

Quantum teleportation

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

T : The set of states \rightarrow the set of states.

$$pT(\rho_1) + (1 - p)T(\rho_2) = T(p\rho_1 + (1 - p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0, 1]$.

$T : \mathcal{H}(V) \rightarrow \mathcal{H}(W)$ must be **linear** (a proof is needed).

Schrödinger picture and Heisenberg picture

T^\dagger : The set of binary measurements \rightarrow the set of binary measurements.

$$\langle T(\rho), P \rangle = \langle \rho, T^\dagger(P) \rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. T^\dagger is an **adjoint** map of T .

$$\begin{aligned} \langle T_3(T_2(T_1(\rho))), P \rangle &= \langle T_2(T_1(\rho)), T_3^\dagger(P) \rangle \\ &= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P)) \rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P))) \rangle \end{aligned}$$

No-cloning theorem

$$\begin{aligned}|0\rangle\langle 0| &\longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ |1\rangle\langle 1| &\longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1|\end{aligned}$$

From the linearity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Axioms for quantum channel

$$T: \mathcal{H}(V) \rightarrow \mathcal{H}(W).$$

- ① Trace-preserving: $\text{Tr}(T(\rho)) = \text{Tr}(\rho)$.
- ② Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.
- ③ Completely positive: $\text{id} \otimes T$ is positive.

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Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$: **transposition** according to $\{|0\rangle, |1\rangle\}$.

The transposition is obviously linear and trace-preserving.

Lemma

*The transposition is **positive**.*

Proof.

For any $A \succeq 0$ and $|\psi\rangle \in \mathbb{C}^2$,

$$\langle \psi | T(A) | \psi \rangle = \langle \psi | A^T | \psi \rangle = \langle \psi | A^* | \psi \rangle = (\langle \psi |^* A | \psi \rangle^*)^* \geq 0$$

□

Positive but not completely positive 2/2

Lemma

The transposition is *not* completely positive.

Proof.

For $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\begin{aligned} & (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|) \\ &= (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T) \left(\frac{1}{2} \left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \right. \right. \\ &\quad \left. \left. + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \right) \\ &= \frac{1}{2} \left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \right. \\ &\quad \left. + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \end{aligned}$$

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |10\rangle \quad |10\rangle \mapsto |01\rangle \quad |11\rangle \mapsto |11\rangle$$

Hence, $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$. $(\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$ is not positive semidefinite.

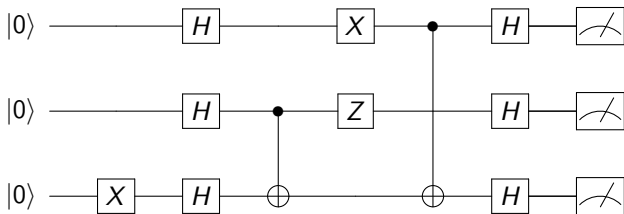
Unitary operations

$$\rho \longmapsto U\rho U^\dagger.$$

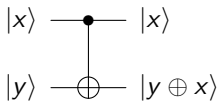
- ① Trace-preserving: $\text{Tr}(U\rho U^\dagger) = \text{Tr}(\rho)$.
- ② Completely positive: $(\text{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only **pure** states and **unitary** operations are used.

Quantum circuit



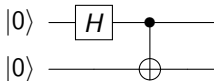
Controlled not



$$\text{CNOT } |x\rangle |y\rangle \mapsto |x\rangle |y \oplus x\rangle$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit



$$\begin{aligned} |0\rangle |0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |0\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle) \end{aligned}$$

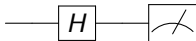
$$\begin{aligned} |x\rangle |y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |y\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |\bar{y}\rangle) = |\Phi_{xy}\rangle. \end{aligned}$$

Measurements

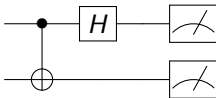
The measurement with the **computational basis** ($|0\rangle, |1\rangle$).



The measurement by ($|+\rangle, |-\rangle$).



The measurement by the Bell basis.



Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)).$$

$$P(a | b) = \frac{1}{P(b)} \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)} \text{Tr}(\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b))P_a).$$

$$\rho_{V|Q_b} := \frac{1}{P(b)} \text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)).$$

$$V \otimes W \cong \mathcal{L}(W, V)$$

$$A_{W \rightarrow V} = \sum_{ij} A_{i,j} |i\rangle_V \langle j|_W$$

$$|A_{W \rightarrow V}\rangle\rangle := \mathcal{M}^{-1}(A_{W \rightarrow V}) = \sum_{ij} A_{i,j} |i\rangle_V |j\rangle_W$$

$$\langle A_{W \rightarrow V}, B_{W \rightarrow V} \rangle = \text{Tr}(A_{W \rightarrow V}^\dagger B_{W \rightarrow V}) = \langle\langle A_{W \rightarrow V} | B_{W \rightarrow V} \rangle\rangle$$

Examples

$$\left| \frac{1}{\sqrt{2}} I_{W \rightarrow V} \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle_V |0\rangle_W + |1\rangle_V |1\rangle_W) = |\Phi_{00}\rangle$$

$$\left| \frac{1}{\sqrt{2}} X_{W \rightarrow V} \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle_V |1\rangle_W + |1\rangle_V |0\rangle_W) = |\Phi_{01}\rangle$$

$$\left| \frac{1}{\sqrt{2}} Z_{W \rightarrow V} \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle_V |0\rangle_W - |1\rangle_V |1\rangle_W) = |\Phi_{10}\rangle$$

$$\left| \frac{1}{\sqrt{2}} (ZX)_{W \rightarrow V} \right\rangle\rangle = \frac{1}{\sqrt{2}} (|0\rangle_V |1\rangle_W - |1\rangle_V |0\rangle_W) = |\Phi_{11}\rangle.$$

Lemma

$$(B_V \otimes C_W) |A_{W \rightarrow V}\rangle\rangle = \left| B_V A_{W \rightarrow V} C_W^T \right\rangle\rangle.$$

Proof.

$$\begin{aligned}
 (B_V \otimes C_W) |A_{W \rightarrow V}\rangle\rangle &= (B_V \otimes C_W) \sum_{i,j} A_{i,j} |i\rangle_V |j\rangle_W \\
 &= \sum_{i,j} A_{i,j} (B_V |i\rangle_V) \otimes (C_W |j\rangle_W) \\
 &\stackrel{\mathcal{M}}{\mapsto} \sum_{i,j} A_{i,j} (B_V |i\rangle_V) (\langle j|_W C_W^\dagger)^* \\
 &= B_V \sum_{i,j} A_{i,j} |i\rangle_V \langle j|_W C_W^T \\
 &= B_V A_{W \rightarrow V} C_W^T \\
 &\stackrel{\mathcal{M}^{-1}}{\mapsto} \left| B_V A_{W \rightarrow V} C_W^T \right\rangle\rangle.
 \end{aligned}$$

Trick

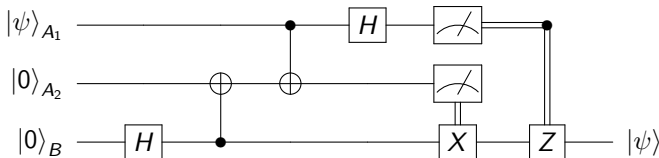
$$\dim(A) = \dim(B).$$

$$\begin{aligned}\mathrm{Tr}_A \left(\rho_A \left| L_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle L_{B \rightarrow A} \right| \right) &= \mathrm{Tr}_A \left(\rho_A \left| L_A I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle L_A I_{B \rightarrow A} \right| \right) \\&= \mathrm{Tr}_A \left(\rho_A L_A \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \left| L_A^\dagger \right. \right) \\&= \mathrm{Tr}_A \left(L_A^\dagger \rho_A L_A \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= \mathrm{Tr}_A \left((L_B^\dagger \rho_B L_B)^T \left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= (L_B^\dagger \rho_B L_B)^T \mathrm{Tr}_A \left(\left| I_{B \rightarrow A} \right\rangle\right\rangle\left\langle\left\langle I_{B \rightarrow A} \right| \right) \\&= (L_B^\dagger \rho_B L_B)^T\end{aligned}$$

Quantum teleportation

Alice sends a qubit ρ_{A_1} to Bob using a classical channel and a shared Bell state $|\Phi\rangle_{A_2B} = \left| \frac{1}{\sqrt{2}} I_{B \rightarrow A_2} \right\rangle\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B)$.

- 1 Alice measure A_1A_2 by the Bell basis
 $\left\{ \left| \frac{1}{\sqrt{2}} I_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} X_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} Z_{A_2 \rightarrow A_1} \right\rangle\rangle, \left| \frac{1}{\sqrt{2}} (ZX)_{A_2 \rightarrow A_1} \right\rangle\rangle \right\}$.
- 2 Send the measurement outcome (2bit) to Bob
- 3 Bob apply the corresponding unitary to B_2 .



Quantum teleportation

Let $A_1 = A_2 = B = \mathbb{C}^d$. For $L_{A_2 \rightarrow A_1}$ satisfying $\text{Tr}(L_{A_2 \rightarrow A_1}^\dagger L_{A_2 \rightarrow A_1}) = 1$,

$$\begin{aligned} & \text{Tr}_{A_1 A_2} \left(\rho_{A_1} \otimes \left| \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right\rangle \left\langle \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right| \left| L_{A_2 \rightarrow A_1} \right\rangle \left\langle L_{A_2 \rightarrow A_1} \right| \right) \\ &= \text{Tr}_{A_2} \left(\left| \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right\rangle \left\langle \frac{1}{\sqrt{d}} I_{B \rightarrow A_2} \right| (L_{A_2}^\dagger \rho_{A_2} L_{A_2})^T \right) \\ &= \frac{1}{d} L_{B \rightarrow A_1}^\dagger \rho_{A_1} L_{B \rightarrow A_1}. \end{aligned}$$

When $L_{A_2 \rightarrow A_1} = \frac{1}{\sqrt{d}} U_{A_2 \rightarrow A_1}$ for some unitary matrix (isometry) $U_{A_2 \rightarrow A_1}$,

$$\frac{1}{d} L_{B \rightarrow A_1}^\dagger \rho_{A_1} L_{B \rightarrow A_1} = \frac{1}{d^2} U_{B \rightarrow A_1}^\dagger \rho_{A_1} U_{B \rightarrow A_1}.$$

The probability that $\left| \frac{1}{\sqrt{d}} U_{A_2 \rightarrow A_1} \right\rangle \left\langle \frac{1}{\sqrt{d}} U_{A_2 \rightarrow A_1} \right|$ is measured is $\frac{1}{d^2}$.

Orthogonal unitaries

When $d = 2$, I , X , Z , ZX are orthogonal unitaries in $\mathcal{L}(\mathbb{C}^2)$.

For $d \geq 3$, does there exist d^2 orthogonal unitaries?

Discrete Weyl operators:

Let ω be a primitive d -th root of unity, i.e., $\omega = \exp\{i2\pi/d\}$.

$$Z := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad X := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Z^d = I,$$

$$X^d = I$$

$$ZX = \omega XZ$$

$$W(s, t) := X^s Z^t$$

Orthogonality of Weyl operators

$$W(s, t) := X^s Z^t.$$

$$W(s, t)^\dagger = Z^{\dagger t} X^{\dagger s} = Z^{-t} X^{-s} = \omega^{st} X^{-s} Z^{-t} = \omega^{st} W(-s, -t)$$

$$W(s, t)W(u, v) = \omega^{tu} W(s + u, t + v) = \omega^{tu - sv} W(u, v)W(s, t)$$

$$\mathrm{Tr}(W(s, t)) = \begin{cases} d, & \text{if } (s, t) = (0, 0) \bmod d \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \langle W(s, t), W(u, v) \rangle &= \mathrm{Tr}(W(s, t)^\dagger W(u, v)) \\ &= \omega^{st} \mathrm{Tr}(W(-s, -t)W(u, v)) \\ &= \omega^{st - tu} \mathrm{Tr}(W(u - s, v - t)) \\ &= \begin{cases} d, & \text{if } (u, v) = (s, t) \bmod d \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

$\left\{ \frac{1}{\sqrt{d}} W(s, t) \mid s, t \in \{0, 1, \dots, d-1\} \right\}$ is **orthonormal basis**.

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W}$ and $Q_b = |\psi_b\rangle_W \langle \psi_b|_W$,

$$\text{Tr}_W(\rho_{V \otimes W} (I_V \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I \otimes |\psi_b\rangle_W \langle \psi_b|_W))$$

From an expression $|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W$,

$$\begin{aligned} & \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W)) \\ &= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V |\psi_j\rangle_W \langle k|_V \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \otimes |\psi_j\rangle_W \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \text{Tr}(|\psi_j\rangle_W \langle \psi_l|_W |\psi_b\rangle_W \langle \psi_b|_W) \\ &= \sum_{i,k} \varphi_{i,b} \varphi_{k,b}^* |i\rangle_V \langle k|_V = \left(\sum_i \varphi_{i,b} |i\rangle_V \right) \left(\sum_k \varphi_{k,b}^* \langle k|_V \right) \end{aligned}$$

$$|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W \mapsto \frac{1}{\sqrt{P(b)}} \sum_i \varphi_{i,b} |i\rangle_V$$

Examples of conditional density operator

For $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^1 \alpha_{i,j} |i\rangle_V |j\rangle_W$, we measure the system W by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$.

if the outcome is 0, the state $\frac{1}{\sqrt{|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2}} \sum_{i=0}^1 \alpha_{i,0} |i\rangle_V$.

if the outcome is 1, the state $\frac{1}{\sqrt{|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2}} \sum_{i=0}^1 \alpha_{i,1} |i\rangle_V$.

Assignments

- ① Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

when $|0\rangle\langle 0|$ is measured at the second system.

- ② Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|+\rangle\langle +|$ is measured at the second system.
- ③ Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|\varphi\rangle\langle \varphi|$ is measured at the second system where $|\varphi\rangle := \alpha|0\rangle + \beta|1\rangle$.