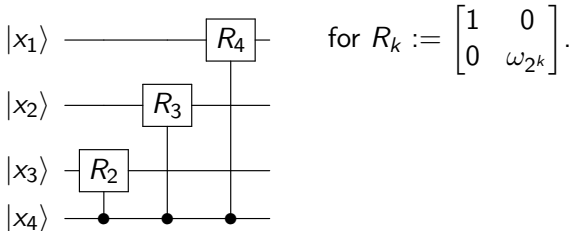
Quantum Fourier transform

$$U_{\text{QFT}(2^n)} = B_n(I_2 \otimes U_{\text{QFT}(2^{n-1})}) \Lambda \left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^x |x\rangle \langle x| \right) (H \otimes I_{2^{n-1}})$$

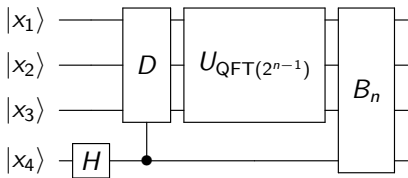


Quantum Fourier transform

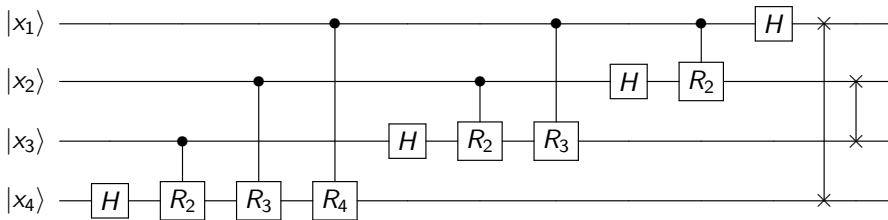
$$\begin{aligned}
 \Lambda \left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^x |x\rangle \langle x| \right) &= \Lambda \left(\sum_{x \in \{0,1\}^n} \omega_{2^n}^{x_1 + 2x_2 + \dots + 2^{n-1}x_n} |x\rangle \langle x| \right) \\
 &= \Lambda \left(\sum_{x \in \{0,1\}^n} \bigotimes_{i=1}^n \omega_{2^n}^{2^{i-1}x_i} |x_i\rangle \langle x_i| \right) = \Lambda \left(\bigotimes_{i=1}^n \sum_{x_i \in \{0,1\}} \omega_{2^n}^{2^{i-1}x_i} |x_i\rangle \langle x_i| \right) \\
 &= \prod_{i=1}^n \Lambda \left(\sum_{x_i \in \{0,1\}} \omega_{2^{n-i+1}}^{x_i} |x_i\rangle \langle x_i| \right)
 \end{aligned}$$



Whole quantum circuit of QFT



Let $R_k := \begin{bmatrix} 1 & 0 \\ 0 & \omega_{2^k} \end{bmatrix}$.



$O(n^2)$ Hadamard and Controlled phase shift gates.

Assignments

- 1 Show that the Fourier basis $\{|\widehat{x}\rangle\}_{x \in \{0,1,\dots,N-1\}}$ is orthonormal.
- 2 Show all eigenvectors and corresponding eigenvalues of

$$X_N := \sum_{j=0}^{N-1} |j+1 \bmod N\rangle \langle j|.$$