# Notes on Kitaev model

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**Abstract**

We summarize main results in [1]. Backgrounds and more information needed to understand this work is also included.

# Contents



# 1 The model and the solution

## 1.1 The model

Defined on a honeycomb lattice with 'even' and 'odd' sublattices and links of type $x, y$ and $z$ (see Fig. 1), the Hamiltonian of the model is

$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z, \tag{1}$$

where $J_x, J_y, J_z$ are model parameters.

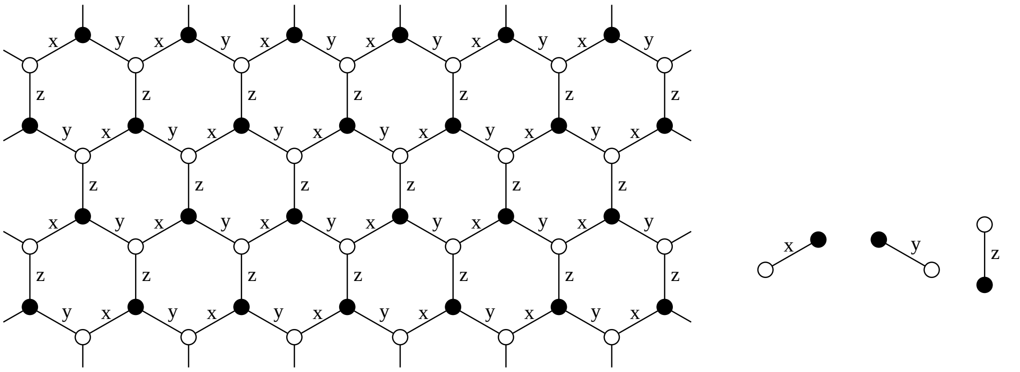


**Figure 1:** Lattice for the model. 'even' lattice is shown in empty circles and 'odd' shown in full circles. Links of types $x, y$ and $z$ are marked and illustrated especially on the right.

For individual terms in the model, we define

$$K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x, & \text{if } (j,k) \text{ is an } x\text{-link;} \\ \sigma_j^y \sigma_k^y, & \text{if } (j,k) \text{ is a } y\text{-link;} \\ \sigma_j^z \sigma_k^z, & \text{if } (j,k) \text{ is a } z\text{-link;} \end{cases} \tag{2}$$

Define operators $W_p$ on plaquettes as

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = K_{12} K_{23} K_{34} K_{45} K_{56} K_{61}. \tag{3}$$

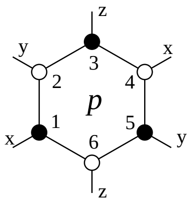



**Figure 2:** Conventions for Eq. 3.

See Fig. 2 for conventions of indices. Remarkably, $W_p$ commutes with all $K_{ij}$ and hence all other $W_{p'}$ and the Hamiltonian. Therefore, for all plaquettes $p$, $W_p$ is a set of 'integrals of motion'. If the lattice has $m$ plaquettes, the total Hilbert space $\mathcal{L}$ can be decomposed into sectors

$$\mathcal{L} = \bigoplus_{w_1,\ldots,w_m} \mathcal{L}_{w_1,\ldots,w_m}. \tag{4}$$

Each $w_p = \pm 1$ is the eigenvalue of $W_p$.

### 1.2 Representing spins by Majorana operators

**¶ A general transformation** Given fermion creation and annihilation operators $a_k^\dagger$ and $a_k$ for $n$ fermionic modes, where $k = 1, \ldots, n$, we can define Majorana operators as

$$c_{2k-1} = a_k + a_k^\dagger, \quad c_{2k} = \frac{a_k - a_k^\dagger}{\mathrm{i}}. \tag{5}$$

They have anti-commutation relations

$$\{c_i, c_j\} = 2\delta_{ij}. \tag{6}$$

We represent one spin by two fermionic modes, i.e., four Majorana modes, denoted as $b^x, b^y, b^z, c$ instead of $c_1, c_2, c_3, c_4$. The Majorana operators act on the 4-dimensional Fock space $\widetilde{\mathcal{M}}$, whereas the Hilbert space of a spin is a two-dimensional subspace $\mathcal{M} \subset \widetilde{\mathcal{M}}$ defined by condition

$$|\xi\rangle \in \mathcal{M} \text{ if and only if } D\,|\xi\rangle = |\xi\rangle\,, \text{ where } D = b^x b^y b^z c. \tag{7}$$

The Pauli operators are represented by $\widetilde{\sigma}^x, \widetilde{\sigma}^y, \widetilde{\sigma}^z$,

$$\widetilde{\sigma}^x = \mathrm{i}b^x c, \quad \widetilde{\sigma}^y = \mathrm{i}b^y c, \quad \widetilde{\sigma}^z = \mathrm{i}b^z c, \tag{8}$$

such that: (1) they preserve the subspace $\mathcal{M}$; (2) when restricted to $\mathcal{M}$, they obey the commutation relations as Pauli operators.

If we identify $b^x, b^y, b^z, c$ with $c_1, c_2, c_3, c_4$, The operator $D$ gets the form

$$D = -\left(2n_1 - 1\right)\left(2n_2 - 1\right), \tag{9}$$

so that $\mathcal{M} = \mathrm{span}\{|01\rangle, |10\rangle\}$. The spin operators under basis $|00\rangle, |10\rangle, |01\rangle, |11\rangle = a_1^\dagger a_2^\dagger |00\rangle$ are

$$\widetilde{\sigma}^x = \left(a_1 + a_1^\dagger\right)\left(a_2 - a_2^\dagger\right) = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, \tag{10}$$

$$\widetilde{\sigma}^y = -\mathrm{i}\left(a_1 - a_1^\dagger\right)\left(a_2 - a_2^\dagger\right) = \begin{pmatrix} & & & \mathrm{i} \\ & & \mathrm{i} & \\ & -\mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix}, \tag{11}$$

$$\widetilde{\sigma}^z = \left(a_2 + a_2^\dagger\right)\left(a_2 - a_2^\dagger\right) = 2n_2 - 1 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{12}$$

which, restricted to subspace spanned by basis $|01\rangle, |10\rangle$, recover the original Pauli matrices.

For a multi-spin system, the spin-Majorana dictionary is

$$\begin{gathered} \widetilde{\sigma}^{\alpha_j} = \mathrm{i} b_j^\alpha c_j, \quad D_j = b_j^x b_j^y b_j^z c_j; \\ |\xi\rangle \in \mathcal{L} \text{ if and only if } D_j |\xi\rangle = |\xi\rangle, \text{ where } D = b^x b^y b^z c; \\ H\left\{\sigma_j^\alpha\right\} \to \widetilde{H}\left\{b_j^\alpha, c_j\right\} = H\left\{\widetilde{\sigma}^{\alpha_j}\right\}. \end{gathered} \tag{13}$$

Here $\mathcal{L}$ is the physical Hilbert space. The resulting Hamiltonian $\widetilde{H}$ commutes with $D_j$, so that $\mathcal{L}$ is invariant under $\widetilde{H}$. Since $D_j$ preserves the physical space while $D_j^2 = 1$, models in the extended space can be viewed to have a $\mathbb{Z}_2$ gauge while $D_j$'s are gauge transformation operators.

¶ **Application to the concrete model** Applying the transformation to Hamiltonian (1), we have

$$K_{ij} \to \widetilde{K}_{ij} = -\mathrm{i}\left(\mathrm{i} b_j^\alpha b_k^\alpha\right) c_j c_k; \tag{14}$$

$$H \to \widetilde{H} = \frac{\mathrm{i}}{4} \sum_{j,k} \widehat{A}_{jk} c_j c_k, \quad \widehat{A}_{jk} = \begin{cases} 2J_{\alpha_{jk}} \widehat{u}_{jk} & \text{if } j \text{ and } k \text{ are connected,} \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

$$\widehat{u}_{jk} = \mathrm{i} b_j^{\alpha_{jk}} b_k^{\alpha_{jk}}. \tag{16}$$

Operators $\widehat{u}_{jk}$ commutes with each other and with the Hamiltonian, so the extended Hilbert space can be decomposed as $\widetilde{\mathcal{L}} = \bigoplus_u \widetilde{\mathcal{L}}_u$, where $u$ represents a collection of eigenvalues $u_{jk} = \pm 1$ of all $\widehat{u}_{jk}$'s. Restricting the Hamiltonian in a subspace $\widetilde{\mathcal{L}}_u$ is achieved by replacing all $\widehat{u}_{jk}$'s with their eigenvalues $u_{jk}$'s, producing $\widetilde{H}_u = (\mathrm{i}/4) \sum_{j,k} A_{jk} c_j c_k$, which is a free fermionic model. The eigenstate $\left|\widetilde{\Psi}_u\right\rangle$ of $\widetilde{H}_u$ can be solved exactly and mapped to the physical space by

$$|\Psi_w\rangle = \prod_j \left(\frac{1 + D_j}{2}\right) \left|\widetilde{\Psi}_u\right\rangle \in \mathcal{L}. \tag{17}$$

Here $w$ is the equivalence class of $u$ under gauge transformations. For a planar lattice it can be represented by

$$w_p = \prod_{(j,k) \in \partial p} u_{jk}, \quad (j \in \text{even sublattice}, k \in \text{odd sublattice}). \tag{18}$$

This coincides with $w_p$'s in Eq. (4).

¶ **Path and loop operators** We may interpret $w_p$ as the magnetic flux of plaquette $p$ in the context of $\mathbb{Z}_2$ gauge fields. We say that the plaquette carries a vortex if $w_p = -1$. For further use, we define a gauge invariant fermionic path operator

$$W(j_0, \dots, j_n) = K_{j_n j_{n-1}} \cdots K_{j_1 j_0} = \left( \prod_{s=1}^{n} -\mathrm{i}\widehat{u}_{j_s j_{s-1}} \right) c_n c_0. \tag{19}$$

If $j_n = j_0$, the path operator is called a Wilson loop, which can be viewed as a generalization of magnetic flux.

## 1.3 Quadratic Hamiltonians

**¶ Convention on $\boldsymbol{H}$** We have reduced the original model to a quadratic Hamiltonian of Majorana fermions,

$$H(A) = \frac{\mathrm{i}}{4} \sum_{j,k} A_{jk} c_j c_k, \tag{20}$$

where $A_{jk}$ is a $2m \times 2m$ antisymmetric matrix. The convention $\mathrm{i}/4$ is chosen so that $-\mathrm{i}H$ is a representation of Lie algebra $\mathfrak{so}(2m)$, that is,

$$[-\mathrm{i}H(A), -\mathrm{i}H(B)] = -\mathrm{i}H([A, B]). \tag{21}$$

And therefore

$$\mathrm{e}^{-\mathrm{i}H(A)} c_k \mathrm{e}^{\mathrm{i}H(A)} = \sum_j c_j Q_{jk}, \tag{22}$$

where $Q = \mathrm{e}^{A}$.

**¶ Spectrum of the Hamiltonian** The Hamiltonian is solved by noticing that there exists $Q \in \mathrm{O}(2m)$ such that

$$A = Q \begin{pmatrix} 0 & \epsilon_1 & & & \\ -\epsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \epsilon_m \\ & & & -\epsilon_m & 0 \end{pmatrix} Q^T, \quad \epsilon_k \geq 0. \tag{23}$$

We can define

$$\left(b_1', b_1'', \dots, b_m', b_m''\right) = \left(c_1, c_2, \dots, c_{2m-1}, c_{2m}\right) Q, \tag{24}$$

and write the Hamiltonian into

$$H = \frac{\mathrm{i}}{2} \sum_{k=1}^{m} \epsilon_k b_k' b_k'' = \sum_{k=1}^{m} \epsilon_k \left( a_k^\dagger a_k - \frac{1}{2} \right), \tag{25}$$

where $a_k^\dagger = \left(b_k' - \mathrm{i}b_k''\right)/2$, $a_k = \left(b_k' + \mathrm{i}b_k''\right)/2$. Then the ground state $|\Psi\rangle$ satisfies for any $k$ that

$$a_k |\Psi\rangle = 0. \tag{26}$$

The ground state energy is given by

$$E = -\frac{1}{2} \sum_{k=1}^{m} \epsilon_k = -\frac{1}{4} \operatorname{tr} |\mathrm{i}A|. \tag{27}$$

Here the absolute value of Hermitian matrix is defined by spectrum.

Note that there are $m$ fermionic modes for a given flux configuration, yielding a Hilbert space of dimension $2^m$. There are $2^m$ flux configurations in total, which gives the total Hilbert space dimension $2^{2m}$ as desired.

¶ **Majorana modes and spectral projector** We now introduce a linear space $\mathbb{C}^{2m}$ consisting of

$$F(x) = \sum_j x_j c_j. \tag{28}$$

If all $x_j$'s are real, we call $F(x)$ (or $x$ itself) Majorana modes. Operators $b'_k, b''_k$ introduced above are called normal modes.

We want to find the subspace $L \in \mathbb{C}^{2m}$ such that if $z \in L$, $F(z)$ annihilates the ground state. This *space of annihilation operators* are spanned by coefficients in $a_1, \ldots, a_m$, thus of dimension $m$ if $A$ is not degenerate. Denote

$$S = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \left(b'_1, b''_1, \ldots, b'_m, b''_m\right), \quad \boldsymbol{c} = \left(c_1, c_2, \ldots, c_{2m-1}, c_{2m}\right), \tag{29}$$

and we have

$$\boldsymbol{a} \equiv (a_1, -\mathrm{i}a_1, \ldots, a_m, -\mathrm{i}a_m) = \boldsymbol{b}\frac{1}{2}(I - \mathrm{i}S) = \boldsymbol{c}\frac{1}{2}\left(I - \mathrm{i}QSQ^T\right)Q \equiv \boldsymbol{c}PQ. \tag{30}$$

Or more compactly,

$$P = \frac{1}{2}\left(I - \operatorname{sgn}(\mathrm{i}A)\right). \tag{31}$$

Therefore, 1) for any $x \in \mathbb{C}^{2m}$, $F(Px) = \boldsymbol{c}Px = \boldsymbol{a}Q^T x$, which annihilates the ground state, and that 2) $P$ has the same rank $m$ as $I - \mathrm{i}S$. So $P$ is the projector onto $L$, called the *spectral projector*. Note that for any $z, z' \in L$, $\sum_j z_j z'_j = 0$ since $P^T P = 0$.

### 1.4 The spectrum and the phase diagram

¶ **Spectrum of the model** For the original model (1), it is argued that the spectrum does not depend on signs of $J_x, J_y, J_z$ since changing a sign can be compensated by a gauge transformation.

According to a theorem proved by Lieb [2], the configuration of $w_p$ that minimizes the ground state energy is that without vortices, i.e., $w_p = 1$ for all plaquettes $p$. We take the standard configuration $u^{\mathrm{std}}_{jk} = 1$ for all links to achieve this vortex-free requirement. Now the Hamiltonian becomes

$$H_{\text{vortex-free}} = \frac{\mathrm{i}}{4}\sum_{j,k} A_{jk} c_j c_k, \quad A_{jk} = 2J_{\alpha_{jk}}. \tag{32}$$

It has a translational symmetry. Choose a unit cell as Fig. (3) shows, we can Fourier transform the Hamiltonian according to

$$a_{\boldsymbol{q},\lambda} = \frac{1}{\sqrt{2m}}\sum_s \mathrm{e}^{-\mathrm{i}(\boldsymbol{q},\boldsymbol{r}_s)c_{s,\lambda}}, \quad \widetilde{A}_{\lambda\mu}(\boldsymbol{q}) = \sum_s \mathrm{e}^{\mathrm{i}(\boldsymbol{q},\boldsymbol{r}_s)} A_{0\lambda,s\mu}. \tag{33}$$

Here $s$ is an index for unit cells and $\lambda, \mu$ are indices for bases. The resulting Hamiltonian is

$$H = \frac{1}{2}\sum_{\boldsymbol{q},\lambda,\mu} \mathrm{i}\widetilde{A}_{\lambda\mu}(\boldsymbol{q})\, a_{-\boldsymbol{q},\lambda} a_{\boldsymbol{q},\mu}, \tag{34}$$

where

$$\mathrm{i}\widetilde{A}(\boldsymbol{q}) = \begin{pmatrix} 0 & \mathrm{i}f(\boldsymbol{q}) \\ -\mathrm{i}f^*(\boldsymbol{q}) & 0 \end{pmatrix}, \quad f(\boldsymbol{q}) = 2\left(J_x \mathrm{e}^{\mathrm{i}(\boldsymbol{q},\boldsymbol{n}_1)} + J_y \mathrm{e}^{\mathrm{i}(\boldsymbol{q},\boldsymbol{n}_2)} + J_z\right). \tag{35}$$

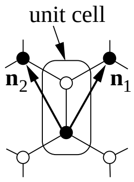



**Figure 3:** Unit cell. The basis of translation group is chosen as $\boldsymbol{n}_1 = (1/2, \sqrt{3}/2)$, $\boldsymbol{n}_2 = (-1/2, \sqrt{3}/2)$.

The spectrum is given by eigenvalues of $\mathrm{i}\widetilde{A}(\boldsymbol{q})$, that is, $\epsilon(q) = \pm|f(\boldsymbol{q})|$. It is a 'double spectrum' because there are pairs of $\pm\epsilon(\boldsymbol{q})$. We can take only non-negative eigenvalues to get a 'single spectrum'.

¶ **The phase diagram** According to whether the spectrum is gapped or gapless, that is, whether there is a point $\boldsymbol{q}$ at which $\epsilon(\boldsymbol{q}) = 0$, there are different phases. The spectrum is gapless if and only if $|J_x|$, $|J_y|$, $|J_z|$ can form a triangle,

$$|J_x| \le |J_y| + |J_z|, \quad |J_z| \le |J_x| + |J_y|, \quad |J_y| \le |J_z| + |J_x|. \tag{36}$$

The phase diagram for $J_x, J_y, J_z \ge 0$ is shown in Fig. 4. If the inequality is strict, there are two points $\boldsymbol{q} = \pm\boldsymbol{q}^*$. At the points $\pm\boldsymbol{q}^*$, the spectrum has conic singularities $\epsilon(\boldsymbol{q}) = \pm\sqrt{g_{\alpha\beta}\delta q_\alpha \delta q_\beta}$, where $\delta\boldsymbol{q} = \boldsymbol{q} \mp \boldsymbol{q}^*$. (YF: Why is it called a singularity?)

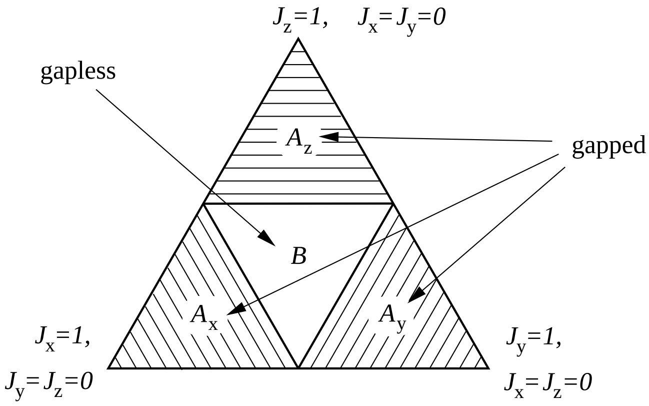



**Figure 4:** Phase diagram of the model. The triangle is the section of the positive octant $J_x, J_y, J_z \ge 0$ by the plane $J_x + J_y + J_z = 1$. Copied from [1].

# 2 Properties of the gapped phases

## 2.1 Perturbation theory study

We study the gapped phase by perturbative calculation. The Hamiltonian is $H = H_0 + V$ where

$$H_0 = -J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z, \quad V = -J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z. \tag{37}$$

We assume that $J_z > 0$ and $|J_x|, |J_y| \ll J_z$. Therefore the model is gapped and we are in a perturbative regime. The ground states, in which case each two spins connected by a $z$-link are aligned and form an effective spin, are highly degenerate with $E_0 = -mJ_z$. We denote the Hilbert subspace of ground states $\mathcal{L}_{\text{eff}}$ and we want to have an effective Hamiltonian $H_{\text{eff}}$ on it. Note that a typical basis in this ground state subspace does not have any well-defined $w_p$ configuration.

¶ **General approach** Denote $\Upsilon : \mathcal{L}_{\text{eff}} \hookrightarrow \mathcal{L}$ the embedding of the ground state subspace to the whole Hilbert space. Its conjugate is defined by $\langle \Upsilon^\dagger A, b \rangle = \langle A, \Upsilon b \rangle$, which is just the projector onto

$\mathcal{L}_{\text{eff}}$. Define Green function in $\mathcal{L}_{\text{eff}}$ to be

$$G(z) = \Upsilon^\dagger (z - H_0 - V)^{-1} \Upsilon. \tag{38}$$

Define self energy $\Sigma(z)$, which is an operator on $\mathcal{L}_{\text{eff}}$, such that

$$G(z) = (z - E_0 - \Sigma(E))^{-1}. \tag{39}$$

With this self energy we can define the effective Hamiltonian as

$$H_{\text{eff}} = E_0 + \Sigma(E_0). \tag{40}$$

To obtain the self energy, we write operators in block matrix form within subspaces $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}_{\text{eff}}^{\perp}$ as

$$z - H_0 - V = \begin{pmatrix} z - E_0 - \Upsilon^\dagger V \Upsilon & -\Upsilon^\dagger V Q \\ -QV\Upsilon & G_0^{-1} - QVQ \end{pmatrix}, \tag{41}$$

where $G_0(z) = Q(z - H_0)^{-1} Q$ and $Q = 1 - \Upsilon^\dagger$. Note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} \left(A - BD^{-1}C\right)^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{pmatrix}. \tag{42}$$

Therefore

$$G(z) = \left(z - E_0 - \Upsilon^\dagger V \Upsilon - \Upsilon^\dagger V Q \left(G_0^{-1} - QVQ\right)^{-1} QV\Upsilon\right)^{-1}, \tag{43}$$

and that

$$\Sigma(z) = z - E_0 - G^{-1}(z) = \Upsilon^\dagger \left(V + VG_0(z)V + VG_0(z)VG_0(z)V + \cdots\right)\Upsilon. \tag{44}$$

This coincides with the matrix approach such that

$$\left\langle a \middle| H_{\text{eff}}^{(1)} \middle| b \right\rangle = \langle a | V | b \rangle, \quad \left\langle a \middle| H_{\text{eff}}^{(2)} \middle| b \right\rangle = {\sum_j}' \frac{\langle a | V | j \rangle \langle j | V | b \rangle}{E_0 - E_j}, \quad \ldots, \tag{45}$$

where the prime means that the summation is carried out over excited states. For a detailed discussion on this perturbation theory, see Sec. 6.1 in [3].

¶ **Apply to our model** Here we calculate $H_{\text{eff}}$ order by order.

1. $H_{\text{eff}}^{(1)} = \Upsilon^\dagger V \Upsilon = 0$.
2. $H_{\text{eff}}^{(2)} = \Upsilon^\dagger V G_0 V \Upsilon = -N\left(J_x^2 + J_y^2\right)/4J_z$, which is constant.
3. $H_{\text{eff}}^{(3)} = 0$.
4. $H_{\text{eff}}^{(4)}$ has non-constant terms from flipping two $x$-links and two $y$-links in a plaquette. Such four flips, if performed clockwise from the northwest $x$-link, give rise to a factor

$$-J_x^2 J_y^2 Q_p \equiv -J_x^2 J_y^2 \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}. \tag{46}$$

We label the four links by 1 to 4 clockwise from the northwest $x$-link. There are $4! = 24$ terms of three kinds:

(a) Flipping in order (1, 3, 2, 4), in which case we can reverse the order of the first two links, the last two links, and the first and last two links as a whole. There are 8 terms of this kind, which gives a factor

$$-8 \times \frac{1}{128 J_z^3} = -\frac{1}{16 J_z^3} \tag{47}$$

considering $G_0$'s contribution.

(b) Flipping in order (1, 2, 3, 4), in which case we can reverse the order of the first two links and the second two links simultaneously, or flip the first and last two links as a whole. There are 8 terms of this kind, which gives a factor

$$-4 \times \frac{1}{64 J_z^3} = -\frac{1}{16 J_z^3}. \tag{48}$$

(c) Other cases, including (2, 1, 3, 4) where we can flip the first two links and the second two links simultaneously, or flip the first and last two links as a whole, and (1, 4, 2, 3) where we can reverse the order of the first two links, the last two links, and the first and last two links as a whole. There are 12 terms of this kind, which gives a factor

$$12 \times \frac{1}{64 J_z^3} = \frac{3}{16 J_z^3}. \tag{49}$$

Finally, we have the lowest order non-constant effect Hamiltonian

$$H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16 J_z^3} \sum_p Q_p. \tag{50}$$

The $J_z < 0$ case is similar with $J_z \to |J_z|$.

### 2.2 Abelian anyons as low energy excitations in gapped phase

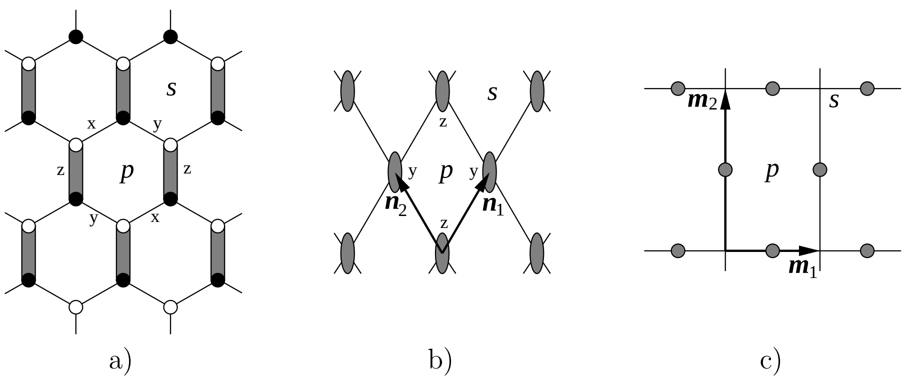



**Figure 5:** Mapping a hexagonal lattice to a square lattice. Plaquette $s$ becomes a vertex and plaquette $p$ becomes a plaquette.

¶ **Mapping to a toric code model** Mapping $z$-links of the hexagonal lattice to links of a square lattice as Fig. 5, the plaquettes then becomes vertices or plaquettes. The effective Hamiltonian can now be written as

$$H_{\text{eff}} = -J_{\text{eff}} \left( \sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p \right), \tag{51}$$

where $J_{\text{eff}} = J_x^2 J_y^2 / \left(16 J_z^3\right)$. It can be transformed to a toric code model by a unitary transformation

$$U = \prod_{\text{horizontal links}} X_j \prod_{\text{vertical links}} Y_k, \tag{52}$$

where $X_j$ is a rotation that maps $y$-axis to $x$-axis and leaves $z$-axis unchanged, and $Y_k$ is a rotation that maps $z \to x$ and $y \to z$. Note that this transformation breaks the translational symmetry. The transformed Hamiltonian is the well studied toric code model,

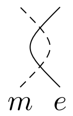
$$H'_{\text{eff}} = U H_{\text{eff}} U^\dagger = -J_{\text{eff}} \left( \sum_{\text{vertices}} A_s + \sum_{\text{plaquettes}} B_p \right), \tag{53}$$

where $A_s = \prod_{\text{star}(s)} \sigma_j^x$, $B_p = \prod_{\text{boundary}(p)} \sigma_j^z$.

**¶ Superselection sectors** A superselection sector is an excitation type defined up to local operations. The toric code model has 4 superselection sectors: 1 (the vacuum), $e$ (the 'electric charge' which lives on a vertex, that is, $A_s |e\rangle = -|e\rangle$ for some $s$, which can be constructed by acting a $\sigma^z$ string linking $s$ to infinity), $m$ (the 'magnetic vortex' which lives on a plaquette, that is, $B_p |m\rangle = -|m\rangle$ for some $p$, which can be constructed by acting a $\sigma^x$ string on dual lattice linking $p$ to infinity), and $\varepsilon = e \times m$. The fusion rules for these superselection sectors are

$$e \times e = m \times m = \varepsilon \times \varepsilon = 1, \quad e \times m = \varepsilon, \quad e \times \varepsilon = m, \quad m \times \varepsilon = e. \tag{54}$$

Braiding rules describe the properties of changing two particles. For example, nothing happens if an $e$-particle moves around another $e$-particle, which means $e$-particle itself is a boson. So does $m$-particle. However, if we move an $e$-particle around an $m$-particle, the overall state of the system is multiplied by $-1$. This can be written as

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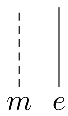
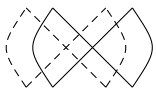



. (55)

The $\varepsilon$-particle is a fermion. We can check this by considering the following process: create two $\varepsilon\varepsilon$-pairs, exchanging two $\varepsilon$-particles chosen from different pairs, and annihilate the two pairs. This process can be represented graphically as



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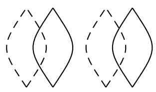


. (56)

Compared to creating two pairs and annihilating them directly, exchanging two $\varepsilon$-particles introduces a minus sign, which indicates that $\varepsilon$-particle is a fermion. Furthermore, braiding an $\varepsilon$-particle with an $e$- or $m$-particle also gievs $-1$.

Tracing back to the original model, we find that the $e$- and $m$-particles are vortices living on alternating rows of hexagons. (YF: Why can properties of the whole model be related to gaps of the sector without vortices?) The gap $2|J_z|$ is solved in a configuration with no vortices, which is much larger than excitation energies $2J_{\text{eff}}$. Fermions in the original model belong to the $\varepsilon$ sector (YF: why?) and can only decay to an $em$ pair if coupled to a zero-temperature heat bath.

# 3 Properties of the gapless phase

## 3.1 Effects of time-reversal symmetry

**¶ Time reversal operators** The time-reversal operator is an anti-unitary operator $T$ such that

$$T\sigma_j^\alpha T^{-1} = -\sigma_j^\alpha. \tag{57}$$

By anti-unitary we mean that

$$\langle T\Psi \,|\, T\Phi \rangle = \langle \Phi \,|\, \Psi \rangle. \tag{58}$$

Therefore, $T$ acts on Majorana operators like

$$Tb_jT^{-1} = b_j, \quad Tc_jT^{-1} = c_j. \tag{59}$$

As a result, $THT^{-1} = H$ for our model. In a bipartite lattice, $T$ commutes with any Wilson loop, so fixing gauge sector (that is, fixing $w_p$) does not break the time reversal symmetry. However, for a non-bipartite lattice, time reversal symmetry may lead to a two-fold degeneracy.

If we want to perform the perturbation theory relative to a gauge sector, we need an operator that preserves $\widehat{u}_{ij}$. Define $T'$ which has the same physical effect as $T$, that is, $T'\sigma_j^\alpha (T')^{-1} = -\sigma_j^\alpha$, as follows,

$$T' d_j (T')^{-1} = \begin{cases} d_j, & \text{if } j \in \text{even sublattice}, \\ -d_j, & \text{if } j \in \text{odd sublattice}, \end{cases} \tag{60}$$

where $d$ can be $b$ or $c$.

**¶ Time-reversal Perturbations do not open a gap** Any $T'$-invariant perturbation cannot contain terms like $\mathrm{i} c_j c_k$ where $j$ and $k$ belong to the same sublattice. Therefore $\widetilde{A}(\boldsymbol{q})$ still has vanishing diagonal terms. The function $f(\boldsymbol{q})$ is a map from a 2D diamond to $\mathbb{C}$, whose zeros are robust against small perturbations.

### 3.2 Effective Hamiltonian under a magnetic field

To study the gapless phase under a perturbation that breaks time-reversal symmetry, let's consider a magnetic field term

$$V = -\sum_j \left( h_x \sigma_j^x + h_y \sigma_j^y + h_z \sigma_j^z \right). \tag{61}$$

The unperturbed Hamiltonian has parameters $J_x = J_y = J_z = J$ for simplicity. We perform the perturbation on the vortex-free sector. Denoting the projector as $\Pi_0$.

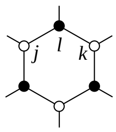
The first order effective Hamiltonian $H_{\text{eff}}^{(1)} = 0$. The second order Hamiltonian does not break the time-reversal symmetry. Consider the third order one

$$H_{\text{eff}}^{(3)} = \Pi_0 V G_0(E_0) V G_0(E_0) V \Pi_0, \tag{62}$$

which is difficult to calculate. We can approximate it by assuming that the intermediate states have energy $\Delta E \sim J$. Then the effective Hamiltonian becomes

$$H_{\text{eff}}^{(3)} \sim -\frac{h_x h_y h_z}{J^2} \sum_{j,k,l} \sigma_j^x \sigma_k^y \sigma_l^z, \tag{63}$$

where the spin triples are arranged as Fig. 6 shows.

a) 

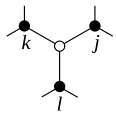
(and symmetry-equivalent), or b) 

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**Figure 6:** Spin triple configurations in $H_{\text{eff}}^{(3)}$. In every related plaquette the perturbation term commutes with $W_p$.

Configuration (a) corresponds to the term $\sigma_j^x \sigma_k^y \sigma_l^z = -\mathrm{i} D_l \widehat{u}_{jl} \widehat{u}_{kl} c_j c_k$, which, in standard gauge and with $D_l$ neglected since we work in physical subspace, becomes $-\mathrm{i} c_j c_k$. Configuration (b) gives a four-fermion term and therefore does not directly influence the spectrum. (YF: Why?) Therefore the effective Hamiltonian is

$$H_{\text{eff}} = \frac{\mathrm{i}}{4} \sum_{j,k} A_{jk} c_j c_k, \tag{64}$$

where $A = 2J$ for solid lined arrows and $2\kappa$ for dashed arrows, and $A$ gets a minus sign if $jk$ is opposite to the arrow's direction. See Fig. 7. Here $\kappa \sim h_x h_y h_z / J^2$.

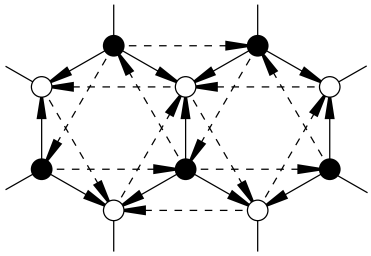

**Figure 7:** Graphical illustration for $A_{jk}$ links.

### 3.3 The spectrum and edge modes

**¶ The spectrum** The matrix $\widetilde{A}(\boldsymbol{q})$ in Eq. (35) now becomes

$$\mathrm{i}\widetilde{A}(\boldsymbol{q}) = \begin{pmatrix} \Delta(\boldsymbol{q}) & \mathrm{i}f(\boldsymbol{q}) \\ -\mathrm{i}f^*(\boldsymbol{q}) & -\Delta(\boldsymbol{q}) \end{pmatrix}, \quad \Delta(\boldsymbol{q}) = 4\kappa\left(\sin(\boldsymbol{q},\boldsymbol{n}_1) + \sin(\boldsymbol{q},-\boldsymbol{n}_2) + \sin(\boldsymbol{q},\boldsymbol{n}_2-\boldsymbol{n}_1)\right), \tag{65}$$

and $f$ remains unchanged, i.e., $f(\boldsymbol{q}) = 2J\left(\mathrm{e}^{\mathrm{i}(\boldsymbol{q},\boldsymbol{n}_1)} + \mathrm{e}^{\mathrm{i}(\boldsymbol{q},\boldsymbol{n}_2)} + 1\right)$. Again, the spectrum is given by eigenvalues of $\mathrm{i}A(\boldsymbol{q})$, which is $\epsilon(\boldsymbol{q}) = \pm\sqrt{|f(\boldsymbol{q})|^2 + \Delta(\boldsymbol{q})^2}$.

The perturbation opens a gap at $\pm\boldsymbol{q}^*$. To calculate this gap, we first define $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ reciprocal to $\boldsymbol{n}_1 = \left(1/2, \sqrt{3}/2\right)$, $\boldsymbol{n}_2 = \left(-1/2, \sqrt{3}/2\right)$ as

$$\boldsymbol{q}_1 = 2\pi\left(1, \frac{1}{\sqrt{3}}\right), \quad \boldsymbol{q}_2 = 2\pi\left(-1, \frac{1}{\sqrt{3}}\right). \tag{66}$$

We can solve from $f(\boldsymbol{q}^*) = 0$ that

$$\boldsymbol{q}^* = \frac{1}{3}\boldsymbol{q}_1 + \frac{2}{3}\boldsymbol{q}_2. \tag{67}$$

The energy gap

$$\Delta = \Delta(\boldsymbol{q}^*) = -\Delta(-\boldsymbol{q}^*) = 6\sqrt{3}\kappa \sim \frac{h_x h_y h_z}{J^2}. \tag{68}$$

The conic singularities are resolved as

$$\epsilon(\boldsymbol{q}) \approx \pm\sqrt{3J^2|\delta\boldsymbol{q}|^2 + \Delta^2}. \tag{69}$$

**¶ The spectral projector and the Chern number** The spectral projector in momentum space is

$$\widetilde{P}(\boldsymbol{q}) = \frac{1}{2}\left(1 - \operatorname{sgn}\left(\mathrm{i}\widetilde{A}(\boldsymbol{q})\right)\right) = \frac{1}{2}\left(1 + (\boldsymbol{m}(\boldsymbol{q}), \boldsymbol{\sigma})\right), \tag{70}$$

where

$$\boldsymbol{m}(\boldsymbol{q}) = \frac{1}{|\epsilon(\boldsymbol{q})|}\left(\operatorname{Im} f(\boldsymbol{q}), \operatorname{Re} f(\boldsymbol{q}), -\Delta(\boldsymbol{q})\right). \tag{71}$$

This operator assign a 1D annihilation space to each point $\boldsymbol{q}$, which forms a complex vector bundle. The first Chern number of this bundle is

$$\begin{aligned} \nu &= \frac{1}{2\pi\mathrm{i}}\int \operatorname{tr}\left(\widetilde{P}\mathrm{d}\widetilde{P}\wedge\mathrm{d}\widetilde{P}\right) \\ &= \frac{1}{2\pi\mathrm{i}}\int \operatorname{tr}\left(\widetilde{P}\left(\frac{\partial\widetilde{P}}{\partial q_x}\frac{\partial\widetilde{P}}{\partial q_y} - \frac{\partial\widetilde{P}}{\partial q_y}\frac{\partial\widetilde{P}}{\partial q_x}\right)\right)\mathrm{d}q_x\mathrm{d}q_y \\ &= \frac{1}{4\pi}\int\left(\frac{\partial\boldsymbol{m}}{\partial q_x}\times\frac{\partial\boldsymbol{m}}{\partial q_y}, \boldsymbol{m}\right)\mathrm{d}q_x\mathrm{d}q_y \\ &= \operatorname{sgn}\Delta = \pm 1. \end{aligned} \tag{72}$$

(YF: Check the general mathematical definition of Chern number and understand why it is compatible with this and that defined by Berry curvature.) We use $B_\nu$ to denote phase $B$ in a magnetic field. In $A$ phases the Chern number is zero.

**¶ Chiral edge modes** Gapless chiral, i.e., propagating only in one direction, edge modes occur when a system has a non-zero Chern number, and the number of modes satisfies

$$\nu_{\text{edge}} = (\#\text{ of left-movers} - \#\text{ of right movers}) = \nu. \tag{73}$$

This edge mode can lead to thermal transport,

$$I = \frac{\pi}{12} c_- T^2, \tag{74}$$

where $c_- = c - \bar{c}$ is the chiral central charge of the conformal field theory (CFT) describing the edge modes [4]. (YF: Details required.)

Another way to derive the transport current is by noting that the 1D chiral edge mode has a spectrum $\epsilon(\boldsymbol{q})$ such that $\epsilon(-q) = -\epsilon(q)$. Therefore the current due to one mode propagating in the positive direction is

$$I_1 = \int_{\epsilon(q)>0} n(q)\,\epsilon(q)\,v(q)\,\frac{\mathrm{d}q}{2\pi} = \int_{\epsilon(q)>0} \frac{\epsilon(q)}{1+\mathrm{e}^{\epsilon(q)/T}} \frac{\mathrm{d}\epsilon}{\mathrm{d}q} \frac{\mathrm{d}q}{2\pi} = \frac{\pi}{24} T^2. \tag{75}$$

As a result, $I = \nu I_1$ and $c_- = \nu/2$.

### 3.4 Non-Abelian anyons

Properties of anyons are described by superselection sectors as for the gapped phases. In this context what we care about is not details (specific positions, local interactions, etc.), but rather topology of fusions and braidings.

**¶ Bulk-edge correspondence and CFT** (YF: Details required.) Anyons in bulk are related to edge modes on the edge, the latter of which is described by a CFT. The properties of anyons can be derived from this CFT. For Chern number $\nu = \pm 1$, the algebraic properties of anyons are the same as the algebraic structure described by Moore and Seiberg [5] in CFT context. For general $\nu$, The topological spin of a vortex is

$$\theta_\sigma = \mathrm{e}^{\mathrm{i}\pi\nu/8}. \tag{76}$$

See Appendix E of [1] for more details of the notion of topological spins.

**¶ Unpaired Majorana modes, some basic fusion and braiding rules and associativity relations** Considering the analogy between a two-layered Kitaev model (1) and a quantum Hall system, Kitaev argued that if $\nu$ is odd, each vortex carries an unpaired Majorana mode. From this statement some basic properties of non-Abelian anyons can be derived.

There are three kinds of sectors, 1 (the vacuum), $\epsilon$ (a fermion) and $\sigma$ (a vortex carrying an unpaired Majorana mode). The fusion rules are

$$\varepsilon \times \varepsilon = 1, \quad \varepsilon \times \sigma = \sigma, \quad \sigma \times \sigma = 1 + \varepsilon, \tag{77}$$

and trivial rules $1 \times x = x$. These fusion rules can also be understood as split rules. Note the form $1+\varepsilon$ in the last equation, which indicates that the protected space of a $\sigma$ pair has two basis $|\psi_1^{\sigma\sigma}\rangle$ and $|\psi_\varepsilon^{\sigma\sigma}\rangle$.

For fusion rules, define $R_z^{xy}$ as Fig. 8. By introducing fermionic and vortex path operators, Kitaev argued that

$$R_1^{\sigma\sigma} = \theta \mathrm{e}^{-\mathrm{i}\alpha\pi/4}, \quad R_\varepsilon^{\sigma\sigma} = \theta \mathrm{e}^{\mathrm{i}\alpha\pi/4}. \tag{78}$$

Here $\theta$ is again topological spin and $\alpha$ is to be determined later.

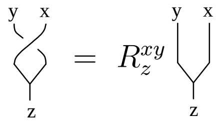

**Figure 8:** Definition of $R_z^{xy}$.

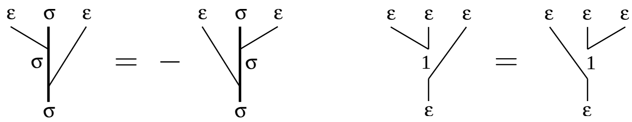

**Figure 9:** Two simple associativity relations.

Things still remains non-trivial when two fusions happen sequentially. This is described by the associativity relations. Using the argument of unpaired Majorana modes and path operators, we have associativity relations as Fig. 9.

¶ **Algebraic consistency** The undetermined parameters in braiding rules and other associativity relations can be determined by algebraic considerations. This algebraic theory is captured by unitary modular category (UMC), which constitutes the algebraic core of topological quantum field theory (TQFT). (YF: Interesting! Seek for a deeper understanding.) This approach is based on axioms that successive fusion and braiding events must commute with each other in certain cases. (YF: Why?)

The first axiom is called pentagon equations. See Fig. 10. One can introduce undetermined parameters in associativity relations and apply the consistency requirements by pentagon equations to determine them.

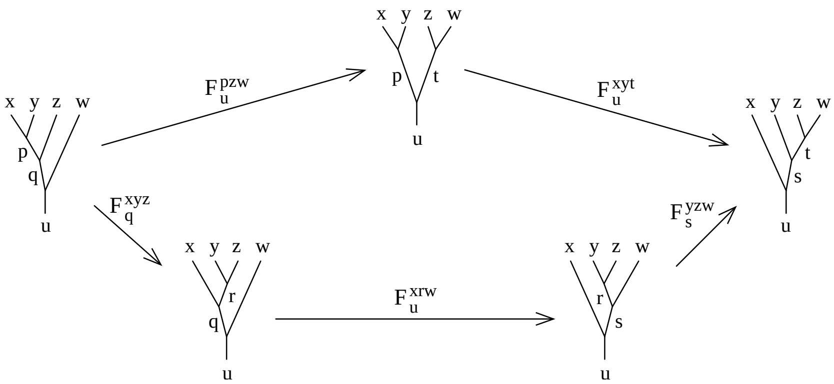

**Figure 10:** Commute diagram for pentagon equations. Successive splittings from the upper channel leads to the same state as from the lower channel.

The second axiom is called hexagon equations which imply consistency requirements on successive fusions and braidings. See Fig. 11. These requirements can also be used to determine the unfixed parameters such as $\alpha$ in Eq. (78).

After applying the algebraic consistency requirements, we get eight solutions. Finally we need to use some general considerations and Eq. (76) to find the correct solutions for different $\nu$'s.

¶ **The sixteen-fold way** For the original Kitaev model, $\nu = \pm 1$ case is enough. However, the approaches above yields a rather general result. We find that the properties of anyons of this model depend on $\nu \mod 16$. Algebraic properties for anyons for even $\nu$ can also be discussed similarly.

For a complete construction of the algebraic theory for anyons above, see Appendix E in [1].

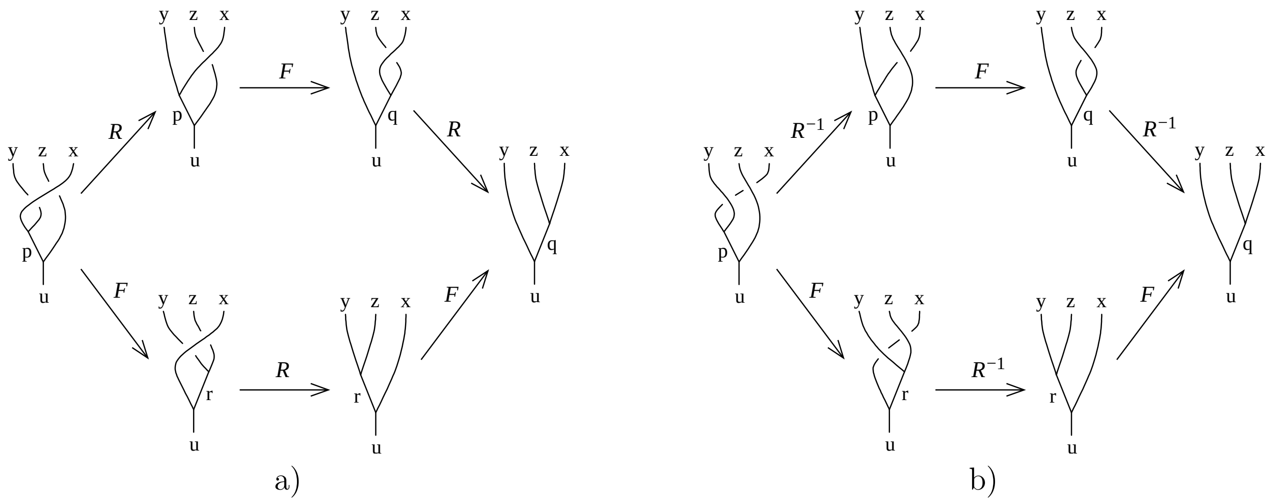


**Figure 11:** Commute diagram for hexagon equations. Successive splittings from the upper channel leads to the same state as from the lower channel.

## References


[1] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Annals of Physics* **321** (Jan., 2006) 2–111.

[2] E. H. Lieb, *Flux Phase of the Half-Filled Band*, *Physical Review Letters* **73** (Oct., 1994) 2158–2161.

[3] J. Kempe, A. Kitaev and O. Regev, *The Complexity of the Local Hamiltonian Problem*, Oct., 2005. 10.48550/arXiv.quant-ph/0406180.

[4] A. Cappelli, M. Huerta and G. R. Zemba, *Thermal transport in chiral conformal theories and hierarchical quantum Hall states*, *Nuclear Physics B* **636** (Aug., 2002) 568–582.

[5] G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Communications in Mathematical Physics* **123** (June, 1989) 177–254.