

# Composite system and entanglement

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# Composite system

- System = Set of states & set of measurements
- **Composite** system = “Product” of systems.
- **Composite** system of a system of a coin (two-dimensional classical system) and a system of a dice (six-dimensional classical system) is twelve-dimensional classical system.
- What is a **composite** system of quantum systems ?

## Tensor product of linear spaces

For linear product spaces  $V$  and  $W$  over a field  $F$  (usually  $\mathbb{R}$  or  $\mathbb{C}$ ), a tensor space  $V \otimes W$  is a linear space spanned by  $v \otimes w$  for all  $v \in V$ ,  $w \in W$ .

- $\forall c \in F, \forall v \in V, \forall w \in W, c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$ .
- $\forall u, v \in V, \forall w \in W, (u + v) \otimes w = u \otimes w + v \otimes w$ .
- $\forall v \in V, \forall w, y \in W, v \otimes (w + y) = v \otimes w + v \otimes y$ .

Let  $(e_i)_i$  be an orthonormal basis of  $V$  and  $(f_j)_j$  be an orthonormal basis of  $W$ . Since  $v \otimes w = (\sum_i v_i e_i) \otimes (\sum_j w_j f_j) = \sum_{i,j} v_i w_j (e_i \otimes f_j)$   
This implies  $\dim(V \otimes W) = \dim(V) \dim(W)$ .

If  $V$  and  $W$  are inner product spaces,  $V \otimes W$  is also a inner product space defined by

$$\langle v \otimes w, u \otimes y \rangle = \langle v, u \rangle \langle w, y \rangle.$$

## Vector representation in tensor product

Let  $V := \mathbb{C}^n$ ,  $W := \mathbb{C}^m$ .

$$e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} 1 \\ \\ i-1 \\ i \\ i+1 \\ \\ n \end{matrix} \in \mathbb{C}^n, \quad f_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} 1 \\ \\ j-1 \\ j \\ j+1 \\ \\ m \end{matrix} \in \mathbb{C}^m$$

$$e_i \otimes f_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} (1, 1) \\ \\ (i, j-1) \\ (i, j) \\ (i, j+1) \\ \\ (n, m) \end{matrix} \in \mathbb{C}^n \otimes \mathbb{C}^m$$

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$$\begin{aligned} v \otimes w &= \left( \sum_i v_i e_i \right) \otimes \left( \sum_j w_j f_j \right) = \sum_{i,j} v_i w_j (e_i \otimes f_j) \\ &= \begin{bmatrix} \vdots \\ v_i w_j \\ \vdots \end{bmatrix} (i, j) = \begin{bmatrix} v_1 w \\ v_2 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{C}^n \otimes \mathbb{C}^m \end{aligned}$$

## Linear spaces

- $\mathcal{L}(V, W)$ : A linear space spanned by linear maps from a linear space  $V$  to a linear space  $W$ .
- $\mathcal{L}(V) := \mathcal{L}(V, V)$ .
- $\mathcal{H}(V)$ : A real linear space spanned by Hermitian operators acting on a complex linear space  $V$ .

## Tensor product of linear maps

$$\mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) \cong \mathcal{L}(V \otimes W, X \otimes Y)$$

since the both sides are complex linear spaces with dimension

$$\dim(V) \dim(W) \dim(X) \dim(Y).$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) &\longrightarrow \mathcal{L}(V \otimes W, X \otimes Y) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nm}B \end{bmatrix}$$

## Tensor product of Hermitian maps

$$\mathcal{H}(V) \otimes \mathcal{H}(W) \cong \mathcal{H}(V \otimes W)$$

since the both sides are real linear spaces with dimension

$$\dim(V)^2 \dim(W)^2.$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{H}(V) \otimes \mathcal{H}(W) &\longrightarrow \mathcal{H}(V \otimes W) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$



## Composite quantum system

A quantum system on a complex linear space  $V$ :

- Set of states =  $\{\omega \in \mathcal{H}(V) \mid \omega \in C_{\geq 0}, \text{Tr}(\omega) = 1\}$ .
- Set of binary measurements =  $\{e \in \mathcal{H}(V) \mid e \in C_{\geq 0}, I - e \in C_{\geq 0}\}$ .

For a quantum systems on  $V$  and  $W$ , a composite system is a quantum system on  $V \otimes W$ .

A useful formula.

$$\begin{aligned}\text{Tr}(A \otimes B) &= \sum_{i,j} \langle i| \otimes \langle j| A \otimes B |i\rangle \otimes |j\rangle \\ &= \sum_{i,j} \langle i| A |i\rangle \langle j| B |j\rangle \\ &= \text{Tr}(A)\text{Tr}(B)\end{aligned}$$

## Tensor product of states

$$\begin{aligned} & (|\psi\rangle \langle\psi| \otimes |\phi\rangle \langle\phi|) (|\nu\rangle \otimes |w\rangle) \\ &= (|\psi\rangle \langle\psi| |\nu\rangle) \otimes (|\phi\rangle \langle\phi| |w\rangle) \\ &= \langle\psi|\nu\rangle \langle\phi|w\rangle |\psi\rangle \otimes |\phi\rangle \end{aligned}$$

On the other hand,

$$\begin{aligned} & (|\psi\rangle \otimes |\phi\rangle) (\langle\psi| \otimes \langle\phi|) (|\nu\rangle \otimes |w\rangle) \\ &= \langle\psi|\nu\rangle \langle\phi|w\rangle |\psi\rangle \otimes |\phi\rangle \end{aligned}$$

Hence,

$$|\psi\rangle \langle\psi| \otimes |\phi\rangle \langle\phi| = (|\psi\rangle \otimes |\phi\rangle) (\langle\psi| \otimes \langle\phi|).$$

We use the notations  $|\psi\phi\rangle := |\psi\rangle |\phi\rangle := |\psi\rangle \otimes |\phi\rangle$ .

For quantum states  $\rho$  and  $\sigma$

$$\begin{aligned} \rho \otimes \sigma &= \left( \sum_j \mu_j |\psi_j\rangle \langle\psi_j| \right) \otimes \left( \sum_k \nu_k |\phi_k\rangle \langle\phi_k| \right) \\ &= \sum_{j,k} \mu_j \nu_k |\psi_j \phi_k\rangle \langle\psi_j \phi_k| \succeq 0 \end{aligned}$$

## Examples: two-qubit system

### Examples of states

- $|0\rangle\langle 0| \otimes |1\rangle\langle 1| = |01\rangle\langle 01|$
- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}I.$
- $\frac{1}{2}(|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|).$
- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) = |\Phi\rangle\langle\Phi|$  for  $|\Phi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle).$

## Separable states & entangled states

A quantum state  $\rho$  in a composite system is said to be **separable** if

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$$

for some probability distribution  $p$  and quantum states  $\{\rho_1^i\}$  and  $\{\rho_2^i\}$  for subsystems.

If a quantum state is not separable, the state is said to be **entangled** state.

In general, it is difficult to determine whether given state is separable or entangled.

## Pure separable states

### Lemma

A pure state  $|\psi\rangle \in V \otimes W$  is separable if and only if there exist pure states  $|\varphi\rangle \in V$  and  $|\phi\rangle \in W$  such that  $|\psi\rangle = |\varphi\rangle |\phi\rangle$ .

### Proof.

$$\begin{aligned} |\psi\rangle \langle\psi| &= \sum_i p_i \rho_i \otimes \sigma_i \\ &= \sum_i p_i \left( \sum_j \lambda_{i,j} |\varphi_{i,j}\rangle \langle\varphi_{i,j}| \right) \otimes \left( \sum_k \gamma_{i,k} |\phi_{i,k}\rangle \langle\phi_{i,k}| \right) \\ &= \sum_\ell q_\ell |\varphi_\ell\rangle \langle\varphi_\ell| \otimes |\phi_\ell\rangle \langle\phi_\ell| \end{aligned}$$

$$\begin{aligned} 1 &= \text{Tr} \left( |\psi\rangle \langle\psi| \left( \sum_i p_i \rho_i \otimes \sigma_i \right) \right) \\ &= \sum_\ell q_\ell |\langle\psi| (|\varphi_\ell\rangle |\phi_\ell\rangle)|^2 \iff |\psi\rangle = e^{i\theta_\ell} |\varphi_\ell\rangle |\phi_\ell\rangle \quad \forall \ell \quad \square \end{aligned}$$

## Isomorphism between $V \otimes W$ and $\mathcal{L}(W, V)$

We consider isomorphism  $\mathcal{M}$  between  $V \otimes W$  and  $\mathcal{L}(W, V)$  defined by

$$\mathcal{M} : V \otimes W \rightarrow \mathcal{L}(W, V)$$

$$|i\rangle_V |j\rangle_W \mapsto |i\rangle_V \langle j|_W$$

where  $(|i\rangle_V)_i$  and  $(|j\rangle_W)_j$  are orthonormal basis of  $V$  and  $W$ , respectively.

$$\begin{aligned} & \mathcal{M}(|\psi\rangle_V |\varphi\rangle_W) \\ &= \mathcal{M}\left(\left(\sum_i \psi_i |i\rangle_V\right) \otimes \left(\sum_j \varphi_j |j\rangle_W\right)\right) \\ &= \sum_{i,j} \psi_i \varphi_j \mathcal{M}(|i\rangle_V |j\rangle_W) \\ &= \sum_{i,j} \psi_i \varphi_j |i\rangle_V \langle j|_W \\ &= \left(\sum_i \psi_i |i\rangle_V\right) \left(\sum_j \varphi_j \langle j|_W\right) = |\psi\rangle_V \langle \varphi|_W^* \end{aligned}$$

## Determine the separability of pure state

$$\begin{aligned} |\psi\rangle \in V \otimes W \text{ is separable} &\iff |\psi\rangle = |\varphi\rangle |\phi\rangle \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\ &\iff \mathcal{M}(|\psi\rangle) = |\varphi\rangle \langle\phi| \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\ &\iff \mathcal{M}(|\psi\rangle) \text{ is rank 1} \end{aligned}$$

$$\begin{aligned} &\mathcal{M}\left(\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right) \\ &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{\sqrt{2}}I \end{aligned}$$

$\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$  is entangled!

# Schmidt decomposition

## Theorem (Schmidt decomposition)

For any pure state  $|\psi\rangle \in V \otimes W$ , there exist *orthonormal systems*  $(|v_i\rangle)_i$  of  $V$  and  $(|w_j\rangle)_j$  of  $W$ , and positive real numbers  $(\lambda_i)_i$  such that

$$|\psi\rangle = \sum_i \lambda_i |v_i\rangle_V |w_i\rangle_W.$$

## Proof.

Let  $A := \mathcal{M}(|\psi\rangle)$ . By the *singular value decomposition*,

$$A = \sum_i \lambda_i |s_i\rangle_V \langle t_i|_W$$

Since  $|\psi\rangle = \mathcal{M}^{-1}(A)$ ,

$$|\psi\rangle = \sum_i \lambda_i |s_i\rangle_V |t_i\rangle_W^*.$$



The number of the terms in the decomposition is called the *Schmidt rank*.



## Measurements on composite system

Set of measurements on the composite system on  $V \otimes W$  is

$$\{(e_1, \dots, e_k) \in \mathcal{H}(V \otimes W) \mid e_1 + \dots + e_k = I, e_j \in C_{\succeq 0} \\ i = 1, 2, \dots, k, k = 1, 2, \dots\}.$$

For measurements  $(P_a \in \mathcal{H}(V))_a$  and  $(Q_b \in \mathcal{H}(W))_b$  for the partial systems,  $(P_a \otimes Q_b \in \mathcal{H}(V \otimes W))_{a,b}$  is a measurement since  $P_a \otimes Q_b \succeq 0$  and

$$\sum_{a,b} P_a \otimes Q_b = \left( \sum_a P_a \right) \otimes \left( \sum_b Q_b \right) = I_V \otimes I_W = I_{V \otimes W}.$$

## Partial trace and reduced density matrix

A probability of outcome of local measurement in a composite system is

$$P_{V \otimes W}(a, b) = \text{Tr}(\rho(P_a \otimes Q_b)).$$

$$\begin{aligned} P_V(a) &= \sum_b P_{V \otimes W}(a, b) = \sum_b \text{Tr}(\rho(P_a \otimes Q_b)) \\ &= \text{Tr} \left( \rho \left( P_a \otimes \sum_b Q_b \right) \right) \\ &= \text{Tr}(\rho(P_a \otimes I)) \\ &= \text{Tr}(\text{Tr}_W(\rho)P_a). \end{aligned}$$

The **partial trace**  $\text{Tr}_W(\rho) \in \mathcal{H}(V)$  is defined by

$$\text{Tr}(\text{Tr}_W(\rho)P) = \text{Tr}(\rho(P \otimes I))$$

for any  $P \in \mathcal{H}(V)$ . Indeed,  $\text{Tr}_W(\cdot)$  is a linear operator from  $\mathcal{H}(V \otimes W)$  to  $\mathcal{H}(V)$  defined by  $\text{Tr}_W(\rho_V \otimes \sigma_W) = \text{Tr}(\sigma_W)\rho_V$ .

## Reduced density matrix from the Schmidt decomposition

For a pure state  $|\psi\rangle \in \mathcal{H}(V) \otimes \mathcal{H}(W)$  with the **Schmidt decomposition**

$$|\psi\rangle = \sum_i \lambda_i |v_i\rangle_V |w_i\rangle_W$$

it is easy to derive a **reduced density matrices**.

$$\begin{aligned} |\psi\rangle \langle\psi| &= \sum_{i,j} \lambda_i \lambda_j |v_i\rangle_V |w_i\rangle_W \langle v_j|_V \langle w_j|_W \\ &= \sum_{i,j} \lambda_i \lambda_j |v_i\rangle_V \langle v_j|_V \otimes |w_i\rangle_W \langle w_j|_W \end{aligned}$$

$$\begin{aligned} \text{Tr}_W(|\psi\rangle \langle\psi|) &= \sum_{i,j} \lambda_i \lambda_j |v_i\rangle_V \langle v_j|_V \text{Tr}(|w_i\rangle_W \langle w_j|_W) \\ &= \sum_i \lambda_i^2 |v_i\rangle_V \langle v_i|_V \end{aligned}$$

$$\text{Tr}_V(|\psi\rangle \langle\psi|) = \sum_i \lambda_i^2 |w_i\rangle_W \langle w_i|_W$$

## Assignments

① Show the Schmidt decomposition of the following pure states

Ⓐ  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

Ⓑ  $\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$

Ⓒ  $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$

Ⓓ  $\cos \theta |00\rangle - \sin \theta |01\rangle + \sin \theta |10\rangle + \cos \theta |11\rangle$

② Show the reduced density matrices for each qubit of the above pure states.