Nonlocality and Tsirelson's bound

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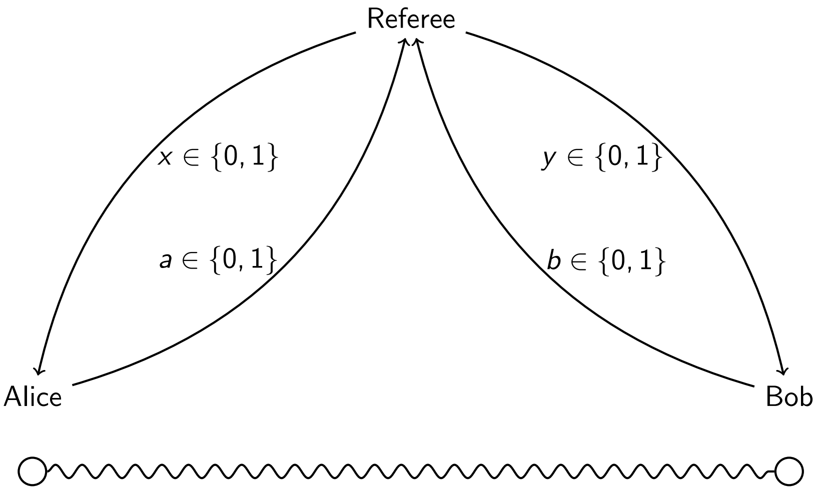
Bell test: CHSH game (1964, 1969)



Alice and Bob win iff $a \oplus b = x \wedge y$.

Bell inequality

a_x: Output of Alice for given x.
b_y: Output of Bob for given y.

$$a_0 \oplus b_0 = 0$$
$$a_1 \oplus b_0 = 0$$
$$a_0 \oplus b_1 = 0$$
$$a_1 \oplus b_1 = 1$$

By adding all equations, we get $0 = 1$, which means there is no solution. Hence, the winning probability 1 cannot be achieved.

Three equalities can be satisfied, so that the largest winning probability is $3/4$ (Bell inequality or CHSH inequality).

If Alice and Bob share quantum states, then the largest winning probability is $(2 + \sqrt{2})/4 \approx 0.854$ (Violation of Bell/CHSH inequality)

Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution $P(a, b \mid x, y)$ is said to be local if

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Quantum physics allow nonlocal behaviors.

Joint probability distribuion

Lemma

There exists probability distributions $P(\lambda)$, $P(a \mid x, \lambda)$ and $P(b \mid y, \lambda)$ such that

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda)$$

if and only if there exists probability distribution $q(a_0, a_1, b_0, b_1)$ such that

$$P(a, b \mid x, y) = \sum_{\substack{a_0, a_1, b_0, b_1 \\ a_x = a,\, b_y = b}} q(a_0, a_1, b_0, b_1).$$

Proof.

$$(\Rightarrow) \qquad q(a_0, a_1, b_0, b_1) := \sum_{\lambda} P(\lambda) P(a_0 \mid x = 0, \lambda) P(a_1 \mid x = 1, \lambda) \cdot P(b_0 \mid y = 0, \lambda) P(b_1 \mid y = 1, \lambda)$$

$$(\Leftarrow) \qquad \lambda = (a_0, a_1, b_0, b_1), \qquad P(\lambda) = q(a_0, a_1, b_0, b_1)$$

$$P(a_x \mid x, (a_0, a_1, b_0, b_1)) = 1$$

Randomness doesn't help

$$\begin{aligned} &\mathbb{E}_{x,y}\left[\mathbb{E}_{a_0,a_1,b_0,b_1}\left[\mathbb{I}\{a_x \oplus b_y = x \wedge y\}\right]\right] \\ = &\,\mathbb{E}_{a_0,a_1,b_0,b_1}\left[\mathbb{E}_{x,y}\left[\mathbb{I}\{a_x \oplus b_y = x \wedge y\}\right]\right]. \end{aligned}$$

There exists $a_0^*, a_1^*, b_0^*, b_1^*$ such that

$$\mathbb{E}_{a_0,a_1,b_0,b_1}\left[\mathbb{E}_{x,y}\left[\mathbb{I}\{a_x \oplus b_y = x \wedge y\}\right]\right] \le \mathbb{E}_{x,y}\left[\mathbb{I}\{a_x^* \oplus b_y^* = x \wedge y\}\right].$$

Einstein–Podolsky–Rosen (EPR) paradox (1935)

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

$\iff$ there exists a joint distribution of (a_0, a_1, b_0, b_1).

$$\Downarrow$$

In quantum physics, a_0, a_1, b_0, b_1 cannot *exists* simultaneously.

$$\Downarrow$$

In quantum physics, position and momentum cannot *exists* simultaneously.

Bell state

Bell state

$$|\Psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

Let $|\psi_\theta\rangle := \cos\theta|0\rangle + \sin\theta|1\rangle$.
Alice measure this state by $\{|\psi_{\theta_A}\rangle, |\psi_{\theta_A+\pi/2}\rangle\}$.
Bob measure this state by $\{|\psi_{\theta_B}\rangle, |\psi_{\theta_B+\pi/2}\rangle\}$.

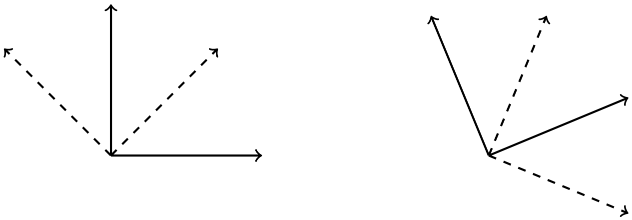
A outcome corresponding to $|\psi_\theta\rangle|\psi_\tau\rangle$ is obtained with probability

$$\begin{aligned}|\langle\psi_\theta|\langle\psi_\tau||\Psi\rangle|^2 &= \left|\langle\psi_\theta|\langle\psi_\tau|\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right|^2 \\ &= \frac{1}{2}|\cos\theta\cos\tau + \sin\theta\sin\tau|^2 = \frac{1}{2}\cos^2(\theta-\tau).\end{aligned}$$

Another proof:

$$\begin{aligned}\langle\psi_\theta|\langle\psi_\tau||\Psi\rangle &= \mathrm{Tr}\left(\mathcal{M}(|\psi_\theta\rangle|\psi_\tau\rangle)^\dagger\mathcal{M}(|\Psi\rangle)\right) \\ &= \mathrm{Tr}\left(|\psi_\tau\rangle\langle\psi_\theta|\frac{1}{\sqrt{2}}I\right) = \frac{1}{\sqrt{2}}\langle\psi_\theta|\psi_\tau\rangle\end{aligned}$$

Quantum strategy

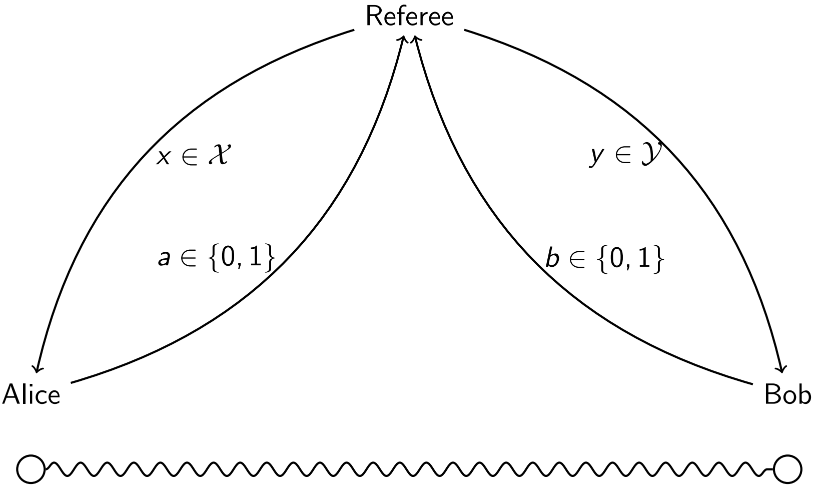


$$\theta_A^{x=0} = 0, \qquad \theta_A^{x=1} = \pi/4, \qquad \theta_B^{y=0} = \pi/8, \qquad \theta_B^{y=1} = -\pi/8$$

For any $x \in \{0,1\}, y \in \{0,1\}$, the winning probability is

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2+\sqrt{2}}{4} \approx 0.854.$$

XOR game



Alice and Bob win iff $a \oplus b = f(x, y)$.

Tsirelson's theorem

The maximum quantum winning probability is

$$
\begin{aligned}
\text{max:} \quad & \sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a \oplus b = f(x,y)}} \operatorname{Tr}(\rho(P_a^{(x)} \otimes Q_b^{(y)})) \\
\text{subject to:} \quad & n \in \mathbb{N} \\
& \rho \in \mathcal{H}(\mathbb{C}^n \otimes \mathbb{C}^n) \\
& \rho \succeq 0 \\
& \operatorname{Tr}(\rho) = 1 \\
& P_a^{(x)},\ Q_b^{(y)} \in \mathcal{H}(\mathbb{C}^n) \qquad \forall x, y, a, b \\
& P_a^{(x)} \succeq 0 \qquad \forall x, a \\
& Q_b^{(y)} \succeq 0 \qquad \forall y, b \\
& P_0^{(x)} + P_1^{(x)} = I \qquad \forall x \\
& Q_0^{(y)} + Q_1^{(y)} = I \qquad \forall y.
\end{aligned}
$$

Tsirelson's theorem

The maximum quantum winning probability is

$$\begin{aligned}
\text{max:} \quad & \sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a\oplus b = f(x,y)}} \langle\psi| P_a^{(x)} \otimes Q_b^{(y)} |\psi\rangle \\
\text{subject to:} \quad & n \in \mathbb{N} \\
& |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \\
& \langle\psi|\psi\rangle = 1 \\
& P_a^{(x)},\ Q_b^{(y)} \in \mathcal{H}(\mathbb{C}^n) \qquad \forall x, y, a, b \\
& P_a^{(x)} \succeq 0 \qquad \forall x, a \\
& Q_b^{(y)} \succeq 0 \qquad \forall y, b \\
& P_0^{(x)} + P_1^{(x)} = I \qquad \forall x \\
& Q_0^{(y)} + Q_1^{(y)} = I \qquad \forall y.
\end{aligned}$$

Binary measurements and observable

Lemma

$$P_0 \succeq 0,\ P_1 \succeq 0,\ P_0 + P_1 = I \iff \exists P,\ I - P^2 \succeq 0, P_a = \frac{I + (-1)^a P}{2}.$$

Proof.

$(\Rightarrow)$ $P = P_0 - P_1$.

$(\Leftarrow)$

$$\frac{I + (-1)^a P}{2} \succeq 0$$
$$\frac{I + P}{2} + \frac{I - P}{2} = I.$$

$\square$

Tsirelson's theorem

By letting $P^{(x)} := P_0^{(x)} - P_1^{(x)}$ and $Q^{(y)} := Q_0^{(y)} - Q_1^{(y)}$, the maximum quantum winning probability is

$$\begin{array}{ll} \text{max:} & \displaystyle\sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a\oplus b = f(x,y)}} \langle\psi| \frac{I + (-1)^a P^{(x)}}{2} \otimes \frac{I + (-1)^b Q^{(y)}}{2} |\psi\rangle \\ \text{subject to:} & n \in \mathbb{N} \\ & |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \\ & \langle\psi|\psi\rangle = 1 \\ & P^{(x)}, Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \qquad \forall x, y \\ & I - (P^{(x)})^2 \succeq 0 \qquad \forall x \\ & I - (Q^{(y)})^2 \succeq 0 \qquad \forall y \end{array}$$

Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \sum_{x,y,a,b} p(x,y) \langle\psi| \frac{I + (-1)^a P^{(x)}}{2} \otimes \frac{I + (-1)^b Q^{(y)}}{2} |\psi\rangle \frac{1 + (-1)^{a+b+f(x,y)}}{2}$$

$$\begin{aligned}
\text{s.t.}: \;& n \in \mathbb{N} \\
& |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \\
& \langle\psi|\psi\rangle = 1 \\
& P^{(x)},\ Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \qquad \forall x, y \\
& I - (P^{(x)})^2 \succeq 0 \qquad \forall x \\
& I - (Q^{(y)})^2 \succeq 0 \qquad \forall y
\end{aligned}$$

Tsirelson's theorem

The maximum quantum winning probability is

$$
\begin{aligned}
\text{max:} \quad & \frac{1}{2}\left(1 + \sum_{x,y} p(x,y) \langle\psi| P^{(x)} \otimes Q^{(y)} |\psi\rangle (-1)^{f(x,y)}\right) \\
\text{subject to:} \quad & n \in \mathbb{N} \\
& |\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \\
& \langle\psi|\psi\rangle = 1 \\
& P^{(x)},\ Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \qquad \forall x, y \\
& I - (P^{(x)})^2 \succeq 0 \qquad \forall x \\
& I - (Q^{(y)})^2 \succeq 0 \qquad \forall y
\end{aligned}
$$

Convexity and extremal points

Lemma

A set

$$C := \left\{ P \in \mathcal{H}(\mathbb{C}^n) \mid I - P^2 \succeq 0 \right\}$$

is a convex set. $P \in C$ is extremal if and only if $P^2 = I$.

Proof.

If P has an eigenvalue in $(-1, +1)$, P can be represented by convex combination of two different points in C.

Assume P has an eigenvalue in ± 1. Let $|\psi\rangle$ be an eigenvector of P for an eigenvalue $\lambda \in \{-1, +1\}$. If $P = pP_0 + (1-p)P_1$ for some $p \in (0, 1)$ and $P_0, P_1 \in C$, $p \langle\psi| P_0 |\psi\rangle + (1-p) \langle\psi| P_1 |\psi\rangle = \lambda$. This means that $\langle\psi| P_0 |\psi\rangle = \langle\psi| P_1 |\psi\rangle = \lambda$. Hence, $|\psi\rangle$ is also an eigenvector of P_0 and P_1. All eigenvectors of P are also eigenvectors of P_0 and P_1 for the same eigenvalue. Hence, $P_0 = P_1 = P$. $\square$

Tsirelson's theorem

The maximum quantum winning probability is

$$\begin{aligned}
\text{max:} \quad & \frac{1}{2}\left(1+\sum_{x,y} p(x,y)\,\langle\psi|\,P^{(x)}\otimes Q^{(y)}\,|\psi\rangle\,(-1)^{f(x,y)}\right) \\
\text{subject to:} \quad & n\in\mathbb{N} \\
& |\psi\rangle\in\mathbb{C}^n\otimes\mathbb{C}^n \\
& \langle\psi|\psi\rangle=1 \\
& P^{(x)},\ Q^{(y)}\in\mathcal{H}(\mathbb{C}^n) \qquad \forall x,y \\
& (P^{(x)})^2=I \qquad \forall x \\
& (Q^{(y)})^2=I \qquad \forall y
\end{aligned}$$

Here, $P^{(x)}$ and $Q^{(y)}$ are unitary. Let

$$|A_x\rangle := (P^{(x)}\otimes I)\,|\psi\rangle$$
$$|B_y\rangle := (I\otimes Q^{(y)})\,|\psi\rangle\,.$$

Then, $\langle\psi|\,P^{(x)}\otimes Q^{(y)}\,|\psi\rangle=\langle A_x|B_y\rangle$.

Tsirelson's theorem

The maximum quantum winning probability is at most (in fact, equal to)

$$
\begin{aligned}
\text{max:} \quad & \frac{1}{2}\left(1+\sum_{x,y} p(x,y)\,\mathrm{Re}\left(\langle A_x|B_y\rangle\right)(-1)^{f(x,y)}\right) \\
\text{subject to:} \quad & n \in \mathbb{N} \\
& |A_x\rangle, |B_y\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \qquad \forall x, y \\
& \langle A_x|A_x\rangle = 1 \qquad \forall x \\
& \langle B_y|B_y\rangle = 1 \qquad \forall y
\end{aligned}
$$

Real vectors

For some orthonormal basis $\{|e_i\rangle\}$, $|A_x\rangle = \sum_{i=1}^{n^2} \alpha_i^{(x)} |e_i\rangle$, $|B_y\rangle = \sum_{i=1}^{n^2} \beta_i^{(y)} |e_i\rangle$. Let

$$v_x := \begin{bmatrix} \mathrm{Re}(\alpha_1^{(x)}) & \mathrm{Im}(\alpha_1^{(x)}) & \mathrm{Re}(\alpha_2^{(x)}) & \cdots & \mathrm{Im}(\alpha_{n^2}^{(x)}) \end{bmatrix} \in \mathbb{R}^{2n^2}$$
$$w_y := \begin{bmatrix} \mathrm{Re}(\beta_1^{(y)}) & \mathrm{Im}(\beta_1^{(y)}) & \mathrm{Re}(\beta_2^{(y)}) & \cdots & \mathrm{Im}(\beta_{n^2}^{(y)}) \end{bmatrix} \in \mathbb{R}^{2n^2}$$

Then,

$$\langle v_x, v_x \rangle = \langle A_x | A_x \rangle$$
$$\langle w_y, w_y \rangle = \langle B_y | B_y \rangle$$
$$\langle v_x, w_y \rangle = \mathrm{Re}(\langle A_x | B_y \rangle).$$

Tsirelson's theorem [Tsirelson 1980]

The maximum quantum winning probability is at most (in fact, equal to)

$$
\begin{aligned}
\text{max:} \quad & \frac{1}{2}\left(1 + \sum_{x,y} p(x,y)\langle v_x, w_y\rangle(-1)^{f(x,y)}\right) \\
\text{subject to:} \quad & n \in \mathbb{N} \\
& v_x \in \mathbb{R}^{2n^2} \qquad \forall x \\
& w_y \in \mathbb{R}^{2n^2} \qquad \forall y \\
& \|v_x\| = 1 \qquad \forall x \\
& \|w_y\| = 1 \qquad \forall y
\end{aligned}
$$

The dimension $2n^2$ is at most $|\mathcal{X}| + |\mathcal{Y}|$.

Tsirelson's bound

SDP for Tsirelson's theorem

The maximum quantum winning probability is at most (in fact, equal to)

$$
\begin{aligned}
\text{max:} \quad & \frac{1}{2}\left(1 + \sum_{x,y} p(x,y) C_{x,y} (-1)^{f(x,y)}\right) \\
\text{subject to:} \quad & C \succeq 0 \\
& C_{z,z} = 1 \qquad \forall z \in \mathcal{X} \cup \mathcal{Y}
\end{aligned}
$$

Tsirelson's bound [Tsirelson 1980]

$$\text{max:} \qquad \frac{1}{2}\left(1 + \frac{1}{4}\sum_{x,y}\langle v_x, w_y\rangle(-1)^{(x\wedge y)}\right)$$

$$\begin{aligned}
\frac{1}{4}\sum_{x,y}\langle v_x, w_y\rangle(-1)^{(x\wedge y)} &= \frac{1}{4}\left(\langle v_0, w_0\rangle + \langle v_0, w_1\rangle + \langle v_1, w_0\rangle - \langle v_1, w_1\rangle\right) \\
&= \frac{1}{4}\left(\langle v_0, w_0 + w_1\rangle + \langle v_1, w_0 - w_1\rangle\right) \\
&\leq \frac{1}{4}\left(\|v_0\|\|w_0 + w_1\| + \|v_1\|\|w_0 - w_1\|\right) \\
&= \frac{1}{4}\left(\|w_0 + w_1\| + \|w_0 - w_1\|\right) \\
&\leq \frac{\sqrt{2}}{4}\sqrt{\|w_0 + w_1\|^2 + \|w_0 - w_1\|^2} \\
&= \frac{\sqrt{2}}{4}\sqrt{2\|w_0\|^2 + 2\|w_1\|^2} = \frac{1}{\sqrt{2}}
\end{aligned}$$

Assignments

1. Let

$$T_{2i-1} := \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes X \otimes \underbrace{Y \otimes \cdots \otimes Y}_{d-i} \in \mathcal{H}(\mathbb{C}^{2^d})$$

$$T_{2i} := \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes Z \otimes \underbrace{Y \otimes \cdots \otimes Y}_{d-i} \in \mathcal{H}(\mathbb{C}^{2^d})$$

for $i = 1, \ldots, d$. Show that $T_i^2 = I$ for all $1 \leq i \leq 2d$ and $T_i T_j = -T_j T_i$ for all $1 \leq i < j \leq 2d$.

2. For any $v, w \in \mathbb{R}^{2d}$ satisfying $\|v\| = \|w\| = 1$,

$$P_v := \sum_{i=1}^{2d} v_i T_i \in \mathcal{H}(\mathbb{C}^{2^d}), \qquad Q_w := \sum_{i=1}^{2d} w_i T_i^T \in \mathcal{H}(\mathbb{C}^{2^d})$$

where ^{T} stands for the transposition. Show that $P_v^2 = Q_w^2 = I$.

3. [Advanced] Let $|\psi\rangle := \frac{1}{2^{d/2}} \sum_{i=1}^{2^d} |i\rangle |i\rangle \in \mathbb{C}^{2^{2d}}$. Show $\langle\psi| P_v \otimes Q_w |\psi\rangle = \langle v, w\rangle$ for any $v, w \in \mathbb{R}^{2d}$ satisfying $\|v\| = \|w\| = 1$.