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„Universal Properties of Higher Idempotent Completions“

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Abstract

"Idempotent completions" of categories are motivated as a generalisation of completions of metric spaces [Law73]. These, in turn, may be generalised to "higher idempotent completions" of "higher categories". For instance, topological quantum field theory motivates "condensation completions" [GJF19] and "orbifold completions" [CR16, CM23]. Completions of metric spaces are characterised by a universal property. Analogously, higher idempotent completions are characterised by "higher universal properties" [Dé22, CM23]. Inspired by [CM23], we set out to supplement details to the proofs of universality given by [Dé22] and [CM23]. Along the way, we showcase the utility and elegance of string diagrams.

Correspondingly, we start off with a string diagrammatic introduction to 2-category theory. Moving on, we review the 2-universal property of [Fio06] and show how it relates to the 3-universal property of [Dé22]. Furthermore, we generalise "split idempotents" as "condensations" of "2-condensation algebras" and "2-orbifold data". Finally, we construct idempotent completions and show that they are characterised by 2-universal properties. We generalise this to show that 2-condensation completions and 2-orbifold completions are characterised by 3-universal properties.

Our discussions supplement various details omitted in the literature. In particular, they highlight that the construction of n-idempotent completions requires that the n-idempotents determine their splittings. Moreover, our discussions suggest that $(n+1)$-universal properties of n-idempotent completions $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ should be shown via the idempotence $\overline{(\ \ )} \circ \overline{(\ \ )} \simeq \overline{(\ \ )}$ of the $(n+1)$-functors $\overline{(\ \ )} : \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}} \longrightarrow \mathbf{nCat}^{\mathbf{ncc}} \subseteq \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$.

Kurzfassung

„Idempotente-Vervollständigungen" von Kategorien sind motiviert als eine Verallgemeinerung der Vervollständigungen von metrischen Räumen [Law73]. Diese wiederum können zu „höheren Idempotetente Vervollständigungen" von „höheren Kategorien" verallgemeinert werden. Zum Beispiel motiviert die topologische Quantenfeldtheorie „Kondensationsvervollständigungen" [GJF19] und „Orbifold-Vervollständigungen" [CR16, CM23]. Vervollständigungen von metrischen Räumen werden durch eine universelle Eigenschaft charakterisiert. Analog dazu werden höhere Idempotente-Vervollständigungen durch „höhere universelle Eigenschaften" charakterisiert [Dé22, CM23]. Inspiriert durch [CM23] beabsichtigen wir, Details zu den Beweisen der Universalität aus [Dé22] und [CM23] zu ergänzen. Dabei demonstrieren wir den Nutzen und die Eleganz von String-Diagrammen.

Entsprechend beginnen wir mit einer string-diagrammatischen Einführung in die 2-Kategorientheorie. Danach behandeln wir die 2-universelle Eigenschaft aus [Fio06] und zeigen, wie sie mit der 3-universelle Eigenschaft von [Dé22] zusammenhängt. Außerdem verallgemeinern wir „gespaltene Idempotente" als „Kondensationen" von „2-Kondensationsalgebren" und „2-Orbifold-Daten". Schließlich konstruieren wir Idempotente-Vervollständigungen und zeigen, dass sie durch 2-universelle Eigenschaften charakterisiert sind. Wir verallgemeinern dies, um zu zeigen, dass 2-Kondensationsvervollständigungen und 2-Orbifold-Vervollständigungen durch 3-universelle Eigenschaften charakterisiert werden.

Unsere Diskussionen supplementieren verschiedene in der Literatur übersprungene Details. Insbesondere heben diese hervor, dass die Konstruktion von n-idempotenten Vervollständigungen erfordert, dass die n-Idempotenten ihre Spaltungen bestimmen. Außerdem legen unsere Diskussionen nahe, dass $(n+1)$-universelle Eigenschaften von n-idempotenten Vervollständigungen $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ durch die Idempotenz $\overline{(\ )} \circ \overline{(\ )} \simeq \overline{(\ )}$ der $(n+1)$-Funktoren $\overline{(\ )} : \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}} \longrightarrow \mathbf{nCat}^{\mathbf{ncc}} \subseteq \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$ gezeigt werden sollte.

Contents

1. Outline

"What is a universal property of a higher idempotent completion?", one may wonder. Well, in short, *universal properties* are a mathematical tool with which mathematical objects are essentially fully – i.e. universally – characterised independently of explicit constructions. The mathematical objects we want to characterise via universal properties are *higher idempotent completions*.

We shall first introduce the motivation of this thesis, namely *orbifolded topological quantum field theories*, in Section 1.1. Hereby, topological quantum field theories (TQFTs) are quantum field theories that investigate topological properties. We sketch how orbifolding a TQFT may be thought of as *gauging* a symmetry of the TQFT, which in turn may be thought of as quotienting out the symmetry.

Next, we give a brief overview of higher idempotent completions in Section 1.2. We sketch *higher idempotents*, i.e. *categorifications* of idempotents, and note that the gaugeable symmetries of Section 1.1 may be interpreted as higher idempotents. Moreover, we discuss completions and how the orbifolding procedure produces higher idempotent completions.

To round off this chapter, we provide an overview of the following chapters in Section 1.3 and list some conventions in Section 1.4. The motivation and background information presented in Chapter 1 is not necessary for understanding the further contents of this thesis. Therefore, readers who are not interested in this overview may skip directly to Section 1.3.[1]

Sketch 1.0.1 (Higher Category Theory)**.** While this thesis will only go into detail on n-categories for $n \geq 3$, this outline does assume some foundational knowledge of category theory. Readers unfamiliar with category theory may prefer to quickly check the definition of a *category* (Definition A.1.2) and some examples, e.g. Examples A.1.5 and A.1.6, before continuing. Checking examples should elucidate that category theory may simply be thought of as a framework for expressing concepts of different fields of mathematics in a common language.

[1] Another reason to skip Chapter 1 is that the following chapters are more polished.

Monoidal categories may be thought of as categories whose objects form a monoid (Definition 2.3.6). The *monoidal category of vector spaces* $\mathbf{Vect}_K$ will be used in Section 1.1, so readers unfamiliar with $\mathbf{Vect}_K$ may want to quickly check Example 2.3.8. Furthermore, *n-categories* may be thought of as categories whose hom-sets are $(n-1)$-categories. For instance, a 2-category is a category whose hom-sets are categories, i.e. a category whose morphisms (*1-morphisms*) have morphisms (*2-morphisms*) between them. Moreover, there are homomorphisms between categories and they are called *functors* (Definition A.1.7). Accordingly, *monoidal functors* and n-functors are homomorphisms between monoidal categories and n-categories, respectively.

1.1. Orbifolding Topological Quantum Field Theories

In this section we will first motivate topological investigations of *quantum field theory* (QFT). By modelling spacetime as a closed manifold we then formalise *topological quantum field theory* (TQFT) as *closed TQFTs*.[2] After that we generalise closed TQFTs to *defect TQFTs* by adding structure to our spacetimes. After that we will introduce *categorical symmetries* and show how they may be *gauged*.

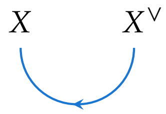
1.1.1. Topological Quantum Field Theory

As the name suggests, *topological quantum field theory* is a topological approximation to quantum field theory. Hereby, QFT is a branch of theoretical physics that examines the structure of the universe at a fundamental level. Meanwhile, approaching QFT topologically means examining global properties by disregarding geometric information. While the universe of course contains lots of interesting geometric information, global information is also interesting. Since geometry obstructs the view of topological information, disregarding the geometry makes the global properties more accessible.

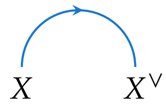
[2] TQFT has two meanings in this sentence. For one, TQFT is a branch of research, and for another, a TQFT is a specific approximation to physics.

Example 1.1.1. Let us consider an example of topological properties. Let

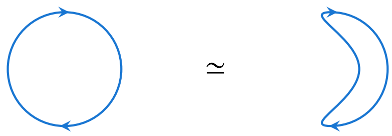
represent the path of a particle X through some spacetime. Further, let



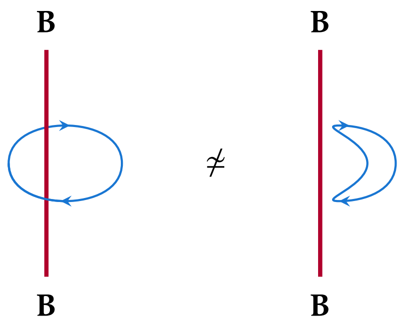
and



denote spontaneous creation and annihilation, respectively, of a particle–antiparticle pair $(X, X^\vee)$. When these particles are sufficiently far apart, they do not affect one another. Then – if there are no other external effects – the paths



are *topologically equivalent*.[3] Interestingly, this implies that $(X, X^\vee)$ may take either path with equal probability, i.e. $P(\bigcirc) = P($☽$)$. However, if we introduce a region **B** that the particles cannot move through without their state being affected, then we can produce topologically non-equivalent processes, e.g.:



(1.1)

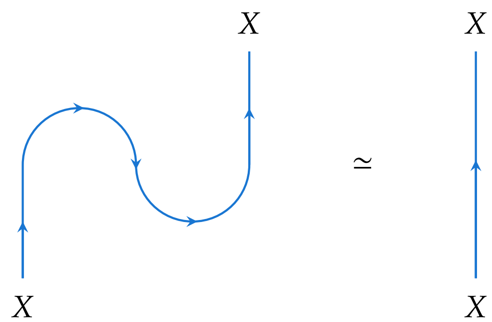
This insight may be used to observe the Aharonov–Bohm effect. If we physically restrict our particles to a 2-dimensional plane, then the processes of (1.1) each represent a 3-dimensional spacetime with time going from bottom to top and space stretching left to right and front to

[3] In this context, topologically equivalent is supposed to mean that there exists a homeomorphism, i.e. a notion of continuous deformation of one path onto the other path.

back. Let the region **B** indicate a magnetic field perpendicular to the plane, let the plane outside of **B** have negligible magnetic field strength, and let the particle X have a charge $\neq 0$. Then the probability of the path ☽ remains equal to the probability (1.1.1) whereas the probability of the path ○ generally changes [Pre04, Sec. 9.2].

In particular, by making convenient choices of input states of (1.1) etc., we can theoretically achieve that the probability of the path ○ becomes 0 and that the probability of the path ☽ becomes arbitrarily close to 1. It is essentially this mechanism that enables a model of computation based on braiding particles [Mel23]. This model of computation is called *topological quantum computing* since it can simulate “regular” quantum computers and vice versa [FLW00, FKW02]. Topological quantum computers are interesting, because while other models of quantum computers are difficult to build due to their high sensitivity to “noise”, topological quantum computers have intrinsic fault tolerance. Perturbations of the paths of (1.1) do not affect their probabilities as we saw in (1.1.1), so this provides intrinsic fault tolerance [Pre04, Sec. 9.11].

Another feature of the topological approach is that $(d+1)$-dimensional spacetimes should not be thought of as d space dimensions and one time dimension, but rather as a $d+1$ spacetime dimensions on equal footing. For instance, this enables the realisation that the processes



are indistinguishable. This particular equivalence is essentially “quantum teleportation” (Remark 5.2.28).

TQFT may be formalised in a variety of ways. For instance, one could model spacetimes as topological spaces and construct concrete *TQFTs* as *functors* mapping topological spaces to algebraic data, e.g. rings or vector spaces. However, usually TQFTs are constructed according to the framework introduced by [Ati88, Sec. 2]; this is also the approach we will follow. This framework constructs TQFTs as process theories, i.e. they consist of processes that take inputs and return outputs. Specifically, inputs and outputs model physical spaces, while the processes model time evolutions. One may view this as a mathematical computer scientist’s approach to theoretical physics. This approach is mathematical in the sense that it utilises *monoidal category theory* to make the TQFTs rigorous. In particular, *monoidal functors* will guarantee compatibility of the time evolution processes.[4]

1.1.2. Closed TQFTs

Assumption 1.1.2. In the following we are only interested in manifolds up to equivalence. Therefore, when we discuss manifolds in this Section 1.1, we will implicitly mean diffeomorphism classes of manifolds![5] In other words, whenever we, for instance, discuss a manifold M, we actually mean the diffeomorphism class $[M]$.

Less outrageously, we shall implicitly also assume that the underlying manifolds are all oriented and smooth. This, for instance, implies that our 2-manifolds (with boundaries) will not include Möbius strips or Klein bottles. However, the orientation also implies some technicalities that we will skip. For instance, we will sketch how to "glue together" manifolds with boundaries. To ensure compatibility one technically needs to introduce orientation preserving boundary parametrisations $U, U' \hookrightarrow \partial M$ [Koc03, Sec. 1.2.11], but we will not concern ourselves with this.

A detailed account of 2-dimensional TQFT covering these technicalities is given by [Koc03].

Sketch 1.1.3. TQFT is formalised by modelling $(d+1)$-dimensional spacetime as a $(d+1)$-*manifold*[6] which we construct from

Physical Spaces: compact d-manifolds[7] and

Time Evolutions: $(d+1)$-*manifolds with boundaries*[8] (MwBs).

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4 Therefore, such approaches to QFT may also be referred to as a *functorial (quantum) field theory*. In other contexts this may be more fitting as this also encompasses extensions of TQFT to geometric quantum field theory.

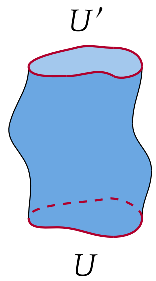
5 Diffeomorphisms are basically smooth homeomorphisms. We do not consider manifolds M, M' but rather diffeomorphism classes of manifolds $[M], [M']$ because this means that we identify topologically equivalent manifolds with one another, i.e. $[M] = [M'] \Leftrightarrow M \simeq M'$.

6 n-manifolds are topological spaces where each point is locally isomorphic to $\mathbb{R}^n$.

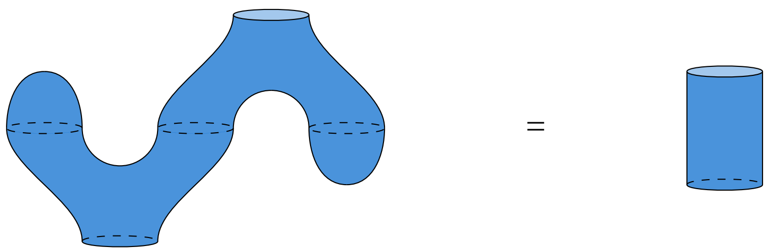
7 Most authors call them "closed manifolds".

8 *n-manifolds with boundaries* are a generalisation of n-manifolds where each point is locally diffeomorphic to either the real plane $\mathbb{R}^n$ or the real half-plane $\mathbb{H}^n$. Some authors include $(d+1)$-manifolds with boundaries when they say "manifolds", but we do not.

Compactness of our manifolds (with boundaries) corresponds to finiteness of our spacetimes. The boundary ∂M of a MwB M corresponds to its input and output, i.e. M has an input space U and an output space U' and fulfils $\partial M = U \sqcup U'$,[9,10] e.g.:

$M =$ 

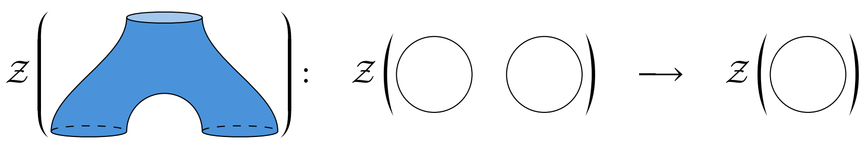
We also write $M : U \longrightarrow U'$ to indicate that we think of M as a morphism from U to U'. The dimensionality $d+1$ implies d space dimensions and one time dimension, but we should think of them simply as $d+1$ spacetime dimensions on equal footing. This allows us, for instance, to make the identification



of diffeomorphism classes.

A *TQFT*[11] is then constructed by assigning algebraic data to these manifolds. Specifically, physical spaces are mapped to "state spaces" and time evolutions are mapped to "time evolution operators" acting on these state spaces. State spaces will be modelled as vector spaces and time evolution operators as linear maps. Such maps $\mathcal{Z}$ should fulfil the following compatibility conditions:

- A time evolution $M : U \longrightarrow U'$ should be mapped to an operator acting on the state spaces $\mathcal{Z}(U)$ and $\mathcal{Z}(U')$, e.g.:

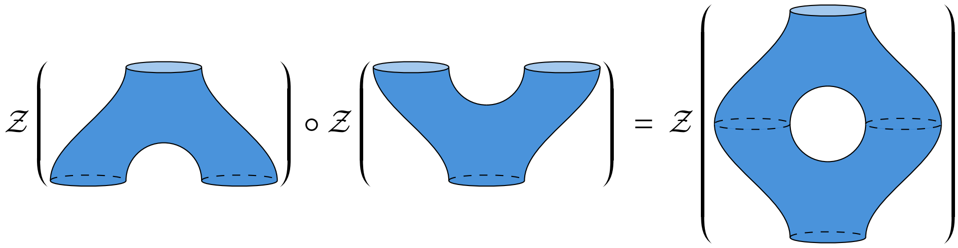


9 The *disjoint union* $U' \sqcup U$ of closed manifolds U and U' may be thought of as the closed manifold consisting of U and U' next to one another.

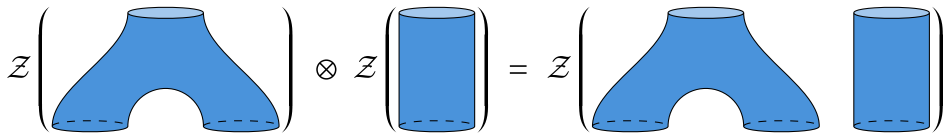
10 Technically $\partial M = U^{\vee} \sqcup U'$, where $(\ \ )^{\vee}$ stands for orientation reversal. This ensures that one may "glue" MwBs together; cf. Assumption 1.1.2.

11 Here "TQFT" refers not to the field TQFT but a specific theory, i.e. a specific model.

- Time evolution operators should be compatible with gluing, e.g.:



- Time evolution operators should be compatible with “spacelike” separation, e.g.:

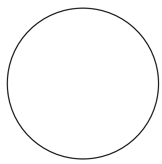


Overall, one may notice that these compatibility requirements imply that the physical spaces and time evolutions form a *monoidal category* and that the maps $\mathcal{Z}$ are *monoidal functors*.

Bordisms

We formalise for 2-dimensional spacetimes the approach we just sketched by introducing the *symmetric*[12] *monoidal category of 2-dimensional oriented bordisms* $\mathbf{Bord}_2$ consisting of

Objects: disjoint unions[13] $U = \bigsqcup^k S^1$ of circles

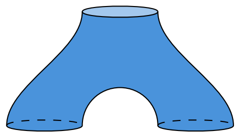
$S^1 \equiv$  ,

Morphisms: 2-manifolds with boundaries, now called *bordisms*, $M : U \longrightarrow U'$ generated by horizontal and vertical composition of the *generating bordisms*

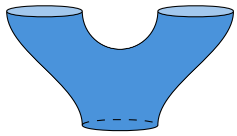
12 “Symmetric” basically means that the bordisms may be “moved through one another without touching”. This implies for instance that all knots constructed from 2-bordisms are trivial. In particular, this means that $\mathbf{Bord}_2$ contains freely generated *symmetric braiding morphisms*. Going forwards we will not make use of the symmetry property, so we will omit introducing the braidings.

13 $\bigsqcup^k S^1$ is just $k \in \mathbb{N}_0$ circles next to one another. In particular, this includes the empty union $\bigsqcup^0 S^1 = \emptyset$.

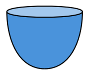
- *Pair of Pants:*

 $: S^1 \sqcup S^1 \longrightarrow S^1$

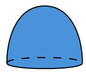
- *Upside Down Pair of Pants:*

 $: S^1 \longrightarrow S^1 \sqcup S^1$

- *Cup:*

 $: \emptyset \longrightarrow S^1$

- *Cap:*

 $: S^1 \longrightarrow \emptyset \quad ,$

Horizontal Composition: disjoint unions $U \sqcup U'$ and $M \sqcup M'$,

Monoidal Unit: $\mathbb{1}_{\mathbf{Bord}_2} := \emptyset$,

Vertical Composition: gluing bordisms, and

Unit Morphisms: cylinders $\mathrm{id}_U := U \times [0,1]$ with $[0,1] \subset \mathbb{R}$ being a closed interval.

Hereby,

- disjoint unions of circles $\bigsqcup^k S^1$ are (up to diffeomorphisms) the only closed 1-manifolds,
- the monoidal unit fulfils

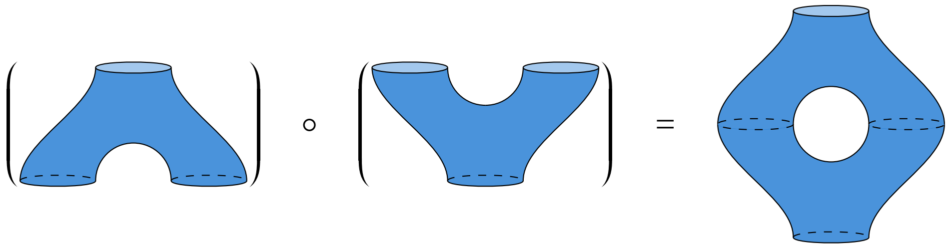
$$\emptyset \sqcup U = U = U \sqcup \emptyset,$$

- "gluing bordisms" means bordisms $M : U \longrightarrow U'$ and $M' : U' \longrightarrow U''$ are glued along their boundary U'

$$M' \circ M := M' \sqcup_{U'} M,^{14} \tag{1.2}$$

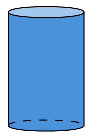
[14] We keep in mind that we are actually considering equivalence classes of manifolds, i.e. (1.2) actually means $[M' \circ M] := [M' \sqcup_{U'} M]$ (Assumption 1.1.2). The same applies to (1.3).

e.g.,[15]

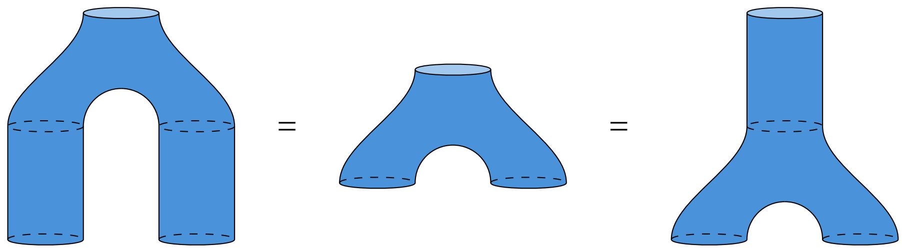


and

- the unit morphisms fulfil
$$M \sqcup_U \mathrm{id}_U = M = \mathrm{id}_{U'} \sqcup_{U'} M \tag{1.3}$$
because they are topologically equivalent. For instance, the unit on a single circle is

$\mathrm{id}_{S^1} :=$  $: S^1 \longrightarrow S^1$

and



are topologically equivalent.

Moreover, one may generalise the category of 2-bordisms $\mathbf{Bord}_2$ to the *category of n-bordisms* $\mathbf{Bord}_n$ consisting of

Objects: compact $(n-1)$-manifolds and

Morphisms: compact n-manifolds with boundaries

[Ati88].

[15] We keep in mind, that we are not actually considering manifolds but diffeomorphism classes of manifolds (Assumption 1.1.2).

TQFTs

n-bordisms enable the definition of *n-dimensional closed TQFTs* as *symmetric monoidal functors*

$$\begin{aligned} \mathcal{Z} : \mathbf{Bord}_n &\longrightarrow \mathbf{Vect}_{\mathbb{C}} \\ U &\longmapsto \mathcal{Z}(U) \\ M &\longmapsto \mathcal{Z}(M). \end{aligned} \tag{1.4}$$

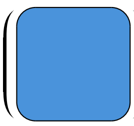
$\mathcal{Z}$ maps $(n-1)$-manifolds U to $\mathbb{C}$-vector spaces $\mathcal{Z}(U)$ and maps n-bordisms $M : U \longrightarrow U'$ to linear maps $\mathcal{Z}(M)$ in a way that is compatible with compositions of $\mathbf{Bord}_n$ and $\mathbf{Vect}_{\mathbb{C}}$:

- $\mathcal{Z}(M) \in \hom_{\mathbf{Vect}_{\mathbb{C}}}(\mathcal{Z}(U), \mathcal{Z}(U'))$,
- $\mathcal{Z}(M' \sqcup M) \cong \mathcal{Z}(M') \otimes_{\mathbb{C}} \mathcal{Z}(M)$,
- $\mathcal{Z}(M \sqcup_{U'} M') \cong \mathcal{Z}(M) \circ \mathcal{Z}(M')$

Observation 1.1.4. If we know $\mathcal{Z}(S^1) \in \mathbf{Vect}_{\mathbb{C}}$, then we know $\mathcal{Z}(U)$ for every object $U \in \mathbf{Bord}_2$, because every compact 1-manifold is a union of circles. Since every such U is of the form $U = \bigsqcup^k S^1$, we see that

$$\mathcal{Z}(U) = \mathcal{Z}\left(\bigsqcup^{k} S^1\right) \cong \bigotimes^{k} \mathcal{Z}(S^1).$$

Observation 1.1.5. An (upside down) pair of pants is diffeomorphic to a disk with two holes, a cylinder are diffeomorphic to a disk with one hole, and a cup/cap is diffeomorphic to a disk, i.e. every bordism is a finite disjoint union of disks which each have a finite numbers of holes. In particular, evaluations such as

$\mathcal{Z}\Bigg($  $\Bigg)$ (1.5)

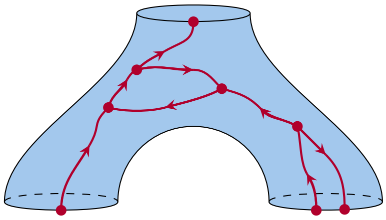
are well-defined $\mathbb{C}$-linear maps. Here and in the following we will interpret disks as cups; this means that, (1.5) corresponds to a morphism $\mathbb{C} \cong \mathcal{Z}(\emptyset) \longrightarrow \mathcal{Z}(S^1)$.[16]

Remark 1.1.6. Other symmetric monoidal categories $\mathcal{C}$ may be used as the codomain of $\mathcal{Z}$ instead of $(\mathbf{Vect}_{\mathbb{C}}, \otimes)$.

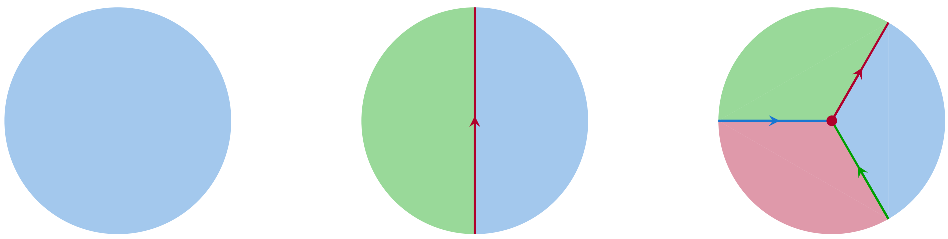
[16] One may just as well interpret disks as caps. Then, (1.5) would correspond to a morphism $\mathcal{Z}(S^1) \longrightarrow \mathbb{C}$. Either choice is as good as the other, since each determines the other.

1.1.3. Defect TQFTs

Our closed TQFTs physically correspond to "vacuum states" without real dynamics [Ati88, p. 182]. While they are not uninteresting [Ati88, Sec. 3], we would like to make our TQFTs more exciting by adding some structure to our bordisms. We do this by *decorating* the bordisms [DKR11] with *point defects*, *line defects* that start and finish at *point defects*, and *surface defects* that fill the areas delimited by the *line defects*, e.g.:



This example only has blue surface defects, red line defects, and red point defects, but we may use multiple colours to indicate that the surfaces, lines, and points may have different types. We note that the neighbourhood of every point on the inside of the bordisms should look like a surface, like an oriented line, or like a point with in- and outgoing lines, e.g.:



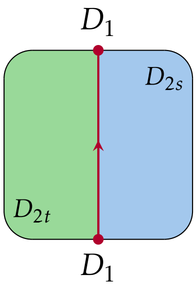
The point in the third diagram has two ingoing lines and one outgoing line, but arbitrary points may have a different number of in and outgoing lines.

Remark 1.1.7. We will stick to 2-dimensional bordisms but the following generalises well to arbitrary dimensions [CRS19, Sec. 2.2.2]. Physically, given an n-dimensional spacetime, line defects could for instance indicate paths of particles and point defects could denote interactions between particles. $(n-1)$-dimensional defects of the $(n-1)$-dimensional space could indicate regions where the dynamics are determined by a certain physical theory, e.g. electromagnetism, and $(n-2)$-dimensional defects of d-dimensional space could indicate boundaries between two such theories. Accordingly, surface defects of 2-dimensional TQFTs are also referred to as *bulk theories* as they may be interpreted as closed TQFTs (Note 1.1.12).

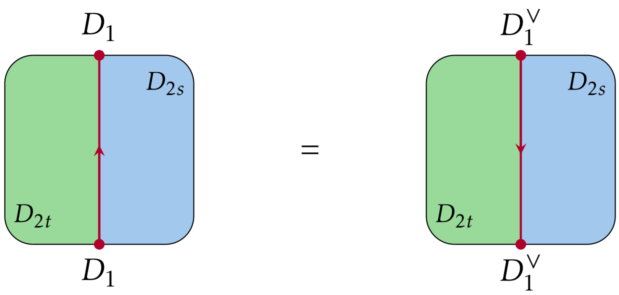
Formally, we define a set of *defects*[17] $\mathbb{D} = (\mathbb{D}_2, \mathbb{D}_1, \mathbb{D}_0, d_1, d_0)$ consisting of

- a set $\mathbb{D}_2$ of *surface defects* (also called *bulk theories*
- a set $\mathbb{D}_1$ of *line defects*,
- a set $\mathbb{D}_0$ of *point defects*,
- a map $d_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_2^2$, and
- a map $d_0 : \mathbb{D}_0 \longrightarrow \bigsqcup_{k \in \mathbb{N}_0} [\mathbb{D}_1 \times \{\cdot, \vee\}]^k$ with *cyclically composable* image, i.e. (1.6).

The maps d_1 and d_0 may be understood as assignments of *sources* and *targets*. Each line defect $D_1 \in \mathbb{D}_1$ has one source and one target given by a pair $d_1(D_1) = (D_{2s}, D_{2t})$. This may be understood as D_1 being a line with an orientation, e.g. the diagram



represents the line defect D_1 with source D_{2s} and target D_{2t}. If we want to specify the *orientation* of D_1, then we write it as D_1^{ε} where $\varepsilon \in \{\cdot, \vee\}$ indicates the direction of D_1. $D_1^{\cdot} := D_1$ is the one orientation and $D_1^{\vee}$ is the reverse orientation:



A tuple[18] of oriented line defects

$$(D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \ldots, D_{1,n}^{\varepsilon_n}) \in (\mathbb{D}_1 \times \{\cdot, \vee\})^n$$

[17] More formally, $\mathbb{D}$ is a set of *defect labels*; *defects* would then just refer to the decorations of decorated bordisms of $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$, which we will introduce momentarily.

[18] Tuples are finite ordered sets.

is *composable* if

$$d_0(D_{1,j}^{\varepsilon_j})_2 = d_0(D_{1,j+1}^{\varepsilon_j+1})_1$$

for all $j \in [1, n-1]$. Given a composable tuple $(D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \dots, D_{1,n}^{\varepsilon_n})$, then

$$(D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \dots, D_{1,n}^{\varepsilon_n})^\vee := (D_{1,n}^{\vee\varepsilon_n}, D_{1,n-1}^{\vee\varepsilon_{n-1}}, \dots, D_{1,1}^{\vee\varepsilon_1}),$$

where $D_1^{\vee\vee} := D_1$. A cycle[19] of oriented line defects

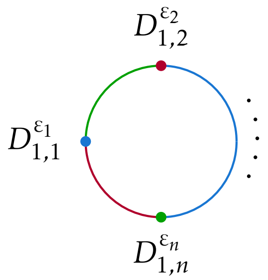
$$[D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \dots, D_{1,n}^{\varepsilon_n}] \in [\mathbb{D}_1 \times \{\cdot, \vee\}]^n$$

is *cyclically composable* if

$$d_0(D_{1,j}^{\varepsilon_j})_2 = d_0(D_{1,j+1}^{\varepsilon_j+1})_1 \tag{1.6}$$

for all $j \in \mathbb{Z}_n$.[20] Thus, the map d_0 assigns a composable cycle to each point defect. The cycle may be empty, in which case the point defect has no incoming and no outgoing lines. A cycle element D_1 indicates an ingoing line D_1 and $D_1^\vee$ indicates an outgoing line D_1.

This allows us to define the *category of oriented 2-bordisms with defects* $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$. Given a circle S^1, we may *decorate* it with a composable cycle $X = [D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \dots, D_{1,n}^{\varepsilon_n}]$ by adding n points to the circle and labelling the points and line segments by the cycle elements $D_{1,j}^{\varepsilon_j}$ and the corresponding surface defects $d_0(D_{1,j}^{\varepsilon_j})$, respectively. We denote such decorated circles by

$S_X^1 :=$ 

. (1.7)

We now define $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$ as the symmetric monoidal category consisting of

Objects: disjoint unions (including the empty union) of decorated circles S_X^1,

Morphisms: bordisms $M : U \longrightarrow U'$ decorated with points, lines, and surfaces that are appropriately labelled by point defects, line defects, and surface defects, respectively [DKR11],

[19] Cycles are tuples without a fixed starting point. For instance $[a, b, c] = [b, c, a]$.
[20] Here, $\mathbb{Z}_n$ denotes whole numbers modulo n. This means that $n \equiv 0$ in $\mathbb{Z}_n$.

HORIZONTAL COMPOSITION: disjoint unions,

MONOIDAL UNIT: $\mathbb{1}_{\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})} := \emptyset$,

VERTICAL COMPOSITION: gluing defect bordisms, and

UNIT MORPHISMS: cylinders over the decorated objects.

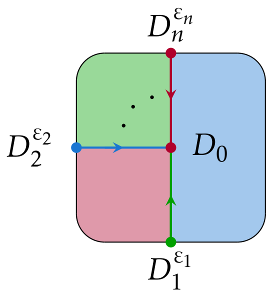
This construction may be generalised further to define the *category of oriented n-bordisms with defects* $\mathbf{Bord}_n$ [CRS19, Sec. 2.2.2]. Analogously to closed TQFTs (1.4), *defect TQFTs* are symmetric monoidal functors

$$\mathcal{Z} : \mathbf{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}\,.$$

Note 1.1.8. If we are given a defect TQFT $\mathcal{Z}$, and know how $\mathcal{Z}$ evaluates bordisms without point defects and how $\mathcal{Z}$ evaluates point defects on disks, then $\mathcal{Z}$ is already fully determined due to functoriality.

Note 1.1.9. Given a defect in $\mathbb{D}$, we may evaluate it with $\mathcal{Z}$ by placing the defect on a disk. For instance, if we are given a point defect $D_0 \in \mathbb{D}_0$ and $d_0(D_0) = [D_{1,1}^{\varepsilon_1}, D_{1,2}^{\varepsilon_2}, \dots, D_{1,n}^{\varepsilon_n}]$, then we may make a slight abuse of notation to define the shorthand

$$\mathcal{Z}(D_0) := \quad \mathcal{Z}\left(\qquad \right).$$



Note 1.1.10. Point defects correspond to vectors. Specifically, if we are given a point defect $D_0 \in \mathbb{D}_0$, then we may interpret it as a bordism $D_0 : \emptyset \longrightarrow S^1_{d_0(D_0)}$. Recalling that there is a correspondence $V \leftrightarrow \{K\text{-linear maps } K \longrightarrow V\}$ for all K-vector spaces V, we see that the $\mathbb{C}$-linear map $\mathcal{Z}(D_0) : \mathbb{C} \longrightarrow \mathcal{Z}(S^1_{d_0(D_0)})$ may be interpreted as the vector $\mathcal{Z}(D_0)(1) \in \mathcal{Z}(S^1_{d_0(D_0)})$.

Note 1.1.11. We may identify objects of $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$ with tuples of composable cycles. In particular, there is a natural map

$$\begin{aligned} \mathbb{D}_0 &\longrightarrow \mathrm{Ob}(\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})) \\ D_0 &\longmapsto S^1_{d_0(D_0)}. \end{aligned}$$

Note 1.1.12. The surface defects/bulk theories of defect TQFTs correspond to closed TQFTs. By picking a surface defect $D_n \in \mathbb{D}_n$ of a defect TQFT $\mathcal{Z} : \mathbf{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ we may construct the isomorphism

$$\varphi_{D_n} : \mathbf{Bord}_n \longrightarrow \mathbf{Bord}_n^{\mathrm{def}}\Big((\{D_n\}, \emptyset, \ldots, \emptyset)\Big).$$

Thus, we may define a closed TQFT

$$\mathcal{Z}_{D_n} := \left(\mathbf{Bord}_n \xrightarrow{\varphi_{D_n}} \mathbf{Bord}_n^{\mathrm{def}}\Big((\{D_n\}, \emptyset, \ldots, \emptyset)\Big) \hookrightarrow \mathbf{Bord}_n^{\mathrm{def}}(\mathbb{D}) \xrightarrow{\mathcal{Z}} \mathbf{Vect}_{\mathbb{C}}\right).$$

1.1.4. Categories of Defects

We may use the defects $\mathbb{D}$ of a defect TQFT $\mathcal{Z} : \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ to "freely generate" (Definition A.1.26) a strict *pivotal* (Remark 1.1.14) 2-category $\langle \mathbb{D} \rangle$ consisting of

Objects: bulk theories $D_2 \in \mathbb{D}_2$,

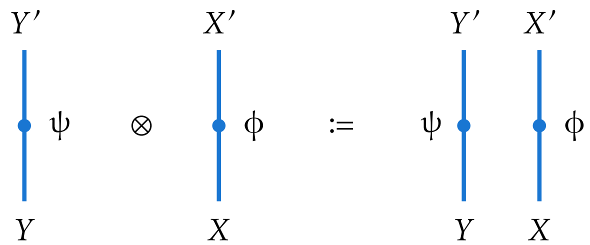
1-Morphisms: composable tuples of line defects,

Unit 1-Morphisms: empty tuples $\mathbb{1}_{D_2} := \emptyset$,

2-Morphisms: freely generated by point defects,

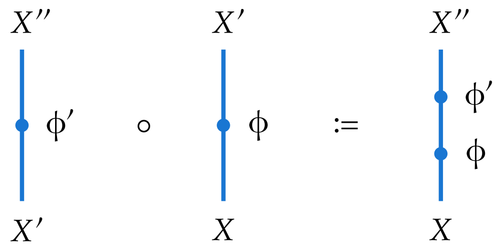
Unit 2-Morphisms: 1_X freely generated,

Horizontal Composition: free horizontal composition, e.g.:



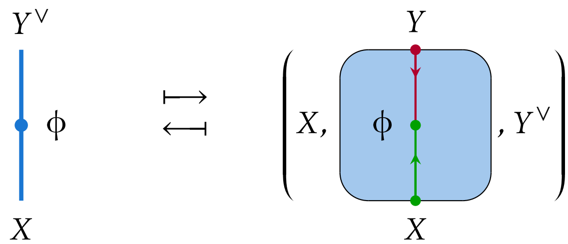
and

Vertical Composition: free vertical composition, e.g.:



Remark 1.1.13. We may equivalently interpret $\langle\mathbb{D}\rangle$ as consisting of disks of $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$ covered with defect data, together with the information about which entities are domains and which are codomains. Specifically,

- objects $D_2 \in \langle\mathbb{D}\rangle$ each simply correspond to a disk decorated with the bulk theory D_2,
- 1-morphisms $(D_{1,1}, \ldots, D_{1,n}) : D_2 \longrightarrow \widetilde{D}_2$ each correspond to a disk decorated with the line defects $(D_{1,1}, \ldots, D_{1,n})$ such that each line defect $D_{1,k}$ "goes from one side to the other", and
- 2-morphisms $\varphi : (D_{1,1}, \ldots, D_{1,n}) \Longrightarrow [\widetilde{D}_{1,1}, \ldots, \widetilde{D}_{1,m}]$ each correspond to a disk decorated with point defects $D_{0,1}, \ldots, D_{0,k}$, which are connected as indicated by φ, and such that $[D_{1,1}, \ldots, D_{1,n}, \widetilde{D}^{\vee}_{1,m}, \ldots, \widetilde{D}^{\vee}_{1,1}]$ is a composable cycle of line defects, e.g.:



Moreover, one may then interpret vertical and horizontal composition of 2-morphisms in $\langle\mathbb{D}\rangle$ as being induced by pairs of pants (Note 1.1.8 or cf. [Car18, Sec. 2.3]). Since pairs of pants are *associative algebras*, vertical and horizontal composition in $\langle\mathbb{D}\rangle$ is well-defined.

Remark 1.1.14. Since the entities of $\langle\mathbb{D}\rangle$ are essentially disks decorated with defect data, the domain X and codomain X' of each 2-morphism must form a composable cycle of line defects $[X, (X')^{\vee}]$. In particular, the 2-morphisms depend only on the cycle $[X, (X')^{\vee}]$ and not on the specific X and $(X')^{\vee}$. This is roughly what $\langle\mathbb{D}\rangle$ being *pivotal* means. In other words, pivotality of $\langle\mathbb{D}\rangle$ corresponds to the existence of orientations $X \longleftrightarrow X^{\vee}$. In particular, pivotality naturally extends to the category $\mathcal{D}_{\mathcal{Z}}$ [e.g. Car18, Sec. 2.3] we are about to meet in Sketch 1.1.15 (and its important special case Remark 1.1.16).

Sketch 1.1.15. On its own, $\langle \mathbb{D} \rangle$ is of course not particularly interesting because it is just a freely generated 2-category. However, we may use $\mathcal{Z}$ to introduce the congruence relation[21]

$$\phi \sim_{\mathcal{Z}} \psi \qquad \Leftrightarrow \qquad \mathcal{Z}(\phi) = \mathcal{Z}(\psi)$$

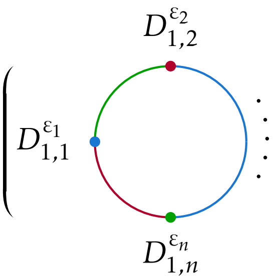
on 2-morphisms $\phi, \psi : X \Longrightarrow Y$ of $\langle \mathbb{D} \rangle$. Here we define $\mathcal{Z}(\phi)$ and $\mathcal{Z}(\psi)$ by evaluating disks with ϕ and ψ (which we do by extending the shorthand of Note 1.1.9).[22] In particular, this congruence relation implies a 2-category of defects

$$\mathcal{D}_{\mathcal{Z}} := \langle \mathbb{D} \rangle \,/\!\sim_{\mathcal{Z}},$$

where we identify the 2-morphisms of $\langle \mathbb{D} \rangle$ according to $\sim_{\mathcal{Z}}$. Therefore, the objects and 1-morphisms of $\mathcal{D}_{\mathcal{Z}}$ are the same as the objects and 1-morphisms of $\langle \mathbb{D} \rangle$, respectively. Meanwhile, the 2-morphisms of $\mathcal{D}_{\mathcal{Z}}$ are linear maps $\mathcal{Z}(\phi) : \mathbb{C} \longrightarrow \mathcal{Z}(S^1_X)$ corresponding to disks $\emptyset \longrightarrow S^1_X$ decorated with defects ϕ evaluated with $\mathcal{Z}$. We may, therefore, identify the 2-morphisms of $\mathcal{D}_{\mathcal{Z}}$ with vectors (Note 1.1.10) by evaluating the linear maps at $1 \in \mathbb{C}$: $\mathcal{Z}(\phi)(1) \in \mathcal{Z}(S^1_X)$.

Remark 1.1.16. Let $\mathbb{D}$ be such that $\mathcal{Z} : \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$ is bijective on disks, i.e. $\hom_{\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})}(\emptyset, S^1_X) \cong \hom_{\mathbf{Vect}_{\mathbb{C}}}(\mathbb{C}, \mathcal{Z}(S^1_X))$ for all composable cycles of line defects $X \in \langle \mathbb{D} \rangle$. Using $\hom_{\mathbf{Vect}_{\mathbb{C}}}(\mathbb{C}, \mathcal{Z}(S^1_X)) \cong \mathcal{Z}(S^1_X)$ we see that such $\mathbb{D}$ implies that the 2-hom-sets of $\mathcal{D}_{\mathcal{Z}}$ correspond to vector spaces in the image of $\mathcal{Z}$. Specifically, $\mathcal{D}_{\mathcal{Z}}(X, Y) \cong \mathcal{Z}(S^1_{[X, Y^\vee]})$, where X and Y are composable tuples of line defects with matching source and target. In other words, we may consider the 2-morphisms of such $\mathcal{D}_{\mathcal{Z}}$ as being the vectors

$$v \in \quad \mathcal{Z}\left(\qquad \right).$$



We may consider such a defect set $\mathbb{D}$ as being maximal w.r.t. $\mathcal{Z}$.

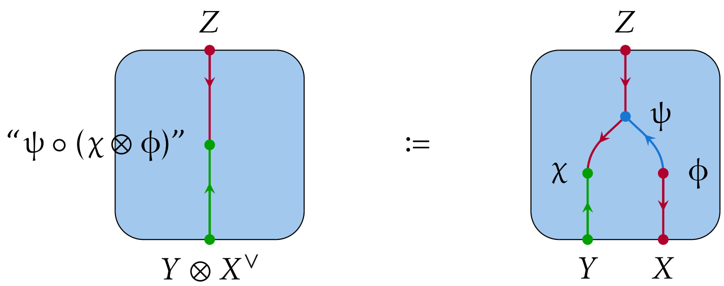
[21] Congruence relations are equivalence relations that preserve the algebraic structure. In other words, congruence relations are equivalence relations that allow well-behaved quotients.

[22] We could also have defined our congruence relation $\sim_{\mathcal{Z}}$ via a 2-functor $F : \langle \mathbb{D} \rangle \longrightarrow \langle (\mathbb{D}_2, \mathbb{D}_1, \bigsqcup \mathcal{Z}(S^1_X)) \rangle$. In that case $\langle \mathbb{D} \rangle \,/\, \mathcal{Z} = \mathbf{im}(F)$.

Categories of defects were introduced by [DKR11], who considered exactly those $\mathcal{D}_{\mathcal{Z}}$ that we just described. The advantage of these maximal defect sets $\mathbb{D}$ is that the defect categories $\mathcal{D}_{\mathcal{Z}}$ are *enriched* (Sketch A.2.2) over $\mathbf{Vect}_{\mathbb{C}}$, i.e. the 2-hom-sets of $\mathcal{D}_{\mathcal{Z}}$ are vector spaces. In particular, string diagrams of $\mathcal{D}_{\mathcal{Z}}$ may be readily interpreted as "correlators" of $\mathcal{Z}$ [DKR11].

Remark 1.1.17. Given a defect TQFT $\mathbf{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$, there should be some kind of strict *n-categories with duals* $\langle\mathbb{D}\rangle$ and $\mathcal{D}_{\mathcal{Z}}$ for all $n \geq 1$. This has been worked out in detail for $n \leq 3$ [CMS20, Sec. 3.4].

Notation 1.1.18. In the following, let us allow labelling line and point defects with $\langle\mathbb{D}\rangle$. In other words, we now do not just label bordisms with defects of $\mathbb{D}$, but also with compositions of defects of $\mathbb{D}$, e.g.:[23]



1.1.5. Categorical Symmetries

Symmetries are an important tool for investigating theories of physics. Classically, a symmetry of a physical theory with underlying state space $\mathcal{H}$ is a group action

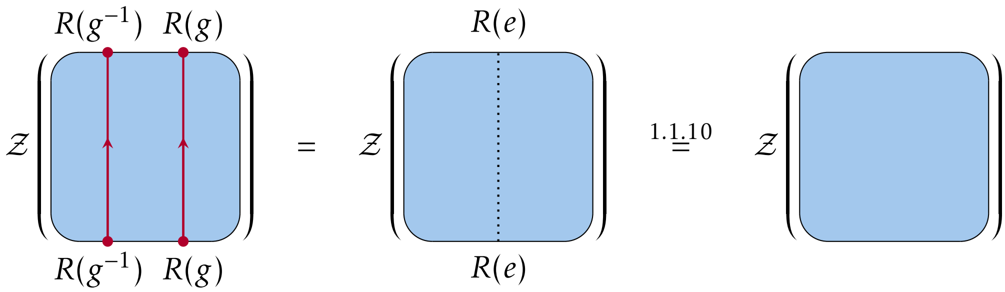
$$\rho : G \longrightarrow \mathrm{End}(\mathcal{H})$$

such that the dynamics of the theory are invariant under ρ. For instance, if the theory is quantum mechanics, then invariance under ρ means that ρ commutes with the Hamiltonian. Now, if this theory is given in terms of a bulk theory $D_n \in \mathbb{D}_n$ of an n-dimensional defect TQFT $\mathcal{Z} : \mathbf{Bord}_n^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$, then the group G may be interpreted as a monoidal $(n-1)$-category $\underline{G}$ whose objects are given by the group elements $g \in G$, and whose higher morphisms are trivial. The symmetry is then a strict n-functor

$$R : \underline{G} \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{Z}}}(D_n),$$

[23] Expressions such as $\psi \circ (\xi \otimes \phi)$ are well-defined as 2-morphisms of a 2-category, but not necessarily well-defined as labels on bordisms because there is no fixed order in which we read the labels on bordisms. However, this will sufficient for our purposes.

i.e. the symmetry consists of $(n-1)$-dimensional defects $R(g) \in \bigsqcup^k \mathbb{D}^k_{n-1}$, which conform with the group law of G (cf. e.g. [BBFT⁺24, State. 2.1]), e.g.:



Evidently, there are various natural generalisations of symmetries. For instance, one may consider group symmetries given by defects of dimension $\leq n-2$ (see e.g. [BBFT⁺24, Def. 2.1]) or lift the strictness requirement of R and replace $\underline{G}$ with an $(n-1)$-group (see e.g. [CH25]). More generally, one may generalise symmetries by defining *categorical symmetries* as monoidal n-functors

$$\mathcal{C} \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{Z}}}(D_n),$$

where $\mathcal{C}$ may be any monoidal $(n-1)$-category.

1.1.6. Orbifold TQFTs

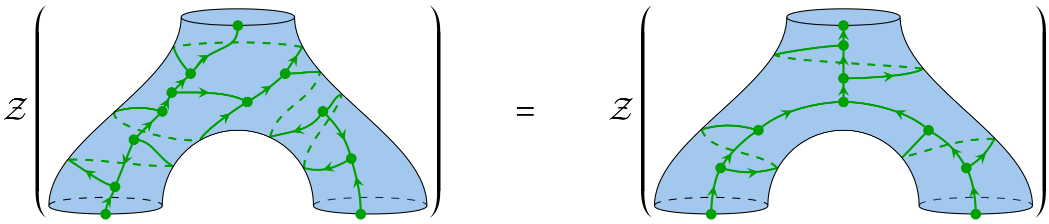
Remark 1.1.19. In mathematics an *orbifold* is (roughly speaking) a quotient $M \,/\, G$ where M is a manifold together with a group action $G \longrightarrow \mathrm{End}(M)$ [ALR07, Def. 1.7 & Cor. 1.24].

In theoretical physics a *gaugeable* G-symmetry of a theory of physics with underlying state space $\mathcal{H}$ may be *gauged* by orbifolding the underlying state space.[24] Then the G-*orbifolded* theory $\mathcal{H} \mathbin{/\!\!/} G$ describes the same physical reality as the original theory despite having a different state space. To generalise orbifolds to defect TQFTs[25] $\mathcal{Z} : \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$, let us assume we are given a bulk theory $D_2 \in \mathbb{D}_2$ and a categorical symmetry $\mathcal{C} \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{Z}}}(D_2)$. To construct a generalised orbifolded theory "$\mathcal{Z}_{D_2} \mathbin{/\!\!/} \mathcal{C}$", we must identify an appropriate generalisation of $\mathcal{C}$-cosets. This was outlined by [FFRS10] in the setting of "CFTs" and worked out by [CR16] in the setting of TQFTs, who showed how to interpret 1-dimensional *defect networks* as 2-dimensional bulk theories. These defect networks must be given by *orbifold data*.

[24] Generally, one needs to also add "twisted sectors" to the orbifolded theory, but we shall not concern ourselves with this.

[25] We only consider dimension 2 here, but the following orbifold construction may be generalised to n-dimensional defect TQFTs [CRS19, Thm. 3.10].

Conceptually, an orbifold datum is a tuple $\mathcal{A}$ consisting of defects of $\mathbb{D}$ such that bordisms covered with a defect network labelled by $\mathcal{A}$ evaluate equally for every defect network $\mathcal{A}$, e.g.:



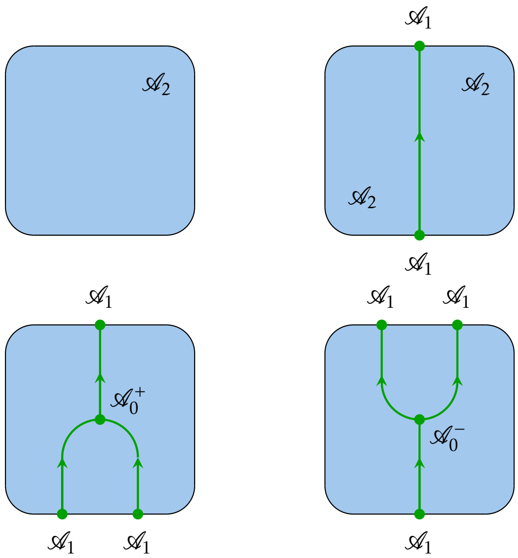
Specifically, an *orbifold datum* is a tuple

$$\mathcal{A} = (\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0^+, \mathcal{A}_0^-)$$

consisting of

- a surface defect $\mathcal{A}_2 \in \mathrm{Ob}(\mathcal{D}_\mathcal{Z}) = \mathbb{D}_2$,
- a composition of line defects $\mathcal{A}_1 \in \mathcal{D}_\mathcal{Z}(\mathcal{A}_2, \mathcal{A}_2)$,
- a composition of point defects $\mathcal{A}_0^+ \in \mathrm{hom}(\mathcal{A}_1 \otimes \mathcal{A}_1, \mathcal{A}_1)$, and
- a composition of point defects $\mathcal{A}_0^- \in \mathrm{hom}(\mathcal{A}_1, \mathcal{A}_1 \otimes \mathcal{A}_1)$,

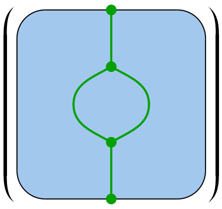
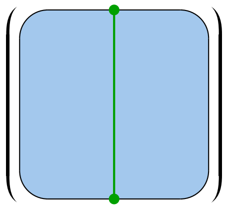
such that arbitrary $\mathcal{A}$*-defect networks* evaluate equally.[26] Graphically, this means that an orbifold datum consists of defect data



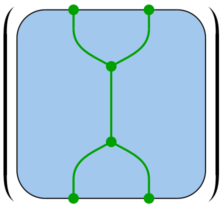
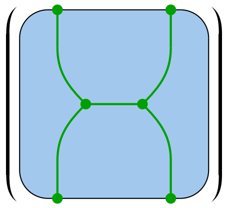
(1.8)

[26] One may naïvely construct orbifold data as elements $\mathcal{A} \in \mathbb{D}_2 \times \mathbb{D}_1 \times \mathbb{D}_0^2$, but this approach would be limited. For instance, the component $\mathcal{A}_1 \in \mathbb{D}_1^k$ would be limited to $k = 1$.

such that

$\mathcal{Z}$  $=$ $\mathcal{Z}$ 

and

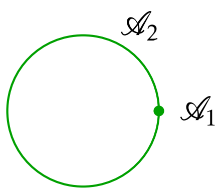
$\mathcal{Z}$  $=$ $\mathcal{Z}$ 

for all choices of labelling by $\mathscr{A}$ [CR16, Sec. 3.3].[27]

Remark 1.1.20. Orbifold data $\mathscr{A}$ may also be defined as *symmetric Δ-separable Frobenius algebras* (Definition 5.2.4) in $\mathcal{D}_{\mathcal{Z}}$ [CR16, Prop. 3.4; FRS02, Lemma 3.9].

Intuition 1.1.21. The idea behind $\mathscr{A}$-defect networks is that if one has already covered a surface $\mathscr{A}_2$ with $\mathscr{A}$, then $\mathcal{Z}$ becomes invariant under adding further $\mathscr{A}$-defects. This is reminiscent of idempotents. As we shall soon see, orbifold data are indeed generalisations of idempotents.

Now, this allows us to define an embedding[28] $(-)^{\mathscr{A}} : \mathbf{Bord}_2 \hookrightarrow \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$ that covers bordisms with $\mathscr{A}$. For one, $(-)^{\mathscr{A}}$ is determined on objects by

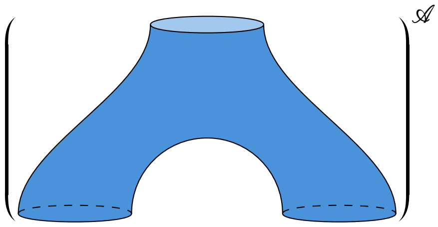
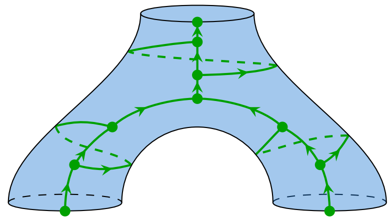
$(S^1)^{\mathscr{A}} := S^1_{[\mathscr{A}_1]} \overset{(1.7)}{=}$ 

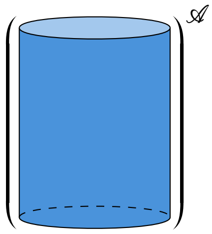
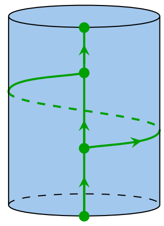
.

[27] These choices must of course be valid, i.e. each node has either one or two ingoing $\mathscr{A}_1$.

[28] $(-)^{\mathscr{A}}$ is not a functor, but $\mathcal{Z}\left((-)^{\mathscr{A}}\right)$ is a functor.

Moreover, $(-)^{\mathscr{A}}$ maps bordisms $M : U \longrightarrow U'$ of $\mathbf{Bord}_2$ to bordisms $M^{\mathscr{A}} : U^{\mathscr{A}} \longrightarrow (U')^{\mathscr{A}}$ covered with $\mathscr{A}$-defect networks,[29] e.g.:

 $:=$ 

$(S^1 \times I)^{\mathscr{A}} \equiv$  $:=$ 

Now, we may notice that the orbifold conditions imply that $\mathcal{Z}((S^1 \times I)^{\mathscr{A}})$ is an idempotent (cf. Lemma 5.2.50). Analogously, $e_U := \mathcal{Z}((U \times I)^{\mathscr{A}})$ is an idempotent for all $U \in \mathbf{Bord}_2$, so there are linear maps[30]

$$\begin{tikzcd} \mathcal{Z}((U \times I)^{\mathscr{A}}) \arrow[loop left, "e_U"] \arrow[r, bend left, "\pi_U"] & \mathbf{im}(e_U) \arrow[l, bend left, "\iota_U"] \end{tikzcd}$$

such that $e_U = \iota_U \circ \pi_U$ and $1_{\mathbf{im}(e_U)} = \pi_U \circ \iota_U$.

We use these components to define an *orbifold TQFT*[31]

$$\mathcal{Z}_{\mathscr{A}} := \left(\begin{array}{ccccccc} \mathbf{Bord}_{2,1} & \xrightarrow{(-)^{\mathscr{A}}} & \mathbf{Bord}_{2,1} & \xrightarrow{\mathcal{Z}} & \mathbf{Vect}_{\mathbb{C}} & \xrightarrow{\pi_{(-)} \circ (-) \circ \iota_{(-)}} & \mathbf{Vect}_{\mathbb{C}} \\ U & \longmapsto & U^{\mathscr{A}} & \longmapsto & \mathcal{Z}(U^{\mathscr{A}}) & \longmapsto & \mathbf{im}(e_U) \\ (U \xrightarrow{M} U') & \longmapsto & M^{\mathscr{A}} & \longmapsto & \mathcal{Z}(M^{\mathscr{A}}) & \longmapsto & \pi_{U'} \circ \mathcal{Z}(M^{\mathscr{A}}) \circ \iota_U \end{array} \right)$$

Remark 1.1.22. The notation $\mathcal{Z}_{\mathscr{A}}$ is very much in line with the notation $\mathcal{D}_{D_2}$ because $\mathcal{Z}_{D_2}$ is a closed TQFT corresponding to the bulk theory D_2 and $\mathcal{Z}_{\mathscr{A}}$ is essentially the same, we just had to lift the defect network $\mathscr{A}$ to the level of a bulk theory. $\mathcal{Z}_{D_2}$ then simply corresponds to orbifolding with the trivial defect network $\mathbb{1}_{D_2}$ (Example 1.1.23).

[29] The $\mathscr{A}$-defect networks need to be such that they sufficiently wrap M. This may be made precise by requiring that the Poincaré dual of the $\mathscr{A}$-network corresponds to a (non-trivial) decomposition of M into simplices.

[30] This works because all idempotents in $\mathbf{Vect}_{\mathbb{C}}$ *split*. If $\mathbf{Vect}_{\mathbb{C}}$ is replaced by a different target category $\mathcal{C}$, then the $\mathscr{A}$-orbifolding construction requires that the e_U splits.

[31] The notation $\mathbf{Vect}_{\mathbb{C}} \xrightarrow{\pi_{(-)} \circ (-) \circ \iota_{(-)}} \mathbf{Vect}_{\mathbb{C}}$ is quite implicit. In particular, it is not a well-defined functor on all of $\mathbf{Vect}_{\mathbb{C}}$ but rather only on $\mathbf{im}(\mathcal{Z}((-)^{\mathscr{A}}) \subset \mathbf{Vect}_{\mathbb{C}}$.

Now, if we are given an orbifold datum $\mathscr{A}$ of a defect TQFT $\mathcal{Z}$ with bulk theory $\mathscr{A}_2 \in \mathbb{D}_2$, then there is a sub-2-category $\langle \mathscr{A} \rangle \subseteq \mathcal{D}_{\mathcal{Z}}$ generated by the orbifold datum, and we may consider the endomorphism category $\mathcal{C}_{\mathscr{A}} := \mathrm{End}_{\langle \mathscr{A} \rangle}(\mathscr{A}_2)$. There is a categorical symmetry

$$\mathcal{C}_{\mathscr{A}} \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{Z}}}(\mathscr{A}_2),$$

so we may regard

$$\mathcal{Z}_{D_2} \mathbin{/\!/} \mathcal{C}_{\mathscr{A}} := \mathcal{Z}_{\mathscr{A}}$$

as the $\mathscr{A}$-orbifolded theory of $\mathscr{A}_2$.[32]

Example 1.1.23. For every surface defect $D_2 \in \mathbb{D}_2$ there is a trivial orbifold datum

$$\mathbb{1}_{D_2} \triangleq \Big(D_2, \underbrace{(\,)}_{= \mathbb{1}_{D_2}}, \underbrace{1_{(\,)}}_{= 1_{\mathbb{1}_{D_2}}}, \underbrace{1_{(\,)}}_{= 1_{\mathbb{1}_{D_2}}}\Big)$$

that may be interpreted as defect networks consisting of nothing. The $\mathbb{1}_{D_2}$-orbifolded TQFT is simply

$$\mathcal{Z}_{\mathcal{D}_2} \mathbin{/\!/} \mathbb{1}_{D_2} = \mathcal{Z}_{\mathbb{1}_{D_2}} = \mathcal{Z}_{\mathcal{D}_2}.$$

1.1.7. Orbifold Defect TQFTs

We just saw how to construct closed TQFTs $\mathcal{Z}_{\mathscr{A}}$ from defect TQFTs $\mathcal{Z}$. These $\mathcal{Z}_{\mathscr{A}}$ may be joined with $\mathcal{Z}$ to construct an *orbifold defect TQFT*[33] $\mathcal{Z}_{\mathrm{orb}}$ that subsumes $\mathcal{Z}$, i.e.:

$$\begin{tikzcd}
\mathbf{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \arrow[r, "\mathcal{Z}"] \arrow[d, hook] & \mathbf{Vect}_{\mathbb{C}} \\
\mathbf{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}_{\mathrm{orb}}) \arrow[ur, "\mathcal{Z}_{\mathrm{orb}}"'] &
\end{tikzcd}$$

[32] This derivation of $\mathcal{Z}_{D_2} \mathbin{/\!/} \mathcal{C}_{\mathscr{A}}$ may feel a bit incomplete. However, the formulation of categorical symmetries is currently a hot topic, so maybe a complete derivation will be formulated in the not-too-distant future. Two fresh articles that further close the gap are [Mü25] and [CH25].

[33] The following orbifold defect TQFT construction should generalise to n-dimensional defect TQFTs (Remark 7.2.15). This has been made rigorous for $n \leq 3$ [CM23, Thm. 6.4].

To construct $\mathcal{Z}_{\mathrm{orb}}$ we first construct $\mathbb{D}_{\mathrm{orb}}$. Naturally, $(\mathbb{D}_{\mathrm{orb}})_2$ consists of bulk theories[34]

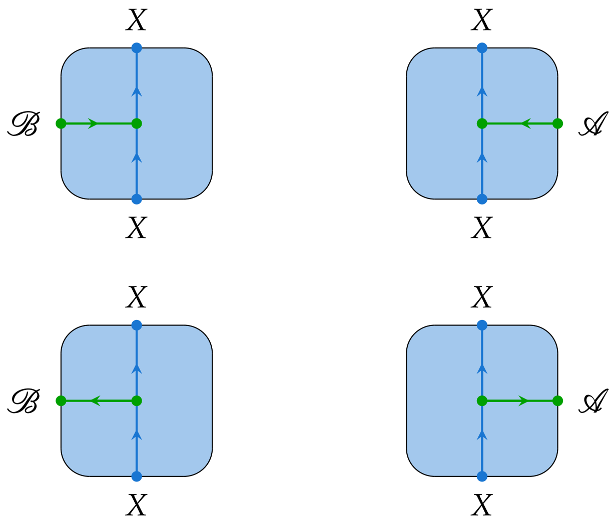
$$(\mathbb{D}_{\mathrm{orb}})_2 := \{\mathscr{A} \mid \mathscr{A} \text{ is orbifold datum in } \mathcal{D}_{\mathcal{Z}}\}.$$

This allows a canonical embedding

$$\begin{aligned} \mathbb{D}_2 &\hookrightarrow (\mathbb{D}_{\mathrm{orb}})_2 \\ D_2 &\longmapsto \mathbb{1}_{D_2} \end{aligned}$$

where $\mathbb{1}_{D_2}$ implicitly indicates the trivial orbifold datum on D_2 (Example 1.1.23).

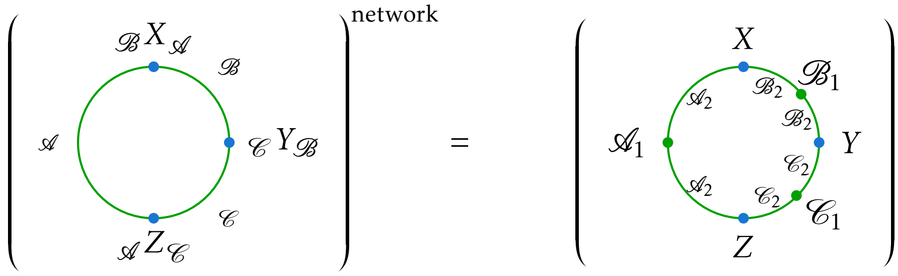
By similar considerations, *bimodules* and *bimodule maps* (Sections 5.2.3 and 6.2) are used to construct $(\mathbb{D}_{\mathrm{orb}})_1$ and $(\mathbb{D}_{\mathrm{orb}})_0$, respectively. Essentially bimodules $({}_{\mathscr{B}}X_{\mathscr{A}} : \mathscr{A} \longrightarrow \mathscr{B}) \in (\mathbb{D}_{\mathrm{orb}})_2$ consist of line defects $X \in \mathbb{D}_1^k$ together with point defects that each nicely connect $\mathscr{A}$ and $\mathscr{B}$ to X:



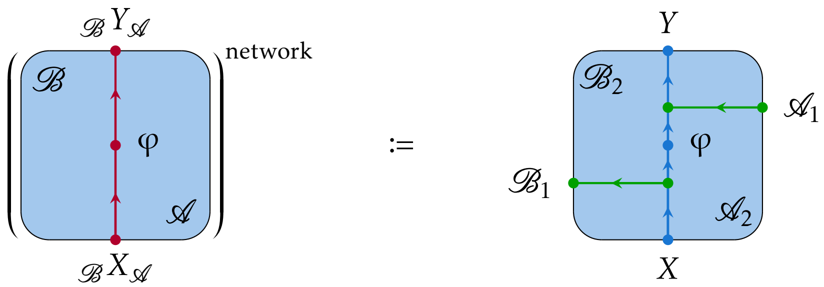
Bimodule maps ${}_{\mathscr{B}}X_{\mathscr{A}} \longrightarrow {}_{\mathscr{B}}Y_{\mathscr{A}}$ are simply point defects $X \longrightarrow Y$ with certain properties. Altogether, this implies a canonical embedding $\mathbb{D} \hookrightarrow \mathbb{D}_{\mathrm{orb}}$ which in turn implies a canonical embedding $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}) \hookrightarrow \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}_{\mathrm{orb}})$.

The bimodule conditions allow us to define a functor $(-)^{\mathrm{network}} : \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}_{\mathrm{orb}}) \longrightarrow \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D})$ determined on objects by mapping line defects ${}_{\mathscr{B}}X_{\mathscr{A}} \in (\mathbb{D}_{\mathrm{orb}})_1$ to their underlying line defects $X \in \mathbb{D}_2^k$ and mapping surface defects $\mathscr{A} \in (\mathbb{D}_{\mathrm{orb}})_2$ to a line defect $\mathscr{A}_1 \in \mathbb{D}_2^l$ with surface defects $\mathscr{A}_2$ on either side, e.g.:

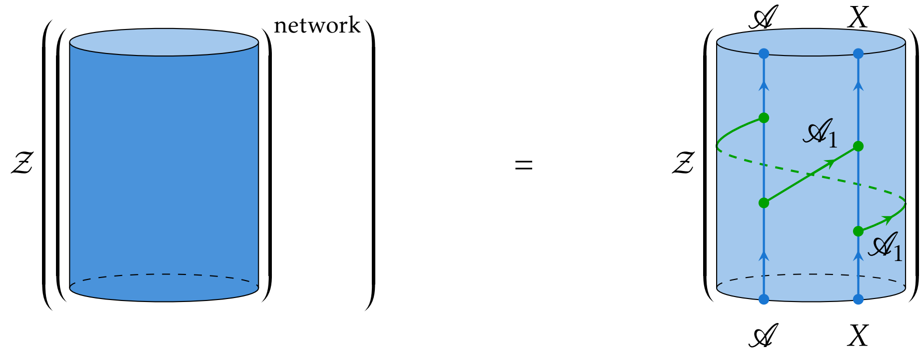
[34] This immediately suggests $(\mathcal{Z}_{\mathrm{orb}})_{\mathscr{A}} = \mathcal{Z}_{\mathscr{A}}$, where the $(\mathcal{Z}_{\mathrm{orb}})_{\mathscr{A}}$ is defined as in Note 1.1.12.



Moreover, defect bordisms $M : U \longrightarrow U'$ in $\mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}_{\mathrm{orb}})$ are mapped to defect bordisms $M^{\mathrm{network}} : U^{\mathrm{network}} \longrightarrow (U')^{\mathrm{network}}$ where line defects ${}_{\mathscr{B}}X_{\mathscr{A}} \in (\mathbb{D}_{\mathrm{orb}})_1$ are again substituted by $X \in \mathbb{D}_2^k$ and where surface defects $\mathscr{A} \in (\mathbb{D}_{\mathrm{orb}})_2$ are substituted by $\mathscr{A}$-defect networks, e.g.:



As previously, the orbifold conditions imply that $e_U := \mathcal{Z}((U \times I)^{\mathrm{network}})$ are idempotents for all $U \in \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}_{\mathrm{orb}})$ and, therefore, again provide the linear maps π_U and ι_U. For instance, Lemma 5.2.50 implies that



is an idempotent.

Therefore, we may define the *orbifold defect TQFT*

$$\begin{aligned}\mathcal{Z}_{\mathrm{orb}} : \mathbf{Bord}_2^{\mathrm{def}}(\mathbb{D}_{\mathrm{orb}}) &\longrightarrow \mathbf{Vect}_{\mathbb{C}}\\ U &\longmapsto \mathbf{im}(e_U)\\ M &\longmapsto \pi_{U'} \circ \mathcal{Z}(M^{\mathrm{network}}) \circ \iota_U .\end{aligned}$$

We see that $\mathcal{Z}_{\mathrm{orb}}$ indeed acts like $\mathcal{Z}$ on bordisms decorated with $\mathbf{im}\big(\mathbb{D} \hookrightarrow \mathbb{D}_{\mathrm{orb}}\big)$ and like $\mathcal{Z}_{\mathscr{A}}$ on surface defects $\mathscr{A} \in (\mathbb{D}_{\mathrm{orb}})_2$.

Remark 1.1.24. By construction $\mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}} \simeq \mathcal{D}_{(\mathcal{Z}_{\mathrm{orb}})_{\mathrm{orb}}}$ ((1.10) & Lemma 6.2.41).

1.2. Higher Idempotent Completions

Higher idempotents are categorifications of idempotents in *higher categories* (Section 1.2.1). Meanwhile, *idempotent completions* are a categorification of *completions* of metric spaces and *higher idempotent completions* are even higher categorifications (Section 1.2.2). Since orbifold data are higher idempotents (Section 1.2.1), it is natural to consider *orbifold completions* $\mathcal{C}_{\mathrm{orb}}$ of higher categories $\mathcal{C}$. In particular, orbifold completions should correspond to orbifold defect TQFTs (Section 1.2.3):

$$(\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}} \simeq \mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}}$$

1.2.1. Higher Idempotents

If we are given a monoid $(M, \circ\,)$, then an element $e \in M$ is *idempotent* if

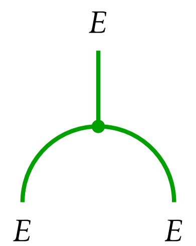
$$e \circ e = e. \tag{1.9}$$

Therefore, endomorphisms in categories may also be idempotent.

Idempotents may be categorified.[35] Since idempotents are endomorphisms, let us consider a 1-endomorphism $E : x \longrightarrow x$ of a 2-category and turn the identity (1.9) into a 2-morphism $\mu : E \otimes E \longrightarrow E$. (E, μ) is a categorification of an idempotent, so we may regard it as being a *2-idempotent*. However, usually one would define 2-idempotence less generally, e.g. one may impose *associativity*, i.e. the condition

$$\mu \circ (\mu \otimes 1_E) = \mu \circ (1_E \otimes \mu).$$

Moreover, denoting (E, μ) graphically as



$:= \mu,$

our interest in 2-idempotents becomes apparent: 2-orbifold data (1.8) are 2-idempotents!

[35] Note that categorification can also mean other procedures than the one used here. It is not a precisely defined concept [LS22, p. 11]. Generally, it means taking a mathematical object/construction, placing it in a category, and then generalising it. If the construction is already part of an n-category, categorification usually means considering generalisations in an $(n + 1)$-category.

Example 1.2.1. Given a field K, an algebra $(A, \cdot)$ over K is a K-vector space A together with a K-linear map $\cdot : A \otimes_K A \longrightarrow A$. Thus, algebras over fields fulfil a very general notion of 2-idempotence. If one requires 2-idempotents to be associative, then instead associative algebras are 2-idempotents.

Similarly, one may categorify idempotents further to *n-idempotents* in *n-categories* for arbitrary $n \in \mathbb{N}$.[36] The most standard notion of n-idempotents is likely given by the *n-condensation algebras* introduced by [GJF19, Def. 2.2.1] to describe "gapped condensed matter systems". Another important notion of n-idempotents is given by the *n-orbifold data* introduced by [CRS19] and worked out in detail for $n = 3$ by [CRS19, CMR$^+$21, CM23]. n-categories are called *higher categories* when $n \geq 2$, so n-idempotents are *higher idempotents* when $n \geq 2$. In this thesis we will be content with 2-idempotents since 3-categories are a bit unwieldy, but it is expected that the discussed constructions should generalise well to arbitrary dimensions (Section 7.2).

In this thesis we will be interested in *2-orbifolds* and the more general *2-condensation algebras* (Nomenclatures 5.2.60 and 5.2.66). However, there are also other contexts in which higher idempotents are of interest; a few reasons mathematicians, physicists, and computer scientists are interested in n-idempotents (in particular at $n = 2$) are:

Higher Idempotent	Application Area
orbifold data	TQFT [CR16, BCP14, Car25]
condensation algebras	TQFT [GJF19], weighted colimits [Kam24]
"spiders"	*ZX-calculus* (quantum computing) [CD08, CK17, vdW20]
"stateful computations"	Haskell (programming language) [HM23, Ex. 3.6]
"open games"	game theory [GHWZ18, GKLF18]
loss functions	machine learning [FST19]
meanings (of words and sentences)	computational linguistics [SCC13]

36 *0-idempotents* should be defined as identities (Sketch 7.2.2).

1.2.2. Higher Idempotent Completions

Intuition 1.2.2. The *completion* of a mathematical objects $\mathcal{O}$ (w.r.t. a property $\mathcal{P}$) is the "smallest" mathematical object $\overline{\mathcal{O}}$ such that

- $\overline{\mathcal{O}}$ has the property $\mathcal{P}$,
- $\mathcal{O} \subseteq \overline{\mathcal{O}}$, and
- $\overline{\overline{\mathcal{O}}} = \overline{\mathcal{O}}$.[37]

Since $\overline{\mathcal{O}}$ is the "smallest" object such that $\overline{\mathcal{O}}$ has $\mathcal{P}$ and such that $\mathcal{O} \subseteq \overline{\mathcal{O}}$, we may think of this as a property that *universally* characterises $\overline{\mathcal{O}}$.

Example 1.2.3. Metric spaces M are *complete* if all Cauchy sequences in M have a limit in M. Given a metric space M we may construct a complete metric space $\overline{M}$ by taking M and adding points corresponding to limits. $\overline{M}$ is the *completion* of M fulfils a corresponding universal property (Example 4.2.2).

Interestingly, there exists a categorification of limits in category theory and even a categorification of completions $\overline{M}$. Specifically, a category $\mathcal{C}$ is called "Cauchy complete" if it "has all absolute limits". This is the case if and only if $\mathcal{C}$ is *idempotent complete* (Section A.2.1).

Idempotents $e \in \mathcal{C}(X, X)$ may *split*, which roughly means that their images exists, i.e. $\mathbf{im}(e) \in \mathcal{C}$. A category $\mathcal{C}$ is called *idempotent complete* if every idempotent $e : X \longrightarrow X$ *splits*. Given a category $\mathcal{C}$ we may construct a category $\overline{\mathcal{C}}$ by taking $\mathcal{C}$ and adding objects and morphisms. The objects one adds to $\mathcal{C}$ to construct $\overline{\mathcal{C}}$ may then be thought of as images $\mathbf{im}(e)$ of idempotents $e : X \longrightarrow X$ in $\mathcal{C}$ and the morphisms one adds are induced by e. One then finds that

- $\overline{\mathcal{C}}$ is idempotent complete,
- $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$, and
- $\overline{\overline{\mathcal{C}}} \simeq \overline{\mathcal{C}}$.

[37] In a sense, completions are a subset of idempotents, since $\overline{(\ \ )} \circ \overline{(\ \ )} = \overline{(\ \ )}$.

Moreover, since $\mathcal{C}$ is constructed by adding only those objects and morphisms that are absolutely necessary, $\overline{\mathcal{C}}$ is universal. Therefore, it is appropriate to call $\overline{\mathcal{C}}$ the *idempotent completion* of $\mathcal{C}$ (Sketch 6.1.7). Idempotent completions may by categorified even further. [CR16] introduced *equivariant completions* and *orbifold completions* of 2-categories. Later [CM23] shows that these completions are universal and introduces *orbifold completions* of 3-categories. Similarly, [DR18, Def. 1.3.1] introduce Δ-*separable Frobenius algebras with units* (Definition 5.2.4) as 2-idempotent and show that they admit a universal completion [DR18, Con. 1.3.9, Prop. A.6.4]. More generally, [GJF19, Thm. 2.3.10] provides extensive intuition towards *n-condensation completions* of n-categories. Subsequently, the universality of *2-condensation completions* has been outlined rigorously by [Dé22]. General n-idempotent completions should of course also be universal (Theorem 7.2.14) – rigorous details on n-condensation completions will be provided by [RZ]. Furthermore, there are many more notions of idempotent completions, e.g. the "Q-system completion" of [CP21, Thm. B] which is a special case of equivariant completions [CPJP22, Def. 3.1].

Remark 1.2.4. One may wonder which notions of n-idempotents could admit universal completions. This is hard to answer in full generality because the answer depends on the notion of n-idempotent and the notion of *n-idempotent splitting*. In this thesis we emphasise that the key requirement should be that n-idempotent splittings are fully determined by their n-idempotents (Remark 7.1.1).

In most of these cases, universality was not the main focus and was therefore not defined in full rigour. The exception is [Dé22], who implicitly uses the approach of [Fio06] to rigorously define 3-universal properties. Therefore, we will derive these higher universal properties in detail and show how they differ from other notions of universality (Remark 4.3.4). Moreover, while [Dé22, A.1] and [CM23, Prop. 4.16] show that 2-condensation completions and 2-orbifold completions, respectively, are universal, they understandably omit some details. Therefore, the goal of this thesis is to show in detail that condensation completions[38] of 2-categories are universal and extend this proof to orbifold completions.

[38] Meanwhile, [Kam24] had also filled in several of those details but that thesis only came to my attention towards the end of this thesis when I had finished writing up the proof. Moreover, they are also provided by [CP21, Thm. B], albeit in a less general setting. Unfortunately, the presentation of [CP21] was not accessible to me when I started this thesis. Interestingly, although their terminology was unfamiliar to me, I was able to see the connection later on, because I noticed that their string diagrams resembled mine – another good reason to use string diagrams!

Remark 1.2.5. We will be considering 2-orbifold completions in the setting of a 3-category of pivotal 2-categories $\mathbf{2Cat^{piv}}$ (Definition 3.3.9). However, other settings may also be considered. Since pivotality is motivated by the orientation of our bordisms (Remark 1.1.14), i.e. by an SO(2)-action, one may instead also construct orbifold completions of more general "O(2)-dagger categories" [CL25, Rem. 2.31].

More generally, it is thought that

- orbifold data correspond to gaugeable symmetries of oriented TQFTs,
- condensation algebras correspond to gaugeable symmetries of "framed" TQFTs, and
- analogue constructions in TQFTs with certain other types of tangential structures also correspond to gaugeable symmetries that should likewise lead to analogue higher idempotent completions [Mü25, Sec. 5.1].

1.2.3. Orbifold Completions

Previously we saw that defect TQFTs may be represented by higher categories:

$$\mathcal{Z} \longmapsto \mathcal{D}_{\mathcal{Z}}$$

Moreover, we saw that there is an orbifold construction

$$\mathcal{Z} \longmapsto \mathcal{Z}_{\mathrm{orb}}$$

that constructs orbifold defect TQFTs $\mathcal{Z}_{\mathrm{orb}}$ from defect TQFTs $\mathcal{Z}$ by enlarging the defects $\mathbb{D}$ of $\mathcal{Z}$ to $\mathbb{D}_{\mathrm{orb}}$. We may put these together to construct higher categories $\mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}}$. $\mathbb{D}_{\mathrm{orb}}$ is constructed from $\mathbb{D}$ by adding orbifold data as bulk theories and adding appropriate line and point defects. Thus, $\mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}}$ should be constructable analogously from $\mathcal{D}_{\mathcal{Z}}$. Considering that $\mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}} \simeq \mathcal{D}_{(\mathcal{Z}_{\mathrm{orb}})_{\mathrm{orb}}}$, one may suspect that this construction should even be a higher idempotent completion. For $n \leq 3$ it has indeed been shown that the *orbifold completion*

$$\mathcal{D} \longmapsto \mathcal{D}_{\mathrm{orb}},$$

exists and that

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{D}_{\mathcal{Z}} \arrow[d, mapsto] \\
\mathcal{Z}_{\mathrm{orb}} \arrow[r, mapsto] & \mathcal{D}_{\mathcal{Z}_{\mathrm{orb}}} \simeq (\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}}
\end{tikzcd} \tag{1.10}$$

commutes by construction [CR16, CM23].

1.3. Overview

Now that we have contextualised this thesis, let us glimpse what lies ahead. In short, Chapters 2 to 4 lay the necessary 2-categorical foundations and Chapter 5 explains *split (2-)idempotents*, Chapter 6 presents *(2-)idempotent completions*, and Chapter 7 summarises (2-)idempotent completions and sketches *n-idempotent completions*.

2. 2-Dimensional Category Theory As motivated, we would like to construct categories that have a second layer of morphisms. Therefore, we first introduce *strict 2-categories* as categories with this extra structure in Section 2.1. Then in Section 2.2 we derive *string diagrams*, which will be an invaluable tool for reasoning in (strict) 2-categories.[39] In Section 2.3 we fully categorify categories to *2-categories* and also introduce *2-functors*, *2-natural transformations*, *modifications*, and *2-equivalences*. Lastly, we introduce the 3-category of 2-categories **2Cat** and some *coherence theorems* in Section 2.4. In particular, the coherence theorems justify string diagrammatic reasoning in non-strict 2-categories and even in **2Cat**.

3. Adjunctions We briefly review *adjunctions*, a generalisation of equivalences. In Section 3.1 we also introduce an adjunction criterion for 1-morphisms that are left and right inverse to one another. We categorify this for 2-categories in Section 3.2. Moreover, in Section 3.3 we introduce *pivotal 2-categories*, 2-categories whose 1-morphisms have coherent left and right adjoints.

4. Universal Properties *Universal properties* enable us to characterise mathematical objects independently of explicit constructions. We start off with *coequalisers* in Section 4.1; coequalisers may be thought of as a generalisation of quotients. We continue with *universal morphisms* in Section 4.2; universal morphisms are maybe the most basic formulation of a universal property and may also be thought of as "local adjunctions". After that we categorify universal morphisms to *universal functors*, which universally characterise categories, in Section 4.3. We further categorify to *universal 2-functors*, which universally characterise 2-categories, in Section 4.4.

5. Categorifying Idempotents We start off with a discussion of *split idempotents* and *split coequalisers* in Section 5.1. In particular, we explain that *splittings* of idempotents are fully determined by

[39] Likewise, TikZiT [Kis] has been an invaluable tool to me. I used TikZiT to create all my diagrams (including the 3d-ish bordism generators in Section 1.1.2).

the idempotent they split. Split coequalisers are useful because they are *absolute*. After that we categorify split idempotents as *split algebras* in Section 5.2. Furthermore, we introduce *modules* as morphisms between algebras and *relative tensor products* as compositions of modules. Split coequalisers then enable us to explicitly construct *absolute* relative tensor products. Finally, we discuss two important forms of split algebras, namely *condensations* and *orbifold condensations*; the latter only exist in pivotal categories. In particular, we explain how condensations are fully determined by the underlying algebras they split.

6. Universal Completions We finally arrive at the heart of this thesis, *(2-)idempotent completions*. First, we show that *Karoubi envelopes* are universal *idempotent completions* in Section 6.1. In particular, we show universality as a corollary of a 2-adjunction. In Section 6.2 we show that *2-Karoubi envelopes* are universal *2-condensation completions* by completing the proof of [Dé22, A.1]. Finally, we extend this proof to *2-orbifold completions*, thereby providing details omitted by [CM23, Prop. 4.16].

7. Conclusion We summarise our discussion of (2-)idempotent completions and tentatively sketch how they may extend to *n-idempotent completions*.

A. Appendix Mainly, we briefly review basic category theory in Section A.1 to make this thesis self-contained. Additionally, we collect some remarks and sketches that may be of interest in Section A.2.

Apart from reading this thesis from beginning to end, one may also read by chapters sorted by “completion types”:

1. Foundations
2. Idempotent Completions
3. 2-Condensation Completions
4. 2-Orbifold Completions
5. Outlook on n-Idempotent Completions

The assignment of sections to “completion types” is depicted in Table 1.1 where the column of containing the blue and red boxes indicate the “completion type” of the Section. The red boxes simply highlight that the completions are the main achievements of this thesis. The differentiation between “Foundations” and “Idempotent Completions” is arguably partially arbitrary. Readers familiar with the discussed topics should of course just start reading the sections they are interested in and go back to previous definitions, statements, etc. when necessary.

Section	Foundations	Idempotent Completions	2-Condensation Completions	2-Orbifold Completions	Outlook on n-Idempotent Completions
A.1. Category Theory	■				
A.2. Supplementary Sketches					■
2.1. Strict 2-Categories	■				
2.2. String Diagrams	■				
2.3. 2-Category Theory			■		
2.4. **2Cat**			■		
3.1. Adjunctions	■				
3.2. 2-Adjunctions		■			
3.3. Pivotality				■	
4.1. Coequalisers			■		
4.2. Universal Morphisms	■				
4.3. Universal Functors		■			
4.4. Universal 2-Functors			■		
5.1.1. Split Idempotents		■			
5.1.2. Split Coequalisers			■		
5.2.1. Algebras			■		
5.2.2. Split Algebras			■		
5.2.3. Modules			■		
5.2.4. Relative Tensor Products			■		
5.2.5. Condensation			■		
5.2.6. Orbifold Condensation				■	
6.1. Idempotent Completions		■			
6.2.1. 2-Condensation Completions			■		
6.2.2. 2-Orbifold Completions				■	
7. Conclusion					■

Table 1.1: Reading by completions

1.4. Conventions

Nomenclature 1.4.1. Some of our nomenclature is displayed in Table 1.2 relative to other common nomenclature.

Notation 1.4.2. In an n-category with $1 \leq n \leq 3$ we denote

- vertical composition of n-morphisms by $\circ$,
- horizontal composition of $(n-1)$-morphisms by $\otimes$, and
- $\square$-composition of $(n-2)$-morphisms by $\square$.

This includes compositions in **Cat** and **2Cat**, e.g. we denote the composition of functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{A}$ by $G \otimes F$.

Notation 1.4.3. Given a (2-)functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and an object or morphism X in $\mathcal{C}$, then $FX := F(X)$.

Remark 1.4.4. We shall avoid the logic aspects of category theory. Instead of defining categories via sets or classes of objects, we shall define them as having *collections* of objects. The implication is that we will not be distinguishing between "small" and "large" categories. Furthermore, we will assume the axiom of choice.

Remark 1.4.5. We define functors to be *forgetful*, when they are fully faithful and canonical (Remark A.1.20).

[43] We use constant functors $\widehat{X}$ to lift objects $X \in \mathcal{C}$ to the level of functors. Likewise we lift morphisms to the level of natural transformations. This allows us to express relations of functors and natural transformations in string diagrams (Observation 2.2.4).

our terms/symbols	**other common terms/symbols**
2-category	bicategory
strict 2-category	2-category
2-functor	pseudofunctor
strict 2-functor	2-functor
2-natural transformation	pseudonatural transformation
strict 2-natural transformation	2-natural transformation
(symmetric) monoidal category	tensor category
algebra	monad, associative algebra
module	algebra
idempotent completion	Cauchy completion
$\otimes$ (horizontal composition of 2-natural transformations in **Cat**)	$\star$
$\otimes$ (composition of functors in **Cat**)	$\circ$
$X \in \mathcal{C}$	$X \in \mathrm{Ob}(\mathcal{C})$
$\mathcal{C}(X, Y)$	$\mathrm{hom}_{\mathcal{C}}(X, Y)$
$\{\star\}$ (category with single object)	$\mathbb{1}$
1_X (unit morphism)	id_X
$\mathbb{1}_a$ (non-strict unit 1-morphism)	id_a
$\widehat{X} : \{\star\} \longrightarrow \mathcal{C}$ (constant functor) [43]	$\Delta_X : \{\star\} \longrightarrow \mathcal{C}$
$\Box$ ($\Box$-composition of modifications in **2Cat**)	$\Box$
$\Box$ ($\Box$-composition of 2-natural transformations in **2Cat**)	$\star$
$\Box$ (composition of 2-functors in **2Cat**)	$\circ$
$\otimes$ (horizontal composition of modifications in **2Cat**)	$\star$
$\otimes$ (composition of 2-natural transformations in **2Cat**)	$\circ$

Table 1.2: Nomenclature

2. 2-Dimensional Category Theory

As motivated, we wish to construct categories with morphisms between morphisms. Therefore, we start off Section 2.1 by noticing that the category of categories **Cat** already has this extra structure (Example 2.1.1) since its hom-sets $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$ extend to categories $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. Therefore, we introduce *strict 2-categories*, categories whose hom-sets are themselves categories. (Note 2.1.3).

In Section 2.2 we will see that adding morphisms between morphisms may be understood as lifting a one-dimensional theory to a 2-dimensional one. In particular, we derive *string diagrams*, a graphical calculus for reasoning in (strict) 2-categories, which will become invaluable to us. Moreover, we give a glimpse of how we may use string diagrams to reason with functors and natural transformations (Observation 2.2.4 & Example 2.2.5).

In Section 2.3 we introduce *2-categories* as a generalisation of strict 2-categories whose horizontal compositions is less strict. Unfortunately, the *associators* of 2-categories cannot be expressed in our formulation of string diagrams, but this will not worry us too much since we will see how to overcome this later on (Assumption 2.3.4). Furthermore, we will categorify various other notions of category theory, e.g. *equivalences* categorify isomorphisms. In particular, we introduce 2-functors and 2-natural transformations and discuss aspects such as composition in Sections 2.3.2 and 2.3.3, respectively. Then, in Section 2.3.4 we introduce and discuss *modifications* as transformations between 2-natural transformations. Lastly, in Section 2.3.5 we discuss invertibility of 2-functors, 2-natural transformations, and modifications.

Finally, in Section 2.4 we superficially discuss **2Cat**, the 3-category of 2-categories (Sketch 2.4.1), and introduce some *coherence theorems*. The coherence theorems are vital to our work since they allow us to perform rigorous calculations in 2-categories using our invaluable string diagrams (Note 2.4.3, Notation 2.4.5).

2.1. Strict 2-Categories

Example 2.1.1. Let us motivate *strict 2-categories* (Definition 2.1.2) using an example. As we have teased, (strict) 2-categories are categories with morphisms between morphisms. As it turns out, there is a very well-known category that is actually a strict 2-category, namely, the *category of categories* **Cat** [JY21, Example 2.3.14]. As a category **Cat** consists of

Objects: categories,

Morphisms: functors,

Composition: composition of functors, and

Units: identity functors.

Given two categories $\mathcal{C}$ and $\mathcal{D}$ we may consider the category $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ consisting of

Objects: functors $\mathcal{C} \longrightarrow \mathcal{D}$,

Morphisms: natural transformations,

Composition: vertical composition of natural transformations, and

Units: identity natural transformations.

Thus,

$$\mathbf{Cat}(\mathcal{C},\mathcal{D}) = \mathrm{Ob}(\mathbf{Fun}(\mathcal{C},\mathcal{D}))$$

for all $\mathcal{C},\mathcal{D} \in \mathbf{Cat}$, i.e. the hom-sets of **Cat** themselves form categories called *Hom categories*. Moreover, we may use horizontal composition of natural transformations (Definition A.1.13) to compose Hom categories $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ and $\mathbf{Fun}(\mathcal{D},\mathcal{A})$ via the functor

$$\begin{aligned} \mathbf{Fun}(\mathcal{D},\mathcal{A}) \times \mathbf{Fun}(\mathcal{C},\mathcal{D}) &\longrightarrow \mathbf{Fun}(\mathcal{C},\mathcal{A}) \\ (G,F) &\longmapsto G \otimes F \\ (\beta,\alpha) &\longmapsto \beta \otimes \alpha. \end{aligned} \tag{2.1}$$

Thus, labelling morphisms as *1-morphisms* and morphisms of the Hom categories as *2-morphisms* we may interpret **Cat** as a strict 2-category consisting of

Objects: categories,

1-Morphisms: functors,

2-Morphisms: natural transformations,

Horizontal Composition: the functors (2.1),

Unit 1-Morphisms: identity functors, *Vertical Composition:* vertical composition of natural transformations $(\beta, \alpha) \longmapsto \beta \circ \alpha$, and

Units: identity natural transformations.

Definition 2.1.2 (Strict 2-Categories)**.** A *strict 2-category* $\mathcal{B}$ consists of

- a collection[44] of *objects* $\mathrm{Ob}(\mathcal{B})$,
- a category $\mathcal{B}(a, b)$ for each pair of objects $a, b \in \mathrm{Ob}(\mathcal{B})$,
- a functor
$$\begin{aligned} \otimes_{c,b,a} : \mathcal{B}(b, c) \times \mathcal{B}(a, b) &\longrightarrow \mathcal{B}(a, c) \\ (Y, X) &\longmapsto Y \otimes X \\ (g, f) &\longmapsto g \otimes f \end{aligned}$$
for each triple of objects $a, b, c \in \mathrm{Ob}(\mathcal{B})$, and
- an object $\mathrm{id}_a \in \mathcal{B}(a, a)$ for all objects $a \in \mathrm{Ob}(\mathcal{B})$

such that the *strict associativity* condition (2.2) and the *strict unitality* condition (2.3) are fulfilled [JY21, Def. 2.3.2 – Expl. 2.3.10].

The categories $\mathcal{B}(a, b)$ are called *Hom categories* and their objects and morphisms are called *1-morphisms* and *2-morphisms*, respectively. The 1-morphisms $\mathrm{id}_a \in \mathcal{B}(a, a)$ are called *unit 1-morphisms*. We denote 1-morphisms $X \in \mathcal{B}(a, b)$ by arrows $X : a \longrightarrow b$ and 2-morphisms $f \in \mathcal{B}(a, b)(X, Y)$ by double arrows $f : X \Longrightarrow Y$. We drop the indices of the functors $\otimes_{c,b,a}$ and call the collection of $\otimes$ either the *horizontal composition* of $\mathcal{B}$. The composition of 2-morphisms induced by the Hom categories is called *vertical composition* and will be denoted by $\circ$. Instead of writing $a \in \mathrm{Ob}(\mathcal{B})$ we will write $a \in \mathcal{B}$.

Strict Associativity: The functors
$$\otimes_{d,c,a}(\otimes_{c,b,a} \times \mathrm{id}_{\mathcal{B}(a,b)}) : \mathcal{B}(c, d) \times \mathcal{B}(b, c) \times \mathcal{B}(a, b) \longrightarrow \mathcal{B}(a, d)$$
and
$$\otimes_{d,c,b}(\mathrm{id}_{\mathcal{B}(c,d)} \otimes_{d,b,a}) : \mathcal{B}(c, d) \times \mathcal{B}(b, c) \times \mathcal{B}(a, b) \longrightarrow \mathcal{B}(a, d)$$
must be equal for all objects $a, b, c, d \in \mathcal{B}$. In other words,
$$Z \otimes (Y \otimes X) = (Z \otimes Y) \otimes X \tag{2.2}$$
for all 1-morphisms $X : a \longrightarrow b$, $Y : b \longrightarrow c$, $Z : c \longrightarrow d$ in $\mathcal{B}$.

[44] We will not be distinguishing between sets and classes (Remark 1.4.4).

Strict Unitality:

$$\otimes_{b,b,a}(\mathrm{id}_{\mathcal{B}(a,b)} \times \mathrm{id}_a) = \mathrm{id}_{\mathcal{B}(a,b)} = \otimes_{b,a,a}(\mathrm{id}_b \times \mathrm{id}_{\mathcal{B}(a,b)})$$

for all 1-morphisms $a, b \in \mathrm{Ob}(\mathcal{B})$. In other words,

$$X \otimes \mathrm{id}_a = X = \mathrm{id}_b \otimes X \tag{2.3}$$

for all 1-morphisms $X \in \mathcal{B}(a, b)$.

Note 2.1.3. Strict 2-categories are a *categorification* of categories in the sense that strict 2-categories $\mathcal{B}$ contain categories consisting of

Objects: objects of $\mathcal{B}$,

Morphisms: 1-morphisms of $\mathcal{B}$,

Composition: horizontal composition of 1-morphisms in $\mathcal{B}$, and

Units: unit 1-morphisms.

The associativity and unitality conditions (2.2) and (2.3) on 1-morphisms match those on morphisms of categories (Definition A.1.2). Therefore, strict 2-categories are categories with extra structure.

Conversely, every category $\mathcal{C}$ may be interpreted as a strict 2-category with trivial 2-morphisms, i.e. a strict 2-category consisting of

Objects: objects of $\mathcal{C}$,

Hom categories: 1-morphisms are morphisms of $\mathcal{C}$ and the only 2-morphisms are units,

Horizontal Composition: 1-morphisms are composed horizontally as they were composed in $\mathcal{C}$ and units compose trivially,

Unit 1-Morphisms: units of $\mathcal{C}$, and

Vertical Composition: units compose trivially.

Example 2.1.4. The trivial category (Example A.1.3) $\{\star\}$ is trivially also a strict 2-category. As a strict 2-category $\{\star\}$ consists of

Object: one object $\star \in \{\star\}$,

1-Morphism: one 1-morphism $\mathrm{id}_\star : \star \longrightarrow \star$,

2-Morphism: one 2-morphism $1_{\mathrm{id}_\star} : \mathrm{id}_\star \Longrightarrow \mathrm{id}_\star$,

Horizontal Composition: $\mathrm{id}_\star \otimes \mathrm{id}_\star = \mathrm{id}_\star$ and $1_{\mathrm{id}_\star} \otimes 1_{\mathrm{id}_\star} = 1_{\mathrm{id}_\star}$,

Unit 1-Morphism: $\mathrm{id}_\star$, and

Vertical Composition: $1_{\mathrm{id}_\star} \circ 1_{\mathrm{id}_\star} = 1_{\mathrm{id}_\star}$.

2.2. String Diagrams

Notation 2.2.1. We may generalise commutative diagrams to encompass the extra structure of 2-categories. Since arrows

$$(X : a \longrightarrow b) \equiv \quad \begin{tikzcd} b & a \arrow[l, "X"'] \end{tikzcd}$$

now represent 1-morphisms, we may now represent 2-morphisms between them via *pasting diagrams* [RV22, Def. B.1.1]:

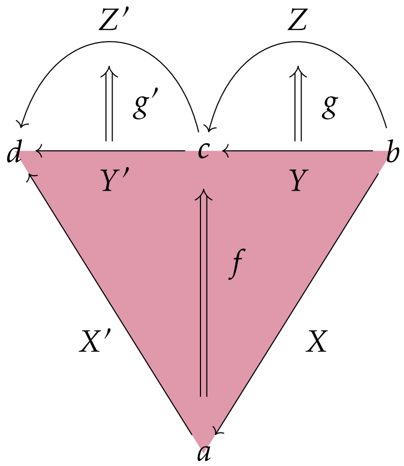
$$(f : X \longrightarrow Y) \equiv \quad \begin{tikzcd}[column sep=huge] b & a \arrow[l, bend right=50, "Y"'] \arrow[l, bend left=50, "X"] \arrow[l, phantom, "\Uparrow f"] \end{tikzcd}$$

Pasting diagrams may also represent 2-morphisms as vertical and horizontal compositions of 2-morphisms:

$$(g' \otimes g) \circ f \equiv \quad \begin{tikzcd}[column sep=large, row sep=large] d & & c \arrow[ll, "Y'"] \arrow[ll, bend right=80, "Z'"'] \arrow[ll, phantom, shift right=4ex, "\Uparrow g'"] & & b \arrow[ll, "Y"] \arrow[ll, bend right=80, "Z"'] \arrow[ll, phantom, shift right=4ex, "\Uparrow g"] \arrow[ddll, "X"] \\ & & \Uparrow f & & \\ & & a \arrow[uull, "X'"] & & \end{tikzcd}$$

Clearly pasting diagrams generalise commutative diagrams since commutative diagrams are simply pasting diagrams whose 2-morphisms are just units.

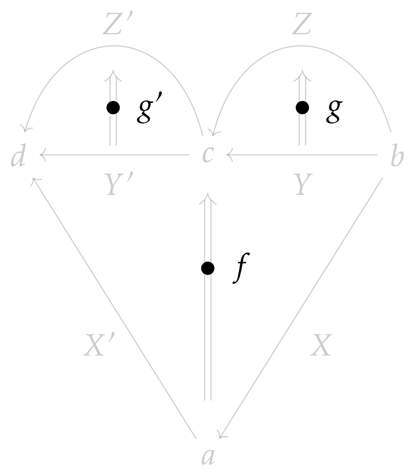
Remark 2.2.2. Pasting diagrams are convenient in the same way commutative diagrams are convenient. However, we will not use them much since their duals, *string diagrams*, are generally more legible (Derivation 2.2.3). Essentially, pasting diagrams do not utilise 2-dimensionality of (strict) 2-categories as well as string diagrams do. Their special case, commutative diagrams, do not have this issue. Categories have a composition that allows chaining together morphisms in a row and commutative diagrams enable us to visualise this well. Meanwhile, strict 2-categories have two dimensions of compositions, but pasting diagrams do not communicate this well, e.g. when one first learns about pasting diagrams, it is not necessarily immediately obvious that the whole red surface of



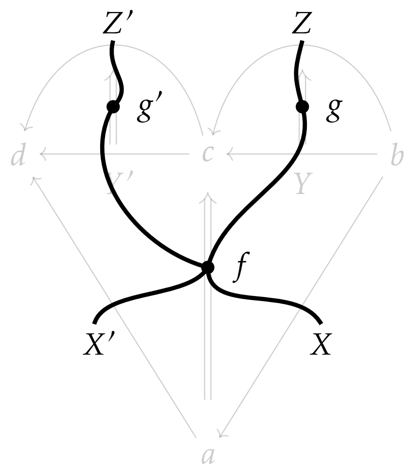
should be thought of as f.

Luckily, the 2-dimensionality is solved by string diagrams. String diagrams eliminate this problem by turning objects into surfaces, keep 1-morphisms 1-dimensional, and turn 2-morphisms into points. This is not just more satisfying cosmetically, it is in particular advantageous in terms of visual clarity since the information concerning 2-morphisms is condensed to smaller regions. This means that it takes less time to process how all the 2-morphisms are linked together which is beneficial since one is generally examining 2-morphisms rather than 1-morphisms and objects. Therefore, string diagrams will become an invaluable tool going forwards.

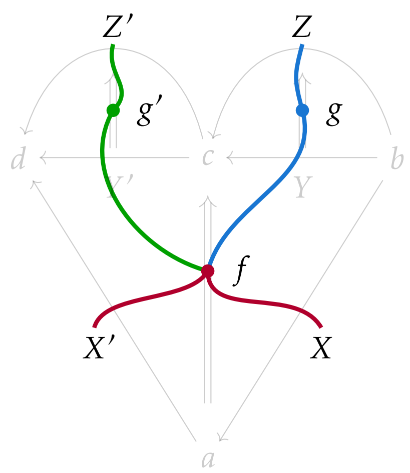
Derivation 2.2.3 (String Diagrams)**.** Given a pasting diagram, we see that it has 0-dimensional objects, 1-dimensional 1-morphisms, and 2-dimensional 2-morphisms. We dualise the pasting diagram by replacing 2-morphisms by dots:



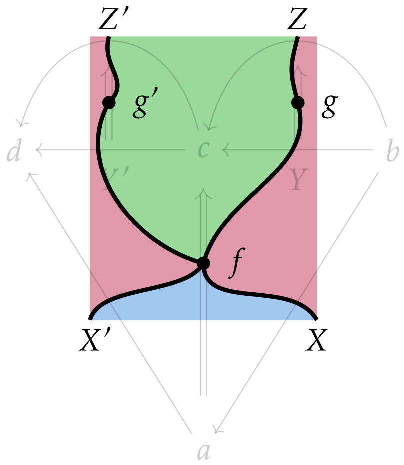
We then draw ingoing and outgoing lines from and to these dots for each 1-morphism in a 2-morphism's source and target, respectively. Connecting these appropriately yields a *string diagram* [JY21, Def. 3.7.2]



with labelled 2-morphisms and source and target 1-morphisms. If we wanted to distinguish our 1- and 2-morphisms better, we could even colour them, e.g.:



If we like, we may even colour the areas between the 1-morphisms to indicate the respective objects:



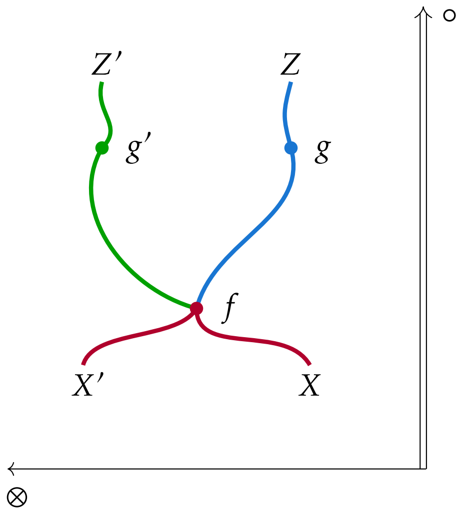
To optimise visual clarity we will usually colour the strings but omit colouring objects. Objects are already implicitly contained in diagrams as source and target of the 1-morphisms and are often not of particular interest. In those cases, where they are of interest, we colour them accordingly. Moreover, as we did in this example, we often omit labelling 1-morphisms that are not the source or target of the depicted 2-morphism. Sometimes we even omit 2-morphism labels.

The colours we use for objects, 1-morphisms, and 2-morphisms are generally chosen in a way that quickly conveys important information. For instance, usually we use green to indicate *algebras*, blue to indicate 1-morphisms, red to indicate functors and natural transformations, and black for general purposes.[45]

As a convention we fix that 1-morphisms in string diagrams are read from right to left while 2-morphisms are read from bottom to top:

[45] Specifically the colours used are #B0002C red, #00A000 green, #1A76D2 blue, #000000 black. I optimised them for accessibility, i.e. for various forms of colour blindnesses, in an iterative process. I viewed my palette and its colour blind appearance on `https://davidmathlogic.com/colorblind` and tested variations to my palette according to suggestions by ChatGPT. Unfortunately, simultaneously optimizing the desired colours for greyscale did not work out well so I prioritized true colours and accessibility.

$$(g' \otimes g) \circ f \equiv$$



The string diagrammatic calculus presented here is quite standard. There are, however, also variations. For instance, [Fra22, Rem. 3.25] adds a notation for *associators*. We will make an extension specific to our use case in Notation 5.2.56.

Observation 2.2.4 (String Diagrams in **Cat**)**.** We may express relations between functors and natural transformations using string diagrams in **Cat** [HM23, Rem. 2.1] [Mar14, p. 8]. Recalling the trivial category $\{\star\}$ (Example A.1.3), let us define *constant functors*[46]

$$\begin{aligned} \widehat{X} : \{\star\} &\longrightarrow \mathcal{C} \\ \star &\longmapsto X \\ 1_\star &\longmapsto 1_X \end{aligned}$$

for every object $X \in \mathcal{C}$. Moreover, we define *constant natural transformations* $\widehat{f} : \widehat{X} \Longrightarrow \widehat{Y}$ with sole components $(\widehat{f})_\star = f$ for every morphism $f \in \mathcal{C}(X, Y)$. We may use them to define an isomorphism

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathbf{Fun}(\star, \mathcal{C}) \\ X &\longmapsto \widehat{X} \\ f &\longmapsto \widehat{f}. \end{aligned}$$

Thus, functors $\widehat{X}$ and natural transformations $\widehat{f}$ clearly fulfil the same relations that objects $X \in \mathcal{C}$ and morphisms $f \in \mathcal{C}(X, Y)$ fulfil. Therefore, **Cat** contains all the information contained in its objects.

46 Usually constant functors are denoted as Δ_X, but we choose to write them as $\widehat{X}$ because this will make our string diagrams in **Cat** – which we will rely on heavily – more readable. Denoting them instead as $X^\star$ would seem more fitting but this seems prone to other confusions.

We note that functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ imply

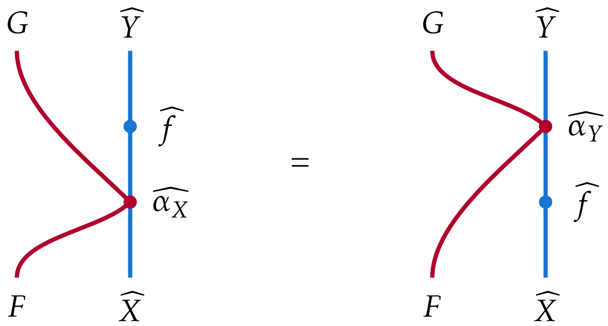
$$\widehat{FX} = F\widehat{X} \qquad \text{and} \qquad \widehat{Ff} = F\widehat{f}$$

for all objects X and morphisms f.

Example 2.2.5. Typically, one would define a natural transformation $\alpha : F \Longrightarrow G$ (Definition A.1.10) between functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ by requiring that

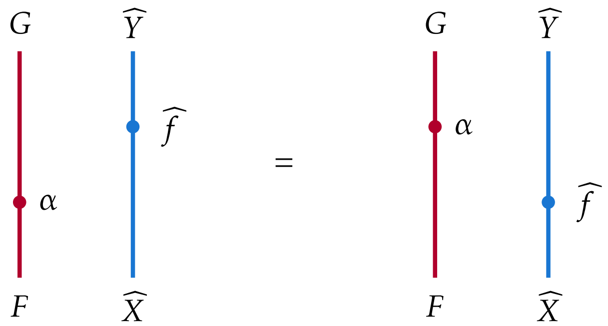
$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \tag{2.4}$$

commutes for all $X, Y \in \mathcal{C}$. In terms of string diagrams the commutative diagram (2.4) becomes[47]



. (2.5)

The naturality condition is, therefore, just the requirement that 2-morphisms on separate 1-morphisms commute:



47 We write $\widehat{\alpha_X}$ and not than $\widehat{\alpha}_X$ because we are interpreting the morphism α_X as a natural transformation $\widehat{X} \Longrightarrow \widehat{Y}$ and $\widehat{\alpha}$ is not defined.

2.3. 2-Category Theory

The philosophy of category theory is that we are not consider objects themselves, but rather only how they relate to other objects. In that spirit, we think of two objects in a category being essentially equal if they are isomorphic and think of two categories being essentially equal if they are equivalent. Therefore, we introduce *2-categories* in Section 2.3.1 as a generalisation of strict 2-categories where the conditions on the horizontal composition are more relaxed to reflect this. We categorify functors and natural transformations accordingly in Sections 2.3.2 and 2.3.3, respectively. *2-natural transformations* permit morphisms between them called *modifications* that we introduce in Section 2.3.4. Lastly, we categorify equivalences in Section 2.3.5.

The definitions and statements we shall introduce in this section are taken from [JY21]. Keep in mind that although our definitions are equivalent to those of [JY21], they may be named and presented differently. For instance, they define unit 1-morphisms $\mathbb{1}_a$ of a 2-category $\mathcal{B}$ not as 1-morphisms of $\mathcal{B}$ but rather as functors $\{\star\} \longrightarrow \mathcal{B}(a,a)$. In terms of nomenclature we will usually align with [CR16, CM23], e.g. what we will here call *2-categories*, *2-functors*, etc. are often called "bicategories", "pseudofunctors", etc. in other – especially older – literature.

2.3.1. 2-Categories

Definition 2.3.1 (2-Categories)**.** A *2-category* $\mathcal{B}$ consists of

- a collection of *objects* $\mathrm{Ob}(\mathcal{B})$,
- a category $\mathcal{B}(a,b)$ for each pair of objects $a, b \in \mathrm{Ob}(\mathcal{B})$,
- a functor
$$\begin{aligned} \otimes_{c,b,a} : \mathcal{B}(b,c) \times \mathcal{B}(a,b) &\longrightarrow \mathcal{B}(a,c) \\ (Y,X) &\longmapsto Y \otimes X \\ (g,f) &\longmapsto g \otimes f \end{aligned}$$
for each triple of objects $a, b, c \in \mathrm{Ob}(\mathcal{B})$,
- a natural isomorphism
$$\alpha^{\mathcal{B}} : \otimes_{c,b,a}(\otimes_{d,c,b} \times \mathrm{id}_{\mathcal{B}(a,b)}) \Longrightarrow \otimes_{d,c,a}(\mathrm{id}_{\mathcal{B}(c,d)} \times \otimes_{c,b,a})$$
for each quadruple of objects $a, b, c, d \in \mathcal{B}$
- a *unit 1-morphism*[48] $\mathbb{1}_a \in \mathcal{B}(a,a)$ for each object $a \in \mathrm{Ob}(\mathcal{B})$, and

[48] Note that we labelled unit morphisms of strict 2-categories by id_a instead of $\mathbb{1}_a$ to emphasise that they act as identities, i.e. $X \otimes \mathrm{id}_a = X = \mathrm{id}_b \otimes X$ (Definition 2.1.2).

- two natural isomorphisms
$$\rho : \otimes_{baa}(\mathrm{id}_{\mathcal{B}(a,b)} \times \mathbb{1}_a) \Longrightarrow \mathrm{id}_{\mathcal{B}(a,b)}$$
and
$$\lambda : \otimes_{baa}(\mathbb{1}_b \times \mathrm{id}_{\mathcal{B}(a,b)}) \Longrightarrow \mathrm{id}_{\mathcal{B}(a,b)}$$
for each pair of objects $a, b \in \mathcal{B}$

such that the *pentagon* axiom (2.7) and the *unitality* axiom (2.6) are fulfilled [JY21, Def. 2.1.3].

The categories $\mathcal{B}(a,b)$ are *Hom categories* and their objects and morphisms are called *1-* and *2-morphisms*, respectively. The 1-morphisms $X \in \mathcal{B}(a,b)$ are denoted as $X : a \longrightarrow b$ and 2-morphisms $f \in \mathcal{B}(a,b)(X,Y)$ are denoted as $f : X \Longrightarrow Y$. The functors $\otimes_{c,b,a}$ are the *horizontal composition* of $\mathcal{B}$ and we will generally drop the indices in favour of improved readability. The collection of natural transformations $\alpha^{\mathcal{B}}$ is called the *associator* of $\mathcal{B}$ and they have 2-isomorphism components
$$\alpha^{\mathcal{B}}_{Z,Y,X} : (Z \otimes Y) \otimes X \Longrightarrow Z \otimes (Y \otimes X)$$
for all 1-morphisms $X : a \longrightarrow b, Y : b \longrightarrow c, Z : c \longrightarrow d$ in $\mathcal{B}$. The collections of natural isomorphisms λ and ρ are called the *left and right unitors*, respectively, and they have 2-isomorphism components
$$\rho_X : X \otimes \mathbb{1}_a \Longrightarrow X$$
and
$$\lambda_X : \mathbb{1}_b \otimes X \Longrightarrow X$$
for all 2-morphisms $X : a \longrightarrow b$ in $\mathcal{B}$.

Unitality:

$$\begin{tikzcd}
(Y \otimes \mathbb{1}_b) \otimes X \arrow[rr, Rightarrow, "\alpha_{Y,\mathbb{1}_b,X}"] \arrow[dr, Rightarrow, "\rho_Y \otimes 1_X"'] & & Y \otimes (\mathbb{1}_b \otimes X) \arrow[dl, Rightarrow, "1_Y \otimes \lambda_X"] \\
& Y \otimes X &
\end{tikzcd} \tag{2.6}$$

must commute for all pairs of 1-morphisms $a \xrightarrow{X} b$ and $b \xrightarrow{Y} c$.

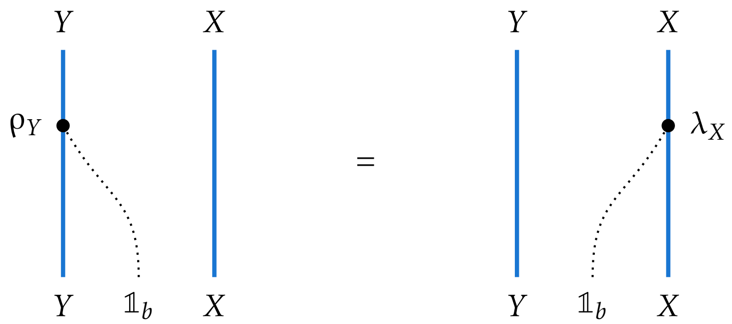
Pentagon:

$$
\begin{tikzcd}[column sep=small]
 & (W \otimes Z) \otimes (Y \otimes X) \arrow[dr, Rightarrow, "\alpha_{W,Z,Y\otimes X}"] & \\
((W \otimes Z) \otimes Y) \otimes X \arrow[ur, Rightarrow, "\alpha_{W\otimes Z,Y,X}"] \arrow[d, Rightarrow, "\alpha_{W,Z,Y} \otimes 1_X"'] & & W \otimes (Z \otimes (Y \otimes X)) \\
(W \otimes (Z \otimes Y)) \otimes X \arrow[rr, Rightarrow, "\alpha_{W,Z\otimes Y,X}"'] & & W \otimes ((Z \otimes Y) \otimes X)) \arrow[u, Rightarrow, "1_W \otimes \alpha_{Z,Y,X}"']
\end{tikzcd}
\tag{2.7}
$$

must commute for all quadruples of 1-morphisms $W : a \longrightarrow b$, $X : b \longrightarrow c$, $Y : c \longrightarrow d$, and $Z : d \longrightarrow e$ in $\mathcal{B}$.

Definition 2.3.2. If the unitors of a 2-category $\mathcal{B}$ are identities, then we say that the unit 1-morphisms of $\mathcal{B}$ are *strict*. Likewise, if the associator of $\mathcal{B}$ consists of identities, then we say the the horizontal composition of $\mathcal{B}$ is *strict*. If $\mathcal{B}$ has strict unit 1-morphisms and a strict horizontal composition, then $\mathcal{B}$ is a strict 2-category (Definition 2.1.2), i.e. we say that $\mathcal{B}$ is *strict*.

Notation 2.3.3. We shall depict unit 1-morphisms $\mathbb{1}_a$ in string diagrams via dotted lines. For instance, if the associator is strict, then we may visualise the unity condition (2.6):



(2.8)

In the following, we will usually omit the labels of the unitors.

Assumption 2.3.4. Standard string diagrams as we have introduced are not able to handle associators but we remedy this by assuming strict horizontal compositions. Generally, we may and will assume that the 2-categories we are working in are strict thanks to the *coherence theorem* (cf. Note 2.4.3). In particular, the assumption that the horizontal compositions are strict will allow us to freely utilise string diagrams for reasoning in 2-categories. The assumption that units are strict will allow us to draw and omit units in our string diagrams as we please. When we want to remind ourselves that units are part of the diagrams we will draw them, but especially when there is already a lot going on we will omit them.

Remark 2.3.5. 2-categories and strict 2-categories both categorify categories, but 2-categories should feel like they do so more naturally. While strict 2-categories are categories with extra structure (Note 2.1.3), 2-categories need not be categories. Nonetheless, since strict 2-categories are 2-categories, every category may be interpreted as a 2-category with trivial 2-morphisms (Note 2.1.3).

Analogously, the following definitions of this section are also categorifications, e.g. 2-functors between categories with trivial 2-morphisms are functors, 2-natural transformations between functors are natural transformations, and modifications between natural transformations are identities.

Definition 2.3.6. A *monoidal category* is a 2-category with only one object [JY21, Def. 1.2.1, Example 2.1.19]. Therefore, we will think of them instead as categories together with extra data given by *horizontal compositions*, *associators*, *monoidal units*, and *unitors*.

Remark 2.3.7. Monoidal categories as such are not of interest to us. However, 2-categories $\mathcal{B}$ contain monoidal categories in the form of endomorphism categories $\mathcal{B}(a,a)$. Moreover, many statements in monoidal categories trivially extend to 2-categories. Therefore, we will sometimes use monoidal categories instead of 2-categories because they involve less notational baggage.

Example 2.3.8. K-vector spaces do not just form a category $\mathbf{Vect}_K$ (Example A.1.6), but even a monoidal category $(\mathbf{Vect}_K, \otimes_K, K)$ consisting of

Objects: K-vector spaces,

Morphisms: K-linear maps,

Horizontal Composition: tensor products of K-vector spaces

$$\begin{aligned}\otimes_K : \mathbf{Vect}_K \times \mathbf{Vect}_K &\longrightarrow \mathbf{Vect}_K\\ (W, V) &\longmapsto W \otimes_K V\\ (g, f) &\longmapsto (g \otimes_K f),\end{aligned}$$

and

Monoidal Unit: $\mathbb{1}_{\mathbf{Vect}_K} := K$.

To see that $(\mathbf{Vect}_K, \otimes_K, K)$ is indeed monoidal, one may check that the associator

$$(U \otimes_K V) \otimes_K W \cong U \otimes_K (V \otimes_K W$$

and the unitors

$$V \otimes_K K \cong V \cong K \otimes_K V$$

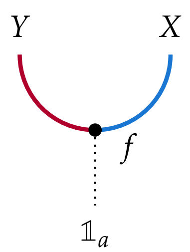
are natural.

Sketch 2.3.9. *Sub-2-categories* are intuitively clear but slightly awkward to define so we shall be a bit hand-wavy here. Given two 2-categories $\mathcal{B}$ and $\mathcal{D}$ we say that $\mathcal{D}$ is a *sub-2-category* $\mathcal{D} \subseteq \mathcal{B}$ of $\mathcal{B}$ if objects, 1-morphisms, unit 1-morphisms, and 2-morphisms of $\mathcal{D}$ are objects, 1-morphisms, unit 1-morphisms, and 2-morphisms of $\mathcal{B}$, $\otimes^{\mathcal{D}}$ factors through $\otimes^{\mathcal{B}}$, and the components of the associator and unitors of $\mathcal{D}$ correspond to the components of the associator and unitors of $\mathcal{B}$ [JY21, Def. 2.1.15].

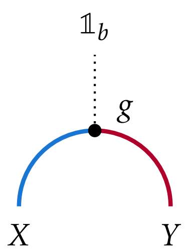
$\mathcal{D}$ is a *full 2-subcategory* of $\mathcal{B}$ if $\mathcal{D}$ inherits Hom categories of $\mathcal{B}$ completely, i.e. if $\mathcal{D}(a, b) = \mathcal{B}(a, b)$ for all pairs of objects $a, b \in \mathcal{D}$.

Definition 2.3.10. If we are given a property $\mathcal{P}$ that categories may have, then a 2-category $\mathcal{B}$ has $\mathcal{P}$ *locally* if all Hom categories of $\mathcal{B}$ have $\mathcal{P}$ [JY21, Def. 2.1.14].

Definition 2.3.11. Two objects $a, b \in \mathcal{B}$ are *equivalent* [JY21, Def. 5.1.18] if there exist 1-morphisms $X : a \rightleftarrows b : Y$ together with 2-isomorphisms[49]

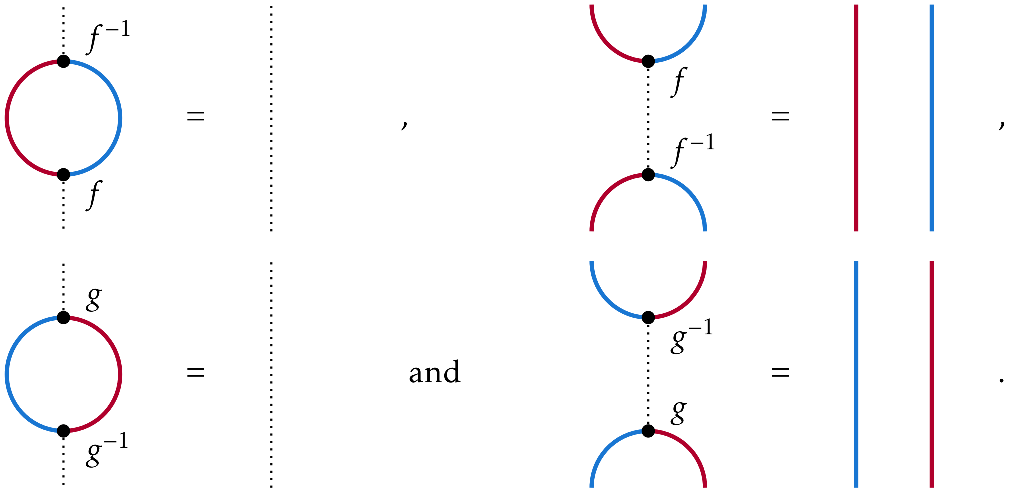


and



[49] Since they are isomorphisms we could of course define either f or g to point in the other direction to make them look more similar. However, this presentation is useful because it is in line with the form of adjunctions (Definition 3.1.2).

such that



We also say that (X, Y, f, g) *forms an equivalence* $a \simeq b$. Moreover, X and Y are *weak inverses* of one another.

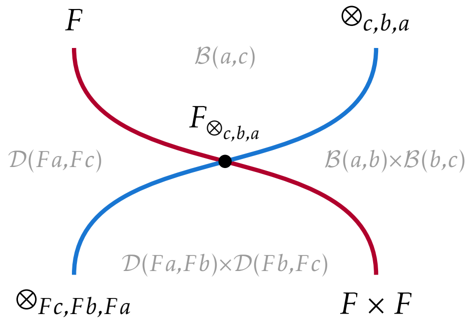
2.3.2. 2-Functors

Definition 2.3.12 (2-Functors)**.** A *2-functor* $F : \mathcal{B} \longrightarrow \mathcal{D}$ is a map

$$\begin{aligned} F : \mathcal{B} &\longrightarrow \mathcal{D} \\ a &\longmapsto Fa \\ X &\longmapsto FX \\ f &\longmapsto Ff \end{aligned}$$

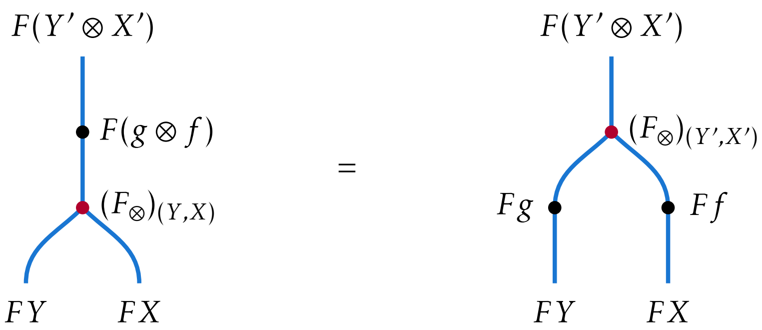
consisting of

- a function
$$\begin{aligned} F^{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{B}) &\longrightarrow \mathrm{Ob}(\mathcal{D}) \\ a &\longrightarrow Fa, \end{aligned}$$
- a functor
$$\begin{aligned} F_{a,b} : \mathcal{B}(a,b) &\longrightarrow \mathcal{D}(Fa, Fb) \\ X &\longmapsto FX \\ f &\longmapsto Ff \end{aligned}$$
for each $a, b \in \mathcal{B}$,
- a 2-isomorphism $F_{\mathbb{1}_a} : F\mathbb{1}_a \Longrightarrow \mathbb{1}_{Fa}$ for each object $a \in \mathcal{B}$, and
- a natural isomorphism



for each triple of objects $a, b, c \in \mathcal{B}$

such that the *associativity* condition (2.9) and the *unity* conditions (2.10) are fulfilled [JY21, Def. 4.1.2 & Def. 4.1.4]. As usual, we drop the indices of $\otimes$ from $F_{\otimes_{c,b,a}}$. Naturality of $F_\otimes$ means that



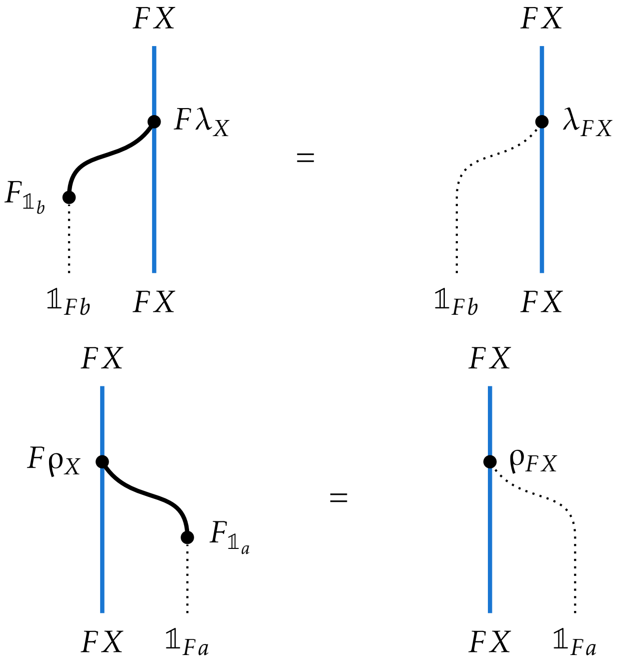
for all 2-morphisms $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ in $\mathcal{B}$.

Associativity:

$$
\begin{tikzcd}
 & (FZ\otimes FY)\otimes FX \arrow[r, Rightarrow, "\alpha^{\mathcal{D}}"] \arrow[dl, Rightarrow, "F_{\otimes_{b,c,d}}\otimes 1_{FX}"'] & FZ\otimes(FY\otimes FX) \arrow[dr, Rightarrow, "1_{FZ}\otimes F_{\otimes_{a,b,c}}"] & \\
F(Z\otimes Y)\otimes FX \arrow[dr, Rightarrow, "F_{\otimes_{a,b,d}}"'] & & & FZ\otimes F(Y\otimes X) \arrow[dl, Rightarrow, "F_{\otimes_{a,c,d}}"] \\
 & F((Z\otimes Y)\otimes X) \arrow[r, Rightarrow, "F\alpha^{\mathcal{B}}"'] & F(Z\otimes(Y\otimes X)) &
\end{tikzcd}
\tag{2.9}
$$

must commute for all triples $X : a \longrightarrow b$, $Y : b \longrightarrow c$, $Z : c \longrightarrow d$ in $\mathcal{B}$.

Unity:



(2.10)

must hold for all 1-morphisms $X : a \longrightarrow b$ in $\mathcal{B}$.

Notation 2.3.13. We will describe how 2-functors $F : \mathcal{B} \longrightarrow \mathcal{D}$ map objects, 1-morphisms, and 2-morphisms of $\mathcal{B}$ either as

$$\begin{aligned} F : \mathcal{B} &\longrightarrow \mathcal{D} \\ a &\longmapsto Fa \\ X &\longmapsto FX \\ f &\longmapsto Ff \end{aligned}$$

or as

$$\begin{aligned} F : \mathcal{B} &\longrightarrow \mathcal{D} \\ a &\longmapsto Fa \\ \mathcal{B}(a,b) &\longrightarrow \mathcal{D}(Fa, Fb). \end{aligned}$$

The latter allows us to emphasise the structure of the Hom categories $\mathcal{D}(Fa, Fb)$ when the functor $\mathcal{B}(a,b) \longrightarrow \mathcal{D}(Fa, Fb)$ is clear from the context.

Definition 2.3.14. A 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ is *strict* if $\mathcal{B}, \mathcal{D}$ are strict[50] and $F_{\mathbb{1}_a}$ and $F_{\otimes_{a,b,c}}$ are identities and identity natural transformations, respectively, for all $a, b, c \in \mathcal{B}$.

[50] As usual our nomenclature differs from [JY21]. What [JY21, Def. 4.1.4] defines as a "2-functor" is what we call a strict 2-functor.

Note 2.3.15. The 2-functoriality conditions (2.9) and (2.10) of strict 2-functors simplify to

- $F\mathbb{1}_a = \mathbb{1}_{Fa}$ for all objects $a \in \mathcal{B}$ and
- $FY \otimes FX = F(Y \otimes X)$ for all composable 1-morphisms X, Y in $\mathcal{B}$.

Definition 2.3.16. Given 2-functors $F = ((F_{a,b})_{a,b\in\mathcal{B}}, F_\otimes, F_\mathbb{1}) : \mathcal{B} \longrightarrow \mathcal{B}'$ and $F' = ((F'_{a,b})_{a,b\in\mathcal{B}'}, F'_\otimes, F'_\mathbb{1}) : \mathcal{B}' \longrightarrow \mathcal{B}''$, then we may define a 2-functor

$$F' \,\square\, F := ((F'_{Fa,Fb} \otimes F_{a,b})_{a,b\in\mathcal{B}}, F'_\otimes \circ F_\otimes, F'_\mathbb{1} \circ F_\mathbb{1})$$

that we call the *composite* of F and G [JY21, Def. 4.1.26 & Lemma 4.1.29].

Example 2.3.17. For every 2-category $\mathcal{B}$ the *identity 2-functor*

$$\begin{aligned} \mathrm{id}_\mathcal{B} : \mathcal{B} &\longrightarrow \mathcal{B} \\ a &\longmapsto a \\ X &\longmapsto X \\ f &\longmapsto f \end{aligned}$$

exists. Its components $F_\mathbb{1}$ and $F_\otimes$ are identities and identity natural transformations, respectively.

Note 2.3.18. Composition of 2-functors is *strictly associative* and *strictly unital*, i.e.

$$(H \,\square\, G) \,\square\, F = H \,\square\, (G \,\square\, F)$$

and

$$\mathrm{id}_D \,\square\, F = F = F \,\square\, \mathrm{id}_\mathcal{B}$$

for all 2-functors $F : \mathcal{B} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{A}$, and $H : \mathcal{A} \longrightarrow \mathcal{C}$ [JY21, Thm. 4.1.30].

Definition 2.3.19. A *monoidal functor* is a 2-functor between monoidal categories. We will again think of them as functors with extra data but also extra conditions [JY21, Def. 1.2.14 & Example 4.1.13].

Example 2.3.20. Given a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ and an object $X \in \mathcal{C}$, then

$$\begin{aligned}(X \otimes (-)) : \mathcal{C} &\longrightarrow \mathcal{C} \\ Y &\longmapsto X \otimes Y \\ f &\longmapsto 1_X \otimes f\end{aligned}$$

is a functor. In general, this functor is not monoidal. It is a functor because the horizontal composition is a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which we may equivalently view as a functor mapping objects to functors, i.e. $\otimes : \mathcal{C} \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathcal{C})$. In that case, $(X \otimes (-))$ is the evaluation of $\otimes : \mathcal{C} \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathcal{C})$ at X.

Note 2.3.21. If $\mathcal{B} \subseteq \mathcal{D}$ then there exists a canonical "embedding" 2-functor:

$$\begin{aligned}\mathcal{B} &\hookrightarrow \mathcal{D} \\ a &\longmapsto a \\ X &\longmapsto X \\ f &\longmapsto f\end{aligned}$$

It is *fully faithful* (Definition 2.3.42) if and only if $\mathcal{B}$ is a full sub-2-category of $\mathcal{D}$.

Definition 2.3.22. The *image* $\mathbf{im}\, F$ of a 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ is the smallest sub-2-category of $\mathcal{D}$ containing all the objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$ that F maps to. In other words, $\mathbf{im}\, F$ is the sub-2-category of $\mathcal{D}$ containing the objects, 1-morphisms, and 2-morphisms that F maps to and also their composites.

Observation 2.3.23. Every 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ factors as

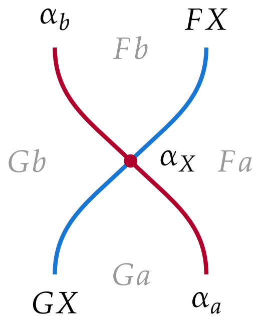
$$F = (\mathcal{B} \xrightarrow{p_F} \mathbf{im}\, F \xhookrightarrow{i_F} \mathcal{D}).$$

2.3.3. 2-Natural Transformations

Definition 2.3.24 (2-Natural Transformations)**.** Let $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ be 2-functors. A *2-natural transformation* $\alpha : F \Rightarrow G$ consists of

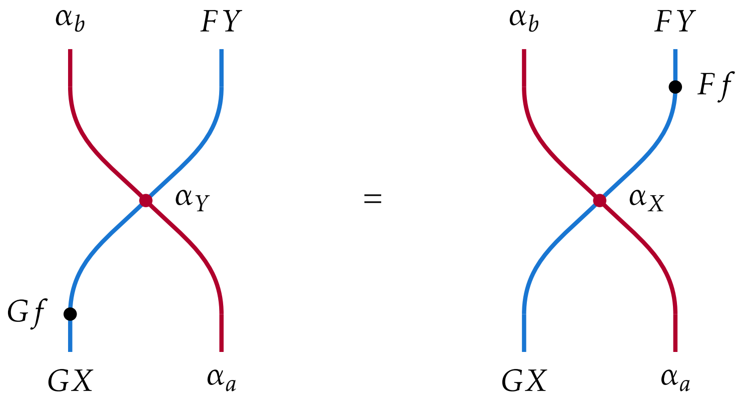
- a 1-morphism $\alpha_a : Fa \longrightarrow Ga$ in $\mathcal{D}$ for each object $a \in \mathcal{B}$ and

- a natural isomorphism $\alpha_{(a,b)} = (\alpha_X)_{X \in \mathcal{B}(a,b)}$ with components



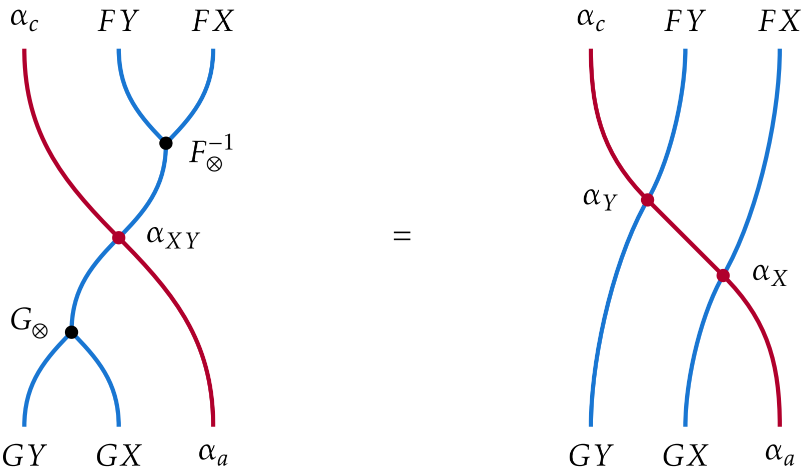
for each category $\mathcal{B}(a,b)$,

such that the *2-naturality* condition (2.3.24) and *unity* condition (2.3.24) are fulfilled [JY21, Def. 4.2.1 & 4.2.3]. Naturality of $\alpha_{a,b}$ is meant concerning the horizontal composition, i.e.



for all 1-morphisms $X, Y : a \longrightarrow b$ and all 2-morphism $f : X \longrightarrow Y$.

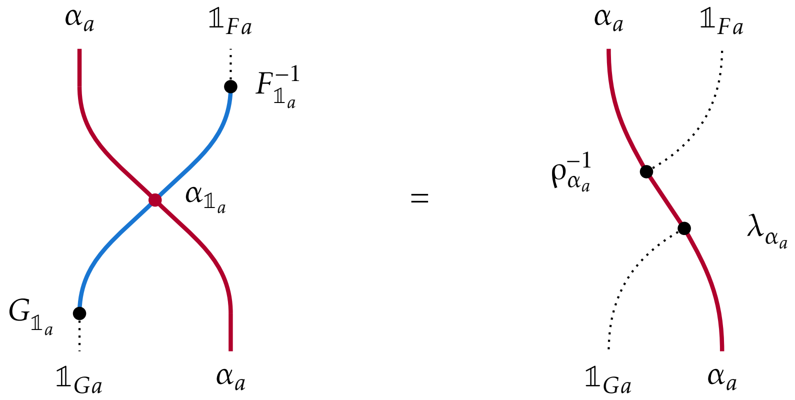
2-Naturality:



(2.11)

for all objects $a, b, c \in \mathcal{B}$ and all 1-morphisms $X : a \longrightarrow b$, $Y : b \longrightarrow c$ in $\mathcal{B}$.

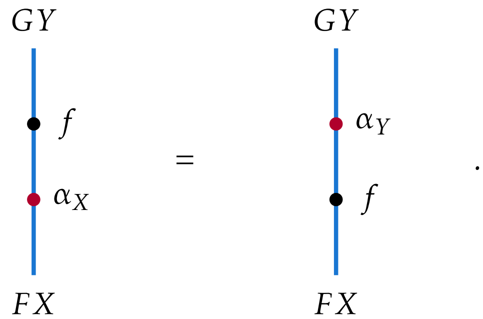
Unity:



(2.12)

for all objects $a \in \mathcal{B}$.

Observation 2.3.25. Let us recall that natural transformations are morphisms that may be "moved past" other morphisms, i.e. if we are given functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a natural transformation $\alpha : F \Longrightarrow G$, and a 2-morphism $f : X \Longrightarrow Y$, then



Similarly, 2-natural transformations may be "moved past" 1-morphisms, i.e. 2-natural transformations may be thought of as morphisms that we may "cross over" other 1-morphisms and "move past" 2-morphisms at will as long as they "end up" in the right place. To make this more precise, let us assume we are given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$, a 2-natural transformation $\alpha : F \Longrightarrow G$, and a string diagram in $\mathcal{B}$. Now, we may create a string diagram in $\mathcal{D}$ by following these steps:

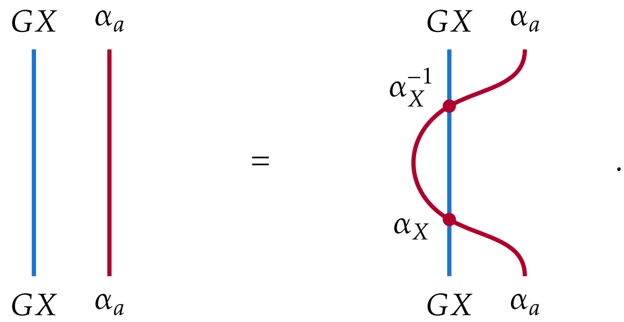
1. Choose a 1-morphism $X : a \longrightarrow b$ in the source and add the 1-morphism α_a to the right of X or instead add α_b to the left of X. Analogously, choose a 1-morphism $Y : c \longrightarrow d$ in the target and add a 1-morphism α_c or α_d. W.l.o.g. let us assume we chose the right side of X and Y, i.e. that we added 1-morphisms α_a and α_c.
2. Draw an arbitrary string from α_a to α_c. This string may cross other 1-morphisms but not 2-morphisms. Draw dots on the intersections between this new string and 1-morphisms W and label them by α_W if the new string crosses W from the right and label intersection by α_W^{-1} otherwise.

3. Relabel all 1-morphisms Y to the right of the new string as FY and all 1-morphisms Z to the left as GZ. Relabel their 2-morphisms f as Ff or Gf accordingly.

In particular, if we are given such a string diagram with α_a connected to α_b, then we may create a new string diagram that is equal to the given one by following these steps:

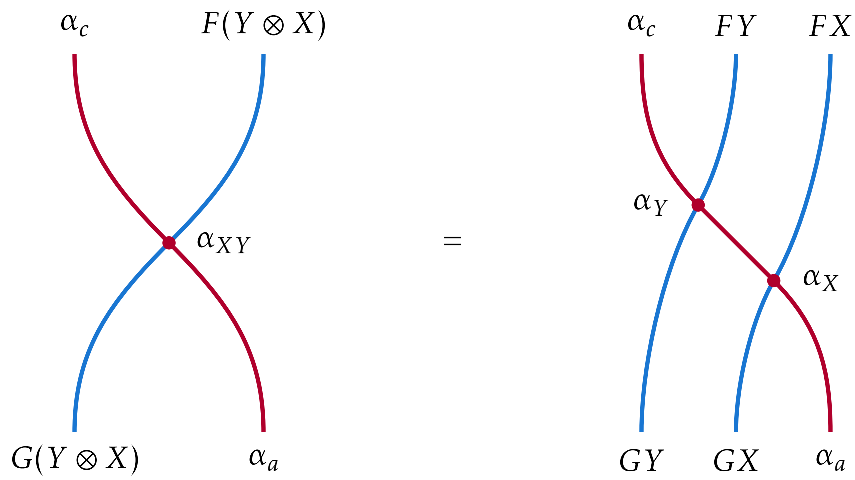
1. draw a new string from α_a to α_b,
2. relabel the morphisms between the new string and its predecessor, and
3. erase the old string.

As a short example, this provides us with the identity



Definition 2.3.26. A 2-natural transformation $\alpha : F \Longrightarrow G$ is *strict* if F and G are strict and α is a natural transformation, i.e. if α's 2-morphism components are identities [JY21, Def. 4.2.3].

Note 2.3.27. If we are given strict 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ and a 2-natural transformation $\alpha : F \Longrightarrow G$, then α's 2-naturality condition (2.3.24) simplifies slightly to



and α's unity condition (2.3.24) simplifies nicely to

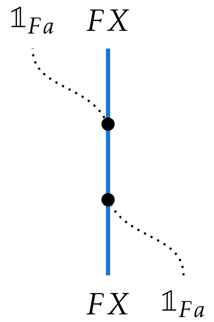
$$\alpha_{\mathbb{1}_a} = 1_{\alpha_a}.$$

Example 2.3.28. For every 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ there is a 2-natural transformation, the *identity 2-transformation* $\mathbb{1}_F : F \Longrightarrow F$, with components [JY21, Prop. 4.2.12]

$$(\mathbb{1}_F)_a \coloneqq \mathbb{1}_{Fa}$$

and

$$(\mathbb{1}_F)_X \coloneqq$$



.

If F is strict, then $\mathbb{1}_F$ is strict since $\mathcal{D}$ is strict for strict F (Definition 2.3.14).

Definition 2.3.29. If we are given two 2-natural transformations

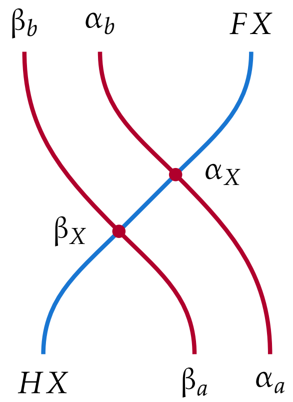
$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{D} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Uparrow\beta"] & \mathcal{D} \end{tikzcd},$$

then their *horizontal composite* [JY21, Def. 4.2.15 & Lemma 4.2.19] is the 2-natural transformation $\beta \otimes \alpha : F \Longrightarrow H$ with components

$$(\beta \otimes \alpha)_a \coloneqq \beta_a \otimes \alpha_a$$

and

$$(\beta \otimes \alpha)_X \coloneqq$$



. (2.13)

Note 2.3.30. Horizontal composition of 2-natural transformations is neither strictly associative nor strictly unital; i.e.

$$(\gamma \otimes \beta) \otimes \alpha \neq \gamma \otimes (\beta \otimes \alpha)$$

and

$$\mathbb{1}_G \otimes \alpha \neq \alpha \neq \alpha \otimes \mathbb{1}_F$$

is possible for 2-natural transformations $\alpha : F \longrightarrow G$, $\beta : G \longrightarrow H$, and $\gamma : H \longrightarrow I$ [JY21, p. 121]. This is also why we denote identity 2-transformations by $\mathbb{1}_F$ rather than id_F.

However, if $\mathcal{D}$ is strict, then

$$\mathbb{1}_G \otimes \alpha = \alpha = \alpha \otimes \mathbb{1}_F$$

is clearly fulfilled. Moreover, inspection of diagram (2.13) reveals that

$$(\gamma \otimes \beta) \otimes \alpha = \gamma \otimes (\beta \otimes \alpha)$$

for strict $\mathcal{D}$. In such cases we may denote identity 2-transformations $\mathbb{1}_F$ by id_F.

Definition 2.3.31.

- If we are given 2-functors

$$B \xrightarrow{F} \mathcal{D} \qquad \text{and} \qquad \mathcal{A} \xrightarrow{H} \mathcal{C}$$

and a 2-natural transformation

$$\begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "G'"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{A} \end{tikzcd} \quad ,$$

then we may define *pre-whiskering* [JY21, Def. 11.1.1 & Lemma 11.1.5] of α with F as the 2-natural transformation

$$\alpha \,\square\, F \equiv \quad \begin{tikzcd} B \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left=50, "G'"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{A} \end{tikzcd}$$

with 1-morphism components

$$(\alpha \,\square\, F)_a \coloneqq \alpha_{Fa}$$

and 2-morphism components

$$(\alpha \,\square\, F)_X \coloneqq \alpha_{FX}. \tag{2.14}$$

- Furthermore, we may define *post-whiskering* [JY21, Def. 11.1.2 & Lemma 11.1.6] of α with H as the 2-natural transformation

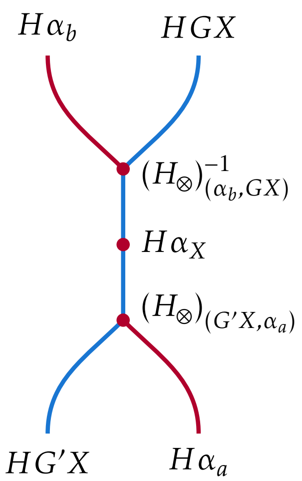
$$H \square \alpha \equiv \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "G'"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow \alpha"] & \mathcal{A} \arrow[r, "H"] & \mathcal{C} \end{tikzcd}$$

with 1-morphism components

$$(H \square \alpha)_a := H\alpha_a$$

and 2-morphism components

$$(H \square \alpha)_X :=$$



. (2.15)

- If we are given 2-natural transformations

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F'"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow \alpha"] & \mathcal{D} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "G'"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow \beta"] & \mathcal{A} \end{tikzcd} ,$$

then their *composite*[51] [JY21, Explanation 11.3.5]

[51] We note that we made a choice here since, in general, $(G' \square \alpha) \circ (\beta \square F) \neq (\beta \square F) \circ (G' \square \alpha)$.

$$\beta \,\square\, \alpha := (G' \,\square\, \alpha) \circ (\beta \,\square\, F) \equiv \left(\begin{tikzcd} \mathcal{B} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G'"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\beta"] & \mathcal{A} \end{tikzcd} \right)$$

$$\circ \left(\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F'"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{D} \arrow[r, "G'"] & \mathcal{A} \end{tikzcd} \right)$$

has 1-morphism components

$$\begin{aligned} (\beta \,\square\, \alpha)_a &= (G' \,\square\, \alpha)_a \otimes (\beta \,\square\, F)_a \\ &= (G'\alpha_a) \otimes \beta_{Fa} \end{aligned}$$

and 2-morphism components

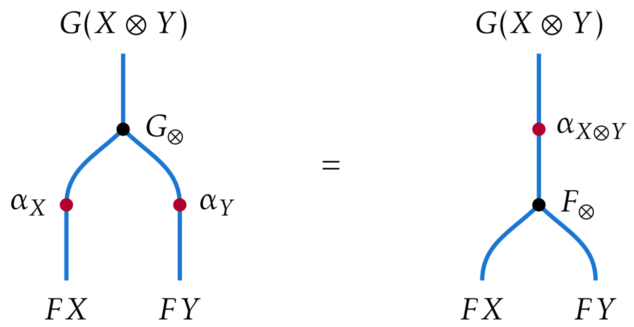
$$\begin{aligned} (\beta \,\square\, \alpha)_X &= (G' \,\square\, \alpha)_X \otimes (\beta \,\square\, F)_X \\ &= (G'\alpha_X) \otimes \beta_{FX}. \end{aligned} \tag{2.16}$$

Note 2.3.32. Horizontal compositions of strict 2-natural transformations are strict; cf. (2.13). This would not be true if our definition of strict 2-natural transformations did not require strict 2-functors and strict 2-categories [JY21, Rem. 4.2.20]. Likewise, whiskering a 2-functor with a strict 2-natural transformation produces a strict 2-natural transformation; cf. (2.14) and (2.15). Thus, the composite of strict 2-natural transformations is strict, too.

Remark 2.3.33. Monoidal natural transformations are *icons* [JY21, Def. 1.2.20, Ex. 4.6.10]. They are a type of 2-natural transformation whose 1-morphism components are unit 1-morphisms and whose 2-morphism components need not be invertible but rather suffice to be any 2-morphisms $\alpha_X : FX \Longrightarrow GX$. The *monoidal naturality* condition (2.17) and the *monoidal unity* condition (2.18) may therefore be retrieved from the 2-naturality condition (2.3.24) and the unity condition (2.3.24).

A *monoidal natural transformation* is a natural transformation $\alpha : F \Longrightarrow G$ between monoidal functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ that fulfils the following conditions.

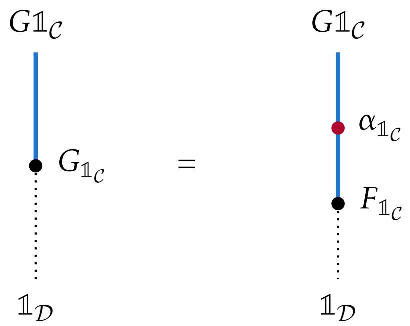
Monoidal Naturality:



(2.17)

for all objects $X, Y \in \mathcal{C}$.

Monoidal Unity:



(2.18)

2.3.4. Modifications

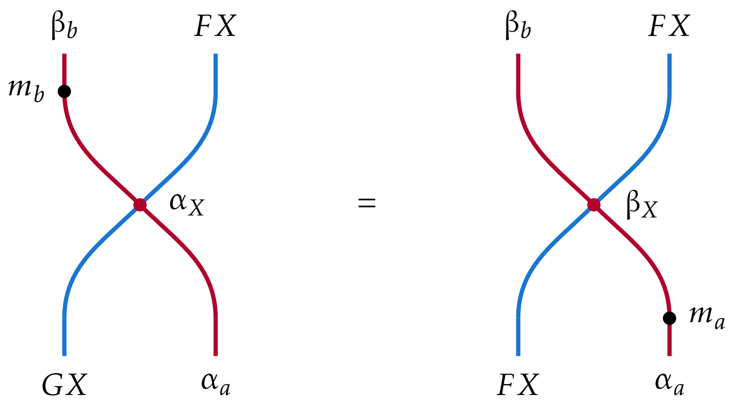
Definition 2.3.34 (Modifications)**.** Given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ and 2-natural transformations $\alpha, \beta : F \Longrightarrow G$, then a *modification*

$$m : \alpha \Rrightarrow \beta$$

consists of a collection of 2-morphisms

$$m_a : \alpha_a \Longrightarrow \beta_a$$

in $\mathcal{D}$ indexed by the objects of $\mathcal{B}$. The relation



(2.19)

must be fulfilled for all objects $a, b \in \mathcal{B}$ and all morphisms $X : a \longrightarrow b$ in $\mathcal{B}$ [JY21, Def. 4.4.1].

Example 2.3.35. Given a 2-natural transformation

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Uparrow\alpha"] & \mathcal{D} \end{tikzcd} \quad ,$$

then the *identity modification* $\mathrm{id}_\alpha : \alpha \Rrightarrow \alpha$ exists [JY21, Def. 4.4.5]. It has components

$$(\mathrm{id}_\alpha)_a = 1_{\alpha_a}$$

for each object $a \in \mathcal{B}$.

Note 2.3.36. We may extend our definition of pasting diagrams (Notation 2.2.1) to incorporate modifications naturally. For instance, given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ and 2-natural transformations $\alpha, \beta : F \Longrightarrow G$, then we may depict modifications $m : \alpha \Rrightarrow \beta$ as

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left=60, "G"] \arrow[r, bend right=60, "F"'] \arrow[r, phantom, "\alpha\Uparrow \overset{m}{\Rrightarrow} \Uparrow\beta"] & \mathcal{D} \end{tikzcd} \quad .$$

Definition 2.3.37.

- If we are given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ and modifications

$$\begin{tikzcd} F \arrow[r, bend left=50, "\beta"] \arrow[r, bend right=50, "\alpha"'] \arrow[r, phantom, "\Uparrow m"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, bend left=50, "\gamma"] \arrow[r, bend right=50, "\beta"'] \arrow[r, phantom, "\Uparrow n"] & G \end{tikzcd} \quad ,$$

  then their *vertical composite* [JY21, Def. 4.4.5] is the modification

$$n \circ m := \quad \begin{tikzcd} F \arrow[rr, bend left=60, "\gamma"] \arrow[rr, bend right=60, "\alpha"'] \arrow[rr, "\beta"] \arrow[rr, phantom, shift left=3ex, "\Uparrow n"] \arrow[rr, phantom, shift right=3ex, "\Uparrow m"] & & G \end{tikzcd}$$

with components

$$(n \circ m)_a := n_a \circ m_a$$

for objects $a \in \mathcal{B}$.

- If we are given 2-functors $F, G, H : \mathcal{B} \longrightarrow \mathcal{D}$ and modifications

$$\begin{tikzcd} F \arrow[r, bend left, "\alpha'"] \arrow[r, bend right, "\alpha"'] \arrow[r, phantom, "\Uparrow m"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, bend left, "\beta'"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Uparrow n"] & H \end{tikzcd} ,$$

then their *horizontal composite* [JY21, Def. 4.4.6] is the modification

$$n \otimes m := \begin{tikzcd} F \arrow[r, bend left, "\alpha'"] \arrow[r, bend right, "\alpha"'] \arrow[r, phantom, "\Uparrow m"] & G \arrow[r, bend left, "\beta'"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Uparrow n"] & H \end{tikzcd}$$

with components

$$(n \otimes m)_a = n_a \otimes m_a$$

for objects $a \in \mathcal{B}$.

- If we are given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ and $F', G' : \mathcal{D} \longrightarrow \mathcal{A}$ and modifications

$$\begin{tikzcd} F \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] \arrow[r, phantom, "\Uparrow m"] & G \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F' \arrow[r, bend left, "\beta'"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\Uparrow m'"] & G' \end{tikzcd} ,$$

then their *composite* [JY21, Def. 11.3.4]

$$m' \,\square\, m := \begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha \Uparrow \overset{m}{\Rrightarrow} \Uparrow \beta"] & \mathcal{D} \arrow[r, bend left, "G'"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\alpha' \Uparrow \overset{m'}{\Rrightarrow} \Uparrow \beta'"] & \mathcal{A} \end{tikzcd}$$

has components[52]

$$(m' \,\square\, m)_a := (G' m_X) \otimes m'_{Fa}.$$

[52] Compare to 2-morphism components of composites of 2-natural transformations (2.16).

Note 2.3.38. Vertical composition of modifications is *strictly associative* and *strictly unital*, i.e.

$$(n \circ m) \circ l = n \circ (m \circ l)$$

and

$$\mathrm{id}_\beta \circ m = m = m \circ \mathrm{id}_\alpha$$

for all modifications $l : \alpha \Rrightarrow \beta$, $m : \beta \Rrightarrow \gamma$, and $n : \gamma \Rrightarrow \delta$ [JY21, Lemma 4.4.8].

2.3.5. 2-Equivalences

Definition 2.3.39. A modification $m : \alpha \Rrightarrow \beta$ is *invertible* if there exists a modification $n : \beta \Rrightarrow \alpha$ such that

$$n \circ m = \mathrm{id}_\alpha$$
$$m \circ n = \mathrm{id}_\beta.$$

In that case there is an *isomorphism* $\alpha \cong \beta$ [JY21, Def. 4.4.2 & Lemma 4.4.9].

Two 2-natural transformations $\alpha : F \Longrightarrow G$ and $\beta : G \Longrightarrow F$ form a *2-natural equivalence* $F \simeq G$ if there are invertible modifications

$$\beta \otimes \alpha \Rrightarrow \mathbb{1}_F$$
$$\alpha \otimes \beta \Rrightarrow \mathbb{1}_G.$$

We also say that α and β are *2-natural equivalences* and that they are *weak inverses* of one another [JY21, Def. 6.2.2].

Two 2-functors $F : \mathcal{B} \rightleftarrows \mathcal{D} : G$ form a *2-equivalence* $\mathcal{B} \simeq \mathcal{D}$ if there are 2-natural equivalences

$$G \mathbin{\square} F \Longrightarrow \mathrm{id}_{\mathcal{B}}$$
$$F \mathbin{\square} G \Longrightarrow \mathrm{id}_{\mathcal{D}}.$$

We also say that F and G are *2-natural equivalences* and that they are *weak inverses* of one another [JY21, Def. 6.2.8].

Note 2.3.40. A modification is invertible if and only if its components are all 2-isomorphisms.

Note 2.3.41. A 2-natural transformation $\alpha : F \Longrightarrow G$ is a 2-natural equivalence if and only if its 1-morphism components $\alpha_a : Fa \longrightarrow Ga$ are all equivalences (Definition 2.3.11).

Definition 2.3.42. A 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ is

- *faithful* if it is fully faithful on Hom categories, i.e. if
$$B(a,b)(X,Y) \cong \mathcal{D}(Fa,Fb)(FX,FY)$$
for all $X, Y : a \longrightarrow b$ in $\mathcal{B}$,
- *essentially full*, if it is essentially surjective on Hom categories, i.e. if for every pair of objects $a, b \in \mathcal{B}$ and every 1-morphism $D : Fa \longrightarrow Fb$ in $\mathcal{D}$ there exists a 1-morphism $B : a \longrightarrow b$ in $\mathcal{B}$ such that $FB \cong D$,
- *fully faithful* if it is essentially full and faithful, i.e. if it is an equivalence on Hom categories, i.e. $\mathcal{B}(a,b) \simeq \mathcal{D}(Fa,Fb)$, and
- *essentially surjective* if for every object $d \in \mathcal{D}$ there exists an object $b \in \mathcal{B}$ such that $Fb \simeq d$.

Definition 2.3.43. We shall say that canonical[53] fully faithful 2-functors are *forgetful* (Remark 1.4.5).

Theorem 2.3.44. *A 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ is a 2-equivalence if and only if F is*

- *essentially surjective,*
- *essentially full, and*
- *faithful.*

If $F : \mathcal{B} \longrightarrow \mathcal{D}$ is a strict 2-functor, then F is a 2-equivalence if F is

- *essentially surjective as a functor on objects and 1-morphisms,*
- *bijective on 1-morphisms, and*
- *faithful.*

Proof. See [JY21, Thm. 7.4.1] for the general statement/construction and [JY21, Thm. 7.5.8] for the strict statement/construction. □

[53] Mathematical constructions being "canonical" w.r.t. a property they fulfil means that they are the "natural" construction. For instance, the embedding of sub-2-categories (Note 2.3.21) is the prototypical forgetful 2-functor. We note that canonical mathematical constructions are "natural" in a unique way, but do not need to be unique w.r.t. the property they fulfil. Thus, one of the reasons that forgetful functors are not precisely defined in literature is because being canonical is not precisely definable.

2.4. 2Cat

Sketch 2.4.1. Disclaimer: We will be working with *3-categories*, but only superficially. Therefore, we will only introduce them superficially and implicitly assume that many notions of category theory and 2-category theory lift to 3-category theory.

3-categories $\mathcal{T}$ are a categorification of 2-categories, each consisting of

- a collection of *objects* $\mathrm{Ob}(\mathcal{T})$,
- a *hom-2-category* $\mathcal{T}(\mathfrak{A}, \mathfrak{B})$ for each pair of objects $\mathfrak{A}, \mathfrak{B} \in \mathrm{Ob}(\mathcal{T})$,
- a 2-functor $\square : \mathcal{T}(\mathfrak{B}, \mathfrak{C}) \times \mathcal{T}(\mathfrak{A}, \mathfrak{B}) \longrightarrow \mathcal{T}(\mathfrak{A}, \mathfrak{C})$ for each triple of objects $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathrm{Ob}(\mathcal{T})$,
- a unit 1-morphism $\mathbf{1}_{\mathfrak{A}}$ for each object $\mathfrak{A} \in \mathrm{Ob}(\mathcal{T})$, and

coherence data that must fulfil certain coherence conditions [Gur13, Def. 4.1]. The objects, 1-morphisms, and 2-morphisms of $\mathcal{T}(\mathfrak{A}, \mathfrak{B})$ are called *1-morphisms*, *2-morphisms*, and *3-morphisms*, respectively. The collection of 2-functors $\square$ is called $\square$-*composition*.

In particular, there exists a 3-category[54] **2Cat** [JY21, Thm. 11.6.25] consisting of

Objects: 2-categories,

1-Morphisms: 2-functors,

2-Morphisms: 2-natural transformations,

3-Morphisms: modifications,

$\square$-***Composition:*** composites of 2-functors, 2-natural transformations, and modifications

$$(G, F) \longmapsto G \square F$$
$$(\beta, \alpha) \longmapsto \beta \square \alpha$$
$$(n, m) \longmapsto n \square m,$$

Horizontal Composition: horizontal composites of 2-natural transformations and modifications

$$(\beta, \alpha) \longmapsto \beta \otimes \alpha$$
$$(n, m) \longmapsto n \otimes m,$$

and

[54] **2Cat** is usually labelled as **Bicat** by authors who refer to 2-categories as "bicategories".

Vertical Composition: vertical composition of modifications

$$(n, m) \longmapsto n \circ m.$$

Moreover, there exists a 3-category $\mathbf{2Cat}^{\mathbf{st}}$ which is the sub-3-category of $\mathbf{2Cat}$ whose objects are strict 2-categories and whose 1-morphisms are strict 2-functors, but whose 2- and 3-morphisms are still 2-natural transformations and modifications, respectively. While there exists a 3-functor $\mathbf{st} : \mathbf{2Cat} \rightleftarrows \mathbf{2Cat}^{\mathbf{st}} : \mathcal{U}$, it is unfortunately not a 3-equivalence [Lac06, Thm. 2].

3-categories that fulfil certain strictness conditions are called *Gray categories*. In particular, $\mathbf{2Cat}^{\mathbf{st}}$ is a Gray category [JY21, Prop. 12.2.21].

Theorem 2.4.2 (Coherence Theorems)**.**

i) Every 2-category $\mathcal{B}$ is 2-equivalent to a strict 2-category $\mathbf{st}\,\mathcal{B}$ [Gur13, Cor. 2.7].

ii) If $\mathcal{D}$ is a strict 2-category, then $\mathbf{2Cat}(\mathcal{B}, \mathcal{D})$ is also a strict 2-category [Gur13, Cor. 2.2].

iii) For every 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ there exists a strict 2-functor $\mathbf{st}\,F : \mathbf{st}\,\mathcal{B} \longrightarrow \mathbf{st}\,\mathcal{D}$ such that

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, shift left, "\simeq"'] & \mathcal{D} \arrow[d, shift left, "\simeq"] \\ \mathbf{st}\,\mathcal{B} \arrow[r, "\mathbf{st}\,F"'] \arrow[u, shift left] & \mathbf{st}\,\mathcal{D} \arrow[u, shift left] \end{tikzcd}$$

commutes [Gur13, Sec. 2.3.3].

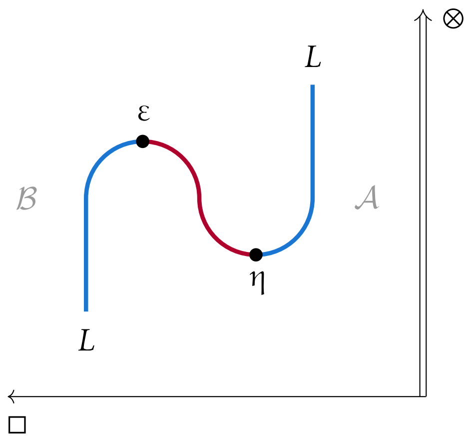
Proof of ii). We just need to show that horizontal composition of 2-natural transformations is strict. We have shown this in Note 2.3.30. □

Note 2.4.3. Whenever convenient we may assume that 2-categories are strict. In particular, we may freely use string diagrams to reason in 2-categories! Moreover, when we are examining properties of 2-functors that are preserved by composition with 2-equivalences, then we may assume that 2-functors are also strict (Theorem 2.4.2).

Remark 2.4.4. Analogously to the string diagrammatic calculus of 2-categories there exist "surface diagrams" for reasoning in 3-categories. Reasoning with surface diagrams is rigorous for Gray categories [BMS24, Thm. 2.32], which is fortunate, because every 3-category is 3-equivalent to a Gray category [Gur13, Cor. 9.15]. Thus, calculations in **2Cat** may be expressed via surface diagrams.

Notation 2.4.5. We will express calculations in **2Cat** using string diagrams as if they were 2-categories.[55] String diagrams in **2Cat** depict 2-categories as surfaces, 2-functors as strings, and 2-natural transformations as vertices; e.g.

$$(\varepsilon \,\square\, L) \otimes (L \,\square\, \eta) \equiv$$



(Definition 3.2.2). Modifications will not directly be expressible in these diagrams. Reasoning in **2Cat** with string diagrams is rigorous since string diagrams are 2-dimensional slices of surface diagrams which in turn are rigorous (Remark 2.4.4).

[55] Moreover, we will express conditions of *universal 2-functors* in terms of surface diagrams in Remark 4.4.5.

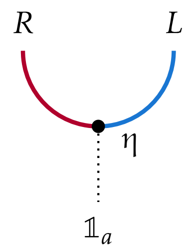
3. Adjunctions

We introduce *adjunctions* in Section 3.1 and discuss adjoints that are left and right inverse to one another (Proposition 3.1.9). Next, we categorify this to *2-adjunctions* in Section 3.2. Later on in Section 6.1 we will use these insights to construct *idempotent completions* from 2-adjunctions. Lastly, we introduce and briefly discuss *pivotal 2-categories*, 2-categories where all 1-morphisms have coherent left and right adjoints, in Section 3.3. Pivotal 2-categories will enable us to define *orbifold data* in Section 5.2.6.

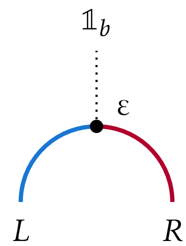
3.1. Adjunctions

Assumption 3.1.1. In this section we fix a 2-category $\mathcal{B}$ and implicitly assume all our objects, 1-morphisms, and 2-morphisms live in $\mathcal{B}$. Moreover, we may assume $\mathcal{B}$ to be strict (Note 2.4.3).

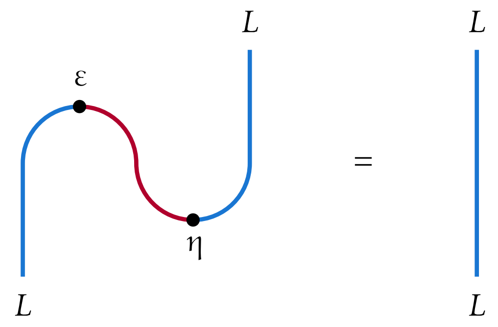
Definition 3.1.2 (Adjunctions)**.** A pair of 1-morphisms $L : a \rightleftarrows b : R$ form an *adjunction* $L \dashv R$ if there exist 2-morphisms



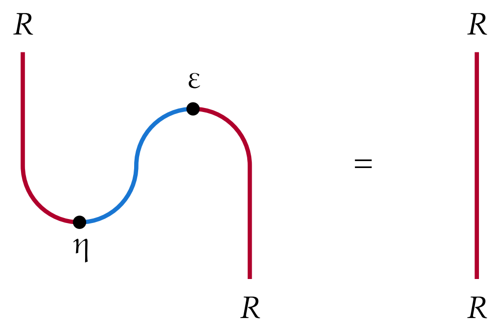
and



such that the *left zigzag identity*



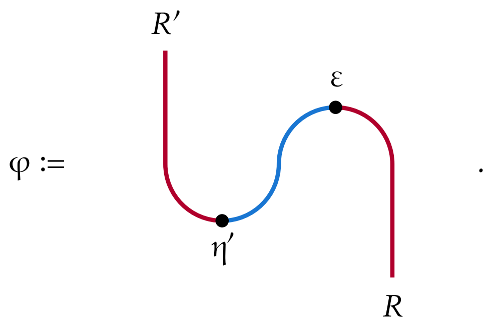
and the *right zigzag identity*



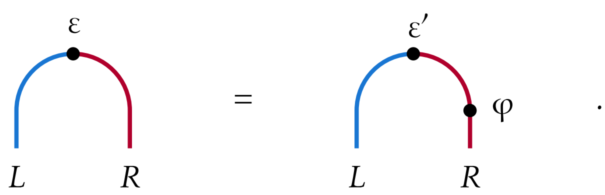
are fulfilled.[56]

We call η the *unit* and ε the counit. We say that L is *left adjoint* to R, that R is *right adjoint* to L, and call the tuple $(L, R, \eta, \varepsilon)$ the *adjunction data*.

Note 3.1.3. Adjoints are unique up to isomorphism, i.e. if we are given two adjunctions $(L, R, \eta, \varepsilon)$ and $(L, R', \eta', \varepsilon')$, then there is a 2-isomorphism $R \Longrightarrow R'$ given by



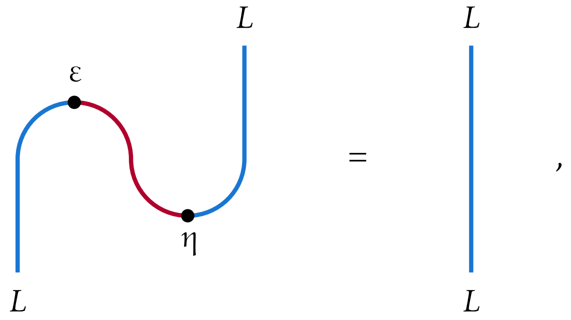
In particular, φ is compatible with the adjunction data, e.g.



56 R for **R**ed and L for b**L**ue ;)

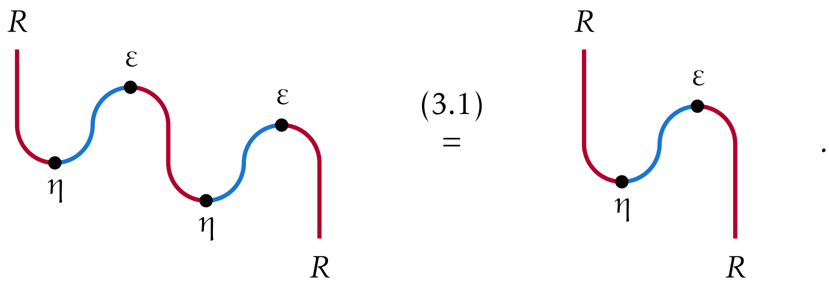
Lemma 3.1.4. *If we are given a zigzag identity, then the second zigzag is an idempotent.*

Proof. For instance, if



(3.1)

then



□

Corollary 3.1.5. *If 1-morphisms $L : a \rightleftarrows b : R$ fulfil a zigzag identity and the second zigzag is a 2-isomorphism, then L and R form an adjunction.*

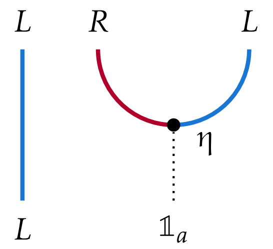
Proof. Idempotent isomorphisms are identities (Lemma 5.1.4). □

Corollary 3.1.6. *If we are given an equivalence $(F, G, \eta, \varepsilon)$ that fulfils one zigzag identity, then $(F, G, \eta, \varepsilon)$ is an adjunction $F \dashv G$ and, analogously, $(G, F, \varepsilon^{-1}, \eta^{-1})$ is an adjunction $G \dashv F$.*

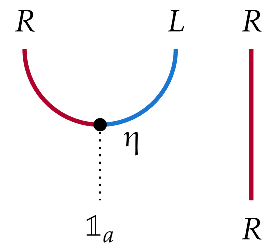
Definition 3.1.7. An adjunction $L \dashv R$ is a *right adjoint right inverse adjunction*[57] if the counit is a 2-isomorphism [RV22, Def. B.4.1].

[57] There does not seem to exist a widely used terminology for this notion. [ML78, p. 94] and [RV22] use this terminology but other authors prefer "reflection" – at least when working in **Cat** – [Lei14, Ex. 2.2.12]. "Right adjoint right inverse adjunction" makes sense because if the counit ε is an isomorphism, then the right adjoint $R : b \longrightarrow a$ is right inverse to $L : a \longrightarrow b$ up to the 2-isomorphism $\varepsilon : L \otimes R \Longrightarrow \mathbb{1}_b$. Conversely, if the right adjoint R is right inverse to the left adjoint L, then one may use Note 3.1.3 to show that the counit must be a 2-isomorphism.

Observation 3.1.8. If we are given a right adjoint right inverse adjunction $L \dashv R$ with unit η, then

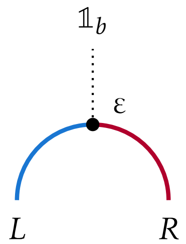


and

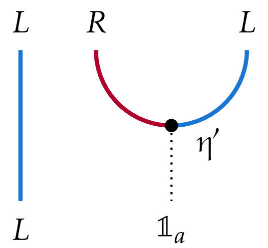


are 2-isomorphisms. Moreover, the converse implication is also true (Proposition 3.1.9).

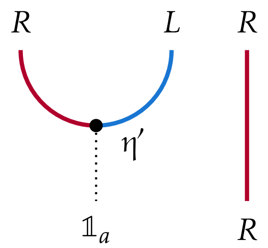
Proposition 3.1.9. *If we are given 2-isomorphisms*



,



, *and*

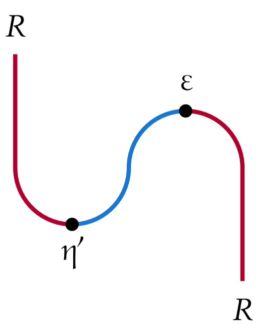


,

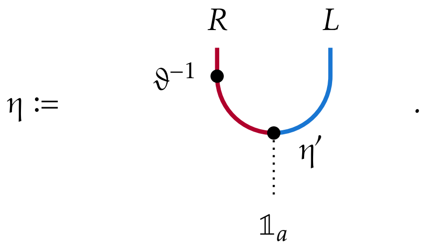
then $L \dashv R$ [RV22, Lemma B.4.2].

Proof. We compose 2-isomorphisms to construct the 2-isomorphism

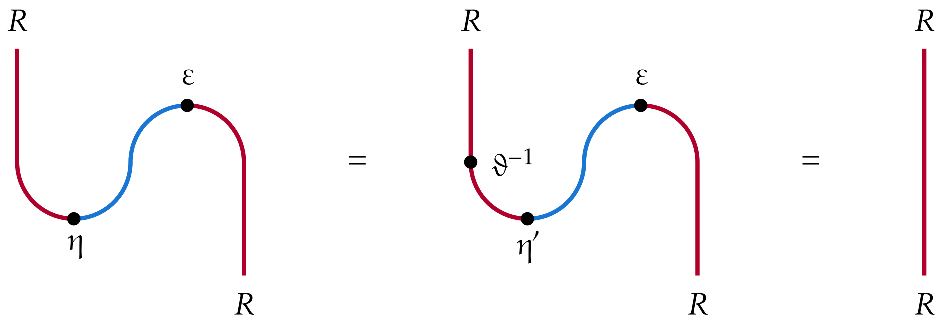
$\vartheta :=$



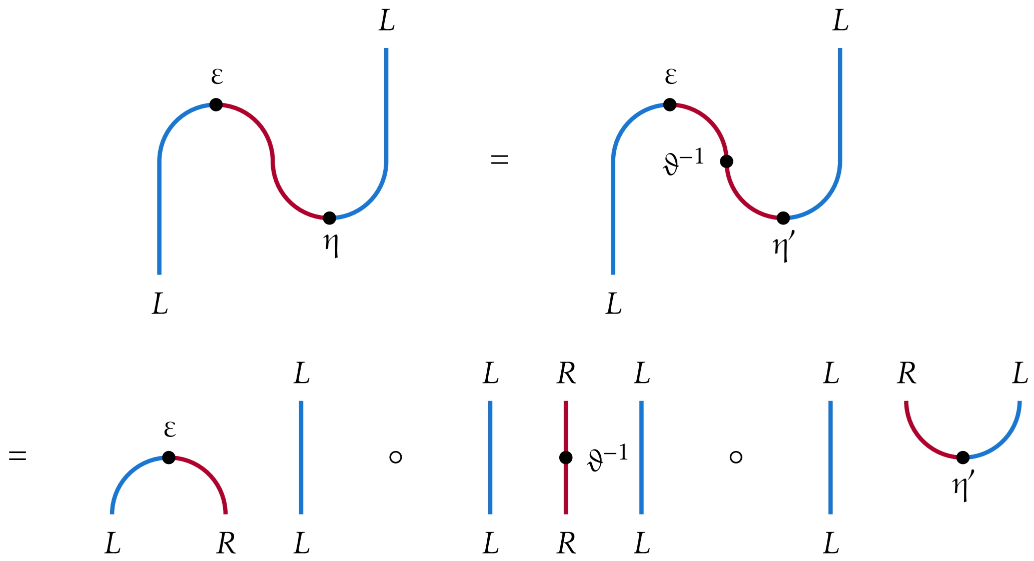
and set



For one, we receive a right zigzag identity:



Moreover, the left zigzag is a 2-isomorphism since it is a composition of 2-isomorphisms:



Thus, the left zigzag identity is also fulfilled (Corollary 3.1.5). □

Corollary 3.1.10. *Equivalences* $(F, G, \eta, \varepsilon)$ *induce* adjoint equivalences, *i.e. equivalences* $(F, G, \eta, \widetilde{\varepsilon})$ *that are simultaneously adjunctions* $F \dashv G \dashv F$.

Remark 3.1.11. The converse is not true, i.e. $F \dashv G \dashv F$ does not imply that F and G form an equivalence.[58] For instance, consider the "walking split idempotent" (Remark 5.2.20). Explicitly, this is a category I_e consisting of two objects $X, \mathbf{im}(e) \in I_e$ and the non-trivial morphisms

$$\begin{tikzcd} X \arrow[loop left, "e"] \arrow[r, bend left, "\pi"] & \mathbf{im}(e) \arrow[l, bend left, "\iota"] \end{tikzcd}$$

such that e is a *split idempotent* (Definition 5.1.9), i.e.

$$\begin{aligned} e &\neq 1_X \\ e &= e \circ e \\ e &= \iota \circ \pi \\ 1_{\mathbf{im}(e)} &= \pi \circ \iota. \end{aligned}$$

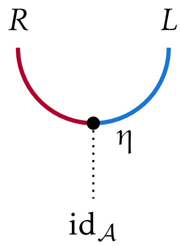
We may think of $\mathbf{im}(e)$ as the image of e (Remark 5.1.13). Moreover, recalling the trivial category $\{\star\}$ (Example A.1.3), we may think of $\{\star\}$ as a full subcategory of I_e by identifying $\star$ with $\mathbf{im}(e)$. In particular, we may construct a functor $F : \{\star\} \hookrightarrow I_e$ that maps $\star \longmapsto \mathbf{im}(e)$. We may also construct a functor a trivial functor $G : I_e \longrightarrow \{\star\}$ (Example A.1.8). Noticing that $G \otimes F = \mathrm{id}_\star$ one may construct adjunctions $F \dashv G$ and $G \dashv F$, but clearly $I_e \not\simeq \{\star\}$.

3.2. 2-Adjunctions

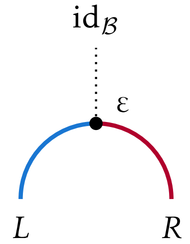
Reminder 3.2.1. We may utilise string diagrams to reason in **2Cat** (Notation 2.4.5).

[58] This is quite fortunate because our section on *pivotal 2-categories* (Section 3.3) would be pointless otherwise.

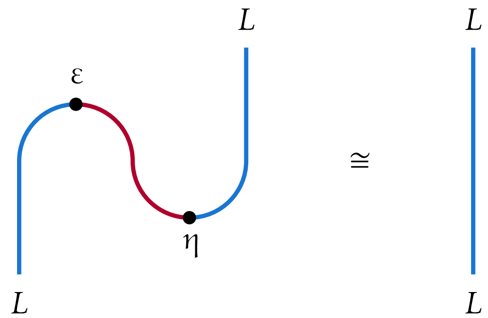
Definition 3.2.2 (2-Adjunctions)**.** A pair of 2-functors[59] $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ form a *2-adjunction* $L \dashv R$ if there exist 2-natural transformations



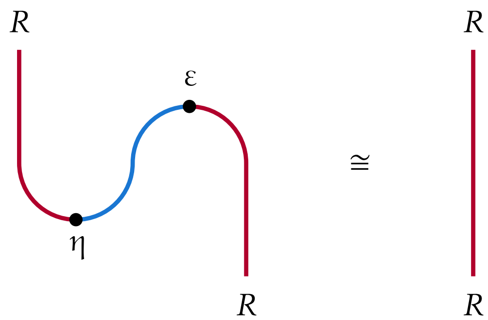
and



such that a *left zigzag modification*



and a *right zigzag modification*



exist.

We call η the *unit* and ε the counit. We say that L is *left 2-adjoint* to R, that R is *right 2-adjoint* to L, and that the tuple $(L, R, \eta, \varepsilon)$ is the *2-adjunction data*.[60]

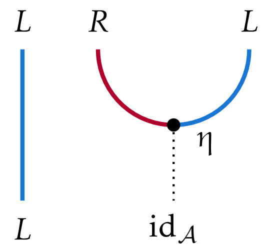
Definition 3.2.3. A 2-adjunction $L \dashv R$ is a *right adjoint right inverse 2-adjunction*[61] if the counit is a 2-natural equivalence.

[59] Naturally, we could consider 1-morphisms in arbitrary 3-categories, but working in **2Cat** is sufficient for our purposes.

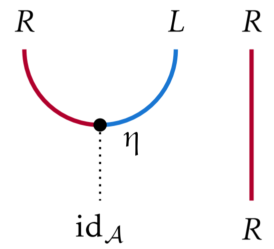
[60] Technically the 2-adjunction data and the 2-adjunction itself should include the zigzag modifications, but we omit them in favour of readability.

[61] I did not find any terminology for this but it clearly fits to the terminology of [RV22] (Definition 3.1.7).

Observation 3.2.4. For right adjoint right inverse 2-adjunctions with unit η it then follows that

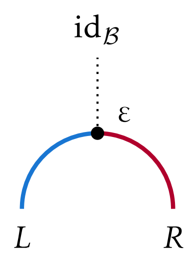


and

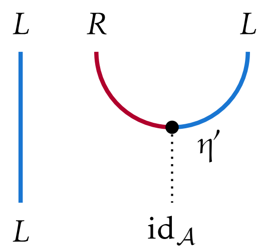


are 2-natural equivalences.

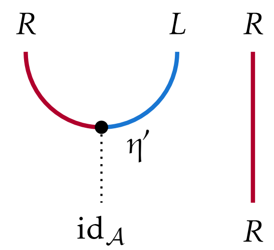
Proposition 3.2.5. *If we are given 2-natural equivalences*



,



, *and*

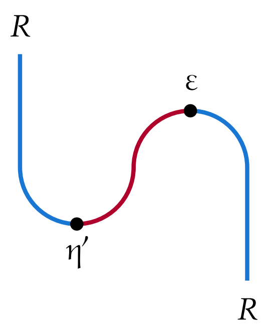


,

then $L \dashv R$.

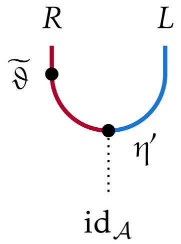
Proof. The idea behind this proof is the same as in Proposition 3.1.9. Thus, we again start by constructing a 2-natural equivalence

$\vartheta :=$



,

denote its weak inverse (Definition 2.3.11) by $\widetilde{\vartheta}$, and set

$\eta :=$ 

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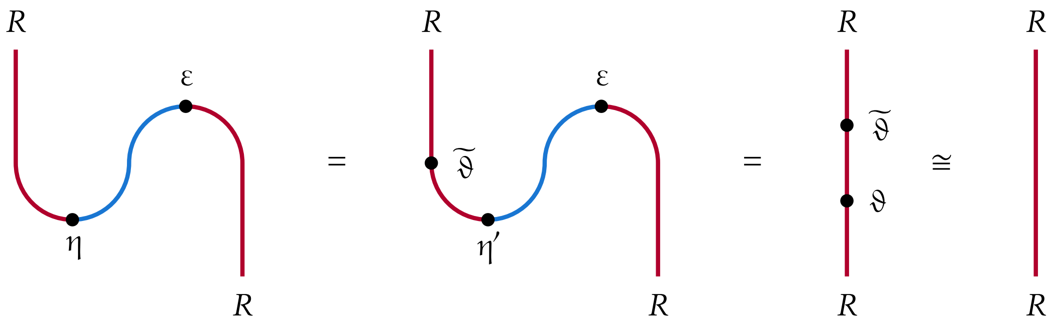
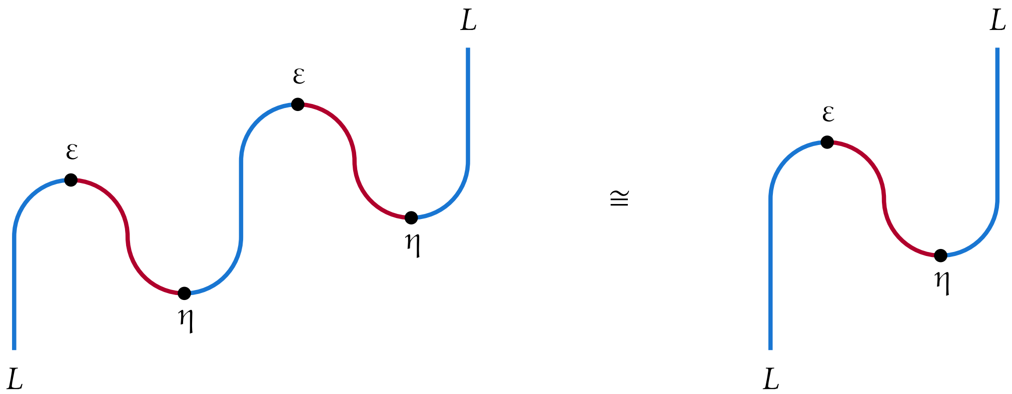
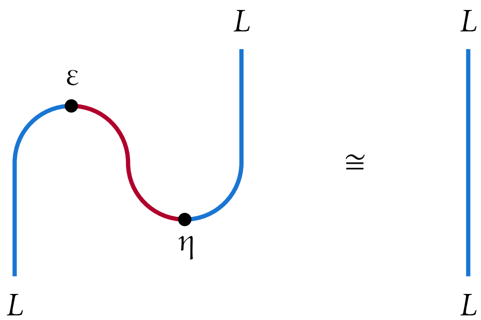
For one, we may see that a right zigzag modification exists:



This implies that the left zigzag is idempotent up to an invertible modification:



Since the left zigzag is idempotent up to isomorphism and also an equivalence, it must be an identity up to isomorphism (Corollary 5.2.18), i.e. the left zigzag modification



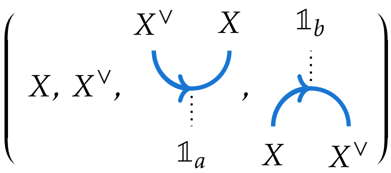
exists. Thus, $L \dashv R$. □

3.3. Pivotality

Remark 3.3.1. *Pivotality* in the setting of monoidal categories was introduced by [JS]. The 2-categorical notion of pivotality that we are about to introduce is the notion used in [CM23, Def. 2.1]. Our notion may also be regarded as "strict pivotality" [JS, Def. 3.1.3]. This strict notion may be less elegant than the general notion but it is convenient because it is quite concrete. Moreover, every pivotal category is monoidally equivalent to a strict pivotal category [JS, Prop. 3.1.4] so utilising this strictification is reasonable.

Assumption 3.3.2. In this section we fix a 2-category $\mathcal{B}$ and implicitly assume all our objects, 1-morphisms, and 2-morphisms live in $\mathcal{B}$. Moreover, we may assume $\mathcal{B}$ to be strict (Note 2.4.3).

Definition 3.3.3. $\mathcal{B}$ is a 2-category *with right adjoints* if it is endowed with a collection of adjunction data

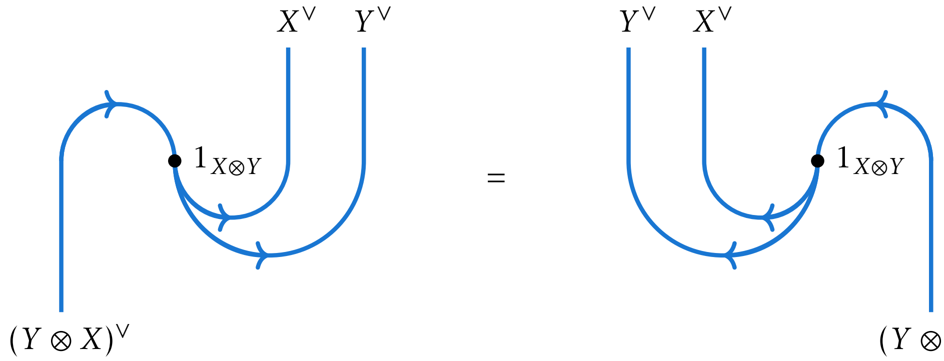


indexed by the objects $a, b \in \mathcal{B}$ and 1-morphisms $X : a \longrightarrow b$.[62]

Analogously, we may define 2-categories *with left adjoints* where we label left adjoints as ${}^{\vee}X : b \longrightarrow a$.

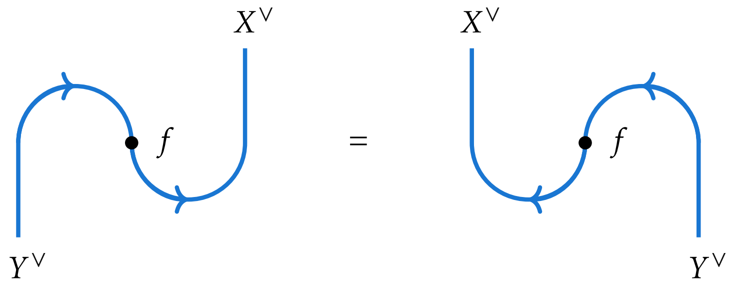
[62] We get away with not "properly" labelling the unit and counit as there is exactly one set of adjunction data for each 1-morphism X in $\mathcal{B}$.

Definition 3.3.4. If $\mathcal{B}$ has left and right adjoints, then $\mathcal{B}$ is *pivotal* if ${}^{\vee}X = X^{\vee}$ and the conditions



(3.2)

and

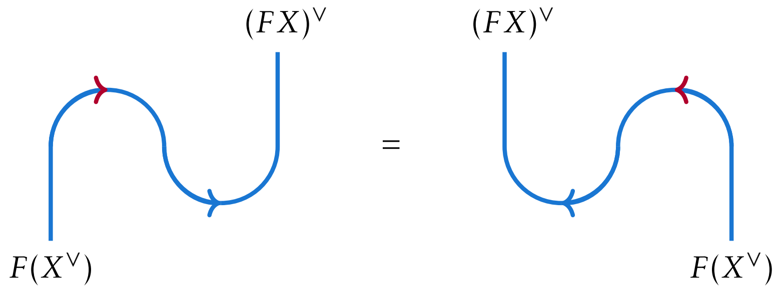


(3.3)

are fulfilled for all objects $a, b, c \in \mathcal{B}$, 1-morphisms $X : a \longrightarrow b$, $Y : b \longrightarrow c$, and 2-morphisms $f : X \Longrightarrow Y$.

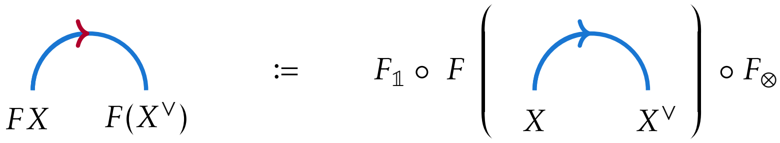
Note 3.3.5. 2-categories with right/left adjoints and pivotal 2-categories are not just 2-categories fulfilling extra properties, they also have extra structure – according to the definitions we are using. This is an important distinction because this means we may have two different pivotal 2-categories that have the same underlying 2-categories. In particular, since the extra structure is given by data of the 2-category itself, there are 2-functors that essentially preserve this extra structure.

Definition 3.3.6. A 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ between pivotal 2-categories $\mathcal{B}$ and $\mathcal{D}$ is *pivotal* if



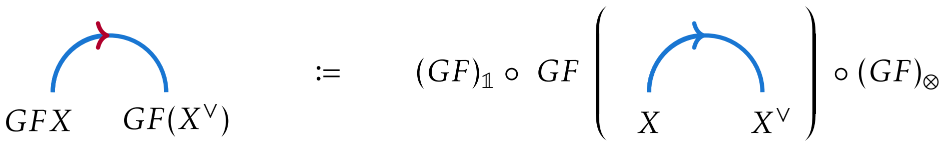
(3.4)

for all 1-morphisms X in $\mathcal{B}$ [CM23, Def. 2.1]. Hereby, the red arrows denote adjunction data of $\mathcal{B}$ in the image of F:

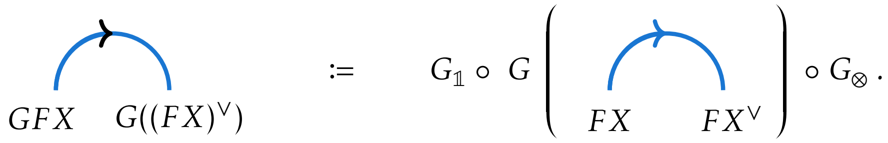


Proposition 3.3.7. *The composite of pivotal functors is itself pivotal.*

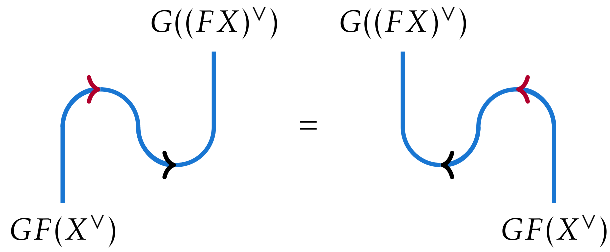
Proof. Let $F : \mathcal{B} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{A}$ be arbitrary 2-functors and let us denote



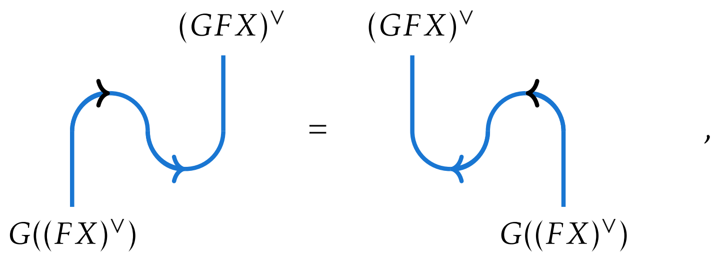
and



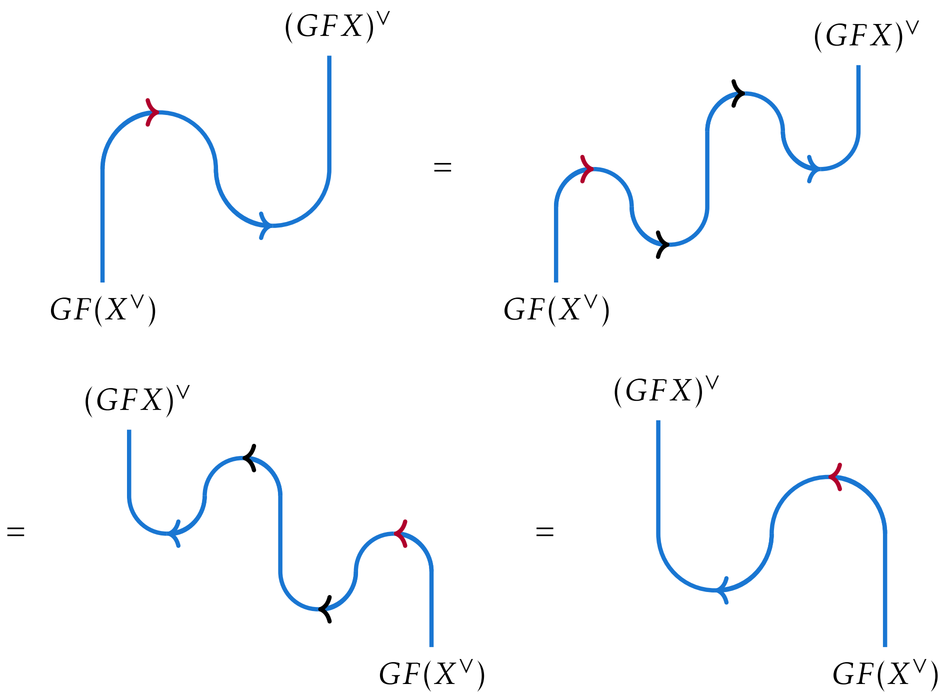
Pivotality of F and G implies



and



respectively. Therefore, $G \square F$ also fulfils the pivotality condition (3.4):



□

Remark 3.3.8. We may interpret strict pivotal 2-categories $\mathcal{B}$ as a variation of 2-categories whose 2-morphisms do not distinguish between input and output. Specifically, the chosen adjunction data of $\mathcal{B}$ induces canonical bijections between 2-morphisms $\mathcal{B}(X, Y)$ and 2-morphisms $\mathcal{B}(\mathbb{1}_a, Y^\vee \otimes X)$. Thus, rather than interpreting 2-morphisms $\varphi : X \Longrightarrow Y$ in $\mathcal{B}$ as having input $X : a \to b$ and output $Y : a \to b$, we may interpret them as equivalence classes $[\varphi] \in$ "hom$(Y^\vee \otimes X)$" belonging to the composable cycle $[Y^\vee \otimes X]$ (Section 1.1.3). If $\mathcal{B}$ is not strict, one has to be more careful about coherently defining $[Y^\vee \otimes X]$, but the interpretation is still valid [FSY23].

Definition 3.3.9. $\mathbf{2Cat}^{\mathbf{piv}}$ is the sub-3-category of $\mathbf{2Cat}$ consisting of[63]

Objects: pivotal 2-categories,

1-Morphisms: pivotal 2-functors,

2-Morphisms: 2-natural transformations, and

3-Morphisms: modifications.

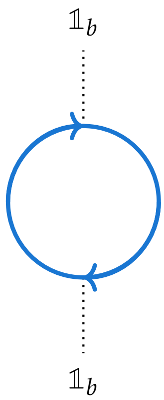
[63] We note that this is likely not the most appropriate definition of $\mathbf{2Cat}^{\mathbf{piv}}$. For instance, pivotality is generally defined via monoidal natural transformations [JS, Def. 3.1.1] so using icons as 2-morphisms of $\mathbf{2Cat}^{\mathbf{piv}}$ may be more fitting than 2-natural transformations (Remark 2.3.33). However, more sophisticated considerations would be necessary to make such judgements. Meanwhile, defining $\mathbf{2Cat}^{\mathbf{piv}}$ as a sub-3-category of $\mathbf{2Cat}$ is convenient for discussing *orbifold completions* (Section 6.2.2). This is implicitly also the definition of $\mathbf{2Cat}^{\mathbf{piv}}$ used in [CM23, Prop. 4.16] for the same purpose.

$\mathbf{2Cat^{piv}}$ is well-defined as a sub-3-category of $\mathbf{2Cat}$ since $\mathbf{2Cat^{piv}}$ is closed under composition of 1-morphisms (Proposition 3.3.7).

Remark 3.3.10. One may introduce "pivotal equivalences" and a pivotal version of 2-equivalences. However, we will not require these definitions for our purposes (Theorem 6.2.43) because 2-equivalences in $\mathbf{2Cat^{piv}}$ are automatically "pivotal 2-equivalences", i.e. if $F : \mathcal{B} \longrightarrow \mathcal{D}$ is a 2-equivalence in $\mathbf{2Cat^{piv}}$, then its weak inverse $G : \mathcal{D} \longrightarrow \mathcal{B}$ is also pivotal.

Definition 3.3.11. If $\mathcal{B}$ is pivotal, then the *right quantum dimension* of a 1-morphism $X : a \longrightarrow b$ is

$\dim_r X :=$



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4. Universal Properties

In mathematics, we are often not interested in explicit constructions of mathematical objects but, rather we are interested in their properties. Moreover, in any given context there will be a notion of being essentially equal, e.g. group isomorphisms in the context of groups. Thus, it is natural to characterise mathematical objects X by properties fulfilled exactly by those objects that are essentially equal to X. Such constructions can be used to make proofs and connections to other mathematical constructions clearer by cutting away the clutter of explicit constructions. Since such properties *universally* characterise mathematical objects, they are called *universal properties*.

We shall present two common universal properties, namely those of *coequalisers* in Section 4.1 and those of *universal morphisms* in Section 4.2.[64] In Section 4.3 we shall categorify universal morphisms to *universal functors* according to the approach of [Fio06]. We will then categorify them further to *universal 2-functors* in Section 4.4 and show that they correspond to the notion utilised by [Dé22].

Example 4.0.1. A *Cartesian product* of two sets $A, B \in \mathbf{Set}$ is a tuple (P, p_A, p_B) in $\mathbf{Set}$ such that there exists a unique morphism $q : Q \dashrightarrow P$ for all tuples (Q, f_A, f_B) in $\mathbf{Set}$ such that

$$\begin{tikzcd} & Q \arrow[dl, "f_A"'] \arrow[d, dashed, "\exists! q"] \arrow[dr, "f_B"] & \\ A & P \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd} \tag{4.1}$$

64 A detailed review of various notions of universal properties is given by [Lei14].

commutes [Lei14, Ex. 5.1.3]. Such a tuple (P, p_A, p_B) always exists in **Set** and is usually defined canonically via $P := A \times B$ together with

$$\begin{aligned} p_A : A \times B &\longrightarrow A \\ (a, b) &\longmapsto a \end{aligned}$$

and

$$\begin{aligned} p_B : A \times B &\longrightarrow B \\ (a, b) &\longmapsto b. \end{aligned}$$

The unique morphism q is then given by

$$\begin{aligned} q : Q &\longrightarrow A \times B \\ c &\longmapsto (f_A(c), f_B(c)). \end{aligned}$$

This leads us to another benefit of universal properties, namely, that they simplify generalisations. For instance, we may utilise the condition (4.1) to define *products* in arbitrary categories $\mathcal{C}$ [Lei14, Def. 5.1.1]. For instance, products in $\mathcal{C} = \mathbf{Vect}_K$ are direct sums, i.e. $V \times W := V \oplus W$ [Lei14, Ex. 5.1.5].

Example 4.0.2. The *tensor product*[65] of vector spaces $V, W \in \mathbf{Vect}_K$ is a vector space $V \otimes W$ together with a bilinear map $\otimes : V \times W \longrightarrow V \otimes W$ that has the property that all bilinear maps $b : V \times W \longrightarrow Z$ uniquely factor as $l \circ \otimes$ for a linear map $l : V \otimes W \dashrightarrow Z$:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "b"'] & V \otimes W \arrow[d, dashed, "\exists! l"] \\ & Z \end{tikzcd}$$

[Lei14, Ex. 0.6]

[65] Monoidal categories are also known as "tensor categories". This is motivated by $\mathbf{Vect}_K$ being a monoidal category with monoidal composition $\otimes : \mathbf{Vect}_K \times \mathbf{Vect}_K \longrightarrow \mathbf{Vect}_K$ (Example 2.3.8).

Example 4.0.3. One may utilise universality to generalise images of maps to images of morphisms. To do this, we recall that we define the image of a map $f : X \longrightarrow Y$ as

$$\mathbf{im}(f) := f(X) = \{y \in Y | \exists x \in X : f(x) = y\}. \tag{4.2}$$

To generalise images we must reformulate the properties of $\mathbf{im}(f)$ in terms of morphisms. To start off with, subsets are categorified as monomorphisms:

$$(\mathbf{im}(f) \subseteq Y) \qquad \rightsquigarrow \qquad (\mathbf{im}(f) \hookrightarrow Y).$$

In particular, in set-theoretic settings we canonically choose

$$\begin{aligned} i : \mathbf{im}(f) &\hookrightarrow Y \\ y &\longmapsto y \end{aligned}$$

and

$$\begin{aligned} \widetilde{f} : X &\longrightarrow \mathbf{im}(f) \\ x &\longmapsto f(x) \end{aligned}$$

to get a decomposition

$$f = \left(X \xrightarrow{\widetilde{f}} \mathbf{im}(f) \xrightarrow{i} Y \right). \tag{4.3}$$

Moreover, $\mathbf{im}(f)$ is the "smallest" set that f factors through, i.e. if f factors through $I' \subseteq Y$, then $\mathbf{im}(f) \subseteq I'$. In particular, there are again canonical inclusions

$$\begin{aligned} i' : I' &\hookrightarrow Y \\ y &\longmapsto y \end{aligned}$$

and

$$\begin{aligned} u : \mathbf{im}(f) &\longrightarrow I' \\ y &\longmapsto y \end{aligned}$$

which fulfil $i = i' \circ u$. Since i' is a monomorphism, there can only by one map u s.t. $i = i' \circ u$.

This is sufficient to generalise images: The *image* of a morphism $f : X \longrightarrow Y$ in an arbitrary category $\mathcal{C}$ is an object $\mathbf{im}(f) \in \mathcal{C}$ together with a monomorphism $i : \mathbf{im}(f) \hookrightarrow Y$ with the universal property that

- there exists $\widetilde{f} : X \longrightarrow \mathbf{im}(f)$ s.t. $f = i \circ \widetilde{f}$ and
- for every monomorphism $i' : I' \hookrightarrow Y$ and morphism $\widetilde{f}' : X \longrightarrow I'$ s.t. $f = i' \circ \widetilde{f}'$ there exists a unique morphism $u : \mathbf{im}(f) \longrightarrow I'$ s.t. $i' \circ u = i$;

see [Mit65, p. 12]. Since such i' are monomorphisms, this implies $\widetilde{f}' = u \circ \widetilde{f}$, which allows us to express this universal property as the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\widetilde{f}"'] \arrow[ddr, "\widetilde{f}'"'] & & Y \\ & \mathbf{im}(f) \arrow[ur, hook, "i"'] \arrow[d, dashed, "\exists! u"] & \\ & I' \arrow[uur, hook, "i'"'] & \end{tikzcd} \quad . \tag{4.4}$$

Note 4.0.4. These examples of universal properties already foreshadow a key technicality of universal properties, namely, the entities that fulfil a given universal property are not objects but rather tuples of objects and morphisms – usually object-morphism pairs.

Moreover, oftentimes the object is implied by the morphism, e.g. the object P of a Cartesian product (P, p_A, p_B) is implied as the domain of p_A and p_B. Therefore, in such cases, the universal property is fulfilled by morphisms alone and, for instance, one may say that "the Cartesian product is a morphism pair (p_A, p_B)".

However, more commonly one will think of universal properties being fulfilled by objects. This is technically imprecise, since there may be multiple universal morphism choices. For instance, if we are given a Cartesian product (P, p_A, p_B) and an automorphism $\varphi : P \longrightarrow P$, then $(P, p_A \circ \varphi, p_B \circ \varphi)$ is also a Cartesian product according to our definition (Example 4.0.1). However, when there is a canonical choice of objects and morphisms fulfilling the universal property, then it is reasonable to think of the property being fulfilled by an object. For instance, usually one agrees that $A \times B$ *is the Cartesian product of* A *and* B for all objects $A, B \in$ **Set** because it is clear from context that one should choose p_A and p_B as in Example 4.0.1.

4.1. Coequalisers

This section may be seen as a small appetiser for the following sections on universal morphisms and their categorifications, but it also prepares our discussion of *split coequalisers* (Section 5.1.2). There we will see that *split idempotents* may be characterised by coequalisers (Lemma 5.1.19) which we will use to construct *absolute coequalisers* (Corollary 5.1.25) as *split coequalisers*. This in turn will be invaluable in Section 5.2.4 where we define *relative tensor products* via coequalisers.

Assumption 4.1.1. In this section we fix an arbitrary category $\mathcal{C}$ and implicitly assume all our objects and morphisms live in $\mathcal{C}$ unless stated otherwise.

Definition 4.1.2. A *fork* is a diagram of the form

$$\begin{tikzcd} X \arrow[r, "i"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Z \end{tikzcd}$$

such that $i \circ f = i \circ f'$. We say that i *equalises* (f, f').

Analogously, a *cofork* is a diagram of the form

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

such that $f \circ p = f' \circ p$. We say that p *coequalises* (f, f').

Definition 4.1.3 (Coequalisers)**.** If we are given the cofork

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Y \arrow[r, "p"] & Z , \end{tikzcd}$$

then (Z, p) is the *coequaliser*[66] of (f, f') if (Z, p) universally equalises (f, f'), i.e. if there exists a unique $\widetilde{\phi} : Z \dashrightarrow W$ such that

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[dr, "\phi"'] & Z \arrow[d, dashed, "\widetilde{\phi}"] \\ & W \end{tikzcd}$$

commutes for all objects W and for all morphisms $\phi : Y \longrightarrow W$ that coequalise (f, f').

66 We will also say things such as a given cofork "is a coequaliser". In that case we implicitly mean that (Z, p) is the coequaliser of (f, f').

Remark 4.1.4. Coequalisers are maybe somewhat more familiar in the form of quotients [Lei14, Rem. 5.2.8]. If the objects of $\mathcal{C}$ are sets and its morphisms are maps then the cofork

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\quad p \quad} Z$$

induces an equivalence relation on Y that includes

$$f(x) \sim f'(x)$$

for all $x \in X$. p then maps to a set Z such

$$p(y) = p(y')$$

when $y \sim y'$.

Now, if the cofork is universal with this property, i.e. when it is a coequaliser, then the equivalence relation is a minimal equivalence relation and p is a surjection (Lemma 4.1.6) that canonically maps onto the quotient $Y/\!\sim$:

$$\begin{aligned} p : Y &\longrightarrow Z \cong Y/\!\sim \\ y &\longmapsto [y] \end{aligned}$$

Lemma 4.1.5. *Let us assume we are given two morphism pairs*

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y$$

and

$$W \underset{g'}{\overset{g}{\rightrightarrows}} Y$$

with the property that for all objects Z and all morphisms $\pi : Y \longrightarrow Z$, π coequalises (f, f') if and only if π coequalises (g, g'). Then,

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\quad \pi \quad} Z \tag{4.5}$$

is a coequaliser if and only if

$$W \underset{g'}{\overset{g}{\rightrightarrows}} Y \xrightarrow{\quad \pi \quad} Z \tag{4.6}$$

is a coequaliser.

Proof. The cofork (4.5) as well as the cofork (4.6) are each coequalisers if and only if there exists a unique $\widetilde{\phi} : Z \dashrightarrow V$ such that

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, "\phi"'] & Z \arrow[d, dashed, "\widetilde{\phi}"] \\ & V \end{tikzcd}$$

commutes for all objects $V \in \mathcal{C}$ and morphisms $\phi \in \mathcal{C}(Y, V)$ that coequalise (f, f') and (g, g'). □

Lemma 4.1.6. *Coequalisers are epimorphisms (Definition 5.1.7).*

Proof. Let (Z, p) in the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Y \arrow[r, "p"] & Z \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & W \end{tikzcd}$$

be a coequaliser of (f, f') and let p equalise (g, g'), i.e. $g \circ p = g' \circ p$. Then $g = g'$ because g is the unique morphism such that

$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[dr, "g' \circ p"'] & Z \arrow[d, dashed, "g"] \\ & W \end{tikzcd}$$

commutes. Thus, p is an epimorphism. □

Definition 4.1.7. A coequaliser in $\mathcal{C}$ is *absolute*, if it is preserved by all functors. Explicitly, a coequaliser

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

is absolute if

$$\begin{tikzcd} FX \arrow[r, shift left, "Ff"] \arrow[r, shift right, "Ff'"'] & FY \arrow[r, "Fp"] & FZ \end{tikzcd}$$

is a coequaliser for all functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and all categories $\mathcal{D}$ [ML78, p. 149].

4.2. Universal Morphisms

Definition 4.2.1 (Universal Morphisms)**.** Given a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ and an object $X \in \mathcal{D}$, a *universal morphism from X to $\mathcal{F}$* is an object-morphism pair

$$(A \in \mathcal{C}, u : X \longrightarrow \mathcal{F}A)$$

such that it fulfils the following *universal property*. Namely, for all objects $A' \in \mathcal{C}$ and all morphisms $f : X \longrightarrow \mathcal{F}A'$ there exists a unique morphism $f' : A \dashrightarrow A'$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & \mathcal{F}A' \\ \mathcal{F}A \arrow[ur, dashed, "\mathcal{F}f'"'] & \end{tikzcd} \tag{4.7}$$

commutes.

Example 4.2.2. Let **Met** be the category consisting of

Objects: metric spaces and

Morphisms: metric preserving maps

and let $\mathbf{Met}^{\mathrm{c}} \subset \mathbf{Met}$ be the full subcategory consisting of complete metric spaces. This implies the forgetful functor

$$U : \mathbf{Met}^{\mathrm{c}} \hookrightarrow \mathbf{Met}\,.$$

Recalling that every metric space $M \in \mathbf{Met}$ has a completion $\overline{M} \in \mathbf{Met}^{\mathrm{c}}$ and an embedding

$$\iota_M : M \hookrightarrow \overline{M},$$

we may see that there is a universal morphism $(\overline{M}, \iota_M)$ from M to U for all $M \in \mathbf{Met}$ [ML78, pp. 56 f.].

Lemma 4.2.3. *$(A \in \mathcal{C}, u : X \longrightarrow \mathcal{F}A)$ is a universal morphism from $X \in \mathcal{D}$ to $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ if and only if*

$$\begin{aligned} \mathcal{F}(-) \circ u : \mathcal{C}(A, A') &\longrightarrow \mathcal{D}(X, \mathcal{F}A') \\ h &\longmapsto \mathcal{F}h \circ u \end{aligned} \tag{4.8}$$

is a bijection for all $A' \in \mathcal{C}$.

Proof. **"$\Rightarrow$":** If $(A \in \mathcal{C}, u : X \longrightarrow \mathcal{F}A)$ is a universal morphism, then we may define a map

$$\begin{aligned} \mathcal{D}(X, \mathcal{F}A') &\longrightarrow \mathcal{C}(A, A') \\ f &\longmapsto f' \end{aligned}$$

via Diagram (4.7). This map is clearly left inverse to $\mathcal{F}(-) \circ u$. Moreover, it is right inverse by uniqueness of the morphisms f'.

"$\Leftarrow$": Conversely, if $\mathcal{F}(-) \circ u$ is a bijection, then

$$\begin{aligned} (\mathcal{F}(-) \circ u)^{-1} : \mathcal{D}(X, \mathcal{F}A') &\longrightarrow \mathcal{C}(A, A') \\ f &\longmapsto f' \end{aligned}$$

exists and produces unique morphisms $f' : A \dashrightarrow A$ such that diagram (4.7) commutes. □

Remark 4.2.4. We will utilise Lemma 4.2.3 as a redefinition of universal morphisms for the remaining two propositions of this section. It may make some parts of the proofs a bit awkward, but it will allow us to generalise them to *2-universal morphisms* (Definition 4.3.1).

Proposition 4.2.5. *Universal morphisms characterise objects up to isomorphism, i.e. if $(A \in \mathcal{C}, u : X \longrightarrow \mathcal{F}A)$ is a universal morphism from $X \in \mathcal{D}$ to $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$, then $\widetilde{A} \in \mathcal{C}$ is part of a universal morphism $(\widetilde{A}, \widetilde{u} : X \longrightarrow \mathcal{F}\widetilde{A})$ from X to $\mathcal{F}$ if and only if $A \cong \widetilde{A}$.*

Proof.

"$\Rightarrow$": Assuming $(\widetilde{A}, \widetilde{u})$ is also a universal morphisms from X to $\mathcal{F}$, let

$$\begin{aligned} \mathcal{G}_{A'} &: \mathcal{D}(X, \mathcal{F}A') \longrightarrow \mathcal{C}(A, A') \quad \text{and} \\ \widetilde{\mathcal{G}}_{A'} &: \mathcal{D}(X, \mathcal{F}A') \longrightarrow \mathcal{C}(\widetilde{A}, A') \end{aligned}$$

be inverses of $\mathcal{F}(-) \circ u$ and $\mathcal{F}(-) \circ \widetilde{u}$, respectively, for all $A' \in \mathcal{C}$. Now, we want to see that

$$\mathcal{G}_{\widetilde{A}}\widetilde{u} : A \rightleftarrows \widetilde{A} : \widetilde{\mathcal{G}}_A u$$

is an isomorphism. The calculation

$$\begin{aligned} \widetilde{\mathcal{G}}_A(u) \circ \mathcal{G}_{\widetilde{A}}(\widetilde{u}) &= \mathcal{G}_A(\mathcal{F}\left(\widetilde{\mathcal{G}}_A(u) \circ \mathcal{G}_{\widetilde{A}}(\widetilde{u})\right) \circ u) \\ &= \mathcal{G}_A(\mathcal{F}\left(\widetilde{\mathcal{G}}_A(u)\right) \circ \mathcal{F}\left(\mathcal{G}_{\widetilde{A}}(\widetilde{u})\right) \circ u) \\ &= \mathcal{G}_A(\mathcal{F}\left(\widetilde{\mathcal{G}}_A(u)\right) \circ \widetilde{u}) \\ &= \mathcal{G}_A(u) \\ &= \mathcal{G}_A(\mathcal{F}(1_A) \circ u) \\ &= 1_A \end{aligned}$$

shows that $\mathcal{G}_{\widetilde{A}}(\widetilde{u})$ is right inverse to $\widetilde{\mathcal{G}}_A(u)$ for all $A' \in \mathcal{C}$. Analogously, the $\mathcal{G}_{\widetilde{A}}(\widetilde{u})$ must also be left inverses.

"$\Leftarrow$": If we are given an isomorphism $f : \widetilde{A} \longrightarrow A$, then

$$\mathcal{F}((-) \circ f) \circ u : \mathcal{C}(\widetilde{A}, A') \longrightarrow \mathcal{D}(X, \mathcal{F}A')$$

is an isomorphism for all $A' \in \mathcal{C}$. Since

$$\mathcal{F}((-) \circ f) \circ u = \mathcal{F}(-) \circ (\mathcal{F}f \circ u)$$

$(\widetilde{A}, \mathcal{F}f \circ u)$ is a universal morphism from X to $\mathcal{F}$. □

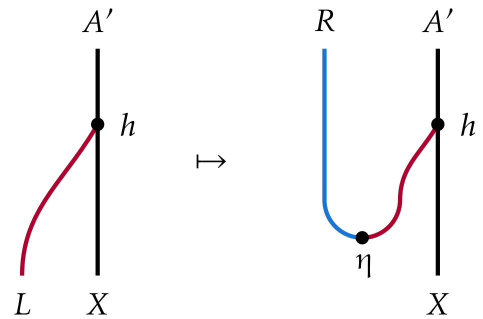
Proposition 4.2.6. *One may interpret universal morphisms as "local adjunctions", i.e. a functor $R : \mathcal{C} \longrightarrow \mathcal{D}$ has a left adjoint if and only if there is a universal morphism from X to R for all $X \in \mathcal{D}$.*

Proof. We shall see that the unit and counit of an adjunction correspond to the maps (4.8) and their inverses, respectively.

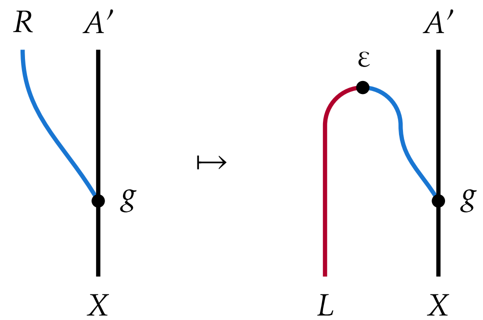
"$\Rightarrow$": Given an adjunction $(L, R, \eta, \varepsilon)$, there is a universal morphism $(LX, \eta_X : X \longrightarrow RLX)$ from X to $R : \mathcal{C} \longrightarrow \mathcal{D}$ for all $X \in \mathcal{D}$. The bijection

$$\begin{aligned} R(-) \circ \eta_X : \mathcal{C}(LX, A') &\longrightarrow \mathcal{D}(X, \mathcal{F}A') \\ h &\longmapsto Rh \circ \eta_X \end{aligned}$$

may be depicted as



for all $A' \in \mathcal{C}$ and $h : LX \longrightarrow A'$ in $\mathcal{C}$. Thus, utilising zigzag identities (Definition 3.1.2), we see that its inverse is given by



for all $A' \in \mathcal{C}$ and $g : X \longrightarrow RA'$ in $\mathcal{D}$.

"$\Leftarrow$": If there exists a universal morphism $(A_X \in \mathcal{C}, u_X : X \longrightarrow RA_X)$ from X to $R : \mathcal{C} \longrightarrow \mathcal{D}$ for all $X \in \mathcal{D}$, then we may use maps

$$\mathcal{G}_X := (R(-) \circ u_X)^{-1} : \mathcal{D}(X, RA_{X'}) \longrightarrow \mathcal{C}(A_X, A_{X'})$$

to define a left adjoint of R as

$$\begin{aligned} L : \mathcal{D} &\longrightarrow \mathcal{C} \\ X &\longmapsto A_X \\ (g : X \longrightarrow X') &\longmapsto \mathcal{G}_X(u_{X'} \circ g). \end{aligned}$$

It fulfils functoriality because

$$\begin{aligned} L(1_X) &= G_X(u_X \circ 1_X) \\ &= G_X(u_X) \\ &= (R(-) \circ u_X)^{-1}(u_X) \\ &= 1_{A_X} \\ &= 1_{LX} \end{aligned}$$

and

$$\begin{aligned} L(h) \circ L(g) &= \mathcal{G}_{X'}(u_{X''} \circ h) \circ \mathcal{G}_X(u_{X'} \circ g) \\ &= \mathcal{G}_X \circ \mathcal{G}_X^{-1} \left(\mathcal{G}_{X'}(u_{X''} \circ h) \circ \mathcal{G}_X(u_{X'} \circ g)\right) \\ &= \mathcal{G}_X(R(-) \circ u_X) \left(\mathcal{G}_{X'}(u_{X''} \circ h) \circ \mathcal{G}_X(u_{X'} \circ g)\right) \\ &= \mathcal{G}_X \left(R \left(\mathcal{G}_{X'}(u_{X''} \circ h) \circ \mathcal{G}_X(u_{X'} \circ g)\right) \circ u_X\right) \\ &= \mathcal{G}_X \left(R(\mathcal{G}_{X'}(u_{X''} \circ h)) \circ R(\mathcal{G}_X(u_{X'} \circ g)) \circ u_X\right) \\ &= \mathcal{G}_X \left(R(\mathcal{G}_{X'}(u_{X''} \circ h)) \circ u_{X'} \circ g\right) \\ &= \mathcal{G}_X \left(u_{X''} \circ h \circ g\right) \\ &= L(h \circ g) \end{aligned}$$

hold for all $X \in \mathcal{D}$ and all $g : X \longrightarrow X'$ and $h : X' \longrightarrow X''$ in $\mathcal{D}$.

Noticing that the target of u_X is $RA_X = RLX$, we construct the unit $\eta : \mathrm{id}_{\mathcal{D}} \Longrightarrow R \circ L$ as

$$\eta_X := (R(-) \circ u_X)(1_{LX}) \equiv u_X$$

for all $X \in \mathcal{D}$. This is a natural transformation because

$$\begin{aligned} RLf \circ \eta_X &= R\mathcal{G}_X(u_{X'} \circ f) \circ u_X \\ &= \mathcal{G}_X^{-1}(\mathcal{G}_X(u_{X'} \circ f)) \\ &= u_{X'} \circ f \\ &= \eta_{X'} \circ \mathrm{id}_{\mathcal{D}} f \end{aligned}$$

for all $f : X \longrightarrow X'$ in $\mathcal{D}$.

We use

$$\mathcal{G}_{RA} : \mathcal{D}(RA, RA) \longrightarrow \mathcal{C}(A_{RA}, A) = \mathcal{C}(LRA, A)$$

to construct the counit $\varepsilon : L \circ R \Longrightarrow \mathrm{id}_{\mathcal{C}}$ as

$$\varepsilon_A \coloneqq \mathcal{G}_{RA}(1_{RA}) \equiv \mathcal{G}_{RA}(R1_A)$$

for all $A \in \mathcal{C}$. It is a natural transformation because

$$\begin{aligned}
h \circ \varepsilon_A &= h \circ \mathcal{G}_{RA}(R1_A) \\
&= \mathcal{G}_{RA} \circ \mathcal{G}_{RA}^{-1}(h \circ \mathcal{G}_{RA}(R1_A)) \\
&= \mathcal{G}_{RA}(Rh \circ R(\mathcal{G}_{RA}(R1_A)) \circ u_{RA}) \\
&= \mathcal{G}_{RA}(Rh \circ \mathcal{G}_{RA}^{-1}(\mathcal{G}_{RA}(R1_A))) \\
&= \mathcal{G}_{RA}(Rh \circ R1_A) \\
&= \mathcal{G}_{RA}(R1_{A'} \circ Rh) \\
&= \mathcal{G}_{RA}(\mathcal{G}_{RA'}^{-1}(\mathcal{G}_{RA'}(R1_{A'})) \circ Rh) \\
&= \mathcal{G}_{RA}(R(\mathcal{G}_{RA'}(R1_{A'})) \circ u_{RA'} \circ Rh) \\
&= \mathcal{G}_{RA}(R(\mathcal{G}_{RA'}(R1_{A'})) \circ \mathcal{G}_{RA}^{-1}(\mathcal{G}_{RA}(u_{RA'} \circ Rh))) \\
&= \mathcal{G}_{RA}(R(\mathcal{G}_{RA'}(R1_{A'})) \circ R(\mathcal{G}_{RA}(u_{RA'} \circ Rh)) \circ u_{RA}) \\
&= \mathcal{G}_{RA} \circ \mathcal{G}_{RA}^{-1}(\mathcal{G}_{RA'}(R1_{A'}) \circ \mathcal{G}_{RA}(u_{RA'} \circ Rh)) \\
&= \mathcal{G}_{RA'}(R1_{A'}) \circ \mathcal{G}_{RA}(u_{RA'} \circ Rh) \\
&= \varepsilon_{A'} \circ LRh
\end{aligned}$$

for all $h : A \longrightarrow A'$ in $\mathcal{C}$.

Lastly, $(L, R, \eta, \varepsilon)$ form an adjunction. The right zigzag identity is quite short:

$$\begin{aligned}
(R\varepsilon_A) \circ (\eta_{RA}) &= R(\mathcal{G}_{RA}(1_{RA})) \circ u_{RA} \\
&= \mathcal{G}_{RA}^{-1}(\mathcal{G}_{RA}(1_{RA})) \\
&= 1_{RA}
\end{aligned}$$

holds for all $A \in \mathcal{C}$. The right zigzag identity is a bit longer:

$$\begin{aligned}
(\varepsilon_{LX}) \circ (L\eta_X) &= (\mathcal{G}_{RLX}(1_{RLX})) \circ \mathcal{G}_X(u_{RLX} \circ u_X) \\
&= \mathcal{G}_X(\mathcal{G}_X^{-1}((\mathcal{G}_{RLX}(1_{RLX})) \circ \mathcal{G}_X(u_{RLX} \circ u_X))) \\
&= \mathcal{G}_X(R(\mathcal{G}_{RLX}(1_{RLX})) \circ R(\mathcal{G}_X(u_{RLX})) \circ u_X)) \\
&= \mathcal{G}_X(R(\mathcal{G}_{RLX}(1_{RLX})) \circ u_{RLX} \circ u_X) \\
&= \mathcal{G}_X(1_{RLX} \circ u_X) \\
&= \mathcal{G}_X(u_X) \\
&= 1_{LX}
\end{aligned}$$

holds for all $X \in \mathcal{D}$. $\square$

4.3. Universal Functors

Since universal morphisms allow us to universally characterise objects in categories, we would like to categorify them to enable us to *2-universally* characterise objects in 2-categories. Following the approach of [Fio06] we define *2-universal morphisms* (Definition 4.3.1) and justify this approach (Remark 4.3.4). Furthermore, we shall see that these 2-universal morphisms correspond to the notion utilised by [Dé22] (Lemma 4.3.6). Lastly, while 2-universal morphisms live in a 2-categorical realm, we will use them solely to characterise categories (Definition 6.1.4), i.e. objects of **Cat**. Thus, it will be sufficient for our purposes to think of them as a construction of regular 1-category theory that we shall call *universal functors* (Definition 4.3.10).

Definition 4.3.1 (2-Universal Morphisms)**.** Given a 2-functor $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{D}$ and an object $x \in \mathcal{D}$, a *2-universal morphism from x to $\mathcal{F}$* is an object-1-morphism pair

$$(a \in \mathcal{B}, U : x \longrightarrow \mathcal{F}a)$$

such that the functor

$$\begin{aligned} \mathcal{F}(-) \otimes U : \mathcal{B}(a, a') &\longrightarrow \mathcal{D}(x, \mathcal{F}a') \\ X &\longmapsto \mathcal{F}X \otimes U \\ f &\longmapsto \mathcal{F}f \otimes 1_U \end{aligned} \tag{4.9}$$

is an equivalence for all $a' \in \mathcal{B}$ [Fio06, Def. 9.4].

Note 4.3.2. The maps $\mathcal{F}(-) \otimes U$ of Definition 4.3.1 are indeed functors since they are each a composition of the functor

$$\mathcal{F} : \mathcal{B}(a, a') \longrightarrow \mathcal{D}(\mathcal{F}a, \mathcal{F}a')$$

and a functor

$$(-) \otimes U : \mathcal{D}(\mathcal{F}a, x') \longrightarrow \mathcal{D}(x, x')$$

for each $x' \in \mathbf{im}\,\mathcal{F} \subseteq \mathcal{D}$ (Example 2.3.20).

Example 4.3.3. We saw that completions $\iota_M : M \hookrightarrow \overline{M}$ of metric spaces M correspond to universal morphisms (Example 4.2.2). This may be categorified by interpreting metric spaces M as categories $\mathcal{M}$. Naturally, their completions $\overline{M}$ may also be interpreted as categories $\overline{\mathcal{M}}$, which leads to a 2-universal morphism $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ (cf. Section A.2.1).

Remark 4.3.4. We just saw why we reformulated the "standard" definition of universal morphisms (Definition 4.2.1) in terms of bijections of hom-sets (Lemma 4.2.3). Namely, it allowed us to categorify universal morphisms as 2-universal morphisms by simply replacing bijections of hom-sets with equivalences of Hom categories. However, one may wonder if this approach has benefits. Therefore, let us consider a different – likely more intuitive – 2-universal property based on *essentially unique* 1-morphisms analogously to Definition 4.2.1. Hereby, a 1-morphism $X : x \longrightarrow x'$ is essentially unique with a certain property if all other 1-morphisms $X' : x \longrightarrow x'$ that fulfil the same property imply a 2-isomorphism $X \Longrightarrow X'$.

Given a 2-functor $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{D}$ and an object $x \in \mathcal{D}$, an "intuitive 2-universal morphism from x to $\mathcal{F}$" is an object-1-morphism pair $(a \in \mathcal{B}, U : x \longrightarrow \mathcal{F}a)$ such that for all objects $a' \in \mathcal{B}$ and all 1-morphisms $X : x \longrightarrow \mathcal{F}a'$ there exists an essentially unique 1-morphism $X' : a \dashrightarrow a'$ such that

$$\begin{tikzcd} x \arrow[r, "X"] \arrow[d, "U"'] & \mathcal{F}a' \\ \mathcal{F}a \arrow[ur, dashed, "\mathcal{F}X'"'] & \end{tikzcd}$$

commutes up to isomorphism.[67]

The "intuitive 2-universal morphisms" indeed fulfil a 2-universal property since if there are two such universal pairs $(a \in \mathcal{B}, U : x \longrightarrow \mathcal{F}a)$ and $(\tilde{a} \in \mathcal{B}, \widetilde{U} : x \longrightarrow \mathcal{F}\tilde{a})$, then there is an equivalence $a \simeq \tilde{a}$ which can be read from the diagram

$$\begin{tikzcd} x \arrow[r, "\widetilde{U}"] \arrow[d, "U"'] & \mathcal{F}\tilde{a} \arrow[dl, dashed, shift left] \\ \mathcal{F}a \arrow[ur, dashed, shift left] & \end{tikzcd} \quad .$$

[67] [CM23] use such a 2-universal property but implicitly keep in mind that coherence conditions should possibly be added [cf. Mü25, p. 15]. This is also the version used by [DR18, Prop. A.6.4].

However, the drawback of these "intuitive 2-universal morphisms" would be that they do not pose many requirements on 2-morphisms. To see the difference, consider the following example. Let $\mathcal{B}$ have two objects and two non-unit 1-morphisms as given by the non-commutative diagram

$$\begin{tikzcd} a \arrow[r, bend left, "Y'"] \arrow[r, bend right, "X'"'] & a' \end{tikzcd}$$

and let $\mathcal{B}$ have no non-trivial 2-morphisms. Meanwhile, let $\mathcal{D}$ have two objects, two non-unit 1-morphism, and a non-trivial 2-morphism as given by the pasting diagram

$$\begin{tikzcd} x \arrow[r, bend left, "Y", ""{name=U, below}] \arrow[r, bend right, "X"', ""{name=D, above}] & x' \arrow[from=D, to=U, Rightarrow, "f"'] \end{tikzcd} .$$

Then, there is clearly a 2-functor $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{D}$ that maps $a \longmapsto x$, $a' \longmapsto x'$, $X' \longmapsto X$, and $Y' \longmapsto Y$ and an "intuitive 2-universal morphism" $(a, \mathbb{1}_x : x \longrightarrow x)$ from x to $\mathcal{F}$. We may visualise our setup in the non-pasting diagram

$$\begin{tikzcd} x \arrow[r, bend left, "Y", ""{name=U, below}] \arrow[r, bend right, "X"', ""{name=D, above}] \arrow[d, "\mathbb{1}_x"'] & \mathcal{F}a' \arrow[from=D, to=U, Rightarrow, "f"'] \\ \mathcal{F}a \arrow[ur, bend left, "Y'"] \arrow[ur, bend right, "X'"'] & \end{tikzcd} .$$

However, in this example $\mathcal{B}(a, a') \not\simeq \mathcal{D}(x, \mathcal{F}a')$.

Remark 4.3.5. 2-universal morphisms characterise objects up to equivalence, i.e. in the setting of Definition 4.3.1 another object $\widetilde{a} \in \mathcal{B}$ is part of a 2-universal morphism $(\widetilde{a}, \widetilde{U} : x \longrightarrow \mathcal{F}\widetilde{a})$ from x to $\mathcal{F}$ if and only if $a \simeq \widetilde{a}$.

Proof. We basically repeat the proof of Proposition 4.2.5:

"$\Rightarrow$": Assuming $(\widetilde{a}, \widetilde{\mathcal{A}})$ is also a 2-universal morphisms from x to $\mathcal{F}$, let

$$\begin{aligned} \mathcal{G}_{a'} &: \mathcal{D}(x, \mathcal{F}a') \longrightarrow \mathcal{C}(a, a') \quad \text{and} \\ \widetilde{\mathcal{G}}_{a'} &: \mathcal{D}(x, \mathcal{F}a') \longrightarrow \mathcal{C}(\widetilde{a}, a') \end{aligned}$$

be weak inverses of $\mathcal{F}(-) \otimes U$ and $\mathcal{F}(-) \otimes \widetilde{U}$, respectively, for all $a' \in \mathcal{B}$. Now the equivalence we are searching is

$$\mathcal{G}_{\widetilde{a}}\widetilde{U} : a \rightleftarrows \widetilde{a} : \widetilde{\mathcal{G}}_a U.$$

The calculation

$$\begin{aligned}
\widetilde{\mathcal{G}}_a(U) \otimes \mathcal{G}_{\widetilde{a}}(\widetilde{U}) &\cong \mathcal{G}_a(\mathcal{F}\left(\widetilde{\mathcal{G}}_a(U) \otimes \mathcal{G}_{\widetilde{a}}(\widetilde{U})\right) \otimes U) \\
&\cong \mathcal{G}_a(\mathcal{F}\left(\widetilde{\mathcal{G}}_a(U)\right) \otimes \mathcal{F}\left(\mathcal{G}_{\widetilde{a}}(\widetilde{U})\right) \otimes U) \\
&\cong \mathcal{G}_a(\mathcal{F}(\widetilde{\mathcal{G}}_a(U)) \otimes \widetilde{U}) \\
&\cong \mathcal{G}_a(U) \\
&\cong \mathcal{G}_a(F(1_a) \otimes U) \\
&\cong \mathbb{1}_a
\end{aligned}$$

analogously also implies $\mathcal{G}_{\widetilde{a}}(\widetilde{u}) \otimes \widetilde{\mathcal{G}}_a(U) \cong \mathbb{1}_{\widetilde{a}}$ which together implies the equivalence.

"$\Leftarrow$": If we are given an equivalence $E : \widetilde{a} \longrightarrow a$, then

$$\mathcal{F}((-) \otimes E) \otimes U : \mathcal{B}(\widetilde{a}, a') \longrightarrow \mathcal{D}(x, \mathcal{F}a')$$

is an equivalence. Since

$$\mathcal{F}((-) \otimes E) \otimes U \cong \mathcal{F}(-) \otimes (\mathcal{F}E \otimes U)$$

$(\widetilde{a}, \mathcal{F}E \otimes U)$ is a 2-universal morphism from x to $\mathcal{F}$. □

Lemma 4.3.6. *If we are given a 2-functor $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{D}$ and an object $x \in \mathcal{D}$, then an object-1-morphism pair $(a \in \mathcal{B}, U : x \longrightarrow \mathcal{F}a)$ is a 2-universal morphism from x to $\mathcal{F}$ if and only if the following conditions are fulfilled:*

(i) for all $a' \in \mathcal{B}$ and all 1-morphisms $X : x \longrightarrow \mathcal{F}a'$ there exists a 1-morphism $\overline{X} \in \mathcal{B}(a, a')$ together with a 2-isomorphism

$$\begin{tikzcd}
x \arrow[rr, "X"] \arrow[d, "U"'] & & \mathcal{F}a' \\
\mathcal{F}a \arrow[urr, "\mathcal{F}\overline{X}"'] \arrow[ur, phantom, "\cong"] & &
\end{tikzcd} \tag{4.10}$$

and

(ii) for all 1-morphisms $Y, Z : a \longrightarrow a'$ and 2-morphisms $f : Y \otimes U \Longrightarrow Z \otimes U$ there exists a unique 2-morphism $\widetilde{f} : Y \Longrightarrow Z$ such that

$$
\begin{tikzcd}
x \arrow[d, "U"] \\
a \arrow[d, bend right=60, "Y"'] \arrow[d, bend left=60, "Z"] \arrow[d, phantom, "\overset{\widetilde{f}}{\Longrightarrow}"] \\
a'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& x \arrow[dl, "U"'] \arrow[dr, "U"] & \\
a \arrow[rr, Rightarrow, "f"] \arrow[dr, "Y"'] & & a \arrow[dl, "Z"] \\
& a' &
\end{tikzcd}
\tag{4.11}
$$

[Fio06, pp. 83 f.] [Dé22, Def. A.1.1].

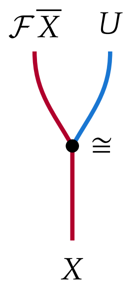
Proof.

(i) Essential surjectivity of $\mathcal{F}(-) \otimes U$ implies that exists a 1-morphism $\overline{X} \in \mathcal{B}(a, a')$ such that $\mathcal{F}X \otimes U \cong \overline{X}$ for all $X \in \mathcal{D}(x, \mathcal{F}a')$.

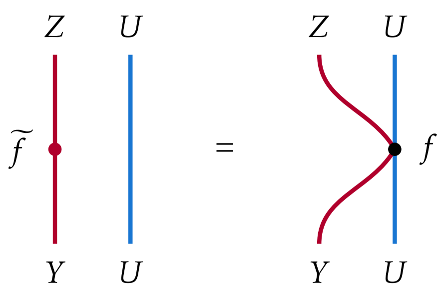
(ii) Fully faithfulness of $\mathcal{F}(-) \otimes U$ implies bijectivity on hom-sets. Then, $\mathcal{F}(-) \otimes U$ induces a unique morphism $\tilde{f} : Y \longrightarrow Z$ in $\mathcal{B}(a, a')$ for each morphism $f : Y \otimes U \longrightarrow Z \otimes U$ in $\mathcal{D}(x, \mathcal{F}a')$. $\square$

Remark 4.3.7. We may reformulate these pasting diagrams in terms of string diagrams.

- Diagram (4.10):

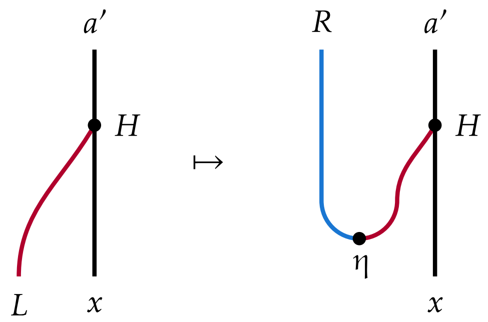


- Diagram (4.11):

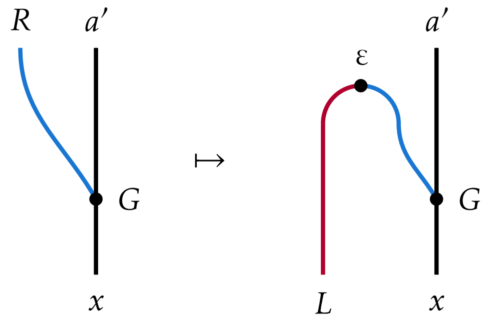


Proposition 4.3.8. *If the 2-functor $R : \mathcal{B} \longrightarrow \mathcal{D}$ has a left adjoint, then there is a 2-universal morphism from x to R for all $x \in \mathcal{D}$.*

Proof. Given a 2-adjunction $(L, R, \eta, \varepsilon)$, there is a 2-universal morphism $(Lx, \eta_x : x \longrightarrow RLx)$ from x to $R : \mathcal{B} \longrightarrow \mathcal{D}$ for all $x \in \mathcal{D}$. The equivalence (4.9) is given by



for all $a' \in \mathcal{B}$ and $H : Lx \longrightarrow a'$ in $\mathcal{B}$. Its weak inverse is given by



for all $a' \in \mathcal{B}$ and $G : X \longrightarrow Ra'$ in $\mathcal{D}$. These functors are weakly inverse to one another as their compositions form zigzags (Definition 3.2.2). □

Remark 4.3.9. One may interpret 2-universal morphisms as "local 2-adjunctions", i.e. the 2-functor $R : \mathcal{B} \longrightarrow \mathcal{D}$ has a left adjoint if and only if there is a 2-universal morphism from x to R for all $x \in \mathcal{D}$. Consult [Fio06, Ch. 9] for a proof in the case of strict 2-categories. Because 2-categories are naturally equivalent to strict 2-categories (Theorem 2.4.2), the statement must also hold for non-strict 2-categories [Dé22, Rem. A.1.2].

Definition 4.3.10. *Universal functors*[68] are 2-universal morphisms of **Cat** or sub-2-categories of **Cat**.

[68] This is not standard terminology. Since I did not find another term for this notion, I introduce this term as it fits well.

4.4. Universal 2-Functors

Analogously to Section 4.3 we further categorify universal functors to *universal 2-functors* and show that our definition corresponds to the definition used by [Dé22]. Universal 2-functors will enable *3-universal* characterisations of 2-categories in Section 6.2.

Definition 4.4.1 (Universal 2-Functor)**.** Given 3-categories $\mathcal{T}, \mathcal{D} \subseteq \mathbf{2Cat}$, a 3-functor $\mathcal{F} : \mathcal{T} \longrightarrow \mathcal{D}$, and a 2-category $\mathcal{X} \in \mathcal{D}$, then a *universal 2-functor from $\mathcal{X}$ to $\mathcal{F}$* is a 2-category-2-functor pair

$$(\mathcal{A} \in \mathcal{T}, U : \mathcal{X} \longrightarrow \mathcal{F}\mathcal{A})$$

such that

$$\begin{aligned} \mathcal{F}(-) \,\Box\, U : \mathcal{T}(\mathcal{A}, \mathcal{A}') &\longrightarrow \mathcal{D}(\mathcal{X}, F\mathcal{A}') \\ F &\longmapsto \mathcal{F}F \,\Box\, U \\ \alpha &\longmapsto \mathcal{F}\alpha \,\Box\, U \\ m &\longmapsto \mathcal{F}m \,\Box\, U \end{aligned}$$

is a 2-equivalence for all $\mathcal{A}' \in \mathcal{T}$.

Remark 4.4.2. Universal 2-functors characterise 2-categories up to 2-equivalence, i.e. in the setting of Definition 4.4.1 another 2-category $\widetilde{\mathcal{A}} \in \mathcal{T}$ is part of a universal 2-functor $(\widetilde{\mathcal{A}}, \widetilde{U} : \mathcal{X} \longrightarrow \mathcal{F}\widetilde{\mathcal{A}})$ from $\mathcal{X}$ to $\mathcal{F}$ if and only if there is a 2-equivalence $\mathcal{A} \simeq \widetilde{\mathcal{A}}$.

Proof. Further generalise the proof of Remark 4.3.5 to 3-categories. □

Remark 4.4.3. It is expected that 3-universal morphisms (and, therefore, also universal 2-functors) are “local 3-adjunctions” as in the 1- and 2-categorical cases (Proposition 4.2.6, Remark 4.3.9) [Dé22, Rem. A.2.3].

Lemma 4.4.4 (cf. [Dé22, Def. 1.2.1])**.** *If we are given a 3-functor $\mathcal{F} : \mathcal{T} \longrightarrow \mathcal{D}$ and a 2-category $\mathcal{X} \in \mathcal{D}$, then a 2-category-2-functor pair $(\mathcal{A} \in \mathcal{T}, U : \mathcal{X} \longrightarrow \mathcal{F}\mathcal{A})$ is a universal 2-functor from $\mathcal{X}$ to $\mathcal{F}$ if and only if the following conditions are fulfilled:*

(i) for all 2-categories $\mathcal{A}' \in \mathcal{T}$ and 2-functors $F : \mathcal{X} \longrightarrow \mathcal{F}\mathcal{A}'$ there exists a 2-functor $\overline{F} : \mathcal{A} \longrightarrow \mathcal{A}'$ together with a 2-natural equivalence

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] \arrow[d, "U"'] & \mathcal{F}\mathcal{A}' \\
\mathcal{F}\mathcal{A} \arrow[ur, "\mathcal{F}\overline{F}"'] \arrow[u, phantom, "\simeq\Nearrow"] &
\end{tikzcd} , \tag{4.12}$$

(ii) for all 2-functors $G, H : \mathcal{F}\mathcal{A} \longrightarrow \mathcal{F}\mathcal{A}'$ and 2-natural transformations $\alpha : G \,\square\, U \Longrightarrow H \,\square\, U$ there exists a 2-natural transformation $\overline{\alpha} : G \Longrightarrow H$ and an invertible modification

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "U"] \\
\mathcal{F}\mathcal{A} \arrow[d, bend right=50, "G"'] \arrow[d, bend left=50, "H"] \arrow[d, phantom, "\overset{\overline{\alpha}}{\Longrightarrow}"] \\
\mathcal{F}\mathcal{A}'
\end{tikzcd}
\quad \overset{\cong}{\Rrightarrow} \quad
\begin{tikzcd}
 & \mathcal{X} \arrow[dl, bend right, "U"'] \arrow[dr, bend left, "U"] & \\
\mathcal{F}\mathcal{A} \arrow[rr, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, bend right, "G"'] & & \mathcal{F}\mathcal{A} \arrow[dl, bend left, "H"] \\
 & \mathcal{F}\mathcal{A}' &
\end{tikzcd} , \tag{4.13}$$

and

(iii) for all 2-functors $G, H : \mathcal{F}\mathcal{A} \longrightarrow \mathcal{F}\mathcal{A}'$, 2-natural transformations $\alpha, \beta : G \Longrightarrow H$, and modifications $m : \alpha \,\square\, U \Rrightarrow \beta \,\square\, U$ there exists a unique modification $\overline{m} : \alpha \Rrightarrow \beta$ such that

$$\begin{tikzcd} \mathcal{X} \arrow[d, "U"] \\ \mathcal{F}\mathcal{A} \arrow[d, bend right=60, "G"'] \arrow[d, bend left=60, "H"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}\ \Downarrow \overline{m}\ \underset{\beta}{\Rightarrow}"] \\ \mathcal{F}\mathcal{A}' \end{tikzcd} \quad = \quad \begin{tikzcd} & \mathcal{X} \arrow[dl, bend right, "U"'] \arrow[dr, bend left, "U"] & \\ \mathcal{F}\mathcal{A} \arrow[dr, bend right, "G"'] \arrow[rr, phantom, "\overset{\alpha \,\square\, U}{\Rightarrow}\ \Downarrow m\ \underset{\beta \,\square\, U}{\Rightarrow}"] & & \mathcal{F}\mathcal{A} \arrow[dl, bend left, "H"] \\ & \mathcal{F}\mathcal{A}' & \end{tikzcd} \tag{4.14}$$

.

Proof.

(i) Implied by essential surjectivity of $\mathcal{F}(-) \,\square\, U$.

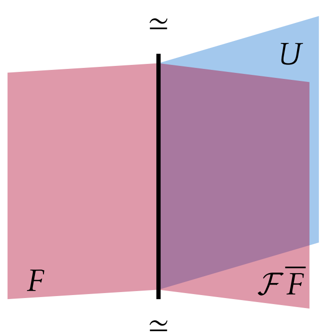
(ii) Essential fullness of $\mathcal{F}(-) \,\square\, U$ implies this equivalence on Hom categories.

(iii) Faithfulness of $\mathcal{F}(-) \,\square\, U$ implies this bijection on 2-morphisms. □

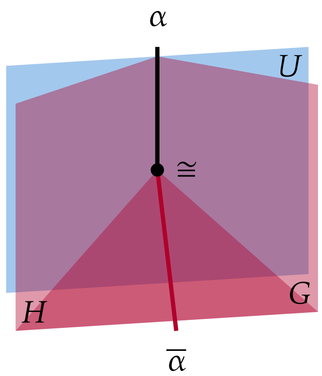
Remark 4.4.5. These pasting diagrams may be reformulated in terms of surface diagrams [BMS24, Def. 2.28]. In these surface diagrams surfaces that are 2-functors that are read from front to back, strings are 2-natural transformations that are read from right to left, and vertices are modifications that are read from bottom to top. The 3-dimensional empty spaces in front of and behind the 2-functors are their domain and codomain 2-categories, respectively.

Here we use red surfaces with two different opacities to indicate differing angles of the same surface. In other words, the borders between different opacities are creases in the red surface. This is supposed to visualise that the red surface is not touching the blue surface except along the black strings and vertices.

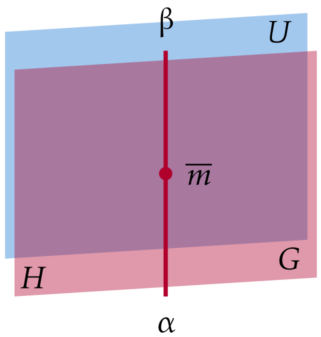
- Diagram (4.12):



- Diagram (4.13):



- Diagram (4.14):



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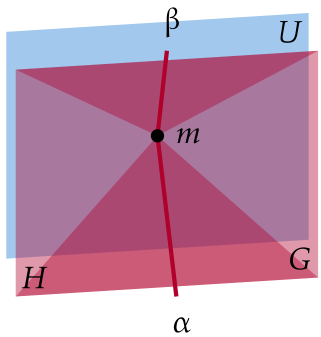


5. Categorifying Idempotents

5.1. Idempotents

Assumption 5.1.1. Let us fix an arbitrary category $\mathcal{C}$ and assume for this section that the objects and morphisms we are working with live in this category unless stated otherwise.

Definition 5.1.2. Endomorphisms $e : X \longrightarrow X$ are *idempotent*[69,70] if $e \circ e = e$.

Example 5.1.3. $1_X : X \longrightarrow X$ is an idempotent for all objects X.

Lemma 5.1.4. *Idempotent isomorphisms are identities.*

Proof. Let $e : X \longrightarrow X$ be an idempotent isomorphism, then

$$e = e \circ e \circ e^{-1} = e \circ e^{-1} = 1_X. \qquad \square$$

Note 5.1.5. Functors map idempotents to idempotents, i.e. if $e : X \longrightarrow X$ is an idempotent, then $Fe : FX \longrightarrow FX$ is an idempotent.

---

[69] “Idempotent” is derived from Latin and means “same power”.

[70] Here “idempotent” is used as an adjective. Alternatively we will also say that idempotent morphisms are *idempotents* (noun).

Preview 5.1.6.

- In Section 5.1.1 we will define *splittings* of idempotents (Definition 5.1.9) and introduce *splitting diagrams* (Observation 5.1.14) as a visual aid. Finally we will see that *splittings* are fully determined by the idempotent they *split* (cf. Note 5.1.17). This foreshadows our later construction of *idempotent completions* (see Sketch 6.1.7). Moreover, our discussion of split idempotents prepares their categorifications in Section 5.2.
- In Section 5.1.2 we will define *split coequalisers* (Definition 5.1.21) and see that they provide a construction of coequalisers from split idempotents (Corollary 5.1.26). In particular, these coequalisers are absolute coequalisers (Corollary 5.1.25). This will later allow us to construct *relative tensor products* as *split relative tensor products* in Section 5.2.4.

5.1.1. Split Idempotents

Definition 5.1.7. A morphism $i : X \longrightarrow Y$ is an *epimorphism* if it is only the "handle" of trivial forks (Definition 4.1.2). In other words, every fork

$$\begin{tikzcd} X \arrow[r, "i"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & Z \end{tikzcd}$$

implies $f = f'$.

Analogously, a morphism $p : Y \longrightarrow Z$ is a *monomorphism* if it is only the "handle" of trivial coforks.

Remark 5.1.8. Monomorphisms may feel familiar, as they are the categorical analogue of injective homomorphisms. One way to think of monomorphisms is by noticing that morphisms $i : X \longrightarrow Y$ induce maps

$$\begin{aligned} i_\star : \mathcal{C}(Z, X) &\longrightarrow \mathcal{C}(Z, Y) \\ f &\longmapsto i \circ f. \end{aligned}$$

By definition, i is a monomorphism if and only if $i_\star$ is injective for all objects $Z \in \mathcal{C}$. This may be stated as i being injective on "generalised elements", i.e. morphisms [Lei14, Def. 5.1.29]. It is therefore natural that morphisms in many common categories such as **Set**, **Grp**, and $\mathbf{Vect}_K$ are monomorphisms precisely if they are injections. However, this does not hold in general, e.g. this does not hold in the category of pointed topological spaces [Bor94, Exs. 1.7.7]

On the other hand, epimorphisms should not feel familiar. Epimorphisms are clearly a dual notion of monomorphisms and should, therefore, intuitively be similar surjections but at the same time cannot be categorifications of surjections because injectivity and surjectivity are not dual notions. In **Set**, epimorphisms and surjections turn out to be equivalent [Lei14, Ex. 5.2.18], but this does not generally hold. For instance, in the category of monoids the canonical inclusion $(\mathbb{N}, +, 0) \hookrightarrow (\mathbb{Z}, +, 0)$ is an epimorphism but clearly not surjective [Lei14, Exe. 5.2.23].

Definition 5.1.9 (Split Idempotents)**.** If we are given two morphisms $\pi : X \longrightarrow Y$ and $\iota : Y \longrightarrow X$ such that

$$\pi \circ \iota = 1_Y$$

then we say that

- $e := \iota \circ \pi : X \longrightarrow X$ is a *split idempotent* or alternatively *e splits*,
- (Y, π, ι) is *splitting data* of *e*,
- Y is a *splitting/retract* of *e*,
- π is a *split epimorphism/retraction/right inverse*, and
- ι is a *split monomorphism/section/left inverse*.

Y may be implicitly denoted by $\mathbf{split}(e)$ as Y is unique up to isomorphism (Lemma 5.1.15).

Note 5.1.10. Split idempotents are idempotents, split monomorphism are monomorphisms, and split epimorphisms are epimorphisms.
Proof that split epimorphisms are epimorphisms. If we are given a split idempotent $e : X \longrightarrow X$ with splitting data (Y, π, ι) and a fork

$$X \xrightarrow{\pi} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z\ ,$$

then the diagram

$$Y \xrightarrow{\iota} X \xrightarrow{\pi} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z$$

provides the fork

$$Y \xrightarrow{1_Y} Y \underset{f'}{\overset{f}{\rightrightarrows}} Z\ ,$$

i.e. $f = f'$. □

Note 5.1.11. While "retraction", "right inverse", and "split epimorphism" are equivalent, we will in the following be using all of these names as they each stress different properties. We call a morphism

- a "retraction" to stress that it splits a given idempotent,
- a "right inverse" to stress that it cancels together with its right inverse, and
- a "split epimorphism" to stress that it is an epimorphism.

This holds analogously for "section", "left inverse", and "split monomorphism".

Example 5.1.12. One of the most familiar examples of an idempotent is the absolute value map

$$\begin{aligned} |\cdot| : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto |x| \end{aligned}$$

which has splitting data given by

$$\left(\mathbb{R}_{\geq 0}, \begin{array}{c} \pi : \mathbb{R} \twoheadrightarrow \mathbb{R}_{\geq 0}, \\ x \longmapsto |x| \end{array} \begin{array}{c} \iota : \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R} \\ y \longmapsto y \end{array} \right)$$

From this example we can also see why π is called a retraction. Topologically, the space $\mathbb{R}$ was retracted to $\mathbb{R}_{\geq 0}$, the image of $|\cdot|$.

Remark 5.1.13. In the literature it is not uncommon to find the splitting $\mathbf{split}(e)$ of an idempotent e being referred to instead as the image $\mathbf{im}(e)$ of e (Example 4.0.3). This is because images are familiar from other mathematical contexts and when $\mathbf{split}(e)$ exists, then $(\mathbf{split}(e), \iota_e)$ fulfil the universal property (4.4), i.e. $\mathbf{im}(e)$ exists and $\mathbf{im}(e) = \mathbf{split}(e)$.

However, it is possible that $\mathbf{im}(e)$ exists while $\mathbf{split}(e)$ does not. For instance, if we are given a "walking idempotent" (Remark 5.2.20), i.e. a category with one object X and one non-trivial morphism $e : X \longrightarrow X$ that is idempotent, then there is no splitting $\mathbf{split}(e)$ but $\mathbf{im}(e) = X$ together with the monomorphism 1_X fulfil the universal property of images (4.4).

Nonetheless, it is reasonable to refer to splittings as images since splittings are implied by images in many typical categories. As we have seen, the image of **Set**-valued maps (4.2) always exists and allows the splitting (4.3). The same holds in some other common categories, in particular it in $\mathbf{vect}_K$.

Observation 5.1.14. Idempotents split if and only if the diagram

$$\begin{tikzcd} X \arrow[loop above, "e"] \arrow[d, bend left, "\pi"] \\ Y \arrow[u, bend left, "\iota"] \arrow[loop below, "1_Y"] \end{tikzcd}$$

commutes for "non-empty paths". Hereby, empty paths are empty compositions of morphisms, i.e. identities that need not be given as a path in the diagram. We must exclude empty paths, because the empty path $1_X : X \longrightarrow X$ is not generally equal to the other paths $X \longrightarrow X$. We will refer to such almost commutative diagrams as *splitting diagrams*.[71]

Lemma 5.1.15. *Splittings are unique up to isomorphism.*

Proof. The statement means that if we are given two sets of splitting data (Y, π, ι) and (Y', π', ι') for an idempotent e, then $Y \cong Y'$. This is true because

$$\begin{tikzcd} Y \arrow[loop left, "1_Y"] \arrow[rr, bend left, "\pi' \circ \iota"] & & Y' \arrow[ll, bend left, "\pi \circ \iota'"] \arrow[loop right, "1_{Y'}"] \end{tikzcd}$$

commutes. □

Lemma 5.1.16. *Functors map splitting data to splitting data.*

Proof. Given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and a split idempotent $e : X \longrightarrow X$ in $\mathcal{C}$ with splitting data (Y, π, ι), then $(FY, F\pi, F\iota)$ splits $Fe : FX \longrightarrow FX$. □

---

[71] "Splitting diagrams" is not standard terminology.

Note 5.1.17. Splittings are fully determined by the idempotents they split. Specifically, morphism factoring through a splitting $Y = \mathbf{split}(e)$ correspond to morphisms factoring through $e : X \longrightarrow X$. For instance, if a morphism $f : V \longrightarrow W$ decomposes as

$$f = \left(V \xrightarrow{g} Y \xrightarrow{h} W \right),$$

then we see that f factors through e:

$$\begin{tikzcd}
 & X \arrow[loop above, "e"] \arrow[d, bend left, "\pi"] & \\
V \arrow[r, "g"'] & Y \arrow[u, bend left, "\iota"] \arrow[r, "h"'] & W
\end{tikzcd} .$$

Conversely, if a morphism $f : V \longrightarrow W$ decomposes as

$$f = \left(V \xrightarrow{\widetilde{g}} X \xrightarrow{e} X \xrightarrow{\widetilde{h}} W \right),$$

then f factors through Y via

$$f = \left(V \xrightarrow{\pi \circ g} Y \xrightarrow{h \circ \iota} W \right).$$

Equivalently,

$$\begin{tikzcd}
\mathcal{C}(X, -) \circ e \arrow[rr, bend right, "(-) \circ \pi"'] & & \mathcal{C}(Y, -) \arrow[ll, bend right, "(-) \circ \iota"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
e \circ \mathcal{C}(-, X) \arrow[rr, bend right, "\iota \circ (-)"'] & & \mathcal{C}(-, Y) \arrow[ll, bend right, "\pi \circ (-)"']
\end{tikzcd}$$

are isomorphisms.

Lemma 5.1.4 may be seen as a corollary of this remark. However, it is important to note that they are not equivalent statements. This remark tells us how the splitting of a given idempotent must look like regardless if it actually splits! Without further context the utility of this distinction may seem unclear, but this insight will be invaluable in Section 6.1 where we will construct categories where all idempotents split (Sketch 6.1.7).

5.1.2. Split Coequalisers

Remark 5.1.18. Since splittings are fully determined (Note 5.1.17), splittings should be universally characterisable (Lemma 5.1.19).

Lemma 5.1.19. *Splittings* (Y, π, ι) *of idempotents* $e : X \longrightarrow X$ *are the same as the coequalisers*

$$\begin{tikzcd} X \arrow[r, shift left, "e"] \arrow[r, shift right, "1_X"'] & X \arrow[r, "\pi"] & Y . \end{tikzcd} \tag{5.1}$$

Proof. Being the same means if either of them exists the other exists too and their data coincide.

"⇒": If we are given splitting data (Y, π, ι) of e and a cofork

$$\begin{tikzcd} X \arrow[r, shift left, "e"] \arrow[r, shift right, "1_X"'] & X \arrow[r, "\phi"] & Z , \end{tikzcd}$$

then

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\phi"'] & Y \arrow[d, "\zeta"] \\ & Z \end{tikzcd}$$

commutes for $\zeta := \phi \circ \iota$. ζ is unique because π is an epimorphism (Lemma 4.1.6).

"⇐": If (5.1) is a coequaliser, then there exists a unique morphism $\iota : Y \longrightarrow X$ s.t.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "e"'] & Y \arrow[d, dashed, "\iota"] \\ & X \end{tikzcd}$$

commutes. Since π coequalises $(e, 1_X)$ we see that

$$\begin{aligned} \pi \circ \iota \circ \pi &= \pi \circ e \\ &= \pi \circ 1_X \\ &= 1_Y \circ \pi. \end{aligned}$$

Since coequalisers are epimorphisms, this implies $\pi \circ \iota = 1_Y$. □

Note 5.1.20. The coequaliser

$$\begin{tikzcd} X \arrow[r, shift left, "e"] \arrow[r, shift right, "1_X"'] & X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is absolute when e is idempotent since functors map splittings to splittings (Lemma 5.1.16).

Definition 5.1.21. A *split coequaliser* is given by morphisms

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[l, bend right=60, "t"'] \arrow[r, "\pi"] & Y \arrow[l, bend right=60, "\iota"'] \end{tikzcd}$$

such that

$$f' \circ t = \iota \circ \pi$$

and such that

$$\begin{tikzcd} W \arrow[loop above, "t \circ f"] \arrow[d, bend left, "f"] \\ X \arrow[u, bend left, "t"] \arrow[loop below, "1_X"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[loop above, "\iota \circ \pi"] \arrow[d, bend left, "\pi"] \\ Y \arrow[u, bend left, "\iota"] \arrow[loop below, "1_Y"] \end{tikzcd}$$

are splittings [ML78, p. 149].

Intuition 5.1.22. One way of thinking of split coequalisers is as sets

$$Y \subseteq X \subseteq W$$

with canonical inclusions ι, t and a map $f' : W \longrightarrow X \subseteq W$ such that

$$f'|_X = \pi : X \twoheadrightarrow Y \subseteq X.$$

Lemma 5.1.23. *If we are given morphisms*

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[l, bend right=80, "t"'] \end{tikzcd}$$

s.t.

$$\begin{tikzcd} W \arrow[loop above, "t \circ f"] \arrow[d, bend left, "f"] \\ X \arrow[u, bend left, "t"] \arrow[loop below, "1_X"] \end{tikzcd}$$

is a splitting diagram, then

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is a cofork/coequaliser if and only if

$$\begin{tikzcd} X \arrow[r, shift left, "1_X"] \arrow[r, shift right, "f' \circ t"'] & X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is a cofork/coequaliser.

Proof. The correspondence of coforks is given by precomposing the coforks with t and f, respectively. The correspondence of coforks implies a correspondence of coequalisers (Lemma 4.1.5). □

Corollary 5.1.24. *Split coequalisers are coequalisers, i.e. the split coequaliser*

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[l, bend right=60, "t"'] \arrow[r, "\pi"] & Y \arrow[l, bend right=60, "\iota"'] \end{tikzcd}$$

contains the coequaliser

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[r, "\pi"] & Y \end{tikzcd} .$$

Proof. Apply Lemma 5.1.23 together with Lemma 5.1.19. ☐

Corollary 5.1.25. *Split coequalisers are absolute coequalisers.*

Proof. Functors preserve relations (Theorem A.1.27). ☐

Corollary 5.1.26. *If we are given morphisms*

$$\begin{tikzcd} W \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & X \arrow[l, bend right=60, "t"'] \end{tikzcd}$$

s.t. $e := f' \circ t$ *is idempotent and*

$$\begin{tikzcd} W \arrow[loop above, "t \circ f"] \arrow[d, bend left, "f"] \\ X \arrow[u, bend left, "t"] \arrow[loop below, "1_X"] \end{tikzcd}$$

is a splitting diagram, then coequalisers of (f, f') *are splittings of* e.

5.2. 2-Idempotents

Preview 5.2.1. As motivated in Section 1.1 we are interested in categorifications of split idempotents. Therefore, this chapter covers the following sections.

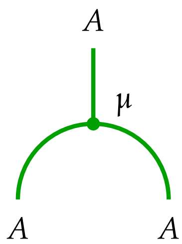
- **Section 5.2.1. Algebras:** We categorify idempotents as *algebras*.
- **Section 5.2.2. Split Algebras:** We categorify split idempotents as *split algebras*.
- **Section 5.2.3. Modules:** We introduce *modules* as 1-morphisms between algebras.
- **Section 5.2.4. Relative Tensor Products:** We introduce *relative tensor products* as a composition of modules and show how to construct relative tensor products as idempotent splittings.
- **Section 5.2.5. Condensation:** We discuss *condensation*, i.e. split Δ-separable Frobenius algebras.
- **Section 5.2.6. Orbifold Condensation:** We extend our discussion of condensation to *orbifold condensations*, i.e. split symmetric Δ-separable Frobenius algebras.

Assumption 5.2.2. We fix a 2-category $\mathcal{B}$ and an endomorphism category $\mathcal{C} := \mathcal{B}(\star, \star)$ for the rest of this chapter. In Section 5.2.1 we assume our objects and morphisms to be living in $\mathcal{C}$ unless stated otherwise. In the remaining section we will assume our objects, 1-morphisms, and 2-morphisms live in $\mathcal{B}$ unless stated otherwise. Additionally, we assume that $\mathcal{B}$ is pivotal in Section 5.2.6.

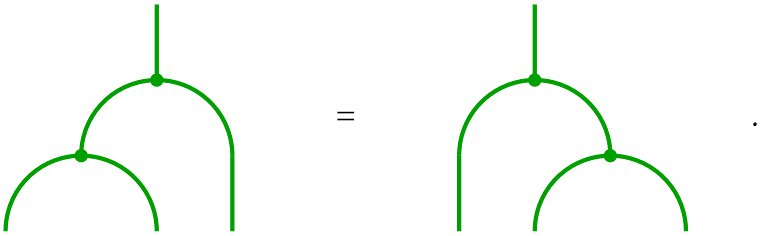
5.2.1. Algebras

Note 5.2.3. While we are interested in categorifications of idempotents in 2-categories, it will suffice to consider the fixed monoidal category $\mathcal{C}$ in this section. The reason is that idempotents $e : X \longrightarrow X$ are endomorphisms in a category $\mathcal{D}$, so idempotents are actually elements of a monoid $e \in \mathcal{D}(X, X)$. Analogously, categorifications of idempotents are 1-endomorphisms $A : a \longrightarrow a$ in a 2-category $\mathcal{A}$, i.e. objects of a monoidal category $\mathcal{A}(a, a)$.

Definition 5.2.4 (Algebras)**.** An *algebra*[72] in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with a morphism

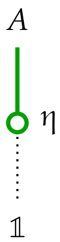


that we call *multiplication*. We require[73] μ to fulfil the *associativity* condition[74]

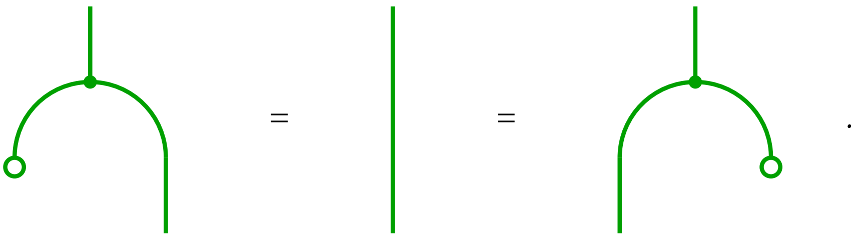


(5.2)

An algebra[75] A is *unital* if it is also equipped with a *unit*



that fulfils



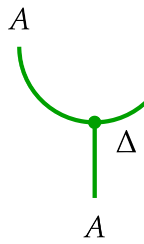
[72] Algebras are commonly also known as "monad". In the literature most relevant to this thesis both are in use [GJF19, Fra22, Dé22, CR16, CM23]. Therefore, we choose "algebra" in alignment with [CR16, CM23].

[73] Some authors do not require this condition and instead call (A, μ) an "associative algebra" if the condition (5.2) is fulfilled.

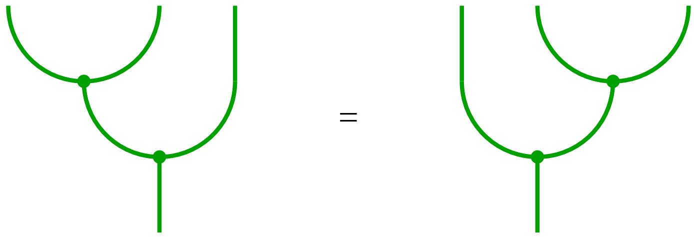
[74] We often suppress labels when it they are implicitly clear. In particular, the multiplication of an algebra is usually implicitly clear.

[75] When we say "an algebra A" it is a shorthand with which we implicitly mean all the algebra data $(A, \mu : A \otimes A \longrightarrow A)$. In particular, if we have not previously specified the multiplication of an algebra A, then we may implicitly denote it by $\mu_A : A \otimes A \longrightarrow A$. While this is imprecise as there could be multiple multiplications for the same object $A \in \mathcal{C}$, this is of no concern as we never consider two algebras that specifically have the same underlying object.

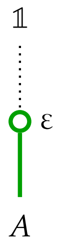
Analogously, a *coalgebra* is an object $A \in \mathcal{C}$ together with a *comultiplication*



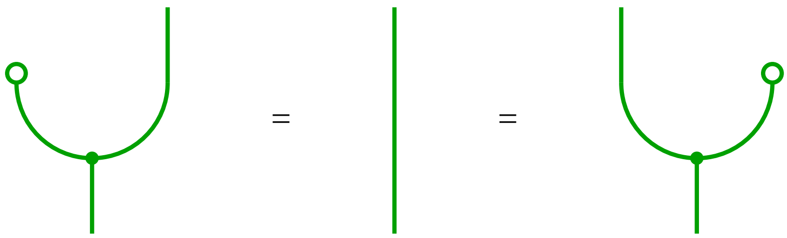
such that (A, Δ) fulfils the *coassociativity condition*



Coalgebras are *counital* if they are equipped with a *counit*

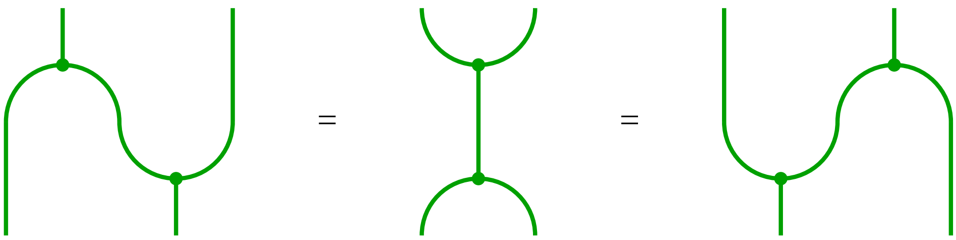


such that

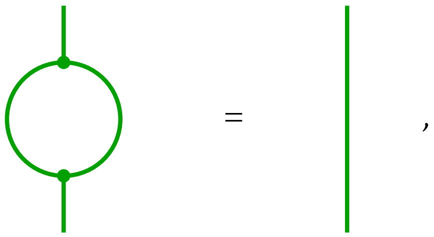


is fulfilled.

A *Frobenius algebra* is an object $A \in \mathcal{C}$ is equipped with both an algebra and a coalgebra structure and such that the *Frobenius relations*

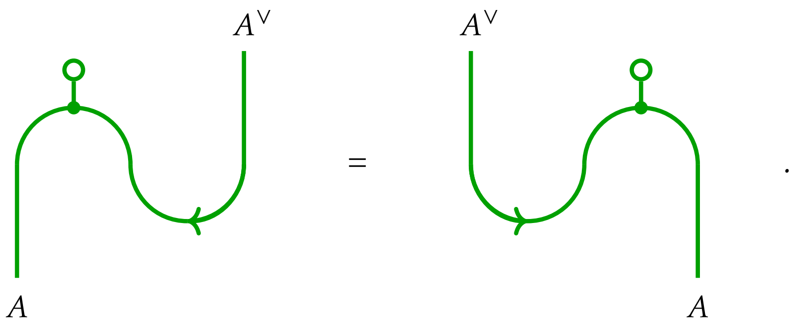


are fulfilled. Moreover, if



then A is called Δ-*separable*. A Frobenius algebra is *unital* if it is unital as an algebra and counital as a coalgebra.

If $\mathcal{C}$ is pivotal[76] (Definition 3.3.4), then a unital Δ-separable Frobenius algebra $A \in \mathcal{C}$ is called *symmetric* if

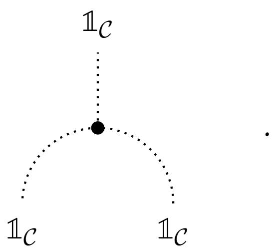


(5.3)

Remark 5.2.5. The following statements usually extend to various forms of algebras even when it is not explicitly stated.

Example 5.2.6. $\mathbb{1}_\mathcal{C}$ is trivially a unital Δ-separable Frobenius algebra with multiplication

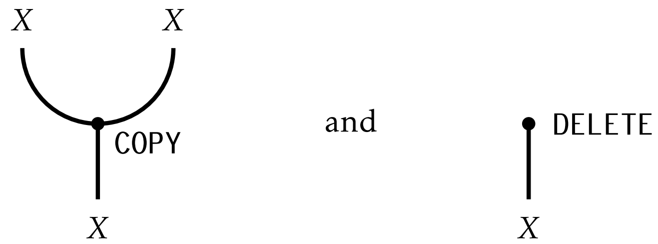
$$\rho_{\mathbb{1}_\mathcal{C}} \equiv \lambda_{\mathbb{1}_\mathcal{C}} \equiv$$



Example 5.2.7. The generators of the oriented bordism category $\mathbf{Bord}_{2,1}$ (Section 1.1.2) form a unital Frobenius algebra.

[76] In Section 3.3 we only defined *pivotality* for 2-categories. However, pivotal categories are simply pivotal 2-categories with a single object, i.e. monoidal categories together with extra structure fulfilling the pivotality conditions.

Example 5.2.8. While this thesis is motivated by TQFT, algebras are also of interest in computer science. As a very concrete example, the operations

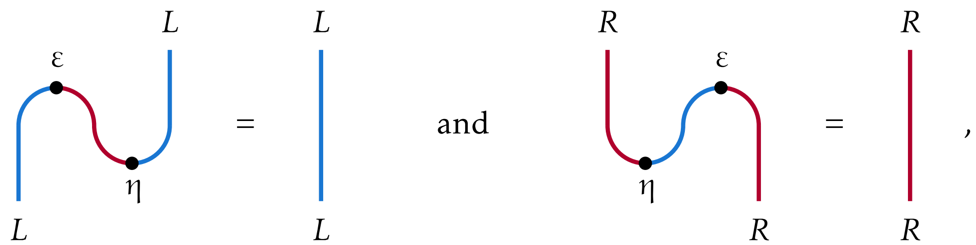


form a counital coalgebra. For instance, if X is a bit, i.e. if $X \in \{0, 1\}$, then

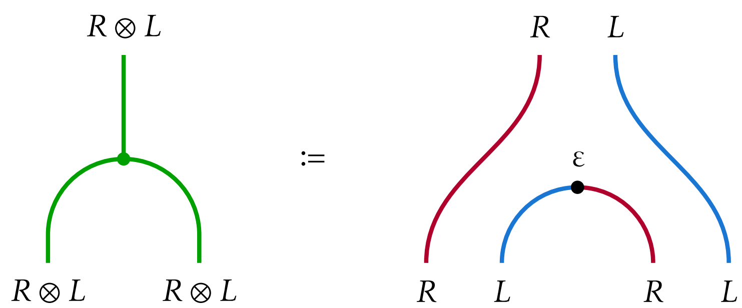
$$\texttt{COPY}(0) = (0, 0) \qquad \texttt{DELETE}(0) = \emptyset$$
$$\texttt{COPY}(1) = (1, 1) \qquad \texttt{DELETE}(1) = \emptyset.$$

However, there are also more general abstractions such as modelling computations. Computations may form an algebra with multiplication given by composition of computations and unit given by identity computations [HM23, Rem. 3.16].

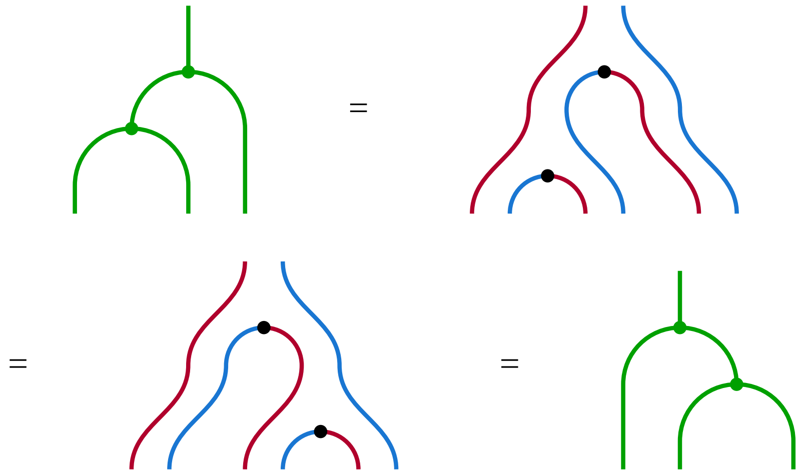
Example 5.2.9. Adjunction data induces unital algebras. For instance, if we are given zigzag identities



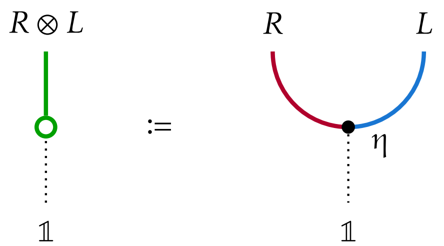
then $R \otimes L$ has a multiplication

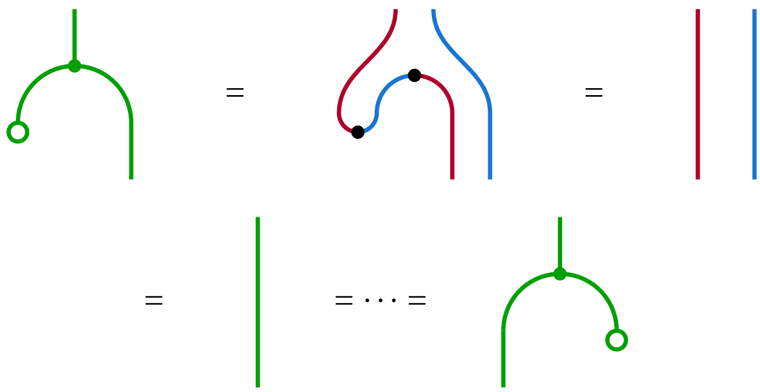


that fulfils associativity:



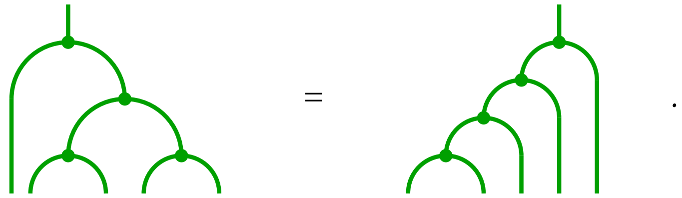
Naturally, the unit of the adjunction $L \dashv R$ is the unit of the algebra $R \otimes L$:





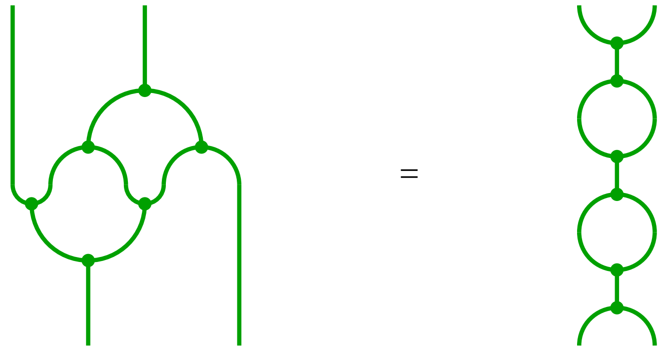
Note 5.2.10. One of the first exercises in group theory is showing that if a group has two identity elements e, e', then $e = e'$. Analogously, if an algebra (A, μ) has two units $\eta, \eta' : \mathbb{1} \longrightarrow A$, then $\eta = \eta'$. In particular, this means that unital algebras are not actually algebras with extra structure but rather algebras that fulfil an extra property.

Note 5.2.11. Given an algebra A, all compositions of μ_A forming a morphism $A^{\otimes n} \longrightarrow A$ are equal. For instance,



Thus, the r.h.s. of this equation is a *normal form* of algebra multiplications for a single "connected component". A composition of μ_A forming a morphism $A^{\otimes n} \longrightarrow A^{\otimes k}$ would have k connected components with a normal form given by writing each connected component in its normal form. Coalgebras have an analogous normal form.

The normal form of Frobenius algebras is essentially the same but depends also on the number of "holes"; for instance:



Compositions with multiple connected components with varying numbers of holes are handled analogously. In particular, we see that Δ-separable Frobenius algebras do not depend on the number of holes and are therefore more similar to algebras and coalgebras in this regard.

Remark 5.2.12. As motivated in Section 1.2, algebras are categorifications of idempotents. Namely, if we are given a 2-category $\mathcal{C}$ that only has trivial 2-morphisms (Remark 2.3.5), then if $A : a \longrightarrow a$ in $\mathcal{C}$ is an algebra, then the multiplication $\mu : A \otimes A \Longrightarrow A$ implies $A = A \otimes A$, i.e. implies idempotence of A.

One may wonder which type of algebra is "the most fitting" categorification of an idempotent. The information discussed in this thesis suggests that Δ-separable Frobenius algebras are the most fitting but in other contexts other algebras, e.g. symmetric Δ-separable Frobenius algebras, will be more fitting. The reasons are:

Unital algebras categorify identities: If $\mathcal{C}$ is still a 2-category with trivial 2-morphisms and $A : a \longrightarrow a$ is a unital algebra, then the unit $\eta : \mathbb{1}_a \Longrightarrow A$ implies $A = \mathbb{1}_a$. Therefore, unital algebras are not just a categorification of idempotents but rather a categorification of identities [GJF19, p. 19]. Thus, non-unital algebras are more fitting.

Unit morphisms are condensation algebras: Every unit morphism $\mathbb{1}_a$ is a Δ-separable Frobenius algebra. In particular, unit morphisms are Δ-separable Frobenius algebras uniquely up to isomorphism. This differs, for instance, from Frobenius algebras $(\mathbb{1}_a, \mu, \Delta)$, where $\mu : \mathbb{1}_a \otimes \mathbb{1}_a \Longrightarrow \mathbb{1}_a$ and $\Delta : \mathbb{1}_a \Longrightarrow \mathbb{1}_a \otimes \mathbb{1}_a$ are arbitrary.

Condensation algebras are not limited: On a related note, condensation algebras are not limited to 2-categories with extra structure. Every non-empty 2-category contains condensation algebras, but, for instance, symmetric Frobenius algebras only exist in pivotal 2-categories.

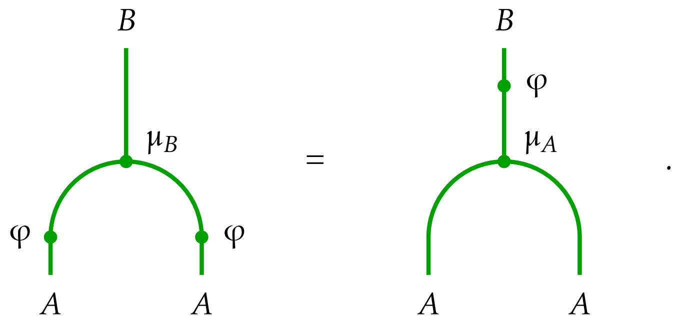
Δ-separable Frobenius algebras feel natural: They are "symmetric" in the sense that "co-Frobenius algebras" are Frobenius algebras. Moreover, Δ-separability provides a satisfying normal form (Note 5.2.11).

Properties of idempotents lift: "Idempotent isomorphisms are identities" (Lemma 5.1.4) lifts to "Δ-separable Frobenius algebras that are equivalences are units" (Lemma 5.2.17). Moreover, splittings of idempotents are fully determined by the idempotent they split (Note 5.1.17) and the same is true for *splittings* of Δ-separable Frobenius algebras in certain 2-categories (Note 5.2.64).

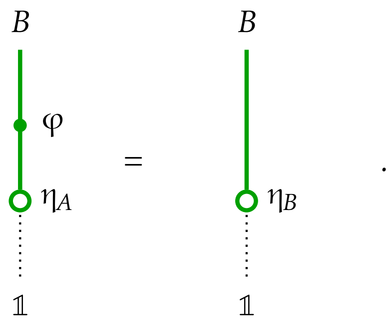
Δ-separable Frobenius algebras are 2-condensation algebras: There exist recursively defined notion of n-idempotents, called *n-condensation algebras* (Sketch 7.2.2) s.t.

- 0-condensation algebras are equalities in sets,
- 1-condensation algebras are idempotents in categories, and
- 2-condensation algebras are Δ-separable Frobenius algebras in 2-categories.

Definition 5.2.13. An *algebra map* is a morphism $\varphi : A \longrightarrow B$ between algebras $A, B \in \mathcal{C}$ with the property



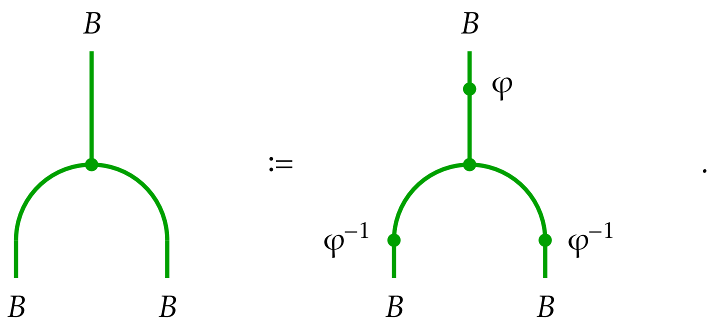
If A is unital we furthermore impose



(5.4)

We define *coalgebra maps* analogously. Naturally, *Frobenius algebra maps* must be algebra maps and coalgebra maps simultaneously.

Example 5.2.14. Given an algebra A and an isomorphism $\varphi : A \longrightarrow B$, then B is an algebra with multiplication



Here, the l.h.s. multiplication represents μ_B and the r.h.s. multiplication represents μ_A. In particular, φ is an *algebra isomorphism*.

Definition 5.2.15. We may define $\mathbf{Alg}(\mathcal{C})$, the *category of algebras in* $\mathcal{C}$, consisting of:

- ***Objects:*** algebras (A, μ_A) in $\mathcal{C}$,
- ***Morphisms:*** algebra maps $\phi : A \longrightarrow B$ in $\mathcal{C}$, and
- ***Composition:*** composition of morphisms in $\mathcal{C}$.

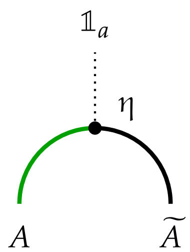
In general $\mathbf{Alg}(\mathcal{C})$ will not be monoidal since morphisms $(A \otimes B) \otimes (A \otimes B) \longrightarrow (A \otimes B)$ need not exist.

Example 5.2.16. A vector space $A \in \mathbf{Vect}_K$ (Example 2.3.8) is a K-algebra exactly if it is an algebra according to Definition 5.2.4. Thus,

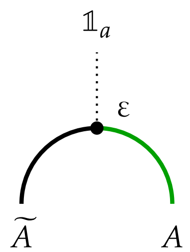
$$\mathbf{Alg}_K \simeq \mathbf{Alg}(\mathbf{Vect}_K).$$

Lemma 5.2.17. *A Δ-separable Frobenius algebra*[77] $A : a \longrightarrow a$ *is an equivalence exactly when* $A \cong \mathbb{1}_a$.

Proof. **"⇒":** Let us assume we are given a Δ-separable Frobenius algebra $(A : a \longrightarrow a, \mu, \Delta)$ and an equivalence given by the 2-isomorphisms



and



.

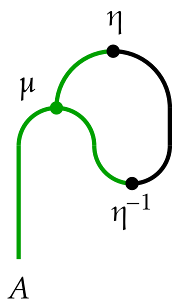
W.l.o.g. we may assume that the equivalence is an adjoint equivalence (Corollary 3.1.10). We draw the weak inverse $\widetilde{A} : a \longrightarrow a$ using black lines although $\widetilde{A}$ is a Δ-separable Frobenius algebra, because we will not need the algebra structure of $\widetilde{A}$.

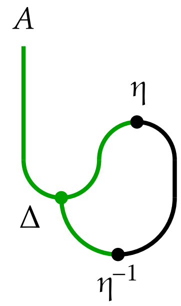
[77] Actually, we do not even require associativity or the Frobenius relations but rather just require Δ-separability.

In the following, we shall construct a 2-isomorphism $A \cong \mathbb{1}_a$ by replacing the equations of Lemma 5.1.4 by 2-morphisms, i.e. we construct a 2-morphism

$$\begin{aligned} A &\Rightarrow A \otimes A \otimes \widetilde{A} \\ &\Rightarrow A \otimes \widetilde{A} \\ &\Rightarrow \mathbb{1}_a \end{aligned} \tag{5.5}$$

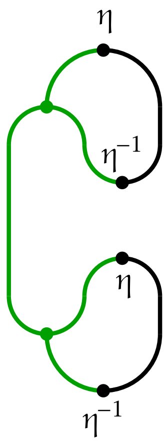
and a 2-morphism in the opposite direction. Explicitly, the 2-morphism sketched in (5.5) and the reversed 2-morphism are given by

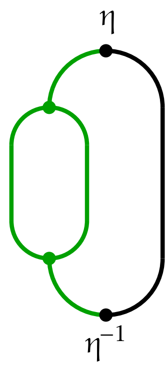
$\phi :=$ 

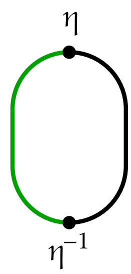
and $\phi_h :=$ 

,

respectively. In the following we want to see that they are inverse to one another. For one, ϕ is left inverse to ϕ_h:

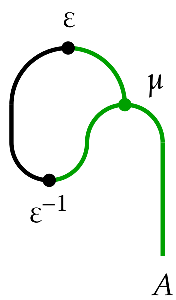


$=$ 

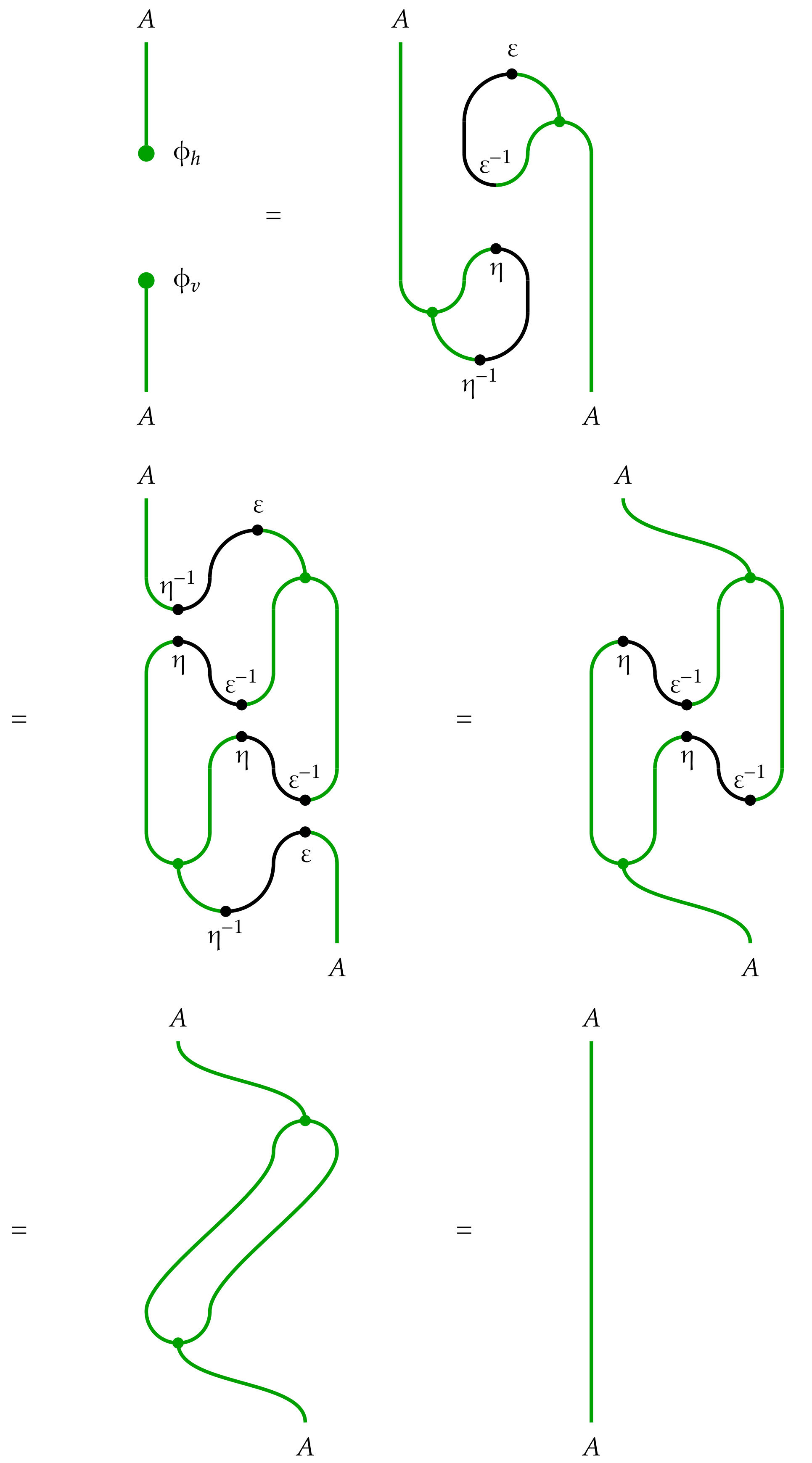
$=$ 

$= 1_{\mathbb{1}_a}$

Next, we define a vertically flipped version of ϕ

$\phi_v :=$ 

and see that it is a right inverse to ϕ_h:


A
φ_h
φ_v
A
=
A
ε
ε^{-1}
η
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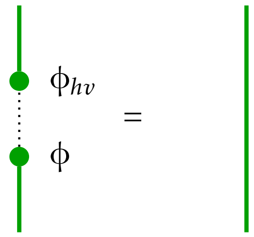
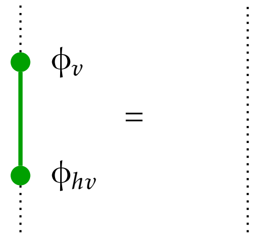
A
=
A
A

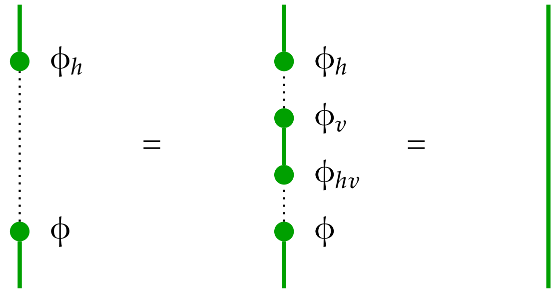
Defining a 2-morphism ϕ_{hv} in line with ϕ, ϕ_h, and ϕ_v we see that due to symmetry we must have



and



which we use to see that ϕ is right inverse to ϕ_h:



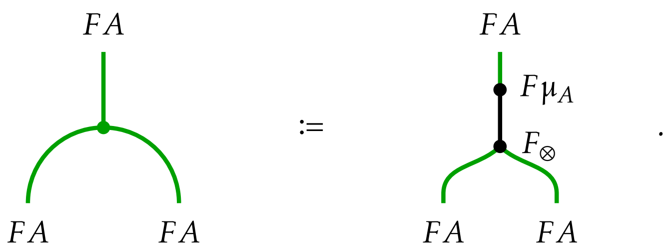
In conclusion, $A \cong \mathbb{1}_a$.

"$\Leftarrow$": Unit 1-morphisms are trivially Δ-separable Frobenius algebras (Example 5.2.6) and equivalences. □

Corollary 5.2.18. *Let us assume we are given a 2-isomorphism $A \otimes A \cong A$.*[78] *If A is an equivalence, then $A \cong \mathbb{1}_a$.*

Proof. Lemma 5.2.17 used Δ-separability of A but did not use associativity of A. Thus, the same proof proves this corollary. □

Proposition 5.2.19. *Monoidal functors map algebras to algebras. Specifically, given a monoidal functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and an algebra $A \in \mathcal{C}$, then FA has the multiplication*



Furthermore, algebra maps are mapped to algebra maps. In particular, F induces a functor $F : \mathbf{Alg}(\mathcal{C}) \longrightarrow \mathbf{Alg}(\mathcal{D})$.

Analogous statements hold for other types of algebras. In particular, pivotal monoidal functors map symmetric algebras to symmetric algebras.

[78] One may think of such a 1-morphism A as a "weak idempotent".

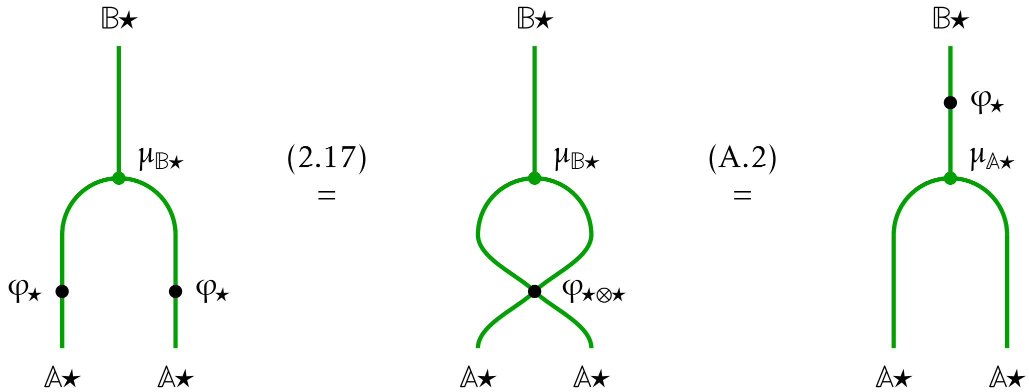
Proof. Associativity of μ_{FA} follows from the associativity condition of μ_A and the associativity condition (2.9) of $F_\otimes$.

In the case of symmetric algebras pivotality of monoidal functors (3.4) ensures fulfilment of the symmetry condition (5.3). $\square$

Remark 5.2.20. Instead of defining algebras as data (A, μ_A) of a monoidal category $\mathcal{C}$, one could define them as monoidal functors. To do this we would define a monoidal category $I_{\mathbf{Alg}}$ generated by a single object $\star \in I_{\mathbf{Alg}}$. Hereby, "generated" means that all objects of $I_{\mathbf{Alg}}$ are horizontal compositions of $\star$ and $\mathbb{1}_{I_{\mathbf{Alg}}}$. The morphisms of $I_{\mathbf{Alg}}$ are generated by a multiplication $\mu : \star \otimes \star \longrightarrow \star$. In this setup, the image of each monoidal functor $\mathbb{A} : I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ corresponds to an algebra $\mathbb{A}\star \in \mathcal{C}$ (Proposition 5.2.19). Conversely, each algebra $A \in \mathcal{C}$ corresponds to a monoidal functor $I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ that maps $\star \longmapsto A$.

This monoidal category $I_{\mathbf{Alg}}$ is called the *walking algebra*. Defining algebras in $\mathcal{C}$ via monoidal functors $I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ corresponds to the approach taken by [GJF19]. The advantage of this "walking construction" approach is that monoidal functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ trivially map walking algebras $\mathbb{A} : I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ to walking algebras $F \circ \mathbb{A} : I_{\mathbf{Alg}} \longrightarrow \mathcal{D}$.

Furthermore, monoidal natural transformations $\varphi : \mathbb{A} \Longrightarrow \mathbb{B}$ between walking algebras $\mathbb{A}, \mathbb{B} : I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ induce algebra maps and vice versa. This is shown by



where we assume strictness of $\mathbb{A}$ and $\mathbb{B}$ to improve readability.

Overall, this correspondence is an equivalence

$$\mathbf{Alg}(\mathcal{C}) \simeq \mathbf{MonFun}(I_{\mathbf{Alg}}, \mathcal{C})$$

where $\mathbf{MonFun}(I_{\mathbf{Alg}}, \mathcal{C})$ denotes the category of monoidal functors $I_{\mathbf{Alg}} \longrightarrow \mathcal{C}$ and monoidal natural transformations between them.

Naturally, one may generalise the "walking" construction to any structure $\mathbb{S}$ given by a collection of data $\mathcal{U}$ internal to a category $\mathcal{C}$. If this collection of data generates a subcategory $\langle \mathcal{U} \rangle \subseteq \mathcal{C}$, then functors $\langle \mathcal{U} \rangle \longrightarrow \mathcal{C}$

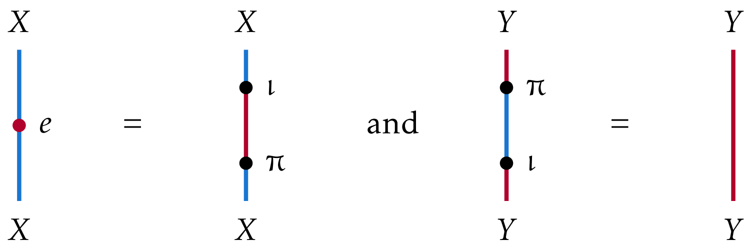
are walking $\mathbb{S}$. In particular, one may extend the walking construction to other types of algebras, split algebras, and modules.

Alternatively, one could define walking algebras as *lax monoidal functors* [JY21, Def. 1.2.14 and Def. 4.1.2] $\{\star\} \longrightarrow \mathcal{C}$, where $\{\star\}$ denotes the trivial monoidal category (Example 2.1.4). Analogously, one could define walking Frobenius algebras as *Frobenius monoidal functors* $\{\star\} \longrightarrow \mathcal{C}$.

5.2.2. Split Algebras

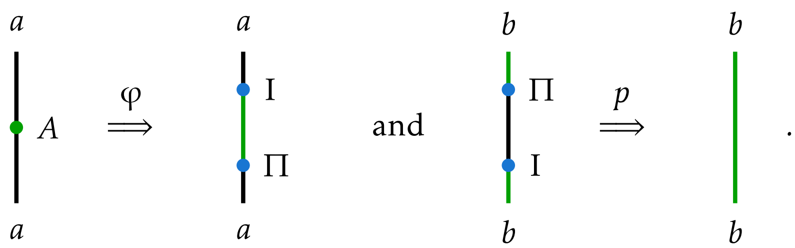
Reminder 5.2.21 (Assumption 5.2.2)**.** In this section and the following sections we fix a 2-category $\mathcal{B}$ and assume our objects, 1-morphisms, and 2-morphisms live in $\mathcal{B}$ unless stated otherwise.

Derivation 5.2.22. We recall that splitting data of an idempotent $e : X \longrightarrow X$ in a category $\mathcal{C}$ is given by data $(Y \in \mathcal{C}, \pi : X \longrightarrow Y, \iota : Y \longrightarrow X)$ such that



(5.6)

are fulfilled. One way of interpreting this is that the idempotent e is a 0-dimensional point that is being "stretched" and, thus, becomes a 1-dimensional interval Y.[79] Therefore, a splitting of an algebra $A : a \longrightarrow a$ in a 2-category $\mathcal{B}$ should consist of an object $b \in \mathcal{B}$, 1-morphisms $\Pi : a \longrightarrow b$ and $\mathrm{I} : b \longrightarrow a$[80], and 2-morphisms

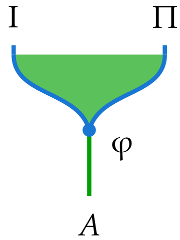


(5.7)

[79] This is motivated as being the opposite of "compressing" ι and π to their composite $\iota \circ \pi = e$. However, there is also an opposing intuition of interpreting Y as an interval completely covered with points e [CPJP22, p. 2]. This interpretation also feels natural in the sense that it corresponds to putting n copies of e on X and then taking a limit $n \longrightarrow \infty$. Having two such opposing interpretations may be thought of as being analogous to interpreting a cylinder as being a geometric prism with 1 face or as being a geometric prism with ∞ faces.

[80] The I is a capital ι.

These diagrams may be thought of as partially dualised pasting diagrams (Derivation 2.2.3). Principally, (5.7) is indeed a categorification of split idempotents because we retrieve the relations (5.6) when φ and p are identities. The directions of φ and p may seem arbitrary at first glance, but we shall give them meaning. To do this, we first rewrite them in our standard string diagrammatic notation as



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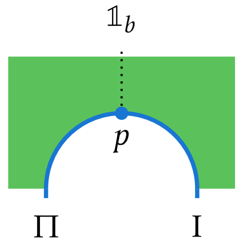
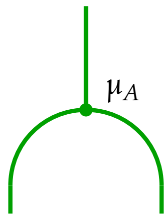


(5.8)

where we colour b green to visualise that the object b may be thought of as the 1-dimensional line A having been "stretched" to a 2-dimensional surface. Now, since $\mathrm{I} \otimes \Pi$ is an algebra (Example 5.2.9) it is natural to require φ to be an isomorphism because this turns φ into an invertible algebra map (Example 5.2.14). In particular, A becomes an algebra with multiplication



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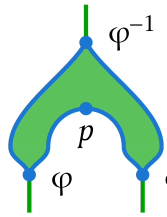
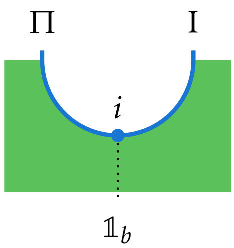


, (5.9)

which may be thought of as "stretching" the 0-dimensional point μ_A to a 2-dimensional surface b.

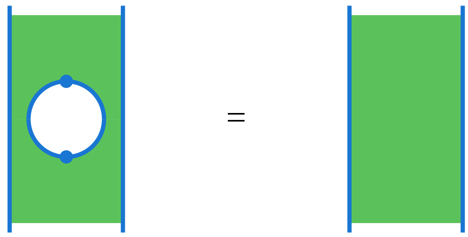
In summary, an algebra $(A : a \longrightarrow a, \mu_A)$ splits if such 1-morphisms Π and I and 2-morphisms (5.8) exist s.t. the relation (5.9) is fulfilled. Analogously, we may split coalgebras via φ and a morphism



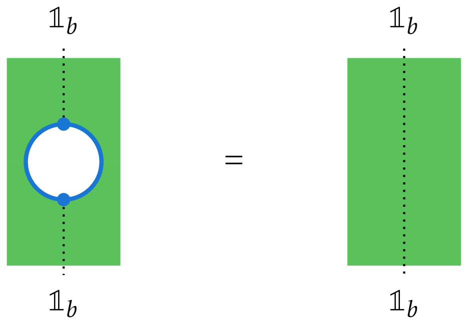
(5.10)

fulfilling a condition analogous to (5.9).

Naturally, splittings of Frobenius algebras should include morphisms (5.8) and (5.10), but there is another condition for Δ-separable Frobenius algebras. Δ-separability already implies

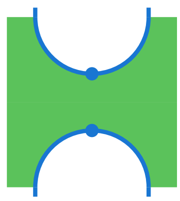


but we shall strengthen this condition further to require

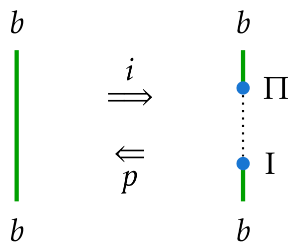


. (5.11)

This strengthening is natural because now



is a split idempotent via $(\mathbb{1}_b, p, i)$, i.e. we have categorified the l.h.s. of (5.6) as an isomorphism and the r.h.s. as the split idempotent:

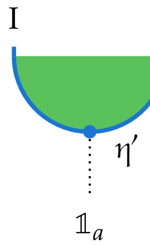


This is satisfying because this gives the categorification of (5.6) more symmetry in the sense that we no longer need to choose directions as in (5.7). In particular, interpreting isomorphisms and split idempotents as categorification of equalities we see that we categorified both the l.h.s. as well as the r.h.s. of (5.6) as idempotents. These considerations may be utilised to recursively construct n-idempotents as *n-condensation algebras* (Sketch 7.2.2).

Definition 5.2.23 (Split Algebras)**.** The *splitting of an algebra* $A : a \longrightarrow a$ is a tuple

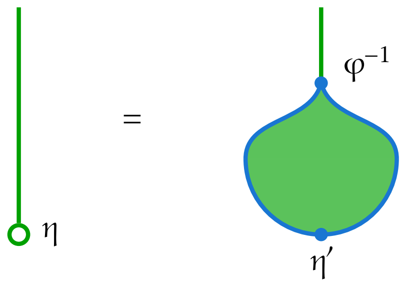
$$(b \in \mathcal{B}, \Pi : a \longrightarrow b, \mathrm{I} : b \longrightarrow a, p : \Pi \otimes \mathrm{I} \Longrightarrow \mathbb{1}_b, \varphi : A \Longrightarrow \mathrm{I} \otimes \Pi)$$

such that $A \cong_\varphi \mathrm{I} \otimes \Pi$ and the multiplication induced by p fulfils equation (5.9). We extend this definition analogously to coalgebras and Frobenius algebras via a morphism $i : \mathbb{1}_b \Longrightarrow \Pi \otimes \mathrm{I}$ fulfilling a condition analogous to fulfils equation (5.9). Additionally, we require splittings of Δ-separable Frobenius algebras to fulfil the condition (5.11). Unital algebras $(A : a \longrightarrow a, \mu, \eta)$ *split* when p is the counit of an adjunction $\Pi \dashv \mathrm{I}$. Labelling the unit of the adjunction $\Pi \dashv \mathrm{I}$ by



,

$\varphi : A \Longrightarrow \mathrm{I} \otimes \Pi$ is of course a unital algebra map (cf. Note 5.2.10):

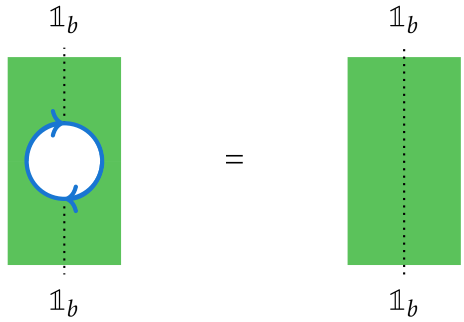


Counital algebras split analogously.

Symmetric Frobenius algebras *split* when they split as unital Frobenius algebras and the splitting data is given by pivotality data of $\mathcal{B}$. Specifically, a symmetric Frobenius algebra $(A : a \longrightarrow a, \mu, \Delta, \eta, \varepsilon)$ splits if there exists a 1-morphism $X : a \longrightarrow b$ and a Frobenius algebra map

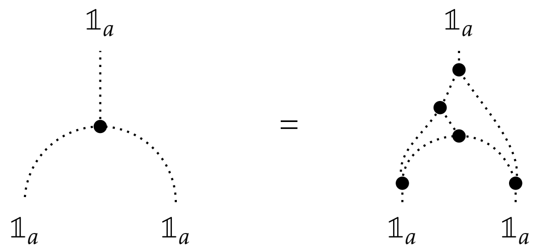
$$A \cong_\varphi X^\vee \otimes X \tag{5.12}$$

such that the 2-morphisms p of (5.8) and η are given by the pivotality data of $X \dashv X^\vee$ and the 2-morphisms i of (5.10) and ε are given by the pivotality data of $X^\vee \dashv X$. If A is also Δ-separable, then A *splits* if there is a 1-morphism $X : a \longrightarrow b$ together with an isomorphism (5.12) such that X has trivial right quantum dimension [CM23, Def. 4.14]. The trivial quantum dimension

$$\dim \Pi =$$ 

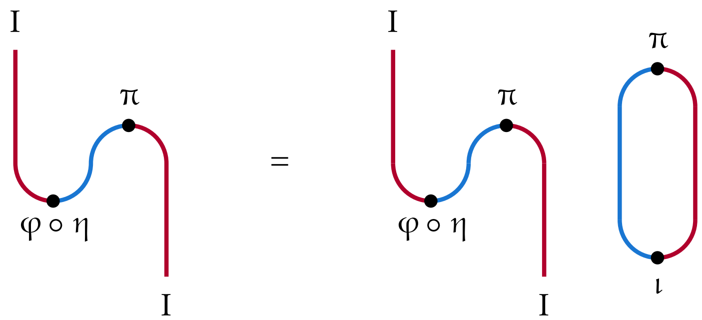
then already implies that A splits as a symmetric Frobenius algebra.

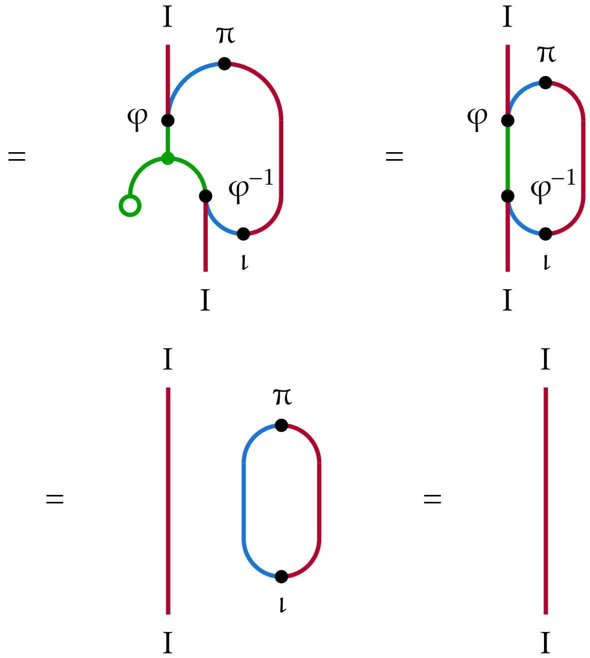
Example 5.2.24. Trivial algebras $\mathbb{1}_a : a \longrightarrow a$ trivially split:



Note 5.2.25. If an algebra A splits both via (b, Π, I, p) as well as $(\widetilde{b}, \widetilde{\Pi}, \widetilde{\mathrm{I}}, \widetilde{p})$, then there is no a priori reason to expect $b \simeq \widetilde{b}$. This is, of course, quite unpleasant, but we may rejoice since this imperfection is remedied for Δ-separable Frobenius algebras when idempotent 2-morphisms split (Lemma 5.2.62).

Remark 5.2.26. Interestingly, unital Δ-separable Frobenius algebras split if they split as Δ-separable Frobenius algebras. If we are given a unital Δ-separable Frobenius algebra A and A splits as a Δ-separable Frobenius algebra with splitting data $(b, \varphi, \Pi, \mathrm{I}, \pi, \iota)$, then





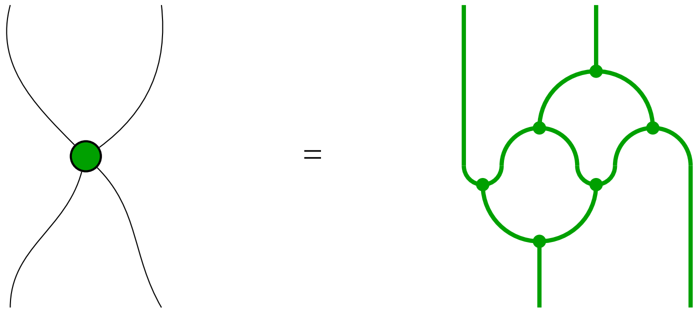
together with analogous calculations shows that A splits as a unital Δ-separable Frobenius algebra.

However, when one tries to extend this idea to Δ-separable symmetric Frobenius algebras, one walks into problems. Utilising Note 3.1.3, one may assume that either π or ι is part of the pivotal monoidal categories adjunction data, but this does not imply that both may be assumed to be part of the adjunction data – quite the contrary. Thus, it seems that Δ-separable symmetric Frobenius algebras need not split if they split as Δ-separable Frobenius algebras.

Proposition 5.2.27. *2-functors map algebra splittings to algebra splittings. Analogous statements hold for splittings of other types of algebras. In particular, pivotal 2-functors map splittings of symmetric algebras to splittings of symmetric algebras.*

Proof. Proof is similar to proving Proposition 5.2.19. □

Remark 5.2.28. Formalising Δ-separable Frobenius algebras with string diagrams is immensely useful not just in TQFT but also in other fields. In particular, they are a powerful tool in quantum computing. For instance, the "ZX-calculus" is a diagrammatic calculus that expresses calculations in quantum computing in terms of string diagrams referred to as "spiders". Spiders are unital Δ-separable Frobenius algebras together with some extra properties. Utilizing the normal form of Δ-separable Frobenius algebras (Note 5.2.11) their multiplications and comultiplications are written as nodes with n incoming and m outgoing lines; e.g. in the case $n, m = 2$:



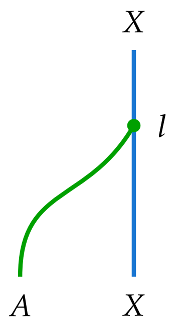
Expressing quantum computations diagrammatically is very useful. For one, this eliminates tedious computations with matrices, and for another, it allows for deeper insights. For example, there is a phenomenon in physics called "quantum teleportation" where arbitrary quantum states are transferred from a location A to a location B by sending a physical quantum state from B to A. This may be found to be a consequence of the zigzag identities provided by unital Frobenius algebras. As another example, one may use string diagrams to express the difference between quantum and classical computations. Roughly, computations may be expressed via algebras and if one of those algebras has a non-trivial splitting, then the computation is quantum [CK17, Sec. 8.6.2].

To demonstrate the educational power of diagrammatic calculi, [DCYP+23][81] conducted a study where they taught 51 pupils aged 15-17 quantum computing using string diagrams. After the eight- week course, consisting of one weekly video lecture, one weekly one-hour tutoring session, and one to two weekly exercise sheets, the pupils outperformed Oxford university graduate students on exam questions taken from Oxford university exams [CKG+24, Coe23].

[81] This publication does not contain the results of the study as this is a preregistration of their study. Preregistration of studies is part of proper scientific practice in social sciences.

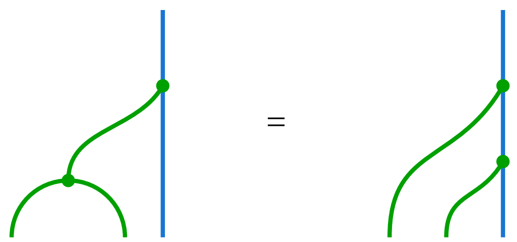
5.2.3. Modules

Definition 5.2.29 (Modules)**.** Given an algebra $A : a \longrightarrow a$ in $\mathcal{B}$, we define *left modules over A* as 1-morphisms $X : b \longrightarrow a$ together with 2-morphisms



(5.13)

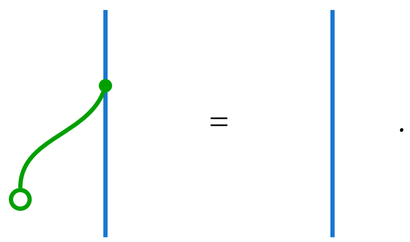
called *left actions* such that



(5.14)

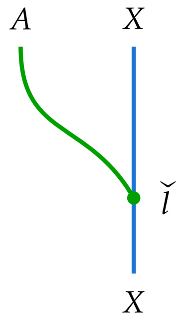
is fulfilled.[82]

We require that *modules over unital algebras* fulfil



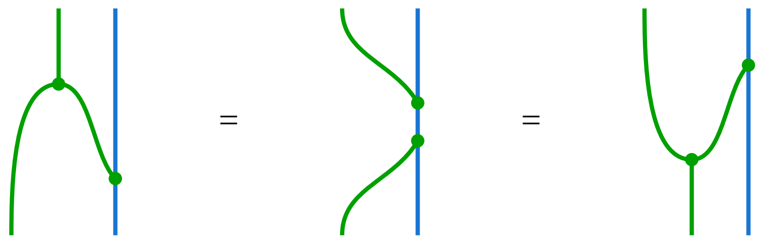
For coalgebras we may analogously define *left comodules* with *coactions*

[82] As with algebras, we tend to not label the 2-morphism components of modules when they are already implied by the context. Another shorthand we use is denoting left A-modules as ${}_AX$. Moreover, if we have not labelled the action of ${}_AX$, then it may be denoted implicitly by l_X. We implicitly extend these shorthands analogously to the following types of modules and their actions. This carries some ambiguity because we will, for instance, label both left modules as well as left Frobenius modules and others by ${}_AX$. However, this will not be an issue because the type of module will be clear from context.

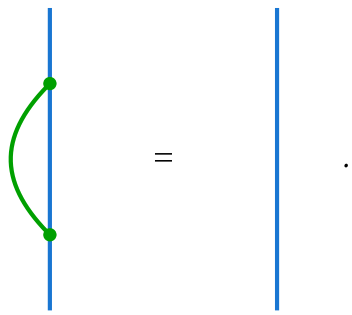


that fulfil a condition analogous to (5.14). Furthermore, *left comodules over counital coalgebras* are defined analogously.

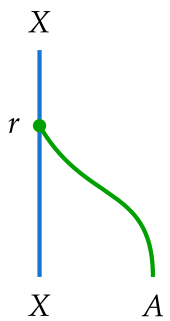
Given a Frobenius algebra ($A : a \longrightarrow a$, , ) and 1-morphism $X : b \longrightarrow a$ that is both a left A-module as well as a left A-comodule, then it is a *left Frobenius A-module* if


(5.15)

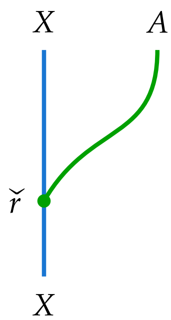
Furthermore, a Frobenius module is *special* if


(5.16)

We may define *right modules* and *right comodules* $X : a \longrightarrow b$ analogously with *right actions* and *right coactions*



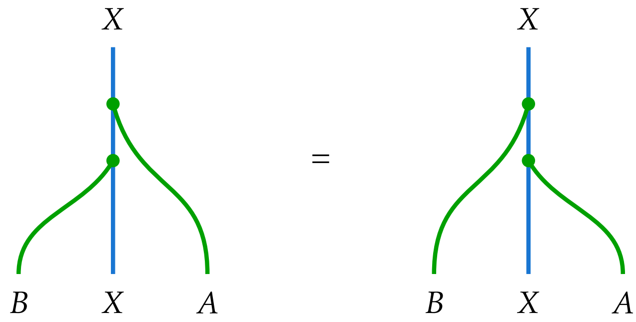
and



,

respectively.

If we are given a second algebra $B : b \longrightarrow b$, then *B-A-bimodules* are 1-morphisms $X : a \longrightarrow b$ that are simultaneously left B-modules and right A-modules with compatible left and right actions, i.e. s.t.



. (5.17)

B-A-bicomodules are defined analogously via left B-comodules and right A-comodules. *Frobenius B-A-bimodules* are defined via left Frobenius B-modules and right Frobenius A-modules and must have compatible left actions and right actions as in (5.17), but left actions and right coactions, left coactions and right actions, and left coactions and right coactions must be compatible analogously. Analogously, Frobenius B-A-bimodules are *special* if they are special as left Frobenius B-modules and right Frobenius A-modules.

Intuition 5.2.30. Modules may be thought of as morphisms between algebras. We will make this more precise when we define the 2-Karoubi envelope (Definition 6.2.8) but for now let us make the following observation. If we are given a split idempotent $e : X \longrightarrow X$ in a category $\mathcal{C}$, then there is, for instance, a correspondence between morphisms $f : Y \longrightarrow \mathbf{split}(e)$ and morphisms $f' : Y \longrightarrow X$ which fulfil

$$f = e \circ f$$

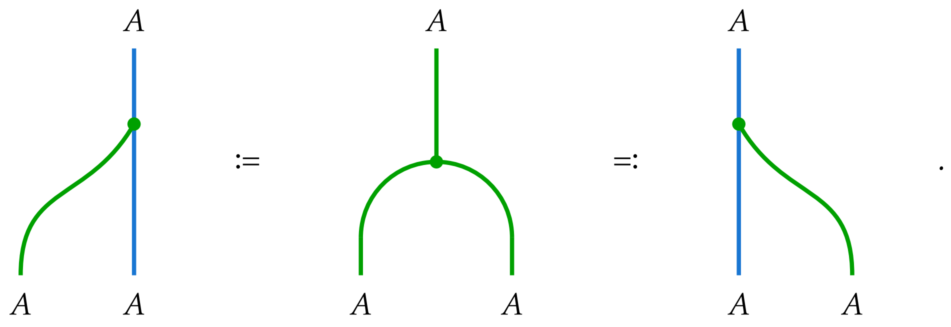
(Note 5.1.17). If we are given a 2-category $\mathcal{D}$ whose 2-morphisms are all identities, and an algebra $E : x \longrightarrow x$ in $\mathcal{D}$, then we already know that E is an idempotent (Remark 5.2.12). If we are given a left E-module ${}_E F$ in $\mathcal{D}$, then the E-action (5.13) implies

$$F = E \otimes F.$$

Thus, we just saw that modules are a categorification of morphisms between idempotents.

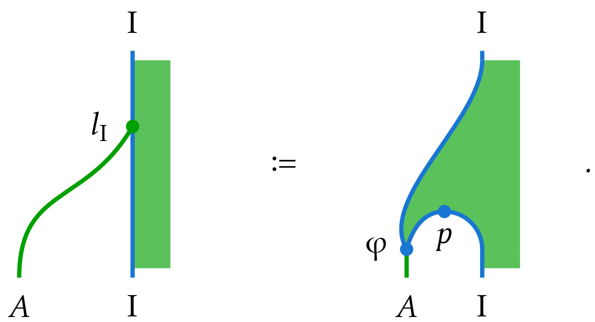
Remark 5.2.31. The following insights usually extend to various forms of modules even when it is not explicitly stated.

Example 5.2.32. Given an algebra, then it is trivially an A-A-bimodule with left and right actions



In particular, Δ-separable Frobenius algebras A are special Frobenius A-A-bimodules.

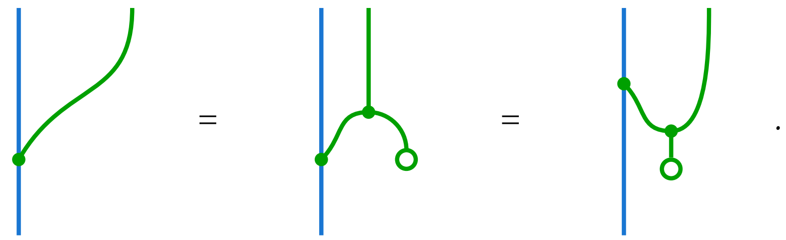
Example 5.2.33. Similarly to Example 5.2.32, if we are given an algebra $A : a \longrightarrow a$ that splits via (b, Π, I, p), then I is a left A-module with left action



(5.18)

Analogously, Π is a right A-module.

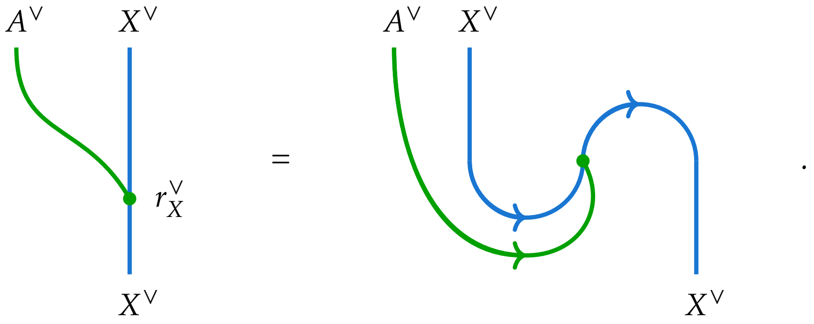
Note 5.2.34. Modules over unital Frobenius algebras are already Frobenius modules. To see this, we assume that we are given a Frobenius module X_A and reformulate its action as



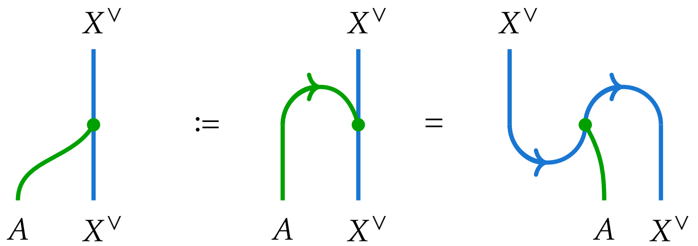
(5.19)

Thus, the A-coaction is already fully determined by the A-action, A's multiplication, and A's unit.

Lemma 5.2.35. *If we are given a right A-module X in a pivotal 2-category, then $X^\vee$ is a left $A^\vee$-comodule with canonical action*

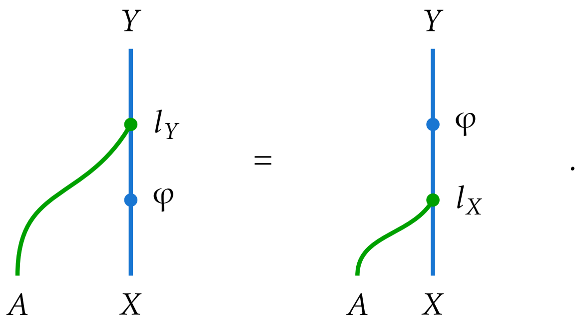


Furthermore, $X^\vee$ is a left A-module:



(5.20)

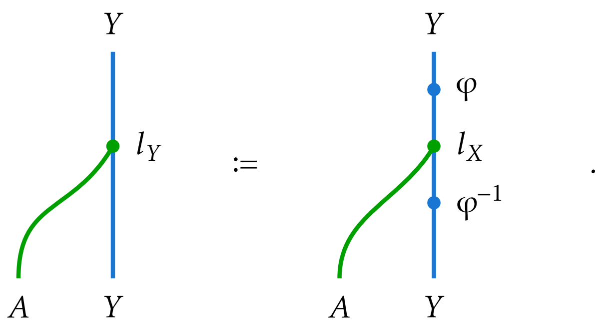
Definition 5.2.36. A *left A-module map* between two left A-modules $X, Y : b \longrightarrow a$ is a 2-morphism $\varphi : X \Longrightarrow Y$ such that



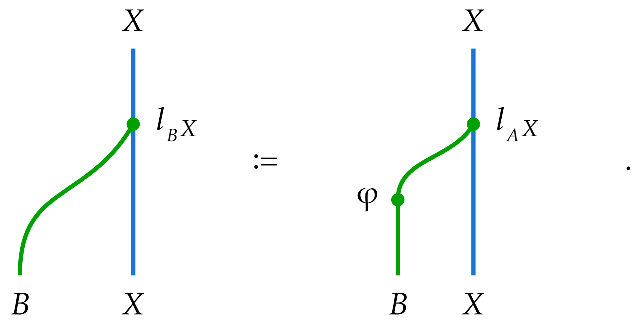
(5.21)

We define other types of module maps analogously.

Observation 5.2.37. Given an A-module X_A and an isomorphism $\varphi : X \Longrightarrow Y$, then Y is an A-module with action



Similarly, if we are given an A-module X_A and an algebra isomorphism $\varphi : B \Longrightarrow A$, then X is a B-module with action



Proposition 5.2.38. *2-functors map modules to modules and module maps to module maps. Analogous statements hold for modules over various types of algebras.*

Proof. Proof is similar to proving Proposition 5.2.19. □

Remark 5.2.39. Interestingly, the module map condition (5.21) expresses naturality of actions (cf. Example 2.2.5). Specifically, if A is an algebra in a monoidal category, then there are endofunctors

$$\mathrm{id}_{{}_A\mathbf{Mod}(\mathcal{C})}, A \otimes (-) : {}_A\mathbf{Mod}(\mathcal{C}) \longrightarrow {}_A\mathbf{Mod}(\mathcal{C})$$

and the module actions are natural transformations $A \otimes (-) \Longrightarrow \mathrm{id}_{{}_A\mathbf{Mod}(\mathcal{C})}$.

However, this does not seem useful to us because this property characterises the module actions and we would be more interested in characterising the module maps. Instead, one may reformulate (5.21) as a naturality condition of module maps. Without context this may seem somewhat contrived, but is actually a natural choice for constructing *n-condensation modules* [GJF19, Def. 2.3.3].

Definition 5.2.40. Given an algebra $A : a \longrightarrow a$ in $\mathcal{B}$ and an object $b \in \mathcal{B}$, then we may define a category of right A-modules $\mathbf{Mod}_A$ consisting of

- ***Objects:*** right A-modules $X : a \longrightarrow b$,
- ***Morphisms:*** right A-modules maps $f : X_A \longrightarrow Y_A$, and
- ***Composition:*** vertical composition of $\mathcal{B}$.

We may likewise define categories of left A-modules ${}_A\mathbf{Mod}$ and B-A-bimodules ${}_B\mathbf{Mod}_A$. Moreover, we could of course analogously construct categories for other types of modules. In particular, we shall denote the categories of special Frobenius modules by $\mathbf{Mod}^{\mathrm{sp}}$.

Proposition 5.2.41.

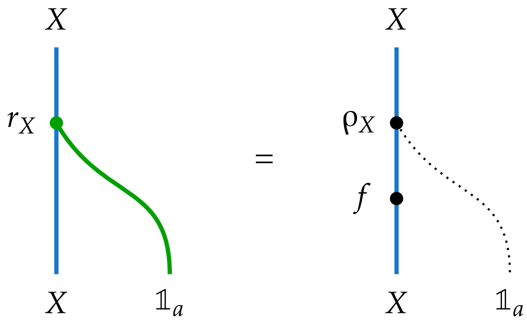
$$\mathcal{B}(a,b) \cong {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a} \tag{5.22}$$

Proof. Every 1-morphism $X : a \longrightarrow b$ is a right $\mathbb{1}_a$-module with right action given by the unitor $\rho_X : X \otimes \mathbb{1}_a \Longrightarrow X$. For modules (X, ρ_X) every 2-morphism $f : X \Longrightarrow Y$ is a right $\mathbb{1}_a$-module map, so

$$\mathcal{B}(a,b) \hookrightarrow \mathbf{Mod}_{\mathbb{1}_a}$$

is a fully faithful functor.

However, not every right $\mathbb{1}_a$-module action must be a unitor. First, we notice that actions may be uniquely decomposed into an endomorphism composed with a unitor, e.g.



with $f := r_X \circ \rho_X^{-1}$. The module condition (5.14) then implies $f \circ f = f$, i.e. that f is an idempotent.

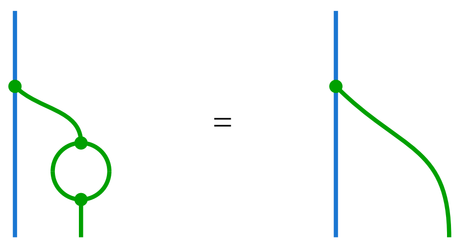
If X is a Frobenius right $\mathbb{1}_a$-module, then there is also a coaction $\check{r} : X \Longrightarrow X \otimes \mathbb{1}_a$ which decomposes as $\check{r} = \rho_X^{-1} \circ g$, where $g : X \Longrightarrow X$ is also an idempotent. The Frobenius condition (5.15) then implies $f = g$.

Moreover, if X is a special Frobenius right $\mathbb{1}_a$-module, then the specialness condition (5.16) implies $f \circ f = 1_X$. Together with idempotence, this implies $f = 1_X$, i.e. all special Frobenius right $\mathbb{1}_a$-modules have actions and coactions given by unitors. Therefore,

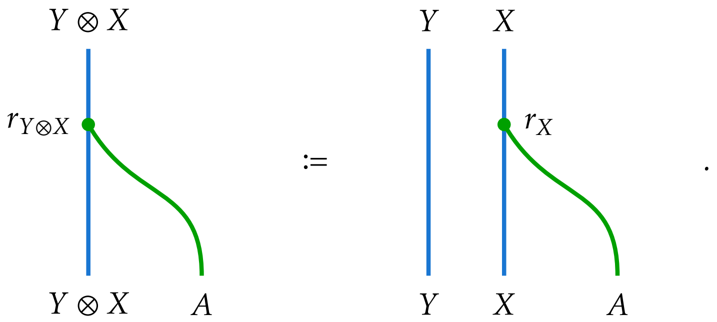
$$\mathcal{B}(a,b) \cong \mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}.$$

This implies the isomorphism (5.22). $\square$

Observation 5.2.42. If we have a Frobenius algebra A and a special Frobenius module X_A, then the action of X makes A behave as if A was Δ-separable, e.g.

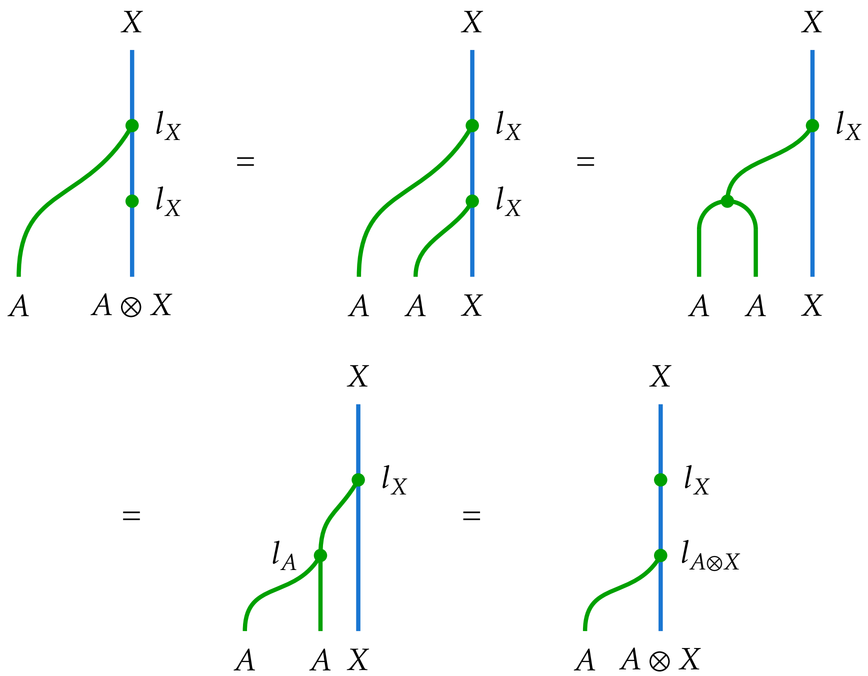


Note 5.2.43. If $A : a \longrightarrow a$ is an algebra, $X : a \longrightarrow b$ is a right A-module, and $Y : b \longrightarrow c$ is a 1-morphism, then $Y \otimes X$ is trivially a right A-module with action

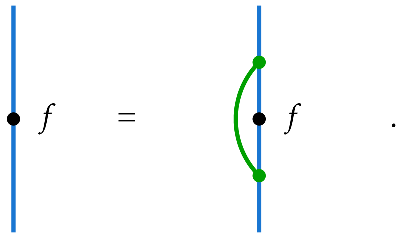


Likewise, if we are given a A-module map $f : X \Longrightarrow X'$ and a 2-morphism $g : Y \Longrightarrow Y'$, then $g \otimes f : Y \otimes X \Longrightarrow Y' \otimes X'$ is trivially a right A-module map.

Observation 5.2.44. Algebra actions $l_X : A \otimes X \Longrightarrow X$ are themselves A-module maps:



Note 5.2.45. The maps of left special Frobenius A-modules $X, Y : b \longrightarrow a$ are precisely those 2-morphisms $f : X \Longrightarrow Y$ such that

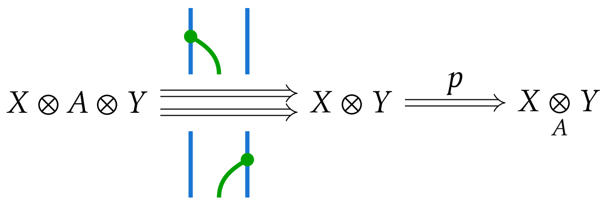


(5.23)

Proof. If f is a module map, then (5.23) is clearly fulfilled. On the other hand, if (5.23) is fulfilled, then the r.h.s. of (5.23) is a left Frobenius A-module map since the algebra actions are module maps (Observation 5.2.44 & Note 5.2.43). $\square$

5.2.4. Relative Tensor Products

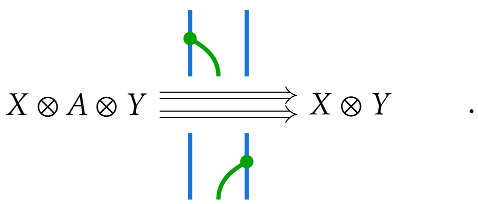
Definition 5.2.46 (Relative Tensor Product)**.** Let us assume we are given an algebra A, a left A-module Y, and a right A-module X. If the coequaliser (Definition 4.1.3)



exists, then it, i.e. $(X \otimes_A Y, p)$, is called the *tensor product of X and Y over A*.

Intuition 5.2.47. By Remark 4.1.4 we see that one may think of a relative tensor product $X \otimes_A Y$ as a construction that quotients $X \otimes Y$ by A. Another way to think about them is that the A-actions behave like unitors (2.8) in the "image" of p.

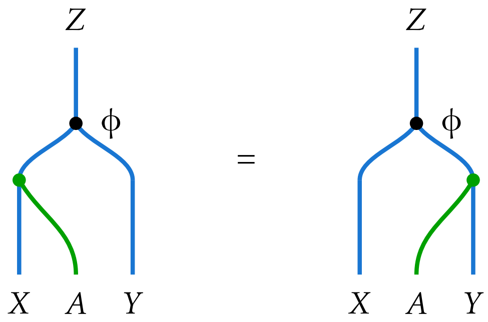
As a first step towards constructing a relative tensor product, we may first search for a 2-morphism[83] $\phi : X \otimes Y \Longrightarrow Z$ that coequalises



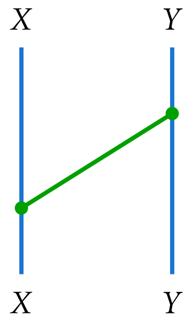
(5.24)

[83] Such morphisms ϕ are also called "balanced" morphisms.

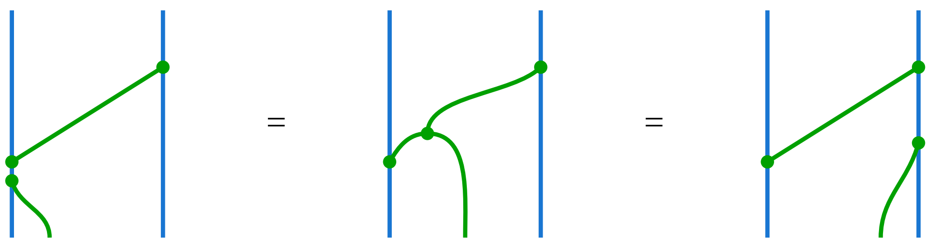
Rewriting this condition graphically as



provides us with some intuition towards the nature of ϕ, namely that ϕ somehow connects X and Y in a way that lets r_X "wander" along X until ϕ where it "jumps" to Y and becomes l_Y. If X and A are Frobenius, one may explore

$\phi :=$  (5.25)

because it has the desired "jumping" property:

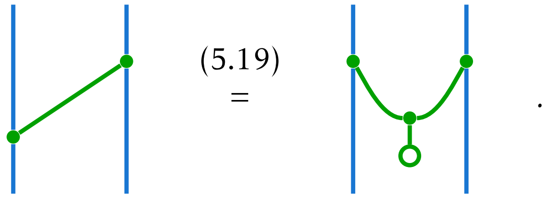


As a next step and since the coequaliser must be an epimorphism (Lemma 4.1.6), we may attempt to construct an epimorphism $\pi : X \otimes Y \Longrightarrow W$ from ϕ. One may wonder what happens if $\phi : X \otimes Y \Longrightarrow X \otimes Y$ is a split idempotent. In that case, we would have a split epimorphism $\pi : X \otimes Y \longrightarrow \mathbf{split}(\phi)$. However, π will still not generally be the relative tensor product as it must fulfil the universality property that there exists a unique 2-morphism such that

$$\begin{tikzcd}
X \otimes Y \arrow[r, Rightarrow, "p"] \arrow[dr, Rightarrow, "\psi"'] & X \underset{A}{\otimes} Y \arrow[d, dashed, Rightarrow] \\
& Z
\end{tikzcd}$$

commutes for all ψ that equalise (5.24). This is not generally satisfied by a split idempotent $(\mathbf{split}(\phi), \pi)$.[84]

Remark 5.2.48. If the the algebra A in (5.25) is unital, then we may rewrite ϕ as

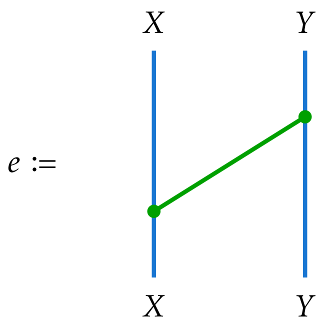


This is the version used in [CR16, Lemma 2.3].

Lemma 5.2.49. *If we are given a relative tensor product $X \otimes_A Y$ and an isomorphism $A \cong B$, then $X \otimes_B Y$ exists and $X \otimes_A Y \cong X \otimes_B Y$.*

Proof. X and Y are trivially left and right B-modules (Observation 5.2.37). It follows that they have the same coequalisers (Lemma 4.1.5). □

Lemma 5.2.50. *Given a Frobenius algebra A, a right Frobenius A-module X, and a left A-module Y, then*



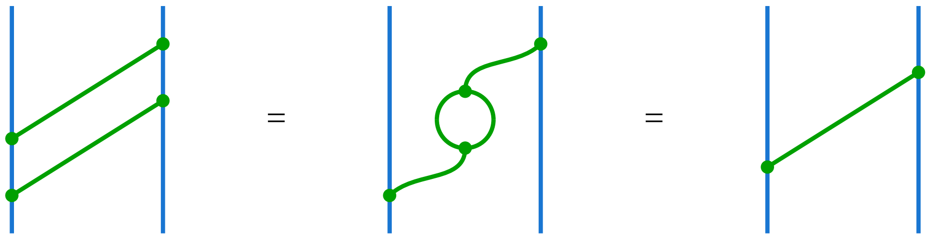
is idempotent if

- *X_A is special,*
- *${}_AY$ is Frobenius and special, or*
- *A is Δ-separable.*

[84] Note that this does not contradict [CR16, Lemma 2.3] as they are working with the special case of unital Δ-separable Frobenius algebras which means that their modules are special Frobenius modules.

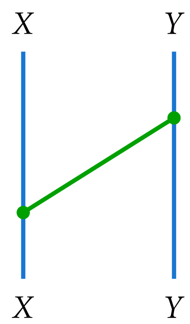
Proof. First, we note that A-actions on special Frobenius modules are as if A is Δ-separable (Observation 5.2.42). Thus, since this is the weaker condition, it is sufficient to check only this condition.

So, assuming A is Δ-separable, then e is idempotent:



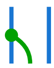
$\square$

Theorem 5.2.51. *Given a Frobenius algebra A, a special right Frobenius A-module X, and a left A-module Y, then the relative tensor product $X \otimes_A Y$ is the splitting of the idempotent*

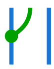
$e :=$ 

if either of them exist [CR16, Rem. 2.4].

Proof. Since X_A is special, e is idempotent (Lemma 5.2.50). Now, if e splits, then plugging the morphisms

$f =$  ,

$f' =$  , and

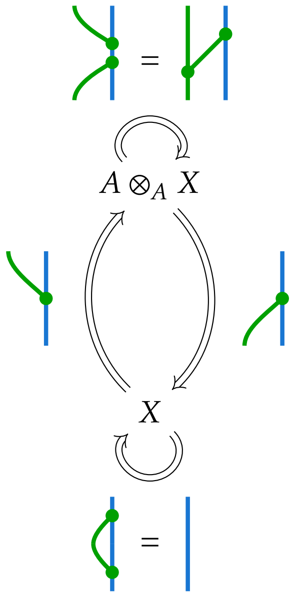
$t =$ 

into Corollary 5.1.26 already finishes the proof. $\square$

Corollary 5.2.52. *Given a Frobenius algebra A and a special right Frobenius A-module X, then we may choose*

$$A \underset{A}{\otimes} X = X. \tag{5.26}$$

Proof. The splitting diagram



(5.27)

implies (5.26) (Theorem 5.2.51). □

Corollary 5.2.53.

$$A \underset{A}{\otimes} A = A \tag{5.28}$$

for Δ-separable Frobenius algebras A.

Definition 5.2.54. If the relative tensor product of Theorem 5.2.51 exists, then we call it a *split relative tensor product*.[85]

If the modules are bimodules, we define a bimodule structure on the split relative tensor product according to Lemma 5.2.58.

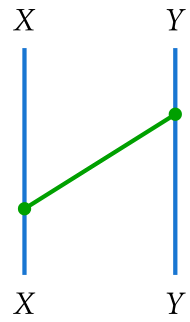
Lemma 5.2.55. *If we are given a split relative tensor product $X \otimes_A Y$, then*

$$F(X \underset{A}{\otimes} Y) \cong FX \underset{FA}{\otimes} FY.$$

Proof. Split coequalisers are absolute coequalisers (Corollary 5.1.25). □

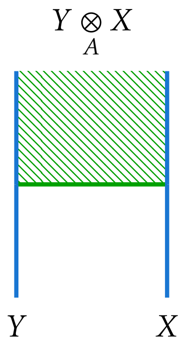
85 To the best of my knowledge there is no specific terminology for this in the literature. However, some authors refer to the special case $X \otimes_A A$ as "Beck coequalisers" so maybe "Beck tensor product" or a similar term might also be fitting..

Notation 5.2.56 (cf. [CM23, Eq. (4.15)]). Recall that all morphisms factoring through a split relative tensor product $X \otimes_A Y$ factor through



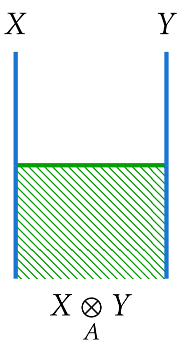
(5.29)

(Note 5.1.17). Given a split relative tensor product $(X \otimes_A Y, \pi, \iota)$, then we may extend our rules of string diagrams to denote its retraction as



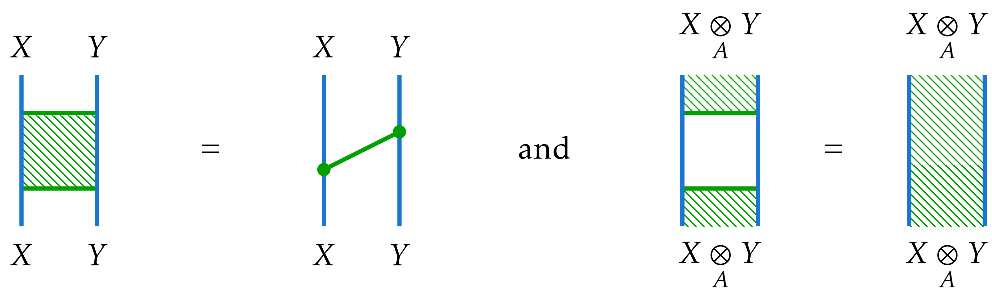
$$:= \pi$$

and its section as

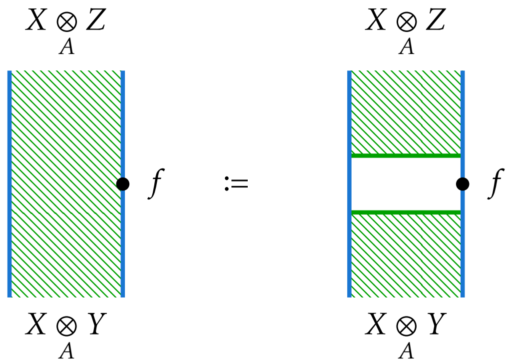


$$:= \iota\,.$$

Rewriting the relations of π and ι (diagram (5.6)) using this notation is visually pleasing:

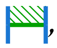
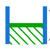
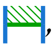


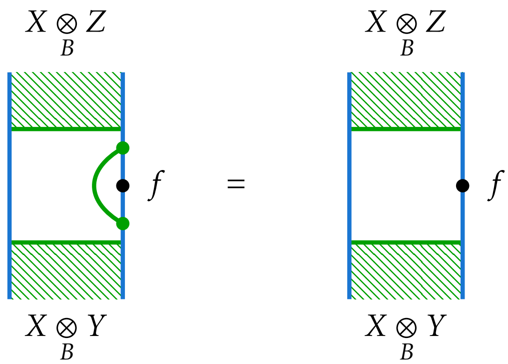
Since morphisms factoring through a split relative tensor product $X \otimes_A Y$ must factor through (5.29) we may identify morphisms of $X \otimes_A Y$ as morphisms of $X \otimes Y$. In particular, if we are given split relative tensor products $X \otimes_A Y$ and $X \otimes_A Z$, then we may use the notation



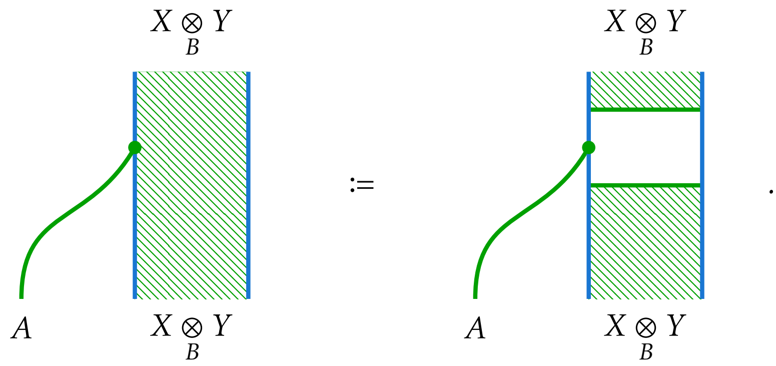
for left A-module maps $f : Y \Longrightarrow Z$. We introduce an analogous notation for right module maps. We restrict to modules maps in line with Note 5.2.57.

This notation may be understood as a teaser of the observation that relative tensor products may induce a horizontal composition (Definition 6.2.8).

Note 5.2.57. The split relative tensor products ($X \otimes_A Y$, , ) and ($X \otimes_A Z$, , ) project 2-morphisms $f : Y \longrightarrow Z$ onto left A-module maps (Note 5.2.45):

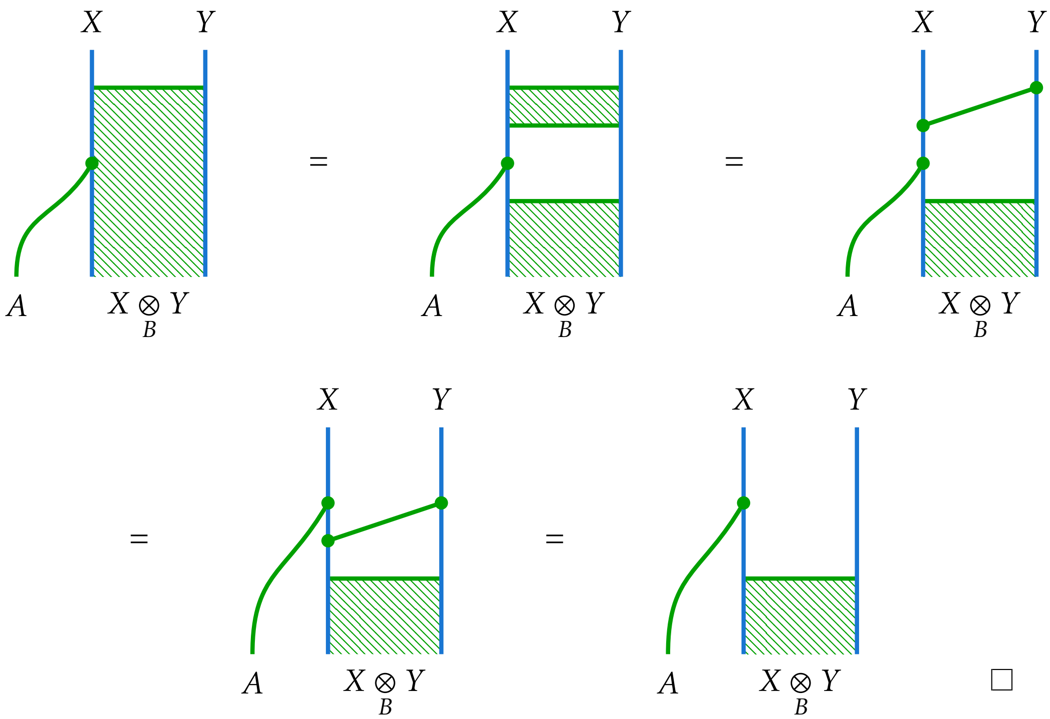


Lemma 5.2.58. *Split relative tensor products of special bimodules have a canonical special bimodule structure. For instance, if we are given bimodules ${}_AX_B$ and ${}_BY_C$, then $X \otimes_B Y$ has a left A-action*



Moreover, the splitting maps  *and*  *of $X \otimes_B Y$ are then bimodule maps.*

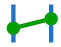
Proof. The special bimodule conditions (Definition 5.2.29) are simple to check and work similarly to the proof that the splitting maps are bimodule maps. The following calculation shows that  is a bimodule map:



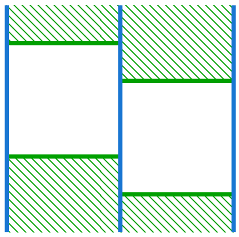
□

Lemma 5.2.59. *If idempotents in $\mathcal{B}$ split and we are given special modules ${}_AX$, ${}_BY_A$, and Z_B, then $(Z \otimes_B Y) \otimes_A X$ and $Z \otimes_B (Y \otimes_A X)$ exist. In particular,*

$$(Z \underset{B}{\otimes} Y) \underset{A}{\otimes} X \cong Z \underset{B}{\otimes} (Y \underset{A}{\otimes} X). \tag{5.30}$$

Proof. Existence follows from splitting idempotents of the form  (Theorem 5.2.51). The isomorphism (5.30) is given by

$$Z \underset{B}{\otimes} (Y \underset{A}{\otimes} X)$$



$$(Z \underset{B}{\otimes} Y) \underset{A}{\otimes} X \qquad . \qquad (5.31)$$

Its inverse is given by horizontally flipping (5.31). □

5.2.5. Condensation

Nomenclature 5.2.60.

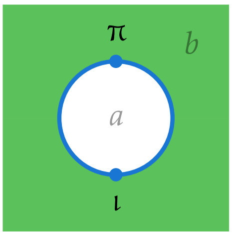
- From now onwards we will refer to Δ-separable Frobenius algebras by the term *2-condensation algebras* ([GJF19, Def. 2.2.1]).[86]
- If a 2-condensation algebra $A : a \longrightarrow a$ splits with splitting data $(b, \Pi, \mathrm{I}, \pi, \iota)$, then we say that *$a$ condenses onto b*. The splitting data is now called a *condensation of a onto b*. Moreover, we say that b is the *condensate* of A ([GJF19, Def. 2.1.1]).
- *2-condensation modules* are special Frobenius modules [GJF19, Def. 2.3.3].
- *2-condensation module maps* are Frobenius module maps.

Lemma 5.2.61. *If idempotents in $\mathcal{B}$ split and $b \in \mathcal{B}$ is a condensate of $A : a \longrightarrow a$, then*

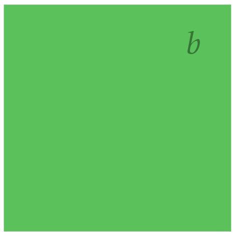
$$\Pi \underset{A}{\otimes} \mathrm{I} \cong \mathbb{1}_b.$$

[86] Keep in mind that there is still a slight difference between our definition and the definition of [GJF19]. For one, their definition utilises a "walking construction" (Remark 5.2.20). For another, [GJF19] think of Δ-separable Frobenius algebras internal to monoidal categories as "2-condensation algebras" and think of Δ-separable Frobenius algebras internal to 2-categories as "2-condensation monads". We, meanwhile, do not make this distinction. We refer to both as "2-condensation algebras" to be more in line with [CR16, CM23].

Proof. Let $(b, \Pi, \mathrm{I}, \pi, \iota)$ be the condensation of a onto b. The relative tensor product $\Pi \otimes_A \mathrm{I}$ exists since the condition

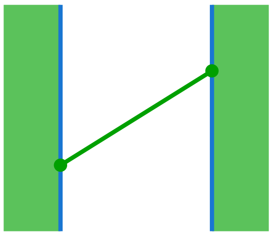


$=$

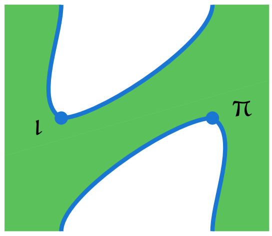


implies that

$e :=$



$\overset{(5.18)}{=}$



is a split idempotent. Thus, $\mathbf{split}(e) = \mathbb{1}_b$ implies

$$\Pi \underset{A}{\otimes} \mathrm{I} \cong \mathbb{1}_b$$

(Theorem 5.2.51). □

Lemma 5.2.62. *If idempotents in $\mathcal{B}$ split and $b, \widetilde{b} \in \mathcal{B}$ are both condensates of a 2-condensation algebra A, then*

$$\widetilde{\Pi} \underset{A}{\otimes} \mathrm{I} : b \rightleftarrows \widetilde{b} : \Pi \underset{A}{\otimes} \widetilde{\mathrm{I}}.$$

is an equivalence [GJF19, Lemma 3.3.2].

Proof.

$$\begin{aligned}
(\Pi \underset{A}{\otimes} \widetilde{\mathrm{I}}) \otimes (\widetilde{\Pi} \underset{A}{\otimes} \mathrm{I}) &\overset{Lemma\ 5.2.59}{=} (\Pi \underset{A}{\otimes} (\widetilde{\mathrm{I}} \otimes \widetilde{\Pi})) \underset{A}{\otimes} \mathrm{I} \\
&\cong (\Pi \underset{A}{\otimes} A) \underset{A}{\otimes} \mathrm{I} \\
&\overset{Lemma\ 5.2.49}{\cong} \Pi \underset{A}{\otimes} \mathrm{I} \\
&\overset{Lemma\ 5.2.61}{\cong} \mathbb{1}_b
\end{aligned}$$

and analogously

$$(\widetilde{\Pi} \underset{A}{\otimes} \mathrm{I}) \otimes (\Pi \underset{A}{\otimes} \widetilde{\mathrm{I}}) \cong \mathbb{1}_{\widetilde{b}}\,.$$

□

Intuition 5.2.63 (behind Lemma 5.2.62)**.** Conceptually, $\Pi \otimes_A (-)$ is left inverse to $\mathrm{I} \otimes (-)$, i.e.

$$\Pi \otimes_A (-) \triangleq \mathrm{I}^{-1} \otimes (-).$$

Extending this idea, we may think of the relative tensor product as quotienting by A (Intuition 5.2.47):

$$(-) \otimes_A (-) \triangleq (-) \otimes A^{-1} \otimes (-).$$

Note 5.2.64. Analogously to the case of split idempotents (Note 5.1.17), condensates b are fully determined by their condensation algebras $A : a \longrightarrow a$ [GJF19, p. 12]. Specifically, every 1-morphism $X : b \longrightarrow x$ induces a special Frobenius right A-module $X \otimes \Pi$ (Example 5.2.33) and every special Frobenius right A-module $Y_A : a \longrightarrow y$ induces a 1-morphism $Y \otimes_A \mathrm{I}$. In particular, this produces a correspondence, i.e.

$$\begin{tikzcd} \mathbf{Mod}_A^{\mathrm{sp}}(a,-) \arrow[rr, bend left, "(-)\underset{A}{\otimes}\Pi"] & & \mathcal{B}(b,-) \arrow[ll, bend left, "(-)\otimes \mathrm{I}"] \end{tikzcd}$$

and

$$\begin{tikzcd} {}_A\mathbf{Mod}^{\mathrm{sp}}(-,a) \arrow[rr, bend left, "\mathrm{I}\underset{A}{\otimes}(-)"] & & \mathcal{B}(-,b) \arrow[ll, bend left, "\Pi\otimes(-)"] \end{tikzcd}$$

are equivalences.

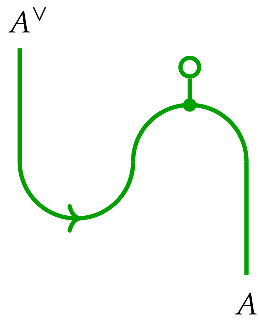
5.2.6. Orbifold Condensation

Reminder 5.2.65 (Assumption 5.2.2)**.** In this section we assume our fixed a 2-category $\mathcal{B}$ to be pivotal. We continue to assume that our objects, 1-morphisms, and 2-morphisms live in $\mathcal{B}$ unless stated otherwise.

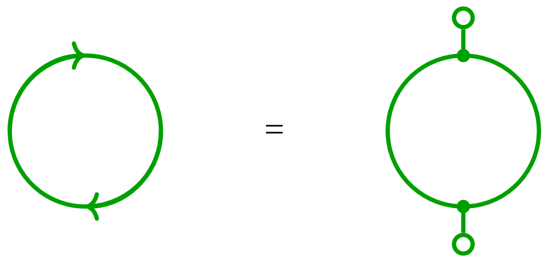
Nomenclature 5.2.66 (Orbifolds)**.**

- *2-orbifold data* are symmetric Δ-separable Frobenius algebras.
- *2-orbifold condensations* are splittings of orbifold data (Definition 5.2.23). Given a 2-condensation algebra A together with a corresponding 2-condensate b we say that they constitute a *condensation of A onto b* [GJF19, Def. 2.1.1].

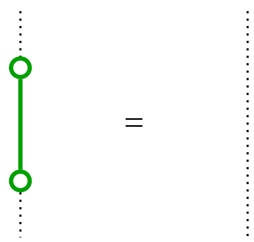
Lemma 5.2.67. *Using the invertible algebra map*



one may see that symmetric Frobenius algebras A have quantum dimensions

$\dim A =$  .

Thus, if

 ,

and A is an orbifold datum, then A has trivial quantum dimension, i.e.

$$\dim A = \mathbb{1}_a.$$

6. Universal Completions

Finally, we have finished our preparation and are now able to turn to the universal properties of *higher idempotent completions*. We shall start off with the *idempotent completion* in Section 6.1. For one, it serves the purpose of building up intuition in preparation of our treatment of *2-idempotent completions* in Section 6.2. However, we will also be able to go more in depth in our treatment of idempotent completions than in our treatment of 2-idempotent completions. This will provide us with conjectures about further properties of 2-idempotent completions and even n-idempotent completions (Section 7.2).

6.1. Idempotent Completions

Preview 6.1.1. We start this section by defining the *idempotent completion* of an arbitrary category $\mathcal{C}$. This should be a category $\overline{\mathcal{C}}$ in which every idempotent splits (Definition 6.1.2) and that "universally completes $\mathcal{C}$" in this regard (Definition 6.1.4). Then, we show how to construct *Karoubi envelopes* $\mathbf{Kar}\,\mathcal{C}$ (Definition 6.1.8) and embeddings $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbf{Kar}\,\mathcal{C}$. Showing that all the idempotents of $\mathbf{Kar}\,\mathcal{C}$ split (Lemma 6.1.13) foreshadows Theorem 6.1.11 which states that the Karoubi envelope $\mathbf{Kar}\,\mathcal{C}$ is an idempotent completion $\overline{\mathcal{C}}$. To prove this, we show that $\iota_{\mathcal{A}}$ has weak inverses $\pi_{\mathcal{A}}$ for categories $\mathcal{A}$ whose idempotents split (Proposition 6.1.15) and that the Karoubi envelope is actually a 2-functor $\mathbf{Kar} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ (Theorem 6.1.16). Lastly, we use our insights to show that $\mathbf{Kar}$ is part of a 2-adjunction (Theorem 6.1.20) which implies that the Karoubi envelope is an idempotent completion.

Definition 6.1.2. A category is called *idempotent complete* if all of its idempotents split. We organise such categories into $\mathbf{Cat}^{\mathbf{ic}}$, the full sub-2-category of $\mathbf{Cat}$ consisting of all the idempotent complete categories.

Sanity Check 6.1.3. $\mathbf{Cat}^{\mathbf{ic}}$ is closed under equivalences as a sub-2-category of $\mathbf{Cat}$.

Proof. The statement means, that if we are given an equivalence $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ in $\mathbf{Cat}$, then $\mathcal{C} \in \mathbf{Cat}^{\mathbf{ic}}$ if and only if $\mathcal{D} \in \mathbf{Cat}^{\mathbf{ic}}$. To see this, let us assume $\mathcal{C} \in \mathbf{Cat}^{\mathbf{ic}}$ and that $e : X \longrightarrow X$ is an arbitrary idempotent in $\mathcal{D}$. Then, Ge is an idempotent in $\mathcal{C}$ (Lemma 5.1.16). By assumption Ge splits via some $Y \in \mathcal{C}$ which implies that FGe splits via FY (Lemma 5.1.16). Therefore, since $FGX \cong X$, e splits via FY, too. □

Definition 6.1.4 (Idempotent Completions)**.** An *idempotent completion* of a category $\mathcal{C}$ is a universal functor[87] $(\overline{\mathcal{C}}, \iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \overline{\mathcal{C}})$ from $\mathcal{C}$ to the forgetful 2-functor $\mathbf{Cat}^{\mathbf{ic}} \longrightarrow \mathbf{Cat}$, i.e. it is a category-functor pair[88] such that

$$(-) \circ \iota_{\mathcal{C}} : \mathbf{Cat}^{\mathbf{ic}}(\overline{\mathcal{C}}, \mathcal{A}) \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathcal{A})$$

is an equivalence for all $\mathcal{A} \in \mathbf{Cat}^{\mathbf{ic}}$ [Dé22, Def. 1.2.1].

Remark 6.1.5. The idempotent completion is indeed a completion, i.e. $\overline{\overline{\mathcal{C}}} \cong \overline{\mathcal{C}}$. To see this, we simply notice that $(\overline{\mathcal{C}}, \mathrm{id}_{\overline{\mathcal{C}}})$ is an idempotent completion of $\overline{\mathcal{C}}$ for all $\mathcal{C} \in \mathbf{Cat}$.

Remark 6.1.6. Idempotent completions are also known as *Cauchy completions* because they categorify the completions of metric spaces w.r.t. to Cauchy sequences (cf. Example 4.3.3 & Section A.2.1).

Sketch 6.1.7. What should a idempotent completion $\overline{\mathcal{C}}$ look like?

Essential surjectivity of $(-) \circ \iota_{\mathcal{C}}$ implies that all functors $\mathcal{C} \longrightarrow \mathcal{A}$ factor through $\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$ (Lemma 4.3.6). Therefore, $\iota_{\mathcal{C}}$ should be faithful. Similarly, fully faithfulness of $(-) \circ \iota_{\mathcal{C}}$ suggests fullness of $\iota_{\mathcal{C}}$. Therefore, we should try to construct $\overline{\mathcal{C}}$ by taking $\mathcal{C}$ and adjoining objects and morphisms in a way that makes all idempotents split. This also makes the construction of $\iota_{\mathcal{C}}$ trivial once we have constructed $\overline{\mathcal{C}}$ since $\iota_{\mathcal{C}}$ is forgetful.

---

87 Some authors define the idempotent completion just as the functor $\iota_{\mathcal{C}}$ and some define it just as the category $\overline{\mathcal{C}} := \mathbf{Kar}\,\mathcal{C}$ (Note 4.0.4).

88 Since the 2-functor $\mathbf{Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{Cat}$ maps $\overline{\mathcal{C}} \longmapsto \overline{\mathcal{C}}$, the functor $\iota_{\mathcal{C}}$ already includes the information about the universal object $\overline{\mathcal{C}}$. Therefore, we may say that $\iota_{\mathcal{C}}$ is the universal functor instead of specifying the pair $(\overline{\mathcal{C}}, \iota_{\mathcal{C}})$.

While doing so, we have to keep in mind that we cannot add objects that would break the correspondence between functors $\mathcal{C} \longrightarrow \mathcal{A}$ and functors $\overline{\mathcal{C}} \longrightarrow \mathcal{A}$. But this should not be a problem if we only add objects and morphisms that correspond to splittings of idempotents in $\mathcal{C}$. This is because functors map splittings to splittings (Lemma 5.1.16). Moreover, this ensures $\overline{\overline{\mathcal{C}}} \cong \overline{\mathcal{C}}$.

Now that we know that we want to add splittings, we still need to consider which morphisms we should add. This is solved by Note 5.1.17 because it tells us that a splitting of an idempotent $e : X \longrightarrow X$ must have hom-sets isomorphic to $\mathcal{C}(X, -) \circ e$ and $e \circ \mathcal{C}(-, X)$. In conclusion, we want to construct $\overline{\mathcal{C}}$ by adding to $\mathcal{C}$ an object X_e for each idempotent $e : X \longrightarrow X$ in $\mathcal{C}$. Next, we add morphisms $\pi : X \longrightarrow X_e$ and $\iota : X_e \longrightarrow X$ together with relations $\iota \circ \pi = e$ and $\pi \circ \iota = 1_{X_e}$. Lastly, we add all morphisms generated by the morphisms of $\mathcal{C}$ and this newly adjoined splitting data.

Definition 6.1.8. Given a 1-category $\mathcal{C}$, its *Karoubi envelope* $\mathbf{Kar}(\mathcal{C})$ is the 1-category consisting of

Objects: pairs (X, e) representing idempotents $e : X \longrightarrow X$ of $\mathcal{C}$,

Morphisms: $(X, e) \longrightarrow (X', e')$ given by triples (e', f, e) such that $e' \circ f = f = f \circ e$,

Composition: $(e'', g, e') \circ (e', f, e) := (e'', g \circ f, e)$, and

Units: $1_{(X,e)} = (e, e, e)$,

It is straightforward to see that $\mathbf{Kar}\,\mathcal{C}$ fulfils associativity and unity.

Remark 6.1.9. We note that the construction of the Karoubi envelope coincides with Sketch 6.1.7. We sketched constructing $\overline{\mathcal{C}}$ from $\mathcal{C}$ by adjoining to $\mathcal{C}$ objects together with their hom-sets, and this is what $\mathbf{Kar}\,\mathcal{C}$ does. Objects $X \in \mathcal{C}$ correspond to objects $(X, 1_X) \in \mathbf{Kar}\,\mathcal{C}$ (Lemma 6.1.10). Objects $(X, e) \in \mathbf{Kar}\,\mathcal{C}$ correspond to splittings of the idempotents $e : X \longrightarrow X$ in $\mathcal{C}$ (Lemma 6.1.13). Utilizing the correspondence between trivial idempotents and objects in this way is quite elegant.

Lemma 6.1.10. *There is a forgetful functor*

$$\begin{aligned} \iota_{\mathcal{C}} : \mathcal{C} &\hookrightarrow \mathbf{Kar}\,\mathcal{C} \\ X &\longmapsto (X, 1_X) \\ (f : X \longrightarrow Y) &\longmapsto (1_Y, f, 1_X) \end{aligned}$$

for all categories $\mathcal{C}$.

In the rest of this Section $\iota_{\mathcal{C}}$ will continue to refer to this functor.

Proof. $\mathcal{C}$ is clearly isomorphic to the full subcategory of $\mathbf{Kar}\,\mathcal{C}$ whose objects are of the form $(X, 1_X)$:

$$\mathbf{Kar}(\mathcal{C})\Big((X, 1_X), (Y, 1_Y)\Big) = \Big\{(1_Y, f, 1_X) \,|\, 1_Y \circ f = f = f \circ 1_X\Big\} \cong \mathcal{C}(X, Y) \qquad \square$$

Theorem 6.1.11. $(\mathbf{Kar}(\mathcal{C}), \iota_{\mathcal{C}})$ *is an idempotent completion of* $\mathcal{C}$ *for every* $\mathcal{C} \in \mathbf{Cat}$.

Example 6.1.12 ([Law89, p. 267])**.** Karoubi envelopes enable a description of manifolds without defining *charts*, *transformation maps*, and *atlases*. Namely, if Man is the category of manifolds and maps between them and Open $\subset$ Man is the full subcategory whose objects are simply open subsets $U \subseteq \mathbb{R}^n$ for any $n \in \mathbb{N}_0$, then Man $\simeq$ $\mathbf{Kar}$ Open implies that Open $\hookrightarrow$ Man is an idempotent completion (Theorem 6.1.11).

For instance, to construct the sphere $S^2 \in$ Man, one may consider $\mathbb{R}^3 \in$ Open and the idempotent $\frac{x}{|x|} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. $(\mathbb{R}^3, \frac{x}{|x|}) \in \mathbf{Kar}$ Open then corresponds to $S^2 \in$ Man.

Lemma 6.1.13. *The Karoubi envelope is idempotent complete. Given an idempotent*

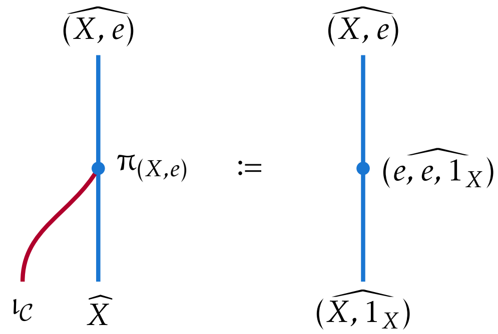
$$(e, e', e) : (X, e) \longrightarrow (X, e) \tag{6.1}$$

in a Karoubi envelope $\mathbf{Kar}\,\mathcal{C}$*, it splits according to the diagram*

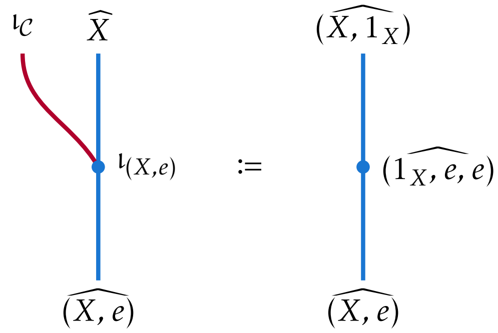
$$\begin{tikzcd}
(X, e) \arrow[loop above, "{(e, e', e)}"] \arrow[d, bend left=60, "{(e', e', e)}"] \\
(X, e') \arrow[u, bend left=60, "{(e, e', e')}"] \arrow[loop below, "{(e', e', e')}"]
\end{tikzcd} \quad . \tag{6.2}$$

Proof. We note that all idempotents in $\mathbf{Kar}\,\mathcal{C}$ must be of the form (6.1) for some idempotent $e : X \longrightarrow X$ in $\mathcal{C}$. Now, the idempotence of (e, e', e) implies that $e' : X \longrightarrow X$ is an idempotent in $\mathcal{C}$. Therefore, $(X, e') \in \mathbf{Kar}(\mathcal{C})$, which in turn shows that the diagram (6.2) is well-defined. Moreover, $1_{(X,e')} = (e', e', e')$ and idempotence of e' in $\mathcal{C}$ implies that the diagram commutes except the empty path $(X, e) \longrightarrow (X, e)$ (Observation 5.1.14). $\square$

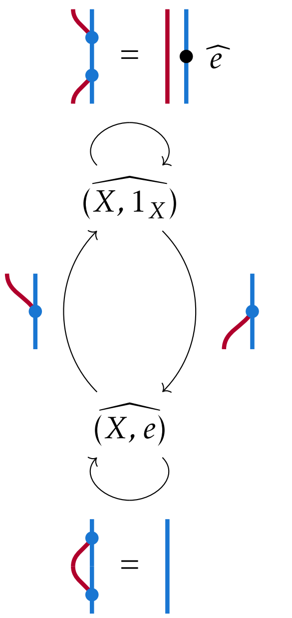
Observation 6.1.14. By setting



and



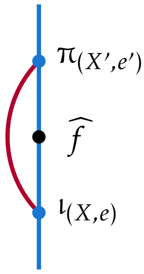
using string diagrams in **Cat** (Observation 2.2.4) we may write splitting diagrams for idempotents $(1_X, e, 1_X)$ in $\mathbf{Kar}\,\mathcal{C}$ as



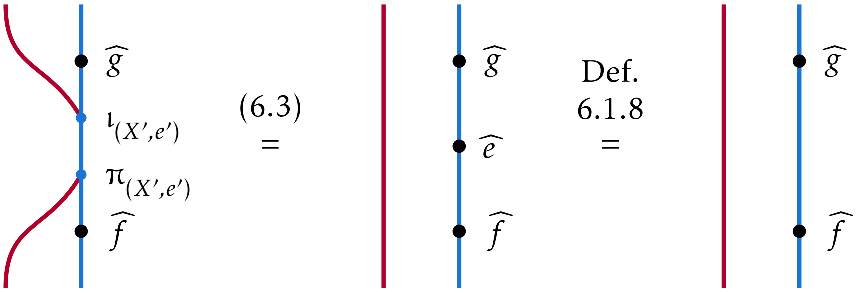
. (6.3)

Moreover, we may decompose arbitrary morphisms in $\mathbf{Kar}\,\mathcal{C}$ as

$$(e', f, e) \mathrel{\hat{=}} \widehat{(e', f, e)} =$$



and retrieve the useful identity for their compositions:



(6.4)

for their compositions.

Proposition 6.1.15. $\iota_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{Kar}\,\mathcal{A}$ *forms an adjoint equivalence together with the functor*

$$\begin{aligned} \pi_{\mathcal{A}} : \mathbf{Kar}\,\mathcal{A} &\longrightarrow \mathcal{A} \\ (X, e) &\longmapsto X_e \\ (e', f, e) &\longmapsto \pi_{e'} \circ f \circ \iota_e \end{aligned}$$

for all $\mathcal{A} \in \mathbf{Cat}^{\mathbf{ic}}$*. Hereby,* (X_e, π_e, ι_e) *denotes splitting data of idempotents* $e : X \longrightarrow X$ *in* $\mathcal{A}$*. In particular,* $\iota_{\mathcal{A}}$ *and* $\pi_{\mathcal{A}}$ *themselves form a split idempotent:*

$$\begin{tikzcd} \mathbf{Kar}\,\mathcal{A} \arrow[loop above, "\iota_{\mathcal{A}} \circ \pi_{\mathcal{A}}"] \arrow[d, bend left, "\pi_{\mathcal{A}}"] \\ \mathcal{A} \arrow[u, bend left, "\iota_{\mathcal{A}}"] \arrow[loop below, "\mathrm{id}_{\mathcal{A}}"] \end{tikzcd}$$

We fix the notation $\pi_{\mathcal{A}}$ for this functor for the rest of this chapter.

Proof. Let us break $\pi_{\mathcal{A}}$ up into two components:

$$\begin{array}{ccccc}
\mathbf{Kar}\,\mathcal{A} & \xrightarrow{\qquad\widetilde{\pi}_{\mathcal{A}}\qquad} & \mathbf{im}(\iota_{\mathcal{A}}) & \xrightarrow{\qquad\iota_{\mathcal{A}}^{-1}\qquad} & \mathcal{A} \\
(X,e) & \longmapsto & (X_e, 1_{X_e}) & \longmapsto & X_e \\
(e', f, e) & \longmapsto & (1_{X'_{e'}}, \pi_{e'} \circ f \circ \iota_e, 1_{X_e}) & \longmapsto & \pi_{e'} \circ f \circ \iota_e
\end{array}$$

As we saw, $\mathbf{im}(\iota_{\mathcal{A}})$ is the full subcategory of $\mathbf{Kar}\,\mathcal{A}$ whose objects are of the form $(X, 1_X) \in \mathbf{Kar}\,\mathcal{A}$ and there is an isomorphism $\iota_{\mathcal{A}}^{-1} : \mathbf{im}(\iota_{\mathcal{A}}) \longrightarrow \mathcal{A}$ (cf. proof of Lemma 6.1.10). Now, since we naturally choose trivial splitting data $(X, 1_X, 1_X)$ for trivial idempotents 1_X, we see that

$$\widetilde{\pi}_{\mathcal{A}}\big|_{\mathbf{im}(\iota_{\mathcal{A}})} : \mathbf{im}(\iota_{\mathcal{A}}) \longrightarrow \mathbf{im}(\iota_{\mathcal{A}}),$$

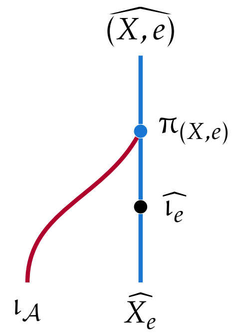
the restriction of $\widetilde{\pi}_{\mathcal{A}}$ to $\mathbf{im}(\iota_{\mathcal{A}})$, is an identity. Thus,

$$\pi_{\mathcal{A}}\big|_{\mathbf{im}(\iota_{\mathcal{A}})} = \iota_{\mathcal{A}}^{-1},$$

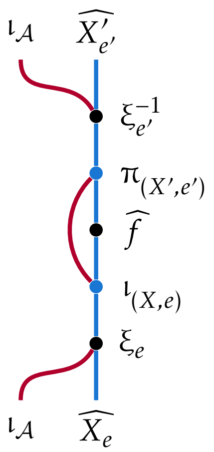
i.e. $\pi_{\mathcal{A}}$ is left inverse to $\iota_{\mathcal{A}}$ and $\widetilde{\pi}_{\mathcal{A}} = \iota_{\mathcal{A}} \circ \pi_{\mathcal{A}}$ is an idempotent.

Interestingly, $\widetilde{\pi}_{\mathcal{A}}$ is not just an idempotent but also an equivalence. To see the equivalence consider that idempotents $(1_X, e, 1_X)$ in $\mathbf{Kar}\,\mathcal{A}$ split both via $(X_e, 1_{X_e})$ and (X, e). This implies an isomorphism

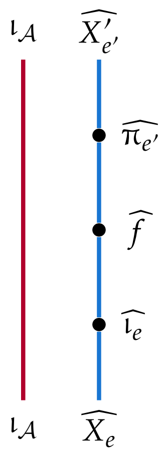
$$\xi_e := (e, e, 1_x) \circ (1_X, \iota_e, 1_{X_e}) \mathrel{\hat{=}}$$



for all idempotents e in $\mathcal{A}$ (cf. Lemma 5.1.15) that we denoted similarly to Observation 6.1.14. Using the calculation



$$\overset{(6.4)}{=}$$

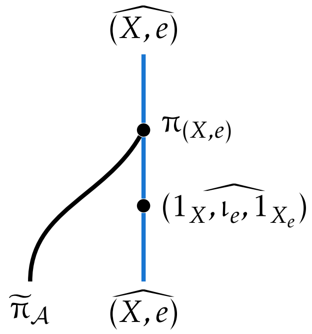


we see that we may rewrite the action of $\widetilde{\pi}_{\mathcal{A}}$ as

$$\widetilde{\pi}_{\mathcal{A}}\mathbb{F} = \xi_{e'}^{-1} \circ \mathbb{F} \circ \xi_e \tag{6.5}$$

for all morphisms $\mathbb{F} = (e', f, e) : (X, e) \longrightarrow (X', e')$ in **Kar** $\mathcal{A}$. Rewriting ξ_e once more, this time as

$\xi_e \triangleq \widehat{\xi_e} =$

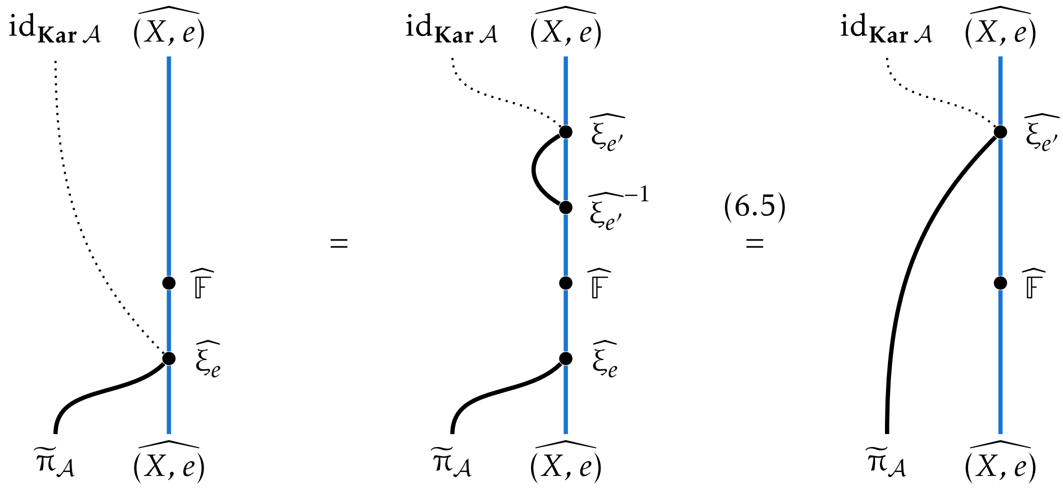


,

we see that $\xi_{\mathcal{A}} : \widetilde{\pi}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathbf{Kar}\,\mathcal{A}}$ with components

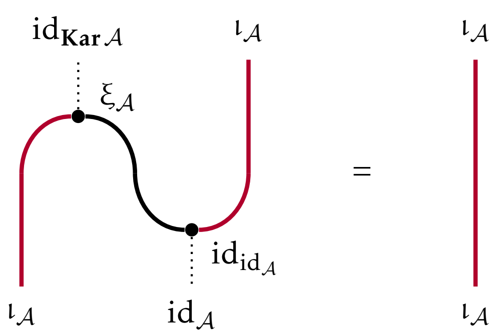
$$(\xi_{\mathcal{A}})_{(X,e)} := \xi_e$$

is a natural isomorphism because



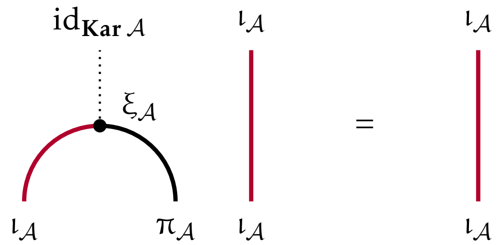
holds for all morphisms $\mathbb{F} : (X, e) \longrightarrow (X', e')$ in **Kar** $\mathcal{A}$ (cf. Diagram (2.5)). Thus, $\widetilde{\pi}_{\mathcal{A}}$ and $\pi_{\mathcal{A}} = \iota_{\mathcal{A}}^{-1} \circ \widetilde{\pi}_{\mathcal{A}}$ are equivalences.

Furthermore, we may see that $\iota_{\mathcal{A}}$ and $\pi_{\mathcal{A}}$ form an adjoint equivalence with left zigzag identity



. (6.6)

The bottom half of the l.h.s. of (6.6) is clearly an identity natural transformation and



is too, as it has components $(\xi_{\mathcal{A}})_{\iota_{\mathcal{A}} X} = (\xi_{\mathcal{A}})_{(X,1_X)} = \xi_{1_X} = 1_{(X,1_X)}$. Now, the second zigzag identity follows automatically from the first zigzag identity because $\iota_{\mathcal{A}}$ and $\pi_{\mathcal{A}}$ are equivalences (Corollary 3.1.6). $\square$

Theorem 6.1.16. *The Karoubi envelope is a strict and faithful 2-functor*

$$\begin{aligned} \mathbf{Kar} : \mathbf{Cat} &\longrightarrow \mathbf{Cat}^{\mathrm{ic}} \\ \mathcal{C} &\longmapsto \mathbf{Kar}\,\mathcal{C} \\ F &\longmapsto \mathbf{Kar}\,F \\ \alpha &\longmapsto \mathbf{Kar}\,\alpha \end{aligned}$$

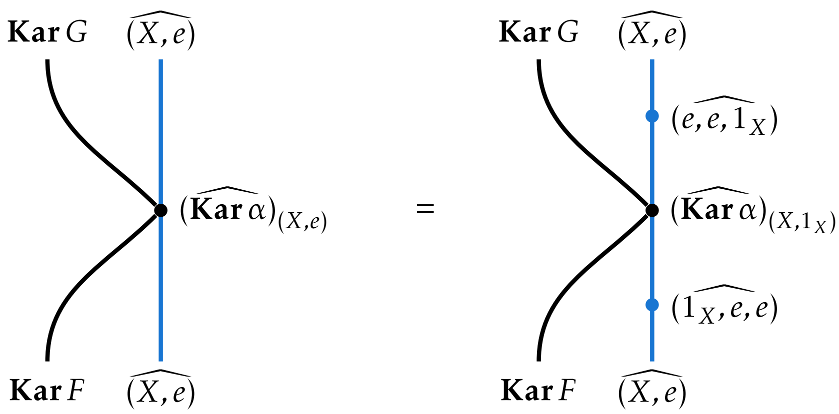
whereby functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ *are mapped to*

$$\begin{aligned} \mathbf{Kar}\,F : \mathbf{Kar}\,\mathcal{C} &\longrightarrow \mathbf{Kar}\,\mathcal{D} \\ (X,e) &\longmapsto (FX, Fe) \\ (e', f, e) &\longmapsto (Fe', Ff, Fe) \end{aligned}$$

and natural transformations $\alpha : F \Longrightarrow G$ *are mapped to*

$$\mathbf{Kar}\,\alpha : \mathbf{Kar}\,F \Longrightarrow \mathbf{Kar}\,G$$

with components



(6.7)

determined by

$$(\mathbf{Kar}\,\alpha)_{(X,1_X)} := \iota_{\mathcal{D}}(\alpha_X) = (G1_X, \alpha_X, F1_X).$$

Proof. Let us first see, that $\mathbf{Kar} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ is a well-defined map. We already know $\mathbf{Kar}\,\mathcal{C} \in \mathbf{Cat}^{\mathbf{ic}}$, so ut renaubs ti cgecj that such $\mathbf{Kar}\,F$ and $\mathbf{Kar}\,\alpha$ are functors and natural transformations, respectively.

$\mathbf{Kar}\,F$ IS A FUNCTOR:

$$\begin{aligned}(\mathbf{Kar}\,F)1_{(X,e)} &= (\mathbf{Kar}\,F)(e,e,e)\\ &= (Fe,Fe,Fe)\\ &= 1_{(FX,Fe)}\\ &= 1_{(\mathbf{Kar}\,F)(X,e)}\end{aligned}$$

for all $(X,e) \in \mathbf{Kar}\,\mathcal{C}$ and

$$\begin{tikzcd}
\big((e'',g,e'),(e',f,e)\big) \arrow[r, mapsto, "\circ_{\mathcal{C}}"] \arrow[dd, mapsto, "\mathbf{Kar}\,F \times \mathbf{Kar}\,F"'] & (e'',g\circ f,e) \arrow[d, mapsto, "\mathbf{Kar}\,F"] \\
 & (Fe'',F(g\circ f),Fe) \arrow[d, phantom, "="] \\
\big((Fe'',Fg,Fe'),(Fe',Ff,Fe)\big) \arrow[r, mapsto, "\circ_{\mathbf{Kar}\,\mathcal{C}}"'] & (Fe'',Fg\circ Ff,Fe)
\end{tikzcd}$$

for all $(e'',g,e'),(e',f,e)$ in $\mathbf{Kar}\,\mathcal{C}$.

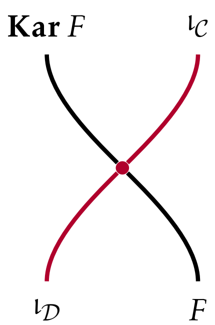
We note that

$$\mathbf{Kar}\,F\big|_{\mathbf{im}(\iota_{\mathcal{C}})} : \mathbf{im}(\iota_{\mathcal{C}}) \longrightarrow \mathbf{im}(\iota_{\mathcal{D}})$$

maps the same way as $F : \mathcal{C} \longrightarrow \mathcal{D}$, i.e.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, hook, "\iota_{\mathcal{D}}"] \\
\mathbf{Kar}\,\mathcal{C} \arrow[r, "\mathbf{Kar}\,F"'] & \mathbf{Kar}\,\mathcal{D}
\end{tikzcd} \tag{6.8}$$

commutes. In string diagrams this means that

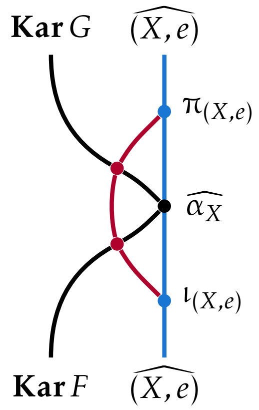


(6.9)

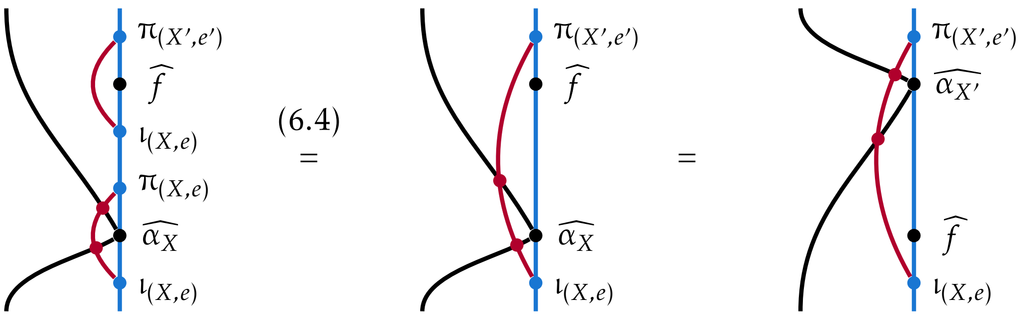
is an identity natural transformation.

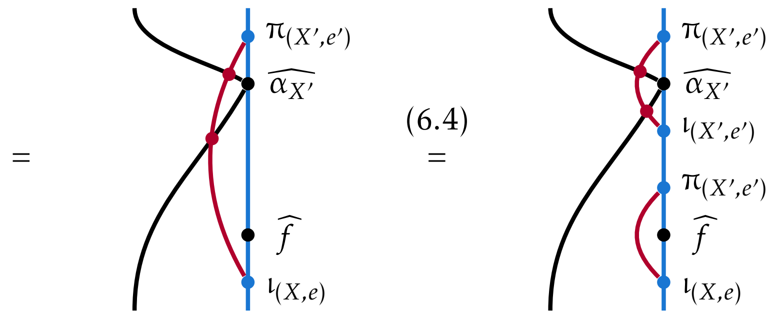
Kar α *is a natural transformation:* We slightly reformulate $\mathbf{Kar}\,\alpha : \mathbf{Kar}\,F \Longrightarrow \mathbf{Kar}\,G$ using Observation 6.1.14 and the identity natural transformation (6.9):

$$(\mathbf{Kar}\,\alpha)_{(X,e)} =$$



The calculation





holds for all (e', f, e) in $\mathbf{Kar}\,\mathcal{C}$ so $\mathbf{Kar}\,\alpha$ is a natural transformation (cf. Dia. (2.5)).

Kar *is a strict and faithful 2-functor:* It is easy to check that **Kar** acts functorially:

$$\mathbf{Kar}\,\mathrm{id}_{\mathcal{C}} = \mathrm{id}_{\mathbf{Kar}\,\mathcal{C}}$$
$$(\mathbf{Kar}\,G) \circ (\mathbf{Kar}\,F) = \mathbf{Kar}(G \circ F)$$

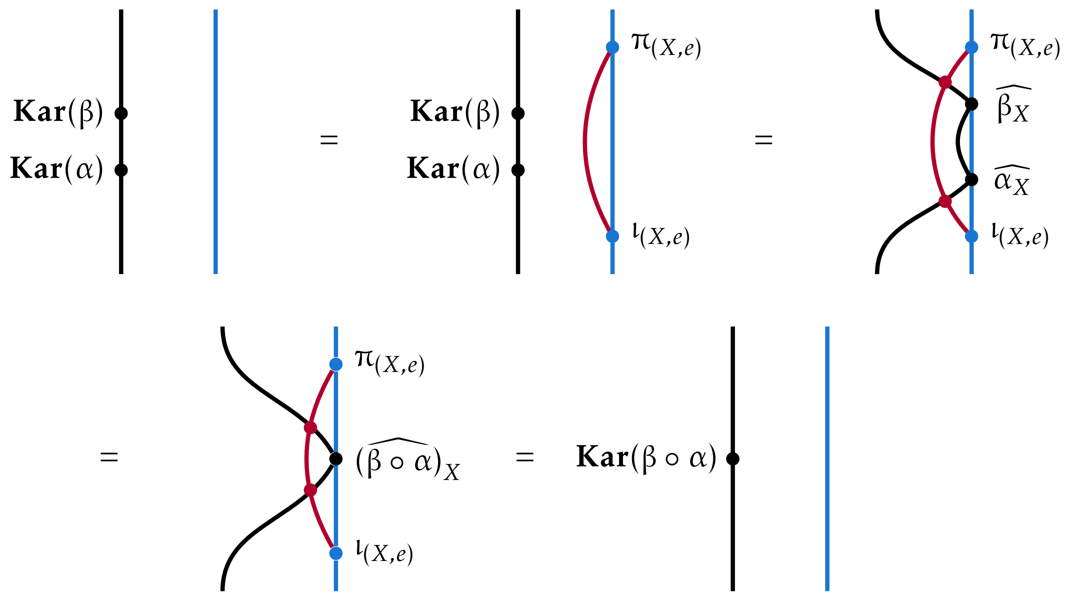
Moreover, we want to see that **Kar** is a functor on Hom categories, i.e. that

$$\begin{aligned} \mathbf{Kar}_{(\mathcal{C},\mathcal{D})} : \mathbf{Cat}(\mathcal{C},\mathcal{D}) &\longrightarrow \mathbf{Cat}^{\mathrm{ic}}(\mathbf{Kar}\,\mathcal{C},\mathbf{Kar}\,\mathcal{D}) \\ F &\longmapsto \mathbf{Kar}\,F \\ \alpha &\longmapsto \mathbf{Kar}\,\alpha \end{aligned}$$

is a functor. It is again easy to check that identities id_F are preserved:

$$\mathbf{Kar}_{(\mathcal{C},\mathcal{D})}\,\mathrm{id}_F = \mathrm{id}_{\mathbf{Kar}\,F}$$

Moreover, composition of natural transformations $\mathbf{Kar}\,\alpha : \mathbf{Kar}\,F \Longrightarrow \mathbf{Kar}\,G$, $\mathbf{Kar}\,\beta : \mathbf{Kar}\,G \Longrightarrow \mathbf{Kar}\,H$ is also preserved:



Thus, $\mathbf{Kar} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathrm{ic}}$ is a strict 2-functor.

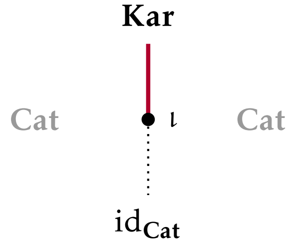
Lastly, $\mathbf{Kar}$ is faithful, i.e.

$$\begin{aligned} \mathbf{Kar}_{(F,G)} : \hom(F,G) &\longrightarrow \hom(\mathbf{Kar}\,F,\mathbf{Kar}\,G) \\ \alpha &\longmapsto \mathbf{Kar}\,\alpha \end{aligned}$$

a bijection for all $F, G : \mathcal{C} \longrightarrow \mathcal{D}$. To see that this a bijection, we note that natural transformations $\beta : \mathbf{Kar}\,F \Longrightarrow \mathbf{Kar}\,G$ are already fully determined by the $\beta_{(X,1_X)} = \beta_{\iota_{\mathcal{C}}X}$ components (6.7). Moreover, the components $\beta_{(X,1_X)}$ are all in the image of $\iota_{\mathcal{D}}$. Therefore, we may construct the inverse

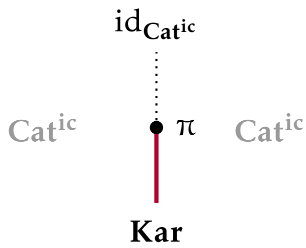
$$\begin{aligned} \mathbf{Kar}^{-1}_{(F,G)} : \hom(\mathbf{Kar}\,F,\mathbf{Kar}\,G) &\longrightarrow \hom(F,G) \\ \beta &\longmapsto \iota_{\mathcal{D}}^{-1} \otimes \beta \otimes \iota_{\mathcal{C}}. \end{aligned}$$ □

Lemma 6.1.17. *Interpreting* **Kar** *as a 2-functor* **Cat** $\longrightarrow$ **Cat** *we may construct a strict 2-natural transformation*



from the collection of all $\iota_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Kar}\,\mathcal{C}$.

Similarly, interpreting **Kar** *as a 2-functor* $\mathbf{Cat}^{\mathbf{ic}} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ *we may construct a 2-natural equivalence*

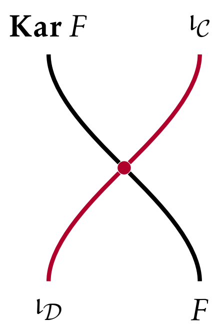


from the collection of all $\pi_{\mathcal{A}}$ $\mathbf{Kar}\,\mathcal{A} \longrightarrow \mathcal{A}$.

Proof. Let us take a small step back first. The 1-morphism components of such 2-natural transformations consist of 1-morphisms in **Cat**, i.e. functors. The 2-morphism components of these 2-natural transformations consist of 2-morphisms in **Cat**, i.e. natural transformations.

The 2-natural transformation ι has 1-morphism components given by the functors $\iota_{\mathcal{C}}$ defined in Lemma 6.1.10. Commutativity of (6.8) shows that $\iota_{\mathcal{C}}$ is already a strict 2-natural transformation, i.e. has identity 2-morphism components

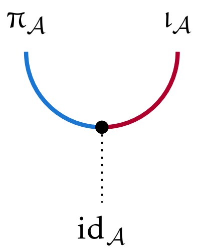
$\iota_F =$



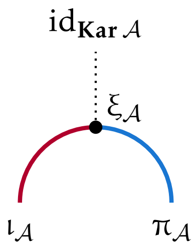
(6.10)

for all $F : \mathcal{C} \longrightarrow \mathcal{D}$ in **Cat**.

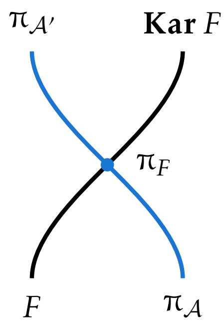
Meanwhile, π is a 2-natural equivalence with 1-morphism components given by the functors $\pi_{\mathcal{A}}$ defined in Proposition 6.1.15. We construct the 2-morphism components of π from the units and counits



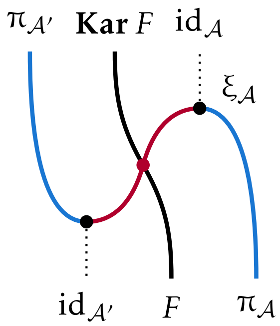
and



of the adjoint equivalences $\iota_{\mathcal{A}} \dashv \pi_{\mathcal{A}}$ (cf. Dia. (6.6)) and identity natural transformations (6.10):

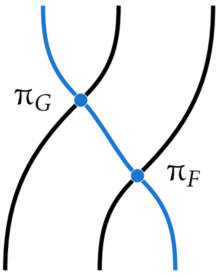


:=



(6.11)

for all functors $F : \mathcal{A} \longrightarrow \mathcal{A}'$ in $\mathbf{Cat}^{\mathbf{ic}}$. π fulfils 2-naturality because



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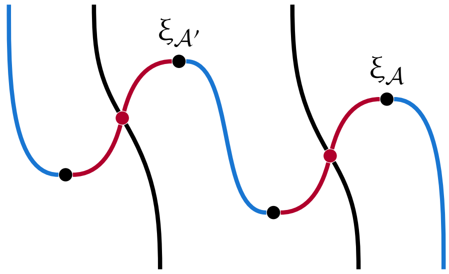
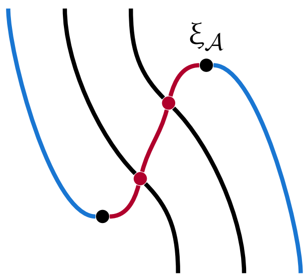


$\overset{(6.6)}{=}$



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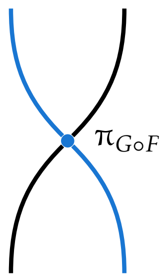
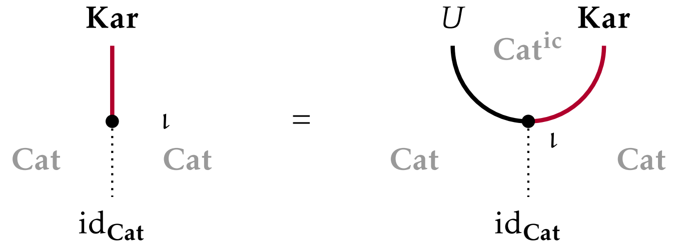


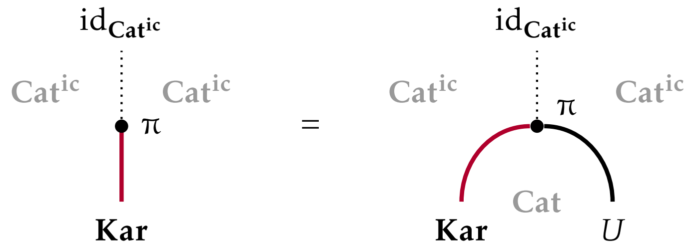
holds for all $F : \mathcal{A} \longrightarrow \mathcal{A}', G : \mathcal{A}' \longrightarrow \mathcal{A}''$. □

Remark 6.1.18. One may consider making π strict by redefining the Karoubi envelope **Kar** (Sketch A.2.7).

Observation 6.1.19. Interpreting **Kar** again as a 2-functor $\mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ and denoting forgetful 2-functor $\mathbf{Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{Cat}$ by U we see that we may reformulate the 2-natural transformations ι and π as

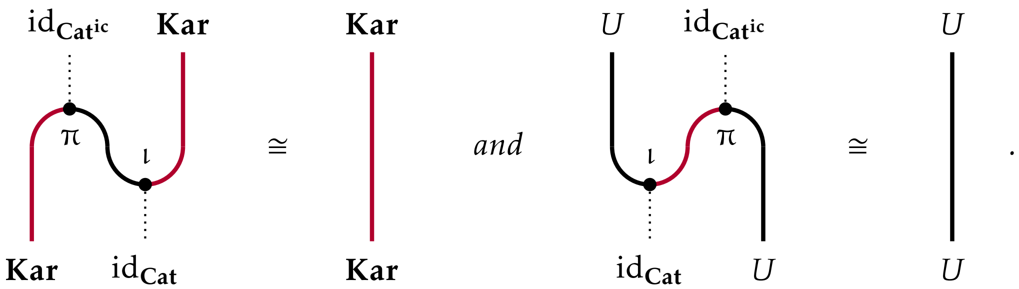


and



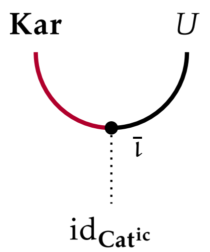
.

Theorem 6.1.20. *The Karoubi envelope* $\mathbf{Kar} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ *is left 2-adjoint to the forgetful 2-functor* $U : \mathbf{Cat}^{\mathbf{ic}} \longrightarrow \mathbf{Cat}$*, i.e.*



Proof. The idea is that the 2-adjunction is a right adjoint right inverse 2-adjunction (Definition 3.2.3). This proof is inspired by Proposition 3.2.5 but slightly simplified.

We notice that ι induces a strict 2-natural equivalence



by restricting along $\mathbf{Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{Cat}$, i.e.

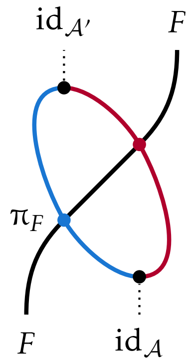
$$\bar{\iota}_{\mathcal{A}} := \iota_{\mathcal{A}}$$

for all $\mathcal{A} \in \mathbf{Cat}^{\mathrm{ic}}$. Moreover, composition $\pi \circ \bar{\iota}$ has 1-morphism components

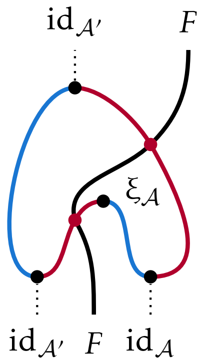
$$(\pi \circ \bar{\iota})_{\mathcal{A}} = \pi_{\mathcal{A}} \circ \iota_{\mathcal{A}} \equiv \mathrm{id}_{\mathcal{A}}$$

and 2-morphism components

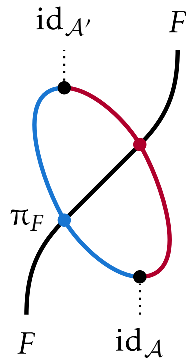
$(\pi \circ \bar{\iota})_F =$



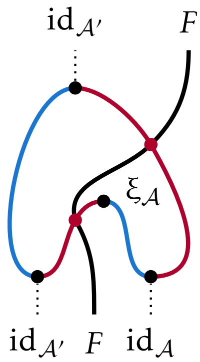
$\overset{(6.11)}{=}$



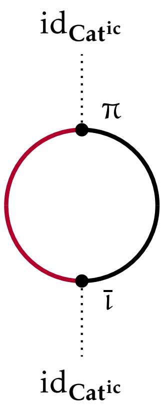
$=$



$\overset{(6.11)}{=}$



In other words, $\bar{\iota}$ is right inverse[89] to π:



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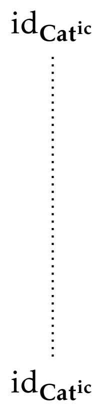
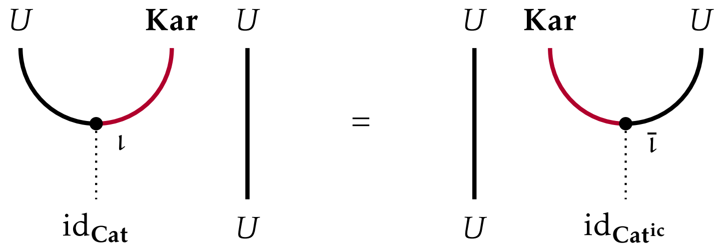


We notice that

89 and actually also weak left inverse

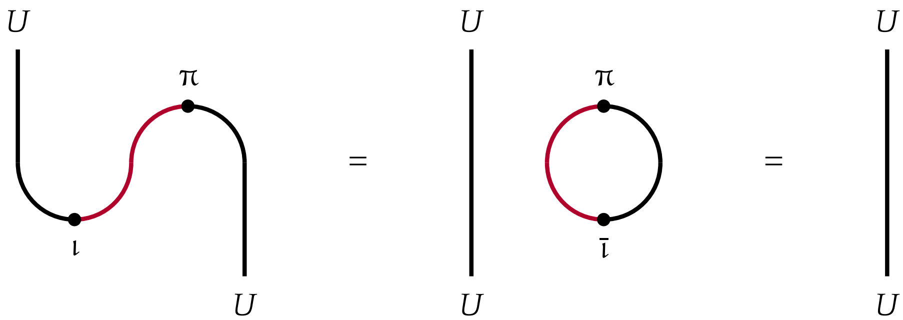


(6.12)

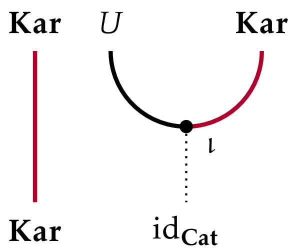
since

$$\begin{aligned}(\iota \otimes U)_{\mathcal{A}} &= \iota_{U\mathcal{A}}\\ &= \iota_{\mathcal{A}}\\ &= \bar{\iota}_{\mathcal{A}}\\ &= U\bar{\iota}_{\mathcal{A}}\\ &= (U \otimes \bar{\iota})_{\mathcal{A}}\end{aligned}$$

for all $\mathcal{A} \in \mathbf{Cat}^{\mathrm{ic}}$ (Definition 2.3.31). This calculation shows that both sides agree on 1-morphism components. Moreover, both sides of the equation represent strict 2-natural transformations (cf. Note 2.3.32) and, therefore, agree on 2-morphism components, too. The equation (6.12) shows us that the right zigzag modification is an identity:



Furthermore, we notice that



(6.13)

has 1-morphism components

$$\begin{aligned}(\mathbf{Kar} \otimes \iota)_{\mathcal{C}} = \mathbf{Kar}\, \iota_{\mathcal{C}} : \mathbf{Kar}\, \mathcal{C} &\longrightarrow \mathbf{Kar}\, \mathbf{Kar}\, \mathcal{C}\\ (X, e) &\longmapsto ((X, 1_X), (1_X, e, 1_X))\\ (e', f, e) &\longmapsto ((1_Y, e', 1_Y), (1_Y, f, 1_X), (1_X, e, 1_X)).\end{aligned}$$

Moreover, we notice that

$$\begin{array}{rcl} \mathbf{im}(\mathbf{Kar}\,\iota_{\mathcal{C}}) & \hookrightarrow \mathbf{Kar}\,\mathbf{Kar}\,\mathcal{C} \xrightarrow{\pi_{\mathbf{Kar}\,\mathcal{C}}} & \mathbf{Kar}\,\mathcal{C} \\ ((X,1_X),(1_X,e,1_X)) & \longmapsto & (X,e) \\ ((1_Y,e',1_Y),(1_Y,f,1_X),(1_X,e,1_X)) & \longmapsto & (e',f,e) \end{array}$$

is left inverse to $\mathbf{Kar}\,\iota_{\mathcal{C}}$ by choosing splitting data $((X,e),(e,e,e'),(e',e,e))$ for idempotents (e',e,e') in $\mathbf{Kar}\,\mathcal{C}$. This is consistent with our previous choices made for $\pi_{\mathcal{A}}$ (Proposition 6.1.15). Therefore, there exist natural isomorphisms

$$\mathbf{Kar}\,\iota_{\mathcal{C}} = \mathbf{Kar}\,\iota_{\mathcal{C}} \circ \pi_{\mathbf{Kar}\,\mathcal{C}} \circ \iota_{\mathbf{Kar}\,\mathcal{C}} \cong \iota_{\mathbf{Kar}\,\mathcal{C}}$$

for all $\mathcal{C} \in \mathbf{Cat}$. In particular, the $\mathbf{Kar}\,\iota_{\mathcal{C}}$ are equivalences which means that the 2-natural transformation (6.13) is a 2-natural equivalence. This implies the left zigzag modification (Proposition 3.2.5). □

Corollary 6.1.21 (Theorem 6.1.11)**.** $\mathbf{Kar}\,\mathcal{C}$ *is the idempotent completion of* $\mathcal{C}$ *for all categories* $\mathcal{C}$*.*

Proof. The 2-adjunction $\mathbf{Kar} \dashv U$ implies that $(\mathbf{Kar}\,\mathcal{C}, \iota_{\mathcal{C}})$ is a universal functor from $\mathcal{C}$ to $U : \mathbf{Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{Cat}$ (Proposition 4.3.8). □

6.2. 2-Idempotent Completions

6.2.1. 2-Condensation Completions

Preview 6.2.1. *2-Condensation completions* are a categorification of idempotent completions. Thus, our discussion of 2-condensation completions will be similar to our discussion of idempotent completions:

- Definition 6.2.5: We define *2-Condensation completions* as a categorification of idempotent completions, i.e. a condensation completion of a 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ is a universal 2-functor

  $$(\overline{\mathcal{B}} \in \mathbf{2Cat}^{\mathbf{cc}}, \iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \overline{\mathcal{B}})$$

  s.t.

  $$(-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathbf{cc}}(\overline{\mathcal{B}}, \mathcal{A}) \longrightarrow \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{A})$$

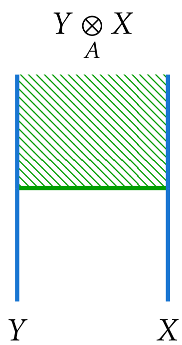
  is a 2-equivalence for all $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$.

- Definition 6.2.8: We define *2-Karoubi envelopes* $\mathbf{2Kar}\,\mathcal{B}$ of 2-categories $\mathcal{B} \in \mathbf{2Cat^{ic}}$ as a categorification of Karoubi envelopes.
- Assumption 6.2.14: We note that we will assume our 2-categories and 2-functors to be strict for the rest of the section.
- Lemma 6.2.15: We discover the forgetful 2-functor $\iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathbf{2Kar}\,\mathcal{B}$.
- Theorem 6.2.16: This is our main theorem and it states that pairs $(\mathbf{2Kar}\,\mathcal{B}, \iota_{\mathcal{B}})$ are condensation completions. We first sketch the proof and then spend the rest of this section completing the proof. We show some extra details along the way and finally conclude this section and the proof of this theorem with Corollary 6.2.28.

Reminder 6.2.2. For one, we recall our Nomenclature 5.2.60:

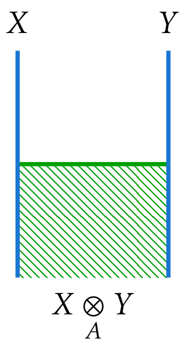
2-Condensation algebras	Δ-separable Frobenius algebras
2-Condensates	splittings of 2-condensation algebras
2-Condensation modules	special Frobenius modules
2-Condensation module maps	Frobenius module maps

For another, we recall that we depict the splitting maps of split relative tensor products $(X \otimes_A Y, \pi, \iota)$ (Notation 5.2.56) as



$:= \pi$

and



$:= \iota$.

Definition 6.2.3.

- If all 2-condensation algebras in a 2-category $\mathcal{B}$ condense, then we say that $\mathcal{B}$ *has all condensates* [GJF19, Def. 2.3.1] or alternatively that $\mathcal{B}$ is *condensation complete*.
- We define a $\mathbf{2Cat}^{\mathbf{ic}}$ as the full sub-3-category of $\mathbf{2Cat}$ containing all locally idempotent complete 2-categories. Moreover, we define $\mathbf{2Cat}^{\mathbf{cc}}$ as the full sub-3-category of $\mathbf{2Cat}^{\mathbf{ic}}$ containing all locally idempotent complete 2-categories that also have all condensates. We note that this provides us with forgetful 2-functors

$$\mathbf{2Cat}^{\mathbf{cc}} \hookrightarrow \mathbf{2Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{2Cat}\,.$$

Sanity Check 6.2.4. $\mathbf{2Cat}^{\mathbf{cc}}$ is closed under 2-equivalences as a sub-3-category of $\mathbf{2Cat}^{\mathbf{ic}}$.

Proof. The statement means, that if we are given a 2-equivalence $F : \mathcal{B} \rightleftarrows \mathcal{D} : G$, then $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{cc}}$ if and only if $\mathcal{D} \in \mathbf{2Cat}^{\mathbf{cc}}$. Therefore, let us assume that $\mathcal{B}$ has all condensates and that we are given an arbitrary condensation algebra $A : a \longrightarrow a$ in $\mathcal{D}$. Then, GA in $\mathcal{B}$ is a condensation algebra (Proposition 5.2.19). By assumption GA condenses onto some $b \in \mathcal{B}$ which implies that FGA condenses onto Fb (Proposition 5.2.27). Therefore, since $FGa \simeq a$, A condenses onto Fb. □

Definition 6.2.5 (Condensation Completion)**.** A *Condensation completion* [Dé22, Def. 1.2.1] of a 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ is a universal 2-functor $(\overline{\mathcal{B}} \in \mathbf{2Cat}^{\mathbf{cc}}, \iota_{\mathcal{B}} : \mathcal{B} \longrightarrow \overline{\mathcal{B}})$ from $\mathcal{B}$ to the forgetful 3-functor $\mathbf{2Cat}^{\mathbf{cc}} \hookrightarrow \mathbf{2Cat}^{\mathbf{ic}}$. In other words, it is a 2-category-2-functor pair (Definition 4.4.1) such that

$$(-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathbf{cc}}(\overline{\mathcal{B}}, \mathcal{A}) \longrightarrow \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{A})$$

is a 2-equivalence for all $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$.

Remark 6.2.6. The Condensation completion is indeed a completion, i.e. $(\mathcal{B}, \mathrm{id}_{\mathcal{B}})$ is a 2-condensation completion of a 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ if and only if $\mathcal{B}$ is condensation complete.

Sketch 6.2.7. The intuition behind the Karoubi envelope (Sketch 6.1.7) will serve us well here again. Analogously to the idempotent completion, the condensation completion $\overline{\mathcal{B}}$ of a 2-category $\mathcal{B}$ should contain $\mathcal{B}$, i.e. there should be a fully faithful 2-functor

$$\iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathbf{2Kar}\,\mathcal{B}.$$

Thus, to construct a condensation completion $\overline{\mathcal{B}}$ of a 2-category $\mathcal{B}$ we will again construct it by "adding" objects and Hom categories to $\mathcal{B}$. Thus, $\mathrm{Ob}(\overline{\mathcal{B}})$ should be $\mathrm{Ob}(\mathcal{B})$ together with an object a_A for every condensation algebra $A : a \longrightarrow a$ in $\mathcal{B}$ that does not yet condense. The Hom categories of $\overline{\mathcal{B}}$ should be the Hom categories of $\mathcal{B}$ and Hom categories induced by the fact that the objects a_A should become condensates of the condensation algebras A. Therefore, $\overline{\mathcal{B}}$ should also contain Hom categories of the form

$$\begin{aligned} \hom(a_A, b) &\simeq \mathbf{Mod}_A(a, b) \\ \hom(a, b_B) &\simeq {}_B\,\mathbf{Mod}(a, b) \\ \hom(a_A, b_B) &\simeq {}_B\,\mathbf{Mod}_A(b, a) \end{aligned}$$

as discussed in Note 5.2.64. Moreover, as discussed in Note 5.2.64 the horizontal composition between these Hom categories should be relative tensor products $\underset{A}{\otimes}$.

As we are about to see, the 2-Karoubi envelope (Definition 6.2.8) satisfies the criteria we have just set out.

Definition 6.2.8 (2-Karoubi Envelope)**.** Given a 2-category $\mathcal{B} \in \mathbf{2Cat^{ic}}$, its *2-Karoubi envelope* $\mathbf{2Kar}(\mathcal{B})$ is the 2-category consisting of:

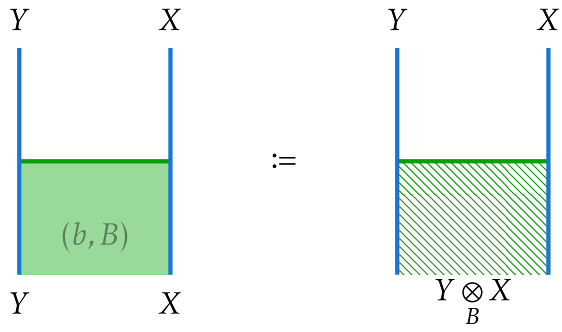
Objects: Condensation algebras (a, A) in $\mathcal{B}$.

Hom categories: Condensation bimodules ${}_BX_A : a \longrightarrow b$ and their maps[90] $f : X \Longrightarrow Y$, i.e.

$$\mathbf{2Kar}(\mathcal{B})((a, A), (b, B)) := {}_B\mathbf{Mod}^{\mathrm{sp}}_A(\mathcal{B}).$$

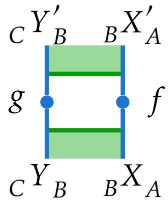
[90] This notation brings a lot of ambiguity. We are already used to the implicit notation of algebras A instead of (A, μ). Here, our notation of B-A-condensation bimodule maps $f : X \Longrightarrow Y$ is even more ambiguous because f is not just a 2-morphism ${}_BX_A \Longrightarrow {}_BY_A$ in $\mathbf{2Kar}\,\mathcal{B}$, but also a 2-morphism ${}_{\mathbb{1}_b}X_{\mathbb{1}_a} \Longrightarrow {}_{\mathbb{1}_b}Y_{\mathbb{1}_a}$ in $\mathbf{2Kar}\,\mathcal{B}$ and even a 2-morphism $X \Longrightarrow A$ in $\mathcal{B}$! This ambiguity will be convenient in our calculations, but we will have to take care to not get confused.

Horizontal composition: Relative tensor products in $\mathcal{B}$; specifically, by setting



we may express the horizontal composition as

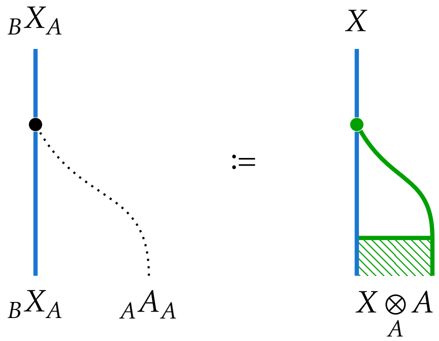
$$\otimes_{(a,A),(b,B),(c,C)} := (-) \underset{B}{\otimes} (-) : {}_C\mathbf{Mod}_B^{\mathrm{sp}} \times {}_B\mathbf{Mod}_A^{\mathrm{sp}} \longrightarrow {}_C\mathbf{Mod}_A^{\mathrm{sp}}$$
$$({}_CY_B, {}_BX_A) \longmapsto {}_C(Y \underset{B}{\otimes} X)_A$$
$$(g, f) \longmapsto$$



.

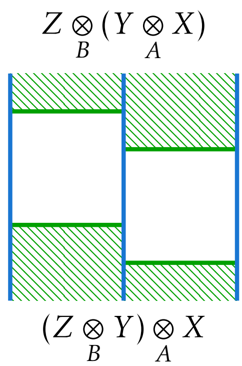
Unit Morphisms: $\mathbb{1}_{(a,A)} := {}_AA_A$.

Unitors: Bimodule actions, e.g.



Associators:

$\alpha_{\mathbf{2Kar}\,\mathcal{B}} :=$



Theorem 6.2.9. *The 2-Karoubi envelope is indeed a 2-category.*

Proof. Consult [Fra22, Thm. 3.17] for this proof. Note that

$$F(Y \otimes_B X) \cong FY \otimes_{FB} X$$

is essential for this proof and the rest of this Section (Lemma 5.2.55). $\square$

Remark 6.2.10. We note that the 2-Karoubi envelope is deserving of its name, as it is a generalisation of the Karoubi envelope. To see this, we assume we are given an arbitrary category $\mathcal{C} \in \mathbf{Cat}$. We recall (Remark 2.3.5) that we may interpret $\mathcal{C}$ as a 2-category $\mathcal{C} \in \mathbf{2Cat}$ whose 2-morphisms are all identities. Immediately, we see that $\mathcal{C} \in \mathbf{2Cat^{ic}}$ which allows us to construct $\mathbf{2Kar}\,\mathcal{C}$. Recalling that condensation algebras in $\mathcal{C}$ are idempotents (Remark 5.2.12), we see that $\mathbf{2Kar}\,\mathcal{C}$ and $\mathbf{Kar}\,\mathcal{C}$ have the same objects. Analogously, one may show that bimodules ${}_{e'}f_e$ in our 2-category $\mathcal{C}$ fulfil $e' \otimes f = f = f \otimes e$. Therefore, $\mathbf{2Kar}\,\mathcal{C} = \mathbf{Kar}\,\mathcal{C}$.

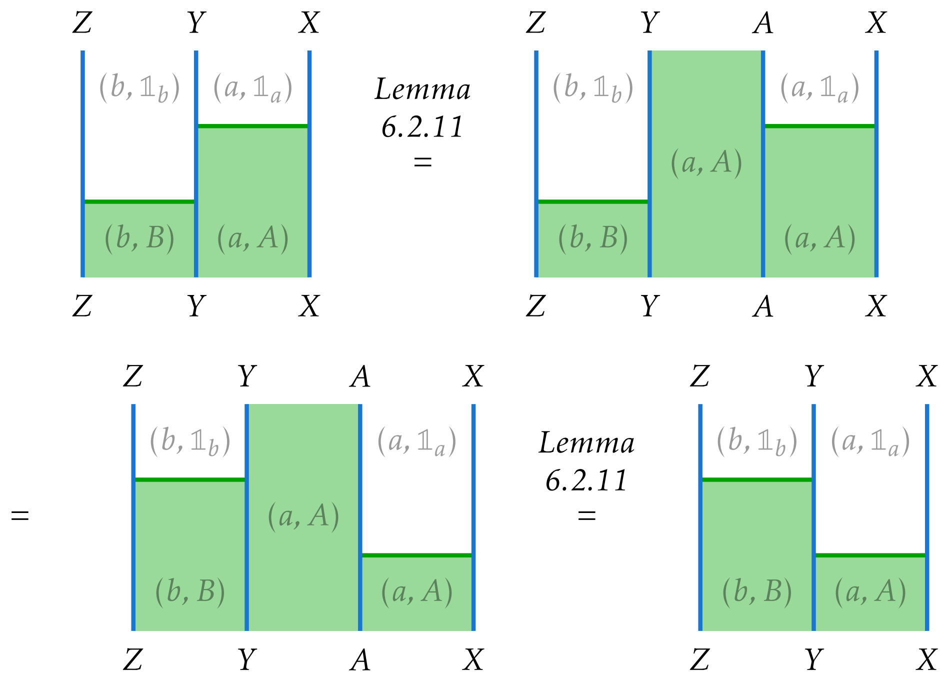
Lemma 6.2.11. *We may choose strict unitors for* $\mathbf{2Kar}\,\mathcal{B}$*, i.e.*

$$X \underset{A}{\otimes} A := X$$

$$A \underset{A}{\otimes} Y := Y$$

(Corollary 5.2.52).

Lemma 6.2.12. *We may choose*



Corollary 6.2.13. *We may choose* $\mathbf{2Kar}\,\mathcal{B}$ *to be strict if* $\mathcal{B}$ *is strict.*

Proof. We may choose strict units (Lemma 6.2.11). Then, Lemma 6.2.12 implies $\alpha_{\mathbf{2Kar}\,\mathcal{B}} = \mathrm{id}$ for $\alpha_{\mathcal{B}} = \mathrm{id}$. □

Assumption 6.2.14. Due to Corollary 6.2.13, we may assume that our 2-categories are strict for the remainder of this chapter. In particular, we may write out or omit units and unitors in our string diagrams depending on what feels more pedagogically useful on a case by case basis.

Lemma 6.2.15. *For all* $\mathcal{B} \in \mathbf{2Cat^{ic}}$ *there exists the forgetful 2-functor*

$$\begin{aligned} \iota_{\mathcal{B}} : \mathcal{B} &\hookrightarrow \mathbf{2Kar}\,\mathcal{B} \\ a &\longmapsto (a, \mathbb{1}_a) \\ \mathcal{B}(a,b) &\longrightarrow {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}(\mathcal{B}). \end{aligned}$$

For the rest of this section we will mean this 2-functor when we write $\iota_{\mathcal{B}}$.

Proof. Recall that

$$\begin{aligned} \mathcal{B}(a,b) &\longrightarrow {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}(\mathcal{B}) \\ (X : a \longrightarrow b) &\longmapsto {}_{\mathbb{1}_b}X_{\mathbb{1}_a} \\ (f : X \longrightarrow Y) &\longmapsto (f : {}_{\mathbb{1}_b}X_{\mathbb{1}_a} \longrightarrow {}_{\mathbb{1}_b}Y_{\mathbb{1}_a}) \end{aligned}$$

is an invertible functor (Proposition 5.2.41). Therefore, the full sub-2-category of $\mathbf{2Kar}\,\mathcal{B}$ containing all the objects $(a, \mathbb{1}_a) \in \mathbf{2Kar}\,\mathcal{B}$ is the same as $\mathcal{B}$ except for different labelling. Noticing that this full sub-2-category is $\mathbf{im}(\iota_{\mathcal{B}})$, we see that there is a canonical isomorphism

$$\mathcal{B} \longrightarrow \mathbf{im}(\iota_{\mathcal{B}})$$

and a forgetful 2-functor

$$\mathbf{im}(\iota_{\mathcal{B}}) \hookrightarrow \mathbf{2Kar}\,\mathcal{B}.$$ □

Theorem 6.2.16. $(\mathbf{2Kar}\,\mathcal{B}, \iota_{\mathcal{B}})$ *is a condensation completion of* $\mathcal{B}$ *for all* $\mathcal{B} \in \mathbf{2Cat^{ic}}$, *i.e.*

$$(-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat^{cc}}(\overline{\mathcal{B}}, \mathcal{A}) \longrightarrow \mathbf{2Cat^{ic}}(\mathcal{B}, \mathcal{A})$$

is a 2-equivalence for all $\mathcal{A} \in \mathbf{2Cat^{cc}}$.

Proof sketch. The rest of this Section will give a detailed proof of this theorem, with Corollary 6.2.28 finally concluding the proof. To ensure clarity, let us prepare our path to Corollary 6.2.28:

(i) Lemma 6.2.17 shows that condensation algebras $A : a \longrightarrow a$ of $\mathcal{B}$ condense in $\mathbf{2Kar}\,\mathcal{B}$ via $\iota_{\mathcal{B}} A = {}_{\mathbb{1}_a}A_{\mathbb{1}_a} = {}_{\mathbb{1}_a}A_A \otimes {}_A A_{\mathbb{1}_a}$. Proposition 6.2.18 shows that condensation algebras ${}_B A_B$ are determined by condensation algebras ${}_{\mathbb{1}_a}A_{\mathbb{1}_a}$ in $\mathbf{2Kar}\,\mathcal{B}$. Thus, $\mathbf{2Kar}\,\mathcal{B} \in \mathbf{2Cat}^{\mathbf{cc}}$.

(ii) In Lemma 6.2.20 we show modifications of $\mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{D})$ correspond to modifications in $\mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{A})$.[91]

(iii) Proposition 6.2.22 shows the 2-functor

$$\begin{aligned} \mathbf{2Kar}_{\mathcal{B},\mathcal{D}} : \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{D}) &\longrightarrow \mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B}, \mathbf{2Kar}\,\mathcal{D}) \\ F &\longmapsto \mathbf{2Kar}\,F \\ \alpha &\longmapsto \mathbf{2Kar}\,\alpha \\ m &\longmapsto \mathbf{2Kar}\,m. \end{aligned}$$

(iv) In Corollary 6.2.23 we deduce that 2-natural transformations of $\mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{D})$ induce 2-natural transformations in $\mathbf{2Cat}^{\mathbf{ic}}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{D})$.

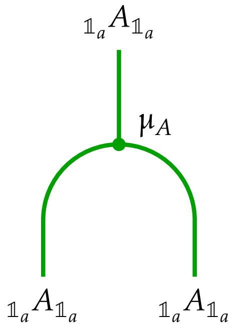
(v) Lemma 6.2.24 hints that the $\iota_{\mathcal{B}}$ are 1-morphism components of a unit $\iota : \mathrm{id}_{\mathbf{2Cat}^{\mathbf{ic}}} \Longrightarrow U \circ \mathbf{2Kar}$. Proposition 6.2.25 shows that $\iota_{\mathcal{A}}$ have left inverses $\pi_{\mathcal{A}}$ for all $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$. We combine this in Corollary 6.2.26 to show that 2-functors $F \in \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{A})$ induce 2-functors $\overline{F} \in \mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{A})$ for all $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$.

(vi) In Corollary 6.2.28 we conclude that we have finally collected all necessary ingredients: (ii) implies that $(-) \circ \iota_{\mathcal{B}}$ is faithful, (iv) implies that $(-) \circ \iota_{\mathcal{B}}$ is essentially full, and (v) implies that $(-) \circ \iota_{\mathcal{B}}$ is essentially surjective. □

[91] This may be where we will see our main tool most clearly. Specifically, the objects $(a, A) \in \mathbf{2Kar}\,\mathcal{B}$ are fully determined by the objects $(a, \mathbb{1}_a) \triangleq a \in \mathcal{B}$. Therefore, when we are dealing with condensates, the "name of the game" will be creating "$\mathcal{B}$-bubbles" in $\mathbf{2Kar}\,\mathcal{B}$ – such as the l.h.s. of equation (6.16) – and then "putting our stuff" into the bubble.

Lemma 6.2.17. *Condensation algebras of $\mathcal{B} \in \mathbf{2Cat}^{\mathrm{ic}}$ condense in $\mathbf{2Kar}\,\mathcal{B}$. For instance, if $(A : a \longrightarrow a, \mu_A, \Delta_A)$ is a condensation algebra in $\mathcal{B}$, then*

$\iota_{\mathcal{B}}\mu_A =$



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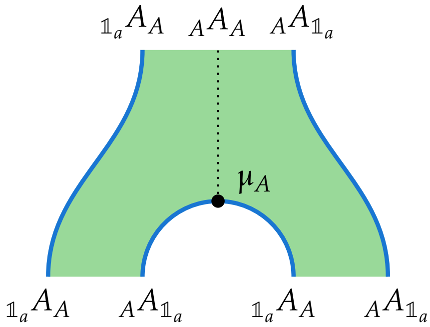


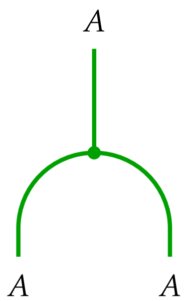
(6.14)

in $\mathbf{2Kar}\,\mathcal{B}$.

Proof. First, let us note that the choice of colours in the string diagrams of equation (6.14) and the following proof is, unsurprisingly, intentional. As usual, we choose green lines when we are considering the algebra structure of a 1-morphism and blue when we are considering its module structure.

Let us start by showing that the r.h.s. of equation (6.14) is well-defined. To that end, we notice that A being a condensation algebra in $\mathcal{B}$ means that A is trivially a module over itself, which, in particular, means that ${}_{\mathbb{1}_a}A_A$, ${}_A A_A = \mathbb{1}_{(a,A)}$, and ${}_A A_{\mathbb{1}_a}$ are 1-morphisms in $\mathbf{2Kar}\,\mathcal{B}$.

Furthermore, considering the calculation



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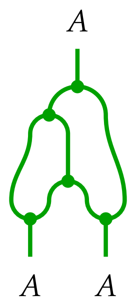
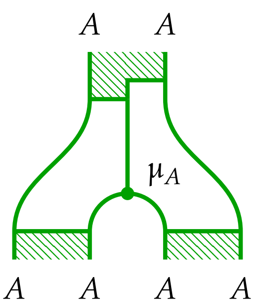


(6.15)

$\overset{(5.28)}{=}$



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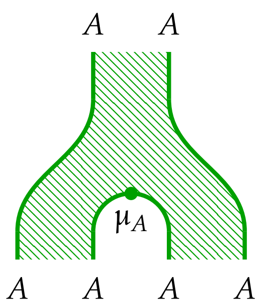
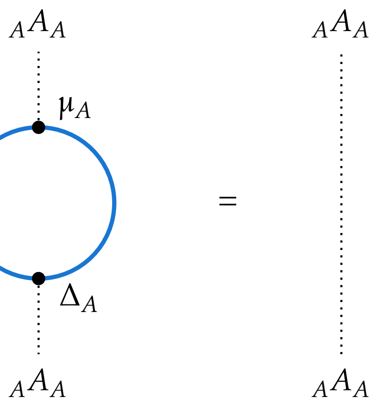


we see that the final diagram of the calculation may be expressed in $\mathbf{2Kar}\,\mathcal{B}$ as the string diagram on the r.h.s. of equation (6.14). $\square$

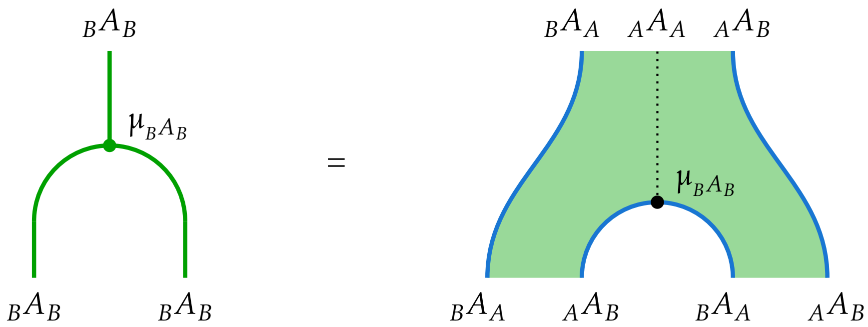
We may obtain an analogous decomposition of Δ_A in $\mathbf{2Kar}\,\mathcal{B}$. Then, by Δ-separability of A



(6.16)

completes the condensation of $(a, \mathbb{1}_a)$ onto (a, A). □

Proposition 6.2.18. *$\mathbf{2Kar}(\mathcal{B})$ has all condensates [Dé22, Prop. 1.2.3]. For instance, the multiplication of a 2-condensation algebra $({}_BA_B : (a, B) \longrightarrow (a, B), \mu_{{}_BA_B}, \Delta_{{}_BA_B})$ in $\mathbf{2Kar}\,\mathcal{B}$ decomposes as*

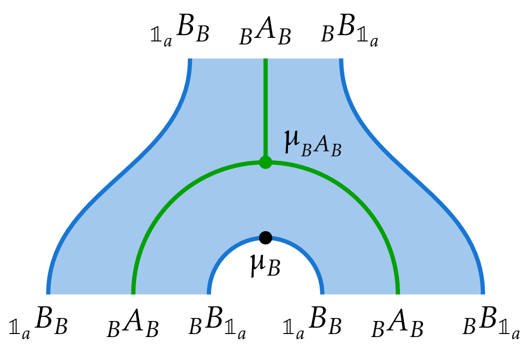


(6.17)

Proof.

"A is a condensation algebra in $\mathcal{B}$": Implicitly, there exists a condensation algebra (B, μ_B, Δ_B) in $\mathcal{B}$. Therefore, we may construct the 2-morphism

$\mu_A :=$

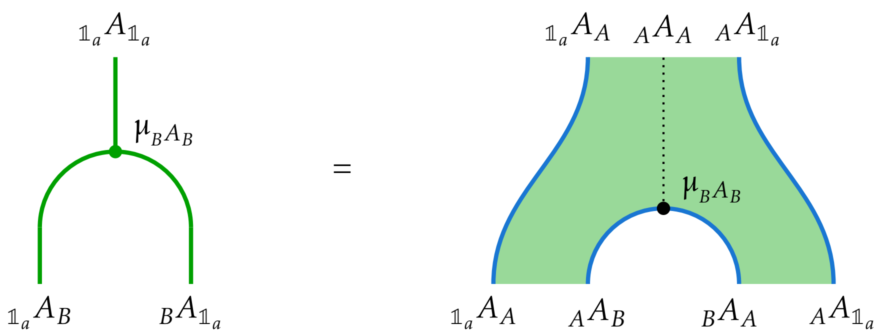


that makes $\iota_{\mathcal{B}} A_{\mathbb{1}_a} A_{\mathbb{1}_a}$ an algebra in **Kar** $\mathcal{B}$. Therefore, (A, μ_a) is an algebra in A. Analogously we may construct a comultiplication Δ_A. Furthermore, A is Frobenius because ${}_B A_B$ is Frobenius. A is Δ-separable because B condenses and ${}_B A_B$ is Δ-separable.

Δ-separability of A follows from the condensation of B (cf. Eq. (6.16)) and Δ-separability of ${}_B A_B$. Since this condensation algebra ${}_{\mathbb{1}_a} B_B \otimes {}_B A_B \otimes {}_B B_{\mathbb{1}_a} = {}_{\mathbb{1}_a} A_{\mathbb{1}_a}$ lies in the image of $\iota_{\mathcal{B}}$, its preimage A must also be a condensation algebra in $\mathcal{B}$.

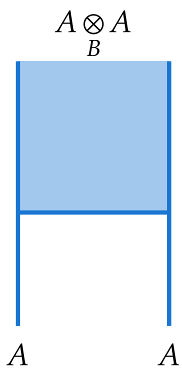
"The r.h.s. of (6.17) **is well-defined":** We have just seen that $(a, A) \in$ **2Kar** $\mathcal{B}$. Moreover, we notice that the 1-morphisms ${}_A A_A = \mathbb{1}_{(a,A)}, {}_A A_B, {}_B A_A$ exist in **2Kar** $\mathcal{B}$. Furthermore, $\mu_{{}_B A_B} : A \otimes A \Longrightarrow A$ is a B-B-condensation bimodule map in $\mathcal{B}$. Therefore, $\mu_{{}_B A_B}$ must also be a A-A-condensation bimodule map in $\mathcal{B}$.

"Equation (6.17) **is true":** We notice that (6.17) holds if



(6.18)

holds. However, (6.18) is just (6.14) precomposed with



.

"2Kar($\mathcal{B}$) has all condensates": We may show that the comultiplication $\Delta_{{}_B A_B}$ decomposes analogously. The split Δ-separability condition follows similarly. □

Notation 6.2.19. For 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ and $\mathcal{D} \in \mathbf{2Cat}$, 2-functors F, 2-natural transformations α, and modifications m in $\mathbf{2Cat}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{D})$ we define

$$\begin{aligned} F_{\mathcal{B}} &:= F \circ \iota_{\mathcal{B}} \\ \alpha_{\mathcal{B}} &:= \alpha \otimes \iota_{\mathcal{B}} \\ m_{\mathcal{B}} &:= m \Box \iota_{\mathcal{B}}. \end{aligned}$$

We may express them graphically via the following diagrams:

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "F_{\mathcal{B}}"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & & \mathcal{D} \\ \mathbf{2Kar}\,\mathcal{B} \arrow[urr, "F"'] & & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[dd, bend right=50, "F_{\mathcal{B}}"'] \arrow[dd, bend left=50, "G_{\mathcal{B}}"] \arrow[dd, phantom, "\overset{\alpha_{\mathcal{B}}}{\Longrightarrow}"] \\ \\ \mathcal{D} \end{tikzcd} \quad := \quad \begin{tikzcd} \mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] \\ \mathbf{2Kar}\,\mathcal{B} \arrow[d, bend right=50, "F"'] \arrow[d, bend left=50, "G"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ D \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{B} \arrow[dd, bend right=60, "F_{\mathcal{B}}"'] \arrow[dd, bend left=60, "G_{\mathcal{B}}"] \arrow[dd, phantom, "\overset{\alpha_{\mathcal{B}}}{\Longrightarrow}\ \Downarrow m_{\mathcal{B}}\ \underset{\beta_{\mathcal{B}}}{\Longrightarrow}"] \\ \\ \mathcal{D} \end{tikzcd} \quad := \quad \begin{tikzcd} \mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] \\ \mathbf{2Kar}\,\mathcal{B} \arrow[d, bend right=60, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}\ \Downarrow m\ \underset{\beta}{\Longrightarrow}"] \\ D \end{tikzcd}$$

Since $\iota_{\mathcal{B}}$ embeds $\mathcal{B}$ into $\mathbf{2Kar}\,\mathcal{B}$, the 2-functor

$$\begin{aligned} (-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{D}) &\longrightarrow \mathbf{2Cat}(\mathcal{B}, \mathcal{D}) \\ F &\longmapsto F_{\mathcal{B}} \\ \alpha &\longmapsto \alpha_{\mathcal{B}} \\ m &\longmapsto m_{\mathcal{B}} \end{aligned}$$

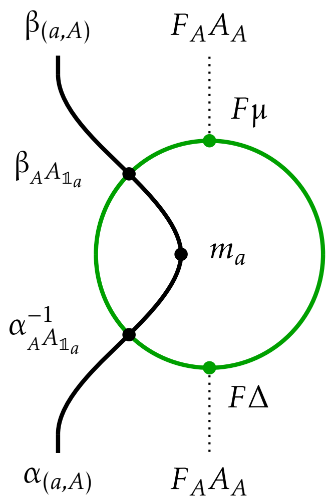
acts as a restriction of $\mathbf{2Kar}\,\mathcal{B}$ onto $\mathcal{B}$, i.e.

- $F_{\mathcal{B}}\, a = F(a, \mathbb{1}_a)$ for all $a \in B$,
- $(\alpha_{\mathcal{B}})_a = \alpha_{(a,\mathbb{1}_a)}$ and $(\alpha_{\mathcal{B}})_X = \alpha_{\mathbb{1}_b X \mathbb{1}_a}$ for all a, b and $X : a \longrightarrow b$ in $\mathcal{B}$, and
- $(m_{\mathcal{B}})_a = m_{(a,\mathbb{1}_a)}$ for all $a \in \mathcal{B}$.

Lemma 6.2.20. *Given 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}, \mathcal{D} \in \mathbf{2Cat}$, for every modification*

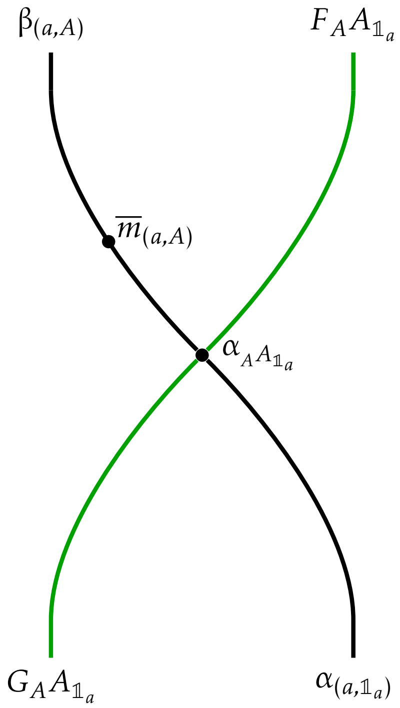
$$\begin{tikzcd}
\mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] & & \mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] \\
\mathbf{2Kar}\,\mathcal{B} \arrow[d, bend right=60, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \overset{m}{\Rrightarrow} & \mathbf{2Kar}\,\mathcal{B} \arrow[d, bend right=60, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, "\overset{\beta}{\Longrightarrow}"] \\
\mathcal{D} & & \mathcal{D}
\end{tikzcd}$$

there exists a unique modification $\overline{m} : \alpha \Rrightarrow \beta$ such that $\overline{m}_{\mathcal{B}} = m$. Its components are given by

$\overline{m}_{(a,A)} =$ 

for all condensation algebras $(a, A) \in \mathbf{2Kar}\,\mathcal{B}$.

Proof. Let us first show the following identity:



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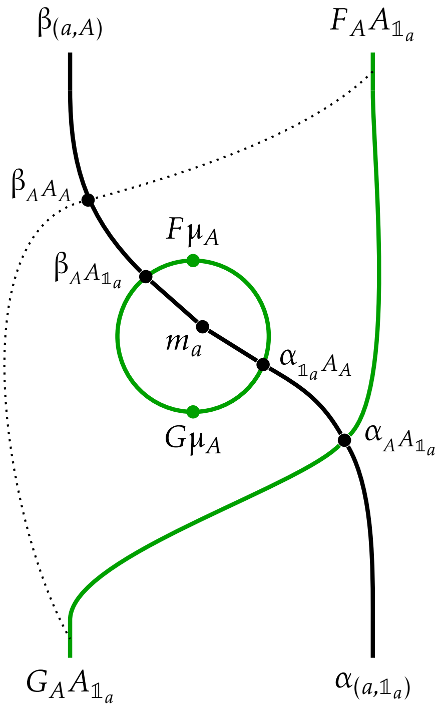
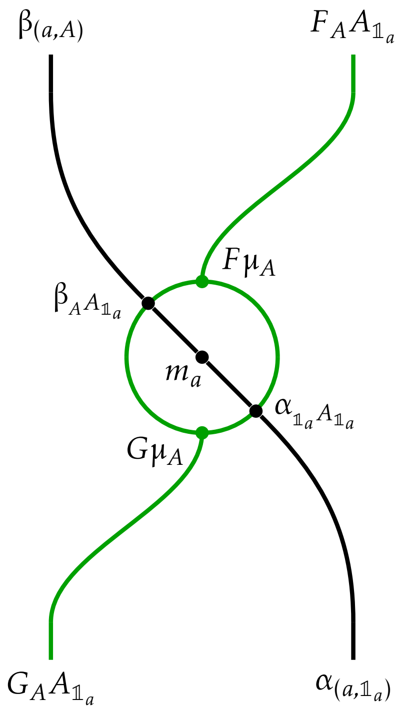


$\overset{(6.15)}{=}$



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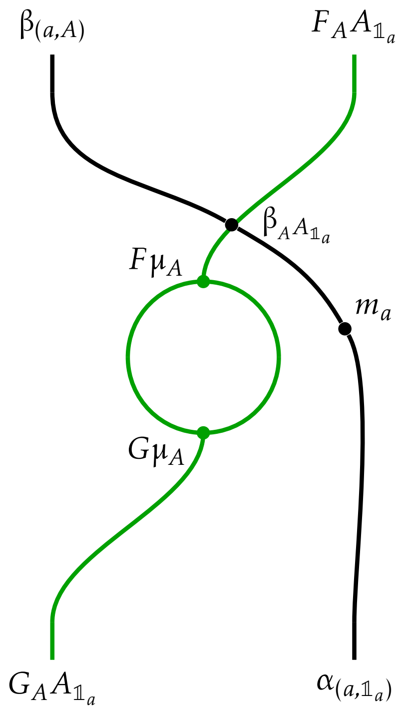
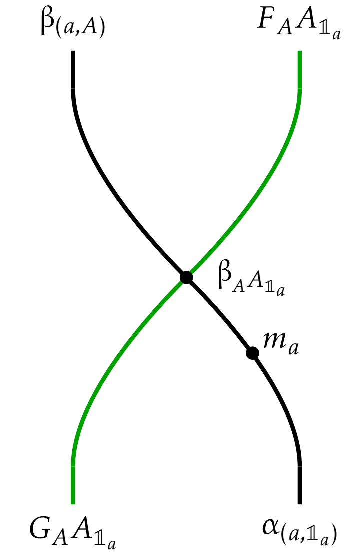
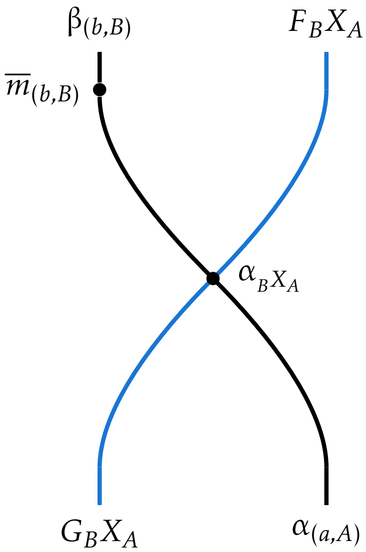


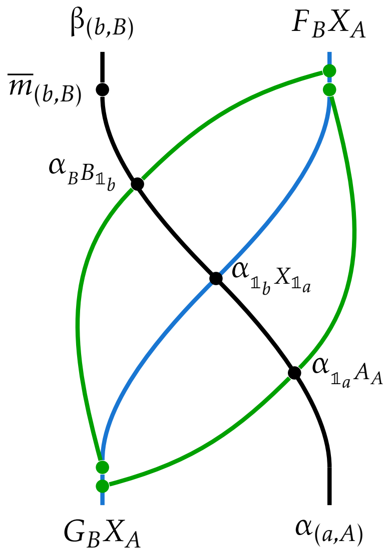
$$= \qquad (6.19)$$



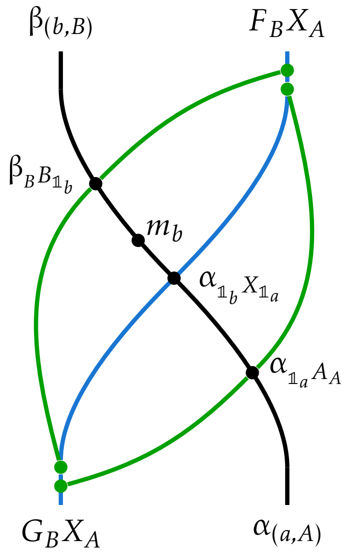
We may use this identity to see that $\overline{m}$ is indeed a modification:



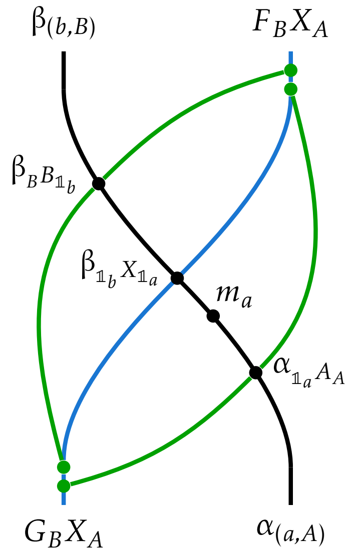
$=$

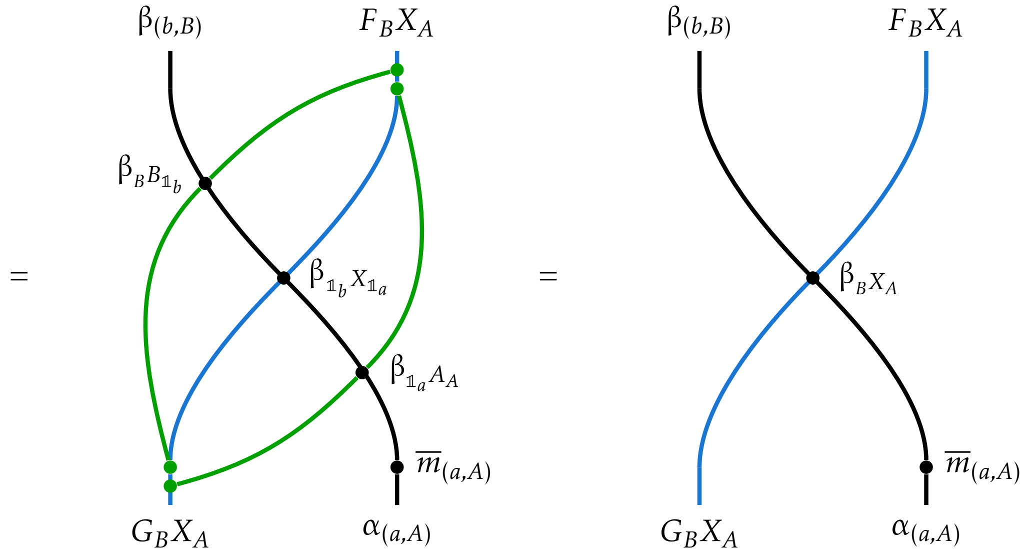


$\overset{(6.19)}{=}$

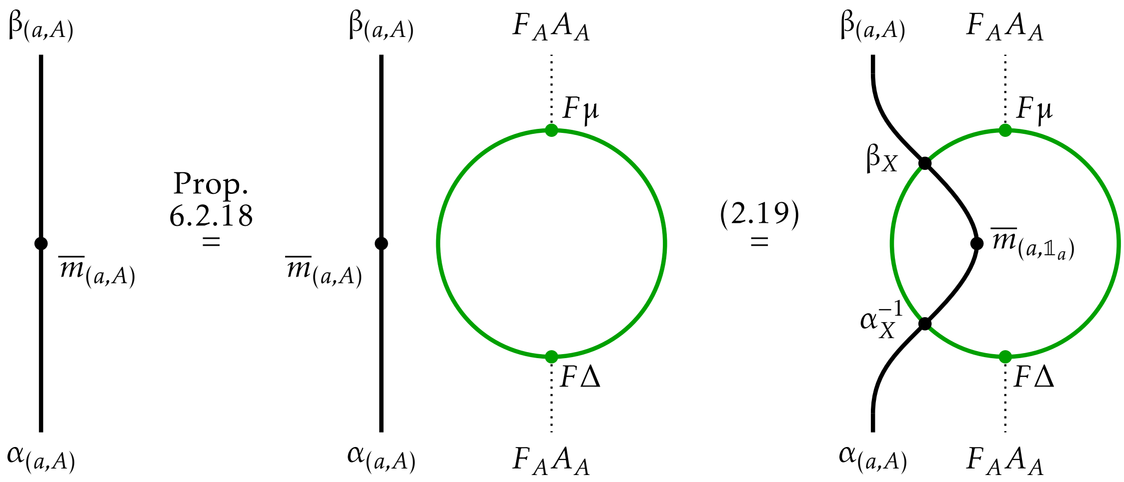


$=$



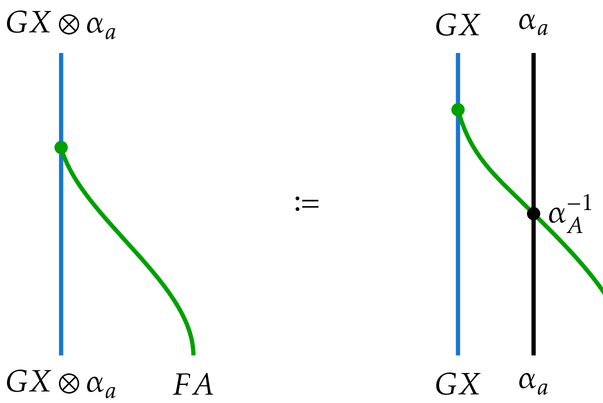


$\overline{m}$ is unique since its components are fully determined by $m = \overline{m}_{\mathcal{B}}$:



□

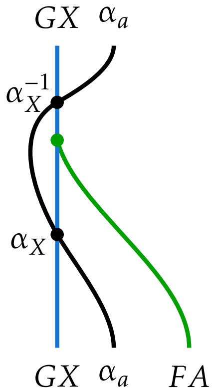
Lemma 6.2.21. *If we are given a 2-natural transformation $\alpha : F \longrightarrow G$ and a right A-module $X : a \longrightarrow b$ in $\mathcal{B}$, then $GX \otimes \alpha_a$ is a right FA-module with action*



. (6.20)

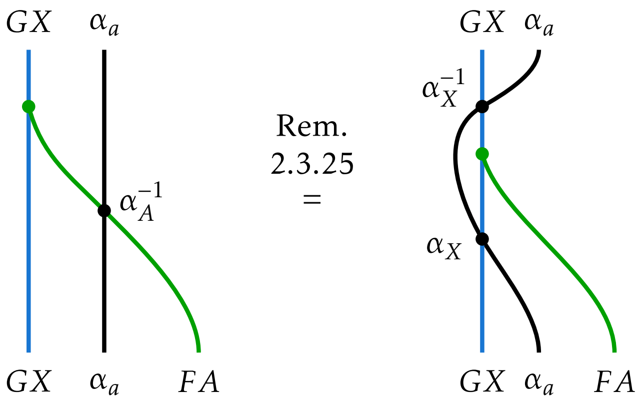
Moreover, $\alpha_X : GX \otimes \alpha_a \Longrightarrow \alpha_a \otimes FX$ is a right A-module map.

Proof. $\alpha_b \otimes FX$ is trivially a right FA-module and since α_X is an isomorphism, α_X must be a module map. Therefore, $GX \otimes \alpha_a$ is right FA-module with action



.

This action coincides with the action (6.20):



□

Proposition 6.2.22. *There is a (strict) fully faithful*[92] *2-functor*

$$\begin{aligned} \mathbf{2Kar}_{\mathcal{B},\mathcal{D}} : \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B},\mathcal{D}) &\longrightarrow \mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B},\mathbf{2Kar}\,\mathcal{D}) \\ F &\longmapsto \mathbf{2Kar}\,F \\ \alpha &\longmapsto \mathbf{2Kar}\,\alpha \\ m &\longmapsto \mathbf{2Kar}\,m \end{aligned}$$

[92] We note that $\mathbf{2Kar}_{\mathcal{B},\mathcal{D}}$ is not forgetful by our definition since it is not quite canonical. $\mathbf{2Kar}_{\mathcal{B},\mathcal{D}}$ is not canonical because $\mathbf{2Kar}\,\alpha$ is not defined canonically. If one wanted, one could remedy this by defining the 1-morphism components of $\mathbf{2Kar}\,\alpha$ via the idempotent splitting (6.22) instead.

for all 2-categories[93] $\mathcal{B}, \mathcal{D} \in \mathbf{2Cat^{ic}}$. *Its image is defined as follows:*

- $\mathbf{2Kar}\, F : \mathbf{2Kar}\, \mathcal{B} \longrightarrow \mathbf{2Kar}\, \mathcal{D}$ *maps (Definition 6.2.8) as*

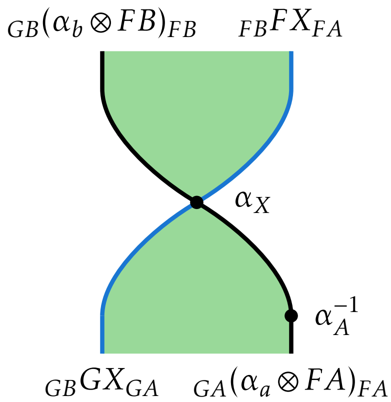
$$\begin{aligned} \mathbf{2Kar}\, \mathcal{B} &\longrightarrow \mathbf{2Kar}\, \mathcal{D} \\ (a, A) &\longmapsto (Fa, FA) \\ {}_B X_A &\longmapsto {}_{FB} F X_{FA} \\ f &\longmapsto Ff. \end{aligned}$$

- $\mathbf{2Kar}\, \alpha : \mathbf{2Kar}\, F \Longrightarrow \mathbf{2Kar}\, G$ *has 1-morphism components*

$$(\mathbf{2Kar}\, \alpha)_{(a,A)} \coloneqq {}_{GA}(\alpha_a \otimes FA)_{FA}$$

and 2-morphism components

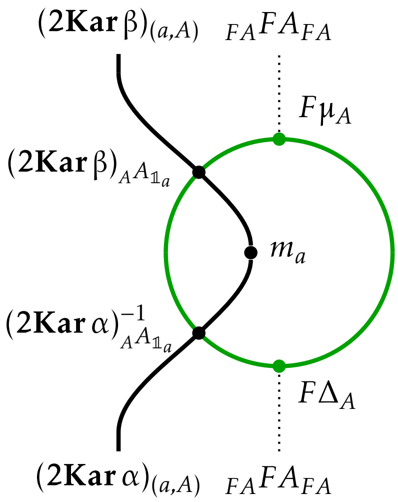
$$(\mathbf{2Kar}\, \alpha)_{{}_B X_A} \coloneqq$$



.

- $\mathbf{2Kar}\, m : \mathbf{2Kar}\, \alpha \Rrightarrow \mathbf{2Kar}\, \beta$ *has components*

$$(\mathbf{2Kar}\, m)_{(a,A)} \coloneqq$$



.

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[93] Naturally, this suggests the existence of a 3-functor $\mathbf{2Kar} : \mathbf{2Cat^{ic}} \longrightarrow \mathbf{2Cat^{cc}}$.

Proof.

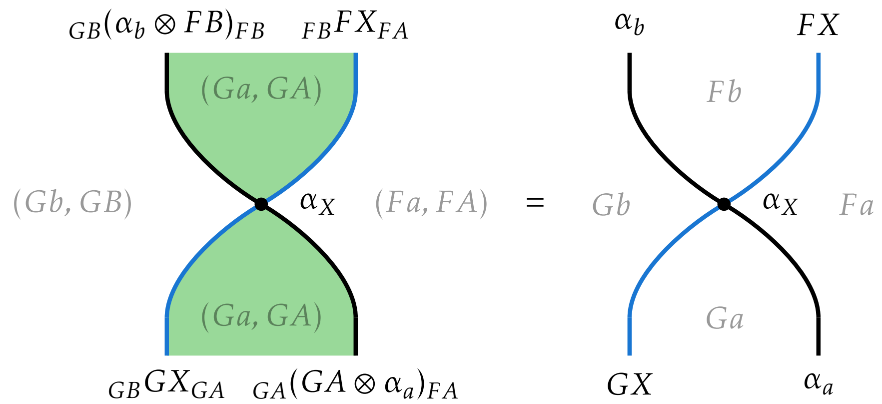
2Kar F is a 2-functor: First, to see that **2Kar** F is well-defined as a map, we notice that F maps condensation algebras, bimodules, and bimodule maps in $\mathcal{B}$ to condensation algebras, bimodules, and bimodule maps, respectively, in $\mathcal{D}$ (Proposition 5.2.38). These are precisely the objects, 1-morphisms, and 2-morphisms of **2Kar** $\mathcal{B}$ and **2Kar** $\mathcal{D}$, respectively.

2Kar α is a 2-natural transformation: The 1-morphism components of **2Kar** α are well-defined, i.e.

$$_{GA}(\alpha_a \otimes FA)_{FA} \in (\mathbf{2Kar}\,\mathcal{B})\big((Fa, FA), (Ga, GA)\big)$$

due to Lemma 6.2.21.

To see that the 2-morphism components are well-defined, we shall inspect the components of $(\mathbf{2Kar}\,\alpha)_{{}_BX_A}$. For one,



(6.21)

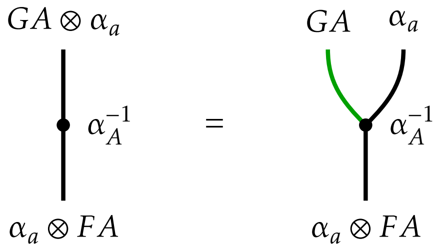
follows from

$$\begin{aligned} {}_{GB}GX_{GA} \otimes {}_{GA}(GA \otimes \alpha_a)_{FA} &= {}_{G\mathbb{1}_b}GX_{GA} \otimes {}_{GA}(GA \otimes \alpha_a)_{F\mathbb{1}_a} \\ &= {}_{G\mathbb{1}_b}GX_{GA} \otimes {}_{GA}GA_{G\mathbb{1}_a} \otimes {}_{G\mathbb{1}_a}(\alpha_a)_{F\mathbb{1}_a} \\ &= {}_{GB}GX_{G\mathbb{1}_a} \otimes {}_{G\mathbb{1}_a}(\alpha_a)_{FA}. \end{aligned}$$

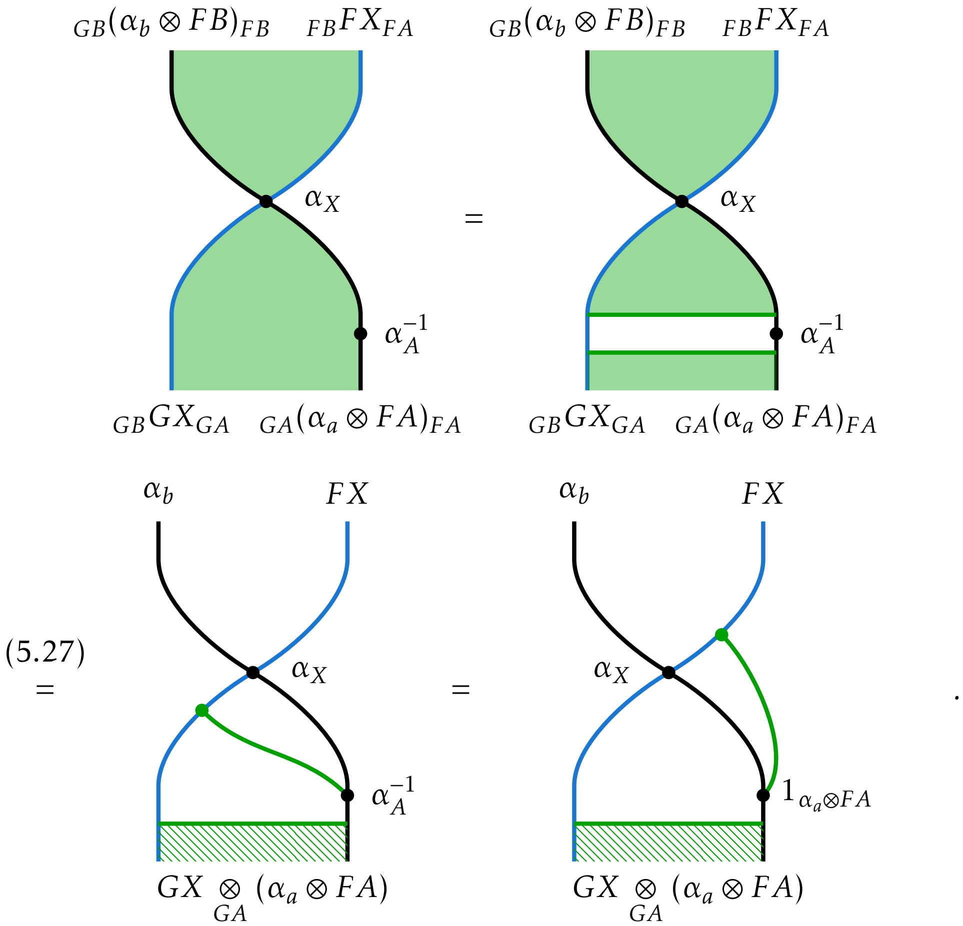
Furthermore, examining source and target of

$$\begin{aligned} 1_{GX} \otimes \alpha_A^{-1} : {}_{GB}GX_{GA} \otimes {}_{GA}(\alpha_a \otimes FA)_{FA} \Longrightarrow {}_{GB}GX_{GA} &\otimes {}_{GA}(GA \otimes \alpha_a)_{FA} \\ &= {}_{GB}(GX \otimes \alpha_a)_{FA} \end{aligned}$$

and using the trivial identity



we see that $(\mathbf{2Kar}\,\alpha)_{{}_BX_A}$ is a well-defined GB-FA-bimodule map in $\mathcal{B}$:

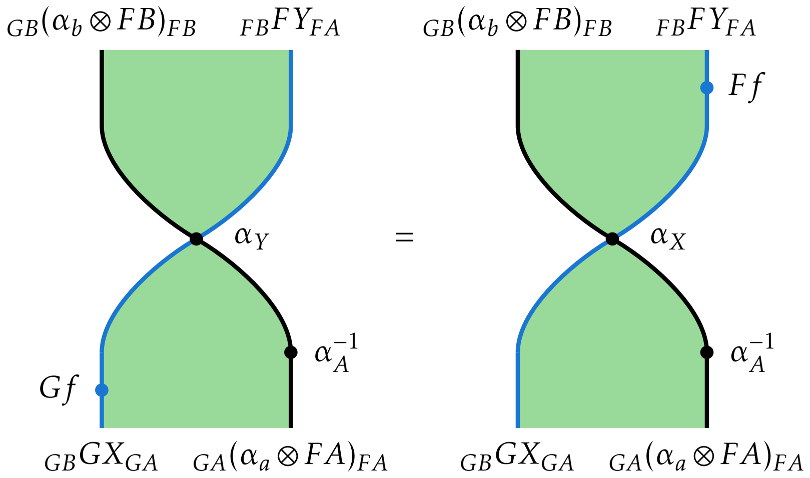


Thus,

$$(\mathbf{2Kar}\,\alpha)_{{}_BX_A} : GX \underset{GA}{\otimes} (\mathbf{2Kar}\,\alpha)_{(a,A)} \Longrightarrow (\mathbf{2Kar}\,\alpha)_{(b,B)} \underset{FB}{\otimes} FX$$

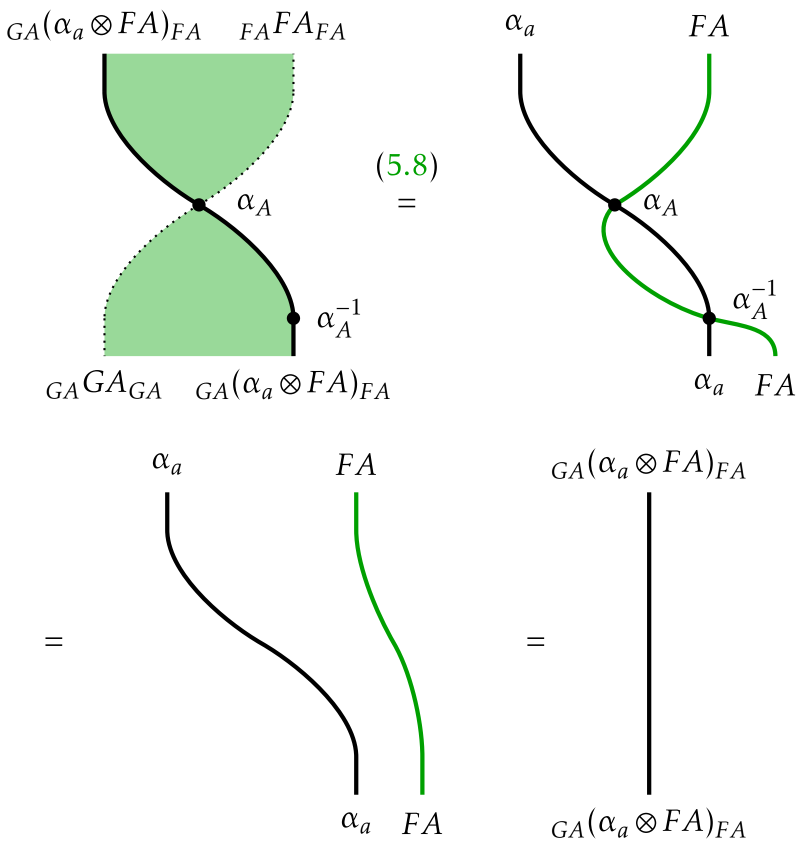
exists in $\mathbf{2Kar}\,\mathcal{B}$ for all ${}_BX_A$.

$\left((\mathbf{2Kar}\,\alpha)_{{}_BX_A}\right)_{{}_BX_A \in (\mathbf{2Kar}\,\mathcal{B})((a,A),(b,B))}$ is a natural transformation because applying (6.21), together with $\alpha_{(a,b)}$ being a natural transformation in $\mathcal{B}$, yields

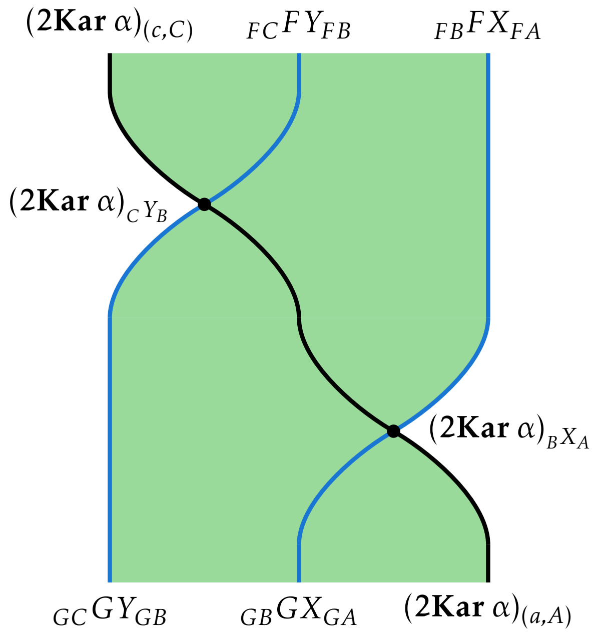


for all B-A-condensation bimodule maps $f : X \longrightarrow Y$. Moreover, the 2-morphism components $(\mathbf{2Kar}\,\alpha)_{{}_B X_A}$ are clearly isomorphisms.

2Kar α fulfils the unity condition:



Furthermore, **2Kar** α fulfils the 2-naturality condition:



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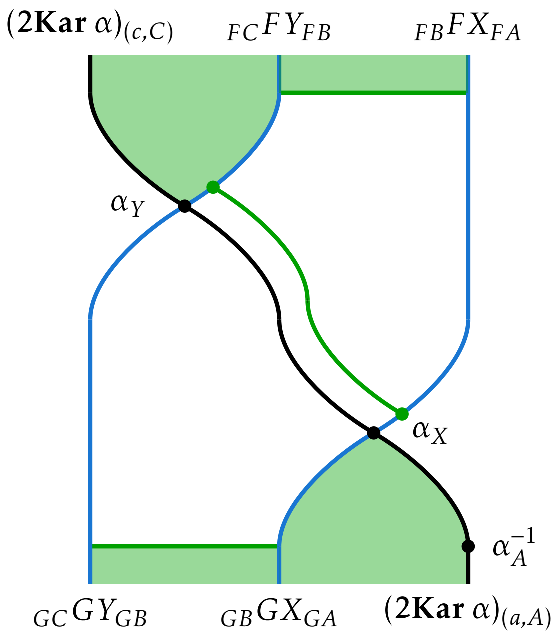

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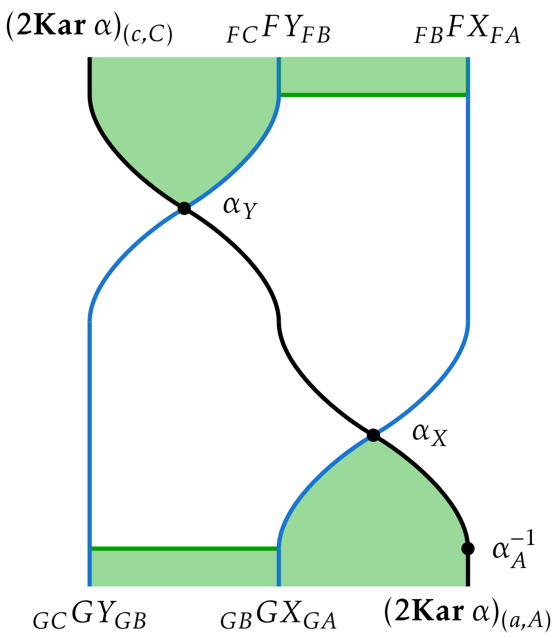

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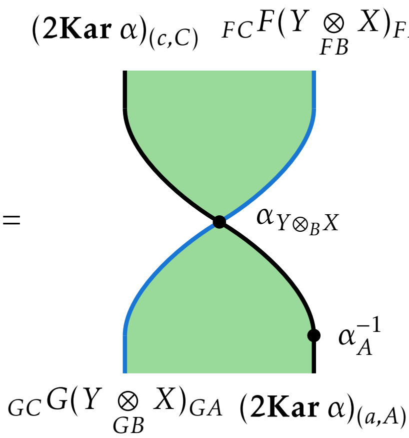


2Kar m is a modification: Analogously to Lemma 6.2.24 we may notice that

$$
\begin{tikzcd}
\mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] \\
\mathbf{2Kar}\,\mathcal{B} \arrow[d, bend right=60, "\mathbf{2Kar}\,F"'] \arrow[d, bend left=60, "\mathbf{2Kar}\,G"] \arrow[d, phantom, "\overset{\mathbf{2Kar}\,\alpha}{\Longrightarrow}"] \\
\mathbf{2Kar}\,\mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{B} \arrow[d, bend right=60, "F"'] \arrow[d, bend left=60, "G"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] \\
\mathcal{D} \arrow[d, "\iota_{\mathcal{D}}"] \\
\mathbf{2Kar}\,\mathcal{D}
\end{tikzcd}
$$

Using this we may apply Lemma 6.2.20 to the modification

$$
\begin{tikzcd}
 & \mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] & \\
 & \mathbf{2Kar}\,\mathcal{B} \arrow[dd, bend right=60, "\mathbf{2Kar}\,F"'] \arrow[dd, bend left=60, "\mathbf{2Kar}\,G"] & \\
 & \arrow[phantom, "\xRightarrow{\mathbf{2Kar}\,\alpha}"] & \\
 & \mathbf{2Kar}\,\mathcal{D} &
\end{tikzcd}
\overset{m_{\mathcal{D}}}{\Rrightarrow}
\begin{tikzcd}
 & \mathcal{B} \arrow[d, "\iota_{\mathcal{B}}"] & \\
 & \mathbf{2Kar}\,\mathcal{B} \arrow[dd, bend right=60, "\mathbf{2Kar}\,F"'] \arrow[dd, bend left=60, "\mathbf{2Kar}\,G"] & \\
 & \arrow[phantom, "\xRightarrow{\mathbf{2Kar}\,\beta}"] & \\
 & \mathbf{2Kar}\,\mathcal{D} &
\end{tikzcd}
$$

$\mathbf{2Kar}_{\mathcal{B},\mathcal{D}}$ is a (strict) forgetful 2-functor: We do not need this property to prove Theorem 6.2.16 so we only sketch this property. Strictness, of course, only holds under the assumption that $\mathcal{B}$ and $\mathcal{D}$ are strict.

One may simply calculate that

$$\mathbf{2Kar}\,\mathbb{1}_F = \mathbb{1}_{\mathbf{2Kar}\,F}$$

and

$$\mathbf{2Kar}(\beta \otimes \alpha) = \mathbf{2Kar}\,\beta \otimes \mathbf{2Kar}\,\alpha$$

follow from strictness of units in $\mathbf{2Kar}\,\mathcal{B}$ and $\mathbf{2Kar}\,\mathcal{D}$. Given 2-functors $F, G : \mathcal{B} \longrightarrow \mathcal{D}$, the 2-functors $\mathbf{2Kar}\,F$, $\mathbf{2Kar}\,G$ exist and there is a correspondence between the 2-natural transformations and modifications.

We note that $\mathbf{2Kar}_{\mathcal{B},\mathcal{D}}$ is not a 2-equivalence because if $\mathcal{D}$ has a condensation algebra $\widetilde{\mathcal{A}} : \widetilde{a} \longrightarrow \widetilde{a}$ that does not condense and $\mathcal{B}$ contains a non-trivial condensation $(A : a \longrightarrow a)$ that does condense, then 2-functors $\mathcal{B} \longrightarrow \mathcal{D}$ cannot map $A \longmapsto \widetilde{\mathcal{A}}$ but 2-functors $\mathbf{2Kar}\,\mathcal{B} \longrightarrow \mathbf{2Kar}\,\mathcal{D}$ can map ${}_{\mathbb{1}_a}A_{\mathbb{1}_a} \longmapsto {}_{\mathbb{1}_{\widetilde{a}}}\widetilde{\mathcal{A}}_{\mathbb{1}_{\widetilde{a}}}$. □

Corollary 6.2.23. *Given 2-categories $\mathcal{B}, \mathcal{D} \in \mathbf{2Cat}^{\mathbf{ic}}$, and a 2-natural transformation*

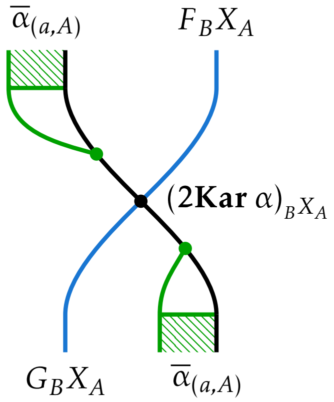
$$
\begin{tikzcd}
 & \mathcal{B} \arrow[dl, "\iota_{\mathcal{B}}"'] \arrow[dr, "\iota_{\mathcal{B}}"] & \\
\mathbf{2Kar}\,\mathcal{B} \arrow[rr, phantom, "\overset{\alpha}{\Longrightarrow}"] \arrow[dr, "F"'] & & \mathbf{2Kar}\,\mathcal{B} \arrow[dl, "G"] \\
 & \mathcal{D} &
\end{tikzcd}
,
$$

then we may define a 2-natural transformation $\overline{\alpha} : F \Longrightarrow G$ with 1-morphism components

$$\overline{\alpha}_{(a,A)} \coloneqq G_A A_{\mathbb{1}_a} \underset{G_{\mathbb{1}_a} A_{\mathbb{1}_a}}{\otimes} (\alpha_a \otimes F_{\mathbb{1}_a} A_A)$$

and 2-morphism components

$\overline{\alpha}_{{}_BX_A} :=$



.

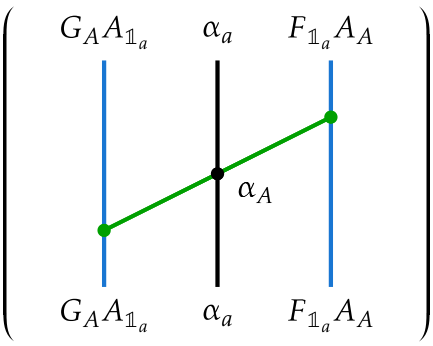
We find that

$$\overline{\alpha}_{\mathcal{B}} = \alpha.$$

Proof. The 2-morphism components of $\overline{\alpha} : F \Longrightarrow G$ may seem sketchy at first glance since they are constructed from 2-morphisms in $\mathbf{2Kar}\,\mathcal{D}$. However, 2-morphisms in $\mathbf{2Kar}\,\mathcal{D}$ are 2-morphisms of $\mathcal{D}$ so this construction is fine.

Moreover, although $\overline{\alpha}$ is presented differently than $\mathbf{2Kar}\,\alpha$ they are essentially the same. To see this notice that

$\overline{\alpha}_{(a,A)} =$ **split** (6.22)



and also that $(\mathbf{2Kar}\,\alpha)_{(a,A)}$ may be represented as in (6.22). Therefore, analogous to Proposition 6.2.22 $\overline{\alpha}$ is a 2-natural transformation $F \Longrightarrow G$.

$\overline{\alpha}_{\mathcal{B}} = \alpha$ follows from the calculations

$$\begin{aligned}
(\overline{\alpha}_{\mathcal{B}})_a &= \overline{\alpha}_{1_{\mathcal{B}}a} \\
&= \overline{\alpha}_{(a,\mathbb{1}_a)} \\
&= G_{\mathbb{1}_a}\mathbb{1}_{a\mathbb{1}_a} \underset{G_{\mathbb{1}_a}\mathbb{1}_{a\mathbb{1}_a}}{\otimes} (\alpha_a \otimes F_{\mathbb{1}_a}\mathbb{1}_{a\mathbb{1}_a}) \\
&= \alpha_a
\end{aligned}$$

and

$$\begin{aligned}(\overline{\alpha}_{\mathcal{B}})_X &= \overline{\alpha}_{\iota_{\mathcal{B}} X} \\ &= \overline{\alpha}_{\mathbb{1}_b X_{\mathbb{1}_a}} \\ &= (\mathbf{2Kar}\,\alpha)_{\mathbb{1}_b X_{\mathbb{1}_a}} \\ &= \alpha_X. \end{aligned}$$ □

Lemma 6.2.24. *The diagram*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & \mathcal{D} \arrow[d, hook, "\iota_{\mathcal{D}}"] \\ \mathbf{2Kar}\,\mathcal{B} \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{2Kar}\,\mathcal{D} \end{tikzcd}$$

commutes[94] *for all 2-categories $\mathcal{B}, \mathcal{D} \in \mathbf{2Cat}^{\mathbf{ic}}$ and 2-functors $F : \mathcal{B} \longrightarrow \mathcal{D}$.*

Proof. Simply inspect where components of $\mathcal{B}$ are mapped by strict 2-functors $F : \mathcal{B} \longrightarrow \mathcal{D}$:

$$\begin{aligned} \mathcal{B} &\longrightarrow \mathbf{2Kar}\,\mathcal{D} \\ a &\longmapsto (Fa, F\mathbb{1}_a) = (Fa, \mathbb{1}_{Fa}) \\ \mathcal{B}(a,b) &\longrightarrow {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}(\mathcal{B}) \end{aligned}$$ □

Proposition 6.2.25. $\iota_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbf{2Kar}\,\mathcal{A}$ *forms an (adjoint) 2-equivalence together with the 2-functor*

$$\begin{aligned} \pi_{\mathcal{A}} : \mathbf{2Kar}\,\mathcal{A} &\longrightarrow \mathcal{A} \\ (a, A) &\longmapsto a_A \\ {}_B\mathbf{Mod}^{\mathrm{sp}}_A(\mathcal{A}) &\longrightarrow \Pi_B \underset{B}{\otimes} \mathcal{A}(a,b) \underset{A}{\otimes} \mathrm{I}_A \end{aligned}$$

for all $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$. Hereby, $(a_A, \mathrm{I}_A, \Pi_A, \iota_A, \pi_A)$ denotes condensation data of condensation algebras $A : a \longrightarrow a$ in $\mathcal{A}$. The 2-morphism components of $\pi_{\mathcal{A}}$ are given by the 2-isomorphisms $\mathrm{I}_A \otimes \Pi_A \Longrightarrow A$. In particular, $\iota_{\mathcal{A}}$ and $\pi_{\mathcal{A}}$ form a split idempotent:

94 This implies that there is a strict 3-natural transformation $\iota : \mathrm{id}_{\mathbf{2Cat}^{\mathbf{ic}}} \Longrightarrow U \circ \mathbf{2Kar}$ where U is the forgetful 3-functor $\mathbf{2Cat}^{\mathbf{cc}} \hookrightarrow \mathbf{2Cat}^{\mathbf{ic}}$.

$$\begin{tikzcd}
\mathbf{2Kar}\,\mathcal{A} \arrow[loop above, "\iota_{\mathcal{A}} \circ \pi_{\mathcal{A}}"] \arrow[d, bend left, "\pi_{\mathcal{A}}"] \\
\mathcal{A} \arrow[u, bend left, "\iota_{\mathcal{A}}"] \arrow[loop below, "\mathrm{id}_{\mathcal{A}}"]
\end{tikzcd}$$

We fix the notation $\pi_{\mathcal{A}}$[95] for this 2-functor for the rest of this Section.

Proof. Let us start by showing that $\iota_{\mathcal{A}}$ is a 2-equivalence before showing that $\iota_{\mathcal{A}}$ and $\pi_{\mathcal{A}}$ form a split idempotent. From this we shall conclude that $\iota_{\mathcal{A}}$ forms the 2-equivalence together with $\pi_{\mathcal{A}}$.

We have already seen that $\iota_{\mathcal{A}}$ is fully faithful so let us show that it is essentially surjective for $\mathcal{A} \in \mathbf{2Cat^{cc}}$. To do so, assume we are given an arbitrary object $(a, A) \in \mathbf{2Kar}\,\mathcal{A}$. We know that $A : a \longrightarrow a$ condenses onto some $(a_A, \Pi_A, \mathrm{I}_A, \pi_A, \iota_A)$ in $\mathcal{A}$ since $\mathcal{A} \in \mathbf{2Cat^{cc}}$. Thus, $(a, \mathbb{1}_a)$ condenses both onto (a, A) and onto $\iota_{\mathcal{A}}\, a_A = (a_A, \mathbb{1}_{a_A})$. Therefore,

$$(a, A) \simeq (a_A, \mathbb{1}_{a_A})$$

by Lemma 5.2.62. Thus, $\iota_{\mathcal{A}}$ is a 2-equivalence and its weak inverse must be given by $\pi_{\mathcal{A}}$ (proof of Theorem 2.3.44).

The idempotent splitting follows from inspecting where objects and Hom categories of $\mathcal{A}$ are mapped. Choosing trivial condensation data $(a, \mathbb{1}_a, \mathbb{1}_a, \rho_{\mathbb{1}_a}, \rho_{\mathbb{1}_a}^{-1})$ for trivial condensation algebras $\mathbb{1}_a$ (Assumption 6.2.14) we see that $\pi_{\mathcal{A}} \circ \iota_{\mathcal{A}}$ maps as:

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{\quad \iota_{\mathcal{A}} \quad} & \mathbf{2Kar}\,\mathcal{A} & \xrightarrow{\quad \pi_{\mathcal{A}} \quad} & \mathcal{A} \\
a & \longmapsto & (a, \mathbb{1}_a) & \longmapsto & a \\
\mathcal{A}(a,b) & \longrightarrow & {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}(\mathcal{A}) & \longrightarrow & \mathbb{1}_b \underset{\mathbb{1}_b}{\otimes} \mathcal{A}(a,b) \underset{\mathbb{1}_a}{\otimes} \mathbb{1}_a
\end{array}$$

The convenient choices

$$\mathbb{1}_b \underset{\mathbb{1}_b}{\otimes} X \underset{\mathbb{1}_a}{\otimes} \mathbb{1}_a = X$$

(Assumption 6.2.14) then imply $\pi_{\mathcal{A}} \circ \iota_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$. □

[95] Moreover, we take care not to confuse the 2-functor $\pi_{\mathcal{A}}$ and the 2-morphism π_A.

Corollary 6.2.26. *The diagram*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & \mathcal{A} \\ \mathbf{2Kar}\,\mathcal{B} \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{2Kar}\,\mathcal{A} \arrow[u, two heads, "\pi_{\mathcal{A}}"'] \end{tikzcd}$$

commutes for all 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$, $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$ and 2-functors[96] *$F : \mathcal{B} \longrightarrow \mathcal{A}$. In particular,*

$$\begin{aligned} \overline{F} : \mathbf{2Kar}\,\mathcal{B} &\longrightarrow \mathcal{A} \\ (a, A) &\longmapsto (Fa)_{FA} \\ {}_B\mathbf{Mod}^{\mathrm{sp}}_A(\mathcal{B}) &\longrightarrow \Pi_{FB} \underset{FB}{\otimes} \mathcal{A}(Fa, Fb) \underset{FA}{\otimes} \mathrm{I}_{FA}, \end{aligned}$$

where $(Fa)_{FA}, \Pi_{FB}, \mathrm{I}_{FA}$ belong to the condensation data of FA and FB in $\mathcal{A}$, has the property

$$\overline{F}_{\mathcal{B}} = F.$$

Remark 6.2.27. The 2-natural equivalence

$$\iota_{\mathcal{A}} \circ \pi_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathbf{2Kar}\,\mathcal{A}}$$

implies the 2-natural equivalence

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, Rightarrow] & \mathcal{D} \\ \mathbf{2Kar}\,\mathcal{A} \arrow[u, two heads, "\pi_{\mathcal{A}}"] \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{2Kar}\,\mathcal{D} \arrow[u, two heads, "\pi_{\mathcal{D}}"'] \end{tikzcd}$$

for all 2-functors $F : \mathcal{A} \longrightarrow \mathcal{D}$ in $\mathbf{2Cat}^{\mathbf{cc}}$. This 2-natural equivalence implies the existence of a 3-natural equivalence $\pi : \mathbf{2Kar} \circ U \Longrightarrow \mathrm{id}_{\mathbf{2Cat}^{\mathbf{cc}}}$.

Corollary 6.2.28 (Theorem 6.2.16)**.** $(\mathbf{2Kar}\,\mathcal{B}, \iota_{\mathcal{B}})$ *is a condensation completion of $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$.*

96 One could reformulate this Corollary slightly more generally by only requiring $\mathcal{A} \in \mathbf{2Cat}$ and $\mathbf{im}\,F \in \mathbf{2Cat}^{\mathbf{cc}}$.

Proof. $\mathbf{2Kar}\,\mathcal{B}$ has all condensates (Proposition 6.2.18). Moreover, (by Lemma 4.4.4)

$$(-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B}, \mathcal{A}) \longrightarrow \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{A})$$

is

- essentially surjective because $\overline{F}_{\mathcal{B}} = F$ for every 2-functor $F : \mathcal{B} \longrightarrow \mathcal{A}$ (Corollary 6.2.26),
- essentially full because $\overline{\alpha}_{\mathcal{B}} = \alpha$ for every 2-natural transformation $\alpha : F_{\mathcal{B}} \Longrightarrow G_{\mathcal{B}}$ (Corollary 6.2.23), and
- faithful because $\overline{m}_{\mathcal{B}} = m$ uniquely for all modifications $\overline{m} : \alpha_{\mathcal{B}} \Rrightarrow \beta_{\mathcal{B}}$ (Lemma 6.2.20). □

6.2.2. 2-Orbifold Completions

Assumption 6.2.29. We may assume our 2-categories to be strict also in this section (Assumption 6.2.14).

Remark 6.2.30. In Section 6.1 we saw that Karoubi envelopes $\mathbf{Kar}\,\mathcal{C}$ universally complete categories $\mathcal{C} \in \mathbf{Cat}$ w.r.t. 1-condensation, i.e. idempotent splitting. In Section 6.2.1 we saw that the 2-Karoubi envelope $\mathbf{2Kar}\,\mathcal{B}$ universally completes 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ w.r.t. 2-condensation (Theorem 6.2.16). Since we introduced further forms of split algebras (Definition 5.2.23), one may wonder if universal completions w.r.t. other algebras and their splittings also exist.

For instance, one may want to consider condensation algebras A with units and/or counits. Specifically, we may consider the full sub-2-categories of $\mathbf{2Kar}\,\mathcal{B}$ whose objects are condensation algebras with units and the full sub-2-category whose objects are unital condensation algebras and denote them as $\mathcal{B}^{\triangledown}$ and $\mathcal{B}_{\mathrm{eq}}$, respectively.[97] Units and counits split if they form adjunctions with the condensation 2-morphisms π_A or ι_A (Definition 5.2.23). It turns out that the units and counits of condensation algebras A also split if A condenses (Remark 5.2.26). Therefore, we may state as a corollary to Theorem 6.2.16 that $\mathcal{B}^{\triangledown}$ and $\mathcal{B}_{\mathrm{eq}}$ are each universal completions w.r.t. condensations with units and w.r.t. unital condensation, respectively.[98]

[97] $\mathcal{B}^{\triangledown}$ is constructed by [DR18, Con. 1.3.9] as a categorification of idempotent/Cauchy completions. Meanwhile, $\mathcal{B}_{\mathrm{eq}}$ is the "equivariant completion" introduced by [CR16, Def. 4.1] as a generalisation of *orbifold completions*.

[98] Interestingly, if $\mathcal{B}$ has left (or right) adjoints, then $\mathcal{B}^{\triangledown} \simeq \mathbf{2Kar}\,\mathcal{B}$ [GJF19, Thm. 3.3.3]. One may wonder if this statement extends to $\mathcal{B}_{\mathrm{eq}}$, but this does not seem to be true (Section A.2.4).

Therefore, we see that there are indeed universal completions w.r.t. other algebras and their splittings. One may wonder how we may extend this framework even further. To do so, let us shortly examine how the completions $\mathcal{B}^\triangledown$ and $\mathcal{B}_{\text{eq}}$ follow from $\mathbf{2Kar}\,\mathcal{B}$ conceptually. The condensation completion should be a 3-functor $\mathbf{2Kar} : \mathbf{2Cat^{ic}} \longrightarrow \mathbf{2Cat^{cc}}$ that is left 3-adjoint to the forgetful 3-functor $\mathbf{2Cat^{cc}} \hookrightarrow \mathbf{2Cat^{ic}}$ (Theorem 6.2.16). By considering full sub-3-categories

$$\mathbf{2Cat}^{\text{eq}} \subset \mathbf{2Cat}^{\triangledown} \subset \mathbf{2Cat^{cc}}$$

we receive forgetful 3-functors $\mathbf{2Cat}^{\text{eq}} \hookrightarrow \mathbf{2Cat^{ic}}$ and $\mathbf{2Cat}^{\triangledown} \hookrightarrow \mathbf{2Cat^{ic}}$. Moreover, $\mathbf{2Kar} : \mathbf{2Cat^{ic}} \longrightarrow \mathbf{2Cat^{cc}}$ should factor through 3-functors

$$\begin{aligned} (-)_{\text{eq}} : \mathbf{2Cat^{ic}} &\longrightarrow \mathbf{2Cat}^{\text{eq}} \\ \mathcal{B} &\longmapsto \mathcal{B}_{\text{eq}} \subseteq \mathbf{2Kar}\,\mathcal{B} \end{aligned}$$

and

$$\begin{aligned} (-)^{\triangledown} : \mathbf{2Cat^{ic}} &\longrightarrow \mathbf{2Cat}^{\triangledown} \\ \mathcal{B} &\longmapsto \mathcal{B}^{\triangledown} \subseteq \mathbf{2Kar}\,\mathcal{B}. \end{aligned}$$

These should be left 3-adjoint to the forgetful 3-functors. In other words, these universal completions exist because these algebras are special cases of condensation algebras but their splittings are not special cases. This makes sense because we used full-determinedness of condensations to construct condensation completions (Sketch 6.2.7) and this full-determinedness must automatically extend to these special cases. Therefore, if we were considering algebras that are more general than condensation algebras, then it should generally not be possible to construct universal completions because their splittings will generally not be fully determined. But what about other special cases of condensation algebras that do not fit into the above framework?

Orbifold data are special cases of condensation algebras, but their condensations are not implied by those of the underlying condensation algebra. Therefore, orbifold condensations are special cases of "regular" condensations. Moreover, orbifold data only exist in pivotal 2-categories, so we must consider the intersection

$$\mathbf{2Cat^{piv,ic}} := \mathbf{2Cat^{ic}} \cap \mathbf{2Cat^{piv}}$$

and the sub-3-category

$$\mathbf{2Cat}^{\text{orb}} \subseteq \mathbf{2Cat}^{\text{eq}} \cap \mathbf{2Cat^{piv}}$$

consisting of *orbifold complete* 2-categories. Clearly, there is a forgetful 3-functor

$$\mathbf{2Cat}^{\text{orb}} \hookrightarrow \mathbf{2Cat^{piv,ic}},$$

but the existence of a left adjoint $(-)_{\mathrm{orb}} : \mathbf{2Cat}^{\mathbf{piv,ic}} \longrightarrow \mathbf{2Cat}^{\mathrm{orb}}$ requires more additional work than $(-)_{\mathrm{eq}} : \mathbf{2Cat}^{\mathbf{ic}} \longrightarrow \mathbf{2Cat}^{\mathrm{eq}}$ and $(-)^{\triangledown} : \mathbf{2Cat}^{\mathbf{ic}} \longrightarrow \mathbf{2Cat}^{\triangledown}$ did. For one, we have to move everything to the pivotal setting:

$$\begin{tikzcd}
 & & \mathbf{2Cat} & \\
 & \mathbf{2Cat}^{\mathbf{ic}} \arrow[ur, hook] & & \mathbf{2Cat}^{\mathbf{piv}} \arrow[ul, hook'] \\
\mathbf{2Cat}^{\mathbf{cc}} \arrow[ur, hook] & & \mathbf{2Cat}^{\mathbf{piv,ic}} \arrow[ul, hook'] \arrow[ur, hook] & \\
 & \mathbf{2Cat}^{\mathrm{orb}} \arrow[ul, hook'] \arrow[ur, hook] & &
\end{tikzcd}$$

For another, we must check that orbifold data in $\mathbf{2Kar}\,\mathcal{B}$ condenses.

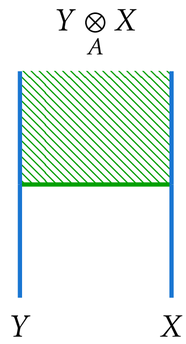
As in the previous section on condensation completions, we will not show the 3-adjunction $(-)_{\mathrm{orb}} : \mathbf{2Cat}^{\mathbf{piv,ic}} \leftrightarrows \mathbf{2Cat}^{\mathrm{orb}}$ but rather just show the universality of the *orbifold completions* $\mathcal{B}_{\mathrm{orb}}$. Specifically, this section shows the following.

- Lemma 6.2.32: $\mathbf{2Kar}\,\mathcal{B}$ is pivotal if $\mathcal{B}$ is pivotal.
- Lemma 6.2.34: Orbifold data in $\mathbf{2Kar}\,\mathcal{B}$ condense.
- Definition 6.2.36: We define *2-orbifold completions* $\mathcal{B}_{\mathrm{orb}}$ of 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$ as the sub-2-category of $\mathbf{2Kar}\,\mathcal{B}$ whose objects are 2-orbifold data of $\mathcal{B}$.
- Lemma 6.2.39: $\iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{orb}}$ is pivotal.
- Lemma 6.2.41: $\pi_{\mathcal{A}} : \mathcal{A}_{\mathrm{orb}} \longrightarrow \mathcal{A}$ is pivotal.
- Theorem 6.2.43: $(\mathcal{B}_{\mathrm{orb}}, \iota_{\mathcal{B}})$ is a universal 2-functor from $\mathcal{B}$ to the forgetful 3-functor $\mathbf{2Cat}^{\mathrm{orb}} \hookrightarrow \mathbf{2Cat}^{\mathbf{piv,ic}}$.

Reminder 6.2.31 (Orbifold Condensation)**.** For one, we recall our Nomenclature 5.2.66:

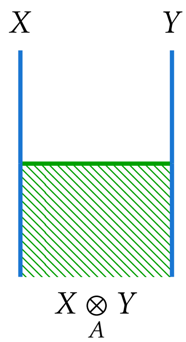
2-orbifold data	symmetric Δ-separable Frobenius algebras
2-orbifold condensates	splittings of symmetric Δ-separable Frobenius algebras
$\mathbf{2Cat}^{\mathbf{piv}}$	full sub-3-category of **2Cat** whose objects are pivotal 2-categories

For another, we recall that we depict split relative tensor products $(X \otimes_A Y, \pi, \iota)$ (Notation 5.2.56) as



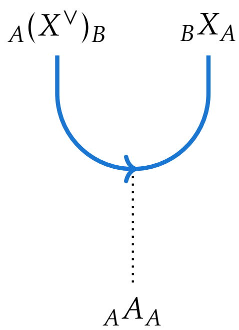
$$:= \pi$$

and

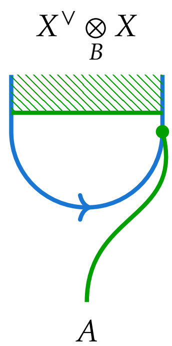


$$:= \iota \quad .$$

Lemma 6.2.32. *The 2-Karoubi envelope of a pivotal 2-category $\mathcal{B} \in$ **2Cat**$^{\mathbf{ic}}$ is again pivotal. We endow **2Kar** $\mathcal{B}$ with a pivotal structure by defining units*[99]



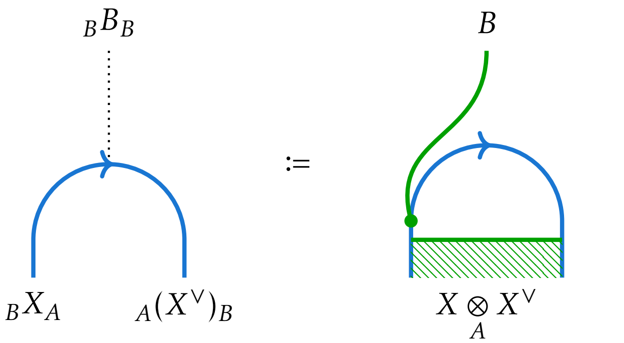
$:=$



(6.23)

[99] Our definition differs from [CR16, Prop. 4.7] as they add a "twist". This twist is not available to us because we are not working with $\mathcal{B}_{\mathrm{eq}}$ but rather with **2Kar** $\mathcal{B}$. However, the proofs are of course analogous because adjoints are unique up to unique isomorphism.

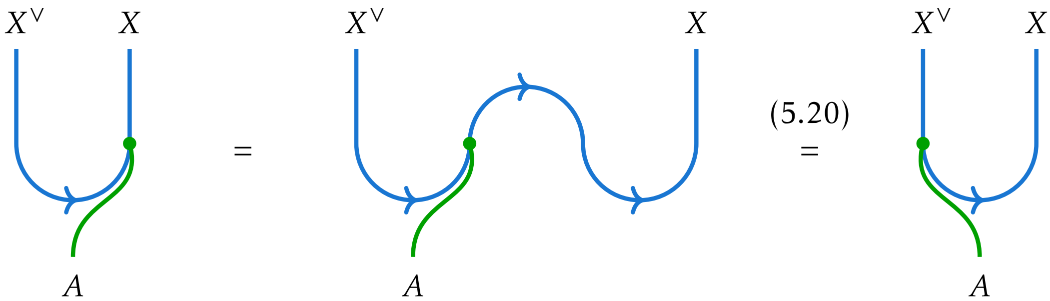
and counits



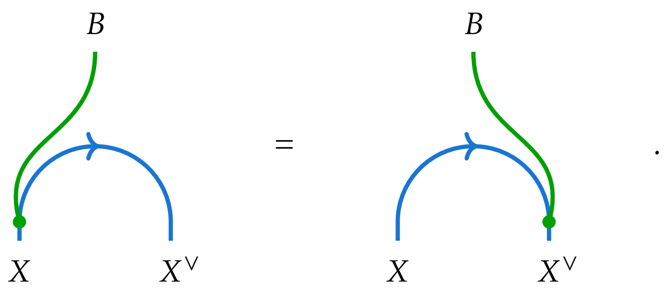
(6.24)

for all 1-morphisms $_B X_A$ in **2Kar** $\mathcal{B}$.

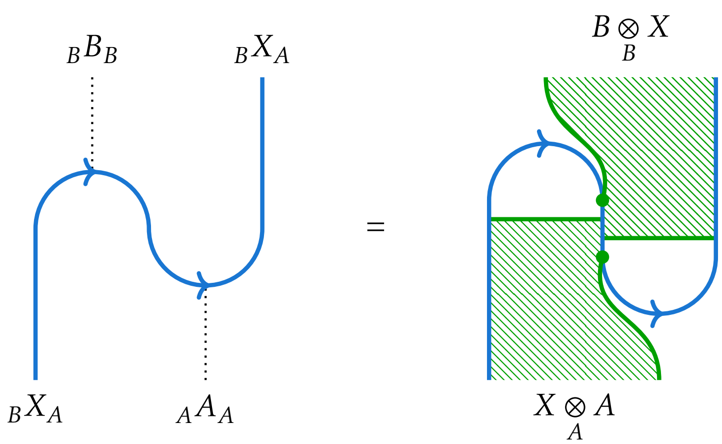
Proof. Clearly, the unit (6.23) is a right A-module map and the counit (6.24) is a left B-module map. Now, Lemma 5.2.35 implies



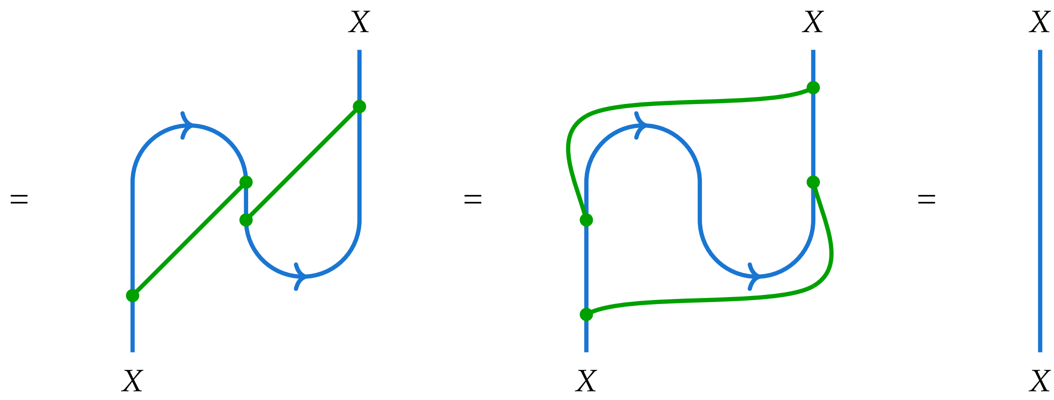
and analogously

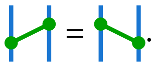


Therefore, the unit (6.23) and counit (6.24) are both A-B-bimodule maps. Thus, these 2-morphisms exist in **2Kar** $\mathcal{B}$. Furthermore,



(6.25)

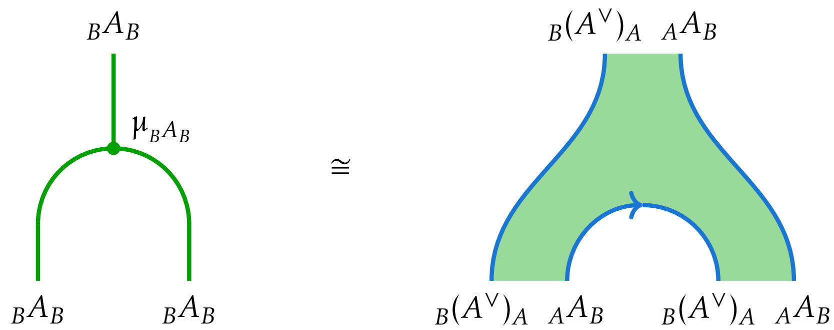


follows from interpreting 2-morphisms of **2Kar** $\mathcal{B}$ as 2-morphisms of $\mathcal{B}$. The other zigzag identity follows similarly using . Thus, **2Kar** $\mathcal{B}$ has adjoints.

Finally, the pivotality conditions (3.2) and (3.3) are fulfilled by a calculation similar to (6.25). Thus, **2Kar** $\mathcal{B}$ is pivotal. □

Note 6.2.33. The pivotal structure of **2Kar** $\mathcal{B}$ is consistent with the pivotal structure of $\mathcal{B}$, i.e. the adjunction data of ${}_{\mathbb{1}_b}X_{\mathbb{1}_a}$ in **2Kar** $\mathcal{B}$ is the adjunction data of $X : a \longrightarrow b$ in $\mathcal{B}$.

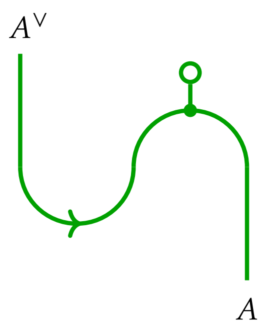
Lemma 6.2.34. *Orbifold data in* **2Kar** $\mathcal{B}$ *condense. For instance, if ${}_BA_B$ is an orbifold datum in* **2Kar** $\mathcal{B}$*, then its multiplication decomposes as*



(6.26)

and its comultiplication decomposes analogously.

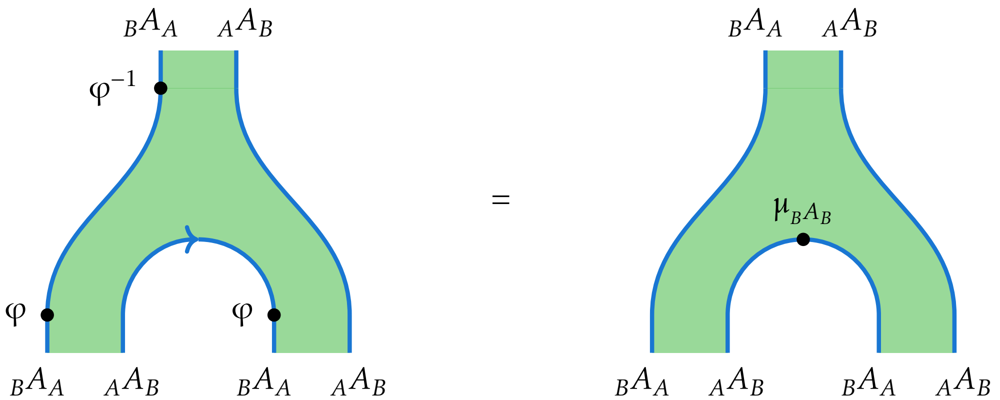
Proof. Interpreting the invertible algebra map

$\varphi :=$ 

(Lemma 5.2.67) as a map

$$\underbrace{A \underset{A}{\otimes} A}_{=A} \Longrightarrow \underbrace{A^\vee \underset{A}{\otimes} A}_{=A^\vee},$$

we see that φ is the 2-isomorphism (6.26) (Lemma 6.2.17):



We decompose the comultiplication analogously. Moreover, the condensation has trivial quantum dimension because $\mathbb{1}_{(a,A)} = {}_AA_A$ (Lemma 5.2.67). □

Definition 6.2.35. $\mathbf{2Cat}^{\mathbf{piv,ic}}$ is the full sub-3-category of $\mathbf{2Cat}^{\mathbf{piv}}$ whose objects are locally idempotent complete. A 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$ is *orbifold complete* if all its orbifold data condense. $\mathbf{2Cat}^{\mathrm{orb}} \subset \mathbf{2Cat}^{\mathbf{piv,ic}}$ is the 3-category of orbifold complete 2-categories.

Definition 6.2.36 (Orbifold Completion[100]). Given a pivotal 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$, we may define the *orbifold completion* $\mathcal{B}_{\mathrm{orb}}$ as the full pivotal sub-2-category of **2Kar** $\mathcal{B}$ whose objects are 2-orbifold data in $\mathcal{B}$. The pivotal structure of $\mathcal{B}_{\mathrm{orb}}$ is inherited from **2Kar** $\mathcal{B}$ (Lemma 6.2.32).

Remark 6.2.37. This definition of orbifold completion looks slightly different to the definition of [CR16, Def. 5.1 & Def. 4.1] as they use unital bimodules while we use special Frobenius bimodules as 1-morphisms. However, this makes no difference as these are the same over unital condensation algebras.

[100] We make a deviation from our discussions of idempotent completions and condensation completions by giving the explicit construction the name “completion” and not naming the universal 2-functor.

Note 6.2.38. If $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ is a pivotal 2-category, then $\mathcal{B}_{\mathrm{orb}}$ is pivotal (Lemma 6.2.32) and orbifold complete (Lemma 6.2.34).

Lemma 6.2.39. *The forgetful 2-functor $\mathcal{B} \hookrightarrow \mathcal{B}_{\mathrm{orb}}$ of Lemma 6.2.15 is pivotal for all $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$. Therefore, by restricting the codomain to $\mathcal{B}_{\mathrm{orb}}$, the 2-functor*

$$\begin{aligned} \iota_{\mathcal{B}} : \mathcal{B} &\hookrightarrow \mathcal{B}_{\mathrm{orb}} \\ a &\longmapsto (a, \mathbb{1}_a) \\ \mathcal{B}(a,b) &\longrightarrow {}_{\mathbb{1}_b}\mathbf{Mod}^{\mathrm{sp}}_{\mathbb{1}_a}(\mathcal{B}), \end{aligned}$$

is pivotal.

For the rest of this section we will mean this 2-functor when we write $\iota_{\mathcal{B}}$.

Proof. We decompose

$$i := \Big(\mathcal{B} \longrightarrow \mathbf{im}(\iota_{\mathcal{B}}) \hookrightarrow \mathbf{2Kar}\,\mathcal{B}\Big)$$

as in Lemma 6.2.15. Then $\mathbf{im}(i) \hookrightarrow \mathbf{2Kar}\,\mathcal{B}$ is trivially pivotal and $\mathcal{B} \longrightarrow \mathbf{im}(i)$ maps adjunction data to adjunction data (Note 6.2.33). □

Lemma 6.2.40. *For every pivotal 2-functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ in $\mathbf{2Cat}^{\mathbf{piv,ic}}$, the 2-functor $\mathbf{2Kar}\,F : \mathbf{2Kar}\,\mathcal{B} \longrightarrow \mathbf{2Kar}\,\mathcal{D}$ is pivotal. Therefore, there exists a pivotal 2-functor*

$$\begin{aligned} F_{\mathrm{orb}} : \mathcal{B}_{\mathrm{orb}} &\longrightarrow \mathcal{D}_{\mathrm{orb}} \\ (a, A) &\longmapsto (Fa, FA) \\ {}_B X_A &\longmapsto {}_{FB}FX_{FA} \\ f &\longmapsto Ff \end{aligned}$$

in $\mathbf{2Cat}^{\mathrm{orb}}$.

Proof. F_{orb} is $\mathbf{2Kar}\,F$ restricted to $\mathcal{B}_{\mathrm{orb}}$. Regarding pivotality, we may first notice that

$$F_{\mathrm{orb}}\Big(({}_B X_A)^\vee\Big) = {}_{FA}\Big(F(X^\vee)\Big)_{FB}$$

and

$$\Big(F_{\mathrm{orb}}({}_B X_A)\Big)^\vee = {}_{FA}\Big((FX)^\vee\Big)_{FB}.$$

Pivotality of F_{orb} then follows from the pivotality of F via a calculation similar to (6.25). □

Lemma 6.2.41. *The 2-equivalence* $\mathbf{2Kar}\,\mathcal{A} \longrightarrow \mathcal{A}$ *of Proposition 6.2.25 is pivotal for all pivotal* $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{cc}}$. *Therefore, by restricting to* $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{orb}}$, *we see that*

$$\begin{aligned} \pi_{\mathcal{A}} : \mathcal{A}_{\mathrm{orb}} &\longrightarrow \mathcal{A} \\ (a, A) &\longmapsto a_A \\ {}_B\mathbf{Mod}^{\mathrm{sp}}_A(\mathcal{A}) &\longrightarrow \Pi_B \underset{B}{\otimes} \mathcal{A}(a, b) \underset{A}{\otimes} \Pi_A^\vee, \end{aligned}$$

with the orbifold condensation data choices $(a_A, \Pi_A, \Pi_A^\vee, \ , \ )$, *is pivotal.*

For the rest of this section we will mean this 2-functor when we write $\pi_{\mathcal{A}}$.

Proof. This 2-functor $\pi_{\mathcal{A}}$ is the 2-functor $\mathbf{2Kar}\,\mathcal{A} \longrightarrow \mathcal{A}$ restricted to $\mathcal{B}_{\mathrm{orb}} \subseteq \mathbf{2Kar}\,\mathcal{B}$. Regarding pivotality we may first calculate that

$$\pi_{\mathcal{A}}\Big(({}_B X_A)^\vee\Big) = \Pi_A \underset{A}{\otimes} {}_A(X^\vee)_B \underset{B}{\otimes} \Pi_B^\vee = \Big(\pi_{\mathcal{A}}({}_B X_A)\Big)^\vee.$$

The pivotality condition (3.4) of $\pi_{\mathcal{A}}$ then follows from the pivotality condition (3.2) of $\mathcal{B}$ via a calculation similar to (6.25). □

Corollary 6.2.42. *Given 2-categories* $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$, $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{orb}}$, *then we may construct the pivotal 2-functor*

$$\begin{aligned} \overline{F} := \pi_{\mathcal{A}} \,\square\, F_{\mathrm{orb}} : \mathcal{B}_{\mathrm{orb}} &\longrightarrow \mathcal{A} \\ (a, A) &\longmapsto (Fa)_{FA} \\ {}_B\mathbf{Mod}^{\mathrm{sp}}_A(\mathcal{B}) &\longrightarrow \Pi_{FB} \underset{FB}{\otimes} \mathcal{A}(Fa, Fb) \underset{FA}{\otimes} \Pi_{FA}^\vee \end{aligned}$$

where $a_{FA}, \Pi_{FB}, \Pi_{FA}^\vee$ *belong to the condensation data of* FA *and* FB *in* $\mathcal{A}$. *In particular,* $\overline{F}$ *has the property*

$$\overline{F} \,\square\, \iota_{\mathcal{B}} = F.$$

Theorem 6.2.43 (Orbifold Completion)**.** *If we are given a pivotal 2-category* $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$, *then* $(\mathcal{B}_{\mathrm{orb}}, \iota_{\mathcal{B}})$ *is a universal 2-functor from* $\mathcal{B}$ *to the forgetful 3-functor* $\mathbf{2Cat}^{\mathrm{orb}} \hookrightarrow \mathbf{2Cat}^{\mathbf{piv,ic}}$.

Proof. In other words, this theorem states that $(\mathcal{B}_{\mathrm{orb}}, \iota_{\mathcal{B}})$ is a 2-category-2-functor pair such that

$$(-) \circ \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathrm{orb}}(\mathcal{B}_{\mathrm{orb}}, \mathcal{A}) \longrightarrow \mathbf{2Cat}^{\mathbf{piv,ic}}(\mathcal{B}, \mathcal{A}) \tag{6.27}$$

is a 2-equivalence for all $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{orb}}$.

Indeed, $\mathcal{B}_{\mathrm{orb}} \in \mathbf{2Cat}^{\mathrm{orb}}$ (Lemma 6.2.34) for $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$, so the 2-functor (6.27) is well-defined. Moreover, (6.27) is

- essentially surjective because there exists a pivotal 2-functor $\overline{F} : \mathcal{B}_{\mathrm{orb}} \longrightarrow \mathcal{D}$ s.t. $\overline{F} \,\square\, \iota_{\mathcal{B}} = F$ for every pivotal 2-functor $F : \mathcal{B} \longrightarrow \mathcal{A}$ (Corollary 6.2.42) and
- essentially fully faithful as in Corollary 6.2.28. $\square$

7. Conclusion

Let us summarise our constructions of idempotent and 2-idempotent completions and then dare an outlook on n-idempotent completions.

7.1. Summary

Idempotent Completions

In Section 5.1.1 we saw that splittings of idempotents $e : X \longrightarrow X$ are not just unique up to isomorphism but even fully determined by e (Note 5.1.17). We then used this insight to construct Karoubi envelopes

$$\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbf{Kar}\,\mathcal{C}$$

in Section 6.1. Extending this insight even further allowed us to see that the Karoubi envelope is actually a 2-functor

$$\mathbf{Kar} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$$

(Theorem 6.1.16) and that $\iota_{\mathcal{C}}$ extends to a 2-natural transformation

$$\iota : \mathrm{id}_{\mathbf{Cat}} \Longrightarrow \mathbf{Kar}\,.$$

Since **Kar** is idempotent up to equivalence there are weak inverses

$$\iota_{\mathcal{A}}^{-1} \mathrel{\hat{=}} \pi_{\mathcal{A}} : \mathbf{Kar}\,\mathcal{A} \longrightarrow \mathcal{A}$$

for all idempotent complete categories $\mathcal{A} \in \mathbf{Cat}^{\mathbf{ic}}$. In particular, these $\pi_{\mathcal{A}}$ are adjoint to the $\iota_{\mathcal{A}}$ which we utilise to construct a 2-natural equivalence

$$\pi : \mathbf{Kar} \Longrightarrow \mathrm{id}_{\mathbf{Cat}^{\mathbf{ic}}}$$

(Lemma 6.1.17). We then used these 2-natural transformations ι and π to see that **Kar** is left 2-adjoint to the forgetful 2-functor $\mathbf{Cat}^{\mathbf{ic}} \hookrightarrow \mathbf{Cat}$. In particular, this means that every Karoubi envelope $\mathbf{Kar}\,\mathcal{C}$ universally completes the category $\mathcal{C} \in \mathbf{Cat}$ w.r.t. idempotents (Theorem 6.1.11).

Condensation Completions

In Section 5.2.5 we saw that splittings of condensation algebras $A : a \longrightarrow a$ are not just unique up to equivalence but even fully determined by A (Note 5.2.64). We then used this insight to construct 2-Karoubi envelopes

$$\iota_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathbf{2Kar}\,\mathcal{B}$$

in Section 6.2. Extending this insight even further allowed us to see that the Karoubi envelope is actually also a 2-functor on Hom categories:

$$\mathbf{2Kar} : \mathbf{2Cat}^{\mathbf{ic}}(\mathcal{B}, \mathcal{D}) \longrightarrow \mathbf{2Cat}^{\mathbf{cc}}(\mathbf{2Kar}\,\mathcal{B}, \mathbf{2Kar}\,\mathcal{D})$$

(Proposition 6.2.22). We may regard this as a step towards showing that **2Kar** is a 3-functor[101] $\mathbf{2Cat}^{\mathbf{ic}} \longrightarrow \mathbf{2Cat}^{\mathbf{cc}}$. Moreover, we constructed weak inverses

$$\iota_{\mathcal{A}}^{-1} \triangleq \pi_{\mathcal{A}} : \mathbf{2Kar}\,\mathcal{A} \longrightarrow \mathcal{A}$$

for condensation complete 2-categories $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{cc}}$. Furthermore, we showed that

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & \mathcal{D} \arrow[d, hook, "\iota_{\mathcal{D}}"] \\ \mathbf{2Kar}\,\mathcal{B} \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{2Kar}\,\mathcal{D} \end{tikzcd}$$

commutes (Lemma 6.2.24) and that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, Rightarrow] & \mathcal{D} \\ \mathbf{2Kar}\,\mathcal{A} \arrow[u, two heads, "\pi_{\mathcal{A}}"] \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{2Kar}\,\mathcal{D} \arrow[u, two heads, "\pi_{\mathcal{D}}"'] \end{tikzcd}$$

is a 2-natural equivalence (Remark 6.2.27) for all $\mathcal{A}, \mathcal{D} \in \mathbf{2Cat}^{\mathbf{cc}}$. This indicates that the $\iota_{\mathcal{B}}$ and $\pi_{\mathcal{A}}$ should form 3-natural transformations, which, moreover, mean that **2Kar** is left 3-adjoint to the forgetful 3-functor $\mathbf{2Cat}^{\mathbf{cc}} \hookrightarrow \mathbf{2Cat}^{\mathbf{ic}}$. Lastly, we combined our results to prove that every 2-Karoubi envelope $\mathbf{2Kar}\,\mathcal{B}$ universally completes the 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ (Corollary 6.2.28).

[101] [Dé22, Rem. A.2.3] notes that this has essentially already been proved by [CP21, Thm. A].

Remark 7.1.1. We see how important it is that condensation algebras fully determine their splittings since this is what allows the construction of condensation completions. In particular, we must keep in mind that this full determination depends on the ambient category, in this case $\mathbf{2Cat}^{\mathbf{ic}}$. Algebras do not fully determine their splittings and, therefore, there are no "algebra completions" (Proposition A.2.8).

Orbifold Completions

In Section 5.2.6 we saw that orbifold data and their condensations are special cases of condensation algebras and their condensations. Therefore, it was natural to suspect that it is possible to construct orbifold completions $\mathcal{B}_{\mathrm{orb}}$ of pivotal 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$ analogously to condensation completions $\mathbf{2Kar}\,\mathcal{D}$ of 2-categories $\mathcal{D} \in \mathbf{2Cat}^{\mathbf{ic}}$. We investigated this in Section 6.2.2. In particular, we showed that the 2-Karoubi envelope $\mathbf{2Kar}\,\mathcal{B}$ of pivotal 2-categories $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{piv,ic}}$ is also pivotal (Lemma 6.2.32) and that orbifold data condense in $\mathbf{2Kar}\,\mathcal{B}$ (Lemma 6.2.34). This allowed us to construct orbifold completions $\mathcal{B}_{\mathrm{orb}} \subseteq \mathbf{2Kar}\,\mathcal{B}$ (Definition 6.2.36) and show that they are universal (Theorem 6.2.43).

7.2. Outlook on n-Idempotent Completions

To construct idempotent completions we utilised that idempotent splittings are fully determined by the idempotents they split. Analogously, we utilised that condensations in $\mathbf{2Cat}^{\mathbf{ic}}$ are fully determined by their condensation algebras to construct condensation completions. While we were able to adapt this construction to orbifold completions, it would not be possible to extend this to 2-idempotents whose splittings are not fully determined as this would contradict universality. Overall, the construction of universal idempotent completions should extend to exactly those flavours of n-idempotents whose splittings are fully determined by the n-idempotent they split. Therefore, let us sketch how an n-condensation completion might be constructed analogously to our discussion of idempotent completions (Section 6.1).

Remark 7.2.1. n-idempotents and their splittings are of course embedded in *n-categories*. However, there are a variety of ways to define n-categories and we are not actually working with n-categories in this thesis. Therefore, we will naïvely think of n-categories as consisting of objects, 1-morphisms, 2-morphisms,..., and n-morphisms without concerning ourselves with coherence data and coherence conditions. This is fitting since this outlook heavily leans on [GJF19, Sec. 2.1–2.3], who use the same approach – albeit for slightly different reasons.

Furthermore, the results of [GJF19] are deep but come at the cost of reduced rigour [GJF25, Rem. 1.0.1].[102] Therefore, we shall declare the following definitions and statements as sketches to reflect this.

Sketch 7.2.2 (Definition)**.** [GJF19] introduce *n-condensation* as a categorification of splitting data. They then derive n-condensation algebras and n-condensation bimodules from n-condensations. These are categorifications of idempotents and morphisms compatible with idempotents, respectively.

- *n-condensations* may be defined recursively. A 0-condensation is an equality between two elements of a set. Given an n-category $\mathcal{C}$ and two objects $a, b \in \mathcal{C}$, then an n-condensation $a \twoheadrightarrow b$ is a pair of 1-morphisms
$$(\Pi : a \longrightarrow b, \mathrm{I} : b \longrightarrow a)$$
together with an $(n-1)$-condensation
$$\Pi \otimes \mathrm{I} \Longrightarrow \mathbb{1}_b$$
[GJF19, Def. 2.1.1].
- An *n-condensation algebra* is then given by a 1-morphism $A : a \longrightarrow a$ that has the properties of the 1-morphism $\mathrm{I} \otimes \Pi$. To make this more precise, if the condensation of the "walking n-condensation" (Remark 5.2.20) is given by the data $(a, b, \Pi : a \longrightarrow b, \mathrm{I} : b \longrightarrow a, \dots)$, then the full sub-$n$-category $\langle a \rangle$ on the object a is the *walking condensation algebra*. An *n-condensation algebra in an n-category $\mathcal{C}$* is then given as an n-functor
$$\langle a \rangle \longrightarrow \mathcal{C}$$
[GJF19, Def. 2.2.1].[103]

[102] Rigorous results are expected to be published soonish by [RZ].
[103] Recently [GJF19] uploaded a new version of this paper to arXiv where they demote the fullness of $\langle a \rangle$ to a conjecture [GJF25, Conj. 2.2.2]. [Fra22, Rem. 3.14] explains why this conjecture only holds up to 2-equivalence for 2-condensation algebras.

- *n-condensation bimodules* may also be constructed from a walking construction. Specifically, one constructs an n-category generated by two n-condensations $(a, b, \Pi : a \longrightarrow b, \mathrm{I} : b \longrightarrow a, \dots)$ and $(a', b', \Pi' : a' \longrightarrow b', \mathrm{I}' : b' \longrightarrow a', \dots)$ together with a free 1-morphism $X : b \longrightarrow b'$. This then induces the *n-condensation bimodule*

$$\mathrm{I}' \otimes \mathrm{X} \otimes \Pi : a \longrightarrow a'.$$

The *walking condensation bimodule* is then the the full sub-n-category $\langle a, a' \rangle$ and an *n-condensation bimodule in an n-category $\mathcal{C}$* is then given by an n-functor

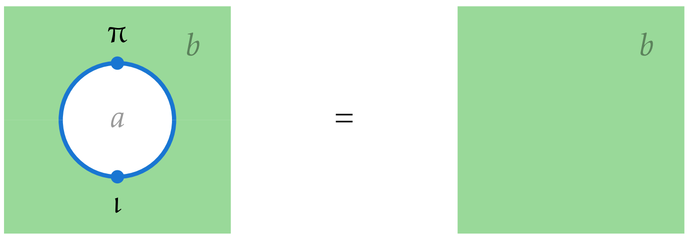
$$\langle a, a' \rangle \longrightarrow \mathcal{C}$$

[GJF19, Def. 2.3.3].[104]

Intuition 7.2.3. Essentially, a 2-condensation in a 2-category $\mathcal{B}$ just consists of data

$$(a \in \mathcal{B}, b, \Pi : a \longrightarrow b, \mathrm{I} : b \longrightarrow a, \pi : \Pi \otimes \mathrm{I} \Longrightarrow \mathbb{1}_b, \iota : \mathbb{1}_b \Longrightarrow \Pi \otimes \mathrm{I})$$

such that



,

i.e. it consists of data to construct disks of a which vanish in b. Analogously, an n-condensation $(a, b, \dots)$ consists of two objects a, b, two 1-morphisms, two 2-morphisms,..., and two n-morphisms constructing n-balls of a which vanish in b; cf. [CDZR25, Sec. 4.3].

The data of n-condensation algebras may be derived explicitly from n-condensations [GJF19, Prop. 2.2.4], but its combinatorial nature does not readily provide intuition. The same is true for n-condensation bimodules [GJF25, Def. 2.3.3].

[104] Recently [GJF19] uploaded a new version of this paper to arXiv where they demote the fullness of $\langle a \rangle$ to a conjecture [GJF25, p. 15].

Remark 7.2.4. *n-condensation bimodule maps* and higher maps between those are not explicitly detailed by [GJF19].[105] However, since n-condensation bimodule maps are 1-morphisms between n-functors $\langle a, a' \rangle \longrightarrow \mathcal{C}$, they should be n-natural transformations that fulfil certain properties. *Higher n-condensation bimodule maps* would then be n-modifications, "n-perturbations", etc. When considering n-functors, n-natural transformations etc., one usually views them as an n-category – specifically a sub$n$-category of **nCat**. However, in this construction, the condensation modules are n-functors corresponding to 1-morphisms in $\mathcal{C}$, n-condensation bimodule maps are n-natural transformations corresponding to 2-morphisms in $\mathcal{C}$, etc. Therefore, n-condensation modules and their maps should form an $(n-1)$-category [GJF25, Rem. 2.3.4].[106]

Conceptually, n-condensation modules A are morphisms of a condensate of A. Therefore, these (higher) maps should also follow from relative tensor products (Sketch 7.2.6).

Sketch 7.2.5 (Proposition)**.** If we are given an n-condensation algebra A that condenses, then there exists a *relative tensor product* $Y \otimes_A X$ [GJF19, Def. 2.3.9] between left n-condensation A-modules Y and right n-condensation A-modules X given by the $(n-1)$-condensation $Y \otimes X \rightrightarrows Y \otimes_A X$ [GJF19, Prop. 2.3.8]. Moreover, the relative tensor products $\otimes_A$ are $(n-1)$-functors [GJF19, Thm. 2.3.10].

[105] Their revision now addresses 2-morphisms between n-condensation modules [GJF25, Rem. 2.3.4].

[106] n-functors consist of a map on objects together with a collection of $(n-1)$-functors together with coherence data. Since n-condensation modules are conceptually 1-morphisms, maybe it would be more fitting to define them not as n-functors $X : \langle a, a' \rangle \longrightarrow \mathcal{C}$, but rather as the $(n-1)$-functors $X_{a,a'} : \langle a, a' \rangle(a, a') \longrightarrow \mathcal{C}(Xa, Xa')$. Not just would $(n-1)$-functors, $(n-1)$-natural transformations etc. elegantly form an $(n-1)$-category, this would also seem more fitting for the following reason. In spirit, n-natural transformations are 1-morphisms, n-modifications are 2-morphisms, etc., but n-condensation bimodule maps are conceptually 2-morphisms, maps between n-condensation bimodule maps are conceptually 3-morphisms, etc. Thus, for instance, defining n-condensation bimodule maps as $(n-1)$-natural transformations on $(n-1)$-functors of the form $\langle a, a' \rangle(a, a') \longrightarrow \mathcal{C}(Xa, Xa')$ seems more fitting. However, presumably there is a good reason that [GJF25] do not make this choice.

Sketch 7.2.6 (Proposition)**.** There is an $(n-1)$-category of n-condensation bimodules and their (higher) maps [GJF19, Thm. 2.3.10]. Given an n-condensation algebra $A : a \longrightarrow a$ with condensate b, then there the $(n-1)$-categories of n-condensation A-modules are $(n-1)$-equivalent to Hom $(n-1)$-categories of b [GJF19, p. 12]. Supposedly, these equivalences would be constructed from the relative tensor products as in Note 5.2.64.

Remark 7.2.7. Sketch 7.2.6 implies that not only are n-condensates in $\mathbf{nCat^{(n\text{-}1)cc}}$ unique up to $(n-1)$-equivalence, they are also fully determined by their n-condensation algebras – just like in the $n \leq 2$ cases (Note 5.2.64, Remark 7.1.1). Given a condensation $(a, b, \Pi, \mathrm{I}, \dots)$, there is an $(n-1)$-equivalence between n-condensation $\mathrm{I} \otimes \Pi$-modules and the Hom $(n-1)$-categories of b.

Sketch 7.2.8 (Definition)**.**

- An n-category is *condensation complete* if all its n-condensation algebras condense.
- Let $\mathbf{nCat^{(n\text{-}1)cc}}$ be the full sub-$(n+1)$-category of $\mathbf{nCat}$ consisting of locally condensation complete n-categories.
- Let $\mathbf{nCat^{ncc}}$ be the full sub-$(n+1)$-category of $\mathbf{nCat^{(n\text{-}1)cc}}$ consisting of condensation complete n-categories.

Sketch 7.2.9 (cf. [GJF19, Thm. 2.3.10])**.** We may use full determinedness of condensations (Remark 7.2.7) to construct an *n-Karoubi envelope* $\mathbf{nKar}\,\mathcal{C}$ for every locally condensation complete n-category $\mathcal{C} \in \mathbf{nCat^{(n\text{-}1)cc}}$. The objects of $\mathbf{nKar}\,\mathcal{C}$ are n-condensation algebras in $\mathcal{C}$ and the Hom $(n-1)$-categories consist of n-condensation bimodules and their maps. The horizontal composition of 1-morphisms is given by a relative tensor product over n-condensation algebras.

Moreover, there is a forgetful n-functor

$$\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbf{nKar}\,\mathcal{C}$$

that maps objects $a \in \mathcal{C}$ to trivial n-condensation algebras $\mathbb{1}_a$, and all n-condensation algebras in $\mathbf{nKar}\,\mathcal{C}$ condense, i.e. $\mathbf{nKar}\,\mathcal{C} \in \mathbf{nCat^{ncc}}$.

For all n-categories $\mathcal{A} \in \mathbf{nCat^{ncc}}$ there is an n-functor

$$\pi_{\mathcal{A}} : \mathbf{nKar}\,\mathcal{A} \longrightarrow \mathcal{A}.$$

Conjecture 7.2.10. *These $\pi_{\mathcal{A}}$ are left inverse to the $\iota_{\mathcal{A}}$, i.e. they form "adjoint n-equivalences" for all $\mathcal{A} \in \mathbf{nCat^{ncc}}$.*

Conjecture 7.2.11. *The n-Karoubi envelope is an $(n+1)$-functor*

$$\begin{aligned} \mathbf{nKar} : \mathbf{nCat^{(n\text{-}1)cc}} &\longrightarrow \mathbf{nCat^{ncc}} \\ \mathcal{C} &\longmapsto \mathbf{nKar}\,\mathcal{C} \\ F &\longmapsto \mathbf{nKar}\,F \\ &\vdots \end{aligned}$$

Conjecture 7.2.12.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "\iota_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, hook, "\iota_{\mathcal{D}}"] \\ \mathbf{nKar}\,\mathcal{C} \arrow[r, "\mathbf{Kar}\,F"'] & \mathbf{nKar}\,\mathcal{D} \end{tikzcd} \tag{7.1}$$

commutes for all n-functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ in $\mathbf{nCat^{(n\text{-}1)cc}}$, which implies a strict[107] *$(n+1)$-natural transformation*

$$\iota : \mathrm{id}_{\mathbf{nCat^{(n\text{-}1)cc}}} \Longrightarrow \mathbf{nKar}\,.$$

Similarly,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, Rightarrow] & \mathcal{D} \\ \mathbf{nKar}\,\mathcal{A} \arrow[u, two heads, "\pi_{\mathcal{A}}"] \arrow[r, "\mathbf{2Kar}\,F"'] & \mathbf{nKar}\,\mathcal{D} \arrow[u, two heads, "\pi_{\mathcal{D}}"'] \end{tikzcd}$$

is an n-natural equivalence for all n-functors $F : \mathcal{A} \longrightarrow \mathcal{D}$ in $\mathbf{nCat^{ncc}}$, which implies an $(n+1)$-natural equivalence

$$\pi : \mathbf{nKar} \Longrightarrow \mathrm{id}_{\mathbf{nCat^{ncc}}}.$$

π is constructed using the $(n+1)$-adjunction data of Conjecture 7.2.10.

[107] Strictness only makes sense if $\mathbf{nKar}\,F$ in Conjecture 7.2.11 is defined analogously to Proposition 6.2.22 and composition of n-functors is strict. In general n-categorical settings strictness does not make sense. In such general settings diagram (7.1) only commutes up to an n-natural transformation

Conjecture 7.2.13. *The n-Karoubi completion* $\mathbf{nKar} : \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}} \longrightarrow \mathbf{nCat}^{\mathbf{ncc}}$ *is left* $(n+1)$*-adjoint to the forgetful* $(n+1)$*-functor* $\mathbf{nCat}^{\mathbf{ncc}} \hookrightarrow \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$*. Specifically, the* $(n+1)$*-adjunction is a* left inverse left $(n+1)$-adjunction *with units given by the* $(n+1)$*-natural transformations* π *and* ι*. In particular, the* $(n+1)$*-adjunction implies that there are* n-condensation completions $\overline{\mathcal{C}} \in \mathbf{nCat}^{\mathbf{ncc}}$ *for all* $\mathcal{C} \in \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$*, i.e.*

$$(-) \,\Box\, \iota_{\mathcal{C}} : \mathbf{nCat}^{\mathbf{ncc}}(\overline{\mathcal{C}}, \mathcal{A}) \longrightarrow \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}(\mathcal{C}, \mathcal{A})$$

is an n-equivalence for all $\mathcal{A} \in \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$.

Theorem 7.2.14. *Every n-category* $\mathcal{C} \in \mathbf{nCat}^{\mathbf{(n\text{-}1)cc}}$ *has an n-condensation completion* $\overline{\mathcal{C}}$ *[RZ].*

Remark 7.2.15. Statements analogous to Theorem 7.2.14 should hold for all other types of n-idempotents in an appropriate sub-$(n+1)$-category $\widetilde{\mathbf{nCat}} \subseteq \mathbf{nCat}$ if their splittings are fully determined in $\widetilde{\mathbf{nCat}}$. In particular, they should extend to orbifold completions with appropriately defined n-orbifold condensation (Sketch A.2.15).[108]

[108] Recently, [Mü25, Exp. 5.4] presented a similar statement; it comes from more of a physicist's perspective, is slightly less general, but much more concrete.

A. Appendix

A.1. Category Theory

This chapter quickly recaps some foundations of category theory that we will use in this thesis. A more substantial introduction is given by [Lei14]. Moreover, readers who enjoy graphical proofs may be interested in the introductions of [Nak23] and [Mar14, HM23].

Definition A.1.1. A *commutative diagram* is a directed graph that expresses that paths with the same start and end point represent equal mathematical objects. For instance, commutativity of

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

implies that $Gf \circ \alpha_X = \alpha_Y \circ Ff$.

Definition A.1.2. A *category* $\mathcal{C}$ consists of:

- a collection of *objects*,
- collections $\mathcal{C}(X, Y)$ of *morphisms* $(f : X \longrightarrow Y)$ – colloquially also called *hom-sets* – for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$,
- and a *composition* given by a collection of functions of the form

  $$\begin{aligned} \circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) &\longrightarrow \mathcal{C}(X, Z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

  for all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

This data must fulfil *associativity* and *unity*. Associativity means that

$$(h \circ g) \circ f = h \circ (g \circ f)$$

for all morphisms $f : X \longrightarrow Y, g : Y \longrightarrow Z, h : Z \longrightarrow W$. Unity means that there exists a *unit morphism* $1_X \in \mathcal{C}(X, X)$ for every object $X \in \mathrm{Ob}(\mathcal{C})$ such that

$$1_Y \circ f = f = f \circ 1_X$$

for any morphism $f : X \longrightarrow Y$.

An *isomorphism* is a morphism that is invertible, i.e. a morphism $f : X \longrightarrow Y$ such that there exists a morphism $g : Y \longrightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

Example A.1.3. As mathematics enthusiasts we like to start with trivial examples. Therefore, we now introduce the trivial category $\{\star\}$. It is defined as a category with one object $\star \in \{\star\}$ and one morphism:

$$\{\star\}(\star, \star) = \{1_\star\}$$

Example A.1.4. Set is the *category of sets* and consists of

Objects: sets,

Morphisms: maps,

Composition: composition of maps, and

Unit Morphisms: identity maps.

Example A.1.5. Grp is the *category of groups* and consists of

Objects: groups,

Morphisms: group homomorphisms,

Composition: composition of group homomorphisms, and

Unit Morphisms: identity group homomorphisms.

Example A.1.6. If we are given a field K, then we may construct $\mathbf{Vect}_K$, the *category of vector spaces over K*, consisting of

Objects: K-vector spaces,

Morphisms: K-linear maps,

Composition: composition of linear maps,

Unit Morphisms: identity maps.

Definition A.1.7. A *functor*

$$\begin{aligned} F : \mathcal{C} &\longrightarrow \mathcal{D} \\ X &\longmapsto F(X) \\ f &\longmapsto F(f) \end{aligned}$$

is a map between categories $\mathcal{C}, \mathcal{D}$ that maps objects $X \in \mathcal{C}$ to objects $F(X) \in \mathcal{D}$ and morphisms $f \in \mathcal{C}(X, Y)$ to morphisms $Ff \in \mathcal{D}(F(X), F(Y))$. By a slight abuse of notation we usually write FX and Ff instead of $F(X)$ and Ff, respectively. F must fulfil

$$Fg \circ_{\mathcal{D}} Ff = F(g \circ_{\mathcal{C}} f) \tag{A.1}$$

for all morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in $\mathcal{C}$. Hereby $\circ_{\mathcal{C}}$ and $\circ_{\mathcal{D}}$ denote the compositions of $\mathcal{C}$ and $\mathcal{D}$, respectively. The *functoriality* condition (A.1) may alternatively be expressed as the condition that

$$\begin{tikzcd} \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \arrow[r, "\circ_{\mathcal{C}}"] \arrow[d, "(F,F)"'] & \mathcal{C}(X,Z) \arrow[d, "F"] \\ \mathcal{D}(FY,FZ) \times \mathcal{D}(FX,FY) \arrow[r, "\circ_{\mathcal{D}}"'] & \mathcal{D}(FX,FZ) \end{tikzcd}$$

commutes.

Example A.1.8. For all categories $\mathcal{C}$ there exists a trivial functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \{\star\} \\ X &\longmapsto \star \\ f &\longmapsto 1_\star. \end{aligned}$$

Example A.1.9. There is a category **Cat** whose objects are categories and whose morphisms are functors. *Composition of functors* $F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{A}$ is given by

$$\begin{aligned} G \circ F : \mathcal{C} &\longrightarrow \mathcal{A} \\ X &\longmapsto GFX \\ f &\longmapsto GFf \end{aligned}$$

and identities are the *identity functors*

$$\begin{aligned} \mathrm{id}_{\mathcal{C}} : \mathcal{C} &\longrightarrow \mathcal{C} \\ X &\longmapsto X \\ f &\longmapsto f. \end{aligned}$$

Definition A.1.10. Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a *transformation* $\alpha : F \Longrightarrow G$ is a collection of morphisms $\alpha_X \in \mathcal{D}(FX, GX)$ such that

$$\alpha_X \in \mathcal{D}(FX, GX)$$

for all $X \in \mathcal{C}$.

The transformation $\alpha : F \Longrightarrow G$ is *natural* if

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \tag{A.2}$$

commutes for all $X, Y \in \mathcal{C}$ and all $f \in \mathcal{C}(X, Y$.

A natural transformation is called a *natural isomorphism* if all components α_X are isomorphisms.

Example A.1.11. For every pair of categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$ there is a category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \longrightarrow \mathcal{D}$ and whose morphisms are natural transformations between them. The composition of natural transformations $\alpha : F \Longrightarrow G$ and $\beta : G \Longrightarrow H$ is $\beta \circ \alpha : F \Longrightarrow G$ with components

$$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X : FX \longrightarrow HX$$

for all $X \in \mathcal{C}$. This composition is called *vertical composition*. There exists a trivial natural transformation $\mathrm{id}_F : F \Longrightarrow F$ with components

$$(\mathrm{id}_F)_X = 1_{FX} : FX \longrightarrow FX$$

for each functor $F : \mathcal{C} \longrightarrow \mathcal{D}$.

Lemma A.1.12. *A natural transformation $\alpha : F \longrightarrow G$ is a natural isomorphism if and only if it is invertible, i.e. if and only if there exits a natural transformation $\beta : G \longrightarrow F$ such that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_G$.*

Definition A.1.13. We may also compose natural transformations *horizontally*. Given four functors $F, G\ \mathcal{C} \longrightarrow \mathcal{D}$ and $F', G' : \mathcal{D} \longrightarrow \mathcal{A}$ and two natural transformations $\alpha : F \Longrightarrow G, \alpha' : F' \Longrightarrow G'$, then we may define their *horizontal composition* as $\alpha' \star \alpha : F'F \Longrightarrow G'G$ with components given by either path of the commutative diagram:

$$\begin{tikzcd}
F'FX \arrow[r, "\alpha'_{FX}"] \arrow[d, "F'\alpha_X"'] & F'FX \arrow[d, "G'\alpha_X"] \\
G'GX \arrow[r, "\alpha'_{GX}"'] & G'GX
\end{tikzcd}$$

Definition A.1.14. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is

- *faithful*, if it is injective on hom-sets, i.e. if
  $$F : \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$$
  is injective for all $X, Y \in \mathcal{C}$.
- *full*, if it is surjective on hom-sets, i.e. if
  $$F : \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$$
  is surjective for all $X, Y \in \mathcal{C}$.
- *fully faithful* if it is bijective on hom-sets.
- *essentially surjective* if there exists an object $X \in \mathcal{C}$ such that $FX \cong Y$ for all $Y \in \mathcal{D}$.

Notation A.1.15. Since fully faithful functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ can be thought of as being injective on objects we may denote them as $F : \mathcal{C} \hookrightarrow \mathcal{D}$. Likewise, we may denote essentially surjective functors as $F : \mathcal{C} \twoheadrightarrow \mathcal{D}$.

Definition A.1.16. Two categories $\mathcal{C}, \mathcal{D}$ are *equivalent* if there exists functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ and natural isomorphisms

$$\alpha : \mathrm{id}_{\mathcal{C}} \Longrightarrow G \circ F$$
$$\beta : \mathrm{id}_{\mathcal{D}} \Longrightarrow F \circ G.$$

We then say that the tuple (F, G, α, β) *forms an equivalence* $\mathcal{C} \simeq \mathcal{D}$. Moreover, we may also say things like F *is an equivalence* and that F and G are *weak inverses* of one another.

Proposition A.1.17. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.*

Definition A.1.18. A category $\mathcal{C}$ is a *subcategory* of a category $\mathcal{D}$ if its objects and hom-sets are subcollections, i.e. if $X \in \mathcal{D}$ for all $X \in \mathcal{C}$ and $\mathcal{C}(X, Y) \subseteq \mathcal{D}(X, Y)$ for all $X, Y \in \mathcal{C}$. Conceptually, $\mathcal{C}$ may be derived by removing objects and morphisms from $\mathcal{D}$. We shall denote subcategories via $\mathcal{C} \subseteq \mathcal{D}$.

Such a subcategory $\mathcal{C}$ is a *full* subcategory of $\mathcal{D}$ if the hom-sets are equal, i.e. if $\mathcal{C}(X, Y) = \mathcal{D}(X, Y)$ for all $X, Y \in \mathcal{C}$. Conceptually, such a full $\mathcal{C}$ may be derived from $\mathcal{D}$ by removing objects and keeping all possible morphisms, i.e. only removing morphisms whose domain or codomain is being removed.

Definition A.1.19. We may call canonical[109] fully faithful functors *forgetful*.

Remark A.1.20. Our definition of "forgetfulness" is only one possible choice that one may make [BS10, p. 15]. We made this choice not because it is the most common but because it is the notion we will need later on. Actually, forgetful functors are often not defined precisely in the literature but rather only described as functors that "forget" some or all of the structure of their domains.

To illustrate, let us consider two examples involving categories of groups and rings. Hereby Grp is the category whose objects and morphisms are groups and group homomorphisms between them. Analogously, Ab and Ring are the categories of abelian groups and rings, respectively.

There is a canonical functor Ab $\hookrightarrow$ Grp since Ab $\subseteq$ Grp. The functor "forgets" the property of being abelian. It is fully faithful since group homomorphisms are defined the same way for abelian and non-abelian groups. Therefore, it is forgetful according to our definition.

Analogously, there is a canonical functor Ring $\longrightarrow$ Grp that maps rings to their underlying additive structures but "forgets" their multiplicative structures. This functor, however, is not fully faithful since ring homomorphisms are a subset of group homomorphisms that is not generally equal. Therefore, it is not forgetful according to our definition although it would be considered to be by other definitions.

[109] "Canonical" is not well-defined. It is supposed to mean distinguished in some sense. This also implies that canonical constructions are unique in some sense.

In both examples the functors only forget structure between morphisms but are still able to distinguish them, i.e. those functors are faithful. However, one may consider other notions of forgetfulness that do not imply that the domain is contained in the codomain. Therefore, we see that forgetfulness contains a philosophical notion that makes a precise general purpose definition impossible.

Note A.1.21. If $\mathcal{C} \subseteq \mathcal{D}$, then there is a forgetful functor

$$\begin{aligned} \mathcal{C} &\hookrightarrow \mathcal{D} \\ X &\longmapsto X \\ f &\longmapsto f. \end{aligned}$$

It is full if and only if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$.

Remark A.1.22. Some category theorists define (full) subcategories as (fully) faithful functors, but for our purposes the given definition will be more suitable.

Definition A.1.23. Given two categories $\mathcal{C}$ and $\mathcal{D}$, then their *Cartesian product* is the category $\mathcal{C} \times \mathcal{D}$ consisting of

Objects: pairs (X, Y), where $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ and

Morphisms: pairs $(f, g) : (X, Y) \longrightarrow (X', Y')$, where $f \in \mathcal{C}(X, X')$ and $g \in \mathcal{D}(Y, Y')$.

Definition A.1.24. If we are given a category $\mathcal{D}$ and a collection U of morphisms in $\mathcal{D}$, then $\langle U \rangle \subseteq \mathcal{D}$ is the smallest category such that morphisms in U also lie in $\langle U \rangle$. $\langle U \rangle$ is then *generated* by U.

Definition A.1.25. The *image* $\mathbf{im}\, F$ of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is the category generated by the image of F viewed as a map. In other words, it is the subcategory $\mathbf{im}\, F \subseteq \mathcal{D}$ generated by all morphisms $f \in \mathcal{D}(X, Y)$, such that $f = Ff'$ for some $f' \in \mathcal{C}(X', Y')$.

Definition A.1.26. Given a directed graph G with collection of edges E, then there is a category $\langle E \rangle$ *freely generated* by the edges of G.

Theorem A.1.27. *Functors preserve diagrams, i.e. given a commutative diagram in a category $\mathcal{C}$ and a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, then if all objects $X \in \mathcal{C}$ in the diagram are replaced by $FX \in \mathcal{D}$ and all morphisms f in the diagram are replaced by Ff, then the resulting diagram is a commutative diagram in $\mathcal{D}$ by functoriality.*

A.2. Supplementary Sketches

A.2.1. Categorifying Metric Space Completions

[Law73] shows that metric spaces may be interpreted as *enriched* categories (Proposition A.2.5). While this may at first sound like an unnecessary complication, it is actually a surprisingly natural reformulation. In particular, this enables the application of (enriched) category theory to the study of metric spaces, leading to new results and to a deeper understanding of known results. For instance, idempotent completions are a generalisation of the completions of metric spaces (Sketch A.2.6).

Definition A.2.1. A *Lawvere metric space* is a set M together with a function

$$d : M \times M \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

such that

- the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ is fulfilled for all $x, y, z \in M$ and
- reflexivity $d(x,x) = 0$ is fulfilled for all $x \in M$.

Sketch A.2.2. Given a monoidal category $(\mathcal{V}, \otimes_{\mathcal{V}}, \mathbb{1}_{\mathcal{V}})$, a *$\mathcal{V}$-enriched category* $\mathcal{C}$ consists of

- a collection of objects $\mathrm{Ob}(\mathcal{C})$,
- a *hom-object* $\mathcal{C}(X,Y) \in \mathcal{V}$ for each pair of objects $X, Y \in \mathcal{C}$,
- a *composition morphism*
$$\circ_{X,Y,Z} \in \mathcal{V}\Big(\mathcal{C}(Y,Z) \otimes_{\mathcal{V}} \mathcal{C}(X,Y), \mathcal{C}(X,Z)\Big) \tag{A.3}$$
for each triple of objects $X, Y, Z \in \mathcal{C}$), and
- an *unit morphism*
$$\mathbb{1}_X \in \mathcal{V}\Big(\mathbb{1}_{\mathcal{V}}, \mathcal{C}(X,X)\Big) \tag{A.4}$$
for each object $X \in \mathcal{C}$,

such that composition is *associative* and *unital* [JY21, Def. 1.3.1].

Remark A.2.3. Enriched categories are a generalisation of categories, i.e. not every enriched category is a category. However, **Set**-enriched categories are precisely categories, i.e. every **Set**-enriched category is a category and vice versa. Interestingly, by interpreting **Cat** as a monoidal category $(\mathbf{Cat}, \times, \{\star\})$ one finds that **Cat**-enriched categories are precisely strict 2-categories.

Note A.2.4. If $(P, \lesssim)$ is a preorder, then one may interpret it as a category $\mathcal{P}$ consisting of

Objects: elements $x \in P$ and

Morphisms: relations $x \lesssim y$, i.e. $|\hom_{\mathcal{P}}(x, y)| = \begin{cases} 1 & \text{if } y \lesssim x, \\ 0 & \text{else.} \end{cases}$

The composition in $\mathcal{P}$ is trivial.

The non-negative real numbers $\mathbb{R}_{\geq 0}$ form both a poset $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \leq)$ and a monoid $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0)$. Therefore, one may interpret them as a monoidal category $\mathcal{R} := (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0)$ consisting of

Objects: non-negative real numbers $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$,

Morphisms: relations $r \leq r'$, i.e. $|\hom_{\mathcal{R}}(r, r')| = \begin{cases} 1 & \text{if } r' \leq r, \\ 0 & \text{else,} \end{cases}$

Horizontal Composition: addition $+ : (r, r') \longmapsto r + r'$, and

Monoidal Unit: $\mathbb{1}_{\mathcal{R}} := 0 \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Vertical and horizontal composition of morphisms both become trivial, but are nonetheless interesting. Vertical composition of morphisms in $\mathcal{R}$ may be expressed as transitivity, i.e. $r \leq r'$ and $r' \leq r''$ implies $r \leq r''$.

Proposition A.2.5. *Lawvere metric spaces (M, d) are the same as $\mathcal{R}$-enriched categories $\mathcal{M}$.*

Proof sketch. Lawvere metric spaces $(M, d : M \times M \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\})$ are interpreted as categories $\mathcal{M}$ consisting of

Objects: points $x \in M$ and

Hom-objects: distances $\mathcal{M}(x, y) := d(x, y) \in \mathcal{R}$.

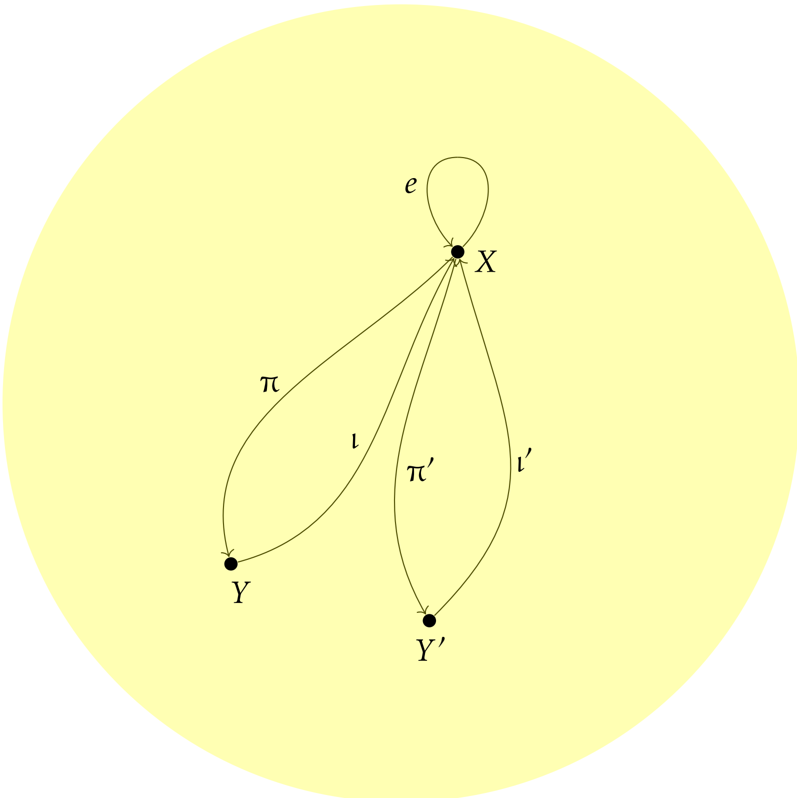
Then, the composition morphisms (A.3) of $\mathcal{M}$ correspond to the triangle inequalities $d(x, z) \leq d(x, y) + d(y, z)$, and the unit morphisms (A.4) of $\mathcal{M}$ correspond to the reflexivity conditions $d(x, x) = \mathbb{1}_{\mathcal{R}} = 0$. □

Sketch A.2.6. Homomorphisms between $\mathcal{V}$-enriched categories are *$\mathcal{V}$-enriched functors*. One may show that functors enriched in $\mathcal{R}$ correspond to *distance-decreasing* maps between metric spaces [Law73, p. 149]. This implies that a metric space (M, d) is complete w.r.t. Cauchy sequences if and only if *$\mathcal{M}$ has all absolute $\mathcal{R}$-weighted colimits* [Str83, p. 379]. Then $\mathcal{M}$ is called *$\mathcal{R}$-Cauchy complete*. Generalising this further, one may find that a category is *(**Set**-)Cauchy complete* if and only if it *has all absolute colimits* [Bor94, Def. 6.5.5]. One may find that absolute colimits correspond to split idempotents [Bor94, cf. Prop. 6.5.7]. Therefore, a category $\mathcal{C}$ is Cauchy complete if and only if all idempotents in $\mathcal{C}$ split. Thus, idempotent completions generalise completions of metric spaces.

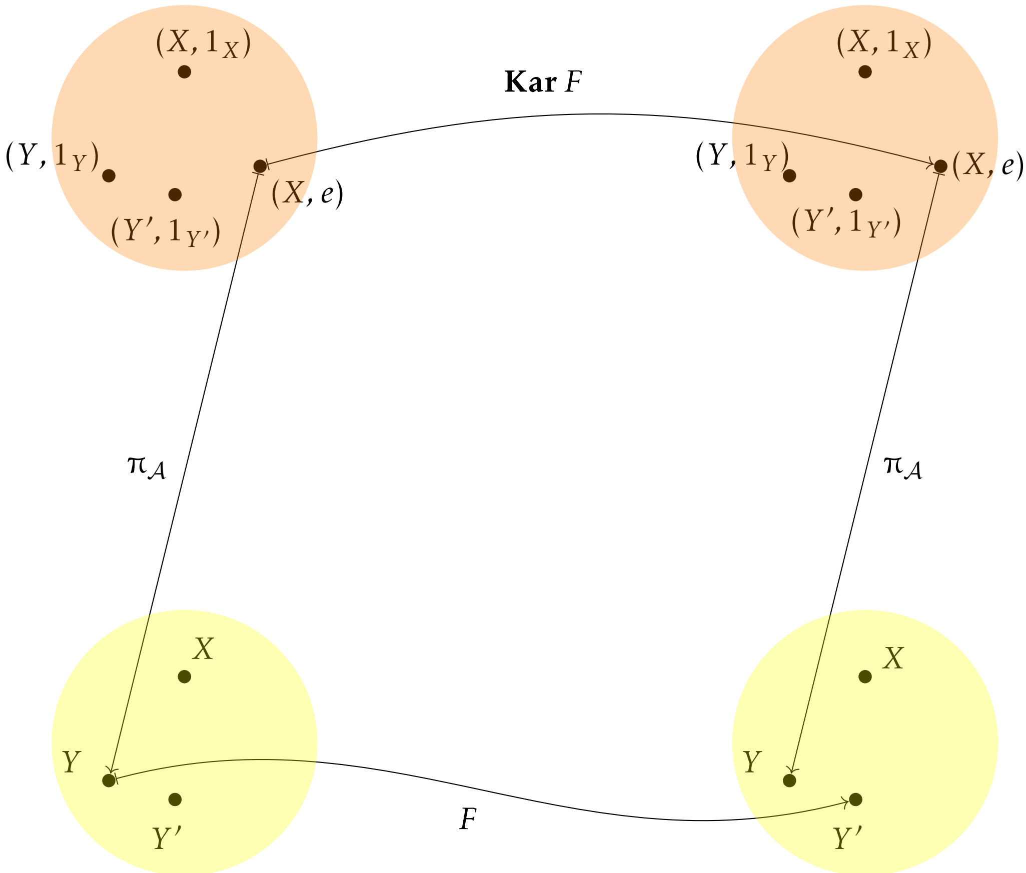
A.2.2. Strictness of $\mathbf{Kar} \dashv U$

Sketch A.2.7. One may wonder if it is possible to make the 2-adjunction $\mathbf{Kar} : \mathbf{Cat} \rightleftarrows \mathbf{Cat}^{\mathbf{ic}} : U$ (of Theorem 6.1.20) strict. In short, it seems plausible to define a strict 2-adjunction $\widetilde{\mathbf{Kar}} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$. However, it seems that this would not generalise to higher categories.

First, to see why $\mathbf{Kar} \dashv U$ is not strict, let us fix the category $\mathcal{A} \in \mathbf{2Cat}^{\mathbf{ic}}$ generated (Definition A.1.24) by the morphisms



where e is idempotent and has two splittings Y and Y'. Then $\pi_{\mathcal{A}}$ depends on the choice of $\pi_{\mathcal{A}}(X, e)$. W.l.o.g. we may choose $\pi_{\mathcal{A}}(X, e) := Y$. Then by considering a functor $F : \mathcal{A} \longrightarrow \mathcal{A}$ that maps $Y \longmapsto Y'$, we see that



implies that

$$\begin{tikzcd} \mathbf{Kar}\,\mathcal{A} \arrow[r, "\mathbf{Kar}\,F"] \arrow[d, two heads, "\pi_{\mathcal{A}}"'] & \mathbf{Kar}\,\mathcal{A} \arrow[d, two heads, "\pi_{\mathcal{A}}"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{A} \end{tikzcd}$$

does not commute. Thus, the 2-natural equivalence $\pi : \mathbf{Kar} \otimes U \Longrightarrow \mathrm{id}_{\mathbf{Cat}^{\mathrm{ic}}}$ is not strict, which implies that the 2-adjunction $\mathbf{Kar} : \mathbf{Cat} \rightleftarrows \mathbf{Cat}^{\mathbf{ic}} : U$ is not strict.

Since we previously ran into trouble having to make a choice $\pi_{\mathcal{A}}(X, e)$, we may attempt to redefine π by defining an "alternative Karoubi envelope" $\widetilde{\mathbf{Kar}} : \mathbf{Cat} \longrightarrow \mathbf{2Cat}^{\mathbf{ic}}$ such that $\widetilde{\mathbf{Kar}}\mathcal{C} \subseteq \mathbf{Kar}\,\mathcal{C}$ for all $\mathcal{C} \in \mathbf{Cat}$, but also $(X, e) \notin \mathbf{Kar}\,\mathcal{A}$. Moreover, $\widetilde{\mathbf{Kar}}\mathcal{C} \in \mathbf{2Cat}^{\mathbf{ic}}$ suggests $\mathbf{im}(\iota_{\mathcal{C}}) \subseteq \widetilde{\mathbf{Kar}}\mathcal{C}$. Recalling that we effectively construct $\mathbf{Kar}\,\mathcal{C}$ by adding splittings to $\mathcal{C}$ for each idempotent in $\mathcal{C}$, we may conclude that we should construct $\widetilde{\mathbf{Kar}}\mathcal{C}$ by only adding splittings $(\widetilde{X}, \widetilde{e})$ for idempotents $\widetilde{e} : \widetilde{X} \longrightarrow \widetilde{X}$ that do not split in $\mathcal{C}$ and that we should only add one splitting $(\widetilde{X}, \widetilde{e})$ for each isomorphism class $[(\widetilde{X}, \widetilde{e})] \subseteq \mathbf{Kar}\,\mathcal{C}$. This guarantees $\widetilde{\mathcal{A}} = \widetilde{\mathbf{Kar}}\widetilde{\mathcal{A}}$ for all $\widetilde{\mathcal{A}} \in \mathbf{Cat}^{\mathbf{ic}}$, which guarantees that $\widetilde{\pi}_{\widetilde{\mathcal{A}}} : \widetilde{\mathbf{Kar}}\widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ is strict.

Theorem 6.1.16 suggests that $\widetilde{\mathbf{Kar}} : \mathbf{Cat} \longrightarrow \mathbf{Cat}^{\mathbf{ic}}$ is a well-defined strict 2-functor. Thus, $\widetilde{\mathbf{Kar}} : \mathbf{Cat} \rightleftarrows \mathbf{Cat}^{\mathbf{ic}} : U$ should be a strict 2-adjunction. However, the utility of this possibility is questionable, because we would want to construct $\widetilde{\mathbf{2Kar}}$ analogously. However, this does not seem feasible since $\widetilde{\mathbf{Kar}}\mathcal{C} \subseteq \mathbf{Kar}\,\mathcal{C}$ is a partially "skeletal" Karoubi envelope. Therefore, it does not seem plausible that this approach would extend to higher categories.

A.2.3. (Non-)Existence of Algebra Completions

Proposition A.2.8. *"Algebra completions" à la condensation completions (Definition 6.2.5) generally do not exist.*

Proof. Let us unpack the statement first. Let $\widetilde{\mathbf{2Cat}}$ be a sub-3-category of $\mathbf{2Cat}$ that we will specify shortly. A 2-category $\mathcal{B} \in \widetilde{\mathbf{2Cat}}$ is "algebra complete" if all algebras in $\mathcal{B}$ split and $\mathbf{2Cat}^{\mathrm{ac}} \subset \widetilde{\mathbf{2Cat}}$ is the full sub-3-category of $\widetilde{\mathbf{2Cat}}$ containing all algebra complete 2-categories. The statement is that universal 2-functors $(\overline{\mathcal{B}} \in \mathbf{2Cat}^{\mathrm{ac}}, \iota_{\mathcal{B}} : \mathcal{B} \longrightarrow \overline{\mathcal{B}})$ from $\mathcal{B} \in \widetilde{\mathbf{2Cat}}$ to the forgetful 3-functor $\mathbf{2Cat}^{\mathrm{ac}} \hookrightarrow \widetilde{\mathbf{2Cat}}$ do not generally exists. We conduct this proof via a counterexample, i.e. we present 2-categories $\mathcal{B} \in \widetilde{\mathbf{2Cat}}$ and $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{ac}}$ such that no $\overline{\mathcal{B}} \in \mathbf{2Cat}^{\mathrm{ac}}$ exists that would make

$$(-) \,\square\, \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathrm{ac}}(\overline{\mathcal{B}}, \mathcal{A}) \longrightarrow \widetilde{\mathbf{2Cat}}(\mathcal{B}, \mathcal{A})$$

a 2-equivalence.

Our counterexample consists of the "walking algebra" $\mathcal{I} \in \widetilde{\mathbf{2Cat}}$ and the "walking algebra with two splittings" $\mathcal{I}'' \in \mathbf{2Cat}^{\mathrm{ac}}$ (Remark 5.2.20). Therefore, $\widetilde{\mathbf{2Cat}}$ can be any sub-3-category of $\mathbf{2Cat}$ that contains $\mathcal{I}$ and $\mathcal{I}''$. Clearly there exists a 2-functor $F : \mathcal{I} \longrightarrow \mathcal{I}''$ that maps the algebra $A : a \longrightarrow a$ in $\mathcal{I}$ to the algebra FA of $\mathcal{I}''$. Let us assume that we are given an algebra completion $(\,\overline{\mathcal{I}} \in \mathbf{2Cat}^{\mathrm{ac}}, \iota_{\mathcal{I}} : \mathcal{I} \longrightarrow \overline{\mathcal{I}}\,)$ of $\mathcal{I}$, then $\iota_{\mathcal{I}}$ must map the algebra of $\mathcal{I}$ to a split algebra in $\overline{\mathcal{I}}$. Since $\overline{\mathcal{I}}$ is the algebra completion of $\mathcal{I}$, there must exist a 2-functor $\overline{F} : \overline{\mathcal{I}} \longrightarrow \mathcal{I}''$ such that $\overline{F} \,\square\, \iota_{\mathcal{I}} = F$. F must map the splitting of the algebra $\iota_{\mathcal{I}} A$ to one of two splittings of the algebra FA. However, this implies that there is also a 2-functor $\widetilde{F} : \overline{\mathcal{I}} \longrightarrow \mathcal{I}''$ that maps to the other splitting of FA and fulfils $\widetilde{F} \,\square\, \iota_{\mathcal{I}} = F$. These 2-functors are displayed in this non-commutative diagram:

$$\begin{tikzcd}
\mathcal{I} \arrow[rr, "F"] \arrow[d, hook, "\iota_{\mathcal{B}}"'] & & \mathcal{I}'' \\
\overline{\mathcal{I}} \arrow[urr, bend left=10, "\overline{F}"] \arrow[urr, bend right=10, "\widetilde{F}"'] & &
\end{tikzcd}$$

We note that $\mathcal{I}''$ does not contain non-trivial equivalences. Therefore, all 2-natural equivalences between 2-functors $\overline{\mathcal{I}} \longrightarrow \mathcal{I}''$ must have unit 1-morphisms as their 1-morphism components. Continuing this realisation one finds

$$\overline{F} \neq \widetilde{F},$$

which implies that

$$(-) \,\square\, \iota_{\mathcal{B}} : \hom(\overline{F}, \widetilde{F}) \longrightarrow \hom(F, F)$$

is not an equivalence. Thus,

$$(-) \,\square\, \iota_{\mathcal{B}} : \mathbf{2Cat}^{\mathrm{ac}}(\,\overline{\mathcal{I}}, \mathcal{I}'') \longrightarrow \widetilde{\mathbf{2Cat}}(\mathcal{I}, \mathcal{I}'')$$

is not essentially full and, therefore, not a 2-equivalence. □

Sketch A.2.9. We keep in mind that Proposition A.2.8 follows our construction of condensation completions. Possibly there exists some setting and (possibly lax) notion of universality that allows algebra completions.

Our proof basically used that a "walking algebra with one splitting" is not 2-equivalent to a walking algebra with two splittings. Therefore, maybe a 2-category $\mathcal{B} \in \widetilde{\mathbf{2Cat}}$ should only be "algebra complete" if all algebras in $\mathcal{B}$ have "essentially unique" splittings. Specifically, we may define $\mathbf{2Cat}^{\mathrm{ac}} \subseteq \widetilde{\mathbf{nCat}}$ as the full sub-3-category consisting of all 2-categories $\mathcal{B} \in \widetilde{\mathbf{nCat}}$ such that

$$(-) \,\square\, \iota_{\mathcal{I}}\ \mathbf{2Cat}^{\mathrm{ac}}(\mathcal{I}', \mathcal{B}) \longrightarrow \widetilde{\mathbf{2Cat}}(\mathcal{I}, \mathcal{B})$$

is a 2-equivalence. Here, $\mathcal{I}'$ denotes the "walking split algebra" and $\iota_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathcal{I}'$ denotes the forgetful 2-functor. Then our counterexample would no longer work since this redefinition implies $\mathcal{I}'' \notin \mathbf{2Cat}^{\mathrm{ac}}$.

However, this redefinition still has some problems. If $\mathcal{B} \in \widetilde{\mathbf{2Cat}}$ contains an algebra with multiple non-equivalent splittings and $\mathcal{B} \hookrightarrow \mathcal{B}'$, then $\mathcal{B}' \notin \mathbf{2Cat}^{\mathrm{ac}}$, which suggests that $\mathcal{B}$ still has no algebra completion. Therefore, we should maybe not redefine $\mathbf{2Cat}^{\mathrm{ac}}$, but rather redefine $\widetilde{\mathbf{2Cat}}$ to contain only 2-categories whose algebras each have either zero splittings or an "essentially unique" splitting. This redefinition of $\widetilde{\mathbf{2Cat}}$ of course also implies that splittings of algebras in $\mathcal{A} \in \mathbf{2Cat}^{\mathrm{ac}}$ are "essentially unique".

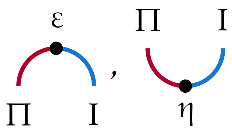
This restriction on $\widetilde{\mathbf{2Cat}}$ effectively forces a version of [GJF19, Thm. 2.3.2] adapted to our setting. Therefore, if we define $\widetilde{\mathbf{2Cat}} \subset \mathbf{2Cat}$ as the full sub-2-category, whose algebras each have either zero splittings or an "essentially unique" splitting, then it becomes more reasonable that algebra completions may exist. However, one would still have to check that the construction of completions $\overline{\mathcal{B}} \in \mathbf{2Cat}^{\mathrm{ac}}$ (Sketch 6.2.7) generalises well to this setting, because we still need $\overline{\mathcal{B}}$ to be universal.

The definition of $\widetilde{\mathbf{2Cat}}$ naturally generalises to arbitrary n-idempotents and their splittings. Essentially, this is a weaker but less vague definition than discussed in Remark 7.2.15, so this could possibly be useful for determining the existence of *n-idempotent completions*.

A.2.4. Unitality of Condensations

Lemma A.2.10 ([GJF19, Prop. 3.1.5]). *If a 2-category $\mathcal{B}$ has a 2-condensation $a \rightharpoonup\!\!\!\rightarrow b$ given by*

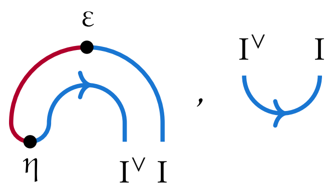
$$\left(a, b, a \xrightarrow{\Pi} b, b \xrightarrow{\mathrm{I}} a, \right.$$



$$\left. \right) \qquad \text{(A.5)}$$

and there is an adjunction $\mathrm{I} \dashv \mathrm{I}^\vee$, then there is also a condensation $a \rightharpoonup\!\!\!\rightarrow b$ whose 2-condensation algebra $\mathrm{I} \otimes \mathrm{I}^\vee$ has a unit:

$$\left(a, b, a \xrightarrow{\mathrm{I}^\vee} b, b \xrightarrow{\mathrm{I}} a, \right.$$



$$\left. \right) \qquad \text{(A.6)}$$

Analogous statements hold for other adjunctions and for constructing counits instead of units.

Note A.2.11. If a 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ has the condensation (A.5), Π has a right adjoint, and we set $A := \mathrm{I} \otimes \Pi$, then there exists a condensation algebra $A^\vee \otimes_A A : a \longrightarrow a$ in $\mathcal{B}$ with a unit that constitutes a condensation $a \rightharpoonup\!\!\!\rightarrow b$.

Corollary A.2.12 ([GJF19, Thm. 3.1.7]). *In particular, if a 2-category $\mathcal{B} \in \mathbf{2Cat}^{\mathbf{ic}}$ has a condensation algebra $A : a \longrightarrow a$ and an adjunction $A \dashv A^\vee$, then there exists in a condensation algebra $A^\vee \otimes_A A : a \longrightarrow a$ with a unit such that $(A, a) \simeq (A^\vee \otimes_A A, a)$ in $\mathbf{2Kar}\,\mathcal{B}$.*

Proposition A.2.13 ([GJF19, Thm. 3.3.3]). *If $\mathcal{B}$ is a 2-category with right adjoints, then $\mathcal{B}^\triangledown \simeq \mathbf{2Kar}\,\mathcal{B}$.*

Proof. As a reminder, $\mathcal{B}^\triangledown$ is the full sub-2-category $\mathcal{B}^\triangledown \subseteq \mathbf{2Kar}\,\mathcal{B}$ whose objects are 2-condensation algebras with units in $\mathcal{B}$ (Remark 6.2.30). Then Corollary A.2.12 implies that $\mathcal{B}^\triangledown \hookrightarrow \mathbf{2Kar}\,\mathcal{B}$ is essentially surjective. □

Sketch A.2.14. Recall that $\mathcal{B}_{\mathrm{eq}}$ is the full sub-2-category $\mathcal{B}_{\mathrm{eq}} \subseteq \mathbf{2Kar}\,\mathcal{B}$ whose objects are 2-condensation algebras with units and counits in $\mathcal{B}$ (Remark 6.2.30). [Fra22, Cor. 3.47] claims that $\mathbf{2Kar}\,\mathcal{B} \simeq \mathcal{B}_{\mathrm{eq}}$ if $\mathcal{B} \in \mathbf{2Cat^{ic}}$ has right (or left) adjoints. The argument given is that condensation algebras $A : a \longrightarrow a$ in $\mathcal{B}$ condense both onto the condensation algebra with unit $(A^\vee \otimes_A A, a)$ and onto the condensation algebra with counit $(A \otimes_A A^\vee, a)$ (cf. Lemma A.2.10). The implied argument is that Lemma A.2.10 would imply the existence of condensation algebras with both units and counits. This would be true, if we had an isomorphism of condensation algebras in $\mathcal{B}$, but we only have an equivalence in our case. Unfortunately, using Lemma A.2.10 to construct units/counits does not preserve already present counits/units, because the construction changes both η as well as ε; cf. (A.6). Therefore, the given argument is insufficient. However, one may try to remedy the situation.

One may wonder if $\mathbf{2Kar}\,\mathcal{B} \simeq \mathcal{B}_{\mathrm{eq}}$ is true for pivotal $\mathcal{B} \in \mathbf{2Cat^{ic}}$. However, calculations in $\mathbf{2Kar}\,\mathcal{B}$ with A and $A^\vee$ can be quite treacherous — e.g. in general $A \otimes_A A^\vee \not\simeq A$ — so let us consider pivotal $\mathcal{B} \in \mathbf{2Cat^{cc}}$. Then $\mathbf{2Kar}\,\mathcal{B} \simeq \mathcal{B}$, so instead of working in $\mathbf{2Kar}\,\mathcal{B}$, we may work in $\mathcal{B}$. So, let us assume that $\mathcal{B}$ is the "walking pivotal 2-condensation", i.e. the 2-category generated by the data (A.5) and freely generated adjoints and where we impose fulfilment of strict pivotality conditions. Then a Frobenius algebra $A : a \to a$ with a unit and a counit and splitting b must consist of adjunction data. However, η and ε are not part of adjunctions, so the multiplication and comultiplication of A are constructed from the freely generated adjunction data. However, there does not seem to be any condition that would impose Δ-separability on such a Frobenius algebra A. Therefore, it seems that in general

$$\mathbf{2Kar}\,\mathcal{B} \not\simeq \mathcal{B}_{\mathrm{eq}}$$

even if $\mathcal{B} \in \mathbf{2Cat^{ic}}$ is pivotal.

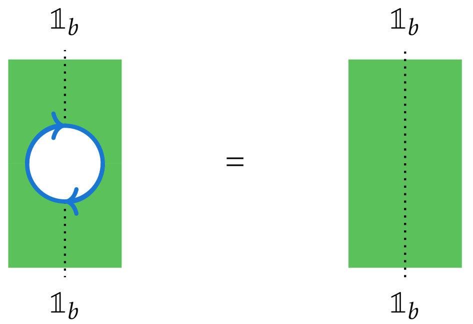
A.2.5. n-Orbifold Data

Sketch A.2.15. It seems reasonable that it should be possible to define n-orbifold data and their condensations analogously to n-condensation algebras and their condensations. Explicitly, condensations are bubbles of duality data that vanish, so if we know what "n-pivotality"[110] is, then we should be able to define *n-orbifold condensations* analogously to n-condensations (Sketch 7.2.2).

For instance, let us assume that such an "n-pivotal" n-category $\mathcal{C}$ must have the property that its k-morphisms each have chosen $n-k$ adjoints for all $k < n$ and that the adjunctions fulfil some coherence conditions. Then, one might expect an *n-orbifold condensation* to be recursively definable:

- A 0-*orbifold condensation* is an equality between two elements of a set.
- Given two objects $a, b \in \mathcal{C}$, then an n-condensation $a \twoheadrightarrow b$ is a 1-morphisms
$$X : a \longrightarrow b$$
together with an $(n-1)$-condensation
$$X \otimes X^{\vee} \twoheadrightarrow\!\!\!\!\Rightarrow \mathbb{1}_b$$
given by the adjunction data.

In other words, an n-orbifold condensation should consist of adjunction data that may form n-balls of a lying in b that vanish (Intuition 7.2.3), e.g.:



[110] This is absolutely just a term I made up for convenience. For strict n-categories, "n"-pivotality naïvely seems achievable. Namely, pivotal 2-categories have 1-morphisms with adjoints and 2-morphisms may be seen as being associated to cycles of 1-morphisms (Remark 3.3.8). Similarly, strict "n-pivotal" n-categories should have k-morphisms associated to $(k-1)$-spheres of $(k-1)$-morphisms. However, generalising this to general n-categories requires at least a definition of general n-categories.

n-orbifold data should then analogously be definable via walking constructions. If $\mathcal{I}$ is the "walking n-orbifold condensation" consisting of two objects $a, b \in \mathcal{I}$ with an n-condensation $a \twoheadrightarrow b$, then the full sub-n-category $\langle a \rangle \subset \mathcal{I}$ should be the "walking n-orbifold datum". If we are given an "n-pivotal" n-category $\mathcal{D}$, an *n-orbifold datum* in $\mathcal{D}$ should be an "n-pivotal" n-functor $\mathcal{I} \longrightarrow \mathcal{D}$.

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