

# Quantum teleportation

Ryuhei Mori

Tokyo Institute of Technology

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# Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

$T$  : The set of states  $\rightarrow$  the set of states.

$$pT(\rho_1) + (1 - p)T(\rho_2) = T(p\rho_1 + (1 - p)\rho_2)$$

for any density matrices  $\rho_1, \rho_2$  and  $p \in [0, 1]$ .

$T : \mathcal{H}(V) \rightarrow \mathcal{H}(W)$  must be **linear** (a proof is needed).

# Schrödinger picture and Heisenberg picture

$T^\dagger$  : The set of binary measurements  $\rightarrow$  the set of binary measurements.

$$\langle T(\rho), P \rangle = \langle \rho, T^\dagger(P) \rangle$$

for any  $\rho \in \mathcal{H}(V)$  and  $P \in \mathcal{H}(W)$ .  $T^\dagger$  is an **adjoint** map of  $T$ .

$$\begin{aligned} \langle T_3(T_2(T_1(\rho))), P \rangle &= \langle T_2(T_1(\rho)), T_3^\dagger(P) \rangle \\ &= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P)) \rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P))) \rangle \end{aligned}$$

## No-cloning theorem

$$\begin{aligned}|0\rangle\langle 0| &\longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ |1\rangle\langle 1| &\longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1|\end{aligned}$$

From the linearity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

## Axioms for quantum channel

$$T: \mathcal{H}(V) \rightarrow \mathcal{H}(W).$$

- ① Trace-preserving:  $\text{Tr}(T(\rho)) = \text{Tr}(\rho)$ .
- ② Positive :  $T(\rho) \succeq 0$  for any  $\rho \succeq 0$ .
- ③ Completely positive:  $\text{id} \otimes T$  is positive.

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## Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$ : **transposition** according to  $\{|0\rangle, |1\rangle\}$ .

The transposition is obviously linear and trace-preserving.

### Lemma

*The transposition is **positive**.*

### Proof.

For any  $A \succeq 0$  and  $|\psi\rangle \in \mathbb{C}^2$ ,

$$\langle\psi| T(A) |\psi\rangle = \langle\psi| A^T |\psi\rangle = \langle\psi| A^* |\psi\rangle = (\langle\psi|^* A |\psi\rangle^*)^* \geq 0$$

□

## Positive but not completely positive 2/2

### Lemma

The transposition is *not* completely positive.

### Proof.

For  $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ,

$$\begin{aligned} & (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|) \\ &= (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T) \left( \frac{1}{2} \left( |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \right. \right. \\ &\quad \left. \left. + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \right) \\ &= \frac{1}{2} \left( |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \right. \\ &\quad \left. + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \end{aligned}$$

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |10\rangle \quad |10\rangle \mapsto |01\rangle \quad |11\rangle \mapsto |11\rangle$$

Hence,  $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$ .  $(\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$  is not positive semidefinite.



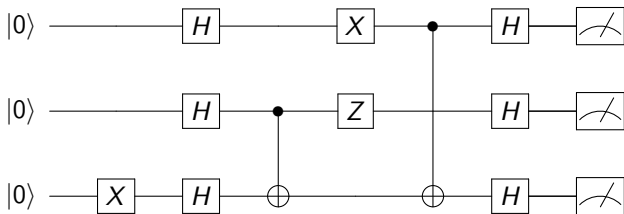
# Unitary operations

$$\rho \longmapsto U\rho U^\dagger.$$

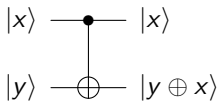
- ① Trace-preserving:  $\text{Tr}(U\rho U^\dagger) = \text{Tr}(\rho)$ .
- ② Completely positive:  $(\text{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$ .

In the most of quantum computing, only **pure** states and **unitary** operations are used.

## Quantum circuit



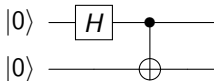
## Controlled not



$$\text{CNOT } |x\rangle |y\rangle \mapsto |x\rangle |y \oplus x\rangle$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

## Bell states and quantum circuit



$$\begin{aligned} |0\rangle |0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |0\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle) \end{aligned}$$

$$\begin{aligned} |x\rangle |y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |y\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |\bar{y}\rangle). \end{aligned}$$

## Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)).$$

$$P(a | b) = \frac{1}{P(b)} \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)} \text{Tr}(\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b))P_a).$$

$$\rho_{V|Q_b} := \frac{1}{P(b)} \text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)).$$

## Trick

$$A_{W \rightarrow V} = \sum_{ij} A_{i,j} |i\rangle_V \langle j|_W$$

$$|A_{W \rightarrow V}\rangle\rangle := \mathcal{M}^{-1}(A_{W \rightarrow V}) = \sum_{ij} A_{i,j} |i\rangle_V |j\rangle_W$$

$$\langle A, B \rangle = \text{Tr}(A^\dagger B) = \langle\langle A | B \rangle\rangle$$

### Examples

$$\left| \frac{1}{\sqrt{2}} I \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle + |1\rangle |1\rangle)$$

$$\left| \frac{1}{\sqrt{2}} X \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle |1\rangle + |1\rangle |0\rangle)$$

$$\left| \frac{1}{\sqrt{2}} Y \right\rangle\rangle = \frac{1}{\sqrt{2}} (-i |0\rangle |1\rangle + i |1\rangle |0\rangle) = -i \frac{1}{\sqrt{2}} (|0\rangle |1\rangle - |1\rangle |0\rangle)$$

$$\left| \frac{1}{\sqrt{2}} Z \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle |0\rangle - |1\rangle |1\rangle)$$

## Lemma

$$(B_V \otimes C_W) |A_{W \rightarrow V}\rangle\rangle = \left| B_V A_{W \rightarrow V} C_W^T \right\rangle\rangle.$$

## Proof.

$$\begin{aligned}
 (B_V \otimes C_W) |A_{W \rightarrow V}\rangle\rangle &= (B_V \otimes C_W) \sum_{i,j} A_{i,j} |i\rangle_V |j\rangle_W \\
 &= \sum_{i,j} A_{i,j} (B_V |i\rangle_V) \otimes (C_W |j\rangle_W) \\
 &\stackrel{\mathcal{M}}{\mapsto} \sum_{i,j} A_{i,j} (B_V |i\rangle_V) (\langle j|_W C_W^\dagger)^* \\
 &= B_V \sum_{i,j} A_{i,j} |i\rangle_V \langle j|_W C_W^T \\
 &= B_V A_{W \rightarrow V} C_W^T \\
 &\stackrel{\mathcal{M}^{-1}}{\mapsto} \left| B_V A_{W \rightarrow V} C_W^T \right\rangle\rangle.
 \end{aligned}$$

## Trick

$$\dim(A) = \dim(B).$$

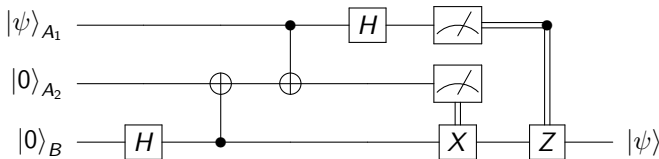
$$\begin{aligned}\mathrm{Tr}_A \left( \rho_A \left| L_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle L_{B \rightarrow A} \right| \right) &= \mathrm{Tr}_A \left( \rho_A \left| L_A I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle L_A I_{B \rightarrow A} \right| \right) \\&= \mathrm{Tr}_A \left( \rho_A L_A \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| L_A^\dagger \right) \\&= \mathrm{Tr}_A \left( L_A^\dagger \rho_A L_A \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= \mathrm{Tr}_A \left( (L_B^\dagger \rho_B L_B)^T \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= (L_B^\dagger \rho_B L_B)^T \mathrm{Tr}_A \left( \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= (L_B^\dagger \rho_B L_B)^T\end{aligned}$$



# Quantum teleportation

Alice sends a qubit  $\rho_{A_1}$  to Bob using a classical channel and a shared Bell state  $|\Phi\rangle_{A_2B} = \left| \frac{1}{\sqrt{2}} I_{B \rightarrow A_2} \right\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A_2} |0\rangle_B + |1\rangle_{A_2} |1\rangle_B)$ .

- 1 Alice measure  $A_1A_2$  by the Bell basis  
 $\left\{ \left| \frac{1}{\sqrt{2}} I_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} X_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} Y_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} Z_{A_2 \rightarrow A_1} \right\rangle\rangle \right\}$ .
- 2 Send the measurement outcome (2bit) to Bob
- 3 Bob apply the corresponding unitary to  $B_2$ .



## Quantum teleportation

Let  $A_1 = A_2 = B = \mathbb{C}^d$ . For  $L_{A_2 \rightarrow A_1}$  satisfying  $\text{Tr}(L_{A_2 \rightarrow A_1}^\dagger L_{A_2 \rightarrow A_1}) = 1$ ,

$$\begin{aligned} & \text{Tr}_{A_1 A_2} \left( \rho_{A_1} \otimes \left| \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right\rangle \left\langle \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right| \left| L_{A_2 \rightarrow A_1} \right\rangle \left\langle L_{A_2 \rightarrow A_1} \right| \right) \\ &= \text{Tr}_{A_2} \left( \left| \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right\rangle \left\langle \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right| (L_{A_2}^\dagger \rho_{A_2} L_{A_2})^T \right) \\ &= \frac{1}{d} L_{B \rightarrow A_1}^\dagger \rho_{A_1} L_{B \rightarrow A_1}. \end{aligned}$$

When  $L_{A_2 \rightarrow A_1} = \frac{1}{\sqrt{d}} U_{A_2 \rightarrow A_1}$  for some unitary matrix (isometry)  $U_{A_2 \rightarrow A_1}$ ,

$$\frac{1}{d} L_{B \rightarrow A_1}^\dagger \rho_{A_1} L_{B \rightarrow A_1} = \frac{1}{d^2} U_{B \rightarrow A_1}^\dagger \rho_{A_1} U_{B \rightarrow A_1}.$$

The probability that  $\left| L_{A_2 \rightarrow A_1} \right\rangle \left\langle L_{A_2 \rightarrow A_1} \right|$  is measured is  $\frac{1}{d^2}$ .

## Orthonormal unitaries

When  $d = 2$ ,  $\frac{1}{\sqrt{2}}I$ ,  $\frac{1}{\sqrt{2}}X$ ,  $\frac{1}{\sqrt{2}}Y$ ,  $\frac{1}{\sqrt{2}}Z$  are orthonormal unitaries in  $\mathcal{L}(\mathbb{C}^2)$ .

For  $d \geq 3$ , does there exist  $d^2$  orthonormal unitaries?

## Conditional density operator for pure state

For  $\rho = |\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W}$  and  $Q_b = |\psi_b\rangle_W \langle \psi_b|_W$ ,

$$\text{Tr}_W(\rho_{V \otimes W} (I_V \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I \otimes |\psi_b\rangle_W \langle \psi_b|_W))$$

From an expression  $|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W$ ,

$$\begin{aligned} & \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W)) \\ &= \text{Tr}_W \left( \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V |\psi_j\rangle_W \langle k|_V \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \text{Tr}_W \left( \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \otimes |\psi_j\rangle_W \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \text{Tr}(|\psi_j\rangle_W \langle \psi_l|_W |\psi_b\rangle_W \langle \psi_b|_W) \\ &= \sum_{i,k} \varphi_{i,b} \varphi_{k,b}^* |i\rangle_V \langle k|_V = \left( \sum_i \varphi_{i,b} |i\rangle_V \right) \left( \sum_k \varphi_{k,b}^* \langle k|_V \right) \end{aligned}$$

$$|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W \mapsto \frac{1}{\sqrt{P(b)}} \sum_i \varphi_{i,b} |i\rangle_V$$

## Examples of conditional density operator

For  $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^1 \alpha_{i,j} |i\rangle_V |j\rangle_W$ , we measure the system  $W$  by  $(|0\rangle\langle 0|, |1\rangle\langle 1|)$ .

if the outcome is 0, the state  $\frac{1}{\sqrt{|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2}} \sum_{i=0}^1 \alpha_{i,0} |i\rangle_V$ .

if the outcome is 1, the state  $\frac{1}{\sqrt{|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2}} \sum_{i=0}^1 \alpha_{i,1} |i\rangle_V$ .

## Assignments

- ① Show the state vector  $|\psi\rangle \in \mathbb{C}^2$  of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

when  $|0\rangle\langle 0|$  is measured at the second system.

- ② Show the state vector  $|\psi\rangle \in \mathbb{C}^2$  of the Bell state when  $|+\rangle\langle +|$  is measured at the second system.
- ③ Show the state vector  $|\psi\rangle \in \mathbb{C}^2$  of the Bell state when  $|\varphi\rangle\langle \varphi|$  is measured at the second system where  $|\varphi\rangle := \alpha|0\rangle + \beta|1\rangle$ .