Gini index as a threshold for tQST

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A relevant point in the tQST protocol is the choice of the threshold, which determine the number of measurements to perform [1]. In this notes we provide a closed formula for the threshold given the diagonal elements of the density matrix.

In [2] the authors analyze different measures of sparsity. Among the many possible sparsity measures the Gini index satisfies the criteria we need to implement a threshold.

Let $\underline{c} = [c_1, c_2, \cdots, c_N]$, and consider the sorted components $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(N)}$. We define the p-norm as:

$$\|\underline{c}\|_p = \left(\sum_i c_i^p \right)^{\frac{1}{p}}, 0 \leq p \leq 1. \tag{1}$$

The Gini index is defined as:

$$\mathrm{GI}\left(\underline{c}\right) = 1 - 2 \sum_{k=1}^{N} \frac{c_{(k)}}{\|\underline{c}\|_1} \left(\frac{N - k + \frac{1}{2}}{N} \right). \tag{2}$$

Let us briefly analyze the range of values that the Gini index can have.
Consider a vector where there is only one non-vanishing element, equal to A, *i.e.*, the most sparse vector. If we sort it as required, the index of the non-vanishing element is N, and $c_{(N)} = A$, $\|\underline{c}\|_1 = A$. Thus:

$$\begin{aligned} \mathrm{GI}\left(\underline{c}\right) &= 1 - 2\frac{A}{A} \left(\frac{N - N + \frac{1}{2}}{N} \right) \\ &= 1 - \frac{1}{N}. \end{aligned} \tag{3}$$

Consider now the least sparse vector, where all the elements are $1/N$. Then we have $c_{(k)}/\|\underline{c}\|_1 = 1/N \ \ \forall k$, and:

$$\mathrm{GI}\left(\underline{c}\right) = 1 - 2\frac{1}{N} \left(\sum_{k=1}^{N} \frac{N - k + \frac{1}{2}}{N} \right) = 0 \tag{4}$$

We conclude that:

$$0 \leq \mathrm{GI}\left(\underline{c}\right) \leq 1 - \frac{1}{N}. \tag{5}$$

1 The Gini index as a threshold for tQST

We now adapt the previous definition of the Gini index to make it a suitable threshold for tQST. First of all, we need to define what is the vector used to compute the Gini index. Based on the tQST protocol, it is the diagonal of the density matrix, which is measured at the beginning of the tomography protocol. Let N be the dimension of the system, equal to the length of the diagonal of the density matrix. We set the threshold:

$$t = \frac{\mathrm{GI}\left(\underline{\rho}\right)}{N-1}, \tag{6}$$

with $\underline{\rho} = (\rho_{11}, \rho_{22}, \cdots, \rho_{NN})$.
In the file `density_matrix_tool.py` we define a function `gini_index` that compute and returns the threshold according to (6).

References

[1] Daniele Binosi, Giovanni Garberoglio, Diego Maragnano, Maurizio Dapor, and Marco Liscidini. A tailor-made quantum state tomography approach, 2024.

[2] Niall P. Hurley and Scott T. Rickard. Comparing measures of sparsity, 2009.