Quantum teleportation

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

$$T : \text{The set of states} \to \text{the set of states.}$$

$$pT(\rho_1) + (1-p)T(\rho_2) = T(p\rho_1 + (1-p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0, 1]$.

$T : \mathcal{H}(V) \to \mathcal{H}(W)$ must be linear (a proof is needed).

Schrödinger picture and Heisenberg picture

$T^\dagger$: The set of binary measurements $\to$ the set of binary measurements.

$$\langle T(\rho), P\rangle = \langle \rho, T^\dagger(P)\rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. $T^\dagger$ is an adjoint map of T.

$$\langle T_3(T_2(T_1(\rho))), P\rangle = \langle T_2(T_1(\rho)), T_3^\dagger(P)\rangle$$
$$= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P))\rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P)))\rangle$$

No-cloning theorem

Theorem

There is no quantum channel $T \colon \mathcal{H}(\mathbb{C}^2) \to \mathcal{H}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ *satisfying* $T(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$ *for any* $|\psi\rangle \in \mathbb{C}^2$.

Proof.

$$|0\rangle\langle 0| \xmapsto{T} |0\rangle\langle 0| \otimes |0\rangle\langle 0|$$
$$|1\rangle\langle 1| \xmapsto{T} |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

From the linearlity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \xmapsto{T} \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

$$|+\rangle\langle +| \xmapsto{T} |+\rangle\langle +| \otimes |+\rangle\langle +|$$
$$|-\rangle\langle -| \xmapsto{T} |-\rangle\langle -| \otimes |-\rangle\langle -|$$

$$\frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|) \xmapsto{T} \frac{1}{2}(|+\rangle\langle +| \otimes |+\rangle\langle +| + |-\rangle\langle -| \otimes |-\rangle\langle -|)$$

This is a contradiction. □

Axioms for quantum channel

$T : \mathcal{H}(V) \to \mathcal{H}(W)$.

1. Trace-preserving: $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$.
2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.
3. Completely positive: $\mathrm{id} \otimes T$ is positive.

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Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$: transposition according to $\{|0\rangle, |1\rangle\}$.

The transposition is obviously linear and trace-preserving.

Lemma

The transposition is positive.

Proof.

For any $A \succeq 0$ and $|\psi\rangle \in \mathbb{C}^2$,

$$\langle\psi| T(A) |\psi\rangle = \langle\psi| A^T |\psi\rangle = \langle\psi| A^* |\psi\rangle = (\langle\psi|^* A |\psi\rangle^*)^* \geq 0 \qquad \square$$

Positive but not completely positive 2/2

Lemma

The transposition is not completely positive.

Proof.

For $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$
\begin{aligned}
&(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|) \\
&= (\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)\Bigg(\frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\
&\qquad + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big)\Bigg) \\
&= \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
&\qquad + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big)
\end{aligned}
$$

$$|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |10\rangle \qquad |10\rangle \mapsto |01\rangle \qquad |11\rangle \mapsto |11\rangle$$

Hence, $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$. $(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$ is not positive semidefinite. □

Unitary operations

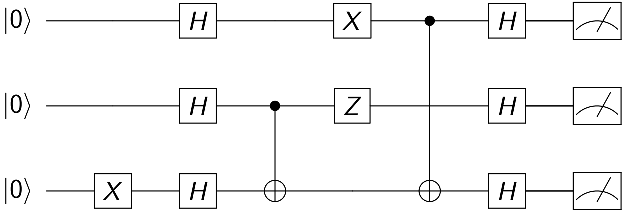
$$\rho \longmapsto U\rho U^\dagger.$$

1. Trace-preserving: $\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}(\rho)$.

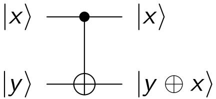
2. Completely positive: $(\mathrm{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only pure states and unitary operations are used.

Quantum circuit



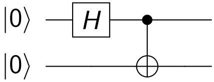
Controlled not



$$\mathrm{CNOT}\,|x\rangle\,|y\rangle \longmapsto |x\rangle\,|y \oplus x\rangle$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit

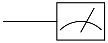


$$
\begin{aligned}
|0\rangle\,|0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)
\end{aligned}
$$

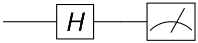
$$
\begin{aligned}
|x\rangle\,|y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle)\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|y\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|\bar{y}\rangle) = |\Phi_{xy}\rangle\,.
\end{aligned}
$$

Measurements

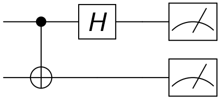
The measurement with the computational basis $(|0\rangle, |1\rangle)$.



The measurement by $(|+\rangle, |-\rangle)$.



The measurement by the Bell basis.



Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \mathrm{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)).$$

$$P(a \mid b) = \frac{1}{P(b)} \mathrm{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)} \mathrm{Tr}(\mathrm{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)) P_a).$$

$$\rho_{V \mid Q_b} := \frac{1}{P(b)} \mathrm{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)).$$

$V \otimes W \cong \mathcal{L}(W, V)$

$$A_{W\to V} = \sum_{i,j} A_{i,j} \left|i\right\rangle_V \left\langle j\right|_W$$

$$\left|A_{W\to V}\right\rangle\!\rangle := \mathcal{M}^{-1}(A_{W\to V}) = \sum_{i,j} A_{i,j} \left|i\right\rangle_V \left|j\right\rangle_W$$

$$\langle A_{W\to V}, B_{W\to V}\rangle = \mathrm{Tr}(A^\dagger_{W\to V} B_{W\to V}) = \langle\!\langle A_{W\to V} \mid B_{W\to V}\rangle\!\rangle$$

Examples

$$\left|\frac{1}{\sqrt{2}} I_{W\to V}\right\rangle\!\!\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_V \left|0\right\rangle_W + \left|1\right\rangle_V \left|1\right\rangle_W\right) = \left|\Phi_{00}\right\rangle$$

$$\left|\frac{1}{\sqrt{2}} X_{W\to V}\right\rangle\!\!\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_V \left|1\right\rangle_W + \left|1\right\rangle_V \left|0\right\rangle_W\right) = \left|\Phi_{01}\right\rangle$$

$$\left|\frac{1}{\sqrt{2}} Z_{W\to V}\right\rangle\!\!\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_V \left|0\right\rangle_W - \left|1\right\rangle_V \left|1\right\rangle_W\right) = \left|\Phi_{10}\right\rangle$$

$$\left|\frac{1}{\sqrt{2}} (ZX)_{W\to V}\right\rangle\!\!\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle_V \left|1\right\rangle_W - \left|1\right\rangle_V \left|0\right\rangle_W\right) = \left|\Phi_{11}\right\rangle.$$

Trick

Lemma

$$(B_V \otimes C_W)|A_{W\to V}\rangle\rangle = \left|B_V A_{W\to V} C_W^T\right\rangle\rangle.$$

Proof.

$$\begin{aligned}
(B_V \otimes C_W)|A_{W\to V}\rangle\rangle &= (B_V \otimes C_W)\sum_{i,j} A_{i,j}\,|i\rangle_V\,|j\rangle_W \\
&= \sum_{i,j} A_{i,j}(B_V\,|i\rangle_V) \otimes (C_W\,|j\rangle_W) \\
&\xmapsto{\mathcal{M}} \sum_{i,j} A_{i,j}(B_V\,|i\rangle_V)(\langle j|_W\, C_W^\dagger)^* \\
&= B_V \sum_{i,j} A_{i,j}\,|i\rangle_V\,\langle j|_W\, C_W^T \\
&= B_V A_{W\to V} C_W^T \\
&\xmapsto{\mathcal{M}^{-1}} \left|B_V A_{W\to V} C_W^T\right\rangle\rangle.
\end{aligned}$$

Trick

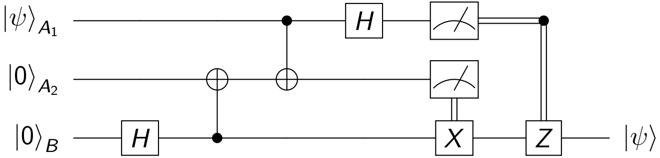
$\dim(A) = \dim(B)$.

$$\begin{aligned}
\mathrm{Tr}_A\left(\rho_A \left| L_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle L_{B\to A} \right|\right.\right.\right) &= \mathrm{Tr}_A\left(\rho_A \left| L_A I_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle L_A I_{B\to A} \right|\right.\right.\right) \\
&= \mathrm{Tr}_A\left(\rho_A L_A \left| I_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle I_{B\to A} \right| L_A^\dagger\right.\right.\right) \\
&= \mathrm{Tr}_A\left(L_A^\dagger \rho_A L_A \left| I_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle I_{B\to A} \right|\right.\right.\right) \\
&= \mathrm{Tr}_A\left((L_B^\dagger \rho_B L_B)^T \left| I_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle I_{B\to A} \right|\right.\right.\right) \\
&= (L_B^\dagger \rho_B L_B)^T \,\mathrm{Tr}_A\left(\left| I_{B\to A} \right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle I_{B\to A} \right|\right.\right.\right) \\
&= (L_B^\dagger \rho_B L_B)^T
\end{aligned}$$

Quantum teleportation

Alice sends a qubit ρ_{A_1} to Bob using a classical channel and a shared Bell state $|\Phi\rangle_{A_2B} = \left|\frac{1}{\sqrt{2}} I_{B\to A_2}\right\rangle\!\!\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B)$.

1. Alice measure $A_1 A_2$ by the Bell basis $\left\{ \left|\frac{1}{\sqrt{2}} I_{A_2\to A_1}\right\rangle\!\!\rangle, \left|\frac{1}{\sqrt{2}} X_{A_2\to A_1}\right\rangle\!\!\rangle, \left|\frac{1}{\sqrt{2}} Z_{A_2\to A_1}\right\rangle\!\!\rangle, \left|\frac{1}{\sqrt{2}} (ZX)_{A_2\to A_1}\right\rangle\!\!\rangle \right\}$.
2. Send the measurement outcome (2bit) to Bob
3. Bob apply the corresponding unitary to B_2.



Quantum teleportation

Let $A_1 = A_2 = B = \mathbb{C}^d$. For $L_{A_2 \to A_1}$ satisfying $\mathrm{Tr}(L^\dagger_{A_2 \to A_1} L_{A_2 \to A_1}) = 1$,

$$
\begin{aligned}
&\mathrm{Tr}_{A_1 A_2}\left(\rho_{A_1} \otimes \left| \frac{1}{\sqrt{d}} I_{B\to A_2} \right\rangle\!\!\right\rangle\!\!\left\langle\!\!\left\langle \frac{1}{\sqrt{d}} I_{B\to A_2} \right| \quad \left| L_{A_2\to A_1} \right\rangle\!\!\right\rangle\!\!\left\langle\!\!\left\langle L_{A_2\to A_1} \right| \right) \\
&= \mathrm{Tr}_{A_2}\left(\left| \frac{1}{\sqrt{d}} I_{B\to A_2} \right\rangle\!\!\right\rangle\!\!\left\langle\!\!\left\langle \frac{1}{\sqrt{d}} I_{B\to A_2} \right| \quad (L^\dagger_{A_2} \rho_{A_2} L_{A_2})^T \right) \\
&= \frac{1}{d} L^\dagger_{B\to A_1} \rho_{A_1} L_{B\to A_1}.
\end{aligned}
$$

When $L_{A_2\to A_1} = \frac{1}{\sqrt{d}} U_{A_2\to A_1}$ for some unitary matrix (isometry) $U_{A_2\to A_1}$,

$$
\frac{1}{d} L^\dagger_{B\to A_1} \rho_{A_1} L_{B\to A_1} = \frac{1}{d^2} U^\dagger_{B\to A_1} \rho_{A_1} U_{B\to A_1}.
$$

The probability that $\left| \frac{1}{\sqrt{d}} U_{A_2\to A_1} \right\rangle\!\!\right\rangle\!\!\left\langle\!\!\left\langle \frac{1}{\sqrt{d}} U_{A_2\to A_1} \right|$ is measured is $\frac{1}{d^2}$.

Orthogonal unitaries

When $d = 2$, I, X, Z, ZX are orthogonal unitaries in $\mathcal{L}(\mathbb{C}^2)$.

For $d \geq 3$, does there exists d^2 orthogonal unitaries?

Discrete Weyl operators:

Let ω be a primitive d-th root of unity, i.e., $\omega = \exp\{i2\pi/d\}$.

$$Z := \sum_{j=0}^{d-1} \omega^j |j\rangle \langle j|, \qquad X := \sum_{j=0}^{d-1} |j+1 \bmod d\rangle \langle j|$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \qquad = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Z^d = I, \qquad X^d = I$$

$$ZX = \omega XZ$$

$$W(s,t) := X^s Z^t$$

Orthogonality of Weyl operators

$$W(s,t) := X^s Z^t.$$

$$W(s,t)^\dagger = Z^{\dagger t} X^{\dagger s} = Z^{-t} X^{-s} = \omega^{st} X^{-s} Z^{-t} = \omega^{st} W(-s,-t)$$

$$W(s,t)W(u,v) = \omega^{tu} W(s+u, t+v) = \omega^{tu-sv} W(u,v)W(s,t)$$

$$\mathrm{Tr}(W(s,t)) = \begin{cases} d, & \text{if } (s,t) = (0,0) \bmod d \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \langle W(s,t), W(u,v)\rangle &= \mathrm{Tr}(W(s,t)^\dagger W(u,v)) \\ &= \omega^{st}\, \mathrm{Tr}(W(-s,-t)W(u,v)) \\ &= \omega^{st-tu}\, \mathrm{Tr}(W(u-s, v-t)) \\ &= \begin{cases} d, & \text{if } (u,v) = (s,t) \bmod d \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$\left\{ \frac{1}{\sqrt{d}} W(s,t) \mid s,\ t \in \{0,1,\ldots,d-1\} \right\}$ is orthonormal basis.

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}$ and $Q_b = |\psi_b\rangle_W\langle\psi_b|_W$,

$$\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b)) = \mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I \otimes |\psi_b\rangle_W\langle\psi_b|_W))$$

From an expression $|\varphi\rangle_{V\otimes W} = \sum_{i,j}\varphi_{i,j}|i\rangle_V|\psi_j\rangle_W$,

$$\begin{aligned}
&\mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)) \\
&= \mathrm{Tr}_W\left(\sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V|\psi_j\rangle_W\langle k|_V\langle\psi_l|_W(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)\right) \\
&= \mathrm{Tr}_W\left(\sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V\langle k|_V \otimes |\psi_j\rangle_W\langle\psi_l|_W(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)\right) \\
&= \sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V\langle k|_V\,\mathrm{Tr}\left(|\psi_j\rangle_W\langle\psi_l|_W|\psi_b\rangle_W\langle\psi_b|_W\right) \\
&= \sum_{i,k}\varphi_{i,b}\varphi^*_{k,b}|i\rangle_V\langle k|_V = \left(\sum_i\varphi_{i,b}|i\rangle_V\right)\left(\sum_k\varphi^*_{k,b}\langle k|_V\right)
\end{aligned}$$

$$|\varphi\rangle_{V\otimes W} = \sum_{i,j}\varphi_{i,j}|i\rangle_V|\psi_j\rangle_W \longmapsto \frac{1}{\sqrt{P(b)}}\sum_i\varphi_{i,b}|i\rangle_V$$

Examples of conditional density operator

For $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^{1} \alpha_{i,j} |i\rangle_V |j\rangle_W$, we measure the system W by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$.

if the outcome is 0, the state $\frac{1}{\sqrt{|\alpha_{0,0}|^2+|\alpha_{1,0}|^2}} \sum_{i=0}^{1} \alpha_{i,0} |i\rangle_V$.

if the outcome is 1, the state $\frac{1}{\sqrt{|\alpha_{0,1}|^2+|\alpha_{1,1}|^2}} \sum_{i=0}^{1} \alpha_{i,1} |i\rangle_V$.

Assignments

1. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state
$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$
when $|0\rangle\langle 0|$ is measured at the second system.

2. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|+\rangle\langle +|$ is measured at the second system.

3. Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|\varphi\rangle\langle \varphi|$ is measured at the second system where $|\varphi\rangle := \alpha|0\rangle + \beta|1\rangle$.