# Developing a Toy Model for Quantum Chaos Theory: Entanglement Entropy of Bipartite System Under Random Real Hamiltonians

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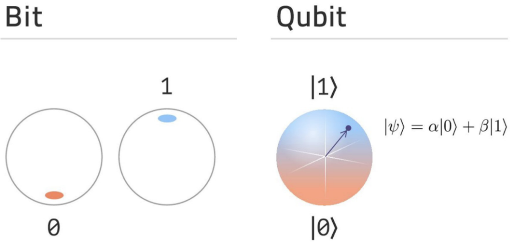
# Overview



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# Thermal Density Matrix

- Density matrix
  - Matrix used to represent the probability of a system being in each possible state

$$\begin{bmatrix} P\big(|0\rangle\big) & v_1 \\ v_1* & P\big(|1\rangle\big) \end{bmatrix} \longrightarrow \begin{bmatrix} P\big(|00\rangle\big) & \dots & \dots & \dots \\ \dots & P\big(|01\rangle\big) & \dots & \dots \\ \dots & \dots & P\big(|10\rangle\big) & \dots \\ \dots & \dots & \dots & P\big(|11\rangle\big) \end{bmatrix} \longrightarrow \begin{matrix} \rho_A = \mathrm{Tr}_B(\rho) = \begin{bmatrix} P\big(|00\rangle\big) + P\big(|01\rangle\big) & \dots \\ \dots & P\big(|10\rangle\big) + P\big(|11\rangle\big) \end{bmatrix} \\ \rho_B = \mathrm{Tr}_A(\rho) = \begin{bmatrix} P\big(|00\rangle\big) + P\big(|10\rangle\big) & \dots \\ \dots & P\big(|01\rangle\big) + P\big(|11\rangle\big) \end{bmatrix} \end{matrix}$$

- Constructed a $N^2 \times N^2$ Hermitian matrix whose elements were drawn from a real Gaussian Distribution.
  - $\mu = 0$ and $\sigma = 1/3$

- Used Thermal Density Matrix:

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$$

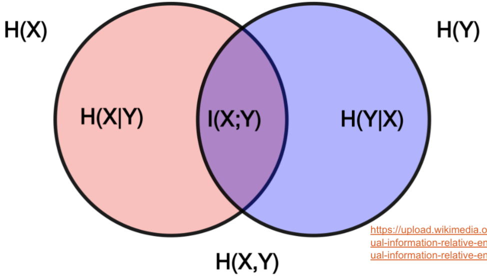
# Calculating Entanglement Entropy

Entropy: $S(\rho) = -\operatorname{Tr}(\rho \ln(\rho))$ or $S(\lambda) = -\sum(\lambda \ln(\lambda))$

- where $\lambda$ are the eigenvalues of the density matrix ($\rho$)

Mutual information (Entanglement) $= \frac{S(\rho_A)+S(\rho_B)-S(\rho_{AB})}{2}$

- $\rho_A$ is the partial trace over subsystem A (density matrix of subsystem B)
- $\rho_B$ is the partial trace over subsystem B (density matrix of subsystem A)

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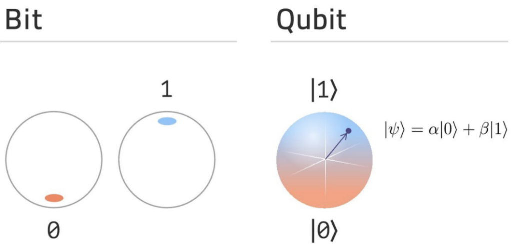


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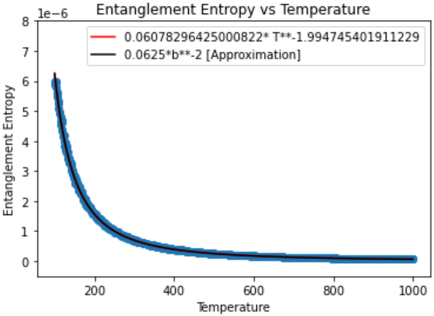
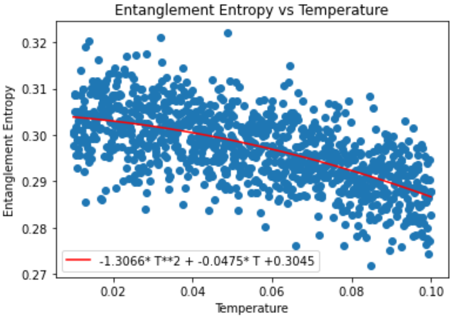
# 2 Qubit (N=2) Subsystems: High Temperature

As $T \to \infty$, system become maximally mixed (all states are equally likely)



# 2 Qubit (N=2) Subsystems: Low Temperature

As $T \to 0$, system become almost pure (100% probability of being in ground state)



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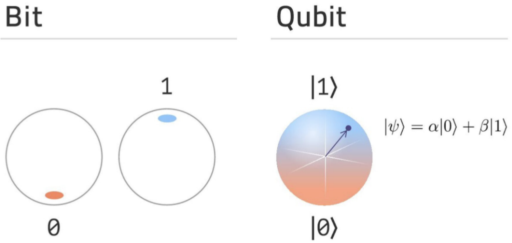


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# 2 Qunit Approximation: High Temperature

- Used $S(\rho) = \sum(-\lambda \ln(\lambda))$ for the approximation
- Also Taylor Expanding the density matrix, we get:

$$\rho(\beta) \approx \frac{1}{4} - \frac{H}{4}\beta + \frac{\mathrm{Tr}(H)}{16}\beta + \frac{H^2}{8}\beta^2 - \frac{\mathrm{Tr}\left(H^2\right)}{32}\beta^2 + \frac{(\mathrm{Tr}(H))^2}{64}\beta^2 - \frac{H\,\mathrm{Tr}(H)}{16}\beta^2$$

- Some important equations used to find entanglement entropy:

$$H \to N^2 \times N^2 \qquad \mathrm{Tr}_B(H) \to N \times N$$

$$\langle \mathrm{Tr}(H) \rangle = 0 \qquad \langle \mathrm{Tr}\left(H^2\right) \rangle = N^4 J^2$$

$$\langle (\mathrm{Tr}(H))^2 \rangle = N^2 J^2 \qquad \langle \mathrm{Tr}_A((\mathrm{Tr}_B(H))^2) \rangle = N^3 J^2$$

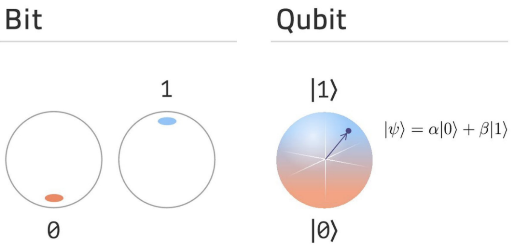
$$\langle S(\rho) \rangle = \ln\left(N^2\right) - \frac{\beta^2}{2}\left(N^2 J^2 - \frac{J^2}{N^2}\right) \qquad \langle S(\mathrm{Tr}_B(\rho)) \rangle = \ln(N) - \frac{\beta^2 J^2}{2} + \frac{\beta^2 J^2}{2N^2}$$

# 2 Qunit Approximation: High Temperature

Final result:

$$\langle S_E \rangle = \Big( -\frac{J^2}{2} + \frac{J^2}{2N^2} + \frac{N^2 J^2}{4} - \frac{J^2}{4N^2} \Big) \beta^2$$

- $J \rightarrow$ Standard deviation of the Hamiltonion
- $N \rightarrow$ Size of the subsystem

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# Wishart Ensemble

## Definition

*A Wishart Ensemble is the set of all $W$ such that:*

$$W = HH^\dagger$$

*where $H$ is a matrix with dimensions $N \times M (M \geq N)$ whose elements are drawn from a Gaussian Distribution.*

- Wishart Matrices are positive semi-definite (N non-negative eigenvalues).
- The spectral density of the Wishart Ensemble gives us the probability distribution of the eigenvalues of all Wishart Matrices.

# What's So Special About Wishart Matrices?

## Theorem

*All subsystem density matrices for which the system is in a random pure state are in the Wishart Ensemble.*

## Proof.

- Ground state eigenvector is a random real unit vector in $N^2$
- Elements of the density matrix are i.i.d. random variables
- $\Psi_0$ represent the system's pure random state
- $\Psi$ is just $\Psi_0$ reshaped
- $\Psi \simeq \Psi_{ij} \simeq \Psi_{kl}$
- Tracing over subsystem A is basically $j = l$
- So $\rho_{ijkj} = \rho_B = \sum(\Psi^{\star}_{ij}\Psi_{kj}) = \Psi\Psi^{\dagger}$

$\square$

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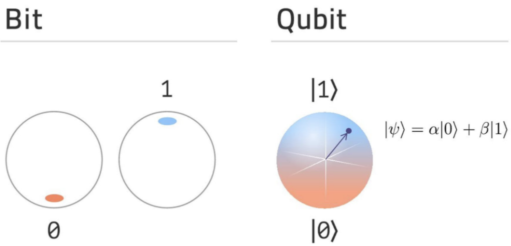


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# 2 Qunit Approximation: Low Temperature

- As $T \rightarrow 0$, the elements of the system's density matrix all become 0, except for the diagonal element corresponding to the ground state.
- Ground state corresponds to where the system has the lowest energy
- The entropy of the whole system is 0 ($\lambda_{groundstate} = 1$)
- The subsystems have entropy
- Therefore the expected value for the Entanglement Entropy is thus given by:

$$\langle S \rangle = N \int_0^L (\lambda \ln(\lambda) f(\lambda)) d\lambda$$

  - $f(\lambda)$ is the spectral density of a Wishart Ensemble

# Spectral Density of a Wishart Ensemble

In general, for large N, the spectral density is given as follows:

$$f(\lambda) = \frac{1}{\beta N} \rho_{MP}\Big(\frac{\lambda}{\beta N}\Big)$$

where $\rho_{MP}(\frac{\lambda}{\beta N})$ is the Marčenko-Pastur scaling function.

$$\rho_{MP}\Big(\frac{\lambda}{\beta N}\Big) = \frac{\beta N}{2\pi\lambda}\sqrt{\Big(\frac{\lambda}{\beta N} - \zeta_-\Big)\Big(\zeta_+ - \frac{\lambda}{\beta N}\Big)}$$

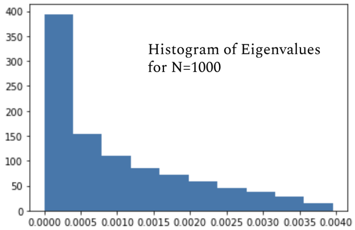
where $\zeta_\pm = (1 \pm (\frac{N}{M})^{-1/2})^2$.

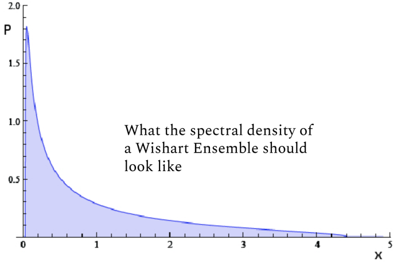
## Spectral Density in Our Case

- In our case since M=N, c=1. So, $\zeta_- = 0$ and $\zeta_+ = 4$:

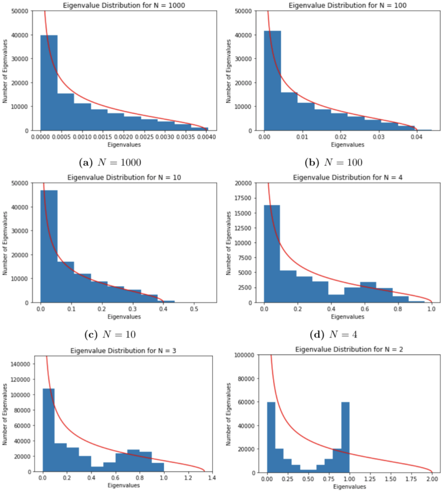
$$\rho_{MP}\left(\frac{\lambda}{\beta N}\right) = \frac{\beta N}{2\pi\lambda}\sqrt{\left(\frac{\lambda}{\beta N}\right)\left(4 - \frac{\lambda}{\beta N}\right)}$$

- Here $\beta$ is the Dyson Index and in our case $\beta = 1$.
- This is still not normalized for our work.



# 2 Qunit Approximation: Zero Temperature




(a) $N = 1000$ (b) $N = 100$ (c) $N = 10$ (d) $N = 4$ (e) $N = 3$ (f) $N = 2$

- Normalized spectral density:

$$f(\lambda) = \frac{N\sqrt{(\frac{4}{N} - \lambda)(\lambda)}}{2\lambda\pi}$$

- Solving the integral:

$$\langle S \rangle = \int_0^{\frac{4}{N}} \frac{-N^2 \ln(\lambda)\sqrt{(\frac{4}{N} - \lambda)(\lambda)}}{2\pi} d\lambda$$

- Final result:

$$\langle S \rangle = \ln(N) - \frac{1}{2}$$

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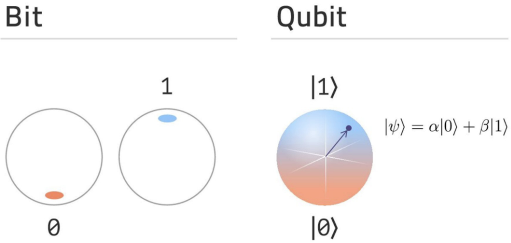


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# Next Steps and Applications

Next Steps:

- Expand for Low Temperature (currently working on)
- Construct a dynamic multi-body model

Applications:

- Quantum Computers