$$H = \sum_{l} h_l \sigma_z^{(l)} + \sum_{l,m} J_{lm} \sigma_z^{(l)} \sigma_z^{(m)}$$

---

Binary variables $\rightarrow \{0,1\}$
$\rightarrow \{+1,-1\}$

bits $\in \{0,1\}$ denoted $x$

bits $\in \{+1,-1\}$ denoted $s$

$$s = 1 - 2x = (-1)^x$$

$$(-1)^0 \rightarrow 1$$

$$(-1)^1 \rightarrow -1$$

Hamiltonian in this picture becomes a function of binary variables. Called pseudo Boolean forms.

$$f : \underbrace{\{0,1\}^n}_{\text{n-long bit string}} \longrightarrow \underbrace{\mathbb{R}}_{\text{real numbers}}$$

$$\tilde{f} : \{+1,-1\}^{\times n} \longrightarrow \mathbb{R}$$

$$\tilde{f} = \sum_{l} h_l S_l + \sum_{l,m} J_{lm} S_l \cdot S_m$$

$S = 1 - 2x \Rightarrow x = \frac{1 - S}{2}$

polarization versus standard bits.

Matrix embedding

$S \rightarrow \sigma_z$

$x \rightarrow |x\rangle\langle x|$

$= \frac{1 + (-1)^x \sigma_z}{2}$

ex. $f(x, y, z) = 1 + x + 2xy + 3xyz$ easy to program

How can I use penalty functions to embed universal boolean gates?

→ Can I use a cost function to emulate the NAND gate?


$z = x \barwedge y$

$f(x,y) = \bar{x}\bar{y} + \bar{x}y + x\bar{y}$

simplify $\Rightarrow$

using $\bar{q} = 1 - q$

| $x$ | $y$ | $z = x \barwedge y$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$1 - xy = f'(x,y)$

$x^2 = x$

idempotent property

$x \wedge y \rightarrow xy$

$\bar{x} = 1 - x$

$\overline{(xy)} = 1 - xy$

$(1-x)(1-y) + \bar{x}y + x\bar{y}$

$= 1 - \cancel{x} - \cancel{y} + \cancel{xy} + \cancel{y} - \cancel{xy} + \cancel{x} - xy$

$= 1 - xy$

$\boxed{S = 1 - 2x}$

$\boxed{x = |x\rangle\langle x| = \dfrac{\mathbb{1} + (-1)^x \sigma_z}{2}}$

let $f = 1 - xy$. find real $h, J$ such that

$$\tilde{f} = \sum_{e=1}^{2} h_e S_e + \frac{1}{2}\sum_{e,m=1}^{2} J_{em} S_e S_m + \text{consant}$$

and evaluate $f, \tilde{f}$ on all inputs.

$f(x,y) = \tilde{F}((-1)^x, (-1)^y)$ solve this equation by finding $h_1$, $h_2$ $J_{12}$

example. $\tilde{f} = 1 - S_1 S_2 \Rightarrow J_{12} \stackrel{?}{=} -1$

constant $= 1$

| $S_1$ | $S_2$ | $\tilde{f}$ |
|---|---|---|
| +1 | +1 | 0 |
| -1 | +1 | 2 |
| +1 | -1 | 2 |
| -1 | -1 | 0 |

$\rightarrow 1 - (-1)\cdot 1 = 2$

$H^\dagger = H \Rightarrow$ eigenvalues hermitian. are real.

$$\sum_{l,m=1}^{2} J_{lm} = J_{11} + J_{12} + J_{21} + J_{22} +$$

$$\sum J_{lm} =$$

$1 \leq l \leq m \leq 2$

$$\cancel{J_{11}} + J_{12} + \cancel{J_{22}}$$

$$f(\vec{x}) = h^T \cdot \vec{x} + \vec{x}^T \cdot J \cdot \vec{x}$$

alternative notation

* optimisation theory

$$\tilde{f}_2 = 1 - S_1 S_2 + S_2 S_3 - 3 S_1 S_3$$

find constant, h, J

$$\frac{1}{2} \sum_{e,m=1}^{3} J_{em} S_e S_m + \sum_{e=1}^{3} h_e S_e + C$$

constant

Translate from polarity ($\pm 1$) to Boolean

$$f(x_1, x_2, x_3) =$$

$$f\left((-1)^{x_1}, (-1)^{x_2}, (-1)^{x_3}\right)$$

$$\boxed{S_k = 1 - 2X_k}$$

$J_{12} \overset{?}{=} J_{21}$

$J_{13} \overset{?}{=} J_{31}$

$J_{23} \overset{?}{=} J_{32}$

$C \overset{?}{=}$

$J_{11} = 0$

$J_{22} = 0$

$J_{33} = 0$

$|x\rangle\langle x|$

$\frac{\mathbb{1} + (-1)^x Z}{2}$

binary $\quad x = \frac{1-s}{2}$

polarisation

$s \to \sigma_z$

Z (hamiltonian)

$$\sum_{l,m=1}^{3} J_{lm} = J_{12} + J_{21} + J_{13} + J_{31} + J_{23} + J_{32}$$

$$\Downarrow$$

$$\sum_{l=1}^{3} \sum_{m=1}^{3} J_{lm}$$

$$f = 1 - x_1 x_2$$

$$\downarrow$$

$$1 - \left[\frac{1-s_1}{2}\right]\left[\frac{1-s_2}{2}\right]$$

$$= 1 - \frac{1}{4}(1-s_1)(1-s_2) =$$

$$\tilde{f} = 1 - \frac{1}{4}(1 - s_1 - s_2 + s_1 s_2)$$

$$\tilde{f}((-1)^{x_1}, (-1)^{x_2}) = f(x_1, x_2)$$

How to turn $f$, $\tilde{f}$ into a hamiltonian in terms of $\sigma_z$?

$$f = 1 - x_1 x_2 \implies \sum_{l,m=1}^{2} (1 - x_1 x_2) |x_1 x_2\rangle\langle x_1 x_2|$$

$$\sum_{e=1}^{3} h_e S_e = h_1 S_1 + h_2 S_2 + h_3 S_3$$

constants

variables

$f(x, y, z) = 0$ when $z = x \bar{\wedge} y$ $\quad x, y, z \in \{0, 1\}$

otherwise $f \geq \Delta \geq 1$

you can use $xyz$ (3-body terms)

find $f$.

| $x$ | $y$ | $z$ | $f$ | |
|---|---|---|---|---|
| 0 | 0 | 0 | $\geq \Delta$ | $\to \bar{x}\bar{y}\bar{z} \cdot \Delta$ |
| 0 | 0 | 1 | 0 | $\leftrightarrow \Delta(1 - \bar{x}\bar{y}z)$ |
| 0 | 1 | 0 | $\geq \Delta$ | |
| 0 | 1 | 1 | 0 | $\to \Delta(1 - \bar{x}yz)$ |
| 1 | 0 | 0 | $\geq \Delta$ | |
| 1 | 0 | 1 | 0 | $\to \Delta(1 - x\bar{y}z)$ |
| 1 | 1 | 0 | 0 | $\to \Delta(1 - xy\bar{z})$ |
| 1 | 1 | 1 | $\geq 0$ | |

NAND

| $x$ | $y$ | $x \bar{\wedge} y$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

| x | y | |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 2 |
| 1 | 0 | -2 |
| 1 | 1 | 3 |

$1\bar{x}\bar{y}$

$$\bar{x}\bar{y} + 2\bar{x}y + 3xy - 2x\bar{y}$$

$3xy$

$2\bar{x}y$ $\quad -2x\bar{y}$

$$f(x, y, z) + \Delta Z$$

#1. code f and validate.

#2. The method I gave you results in a 3-body term; eg $xyz$. Understand and solve the constraint problem.

#3. Know how to embed into gates.

$$\tilde{f}(S_1, S_2) = 1 - S_1 - S_2 + 2S_1S_2$$

$$\tilde{f}(-1, 1) = 1 \cancel{+1} \cancel{-1} + 2(-1)(1) = 1 - 2 = -1$$

$c = 1$

$h_1 = -1$

$h_2 = -1$

$J_{12} = 2$

$$\tilde{f} = \sum_{l=1}^{2} h_l S_l + \frac{1}{2} \sum_{l,m=1}^{2}{}' J_{lm} S_l S_m$$

$$\sum_{1 \le l \le m \le 2}{}' J_{lm} S_l S_m$$