How To Prove It: A Structured Approach

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0.1.1 Recapitulation

High school mathematics is concerd mostly with solving equations and computing answers to numerical questions. College mathematics deals with a wider variety of questions, involving not only numbers, but also sets, functions, and other mathematical objects. What ties them together is the use of deductive reasoning to find the answers to questions.

Deductive reasoning uses general ideas to come to a specific conclusion. Deductive reasoning in mathematics is usually presented in the form of a *proof*.

A number is *prime* of it cannot be written as a product of two smaller positive integers. If it can be written as a product of two smaller positive integers, then it is *composite*

A *conjecture* is a conclusion that is proffered on a tentative basis without proof.

Conjecture 1. *Suppose n is an integer larger than 1 and n is prime. Then $2^n - 1$ is prime.*

Conjecture 2. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

A *counterexample* is a specific instance that demonstrates the falsity of a general statement, argument or theory.

The existence of even one counterexample establishes that a conjecture is incorrect. However, failure to find a counterexample to a conjecture does not show that the conjecture is correct.

We can never be sure that the conjecture is correct if we only check examples. No matter how many examples we check, there is always the possibility that the next one will be the first counterexample.

Once a conjecture has been proven, we can call it a *theorem.*

Theorem 3. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime*

Prime numbers of the form $2^n - 1$ are called *Mersenne primes*. Although many Mersenne primes have been found, it is still not know if there are infinitely many of them.

A positive integer n is said to be *perfect* if n is equal to the sum of all positive integers smaller than n that divide n.

For any two integers m and n, we say that m *divides* n if n is divisible by m; in other words, if there is an integer q such that $n = qm$.

For any positive integer n, the product of all integers from 1 to n is called n *factorial* and is denoted $n!$.

Theorem 4. *For every positive integer n, there is a sequence of n consecutive positive integers containing no primes*

Pairs of primes that differ by only two are called *twin primes*

0.1.2 Problems

1. (a)

$$2^{ab} - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + ... + 2^{(a-1)b})$$
$$n = ab = 15; a = 3; b = 5$$
$$2^{(3)(5)} = (2^5 - 1)(1 + 2^5 + 2^{10}) = (31) \cdot (1057)$$

(b)

$$2^{ab} - 1 = (2^b - 1)(1 + 2^b + 2^{2b} + ... + 2^{(a-1)b})$$
$$n = ab = 32767; a = 1057; b = 31$$
$$2^{1057 \cdot 31} - 1 = (2^{31} - 1)(1 + 2^{31} + ... + 2^{1056 \cdot 31})$$
$$1 < 2^{31} - 1 < 2^{32767} - 1$$

2.

$3^n - 1$		
$n=1$	$3^1 - 1 = 2$	Prime
$n=2$	$3^2 - 1 = 8$	Not Prime
$n=3$	$3^3 - 1 = 26$	Not Prime
$n=4$	$3^4 - 1 = 80$	Not Prime
$n=5$	$3^5 - 1 = 242$	Not Prime
$n=6$	$3^6 - 1 = 728$	Not Prime
$n=7$	$3^7 - 1 = 2186$	Not Prime
$n=8$	$3^8 - 1 = 6560$	Not Prime
$n=9$	$3^9 - 1 = 19682$	Not Prime
$n=10$	$3^{10} - 1 = 59048$	Not Prime

For all $n > 1$, $3^n - 1$ is not a prime

$3^n - 2^n$		
$n=1$	$3^1 - 2^1 = 1$	Prime
$n=2$	$3^2 - 2^2 = 5$	Prime
$n=3$	$3^3 - 2^3 = 19$	Prime
$n=4$	$3^4 - 2^4 = 65$	Not Prime
$n=5$	$3^5 - 2^5 = 211$	Prime
$n=6$	$3^6 - 2^6 = 665$	Not Prime
$n=7$	$3^7 - 2^7 = 2059$	Not Prime
$n=8$	$3^8 - 2^8 = 6305$	Not Prime
$n=9$	$3^9 - 2^9 = 19171$	Not Prime
$n=10$	$3^{10} - 2^10 = 58025$	Not Prime

If n is not prime, then $3^n - 2^n$ is not prime. If n is prime, then $3^n - 2^n$ is either prime or composite.

3. (a)

$$m = p_1 \cdot p_2 \cdot ... \cdot p_n + 1$$
$$m = 2 \cdot 3 \cdot 5 \cdot 7 + 1$$
$$m = 211$$

(b)

$$m = p_1 \cdot p_2 \cdot ... \cdot p_n + 1$$

$$m = 2 \cdot 5 \cdot 11 + 1$$

$$m = 111 = 3 \cdot 37$$

4.

$$x = (n+1)! + 2; n = 5$$

$$x = (5+1)! + 2 = 6! + 2 = 722$$

$$[722, 723, 724, 725, 726]$$

5.

$n = 3$	$2^{n-1}(2^n - 1) = 2^{3-1}(2^3 - 1) = 28$
$n = 5$	$2^{n-1}(2^n - 1) = 2^{5-1}(2^5 - 1) = 496$

6.

$$mod(n,3) = \begin{cases} 0, \text{n is divisible by 3;} \\ 1, \text{n + 2 is divisible by 3} \\ 2, \text{n + 4 is divisible by 3} \end{cases} \tag{1}$$

7.

$$220 : [1, 2, 4, 5, 10, 11, 22, 44, 55, 110]$$

$$1 + 2 + 4 + 5 + 10 + 11 + 22 + 44 + 55 + 110 = 284$$

$$284 : [1, 2, 4, 71, 142]$$

$$1 + 2 + 4 + 71 + 142 = 220$$

Chapter 1

Sentential Logic

1.1 Deductive Reasoning and Logical Connectives

1.1.1 Recapitulation

Profs play a central role in mathematics, and deductive reasoning is the foundation on which proofs are based.

We arrive at a *conclusion* from the assumption that some other statements, called *premises*, are true.

We will say that an argument is *valid* if the premises cannot all be true without the conclusion being true as well. And is *invalid* otherwise.

Connective symbols are symbols used to combine statements to form more complex statements.

Symbol	Meaning
$\vee$	or
$\wedge$	and
$\neg$	not

The statement $P \vee Q$ is called the disjunction of P and Q, the statement $P \wedge Q$ is called the conjunction of P and Q, and $\neg P$ is called the negation of P.

1.1.2 Problems

1. (a)

$$P : \text{We'll have reading assignment}$$

$$Q : \text{We'll have homework problems}$$

$$R : \text{We'll have a test}$$

$$(P \vee Q) \wedge \neg(Q \wedge R)$$

(b)

$$P : \text{You will go skiing}$$
$$Q : \text{There will snow}$$
$$\neg P \vee (P \wedge \neg Q)$$

(c)

$$\neg((\sqrt{7} < 2) \vee (\sqrt{7} = 2))$$

2. (a)

$$P : \text{John is telling the truth}$$
$$Q : \text{Bill is telling the truth}$$
$$(P \wedge Q) \vee (\neg P \wedge \neg Q)$$

(b)

$$P : \text{I'll have fish}$$
$$Q : \text{I'll have chicken}$$
$$R : \text{I'll have mashed potatoes}$$
$$(P \vee Q) \wedge \neg(P \wedge R)$$

(c)

$$P : 3 \text{ is a common divisor of } 6$$
$$Q : 3 \text{ is a common divisor of } 9$$
$$R : 3 \text{ is a common divisor of } 15$$
$$P \wedge Q \wedge R$$

3. (a)

$$\neg(P \wedge Q)$$

(b)

$$\neg P \wedge \neg Q$$

(c)

$$\neg P \vee \neg Q$$

(d)

$$\neg(P \vee Q)$$

4. (a)

$$(P \wedge Q) \vee (R \wedge S)$$

(b)

$$(P \vee R) \wedge (Q \vee S)$$

(c)

$$\neg(P \vee R) \wedge \neg(Q \vee S)$$

(d)

$$\neg((P \wedge R) \vee (Q \wedge S))$$

5. (a) Well-formed
   (b) Not well-formed
   (c) Well-formed
   (d) Not well-formed

6. (a) I won't buy the pants without a shirt
   (b) I won't buy the pants nor the shirt
   (c) Either I won't buy the pants or I won't buy the shirt

7. (a) Either Steve or George is happy, and Either Steve or George is not happy
   (b) Either Steve is happy, or George is happy and Steve isn't happy or George is not happy
   (c) Either Steve is happy, or George is happy and Either Stever or George are not happy

8. (a) Either taxes will go up or The deficit will go up
   (b) The taxes and the deficit won't go up and it is not the case that the taxes and the deficit won't go up
   (c) Either the taxes will go up and the deficit won't go up, or the deficit will go up and the taxes won't go up

9. (a) Conclusion: Pete will win the chemistry prize
   (b) Conclusion: We will not have both beef as a main course and peas as a vegetable.
   (c) Conclusion: Either John is telling the truth or Sam is lying.
   (d) Conclusion: Sales and expenses will not both go up.

1.2 Truth Tables

1.2.1 Recapitulation

When we evaluate the truth or falsity of a statement, we assign to it one of the labels *true* or *false*, and this label is called its *truth value*.

A *Truth table* is a table in which each of its rows shows one of the possible combinations of truth values for a statement or a compound statement.

To verify the validity of arguments, we can arrage the truth values of the premises and the conclusion in a truth table, in the rows where the premises are all true it must follow that the conclusion is also true, thus the argument is valid, otherwise it is invalid.

Equivalent formulas always have the same truth value no matter what the truth value of those statements are.

Formulas that are always true, are called *tautologies*, formulas that are always false are called *contradictions*.

1.2.2 Problems

1. (a)

P	Q	$\neg P \vee Q$
F	F	T
F	T	T
T	F	F
T	T	T

   (b)

S	G	$(S \vee G) \wedge (\neg S \vee \neg G)$
F	F	F
F	T	T
T	F	T
T	T	F

2. (a)

P	Q	$\neg(P \wedge (Q \vee \neg P))$
F	F	T
F	T	T
T	F	T
T	T	F

   (b)

P	Q	R	$(P \vee Q) \wedge (\neg P \vee R)$
F	F	F	F
F	F	T	F
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	F
T	T	T	T

3. (a)

P	Q	$P \oplus Q$
F	F	F
F	T	T
T	F	T
T	T	F

   (b)

P	Q	$(P \vee Q) \wedge \neg(P \wedge Q)$
F	F	F
F	T	T
T	F	T
T	T	F

4.

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

5. (a)

P	Q	$P \downarrow Q$
F	F	T
F	T	F
T	F	F
T	T	F

(b)

$$P \downarrow Q \equiv \neg P \wedge \neg Q \equiv \neg(P \vee Q)$$

(c)

$$\neg P \equiv P \downarrow P$$

$$P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$$

$$P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$$

6. (a)

P	Q	$P \mid Q$
F	F	T
F	T	T
T	F	T
T	T	F

(b)

$$\neg(P \wedge Q)$$

(c)

$$\neg P \equiv P \mid P$$

$$P \vee Q \equiv (P \mid P) \mid (Q \mid Q)$$

$$P \wedge Q \equiv (P \mid Q) \mid (P \mid Q)$$

7. (a) Valid
   (b) Invalid
   (c) Valid
   (d) Invalid

8.

$$(a) \equiv (c), (b) \equiv (e)$$

9. (b) is a contradiction; (c) is a tautology

10. Not needed

11. (a)

$$\neg(\neg P \wedge \neg Q) \equiv P \vee Q$$

(b)

$$(P \wedge Q) \vee (P \wedge \neg Q) \equiv P$$

(c)

$$\neg(P \wedge \neg Q) \vee (\neg P \wedge Q) \equiv \neg P \vee Q$$

12. (a)
$$\neg(\neg P \lor Q) \lor (P \land \neg R \equiv P \land \neg(Q \land R)$$
(b)
$$\neg(\neg P \land Q) \lor (P \land \neg R) \equiv \neg Q \lor P$$
(c)
$$(P \land R) \lor (\neg R \land (P \lor Q))\ equiv \neg P \land \neg Q$$
13. Not needed
14. Not needed
15.
$$2^n$$
16.
$$P \lor \neg Q$$
17.
$$(P \land \neg Q) \lor (\neg P \land Q)$$
18. (1) That it is valid; (2) That it is invalid if there is a combination where all premises are true; (3) Could be either valid or invalid for a tautology, if it is a contradiction then it is valid

1.3 Variables and Sets

1.3.1 Recapitulation

It is often necesary to make statements about objects that are represented by letters called *variables.*

A notation like $P(x)$ can be interpreted as a statement P about the variable x.

In a statement that contains variables we cannot describe the statement as being simply true or false. Its truth value might depend on the values of the variables involved.

A *set* is a collection of objects. The objects in the collection are called the *elements* of the set.

We use the symbol $\in$ to mean *is an element of.* To say that an element is not part of a set we use the symbol $\notin$.

The following notations for sets are valid:

- $A = \{1, 2, 3\}$ is set that contains the numbers 1, 2 and 3
- $B = \{2, 3, 5, 7, 11, 13, 17, ...\}$ is a set that contains all of the prime numbers
- $C = \{x \mid x \text{ is a prime number}\}$ is a set that defines an *elementhood test* for the set, any value of x that passes the test is an element of the set.

Free variables in a statement stand for objects that the statement says something about. *Bound* variables are simply letters that are used as a convenience to help express an idea and should not be thought of as standing for any particular object. A bound variable can always be replaced by a new variable without chaning the meaning of the statement, and often the statement can be rephrased so that the bound variables are eliminated altogether.

To distinguish the values of a statement that contains free variables that make the statement true from those that make it false, we form the set of values of the free variables for which the statement is true. We call this the *truth set* of the statement.

For a statement, the set of all the objects that represent the free variables is called the *universe of discourse* for the statement. And we say that the variables *range over* this universe.

A set without any elements is called an *empty set*, or the *null set*, and it is often denoted by $\emptyset$

1.3.2 Problems

1. (a)

$$P(x,y) : x \text{ is divisible by } y$$

$$P(6,3) \wedge P(9,3) \wedge P(15,3)$$

   (b)

$$P(x,y) : x \text{ is divisible by } y$$

$$P(x,2) \wedge P(x,3) \wedge \neg P(x,4)$$

   (c)

$$N(x) : x \text{ is a natural number}$$

$$P(x) : x \text{ is a prime number}$$

$$N(x) \wedge N(y) \wedge (P(x) \oplus P(y)) \equiv N(x) \wedge N(y) \wedge ((P(x) \wedge \neg P(y)) \vee (P(y) \wedge \neg P(x))$$

2. (a)

$$P(x) : x \text{ is a man}$$

$$Q(x,y) : x \text{ is taller than } y$$

$$P(x) \wedge P(y) \wedge (Q(x,y) \vee Q(y,x))$$

   (b)

$$P(x) : x \text{ has brown eyes}$$

$$Q(x) : x \text{ has red hair}$$

$$(P(x) \vee P(y)) \wedge (Q(x) \vee Q(y))$$

(c)

$$P(x) : x \text{ has brown eyes}$$

$$Q(x) : x \text{ has red hair}$$

$$(P(x) \wedge Q(x)) \vee (P(y) \wedge Q(y))$$

3. (a)

$$\{x \mid x \text{ is a planet in the solar system}\}$$

(b)

$$\{x \mid x \text{ is an Ivy League School}\}$$

(c)

$$\{x \mid x \text{ is a state in the US}\}$$

(d)

$$\{x \mid x \text{ is a province in Canada}\}$$

4. (a)

$$\{x \mid x \text{ is the square of a positive integer}\}$$

(b)

$$\{x \mid \text{ x is a power of 2}\}$$

(c)

$$\{x \mid x \in \mathbb{Z} \wedge 10 \leq x \leq 19\}$$

5. (a)

$$-3 \in \{x \in \mathbb{R} \mid x < 6\}; \text{True statement, x is bound}$$

(b)

$$4 \in \{x \in \mathbb{R}^- \mid x < 6\}; \text{False statement, x is bound}$$

(c)

$$\neg(5 \notin \{x \in \mathbb{R} \mid x < \frac{13-c}{2}\})\text{c is free, x is bound}$$

6. (a)

$$(w \in \mathbb{R}) \wedge (13 - 2w > c); \text{w and c are free, x is bound}$$

(b)

$$(4 \in \mathbb{R}) \wedge (5 \text{ is a prime number})); \text{Statement is true, x and y are bound}$$

(c)

$$(4 \text{ is a prime number}) \wedge (5 > 1); \text{Statement is false, x and y are bound}$$

7. (a)

$$\{-1, \frac{1}{2}\}$$

(b)

$$\{\frac{1}{2}\}$$

(c)

$$\{-1\}$$

(d)

$$\emptyset$$

8. (a) People that Elizabeth Taylor was married to

(b)

$$\{\wedge, \vee, \neg\}$$

(c)

$$\{\text{Daniel J. Velleman}\}$$

9. (a)

$$\{1, 3\}$$

(b)

$$\emptyset$$

(c)

$$\{x \in \mathbb{R} \mid 5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}\}$$
$$\{x \in \mathbb{R} \mid 5 \in \mathbb{R} \wedge x^2 + 25 < 50\}$$
$$\{x \in \mathbb{R} \mid x^2 < 25\}$$
$$\{x \in \mathbb{R} \mid -5 < x < 5\}$$
$$\{-4, -3, 2, 3.9, 4.9999, ...\}$$

1.4 Operations on Sets

1.4.1 Recapitulation

Definition *The intersection of two sets A and B is the set $A \cap B$ defined as*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Definition *The union of two sets A and B is the set $A \cup B$ defined as*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Definition *The difference of A and B is the set $A \setminus B$ defines as*

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Sometimes it is helpful when working with operations on sets to draw pictures of the results of these operations, the most common diagram for this is a *Venn Diagram.*

Definition *The symmetric difference of A and B is the set $A\triangle B$ defines as*

$$A\triangle B = \{x \mid x \in A \text{ and } x \in B \text{ and } x \notin A \cap B\}$$

$$A\triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

Set theory operations are related to the logical connectives, but it is important to remember that although they are related, they are not interchangeable. The logical connectives can only be used to combine statements, whereas the set theory operations must be used to combine sets.

Definition *Supossing that A and B are sets. We will say that A is a subset of B if every element of A is also an element of B. We write $A \subseteq B$ to mean that A is a subset of B. A and B are said to be disjoint if they have no elements in common, that is if $A \cap B = \emptyset$*

Theorem *For any sets A and B, $(A \cup B) \setminus B \subseteq A$*

1.4.2 Problems

1. The sets $A \cap B$ and $(A \cup B) \setminus C$ are subsets of $A \cup (B \setminus C)$

   (a)
   $$A \cap B = \{3, 12\}$$

   (b)
   $$(A \cup B) \setminus C = \{1, 12, 20, 35\}$$

   (c)
   $$A \cup (B \setminus C) = \{1, 3, 12, 20, 35\}$$

2. The sets c is a subset of a, b is disjoint from a and b

   (a)
   $$A \cup B = \{UnitedStates, Germany, China, Australia, France, India, Brazil\}$$

   (b)
   $$(A \cap B) \setminus C = \emptyset$$

   (c)
   $$(B \cap C) \setminus A = \{France\}$$

3. Yes

4. Unnecesary

5. (a)

$$A \setminus (A \cap B) = x \in A \land x \notin (A \cap B) = x \in A \land \neg(x \in A \land x \in B)$$

$$x \in A \land (x \notin A \lor x \notin B)$$

$$(x \in A \land x \notin A) \lor (x \in A \land x \notin B)$$

$$(x \in A \land x \notin B)$$

$$x \in A \setminus B$$

(b)

$$A \cup (B \cap C) = x \in A \lor (x \in B \land x \in C)$$

$$(x \in A \lor x \in B) \land (x \in A \lor x \in C)$$

$$(A \cup B) \cap (A \cup C)$$

6. Unnecesary

7. (a)

$$(A \cup B) \setminus C = (x \in A \lor x \in B) \land x \notin C$$

$$(x \in A \land x \notin C) \lor (x \in B \land x \notin C)$$

$$(A \setminus C) \cup (B \setminus C)$$

(b)

$$A \cup (B \setminus C) = x \in A \lor (x \in B \land x \notin C)$$

$$(x \in A \lor x \in B) \land (x \in A \lor x \notin C)$$

$$(A \cup B) \cap (A \setminus C)$$

$$(A \cup B) \setminus (C \setminus A)$$

8. (a)

$$(A \setminus B) \cap C = (A \cap C) \setminus B$$

$$(x \in A \land x \notin B) \land x \in C$$

$$(x \in A \land (x \notin B)) \land x \in C$$

$$(x \in A \land x \in C) \land x \notin B$$

$$(A \cap C) \setminus B$$

(b)

$$(A \cap B) \setminus B = \emptyset$$

$$(x \in A \land x \in B) \land x \notin B$$

$$x \in A \land (x \in B \land x \notin B)$$

$$\emptyset$$

(c)

$$A \setminus (A \setminus B) = A \cap B$$

$$x \in A \land \neg(x \in A \land \neg x \in B)$$

$$x \in A \land (x \notin A \lor x \in B)$$

$$(x \in A \land x \notin A) \lor (x \in A \land x \in B)$$

$$\emptyset \cup (A \cap B)$$

$$A \cap B$$

9. (a) (d) (e), (b) (c)

   (a)

$$(x \in A \land x \notin B) \land x \notin C$$

   (b)

$$x \in A \land (x \notin B \lor x \in C)$$

   (c)

$$(x \in A \land x \notin B) \lor (x \in A \land x \in C)$$

   (d)

$$(x \in A \land x \notin B) \land (x \in A \land x \notin C)$$

   (e)

$$x \in A \land (x \notin B \land x \notin C)$$

10. (a)

$$A = \{1, 2\}, B = \{2, 3\}$$

   (b)

$$(x \in A \lor x \in B) \land x \notin B$$

$$(x \notin B \land x \in A) \lor (x \notin B \land x \in B)$$

$$(x \notin B \land x \in A)$$

$$(A \setminus B)$$

11. No, since $(A \setminus B) \cup B = A \cup B$. $A \setminus B \subseteq (A \setminus B) \cup B$ and $(A \setminus B) \cup B \subseteq A \cup B$

12. (a) $(A \cap D) \setminus (B \cup C)$ cannot be represented.
    (b) Yes, with irregular forms.

13. (a) $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$
    (b) $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \{3, 4, 5\}$

14. Unnecesary

15. Unnecesary

16. Unnecesary
17. (a)
$$C \setminus B$$
(b)
$$C \setminus B$$
(c)
$$A \cup B$$

1.5 The Conditional and Biconditional Connectives

1.5.1 Recapitulation

The if-then logical connective, represented by the symbol $\rightarrow$, is used to form conditional statements that expresses a relationship between the antecedent and its consequent.

The formulas $P/rightarrowQ$, $\neg P \vee Q$ and $\neg(P \wedge \neg Q)$are equivalent.

The *converse* of $P \rightarrow Q$ is $Q \rightarrow P$, its *contrapositive* is $\neg Q \rightarrow \neg P$ and its *inverse* is $\neg P \rightarrow \neg Q$

The statement of the form $P \iff Q$ is called a biconditional statement, and it is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$

1.5.2 Problems

1. 1

(a)
$$(P \vee \neg Q) \rightarrow \neg R$$
(b)
$$(P \wedge Q) \rightarrow R$$
(c)
$$(P \rightarrow R) \wedge (Q \rightarrow R)$$
(d)
$$\neg P \rightarrow (Q \rightarrow R)$$

2. (a)
$$H \rightarrow (P \wedge A)$$
(b)
$$M \rightarrow (C \wedge D)$$
(c)
$$\neg S \rightarrow D$$

(d)

$$(P \lor Q) \to \neg R$$

3. (a)

$$R \to (W \land \neg S)$$

(b) Converse.

$$W \land \neg S \to R$$

(c) Equivalent.

$$R \to (W \land \neg S)$$

(d) Converse.

$$(W \land \neg S) \to R$$

(e) Equivalent.

$$(S \lor \neg W) \to \neg R$$

(f) Equivalent.

$$(R \to W) land (R \to \neg S)$$

(g) Converse.

$$(W \to R) \lor (\neg S \to R)$$

4. (a) Valid
   (b) Valid
   (c) Invalid

5. (a) Invalid
   (b) Valid

6. (a) Same truth table
   (b) Same truth table

7. (a)

$$(P \to R) \land (Q \to R)$$
$$(\neg P \lor R) \land (\neg Q \lor R)$$
$$(\neg P \land \neg Q) \lor R$$
$$\neg(P \lor Q) \lor R$$
$$(P \lor Q) \to R$$

(b)

$$(P \to R) \lor (Q \to R)$$
$$(\neg P \lor R) \lor (\neg Q \lor R)$$
$$(\neg P \lor \neg Q) \lor R$$
$$\neg(P \land Q) \lor R$$
$$(P \land Q) \to R$$

8. (a) Same truth table
   (b)
$$(P \to Q) \vee (Q \to R)$$
$$(\neg P \vee Q) \vee (\neg Q \vee R)$$
$$(\neg P \vee R) \vee (\neg Q \vee Q)$$
$$(\neg P \vee R) \vee T$$
$$T$$

9.
$$P \wedge Q$$
$$\neg(\neg P \vee \neg Q)$$
$$\neg(P \to \neg Q)$$

10.
$$P \iff Q$$
$$(P \to Q) \wedge (Q \to P)$$
$$\neg(\neg(P \to Q) \vee \neg(Q \to P))$$
$$\neg((P \to Q) \to \neg(Q \to P))$$

11. (a) Same truth table
    (b) Same truth table

12.
$$(a) \equiv (b) \equiv (d); (c) \equiv (e)$$

Chapter 2

Quantificational Logic

2.1 Quantifiers

2.1.1 Recapitulation

We use *quantifiers* to express how many values of a varialble make a statement true or if a statement is true for at least one value.

The symbol $\forall$ is called the *universal quantifier*, because the statement $\forall x P(x)$ says that $P(x)$ is *universally* true.

The symbol $\exists$ is called the *existentail quantifer*, because the statement $\exists x P(x)$ says that there is at least one value of x for which $P(x)$ is true.

In general, if x is a free variable in some statement $P(x)$, it is a bound variable in the satements $\forall x P(x)$ and $\exists x P(x)$. For this reason we say that the quantifiers *bind* a variable. This means that a variable that is bound by a quantifier can always be replaced with a new variable without changing the meaning of the statement, and it is ofter possible to paraphrase the statemnt without mentioning the bound variable at all.

2.1.2 Problems

1. 1

   (a)
   $$\forall x(\exists y F(x,y) \to S(x))$$

   (b)
   $$\neg\exists x(C(x) \land \forall y(D(y) \to S(x,y)))$$

   (c)
   $$\forall x((x \neq Mary) \to L(x, Mary))$$

   (d)
   $$\exists x(P(x) \land S(Jane, x)) \land \exists y(P(y) \land S(Roger, y))$$

(e)
$$\exists x(P(x) \wedge S(Jane, x) \wedge S(Roger, x))$$

2. (a)
$$\forall x(R(x) \rightarrow \exists y(U(y, x) \wedge R(y)))$$

(b)
$$\exists x(D(x) \wedge M(x)) \rightarrow \forall x(\exists y(F(x, y) \wedge D(y)) \rightarrow Q(x))$$

(c)
$$\neg \exists x F(x) \rightarrow \forall x(A(x) \rightarrow \exists y(D(y) \wedge T(x, y)))$$

(d)
$$\exists x D(x) \rightarrow D(Jones)$$

(e)
$$D(Jones) \rightarrow \forall x D(x)$$

3. (a) x and y
$$\forall z((z > x) \rightarrow (z > y))$$

(b) No free variables
$$\forall a(\exists x(ax^2 + 4x - 2 = 0) \iff a \geq -2)$$

(c) No free variables
$$\forall x((x^3 - x < 3) \rightarrow (x < 10))$$

(d) w
$$(\exists x(x^2 + 5x = w) \wedge existsy(4 - y^2 = w)) \rightarrow (-10 < w < 10)$$

4. (a) For all people, if that person is a man and is not married to someone then that person is unhappy. (All unmarried men are unhappy.)
   (b) There is a person that has a child, and a sister.

5. (a) If a number is a prime and is not 2 then the number is odd.
   (b) There is a perfect number that is larger than all the other perfect numbers.

6. (a) There is no number that is the solution of both $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. True
   (b) There is no real solution to both $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. True
   (c) There is no real solution to $x^2 + 2x + 3 = 0$ and $x^2 + 2x - 3 = 0$. False

7. (a) False.

(b) True.
(c) False.
(d) True.
(e) True.

8. (a) True.
(b) False.
(c) False.
(d) True.
(e) True.
(f) False.

9. (a) True.
(b) False.
(c) True.
(d) False.
(e) True.
(f) True.

10. (a) True.
(b) False.
(c) False.
(d) True.
(e) True.
(f) True.

2.2 Equivalentces Involving Quantifiers

2.2.1 Recapitulation

Quantifier NEgation laws

$$\neg\exists x P(x) \text{ is equivalent to } \forall x \neg P(x)$$

$$\neg\forall x P(x) \text{ is equivalent to } \exists x \neg P(x)$$

Reversing the order of two quantifiers can sometimes change the meaning of a formula. However, if the quantifiers are the same type, they can be switched without affecting the meaning of the formula.

To indicate that there exists *exactly one* element, we use $\exists!$.

$$\exists! x P(x) \equiv \exists x(P(x) \wedge \neg\exists y(P(y) \wedge y \neq x))$$

Sometimes it is useful to write the truth set of $P(x)$ as $\{x \in U \mid P(x)\}$. Similarly, instead of writing $\forall x P(x)$ we can write $\forall x \in U, P(x)$ which is equivalent to $\forall x(x \in A \to P(x))$. Similarly for $\exists x \in U, P(x)$, we can write $\exists x(x \in A \wedge P(x))$

Quantifiers in formulas where the set of discourse is specified are sometimes called *bounded quantifiers*, because they place *bounds on which values of* x are to be considered.

A statement is *vacuously true* if the set of discourse is the empty set.

The empty set is a subset of every set.

The universal quantifier distributes over conjunction $\forall x(E(x) \wedge T(x)) \equiv \forall x E(x) \wedge \forall x T(x)$

The existential quantifier distributes over disjunction $\exists x(E(x) \vee T(x)) \equiv \exists x E(x) \vee \exists x T(x)$

2.2.2 Problems

1. (a)
$$\neg(\forall x(M(x) \to \exists y(F(x,y) \wedge H(y))))$$
$$\exists x \neg(M(x) \to \exists y(F(x,y) \wedge H(y)))$$
$$\exists x(M(x) \wedge \neg existsy(F(x,y) \wedge H(y)))$$
$$\exists x(M(x) \wedge \forall y(\neg F(x,y) \vee \neg H(y)))$$
$$\exists x(M(x) \wedge \forall y(F(x,y) \to \neg H(y)))$$

   (b)
$$\neg(\forall x \exists y(R(x,y) \wedge \forall z(\neg L(y,z))))$$
$$\exists x \forall y(\neg R(x,y) \vee \exists z(L(y,z)))$$
$$\exists x \forall y(R(x,y) \to \exists z(L(y,z)))$$

   (c)
$$\neg(A \cup B \subseteq C \setminus D)$$
$$\neg \forall x((x \in A \vee x \in B) \to (x \in C \wedge x \notin D))$$
$$\exists x \neg((x \in A \vee x \in B) \to (x \in C \wedge x \notin D))$$
$$\exists x((x \in A \vee x \in B) \wedge \neg(x \in C \wedge x \notin D))$$
$$\exists x((x \in A \vee x \in B) \wedge (x \notin C \vee x \in D))$$

   (d)
$$\neg(\exists x \forall y(y > x \to \exists z(z^2 + 5z = y)))$$
$$\forall x \exists y(y > x \wedge \forall z(z^2 + 5z \neq y))$$

2. (a)
$$\neg(\exists x(F(x) \wedge \neg \exists y(R(x,y))))$$
$$\forall x(\neg F(x) \vee \exists y(R(x,y)))$$
$$\forall x(F(x) \to \exists y(R(x,y)))$$

(b)

$$\neg(\forall x \exists y(L(x,y)) \wedge \neg\exists x \forall y(L(x,y)))$$
$$\exists x \forall y(\neg L(x,y)) \vee \exists x \forall y(L(x,y))$$

(c)

$$\neg(\forall a \in A \exists b \in B(a \in C \iff b \in C))$$
$$\exists a \in A \forall b \in B \neg(a \in C \iff b \in C)$$
$$\exists a \in A \forall b \in B((a \notin C \vee b \notin C) \wedge (a \in C \vee b \in C))$$

(d)

$$\neg(\forall y > 0 \exists x(ax^2 + bx + c = y))$$
$$\exists y > 0 \forall x(ax^2 + bx + c \neq y)$$

3. (a) True. All Natural numbers smaller than 7 can be represented as the sum of the squares of 3 natural numbers
   (b) False. Equation has two solutions in the natural numbers
   (c) True. Only solution in the natural numbers is 5
   (d) True. 5 solves both equations.

4.

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$
$$P(x) \equiv \neg\neg P(x)$$
$$\neg\forall x \neg\neg P(x))$$
$$\exists x \neg P(x))$$

5.

$$\neg\exists x \in A P(x)$$
$$\neg\exists x(x \in A \wedge P(x))$$
$$\forall x \neg(x \in A \wedge P(x))$$
$$\forall x(x \notin A \vee \neg P(x))$$
$$\forall x(x \in A \to \neg P(x))$$
$$\forall x \in A \neg P(x)$$

6.

$$\exists x(P(x) \vee Q(x))$$
$$\neg\neg(\exists x(P(x) \vee Q(x)))$$
$$\neg(\forall x \neg(P(x) \vee Q(x)))$$
$$\neg(\forall x(\neg P(x) \wedge \neg Q(x)))$$
$$\neg(\forall x \neg P(x) \wedge \forall x \neg Q(x))$$
$$\exists x P(x) \vee \exists x Q(x)$$

7.
$$\exists x(P(x) \to Q(x))$$
$$\exists x(\neg P(x) \lor Q(x))$$
$$\exists x \neg P(x) \lor \exists x Q(x)$$
$$\neg \forall x P(x) \lor \exists x Q(x)$$
$$\forall x P(x) \to \exists x Q(x)$$

8.
$$(\forall x \in A P(x)) \land (\forall x \in B P(x))$$
$$\forall x(x \in A \to P(x)) \land \forall x(x \in B \to P(x))$$
$$\forall x(x \in A \to P(x) \land (x \in B \to P(x)))$$
$$\forall x((x \in A \land x \in B) \to P(x))$$
$$\forall x \in A \cup B, P(x)$$

9. No, first statement is true for every person with any of those two conditions, second one is true when for any person that with a condition or other person with that other condition. That is, the first statement operate over the same subject while the second one doesn't.

10. (a)
$$\exists x \in A, P(x) \lor \exists x \in B, P(x)$$
$$\exists x(x \in A \land P(x)) \lor \exists x(x \in B \land P(x))$$
$$\exists x((x \in A \land P(x)) \lor (x \in B \land P(x)))$$
$$\exists x(x \in A \lor x \in B) \land P(x)$$
$$\exists x \in A \cup B, P(x)$$

   (b) No. The first statement says that there is an element in each of the sets that has the property, the second statement says that there is an element in both sets that has the property.

11.
$$A \subseteq B \equiv \forall x(x \in A \to x \in B)$$
$$A \setminus B \equiv \neg \exists x(x \in A \land x \notin B)$$
$$\neg \exists x(x \in A \land x \notin B) \equiv \forall x(x \notin A \lor x \in B)$$
$$\equiv \forall x(x \in A \to x \in B)$$

12.

$$C \subseteq A \cup B$$
$$\forall x(x \in C \to (x \in A \lor x \in B))$$
$$\forall x(x \notin C \lor (x \in A \lor x \in B))$$
$$\forall x((x \notin C \lor x \in A) \lor x \in B)$$
$$\forall x((x \in C \land x \notin A) \to x \in B)$$
$$(C \setminus A) \subseteq B$$

13. (a)

$$A \subseteq B$$
$$\forall x(x \in A \to x \in B)$$
$$\forall x((x \in A \lor x \in B) \iff x \in B)$$
$$A \cup B = B$$

(b)

$$A \subseteq B$$
$$\forall x(x \in A \to x \in B)$$
$$\forall x((x \in A \land x \in B) \iff x \in A)$$
$$A \cap B = B$$

14.

$$\neg\exists x(x \in A \land x \in B)$$
$$\forall x(x \notin A \lor x \notin B)$$
$$\forall x(x \in A \to x \notin B)$$
$$\forall x((x \in A \land x \notin B) \iff x \in A)$$
$$A \setminus B = A$$

15. (a) x is a teacher who has exactly one student.
  (b) There exists a teacher x which teaches exactly one student.
  (c) There exists a single teacher that has at least one student.
  (d) There exists a single student that has a single teacher.
  (e) There exists a single teacher that teaches a single student.
  (f) There exists a single teacher and a single student that is taught by them.

2.3 More Operations on Sets

2.3.1 Recapitulation

A set defined in the form $P = \{p_i \mid i \in I\}$, where each element p_i in the set is identified by a number $i \in I, I = \{i \in \mathbb{N} \mid 1 \leq i \leq x\}$ called the *index* of the element, is called an *indexed family* and I is called the *index set.*

$$S = \{P(a) \mid a \in A\}$$
$$S = \{x \mid \exists a \in A, (x = P(a))\}$$

Although the indices for an indexed famility are often numbers, they need not to be.

Sets can have other sets as elements. Sets that contain other sets as elements are sometimes called *families* of sets.

Definition *Suppose that A is a set. The power set of A, denoted by $\mathscr{P}(A)$, is the set whose elements are all the subsets of A. In other words,*

$$\mathscr{P}(A) = \{x \mid x \subseteq A\}$$

Definition *Suppose that $\mathscr{F}$ is a family of sets. Then the intersection and union of $\mathscr{F}$ are the sets $\cap\mathscr{F}$ and $\cup\mathscr{F}$ defined as follows*

$$\cap\mathscr{F} = \{x \mid \forall A \in \mathscr{F}(x \in A)\} = \{x \mid \forall A(A \in \mathscr{F} \to x \in A)\}$$

$$\cap\mathscr{F} = \{x \mid \exists A \in \mathscr{F}(x \in A)\} = \{x \mid \exists A(A \in \mathscr{F} \wedge x \in A)\}$$

Supposing $\mathscr{F} = \{A_i \mid i \in I\}$,

$$\bigcap \mathscr{F} = \bigcap_{i \in I} A_i = \{x \mid \forall i \in I(x \in A_i)\}$$

$$\bigcup \mathscr{F} = \bigcup_{i \in I} A_i = \{x \mid \exists i \in I(x \in A_i)\}$$

2.3.2 Problems

1. (a)

$$\mathscr{F} \subseteq \mathscr{P}(A)$$
$$\forall x(x \in \mathscr{F} \to x \in \mathscr{P}(\mathscr{A}))$$
$$\forall x(x \in \mathscr{F} \to \forall y(y \in x \to y \in A))$$

   (b)

$$A \subseteq \{2n + 1 \mid n \in \mathbb{N}\}$$
$$\forall x(x \in A \to x \in \{2n + 1 \mid n \in \mathbb{N}\})$$
$$\forall x(x \in A \to \exists n \in \mathbb{N}(x = 2n + 1))$$

(c)

$$\{n^2+n+1 \mid n \in \mathbb{N}\} \subseteq \{2n+1 \mid n \in \mathbb{N}\}$$
$$\forall x(x \in \{n^2+n+1 \mid n \in \mathbb{N}\} \to x \in \{2n+1 \mid n \in \mathbb{N}\})$$
$$\forall x(\exists n \in \mathbb{N}(x = n^2+n+1) \to \exists n \in \mathbb{N}(x = 2n+1))$$

(d)

$$\mathscr{P}(\bigcup_{i \in I} A_i) \not\subseteq \bigcup_{i \in I} \mathscr{P}(A_i)$$
$$\exists x(x \in \mathscr{P}(\bigcup_{i \in I} A_i) \wedge x \notin \bigcup_{i \in I} \mathscr{P}(A_i))$$
$$\exists x(x \subseteq \bigcup_{i \in I} A_i \wedge \neg\exists i \in I(x \in \mathscr{P}(A_i)))$$
$$\exists x(x \subseteq \bigcup_{i \in I} A_i \wedge \forall i \in I(x \notin \mathscr{P}(A_i)))$$
$$\exists x(\forall y(y \in x \to y \in \bigcup_{i \in I} A_i) \wedge \forall i \in I(x \not\subseteq A_i))$$
$$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \wedge \forall i \in I(\exists y(y \in x \wedge y \notin A_i)))$$

2. (a)

$$x \in \bigcup \mathscr{F} \setminus \bigcup \mathscr{G}$$
$$\exists A \in \mathscr{F}(x \in A) \wedge \forall B \in \mathscr{G}(x \notin B)$$

(b)

$$\{x \in B \mid x \notin C\} \in \mathscr{P}(A)$$
$$\{x \in B \mid x \notin C\} \subseteq A$$
$$\forall x((x \in B \wedge x \notin C) \to x \in A)$$

(c)

$$x \in \bigcap_{i \in I}(A_i \cup B_i)$$
$$x \in \{x \mid \forall i \in I(x \in (A_i \cup B_i))\}$$
$$\forall i \in I(x \in A_i \vee x \in B_i))$$

(d)

$$x \in (\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$$
$$x \in \{x \mid \forall i \in I(x \in A_i)\} \cup \{x \mid \forall i \in I(x \in B_i)\}$$
$$\forall i \in I(x \in A_i) \vee \forall i \in I(x \in B_i)$$

3. The power set of a set that contains the empty set $\mathscr{P}(\{\emptyset\}) = \{\{\emptyset\}, \emptyset\}$

4.
$$\cap\mathscr{F} = \{\text{red}\}$$
$$\cup\mathscr{F} = \{\text{red}, \text{green}, \text{blue}, \text{orange}, \text{purple}\}$$

5.
$$\cap\mathscr{F} = \emptyset$$
$$\cup\mathscr{F} = \{3, 5, 7, 12, 16, 23\}$$

6. (a)
$$A_2 = \{1, 2, 3, 4\}; A_3 = \{2, 3, 4, 6\}; A_4 = \{3, 4, 6, 8\}; A_5 = \{4, 5, 6, 10\}$$

   (b)
$$\cap_{i\in I} A_i = \{4\}$$
$$\cup_{i\in I} A_i = \{1, 2, 3, 4, 5, 6, 8, 10\}$$

7. Unnecesary

8. (a)
$$A_2 = \{2, 4\}; A_3 = \{3, 6\}$$
$$B_2 = \{2, 3\}; B_3 = \{3, 4\}$$

   (b)
$$\cap_{i\in I}(A_i \cup B_i) = \{2, 3, 4\} \cap \{3, 4, 6\} = \{3, 4\}$$
$$(\cap_{i\in I} A_i) \cup (\cap_{i\in I} B_i) = \emptyset \cup \{3\} = \{3\}$$

   (c) Not equivalent

9. (a) Not equivalent
$$x \in \cup_{i\in I}(A_i \setminus B_i) \equiv \exists i \in I(x \in A_i \setminus B_i)$$
$$x \in (\cup_{i\in I} A_i) \setminus (\cup_{i\in I} B_i) \equiv \exists i \in Ix \in A_i \land \forall i \in Ix \notin B_i$$
$$x \in (\cup_{i\in I} A_i) \setminus (\cap_{i\in I} B_i) \equiv \exists i \in Ix \in A_i \land \exists i \in Ix \notin B_i$$

   (b) Not equivalent
$$A_2 = \{2, 4\}; A_3 = \{3, 6\}$$
$$B_2 = \{2, 3\}; B_3 = \{3, 4\}$$
$$\cup_{i\in I}(A_i \setminus B_i) = \{4\} \cup \{6\} = \{4, 6\}$$
$$(\cup_{i\in I} A_i) \setminus (\cup_{i\in I} B_i) = \{2, 3, 4, 6\} \setminus \{2, 3, 4\} = \{6\}$$
$$(\cup_{i\in I} A_i) \setminus (\cap_{i\in I} B_i) = \{2, 3, 4, 6\} \setminus \{3\} = \{2, 4, 6\}$$

10.

$$I = \{1,2\}, A_1 = \{a,b\}, A_2 = \{c,d\}, B_1 = \{a,e\}, B_2 = \{b,f\}$$

$$\cup_{i\in I}(A_i \cap B_i) = \{a\}$$

$$(\cup_{i\in I}A_i) \cap (\cup_{i\in I}B_i) = \{a,b\}$$

11.

$$x \in \mathscr{P}(A \cap B)$$

$$x \subseteq A \cap B$$

$$\forall y(y \in x \to y \in A \cap B)$$

$$\forall y(y \in x \to (y \in A \land y \in B)$$

$$\forall y(y \in x \to y \in A) \land \forall y(y \in x \to y \in B)$$

$$x \in \mathscr{P}(A) \cap x \in \mathscr{P}(B)$$

12.

$$A = \{a\}; B = \{b\}$$

$$\mathscr{P}(A \cup B) = \{a, b, \{a,b\}\}$$

$$\mathscr{P}(A) \cup \mathscr{P}(B) = \{a,b\}$$

13. (a)

$$\cup_{i\in I}(A_i \cup B_i) = (\cup_{i\in I}A_i) \cup (\cup_{i\in I}A_i)$$

$$x \in \cup_{i\in I}(A_i \cup B_i)$$

$$\exists i \in I(x \in A_i \cup B_i)$$

$$\exists i \in I(x \in A_i \lor x \in B_i)$$

$$\exists i \in I(x \in A_i) \lor \exists i \in I(x \in B_i)$$

$$x \in (\cup_{i\in I}A_i) \cup (\cup_{i\in I}B_i)$$

(b)

$$(\cap\mathscr{F}) \cap (\cap\mathscr{G})$$

$$x \in (\cap\mathscr{F}) \cap (\cap\mathscr{G})$$

$$\forall A \in \mathscr{F}(x \in A) land \forall A \in \mathscr{G}(x \in A)$$

$$\forall A \in \mathscr{F} \cup \mathscr{G}(x \in A)$$

$$\cap(\mathscr{F} \cup \mathscr{G})$$

(c)

$$\cap_{i\in I}(A_i \setminus B_i)$$

$$\forall i \in I(x \in A_i \land x \notin B_i)$$

$$\forall i \in I(x \in A_i) \land \neg\exists i \in Ix \in B_i)$$

$$(\cap_{i\in I}A_i) \setminus (\cup_{i\in I}B_i)$$

14. (a)
$$B_3 = \cup_{i\in I} A_{i,3} = A_{1,3} \cup A_{2,3} = \{1,2,3,4,5\}$$
$$B_4 = \cup_{i\in I} A_{i,4} = A_{1,4} \cup A_{2,4} = \{1,2,4,5,6\}$$

(b)
$$\cap_{j\in J} B_j = \cap_{j\in J}(\cup_{i\in I} A_{i,j})$$
$$\cap_{j\in J}(A_{1,j} \cup A_{2,j})$$
$$(A_{1,3} \cup A_{2,3}) \cap (A_{1,4} \cup A_{2,4})$$
$$\{1,2,4,5\}$$

(c) Not equal
$$\cup_{i\in I}(\cap_{j\in J} A_{i,j})$$
$$\cup_{i\in I}(A_{i,3} \cap A_{i,4})$$
$$(A_{1,3} \cap A_{1,4}) \cup (A_{2,3} \cap A_{2,4})$$
$$\{1,4\} \cup \{2\}$$
$$\{1,2,4\}$$

(d) Not equivalent
$$x \in \cap_{j\in J}(\cup_{i\in I} A_{i,j})$$
$$\forall j \in J(x \in \cup_{i\in I} A_{i,j})$$
$$\forall j \in J(\exists i \in I(x \in A_{i,j}))$$
$$-$$
$$x \in \cup_{i\in I}(\cap_{j\in J} A_{i,j})$$
$$\exists i \in I(x \in \cap_{j\in J} A_{i,j})$$
$$\exists i \in I(\forall j \in J(x \in A_{i,j}))$$

15. (a)
$$\mathscr{F} = \emptyset$$
$$x \in \cup \mathscr{F}$$
$$\exists A \in \mathscr{F}(x \in A)$$

False, since there is no set in $\mathscr{F}$

(b)
$$\mathscr{F} = \emptyset$$
$$x \in \cap \mathscr{F}$$
$$\forall A \in \mathscr{F}(x \in A)$$

True, since x is in every set of F, vacuously true

16. (a)
$$R = \{A \in U \mid A \notin A\}$$
$$\forall R \in U(R \in R \iff R \notin R)$$

(b) There can't be a set that contains all sets.

Chapter 3

Proofs

3.1 Proof Strategies

3.1.1 Recapitulation

Mathematicians usually state the answer to a mathematical question in the form of a *theorem* that says that if certain assumptions called *hypotheses* of the theorem are true, then some conclusion must also be true.

Often the hypotheses and conclusion contain free variables, and in this case it isunderstood that these variables can stand for any elements of the universe of discourse. An assignment of particular values to these variables is called an *instance* of the theorem, and in order for the theorem to be correct it must be the case that for every instance of the theorem that makes the hypotheses come out true, the conclusion is also true. If there is even one instance in which the hypotheses are true but the conclusion is false, then the theorem is incorrect. Such an instance is called a *counterexample* to the theorem.

If a counterexample for a theorem is found then we can be sure that the theorem is incorrect, but the only way to know that a theorem is correct is to prove it. A proof of a theorem is simply a deductive argument whose premises are the hypotheses of the theorem and whose conclusion is the conclusion of the theorem. Throught the proof, we think of any free variables in the hypotheses and conclusion of the theorem as standing for some particular but unspecified elements of the universe of discourse.

Never assert anything until you can justify it completely. This is the primary purpose of any proof: to provide a guarantee that the conclusion is true if the hypotheses are.

We will refer to the statements that are known or assumed to be true at some point in the course of figuring out a proof as *givens*, and the statemtn that remains to be provenn at a point as the *goal*.

When we are starting to figure out a proof, the givens will be just the hypotheses of the theorem we are proving, but they may later include other statements that have been inferred from the hypotheses or added as new assumptions as the result of some transformation of the problem. The goal will initially be the conclusion of the theorem, but it may be changed several times in the course of figuring out a proof.

To prove a goal of the form $P \to Q$:
Assume P is true and then prove Q or assume Q is false and then prove that P is false.

3.1.2 Problems

1. (a) Hypotheses: $n \in \mathbb{Z}$, $n > 1$, n is not prime
   Conclusion: $2^n - 1$ is not prime
   - Yes the hypotheses are true when $n = 6$ - That $2^6 - 1$ is not prime
   - Yes $2^6 - 1 = 63$ is not prime.

   (b) $2^1 5 - 1 = 32767$ is not prime

   (c) Nothing, 11 is prime, thus one of the hypotheses is not true.

2. (a) Hypotheses: $b^2 > 4ac$ Conclusion: The quadratic equation $ax^2+bx+c = 0$ has exactly two real solutions

   (b) Because x is not a free variable.

   (c) $a = 2; b = -5; c = 3; b^2 > 4ac \equiv (-5)^2 > 4(2)(3) \equiv 25 > 24$
   So $ax^2 + bx + c = 0$ has exactly two real solutions $x_1 = \frac{3}{2}; x_2 = 1$

   (d) $a = 2; b = 4; c = 3; b^2 > 4ac \equiv (4)^2 > 4(2)(3) \equiv 16 > 24$
   Since the hypotheses is not true we cannot conclude anything from the theorem.

3. Hypotheses: n is a natural number larger than 2, n is not a prime number Conclusion: $2n+13$ is not a prime number Counterexample: $n = 8; 2(8)+13 = 29$ which is not a prime number

4. *Proof.* Suppose $0 < a < b$. Then $b-a > 0$. Multiplying $b-a > 0$ by $b+a$, we obtain $(b + a) \cdot (b - a) > (b + a) \cdot 0$ which is the same as $b^2 - a^2 > 0$. Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore if $0 < a < b$ then $a^2 < b^2$

5. *Proof.* Supposing that $a < b < 0$. Then multiplying the negative number a into the inequality $a < b$, we obtain $a^2 > ab$, and now multiplying the negative number b into the inequality $a < b$, we obtain $ab > b^2$. Since $a^2 > ab > b^2$, it follows that $a^2 > b^2$. Therefore if $a < b < 0$ then $a^2 > b^2$

6. *Proof.* Suppose that $0 < a < b$. Then multiplying the inequality $a < b$ by $\frac{1}{ab}$, we obtain $\frac{1}{b} < \frac{1}{a}$. Thus if $0 < a < b$ then $\frac{1}{b} < \frac{1}{a}$

7. *Proof.* Suppose that $a^3 > a$. Multiplying a^2 to $a^3 > a$ we get $a^5 > a^3$. Since $a^5 > a^3 > a$, it follows that $a^5 > a$. Therefore if $a^3 > a$ then $a^5 > a$.

8. *Proof.* Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$ and $x \notin D$. Then if $x \notin B$ and $x \in A$ then $x \in A \setminus B$. Since $A \setminus B \subseteq C \cap D$ then $x \in C \cap D$ thus $x \in C \wedge x \in D$, Therefore $x \in D$. Thus if $x \in D$ then $x \notin B$

9. *Proof.* Suppose that $A \cap B \subseteq C \setminus D$ and $x \in A$. If $x \in B$, then since $x \in A$ means that $x \in A \cap B$, since $A \cap B \subseteq C \setminus D$ then $x \in C \setminus D$ thus $x \in C \wedge x \notin D$. Therefore if $x \in B$ then $x \notin D$

10. *Proof.* Suppose that a and b are real numbers. If $\frac{a+b}{2} \geq b$. Then, $a+b \geq 2b$ and $a \geq b$. Therefore if $\frac{a+b}{2} \geq b$ then $a \geq b$

11. *Proof.* Suppose that $x = 8$. Then $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{\sqrt[3]{8}+5}{8^2+6} = \frac{2+5}{64+6} = \frac{7}{70} = \frac{1}{7} \neq \frac{1}{x} = \frac{1}{x}$. Thus if $x = 8$ then $\frac{\sqrt[3]{x}+5}{x^2+6} \neq \frac{1}{x}$

12. *Proof.* Suppose that $ac \geq bd$. Then dividing $ac \geq bd$ by d, $\frac{ac}{d} \geq b$. Combining $\frac{ac}{d} \geq b$ with $a < b$, we obtain $a < b \leq \frac{ac}{d}$, thus $a < \frac{ac}{d}$ and $1 < \frac{c}{d}$ so $c > d$

13. *Proof.* Suppose that $x > 1$, then simplifying $3x + 2y \leq 5$ for x, we obtain that $x \leq \frac{5-2y}{3}$, so $1 < x < \frac{5-2y}{3}$, then $1 < \frac{5-2y}{3}$, simplifying for y we obtain that $y < 1$

14. *Proof.* Suppose that $x^2 + y = -3$ and that $2x - y = 2$. Adding the equiations we obtain $x^2 + 2x = -1$, rewriting as $(x+1)^2 = 0$, we see that $x = -1$

15. *Proof.* Suppose that $x > 3$ and $y < 2$. Thus $x^2 > 9$ and $-2y > -4$, adding the inequalities we obtain $x^2 - 2y < 5$

16. (a) We are supposing that the conclusion it's true, we cannot infer anything from the hypotheses.

    (b) *Proof.* Supposing that $\frac{2x-5}{x-4} = 3$, then $(2x-5) = 3(x-4)$, simplifying $-x + 7 = 0$, which means that $x = 7$

17. (a) The other possible value of x that can make the equation true is being ignored.

    (b) $x = -3; y = 1$

3.2 Proofs Involving Negations and Conditionals

3.2.1 Recapitulation

Usually it's easier to prove a positive statement than a negative statement, so it is often helpful to reexpress a goal of the form $\neg P$ before proving it.

To prove a goal of the form $\neg P$:
If possible, reexpress the goal in some other form and then use one of the proof strategies for this other goal form.

Sometimes a goal of the form $\neg P$ cannot be reexpressed as a positive statement. In this case it is usually best to do a *proof by contradiction.* Start by assumming that P is true, and try to use this assumption to prove something that you know is false. Often this is done by proving a statement that contradicts one of the givens. Because you know that the statement you have proven is false, the assumption that P was true must have been incorrect. The only remaining possibility then is that P is false.

To prove a goal of the form $\neg P$:
Assume P is true and try to reach a contradiction. Once you have reached a contradiction, you can conclude that P must be false.

To use a given of the form $\neg P$:
If you're doing a proof by contradiction, try making P your goal. If you can prove P, then the proof will be complete, because P contradicts the given $\neg P$. If you're not doing a proof by contradiction, then if possible, reexpress this given in some other form.

To use a given of the form $P \to Q$:
If you are also given P, or if you can prove that P is true, then you can use this given to conclude that Q is true. Since it is equivalent to $\neg Q \to \neg P$, if you can prove that Q is false, you can use this given to conclude that P is false.

Modus Ponens If you know that both P and $P \to Q$ are true, you can conclude that Q must also be true.

$$P$$

$$P \to Q$$

$$\therefore Q$$

Modus Tollens If you know that $P \to Q$ is true and Q is false, you can conclude that P must also be false.

$$P \to Q$$

$$\neg Q$$

$$\therefore \neg P$$

3.2.2 Problems

1. (a) Suppose that P. $P \to Q$ then Q. $Q \to R$ then R, therefore $P \to R$

   (b) Suppose that P. Suppose that $\neg R$ then $P \to \neg Q$, so $\neg Q$, there fore $\neg Q$ and $P \to (Q \to R)$

2. (a) Suppose that P. Then Q, Since Q then $\neg R$, therefore $P \to \neg R$

   (b) Suppose that Q. Assumming that $Q \to \neg P$ then $\neg P$ but P, therefore $\neg Q \to \neg P$.

3. Suppose $x \in A$, since $A \subseteq C$, then $x \in C$, but since $B \cap C = \emptyset$, then $x \notin B$.

4. Suppose $x \in C$, since $A \setminus B \cap C = \emptyset$ means that $x \notin A \setminus B$, since $x \in A$, then $x \in B$.

5. Suppose that $x \in A \setminus B$ and $x \in B \setminus C$, since $x \in A \setminus B$, $x \in A$ and $x \notin B$ and since $x \in B \setminus C$, $x \in B$ and $x \notin C$

6. Suppose that $A \cap C \subseteq B$ and $a \in C$. Suppose that $a \in A\ setminus B$, then $a \in A$ and $a \notin B$, since $a \in A$ and $a \in C$ then $a \in A \cap C$, since $A \cap C \subseteq B$, then $a \in B$ but $a \notin B$, thus $a \notin A \setminus B$

7. Suppose that $A \subseteq B$, $a \in A$ and $a \notin B \setminus C$. Suppose that $a \notin C$, then $a \notin B$ and since $a \subseteq B$, then $a \notin A$ but $a \in A$, therefore $a \notin C$

8. Suppose that $y + x = 2y - x$ and that x and y are not both zero. Suppose that $y = 0$, then $x = 0$ but both x and y cannot be both zero, thus $y \neq 0$

9. Suppose that $a < \frac{1}{a} < b < \frac{1}{b}$. Suppose that $a \geq -1$, then $-1 \geq a < 0$ and $0 < a$, then since $a \geq -1$ means that $a \geq \frac{1}{a}$ but $a < \frac{1}{a}$ leads to a contradiction, next for $0 < a$ means that $a \in (0, 1)$, but $1 < \frac{1}{a} < b$ means that $b > 1$ but $b < \frac{1}{b}$ leading to a contradiction, thus $a < -1$

10. Suppose that $x^2 y = 2x + y$ and that $y \neq 0$. Suppose that $x = 0$, then $y = 0$ but $t \neq 0$, leading to a contradiction, thus $x \neq 0$

11. Suppose that $x \neq 0$ and $y = \frac{3x^2+2y}{x^2+2}$ and that $y \neq 3$, then simplifying $y = \frac{3x^2+2y}{x^2+2}$ for y leads to $y = 3$ but $y \neq 3$, thus leading to a contradiction, therefore $y = 3$

12. (a) Negating the conclusion of the theorem would mean that $x = 3$ **or** $y = 8$.

    (b) $x = 3, y = 7$

13. (a) If an element is not the a subset of a set, it doesnt mean that it isn't in the set.

    (b) $A = 1, 2, 3, B = 4, C = 1, 2, 3, 4$, $3 \in A$ but $3 \notin B$

14.

P	Q	$P \to Q$	$\neg Q$	$\neg P$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	F

15.

P	Q	R	$P \to (Q \to R)$	$\neg R \to (P \to \neg Q)$
F	F	F	T	T
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	F	F
T	T	T	T	T

16. (a)

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	T
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	F	T	T
T	T	F	T	F	F
T	T	T	T	T	T

(b) Same truth table but with premise and conclusion reversed

17. (a)

P	Q	R	$P \to Q$	$R \to \neg Q$	$P \to \neg R$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	F	T
T	F	F	F	T	T
T	F	T	F	T	F
T	T	F	T	T	T
T	T	T	T	F	F

(b)

P	Q	$Q \to \neg(Q \to \neg P)$
T	F	T
T	T	T

18. No, to move the hypotheses would create a completely different statement. As proof the following is a counter example $x = -3$.

3.3 Proofs Involving Quantifiers

3.3.1 Recapitulation

To prove a goal of the form $\forall x P(x)$:
Let x stand for an arbitrary object and prove $P(x)$. The letter x must be a new variable in the proof. If x is already being used in the proof to stand for something, then you must choose an unused variable, say y, to stand for the arbitrary object, and prove $P(y)$

To prove a goal of the form $\exists x P(x)$:
Try to find a value of x for which you think $P(x)$ will be true. Then start your proof with "Let $x =$ (the decided value) and proceed to prove $P(x)$ for this value of x. Once again, x should be a new variable. If the letter x is already being used in the proof for some other purpose, then you should choose an unused variable, say y, and rewrite the goal in the equivalent form $\exists y P(y)$. Now proceed as before by starting your proof with "Let $y =$ (the decided value)" and prove $P(y)$.

To use a given of the form $\exists x P(x)$:
Introduce a new variable x_0 into the proof to stand for an object fir which $P(x_0)$ is true. This means that you can now assume that $P(x_0)$ is true. Logicians call this rule of inference *existential instantiation.*

To use a given of the form $\forall x P(x)$:
You can plug in any value, say a, for x and use this given to conclude that $P(a)$ is true. This rule is called *universal instantiation.*

3.3.2 Problems

1. Prove that if $\exists x(P(x) \to Q(x))$ then $\forall x P(x) \to \exists x Q(x)$
   Since $\exists x(P(x) \to Q(x))$, choosing $x = x_0$, then $P(x_0) \to Q(x_0)$, supposing that $\forall x P(x)$, then in particular $P(x_0)$ thus $Q(x_0)$, since we have found a value for which $Q(x)$ holds, we can conclude that $\exists x Q(x)$, thus $\forall x P(x) \to \exists x Q(x)$

2. Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$. Since $A \cap B \setminus C = \emptyset$ then for an arbitrary element x of $A \cap B$, Then $x \in A$ and $x \in B$. Supposing that $x \notin C$ then since $x \in B$ and $x \notin C$ it follows that $x \in B \setminus C$, but since we have $x \in A$, therefore $x \in C$. Since x was arbitrary, we conclude that $A \cap B \subseteq C$

3. Supposing that $A \subseteq B \setminus C$ is true, then assuming that $A \cap C \neq \emptyset$, then for some x such that $x \in A$ and $x \in C$, since $A \subseteq B \setminus C$ means that $x \in A$ then $x \in B$ and $x \notin C$, but $x \in C$. Thus $A \cap C = \emptyset$

4. Let X be an arbitrary element of $\mathscr{P}(A)$, then $X \subseteq A$, supposing that $x \in X$, then since $X \subseteq A$, $x \in A$, and therefore $x \in \mathscr{P}(A)$ which means that $X \subseteq \mathscr{P}(A)$, so $X \in \mathscr{P}(\mathscr{P}(A))$. Since X was arbitrary, we can conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$

5. (a) $A = \emptyset$
   (b) $A = 1$

6. (a) Supposing that $x \neq 1$, then let $y = \frac{2x+1}{x-1}$, then $\frac{y+1}{y-2} = x$
   (b) Suppoing that y is a real number such that $x = \frac{2y+1}{y-1}$. Suppose that $x = 1$, then $y + 1 = y - 2$, and therefore $1 = -2$, which is a contradiction, thus $x \neq 1$

7. Supposing that $y = \frac{x+\sqrt[2]{x^2-4}}{2}$, then $y + 1/y = x$.

8. Supposing that $A \in \mathscr{F}$ then $A \subseteq \bigcup \mathscr{F}$ means that $\forall x(x \in A \to \exists B \in \mathscr{F}(x \in B))$. Supposing that x is an arbitrary element of A, then let $B = A$, means that $x \in A$. Thus $A \subseteq \bigcup \mathscr{F}$

9. Supposing that $A \in \mathscr{F}$, then $\bigcap \mathscr{F} \subseteq A$ means that $\forall x(x \in \bigcap \mathscr{F} \to x \in A)$. Supposing that $xin \bigcap \mathscr{F}$ which means that $\forall B \in \mathscr{F}(x \in B)$, Supposing that B is an arbitrary subset of $\mathscr{F}$, then $x \in A$. Thus $\bigcap \mathscr{F} \subseteq A$

10. $B \subseteq \bigcap \mathscr{F}$ means that $\forall x(x \in B \to x \in \bigcap \mathscr{F})$. Supposing that x is an arbitrary element of B then $x \in \bigcap \mathscr{F}$ means that $\forall C \in \mathscr{F}(x \in C)$. Letting C be an arbitrary element of $\mathscr{F}$, then $x \in C$, since $C \in \mathscr{F}$, then from $\forall A \in \mathscr{F}(B \subseteq A)$ then $B \subseteq C$, thus since $x \in C$, $C \in \mathscr{F}$, and $B \subseteq C$, we obtain that $B \subseteq \bigcap \mathscr{F}$

11. Supposing that $\emptyset \in \mathscr{F}$. Supposing that $\bigcap \mathscr{F} \neq 0$, then we can choose some x such that $x \in \bigcap \mathscr{F}$. Since $x \in \bigcap \mathscr{F}$, and $\emptyset \in \mathscr{F}$, $x \in \emptyset$. This is a contradiction, since $\emptyset$ has no elements.

12. Supposing that X is an arbitrary element of $\mathscr{F}$, since $\mathscr{F} \subseteq \mathscr{G}$, then $X \in \mathscr{G}$. Choosing some x such that $x \in \bigcup \mathscr{F}$, since $X \in \mathscr{F}$, $x \in X$, but since $X \in \mathscr{G}$ then $x \in \bigcup \mathscr{G}$

13. Supposing that $\mathscr{F} \subseteq \mathscr{G}$ and let x be an arbitrary element of $\bigcap \mathscr{G}$. Supposing that $X \in \mathscr{F}$. Since $\mathscr{F} \subseteq \mathscr{G}$, it follows that $X \in \mathscr{G}$. Then by the definition of $\bigcap \mathscr{G}$, since $x \in \bigcap \mathscr{G}$ and $X \in \mathscr{G}$ then $x \in X$. Since X was an arbitrary element of $\mathscr{F}$, we conclude that $\forall X \in \mathscr{F}(x \in A)$, which means that $x \in \bigcap \mathscr{F}$. Since x was an arbitrary element of $\bigcap \mathscr{G}$, this shows that $\bigcap \mathscr{G} \subseteq \bigcap \mathscr{F}$

14. Supposing that $x \in \cup_{i \in I} \mathscr{P}(A_i)$. Choosing some $i \in I$ such that $x \in \mathscr{P}(A_i)$, which means $x \subseteq A_i$. Let a be an arbitrary element of x, then $a \in A_i$, therefore $a \in \cup_{i \in I} A_i$. Since a was an arbitrary element of x it follows that $x \subseteq \cup_{i \in I} A_i$ which means that $x \in \mathscr{P}(\cup_{i \in I} A_i)$. Thus $\cup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}(\cup_{i \in I} A_i)$

15. Suppose that $i \in I$. Let x be an arbitrary element of $\cap_{i \in I} A_i$, then $x \in A_i$. This means that $\cap_{i \in I} A_i \subseteq A_i$, thus $\cap_{i \in I} A_i \in \mathscr{P}(A_i)$. Since i was arbitrary $\cap_{i \in I} A_i \in \cap_{i \in I} \mathscr{P}(A_i)$

16. Suppose that $F \subseteq \mathscr{P}(B)$, then let y be an arbitrary element of $\cup \mathscr{F}$ such that $y \in x$, and let x be an arbitrary element of $\mathscr{F}$, since $x \in \mathscr{F}$, then $x \in \mathscr{P}(B)$ which means that $x \subseteq B$, since $y \in x$ then $y \in B$ but since y was an arbitrary element of $\cup \mathscr{F}$ then $\cup \mathscr{F} \subseteq B$

17. Let x be an arbitrary element of $\cup \mathscr{F}$. Then choose some $A \in \mathscr{F}$ such that $x \in A$. Since every element of A is a subset of every element of $\mathscr{G}$, then for an arbitrary $B \in \mathscr{G}$, $A \subseteq B$, thus $x \in B$, but B was an arbitrary element of $\mathscr{G}$ which means that $\forall B \in \mathscr{G}(x \in B)$ so $x \in \cap \mathscr{G}$. Thus $\cup \mathscr{G} \subseteq \cap \mathscr{G}$

18. (a) Supposing that $a \mid b$ and $a \mid c$, then let $b = ma$ and $c = na$ for some integers m and n. Then $b + c = ma + na = (m + n)a$, since $m + n$ is an integer, this means that $a \mid b + c$

    (b) Supposing that $ac \mid bc$ and $c \neq 0$. Choosing an integer m such that $bc = acm$, then since $c \neq 0$, $b = am$, since m is an integer $a \mid b$

19. (a) Let $z = (y - x)/2$, then $x + z = x + \frac{y-x}{2} = y - \frac{y-x}{2} = y - z$.

    (b) No, counterexample $x = 1, y = 2$, then $z = 1/2$.

20. The negation of an universal quantifier should be an existential quantifier.

21. (a) It is assuming that elements in B are elements in A. Which given the constraints, might not be the case.

    (b) $A = \{1, 2\}; B = \{0, 1, 2\}$

22. x needs to be instantiated before y, thus it cannot be defined in terms of y.

23. (a) The emptyset is a set that could have every one of it's elements in both A and B, however that won't mean that $\cup\mathscr{F}$ and $\cup\mathscr{F}$ are not disjoint.

    (b) $\mathscr{F} = \{\emptyset, \{1\}\}; \mathscr{G} = \{\emptyset, \{2\}\}$

24. (a) The proof is reducing the scope of the theorem to a single real number, that is, it is assuming that the values of x and y are equal, but they might not be.

    (b) Incorrect. $x = 0, y = 1$

25. For an arbitrary x, let $y = 2x$ and for an arbitrary z, then $yz = 2xz = (x + z)^2 - (x^2 + z^2) = x^2 + 2xz + z^2 - x^2 - z^2 = 2xz = yz$.

26. (a) Introduce a new variable x without specifying it's value. Introduce a new variable x specifying a value to be assigned to it.

    (b) When using proof by contradiction, a goal with a quantifier is converted to a given of the other kind.

3.4 Proofs Involving Conjunctions and Biconditionals

3.4.1 Recapitulation

A goal of the form $P \land Q$ is treated as two separate goals. The same is true of gives of the same form.

To prove a goal of the form $P \land Q$:
Prove P and Q separately.

To use a given of the form $P \wedge Q$:
Treat this given as two separate givens: P, and Q.

To deal with statements of the for $P \iff Q$, we convert it into its equivalent form of $(P \to Q) \wedge (Q \to P)$. Thus a biconditional is transform as two separate givens or goals.

To prove a goal of the form $P \iff Q$:
Prove $P \to Q$ and $Q \to P$ separately.

To use a give of the form $P \iff Q$:
Treat this as two separate givens: $P \to Q$ and $Q \to P$.

3.4.2 Problems

1. ($\to$) Supposing that $\forall x(P(x) \wedge Q(x))$, let y be arbitrary. Then $P(y) \wedge Q(y)$, since $P(y)$ for an arbitrary y, $\forall x P(x)$, similarly since $Q(y)$ for an arbitrary y, $forally Q(y)$. Thus, $\forall x P(x) \wedge \forall x Q(x)$ ($\leftarrow$) Supposing that $\forall x P(x) \wedge \forall x Q(x)$, let y be arbitrary. Then since $\forall x P(x)$, $P(y)$, similarly for $forallx Q(x)$, $Q(x)$, Thus $P(y) \wedge P(y)$ and since y was arbitrary, it follows that $\forall x(P(x) \wedge Q(x))$

2. Supposing that $A \subseteq B$ and $A \subseteq C$ are true. Let x be an arbitrary element of A. Since $A \subseteq B$, $x \in B$ and similarly since $A \subseteq C$, $x \in C$. Thus $x \in B \cap C$, but since x was an arbitrary element of A, then $A \subseteq B \cap C$

3. Supposing that $A \subseteq B$. Let C be an arbitrary set. Let x be an arbitrary element of $C \setminus B$. Since $x \in C \setminus B$, then $x \in C$ and $x \notin B$. If $x \in A$, then since $A \subseteq B$, $x \in B$ but $x \notin B$, thus $x \notin A$. Since $x \in C$ and $x \notin A$, $x \in C \setminus A$. But x was an arbitrary element of $C \setminus B$. Thus $C \setminus B \subseteq C \setminus A$

4. Supposing that $A \subseteq B$ and that $A \nsubseteq C$. Let x be an arbitrary element of A, then since $A \nsubseteq C$, $x \notin C$, and since $A \subseteq B$, $a \in B$. Thus since $x \in B$ but $x \notin C$, $B \nsubseteq C$

5. Suppose that $A \subseteq B \setminus C$ and that $A \neq \emptyset$. Since $A \neq \emptyset$, let x be an arbitrary element of A. Since $A \subseteq B \setminus C$, $x \in B \setminus C$, which means that $x \in B$ and $x \notin C$. Thus since $x \in B$ but $x \notin C$, $B \nsubseteq C$

6. Let x be an arbitrary element of $A \setminus (B \cap C)$
   $x \in A \setminus (B \cap C)$ iff $x \in A \wedge x \notin B \cap C$
   iff $x \in A \wedge \neg(x \in B \wedge x \in C)$
   iff $x \in A \wedge (x \notin B \vee x \notin C)$
   iff $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$
   iff $x \in (A \setminus B) \cup (A \setminus C)$

7. Suppose that $\mathscr{P}(A \cap B)$ Let X be an arbitrary set such that $X \in \mathscr{P}(A \cap B)$ and let x be an arbitrary element of X. Since $X \in \mathscr{P}(A \cap B)$, $X \subseteq (A \cap B)$, and since $x \in X$, $x \in A \wedge x \in B$. Since x was arbitrary this means that $X \subseteq A$ and $X \subseteq B$, which in turn means that $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. Now supposing that $\mathscr{P}(A) \cap \mathscr{B}$. Let X be an arbitrary

element of $\mathscr{P}(A) \cap \mathscr{B}$. This means that $X \in \mathscr{P}(A) \wedge X \in \mathscr{P}(B)$, let x be an arbitrary element of X. Then $X \subseteq A$, which means that $x \in A$, similarly since $X \in \mathscr{P}(B)$, $x \in B$, thus $x \in A \cap B$, which means that $X \in \mathscr{P}(A \cap B)$

8. Supposing that $A \subseteq B$. Let X be an arbitrary element of $\mathscr{P}(A)$, then $X \subseteq A$. Let x be an arbitrary element of X, such that $x \in A$. Since $A \subseteq B$, $x \in B$. Since x is an arbitrary element of X, it follows that $X \in \mathscr{P}(B)$. Thus $\mathscr{P}(A) \subseteq \mathscr{B}$. Now suppose that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Let X be an arbitrary element of $\mathscr{P}(A)$ and let x be an arbitrary element of X, then $x \in A$, suppose that $A \nsubseteq B$, thus $x \not inB$. Since $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, $X \in \mathscr{P}(B)$, which means that $x \in B$, but $x \notin B$, which is a contradiction, thus $A \subseteq B$

9. Suppose that x and y are odd integers. This means that for some integer k, $x = 2k + 1$, and that for some integer m, $y = 2m + 1$. Multiplying x with y we obtain that $xy = (2k + 1)(2m + 1) = (4km + 2k + 2m + 1)$. Simplifying we see that $(4km + 2k + 2m + 1) = 2(2km + k + m) + 1$, since $2km + k + m$ is an integer xy is odd.

10. Suppose that x and y are odd integers. Then for some integers k and m, $x = 2k + 1$ and $y = 2m + 1$. Subtracting y from x, we see that $x - y = (2k + 1) - (2m + 1) = 2k - 2m + 1 - 1 = 2k - 2m = 2(k - m)$. Since $k - m$ is an integer, by the definition of even number $x - y$ is even.

11. Let n be an arbitrary integer. Suppose that n^3 is even. Suppose that n is odd, then for some integer k, $n = 2k+1$. Then raising n to the 3rd power, $n^3 = (2k+1)^3 = 8k^3+4k^2+2k+1 = 2(4k^3+2k^2+k)+1$. Since $4k^3+2k^2+k$ is an integer, by the definition of odd number n^3 is odd, but n^3 is even. Thus n is even. Now suppose that n is even. Then for some integer k, $n = 2k$, raising n to the 3rd power we see that $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3$ is an integer, then by the definition of even number, n^3 is even.

12. (a) The proof is using the same integer k for the definition of m and n, when they might be different.
    (b) Counterexample $m = 6, n = 1$

13. Let x be an arbitrary real number. Suppose that $\exists x \in \mathbb{R}(x + y = xy)$. Let y_0 be some real number such that $x + y_0 = xy_0$. Suppose that $x = 1$, then $1 + y_0 = 1 \cdot y_0 = y_0$. So $1 = 0$, which is a contradiction, this $x \neq 1$. Now suppose that $x \neq 1$. Let $y = \frac{x}{x-1}$. Then $x + y = x\frac{x}{x-1} = \frac{x^2}{x-1} = x \cdot \frac{x}{x-1} = xy$

14. Let $z = 1$, let x be an arbitrary positive real number. Suppose that $\exists y \in \mathbb{R}(y - x = y/x)$. Let $y = y_0$ such that $y_0 - x = y_0/x$. Suppose that $x = z = 1$, then $y_0 - 1 = y_0/1 = y_0$, thus $-1 = 0$, which is a contradiction, thus $x \neq z$. Now suppose that $x \neq z = 1$. Let $y = \frac{x^2}{x-1}$, then $y - x = \frac{x^2}{x-1} - x = \frac{x^2-x^2+x}{x-1} = \frac{x}{x-1} = \frac{1}{x} \cdot \frac{x^2}{x-1} = y/x$

15. Let x be an arbitrary element of $\cup\{A \setminus B \mid A \in \mathscr{F}\}$, then choose some A in $\mathscr{F}$, such that $x \in A \setminus B$, which means that $x \in A$ and $x \notin B$. Since $x \in A$ and $x \notin B$, $A \nsubseteq B$, thus $A \notin \mathscr{P}(B)$. Since $A \in \mathscr{F}$ and $A \notin \mathscr{P}(B)$, $A \in \mathscr{F} \setminus \mathscr{P}(B)$. And since $x \in A$, $x \in \mathscr{F} \setminus \mathscr{P}(B)$, since x was an arbitrary element of $\cup\{A \setminus B \mid A \in \mathscr{F}\}$, $\cup\{A \setminus B \mid A \in \mathscr{F}\} \subseteq \cup(\mathscr{F} \setminus \mathscr{P}(B))$

16. Suppose that $\cup\mathscr{F}$ and $\cap\mathscr{G}$ are not disjoint. Then we can choose some x such that $x \in \cup\mathscr{F}$ and $x \in \cap\mathscr{G}$. Then for some $A \in \mathscr{F}$, $x \in A$. Then for some $B \in \mathscr{G}$ such that $A \cap B = \emptyset$, since $x \in A$, $x \notin B$. But since $x \in \cap\mathscr{G}$ means that $x \in B$, we arrive at a contradiction, thus $\cup\mathscr{F}$ and $\cap\mathscr{G}$ are disjoint.

17. Let A be an arbitrary set. Suppose $x \in A$. Since $A \subseteq A$, $A \in \mathscr{P}(A)$, which means that $x \in \cup\mathscr{P}(A)$, since x was an arbitrary element of A, $A \subseteq \mathscr{P}(A)$. Now supposing that $x \in \cup\mathscr{P}(A)$. Then choosing some $X \in \mathscr{P}(A)$, such that $x \in X$. Since $X \in \mathscr{P}(A)$, $X \subseteq A$, since $x \in X$, $x \in A$. Since x was arbitrary, we can conclude that $\cup\mathscr{P}(A) \subseteq A$. Since $A \subseteq \mathscr{P}(A)$ and $\cup\mathscr{P}(A) \subseteq A$, $A = \mathscr{P}(A)$.

18. (a) Let x be an arbitrary element of $\cup(\mathscr{F} \cap \mathscr{G})$. Choose some X in $\mathscr{F} \cap \mathscr{G}$ such that $x \in X$. Since $X \in \mathscr{F} \cap \mathscr{G}$, $X \in \mathscr{F}$ and $X \in \mathscr{G}$, thus $X \subseteq \cup\mathscr{F}$ which means that $x \in \cup\mathscr{F}$, similarly we see that $x \in \cup\mathscr{G}$. Thus $x \in (\cup\mathscr{F}) \cap (\cup\mathscr{G})$. Since x was arbitrary $\cup(\mathscr{F} \cap \mathscr{G}) \subseteq (\cup\mathscr{F}) \cap (\cup\mathscr{G})$

    (b) The proof uses the same set A for both $\mathscr{F}$ and $\mathscr{G}$ which might not be the case. We cannot assume that both sets are the same.

    (c) $\mathscr{F} = \{\{1\}, \{2\}\}$ and $\mathscr{G}\{\{1\}, \{1, 2\}\}$

19. Suppose that $(\cup\mathscr{F}) \cap (\cup\mathscr{G}) \subseteq \cup(\mathscr{F} \cap \mathscr{G})$. Let $A \in \mathscr{F}$ and $B \in \mathscr{G}$. Suppose that $x \in A \cap B$. Then $x \in A$ and $x \in B$, thus $x \in \cup\mathscr{F}$ and $x \in \cup\mathscr{G}$, which means that $x \in (\cup\mathscr{F}) \cap (\cup\mathscr{G})$, since x was an arbitrary element of $A \cap B$, then $A \cap B \subseteq \cup(\mathscr{F} \cap \mathscr{G})$.

    Now suppose that $\forall A \in \mathscr{F} \forall B \in \mathscr{G}(A \cap B \subseteq \cup(\mathscr{F} \cap \mathscr{G}))$. Let A and B be arbitrary elements of $\mathscr{F}$ and $\mathscr{G}$ respectively. Let x be an arbitrary element of $(\cup\mathscr{F}) \cap (\cup\mathscr{G})$, which means that $x \in \cup\mathscr{F}$ and $x \in \cup\mathscr{G}$, thus $x \in A$ and $x \in B$, since $x \in A \cap B$, and $A \cap B \subseteq \cup(\mathscr{F} \cap \mathscr{G})$, $x \in \cup(\mathscr{F} \cap \mathscr{G})$, since x was an arbitrary element of $(\cup\mathscr{F}) \cap (\cup\mathscr{G})$, $(\cup\mathscr{F}) \cap (\cup\mathscr{G}) \subseteq \cup(\mathscr{F} \cap \mathscr{G})$

20. Supposing that $\cup\mathscr{F}$ and $\cup\mathscr{G}$ are disjoint. Suppose that $\exists A \in \mathscr{F} \exists B \in \mathscr{G}(A \cap B \neq \emptyset)$. Then choosing some $A \in \mathscr{F}$ and $B \in \mathscr{G}$. Let $x \in A \cap B$, which means that $x \in A$ and $x \in B$ which means that $x \in \cup\mathscr{F}$ and $x \in \cup\mathscr{G}$. But $\cup\mathscr{F}$ and $\cup\mathscr{G}$ are disjoint. Since we arrived at a contradiction, $\forall A \in \mathscr{F} \forall B \in \mathscr{G}(A \cap B = \emptyset)$ Now supposing that $\forall A \in \mathscr{F} \forall B \in \mathscr{G}(A \cap B = \emptyset)$, suppose that $\cup\mathscr{F}$ and $\cup\mathscr{G}$ are not disjoint. Let x be an arbitrary element of $(\cup\mathscr{F}) \cap (\cup\mathscr{G})$, which means that $x \in \cup\mathscr{F}$ and $x \in \cup\mathscr{G}$. So there exists sets $A \in \mathscr{F}$ and $B \in \mathscr{G}$, such that $x \in A$ and $x \in B$ but A and B are disjoint. Thus $\forall A \in \mathscr{F} \forall B \in \mathscr{G}(A \cap B = \emptyset)$

21. (a) Let $x \in (\cup\mathscr{F}) \setminus (\cup\mathscr{G})$, thus $x \in \cup\mathscr{F}$ and $x \notin (\cup\mathscr{G})$. Then for some $A \in \mathscr{F}$, $x \in A$. Supposing that $A \in \mathscr{G}$, then since $x \in A$, $x \in \cup\mathscr{G}$, but $x \notin \cup\mathscr{G}$, thus $A \notin \mathscr{G}$, Which means that $A \in \mathscr{F} \setminus \mathscr{G}$, thus $x \in \cup(\mathscr{F} \setminus \mathscr{G})$

    (b) The fact that A is not in $\mathscr{G}$, doesn't meant that there is no other set in $\mathscr{G}$ that contains x, thus we cannot derive that $x \notin \cup\mathscr{G}$ from $A \notin \mathscr{G}$ and $A \notin \mathscr{G}$

    (c) Supposing that $\cup(\mathscr{F} \setminus \mathscr{G}) \subseteq (\cup\mathscr{F}) \cap (\cup\mathscr{G})$. Suppose that $\exists A \in (\mathscr{F} \setminus \mathscr{G}) \exists B \in \mathscr{G}(A \cap B \neq \emptyset)$. Choosing some $A \in \mathscr{F}$ and some $B \in \mathscr{G}$ such that $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $A \in (\mathscr{F} \setminus \mathscr{G})$, $x \in \cup(\mathscr{F} \setminus \mathscr{G})$ and since $\cup(\mathscr{F} \setminus \mathscr{G}) \subseteq (\cup\mathscr{F}) \setminus (\cup\mathscr{G})$, $x \in (\cup\mathscr{F}) \setminus (\cup\mathscr{G})$, thus $x \in \cup\mathscr{F}$ and $x \notin \cup\mathscr{G}$. Since $x \notin \cup\mathscr{G}$, but $B \in \mathscr{G}$ and $x \in B$, $x \in \cup\mathscr{G}$. Since we arribed at a contradiction $\forall A \in (\mathscr{F} \setminus \mathscr{G}) \forall B \in \mathscr{G}(A \cap B = \emptyset)$. Now suppose that $\forall A \in (\mathscr{F} \setminus \mathscr{G}) \forall B \in \mathscr{G}(A \cap B = \emptyset)$. Let A and B be arbitrary. Let x be an arbitrary element of $\cup(\mathscr{F} \setminus \mathscr{G})$. Choosing some $A \in \mathscr{F} \setminus \mathscr{G}$ such that $x \in A$. Thus $A \in \mathscr{F}$ and $A \notin \mathscr{G}$. Then since $A \in \mathscr{F}$, $x \in \cup\mathscr{F}$, and since $A \cap B = \emptyset$, $x \notin B$, thus $x \notin \cup\mathscr{G}$, then $x \in (\cup\mathscr{F}) \setminus (\cup\mathscr{G})$. Since x was an arbitrary element of $\cup(\mathscr{F} \setminus \mathscr{G})$, $\cup(\mathscr{F} \setminus \mathscr{G}) \subseteq (\cup\mathscr{F}) \setminus (\cup\mathscr{G})$

    (d) $\mathscr{F} = \{\{1\}, \{1, 2\}\}; \mathscr{G} = \{\{1\}, \{2\}\}$

22. Supposing that $\cup\mathscr{F} \not\subseteq \cup\mathscr{G}$. Then suppose that $\forall A \in \mathscr{F} \exists B \in \mathscr{G}(A \subseteq B)$. Let A be an arbitrary element of $\mathscr{F}$ and for some $B \in \mathscr{G}$ such that $A \subseteq B$. Let x be an arbitrary element of A, then since $A \subseteq B$, $x \in A \wedge x \in B$. Since $x \in A$ and $A \in \mathscr{F}$, $x \in \cup\mathscr{F}$, similarly, $x \in \cup\mathscr{G}$. Since x was arbitrary this means that $\cup\mathscr{F} \subseteq \cup\mathscr{G}$, but $\cup\mathscr{F} \not\subseteq \cup\mathscr{G}$. Since we arrived at a contradiction $\cup\mathscr{F} \not\subseteq \cup\mathscr{G} \to \exists A \in \mathscr{F} \forall B \in \mathscr{G}(A \not\subseteq B)$

23. (a) Direct proofs using existential instantiation on both directions.

    (b) Let x be arbitrary. Suppose that $x \in B \setminus (\cup_{i \in I} A_i)$, then $x \in B$ and $x \not{i}n(\cup_{i \in I} A_i)$. Let i be an arbitrary element of I, then $x \notin A_i$. Since $x \in B$, and $x \notin A_i$, $x \in B \setminus A_i$. Since i was arbitrary $x \in \cap_{i \in I}(B \setminus A_i)$. Thus $B \setminus \cup_{i \in I} A_i \subseteq \cap_{i \in I}(B \setminus A_i)$. Now suppose that $x \in \cap_{i \in I}(B \setminus A_i)$. Let i be an arbitrary element of I, then $x \in B \setminus A_i$. Thus $x \in B$ and $x \notin A_i$. Since i was arbitrary $x \notin \cup_{i \in I} A_i$, and since $x \in B$, $x \in B \setminus (\cup_{i \in I} A_i)$. Thus $B \setminus (\cup_{i \in I} A_i) = \cap_{i \in I}(B \setminus A_i)$

    (c) $B \setminus (\cap_{i \in I} A_i) = \cup_{i \in I}(B \setminus A_i)$.
    Let x be arbitrary. Suppose that $x \in B \setminus (\cap_{i \in I} A_i)$. Then $x \in B$ and $x \notin \cap_{i \in I} A_i$. Choosing some $i \in I$ such that $x \notin A_i$. Then since $x \in B$ and $x \notin A_i$, $x \in B \setminus A_i$. Thus $x \in \cup_{i \in I}(B \setminus A_i)$. Since x was arbitrary $\cup_{i \in I}(B \setminus A_i)$
    Now suppose that $x \in \cup_{i \in I}(B \setminus A_i)$. Choosing some $i \in I$ such that $x \in B \setminus A_i$. Then $x \in B$ and $x \notin A_i$. Since A_i was specific, then $x \notin \cap_{i \in I} A_i$. Now since $x \in B$ and $x \notin \cap_{i \in I} A_i$, $x \in B \setminus (\cap_{i \in I} A_i)$. Since x was arbitrary $B \setminus (\cap_{i \in I} A_i)$.

24. (a) Let x be arbitrary. Suppose that $x \in \cup_{i \in I}(A_i \setminus B_i)$. Choose some $i \in I$ such that $x \in A_i \setminus B_i$, Then $x \in A_i$ and $x \notin B_i$. Since $x \in A_i$, $x \in \cup_{i \in I} A_i$, similarly, since $x \notin B_i$, $x \notin \cap_{i \in I} B_i$. Thus $x \in (\cup_{i \in I} A_i) \setminus (\cap_{i \in I} B_i)$. Since x was arbitrary, then $\cup_{i \in I}(A_i \setminus B_i) \subseteq (\cup_{i \in I} A_i) \setminus (\cap_{i \in I} B_i)$

    (b) $I = \{1, 2\}; A_1 = B_1 = \{1\}; A_2 = B_2 = \{2\}$

25. (a) Let x be arbitrary. Suppose that $x \in \cup_{i \in I}(A_i \cap B_i)$. Choosing some $i \in I$ such that $x \in A_i \cap B_i$ then $x \in A_i$ and $x \in B_i$. Since $x \in A_i$, $x \in \cup_{i \in I} A_i$, similarly, since $x \in B_i$, $x \in \cup_{i \in I} B_i$. Since $x \in \cup_{i \in I} A_i$ and $x \in \cup_{i \in I} B_i$, $x \in (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i)$. Since x was arbitrary, $\cup_{i \in I}(A_i \cap B_i) \subseteq (\cup_{i \in I} A_i) \cap (\cup_{i \in I} B_i)$

26. Let a and b be arbitrary integers. Choosing $c = ab$, then $a \mid c$ and $b \mid c$.

27. (a) Let n be an arbitrary integer. Suppose that $15 \mid n$, then for some integer k, $n = 15k$. Then since $n = (3)(5)k$, and 3 and 5 are integers, then $3 \mid n$ and $5 \mid n$. Now suppose that $3 \mid n$ and $5 \mid n$. Then for some integers $k$and m, $n = 3k = 5m$. This means that $k = 5m/3$. Since k is an integer, $5m/3$ must be an integer, therefore $3 \mid m$. Let $m = 3j$, for some integer j, then $n = 5m = 5(3j) = 15j$, thus $15 \mid n$

3.5 Proofs Involving Disjunctions

3.5.1 Recapitulation

Supposing that one of our givens has the form $P \vee Q$. This given tells us that either P or Q is true, but it doesn't tell us which. Thus, there are two possibilities that we must take into account.

One way to do the proof would be to consider these two possibilities in turn. In other words, first assume that P is true and use this assumption to prove our goal. Then assume Q is true and give another proof that the goal is true.

The two possibilities that are considered separately in this type of proof are called *cases*. The given $P \vee Q$ justifies the use of these two cases by guaranteeing that these cases cover all of the possibilities. Mathematicians say in this situation that the cases are *exhaustive*.

To use a given of the form $P \vee Q$:

Break your proof into cases. For case 1, assume that P is true and use this assumption to prove the goal. For case 2, assume that Q is true and give another proof of the goal. If you are also given $\neg P$, or you can prove that P is false, then you can use this given to conclude that Q is true. Similarly, if you are given $\neg Q$ or can prove that Q is false, then you can conclude that P is true.

Proof by cases is sometimes also helpful if you are proving a goal of the form $P \vee Q$. If you can prove P in some cases and Q in others, then as long as your cases are exhaustive you can conclude that $P \vee Q$ is true.

To prove a goal of the form $P \vee Q$:
Break your proof into cases. In each case, either prove P or Q. If P is true, then clearly the goal $P \vee Q$ is true, so you only need to worry about the case in which P is false. You can complete the proof in this case by proving that Q is true.

3.5.2 Problems

1. Let x be arbitrary. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \vee x \in C$.
   (Case 1) Suppose that $x \in B$. Since $x \in A \wedge x \in B$, $x \in (A \cap B) \cup C$.
   (Case 2) Suppose that $x \in C$. Since $x \in A \wedge x \in C$, $x \in (A \cap B) \cup C$.

   Since x was arbitrary, $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

2. Let x be arbitrary. Suppose that $x \in (A \cup B) \setminus C$. Then $(x \in A \vee x \in C) \wedge x \notin C$.
   (Case 1) Suppose that $x \in A$. Since $x \in A \wedge x \notin C$. $x \in A \cup (B \setminus C)$
   (Case 2) Suppose that $x \in B$. Since $x \in A \wedge x \in B$. $x \in A \cup (B \setminus C)$

   Since x was arbitrary, $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

3. Let x be arbitrary.
   ($\rightarrow$)Suppose that $x \in A \setminus (A \setminus B)$. Then $x \in A \wedge x \in B$, thus $x \in A \cap B$.
   ($\rightarrow$)Suppose that $x \in A \cap B$. Then $x \in A \wedge x \in B$, thus $x \in A \setminus (A \setminus B)$.

   Since x was arbitrary, $A \setminus (A \setminus B) = A \cap B$

4. Let x be arbitrary.
   ($\rightarrow$)Suppose $x \in A \setminus (B \setminus C)$, this means that $x \in A \wedge (x \notin B \vee x \in C)$.
   (Case 1) Suppose that $x \notin B$. Then $x \in A \wedge x \notin B$, thus $x \in (A \setminus B) \cup (A \cap C)$.
   (Case 2) Suppose that $x \in C$. Then $x \in A \wedge x \in C$, thus $x \in (A \setminus B) \cup (A \cap C)$.
   ($\rightarrow$)Suppose that $x \in (A \setminus B) \cup (A \cap C)$, this means that $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$
   (Case 1) Suppose that $x \in A \wedge x \notin B$. Then $x \in A \setminus (B \setminus C)/$
   (Case 2) Suppose that $x \in A \wedge x \in C$. Then $x \in A \setminus (B \setminus C)$

   Since x was arbitrary, $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

5. Let x be arbitrary. Suppose $x \in A$.
   (Case 1) Suppose that $x \in C$. Then since $x \in A \cap C$, $x \in B \cap C$, thus $x \in B$.
   (Case 2) Suppose that $x \notin C$. Then since $x \in A \cup C$, $x \in B \cup C$, thus $x \in B$.

   Since x was arbitrary $A \subseteq B$.

6. Let x be arbitrary. Suppose that $A \triangle B \subseteq A$. Suppose that $B \not\subseteq A$. Then choosing some $x \in B$ such that $x \notin A$. Since $x \in B \wedge x \notin A$, $x \in B \setminus A$,

thus $x \in A \triangle B$, since $A \triangle B \subseteq A$, $x \in A$, but $x \notin A$. Since we arrived at a contradiction, $B \subseteq A$.

7. Let x be arbitrary
(→)Suppose that $A \cup C \subseteq B \cup C$. Suppose that $x \in A \setminus C$, then $x \in A$ and $x \notin C$ which means that $x \in A \cup C$, and since $A \cup C \subseteq B \cup C$, $x \in B \cup C$. Thus, since $x \notin C$ and $x \in B$, $x \in B \setminus C$.
(→)Suppose that $A \setminus C \subseteq B \setminus C$. Suppose that $x \in A \cup C$.
(Case 1) Suppose that $x \in C$. Then $x \in B \cup C$.
(Case 2) Suppose that $x \notin C$. Then since $x \in A$ and $x \notin C$, $x \in A \setminus C$, since $A \setminus C \subseteq B \setminus C$, $x \in B \setminus C$, and since $x \notin C$, $x \in B$. Thus $x \in B \cup C$.

8. Let x be arbitrary. Suppose that $x \in \mathscr{P}(A) \cup \mathscr{P}(B)$.
(Case 1) Suppose that $x \in \mathscr{P}(A)$. Let y be an arbitrary element of x. Then $y \in A$, so $y \in A \cup B$, which means that $x \subseteq A \cup B$, thus $x \in \mathscr{P}(A \cup B)$
(Case 2) By a similar argument supposing that $x \in \mathscr{P}(B)$, we arrive at $x \in \mathscr{P}(A \cup B)$. Since x was arbitrary $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$

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3.6 Existence and Uniqueness Proofs

3.6.1 Recapitulation

3.6.2 Problems

3.7 More Examples of Proofs

3.7.1 Recapitulation

3.7.2 Problems

Chapter 4

Relations

Chapter 5

Functions

Chapter 6

Mathematical Induction

Chapter 7

Number Theory

Chapter 8

Infinite Sets