Composite system and entanglement

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Composite system

- System = Set of states & set of measurements
- Composite system = "Product" of systems.
- Composite system of a system of a coin (two-dimentional classical system) and a system of a dice (six-dimentional classical system) is twelve-dimentional classical system.
- What is a composite system of quantum systems ?

Tensor product of linear spaces

For linear product spaces V and W over a field F (usually $\mathbb{R}$ or $\mathbb{C}$), a tensor space $V \otimes W$ is a linear space spanned by $v \otimes w$ for all $v \in V$, $w \in W$.

- $\forall c \in F$, $\forall v \in V$, $\forall w \in W$, $c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.
- $\forall u, v \in V$, $\forall w \in W$, $(u + v) \otimes w = u \otimes w + v \otimes w$.
- $\forall v \in V$, $\forall w, y \in W$, $v \otimes (w + y) = v \otimes w + v \otimes y$.

Let $(e_i)_i$ be an orthonormal basis of V and $(f_j)_j$ be an orthonormal basis of W. Since $v \otimes w = (\sum_i v_i e_i) \otimes (\sum_j w_j f_j) = \sum_{i,j} v_i w_j (e_i \otimes f_j)$
This implies $\dim(V \otimes W) = \dim(V) \dim(W)$.

If V and W are inner product spaces, $V \otimes W$ is also a inner product space defined by

$$\langle v \otimes w, u \otimes y \rangle = \langle v, u \rangle \langle w, y \rangle.$$

Vector representation in tensor product

Let $V := \mathbb{C}^n$, $W := \mathbb{C}^m$.

$$e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ \\ i-1 \\ i \\ i+1 \\ \\ n \end{array} \in \mathbb{C}^n, \qquad f_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ \\ j-1 \\ j \\ j+1 \\ \\ m \end{array} \in \mathbb{C}^m$$

$$e_i \otimes f_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} (1,1) \\ \\ (i, j-1) \\ (i,j) \\ (i, j+1) \\ \\ (n, m) \end{array} \in \mathbb{C}^n \otimes \mathbb{C}^m$$

Vector representation in tensor product

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$$
e_i \otimes f_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} (1,1) \\ \\ (i,j-1) \\ (i,j) \\ (i,j+1) \\ \\ (n,m) \end{array} \in \mathbb{C}^n \otimes \mathbb{C}^m
$$

$$
v \otimes w = \left(\sum_i v_i e_i\right) \otimes \left(\sum_j w_j f_j\right) = \sum_{i,j} v_i w_j (e_i \otimes f_j)
$$

$$
= \begin{bmatrix} \vdots \\ v_i w_j \\ \vdots \end{bmatrix} \begin{array}{c} \\ (i,j) \\ \\ \end{array} = \begin{bmatrix} v_1 w \\ v_2 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{C}^n \otimes \mathbb{C}^m
$$

Linear spaces

- $\mathcal{L}(V, W)$: A linear space spanned by linear maps from a linear space V to a linear space W.

- $\mathcal{L}(V) := \mathcal{L}(V, V)$.

- $\mathcal{H}(V)$: A real linear space spanned by Hermitian operators acting on a complex linear space V.

Tensor product of linear maps

$$\mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) \cong \mathcal{L}(V \otimes W, X \otimes Y)$$

since the both sides are complex linear spaces with dimension

$$\dim(V)\dim(W)\dim(X)\dim(Y).$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) &\longrightarrow \mathcal{L}(V \otimes W, X \otimes Y) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nm}B \end{bmatrix}$$

Tensor product of Hermitian maps

$$\mathcal{H}(V) \otimes \mathcal{H}(W) \cong \mathcal{H}(V \otimes W)$$

since the both sides are real linear spaces with dimension

$$\dim(V)^2 \dim(W)^2.$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{H}(V) \otimes \mathcal{H}(W) &\longrightarrow \mathcal{H}(V \otimes W) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

$$\begin{aligned} \langle x \otimes y, (A \otimes B)(v \otimes w) \rangle &= \langle x \otimes y, Av \otimes Bw \rangle \\ &= \langle x, Av \rangle \langle y, Bw \rangle \\ &= \langle Ax, v \rangle \langle By, w \rangle \\ &= \langle Ax \otimes By, v \otimes w \rangle \\ &= \langle (A \otimes B)(x \otimes y), v \otimes w \rangle. \end{aligned}$$

Composite quantum system

A quantum system on a complex linear space V:

- Set of states $= \{\omega \in \mathcal{H}(V) \mid \omega \in C_{\succeq 0}, \mathrm{Tr}(\omega) = 1\}$.
- Set of binary measurements $= \{e \in \mathcal{H}(V) \mid e \in C_{\succeq 0}, I - e \in C_{\succeq 0}\}$.

For a quantum systems on V and W, a composite system is a quantum system on $V \otimes W$.

A useful formula.

$$\begin{aligned}
\mathrm{Tr}(A \otimes B) &= \sum_{i,j} (\langle i| \otimes \langle j|)(A \otimes B)(|i\rangle \otimes |j\rangle) \\
&= \sum_{i,j} \langle i| A |i\rangle \langle j| B |j\rangle \\
&= \mathrm{Tr}(A)\mathrm{Tr}(B)
\end{aligned}$$

Tensor product of states

$$
\begin{aligned}
&(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|)(|v\rangle \otimes |w\rangle) \\
&= (|\psi\rangle\langle\psi|\,|v\rangle) \otimes (|\phi\rangle\langle\phi|\,|w\rangle) \\
&= \langle\psi|v\rangle\langle\phi|w\rangle\, |\psi\rangle \otimes |\phi\rangle
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(|\psi\rangle \otimes |\phi\rangle)(\langle\psi| \otimes \langle\phi|)(|v\rangle \otimes |w\rangle) \\
&= \langle\psi|v\rangle\langle\phi|w\rangle\, |\psi\rangle \otimes |\phi\rangle
\end{aligned}
$$

Hence,

$$
|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| = (|\psi\rangle \otimes |\phi\rangle)(\langle\psi| \otimes \langle\phi|).
$$

We use the notations $|\psi\phi\rangle := |\psi\rangle|\phi\rangle := |\psi\rangle \otimes |\phi\rangle$.
For quantum states ρ and σ

$$
\begin{aligned}
\rho \otimes \sigma &= \left(\sum_j \mu_j |\psi_j\rangle\langle\psi_j|\right) \otimes \left(\sum_k \nu_k |\phi_k\rangle\langle\phi_k|\right) \\
&= \sum_{j,k} \mu_j \nu_k |\psi_j\phi_k\rangle\langle\psi_j\phi_k| \succeq 0
\end{aligned}
$$

Examples: two-qubit system

Examples of states

- $|0\rangle\langle 0| \otimes |1\rangle\langle 1| = |01\rangle\langle 01|$
- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}I$.
- $\frac{1}{2}(|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|)$.
- $\frac{1}{2}(|0\rangle\langle 0|\otimes|0\rangle\langle 0|+|0\rangle\langle 1|\otimes|0\rangle\langle 1|+|1\rangle\langle 0|\otimes|1\rangle\langle 0|+|1\rangle\langle 1|\otimes|1\rangle\langle 1|) = |\Phi\rangle\langle\Phi|$ for $|\Phi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$.

Separable states & entangled states

A quantum state ρ in a composite system is said to be separable if

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$$

for some probability distribution p and quantum states $\{\rho_1^i\}$ and $\{\rho_2^i\}$ for subsystems.

If a quantum state is not separable, the state is said to be entangled state.

In general, it is difficult to determine whether given state is separable or entangled.

Pure separable states

Lemma

A pure state $|\psi\rangle \in V \otimes W$ is separable if and only if there exist pure states $|\varphi\rangle \in V$ and $|\phi\rangle \in W$ such that $|\psi\rangle = |\varphi\rangle |\phi\rangle$.

Proof.

$$
\begin{aligned}
|\psi\rangle \langle\psi| &= \sum_i p_i \rho_i \otimes \sigma_i \\
&= \sum_i p_i \left(\sum_j \lambda_{i,j} |\varphi_{i,j}\rangle \langle\varphi_{i,j}| \right) \otimes \left(\sum_k \gamma_{i,k} |\phi_{i,k}\rangle \langle\phi_{i,k}| \right) \\
&= \sum_\ell q_\ell |\varphi_\ell\rangle \langle\varphi_\ell| \otimes |\phi_\ell\rangle \langle\phi_\ell|
\end{aligned}
$$

$$
\begin{aligned}
1 &= \mathrm{Tr}\left(|\psi\rangle \langle\psi| \left(\sum_i p_i \rho_i \otimes \sigma_i \right) \right) \\
&= \sum_\ell q_\ell |\langle\psi| (|\varphi_\ell\rangle |\phi_\ell\rangle)|^2 \iff |\psi\rangle = e^{i\theta_\ell} |\varphi_\ell\rangle |\phi_\ell\rangle \qquad \forall \ell
\end{aligned}
$$

□

Isomorphism between $V \otimes W$ and $\mathcal{L}(W, V)$

We consider isomporphism $\mathcal{M}$ between $V \otimes W$ and $\mathcal{L}(W, V)$ defined by

$$\begin{aligned} \mathcal{M} : V \otimes W &\to \mathcal{L}(W, V) \\ |i\rangle_V |j\rangle_W &\mapsto |i\rangle_V \langle j|_W \end{aligned}$$

where $(|i\rangle_V)_i$ and $(|j\rangle_W)_j$ are orthonormal basis of V and W, respectively.

$$\begin{aligned} &\mathcal{M}(|\psi\rangle_V |\varphi\rangle_W) \\ &= \mathcal{M}\left(\left(\sum_i \psi_i |i\rangle_V\right) \otimes \left(\sum_j \varphi_j |j\rangle_W\right)\right) \\ &= \sum_{i,j} \psi_i \varphi_j \mathcal{M}(|i\rangle_V |j\rangle_W) \\ &= \sum_{i,j} \psi_i \varphi_j |i\rangle_V \langle j|_W \\ &= \left(\sum_i \psi_i |i\rangle_V\right)\left(\sum_j \varphi_j \langle j|_W\right) = |\psi\rangle_V \langle \varphi|_W^* \end{aligned}$$

Determine the separability of pure state

$$
\begin{aligned}
|\psi\rangle \in V \otimes W \text{ is separable } &\iff |\psi\rangle = |\varphi\rangle |\phi\rangle \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\
&\iff \mathcal{M}(|\psi\rangle) = |\varphi\rangle \langle\phi| \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\
&\iff \mathcal{M}(|\psi\rangle) \text{ is rank 1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}\left(\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right) \\
&= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{\sqrt{2}} I
\end{aligned}
$$

$\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ is entangled!

Scmidt decomposition

Theorem (Schmidt decomposition)

For any pure state $|\psi\rangle \in V \otimes W$, there exist orthonormal systems $(|v_i\rangle)_i$ of V and $(|w_j\rangle)_j$ of W, and positive real numbers $(\lambda_i)_i$ such that

$$|\psi\rangle = \sum_i \lambda_i \, |v_i\rangle_V \, |w_i\rangle_W \, .$$

Proof.

Let $A := \mathcal{M}(|\psi\rangle)$. By the singular value decomposition,

$$A = \sum_i \lambda_i \, |s_i\rangle_V \, \langle t_i|_W$$

Since $|\psi\rangle = \mathcal{M}^{-1}(A)$,

$$|\psi\rangle = \sum_i \lambda_i \, |s_i\rangle_V \, |t_i\rangle_W^* \, . \qquad \square$$

The number of the terms in the decomposition is called the Schmidt rank.

Measurements on composite system

Set of measurements on the composite sytem on $V \otimes W$ is

$$\{(e_1, \ldots, e_k) \in \mathcal{H}(V \otimes W) \mid e_1 + \cdots + e_k = I, e_j \in C_{\succeq 0}$$
$$i = 1, 2, \ldots, k,\ k = 1, 2, \ldots\}.$$

For measurements $(P_a \in \mathcal{H}(V))_a$ and $(Q_b \in \mathcal{H}(W))_b$ for the partial systems, $(P_a \otimes Q_b \in \mathcal{H}(V \otimes W))_{a,b}$ is a measurement since $P_a \otimes Q_b \succeq 0$ and

$$\sum_{a,b} P_a \otimes Q_b = \left(\sum_a P_a\right) \otimes \left(\sum_b Q_b\right) = I_V \otimes I_W = I_{V \otimes W}.$$

Partial trace and reduced density matrix

A probability of outcome of local measurement in a composite system is

$$P_{V\otimes W}(a,b) = \mathrm{Tr}(\rho(P_a \otimes Q_b)).$$

$$\begin{aligned}
P_V(a) = \sum_b P_{V\otimes W}(a,b) &= \sum_b \mathrm{Tr}(\rho(P_a \otimes Q_b)) \\
&= \mathrm{Tr}\left(\rho\left(P_a \otimes \sum_b Q_b\right)\right) \\
&= \mathrm{Tr}\left(\rho\left(P_a \otimes I\right)\right) \\
&= \mathrm{Tr}(\mathrm{Tr}_W(\rho)P_a).
\end{aligned}$$

The partial trace $\mathrm{Tr}_W(\rho) \in \mathcal{H}(V)$ is defined by

$$\mathrm{Tr}(\mathrm{Tr}_W(\rho)P) = \mathrm{Tr}\left(\rho\left(P \otimes I\right)\right)$$

for any $P \in \mathcal{H}(V)$. Indeed, $\mathrm{Tr}_W(\cdot)$ is a linear operator from $\mathcal{H}(V \otimes W)$ to $\mathcal{H}(V)$ defined by $\mathrm{Tr}_W(\rho_V \otimes \sigma_W) = \mathrm{Tr}(\sigma_W)\rho_V$.

Reduced density matrix from the Schmidt decomposition

For a pure state $|\psi\rangle \in \mathcal{H}(V) \otimes \mathcal{H}(W)$ with the Schmidt decomposition

$$|\psi\rangle = \sum_i \lambda_i \left|v_i\right\rangle_V \left|w_i\right\rangle_W$$

it is easy to derive a reduced density matrices.

$$\begin{aligned} |\psi\rangle\langle\psi| &= \sum_{i,j} \lambda_i \lambda_j \left|v_i\right\rangle_V \left|w_i\right\rangle_W \left\langle v_j\right|_V \left\langle w_j\right|_W \\ &= \sum_{i,j} \lambda_i \lambda_j \left|v_i\right\rangle_V \left\langle v_j\right|_V \otimes \left|w_i\right\rangle_W \left\langle w_j\right|_W \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_W(|\psi\rangle\langle\psi|) &= \sum_{i,j} \lambda_i \lambda_j \left|v_i\right\rangle_V \left\langle v_j\right|_V \mathrm{Tr}(\left|w_i\right\rangle_W \left\langle w_j\right|_W) \\ &= \sum_i \lambda_i^2 \left|v_i\right\rangle_V \left\langle v_i\right|_V \\ \mathrm{Tr}_V(|\psi\rangle\langle\psi|) &= \sum_i \lambda_i^2 \left|w_i\right\rangle_W \left\langle w_i\right|_W \end{aligned}$$

Assignments

1. Show the Schmidt decomposition of the following pure states
   - A. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
   - B. $\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$
   - C. $\frac{1}{2}(|00\rangle - i\,|01\rangle + i\,|10\rangle + |11\rangle)$
   - D. $\frac{1}{\sqrt{2}}(\cos\theta\,|00\rangle - \sin\theta\,|01\rangle + \sin\theta\,|10\rangle + \cos\theta\,|11\rangle)$
2. Show the reduced density matrices for each qubit of the above pure states.