How To Prove It: A Structured Approach

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Chapter 0

Introduction

0.1 Introduction

0.1.1 Recapitulation

High school mathematics is concerd mostly with solving equations and computing answers to numerical questions. College mathematics deals with a wider variety of questions, involving not only numbers, but also sets, functions, and other mathematical objects. What ties them together is the use of deductive reasoning to find the answers to questions.

Deductive reasoning uses general ideas to come to a specific conclusion. Deductive reasoning in mathematics is usually presented in the form of a *proof*.

A number is *prime* of it cannot be written as a product of two smaller positive integers. If it can be written as a product of two smaller positive integers, then it is *composite*

A *conjecture* is a conclusion that is proffered on a tentative basis without proof.

Conjecture 1. *Suppose n is an integer larger than 1 and n is prime. Then $2^n - 1$ is prime.*

Conjecture 2. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

A *counterexample* is a specific instance that demonstrates the falsity of a general statement, argument or theory.

The existence of even one counterexample establishes that a conjecture is incorrect. However, failure to find a counterexample to a conjecture does not show that the conjecture is correct.

We can never be sure that the conjecture is correct if we only check examples. No matter how many examples we check, there is always the possibility that the next one will be the first counterexample.

Once a conjecture has been proven, we can call it a *theorem*.

Theorem 3. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime*

Prime numbers of the form $2^n - 1$ are called *Mersenne primes*. Although many Mersenne primes have been found, it is still not know if there are infinitely many of them.

A positive integer n is said to be *perfect* if n is equal to the sum of all positive integers smaller than n that divide n.

For any two integers m and n, we say that m *divides* n if n is divisible by m; in other words, if there is an integer q such that $n = qm$.

For any positive integer n, the product of all integers from 1 to n is called n *factorial* and is denoted $n!$.

Theorem 4. *For every positive integer n, there is a sequence of n consecutive positive integers containing no primes*

Pairs of primes that differ by only two are called *twin primes*

0.1.2 Problems

1. (a)

$$2^{ab} - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + ... + 2^{(a-1)b})$$
$$n = ab = 15; a = 3; b = 5$$
$$2^{(3)(5)} = (2^5 - 1)(1 + 2^5 + 2^{10}) = (31) \cdot (1057)$$

(b)

$$2^{ab} - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + ... + 2^{(a-1)b})$$
$$n = ab = 32767; a = 1057; b = 31$$
$$2^{1057 \cdot 31} - 1 = (2^{31} - 1)(1 + 2^{31} + ... + 2^{1056 \cdot 31})$$
$$1 < 2^{31} - 1 < 2^{32767} - 1$$

2.

$3^n - 1$		
$n = 1$	$3^1 - 1 = 2$	Prime
$n = 2$	$3^2 - 1 = 8$	Not Prime
$n = 3$	$3^3 - 1 = 26$	Not Prime
$n = 4$	$3^4 - 1 = 80$	Not Prime
$n = 5$	$3^5 - 1 = 242$	Not Prime
$n = 6$	$3^6 - 1 = 728$	Not Prime
$n = 7$	$3^7 - 1 = 2186$	Not Prime
$n = 8$	$3^8 - 1 = 6560$	Not Prime
$n = 9$	$3^9 - 1 = 19682$	Not Prime
$n = 10$	$3^{10} - 1 = 59048$	Not Prime

For all $n > 1$, $3^n - 1$ is not a prime

$3^n - 2^n$		
$n = 1$	$3^1 - 2^1 = 1$	Prime
$n = 2$	$3^2 - 2^2 = 5$	Prime
$n = 3$	$3^3 - 2^3 = 19$	Prime
$n = 4$	$3^4 - 2^4 = 65$	Not Prime
$n = 5$	$3^5 - 2^5 = 211$	Prime
$n = 6$	$3^6 - 2^6 = 665$	Not Prime
$n = 7$	$3^7 - 2^7 = 2059$	Not Prime
$n = 8$	$3^8 - 2^8 = 6305$	Not Prime
$n = 9$	$3^9 - 2^9 = 19171$	Not Prime
$n = 10$	$3^{10} - 2^10 = 58025$	Not Prime

If n is not prime, then $3^n - 2^n$ is not prime. If n is prime, then $3^n - 2^n$ is either prime or composite.

3. (a)

$$m = p_1 \cdot p_2 \cdot ... \cdot p_n + 1$$
$$m = 2 \cdot 3 \cdot 5 \cdot 7 + 1$$
$$m = 211$$

(b)

$$m = p_1 \cdot p_2 \cdot ... \cdot p_n + 1$$
$$m = 2 \cdot 5 \cdot 11 + 1$$
$$m = 111 = 3 \cdot 37$$

4.

$$x = (n+1)! + 2; n = 5$$
$$x = (5+1)! + 2 = 6! + 2 = 722$$
$$[722, 723, 724, 725, 726]$$

5.

$n = 3$	$2^{n-1}(2^n - 1) = 2^{3-1}(2^3 - 1) = 28$
$n = 5$	$2^{n-1}(2^n - 1) = 2^{5-1}(2^5 - 1) = 496$

6.

$$mod(n,3) = \begin{cases} 0, \text{n is divisible by 3;} \\ 1, \text{n + 2 is divisible by 3} \\ 2, \text{n + 4 is divisible by 3} \end{cases} \tag{1}$$

7.

$$220 : [1, 2, 4, 5, 10, 11, 22, 44, 55, 110]$$
$$1 + 2 + 4 + 5 + 10 + 11 + 22 + 44 + 55 + 110 = 284$$
$$284 : [1, 2, 4, 71, 142]$$
$$1 + 2 + 4 + 71 + 142 = 220$$

Chapter 1

Sentential Logic

1.1 Deductive Reasoning and Logical Connectives

1.1.1 Recapitulation

Profs play a central role in mathematics, and deductive reasoning is the foundation on which proofs are based.

We arrive at a *conclusion* from the assumption that some other statements, called *premises*, are true.

We will say that an argument is *valid* if the premises cannot all be true without the conclusion being true as well. And is *invalid* otherwise.

Connective symbols are symbols used to combine statements to form more complex statements.

Symbol	Meaning
$\vee$	or
$\wedge$	and
$\neg$	not

The statement $P \vee Q$ is called the disjunction of P and Q, the statement $P \wedge Q$ is called the conjunction of P and Q, and $\neg P$ is called the negation of P.

1.1.2 Problems

1. (a)

$$P : \text{We'll have reading assignment}$$

$$Q : \text{We'll have homework problems}$$

$$R : \text{We'll have a test}$$

$$(P \vee Q) \wedge \neg(Q \wedge R)$$

(b)

$$P: \text{You will go skiing}$$
$$Q: \text{There will snow}$$
$$\neg P \vee (P \wedge \neg Q)$$

(c)

$$\neg((\sqrt{7} < 2) \vee (\sqrt{7} = 2))$$

2. (a)

$$P: \text{John is telling the truth}$$
$$Q: \text{Bill is telling the truth}$$
$$(P \wedge Q) \vee (\neg P \wedge \neg Q)$$

(b)

$$P: \text{I'll have fish}$$
$$Q: \text{I'll have chicken}$$
$$R: \text{I'll have mashed potatoes}$$
$$(P \vee Q) \wedge \neg(P \wedge R)$$

(c)

$$P: 3 \text{ is a common divisor of } 6$$
$$Q: 3 \text{ is a common divisor of } 9$$
$$R: 3 \text{ is a common divisor of } 15$$
$$P \wedge Q \wedge R$$

3. (a)

$$\neg(P \wedge Q)$$

(b)

$$\neg P \wedge \neg Q$$

(c)

$$\neg P \vee \neg Q$$

(d)

$$\neg(P \vee Q)$$

4. (a)

$$(P \wedge Q) \vee (R \wedge S)$$

(b)

$$(P \vee R) \wedge (Q \vee S)$$

(c)

$$\neg(P \vee R) \wedge \neg(Q \vee S)$$

(d)

$$\neg((P \wedge R) \vee (Q \wedge S))$$

5. (a) Well-formed
   (b) Not well-formed
   (c) Well-formed
   (d) Not well-formed

6. (a) I won't buy the pants without a shirt
   (b) I won't buy the pants nor the shirt
   (c) Either I won't buy the pants or I won't buy the shirt

7. (a) Either Steve or George is happy, and Either Steve or George is not happy
   (b) Either Steve is happy, or George is happy and Steve isn't happy or George is not happy
   (c) Either Steve is happy, or George is happy and Either Stever or George are not happy

8. (a) Either taxes will go up or The deficit will go up
   (b) The taxes and the deficit won't go up and it is not the case that the taxes and the deficit won't go up
   (c) Either the taxes will go up and the deficit won't go up, or the deficit will go up and the taxes won't go up

9. (a) Conclusion: Pete will win the chemistry prize
   (b) Conclusion: We will not have both beef as a main course and peas as a vegetable.
   (c) Conclusion: Either John is telling the truth or Sam is lying.
   (d) Conclusion: Sales and expenses will not both go up.

1.2 Truth Tables

1.2.1 Recapitulation

When we evaluate the truth or falsity of a statement, we assign to it one of the labels *true* or *false*, and this label is called its *truth value*.

A *Truth table* is a table in which each of its rows shows one of the possible combinations of truth values for a statement or a compound statement.

To verify the validity of arguments, we can arrage the truth values of the premises and the conclusion in a truth table, in the rows where the premises are all true it must follow that the conclusion is also true, thus the argument is valid, otherwise it is invalid.

Equivalent formulas always have the same truth value no matter what the truth value of those statements are.

Formulas that are always true, are called *tautologies*, formulas that are always false are called *contradictions*.

1.2.2 Problems

1. (a)

P	Q	$\neg P \vee Q$
F	F	T
F	T	T
T	F	F
T	T	T

(b)

S	G	$(S \vee G) \wedge (\neg S \vee \neg G)$
F	F	F
F	T	T
T	F	T
T	T	F

2. (a)

P	Q	$\neg(P \wedge (Q \vee \neg P))$
F	F	T
F	T	T
T	F	T
T	T	F

(b)

P	Q	R	$(P \vee Q) \wedge (\neg P \vee R)$
F	F	F	F
F	F	T	F
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

3. (a)

P	Q	$P \oplus Q$
F	F	F
F	T	T
T	F	T
T	T	F

(b)

P	Q	$(P \vee Q) \wedge \neg(P \wedge Q)$
F	F	F
F	T	T
T	F	T
T	T	F

4.

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

5. (a)

P	Q	$P \downarrow Q$
F	F	T
F	T	F
T	F	F
T	T	F

(b)

$$P \downarrow Q \equiv \neg P \wedge \neg Q \equiv \neg(P \vee Q)$$

(c)

$$\neg P \equiv P \downarrow P$$
$$P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$$
$$P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$$

6. (a)

P	Q	$P \mid Q$
F	F	T
F	T	T
T	F	T
T	T	F

(b)

$$\neg(P \wedge Q)$$

(c)

$$\neg P \equiv P \mid P$$
$$P \vee Q \equiv (P \mid P) \mid (Q \mid Q)$$
$$P \wedge Q \equiv (P \mid Q) \mid (P \mid Q)$$

7. (a) Valid
   (b) Invalid
   (c) Valid
   (d) Invalid

8.

$$(a) \equiv (c), (b) \equiv (e)$$

9. (b) is a contradiction; (c) is a tautology

10. Not needed

11. (a)

$$\neg(\neg P \wedge \neg Q) \equiv P \vee Q$$

(b)

$$(P \wedge Q) \vee (P \wedge \neg Q) \equiv P$$

(c)

$$\neg(P \wedge \neg Q) \vee (\neg P \wedge Q) \equiv \neg P \vee Q$$

12. (a)
$$\neg(\neg P \vee Q) \vee (P \wedge \neg R \equiv P \wedge \neg(Q \wedge R)$$
(b)
$$\neg(\neg P \wedge Q) \vee (P \wedge \neg R) \equiv \neg Q \vee P$$
(c)
$$(P \wedge R) \vee (\neg R \wedge (P \vee Q)) \ equiv \neg P \wedge \neg Q$$
13. Not needed
14. Not needed
15.
$$2^n$$
16.
$$P \vee \neg Q$$
17.
$$(P \wedge \neg Q) \vee (\neg P \wedge Q)$$
18. (1) That it is valid; (2) That it is invalid if there is a combination where all premises are true; (3) Could be either valid or invalid for a tautology, if it is a contradiction then it is valid

1.3 Variables and Sets

1.3.1 Recapitulation

It is often necesary to make statements about objects that are represented by letters called *variables.*

A notation like $P(x)$ can be interpreted as a statement P about the variable x.

In a statement that contains variables we cannot describe the statement as being simply true or false. Its truth value might depend on the values of the variables involved.

A *set* is a collection of objects. The objects in the collection are called the *elements* of the set.

We use the symbol $\in$ to mean *is an element of.* To say that an element is not part of a set we use the symbol $\notin$.

The following notations for sets are valid:

- $A = \{1, 2, 3\}$ is set that contains the numbers 1, 2 and 3
- $B = \{2, 3, 5, 7, 11, 13, 17, ...\}$ is a set that contains all of the prime numbers
- $C = \{x \mid x \text{ is a prime number}\}$ is a set that defines an *elementhood test* for the set, any value of x that passes the test is an element of the set.

Free variables in a statement stand for objects that the statement says something about. *Bound* variables are simply letters that are used as a convenience to help express an idea and should not be thought of as standing for any particular object. A bound variable can always be replaced by a new variable without chaning the meaning of the statement, and often the statement can be rephrased so that the bound variables are eliminated altogether.

To distinguish the values of a statement that contains free variables that make the statement true from those that make it false, we form the set of values of the free variables for which the statement is true. We call this the *truth set* of the statement.

For a statement, the set of all the objects that represent the free variables is called the *universe of discourse* for the statement. And we say that the variables *range over* this universe.

A set without any elements is called an *empty set*, or the *null set*, and it is often denoted by $\emptyset$

1.3.2 Problems

1. (a)

$$P(x, y) : x \text{ is divisible by } y$$

$$P(6,3) \wedge P(9,3) \wedge P(15,3)$$

(b)

$$P(x, y) : x \text{ is divisible by } y$$

$$P(x,2) \wedge P(x,3) \wedge \neg P(x,4)$$

(c)

$$N(x) : x \text{ is a natural number}$$

$$P(x) : x \text{ is a prime number}$$

$$N(x) \wedge N(y) \wedge (P(x) \oplus P(y)) \equiv N(x) \wedge N(y) \wedge ((P(x) \wedge \neg P(y)) \vee (P(y) \wedge \neg P(x))$$

2. (a)

$$P(x) : x \text{ is a man}$$

$$Q(x, y) : x \text{ is taller than } y$$

$$P(x) \wedge P(y) \wedge (Q(x, y) \vee Q(y, x))$$

(b)

$$P(x) : x \text{ has brown eyes}$$

$$Q(x) : x \text{ has red hair}$$

$$(P(x) \vee P(y)) \wedge (Q(x) \vee Q(y))$$

(c)

$$P(x): x \text{ has brown eyes}$$
$$Q(x): x \text{ has red hair}$$
$$(P(x) \wedge Q(x)) \vee (P(y) \wedge Q(y))$$

3. (a)
$$\{x \mid x \text{ is a planet in the solar system}\}$$
(b)
$$\{x \mid x \text{ is an Ivy League School}\}$$
(c)
$$\{x \mid x \text{ is a state in the US}\}$$
(d)
$$\{x \mid x \text{ is a province in Canada}\}$$

4. (a)
$$\{x \mid x \text{ is the square of a positive integer}\}$$
(b)
$$\{x \mid \text{ x is a power of 2}\}$$
(c)
$$\{x \mid x \in \mathbb{Z} \wedge 10 \leq x \leq 19\}$$

5. (a)
$$-3 \in \{x \in \mathbb{R} \mid x < 6\}; \text{True statement, x is bound}$$
(b)
$$4 \in \{x \in \mathbb{R}^- \mid x < 6\}; \text{False statement, x is bound}$$
(c)
$$\neg(5 \notin \{x \in \mathbb{R} \mid x < \frac{13-c}{2}\})\text{c is free, x is bound}$$

6. (a)
$$(w \in \mathbb{R}) \wedge (13 - 2w > c); \text{w and c are free, x is bound}$$
(b)
$$(4 \in \mathbb{R}) \wedge (\text{5 is a prime number})); \text{Statement is true, x and y are bound}$$
(c)
$$(\text{4 is a prime number}) \wedge (5 > 1); \text{Statement is false, x and y are bound}$$

7. (a)
$$\{-1, \frac{1}{2}\}$$

(b)

$$\{\frac{1}{2}\}$$

(c)

$$\{-1\}$$

(d)

$$\emptyset$$

8. (a) People that Elizabeth Taylor was married to

(b)

$$\{\wedge, \vee, \neg\}$$

(c)

$$\{\text{Daniel J. Velleman}\}$$

9. (a)

$$\{1, 3\}$$

(b)

$$\emptyset$$

(c)

$$\{x \in \mathbb{R} \mid 5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}\}$$
$$\{x \in \mathbb{R} \mid 5 \in \mathbb{R} \wedge x^2 + 25 < 50\}$$
$$\{x \in \mathbb{R} \mid x^2 < 25\}$$
$$\{x \in \mathbb{R} \mid -5 < x < 5\}$$
$$\{-4, -3, 2, 3.9, 4.9999, ...\}$$

1.4 Operations on Sets

1.4.1 Recapitulation

Definition *The intersection of two sets A and B is the set $A \cap B$ defined as*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Definition *The union of two sets A and B is the set $A \cup B$ defined as*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Definition *The difference of A and B is the set $A \setminus B$ defines as*

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Sometimes it is helpful when working with operations on sets to draw pictures of the results of these operations, the most common diagram for this is a *Venn Diagram.*

Definition *The symmetric difference of A and B is the set* $A \triangle B$ *defines as*

$$A \triangle B = \{x \mid x \in A \text{ and } x \in B \text{ and } x \notin A \cap B\}$$

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

Set theory operations are related to the logical connectives, but it is important to remember that although they are related, they are not interchangeable. The logical connectives can only be used to combine statements, whereas the set theory operations must be used to combine sets.

Definition *Supossing that A and B are sets. We will say that A is a subset of B if every element of A is also an element of B. We write* $A \subseteq B$ *to mean that A is a subset of B. A and B are said to be disjoint if they have no elements in common, that is if* $A \cap B = \emptyset$

Theorem *For any sets A and B,* $(A \cup B) \setminus B \subseteq A$

1.4.2 Problems

1. The sets $A \cap B$ and $(A \cup B) \setminus C$ are subsets of $A \cup (B \setminus C)$

   (a)
   $$A \cap B = \{3, 12\}$$

   (b)
   $$(A \cup B) \setminus C = \{1, 12, 20, 35\}$$

   (c)
   $$A \cup (B \setminus C) = \{1, 3, 12, 20, 35\}$$

2. The sets c is a subset of a, b is disjoint from a and b

   (a)
   $$A \cup B = \{UnitedStates, Germany, China, Australia, France, India, Brazil\}$$

   (b)
   $$(A \cap B) \setminus C = \emptyset$$

   (c)
   $$(B \cap C) \setminus A = \{France\}$$

3. Yes

4. Unnecesary

5. (a)

$$A \setminus (A \cap B) = x \in A \wedge x \notin (A \cap B) = x \in A \wedge \neg(x \in A \wedge x \in B)$$
$$x \in A \wedge (x \notin A \vee x \notin B)$$
$$(x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B)$$
$$(x \in A \wedge x \notin B)$$
$$x \in A \setminus B$$

(b)

$$A \cup (B \cap C) = x \in A \vee (x \in B \wedge x \in C)$$
$$(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$$
$$(A \cup B) \cap (A \cup C)$$

6. Unnecesary

7. (a)

$$(A \cup B) \setminus C = (x \in A \vee x \in B) \wedge x \notin C$$
$$(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$$
$$(A \setminus C) \cup (B \setminus C)$$

(b)

$$A \cup (B \setminus C) = x \in A \vee (x \in B \wedge x \notin C)$$
$$(x \in A \vee x \in B) \wedge (x \in A \vee x \notin C)$$
$$(A \cup B) \cap (A \setminus C)$$
$$(A \cup B) \setminus (C \setminus A)$$

8. (a)

$$(A \setminus B) \cap C = (A \cap C) \setminus B$$
$$(x \in A \wedge x \notin B) \wedge x \in C$$
$$(x \in A \wedge (x \notin B)) \wedge x \in C$$
$$(x \in A \wedge x \in C) \wedge x \notin B$$
$$(A \cap C) \setminus B$$

(b)

$$(A \cap B) \setminus B = \emptyset$$
$$(x \in A \wedge x \in B) \wedge x \notin B$$
$$x \in A \wedge (x \in B \wedge x \notin B)$$
$$\emptyset$$

(c)

$$A \setminus (A \setminus B) = A \cap B$$

$$x \in A \land \neg(x \in A \land \neg x \in B)$$

$$x \in A \land (x \notin A \lor x \in B)$$

$$(x \in A \land x \notin A) \lor (x \in A \land x \in B)$$

$$\emptyset \cup (A \cap B)$$

$$A \cap B$$

9. (a) (d) (e), (b) (c)

   (a)
   $$(x \in A \land x \notin B) \land x \notin C$$

   (b)
   $$x \in A \land (x \notin B \lor x \in C)$$

   (c)
   $$(x \in A \land x \notin B) \lor (x \in A \land x \in C)$$

   (d)
   $$(x \in A \land x \notin B) \land (x \in A \land x \notin C)$$

   (e)
   $$x \in A \land (x \notin B \land x \notin C)$$

10. (a)
   $$A = \{1, 2\}, B = \{2, 3\}$$

   (b)
   $$(x \in A \lor x \in B) \land x \notin B$$
   $$(x \notin B \land x \in A) \lor (x \notin B \land x \in B)$$
   $$(x \notin B \land x \in A)$$
   $$(A \setminus B)$$

11. No, since $(A \setminus B) \cup B = A \cup B$. $A \setminus B \subseteq (A \setminus B) \cup B$ and $(A \setminus B) \cup B \subseteq A \cup B$

12. (a) $(A \cap D) \setminus (B \cup C)$ cannot be represented.
    (b) Yes, with irregular forms.

13. (a) $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$
    (b) $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \{3, 4, 5\}$

14. Unnecesary

15. Unnecesary

16. Unnecesary
17. (a) $$C \setminus B$$
    (b) $$C \setminus B$$
    (c) $$A \cup B$$

1.5 The Conditional and Biconditional Connectives

1.5.1 Recapitulation

The if-then logical connective, represented by the symbol $\rightarrow$, is used to form conditional statements that expresses a relationship between the antecedent and its consequent.

The formulas $P/rightarrowQ$, $\neg P \vee Q$ and $\neg(P \wedge \neg Q)$are equivalent.

The *converse* of $P \rightarrow Q$ is $Q \rightarrow P$, its *contrapositive* is $\neg Q \rightarrow \neg P$ and its *inverse* is $\neg P \rightarrow \neg Q$

The statement of the form $P \iff Q$ is called a biconditional statement, and it is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$

1.5.2 Problems

1. 1
   (a) $$(P \vee \neg Q) \rightarrow \neg R$$
   (b) $$(P \wedge Q) \rightarrow R$$
   (c) $$(P \rightarrow R) \wedge (Q \rightarrow R)$$
   (d) $$\neg P \rightarrow (Q \rightarrow R)$$
2. (a) $$H \rightarrow (P \wedge A)$$
   (b) $$M \rightarrow (C \wedge D)$$
   (c) $$\neg S \rightarrow D$$

(d)

$$(P \vee Q) \to \neg R$$

3. (a)

$$R \to (W \wedge \neg S)$$

(b) Converse.

$$W \wedge \neg S \to R$$

(c) Equivalent.

$$R \to (W \wedge \neg S)$$

(d) Converse.

$$(W \wedge \neg S) \to R$$

(e) Equivalent.

$$(S \vee \neg W) \to \neg R$$

(f) Equivalent.

$$(R \to W) land (R \to \neg S)$$

(g) Converse.

$$(W \to R) \vee (\neg S \to R)$$

4. (a) Valid
   (b) Valid
   (c) Invalid
5. (a) Invalid
   (b) Valid
6. (a) Same truth table
   (b) Same truth table
7. (a)

$$(P \to R) \wedge (Q \to R)$$
$$(\neg P \vee R) \wedge (\neg Q \vee R)$$
$$(\neg P \wedge \neg Q) \vee R$$
$$\neg(P \vee Q) \vee R$$
$$(P \vee Q) \to R$$

(b)

$$(P \to R) \vee (Q \to R)$$
$$(\neg P \vee R) \vee (\neg Q \vee R)$$
$$(\neg P \vee \neg Q) \vee R$$
$$\neg(P \wedge Q) \vee R$$
$$(P \wedge Q) \to R$$

8. (a) Same truth table

   (b)

$$(P \to Q) \vee (Q \to R)$$
$$(\neg P \vee Q) \vee (\neg Q \vee R)$$
$$(\neg P \vee R) \vee (\neg Q \vee Q)$$
$$(\neg P \vee R) \vee T$$
$$T$$

9.

$$P \wedge Q$$
$$\neg(\neg P \vee \neg Q)$$
$$\neg(P \to \neg Q)$$

10.

$$P \iff Q$$
$$(P \to Q) \wedge (Q \to P)$$
$$\neg(\neg(P \to Q) \vee \neg(Q \to P))$$
$$\neg((P \to Q) \to \neg(Q \to P))$$

11. (a) Same truth table

    (b) Same truth table

12.

$$(a) \equiv (b) \equiv (d); (c) \equiv (e)$$

Chapter 2

Quantificational Logic

2.1 Quantifiers

2.1.1 Recapitulation

We use *quantifiers* to express how many values of a varialble make a statement true or if a statement is true for at least one value.

The symbol $\forall$ is called the *universal quantifier*, because the statement $\forall x P(x)$ says that $P(x)$ is *universally* true.

The symbol $\exists$ is called the *existentail quantifer*, because the statement $\exists x P(x)$ says that there is at least one value of x for which $P(x)$ is true.

In general, if x is a free variable in some statement $P(x)$, it is a bound variable in the satements $\forall x P(x)$ and $\exists x P(x)$. For this reason we say that the quantifiers *bind* a variable. This means that a variable that is bound by a quantifier can always be replaced with a new variable without changing the meaning of the statement, and it is ofter possible to paraphrase the statemnt without mentioning the bound variable at all.

2.1.2 Problems

1. 1

   (a)
   $$\forall x(\exists y F(x, y) \to S(x))$$

   (b)
   $$\neg\exists x(C(x) \land \forall y(D(y) \to S(x, y)))$$

   (c)
   $$\forall x((x \neq Mary) \to L(x, Mary))$$

   (d)
   $$\exists x(P(x) \land S(Jane, x)) \land \exists y(P(y) \land S(Roger, y))$$

(e)

$$\exists x(P(x) \land S(Jane, x) \land S(Roger, x))$$

2. (a)

$$\forall x(R(x) \to \exists y(U(y, x) \land R(y)))$$

(b)

$$\exists x(D(x) \land M(x)) \to \forall x(\exists y(F(x, y) \land D(y)) \to Q(x))$$

(c)

$$\neg \exists x F(x) \to \forall x(A(x) \to \exists y(D(y) \land T(x, y)))$$

(d)

$$\exists x D(x) \to D(Jones)$$

(e)

$$D(Jones) \to \forall x D(x)$$

3. (a) x and y

$$\forall z((z > x) \to (z > y))$$

(b) No free variables

$$\forall a(\exists x(ax^2 + 4x - 2 = 0) \iff a \geq -2)$$

(c) No free variables

$$\forall x((x^3 - x < 3) \to (x < 10))$$

(d) w

$$(\exists x(x^2 + 5x = w) \land existsy(4 - y^2 = w)) \to (-10 < w < 10)$$

4. (a) For all people, if that person is a man and is not married to someone then that person is unhappy. (All unmarried men are unhappy.)
   (b) There is a person that has a child, and a sister.

5. (a) If a number is a prime and is not 2 then the number is odd.
   (b) There is a perfect number that is larger than all the other perfect numbers.

6. (a) There is no number that is the solution of both $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. True
   (b) There is no real solution to both $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. True
   (c) There is no real solution to $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. False

7. (a) False.

(b) True.

(c) False.

(d) True.

(e) True.

8. (a) True.

    (b) False.

    (c) False.

    (d) True.

    (e) True.

    (f) False.

9. (a) True.

    (b) False.

    (c) True.

    (d) False.

    (e) True.

    (f) True.

10. (a) True.

    (b) False.

    (c) False.

    (d) True.

    (e) True.

    (f) True.

2.2 Equivalentces Involving Quantifiers

2.2.1 Recapitulation

Quantifier NEgation laws

$$\neg\exists x P(x) \text{ is equivalent to } \forall x \neg P(x)$$

$$\neg\forall x P(x) \text{ is equivalent to } \exists x \neg P(x)$$

Reversing the order of two quantifiers can sometimes change the meaning of a formula. However, if the quantifiers are the same type, they can be switched without affecting the meaning of the formula.

To indicate that there exists *exactly one* element, we use $\exists!$.

$$\exists! x P(x) \equiv \exists x (P(x) \land \neg\exists y (P(y) \land y \neq x))$$

Sometimes it is useful to write the truth set of $P(x)$ as $\{x \in U \mid P(x)\}$. Similarly, instead of writing $\forall x P(x)$ we can write $\forall x \in U, P(x)$ which is equivalent to $\forall x(x \in A \to P(x))$. Similarly for $\exists x \in U, P(x)$, we can write $\exists x(x \in A \wedge P(x))$

Quantifiers in formulas where the set of discourse is specified are sometimes called *bounded quantifiers*, because they place *bounds on which values of* x are to be considered.

A statement is *vacuously true* if the set of discourse is the empty set.

The empty set is a subset of every set.

The universal quantifier distributes over conjunction $\forall x(E(x) \wedge T(x)) \equiv \forall x E(x) \wedge \forall x T(x)$

The existential quantifier distributes over disjunction $\exists x(E(x) \vee T(x)) \equiv \exists x E(x) \vee \exists x T(x)$

2.2.2 Problems

1. (a)

$$\neg(\forall x(M(x) \to \exists y(F(x,y) \wedge H(y))))$$
$$\exists x \neg(M(x) \to \exists y(F(x,y) \wedge H(y)))$$
$$\exists x(M(x) \wedge \neg existsy(F(x,y) \wedge H(y)))$$
$$\exists x(M(x) \wedge \forall y(\neg F(x,y) \vee \neg H(y)))$$
$$\exists x(M(x) \wedge \forall y(F(x,y) \to \neg H(y)))$$

(b)

$$\neg(\forall x \exists y(R(x,y) \wedge \forall z(\neg L(y,z))))$$
$$\exists x \forall y(\neg R(x,y) \vee \exists z(L(y,z)))$$
$$\exists x \forall y(R(x,y) \to \exists z(L(y,z)))$$

(c)

$$\neg(A \cup B \subseteq C \setminus D)$$
$$\neg \forall x((x \in A \vee x \in B) \to (x \in C \wedge x \notin D))$$
$$\exists x \neg((x \in A \vee x \in B) \to (x \in C \wedge x \notin D))$$
$$\exists x((x \in A \vee x \in B) \wedge \neg(x \in C \wedge x \notin D))$$
$$\exists x((x \in A \vee x \in B) \wedge (x \notin C \vee x \in D))$$

(d)

$$\neg(\exists x \forall y(y > x \to \exists z(z^2 + 5z = y)))$$
$$\forall x \exists y(y > x \wedge \forall z(z^2 + 5z \neq y))$$

2. (a)

$$\neg(\exists x(F(x) \wedge \neg \exists y(R(x,y))))$$
$$\forall x(\neg F(x) \vee \exists y(R(x,y)))$$
$$\forall x(F(x) \to \exists y(R(x,y)))$$

(b)

$$\neg(\forall x \exists y (L(x,y)) \wedge \neg \exists x \forall y (L(x,y)))$$

$$\exists x \forall y (\neg L(x,y)) \vee \exists x \forall y (L(x,y))$$

(c)

$$\neg(\forall a \in A \exists b \in B (a \in C \iff b \in C))$$

$$\exists a \in A \forall b \in B \neg (a \in C \iff b \in C)$$

$$\exists a \in A \forall b \in B ((a \notin C \vee b \notin C) \wedge (a \in C \vee b \in C))$$

(d)

$$\neg(\forall y > 0 \exists x (ax^2 + bx + c = y))$$

$$\exists y > 0 \forall x (ax^2 + bx + c \neq y)$$

3. (a) True. All Natural numbers smaller than 7 can be represented as the sum of the squares of 3 natural numbers
   (b) False. Equation has two solutions in the natural numbers
   (c) True. Only solution in the natural numbers is 5
   (d) True. 5 solves both equations.

4.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$P(x) \equiv \neg\neg P(x)$$

$$\neg \forall x \neg\neg P(x))$$

$$\exists x \neg P(x))$$

5.

$$\neg \exists x \in A P(x)$$

$$\neg \exists x (x \in A \wedge P(x))$$

$$\forall x \neg (x \in A \wedge P(x))$$

$$\forall x (x \notin A \vee \neg P(x))$$

$$\forall x (x \in A \rightarrow \neg P(x))$$

$$\forall x \in A \neg P(x)$$

6.

$$\exists x (P(x) \vee Q(x))$$

$$\neg\neg(\exists x (P(x) \vee Q(x)))$$

$$\neg(\forall x \neg (P(x) \vee Q(x)))$$

$$\neg(\forall x (\neg P(x) \wedge \neg Q(x)))$$

$$\neg(\forall x \neg P(x) \wedge \forall x \neg Q(x))$$

$$\exists x P(x) \vee \exists x Q(x)$$

7.

$$\exists x(P(x) \to Q(x))$$
$$\exists x(\neg P(x) \lor Q(x))$$
$$\exists x \neg P(x) \lor \exists x Q(x)$$
$$\neg \forall x P(x) \lor \exists x Q(x)$$
$$\forall x P(x) \to \exists x Q(x)$$

8.

$$(\forall x \in A P(x)) \land (\forall x \in B P(x))$$
$$\forall x(x \in A \to P(x)) \land \forall x(x \in B \to P(x))$$
$$\forall x(x \in A \to P(x) \land (x \in B \to P(x)))$$
$$\forall x((x \in A \land x \in B) \to P(x))$$
$$\forall x \in A \cup B, P(x)$$

9. No, first statement is true for every person with any of those two conditions, second one is true when for any person that with a condition or other person with that other condition. That is, the first statement operate over the same subject while the second one doesn't.

10. (a)

$$\exists x \in A, P(x) \lor \exists x \in B, P(x)$$
$$\exists x(x \in A \land P(x)) \lor \exists x(x \in B \land P(x))$$
$$\exists x((x \in A \land P(x)) \lor (x \in B \land P(x)))$$
$$\exists x(x \in A \lor x \in B) \land P(x)$$
$$\exists x \in A \cup B, P(x)$$

(b) No. The first statement says that there is an element in each of the sets that has the property, the second statement says that there is an element in both sets that has the property.

11.

$$A \subseteq B \equiv \forall x(x \in A \to x \in B)$$
$$A \setminus B \equiv \neg \exists x(x \in A \land x \notin B)$$
$$\neg \exists x(x \in A \land x \notin B) \equiv \forall x(x \notin A \lor x \in B)$$
$$\equiv \forall x(x \in A \to x \in B)$$

12.

$$C \subseteq A \cup B$$

$$\forall x(x \in C \to (x \in A \lor x \in B))$$

$$\forall x(x \notin C \lor (x \in A \lor x \in B))$$

$$\forall x((x \notin C \lor x \in A) \lor x \in B)$$

$$\forall x((x \in C \land x \notin A) \to x \in B)$$

$$(C \setminus A) \subseteq B$$

13. (a)

$$A \subseteq B$$

$$\forall x(x \in A \to x \in B)$$

$$\forall x((x \in A \lor x \in B) \iff x \in B)$$

$$A \cup B = B$$

(b)

$$A \subseteq B$$

$$\forall x(x \in A \to x \in B)$$

$$\forall x((x \in A \land x \in B) \iff x \in A)$$

$$A \cap B = B$$

14.

$$\neg \exists x(x \in A \land x \in B)$$

$$\forall x(x \notin A \lor x \notin B)$$

$$\forall x(x \in A \to x \notin B)$$

$$\forall x((x \in A \land x \notin B) \iff x \in A)$$

$$A \setminus B = A$$

15. (a) x is a teacher who has exactly one student.
    (b) There exists a teacher x which teaches exactly one student.
    (c) There exists a single teacher that has at least one student.
    (d) There exists a single student that has a single teacher.
    (e) There exists a single teacher that teaches a single student.
    (f) There exists a single teacher and a single student that is taught by them.

2.3 More Operations on Sets

2.3.1 Recapitulation

A set defined in the form $P = \{p_i \mid i \in I\}$, where each element p_i in the set is identified by a number $i \in I, I = \{i \in \mathbb{N} \mid 1 \leq i \leq x\}$ called the *index* of the element, is called an *indexed family* and I is called the *index set.*

$$S = \{P(a) \mid a \in A\}$$

$$S = \{x \mid \exists a \in A, (x = P(a))\}$$

Although the indices for an indexed famility are often numbers, they need not to be.

Sets can have other sets as elements. Sets that contain other sets as elements are sometimes called *families* of sets.

Definition *Suppose that A is a set. The power set of A, denoted by $\mathscr{P}(A)$, is the set whose elements are all the subsets of A. In other words,*

$$\mathscr{P}(A) = \{x \mid x \subseteq A\}$$

Definition *Suppose that $\mathscr{F}$ is a family of sets. Then the intersection and union of $\mathscr{F}$ are the sets $\cap\mathscr{F}$ and $\cup\mathscr{F}$ defined as follows*

$$\cap\mathscr{F} = \{x \mid \forall A \in \mathscr{F}(x \in A)\} = \{x \mid \forall A(A \in \mathscr{F} \to x \in A)\}$$

$$\cap\mathscr{F} = \{x \mid \exists A \in \mathscr{F}(x \in A)\} = \{x \mid \exists A(A \in \mathscr{F} \land x \in A)\}$$

Supposing $\mathscr{F} = \{A_i \mid i \in I\}$,

$$\bigcap \mathscr{F} = \bigcap_{i \in I} A_i = \{x \mid \forall i \in I(x \in A_i)\}$$

$$\bigcup \mathscr{F} = \bigcup_{i \in I} A_i = \{x \mid \exists i \in I(x \in A_i)\}$$

2.3.2 Problems

1. (a)

$$\mathscr{F} \subseteq \mathscr{P}(A)$$

$$\forall x(x \in \mathscr{F} \to x \in \mathscr{P}(\mathscr{A}))$$

$$\forall x(x \in \mathscr{F} \to \forall y(y \in x \to y \in A))$$

(b)

$$A \subseteq \{2n + 1 \mid n \in \mathbb{N}\}$$

$$\forall x(x \in A \to x \in \{2n + 1 \mid n \in \mathbb{N}\})$$

$$\forall x(x \in A \to \exists n \in \mathbb{N}(x = 2n + 1))$$

(c)

$$\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\}$$

$$\forall x(x \in \{n^2+n+1 \mid n \in \mathbb{N}\} \to x \in \{2n+1 \mid n \in \mathbb{N}\})$$

$$\forall x(\exists n \in \mathbb{N}(x = n^2+n+1) \to \exists n \in \mathbb{N}(x = 2n+1))$$

(d)

$$\mathscr{P}(\bigcup_{i \in I} A_i) \not\subseteq \bigcup_{i \in I} \mathscr{P}(A_i)$$

$$\exists x(x \in \mathscr{P}(\bigcup_{i \in I} A_i) \wedge x \notin \bigcup_{i \in I} \mathscr{P}(A_i))$$

$$\exists x(x \subseteq \bigcup_{i \in I} A_i \wedge \neg\exists i \in I(x \in \mathscr{P}(A_i)))$$

$$\exists x(x \subseteq \bigcup_{i \in I} A_i \wedge \forall i \in I(x \notin \mathscr{P}(A_i)))$$

$$\exists x(\forall y(y \in x \to y \in \bigcup_{i \in I} A_i) \wedge \forall i \in I(x \not\subseteq A_i))$$

$$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \wedge \forall i \in I(\exists y(y \in x \wedge y \notin A_i)))$$

2. (a)

$$x \in \bigcup \mathscr{F} \setminus \bigcup \mathscr{G}$$

$$\exists A \in \mathscr{F}(x \in A) \wedge \forall B \in \mathscr{G}(x \notin B)$$

(b)

$$\{x \in B \mid x \notin C\} \in \mathscr{P}(A)$$

$$\{x \in B \mid x \notin C\} \subseteq A$$

$$\forall x((x \in B \wedge x \notin C) \to x \in A)$$

(c)

$$x \in \bigcap_{i \in I}(A_i \cup B_i)$$

$$x \in \{x \mid \forall i \in I(x \in (A_i \cup B_i))\}$$

$$\forall i \in I(x \in A_i \vee x \in B_i))$$

(d)

$$x \in (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$$

$$x \in \{x \mid \forall i \in I(x \in A_i)\} \cup \{x \mid \forall i \in I(x \in B_i)\}$$

$$\forall i \in I(x \in A_i) \vee \forall i \in I(x \in B_i)$$

3. The power set of a set that contains the empty set $\mathscr{P}(\{\emptyset\}) = \{\{\emptyset\}, \emptyset\}$

4.
$$\cap\mathscr{F} = \{\text{red}\}$$
$$\cup\mathscr{F} = \{\text{red}, \text{green}, \text{blue}, \text{orange}, \text{purple}\}$$

5.
$$\cap\mathscr{F} = \emptyset$$
$$\cup\mathscr{F} = \{3, 5, 7, 12, 16, 23\}$$

6. (a)
$$A_2 = \{1, 2, 3, 4\}; A_3 = \{2, 3, 4, 6\}; A_4 = \{3, 4, 6, 8\}; A_5 = \{4, 5, 6, 10\}$$

(b)
$$\cap_{i \in I} A_i = \{4\}$$
$$\cup_{i \in I} A_i = \{1, 2, 3, 4, 5, 6, 8, 10\}$$

7. Unnecesary

8. (a)
$$A_2 = \{2, 4\}; A_3 = \{3, 6\}$$
$$B_2 = \{2, 3\}; B_3 = \{3, 4\}$$

(b)
$$\cap_{i \in I}(A_i \cup B_i) = \{2, 3, 4\} \cap \{3, 4, 6\} = \{3, 4\}$$
$$(\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i) = \emptyset \cup \{3\} = \{3\}$$

(c) Not equivalent

9. (a) Not equivalent
$$x \in \cup_{i \in I}(A_i \setminus B_i) \equiv \exists i \in I(x \in A_i \setminus B_i)$$
$$x \in (\cup_{i \in I} A_i) \setminus (\cup_{i \in I} B_i) \equiv \exists i \in I x \in A_i \wedge \forall i \in I x \notin B_i$$
$$x \in (\cup_{i \in I} A_i) \setminus (\cap_{i \in I} B_i) \equiv \exists i \in I x \in A_i \wedge \exists i \in I x \notin B_i$$

(b) Not equivalent
$$A_2 = \{2, 4\}; A_3 = \{3, 6\}$$
$$B_2 = \{2, 3\}; B_3 = \{3, 4\}$$
$$\cup_{i \in I}(A_i \setminus B_i) = \{4\} \cup \{6\} = \{4, 6\}$$
$$(\cup_{i \in I} A_i) \setminus (\cup_{i \in I} B_i) = \{2, 3, 4, 6\} \setminus \{2, 3, 4\} = \{6\}$$
$$(\cup_{i \in I} A_i) \setminus (\cap_{i \in I} B_i) = \{2, 3, 4, 6\} \setminus \{3\} = \{2, 4, 6\}$$

10.

$$I = \{1,2\}, A_1 = \{a,b\}, A_2 = \{c,d\}, B_1 = \{a,e\}, B_2 = \{b,f\}$$
$$\cup_{i\in I}(A_i \cap B_i) = \{a\}$$
$$(\cup_{i\in I}A_i) \cap (\cup_{i\in I}B_i) = \{a,b\}$$

11.

$$x \in \mathscr{P}(A \cap B)$$
$$x \subseteq A \cap B$$
$$\forall y(y \in x \to y \in A \cap B)$$
$$\forall y(y \in x \to (y \in A \wedge y \in B)$$
$$\forall y(y \in x \to y \in A) \wedge \forall y(y \in x \to y \in B)$$
$$x \in \mathscr{P}(A) \cap x \in \mathscr{P}(B)$$

12.

$$A = \{a\}; B = \{b\}$$
$$\mathscr{P}(A \cup B) = \{a, b, \{a,b\}\}$$
$$\mathscr{P}(A) \cup \mathscr{P}(B) = \{a,b\}$$

13. (a)

$$\cup_{i\in I}(A_i \cup B_i) = (\cup_{i\in I}A_i) \cup (\cup_{i\in I}A_i)$$
$$x \in \cup_{i\in I}(A_i \cup B_i)$$
$$\exists i \in I(x \in A_i \cup B_i)$$
$$\exists i \in I(x \in A_i \vee x \in B_i)$$
$$\exists i \in I(x \in A_i) \vee \exists i \in I(x \in B_i)$$
$$x \in (\cup_{i\in I}A_i) \cup (\cup_{i\in I}B_i)$$

(b)

$$(\cap\mathscr{F}) \cap (\cap\mathscr{G})$$
$$x \in (\cap\mathscr{F}) \cap (\cap\mathscr{G})$$
$$\forall A \in \mathscr{F}(x \in A) land \forall A \in \mathscr{G}(x \in A)$$
$$\forall A \in \mathscr{F} \cup \mathscr{G}(x \in A)$$
$$\cap(\mathscr{F} \cup \mathscr{G})$$

(c)

$$\cap_{i\in I}(A_i \setminus B_i)$$
$$\forall i \in I(x \in A_i \wedge x \notin B_i)$$
$$\forall i \in I(x \in A_i) \wedge \neg\exists i \in Ix \in B_i)$$
$$(\cap_{i\in I}A_i) \setminus (\cup_{i\in I}B_i)$$

14. (a)

$$B_3 = \cup_{i\in I} A_{i,3} = A_{1,3} \cup A_{2,3} = \{1,2,3,4,5\}$$
$$B_4 = \cup_{i\in I} A_{i,4} = A_{1,4} \cup A_{2,4} = \{1,2,4,5,6\}$$

(b)

$$\cap_{j\in J} B_j = \cap_{j\in J}(\cup_{i\in I} A_{i,j})$$
$$\cap_{j\in J}(A_{1,j} \cup A_{2,j})$$
$$(A_{1,3} \cup A_{2,3}) \cap (A_{1,4} \cup A_{2,4})$$
$$\{1,2,4,5\}$$

(c) Not equal

$$\cup_{i\in I}(\cap_{j\in J} A_{i,j})$$
$$\cup_{i\in I}(A_{i,3} \cap A_{i,4})$$
$$(A_{1,3} \cap A_{1,4}) \cup (A_{2,3} \cap A_{2,4})$$
$$\{1,4\} \cup \{2\}$$
$$\{1,2,4\}$$

(d) Not equivalent

$$x \in \cap_{j\in J}(\cup_{i\in I} A_{i,j})$$
$$\forall j \in J(x \in \cup_{i\in I} A_{i,j})$$
$$\forall j \in J(\exists i \in I(x \in A_{i,j}))$$

-

$$x \in \cup_{i\in I}(\cap_{j\in J} A_{i,j})$$
$$\exists i \in I(x \in \cap_{j\in J} A_{i,j})$$
$$\exists i \in I(\forall j \in J(x \in A_{i,j}))$$

15. (a)

$$\mathscr{F} = \emptyset$$
$$x \in \cup\mathscr{F}$$
$$\exists A \in \mathscr{F}(x \in A)$$

False, since there is no set in $\mathscr{F}$

(b)

$$\mathscr{F} = \emptyset$$
$$x \in \cap\mathscr{F}$$
$$\forall A \in \mathscr{F}(x \in A)$$

True, since x is in every set of F, vacuously true

16. (a)

$$R = \{A \in U \mid A \notin A\}$$
$$\forall R \in U(R \in R \iff R \notin R)$$

(b) There can't be a set that contains all sets.

Chapter 3

Proofs

3.1 Proof Strategies

3.1.1 Recapitulation

3.1.2 Problems

3.2 Proofs Involving Negations and Conditionals

3.2.1 Recapitulation

3.2.2 Problems

3.3 Proofs Involving Quantifiers

3.3.1 Recapitulation

3.3.2 Problems

3.4 Proofs Involving Conjunctions and Biconditionals

3.4.1 Recapitulation

3.4.2 Problems

3.5 Proofs Involving Disjunctions

3.5.1 Recapitulation

3.5.2 Problems

3.6 Existence and Uniqueness Proofs

3.6.1 Recapitulation

3.6.2 Problems

3.7 More Examples of Proofs

3.7.1 Recapitulation

Chapter 4

Relations

Chapter 5

Functions

Chapter 6

Mathematical Induction

Chapter 7

Number Theory

Chapter 8

Infinite Sets