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Foundations of Algebraic Analysis

By

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Translated by Goro Kato

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Contents

Preface

Prior to its founding in 1963, the Research Institute for Mathematical Sciences (to which we are gratefully indebted for support) was the focus of divers discussions concerning goals. One of the more modest goals was to set up an institution that would create a "Courant-Hilbert" for a new age.[1] Indeed, our intention here—even though this book is small in scale and only the opening chapter of our utopian "Treatise of Analysis"—is to write just such a "Courant-Hilbert" for the new generation. Each researcher in this field may have his own definition of "algebraic analysis," a term included in the title of this book. On the other hand, algebraic analysts may well share a common attitude toward the study of analysis: the essential use of algebraic methods such as cohomology theory. This characterization is, of course, too vague: one can observe such common trends whenever analysis has made serious reformations. Professor K. Oka, for example, once spoke of the "victory of abstract algebra" in regard to his theory of ideals of undetermined domains.[2] Furthermore, even Leibniz's main interest, in the early days of analysis, seems to have been in the algebraization of infinitesimal calculus. As used in the title of our book, however, "algebraic analysis" has a more special meaning, after Professor M. Sato: it is that analysis which holds onto substance and survives the shifts of fashion in the field of analysis, as Euler's mathematics, for example, has done. In this book, as the most fruitful result of our philosophy, we pay particular attention to the microlocal theory of linear partial differential equations, i.e. the new thinking on the local analysis on contangent bundles. We hope that the fundamental ideas that appear in this book will in the near future become the conventional wisdom

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vols. 1 and 2 (Interscience, 1953 and 1962). These two volumes seem to reflect the strong influence of the Courant Institute; the countervailing influence must be strong as well.
[2] Quoted by Professor Y. Akizuki in *Sūgaku 12* (1960), 159. A general theory of ideals of undetermined domains has been reorganized by H. Cartan and Serre and is now called the theory of coherent sheaves (see Hitotumatu [1]).

among analysts and theoretical physicists, just as the Courant-Hilbert treatise did.

Despite our initial determination and sense of purpose, the task of writing was a heavy burden for us. It has been a time-consuming project, while our first priority has been to be at the front of the daily rapid progress in this field. Thus, we cannot deny the existence of minor areas that do not yet meet with our full satisfaction. Still, a proverb says, "Striving for the best is an enemy of the good." We are content, then, to publish our book in this form, hoping that the intelligent reader will benefit despite several defects, and expecting that this will become the first part of our "Treatise of Analysis." We would also like to emphasize that our comparison of this book with "Courant-Hilbert" is only a goal, and that we do not pretend to equate the maturity of this book with that of Courant and Hilbert's. Theirs is the crystallization of the great scholar Courant's extended effort. Therefore, we would appreciate hearing the critical reader's opinions on the content of this book, for the purpose of improvement.

Let us turn to the content of each chapter. In Chapter I, §1, a review of cohomology theory is given, with which we define the sheaf of hyperfunctions. Since students of analysis nowadays seem to be given little opportunity to learn cohomology theory, despite its importance, we have prepared a rather comprehensive treatment of sheaf cohomology theory as an introduction to notions and notations used in later chapters. One may skip this material if it is familiar. The main purpose of Chapter I, §2, is to present the mathematical formulation, via the Čech cohomology group, of the idea that "hyperfunctions are boundary values of holomorphic functions." The reader can then obtain the explicit description of a hyperfunction by combining this with the results in §3 of Chapter II.

In Chapter II, §1, the sheaf of microfunctions is constructed on a cotangent bundle, by which the stage for our main theme, microlocal analysis, is established. After some preparation of the theory of holomorphic functions of several complex variables, in §2, the properties of microfunctions will be studied in detail in §3. Furthermore, in §4, specific examples will be treated.

In §1 and §2 of Chapter III, where we basically followed Sato, Kawai, and Kashiwara [1] (hereafter SKK [1]), fundamental operations on microfunctions are discussed. However, the approach taken in SKK [1] may not be suited to the novice; hence the method of description has been changed. There it was necessary to prove a certain lemma (Proposition 3.1.1) directly, which is technical and intricate and could be tiresome for the reader. Because of the introductory nature of this book, therefore, we decided to treat this lemma as an "axiom," so to speak, and to proceed to what follows from it. In §4 through §6, elliptic and hyperbolic differential equations are treated explicitly to show how effectively microfunction theory applies to the theory of linear partial differential equations. These three sections also serve as preparation for the theory of microdifferential equations considered in Chapter IV. Prior to these three sections, we discuss (in §3) the analyticity of Feynman integrals. This section has a somewhat different flavor than other sections; it is intended as an invitation to a new trend in mathematical physics: namely, the study of theoretical physics through methods of algebraic analysis. We also thought that it might be a good exercise to go through the operations on microfunctions. In §7, we prove the flabbiness of the microfunction sheaf; and, in §8, a hyperfunction containing holomorphic parameters is discussed. The last two sections are intended to take into account some important properties of microfunctions not covered by the previous sections.

In Chapter IV, we discuss the theory of microdifferential equations, the most effective application of microfunction theory. In §1, we define a microdifferential operator, and the fundamental properties are given. "Qauntized contact transformations" of microdifferential operators are treated in §2. A quantized contact transformation is an extremely important notion, one that revolutionized the theory of linear differential equations. The reader may be astonished to see how easily one can obtain profound results with the structures of solutions of linear (micro)-differential equations by combining microfunction theory with the theory of quantized contact transformations. This point should be considered as the quintessence of microlocal analysis. As in Chapter III, we proceed in Chapter IV in a manner accesible to the reader rather than in the most logical order, which may be less accessible. For example, in §1 we chose the plane-wave decomposition of the δ-function as a starting point for the introduction of microdifferential operators, and in §2 we restricted our discussion to those contact transformations which have generating functions. We decided not to present our more "algebro-analytic" treatments of the above topics until we write a treatise on microdifferential equations centered around the theory of holonomic systems. Likewise, so that the essence of the theory might be plain to the reader, we did not aim at full generality in §3.

As we close this preface, we would like to express our most sincere gratitude to our teacher Professor Mikio Sato, who indeed provided almost all the essential ideas this book contains. We hope that this book will succeed in imparting the emanation of Professor Sato's throbbing mathematics. It is quite fortunate that authors Kashiwara and Kawai, just at the point when they were choosing their specialities, were able to

attend Professor Hikosaburo Komatsu's introductory lectures in hyperfunction theory.[3] This book might be thought of as a report to Professor Komatsu ten years later. Furthermore, activity centered around Professor Sato and the authors' works has received warm encouragement and support from Professors Kōsaku Yosida and Yasuo Akizuki. Two graduate students at Kyoto University, Mr. Kimio Ueno and Mr. Akiyoshi Yonemura, have read our manuscript and have given beneficial advice. Mr. Yonemura and a graduate student at Sophia University, Mr. Masatoshi Noumi, helped us read the proofs; we would like to take this opportunity to offer our sincere thanks. During the preparation of this book, one or another of us was affiliated with the Research Institute for Mathematical Sciences, Kyoto University; the Department of Mathematics, Nagoya University; the Miller Institute for Basic Research in Science, University of California–Berkeley; the Mathematics Department, Harvard University; the Institute for Advanced Study, Princeton; the Department of Mathematics, Université Paris–Nord; and the Department of Mathematics, Massachusetts Institute of Technology. We thank these institutions and their members for their hospitality during our stay. Last, but not least, we would like to express our profound gratitude to Professor Seizō Itō, who not only gave us the opportunity to write this book, but also kept us from proceeding too slowly. We would again like to apologize to Professor Itō for our delay. Without his warm encouragement, in fact, it is doubtful that this book could ever have been published.

August of the coming-of-age year [1978] of hyperfunction theory[4]

The Authors

[3] *Sato's Hyperfunction Theory and Linear Partial Differential Equations with Constant Coefficients*, Seminar Notes 22 (University of Tokyo). At the time (1968), the above lecture note was at the highest level in the field, rather than at the introductory level.

[4] It was in 1958 that Professor Sato published his outline of hyperfunction theory.

Notations

(I) Sheaf Theory ($\mathscr{F}$ and $\mathscr{G}$ denote sheaves)

(II) Manifold Theory (M and N denote manifolds; however X and Y are sometimes used instead of M and N)

$$\begin{tikzcd} & DM \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ \sqrt{-1}SM \arrow[dr, "\tau"'] & & \sqrt{-1}S^*M \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$ 40

$$\begin{tikzcd} N \underset{M}{\times} \sqrt{-1}S^*M - \sqrt{-1}S^*_NM \arrow[r, "\tilde{\omega}_f"] \arrow[d, "\rho_f"] & \sqrt{-1}S^*M \\ \sqrt{-1}S^*N & \end{tikzcd}$$ 105

where $f: N \to M$

(III) Hyperfunction Theory

$\mathscr{A}$	39	$b(\varphi)$	28, 78, 80
$\mathscr{A}^*$	81	sp	50
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$\mathscr{E}_{(\lambda)}$	195	$N_l^\varphi(P; t)$	212
$\mathscr{E}(\lambda)$	195	$\sigma(P)$	139, 208
$\mathscr{E}^\infty$	195	D^α	139
$\mathscr{E}$	195		

(V) Others

$H^n(K)$	9	Z°	79
$H^k(A^\bullet \xrightarrow{f} B^\bullet)$	61	$A(t) \ll B(t)$	213

Foundations of Algebraic Analysis

CHAPTER I

Hyperfunctions

§1. Sheaf Theory

Recall some of the basic concepts from sheaf theory.

Definition 1.1.1. *A presheaf $\mathscr{F}$ over a topological space X associates with each open set U of X an abelian group $\mathscr{F}(U)$, such that there exists an abelian group homomorphism $\rho_{V,U}:\mathscr{F}(U) \to \mathscr{F}(V)$ for open sets $U \supset V$ with the following axioms:*

(1) $\rho_{U,U} = \mathrm{id}_U$ (= *the identity map on* $\mathscr{F}(U)$)

(2) *For* $V_1 \subset V_2 \subset V_3$, *open sets of* X, *we have*

$$\rho_{V_1,V_2} \circ \rho_{V_2,V_3} = \rho_{V_1,V_3}.$$

The homomorphism $\rho_{V,U}$ is called the restriction map, and for $s \in \mathscr{F}(U)$ $\rho_{V,U}(s)$ is often denoted by $s|_V$.

Definition 1.1.2. *Let $\mathscr{F}$ be a presheaf over X. The stalk of the presheaf $\mathscr{F}$ at $x \in X$ is defined as $\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$, where $\varinjlim$ denotes the inductive limit, where an equivalence relation "$\sim$" on $\bigcup_{x \in U} \mathscr{F}(U)$ is defined as follows: $s_1 \sim s_2$, for $s_1 \in F(U)$ and $s_2 \in F(V)$, if and only if there exists a sufficiently small open set $W \subset U \cap V$ such that $s_1|_W = s_2|_W$. Therefore a canonical map is induced: $\mathscr{F}(U) \to \mathscr{F}_x$ for $x \in U$. The image of $s \in \mathscr{F}(U)$ under the canonical map is denoted by s_x. Hence we have $(s_1)_x = (s_2)_x$ if and only if there exists an open set V such that $x \in V \subset U$ and such that $s_1|_V = s_2|_V$.*

Definition 1.1.3. *A presheaf $\mathscr{F}$ over X is said to be a sheaf if the following axioms are satisfied: it is given an open covering $\{U_i\}_{i \in I}$ of U in X, $U = \bigcup_{i \in I} U_i$.*

(a) *Let $s \in \mathscr{F}(U)$. If $s|_{U_i} = 0$ for each $i \in I$, then $s = 0$.*
(b) *Suppose that for each $i \in I$ there exists $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for $i, j \in I$. Then there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.*

Definition 1.1.4. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ are presheaves. Then $f:\mathscr{F} \to \mathscr{G}$ is said to be a morphism if for each open set U the morphism $f(U):\mathscr{F}(U) \to \mathscr{G}(U)$ is an abelian group homomorphism and if open sets U and V are given such that $U \supset V$, then the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f(U)} & \mathscr{G}(U) \\ {\scriptstyle \rho_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho_{V,U}} \\ \mathscr{F}(V) & \xrightarrow{f(V)} & \mathscr{G}(V) \end{array}$$

Hence there is induced a homomorphism on each stalk, $f_x:\mathscr{F}_x \to \mathscr{G}_x$.

Definition 1.1.5. *Let $\mathscr{F}$ be a presheaf over X. A sheaf $\mathscr{F}'$ is said to be the sheaf associated to the presheaf $\mathscr{F}$ (or $\mathscr{F}'$ is the sheafication of $\mathscr{F}$, or $\mathscr{F}'$ is the induced sheaf from the presheaf $\mathscr{F}$) if for each open set U of X the presheaf $\mathscr{F}'(U)$ (which is actually a sheaf) associates all the maps: $U \xrightarrow{s} \bigcup_{x \in U} \mathscr{F}_x$ such for each $X \in U$ there exists a neighborhood U' of x and $s' \in \mathscr{F}(U')$ such that $s(x') = s'_{x'}$ is true for any x' in U'.*

For a given morphism from a presheaf $\mathscr{F}$ into a sheaf $\mathscr{G}$ there is induced a unique morphism from $\mathscr{F}'$ into $\mathscr{G}$. Note that $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic on each stalk.

Definition 1.1.6. *Let $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ be sheaves over a topological space X. A sequence $\mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}''$ is said to be exact if $\mathscr{F}'_x \xrightarrow{f'_x} \mathscr{F}_x \xrightarrow{f_x} \mathscr{F}''_x$ is an exact sequence, i.e.* $\operatorname{Ker} f_x = \operatorname{Im} f'_x$, *at each $x \in X$.*

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over X, and let $f:\mathscr{F} \to \mathscr{G}$ be a morphism. Then the presheaf assignment of U, an open subset, to $\operatorname{Ker}(\mathscr{F}(U) \xrightarrow{f(U)} \mathscr{G}(U))$ *is a sheaf, denoted by* $\operatorname{Ker}(f)$. *One also has the presheaf* $\operatorname{Coker}(\mathscr{F}(U) \xrightarrow{f(U)} \mathscr{G}(U)) = \mathscr{G}(U)/\operatorname{Im} f(U)$. *This presheaf is not a sheaf in general. The sheaf associated to this presheaf is denoted by* $\operatorname{Coker}(f)$. *Then, by definition, we have the exact sequence of sheaves*

$$0 \to \operatorname{Ker}(f) \to \mathscr{F} \xrightarrow{f} \mathscr{G} \to \operatorname{Coker}(f) \to 0.$$

In the case where $\operatorname{Ker}(f) = 0$, *we often write $\mathscr{G}/\mathscr{F}$ instead of* $\operatorname{Coker}(f)$ *by identifying $\mathscr{F}$ with* $\operatorname{Im} f$.

Definition 1.1.7. *Let X and Y be topological spaces, and let $f:X \to Y$ be a continuous map. For a sheaf $\mathscr{F}$ over X the presheaf assignment of an open subset U of Y to $\mathscr{F}(f^{-1}(U))$ is a sheaf over Y. This sheaf is called the direct image of $\mathscr{F}$ under the continuous map f, denoted by $f_*(F)$. For a sheaf $\mathscr{G}$ on Y there can be defined the presheaf $\varinjlim_{V \supset f(U)} \mathscr{G}(V)$ for an open set U of X. Generally this presheaf is not a sheaf. The associated sheaf is called the inverse image of $\mathscr{G}$ under f, denoted by $f^{-1}(\mathscr{G})$. Suppose that S is an arbitrary subset of X, and let $i_S:S \to X$ be the imbedding map. Then* *the inverse image* $j_S^{-1}(\mathscr{F})$ *of the sheaf* $\mathscr{F}$ *is called the restriction of* $\mathscr{F}$ *to* S, *and we often denote it by* $\mathscr{F}|_S$.

If $\mathscr{G}$ is a sheaf over Y, then there exists a natural morphism $\mathscr{G} \to f_*(f^{-1}(\mathscr{G}))$. Notice that $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$ and that giving a morphism $\mathscr{G} \to f_*\mathscr{F}$ for a sheaf $\mathscr{F}$ over X is equivalent to giving a morphism $f^{-1}\mathscr{G} \to \mathscr{F}$.

Definition 1.1.8. *Let* $\mathscr{F}$ *be a sheaf over a topological space* X, *and let* U *be an open subset of* X. *The subset* $\{x \in U \mid s_x \neq 0\}$ *for* $s \in \mathscr{F}(U)$ *is called the support of* s, *denoted by* $\operatorname{supp}(s)$. *Note that* $\operatorname{supp}(s)$ *is closed in* U.

Definition 1.1.9. *Let* $\mathscr{F}$ *be a sheaf over a topological space* X, *and let* S *be a locally closed subset of* X; *i.e. it is the intersection of an open set and a closed set in* X. *Then define* $\Gamma_S(X, \mathscr{F}) = \{s \in \mathscr{F}(U) \mid \operatorname{supp}(s) \subset S\}$, *where* U *is an open set in* X *such that* S *is closed in* U.

The definition above is independent of the choice of U.

Proof. Let U_1 and U_2 be such open sets; then $U_1 \cap U_2$ contains S as a closed subset. Therefore one can assume that $S \subset U_1 \subset U_2 \subset X$ and that S is closed in U_1 and U_2. Define a map φ from $\{s \in \mathscr{F}(U_2) \mid \operatorname{supp}(s) \subset S\}$ to $\{s \in \mathscr{F}(U_1) \mid \operatorname{supp}(s) \subset S\}$ by $\varphi(s) = s|_{U_1}$. Then φ is bijective. Therefore $\Gamma_S(X, \mathscr{F})$ is independent of the choice of U.

In the case that $S = X$, we denote $\Gamma_S(X, \mathscr{F})$ with $\Gamma(X, \mathscr{F})$, whose elements are called the global sections of $\mathscr{F}$, i.e. $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$. Generally we also denote $\mathscr{F}(U)$ with $\Gamma(U, \mathscr{F})$ for an open set U in X, whose elements are called the sections of $\mathscr{F}$ over U.

Definition 1.1.10. *Let* $\mathscr{F}$ *be a sheaf over a topological space* X, *and let* S *be a locally closed subset of* X. *We denote the sheaf associated to a presheaf* $\Gamma_{S \cap U}(U, \mathscr{F}) = \{s \in \mathscr{F}(U) \mid \operatorname{supp}(s) \subset S \cap U\}$, *for an open set* U *of* X, *by* $\Gamma_S(\mathscr{F})$.

Definition 1.1.11. *A sheaf* $\mathscr{F}$ *over a topological space* X *is said to be flabby if for an arbitrary open set* U *the homomorphism* $\rho_{U,X}: \mathscr{F}(X) \to \mathscr{F}(U)$ *is an epimorphism. Therefore, for a flabby sheaf* $\mathscr{F}$ *any section of* $\mathscr{F}$ *over* U *can be extended to a section over* X.

Proposition 1.1.1. *Let* $\mathscr{F}$ *be a flabby sheaf over a topological space* X, *and let* S *be a locally closed subset of* X. *Then* $\Gamma_S(\mathscr{F})$ *is a flabby sheaf.*

Proof. Let U_1 be an open set such that S is closed in U_1. Then, for any open set U of X, the set $U_1 \cap U$ is open in X and contains $S \cap U$ as a closed set. Let s be an element of $\Gamma_S(\mathscr{F})(U) = \Gamma_{S \cap U}(U, \mathscr{F})$; then $s \in \mathscr{F}(U_1 \cap U)$ and $\operatorname{supp}(s) \subset S \cap U$. Therefore we have $s|_{(U_1 - S) \cap (U_1 \cap U)} = 0$. Then there exists a unique $s' \in \mathscr{F}((U_1 - S) \cup (U_1 \cap U))$ such that $s'|_{(U_1 - S)} = 0$ and $s'|_{U_1 \cap U} = s$. Since the sheaf $\mathscr{F}$ is flabby, s' can be extended to a section $\tilde{s} \in \mathscr{F}(U_1)$. Then we have $\tilde{s}|_{(U_1 - S)} = 0$. Hence $\operatorname{supp}(\tilde{s}) \subset S$, i.e. $\tilde{s} \in \Gamma_S(X, \mathscr{F}) = \Gamma_S(\mathscr{F})(X)$.

Proposition 1.1.2. *Let $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ be sheaves over a topological space X, let U be an open set, and let S be a locally closed set in X.*

(1) *If* $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}''$ *is an exact sequence of sheaves, then*
 (i) $0 \to \mathscr{F}'(U) \xrightarrow{f'(U)} \mathscr{F}(U) \xrightarrow{f(U)} \mathscr{F}''(U)$ *and*
 (ii) $0 \to \Gamma_S(X, \mathscr{F}') \to \Gamma_S(X, \mathscr{F}) \to \Gamma_S(X, \mathscr{F}'')$ *are exact.*

(2) If $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *is a flabby sheaf, then*
 (i) $0 \to \mathscr{F}'(U) \xrightarrow{f'(U)} \mathscr{F}(U) \xrightarrow{f(U)} \mathscr{F}''(U) \to 0$ *and*
 (ii) $0 \to \Gamma_S(X, \mathscr{F}') \to \Gamma_S(X, \mathscr{F}) \to \Gamma_S(X, \mathscr{F}'') \to 0$ *are exact.*

Proof. (1.i) First we will show that $f'(U)$ is a monomorphism. Suppose that $f'(U)s' = 0$ for $s' \in \mathscr{F}'(U)$. Then $f'_x s'_x = 0$ for each x in U. Therefore $s'_x = 0$; i.e. there exists a neighborhood $V(x)$ of x such that $s'|_{V(x)} = 0$. By the definition of a sheaf, we know that $s' = 0$. Therefore $f'(U)$ is monomorphic. Next we will prove that $\operatorname{Im} f'(U) \subset \operatorname{Ker} f(U)$. Since $(f_x \circ f'_x)s'_x = 0$ for $s' \in \mathscr{F}(U)$, for each x one can find a neighborhood $V(x)$ of x such that $f(U)f'(U)s'|_{V(x)} = 0$. Therefore, since $\mathscr{F}''$ is a sheaf we have $f(U)f'(U)s' = 0$. It remains to be proved that $\operatorname{Im} f'(U) \supset \operatorname{Ker} f(U)$. Let $s \in \mathscr{F}(U)$ such that $f(U)s = 0$. Then, for each $x \in U$, $f_x s_x = 0$ holds. By the exactness there exists $s'_x \in \mathscr{F}'_x$ such that $f'_x s'_x = s_x$. This implies that $f'(V(x))s'(x) = s|_{V(x)}$ for some $s'(x) \in \mathscr{F}(V(x))$ in some neighborhood $V(x)$ of x such that $V(x) \subset U$. Since $f'(V(x))$ is a monomorphism, $s'(x)$ is unique. Therefore we have $s'(x)|_{V(x) \cap V(y)} = s'(y)|_{V(x) \cap V(y)}$. By the sheaf axiom, we have $s' \in \mathscr{F}'(U)$ and $s'|_{V(x)} = s'(x)$. Then $f'(U)s' = s$.

Next we will give a proof of (1.ii). Let U be an open set in X such that S is closed in U. It is to be shown that $\operatorname{supp}(s') \subset S$ for the s', as in the above, provided that $\operatorname{supp}(s) \subset S$ for an $s \in \mathscr{F}(U)$. Note that $f'(U - S)s'|_{(U-S)} = s|_{(U-S)} = 0$ and that $f'(U - S)$ is a monomorphism. Therefore $s'|_{(U-S)} = 0$, i.e. $\operatorname{supp}(s') \subset S$.

It suffices to show that $f(U)$ is an epimorphism to prove (2.i). Let $s'' \in \mathscr{F}''(U)$, and let $\mathscr{M} = \{(s, V) \mid V$ is an open subset of U, $s \in \mathscr{F}(V)$ and $f(V)s = s''|_V\}$. Then define an order relation, denoted with $\succ$, in $\mathscr{M}$ as follows: let (s_1, V_1) and (s_2, V_2) be elements of $\mathscr{M}$. The expression $(s_1, V_1) \succ (s_2, V_2)$ holds if and only if $V_1 \supset V_2$ and $s_1|_{V_2} = s_2$. Then $\mathscr{M}$ is a non-empty, inductively ordered set. Therefore there exists a maximal element in $\mathscr{M}$ by Zorn's lemma. Let (s, V) be a maximal element. $V = U$ is left to be proved. Suppose $V \neq U$, and let $x \in U - V$. Then there exists a neighborhood $V(x)$ and $s(x) \in \mathscr{F}(V(x))$ such that $(s(x), V(x)) \in \mathscr{M}$. Then notice that $f(V \cap V(x))(s - s(x))|_{V \cap V(x)} = 0$. One can then find $s' \in \mathscr{F}'(V \cap V(x))$ such that $f'(V \cap V(x))s' = (s - s(x))|_{V \cap V(x)}$ by (1.i). The flabbiness of $\mathscr{F}'$ implies that there exists $\tilde{s}' \in \mathscr{F}'(V(x))$ such that $\tilde{s}'|_{V \cap V(x)} = s'$. Define $\tilde{s} \in \mathscr{F}(V \cup V(x))$ as $\tilde{s}|_V = s$ and $\tilde{s}|_{V(x)} = s(x) + f'(V(x))\tilde{s}'$. Then $(\tilde{s}, V \cup V(x)) \in \mathscr{M}$, which contradicts the maximality of (s, V) in $\mathscr{M}$. Therefore $V = U$; that is, $f(U)$ is an epimorphism. (2.ii) can be proved similarly. Let $\mathscr{M}' = \{(s, U) | s \in \Gamma_{S \cap U}(U, \mathscr{F}), U$ is an open set such that $f(U)s = s''|_U\}$. Then take a maximal element of $\mathscr{M}'$ to be (s, V) satisfying that $(s, V) \succ (0, X - \operatorname{supp}(s''))$.

Remark. Conversely, if $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U)(\to 0)$ is exact for any open set U, then it is plain that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''(\to 0)$ is exact.

Corollary 1. *Let* $0 \to \mathscr{F}' \xrightarrow{f'} \mathscr{F} \xrightarrow{f} \mathscr{F}'' \to 0$ *be an exact sequence of sheaves.*

(1) *If $\mathscr{F}'$ and $\mathscr{F}$ are flabby sheaves, then $\mathscr{F}''$ is a flabby sheaf.*

(2) *If $\mathscr{F}'$ and $\mathscr{F}''$ are flabby sheaves, then $\mathscr{F}$ is a flabby sheaf.*

Proof. Since $\mathscr{F}'$ is a flabby sheaf, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F}'(X) & \xrightarrow{f'(X)} & \mathscr{F}(X) & \xrightarrow{f(X)} & \mathscr{F}''(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho'_{U,X}} & & \downarrow{\scriptstyle \rho_{U,X}} & & \downarrow{\scriptstyle \rho''_{U,X}} & & \\
0 & \longrightarrow & \mathscr{F}'(U) & \xrightarrow{f'(U)} & \mathscr{F}(U) & \xrightarrow{f(U)} & \mathscr{F}''(U) & \longrightarrow & 0
\end{array}$$

with exact rows, where U is any open set in X. To prove (1), first notice that $\rho_{U,X}$ is an epimorphism since $\mathscr{F}$ is flabby. On the other hand, $f(U)$ is an epimorphism. Therefore, for any $s'' \in \mathscr{F}''(U)$, there exists an $s \in \mathscr{F}(X)$ such that $s'' = f(U)\rho_{U,X}(s) = \rho''_{U,X}(f(X)s)$. This implies that $\rho''_{U,X}$ is an epimorphism; i.e. $\mathscr{F}''$ is a flabby sheaf. Next we will prove (2). Let $s \in \mathscr{F}(U)$. Since $\rho''_{U,X}$ and $f(X)$ are both epimorphisms, one can find $\tilde{s} \in \mathscr{F}(X)$ such that $\rho''_{U,X}f(X)\tilde{s} = f(U)s$. By commutativity we have $\rho''_{U,X}f(X)\tilde{s} = f(U)\rho_{U,X}\tilde{s}$. Therefore $f(U)(s - \rho_{U,X}\tilde{s}) = 0$. Then note that $\rho'_{U,X}$ is an epimorphism. Hence there exists $\tilde{s}' \in \mathscr{F}'(X)$ such that $s - \rho_{U,X}\tilde{s} = f'(U)\rho'_{U,X}\tilde{s}' = \rho_{U,X}f'(X)\tilde{s}'$. That is, $s = \rho_{U,X}(\tilde{s} + f'(X)\tilde{s}')$, showing that $\rho_{U,X}$ is an epimorphism. Therefore $\mathscr{F}$ is a flabby sheaf.

Corollary 2. *Suppose that* $0 \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^r \to \mathscr{G} \to 0$ *is an exact sequence of sheaves and that each* $\mathscr{F}^i$, $i = 0, 1, \ldots, r$, *is a flabby sheaf. Then $\mathscr{G}$ is a flabby sheaf, and the following sequences are exact:*

$$0 \to \Gamma_S(X, \mathscr{F}^0) \to \Gamma_S(X, \mathscr{F}^1) \to \cdots \to \Gamma_S(X, \mathscr{F}^r) \to \Gamma_S(X, \mathscr{G}) \to 0$$

and

$$0 \to \mathscr{F}^0(U) \to \mathscr{F}^1(U) \to \cdots \to \mathscr{F}^r(U) \to \mathscr{G}(U) \to 0,$$

where S is a locally closed subset of X, and where U is an open subset of X.

Proof. First split the given long exact sequence into short exact sequences, as follows:

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & & 0 \\
 & & & \mathscr{G}^1 \arrow[ur] \arrow[dr] & & & & \mathscr{G}^3 \arrow[ur] \arrow[dr] & & & & & \mathscr{G} \arrow[ur] & \\
0 \arrow[r] & \mathscr{F}^0 \arrow[r] & \mathscr{F}^1 \arrow[ur] \arrow[rr] & & \mathscr{F}^2 \arrow[rr] \arrow[dr] & & \mathscr{F}^3 \arrow[ur] \arrow[rr] & & \cdots \arrow[r] & \mathscr{F}^{r-1} \arrow[rr] \arrow[dr] & & \mathscr{F}^r \arrow[ur] & & \\
 & & & & & \mathscr{G}^2 \arrow[ur] \arrow[dr] & & & & & \mathscr{G}^{r-1} \arrow[ur] \arrow[dr] & & & \\
 & & & & 0 \arrow[ur] & & 0 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

By Corollary 1, $\mathscr{G}^1$ is a flabby sheaf and, since $\mathscr{F}^2$ is flabby, $\mathscr{G}^2$ is a flabby sheaf. Therefore, by repeating this process, we can conclude that $\mathscr{G}$ is a flabby sheaf. These short exact sequences provide the short exact sequences

$$0 \to \Gamma_S(X, \mathscr{F}^0) \to \Gamma_S(X, \mathscr{F}^1) \to \Gamma_S(X, \mathscr{G}^1) \to 0,$$

$$0 \to \Gamma_S(X, \mathscr{G}^i) \to \Gamma_S(X, \mathscr{F}^{i+1}) \to \Gamma_S(X, \mathscr{G}^{i+1}) \to 0 \qquad \text{for } 1 \leqq i \leqq r-2,$$

and

$$0 \to \Gamma_S(X, \mathscr{G}^{r-1}) \to \Gamma_S(X, \mathscr{F}^r) \to \Gamma_S(X, \mathscr{G}) \to 0.$$

The latter assertion follows plainly from these short exact sequences.

Definition 1.1.12. *An exact sequence* $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ *is said to be a flabby resolution of a sheaf* $\mathscr{F}$ *if each* $\mathscr{L}^i$, $i = 0, 1, \ldots,$ *is a flabby sheaf.*

Proposition 1.1.3. *For an arbitrary sheaf* $\mathscr{F}$ *over a topological space* X *there exists a flabby sheaf* $\mathscr{L}$ *such that* $0 \to \mathscr{F} \to \mathscr{L}$ *is an exact sequence. Therefore there exists a flabby resolution of* $\mathscr{F}$.

Proof. Consider a sheaf $\mathscr{C}^0F$ which associates with each open set U of X an abelian group $\left\{s \,|\, U \xrightarrow{s} \bigcup_{x \in U} \mathscr{F}_x\right.$, where s is an arbitrary mapping such that $s(x) \in \mathscr{F}_x$ holds$\left.\right\}$. It is plain that $\mathscr{C}^0(\mathscr{F})$ is a flabby sheaf, and then the sequence $0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{F})$ is exact. Define a sheaf $\mathscr{Z}^0(\mathscr{F})$ so that the sequence $0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{F}) \to \mathscr{Z}^0(\mathscr{F}) \to 0$ is exact. Similarly, we also have the following exact sequence: $0 \to \mathscr{Z}^0(\mathscr{F}) \to \mathscr{C}^0(\mathscr{Z}^0(\mathscr{F})) \to \mathscr{Z}^1(\mathscr{F})$. If one defines $\mathscr{C}^n(\mathscr{F})$ as $\mathscr{C}^0(\mathscr{Z}^{n-1}(\mathscr{F}))$, the sequence $0 \to \mathscr{F} \to \mathscr{C}^0(\mathscr{F}) \to \mathscr{C}^1(\mathscr{F}) \to \cdots$ gives a flabby resolution of $\mathscr{F}$.

Note. The flabby resolution constructed above is said to be the canonical flabby resolution of $\mathscr{F}$.

Definition 1.1.13. *Let K^n be an abelian group, and let $d_n: K^n \to K^{n+1}$ be a homomorphism such that $d_{n+1} \circ d_n = 0$ for each $n = 0, 1, 2, \ldots$. Then the pair $K = \{K^n, d_n\}_{n=0,1,2,\ldots}$ is called a cochain complex. By definition,* $\operatorname{Im} d_{n-1} \subset \operatorname{Ker} d_n$ *holds. An element of* $\operatorname{Ker} d_n$ *is called an* nth *cocycle, and an element of* $\operatorname{Im} d_{n-1}$ *is called an* nth *coboundary. The quotient group* $\operatorname{Ker} d_n/\operatorname{Im} d_{n-1}$ *is said to be the* nth *cohomology group, which is denoted by $H^n(K)$.*

Let K and K′ be cochain complexes. We call $K \xrightarrow{f} K'$ a morphism of cochain complexes provided that, for each n, $K^n \xrightarrow{f_n} K'^n$ is a homomorphism and the diagram

$$\begin{array}{ccccccccccccc}
\longrightarrow & K^0 & \xrightarrow{d_0} & K^1 & \xrightarrow{d_1} & \cdots & \longrightarrow & K^{n-1} & \xrightarrow{d_{n-1}} & K^n & \xrightarrow{d_n} & K^{n+1} & \xrightarrow{d_{n+1}} \\
 & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & {\scriptstyle f_{n-1}} & & \downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & \\
\longrightarrow & K'^0 & \xrightarrow{d'_0} & K'^1 & \xrightarrow{d'_1} & \cdots & \longrightarrow & K'^{n-1} & \xrightarrow{d'_{n-1}} & K'^n & \xrightarrow{d'_n} & K'^{n+1} & \xrightarrow{d'_{n+1}}
\end{array}$$

is commutative. Let K′, K, and *K″ be cochain complexes. A sequence*

$$0 \to K' \xrightarrow{f'} K \xrightarrow{f} K'' \to 0$$

is said to be exact if, for each n,

$$0 \to K'^n \xrightarrow{f'_n} K^n \xrightarrow{f_n} K''^n \to 0$$

is an exact sequence. That is, all the vertical sequences are exact in the diagram

$$\begin{array}{ccccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots\longrightarrow & K'^{n-2} & \xrightarrow{d'_{n-2}} & K'^{n-1} & \xrightarrow{d'_{n-1}} & K'^n & \xrightarrow{d'_n} & K'^{n+1} & \longrightarrow\cdots \\
 & \downarrow{\scriptstyle f'_{n-2}} & & \downarrow{\scriptstyle f'_{n-1}} & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f'_{n+1}} & \\
\cdots\longrightarrow & K^{n-2} & \xrightarrow{d_{n-2}} & K^{n-1} & \xrightarrow{d_{n-1}} & K^n & \xrightarrow{d_n} & K^{n+1} & \xrightarrow{d_{n+1}}\cdots \\
 & \downarrow{\scriptstyle f_{n-2}} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & \\
\cdots\longrightarrow & K''^{n-2} & \xrightarrow{d''_{n-2}} & K''^{n-1} & \xrightarrow{d''_{n-1}} & K''^n & \xrightarrow{d''_n} & K''^{n+1} & \xrightarrow{d''_{n+1}}\cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Notice that the nth cocycles of K' are mapped into the nth cocycles of K and similarly for the nth coboundaries. Therefore f'_n induces a homomorphism f'^*_n on the cohomology groups $H^n(K') \to H^n(K)$. Likewise there is induced a homomorphism $f^*_n: H^n(K) \to H^n(K'')$. Furthermore, since f_n is an epimorphism, for an nth cocycle z''_n of K'' there exists $z_n \in K^n$ such that $f_n(z_n) = z''_n$. Therefore one has $f_{n+1}(d_n z_n) = d''_n f_n(z_n) = d''_n z''_n = 0$, which implies that $f'_{n+1}(z'_{n+1}) = d_n z_n$ for some z'_{n+1} in K'^{n+1}. Notice that z'_{n+1} is uniquely determined modulo $\operatorname{Im} d'_n$, the $(n+1)$-coboundaries for the given

z''_n. Therefore one has a well-defined homomorphism $h_n^*: H^n(K'') \to H^{n+1}(K')$. Then we have the following important proposition.

Proposition 1.1.4. *Let $0 \to K' \to K \to K'' \to 0$ be an exact sequence of cochain complexes. Then the induced sequence of cohomology groups $0 \to H^0(K') \to H^0(K) \to H^0(K'') \to H^1(K') \to \cdots$ is exact.*

One can plainly prove this assertion from definitions.

Corollary (Nine Lemma). *Suppose that in the commutative diagram below the three vertical sequences are exact and L is a cochain complex. If any two of the horizontal sequences are exact, then the remaining horizontal sequence is exact.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K^0 & \to & K^1 & \to & K^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L^0 & \to & L^1 & \to & L^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M^0 & \to & M^1 & \to & M^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Since the second horizontal sequence is a cochain complex, all three of the horizontal sequences are cochain complexes. By Proposition 1.1.4 we have the long exact sequence $0 \to H^0(K) \to H^0(L) \to H^0(M) \to H^1(K) \to \cdots \to H^2(M) \to 0$. For instance, if the first and second horizontal sequences are exact, then one has $H^i(K) = H^i(L) = 0$, $i = 0, 1, 2$. Then $H^j(M)$ must be trivial for $j = 0, 1, 2$; i.e. the third sequence is exact.

Remark. The assumption that L is a cochain complex in Proposition 1.1.4 is necessary for this claim. In fact, one can have additive groups with the property $a + a = b + b = 0$ and the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & \{0, a\} & \longrightarrow & \{0, a\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \{0, b\} & \to & \{0, b, a, a+b\} & \xrightarrow{\varphi} & \{0, a\} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \{0, b\} & \to & \{0, b\} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where φ is defined as $\varphi(b) = \varphi(a) = a$, $\varphi(0) = \varphi(a + b) = 0$. Then φ is a homomorphism, but the second horizontal sequence is not a cochain complex.

The following lemma is also well-known.

Five Lemma.

$$\begin{array}{ccccccccc} K_1 & \to & K_2 & \to & K_3 & \to & K_4 & \to & K_5 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ L_1 & \to & L_2 & \to & L_3 & \to & L_4 & \to & L_5 \end{array}$$

Suppose that the horizontal sequences are exact in the above diagram. If h_2 and h_4 are isomorphisms, h_1 is an epimorphism, and h_5 is a monomorphism, then h_3 is an isomorphism.

Proving this lemma is left to the reader.

Definition 1.1.14. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let S be a locally closed subset of X. The* jth *relative cohomology of $\mathscr{F}$ with supports in S, denoted by $H^j_S(X, \mathscr{F})$, is defined as follows. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Then one obtains the complex $\{\Gamma_S(X, \mathscr{L}^*)\}$. Define*

$$H^j_S(X, \mathscr{F}) = \frac{\operatorname{Ker}(\Gamma_S(X, \mathscr{L}^j) \to \Gamma_S(X, \mathscr{L}^{j+1}))}{\operatorname{Im}(\Gamma_S(X, \mathscr{L}^{j-1}) \to \Gamma_S(X, \mathscr{L}^j))}.$$

Note that this definition is independent of the choice of the flabby resolution $\{L^*\}$ by Theorem 1.1.1 below.

Theorem 1.1.1. *The $H^j_S(X, \mathscr{F})$ is determined canonically by any flabby resolution of $\mathscr{F}$.*

We begin with lemmas.

Lemma 1. *Let $f\colon \mathscr{F} \to \mathscr{G}$ be a morphism, and let $0 \to \mathscr{F} \to \mathscr{L}^0 \xrightarrow{g_0} \mathscr{L}^1 \xrightarrow{g_1} \cdots$ be an exact sequence of sheaves. Then there exists a flabby resolution of $\mathscr{G}$, $0 \to \mathscr{G} \to \mathscr{M}^0 \to \cdots$, such that for each $j \geqq 0$, there exists a morphism $f_j\colon \mathscr{L}^j \to \mathscr{M}^j$ so that the diagram*

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{F} & \to & \mathscr{L}^0 & \xrightarrow{g_0} & \mathscr{L}^1 & \xrightarrow{g_1} & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \to & \mathscr{G} & \to & \mathscr{M}^0 & \longrightarrow & \mathscr{M}^1 & \longrightarrow & \cdots \end{array}$$

is commutative.

Proof. We prove this lemma inductively on j. Suppose that we have constructed $\mathscr{M}^j$ and f_j for $0 \leqq j \leqq k$ as claimed.

$$\begin{array}{ccccccccccccccc} 0 & \to & \mathscr{F} & \to & \mathscr{L}^0 & \to & \mathscr{L}^1 & \to & \cdots & \to & \mathscr{L}^k & \xrightarrow{g_k} & \mathscr{L}^{k+1} & \to \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_k} & & & \\ 0 & \to & \mathscr{G} & \to & \mathscr{M}^0 & \to & \mathscr{M}^1 & \to & \cdots & \to & \mathscr{M}^k & & & \end{array}$$

Let $\mathscr{X}$ be a sheaf so that the sequence $0 \to \mathscr{G} \to \mathscr{M}^0 \to \cdots \to \mathscr{M}^k \xrightarrow{h} \mathscr{X} \to 0$ is exact. Then define a morphism $\mathscr{L}^k \to \mathscr{X} \oplus \mathscr{L}^{k+1}$ by $x \mapsto ((h \circ f_k)(x), -g_k(x))$. Then we have the exact sequence, $0 \to \mathscr{X} \xrightarrow{\iota} \mathrm{Coker}(\mathscr{L}^k \to \mathscr{X} \oplus \mathscr{L}^{k+1})$. We also have the commutative diagram

$$\begin{tikzcd}
\mathscr{L}^k \arrow[r, "g_k"] \arrow[d, "f_k"] & \mathscr{L}^{k+1} \arrow[d, "f'_{k+1}"] \\
\mathscr{M}^k \arrow[r, "\iota \circ h"] & \mathrm{Coker}(\mathscr{L}^k \to \mathscr{X} \oplus \mathscr{L}^{k+1})
\end{tikzcd}$$

where $f'_{k+1}: \mathscr{L}^{k+1} \to \mathscr{X} \oplus \mathscr{L}^{k+1}$ is the canonical morphism defined by $f'_{k+1}(x) = (0, x)$. Let $\mathscr{M}^{k+1}$ be a flabby sheaf such that $0 \to \mathrm{Coker}(\mathscr{L}^k \to \mathscr{X} \oplus \mathscr{L}^{k+1}) \xrightarrow{\iota'} \mathscr{M}^{k+1}$ is an exact sequence. Then we obtain the commutative diagram

$$\begin{tikzcd}
\mathscr{L}^k \arrow[r, "g_k"] \arrow[d, "f_k"] & \mathscr{L}^{k+1} \arrow[d, "\iota' \circ f_{k+1} = f_{k+1}"] \\
\mathscr{M}^k \arrow[r, "\iota' \circ \iota \circ h"'] & \mathscr{M}^{k+1}
\end{tikzcd}$$

Lemma 2. *Suppose that* $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ *and* $0 \to \mathscr{F} \to \mathscr{L}'^0 \to L'^1 \to \cdots$ *are flabby resolutions of a sheaf* $\mathscr{F}$. *Then there exists a flabby resolution of* $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{L}''^0 \to \mathscr{L}''^1 \to \cdots$, *such that the diagrams*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{L}^0 \arrow[r] \arrow[d] & \mathscr{L}^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{L}''^0 \arrow[r] & \mathscr{L}''^1 \arrow[r] & \cdots
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{L}'^0 \arrow[r] \arrow[d] & \mathscr{L}'^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{L}''^0 \arrow[r] & \mathscr{L}''^1 \arrow[r] & \cdots
\end{tikzcd}$$

are commutative.

Proof. Apply Lemma 1 to the case where $f: \mathscr{F} \oplus \mathscr{F} \to \mathscr{F}$, defined by $f(x, y) = x + y$, and the flabby resolution of $\mathscr{F} \oplus \mathscr{F}$, $0 \to \mathscr{F} \oplus \mathscr{F} \to \mathscr{L}^0 \oplus \mathscr{L}'^0 \to \cdots$. Then take a flabby resolution of $\mathscr{F}$ such that

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \oplus \mathscr{F} \arrow[r] \arrow[d, "f"] & \mathscr{L}^0 \oplus \mathscr{L}'^0 \arrow[r] \arrow[d] & \mathscr{L}' \oplus \mathscr{L}'^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{L}''^0 \arrow[r] & \mathscr{L}''^1 \arrow[r] & \cdots
\end{tikzcd}$$

is a commutative diagram.

Proof of Theorem 1.1.1. It is to be proved that $H^j(\Gamma_S(X, \mathscr{L}^*))$ and $H^j(\Gamma_S(X, \mathscr{L}'^*))$ are isomorphic to $H^j(\Gamma_S(X, \mathscr{L}''^*))$ in order to claim $H^j(\Gamma_S(X, \mathscr{L}^*)) \simeq H^j(\Gamma_S(X, \mathscr{L}'^*))$. Therefore one can assume that there exists h_i, $i = 0, 1, 2, \ldots$, such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[d, equal] & \mathscr{L}^0 \arrow[r, "f_0"] \arrow[d, "h_0"] & \mathscr{L}^1 \arrow[r] \arrow[d, "h_1"] & \cdots \arrow[r] & \mathscr{L}^{i-1} \arrow[r, "f_{i-1}"] \arrow[d, "h_{i-1}"] & \mathscr{L}^i \arrow[r, "f_i"] \arrow[d, "h_i"] & \mathscr{L}^{i+1} \arrow[r] \arrow[d, "h_{i+1}"] & \cdots \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{L}'^0 \arrow[r, "f'_0"] & \mathscr{L}'^1 \arrow[r] & \cdots \arrow[r] & \mathscr{L}'^{i-1} \arrow[r, "f'_{i-1}"] & \mathscr{L}'^i \arrow[r, "f'_i"] & \mathscr{L}'^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

is commutative. Then define a morphism $\mathscr{L}^i \oplus \mathscr{L}'^{i-1} \to \mathscr{L}^{i+1} \oplus \mathscr{L}'^i$ by $(x, y) \mapsto (f_i(x), f'_{i-1}(y) + (-1)^i h_i(x))$. Then we have the exact sequence of flabby sheaves, $0 \to \mathscr{L}^0 \to \mathscr{L}^1 \oplus \mathscr{L}'^0 \to \mathscr{L}^2 \oplus \mathscr{L}'^1 \to \cdots$. Furthermore, $0 \to \mathscr{L}'^i \to \mathscr{L}^{i+1} \oplus \mathscr{L}'^i \to \mathscr{L}^{i+1} \to 0$, $i = 0, 1, \ldots$, is exact. Therefore, by Proposition 1.1.2 and its Corollary 2, all the vertical sequences are exact and the second horizontal sequence is exact in the diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \Gamma(X, \mathscr{L}'^0) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma_S(X, \mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_S(X, \mathscr{L}^1 \oplus \mathscr{L}'^0) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma_S(X, \mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_S(X, \mathscr{L}^1) \arrow[r] \arrow[d] & \cdots \\ & 0 & 0 & \end{tikzcd}$$

Hence we have, by Proposition 1.1.4, the exact sequence

$$0 \to H^0(\Gamma_S(X, \mathscr{L}^*)) \to H^0(\Gamma_S(X, \mathscr{L}'^*)) \to 0 \to H^1(\Gamma_S(X, \mathscr{L}^*)) \to H^1(\Gamma_S(X, \mathscr{L}'^*)) \to 0 \to \cdots.$$

Thus, we conclude, $H^j(\Gamma_S(X, \mathscr{L}^*)) \cong H^j(\Gamma_S(X, \mathscr{L}'^*))$.

Definition 1.1.15. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let S be a locally closed subset of X. A sheaf $\mathscr{H}^j_S(\mathscr{F})$, $j = 0, 1, 2, \ldots$, over X is defined as follows:*

$$\mathscr{H}^j_S(\mathscr{F}) = \mathscr{H}^j(\Gamma_S(\mathscr{L}^*)) = \frac{\operatorname{Ker}(\Gamma_S(\mathscr{L}^j) \to \Gamma_S(\mathscr{L}^{j+1}))}{\operatorname{Im}(\Gamma_S(\mathscr{L}^{j-1}) \to \Gamma_S(\mathscr{L}^j))}.$$

The sheaf $\mathscr{H}^j_S(\mathscr{F})$ *is said to be the* jth *derived sheaf of* $\mathscr{F}$ *with support in* S.

Note. For $x \notin S$ the stalk $\mathscr{H}^j_S(\mathscr{F})_x = 0$. Therefore the sheaf $\mathscr{H}^j_S(\mathscr{F})$ is concentrated on the set S. In view of this, $\mathscr{H}^j_S(\mathscr{F})|_S$ can be denoted simply by $\mathscr{H}^j_S(\mathscr{F})$.

Proposition 1.1.5. *If $\mathscr{H}^j_S(\mathscr{F}) = 0$ for $j < k$, then $\mathscr{H}^k_S(\mathscr{F})(U) = H^k_{S\cap U}(U, \mathscr{F})$.*

Proof. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. By hypothesis, the sequence $0 \to \Gamma_S(\mathscr{L}^0) \to \Gamma_S(\mathscr{L}^1) \to \cdots \to \Gamma_S(\mathscr{L}^{k-1}) \to \mathscr{I} \to 0$ is exact, where $\mathscr{I} = \mathrm{Im}(\Gamma_S(\mathscr{L}^{k-1}) \to \Gamma_S(\mathscr{L}^k))$. Generally we note the following. Let $\mathscr{F}_1 \xrightarrow{\varphi} \mathscr{F}_2$ be a morphism of sheaves, and let U be an open set of X. Let $\mathrm{Im}\,\varphi$ be the sheaf associated to the presheaf $\mathrm{Im}(\mathscr{F}_1(U) \xrightarrow{\varphi(U)} \mathscr{F}_2(U)) = \mathrm{Im}(\varphi(U))$. Then one plainly has $(\mathrm{Im}\,\varphi)(U) \supset \mathrm{Im}(\varphi(U))$. Hence, in particular, $\mathscr{I}(U) \supset \mathrm{Im}(\Gamma_{S\cap U}(U, \mathscr{L}^{k-1}) \to \Gamma_{S\cap U}(U, \mathscr{L}^k))$. By Proposition 1.1.1, $\Gamma_S(\mathscr{L}^k)$ is a flabby sheaf. Therefore, by Corollary 2 of Proposition 1.1.2, $\mathscr{I}$ is flabby. Then $\Gamma_S(\mathscr{L}^{k-1})(U) = \Gamma_{S\cap U}(U, \mathscr{L}^{k-1}) \to \mathscr{I}(U) \to 0$ is an exact sequence. Hence we have $\mathscr{I}(U) = \mathrm{Im}(\Gamma_{S\cap U}(U, \mathscr{L}^{k-1}) \to \Gamma_{S\cap U}(U, \mathscr{L}^k))$. On the other hand, let $0 \to \mathscr{Z} \to \Gamma_S(\mathscr{L}^k) \to \Gamma_S(\mathscr{L}^{k+1})$ be an exact sequence. Then, by Proposition 1.1.2(1), $\mathscr{Z}(U) = \mathrm{Ker}(\Gamma_{S\cap U}(U, \mathscr{L}^k) \to \Gamma_{S\cap U}(U, \mathscr{L}^{k+1}))$. Therefore, $H^k_{S\cap U}(U, \mathscr{F}) = \mathscr{Z}(U)/\mathscr{I}(U)$. By definition, $0 \to \mathscr{I} \to \mathscr{Z} \to \mathscr{H}^k_S(\mathscr{F}) \to 0$ is exact. Since $\mathscr{I}$ is flabby, the sequence $0 \to \mathscr{I}(U) \to \mathscr{Z}(U) \to \mathscr{H}^k_S(\mathscr{F})(U) \to 0$ is also exact. That is, $\mathscr{H}^k_S(\mathscr{F})(U) = \mathscr{Z}(U)/\mathscr{I}(U) = H^k_{S\cap U}(U, \mathscr{F})$.

Theorem 1.1.2. *Let $\mathscr{F}$, $\mathscr{F}'$, and $\mathscr{F}''$ be sheaves over a topological space X; let U be an open set of X; let S be a locally closed subset of X; and let S' be a closed subset of S.*

(1) *Suppose $S \subset U \subset X$ holds, then we have*

$$H^j_S(X, \mathscr{F}) = H^j_S(U, \mathscr{F}), \qquad j = 0, 1, 2, \ldots.$$

(2) *We have $H^0_S(X, \mathscr{F}) = \Gamma_S(X, \mathscr{F})$.*

(3) *The sequence*

$$0 \to H^0_{S'}(X, \mathscr{F}) \to H^0_S(X, \mathscr{F}) \to H^0_{S-S'}(X, \mathscr{F}) \to H^1_{S'}(X, \mathscr{F}) \to \cdots$$

is exact. In particular, for a closed set Z of X, we have the exact sequence

$$0 \to H^0_Z(X, \mathscr{F}) \to H^0(X, \mathscr{F}) \to H^0(X - Z, \mathscr{F}) \to H^1_Z(X, \mathscr{F}) \to \cdots.$$

(4) *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, we have the induced long sequences* (i) *and* (ii):
 (i) $0 \to H^0_S(X, \mathscr{F}') \to H^0_S(X, \mathscr{F}) \to H^0_S(X, \mathscr{F}'') \to H^1_S(X, \mathscr{F}) \to \cdots$
 (ii) $0 \to \mathscr{H}^0_S(\mathscr{F}') \to \mathscr{H}^0_S(\mathscr{F}) \to \mathscr{H}^0_S(\mathscr{F}'') \to \mathscr{H}^1_S(\mathscr{F}') \to \cdots$.

Proof.

(1) is plainly true by Definition 1.1.9.

(2) Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Since $\Gamma_S(X, \cdot)$ is a left exact functor, it follows that $H^0_S(X, \mathscr{F}) = \Gamma_S(X, \mathscr{F})$.

(3) Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma_{S'}(U, \mathscr{L}^0) & \longrightarrow & \Gamma_{S'}(U, \mathscr{L}^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma_S(X, \mathscr{L}^0) & \longrightarrow & \Gamma_S(U, \mathscr{L}^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma_{S-S'}(U, \mathscr{L}^0) & \to & \Gamma_{S-S'}(U, \mathscr{L}^1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where U is an open set of X so that S is closed in the relative topology of U. Since the vertical sequences are exact, one obtains the long exact sequence induced by taking cohomologies.

(4) Let $\mathscr{C}^*(\mathscr{F}')$, $\mathscr{C}^*(\mathscr{F})$, and $\mathscr{C}^*(\mathscr{F}'')$ be canonical flabby resolutions of $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ respectively. For each open set U we have the sequence

$$0 \to \mathscr{C}^0(\mathscr{F}')(U) \to \mathscr{C}^0(\mathscr{F})(U) \to \mathscr{C}^0(\mathscr{F}'')(U) \to 0$$

which is exact by Proposition 1.1.2. By the Remark that follows Proposition 1.1.2, the first and the second horizontal sequences are exact in the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{F}' & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{C}^0(\mathscr{F}') & \to & \mathscr{C}^0(\mathscr{F}) & \to & \mathscr{C}^0(\mathscr{F}'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{Z}^0(\mathscr{F}') & \to & \mathscr{Z}^0(\mathscr{F}) & \to & \mathscr{Z}^0(\mathscr{F}'') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore, Nine Lemma implies that the third sequence is also exact. By repeated use of this argument, one has the exact sequence, $0 \to \mathscr{C}^n(\mathscr{F}') \to \mathscr{C}^n(\mathscr{F}) \to \mathscr{C}^n(\mathscr{F}'') \to 0$, for each n. Since $\mathscr{C}^n(\mathscr{F}')$ is flabby, by Proposition 1.1.2,

$$0 \to \Gamma_S(X, \mathscr{C}^n(\mathscr{F}')) \to \Gamma_S(X, \mathscr{C}^n(\mathscr{F})) \to \Gamma_S(X, \mathscr{C}^n(\mathscr{F}'')) \to 0$$

is an exact sequence for each n.

All the vertical sequences are exact in the commutative diagram

$$\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
0 \to & \Gamma_S(X, \mathscr{C}^0(\mathscr{F}')) & \to & \Gamma_S(X, \mathscr{C}^1(\mathscr{F}')) & \to \cdots \\
 & \downarrow & & \downarrow & \\
0 \to & \Gamma_S(X, \mathscr{C}^0(\mathscr{F})) & \to & \Gamma_S(X, \mathscr{C}^1(\mathscr{F})) & \to \cdots \\
 & \downarrow & & \downarrow & \\
0 \to & \Gamma_S(X, \mathscr{C}^0(\mathscr{F}'')) & \to & \Gamma_S(X, \mathscr{C}^1(\mathscr{F}'')) & \to \cdots \\
 & \downarrow & & \downarrow & \\
 & 0 & & 0 &
\end{array}$$

Therefore one obtains (4.i). Since $\mathscr{H}^j_S(\mathscr{F})_x = \varinjlim_{x \in U} H^j_{S \cap U}(U, \mathscr{F})$ and $\varinjlim$ is an exact functor, (4.ii) follows immediately from (4.i).

Definition 1.1.16. *A sheaf $\mathscr{F}$ is said to be of flabby dimension $\leqq r$, denoted by flabby dim $\mathscr{F} \leqq r$, if there exists an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^r \to 0$$

such that each $\mathscr{L}^i$, $i = 0, 1, \ldots, r$, is a flabby sheaf.

Note. Observe that $\mathscr{F}$ is a flabby sheaf if and only if flabby dim $\mathscr{F} \leqq 0$. When flabby dim $\mathscr{F} \leqq r$ holds, we sometimes say that $\mathscr{F}$ has a flabby resolution of length r.

Theorem 1.1.3. *Let $\mathscr{F}$ be a sheaf over a topological space X. Then the following statements are equivalent:*

(1) flabby dim $\mathscr{F} \leqq r$
(2) $H^{r+1}_S(X, \mathscr{F}) = 0$ *for an arbitrary closed set S in X*
(2′) $H^j_S(X, \mathscr{F}) = 0$ *for $j > r$ and an arbitrary, locally closed subset S in X*
(3) $H^{r+1}_S(\mathscr{F}) = 0$ *for an arbitrary closed subset S in X*
(4) $H^r(X, \mathscr{F}) \to H^r(U, \mathscr{F})$ *is an epimorphism for any open set U of X.*

Proof. By definitions, (1) → (2′) → (2) and (1) → (3) follow plainly. We will show (2) → (4) → (1) and (3) → (1).

(2) → (4): Let $S = X$ and let $S' = X - U$ in (3) of Theorem 1.1.2. Then the long exact sequence becomes

$$\cdots \to H^r(X, \mathscr{F}) \to H^r(U, \mathscr{F}) \to H^{r+1}_{X-U}(X, \mathscr{F}) \to \cdots.$$

The third term $H^{r+1}_{X-U}(X, \mathscr{F})$ 0 by (2). Therefore one concludes (4).

(4) → (1): Take a flabby resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{L}^0 \xrightarrow{f_0} \mathscr{L}^1 \xrightarrow{f_1} \cdots \to \mathscr{L}^{r-1} \xrightarrow{f_{r-1}} \mathscr{L}^r \to \cdots$. Let $\mathscr{G} = \operatorname{Im} f_{r-1}$. Then the sequence, $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^{r-1} \to \mathscr{G} \to 0$, is exact. It suffices to prove that $\mathscr{G}$ is a flabby sheaf. Notice that $0 \to \mathscr{G} \to \mathscr{L}^r \to \mathscr{L}^{r+1} \to \cdots$ is an exact sequence. Therefore

$$H^r(U, \mathscr{F}) = \frac{\operatorname{Ker}(\mathscr{L}^r(U) \to \mathscr{L}^{r+1}(U))}{\operatorname{Im}(\mathscr{L}^{r-1}(U) \to \mathscr{L}^r(U))} = \frac{\mathscr{G}(U)}{f_{r-1}(U)\mathscr{L}^{r-1}(U)}.$$

Statement (4) implies that

$$\frac{\mathscr{G}(X)}{f_{r-1}(X)\mathscr{L}^{r-1}(X)} \to \frac{\mathscr{G}(U)}{f_{r-1}(U)\mathscr{L}^{r-1}(U)}$$

is an epimorphism. Hence, for $u \in \mathscr{G}(U)$ there exist $\tilde{u} \in \mathscr{G}(X)$ and $s \in \mathscr{L}^{r-1}(U)$ such that $u = \tilde{u}|_U + f_{r-1}(U)s$. Since $\mathscr{L}^{r-1}$ is flabby, one can find $\tilde{s} \in \mathscr{L}^{r-1}(X)$ so that $f_{r-1}(U)s = f_{r-1}(U)(\tilde{s}|_U) = f_{r-1}(X)\tilde{s}|_U$. Then $u = (\tilde{u} + f(X)\tilde{s})|_U$ holds; i.e. $\mathscr{G}$ is a flabby sheaf.

(3) → (1): We will give a proof by induction on r. First, when $r = 0$, one must show that $\mathscr{F}$ is a flabby sheaf provided that $\mathscr{H}^1_S(\mathscr{F}) = 0$. Let U_0 be an open set in X, and let $s_0 \in \mathscr{F}(U_0)$. Define an order relation $\succ$ in the set $\mathscr{M} = \{(U', s') \mid U' \supset U_0, s' \in \mathscr{F}(U'), s|_{U_0} = s_0\}$, where $(U, s) \succ (U', s')$ if and only if $U \supset U'$ and $s \mid U' = s'$. Then $\mathscr{M}$ is an inductively ordered set. By Zorn's lemma, there exists a maximal element (U_1, s_1). We claim $U_1 = X$. Suppose $U_1 \neq X$, and then let $x \in X - U_1$. For the closed set $S = X - U_1$, we have $\mathscr{H}^1_S(\mathscr{F})_x = \varinjlim_{x \in V} H^1_{S \cap V}(V, \mathscr{F}) = 0$ by the hypothesis. By (3) in Theorem 1.1.2, for any open set V containing x, the sequence

$$0 \to H^0_{S \cap V}(V, \mathscr{F}) \to H^0(V, \mathscr{F}) \to H^0(V - S, \mathscr{F}) \to H^1_{S \cap V}(V, \mathscr{F})$$

is exact. Taking the inductive limit of each term in this exact sequence, there exists $\tilde{s} \in H^0(V, \mathscr{F}) = \mathscr{F}(V)$ so that $\tilde{s}|_{(V-S)} = s_1|_{V \cap U_1}$ for a sufficiently small V. Therefore, if one lets $s_2|_V = \tilde{s}$ and $s_2|_{U_1} = s_1$, then $(U_1 \cup V, s_2) \in \mathscr{M}$. This contradicts the maximality of (U_1, s_1). Hence one has $s_1 \in \mathscr{F}(X)$ and $s_1|_{U_0} = s_0$, which shows the flabbiness of $\mathscr{F}$.

Next, suppose the case $r = r_0$ is true; then we will prove the statement (1) for $r = r_0 + 1$. Let $\mathscr{L}^0$ be a flabby sheaf such that the sequence, $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{G} \to 0$ is exact. Then, by (4) in Theorem 1.1.2, the sequence

$$0 = \mathscr{H}^{r_0}_S(\mathscr{L}^0) \to \mathscr{H}^{r_0}_S(\mathscr{G}) \to \mathscr{H}^{r_0+1}_S(\mathscr{F}) \to \mathscr{H}^{r_0+1}_S(\mathscr{L}^0) = 0$$

is exact. Therefore $\mathscr{H}^{r_0+1}_S(\mathscr{F}) = 0$ implies $\mathscr{H}^{r_0}_S(\mathscr{G}) = 0$. Then, by the inductive assumption, one has a flabby resolution

$$0 \to \mathscr{G} \to \mathscr{L}'^1 \to \mathscr{L}'^2 \to \cdots \to \mathscr{L}'^{r_0+1} \to 0.$$

Therefore $\mathscr{F}$ has the flabby resolution of length $r_0 + 1$

$$0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^{r_0+1} \to 0.$$

Definition 1.1.17. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let S be a closed subset of X. Then S is said to be purely r-codimensional with respect to $\mathscr{F}$ if*

$$\mathscr{H}^j_S(\mathscr{F}) = 0 \qquad \text{for } j \neq r.$$

Definition 1.1.18. *Let X be a topological space, and let A be an additive group. For an open subset U of X, consider the presheaf $A_X(U) = \{$mappings from U to A which are locally constant, i.e. all the continuous mappings for the discrete topology in $A\}$. The sheaf A_X associated to this presheaf is called the constant sheaf. Notice that for each $x \in X$ one has $(A_X)_x = A$.*

Definition 1.1.19. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves over a topological space X, and let U be an open subset of X. We denote the sheaf associated to the presheaf $\mathscr{F}(U) \underset{\mathbf{Z}}{\otimes} \mathscr{G}(U)$ by $\mathscr{F} \underset{\mathbf{Z}}{\otimes} \mathscr{G}$. Then we have $(\mathscr{F} \underset{\mathbf{Z}}{\otimes} \mathscr{G})_x = \mathscr{F}_x \underset{\mathbf{Z}}{\otimes} \mathscr{G}_x$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of vector spaces over $\mathbf{C}$, $\mathscr{F} \underset{\mathbf{C}}{\otimes} \mathscr{G}$ is defined similarly. Also note that $(\mathscr{F} \underset{\mathbf{C}}{\otimes} \mathscr{G})_x = \mathscr{F}_x \underset{\mathbf{C}}{\otimes} \mathscr{G}_x$. We may denote $\underset{\mathbf{Z}}{\otimes}$ and $\underset{\mathbf{C}}{\otimes}$ simply by $\otimes$ when there is no fear of confusion.*

We are now ready to define the sheaf $\mathscr{B}_M$ of hyperfunctions on a real analytic manifold M. We begin with preliminary notions.

Definition 1.1.20. *Let M be an n-dimensional real analytic manifold. If X is a complex manifold of dimension n containing M such that locally $M \cong \mathbf{R}^n \hookrightarrow \mathbf{C}^n \cong X$, then X is said to be a complexification of M. That is, there exists a neighborhood Ω of each point $x \in X$ and an injective holomorphic map $f:\Omega \to \mathbf{C}^n$ such that $\Omega \cap M = f^{-1}(\mathbf{R}^n)$. Equivalently, M is a real analytic submanifold of X such that, for $x \in M$, $T_xX = T_xM \oplus \sqrt{-1}T_xM$ holds, where we denote the tangent vector space at $x \in M$ by T_xM.*

Remark. If M is paracompact, then the complexification X is unique in the following sense: suppose X_1 and X_2 are complexifications of M; then there exists a complexification X_3 of M such that X_3 is an open subset in X_1 and X_2 (see Bruhat and Whitney [1]).

Definition 1.1.21. *Let $\mathbf{Z}_X$ be the constant sheaf on X such that each stalk is $\mathbf{Z}$, with $\mathbf{Z}$ being the ring of rational integers. The orientation sheaf ω_M over an n-dimensional (real analytic) manifold M is defined as $\mathscr{H}^n_M(\mathbf{Z}_X)$. If an open subset U is oriented, then $\omega_M(U) \cong \mathbf{Z}_M(U)$; see §2. The sign in this isomorphism depends upon the orientation on M. Note that giving a section of ω_M is equivalent to giving an orientation.*

Definition 1.1.22. *Let X be a complex manifold. We denote the sheaf of holomorphic functions on X by $\mathcal{O}_X$. That is, for an open subset U of X, it is the sheaf associated to the presheaf $\mathcal{O}(U) = \{$*holomorphic functions defined on $U\}$.

Definition 1.1.23. *Let M be an n-dimensional real analytic manifold; let X be a complexification of M; let $\mathcal{O}_X$ be the sheaf of holomorphic functions on M; and let ω_M be the orientation sheaf on M. Then we define the sheaf $\mathcal{B}_M$ of hyperfunctions on M by $\mathcal{B}_M = \mathcal{H}^n_M(\mathcal{O}_X) \underset{\mathbf{Z}}{\otimes} \omega_M$, and the sections of $\mathcal{B}_M$ are called hyperfunctions.*

In the case when M is oriented one has $\omega_M = \mathbf{Z}_M$. Therefore $\mathcal{B}_M = \mathcal{H}^n_M(\mathcal{O}_X) \otimes \omega_M = \mathcal{H}^n_M(\mathcal{O}_X)$. Let the open set X_U of X contain $U \subset M$ as a closed set. Then the sections of $\mathcal{B}_M$ over U can be written as $H^n_U(X_U, \mathcal{O}_X)$, by Proposition 1.1.5, provided that M is purely n-codimensional; see Theorem 2.2.1 for proof. In the next section we will examine $H^n_U(X_U, \mathcal{O}_X)$ via Cech cohomology theory. Using this theory, one can write hyperfunctions as sums of boundary values of holomorphic functions.

We will close this section with the proof of the flabbiness of the sheaf $\mathcal{B}_M$. We begin with a generalization of Proposition 1.1.5.

Proposition 1.1.6. *Let $\mathcal{F}$ be a sheaf over a topological space X, and let S and Z be closed sets of X such that $Z \subset S$. Suppose S is purely k-codimensional with respect to $\mathcal{F}$, i.e. $\mathcal{H}^j_S(\mathcal{F}) = 0$ for $j \neq k$; then*

$$H^j_Z(X, \mathcal{H}^k_S(\mathcal{F})) = H^{j+k}_Z(X, \mathcal{F})$$

holds for $j = 0, 1, 2, \ldots$.

Proof. First notice that one has $\Gamma_Z(X, \Gamma_S(\mathcal{L}^i)) = \Gamma_Z(X, \mathcal{L}^i)$, which is the case $j = 0$. Notations being the same as in Proposition 1.1.5, since the sequence $0 \to \mathcal{I} \to \mathcal{Z} \to \mathcal{H}^k_S(\mathcal{F}) \to 0$ is exact and $\mathcal{I}$ is flabby, together with (4) in Theorem 1.1.2, one obtains the exact sequence $0 = H^1_Z(X, \mathcal{I}) \to H^1_Z(X, \mathcal{Z}) \to H^1_Z(X, \mathcal{H}^k_S(\mathcal{F})) \to H^2_Z(X, \mathcal{I}) = 0 \to \cdots$. Hence $H^j_Z(X, \mathcal{H}^k_S(\mathcal{F})) = H^j_Z(X, \mathcal{Z})$ holds for $j \geqq 1$. On the other hand, by the assumption, for $j > k$, $H^j_S(\mathcal{F}) = 0$ holds. Therefore,

$$0 \to \mathcal{Z} \to \Gamma_S(\mathcal{L}^k) \to \Gamma_S(\mathcal{L}^{k+1}) \to \cdots$$

is a flabby resolution of $\mathcal{Z}$. Then one has

$$H^j_Z(X, \mathcal{Z}) = \frac{\operatorname{Ker}(\Gamma_Z(X, \mathcal{L}^{j+k}) \to \Gamma_Z(X, \mathcal{L}^{j+k+1}))}{\operatorname{Im}(\Gamma_Z(X, \mathcal{L}^{j+k-1}) \to \Gamma_Z(X, \mathcal{L}^{j+k}))} = H^{j+k}_Z(X, \mathcal{F}).$$

Therefore $H^j_Z(X, \mathcal{H}^k_S(\mathcal{F})) = H^{j+k}_Z(X, \mathcal{F})$ holds for $j = 0, 1, 2, \ldots$.

We quote a fundamental theorem from the theory of holomorphic functions of several variables.

Theorem 1.1.4 (Malgrange [1]). *Let X be a complex manifold, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Then*

$$\text{flabby dim } \mathcal{O}_X \leqq \dim X.$$

Note. In fact we have the equality flabby dim $\mathcal{O}_X = \dim X$, as will be seen in §2.

Proposition 1.1.6 and the fact that an n-dimensional real analytic manifold M is purely n-codimensional with respect to $\mathcal{O}_X$ provide $H^1_Z(X, \mathscr{H}^n_M(\mathcal{O}_X)) = H^{n+1}_Z(X, \mathcal{O}_X)$ for any closed set Z of M. Theorem 1.1.3 and Theorem 1.1.4 imply $H^1_Z(X, \mathscr{H}^n_M(\mathcal{O}_X)) = H^{n+1}_Z(X, \mathcal{O}_X) = 0$. Hence, again by Theorem 1.1.3, one concludes that flabby dim $\mathscr{H}^n_M(\mathcal{O}_X) \leqq 0$; i.e. $\mathscr{H}^n_M(\mathcal{O}_X)$ is a flabby sheaf. Let $M = \bigcup_{i \in I} U_i$ be an open covering, where U_i is an orientable open set; then $\mathscr{B}_M|_{U_i} = \mathscr{H}^n_M(\mathcal{O}_X)|_{U_i}$, as we noted before. In general, if $\mathscr{F}$ is a sheaf on M and if $M = \bigcup_{i \in I} U_i$ is an open covering, then $\mathscr{F}|_{U_i}$ being flabby for each $i \in I$ is equivalent to $\mathscr{F}$ being a flabby sheaf, by Theorem 1.1.3. Because, by Theorem 1.1.3, flabby dim $\mathscr{F} \leqq r$ and the triviality of $\mathscr{H}^{r+1}_S(\mathscr{F})$ for an arbitrary closed S are equivalent statements. Then note that the triviality of $\mathscr{H}^{r+1}_S(\mathscr{F})$ is a local property, since $\mathscr{H}^{r+1}_S(\mathscr{F})$ is a sheaf. That is, $\mathscr{H}^{r+1}_S(\mathscr{F}) = 0$ if and only if $\mathscr{H}^{r+1}_S(\mathscr{F})|_{U_i} = 0$ for each U_i.

Hence we now conclude the following theorem.

Theorem 1.1.5. *The sheaf $\mathscr{B}_M$ of hyperfunctions on a real analytic manifold M is a flabby sheaf.*

This theorem, together with the general theory of systems of linear differential equations, provides us with explicit flabby resolutions of the constant sheaf $\mathbf{C}_X$ and the sheaf $\mathcal{O}_X$ of holomorphic functions via differential operators (see Komatsu [1]). This is an important and interesting result from the analyst's point of view.

Exercise 1. The presheaf of all the continuous and bounded functions on each open set of $\mathbf{R}^n$ is not a sheaf. Find the sheaf associated to this presheaf.

Exercise 2. Consider the presheaf of all the real-valued, locally constant functions on each open set of $\mathbf{R}^n$ modulo constant functions on $\mathbf{R}^n$. This presheaf is not a sheaf. Find the sheaf associated to the presheaf.

Exercise 3. Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous map.

(1) Suppose $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves; then prove that $0 \to f_*\mathscr{F}' \to f_*\mathscr{F} \to f_*\mathscr{F}''$ is exact.

(2) Suppose $\mathscr{F}$ is a flabby sheaf over X; then prove that $f_*\mathscr{F}$ is a flabby sheaf over Y.

(3) Suppose $\mathscr{G}' \to \mathscr{G} \to \mathscr{G}''$ is an exact sequence of sheaves over Y; then prove that $f^{-1}\mathscr{G}' \to f^{-1}\mathscr{G} \to f^{-1}\mathscr{G}''$ is exact over X.

Exercise 4. Suppose that, in the commutative diagram below, all the vertical sequences are exact and that the second and third horizontal sequences are exact. Then prove that there are induced maps $A' \to A \to A''$ and $C' \to C \to C''$ such that they are both exact.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A' & & A & & A'' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & K' & \to & K & \to & K'' \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L' & \to & L & \to & L'' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C' & & C & & C'' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

§2. Hyperfunctions as Boundary Values of Holomorphic Functions

We will examine the orientation sheaf $\omega_M = \mathscr{H}^n_M(\mathbf{Z}_X)$ and the sheaf of hyperfunctions $\mathscr{B}_M = \mathscr{H}^n_M(\mathscr{O}_X) \otimes \omega_M$ via the Čech cohomology theory. The Čech theory is useful for the explicit presentation of hyperfunctions. In fact, M. Sato seems to have been "naturally" led to the notion of relative cohomology groups expressed in terms of covering, i.e. Čech cohomology groups—independently of Grothendieck—when he tried to find the correct formulation of the idea that a hyperfunction of several variables is a tensor product of hyperfunctions of one variable.

Let X be a topological space, and let Z be a closed subset of X. Suppose $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $X - Z = \bigcup_{\lambda \in \Lambda'} U_\lambda$, where $\Lambda' \subset \Lambda$, are open coverings of X and $X - Z$. We will denote these open coverings by $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ and $\mathscr{U}' = \{U_\lambda\}_{\lambda \in \Lambda'}$ respectively. For a sheaf $\mathscr{F}$ over X the Čech cohomology groups are defined as follows. First define

$$C^k(\mathscr{U}, \mathscr{F}) = \{\varphi = \{\varphi_{\lambda_0, \ldots, \lambda_k}\}_{(\lambda_0, \ldots, \lambda_k) \in \Lambda^{k+1}} \mid \varphi_{\lambda_0, \ldots, \lambda_k} \in \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_k}, \mathscr{F}) \text{ and }$$

$$\varphi_{\lambda_0, \ldots, \lambda_k} = -\varphi_{\lambda_0, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_k}\}.$$

Then notice $C^k(\mathscr{U}, \mathscr{F}) \subset \bigoplus_{(\lambda_0, \ldots, \lambda_k) \in \Lambda^{k+1}} \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_k}, \mathscr{F})$. Furthermore we define

$$C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = \{\varphi = \{\varphi_{\lambda_0, \ldots, \lambda_k}\}_{(\lambda_0, \ldots, \lambda_k) \in \Lambda^{k+1}} \in C^k(\mathscr{U}, \mathscr{F}) \mid \text{if } (\lambda_0, \ldots, \lambda_k) \in \Lambda'^{k+1}, \text{ then } \varphi_{\lambda_0, \ldots, \lambda_k} = 0\}.$$

Then define a map $\delta: C^k(\mathscr{U}, \mathscr{F}) \to C^{k+1}(\mathscr{U}, \mathscr{F})$ by

$$\delta(\{\varphi_{\lambda_0, \ldots, \lambda_k}\}) = \{\psi_{\lambda_0, \ldots, \lambda_{k+1}}\}_{(\lambda_0, \ldots, \lambda_{k+1}) \in \Lambda^{k+2}},$$

where

$$\psi_{\lambda_0, \ldots, \lambda_{k+1}} = \sum_{i=0}^{k+1} (-1)^i \varphi_{\lambda_0, \ldots, \lambda_{i-1}, \lambda_{i+1}, \lambda_{k+1}}|_{U_{\lambda_0} \cap \cdots \cap U_{\lambda_{k+1}}}.$$

Note that one has $\delta^2 = 0$. Hence there is induced a cochain complex

$$0 \to C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathscr{U}, \mathscr{F}) \to \cdots.$$

Definition 1.2.1. *The* k*th Čech cohomology groups,* $H^k(\mathscr{U}, \mathscr{F})$ *and* $H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F})$, *are defined as*

$$H^k(\mathscr{U}, \mathscr{F}) = H^k(C^*(\mathscr{U}, \mathscr{F})) = \frac{\operatorname{Ker}(C^k(\mathscr{U}, \mathscr{F}) \to C^{k+1}(\mathscr{U}, \mathscr{F}))}{\operatorname{Im}(C^{k-1}(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{U}, \mathscr{F}))}$$

and

$$H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = H^k(C^*(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}))$$
$$= \frac{\operatorname{Ker}(C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to C^{k+1}(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}))}{\operatorname{Im}(C^{k-1}(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}))}.$$

For each $k \geqq 0$ one has the exact sequence

$$0 \to C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to C^k(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{U}'; \mathscr{F}) \to 0.$$

Therefore the long exact sequence

$$0 \to H^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to H^0(\mathscr{U}, \mathscr{F}) \to H^0(\mathscr{U}', \mathscr{F})$$
$$\to H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to \cdots$$

is induced.

Next we will prove

$$H^k_Z(X, \mathscr{F}) \cong H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}),$$

provided that for each $k \geqq 0$ and arbitrary $\lambda_0, \ldots, \lambda_r$

$$H^k(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}) = 0$$

By this theorem of Leray, one can present explicitly the sections of the hyperfunction sheaf $\mathscr{B}_M$. We will begin with lemmas.

Lemma 1. $H^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = \Gamma_Z(X, \mathscr{F})$.

Proof. By definition one has

$$H^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = \{\{\varphi_\lambda\}_{\lambda\in\Lambda} \in C^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \mid \delta(\{\varphi_\lambda\}_{\lambda\in\Lambda}) = 0\},$$

where $\delta(\{\varphi_\lambda\}_{\lambda\in\Lambda}) = \{(\varphi_\mu|_{U_\lambda\cap U_\mu} - \varphi_\lambda|_{U_\lambda\cap U_\mu})\}_{(\lambda,\mu)\in\Lambda^2}$. Therefore, by the definition of sheaf, there exists a unique $\varphi \in \Gamma(X, \mathscr{F})$ such that $\varphi|_{U_\lambda} = \varphi_\lambda$. Since $\{\varphi_\lambda\}_{\lambda\in\Lambda} \in C^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F})$, $\varphi|_{(X-Z)} = 0$ holds. This implies $\varphi \in \Gamma_Z(X, \mathscr{F})$. Conversely, if φ belongs to $\Gamma_Z(X, \mathscr{F})$, then clearly $\varphi \in H^0(\mathscr{U} \bmod \mathscr{U}', \mathscr{F})$.

Lemma 2. *If $\mathscr{F}$ is a flabby sheaf, then $H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = 0$ holds for any integer $k > 0$.*

Proof. In Theorem 1.1.2, (2) and (3) imply that the sequence

$$0 \to \Gamma_Z(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Z, \mathscr{F}) \to H_Z^1(X, \mathscr{F})$$

is exact. Furthermore, $H_Z^1(X, \mathscr{F}) = 0$ since $\mathscr{F}$ is flabby by Theorem 1.1.3. Lemma 1 implies that

$$\begin{aligned}0 &\to \Gamma_Z(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Z, \mathscr{F}) \to H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{F})\\ &\to H^1(\mathscr{U}, \mathscr{F}) \to H^1(\mathscr{U}', F) \to H^2(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to \cdots\end{aligned}$$

is an exact sequence. As we will show below, $H^1(\mathscr{U}, \mathscr{F}) = 0$; then $H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = 0$ is true. For the case $k \geqq 2$, $H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = 0$ follows simply from $H^k(\mathscr{U}, \mathscr{F}) = 0$, and $H^k(\mathscr{U}', \mathscr{F}) = 0$ for $k \geqq 1$. It suffices to prove that $H^k(\mathscr{U}, \mathscr{F})$ and therefore $H^k(\mathscr{U}', \mathscr{F})$ vanish for $k \geqq 1$. Let $f = \{f_{\lambda_0,\ldots,\lambda_k}\} \in C^k(\mathscr{U}, \mathscr{F})$ such that $\delta f = 0$. Consider the set $\mathscr{M} = \{(g, U) \mid U$ is an open set in X, $g \in C^{k-1}(\mathscr{U} \cap U, \mathscr{F})$ such that $\delta g = f|_U$, where $\mathscr{U} \cap U = \{U_\lambda \cap U\}$ and $f|_U$ is the image under the map $C^k(\mathscr{U}, \mathscr{F}) \to C^k(\mathscr{U} \cap U, \mathscr{F})\}$. First of all, $\mathscr{M} \neq \varnothing$ will be shown. If one lets $h_{\lambda_0,\ldots,\lambda_{k-1}} = f_{\lambda,\lambda_0,\ldots,\lambda_{k-1}} \in \Gamma(U_\lambda \cap U_{\lambda_0} \cap \cdots \cap U_{\lambda_{k-1}}, \mathscr{F})$, i.e. $h = \{h_{\lambda_0,\ldots,\lambda_{k-1}}\}_{(\lambda_0,\ldots,\lambda_{k-1})\in\Lambda^k} \in C^{k-1}(\mathscr{U} \cap U_\lambda, \mathscr{F})$, then $(\delta h)_{\lambda_0,\ldots,\lambda_k} = \sum_{i=0}^{k} (-1)^i h_{\lambda_0,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_k} = \sum (-1)^i f_{\lambda,\lambda_0,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_k}$. By the assumption, one has $(\delta f)_{\lambda,\lambda_0,\ldots,\lambda_k} = f_{\lambda_0,\ldots,\lambda_k} - \sum_{i=0}^{k} (-1)^i \times f_{\lambda,\lambda_0,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_k} = 0$. Therefore $\delta h = f|_{U_\lambda}$, i.e. $(h, U_\lambda) \in \mathscr{M}$. So $\mathscr{M} \neq \varnothing$. Next we will define an order relation in $\mathscr{M}$. Define $(g_1, U_1) \succ (g_2, U_2)$ if and only if $U_1 \supset U_2$ and $g_1|_{U_2} = g_2$. Then $\mathscr{M}$ is an inductively ordered set. Therefore there exists a maximal element in $\mathscr{M}$ by Zorn's lemma. Let (g, U) be a maximal element; then one is to show $U = X$. Suppose $U \neq X$, and let $x \in X - U$. Since $X = \bigcup_{\lambda\in\Lambda} U_\lambda$, there exists U_λ such that $x \in U_\lambda$. Then, as it was described before, $(h, U_\lambda) \in \mathscr{M}$. Therefore one has $\delta h|_{U\cap U_\lambda} = f|_{U\cap U_\lambda} = \delta g|_{U\cap U_\lambda}$. Then $\delta(h|_{U\cap U_\lambda} - g|_{U\cap U_\lambda}) = 0$. Note

$h|_{U \cap U_\lambda} - g|_{U \cap U_\lambda} \in C^{k-1}(\mathscr{U} \cap U \cap U_\lambda, \mathscr{F})$. When $k = 1$, by letting $Z = X$ in Lemma 1, $h|_{U \cap U_\lambda} - g|_{U \cap U_\lambda} \in H^0(\mathscr{U} \cap U \cap U_\lambda, \mathscr{F}) = \Gamma(U \cap U_\lambda, \mathscr{F})$. Since $\mathscr{F}$ is a flabby sheaf, there exists $s \in \Gamma(U_\lambda, \mathscr{F})$ such that $s|_{U \cap U_\lambda} = h|_{U \cap U_\lambda} - g|_{U \cap U_\lambda}$. Let $g'|_U = g$ and $g'|_{U_\lambda} = h - s$; then $\delta(h - s) = \delta h = f|_{U_\lambda}$. Therefore $(g', U \cup U_\lambda) \in \mathscr{M}$, which contradicts the choice of (g, U) in $\mathscr{M}$. Hence $X = U$. When $k > 1$, we give a proof by induction on k. The inductive assumption $H^{k-1}(\mathscr{U} \cap U \cap U_\lambda, \mathscr{F}) = 0$ implies that there exists $s \in C^{k-2}(\mathscr{U} \cap U \cap U_\lambda, \mathscr{F})$ such that $\delta s = h|_{U \cap U_\lambda} - g|_{U \cap U_\lambda}$. The flabbiness of $\mathscr{F}$ implies that there exists $\tilde{s} \in C^{k-2}(\mathscr{U} \cap U_\lambda, \mathscr{F})$ such that $\tilde{s}|_{U \cap U_\lambda} = s$. Define $g' \in C^{k-1}(\mathscr{U} \cap (U \cap U_\lambda), \mathscr{F})$ to be $g'|_U = g$ and $g'|_{U_\lambda} = h - \delta\tilde{s}$. Then $g|_{U \cap U_\lambda} = (h - \delta\tilde{s})|_{U \cap U_\lambda}$. Therefore g' is well defined and $\delta g'|_U = \delta g|_U = f|_U$ and $\delta g'|_{U_\lambda} = \delta(h - \delta\tilde{s})|_{U_\lambda} = \delta h|_{U_\lambda} = f|_{U_\lambda}$. Then $(g', U \cup U_\lambda) \in \mathscr{M}$, contradicting the maximality of (g, U). Hence one has $g \in C^{k-1}(\mathscr{U}, \mathscr{F})$ such that $\delta g = f$.

We are now ready to prove the theorem of Leray which is fundamental to our theory.

Theorem 1.2.1 (Leray). *Suppose $H^j(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}) = 0$ for an arbitrary integer $j > 0$ and arbitrary $\lambda_0, \ldots, \lambda_r$. Then*

$$H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = H^k_Z(X, \mathscr{F}).$$

Remark. The covering that satisfies the condition of this theorem is sometimes called a Leray covering.

Proof. Lemma 1 is the case when $k = 0$ in this theorem. Therefore, let $k \geqq 1$. Let $0 \to \mathscr{F} \to \mathscr{L} \to \mathscr{G} \to 0$ be an exact sequence of sheaves such that $\mathscr{L}$ is a flabby sheaf. Then there is induced the long exact sequence

$$\begin{aligned} 0 &\to \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}) \to \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{L}) \\ &\to \Gamma(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{G}) \to H^1(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}) \to \cdots. \end{aligned}$$

By the assumption, the sequence

$$0 \to C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \to C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{L}) \to C^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{G}) \to 0$$

is exact. On the other hand, Lemma 1 implies that the sequence

$$\begin{aligned} 0 &\to \Gamma_Z(X, \mathscr{F}) \to \Gamma_Z(X, \mathscr{L}) \to \Gamma_Z(X, \mathscr{G}) \to H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \\ &\to H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{L}) \to H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{G}) \to H^2(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \\ &\to H^2(\mathscr{U} \bmod \mathscr{U}', \mathscr{L}) \to \cdots \end{aligned}$$

is exact. Note that $H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{L})$ and $H^2(\mathscr{U} \bmod \mathscr{U}', \mathscr{L})$ are both trivial by Lemma 2. Therefore one has the isomorphisms

$$H^k(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) \simeq \begin{cases} \operatorname{Coker}(\Gamma_Z(X, \mathscr{L}) \to \Gamma_Z(X, \mathscr{G})) & \text{for } k = 1 \\ H^{k-1}(\mathscr{U} \bmod \mathscr{U}', \mathscr{G}) & \text{for } k > 1 \end{cases}.$$

Similarly, by Theorem 1.1.3,

$$H^k_Z(X, \mathscr{F}) = \begin{cases} \mathrm{Coker}(\Gamma_Z(X, \mathscr{L}) \to \Gamma_Z(X, \mathscr{G}) & \text{for } k = 1 \\ H^{k-1}_Z(X, \mathscr{G}) & \text{for } k > 1 \end{cases}$$

Hence $H^1(\mathscr{U} \bmod \mathscr{U}', \mathscr{F}) = H^1_Z(X, \mathscr{F})$ holds. When $k > 1$, we prove the assertion by induction on k. Let $j > 0$ and r be arbitrary. One has the exact sequence

$$H^j(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{L}) \to H^j(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{G}) \\ \to H^{j+1}(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}).$$

Note that $H^j(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{L}) = 0$ since $\mathscr{L}$ is flabby and that $H^{j+1}(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{F}) = 0$ by the assumption. Therefore one concludes $H^j(U_{\lambda_0} \cap \cdots \cap U_{\lambda_r}, \mathscr{G}) = 0$ for arbitrary $j > 0$ and $\lambda_0, \ldots, \lambda_r$. Now by the inductive assumption one has

$$H^{k-1}(\mathscr{U} \bmod \mathscr{U}', \mathscr{G}) = H^{k-1}_Z(X, \mathscr{G}).$$

Therefore one finally obtains

$$H^k(U \bmod \mathscr{U}', \mathscr{F}) = H^k_Z(X, \mathscr{F}).$$

We will apply this theorem to our theory of hyperfunctions. First we will recall some of the most fundamental results from the theory of holomorphic functions of several variables in order to prove that there is a covering satisfying the assumption of Theorem 1.2.1.

Definition 1.2.2. *Let X be a paracompact complex manifold. Then X is said to be a Stein manifold if* (i) *and* (ii) *are satisfied.*

(i) *Holomorphically convex condition: let K be a compact set in X; then $\hat{K} = \{p \in X \,|\, |f(p)| \leq \sup_{x \in K} |f(x)|$ for any $f \in \Gamma(X, \mathscr{O}_X)\}$ is a compact set.*

(ii) *Holomorphically separable condition: for any distinct points p and q in X, there exists $f \in \Gamma(X, \mathscr{O}_X)$ such that $f(p) \neq f(q)$.*

Examples. The space $\mathbf{C}^n$ is a Stein manifold. For a Stein manifold X and holomorphic function $f(x)$ on X, $\{x \in X \,|\, \mathrm{Im}\, f(x) > 0\}$ and $\{x \in X \,|\, |f(x)| > 1\}$ are both Stein manifolds. The direct product and the intersection of two Stein manifolds are also Stein manifolds. A closed analytic submanifold of a Stein manifold is a Stein manifold. A 1-dimensional complex manifold without compact components is a Stein manifold.

The following theorems are crucial. In particular, Theorem 1.2.2 seems to be one of the most profound results in the field of analysis in this century.

Theorem 1.2.2 (Oka-Cartan). *If X is a Stein manifold, then $H^i(X, \mathscr{O}_X) = 0$ for any integer $i > 0$.*

Theorem 1.2.3 (Grauert [1]). *A paracompact real analytic manifold has complex neighborhoods which are Stein manifolds.*

Consult, for example, Hitotumatu [1] for proofs. Let M be an n-dimensional real analytic manifold, and let X be a complexification. If an open subset U of M is oriented, then $\mathscr{B}_M \cong \mathscr{H}^n_M(\mathscr{O}_X)$ holds over U since $\omega_M|_U = \mathbf{Z}_U$. Let Ω be an open set of X containing U as a closed set. Then one has $\mathscr{B}_M(U) = H^n_U(\Omega, \mathscr{O}_X)$ as we showed in §1. Furthermore, by Theorem 1.2.3, the open set Ω of X can be taken to be a Stein manifold. Suppose that real analytic functions $f_0, f_1, \ldots, f_n$ on M satisfy the conditions:

(1) f_j is real-valued on M for each j, $0 \leqq j \leq n$, and

(2) for each point $x \in M$ the convex hull of $\{df_0(x), \ldots, df_n(x)\} \subset T^*_x M$, i.e. $\left\{\sum_{i=0}^{n} t_i\, df_i(x) \,\middle|\, \sum_{i=0}^{n} t_i = 1, t_i \geqq 0\right\}$, is a neighborhood of the origin in $T^*_x M$. In the case $M = \mathbf{R}^n$, if the convex hull of $\{\xi_0, \ldots, \xi_n\} \subset M$ is a neighborhood of the origin, then one can take $f_j = \langle \xi_j, x\rangle$. Let $V_j = \{z \in \Omega \mid \operatorname{Im} f_j > 0\}$ for $j = 0, 1, \ldots, n$. Then V_j is a Stein manifold. Since $\operatorname{Im} f_j = 0$ on U, one has $\left(\bigcup_{j=0}^{n} V_j\right) \cap U = \varnothing$.

Lemma 1. $\left(\bigcup_{j=0}^{n} V_j\right) \cup U$ *is a neighborhood of* U.

Proof. Let $x_0 \in U$, and let $x + \sqrt{-1}y$ be sufficiently near the point x_0. When $x + \sqrt{-1}y$ does not belong to $\bigcup_{j=0}^{n} V_j$, we must show $y = 0$, i.e. $x \in U$. Consider the Taylor expansion $f_j(x + \sqrt{-1}y) = f_j(x) + \sqrt{-1}\langle y, df_j(x)\rangle$ + (terms of degree greater than 2 in y). Since y is sufficiently small and $x + \sqrt{-1}y \notin V_j$, one has $\langle y, df_j(x)\rangle \leqq 0$ for each j. On the other hand, the convex hull of df_j, $j = 0, \ldots, n$, is a neighborhood of the origin. Therefore, if $y \neq 0$, $\left\langle y, \sum_{j=0}^{n} t_j\, df_j(x)\right\rangle > 0$ holds for some $t_j > 0, j = 0, \ldots, n$, $\sum_{j=0}^{n} t_j = 1$. Then $\left\langle y, \sum_{j=0}^{n} t_j\, df_j(x)\right\rangle = \sum_{j=0}^{n} t_j\langle y, df_j(x)\rangle \leqq 0$ is contradictory.

From Theorem 1.2.3, U has a fundamental neighborhood system consisting of Stein manifolds. Such a neighborhood of U is called a Stein neighborhood. Therefore there exists a Stein neighborhood Ω' such that $\left(\bigcup_{j=0}^{n} V_j\right) \cup U \supset \Omega'$ holds. Replacing V_j by $V_j \cap \Omega'$ and Ω' by Ω, one has $\Omega = \left(\bigcup_{j=0}^{n} V_j\right) \cup U$. Then $\mathscr{V}' = \{\Omega, V_0, \ldots, V_n\}$ and $\mathscr{V}'' = \{V_0, V_1, \ldots, V_n\}$ are Stein open coverings of Ω and $\Omega - U$ respectively. Then, by Oka-

Cartan Theorem 1.2.2, for any integer $j > 0$ and arbitrary $\lambda_0, \ldots, \lambda_r$, $H^j(V_{\lambda_0} \cap \cdots \cap V_{\lambda_r}, \mathscr{O}_X) = 0$ holds. Therefore Leray Theorem 1.2.1 implies $\mathscr{B}_M(U) = H^n_U(\Omega, \mathscr{O}_X) = H^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X)$.

Recall the following definition:

$$H^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) = \frac{\operatorname{Ker}(C^n(V \bmod \mathscr{V}', \mathscr{O}_X) \to C^{n+1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X))}{\operatorname{Im}(C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) \to C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X))}.$$

Lemma 2. $C^{n+1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) = 0.$

Proof. As $C^{n+1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) \to C^{n+1}(\mathscr{V}, \mathscr{O}_X)$ is a monomorphism, it is sufficient to show $C^{n+1}(\mathscr{V}, \mathscr{O}_X) = 0$. $C^{n+1}(\mathscr{V}, \mathscr{O}_X) = \{\varphi \in \Gamma(\Omega \cap V_0 \cap \cdots \cap V_n, \mathscr{O}_X)\} = \{\varphi \in \Gamma(V_0 \cap \cdots \cap V_n, \mathscr{O}_X)\}$. When Ω is a sufficiently small neighborhood of U, we will prove $V_0 \cap \cdots \cap V_n = \varnothing$. Let $x + \sqrt{-1}y \in V_0 \cap \cdots \cap V_n$. Then one obtains $\langle y, df_j(x)\rangle > 0$ as in the proof of Lemma 1. Since $y \neq 0$, there exists $t_j > 0$, $j = 0, 1, \ldots, n$, $\sum_{j=0}^n t_j = 1$ such that $\left\langle y, \sum_{j=0}^n t_j\, df_j(x)\right\rangle \leqq 0$. Then one has $\left\langle y, \sum_{j=0}^n t_j\, df_j(x)\right\rangle = \sum_{j=0}^n t_j \langle y, df_j(x)\rangle > 0$, which is a contradiction. Hence we conclude $C^{n+1}(\mathscr{V}, \mathscr{O}_X) = 0$.

Therefore from this lemma we have

$$\mathscr{B}_M(U) = \frac{C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X)}{\operatorname{Im}(C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) \to C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X))}.$$

We now compute $C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X)$.

$$\begin{aligned} C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) &= \Big\{(\varphi_{0,1,\ldots,n}, \varphi_{\Omega,1,\ldots,n}, \ldots, \varphi_{\Omega,0,1,\ldots,n-1}) \\ &\qquad \in \mathscr{O}_X\Big(\bigcap_{j=0}^n V_j\Big) \oplus \bigoplus_{j=0}^n \mathscr{O}_X\Big(\Omega \cap \bigcap_{k \neq j} V_k\Big) \Big| \varphi_{0,1,\ldots,n} = 0\Big\} \\ &= \bigoplus_{j=0}^n \mathscr{O}_X\Big(\Omega \cap \bigcap_{k \neq j} V_k\Big) = \bigoplus_{j=0}^n \mathscr{O}_X\Big(\bigcap_{k \neq j} V_k\Big). \end{aligned}$$

Denote $W_{\hat{j}} = \bigcap_{k \neq j} V_k$. Then $C^n(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X) = \bigoplus_{j=0}^n \mathscr{O}_X(W_{\hat{j}})$. We also abbreviate $\varphi_{\Omega, j_1, \ldots, j_n} \in \mathscr{O}(W_{\hat{j}_0})$ as $\varphi_{\hat{j}_0}$, where the sign of the permutation

$$\begin{pmatrix} 0, & 1, & \ldots, & n \\ j_0, & j_1, & \ldots, & j_n \end{pmatrix} \text{ is } +1.$$

In the last place we will compute $C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X)$. Let $\{\varphi_{\lambda_0, \ldots, \lambda_{n-1}}\} \in C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathscr{O}_X)$. If $\varphi_{\lambda_0, \ldots, \lambda_{n-1}} \in \Gamma(V_{\lambda_0} \cap \cdots \cap V_{\lambda_{n-1}}, \mathscr{O}_X)$, then $\varphi_{\lambda_0, \ldots, \lambda_{n-1}} = 0$ by the definition of "mod $\mathscr{V}'$." Therefore we only need to consider the type $\varphi_{\Omega,i_0,\ldots,i_{n-2}} \in \Gamma(\Omega \cap V_{i_0} \cap \cdots \cap V_{i_{n-2}}, \mathcal{O}_X)$. We abbreviate this as $\varphi_{\widehat{i_{n-1},i_n}}$, where sign

$$\begin{pmatrix} 0, & 1, & \ldots, & n \\ i_0, & i_1, & \ldots, & i_n \end{pmatrix} = +1.$$

Then $\varphi_{\widehat{j,k}} \in \mathcal{O}_X\left(\Omega \cap \bigcap_{l \neq j,k} V_l\right) = \mathcal{O}_X\left(\bigcap_{l \neq j,k} V_l\right) = \mathcal{O}_X(W_{\widehat{j,k}})$, where $W_{\widehat{j,k}} = \bigcap_{l \neq j,k} V_l$, satisfying $\varphi_{\widehat{j,k}} = -\varphi_{\widehat{k,j}}$. That is, $C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}_X) = \bigoplus'_{\substack{j,k \\ j \neq k}} \mathcal{O}_X(W_{\widehat{j,k}})$, where $\bigoplus'$ denotes the alternating sum; i.e. $\{\varphi_{\widehat{j,k}}\} \in \bigoplus' \mathcal{O}_X(W_{\widehat{j,k}})$ if and only if $\varphi_{\widehat{j,k}} = -\varphi_{\widehat{k,j}}$ holds for arbitrary j and k.

Finally we have the isomorphism

$$\mathscr{B}_M(U) \cong \frac{\bigoplus_{j=0}^{n} \mathcal{O}_X(W_{\hat{j}})}{\operatorname{Im}\left(\bigoplus'_{j,k} \mathcal{O}_X(W_{\widehat{j,k}}) \xrightarrow{\delta} \bigoplus_{j=0}^{n} \mathcal{O}_X(W_{\hat{j}})\right)}.$$

Next we will compute the image of δ, i.e. $(\delta\varphi)_{\hat{j}}$ for $\varphi = \{\varphi_{\widehat{j,k}}\} \in \bigoplus'_{j,k} \mathcal{O}_X(W_{\widehat{j,k}})$. By definition, $(\delta\varphi)_{\hat{j}_n} = (\delta\varphi)_{\Omega,j_0,\ldots,j_{n-1}} = \varphi_{j_0,\ldots,j_{n-1}} - \varphi_{\Omega,j_1,\ldots,j_{n-1}} + \varphi_{\Omega,j_0,j_2,\ldots,j_{n-1}} - \cdots$. As before, $\varphi_{j_0,\ldots,j_{n-1}} = 0$. Let $-\varphi_{\Omega,j_1,\ldots,j_{n-1}} = \varphi_{\widehat{j_0,j_n}}$, where sign $\begin{pmatrix} 0, & 1, & \ldots, & n \\ j_n, & j_0, & \ldots, & j_{n-1} \end{pmatrix} = 1$, $\varphi_{\Omega,j_0,j_2,\ldots,j_{n-1}} = \varphi_{\widehat{j_1,j_n}}$ and so on. Then $(\delta\varphi)_{\hat{j}_n} = \varphi_{\widehat{j_0,j_n}} + \varphi_{\widehat{j_1,j_n}} + \cdots + \varphi_{\widehat{j_{n-1},j_n}} (+\varphi_{\widehat{j_n,j_n}}(=0))$. Therefore $(\delta\varphi)_{\hat{j}} = \sum_{k=0}^{n} \varphi_{\widehat{k,j}}$; i.e. $\operatorname{Im} \delta = \delta(\{\varphi_{\widehat{j,k}}\}) = \left\{\sum_{k=0}^{n} \varphi_{\widehat{k,j}}\right\}_j$. For $\varphi \in \mathcal{O}_X(W_{\hat{j}})$ the boundary value $b(\varphi)$ of φ is defined by the image of the composite map

$$\begin{array}{ccccc}
\mathcal{O}_X(W_{\hat{j}}) & \longrightarrow & \bigoplus_{l=0}^{n} \mathcal{O}_X(W_{\hat{l}}) & \longrightarrow & \dfrac{\bigoplus_{l=0}^{n} \mathcal{O}_X(W_{\hat{l}})}{\operatorname{Im}\left(\bigoplus'_{l,k} \mathcal{O}_X(W_{\widehat{l,k}}) \to \bigoplus_{l=0}^{n} \mathcal{O}_X(W_{\hat{l}})\right)} = \mathscr{B}_M(U \\
\Cup & & \Cup & & \Cup \\
\varphi & \longmapsto & (0, \ldots, 0, \phi_{\hat{l}}, 0, \ldots 0) & \longmapsto & b(\varphi)
\end{array}$$

Therefore $\mathscr{B}_M(U) \cong \sum_{j=0}^{n} b(\mathcal{O}_X(W_{\hat{j}}))$.

Note. We will investigate further the notion of boundary values of holomorphic functions in §3 of the next chapter.

If U is an oriented open set, then the hyperfunctions on U can be expressed as the sum of boundary values of holomorphic functions which are defined on $(n + 1)$ angular domains (see Figure 1.2.1) when $n = 2$.

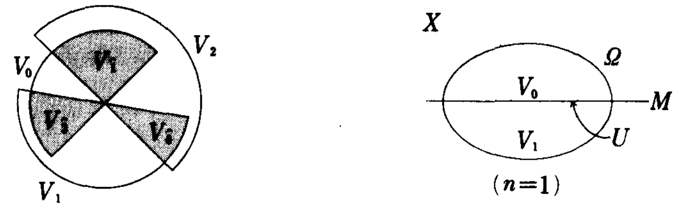

Figure 1.2.1

Notice that for $g_{\hat{j}} \in \mathcal{O}_X(W_{\hat{j}})$, $0 \leqq j \leqq n$, $\sum_{j=0}^{n} b(g_{\hat{j}}) = 0$ holds if and only if $g_{\hat{j}} = \sum_{k=0}^{n} f_{\widehat{j,k}}$, $0 \leqq j \leqq n$, for $f_{\widehat{j,k}} \in \mathcal{O}(W_{\widehat{j,k}})$ such that $f_{\widehat{j,k}} = -f_{\widehat{k,j}}$. That is, two boundary values of holomorphic functions define the same hyperfunction if and only if the difference of the boundary values is a coboundary. That was the reason for introducing the notion of relative cohomology.

For example, when $n = 1$, let $M = \mathbf{R}$, $X = \mathbf{C}$, and $f_0(x) = x, f_1(x) = -x$. From the definition

$$V_0 = \{z \in \Omega \mid \operatorname{Im} z > 0\}$$

and

$$V_1 = \{z \in \Omega \mid \operatorname{Im} z < 0\}.$$

Then $W_{\hat{0}} = V_1$, $W_{\hat{1}} = V_0$ (see Figure 1.2.1). Note that $C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}_X) = C^0(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}_X) = \{(\varphi_\Omega, \varphi_0, \varphi_1) \in \mathcal{O}_X(\Omega) \oplus \mathcal{O}(V_0) \oplus \mathcal{O}(V_1) \mid \varphi_0 = \varphi_1 = 0\} = \mathcal{O}_X(\Omega)$. Therefore

$$\mathscr{B}_M(U) = \frac{\mathcal{O}_X(W_{\hat{0}}) \oplus \mathcal{O}_X(W_{\hat{1}})}{\operatorname{Im}(\mathcal{O}_X(\Omega) \to \mathcal{O}_X(W_{\hat{0}}) \oplus \mathcal{O}(W_{\hat{1}}))}$$

$$= \frac{\mathcal{O}_X(V_1) \oplus \mathcal{O}_X(V_0)}{\operatorname{Im}(\mathcal{O}_X(\Omega) \to \mathcal{O}_X(V_1) \oplus \mathcal{O}_X(V_0))}.$$

Suppose that $g_+ \in \mathcal{O}(V_0)$ and $g_- \in \mathcal{O}(V_1)$ have the same boundary value, i.e. $b(g_+) = b(g_-)$, then $g_+ - g_- = (g_{\hat{0}}, g_{\hat{1}}) \in \mathcal{O}(W_{\hat{0}}) \oplus \mathcal{O}(W_{\hat{1}})$, where $g_{\hat{0}} = -g_-$, $g_{\hat{1}} = g_+$, is a coboundary. Therefore, there exists $f = f_{\widehat{0,1}} = -f_{\widehat{1,0}} \in \mathcal{O}_X(\Omega)$ such that $g_{\hat{0}} = f_{\widehat{0,0}} + f_{\widehat{1,0}} = -f$ and $g_{\hat{1}} = f_{\widehat{0,1}} + f_{\widehat{1,1}} = f$. This means that there is $f \in \mathcal{O}_X(\Omega)$ such that $g_+ = f|_{V_0}$ and $g_- = f|_{V_1}$ hold. Hence one obtains $\mathscr{B}_M(U) = \mathcal{O}_X(\Omega - U)/\mathcal{O}_X(\Omega)$.

Remark. We will compute $\mathscr{B}_{\mathbf{R}^n} = \mathscr{H}^n_{\mathbf{R}^n}(\mathcal{O}_{\mathbf{C}^n})$, hyperfunctions on $\mathbf{R}^n$, expressing hyperfunctions as the sum of 2^n boundary values rather than as the sum of boundary values from $(n+1)$ angular domains.

Let $U_{k,+} = \{z \in \mathbf{C}^n \mid \operatorname{Im} z_k > 0\}$, and let $U_{k,-} = \{z \in \mathbf{C}^n \mid -\operatorname{Im} z_k > 0\}$ for $k = 1, 2, \ldots, n$. Then $\{U_{k,\pm}\}_{k=1,2,\ldots,n}$ is a Stein covering of $\mathbf{C}^n - \mathbf{R}^n$.

Denote $\mathscr{V}' = \{U_{k,\pm}\}_{k=1,2,\ldots,n}$ and $\mathscr{V} = \mathscr{V}' \cup \{\mathbf{C}^n\}$. Then we have

$$\begin{aligned} H^n_{\mathbf{R}^n}(\mathbf{C}^n, \mathcal{O}) &= H^n(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) \\ &= \frac{\operatorname{Ker}(C^n(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) \xrightarrow{\delta} C^{n+1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}))}{\operatorname{Im}(C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) \xrightarrow{\delta} C^n(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}))}. \end{aligned}$$

One has $C^{n+1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) = 0$ as before, $C^n(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) = \bigoplus_{\epsilon_1=\pm,\ldots,\epsilon_n=\pm} \mathcal{O}(U_{1\epsilon_1} \cap \cdots \cap U_{n\epsilon_n})$, $C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}) = \bigoplus_{k,\epsilon_k} \mathcal{O}(\hat{k}, \epsilon_{\hat{k}})$, where $\mathcal{O}(\hat{k}, \epsilon_{\hat{k}}) = \mathcal{O}(U_{1,\epsilon_1} \cap \cdots \cap U_{k-1,\epsilon_{k-1}} \cap U_{k+1,\epsilon_{k+1}} \cap \cdots \cap U_{n,\epsilon_n})$, $\epsilon_{\hat{k}} = (\epsilon_1, \ldots, \epsilon_{k-1}, \epsilon_{k+1}, \ldots, \epsilon_n)$.

Next we compute the coboundary. Let

$$\psi = \{\psi(\hat{k}, \epsilon_{\hat{k}})\} \in C^{n-1}(\mathscr{V} \bmod \mathscr{V}', \mathcal{O}).$$

Then $(\delta\psi)_{\epsilon_1,\ldots,\epsilon_n} = \sum_{k=1}^{n} (-1)^k \psi(\hat{k}, \epsilon_{\hat{k}})$, $(\epsilon_{\hat{k}} = (\epsilon_1, \ldots, \epsilon_{k-1}, \epsilon_{k+1}, \ldots, \epsilon_n))$. We denote the element $\{\psi_{\epsilon_1,\ldots,\epsilon_n}\} \bmod \{\sum (-1)^k \psi(\hat{k}, \epsilon_{\hat{k}})\}$ of $H^n_{\mathbf{R}^n}(\mathbf{C}^n, \mathcal{O})$ by $\sum_\epsilon \epsilon_1 \cdots \epsilon_n b(\psi_{\epsilon_1,\ldots,\epsilon_n}) = \sum_\epsilon b(g_\epsilon)$. Then notice that if $\sum_\epsilon b(g_\epsilon) = 0$, then $\epsilon_1 \cdots \epsilon_n g_\epsilon = \sum_k (-1)^k \varphi_{k,\epsilon_1 \cdots \epsilon_{k-1}\epsilon_{k+1} \cdots \epsilon_n}$ for $\varphi(\hat{k}, \epsilon_{\hat{k}}) \in \mathcal{O}(\hat{k}, \epsilon_{\hat{k}})$. Therefore $g_\epsilon = \sum_k (-1)^k \epsilon_1 \cdots \epsilon_n \varphi_{k,\epsilon_1 \cdots \epsilon_{k-1}\epsilon_{k+1} \cdots \epsilon_n}$. If one lets

$$h_{\epsilon_{\hat{k}}} = \epsilon_1 \cdots \epsilon_{k-1}\epsilon_{k+1} \cdots \epsilon_n \varphi_{k,\epsilon_1 \cdots \epsilon_{k-1}\epsilon_{k+1} \cdots \epsilon_n},$$

then one has $g_\epsilon = \sum_k (-1)^k \epsilon_k h_{\epsilon_{\hat{k}}}$. Therefore, if $\sum_\epsilon b(g_\epsilon) = 0$, then $g_\epsilon = \sum_k (-1)^k \epsilon_k h_{\epsilon_{\hat{k}}}$ holds. This is fundamental when one expresses hyperfunctions explicitly. Hyperfunctions defined over an open set in $\mathbf{R}^n$ can be treated similarly.

We will treat the orientation sheaf $\omega_M = \mathscr{H}^n_M(\mathbf{Z}_X)$ via covering cohomology as we did for the sheaf of hyperfunctions $\mathscr{H}^n_M(\mathcal{O}_X)$. Then we will show that ω_M is locally isomorphic to $\mathbf{Z}_M$. This implies that the hyperfunction sheaf $\mathscr{B}_M$ is locally isomorphic to $\mathscr{H}^n_M(\mathcal{O}_X)$ provided that M is oriented.

Some preliminary notions are necessary.

Definition 1.2.3. *Let X and Y be topological spaces, and let $f_i: X \to Y$ be continuous maps for $i = 0, 1$. Then f_0 and f_1 are said to be homotopic if there exists a continuous map $F: X \times I \to Y$, where $I = [0, 1]$, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We call the continuous map F a homotopy between f_0 and f_1, denoted by $f_0 \simeq f_1$.*

Two topological spaces X and Y are said to have the same homotopy type if there exist continuous maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

A topological space is said to be contractible to a point x_0 if there exists a continuous map $F: X \times I \to X$ such that, for any point $x \in X$ and any $t \in I$, $F(x, 0) = x$, $F(x, 1) = x_0$ and $F(x_0, t) = x_0$.

Remark. If X is contractible to a point, then X has the same homotopy type as a point.

The next theorem indicates that cohomology groups with coefficients in a constant sheaf are homotopy invariant.

Theorem 1.2.4. *Let X and Y be topological spaces, and let M be an additive group. If X and Y have the same homotopy type, then $H^k(X, M) \cong H^k(Y, M)$ for any integer $k \geqq 0$. In particular, when X is contractible, one has*

$$H^k(X, M) = \begin{cases} 0 & \text{for } k \neq 0 \\ M & \text{for } k = 0 \end{cases}$$

where M is regarded as a constant sheaf in this theorem.

The proof of this theorem will be given in §2, Chapter II.

Theorem 1.2.5. *Let M be an additive group. Then*

$$\mathscr{H}^k_{\mathbf{R}^n \times \{0\}}(M_{\mathbf{R}^{n+l}}) = \begin{cases} M_{\mathbf{R}^n} & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}$$

holds. In particular, when $k \neq n$, $\mathscr{H}^k_M(\mathbf{Z}_X) = 0$.

Proof. Let $A_{n,l}$ be the statement of Theorem 1.2.5. Then we will give a proof by induction on n and l. We also denote $A_{n,l} \Rightarrow A_{n',l'}$ when the statement $A_{n,l}$ implies the statement $A_{n',l'}$. Let U and V be open balls in $\mathbf{R}^n$ and $\mathbf{R}^l$ respectively. Then we must show

$$H^k_{U \times \{0\}}(U \times V, M) = \begin{cases} M & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases}.$$

Since U is contractible, Theorem 1.2.4 implies the isomorphisms $H^k(U \times V, M) \cong H^k(V, M)$ and $H^k(U \times (V - \{0\}), M) \cong H^k(V - \{0\}, M)$. Therefore one obtains

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(U \times V, M) \arrow[r] \arrow[d, equal] & H^{k-1}(U \times (V - \{0\}), M) \arrow[r] \arrow[d, equal] & H^k_{U \times \{0\}}(U \times V, M) \arrow[r] & {} \\
\cdots \arrow[r] & H^{k-1}(V, M) \arrow[r] & H^{k-1}(V - \{0\}, M) \arrow[r] & H^k_{\{0\}}(V, M) \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^k(U \times V, M) \arrow[r] \arrow[d, equal] & H^k(U \times (V - \{0\}), M) \arrow[r] \arrow[d, equal] & \cdots. \\
H^k(V, M) \arrow[r] & H^k(V - \{0\}, M) \arrow[r] & \cdots.
\end{tikzcd}$$

By Five Lemma, one concludes the isomorphism $H^k_{U\times\{0\}}(U\times V, M)\cong H^k_{\{0\}}(V, M)$. Hence, for an arbitrary l, the implication $A_{0,l}\Rightarrow A_{n,l}$ is true. Next we will prove the implication $A_{0,1}\Rightarrow A_{0,l}$. Then we need to show the implication $A_{0,l-1}\Rightarrow A_{0,l}$. But since $A_{0,l-1}\Rightarrow A_{1,l-1}$ is true, it suffices to prove the implication $A_{1,l-1}\Rightarrow A_{0,l}$. Since $\mathbf{R}\times\{0\}$ is purely $(l-1)$-codimensional with respect to $M_{\mathbf{R}^l}$, $A_{1,l-1}$ implies

$$H^k_{\{0\}}(\mathbf{R}^l, M) = H^{k-(l-1)}_{\{0\}}(\mathbf{R}\times\{0\}, \mathscr{H}^{l-1}_{\mathbf{R}\times\{0\}}(M)) =$$

$$H^{k-l+1}_{\{0\}}(\mathbf{R}, M) = \begin{cases} M & \text{for } k-l+1=1, \text{ i.e. } k=l \\ 0 & \text{for } k-l+1\neq 1, \text{ i.e. } k\neq l\end{cases}$$

by Proposition 1.1.6. This proves $A_{0,l}$, provided that the last equality is true, i.e. $A_{0,1}$ (which remains to be proved). The exact sequence, since $\mathbf{R}$, $\mathbf{R}^+$, and $\mathbf{R}^-$ are contractible,

$$\cdots\to H^{k-1}(\mathbf{R}-\{0\}, M)\to H^k_{\{0\}}(\mathbf{R}, M)\to H^k(\mathbf{R}, M)\to\cdots$$

gives

$$H^k(\mathbf{R}, M) = \begin{cases} M & \text{for } k=0 \\ 0 & \text{for } k\neq 0\end{cases}$$

and

$$H^{k-1}(\mathbf{R}-\{0\}, M) = H^{k-1}(\mathbf{R}^+, M)\oplus H^{k-1}(\mathbf{R}^-, M) = \begin{cases} M^2 & \text{for } k=1 \\ 0 & \text{for } k\neq 1\end{cases}.$$

Therefore one has $H^k_{\{0\}}(\mathbf{R}, M)=0$ for $k\neq 0, 1$. We now treat the case where $k=0, 1$. Consider the exact sequence

$$\begin{array}{ccccccccc} 0 \to H^0_{\{0\}}(\mathbf{R}, M) & \to & H^0(\mathbf{R}, M) & \xrightarrow{\varphi} & H^0(\mathbf{R}-\{0\}, M) & \to & H^1_{\{0\}}(\mathbf{R}, M) \to 0 \\ & & \| & & \| & & \\ & & M & & M\oplus M & & \end{array}$$

where $\varphi(x)=(x, x)$. Then $H^0_{\{0\}}(\mathbf{R}, M)=0$ since φ is a monomorphism, and therefore $H^1_{\{0\}}(\mathbf{R}, M) = (M\oplus M)/M\cong M$.

We will examine the sections of the sheaf $\omega_M = \mathscr{H}^n_M(\mathbf{Z}_X)$, utilizing Theorems 1.2.4 and 1.2.5. Let $\xi_0, \ldots, \xi_n$ be $(n+1)$ vectors in $\mathbf{R}^n$ such that their convex hull is a neighborhood of the origin, and let $\{U, x=(x_1, \ldots, x_n)\}$ be a connected local-coordinate system of $P\in M$. Let $f_j(x)=\langle\xi_j, x\rangle$ for $0\leq j\leq n$. Then $f_0(x), \ldots, f_n(x)$ are real-valued analytic functions on M, and for each $x\in M$ the convex hull of $df_0(x), \ldots, df_n(x)$ is a neighborhood of the origin in T^*_xM. Let Ω be a contractible neighborhood in X containing U as a closed set. Then the set $V_j = \{z\in\Omega \mid \operatorname{Im} f_j(z)>0\}$ for each j, $0\leqq j\leqq n$, is also contractible. Hence, by Theorem 1.2.4, $\mathscr{V}=\{\Omega, V_0, \ldots, V_n\}$ and $\mathscr{V}'=\{V_0, \ldots, V_n\}$ are Leray coverings of Ω and $\Omega-U$ respectively; i.e. they satisfy the condition of Theorem 1.2.1. Then

Theorem 1.2.5, Proposition 1.1.5, and Theorem 1.2.1 imply

$$\omega_M(U) = H^n_{M\cap U}(\Omega, \mathbf{Z}_X) = \frac{\bigoplus_{j=0}^{n} \Gamma(W_{\hat{j}}, \mathbf{Z}_X)}{\delta\left(\bigoplus'_{j,k} \Gamma(W_{\widehat{j,k}}, \mathbf{Z}_X)\right)},$$

where $W_{\hat{j}} = \bigcap_{k\neq j} V_k$, $W_{\widehat{j,k}} = \bigcap_{l\neq j,k} V_l$ and $\bigoplus'$ is the alternative sum. Since $W_{\hat{j}}$ and $W_{\widehat{j,k}}$ are connected, $\Gamma(W_{\hat{j}}, \mathbf{Z}_X) \cong \mathbf{Z}$ and $\Gamma(W_{\widehat{j,k}}, \mathbf{Z}_X) \cong \mathbf{Z}$ hold. Consider the map $\varphi: \bigoplus_{j=0}^{n} \Gamma(W_{\hat{j}}, \mathbf{Z}_X) \cong \mathbf{Z}^{n+1} \ni (s_i) \mapsto \sum_{i=0}^{n} s_i \in \mathbf{Z}$. Notice that φ is an epimorphism. We will show next that the kernel of φ is $\delta\left(\bigoplus'_{j,k} \Gamma(W_{\widehat{j,k}}, \mathbf{Z}_X)\right)$.

Lemma. *Suppose that integers $s_0, \ldots, s_n$ satisfy $\sum_{i=0}^{n} s_i = 0$. Then there exist integers $s_{j,k}$, where $j, k = 0, \ldots, n$, such that $s_{j,k} = -s_{k,j}$ and $s_i = \sum_{k=0}^{n} s_{k,i}$ for $0 \leqq i \leqq n$. Note that the converse is also true.*

Proof. When $n = 1$, we have $s_0 + s_1 = 0$. Let $s_{1,0} = -s_{0,1} = s_0$. Then $s_0 = s_{0,0} + s_{1,0}$ and $s_1 = s_{0,1} + s_{1,1}$. Notice that $s_{0,0} = s_{1,1} = 0$. Next, assume that $\sum_{i=0}^{n} s_i = 0$ holds. Let $s'_0 = s_0 + s_n$, $s'_1 = s_1, \ldots, s'_{n-1} = s_{n-1}$. Then $\sum_{i=0}^{n-1} s'_i = 0$ holds. Therefore, by the inductive assumption, there exist integers $s_{j,k}$ where $j, k = 0, \ldots, n-1$, such that $s_{j,k} = -s_{k,j}$ and $s'_i = \sum_{k=0}^{n-1} s_{k,i}$ for $i = 0, 1, \ldots, n-1$. Let $s_{0,n} = -s_{n,0} = s_n$ and $s_{n,j} = -s_{j,n} = 0$ for $j = 1, \ldots, n$. Then $s_i = \sum_{k=0}^{n} s_{k,i}$ for $0 \leqq i \leqq n$. The converse is plainly true.

This lemma shows that the kernel of φ is $\delta\left(\bigoplus'_{j,k} \Gamma(W_{\widehat{j,k}}, \mathbf{Z}_X)\right)$. Therefore we conclude $H^n_U(\Omega, \mathbf{Z}_X) \cong \mathbf{Z}$; i.e. if U is a connected coordinate neighborhood, $\omega_M(U) \cong \mathbf{Z}$ holds. Let $(y_1, \ldots, y_n)$ be another local coordinate system; then there is induced an isomorphism $\mathbf{Z} \to \mathbf{Z}$. Notice that this isomorphism is either $1_{\mathbf{Z}}$ or $-1_{\mathbf{Z}}$, since the only automorphisms : $\mathbf{Z} \to \mathbf{Z}$ are $\pm 1_{\mathbf{Z}}$ when $\det(\partial(y_i)/\partial(x_j)) > 0$ and $\det(\partial(y_i)/\partial(x_j)) < 0$ respectively.

Let U be a connected open set of M. If there is an $s \in \omega_M(U)$ such that $s \neq 0$, then s is not zero at each point x in U; i.e. $s_x \neq 0$. Let $U = \bigcup_{i\in I} U_i$, where U_i is a connected coordinate neighborhood; let

$(x_i^1, \ldots, x_i^n)$ be the local coordinate system of U_i; and let the orientation $\mathscr{or}$ be such that $(x_i^1, \ldots, x_i^n)$ is a positive local coordinate system with respect to $\mathscr{or}$. Then $s|_{U_i}$ defines an element $\varphi_i(s)$ in $\mathbf{Z}$. Suppose that $U_i \cap U_j \neq \varnothing$; then $\varphi_i(s) = \pm\varphi_j(s)$, since ± 1 are the only automorphisms of $\mathbf{Z}$. Since U is connected, $\varphi_i(s) = \epsilon_i c$ for some $c \in \mathbf{Z}$, where $\epsilon_i = \pm 1$. So $s_x = c \neq 0$. Let $\widetilde{\mathscr{or}}_{U_i}$ be another orientation such that $\widetilde{\mathscr{or}}_{U_i} = \epsilon_i \mathscr{or}_{U_i}$. Then one has $\widetilde{\mathscr{or}}_{U_i}|_{U_i \cap U_j} = \widetilde{\mathscr{or}}_{U_j}|_{U_i \cap U_j}$. Let $(x_i^1, \ldots, x_i^n)$ be a positive local coordinate system with respect to $\widetilde{\mathscr{or}}_{U_i}$. Then $\det(\partial(x_i^l)/\partial(x_j^k))_{1 \leqq l,k \leqq n} > 0$ holds, provided $U_i \cap U_j \neq 0$. Therefore U is orientable. Furthermore, if U is a paracompact space, one can show that there is a continuous n-form on U, using a partition of unity on U. On the other hand, if U is non-orientable, then $\omega_M(U) = 0$. That is, giving a non-zero section of ω_M over U is equivalent to giving an orientation on U.

Hence, so long as a local coordinate system is fixed, or M is orientable, or one considers locally, then there exist isomorphisms $\mathscr{B}_M = \mathscr{H}_M^n(\mathscr{O}_X) \underset{\mathbf{Z}_M}{\otimes} \omega_M \cong \mathscr{H}_M^n(\mathscr{O}_X)$ and $\mathscr{B}_M(U) \cong H_U^n(X, \mathscr{O}_X)$.

CHAPTER II

Microfunctions

§1. Definition of Microfunctions

Let M be a manifold, and let N be a submanifold of M. We always assume that a submanifold is regular; i.e. its topology is provided with the topology as a subspace. Equivalently, $N \cap U_j = \{x_j^1 = \cdots = x_j^l = 0\}$ for a local coordinate system $\{U_j, (x_j^1, \ldots, x_j^m)\}$ of M; see, for example, Matsushima [1].

Definition 2.1.1.

(a) *For $x \in M$ we denote the tangent space of M at x by T_xM. The tangent bundle of M is denoted by $TM = \bigcup_{x \in M} T_xM$.*

(b) *Let $f: X \to Y$ and $g: X' \to Y$ be mappings. The fibre product, denoted by $X \underset{Y}{\times} X'$, of X and X' is defined by*

$$X \underset{Y}{\times} X' = \{(x, x') \in \mathrm{X} \times \mathrm{X}' \,|\, f(x) = g(x')\}.$$

(c) *Let N be a submanifold of M. Then*

$$\mathrm{Coker}(TN \to N \underset{M}{\times} TM) = \bigcup_{x \in N} T_xM/T_xN$$

is called the normal bundle of N in M, denoted by T_NM.

(d) *The cotangent space T_x^*M is the dual vector space of T_xM, and $T^*M = \bigcup_{x \in M} T_x^*M$ is called the cotangent bundle of M.*

(e) $\mathrm{Ker}(N \underset{M}{\times} T^*M \to T^*N) = \bigcup_{x \in N} \{\eta \in T_x^*M \,|\, \langle \eta, T_xN \rangle = 0\}$ *is called the conormal bundle on N with respect to M, denoted by T_N^*M.*

(f) *Let $\mathbf{R}_+^\times$ be the multiplicative group of positive real numbers. The tangent sphere bundle of M, denoted by SM, is defined by*

$$(TM - M)/\mathbf{R}_+^\times = \bigcup_{x \in M} (T_xM - \{0\})/\mathbf{R}_+^\times.$$

Similarly, the cotangent sphere bundle S^*M is defined as

$$(T^*M - M)/\mathbf{R}_+^\times = \bigcup_{x \in M} (T_x^*M - \{0\})/\mathbf{R}_+^\times.$$

Furthermore, $(T_N M - N)/\mathbf{R}_+^\times = \bigcup_{x \in N} ((T_x M/T_x N) - \{0\})/\mathbf{R}_+^\times$ *is said to be the normal sphere bundle on* N *with respect to* M *and is denoted by* $S_N M$. *The conormal sphere bundle, denoted by* $S_N^* M$, *is defined as*

$$(T_N^* M - N)\mathbf{R}_+^\times = \bigcup_{x \in N} \{\eta \in T_x^* M \,|\, \langle \eta, T_x N\rangle = 0\}/\mathbf{R}_+^\times.$$

Note. M can be identified with zero sections of TM and T^*M, $\{(x, \xi) \in TM \,|\, \xi = 0\}$ and $\{(x, \eta) \in T^*M \,|\, \eta = 0\}$ respectively. Therefore, we often denote M for zero sections of TM or T^*M.

We will define the notion of real monoidal transform of M with center N. Suppose that $M = \bigcup_{j \in J} U_j$ and the local coordinates $(x_j^1, \ldots, x_j^m)$ of U_j satisfy $U_j \cap N = \{x_j \,|\, x_j^1 = \cdots = x_j^l = 0\}$, where $l = \operatorname{codim}_M N$. One can assume that the equations of a coordinate transformation

$$\left.\begin{aligned} x_j^\nu &= f_{jk}^\nu(x_k), & \nu &= l+1, \ldots, m \\ x_j^\nu &= \sum_{\mu=1}^{l} x_k^\mu g_{jk,\mu}^\nu(x_k), & \nu &= 1, \ldots, l \end{aligned}\right\} \tag{2.1.1}$$

hold between local coordinates. Then define $U'_j = \{(x_j, \xi_j) = (x_j^1, \ldots, x_j^m; \xi_j^1, \ldots, \xi_j^l) \in U_j \times (\mathbf{R}^l - \{0\}) \,|\, x_j^\nu \xi_j^\mu = x_j^\mu \xi_j^\nu$ and $x_j^\nu \xi_j^\nu \geqq 0$ for $\mu, \nu = 1, \ldots, l\}$. Notice that, if $(x_j, \xi_j) \in U'_j$ and $x_j \notin N \cap U_j$, then there exists a positive real number t such that $\xi_j = (tx_j^1, tx_j^2, \ldots, tx_j^l)$; and if $x_j \in N \cap U_j$, then no condition is needed on $\xi_j \in \mathbf{R}^l - \{0\}$. If one defines a map $\mathbf{R}_+^\times \times U'_j \to U'_j$ by $((x_j, \xi_j), t) \mapsto (x_j, t\xi_j)$, then $\mathbf{R}_+^\times$ acts on U'_j. Then the transitivity of this action defines an equivalence relation in U'_j and we denote the quotient by $\tilde{U}_j$; i.e. $\tilde{U}_j = U'_j/\mathbf{R}_+^\times$. We will paste together $\tilde{U}_j$ as follows: points $(x_j, \xi_j) \in \tilde{U}_j$ and $(x_k, \xi_k) \in \tilde{U}_k$ are identified if and only if x_j and x_k satisfy the equations in (2.1.1) and $\xi_j^\nu = \sum_{\mu=1}^{l} \xi_k^\mu g_{jk,\mu}^\nu(x_k)$ for $\nu = 1, \ldots, l$. The manifold obtained in this manner is denoted by $\widetilde{{}^N M}$, which is called the real monoidal transform of M with center N. We remark that, as a set, $\widetilde{{}^N M} = (M - N) \sqcup S_N M$, where $\sqcup$ indicates a disjoint union.

Let M be a real analytic manifold, and let X be a complexification of M. Then $TX|_M = TM \oplus \sqrt{-1}TM$ holds; i.e. $T_x X = T_x M \oplus \sqrt{-1}T_x M$ for $x \in M$. Therefore, one has $T_M X = \sqrt{-1}TM$ ($\cong TM$), which implies $S_M X \cong \sqrt{-1}SM$. Let $x \in M$ and $v \in T_x M - \{0\}$; then one may consider $\sqrt{-1}v \in (T_M X)_x$. Let $x + \sqrt{-1}v0$ be the point in $\sqrt{-1}SM$ ($= S_M X$) which corresponds to $\sqrt{-1}v \in (T_M X)_x$. Note that we have $\widetilde{{}^M X} = (X - M) \sqcup \sqrt{-1}SM$ as a set. In particular, if $M = \mathbf{R}^n$ and $X = \mathbf{C}^n$, then $\widetilde{{}^M X} = (\mathbf{C}^n - \mathbf{R}^n) \sqcup \sqrt{-1}S^{n-1} \times \mathbf{R}^n$. Therefore, if $x \in M = \mathbf{R}^n$, $t \in \mathbf{R}_+^\times$, and $v \in \mathbf{R}^n - \{0\}$, then $x + \sqrt{-1}tv$ is a point in $\widetilde{{}^M X}$ and $\lim_{t \to +0} (x + \sqrt{-1}tv) = x + \sqrt{-1}v0 \in \widetilde{{}^M X}$. This indicates that the real monoidal transform $\widetilde{{}^M X}$

consists of all the possible directions toward M from $X - M$. Let

$$A_\epsilon = \{x + \sqrt{-1}v0 \in \sqrt{-1}SM \,|\, |x - x_0| < \epsilon \text{ and } |v - v_0| < \epsilon\}$$

and

$$B_\epsilon = \{x + \sqrt{-1}tv \in \widetilde{{}^M X} - \sqrt{-1}SM = X - M \,|\, 0 < t < \epsilon, |x - x_0| < \epsilon \text{ and } |v - v_0| < \epsilon\}.$$

Then one can take $\{A_\epsilon \cup B_\epsilon\}$ as a base of $x_0 + \sqrt{-1}v_0 0 \in \widetilde{{}^M X}$. See Figure 2.1.1, below.

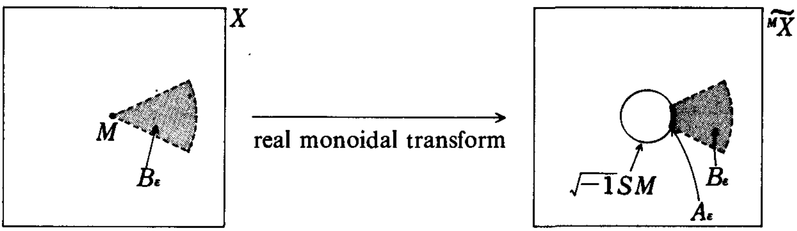

Figure 2.1.1

Definition 2.1.2. *Denote the natural embeddings $X - M \hookrightarrow X$ and $\widetilde{{}^M X} - \sqrt{-1}SM \hookrightarrow \widetilde{{}^M X}$ by ϵ and $\tilde{\epsilon}$ respectively. Let τ be the projection map from $\widetilde{{}^M X}$ onto X (see Figure 2.1.2), and let $\tilde{\tau}$ be the natural embedding $\sqrt{-1}SM \hookrightarrow \widetilde{{}^M X}$.*

$$\begin{tikzcd}
\widetilde{{}^M X} \arrow[d, "\tau"] & \widetilde{{}^M X} - \sqrt{-1}SM \arrow[l, hook', "\tilde{\epsilon}"'] \arrow[d, equal] \\
X & X - M \arrow[l, hook', "\epsilon"']
\end{tikzcd}$$

Figure 2.1.2

Then sheaves $\tilde{\mathcal{O}}$ over $\widetilde{{}^M X}$ and $\tilde{\mathscr{A}}$ over $\sqrt{-1}SM$ are defined by $\tilde{\mathcal{O}} = \tilde{\epsilon}_* \epsilon^{-1} \mathcal{O}_X$ and $\tilde{\mathscr{A}} = \tilde{\mathcal{O}}|_{\sqrt{-1}SM} = \tilde{\tau}^{-1}\tilde{\mathcal{O}}$ respectively, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.

Remark. Let $x + \sqrt{-1}v0 \in \sqrt{-1}SM$; then

$$\begin{aligned}
\tilde{\mathscr{A}}_{x+\sqrt{-1}v0} = \tilde{\mathcal{O}}_{x+\sqrt{-1}v0} &= \varinjlim_{x+\sqrt{-1}v0 \in \tilde{U}} \tilde{\mathcal{O}}(\tilde{U}) = \varinjlim_{x+\sqrt{-1}v0 \in \tilde{U}} (\epsilon^{-1}\mathcal{O}_X)(\tilde{\epsilon}^{-1}(\tilde{U})) \\
&= \varinjlim_{x+\sqrt{-1}v0 \in \tilde{U}} \mathcal{O}_X(\tilde{U} - \sqrt{-1}SM).
\end{aligned}$$

Therefore $\tilde{\mathscr{A}}$ may be considered as the sheaf of boundary values of holomorphic functions, where v indicates the direction along which the boundary value is taken.

Proposition 2.1.1. *$\sqrt{-1}SM$ is purely 1-codimensional with respect to $\tau^{-1}\mathcal{O}_X$; i.e. $\mathscr{H}^k_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X) = 0$ for $k \neq 1$.*

Proof. Since the assertion is local in nature, one can assume $M = \mathbf{R}^n$, $X = \mathbf{C}^n$, and $x_0 = 0$. Therefore, we must show $\mathscr{H}^k_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X)_{0+\sqrt{-1}v_00} = 0$ for $k \neq 1$, where $v_0 \in \mathbf{R}^n - \{0\}$. First note

$$\mathscr{H}^k_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X)_{0+\sqrt{-1}v_00} = \varinjlim_{0+\sqrt{-1}v_00 \in \tilde{U}} H^k_{\sqrt{-1}SM\cap\tilde{U}}(\tilde{U}, \tau^{-1}\mathcal{O}_X).$$

There is induced the long exact sequence, by Theorem 1.1.2 (3),

$$\begin{aligned} \cdots &\to H^{k-1}(\tilde{U}, \tau^{-1}\mathcal{O}_X) \to H^{k-1}(\tilde{U} - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X) \\ &\to H^k_{\sqrt{-1}SM\cap\tilde{U}}(\tilde{U}, \tau^{-1}\mathcal{O}_X) \to H^k(\tilde{U}, \tau^{-1}\mathcal{O}_X) \\ &\to H^k(\tilde{U} - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X) \to \cdots. \end{aligned} \tag{2.1.2}$$

If one lets $A_\epsilon = \{x + \sqrt{-1}v0 \mid |x| < \epsilon$ and $|v - v_0| < \epsilon\}$, $B_\epsilon = \{x + \sqrt{-1}tv \mid 0 < t < \epsilon, |v - v_0| < \epsilon$ and $|x| < \epsilon\}$, and $\tilde{U}_\epsilon = A_\epsilon \cup B_\epsilon$, then $\{\tilde{U}_\epsilon\}$ is a fundamental neighborhood system. Since $\tilde{U}_\epsilon - \sqrt{-1}SM = B_\epsilon$ is a convex set in $\mathbf{C}^n$, it in particular is a Stein manifold. Therefore, $H^k(\tilde{U}_\epsilon - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X) = 0$ for $k \neq 0$ by the Oka-Cartan theorem (Theorem 1.2.2). We need the following lemma to complete the proof of Proposition 2.1.1.

Lemma. *Let $\mathscr{F}$ be a sheaf over a topological space X, and let $x \in X$. Then one has*

$$\varinjlim_{x \in U} H^k(U, \mathscr{F}) = \begin{cases} 0 & \text{for } k \neq 0 \\ \mathscr{F}_x & \text{for } k = 0. \end{cases}$$

Proof of Lemma. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Then $H^k(U, \mathscr{F}) = H^k(\Gamma(U, \mathscr{L}^*))$. Note that taking cohomology of the cochain complex commutes with the direct limit. This implies

$$\varinjlim_{x \in U} H^k(U, \mathscr{F}) = H^k\left(\varinjlim_{x \in U} \Gamma(U, \mathscr{L}^*)\right) = H^k(\mathscr{L}^*_x) = \begin{cases} 0, & k \neq 0 \\ \mathscr{F}_x, & k = 0. \end{cases}$$

This lemma shows $\varinjlim_{0+\sqrt{-1}v_00 \in \tilde{U}} H^k(\tilde{U}, \tau^{-1}\mathcal{O}_X) = 0$ for $k > 0$, and then $\varinjlim_{0+\sqrt{-1}v_00 \in \tilde{U}} H^0(\tilde{U}, \tau^{-1}\mathcal{O}_X) = (\tau^{-1}\mathcal{O}_X)_{0+\sqrt{-1}v_00} = \mathcal{O}_{X,0}$. Taking the direct limit as $\epsilon \to 0$ in (2.1.2) and replacing $\tilde{U}$ by $\tilde{U}_\epsilon$ provides the following exact sequences

$$\left.\begin{aligned}
&0 \to \mathscr{H}^0_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X)_{0+\sqrt{-1}v_0 0} \to \mathcal{O}_{X,0}\\
&\to \varinjlim_{0+\sqrt{-1}v_0 0 \in \tilde{U}} \mathcal{O}_X(\tilde{U} - \sqrt{-1}SM)\\
&\to \mathscr{H}^1_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X)_{0+\sqrt{-1}v_0 0} \to 0\\
&\text{and}\\
&0 \to \mathscr{H}^k_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X) \to 0 \qquad (\text{for } k \geqq 2).
\end{aligned}\right\} \tag{2.1.3}$$

Hence $\mathscr{H}^k_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X) = 0$ for $k \geq 2$. By the uniqueness of the continuation of holomorphic functions, one has the monomorphism $\mathcal{O}_{X,0} \to \varinjlim_{0+\sqrt{-1}v_0 0 \in \tilde{U}} \mathcal{O}_X(\tilde{U} - \sqrt{-1}SM)$, which implies $\mathscr{H}^0_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X) = 0$.

Definition 2.1.3. *Define the sheaf $\mathscr{Q}$ over $\sqrt{-1}SM$ by $\mathscr{Q} = \mathscr{H}^1_{\sqrt{-1}SM}(\tau^{-1}\mathcal{O}_X)$.* Denote the sheaf of real analytic functions on M by $\mathscr{A}$ (or $\mathscr{A}_M$). I.e., $\mathscr{A} = \mathcal{O}_X|_M$.

Proposition 2.1.2. *The sequence of sheaves over $\sqrt{-1}SM$,*

$$0 \to \tau^{-1}\mathscr{A} \to \tilde{\mathscr{A}} \to \mathscr{Q} \to 0$$

is exact.

Proof. Notice that $\mathcal{O}_{X,0} = \mathscr{A}_{M,0} = (\tau^{-1}\mathscr{A})_{0+\sqrt{-1}v_0 0}$ and

$$\varinjlim_{0+\sqrt{-1}v_0 0 \in \tilde{U}} \mathcal{O}_X(\tilde{U} - \sqrt{-1}SM) = \tilde{\mathscr{A}}_{0+\sqrt{-1}v_0 0};$$

see the remark following Definition 2.1.2. Then the exactness of this proposition follows from the exact sequence in (2.1.3).

Remark. The exact sequence in Proposition 2.1.2 indicates that the sheaf $\mathscr{Q}$ may be considered to reflect the irregularity of the sheaf $\tilde{\mathscr{A}}$ of boundary values.

Since $T^*X|_M = T^*M \oplus \sqrt{-1}T^*M$, i.e. for $x \in M$, $T^*_xX = T^*_xM \oplus \sqrt{-1}T^*_xM$, one has $T^*_MX = \sqrt{-1}T^*M$ and $S^*_MX = \sqrt{-1}S^*M$. Let $x \in M$, and let $\eta \in T^*_xM - \{0\}$; then denote the point by $(x, \sqrt{-1}\eta\infty)$ in $\sqrt{-1}S^*M$ corresponding to the point $\sqrt{-1}\eta$. The symbol ∞ is used to suggest that the point $(x, \sqrt{-1}\eta\infty)$ is dual to $(x, \sqrt{-1}\xi) \in \sqrt{-1}TM$, which is denoted by $x + \sqrt{-1}\xi 0$. When we emphasize the point as being on a cotangent bundle, we write $(x, \sqrt{-1}\langle\eta, dx\rangle\infty)$ instead of $(x, \sqrt{-1}\eta\infty)$.

Definition 2.1.4. *Denote* $DM = \{(x+\sqrt{-1}v0,\ (x, \sqrt{-1}\eta\infty)) \in \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}S^*M \mid \langle v, \eta\rangle \leqq 0\}$.

Definition 2.1.5. *The antipodal mapping* $a:\sqrt{-1}S^*M \to \sqrt{-1}S^*M$ *is defined by* $a(x, \sqrt{-1}\eta\infty) = (x, -\sqrt{-1}\eta\infty)$. *If* $\mathscr{F}$ *is a sheaf over* $\sqrt{-1}S^*M$, *we define* $\mathscr{F}^a = a_*\mathscr{F}$ $(= a^{-1}\mathscr{F})$. *The sheaf* $\mathscr{F}^a$ *is called the antipodal image of* $\mathscr{F}$. Note that $\mathscr{F}^a_{(x,\sqrt{-1}\eta\infty)} = \mathscr{F}_{(x,-\sqrt{-1}\eta\infty)}$.

Definition 2.1.6. *Let X and Y be topological spaces, and let* $f:X \to Y$ *be a continuous map. For a sheaf* $\mathscr{F}$ *over X and an open set U in Y, the assignment* $H^k(f^{-1}(U), \mathscr{F})$ *is a presheaf. The associated sheaf is denoted by* $R^kf_*(\mathscr{F})$.

Remark. When $k = 0$, $R^0f_*(\mathscr{F})$ is isomorphic to $f_*\mathscr{F}$.

Definition 2.1.7. *Let X and Y be topological spaces. A continuous map* $f:X \to Y$ *is said to be purely r-dimensional with respect to a sheaf* $\mathscr{F}$ *over X if* $R^jf_*\mathscr{F} = 0$ *for* $j \neq r$.

Proposition 2.1.2′. *The map* $\tau:DM \to \sqrt{-1}S^*M$ *is purely* (*n*-1)-*dimensional with respect to* $\pi^{-1}\mathscr{Q}$; *i.e.* $R^k\tau_*\pi^{-1}\mathscr{Q} = 0$ *for* $k \neq n-1$ (*see Figure* 2.1.3).

$$\begin{tikzcd} & DM \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & \\ \sqrt{-1}SM \arrow[dr, "\tau"'] & & \sqrt{-1}S^*M \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

Figure 2.1.3

The proof of Proposition 2.1.2′ is lengthy. In this section we will prove Proposition 2.1.2′ by momentarily accepting a theorem on the triviality of cohomology groups. See the remark following Proposition 2.1.6. The proofs for the needed theorem and the purely *n*-codimensionality of M with respect to $\mathcal{O}_X$ will be given in the next section.

We are now in the position to give the definition of microfunctions.

Definition 2.1.8. *The sheaf* $\mathscr{C}_M$ *is defined as follows*:

$$\mathscr{C}_M = (R^{n-1}\tau_*\pi^{-1}\mathscr{Q})^a \otimes \pi^{-1}\omega_M.$$

The sections of the sheaf $\mathscr{C}_M$ *are called microfunctions. The stalk* $\mathscr{C}_{M,(x,\sqrt{-1}\xi\infty)} = (R^{n-1}\tau_*\pi^{-1}Q)_{(x,-\sqrt{-1}\xi\infty)} \otimes \omega_{M,x}$ (*see Figure* 2.1.3 *for the maps* τ *and* π).

We begin with preliminaries for the proof of Proposition 2.1.2′.

Definition 2.1.9. *Let* $\mathscr{F}$ *be a sheaf over a topological space X. Define a topology on the set* $F = \bigcup_{x \in X} \mathscr{F}_x$ *with the fundamental neighborhood system*

$\{s_x \mid x \in U\}$ for open sets, where U is an open set in X and $s \in \Gamma(U, \mathcal{F})$. Then, F is called the sheaf space of $\mathcal{F}$. Define a map $\varpi: F \to X$ by $\varpi(x') = x$ for $x' \in \mathcal{F}_x$. Then it is plain to see that ϖ is continuous.

Now we have the following facts.

(a) The map ϖ is a locally homeomorphism; i.e., for $y \in F$ there exists a neighborhood U of y such that, for some neighborhood V of $\varpi(y)$, $\varpi: U \to V$ is a homeomorphism. This statement (a) follows immediately from the definition of the topology on F.

(b) *Let* $\mathcal{G}(U) = \{f \mid f: U \to F$ is continuous and $\varpi \circ f = \mathrm{id}_U\}$. Then $\mathcal{G}(U) \cong \mathcal{F}(U)$ holds, where the map is given by $f(x) = s_X$ for $s \in \mathcal{F}(U)$. In fact, since f is continuous, there exists a neighborhood U' of x such that for $x' \in U'$ one has $f(x') = s'_{x'}$ for some $s' \in \mathcal{F}(U')$.

Therefore $\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{G}(U)$; i.e. the sequence $0 \to \mathcal{F} \to \mathcal{G} \to 0$ is exact. Then, $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to 0$ is exact (see Proposition 1.1.2 (1.i)).

Conversely, let F and X be topological spaces, and let $\varpi: F \to X$ be a locally homeomorphism which is onto. For an open set U in X, the assignment $\{f \mid f: U \to F$ is continuous such that $\varpi \circ f = \mathrm{id}_U\}$ induces a sheaf $\mathcal{F}$. Notice that the sheaf space of $\mathcal{F}$ is then F. Therefore there is a one-to-one correspondence between sheaf spaces and sheaves. In particular, one can define the sections over an arbitrary subset A by $\{f \mid f: A \to F$ is continuous and $\varpi \circ f = \mathrm{id}_U\}$.

Let $f: X \to Y$ be a continuous map, let $\mathcal{F}$ be a sheaf over Y, and let F be the sheaf space of $\mathcal{F}$. Then the sheaf space of $f^{-1}\mathcal{F}$ is given as the fibre product $X \times_Y F = \{(x, x') \in X \times F \mid f(x) = \varpi(x')\}$. See Figure 2.1.4., below:

$$\begin{tikzcd}
X \times_Y F \arrow[r, "\tilde{f}"] \arrow[d, "\varpi'"] & F \arrow[d, "\varpi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Figure 2.1.4

Let Z be a closed set in Y, and let $s \in \Gamma_Z(Y, \mathcal{F})$. Then, $s: Y \to F$ is continuous such that, for $y \notin Z$, $s(y) = 0$ in $\mathcal{F}_y$. Define a map $\tilde{s}: X \to X \times F$ by $\tilde{s}(x) = (x, (s \circ f)(x))$; then $\tilde{s}$ is continuous and $\varpi \circ \tilde{s} = \mathrm{id}_X$. Therefore, $\tilde{s} \in \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathcal{F})$. That is, there is a canonical map $\Gamma_Z(Y, \mathcal{F}) \to \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathcal{F})$. Let $0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \cdots$ be a flabby resolution of $\mathcal{F}$; then the sequence $0 \to f^{-1}\mathcal{F} \to f^{-1}\mathcal{L}^0 \to f^{-1}\mathcal{L}^1 \to \cdots$ is exact. Note that $f^{-1}\mathcal{L}^i$, $i = 0, 1, \ldots$, is not a flabby sheaf, in general. From Lemma 1 in the proof of Theorem 1.1.1, there exists a flabby resolution

$\mathscr{M}^*$ of $f^{-1}\mathscr{F}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & f^{-1}\mathscr{F} \arrow[r] \arrow[d, equal] & f^{-1}\mathscr{L}^0 \arrow[r] \arrow[d] & f^{-1}\mathscr{L}^1 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & f^{-1}\mathscr{F} \arrow[r] & \mathscr{M}^0 \arrow[r] & \mathscr{M}^1 \arrow[r] & \cdots \end{tikzcd}$$

is commutative. Therefore the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_Z(Y, \mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_Z(Y, \mathscr{L}^1) \arrow[r] \arrow[d] & \Gamma_Z(Y, \mathscr{L}^2) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^1) \arrow[r] \arrow[d] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^2) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Gamma_{f^{-1}(Z)}(X, \mathscr{M}^0) \arrow[r] & \Gamma_{f^{-1}(Z)}(X, \mathscr{M}^1) \arrow[r] & \Gamma_{f^{-1}(Z)}(X, \mathscr{M}^2) \arrow[r] & \cdots \end{tikzcd}$$

is induced. Hence there is induced a map from $H^k_Z(Y, \mathscr{F})$ to $H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$. Then there are the long exact sequences such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^k_Z(Y, \mathscr{F}) \arrow[r] \arrow[d] & H^k(Y, \mathscr{F}) \arrow[r] \arrow[d] & H^k(Y - Z, \mathscr{F}) \arrow[r] \arrow[d] & {} \\ \cdots \arrow[r] & H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F}) \arrow[r] & H^k(X, f^{-1}\mathscr{F}) \arrow[r] & H^k(X - f^{-1}(Z), f^{-1}\mathscr{F}) \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} H^{k+1}_Z(Y, \mathscr{F}) \arrow[r] \arrow[d] & \cdots \\ H^{k+1}_{f^{-1}(Z)}(X, f^{-1}\mathscr{F}) \arrow[r] & \cdots \end{tikzcd}$$

commutes. This proves (1) of Proposition 2.1.3, below.

Proposition 2.1.3. *Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous map. Suppose that $\mathscr{F}$ is a sheaf over Y, and that Z is a closed set of Y.*

(1) *Then there exists a map $H^k_Z(Y, \mathscr{F}) \to H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$ such that the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & H^k_Z(Y, \mathscr{F}) \arrow[r] \arrow[d] & H^k(Y, \mathscr{F}) \arrow[r] \arrow[d] & H^k(Y - Z, \mathscr{F}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F}) \arrow[r] & H^k(X, f^{-1}\mathscr{F}) \arrow[r] & H^k(X - f^{-1}(Z), f^{-1}\mathscr{F}) \arrow[r] & \cdots \end{tikzcd}$$

is commutative.

(2) *If f is an open map which is onto and if each fibre $f^{-1}(y)$ for $y \in Y$ is connected, then $\Gamma_Z(Y, \mathscr{F})$ and $\Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$ are isomorphic.*

(3) *In the case where f is a homeomorphism, $H^k_Z(Y, \mathscr{F}) = H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$ holds for $k = 0, 1, 2, \ldots$.*

Proof. Let $\tilde{s}$ be the image of $s \in \Gamma_Z(Y, \mathscr{F})$ under the above-constructed canonical map $\Gamma_Z(Y, \mathscr{F}) \to \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$: Denote the sheaf space of $\mathscr{F}$ by F. Then recall that $\tilde{s}: X \to X \underset{Y}{\times} F$ is defined by $\tilde{s}(x) = (x, (s \circ f)(x))$ for $s: Y \to F$. Therefore one may regard $\tilde{s}: X \to F$. Since f is onto for an arbitrary $y \in Y$, there exists $x \in X$ such that $y = f(x)$. Suppose $\tilde{s}_1 = \tilde{s}_2$; then $s_1(y) = (s_1 \circ f)(x) = \tilde{s}_1(x) = \tilde{s}_2(x) = (s_2 \circ f)(x) = s_2(y)$ holds, i.e. $s_1 = s_2$. Therefore the map $s \mapsto \tilde{s}$ is a monomorphism. Let $\tilde{s}: X \to F$. Then define $s: Y \to F$ by $s(y) = \tilde{s}(x)$, where $y = f(x)$. Next we will show that this definition does not depend upon the choice of x. Fix $x_0 \in X$ such that $f(x_0) = y$. Then the set $\{x \in f^{-1}(y) \mid \tilde{s}(x) = \tilde{s}(x_0)\} = \tilde{s}^{-1}(\tilde{s}(x_0))$ is a closed set (see Figure 2.1.5),

$$\begin{tikzcd}
X \underset{Y}{\times} F \arrow[r] \arrow[d, "\varpi"'] & F \arrow[d, shift left, "\varpi"'] \\
X \arrow[ur, "\tilde{s}"] \arrow[r, "f"] & Y \arrow[u, shift left, "s"']
\end{tikzcd}$$

Figure 2.1.5

since $\tilde{s}$ is continuous. Since ϖ is a local homeomorphism, $f^{-1}(y) \xrightarrow{\tilde{s}} \varpi^{-1}(y)$ is continuous and $\varpi^{-1}(y)$ is discrete, therefore the inverse image $\tilde{s}^{-1}(\tilde{s}(x_0))$ of a point in $\varpi^{-1}(y)$ is open. By the assumption, $f^{-1}(y)$ is connected; thus $f^{-1}(y) = \tilde{s}^{-1}(\tilde{s}(x_0))$. Hence $s: Y \to F$ is uniquely determined. We will show next that s is continuous. Let U be an open set in F. Then one has $s^{-1}(U) = f(f^{-1}(s^{-1}(U))) = f(\tilde{s}^{-1}(U))$, and $\tilde{s}^{-1}(U)$ is open. Since f is an open map by assumption, the set $s^{-1}(U) = f(\tilde{s}^{-1}(U))$ is open. Therefore s is continuous.

Lastly we will prove (3). Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Since f is a homeomorphism, $0 \to f^{-1}\mathscr{F} \to f^{-1}\mathscr{L}^0 \to f^{-1}\mathscr{L}^1 \to \cdots$ is a flabby resolution of $f^{-1}\mathscr{F}$. From (2), one has the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_Z(Y, \mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_Z(Y, \mathscr{L}^1) \arrow[r] \arrow[d] & \Gamma_Z(Y, \mathscr{L}^2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^0) \arrow[r] \arrow[d] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^1) \arrow[r] \arrow[d] & \Gamma_{f^{-1}(Z)}(X, f^{-1}\mathscr{L}^2) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

Therefore $H^k_Z(Y, \mathscr{F}) \cong H^k_{f^{-1}(Z)}(X, f^{-1}\mathscr{F})$ for $k = 0, 1, 2, \ldots$.

Proposition 2.1.4. *Let X be a topological space, and assume that all the open subsets are paracompact and Hausdorff. Then, for an arbitrary subset S of X, there exists the isomorphism $H^k(S, \mathscr{F}|_S) \cong \varinjlim_{S \subset U} H^k(U, \mathscr{F})$.*

Note. This proposition still holds if one assumes that X is paracompact and that S is closed in X. For example, if X is a metric space, the hypotheses of this proposition are satisfied.

Proof. As the first step, we will prove $\Gamma(S, \mathscr{F}|_S) \cong \varinjlim_{S \subset W} \Gamma(W, \mathscr{F})$. Let $s_1 \in \Gamma(W_1, \mathscr{F})$ and $s_2 \in \Gamma(W_2, \mathscr{F})$ such that $s_1|_S = s_2|_S$ in $\Gamma(S, \mathscr{F}|_S)$. Denote the embedding $S \hookrightarrow X$ by j. Then, for any $x \in S$, one has $(\mathscr{F}|_S)_x = (j^{-1}\mathscr{F})_x = \mathscr{F}_x$. Therefore $(s_1)_x = (s_2)_x$ holds for $x \in S$. This implies that there exists a neighborhood of x, $W(x)(\subset W_1 \cap W_2)$, so that $s_1|_{W(x)} = s_2|_{W(x)}$ is true. Then $W = \bigcup_{x \in S} W(x)$ is a neighborhood of S such that $s_1|_W = s_2|_W$ and $W \subset W_1 \cap W_2$. Therefore the map $\varinjlim_{S \subset W} \Gamma(W, \mathscr{F}) \to \Gamma(S, \mathscr{F}|_S)$ is monomorphic. Next we will show that this map is an epimorphism. Let $s \in \Gamma(S, \mathscr{F}|_S)$ and $x \in S$; then $s_x \in (j^{-1}\mathscr{F})_x = \mathscr{F}_x$. So there is a neighborhood of x, $W(x)$, such that for $x \in S \cap W(x)$ one has $\tilde{s}_x = s_x$ for some $\tilde{s} \in \Gamma(W(x), \mathscr{F})$. Then $S \subset \bigcup_{x \in S} W(x)$ is a covering of S. Therefore one obtains a locally finite refinement $\{U_i\}_{i \in I}$. Then there exists $s_i \in \Gamma(U_i, \mathscr{F})$ so that $(s_i)_x = s_x$ holds for $x \in S \cap U_i$. Furthermore, one may assume $X = \bigcup_i U_i$ without loss of generality. Since X is paracompact and Hausdorff, X is a normal space. Therefore there can be found a covering $\{V_i\}$ such that $\bar{V}_i \subset U_i$ and $X = \bigcup_i V_i$, since $\{U_i\}$ is a locally finite covering of X. Define $W = \{x \in X \,|\, (s_i)_x = (s_j)_x \text{ for } x \in \bar{V}_i \cap \bar{V}_j\}$. We will show that W is a neighborhood of S. The assumption of being locally finite implies that one finds a neighborhood U of x and $(i_1, \ldots, i_N)$ with the property $U \cap \bar{V}_i = \varnothing$ for $i \neq i_1, \ldots, i_N$. In the case where $x \notin \bar{V}_i$, replace U by $U - \bar{V}_i$ so that one may have $x \in U \cap \bar{V}_i$ for $i = i_1, \ldots, i_N$. For a sufficiently small neighborhood V of x, $s_{i_\mu}|_V = s_{i_\nu}|_V$ for $\mu, \nu = 1, \ldots, N$. Then there exists $\tilde{s} \in \Gamma(\Omega, \mathscr{F})$ for some open set Ω, $S \subset \Omega \subset W$, such that $\tilde{s}|_{V_i \cap \Omega} = s_i|_{V_i \cap \Omega}$ and $\tilde{s}_x = s_x$ for $x \in S$. Hence, we have proved $\Gamma(S, \mathscr{F}|_S) \cong \varinjlim_{S \subset W} \Gamma(W, \mathscr{F})$.

We will prove the case of higher cohomology groups next. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Let V be an arbitrary open set in X. Then $\Gamma(V \cap S, \mathscr{L}^i|_S) = \varinjlim_{V \cap S \subset W} \Gamma(W, \mathscr{L}^i)$. Therefore, $s \in \Gamma(V \cap S, \mathscr{L}^i|_S)$ can be extended to a section of $\mathscr{L}^i$ over an open set W containing $V \cap S$; and since $\mathscr{L}^i$ is flabby, it can be extended to a section over X, whose restriction to $V \cap S$ of the image in $\Gamma(S, \mathscr{L}^i|_S)$ is s. Therefore, $0 \to \mathscr{F}|_S \to \mathscr{L}^0|_S \to \mathscr{L}^1|_S \to \cdots$ gives a flabby resolution of $\mathscr{F}|_S$. Hence one has isomorphisms $\varinjlim_{S \subset U} H^k(U, \mathscr{F}) = \varinjlim_{S \subset U} H^k(\Gamma(U, \mathscr{L}^*)) \cong$

$$H^k\left(\varinjlim_{S \subset U} \Gamma(U, \mathscr{L}^*)\right) \cong H^k(\Gamma(S, \mathscr{L}^*|_S)) = H^k(S, \mathscr{F}|_S) \text{ for } k = 0, 1, 2, \ldots.$$

Corollary. *Let X be a topological space whose open subsets are paracompact and Hausdorff. Let $\mathscr{F}$ be a sheaf over X, and let $f: X \to Y$ be a closed continuous map. Then for each $y \in Y$, $R^k f_*(\mathscr{F})_y = H^k(f^{-1}(y), \mathscr{F}|_{f^{-1}(y)})$ for $k = 0, 1, 2, \ldots$.*

Proof. Let U be an open neighborhood of $f^{-1}(y)$. Since f is a closed map, $V = Y - f(X - U)$ is an open neighborhood of y such that $f^{-1}(V) \subset U$ holds. Therefore, by Proposition 2.1.4, $H^k(f^{-1}(y), \mathscr{F}|_{f^{-1}(y)}) \cong \varinjlim_{f^{-1}(y) \subset U} H^k(U, \mathscr{F}) = \varinjlim_{y \in V} H^k(f^{-1}(V), \mathscr{F}) = R^k f_*(\mathscr{F})_y$.

Definition 2.1.10. *A continuous map $f: X \to Y$ is said to be proper if conditions* (1) *and* (2) *are satisfied*:

(1) *For each $y \in Y$ the fibre $f^{-1}(y)$ is compact.*

(2) *f is a closed map; i.e. the image of any closed set in X under f is always a closed set in Y.*

Remark. The above corollary holds if X and Y are Hausdorff and if $f: X \to Y$ is a proper map.

Proposition 2.1.5. *The map $\tau: DM \to \sqrt{-1}S^*M$ is proper.*

In order to prove this proposition we need several lemmas.

Lemma 1. *Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous map. Then the following* (1) *and* (2) *are equivalent*:

(1) *f is a closed map.*

(2) *For each $y \in Y$, let U be a neighborhood of $f^{-1}(y)$. Then there exists a neighborhood V of y such that $f^{-1}(V) \subset U$. That is, $\{f^{-1}(V) \mid V$ is an element of a fundamental neighborhood system $V \ni y\}$ is a fundamental neighborhood system for $f^{-1}(y)$.*

Proof. Let $V = Y - f(X - U)$; then (2) follows plainly from (1). Let A be a closed set in X. If $y \notin f(A)$, then $A \cap f^{-1}(y) = \varnothing$ and $(X - A) \supset f^{-1}(y)$. Therefore, there exists a neighborhood V of y so that $f^{-1}(V) \subset X - A$, i.e. $V \cap f(A) = \varnothing$, which means that $f(A)$ is closed.

Lemma 2. *In the Figure* 2.1.6, *if f is a proper map, then f' is proper, where $X \times_Y Y'$ indicates the fibre product of $f: X \to Y$ and $g: Y' \to Y$, where g' and f' are defined as $g'(x, y') = x$ and $f'(x, y') = y'$, respectively, for $(x, y') \in X \times_Y Y'$.*

$$\begin{tikzcd} X & X \times_Y Y' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y & Y' \arrow[l, "g"] \arrow[from=1-1, to=2-1, "f"] \end{tikzcd}$$

Figure 2.1.6

Proof. For $y' \in Y'$ one has the isomorphism $f'^{-1}(y') = \{(x, y') \in X \times_Y Y' | f(x) = g(y')\} \cong f^{-1}(g(y'))$. Since f is a proper map, $f'^{-1}(y')$ is compact. Hence it suffices to prove f' as a closed map. Let $y' \in Y'$, and let U be a neighborhood of $f'^{-1}(y')$. Then, by Lemma 1, it is sufficient to prove that $f'^{-1}(U') \subset U$ for some neighborhood U' of y'. Let $x' \in f'^{-1}(y')$; then let $x = g'(x')$ and $y = g(y')$. Then one can find neighborhoods $W(x')$ of $x = g'(x')$ and $V(x')$ of y' such that $g'^{-1}(W(x')) \cap f'^{-1}(V(x')) \subset U$. Note $\bigcup_{x' \in f'^{-1}(y')} W(x') \supset f^{-1}(y)$. Since $f^{-1}(y)$ is compact, there are finitely many $x_1, \ldots, x_N \in f'^{-1}(y')$ such that $W = \bigcup_{i=1}^{N} W(x_i) \supset f^{-1}(y)$. Then define $V = \bigcap_{i=1}^{N} V(x_i)$. Notice that V is also a neighborhood of y' and that $f'^{-1}(y') \subset g'^{-1}(W) \cap f'^{-1}(V) \subset U$. Since f is a closed map, there exists a neighborhood V' of y such that $f^{-1}(y) \subset f^{-1}(V') \subset W$. Therefore $g'^{-1}(W) \supset g'^{-1}f^{-1}(V') = f'^{-1}g^{-1}(V')$. Let $U' = V \cap g^{-1}(V')$. Then U' is a neighborhood of y' with the property $f'^{-1}(U') \subset U$.

Lemma 3. *Let $f: X \to Y$ be a proper map and Y be compact. Then X is compact.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an arbitrary open covering of X. For each $y \in Y$, $f^{-1}(y)$ is compact and $f^{-1}(y) \subset \bigcup_{i \in I} U_i$. Therefore there exists a finite subset $I(y)$ of I such that $f^{-1}(y) \subset \bigcup_{i \in I(y)} U_i$. Since f is a closed map, by Lemma 1 there is a neighborhood $V(y)$ of y so that $f^{-1}(V(y)) \subset \bigcup_{i \in I(y)} U_i$. Since $\bigcup_{y \in Y} V(y)$ is an open covering of Y and Y is compact, $Y = \bigcup_{i=1}^{N} V(y_i)$ for finitely many points $y_1, y_2, \ldots, y_N$ in Y. Then $X = \bigcup_{i \in \left(\bigcup_{i=1}^{N} I(y_i)\right)} U_i$ is a finite covering of X.

Lemma 4. *If $f: X \to Y$ and $g: Y \to Z$ are proper maps, then $g \circ f: X \to Z$ is proper.*

Proof. If f and g are closed maps, then the composition $g \circ f$ is a closed map. Therefore it suffices to prove that $f^{-1}(g^{-1}(z))$ is compact for each $z \in Z$. By Lemma 3, it is enough to show that $f: f^{-1}(g^{-1}(z)) \to g^{-1}(z)$ is proper. This is plainly so from Lemma 2.

Lemma 5. *$\sqrt{-1}SM \xrightarrow{\tau} M$ is a proper map.*

Proof. From Lemma 1, the map τ being proper is local in nature. For a continuous map $f: X \to Y$ and an open covering of Y, $Y = \bigcup_{i \in I} U_i$, it suffices to prove that $f: f^{-1}(U_i) \to U_i$ is proper for each $i \in I$. Hence, from the beginning, one may assume $\sqrt{-1}SM \cong S^{n-1} \times M$ to prove this assertion. For a point x in M, the map $S^{n-1} \to x$ is a proper map. Therefore, by Lemma 2, τ is a proper map.

We are now ready to prove Proposition 2.1.5. Consider the commutative diagram

$$\begin{tikzcd}
DM \arrow[r, hook, "\imath"] \arrow[dr, "\tau"'] & \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}S^*M \arrow[r] \arrow[d, "p_1"] & \sqrt{-1}SM \arrow[d, "p_2"] \\
& \sqrt{-1}S^*M \arrow[r, "\pi"] & M
\end{tikzcd}$$

In this commutative diagram, p_2 is a proper map by Lemma 5. Therefore, by Lemma 2, p_1 is a proper map. Then, by Lemma 4, $\tau = p_1 \circ \imath$ is proper, since $\imath$ is proper.

Combining the corollary of Proposition 2.1.4 and Proposition 2.1.5, we have

$$(R^k\tau_*\pi^{-1}\mathscr{Q})_{(x_0,\sqrt{-1}\xi_0\infty)} = H^k(\tau^{-1}(x_0, \sqrt{-1}\xi_0\infty), \pi^{-1}\mathscr{Q}|_{\tau^{-1}(x_0,\sqrt{-1}\xi_0\infty)}).$$

The paracompactness condition on M, below, is satisfied at least locally. We will rephrase this cohomology group into the terms of holomorphic functions (see Proposition 2.1.6, below).

First note that one has the homeomorphism $\pi: \tau^{-1}(x_0, \sqrt{-1}\xi_0\infty) \xrightarrow{\sim} \pi\tau^{-1}(x_0, \sqrt{-1}\xi_0\infty)$. Therefore, by (3) of Proposition 2.1.3, $H^k(\tau^{-1}(x_0, \sqrt{-1}\xi_0\infty), \pi^{-1}\mathscr{Q}|_{\tau^{-1}(x_0,\sqrt{-1}\xi_0\infty)}) = H^k(F, \mathscr{Q}|_F)$, where $F = \pi\tau^{-1}(x_0, \sqrt{-1}\xi_0\infty) = \{x + \sqrt{-1}v0 \in \sqrt{-1}SM \mid \langle v, \xi_0 \rangle \leqq 0\}$. On the other hand, Proposition 2.1.4 implies $H^k(F, \mathscr{Q}) = \varinjlim_{F \subset \tilde{U} \subset \widetilde{MX}} H^k(\tilde{U}, \mathscr{H}^1_{\sqrt{-1}SM}(\tau^{-1}\mathscr{O}_X))$, which can also be written as $\varinjlim_{F \subset \tilde{U}} H^{k+1}_{\sqrt{-1}SM \cap \tilde{U}}(\tilde{U}, \tau^{-1}\mathscr{O}_X)$ by Propositions 2.1.1 and 1.1.6. Since the question is local in nature, one may assume $M = \mathbf{R}^n$ without loss of generality. We will construct a neighborhood $\tilde{U}_{(\xi,\epsilon)}$ of F. Let $\xi_1, \xi_2, \ldots, \xi_n$ be n real-vectors so that the convex hull of $\xi_0, \xi_1, \ldots, \xi_n$ contains a neighborhood of the origin. For $\xi = (\xi_1, \ldots, \xi_n)$ and $\varepsilon > 0$, define $\tilde{U}_{(\xi,\epsilon)} = \{x + \sqrt{-1}y \in X \mid |x - x_0| < \epsilon,\ |y| < \epsilon$ and

$\langle y, \xi_j\rangle > 0$ for some j, $1 \leqq j \leqq n\} \cup \{x + \sqrt{-1}v0 \in \sqrt{-1}SM \,|\, |x - x_0| < \epsilon$ and $\langle v, \xi_j\rangle > 0$ for some j, $1 \leqq j \leqq n\}$. Then $\tilde{U}_{(\xi,\epsilon)}$ is a neighborhood of F. When one takes $\xi_1, \ldots, \xi_n$ in a neighborhood of $-\xi_0$ and ϵ in a neighborhood of zero, $\{\tilde{U}_{(\xi,\epsilon)}\}$ forms a fundamental neighborhood system of F (see Figure 2.1.7 for $n = 2$). Note that when $\xi_j \to -\xi_0$ and $\varepsilon \to 0$, $\tilde{U}_{(\xi,\epsilon)}$ goes to F.

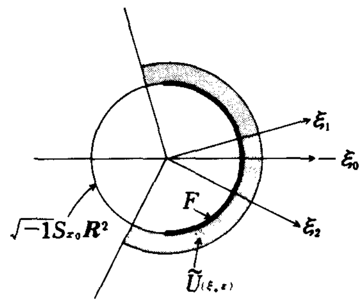

Figure 2.1.7. Cross-section of ${}^{\mathbf{R}^2}\widetilde{\mathbf{C}^2}$ at x_0

Lemma. *Let* $U_\epsilon = \{x + \sqrt{-1}y \in X \,|\, |y| < \epsilon,\ |x - x_0| < \epsilon\}$, *and let* $Z_{(\xi,\epsilon)} = \{x + \sqrt{-1}y \in U_\epsilon \,|\, \langle y, \xi_j\rangle \leqq 0$ *for any* j, $1 \leqq j \leqq n\}$. *Then* $\tilde{U}_{(\xi,\epsilon)} \cap \tau^{-1}(Z_{(\xi,\epsilon)}) = \tilde{U}_{(\xi,\epsilon)} \cap \sqrt{-1}SM$ *holds, where* $\tau: {}^{M}\tilde{X} \to X$.

Proof. Let $x + \sqrt{-1}v0 \in \tilde{U}_{(\xi,\epsilon)} \cap \sqrt{-1}SM$. Then $x \in Z_{(\xi,\epsilon)}$. Therefore $x + \sqrt{-1}v0 \in (\sqrt{-1}SM)_x = \tau^{-1}(x) \subset \tau^{-1}(Z_{(\xi,\epsilon)})$. That is, $\tilde{U}_{(\xi,\epsilon)} \cap \sqrt{-1}SM \subset \tilde{U}_{(\xi,\epsilon)} \cap \tau^{-1}(Z_{(\xi,\epsilon)})$ holds. Suppose $x + \sqrt{-1}y \notin \sqrt{-1}SM$ and $x + \sqrt{-1}y \in \tilde{U}_{(\xi,\epsilon)} \cap \tau^{-1}(Z_{(\xi,\epsilon)})$. Then it would imply that $\langle y, \xi_j\rangle > 0$ for some j and $\langle y, \xi_j\rangle \leqq 0$ for all j, which is a contradiction.

Let $X = \tilde{U}_{(\xi,\epsilon)}$, $Y = U_\epsilon$, $f = \tau|_{\tilde{U}_{(\xi,\epsilon)}}$, $Z = Z_{(\xi,\epsilon)}$, and let $\mathscr{F} = \mathcal{O}_X$ in (1) of Proposition 2.1.3. Note $f^{-1}(Z) = \sqrt{-1}SM \cap \tilde{U}_{(\xi,\epsilon)}$ from the above lemma. Then we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(U_\epsilon, \mathcal{O}_X) \arrow[r] \arrow[d] & H^{k-1}(U_\epsilon - Z_{(\xi,\epsilon)}, \mathcal{O}_X) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & H^{k-1}(\tilde{U}_{(\xi,\epsilon)}, \tau^{-1}\mathcal{O}_X) \arrow[r] & H^{k-1}(\tilde{U}_{(\xi,\epsilon)} - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^k_{Z_{(\xi,\epsilon)}}(U_\epsilon, \mathcal{O}_X) \arrow[r] \arrow[d] & H^k(U_\epsilon, \mathcal{O}_X) \arrow[r] \arrow[d] & H^k(U_\epsilon - Z_{(\xi,\epsilon)}, \mathcal{O}_X) \arrow[r] \arrow[d] & {} \\
H^k_{M \cap \tilde{U}_{(\xi,\epsilon)}}(\tilde{U}_{(\xi,\epsilon)}, \tau^{-1}\mathcal{O}_X) \arrow[r] & H^k(\tilde{U}_{(\xi,\epsilon)}, \tau^{-1}\mathcal{O}_X) \arrow[r] & H^k(\tilde{U}_{(\xi,\epsilon)} - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X) \arrow[r] & {}
\end{tikzcd}$$ (2

Since $\tilde{U}_{(\xi,\epsilon)} - \sqrt{-1}SM \cong U_\epsilon - Z_{(\xi,\epsilon)}$, from (3) of Proposition 2.1.3, one has

$$H^k(U_\epsilon - Z_{(\xi,\epsilon)}, \mathcal{O}_X) = H^k(\tilde{U}_{(\xi,\epsilon)} - \sqrt{-1}SM, \tau^{-1}\mathcal{O}_X).$$

Furthermore, one obtains

$$\varinjlim_{x_0 \in U_\epsilon} H^k(U_\epsilon, \mathcal{O}_X) = \begin{cases} 0 & \text{for } k \neq 0 \\ \mathcal{O}_{X,x_0} & \text{for } k = 0 \end{cases}$$

by the lemma in the proof of Proposition 2.1.1. Notice $\varinjlim_{F \subset \tilde{U}} H^k(\tilde{U}, \tau^{-1}\mathcal{O}_X) = H^k(F, \tau^{-1}\mathcal{O}_X|_F)$ and $\tau(F) = x_0$. Therefore, $\tau^{-1}\mathcal{O}_X|_F$ is a constant sheaf on F. Since F is homeomorphic to a demisphere, F is contractible. Hence, by Theorem 1.2.4,

$$H^k(F, \tau^{-1}\mathcal{O}_X|_F) = \begin{cases} 0 & \text{for } k \neq 0 \\ \mathcal{O}_{X,x_0} & \text{for } k = 0 \end{cases}$$

holds. Taking the direct limit in the commutative diagram (2.1.4), Five Lemma implies

$$\varinjlim_{F \subset \tilde{U}} H^k_{\tilde{U} \cap \sqrt{-1}SM}(\tilde{U}, \tau^{-1}\mathcal{O}_X) \cong \varinjlim_{\xi,\epsilon} H^k_{Z_{(\xi,\epsilon)}}(U_\epsilon, \mathcal{O}_X) = \varinjlim_{\xi} \mathscr{H}^k_{Z_\xi}(\mathcal{O}_X)_x,$$

where $Z_\xi = \{x + \sqrt{-1}y \in X \mid \langle y, \xi_j \rangle \leqq 0 \text{ for } j = 1, 2, \ldots, n\}$. Notice $Z_{(\xi,\epsilon)} = Z_\xi \cap U_\epsilon$. Therefore we could rewrite $(R^k\tau_*\pi^{-1}\mathscr{Q})_{(x_0,\sqrt{-1}\xi\infty)}$ in terms of the cohomology group with coefficients in $\mathcal{O}_X$. This clarifies the relationship to hyperfunctions. More precisely, we have the following.

Proposition 2.1.6.

(1) *For an arbitrary* k

$$(R^k\tau_*\pi^{-1}\mathscr{Q})_{(x_0,\sqrt{-1}\xi_0\infty)} = \varinjlim_{\xi} \mathscr{H}^{k+1}_{Z_\xi}(\mathcal{O}_X)_{x_0}$$

holds.

(2) *A canonical epimorphism* $sp: \mathscr{B}_{M,x_0} \to \mathscr{C}_{M,(x_0,-\sqrt{-1}\xi_0\infty)}$ *exists.*

Remark. We will prove $\mathscr{H}^k_{Z_\xi}(\mathcal{O}_X)_{x_0} = 0$ for $k \neq n$ in the next section. This statement, with (1), proves Proposition 2.1.2′.

Proof. (1) has been proved; we will prove (2) here. It is sufficient to prove that the composite map

$$\mathscr{B}_{M,x_0} = \mathscr{H}^n_M(\mathcal{O}_X)_{x_0} \to \mathscr{H}^n_{Z_\xi}(\mathcal{O}_X)_{x_0} \to \varinjlim_{\xi} \mathscr{H}^n_{Z_\xi}(\mathcal{O}_X)_{x_0} = \mathscr{C}_{M,(x_0,-\sqrt{-1}\xi_0\infty)}$$

is an epimorphism. Notice that

$$\mathscr{H}^n_M(\mathcal{O}_X)_{x_0} = \varinjlim_{\epsilon} H^n_{M \cap U_\epsilon}(U_\epsilon, \mathcal{O}_X)$$

and

$$\mathscr{H}^n_{Z_\xi}(\mathcal{O}_X)_{x_0} = \varinjlim_{\epsilon} H^n_{Z_\xi \cap U_\epsilon}(U_\epsilon, \mathcal{O}_X)$$

hold. We first consider the map before taking the direct limit. Let $U_j = \{x + \sqrt{-1}y \in X \mid \langle y, \xi_j\rangle > 0\}$ for $j = 0, 1, \ldots, n$. Let the open coverings $\mathscr{U}$, $\mathscr{U}'$, and $\mathscr{U}''$ of U_ϵ, $U_\epsilon - U_\epsilon \cap M$, and $U_\epsilon - U_\epsilon \cap Z_\xi$ be as follows: $\mathscr{U} = \{U_\epsilon, U_\epsilon \cap U_0, U_\epsilon \cap U_1, \ldots, U_\epsilon \cap U_n\}$; $\mathscr{U}' = \{U_\epsilon \cap U_0, U_\epsilon \cap U_1, \ldots, U_\epsilon \cap U_n\}$; and $\mathscr{U}'' = \{U_\epsilon \cap U_1, \ldots, U_\epsilon \cap U_n\}$. Similarly as in §2, Chapter I, one has

$$C^{n+1}(\mathscr{U} \bmod \mathscr{U}', \mathcal{O}_X) = C^{n+1}(\mathscr{U} \bmod \mathscr{U}'', \mathcal{O}_X) = 0,$$

$$C^n(\mathscr{U} \bmod \mathscr{U}', \mathcal{O}_X) = C^n(\mathscr{U} \bmod \mathscr{U}'', \mathcal{O}_X) = \bigoplus_{j=0}^{n} \mathcal{O}_X(U_{\hat{j}}),$$

$$C^{n-1}(\mathscr{U} \bmod \mathscr{U}', \mathcal{O}_X) = \bigoplus_{j,k}{}' \mathcal{O}_X(U_{\widehat{j,k}}),$$

and

$$C^{n-1}(\mathscr{U} \bmod \mathscr{U}'', \mathcal{O}_X) = \bigoplus_{j,k}{}' \mathcal{O}_X(U_{\widehat{j,k}}) \oplus \bigoplus_{j=1}^{n} \mathcal{O}_X(U_{\hat{j}})$$

where $U_{\hat{j}} = \bigcap_{k \neq j} U_k$ and $U_{\widehat{j,k}} = \bigcap_{l \neq j,k} U_l$. Therefore the canonical epimorphism

$$H^n_{M \cap U_\epsilon}(U_\epsilon, \mathcal{O}_X) = \frac{\bigoplus_{j=0}^{n} \mathcal{O}_X(U_{\hat{j}})}{\bigoplus_{j,k}{}' \mathcal{O}_X(U_{\widehat{j,k}})} \to \frac{\bigoplus_{j=0}^{n} \mathcal{O}_X(U_{\hat{j}})}{\bigoplus_{j,k}{}' \mathcal{O}_X(U_{\widehat{j,k}}) \oplus \bigoplus_{j=1}^{n} \mathcal{O}_X(U_{\hat{j}})}$$
$$= H^n_{Z_\xi \cap U_\epsilon}(U_\epsilon, \mathcal{O}_X)$$

is induced. Therefore the map $\mathscr{B}_{M,x_0} \to \mathscr{C}_{M,(x_0, -\sqrt{-1}\xi_0\infty)}$, obtained by taking the direct limit of the above epimorphism, is an epimorphism.

Definition 2.1.11. *The epimorphism sp is called the spectrum map (see Definition 2.3.3).*

Note. For $u \in \mathscr{B}_{M,x_0}$ we have

$$\mathrm{sp}(u)_{(x_0, -\sqrt{-1}\xi_0\infty)} = 0 \quad \text{if and only if} \quad u = \sum_{j=1}^{n} b(\varphi_j), \tag{2.1.5}$$

where $\varphi_j \in \mathcal{O}_X(U_{\hat{j}})$.

§2. Vanishing Theorems of Relative Cohomology Groups, Pure n-Codimensionality of $\mathbf{R}^n$ with respect to $\mathcal{O}_{\mathbf{C}^n}$ etc.

We will prove in this section that a real analytic manifold M is purely n-codimensional with respect to the sheaf $\mathcal{O}_X$ of holomorphic functions on a complexification X of M. That is, we show $\mathscr{H}^k_M(\mathcal{O}_X) = 0$ for $k \neq n$. We will also prove $\mathscr{H}^k_{Z_\xi}(\mathcal{O}_X)_{x_0} = 0$ for $k \neq n$, which was assumed in the proof of Proposition 2.1.6. These complete the proof of Proposition 2.1.2′ and Theorem 1.2.4.

Proposition 2.2.1. *Let Y be a complex manifold, and let Z be a closed set in Y. If $H^k_{V \times Z}(V \times Y, \mathcal{O}_{V \times Y}) = 0$ for an arbitrary complex manifold V and for $k < r$, then for any compact set $K \subset \mathbf{C}$, $H^k_{W \times K \times Z}(W \times \mathbf{C} \times Y, \mathcal{O}_{W \times \mathbf{C} \times Y)} = 0$ holds for an arbitrary complex manifold W and for $k < r + 1$.*

Proof. We will prove that, for $k < r + 1$, $H^k_{K \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) = 0$. As we will point out later in the proof, the general case can be reduced to this special case. The long exact sequence (3) in Theorem 1.1.2 implies that

$$\begin{aligned} \cdots &\to H^{k-1}_{\mathbf{C} \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \to H^{k-1}_{(\mathbf{C}-K) \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \\ &\to H^k_{K \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \to H^k_{\mathbf{C} \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \to \cdots \end{aligned}$$

is exact. Notice that $H^{k-1}_{(\mathbf{C}-K) \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \cong H^{k-1}_{(\mathbf{C}-K) \times Z}((\mathbf{C} - K) \times Y, \mathcal{O}_{(\mathbf{C}-K) \times Y})$ by (1) of Theorem 1.1.2. Therefore for $k < r$ one obtains $H^k_{K \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) = 0$ from this long exact sequence.

Next we will prove the case when $k = r$. Let $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, a projective line, and let $\mathscr{I} = \{\varphi \in \mathcal{O}_{\mathbf{P}^1 \times Y} \mid \varphi(\infty, y) = 0\}$. Then one has $\mathscr{I}|_{\mathbf{C} \times Y} = \mathcal{O}_{\mathbf{C} \times Y}$ and $\mathscr{I}|_{(\mathbf{P}^1 - \{0\}) \times Y} \cong \mathcal{O}_{(\mathbf{P}^1 - \{0\}) \times Y}$ with respect to the origin $\{0\}$ via the map $\varphi(x, y) \mapsto x\varphi(x, y)$ (the inverse map being $(1/x)\psi(x, y) \leftarrow\!\shortmid \psi(x, y)$). We identify $\mathscr{I}|_{(\mathbf{P}^1 - \{0\}) \times Y}$ with $\mathcal{O}_{(\mathbf{P}^1 - \{0\}) \times Y}$ under the assumption $0 \in K$ so that $\mathscr{I}|_{(\mathbf{P}^1 - K) \times Y} = \mathcal{O}_{(\mathbf{P}^1 - K) \times Y}$, $\{0\} \subset K \subset \mathbf{P}^1$. Then, one has $H^{r-1}_{(\mathbf{P}^1 - K) \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) = H^{r-1}_{(\mathbf{P}^1 - K) \times Z}((\mathbf{P}^1 - K) \times Y, \mathcal{O}_{(\mathbf{P}^1 - K) \times Y}) = 0$. On the other hand, $H^r_{K \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) = H^r_{K \times Z}(\mathbf{C} \times Y, \mathscr{I}|_{\mathbf{C} \times Y}) = H^r_{K \times Z}(\mathbf{P}^1 \times Y, \mathscr{I})$ holds. Furthermore, the long exact sequence

$$\begin{aligned} \cdots &\to H^{r-1}_{(\mathbf{P}^1 - K) \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) \to H^r_{K \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) \\ &\to H^r_{\mathbf{P}^1 \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) \to \cdots \end{aligned}$$

implies that $H^r_{K \times Z}(\mathbf{C} \times Y, \mathcal{O}_{\mathbf{C} \times Y}) \to H^r_{\mathbf{P}^1 \times Z}(\mathbf{P}^1 \times Y, \mathscr{I})$ is a monomorphism. Hence, it suffices to prove $H^r_{\mathbf{P}^1 \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$ to complete the proof of Proposition 2.2.1. We need two lemmas.

Lemma 1. *Let $f: \mathbf{P}^1 \times Y \to Y$ be the projection map. Then $R^k f_* \mathscr{I} = 0$ for $k \geqq 0$.*

Proof. It is sufficient to prove $R^k f_*(\mathscr{I})_y = \varinjlim_{y \in U} H^k(\mathbf{P}^1 \times U, \mathscr{I}) = 0$ for each $y \in Y$. Since one can take a base of y consisting of Stein manifolds, one is to prove that, for any Stein manifold Y, $H^k(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$ for $k \geqq 0$. Let $U_0 = (\mathbf{P}^1 - \{\infty\}) \times Y$, and let $U_1 = (\mathbf{P}^1 - \{0\}) \times Y$. Then $\mathbf{P}^1 \times Y = U_0 \cup U_1$ holds. Since U_0 and U_1 are Stein manifolds, Oka and Cartan's theorem (Theorem 1.2.2) says that $\mathscr{U} = \{U_0, U_1\}$ is a Leray covering for the sheaf $\mathscr{I}$. We can compute the cohomology groups $H^k(\mathbf{P}^1 \times Y, \mathscr{I})$ using the covering $\mathscr{U} = \{U_0, U_1\}$ by Leray's theorem (Theorem 1.2.1).

Note that $C^0(\mathscr{U}, \mathscr{I}) = \Gamma(U_0, \mathscr{I}) \oplus \Gamma(U_1, \mathscr{I})$, $C^1(\mathscr{U}, \mathscr{I}) = \Gamma(U_0 \cap U_1, \mathscr{I})$ and that $C^k(\mathscr{U}, \mathscr{I}) = 0$ for $k \geqq 2$. So $H^k(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$ for $k \geqq 2$. Next we will prove $H^0(\mathbf{P}^1 \times Y, \mathscr{I}) = \Gamma(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$. Let $\varphi(x, y) \in \Gamma(\mathbf{P}^1 \times Y, \mathscr{I})$, and let y be fixed. Then φ is a holomorphic function on the compact manifold $\mathbf{P}^1$. Therefore φ must be a constant function in x by the maximum principle. Since $\varphi(\infty, y) = 0$, $\varphi(x, y) = 0$ holds. So we have $H^0(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$. Lastly we will show $H^1(\mathbf{P}^1 \times Y, \mathscr{I}) = \Gamma(U_0 \cap U_1, \mathscr{I})/[\delta(\Gamma(U_0, \mathscr{I}) \oplus \Gamma(U_1, \mathscr{I}))] = 0$. Let $\varphi(x, y) \in \Gamma(U_0 \cap U_1, \mathscr{I})$. Then φ is holomorphic in $\{(x, y) | 0 < |x| < \infty,\ y \in Y\}$. Let the Laurent expansion of $\varphi(x, y)$ be $\sum_{n=-\infty}^{\infty} a_n(y)x^n$. Notice that $\varphi_0(x, y) = \sum_{n \geqq 0} a_n(y)x^n$ is holomorphic in U_0 and that $\varphi_1(x, y) = \sum_{n<0} a_n(y)x^n$ is holomorphic in U_1. Furthermore, $\varphi(x, y) = \varphi_0(x, y) + \varphi_1(x, y)$ holds; i.e. $\varphi(x, y) \in \delta(\Gamma(U_0, \mathscr{I}) \oplus \Gamma(U_1, \mathscr{I}))$. This impiles $H^1(\mathbf{P}^1 \times Y, \mathscr{I}) = 0$.

Lemma 2. *Let X and Y be topological spaces, and let $\mathscr{F}$ be a sheaf over X. If a continuous map $f: X \to Y$ is purely r-dimensional with respect to $\mathscr{F}$, i.e. $R^k f_* \mathscr{F} = 0$ for $k \neq r$, then $H^k_{f^{-1}(Z)}(X, \mathscr{F}) = H^{k-r}_Z(Y, R^r f_*(\mathscr{F}))$ for an arbitrary locally closed subset Z of Y.*

Proof. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Recall that the sheaf $R^k f_* \mathscr{F}$ is associated to the presheaf $H^k(f^{-1}(U), \mathscr{F}) = H^k(\Gamma(f^{-1}(U), \mathscr{L}^\bullet)) = H^k(\Gamma(U, f_* \mathscr{L}^\bullet))$ for an open set U in Y. Therefore, one has

$$\mathscr{H}^k(f_* \mathscr{L}^\bullet) = \frac{\operatorname{Ker}(f_* \mathscr{L}^k \to f_* \mathscr{L}^{k+1})}{\operatorname{Im}(f_* \mathscr{L}^{k-1} \to f_* \mathscr{L}^k)}.$$

Let $f_* \mathscr{L}^{r-1} \xrightarrow{\varphi} f_* \mathscr{L}^r \xrightarrow{\psi} f_* \mathscr{L}^{r+1}$, and let $\mathscr{I} = \operatorname{Im} \varphi$ and $\mathscr{Z} = \operatorname{Ker} \psi$. Then

$$0 \to f_* \mathscr{L}^0 \to f_* \mathscr{L}^1 \to \cdots \to f_* \mathscr{L}^{r-1} \to \mathscr{I} \to 0$$

is an exact sequence of flabby sheaves. Noticing that $\Gamma_Z(Y, f_* \mathscr{L}^k) = \Gamma_{f^{-1}(Z)}(X, \mathscr{L}^k)$, one obtains the exact sequence

$$0 \to \Gamma_{f^{-1}(Z)}(X, \mathscr{L}^0) \to \cdots \to \Gamma_{f^{-1}(Z)}(X, \mathscr{L}^{r-1}) \to \Gamma_Z(Y, \mathscr{I}) \to 0.$$

From this exact sequence, $H^k_{f^{-1}(Z)}(X, \mathscr{F}) = 0$ for $k \leqq r - 1$ holds. In the case where $k \geqq r$, one has the exact sequence $0 \to \mathscr{I} \to \mathscr{Z} \to R^r f_*(\mathscr{F}) \to 0$ and the flabby resolution of $\mathscr{Z}$

$$0 \to \mathscr{Z} \to f_* \mathscr{L}^r \to f_* \mathscr{L}^{r+1} \to \cdots.$$

Then the methods in the proof of Proposition 1.1.5 and in the one of Proposition 1.1.6 can be applied to complete the proof.

By Lemmas 1 and 2, for an arbitrary k, one obtains

$$H^k_{\mathbf{P}^1 \times Z}(\mathbf{P}^1 \times Y, \mathscr{I}) = H^{k-r}_Z(Y, R^r f_*(\mathscr{I})) = H^{k-r}_Z(Y, 0) = 0.$$

This implies $H^r_{k \times Z}(\mathbf{C} \times Y, \mathscr{O}_{\mathbf{C} \times Y}) = 0$, as we noted previously. Having proved the above, one obtains

$$H^k_{K \times Z}(\mathbf{C} \times Y, \mathscr{O}_{\mathbf{C} \times Y}) = 0 \qquad \text{for } k < r + 1.$$

By replacing Y with $Y \times W$, and Z with $W \times Z$, Proposition 2.2.1 follows.

Corollary. *Let Ω be a connected open set in $\mathbf{C}$; let Z be a closed set in Ω, $Z \subsetneqq \Omega$; and let K_j be a compact set in $\mathbf{C}$ for each j, $1 \leqq j \leqq n$. Then, for $k \neq n + 1$, $H^k_{Z \times K_1 \times \cdots \times K_n}(\Omega \times \mathbf{C}^n, \mathscr{O}_{\mathbf{C}^{n+1}}) = 0$ holds.*

Proof. If a holomorphic function φ in Ω is zero on the open set $\Omega - Z$, then φ is zero in Ω by analytic continuation. Let V be any complex manifold. If a holomorphic function $\varphi(x, y)$ in $V \times \Omega$, where $x \in V$ and $y \in \Omega$, has a value of zero in $V \times (\Omega - Z)$, then $\varphi(x, y) = 0$ for each $x \in V$. Therefore, $\varphi = 0$ in $V \times \Omega$. That is, $H^0_{V \times Z}(V \times \Omega, \mathscr{O}_{V \times \Omega}) = \Gamma_{V \times Z}(V \times \Omega, \mathscr{O}_{V \times \Omega}) = 0$. Then, by Proposition 2.2.1, $H^k_{W \times K_1 \times Z}(W \times \mathbf{C} \times Y, \mathscr{O}_{W \times \mathbf{C} \times Y}) = 0$ is true for $k < 2$. Replacing $\mathbf{C} \times Y$ by Y and $K_1 \times Z$ by Z as the initial step, Proposition 2.2.1 can be applied inductively. Hence one has $H^k_{Z \times K_1 \times \cdots \times K_n}(\Omega \times \mathbf{C}^n, \mathscr{O}_{\mathbf{C}^{n+1}}) = 0$ for $k < n + 1$. When $k > n + 1$, Malgrange's theorem (Theorem 1.1.4) implies flabby dim $\mathscr{O}_{\mathbf{C}^{n+1}} \leqq n + 1$. This completes the proof.

Definition 2.2.1. *A subset K of $\mathbf{C}^n$ is said to be an analytic polyhedron if there exists $f_1, \ldots, f_N \in \mathscr{O}(\mathbf{C}^n)$ such that $K = \{z \in \mathbf{C}^n \,|\, |f_1(z)| \leqq 1, \ldots, |f_N(z)| \leqq 1\}$.*

Proposition 2.2.2. *Let K_1 and K_2 be compact analytic polyhedrons in $\mathbf{C}^n$. Then $H^k_{K_1 - K_2}(\mathbf{C}^n, \mathscr{O}_{\mathbf{C}^n}) = 0$ for $k \neq n$.*

Proof. Since one can assume $K_1 \supset K_2$ without loss of generality, one may let $K_1 = \{z \in \mathbf{C}^n \,|\, |f_j(z)| \leqq 1$ for $j = 1, \ldots, N\}$ and $K_2 = \{z \in \mathbf{C}^n \,|\, |f_j(z)| \leqq 1$ for $j = 1, \ldots, N, \ldots, N + N'\}$. Define $K'_j = \{z \in \mathbf{C}^n \,|\, |f_l(z)| \leqq 1$ for $l = 1, \ldots, N + j\}$, where $1 \leqq j \leqq N'$. By the long exact sequence (see (3) in Theorem 1.1.2)

$$\cdots \to H^k_{K'_j - K'_{j+1}}(\mathbf{C}^n, \mathscr{O}_{\mathbf{C}^n}) \to H^k_{K_1 - K'_{j+1}}(\mathbf{C}^n, \mathscr{O}_{\mathbf{C}^n}) \to H^k_{K_1 - K'_j}(\mathbf{C}^n, \mathscr{O}_{\mathbf{C}^n}) \to \cdots,$$

it is sufficient to prove the case where $N' = 1$. The general case follows inductively from this long exact sequence. Therefore we may now let $K_1 = \{z \in \mathbf{C}^n \| f_j(z)| \leqq 1 \text{ for } j = 1, \ldots, N-1\}$ and $K_2 = \{z \in \mathbf{C}^n \| f_j(z)| \leqq 1$ for $j = 1, \ldots, N\}$. Furthermore, since K_1 and K_2 are compact, one may let $f_j(z) = z_j$, $1 \leqq j \leq n \leqq N-1$.

Define a map $F: X = \mathbf{C}^n \to Y = \mathbf{C}^N$ by $F(x) = (x_1, \ldots, x_n, f_{n+1}(x), \ldots, f_N(x))$, where $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$. Choose a real number a so that $\max_{z \in K_1} (|f_N(z)|, 1) < a$. Then define $\tilde{K}_1$ and $\tilde{K}_2$, compact sets in Y, as follows:

$$\tilde{K}_1 = \{y \in Y \| y_j| \leqq 1 \text{ for } j = 1, \ldots, N-1 \text{ and } |y_N| \leqq a\}$$
$$\tilde{K}_2 = \{y \in Y \| y_j| \leqq 1 \text{ for } j = 1, \ldots, N\}.$$

Note that $F^{-1}(\tilde{K}_1) = K_1$ and $F^{-1}(\tilde{K}_2) = K_2$ hold.

Generally speaking, let X and Y be topological spaces, and let a closed continuous map $F: X \to Y$ be injective. Then, for a locally closed set S in Y and a sheaf $\mathscr{F}$ over X,

$$H^k_{F^{-1}(S)}(X, \mathscr{F}) \cong H^k_S(Y, F_*\mathscr{F}) \text{ holds for } k = 0, 1, \ldots. \tag{*}$$

Note that $R^kF_*(\mathscr{F})_y = \varinjlim_{y \in U} H^k(F^{-1}(U), \mathscr{F})$ for $y \in Y$. Since F is a closed map, $\varinjlim_{y \in U} H^k(F^{-1}(U), \mathscr{F}) = \varinjlim_{F^{-1}(y) \subset V} H^k(V, \mathscr{F})$. But F is injective. Therefore $F^{-1}(y)$ is either a point or an empty set. Either of these cases implies that the map F is purely 0-dimensional with respect to $\mathscr{F}$. Therefore the isomorphism in (*) exists by Lemma 2 in the proof of Proposition 2.2.1. Hence, in our case, one has the isomorphism $H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) \cong H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, F_*\mathcal{O}_X)$. It is sufficient to prove that $H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, F_*\mathcal{O}_X) = 0$ holds for $k \neq n$.

Lemma 1. *Let $\mathcal{O}^l_Y$ be the direct sum of l copies of the sheaf $\mathcal{O}_Y$ of holomorphic functions in $Y = \mathbf{C}^N$. Then there exists an exact sequence*

$$0 \leftarrow F_*\mathcal{O}_X \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_Y^{N-n} \leftarrow \mathcal{O}_Y^{\binom{N-n}{2}} \leftarrow \cdots \leftarrow \mathcal{O}_Y^{\binom{N-n}{N-n}} \leftarrow 0.$$

Proof. For the coordinate system $(y_1, \ldots, y_N)$ of Y, define a new coordinate system $(w_1, \ldots, w_N)$ by $w_j = y_j$ for $1 \leqq j \leqq n$, and $w_j = y_j - f_j(y_1, \ldots, y_n)$ for $n+1 \leqq j \leqq N$. Then $F(X) = \{(w_1, \ldots, w_N) \in \mathbf{C}^N | w_{n+1} = \cdots = w_N = 0\}$. If $y \notin F(X)$, then $(F_*\mathcal{O}_X)_y = 0$; and if $y \in F(X)$, then $(F_*\mathcal{O}_X)_y = (\mathcal{O}_{F(X)})_y$. Associate $\varphi(w_1, \ldots, w_N) \in \mathcal{O}_Y$ with $\varphi|_{w_{n+1} = \cdots = w_N = 0} \in F_*\mathcal{O}_X$. Therefore one sees the exactness of $0 \leftarrow F_*\mathcal{O}_X \leftarrow \mathcal{O}_Y$. Next consider the map $\{\varphi_j\}_{n+1 \leq j \leq N} \mapsto \sum_{j=n+1}^{N} \varphi_j w_j$. This defines a map from $\mathcal{O}_Y^{N-n}$ to $\mathcal{O}_Y$, inducing the exact sequence $F_*\mathcal{O}_X \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_Y^{N-n}$. Note that an arbitrary element in $\mathcal{O}_Y^{\binom{N-n}{k}}$ can be described as

$\{\varphi_{i_1,\ldots,i_k}\}_{n+1\leq i_1,\ldots,i_k\leq N}$ with an alternating index; i.e. $\varphi_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (\operatorname{sgn}\sigma)\varphi_{i_1,\ldots,i_k}$ for a permutation σ of k letters $i_1,\ldots,i_k$. We will define a map δ

$$\mathcal{O}_Y^{\binom{N-n}{k}} \xrightarrow{\delta} \mathcal{O}_Y^{\binom{N-n}{k-1}}.$$

Let $(\delta\varphi)_{i_1,\ldots,i_{k-1}} = \sum_{j=n+1}^{N} \varphi_{i_1,\ldots,i_{k-1},j}w_j$ and $\delta(\{\varphi_{i_1,\ldots,i_k}\}) = \{(\delta\varphi)_{i_1,\ldots,i_{k-1}}\}$. Since $\varphi_{i_1,\ldots,i_{k-2},j,l} + \varphi_{i_1,\ldots,i_{k-2},l,j} = 0$, one has $(\delta^2\varphi)_{i_1,\ldots,i_{k-2}} = \sum_{j=n+1}^{N} (\delta\varphi)_{i_1,\ldots,i_{k-2},j}w_j = \sum_{j,l} \varphi_{i_1,\ldots,i_{k-2},j,l}w_jw_l = 0$. That is, $\delta^2 = 0$. Next, suppose $\delta\varphi = 0$; i.e. $\sum_{j=n+1}^{N} \varphi_{i_1,\ldots,i_{k-1},j}w_j = 0$. We will show that there exists h such that $\delta h = \varphi$. If $y \notin F(X)$, a direct computation can be applied. We will prove the case when $y \in F(X)$.

We will give a proof by induction on N. If $N = n + 1$, i.e. $k = 1$, then $\varphi_{n+1}w_{n+1} = 0$ holds. Then $\varphi_{n+1} = 0$ on the open set $\{w \mid w_{n+1} \neq 0\}$. By the analytic continuation, $\varphi_{n+1} = 0$, securing the exact sequence $\mathcal{O}_Y \xleftarrow{\delta} \mathcal{O}_Y \leftarrow 0$. Next assume that the claim is true for the case $N - 1$.

$$\sum_{j=n+1}^{N} \varphi_{i_1,\ldots,i_{k-1},j}w_j = 0$$

implies, in particular, $\sum_{j=n+1}^{N-1} (\varphi_{i_1,\ldots,i_{k-1},j}w_j|_{\{w_N=0\}}) = 0$. From the inductive assumption, one obtains

$$\varphi_{i_1,\ldots,i_{k-1},i_k} = \sum_{j=n+1}^{N-1} \psi_{i_1,\ldots,i_k,j}w_j + w_Ng_{i_1,\ldots,i_k},$$

where $i_1,\ldots,i_k < N$, for some $\psi_{i_1,\ldots,i_k,j}$ and $g_{i_1,\ldots,i_k}$. Therefore, $0 = \sum_{j=n+1}^{N} \varphi_{i_1,\ldots,i_{k-1},j}w_j = \varphi_{i_1,\ldots,i_{k-1},N}w_N + w_N \sum_{l=n+1}^{N-1} g_{i_1,\ldots,i_{k-1},l}w_l$ enables one to get $\varphi_{i_1,\ldots,i_{k-1},N} = -\sum_{j=n+1}^{N-1} g_{i_1,\ldots,i_{k-1},j}w_j$. Then define $h_{i_1,\ldots,i_{k+1}}$, $i_1 < \cdots < i_{k+1}$, to be $\psi_{i_1,\ldots,i_{k+1}}$ if $i_{k+1} < N$, and to be $g_{i_1,\ldots,i_{k-1},i_k}$ if $i_{k+1} = N$. Hence, for $i_1 < \cdots < i_k < N$, $(\delta h)_{i_1,\ldots,i_k} = \sum_{j=n+1}^{N} h_{i_1,\ldots,i_k,j}w_j = \sum_{j=n+1}^{N-1} \psi_{i_1,\ldots,i_k,j}w_j + w_Ng_{i_1,\ldots,i_k} = \varphi_{i_1,\ldots,i_k}$ holds; and also, for $i_k = N$,

$$\begin{aligned}(\delta h)_{i_1,\ldots,i_{k-1},N} &= \sum_{j=n+1}^{N-1} h_{i_1,\ldots,i_k,j}w_j = -\sum_{j=n+1}^{N-1} h_{i_1,\ldots,i_{k-1},j,N}w_j \\ &= -\sum_{j=n+1}^{N-1} g_{i_1,\ldots,i_{k-1},j}w_j = \varphi_{i_1,\ldots,i_{k-1},N}\end{aligned}$$

holds. This proves $\delta h = \varphi$; therefore one obtains the exactness of the sequence

$$\mathcal{O}_Y^{\binom{N-n}{k-1}} \xleftarrow{\delta} \mathcal{O}_Y^{\binom{N-n}{k}} \xleftarrow{\delta} \mathcal{O}_Y^{\binom{N-n}{k+1}}.$$

Lemma 2. *Let $\mathscr{F}$ be a sheaf over $Y = \mathbf{C}^N$. If the sequence*

$$0 \leftarrow \mathscr{F} \leftarrow \mathcal{O}_Y^{l_0} \leftarrow \mathcal{O}_Y^{l_1} \leftarrow \cdots \leftarrow \mathcal{O}_Y^{l_r} \leftarrow 0$$

is exact, then for $k < N - r$ one has

$$H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathscr{F}) = 0.$$

Proof. First we will prove $H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathcal{O}_Y) = 0$ for $k \neq N$. Let $\Omega = \{y_N \in \mathbf{C} \mid |y_N| > 1\}$, let $Z = \{y_N \in \mathbf{C} \mid 1 < |y_N| \leqq a\}$, and let $K_j = \{y_j \in \mathbf{C} \mid |y_j| \leqq 1\}$. Then $\tilde{K}_1 - \tilde{K}_2 = Z \times K_1 \times \cdots \times K_{N-1}$ holds. By Corollary of Proposition 2.2.1,

$$H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathcal{O}_Y) = H^k_{Z \times K_1 \times \cdots \times K_{N-1}}(\Omega \times \mathbf{C}^{N-1}, \mathcal{O}_{\mathbf{C}^N}) = 0 \qquad \text{for } k \neq N.$$

Therefore, one obtains $H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathcal{O}^{l_0}) = 0$ for $k \neq N$, proving the case where $r = 0$ in this lemma. We will prove the general case by induction on r. The given exact sequence implies that the sequences

$$0 \leftarrow \mathscr{F} \leftarrow \mathcal{O}_Y^{l_0} \leftarrow \mathscr{G} \leftarrow 0 \quad \text{and} \quad 0 \leftarrow \mathscr{G} \leftarrow \mathcal{O}_Y^{l_1} \leftarrow \cdots \leftarrow \mathcal{O}_Y^{l_r} \leftarrow 0$$

are exact. By the hypothesis, for $k < N - (r - 1) = N - r + 1$,

$$H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathscr{G}) = 0$$

holds. Therefore, for $k < N - r$, the exact sequence

$$H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathcal{O}_Y^{l_0}) \to H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathscr{F}) \to H^{k+1}_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathscr{G})$$

implies that $H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, \mathscr{F}) = 0$ holds.

Combining these two lemmas, one concludes that $H^k_{\tilde{K}_1 - \tilde{K}_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = H^k_{\tilde{K}_1 - \tilde{K}_2}(Y, F_* \mathcal{O}_X) = 0$ for $k < n$. On the other hand, Malgrange's theorem (Theorem 1.1.4) implies flabby dim $\mathcal{O}_{\mathbf{C}^n} \leqq n$. From this, one has

$$H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0 \qquad \text{for } k > n.$$

Finally, $H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0$ for $k \neq n$ is acquired.

Remark. If $f_{n+1}(z) = 2$ in Proposition 2.2.2, then $K_2 = \varnothing$. Hence, for K_1 satisfying the condition of Proposition 2.2.2,

$$H^k_{K_1}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0$$

holds for $k \neq n$.

Now we come to the main theorems.

Theorem 2.2.1. *Let M be a real analytic manifold, and let X be a complexification of M. Then M is purely n-codimensional with respect to $\mathcal{O}_X$.*

Proof. The assertion being local in nature, one may assume $M = \mathbf{R}^n$ and $X = \mathbf{C}^n$ without loss of generality. It is sufficient to prove that at the origin

$$\mathscr{H}^k_{\mathbf{R}^n}(\mathcal{O}_{\mathbf{C}^n})_0 = 0$$

holds for $k \neq n$. Note that $\operatorname{Im} f \geqq 0$ for $f \in \mathcal{O}_{\mathbf{C}^n}$ is equivalent to the condition $|e^{\sqrt{-1}f(z)}| \leqq 1$. Therefore $\mathbf{R} = \{z \in \mathbf{C} \,|\, |e^{\sqrt{-1}z}| \leqq 1 \text{ and } |e^{-\sqrt{-1}z}| \leqq 1\}$. Then notice that for an arbitrary polynomial $\varphi(z)$ the sets $K_1 = \mathbf{R}^n \cap \{z \in \mathbf{C}^n \,|\, |z_1| \leqq 1, \ldots, |z_n| \leqq 1\}$ and $K_2 = K_1 \cap \{z \in \mathbf{C}^n \,|\, \operatorname{Im} \varphi(z) \leqq 0\}$ are both compact analytic polyhedrons. Therefore, by Proposition 2.2.2, one has

$$H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0$$

for $k \neq n$. Let the polynomial φ be $\varphi(z) = z_1 + \sqrt{-1} - 2\sqrt{-1}(z_1^2 + \cdots + z_n^2)$. Then $K_1 - K_2 = \mathbf{R}^n \cap \{z \in \mathbf{C}^n \,|\, \operatorname{Im} \varphi > 0\}$. This is so because if, for $z \in \mathbf{R}^n$, $\operatorname{Im} \varphi(z) = 1 - 2(z_1^2 + \cdots + z_n^2) > 0$, then particularly $|z_j| < 1/\sqrt{2}$ for $j = 1, \ldots, n$. Consequently, $z \in K_1$. Define $\Omega = \{z \in \mathbf{C}^n \,|\, |z_1| < 1, \ldots, |z_n| < 1 \text{ and } \operatorname{Im} \varphi > 0\}$. Then $K_1 - K_2 = \Omega \cap \mathbf{R}^n$ holds. Hence one has

$$H^k_{\mathbf{R}^n \cap \Omega}(\Omega, \mathcal{O}_{\mathbf{C}^n}) = H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0$$

for $k \neq n$. By definition

$$\mathscr{H}^k_{\mathbf{R}^n}(\mathcal{O}_{\mathbf{C}^n})_0 = \varinjlim_{0 \in U} H^k_{\mathbf{R}^n \cap U}(U, \mathcal{O}_{\mathbf{C}^n}).$$

Let $U = a\Omega$ for $a > 0$. Then for $k \neq n$

$$H^k_{\mathbf{R}^n \cap U}(U, \mathcal{O}_{\mathbf{C}^n}) = 0.$$

Since $\{a\Omega\}_{a>0}$ is a fundamental neighborhood system at the origin, one obtains

$$\mathscr{H}^k_{\mathbf{R}^n}(\mathcal{O}_{\mathbf{C}^n}) = 0 \qquad \text{for } k \neq n$$

as $a \to 0$.

Theorem 2.2.2. *Let $Z = \{z \in \mathbf{C}^n \,|\, \operatorname{Im} z_i \geqq 0 \text{ for } 1 \leqq i \leqq n\}$. Then*

$$\mathscr{H}^k_Z(\mathcal{O}_{\mathbf{C}^n})_0 = 0$$

holds for $k \neq n$.

Proof. Let $\Omega \subset \mathbf{C}^n$ be a sufficiently small neighborhood of the origin 0. Then it is sufficient to prove

$$H^k_{Z \cap \Omega}(\Omega, \mathcal{O}_{\mathbf{C}^n}) = 0 \qquad \text{for } k \neq n.$$

Define $\varphi(z) = -(z_1 + \cdots + z_n) + \sqrt{-1}a - b\sqrt{-1}(z_1^2 + \cdots + z_n^2)$. Let $K_1 = Z \cap \{z \in \mathbf{C}^n \mid |z_j| \leqq 1 \text{ for } j = 1, \ldots, n\}$, and let $K_2 = K_1 \cap \{z \in \mathbf{C}^n \mid \operatorname{Im} \varphi(z) \leqq 0\}$. Notice that K_1 and K_2 are compact analytic polyhedrons. Therefore, by Proposition 2.2.2, one has

$$H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0 \qquad \text{for } k \neq n.$$

Let $\Omega = \{z \in \mathbf{C}^n \mid \operatorname{Im} \varphi(z) > 0,\ |z_j| < 2\sqrt{a} \text{ for } j = 1, \ldots, n\}$. Then $(K_1 - K_2) \supset \Omega \cap Z$ holds for sufficiently small $a > 0$. Next we will show that one can choose a and b so that $K_1 - K_2 = \Omega \cap Z$. It is sufficient to prove that $|z_j| < 2\sqrt{a}$ holds for $j = 1, \ldots, n$ on the set $K_1 - K_2$. Note that

$$\operatorname{Im} \varphi(z) = -b(x_1^2 + \cdots + x_n^2) + a - \{y_1(1 - by_1) + \cdots + y_n(1 - by_n)\},$$

where $z_j = x_j + \sqrt{-1}y_j$, $1 \leqq j \leqq n$. Let $b = 1/2$. Then

$$0 < \operatorname{Im} \varphi(z) \leqq a - \tfrac{1}{2}(x_1^2 + \cdots + x_n^2) - \tfrac{1}{2}(y_1 + \cdots + y_n).$$

Therefore, one obtains

$$\tfrac{1}{2}(x_1^2 + \cdots + x_n^2) + \tfrac{1}{2}(y_1 + \cdots + y_n) \leqq a.$$

By the assumption, $\operatorname{Im} z_j = y_j \geqq 0$ for $j = 1, \ldots, n$. Then $x_j^2 \leqq 2a$ and $0 \leqq y_j \leqq 2a$, $1 \leqq j \leqq n$, hold. Therefore, for a sufficiently small real number $a > 0$, one has $|z_j| < 2\sqrt{a}$ for $j = 1, \ldots, n$. Note that Ω forms a fundamental neighborhood system of the origin as $a \to 0$. Hence, if $k \neq n$, then

$$H^k_{\Omega \cap U}(\Omega, \mathcal{O}_{\mathbf{C}^n}) = H^k_{K_1 - K_2}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n}) = 0.$$

Taking the direct limit over Ω as $a \to 0$, one has

$$\mathcal{H}^k_Z(\mathcal{O}_{\mathbf{C}^n}) = 0 \qquad \text{for } k \neq n.$$

Proposition 2.2.3. *Let the sets K_1 and K_2 be the same as in Proposition 2.2.2. Then, for an arbitrary complex manifold W,*

$$H^k_{(K_1 - K_2) \times W}(\mathbf{C}^n \times W, \mathcal{O}_{\mathbf{C}^n \times W}) = 0 \qquad \text{for } k < n.$$

Proof. The proof of Proposition 2.2.3 is quite similar to the one of Proposition 2.2.2. Replace $H^k_{K'_j - K'_{j+1}}(\mathbf{C}^n, \mathcal{O}_{\mathbf{C}^n})$ with $H^k_{(K'_j - K'_{j+1}) \times W}(\mathbf{C}^n \times W, \mathcal{O}_{\mathbf{C}^n \times W})$; $X = \mathbf{C}^n$ with $\mathbf{C}^n \times W$; and $Y = \mathbf{C}^N$ with $\mathbf{C}^N \times W$. The details are left for the reader.

As a byproduct of the discussion in this section, we will give a proof of Theorem 1.2.4 in Chapter I, i.e. the homotopy invariance of the cohomology groups with coefficient in a constant sheaf. Let X and Y be topological spaces, and let M be an additive group. Proposition 2.1.3 implies that for continuous maps $f_\nu: X \to Y$, $\nu = 0, 1$, the maps $f_\nu^*: H^k(Y, M_Y) \to H^k(X, f_\nu^{-1}M_Y) = H^k(X, M_X)$, $\nu = 0, 1$, are induced. Then we will prove that $f_0^* = f_1^*$ holds if f_0 and f_1 are homotopic. There exists a continuous map $F: X \times I \to Y$, where $I = [0, 1]$ such that $F(x, v) = f_v(x)$ for $v = 0, 1$. Define a map $\iota_v: X \to X \times I$ by $\iota_v(x) = (x, v)$, $v = 0, 1$. Then $f_v = F \circ \iota_v$. This implies that $f_v^* = \iota_v^* \circ F^*$, where

$$H^k(Y, M_Y) \xrightarrow{F^*} H^k(X \times I, M_{X \times I}) \underset{\iota_1^*}{\overset{\iota_0^*}{\rightrightarrows}} H^k(X, M_X).$$

In order to claim $f_0^* = f_1^*$, it is sufficient to prove $\iota_0^* = \iota_1^*$. Let p be the projection: $X \times I \to X$. Then, notice that $p \circ \iota_v$, $v = 0, 1$, are identity maps on X. Therefore one has $\iota_v^* \circ p^* = \mathrm{id}$ for $v = 0, 1$. If p^* is an isomorphism, then one obtains $\iota_0^* = \iota_1^*$. But it is sufficient to show

$$R^k p_*(M_{X \times I}) = \begin{cases} M_X & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

to claim that p^* is an isomorphism. This is because, from Lemma 2 in the proof of Proposition 2.2.1, one has

$$H^k(X \times I, M_{X \times I}) \cong H^{k-0}(X, R^0 p_*(M_{X \times I})) = H^k(X, M_X).$$

We will compute $R^k p_*(M_{X \times I})$ below. Since p is a proper map, the Corollary of Proposition 2.1.4 can be applied. Hence, one has for $x \in X$

$$R^k p_*(M_{X \times I})_x = H^k(I, M) \qquad \text{for } k = 0, 1, \ldots.$$

Consequently, it suffices to prove the following lemma.

Lemma. *Let I and M be as the above. Then*

$$H^k(I, M) = \begin{cases} M & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

holds.

Proof. Since I is connected, $H^0(I, M) = M$. Next, when $k > 1$, assume that $s \neq 0$ for $s \in H^k(I, M)$. Let the set $\mathscr{M} = \{[a, b] \subset I \mid s|_{[a,b]} \neq 0\}$. Define an order relation $\succ$ in $\mathscr{M}$ as follows: $[a, b] \succ [a', b']$ if and only if $[a, b] \subset [a', b']$. Note that $\mathscr{M}$ is an inductively ordered set. Let $\{F_j = [a_j, b_j]\}$ be a totally ordered subset of $\mathscr{M}$. Then $F = \bigcap F_j$ is a closed interval in I. Since $H^k(F, M) = \varinjlim_j H^k(F_j, M)$, $s|_F \neq 0$ holds. Therefore there exists a maximal element $[a, b]$ (i.e. a minimal interval in our case) by Zorn's lemma. One can choose a real number c such that $a < c < b$, since $a = b$ would imply $s|_{[a,b]} = 0$. Then the sequence

$$0 \to \Gamma([a, b], \mathscr{L}) \xrightarrow{i} \Gamma([a, c], \mathscr{L}) \oplus \Gamma([c, b], \mathscr{L})$$

and

$$\Gamma([a, c], \mathscr{L}) \oplus \Gamma([c, b], \mathscr{L}) \xrightarrow{j} \Gamma(\{c\}, \mathscr{L}) \to 0$$

are exact if $\mathscr{L}$ is either a constant sheaf or a flabby sheaf, where $i(s) = (s|_{[a,c]}, s|_{[c,b]})$ and $j(s_1, s_2) = (s_1|_c - s_2|_c)$. Consequently, there is induced the long exact sequence

$$\cdots \to H^{k-1}(\{c\}, M) \to H^k([a,b], M) \to H^k([a,c], M) \oplus H^k([c,b], M] \\ \to H^k(\{c\}, M) \to \cdots.$$

In the case where $k > 1$, one has $H^{k-1}(\{c\}, M) = H^k(\{c\}, M) = 0$. Then there exists the isomorphism

$$\begin{array}{ccc} H^k([a,b], M) & \cong & H^k([a,c], M) \oplus H^k([c,b], M) \\ \cup\!\!\! \mid & & \cup\!\!\! \mid \\ s|_{[a,b]} & \longmapsto & s|_{[a,c]} \oplus s|_{[c,b]} \end{array}$$

for each $k > 1$. Since $[a, b]$ is maximal, one must have

$$s|_{[a,b]} \neq 0 \quad \text{and} \quad s|_{[a,c]} = s|_{[c,b]} = 0,$$

contradicting the above isomorphism. Lastly, if $k = 1$, one obtains $H^1([a, b], M) = 0$ from the exact sequence

$$0 \to H^0([a,b], M) \to H^0([a,c], M) \oplus H^0([c,b], M) \to H^0(\{c\}, M) \to 0.$$

Let X and Y have the same homotopy type. That is, there are continuous maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \cong 1_Y$ hold. By the above result, we have $f^* \circ g^* = 1$ and $g^* \circ f^* = 1$, where $H^k(X, M) \underset{f^*}{\overset{g^*}{\rightleftarrows}} H^k(Y, M)$ such that g^* and f^* are inverse maps to each other. Therefore $H^k(X, M) \cong H^k(Y, M)$ is true for $k = 0, 1, 2, \ldots$. In particular, if X is contractible to a point x_0,

$$H^k(X, M) \cong H^k(x_0, M) \qquad \text{holds for } k \geqq 0.$$

Since any sheaf over a point is by definition a flabby sheaf, $H^k(x_0, M) = 0$ for $k > 0$ and $H^0(x_0, M) = M$. This completes the proof of Theorem 1.2.4.

§3. Fundamental Exact Sequences

We will fix a real analytic manifold M throughout this section. Therefore we will simply use $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ instead of $\mathscr{A}_M$, $\mathscr{B}_M$, and $\mathscr{C}_M$. The most fundamental exact sequence on M

$$0 \to \mathscr{A} \to \mathscr{B} \to \pi_*\mathscr{C} \to 0$$

in the theory of microfunctions will be established in this section. This exact sequence tells us that the "sheaf $\mathscr{B}/\mathscr{A}$ of the irregularity" is isomorphic to $\pi_*\mathscr{C}$. In other words, the structure of $\mathscr{B}/\mathscr{A}$ can be analyzed sharply on $\sqrt{-1}S^*M$. It will be proved that there exists the exact sequence

$$0 \to \tilde{\mathscr{A}} \to \tau^{-1}\mathscr{B} \to \pi_*\tau^{-1}\mathscr{C} \to 0$$

on $\sqrt{-1}SM$. This exact sequence clarifies the inner relationship between the sheaf $\mathscr{C}$ and the notion of the "boundary values of holomorphic functions." Important for the applications are the theorems on singularity spectrum (Theorems 2.3.4 and 2.3.5) which will be proved as its consequence.

We will begin with the notion of generalized relative cohomology groups $H^k(A^\bullet \xrightarrow{f} B^\bullet)$.

Definition 2.3.1. *Let $A^\bullet = \{\cdots \to A^n \xrightarrow{d_A^n} A^{n+1} \to \cdots\}$ and $B^\bullet = \{\cdots \to B^n \xrightarrow{d_A^n} B^{n+1} \to \cdots\}$ be cochain complexes of abelian groups. Let $f: A^\bullet \to B^\bullet$ be a morphism, i.e. $f = \{f_n\}$, where $f_n: A^n \to B^n$ is a homomorphism with the property $f_{n+1} \circ d_A^n = d_B^n \circ f_n$ for any integer n.*

Define $C^\bullet = C(A^\bullet \to B^\bullet)$ by $C^n = A^n \oplus B^{n-1}$, and define $d_C^n: A^n \oplus B^{n-1} \to A^{n+1} \oplus B^n$ by $d_C^n(x, y) = (d_A^n x, d_B^{n-1} y + (-1)^n f_n(x))$ for each n. Then $d_C^{n+1} \circ d_C^n = 0$ holds. Therefore $C^\bullet = \{\cdots \to C^n \xrightarrow{d_C^n} C^{n+1} \to \cdots\}$ is a cochain complex. The cohomology groups $H^k(C^\bullet)$ of the cochain complex $C^\bullet$ are called the generalized relative cohomology groups.

Note. We often denote $H^k(C^\bullet)$ by $H^k(A^\bullet \to B^\bullet)$. Propositions 2.3.1 and 2.3.2 (particularly (1) of Proposition 2.3.2) explain why $H^k(A^\bullet \to B^\bullet)$ is called a generalized relative cohomology group.

Proposition 2.3.1.

(1) *There is induced the long exact sequence*

$$\cdots \to H^k(A^\bullet \xrightarrow{f} B^\bullet) \to H^k(A^\bullet) \to H^k(B^\bullet) \to H^{k+1}(A^\bullet \xrightarrow{f} B^\bullet) \to \cdots.$$

(2) *For $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$, the sequence*

$$\cdots \to H^k(A^\bullet \xrightarrow{f} B^\bullet) \to H^k(A^\bullet \xrightarrow{g \circ f} C^\bullet) \to H^k(B^\circ \xrightarrow{g} C^\bullet)$$
$$\to H^{k+1}(A^\bullet \xrightarrow{f} B^\bullet) \to \cdots$$

is exact.

Proof. Define the cochain complex $B^\bullet[k]$ by $(B^\bullet[k])^n = B^{n+k}$. Then one has the exact sequence

$$0 \to B^\bullet[-1] \to C(A^\bullet \to B^\bullet) \to A^\bullet \to 0.$$

The long exact sequence of cohomology groups induced from this exact sequence of cochain complexes is the one sought.

Next we will prove (2). Let $X^\bullet = C(A^\bullet \xrightarrow{g \circ f} C^\bullet)$, and let $Y^\bullet = C(B^\bullet \xrightarrow{g} C^\bullet)$. Then define a morphism $H: X^\bullet \to Y^\bullet$ as $h_n(x, y) = (f_n(x), y) \in B^n \oplus C^{n-1} = Y^n$ for $(x, y) \in A^n \oplus C^{n-1} = X^n$. If one lets $Z^\bullet = C(X^\bullet \xrightarrow{h} Y^\bullet)$, then from (1) one obtains the exact sequence

$$\cdots \to H^k(X^\bullet \to Y^\bullet) \to H^k(A^\bullet \to C^\bullet) \to H^k(B^\bullet \to C^\bullet) \to \cdots.$$

Therefore one must show $H^k(X^\bullet \xrightarrow{h} Y^\bullet) = H^k(A^\bullet \to B^\bullet)$ to complete the proof.

Let $((x, y), (x', y')) \in Z^n = (A^n \oplus C^{n-1}) \oplus (B^{n-1} \oplus C^{n-2})$. Then $d_Z^n((x, y), (x', y')) = ((d_A^n x, d_C^{n-1} y + (-1)^n(g_n \circ f_n)(x)), (d_B^{n-1} x' + (-1)^n f_n(x), (-1)^n y + d_C^{n-2} y' + (-1)^{n-1} g_{n-1}(x')))$. Letting $W^\bullet = C(A^\bullet \to B^\bullet)$, one defines $F_n: Z_n \to W_n$ by $F_n(((x, y), (x', y'))) = (x, x')$. Then $F: Z^\bullet \to W^\bullet$ is a morphism. Define $G_n: W^n \to Z^n$ by $G_n(x, x') = ((x, g_{n-1}(x')), (x', 0))$. Then $G: W^\bullet \to Z^\bullet$ is a morphism. Notice that $F \circ G$ is the identity map on $W^\bullet$. Hence, the induced maps on cohomology groups $H^n(W^\bullet) \xrightarrow{G^*} H^n(Z^\bullet) \xrightarrow{F^*} H^n(W^\bullet)$ imply that $F^* \circ G^*: H^n(W^\bullet) \to H^n(W^\bullet)$ is the identity map.

Let $s_n: Z^n \to Z^{n-1}$ be the map defined by

$$s(((x, y), (x', y'))) = ((0, (-1)^n y'), (0, 0)).$$

Then in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Z^{n-1} \arrow[r, "d_Z^{n-1}"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "G \circ F"] & Z^n \arrow[r, "d_Z^n"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "G \circ F"] \arrow[dl, "s_n"] & Z^{n+1} \arrow[r] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "G \circ F"] \arrow[dl, "s_{n+1}"] & \cdots \\
\cdots \arrow[r] & Z^{n-1} \arrow[r] & Z^n \arrow[r] & Z^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

$s_{n+1} \circ d_Z^n + d_Z^{n-1} \circ s_n = (G \circ F)_n - \mathrm{id}_{Z^n}$ holds. Therefore one has $G^* \circ F^* = \mathrm{id}_Z^*: H^n(Z^\bullet) \to H^n(Z^\bullet)$. Consequently, one obtains

$$H^k(X^\bullet \to Y^\bullet) = H^k(Z^\bullet) \cong H^k(W^\bullet) = H^k(A^\bullet \to B^\bullet).$$

A few lemmas are needed to define a generalized relative cohomology of sheaves.

Lemma 1. *Let X and Y be topological spaces, and let $f: X \to Y$ be a continuous map. Suppose that a sheaf $\mathscr{F}$ over Y is given; then there exists a canonical morphism $\mathscr{F} \to f_* f^{-1}\mathscr{F}$.*

Proof. Recall that the sheaf $f^{-1}\mathscr{F}$ is associated to the presheaf $\varinjlim_{U \supset f(V)} \mathscr{F}(U)$ for an open set V in X. Then, for an open set U of Y, there are induced maps

$$\mathscr{F}(U) \to \varinjlim_{U' \supset f(f^{-1}(U))} \mathscr{F}(U') \to (f^{-1}\mathscr{F})(f^{-1}(U)) = f_* f^{-1}\mathscr{F}(U),$$

which define the morphism $\mathscr{F} \to f_* f^{-1}\mathscr{F}$.

Lemma 2. *Let $\mathscr{G}$ and $\mathscr{F}$ be sheaves over topological spaces X and Y, respectively, and let $f: X \to Y$ be a continuous map. Then there is a natural bijection between the morphisms $\mathscr{F} \to f_*\mathscr{G}$ and the morphisms $f^{-1}\mathscr{F} \to \mathscr{G}$.*

Proof. Suppose that a morphism $f^{-1}\mathscr{F} \to \mathscr{G}$ is given. Then one has the morphism $f_* f^{-1}\mathscr{F} \to f_*\mathscr{G}$. Lemma 1 implies that a morphism $\mathscr{F} \to f_*\mathscr{G}$

exists. Conversely, assume that a morphism $\mathscr{F} \to f_*\mathscr{G}$ is given. That is, for an open set U in Y, the map $\mathscr{F}(U) \to \mathscr{G}(f^{-1}(U))$ is given. For an open set V in X with the property $f(V) \subset U$, the homomorphisms $\mathscr{F}(U) \to \mathscr{G}(f^{-1}(U)) \to \mathscr{G}(V)$ are induced, where we note that $f(V) \subset U$ implies $V \subset f^{-1}(U)$. Consequently, one obtains the homomorphism

$$\varinjlim_{f(V) \subset U} \mathscr{F}(U) \to \mathscr{G}(V) \qquad \text{for } V \text{ in } X.$$

Therefore, the morphism $f^{-1}\mathscr{F} \to \mathscr{G}$ exists.

Lemma 3. *Let $\mathscr{G}$ and $\mathscr{F}$ be sheaves over topological spaces X and Y respectively, let $f: X \to Y$ be a continuous map, and let $\rho: \mathscr{F} \to f_*\mathscr{G}$ be a morphism. For a flabby resolution of $\mathscr{F}$, $0 \to \mathscr{F} \to \mathscr{L}^\bullet$, there exist a flabby resolution of $\mathscr{G}$, $0 \to \mathscr{G} \to \mathscr{M}^\bullet$, and a morphism $\rho_k: \mathscr{L}^k \to f_*\mathscr{M}^k$ for each $k \geqq 0$, such that the diagram*

$$\begin{array}{ccccccccc}
0 \to & \mathscr{F} & \longrightarrow & \mathscr{L}^0 & \longrightarrow & \mathscr{L}^1 & \longrightarrow & \mathscr{L}^2 & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho_0} & & \downarrow{\scriptstyle\rho_1} & & \downarrow{\scriptstyle\rho_2} & \\
0 \to & f_*\mathscr{G} & \to & f_*\mathscr{M}^0 & \to & f_*\mathscr{M}^1 & \to & f_*\mathscr{M}^2 & \to \cdots
\end{array}$$

commutes.

Proof. For a flabby resolution $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ of $\mathscr{F}$, the sequence $0 \to f^{-1}\mathscr{F} \to f^{-1}\mathscr{L}^0 \to f^{-1}\mathscr{L}^1 \to \cdots$ is exact. Then, by Lemma 2, one has a morphism $f^{-1}\mathscr{F} \to \mathscr{G}$. In the proof of Theorem 1.1.1, Lemma 1 implies that there exists a flabby resolution of $\mathscr{G}$, $0 \to \mathscr{G} \to \mathscr{M}^\bullet$, making the diagram

$$\begin{array}{ccccccc}
0 \to & f^{-1}\mathscr{F} & \to & f^{-1}\mathscr{L}^0 & \to & f^{-1}\mathscr{L}^1 & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{M}^0 & \longrightarrow & \mathscr{M}^1 & \longrightarrow \cdots
\end{array}$$

commutative. Hence one obtains the commutative diagram

$$\begin{array}{ccccccc}
0 \to & f_*f^{-1}\mathscr{F} & \to & f_*f^{-1}\mathscr{L}^0 & \to & f_*f^{-1}\mathscr{L}^1 & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & f_*\mathscr{G} & \longrightarrow & f_*\mathscr{M}^0 & \longrightarrow & f_*\mathscr{M}^1 & \longrightarrow \cdots.
\end{array}$$

Then, one can complete the proof by the use of Lemma 1.

Definition 2.3.2. *Let the notations be the same as in Lemma 3. Abbreviate "$\rho: \mathscr{F} \to f_*\mathscr{G}$" as "$\mathscr{F} \xrightarrow{\rho} \mathscr{G}$." Then define*

$$H^k(X \xrightarrow{f} Y, \mathscr{G} \xleftarrow{\rho} \mathscr{F}) = H^k(\Gamma(Y, \mathscr{L}^\bullet) \to \Gamma(X, \mathscr{M}^\bullet))$$

and

$$\mathscr{D}ist^k_f(\mathscr{F} \xrightarrow{\rho} \mathscr{G}) = \mathscr{H}^k(\mathscr{L}^\bullet \to f_*\mathscr{M}^\bullet).$$

In particular, if $\mathscr{G} = f^{-1}\mathscr{F}$ *holds, we define*

$$H^k(X \to Y, \mathscr{F}) = H^k(X \to Y, f^{-1}\mathscr{F} \leftarrow \mathscr{F})$$

and

$$\mathscr{D}\mathit{ist}^k_f(\mathscr{F}) = \mathscr{D}\mathit{ist}^k_f(\mathscr{F} \to f^{-1}\mathscr{F}).$$

The continuous map f is said to be purely r-codimensional with respect to $\mathscr{F}$ if $\mathscr{D}\mathit{ist}^k_f(\mathscr{F}) = 0$ for $k \neq r$.

Remark. The sheaf $\mathscr{D}\mathit{ist}^n_f(\mathscr{F} \to \mathscr{G})$ over Y is associated to the presheaf $H^n(f^{-1}(U) \to U, \mathscr{G} \xleftarrow{\rho} \mathscr{F})$ for an open set U in Y.

The cohomology groups defined above are generalizations of $H^k_Z(X, \mathscr{F})$ and $\mathscr{H}^k_Z(\mathscr{F})$; i.e. we have:

Proposition 2.3.2. *Suppose that X is an open set in Y. The imbedding: $X \hookrightarrow Y$ is denoted by f. Then, one has the isomorphisms*

(1) $H^n(X \to Y, \mathscr{F}) \cong H^n_{Y-X}(Y, \mathscr{F})$ *and*

(2) $\mathscr{D}\mathit{ist}^n_f(\mathscr{F}) = \mathscr{H}^n_{Y-X}(\mathscr{F})$, *for* $n \geqq 0$.

We need a lemma.

Lemma. *If $0 \to E^\bullet \xrightarrow{e} A^\bullet \xrightarrow{f} B^\bullet \to 0$ is an exact sequence, then $H^n(E^\bullet) \cong H^n(A^\bullet \xrightarrow{f} B^\bullet)$ holds for $k \geqq 0$.*

Proof. Define a homomorphism $g_n: E^n \to A^n \oplus B^{n-1}$ by $g_n(x) = (e_n(x), 0)$. Then $g: E^\bullet \to C^\circ \underset{\text{def}}{=} C(A^\bullet \xrightarrow{f} B^\bullet)$ is a morphism. Therefore, from the exact sequence

$$0 \to B^\bullet[-1] \xrightarrow{h} C^\bullet \xrightarrow{k} A^\bullet \to 0,$$

one obtains the commutative diagram

$$\begin{tikzcd}
\to H^{n-1}(A^\bullet) \arrow[r, "f^*"] \arrow[d, equal] & H^{n-1}(B^\bullet) \arrow[r] \arrow[d, "(-1)^{n-1}\mathrm{id}", "\wr"'] & H^n(E^\bullet) \arrow[r, "e^*"] \arrow[d, "g^*"] & H^n(A^\bullet) \arrow[r, "f^*"] \arrow[d, equal] & H^n(B^\bullet) \to \cdot \arrow[d, "(-1)^n\mathrm{id}", "\wr"'] \\
\to H^{n-1}(A^\bullet) \arrow[r] & H^{n-1}(B^\bullet) \arrow[r, "-h^*"] & H^n(C^\bullet) \arrow[r, "k^*"] & H^n(A^\bullet) \arrow[r] & H^n(B^\bullet) \to \cdot
\end{tikzcd}$$

One concludes $H^n(E^\bullet) \cong H^n(C^\bullet) = H^n(A^\bullet \xrightarrow{f} B^\circ)$ by Five Lemma.

Proof of Proposition 2.3.2. Let $0 \to \mathscr{F} \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots$ be a flabby resolution of $\mathscr{F}$. Then $0 \to f^{-1}\mathscr{F} \to f^{-1}\mathscr{L}^0 \to f^{-1}\mathscr{L}^1 \to \cdots$ is a flabby resolution of $f^{-1}\mathscr{F}$, since $f^{-1}\mathscr{L}^k = \mathscr{L}^k|_X$ is flabby. Notice that, since $\mathscr{L}^k$ is flabby for each $k \geqq 0$, the sequence

$$0 \to \Gamma_{Y-X}(Y, \mathscr{L}^\bullet) \to \Gamma(Y, \mathscr{L}^\bullet) \to \Gamma(X, \mathscr{L}^\bullet|_X) \to 0$$

is exact. The above lemma implies the isomorphism

$$H^n(X \to Y, \mathscr{F}) = H^n(\Gamma(Y, \mathscr{L}^\bullet) \to \Gamma(X, \mathscr{L}^\bullet)) \cong H^n(\Gamma_{Y-X}(Y, \mathscr{L}^\bullet))$$
$$= H^n_{Y-X}(Y, \mathscr{F}).$$

In order to prove (2), first note that one has the exact sequence

$$0 \to \Gamma_{Y-X}(\mathscr{L}^\bullet) \to \mathscr{L}^\bullet \to f_* f^{-1} \mathscr{L}^\bullet \to 0.$$

Hence, by the Lemma above, one obtains

$$\mathscr{D}ist^n_f(\mathscr{F}) = \mathscr{H}^n(\mathscr{L}^\bullet \to f_* f^{-1} \mathscr{L}^\bullet) \cong \mathscr{H}^n(\Gamma_{Y-X}(\mathscr{L}^\bullet)) = \mathscr{H}^n_{Y-X}(\mathscr{F}).$$

Proposition 2.3.3.

(1) *Let $\mathscr{G}$ and $\mathscr{F}$ be sheaves over topological spaces X and Y, respectively, and let $f: X \to Y$ be a continuous map. If a morphism $\rho: \mathscr{F} \to f_* \mathscr{G}$ is given, then there is induced the long exact sequence*

(i) $\cdots \to H^k(X \to Y, \mathscr{G} \leftarrow \mathscr{F}) \to H^k(Y, \mathscr{F}) \to H^k(X, \mathscr{G}) \to \cdots$

and, in particular, one has the following:

$$\cdots \to H^k(X \to Y, \mathscr{F}) \to H^k(Y, \mathscr{F}) \to H^k(X, f^{-1}\mathscr{F}) \to \cdots.$$

(ii) *The sequence*

$$0 \to \mathscr{D}ist^0_f(\mathscr{F} \to \mathscr{G}) \to \mathscr{F} \xrightarrow{\rho} f_* \mathscr{G} \to \mathscr{D}ist^1_f(\mathscr{F} \to \mathscr{G}) \to 0$$

is exact, and for $k \geqq 2$, $\mathscr{D}ist^k_f(\mathscr{F} \to \mathscr{G}) = R^{k-1} f_* \mathscr{G}$ holds. In particular, the sequence

$$0 \to \mathscr{D}ist^0_f(\mathscr{F}) \to \mathscr{F} \to f_* f^{-1} \mathscr{F} \to \mathscr{D}ist^1_f(\mathscr{F}) \to 0$$

is exact, and $\mathscr{D}ist^k_f(\mathscr{F}) = R^{k-1} f_*(f^{-1}\mathscr{F})$ for $k \geqq 2$.

(2) *Let $\mathscr{G}$, $\mathscr{F}$, and $\mathscr{H}$ be sheaves over topological spaces X, Y, and Z respectively; and let $f: X \to Y$ and $g: Y \to Z$ be continuous maps. Suppose that morphisms $\mathscr{F} \to f_* \mathscr{G}$ and $\mathscr{H} \to g_* \mathscr{F}$ are given. Then there are induced long exact sequences*

$$\cdots \to H^k(Y \to Z, \mathscr{F} \leftarrow \mathscr{H}) \to H^k(X \to Z, \mathscr{G} \leftarrow \mathscr{H})$$
$$\to H^k(X \to Y, \mathscr{G} \leftarrow \mathscr{F}) \to \cdots,$$

particularly

$$\cdots \to H^k(Y \to Z, \mathscr{H}) \to H^k(X \to Z, \mathscr{H}) \to H^k(X \to Y, g^{-1}\mathscr{H}) \to \cdots.$$

Proof. Definitions and Proposition 2.3.1 imply these claims plainly.

We apply (2) of Proposition 2.3.3 to the case $(X - M) \overset{\iota}{\hookrightarrow} \widetilde{{}^M X} \xrightarrow{\tau} X$ and the sheaf $\mathscr{O}_X$ over X. We have the following exact sequence:

$$\cdots \to H^k(\widetilde{{}^M X} \to X, \mathscr{O}_X) \to H^k((X - M) \hookrightarrow X, \mathscr{O}_X)$$
$$\to H^k((X - M) \hookrightarrow \widetilde{{}^M X}, \tau^{-1}(\mathscr{O}_X)) \to \cdots.$$

We will compute each term. From Proposition 2.3.2 we have

$$H^k((X - M) \hookrightarrow X, \mathcal{O}_X) = H^k_M(X, \mathcal{O}_X).$$

Recall that M is purely n-codimensional with respect to $\mathcal{O}_X$. Therefore, by Proposition 1.1.5,

$$H^k_M(X, \mathcal{O}_X) = 0 \qquad \text{holds for } k < n.$$

On the other hand, flabby dim $\mathcal{O}_X \leqq n$ (Theorem 1.1.4) and Theorem 1.1.3 imply that for $k > n$ we have

$$H^k_M(X, \mathcal{O}_X) = 0.$$

Consequently, we obtain

$$H^k((X - M) \hookrightarrow X, \mathcal{O}_X) = 0 \qquad \text{for } k \neq n.$$

Similarly, using Propositions 2.1.1 and 1.1.6, we have the following:

$$H^k((X - M) \hookrightarrow \widetilde{^M X}, \tau^{-1}\mathcal{O}_X) = H^k_{\sqrt{-1}SM}(\widetilde{^M X}, \tau^{-1}\mathcal{O}_X) = H^{k-1}(\sqrt{-1}SM, \mathcal{Q}).$$

We will prove a few propositions in order to compute $H^k(\widetilde{^M X} \to X, \mathcal{O}_X)$.

Proposition 2.3.4. *Let X and Y be topological spaces, let $f: X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf over Y. If the continuous map f is purely l-codimensional with respect to $\mathcal{F}$, then for an arbitrary integer $k \geqq 0$*

$$H^k(X \to Y, \mathcal{F}) = H^{k-l}(Y, \mathcal{D}ist^l_f(\mathcal{F}))$$

holds.

Proof. Let $0 \to \mathcal{F} \to \mathcal{L}^\bullet$ and $0 \to f^{-1}\mathcal{F} \to \mathcal{M}^\bullet$ be flabby resolutions of $\mathcal{F}$ and $f^{-1}\mathcal{F}$, and let $\mathcal{N}^\bullet = C(\mathcal{L}^\bullet \to f_*\mathcal{M}^\bullet)$. Then $\mathcal{D}ist^k_f(\mathcal{F}) = \mathcal{H}^k(\mathcal{N}^\bullet)$ holds. Note that $\mathcal{N}^\bullet$ is a cochain complex of flabby sheaves. By the hypothesis one has the exact sequences

$$0 \to \mathcal{N}^0 \to \mathcal{N}^1 \to \cdots \to \mathcal{N}^{l-1} \to \mathcal{I} \to 0,$$
$$0 \to \mathcal{I} \to \mathcal{Z} \to \mathcal{D}ist^l_f(\mathcal{F}) \to 0, \quad \text{and}$$
$$0 \to \mathcal{Z} \to \mathcal{N}^l \to \mathcal{N}^{l+1} \to \cdots.$$

One can complete the proof in a similar manner to the proofs of Proposition 1.1.5 or Proposition 1.1.6.

Proposition 2.3.5. *Let X and Y be topological spaces. Suppose that all the open sets in X or in Y are paracompact and Hausdorff. Let $\mathcal{F}$ be a sheaf over Y. If a continuous map $f: X \to Y$ is a closed map, then for an arbitrary $y \in Y$, there exists an isomorphism*

$$\mathcal{D}ist^k_f(\mathcal{F})_y = H^k(f^{-1}(y) \to \{y\}, \mathcal{F}_y).$$

Remark. The above proposition also holds under the assumption that X and Y are Hausdorff spaces and $f: X \to Y$ is a proper map.

Proof. Since f is a closed map, Proposition 2.1.4 can be applied. One has

$$\varinjlim_{y \in U} H^k(U, \mathscr{F}) = H^k(\{y\}, \mathscr{F}_y) \quad \text{and}$$

$$\varinjlim_{y \in U} H^k(f^{-1}(U), f^{-1}(\mathscr{F})) = H^k(f^{-1}(y), \mathscr{F}_y).$$

Take the direct limit of the diagram

$$\begin{array}{ccccccc}
\cdot \longrightarrow H^{k-1}(U, \mathscr{F}) & \longrightarrow & H^{k-1}(f^{-1}(U), f^{-1}\mathscr{F}) & \longrightarrow & H^k(f^{-1}(U) \to U, \mathscr{F}) \to \\
\downarrow & & \downarrow & & \downarrow \\
\cdot \to H^{k-1}(\{y\}, \mathscr{F}_y) & \longrightarrow & H^{k-1}(f^{-1}(y), \mathscr{F}_y) & \longrightarrow & H^k(f^{-1}(y) \to \{y\}, \mathscr{F}) \to
\end{array}$$

$$\begin{array}{ccc}
H^k(U, \mathscr{F}) & \longrightarrow & H^k(f^{-1}(U), f^{-1}\mathscr{F}) \to \cdots \\
\downarrow & & \downarrow \\
H^k(\{y\}, \mathscr{F}_y) & \longrightarrow & H^k(f^{-1}(y), \mathscr{F}_y) \longrightarrow \cdots.
\end{array}$$

Then one obtains, by Five Lemma,

$$\mathscr{D}\!\mathit{ist}^k_f(\mathscr{F})_y = \varinjlim_{y \in U} H^k(f^{-1}(U) \to U, \mathscr{F}) = H^k(f^{-1}(y) \to \{y\}, \mathscr{F}).$$

Proposition 2.3.6. *Let G be a constant sheaf on the n-sphere S^n. Then the following* (1), (2), *and* (3) *hold.*

(1) $$H^k(S^0, G) = \begin{cases} G \oplus G & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

and for $n > 0$,

$$H^k(S^n, G) = \begin{cases} G & \text{for } k = 0, n \\ 0 & \text{for } k \neq 0, n. \end{cases}$$

(2) $$H^k(S^n \to \{x_0\}, G) = \begin{cases} G & \text{for } k = n + 1 \\ 0 & \text{for } k \neq n + 1. \end{cases}$$

(3) *For an arbitrary k*

$$H^k(\{y_0\} \to \{x_0\}, G) = 0$$

holds.

Proof. Note that S^0 consists of two points. Since any sheaf on two points is a flabby sheaf by definition, the first part of (1) is immediate. Next suppose $n \geqq 1$. Let D_1^n and D_2^n be closed hemispheres of S^n such that $D_1^n \cap D_2^n = S^{n-1}$ holds. Then $H^k(D_i^n, G) = 0$ holds for $k > 0$ since D_i ($i = 1, 2$) is contractible. If $\mathscr{S}$ is a flabby sheaf, the sequence

$$0 \to \Gamma(S^n, \mathscr{S}) \xrightarrow{\varphi} \Gamma(D_1^n, \mathscr{S}) \oplus \Gamma(D_2^n, \mathscr{S}) \xrightarrow{\psi} \Gamma(S^{n-1}, \mathscr{S}) \to 0$$

is exact, where $\varphi(s) = (s|_{D_1^n}, s|_{D_2^n})$ and $\psi(s_1, s_2) = s_1|_{S^{n-1}} - s_2|_{S^{n-1}}$ (see Proposition 2.1.4). Therefore, by considering a flabby resolution of the constant sheaf G, one obtains the long exact sequence

$$\cdots \to H^k(S^n, G) \to H^k(D_1^n, G) \oplus H^k(D_2^n, G) \to H^k(S^{n-1}, G) \to \cdots.$$

In the case where $n = 1$, define $i: G \to G \oplus G$ by $i(x) = (x, x)$, and define $j: G \oplus G \to G \oplus G$ by $j(x, y) = (x - y, x - y)$. Then one has the exact sequence

$$0 \to G \xrightarrow{i} G \oplus G \xrightarrow{j} G \oplus G \to H^1(S^1, G) \to 0.$$

When $k \geqq 2$, $0 \to H^k(S^1, G) \to 0$ is exact. Hence the assertion for $n = 1$ follows. Next, in the case where $n \geqq 2$, define a map $j': G \oplus G \to G$ by $j'(x, y) = x - y$. For $k = 1$, one has the exact sequence

$$0 \to G \xrightarrow{i} G \oplus G \xrightarrow{j'} G \to H^1(S^n, G) \to 0.$$

For $k \geqq 2$ the sequence

$$0 \to H^{k-1}(S^{n-1}, G) \to H^k(S^n, G) \to 0$$

is exact, from which the proof can be completed inductively.

To prove (2), first notice that the sequence

$$\cdots \to H^k(S^n \to \{x_0\}, G) \to H^k(\{x_0\}, G) \to H^k(S^n, G) \to \cdots$$

is exact, and notice the fact that $H^k(\{x_0\}, G) = 0$ for $k \neq 0$. Then (2) follows immediately from (1).

Lastly we will prove (3). Note that (3) holds when $k \geqq 2$ from the exact sequence

$$\cdots \to H^k(\{y_0\} \to \{x_0\}, G) \to H^k(\{x_0\}, G) \to H^k(\{y_0\}, G) \to \cdots.$$

For $k = 0$ and 1, one must only observe the exactness of the following sequence to complete the proof:

$$0 \to H^0(\{y_0\} \to \{x_0\}, G) \to G \rightrightarrows G \to H^1(\{y_0\} \to \{x_0\}, G) \to 0.$$

We will state a theorem from Grauert (though we will not give a proof) that is fundamental to our subsequent discussion.

Theorem 2.3.1 (Grauert [1]). *Let $\mathscr{A}$ be sheaf of real analytic functions on a real analytic manifold M, and let ω be the orientation sheaf. Then*

$$H^k(M, \mathscr{A} \otimes \omega) = 0$$

holds for $k \neq 0$.

This theorem gives us the following proposition.

Proposition 2.3.7.

$$H^k(\widetilde{{}^M X} \xrightarrow{\tau} X, \mathcal{O}_X) = \begin{cases} 0 & \text{for } k \neq n \\ (\mathscr{A} \otimes \omega)(M) & \text{for } k = n \end{cases}$$

holds.

Proof. Since τ is a proper map, one has the isomorphism

$$\mathscr{D}ist_\tau^k(\mathcal{O}_X)_x = H^k(\tau^{-1}(x) \to \{x\}, \mathcal{O}_{X,x})$$

from Proposition 2.3.5.

Let $x \in X - M$; then $\tau^{-1}(x) \cong \{x\}$. Hence one has $\mathscr{D}ist_\tau^k(\mathcal{O}_{X,x}) = 0$ by (3) of Proposition 2.3.6. If $x \in M$, then $\tau^{-1}(x) = S^{n-1}$. One obtains $\mathscr{D}ist_\tau^k(\mathcal{O}_{X,x}) = 0$ for $k \neq n$ and $\mathscr{D}ist_\tau^n(\mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$ from (2) of Proposition 2.3.6. Note that the isomorphism $H^n(S^{n-1} \to \{x\}, G) \cong G$ depends upon the orientation. Therefore, $\mathscr{D}ist_\tau^k(\mathcal{O}_X) = 0$ for $k \neq n$ and $\mathscr{D}ist_\tau^n(\mathcal{O}_X) = \mathscr{A} \otimes \omega$ hold. Proposition 2.3.4 provides

$$H^k(\widetilde{{}^M X} \xrightarrow{\tau} X, \mathcal{O}_X) = H^{k-n}(X, \mathscr{A} \otimes \omega) = H^{k-n}(M, \mathscr{A} \otimes \omega).$$

Hence, the above theorem from Grauert (Theorem 2.3.1) now completes the proof.

We will give a summary of what we have found. In the exact sequence

$$\begin{aligned} \cdots &\to H^k(\widetilde{{}^M X} \to X, \mathcal{O}_X) \to H^k((X - M) \hookrightarrow X, \mathcal{O}_X) \\ &\to H^k((X - M) \hookrightarrow \widetilde{{}^M X}, \tau^{-1}\mathcal{O}_X) \to \cdots, \end{aligned}$$

we have obtained

$$H^k(\widetilde{{}^M X} \xrightarrow{\tau} X, \mathcal{O}_X) = \begin{cases} 0 & \text{for } k \neq n \\ (\mathscr{A} \otimes \omega)(M) & \text{for } k = n \end{cases}$$

$$H^k((X - M) \hookrightarrow X, \mathcal{O}_X) = \begin{cases} 0 & \text{for } k \neq n \\ H_M^n(X, \mathcal{O}_X) & \text{for } k = n \end{cases}$$

and

$$H^k((X - M) \hookrightarrow \widetilde{{}^M X}, \tau^{-1}\mathcal{O}_X) = H^{k-1}(\sqrt{-1}SM, \mathscr{Q}).$$

Consequently we have the following exact sequence:

$$\begin{aligned} 0 &\to H^{n-2}(\sqrt{-1}SM, \mathscr{Q}) \to (\mathscr{A} \otimes \omega)(M) \to H_M^n(X, \mathcal{O}_X) \\ &\to H^{n-1}(\sqrt{-1}SM, \mathscr{Q}) \to 0 \end{aligned} \tag{2.3.1}$$

and

$$H^k(\sqrt{-1}SM, \mathscr{Q}) = 0 \qquad \text{for } k \neq n - 1, n - 2. \tag{2.3.2}$$

Note that (2.3.1) and (2.3.2) still hold when M is replaced by an arbitrary open subset of M, and note that we have $H^k(\sqrt{-1}SM, \mathscr{Q}) = H^k(\tau^{-1}(M), \mathscr{Q})$.

Therefore, we have the localized version of (2.3.1) and (2.3.2) as follows. The sequence

$$0 \to R^{n-2}\tau_*\mathscr{Q} \to \mathscr{A} \otimes \omega \to \mathscr{H}^n_M(\mathscr{O}_X) \to R^{n-1}\tau_*\mathscr{Q} \to 0 \qquad (2.3.3)$$

is exact, and we have

$$R^k\tau_*\mathscr{Q} = 0 \qquad \text{for } k \neq n-1, n-2. \qquad (2.3.4)$$

Since $\omega \underset{\mathbf{Z}}{\otimes} \omega = \mathbf{Z}_M$, we also obtain the exact sequence

$$0 \to R^{n-2}\tau_*\mathscr{Q} \otimes \omega \to \mathscr{A} \to \mathscr{B} \to R^{n-1}\tau_*\mathscr{Q} \otimes \omega \to 0. \qquad (2.3.5)$$

We will compute the first and the last terms of the exact sequence (2.3.5).

Proposition 2.3.8.

$$R^k\tau_*\mathscr{Q} = \begin{cases} 0 & \textit{for } k \neq n-1 \\ \pi_*\mathscr{C}^a \otimes \omega & \textit{for } k = n-1 \end{cases}$$

holds.

We need a lemma to prove this proposition.

Lemma. *Let X, Y, and Z be topological spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous maps. Suppose that $\mathscr{F}$ is a sheaf over X and that the continuous map f is purely l-dimensional with respect to $\mathscr{F}$. Then, for any integer k,*

$$R^k(g \circ f)_*\mathscr{F} = R^{k-l}g_*(R^lf_*\mathscr{F})$$

holds.

Proof. By Lemma 2 in the proof of Proposition 2.2.1, $H^k(X, \mathscr{F}) = H^{k-l}(Y, R^lf_*\mathscr{F})$ holds. This implies that for an open set U of Z there exists the isomorphism $H^k(f^{-1}(g^{-1}(U)), \mathscr{F}) = H^{k-l}(g^{-1}(U), R^lf_*\mathscr{F})$. Notice that the sheaf $R^k(g \circ f)_*\mathscr{F}$ over Z is the sheaf associated to the presheaf $H^k(f^{-1}(g^{-1}(U)), \mathscr{F})$ and that $R^{k-l}g_*(R^lf_*\mathscr{F})$ is the sheaf associated to $H^{k-l}(g^{-1}(U), R^lf_*(\mathscr{F}))$.

Note that $\pi^{-1}(x_0 + \sqrt{-1}v0)$, $x_0 + \sqrt{-1}v0 \in \sqrt{-1}SM$, is contractible. Hence we have

$$\begin{aligned} R^k\pi_*(\pi^{-1}\mathscr{Q})_{x+\sqrt{-1}v0} &= H^k(\pi^{-1}(x + \sqrt{-1}v0), \mathscr{Q}_{x+\sqrt{-1}v0}) \\ &= \begin{cases} 0 & \text{for } k \neq 0 \\ \mathscr{Q}_{x+\sqrt{-1}v0} & \text{for } k = 0 \end{cases}; \end{aligned}$$

i.e.

$$R^k\pi_*(\pi^{-1}\mathscr{Q}) = \begin{cases} 0 & \text{for } k \neq 0 \\ \mathscr{Q} & \text{for } k = 0 \end{cases}.$$

By the above lemma, $R^k\tau_*\mathscr{Q} = R^k(\tau \circ \pi)_*(\pi^{-1}\mathscr{Q}) = R^k(\pi \circ \tau)_*(\pi^{-1}\mathscr{Q})$. On the other hand, Proposition 2.1.2′ implies

$$R^k\tau_*(\pi^{-1}\mathscr{Q}) = \begin{cases} \mathscr{C}^a \otimes \omega & \text{for } k = n-1 \\ 0 & \text{for } k \neq n-1 \end{cases}.$$

Again by the preceding lemma, one obtains

$$R^k(\pi \circ \tau)_*(\pi^{-1}\mathscr{Q}) = R^{k-n+1}\pi_*(\mathscr{C}^a \otimes \omega).$$

Hence, $R^k\tau_*\mathscr{Q} = (R^{k-n+1}\pi_*\mathscr{C}^a) \otimes \omega$. For $k = n-2$ and $k = n-1$, $R^{n-2}\tau_*\mathscr{Q} = 0$ and $R^{n-1}\tau_*\mathscr{Q} = \pi_*\mathscr{C}^a \otimes \omega$ hold respectively.

Hence we assert the following theorem.

Theorem 2.3.2.

(1) *There exists the exact squence*

$$0 \to \mathscr{A} \to \mathscr{B} \xrightarrow{\text{sp}} \pi_*\mathscr{C} \to 0$$

on M.

(2) *For* $k \neq 0$, $R^k\pi_*\mathscr{C} = 0$ *holds.*

(3) *The sequence*

$$0 \to \mathscr{A}(M) \to \mathscr{B}(M) \xrightarrow{\text{sp}} \mathscr{C}(\sqrt{-1}S^*M) \to 0$$

is exact.

Proof. Note $\pi_*\mathscr{C}^a = \pi_*\mathscr{C}$. Then the exact sequence (2.3.5) and Proposition 2.3.8 imply the short exact sequence in (1). The assertion (2) has been shown in the proof of Proposition 2.3.8.

There is induced the exact sequence

$$0 \to \mathscr{A}(M) \to \mathscr{B}(M) \to \mathscr{C}(\sqrt{-1}S^*M) \to H^1(M, \mathscr{A})$$

from the short exact sequence in (1). Then Grauert's theorem (Theorem 2.3.1) completes the proof of (3).

Definition 2.3.3. *For* $u \in \mathscr{B}(M)$, $\text{sp}(u) \in \pi_*\mathscr{C}(M) = \mathscr{C}(\sqrt{-1}S^*M)$ *is said to be the spectrum of u. The support of* sp(u), *denoted by* S.S. u, *is called the singularity spectrum of u.*

Corollary. *For* $u \in \mathscr{B}(M)$ S.S. $u = \varnothing$ *holds if and only if u is a real analytic function on M.*

Proof. This is plain from the exact sequence (3) of Theorem 2.3.2.

Note. The terminology "singular spectrum" was coined by Boutet de Monvel. "Spectrum" originally meant the light decomposed according to the frequency or, more mathematically speaking, the support of the Fourier transform of a function. Hence this terminology gives picture of what it means (cf. Example 2.4.6 in the subsequent section). One problem with this terminology is that it becomes confounded with the existing terminology in spectral analysis of linear operators; and, in this respect, the terminology "singularity spectrum" proposed by Komatsu is preferable. Although some other terminologies are also proposed, here we use the terminology "singularity spectrum," which seems to be the most euphonious, self-explanatory, and commonly used.

We note that, for a distribution u, Hörmander [2] introduced a notion similar to S.S. u, in order to analyze the singularity structure of solutions of linear differential equations, and named it the (analytic) wave-front set of u. Bros and Iagolnitzer (Iagolnitzer [1] and the references cited therein) also introduced a similar notion "essential support" of a distribution u, starting from some physical motivation. When u is a distribution—as Bros and Iagolnitzer; Bony and Schapira; Kataoka; Nishiwada; and Hill showed (1975–1976) independently—these three concepts, i.e. the singularity spectrum, the analytic wave-front set, and the essential support, coincide. As we do not need this result in this book, we will not discuss it any further, but refer the reader to Bony [2] and references cited there.

We will construct a morphism $b:\tilde{\mathscr{A}} \to \tau^{-1}\mathscr{B}$ such that $0 \to \tilde{\mathscr{A}} \xrightarrow{b} \tau^{-1}\mathscr{B} \to \pi_* \tau^{-1}\mathscr{C} \to 0$ is an exact sequence on $\sqrt{-1}SM$.

Proposition 2.3.9. *Let π be the natural projection from $DM \underset{\sqrt{-1}S^*M}{\times} DM$ to $\sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM$, and let $\mathscr{F}$ be a sheaf on $\sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM$. Then*

$$\mathscr{D}\mathit{ist}_\pi^k(\mathscr{F}) = \begin{cases} 0 & \text{for } k \neq n-1 \\ \mathscr{F}|_{\Delta^a_{\sqrt{-1}SM}} \otimes \omega & \text{for } k = n-1 \end{cases}$$

holds, where the antidiagonal set $\Delta^a_{\sqrt{-1}SM} = \{(x + \sqrt{-1}v0,\ x - \sqrt{-1}v0) \in \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM\}$.

Proof. Since π is proper, for $x(v_1, v_2) = (x + \sqrt{-1}v_1 0,\ x + \sqrt{-1}v_2 0) \in \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM$ one has

$$\mathscr{D}\mathit{ist}_\pi^k(\mathscr{F})_{x(v_1,v_2)} = H^k(\pi^{-1}(x(v_1, v_2)) \to \{x(v_1, v_2)\}, \mathscr{F}_{x(v_1,v_2)})$$

by Proposition 2.3.5. On the other hand,

$$\pi^{-1}(x(v_1, v_2)) = \{\xi \in S^{n-1} \mid \langle v_1, \xi\rangle \leqq 0 \text{ and } \langle v_2, \xi\rangle \leqq 0\}.$$

Then, for $v_1 \neq -v_2$, $\pi^{-1}(x(v_1, v_2))$ is contractible to a point (see Figure 2.3.1(a)). Therefore one obtains

$$\mathscr{D}\mathit{ist}_\pi^k(\mathscr{F})_{x(v_1,v_2)} = 0 \quad \text{for } v_1 \neq -v_2.$$

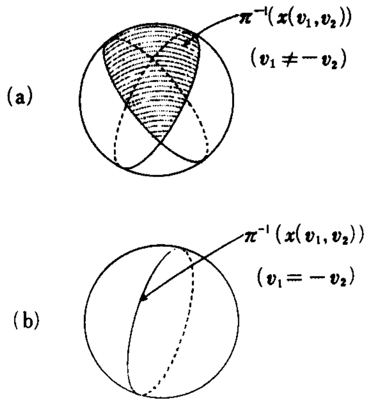

Figure 2.3.1

If $v_1 = -v_2$, one has $\pi^{-1}(x(v_1, v_2)) = S^{n-2}$ (see Figure 2.3.1(b)). Then

$$\mathscr{D}ist_\pi^k(\mathscr{F})_{x(v_1,v_2)} = \begin{cases} 0 & \text{for } k \neq n-1 \\ \mathscr{F}_{x(v_1,v_2)} \otimes \omega & \text{for } k = n-1 \end{cases}$$

by (2) of Proposition 2.3.6.

Proposition 2.3.10. *Let X, Y, and Z be topological spaces; let $f: X \to Y$ and $g: Y \to Z$ be continuous maps; and let $\mathscr{F}$ be a sheaf over Y. If f is purely l-codimensional with respect to $\mathscr{F}$, then there is induced the long exact sequence*

$$\begin{aligned} \cdots &\to R^{k-l} g_* \mathscr{D}ist_f^l(\mathscr{F}) \to R^k g_* \mathscr{F} \to R^k(g \circ f)_*(f^{-1}\mathscr{F}) \\ &\to R^{k-l+1} g_* \mathscr{D}ist_f^l(\mathscr{F}) \to \cdots. \end{aligned}$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{L}^\bullet$ and $0 \to f^{-1}\mathscr{F} \to \mathscr{M}^\bullet$ be flabby resolutions of $\mathscr{F}$ and $f^{-1}\mathscr{F}$ respectively. Let $\mathscr{N}^\bullet = C(\mathscr{L}^\bullet \to f_*\mathscr{M}^\bullet)$. Then one has the following:

$$\begin{cases} \mathscr{D}ist_f^k(\mathscr{F}) = \mathscr{H}^k(\mathscr{N}^\bullet) \\ R^k g_* \mathscr{F} = \mathscr{H}^k(g_* \mathscr{L}^\bullet) \\ R^k(g \circ f)_*(f^{-1}\mathscr{F}) = \mathscr{H}^k((g \circ f)_* \mathscr{M}^\bullet). \end{cases}$$

The exact sequence $0 \to f_*\mathscr{M}^{k-1} \to \mathscr{N}^k \to \mathscr{L}^k \to 0$ induces the exact sequence $0 \to g_* f_* \mathscr{M}^{k-1} \to g_* \mathscr{N}^k \to g_* \mathscr{L}^k \to 0$, since $f_*\mathscr{M}^{k-1}$ is flabby. Hence one obtains the long exact sequence

$$\cdots \to \mathscr{H}^k(g_* \mathscr{N}^\bullet) \to R^k g_* \mathscr{F} \to R^k(g \circ f)_*(f^{-1}\mathscr{F}) \to \mathscr{H}^{k+1}(g_* \mathscr{N}^\bullet) \to \cdots,$$

where one notes $g_* f_* \mathscr{M}^{k-1} = (g \circ f)_* \mathscr{M}^{k-1}$. One needs to prove the isomorphism

$$\mathscr{H}^k(g_* \mathscr{N}^\bullet) = R^{k-1} g_* \mathscr{D}\mathit{ist}_f^l(\mathscr{F})$$

under the hypothesis $\mathscr{D}\mathit{ist}_f^k(\mathscr{F}) = 0$ for $k \neq l$.

Since $\mathscr{H}^k(\mathscr{N}^\bullet) = 0$ for $k \neq l$, there exists an exact sequence

$$0 \to \mathscr{N}^0 \to \mathscr{N}^1 \to \cdots \to \mathscr{N}^{l-1} \to \mathscr{N}^l \to \mathscr{Z} \to 0,$$

where $\mathscr{Z} = \mathscr{N}^l/\mathrm{Im}(\mathscr{N}^{l-1} \to \mathscr{N}^l)$. Notice that $\mathscr{N}^i$ and $\mathscr{Z}$ are flabby. Therefore, the following sequence is also exact:

$$0 \to g_* \mathscr{N}^0 \to g_* \mathscr{N}^1 \to \cdots \to g_* \mathscr{N}^l \to g_* \mathscr{Z} \to 0.$$

This implies that $\mathscr{H}^k(g_* \mathscr{N}^\bullet) = 0 = R^{k-l} g_* \mathscr{D}\mathit{ist}_f^l(\mathscr{F})$ holds for $k < l$. Next, notice that

$$0 \to \mathscr{H}^l(\mathscr{N}^\bullet) \to \mathscr{Z} \to \mathscr{N}^{l+1} \to \cdots$$

gives a flabby resolution of $\mathscr{H}^l(\mathscr{N}^\bullet)$. For $k \geqq l + 2$ one has

$$R^{k-l} g_* \mathscr{H}^l(\mathscr{N}^\bullet) = \mathscr{H}^k(g_* \mathscr{N}^\bullet).$$

For $k = l + 1$ $\mathrm{Im}(g_* \mathscr{Z} \to g_* \mathscr{N}^{l+1}) = \mathrm{Im}(g_* \mathscr{N}^l \to g_* \mathscr{N}^{l+1})$ holds. Hence this case can be treated as the above. In the case where $k = l$, from the exact sequence

$$0 \to g_* \mathscr{H}^l(\mathscr{N}^\bullet) \to g_* \mathscr{Z} \to g_* \mathscr{N}^{l+1},$$

one obtains

$$\begin{aligned} R^0 g_* \mathscr{D}\mathit{ist}_f^l(\mathscr{F}) \cong g_* \mathscr{H}^l(\mathscr{N}^\bullet) &= \mathrm{Ker}(g_* \mathscr{Z} \to g_* \mathscr{N}^{l+1}) \\ &= \frac{\mathrm{Ker}(g_* \mathscr{N}^l \to g_* \mathscr{N}^{l+1})}{\mathrm{Im}(g_* \mathscr{N}^{l-1} \to g_* \mathscr{N}^l)} = \mathscr{H}^l(g_* \mathscr{N}^\bullet). \end{aligned}$$

Proposition 2.3.11. *Let X, Y, and Y' be topological spaces, and let $\mathscr{F}$ be a sheaf over X. Suppose that $f: X \to Y$, $g: Y' \to Y$, $f': X' = X \times_Y Y' \to Y'$, and $g': X' \to X$ are continuous maps such that the diagram*

$$\begin{tikzcd} X & X' = X \times_Y Y' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] & Y' \arrow[l, "g"] \end{tikzcd}$$

is commutative. Further assume that f and *f' are closed maps, and that all the open subsets of X or X' are paracompact. Then one has*

$$R^k f'_* g'^{-1} \mathscr{F} \cong g^{-1} R^k f_* \mathscr{F}.$$

Remark. In order to claim the isomorphism above, one may assume that four of those topological spaces are Hausdorff and that f is proper (hence f' is proper).

Proof. The corollary of Proposition 2.1.4 implies

$$(R^k f'_* g'^{-1}\mathscr{F})_{y'} = H^k(f'^{-1}(y'), g'^{-1}\mathscr{F}|_{f'^{-1}(y')}).$$

Let $y = g(y')$. Since g' is a bijection from $f'^{-1}(y')$ onto $f^{-1}(y)$, one obtains

$$\begin{aligned} H^k(f'^{-1}(y'), g'^{-1}\mathscr{F}|_{f'^{-1}(y')}) &= H^k(f^{-1}(y), \mathscr{F}|_{f^{-1}(y)}) = (R^k f_*\mathscr{F})_y \\ &= (g^{-1}R^k f_*\mathscr{F})_{y'} \end{aligned}$$

Proposition 2.3.12. *Consider the following commutative diagram:*

$$\begin{tikzcd}
 & & DM \underset{\sqrt{-1}S^*M}{\times} DM \arrow[dll, "\tau_1"'] \arrow[d, "\pi"] \arrow[drr, "\tau_2"] & & \\
DM \arrow[dd, "\pi"] \arrow[drr, "\tau"] & & \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM \arrow[ddll, "\tau_1"'] \arrow[ddrr, "\tau_2"] & & DM \arrow[dd, "\pi"] \arrow[dll, "\tau"'] \\
 & & \sqrt{-1}S^*M \arrow[dd, "\pi"] & & \\
\sqrt{-1}SM \arrow[drr, "\tau"'] & & & & \sqrt{-1}SM \arrow[dll, "\tau"] \\
 & & M & &
\end{tikzcd}$$

where τ and π are natural maps $\tau:\sqrt{-1}SM \to M$ and $\pi:\sqrt{-1}S^*M \to M$, respectively, and where τ_1 and τ_2 are projections on the first and the second components respectively. Let $\mathscr{F}$ be a sheaf on $\sqrt{-1}SM$. Then

(1) $$0 \to \tau^{-1}R^{n-2}\tau_*\mathscr{F} \to R^{n-2}(\pi\circ\tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{F}) \to \mathscr{F}^a \otimes \omega \to \tau^{-1}R^{n-1}\tau_*\mathscr{F} \to R^{n-1}(\pi\circ\tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{F}) \to 0$$

is an exact sequence.

(2) $$\tau^{-1}R^k\tau_*\mathscr{F} = R^k(\pi\circ\tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{F})$$

holds for $k \neq n-1, n-2$.

Proof. For the sheaf $\tau_1^{-1}\mathscr{F}$ on $\sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM$, apply Propositions 2.3.9 and 2.3.10 to $DM \underset{\sqrt{-1}S^*M}{\times} DM \xrightarrow{\pi} \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM \xrightarrow{\tau_2} \sqrt{-1}SM$. Then one obtains the exact sequence

$$\cdots \to R^{k-n+1}\tau_{2*}(\tau_1^{-1}\mathscr{F}|_{\Delta^a_{\sqrt{-1}SM}} \otimes \omega) \to R^k\tau_{2*}\tau^{-1}\mathscr{F} \to R^k(\tau_2\circ\pi)_*(\pi^{-1}\tau_1^{-1}\mathscr{F}) \to \cdots.$$

We will compute each term in this sequence. Notice that $\tau_i|_{\Delta^a_{\sqrt{-1}SM}}$, $i = 1$ and 2, are homeomorphisms onto $\sqrt{-1}SM$ and that $\tau_2 \circ \tau_1^{-1}$ is the antipodal mapping $a: \sqrt{-1}SM \to \sqrt{-1}SM$, defined by $a(x + \sqrt{-1}v0) = x - \sqrt{-1}v0$. Therefore one finds

$$R^{k-n+1}\tau_{2*}(\tau_1^{-1}\mathscr{F}|_{\Delta^a_{\sqrt{-1}SM}} \otimes \omega) = R^{k-n+1}\tau_{2*}(\tau_1^{-1}\mathscr{F}|_{\Delta^a_{\sqrt{-1}SM}}) \otimes \omega$$
$$= \begin{cases} 0 & \text{for } k \neq n-1 \\ \mathscr{F}^a \otimes \omega & \text{for } k = n-1, \end{cases}$$

where $\mathscr{F}^a \underset{\text{def}}{=} a_*\mathscr{F} = a^{-1}\mathscr{F}$.

From the commutative diagram

$$\begin{array}{ccc} \sqrt{-1}SM & \xleftarrow{\tau_1} & \sqrt{-1}SM \underset{M}{\times} \sqrt{-1}SM \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau_2} \\ M & \xleftarrow{\quad\tau\quad} & \sqrt{-1}SM, \end{array}$$

one obtains

$$R^k\tau_{2*}\tau_1^{-1}\mathscr{F} = \tau^{-1}R^k\tau_*\mathscr{F}$$

by Proposition 2.3.11. The commutativity of the diagram in Proposition 2.3.12 implies

$$R^k(\tau_2 \circ \pi)_*(\pi^{-1}\tau_1^{-1}\mathscr{F}) = R^k(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{F}).$$

Hence these complete the proof of (2).

Proposition 2.3.13.

(1) *The sequence*

$$0 \to \mathscr{Q} \to \tau^{-1}\pi_*\mathscr{C} \to \pi_*\tau^{-1}\mathscr{C} \to 0$$

is exact on $\sqrt{-1}SM$.

(2) $$R^k\pi_*\tau^{-1}\mathscr{C} = 0 \qquad \text{for } k \neq 0.$$

Proof. Replace $\mathscr{F}$ in Proposition 2.3.12 by $\mathscr{Q} = \mathscr{H}^1_{\sqrt{-1}SM}(\tau^{-1}\mathscr{O}_X)$). First compute $R^k(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{Q})$. Consider the commutative diagram

$$\begin{array}{ccc} DM & \xleftarrow{\tau_1} & DM \underset{\sqrt{-1}S^*M}{\times} DM \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau_2} \\ \sqrt{-1}S^*M & \xleftarrow{\quad\tau\quad} & DM \end{array}$$

Then, for the sheaf $\pi^{-1}\mathscr{Q}$ on DM, Propositions 2.3.11 and 2.1.2′ give

$$R^k\tau_{2*}(\tau_1^{-1}\pi^{-1}\mathscr{Q}) = \tau^{-1}R^k\tau_*\pi^{-1}\mathscr{Q} = \begin{cases} 0 & \text{for } k \neq n-1 \\ \tau^{-1}\mathscr{C}^a \otimes \omega & \text{for } k = n-1. \end{cases}$$

By the lemma in the proof of Proposition 2.3.8,

$$\begin{aligned} R^k(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{Q}) &= R^{k-n+1}\pi_* R^{n-1}\tau_{2*}(\tau_1^{-1}\pi^{-1}\mathscr{Q}) \\ &= R^{k-n+1}\pi_*\tau^{-1}\mathscr{C}^a \otimes \omega. \end{aligned}$$

In particular, one has

$$R^{n-2}(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{Q}) = 0$$

and

$$R^{n-1}(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{Q}) = \pi_*\tau^{-1}\mathscr{C}^a \otimes \omega.$$

On the other hand, from Proposition 2.3.8,

$$\tau^{-1}R^k\tau_*\mathscr{Q} = \begin{cases} 0 & \text{for } k \neq n-1 \\ \tau^{-1}\pi_*\mathscr{C}^a \otimes \omega & \text{for } k = n-1 \end{cases}.$$

Hence, (1) of Proposition 2.3.12 implies that

$$0 \to \mathscr{Q}^a \otimes \omega \to \tau^{-1}\pi_*\mathscr{C}^a \otimes \omega \to \pi_*\tau^{-1}\mathscr{C}^a \otimes \omega \to 0$$

is exact. Since $\omega \otimes \omega = \mathbf{Z}_M$, (1) of Proposition 2.3.13 is proved.

For $k \neq n-1$, $n-2$, one has the following from (2) of Proposition 2.3.12:

$$\tau^{-1}R^k\tau_*\mathscr{Q} = R^k(\pi \circ \tau_2)_*(\tau_1^{-1}\pi^{-1}\mathscr{Q}) = R^{k-n+1}\pi_*\tau^{-1}\mathscr{C}^a \otimes \omega.$$

Then, by Proposition 2.3.8, one obtains (2).

Proposition 2.3.14. *There is a canonical morphism $b: \tilde{\mathscr{A}} \to \tau^{-1}\mathscr{B}$ over $\sqrt{-1}SM$.*

Proof. From the exact sequence (1) of Proposition 2.3.12, there exists the morphism $\tilde{\mathscr{A}}^a \otimes \omega \to \tau^{-1}R^{n-1}\tau_*\tilde{\mathscr{A}}$ for $\mathscr{F} = \tilde{\mathscr{A}}$. Notations being the same as in Definition 2.1.2, recall the diagram

$$\begin{array}{ccc} \widetilde{^M X} & \xleftarrow{\tilde{\epsilon}} & \widetilde{^M X} - \sqrt{-1}SM \\ \Big\downarrow{\scriptstyle \tau} & & \Big\| \\ X & \xleftarrow{\epsilon} & X - M \end{array}$$

and recall the fact that $R^k\tilde{\epsilon}_*(\mathscr{O}_X|_{X-M}) = 0$ for $k \neq 0$. Then one obtains

$$\begin{aligned} R^{n-1}\tau_*\tilde{\mathscr{A}} &= R^{n-1}\tau_*(\tilde{\epsilon}_*\mathscr{O}_X|_{X-M}) = R^{n-1}(\tau \circ \tilde{\epsilon})_*(\mathscr{O}_X|_{X-M}) \\ &= R^{n-1}\epsilon_*(\mathscr{O}_X|_{X-M}). \end{aligned}$$

There is induced a canonical morphism $R^{n-1}\epsilon_*(\mathscr{O}_X|_{X-M}) \to \mathscr{H}^n_M(\mathscr{O}_X)$ from taking the direct limit of the sequence

$$\begin{aligned} \cdots &\to H^{n-1}(U, \mathscr{O}_X) \to H^{n-1}(U - M, \mathscr{O}_X) \to H^n_{U \cap M}(U, \mathscr{O}_X) \\ &\to H^n(U, \mathscr{O}_X) \to \cdots; \end{aligned}$$

furthermore, this morphism is an isomorphism for $n > 1$. Since $\tau^{-1}R^{n-1}\tau_*\tilde{\mathscr{A}} = \tau^{-1}R^{n-1}\epsilon_*(\mathcal{O}_X|_{X-M})$, one obtains the canonical morphism: $\tilde{\mathscr{A}}^a \otimes \omega \to \tau^{-1}\mathscr{H}^n_M(\mathcal{O}_X)$ by composing the above-constructed morphisms. Consequently, the functor ${}^a \otimes \omega$ induces

$$b : \tilde{\mathscr{A}} \to \tau^{-1}\mathscr{B}.$$

Remark. Proposition 2.3.14 asserts that a boundary value of a holomorphic function defines a hyperfunction. The terminology, "a boundary value of a holomorphic function," always has the connotation as that in Proposition 2.3.14. Note that, in spite of the wording, the (pointwise) "value" does not necessarily exist.

Theorem 2.3.3. *The sequence on* $\sqrt{-1}SM$

$$0 \to \tilde{\mathscr{A}} \xrightarrow{b} \tau^{-1}\mathscr{B} \to \pi_*\tau^{-1}\mathscr{C} \to 0$$

is exact.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau^{-1}\mathscr{A} & = & \tau^{-1}\mathscr{A} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\mathscr{A}} & \xrightarrow{b} & \tau^{-1}\mathscr{B} & \longrightarrow & \pi_*\tau^{-1}\mathscr{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{Q} & \longrightarrow & \tau^{-1}\pi_*\mathscr{C} & \longrightarrow & \pi_*\tau^{-1}\mathscr{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In this diagram, all the vertical sequences and the first and the third horizontal sequences are exact as the result of Proposition 2.1.2, Theorem 2.3.2, and Proposition 2.3.13. Notice that the third vertical exact sequence makes the second horizontal sequence a cochain complex. Now, Nine Lemma implies that the second horizontal sequence, $0 \to \tilde{\mathscr{A}} \xrightarrow{b} \tau^{-1}\mathscr{B} \to \pi_*\tau^{-1}\mathscr{C} \to 0$, is exact.

We shall investigate the interplay of the singularity spectrum of a hyperfunction and the boundary value of the hyperfunction (Theorem 2.3.4).

Definition 2.3.4. *A subset Z of $\sqrt{-1}SM$ is said to be convex if each fibre $\tau^{-1}(x) \cap Z$ of $\tau : \sqrt{-1}SM \to M$ is convex.*

Note. A proper subset A of S^{n-1} is said to be convex if the inverse image of A under the projection $\tilde{\omega} : \mathbf{R}^n - \{0\} \to S^{n-1} \cong (\mathbf{R}^n - \{0\})/\mathbf{R}^\times_+$ is convex. Note that S^{n-1} is itself convex by definition.

Similarly, a subset Z of $\sqrt{-1}SM$ is said to be *properly convex* if each fibre $\tau^{-1}(x) \cap Z$ is properly convex, where a convex subset A of S^{n-1} is called properly convex if the inverse image of A under the map $\varpi: \mathbf{R}^n - \{0\} \to S^{n-1}$ does not contain a line, $\cong \mathbf{R}$. A properly convex set is sometimes called a convex set without containing a line.

We similarly define a (properly) convex subset of $\sqrt{-1}S^*M$ as we defined the (properly) convex subset of $\sqrt{-1}SM$. The smallest convex set containing Z (the intersection of convex sets containing Z) is called the *convex hull* of Z.

Definition 2.3.5. *The polar set Z° of a subset $Z \subset \sqrt{-1}SM$ is defined as $Z^\circ = \{(x, \sqrt{-1}\xi_x\infty) \in \sqrt{-1}S^*M | \langle \xi_x, v_x \rangle > 0$ for an arbitary point $x + \sqrt{-1}v_x 0 \in Z\}$. The polar set of a subset of $\sqrt{-1}S^*M$ is defined similarly.*

Remark. The polar set Z° is always convex for any Z. Notice also that the correspondence between open convex sets U in $\sqrt{-1}SM$ and polar sets U° in $\sqrt{-1}S^*M$ is injective. Furthermore, any closed convex set Z can be expressed as the polar set U° of an open convex set U in $\sqrt{-1}SM$. Note that $U^{\circ\circ}$ is the convex hull of U.

The polar set of a convex subset A of $\sqrt{-1}SM$ (or of $\sqrt{-1}S^*M$) is non-empty if and only if A is properly convex.

Theorem 2.3.4. *Let U be an open set in $\sqrt{-1}SM$ and let $U \cap \tau^{-1}(x)$ be a non-empty connected set for each $x \in M$. Then*

(1) *the restriction map $\Gamma(V, \tilde{\mathscr{A}}) \to \Gamma(U, \tilde{\mathscr{A}})$ is bijective, where $V = U^{\circ\circ}$, and*
(2) *the sequence $0 \to \tilde{\mathscr{A}}(U) \xrightarrow{b} \mathscr{B}(M) \xrightarrow{\mathrm{sp}} \mathscr{C}(\sqrt{-1}S^*M - U^\circ)$ is exact. In other words, for $\varphi \in \Gamma(U, \tilde{\mathscr{A}})$ one has S.S. $b(\varphi) \subset U^\circ$; and, if S.S. $u \subset U^\circ$ for $u \in \mathscr{B}(M)$, then $u = b(\varphi)$ for a unique $\varphi \in \Gamma(U, \tilde{\mathscr{A}})$.*

Proof. Since the sequence $0 \to \tilde{\mathscr{A}} \to \tau^{-1}\mathscr{B} \to \pi_*\tau^{-1}\mathscr{C} \to 0$ is exact, from Theorem 2.3.3, the following commutative diagram is obtained:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{\mathscr{A}}(V) & \longrightarrow & \tau^{-1}\mathscr{B}(V) & \longrightarrow & \pi_*\tau^{-1}\mathscr{C}(V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{\mathscr{A}}(U) & \longrightarrow & \tau^{-1}\mathscr{B}(U) & \longrightarrow & \pi_*\tau^{-1}\mathscr{C}(U)
\end{array}$$

By the assumption, $\tau|_U$ and $\tau|_V$ are surjective open maps and the fibres are connected. Then one has

$$\tau^{-1}\mathscr{B}(V) = \tau^{-1}\mathscr{B}(U) = \mathscr{B}(M)$$

by (2) of Proposition 2.1.3. Hence the above diagram becomes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{\mathscr{A}}(V) & \longrightarrow & \mathscr{B}(M) & \longrightarrow & \pi_*\tau^{-1}\mathscr{C}(V) \\
 & & \downarrow & & \| & & \downarrow \\
0 & \longrightarrow & \tilde{\mathscr{A}}(U) & \longrightarrow & \mathscr{B}(M) & \longrightarrow & \pi_*\tau^{-1}\mathscr{C}(U)
\end{array}$$

This implies that the restriction $\tilde{\mathscr{A}}(V) \to \tilde{\mathscr{A}}(U)$ is a monomorphism. In order to show this restriction map is also onto, it is sufficient to prove that $\pi_*\tau^{-1}\mathscr{C}(V) \to \pi_*\tau^{-1}\mathscr{C}(U)$ is injective. Since the map

$$\tau:\pi^{-1}V \to \tau\pi^{-1}V = \sqrt{-1}S^*M - V^\circ = \sqrt{-1}S^*M - U^\circ$$

is surjective and open, and since each fibre is connected, (2) of Proposition 2.1.3 implies

$$\pi_*\tau^{-1}\mathscr{C}(V) = (\tau^{-1}\mathscr{C})(\pi^{-1}V) = \mathscr{C}(\tau\pi^{-1}V) = \mathscr{C}(\sqrt{-1}S^*M - U^\circ).$$

On the other hand, $\tau:\pi^{-1}U \to (\sqrt{-1}S^*M - U^\circ)$ is an epimorphism. Then it is plain that

$$\mathscr{C}(\sqrt{-1}S^*M - U^\circ) \to (\tau^{-1}\mathscr{C})(\pi^{-1}U) = \pi_*\tau^{-1}\mathscr{C}(U)$$

is a monomorphism. This completes the proof of (1), and also yields the exact sequence

$$0 \to \tilde{\mathscr{A}}(U) \to \mathscr{B}(M) \to \mathscr{C}(\sqrt{-1}S^*M - U^\circ).$$

Remark 1. Since the map $\tau:\sqrt{-1}SM \to M$ is smooth (the induced map by τ on the tangent space is surjective), locally τ is a projection. Consequently, it is an open map. Therefore, if each fibre is connected (including the case of the empty set), then statement (2) of Theorem 2.3.4 can be rephrased as the sequence

$$0 \to \tilde{\mathscr{A}}(U) \xrightarrow{b} \mathscr{B}(\tau(U)) \xrightarrow{\mathrm{sp}} \mathscr{C}(\sqrt{-1}S^*M - U^\circ)$$

is exact.

Remark 2. We mention the following fact without proof: for an arbitrary open convex set V in $\sqrt{-1}SM$,

$$H^k(V, \tilde{\mathscr{A}}) = 0 \qquad \text{for } k \neq 0.$$

Hence, this fact and (1) of Theorem 2.3.4 indicate that the convex sets in $\sqrt{-1}SM$ play the role of Stein manifolds for the theory of several complex variables.

We need the following definition to rewrite Theorem 2.3.4 into a more applicable form.

Definition 2.3.6. *Let D be an open set in $X—M$. The open set D is said to be a conoidal neighborhood of $x_0 + \sqrt{-1}v0$ (of $U \subset \sqrt{-1}SM$) if $D \cup \sqrt{-1}SM$ is a neighborhood of $x_0 + \sqrt{-1}v0$ (of U). Denote the boundary value of $\varphi \in \mathscr{O}_X(D)$ by $b_D(\varphi)$. We denote the hyperfunction corresponding to $\varphi \in \tilde{\mathscr{A}}(U)$ by $b_U(\varphi)$ provided that each fibre of U is connected. We also write $b(\varphi; D)$ and $b(\varphi; U)$ instead of $b_D(\varphi)$ and $b_U(\varphi)$ respectively, or even $b(\varphi)$ when there is no fear of confusion.*

Note that

$$\tilde{\mathscr{A}}(U) = \varinjlim_{\substack{D \text{ runs through} \\ \text{the set of conoidal} \\ \text{neighborhoods of } U}} \mathcal{O}_X(D).$$

This and Definition 2.3.6 imply the following theorem.

Theorem 2.3.5. *Let M be a real analytic manifold and X be a complexification of M. Let D be an open set in X—M and U be an open set in $\sqrt{-1}SM$ such that each fibre is connected. If D is a conoidal neighborhood of U, then the boundary value of $f(z) \in \mathcal{O}_X(D)$ determines a hyperfunction $f(x) \in \mathscr{B}_M(\tau(U))$ uniquely and such that the singularity spectrum* S.S. $f(x)$ *is contained in the polar set $U°$. Conversely, if the singularity spectrum* S.S. $u(x)$ *of a hyperfunction $u(x)$ on M is contained in a closed convex set Z in $\sqrt{-1}S^*M$, then there exists a conoidal neighborhood D of the polar set $Z°$ such that $u(x) = b_D(f(z))$ for some $f(z) \in \mathcal{O}_X(D)$.*

It is clear that Theorem 2.4.4 implies Theorem 2.3.5. Notice that the larger U is, the smaller the polar set $U°$ becomes. Therefore, Theorem 2.3.5 says that if a hyperfunction is defined by a boundary value from the larger U, then the singularity spectrum is smaller. In particular, if the convex hull of U is the whole $\sqrt{-1}SM$, then the corresponding hyperfunction is a real analytic function, provided $n \neq 1$ (if $n = 1$, fibres cannot be connected). In that case, we have $U° = (\sqrt{-1}SM)° = \varnothing$ which implies S.S. $u = \varnothing$.

Since the sequence

$$\pi^{-1}\mathscr{B} \xrightarrow{\text{sp}} \mathscr{C} \to 0$$

is exact, from Proposition 2.1.6, we have the exact sequence

$$0 \to \mathscr{A}^* \to \pi^{-1}\mathscr{B} \xrightarrow{\text{sp}} \mathscr{C} \to 0,$$

where the sheaf $\mathscr{A}^*$ on $\sqrt{-1}S^*M$ is defined as $\mathscr{A}^* = \mathrm{Ker}(\pi^{-1}\mathscr{B} \to \mathscr{C})$. Then $u \in \mathscr{A}^*_{(x,\sqrt{-1}\xi\infty)}$ can be expressed as

$$u = \sum_j b(\varphi_j),$$

where $\varphi_j \in \Gamma(U_j, \tilde{\mathscr{A}})$ and $(x, \sqrt{-1}\xi\infty) \notin U_j°$. In this case, u is said to be *micro-analytic* at $(x, \sqrt{-1}\xi\infty)$.

§4. Examples

In this section we will denote the imaginary unit $\sqrt{-1}$ by i. For $\varphi(z) \in \tilde{\mathscr{A}}_{x+i0}$ we denote $b(\varphi(z))$ by $\varphi(x + i0)$.

Example 2.4.1.

$$\text{S.S.}(x+i0)^\lambda = \begin{cases} \{(0, i\,dx\infty)\} & \text{for } \lambda \neq 0, 1, \ldots \\ \varnothing & \text{for } \lambda = 0, 1, 2, \ldots \end{cases}$$

holds.

Since $(x + i0)^\lambda$ is real-analytic at each $x \neq 0$, we will focus on the fibre at $x = 0$. On the other hand, if $\lambda = 0, 1, 2, \ldots$, then $(x + i0)^\lambda$ is real-analytic at $x = 0$ as well. Therefore $\text{S.S.}(x + i0)^\lambda = \varnothing$ for $\lambda = 0, 1, 2, \ldots$. Notice that $(x + i0)^\lambda$ is not real-analytic at $x = 0$ if λ is neither a positive integer nor zero. Hence, the singularity spectrum is not an empty set and is contained in $\{(0, i\,dx\infty), (0, -i\,dx\infty)\}$. But, by Theorem 2.3.4, $(0, -i\,dx\infty) \notin \text{S.S.}(x + i0)^\lambda$ implies $\text{S.S.}(x + i0)^\lambda = \{(0, i\,dx\infty)\}$.

When $(x+i0)^\lambda$ is regarded as a microfunction, i.e. if we consider $\text{sp}(x + i0)^\lambda$, it has zeros of order one at $\lambda = 0, 1, 2, \ldots$, (see §8, Chapter III, for hyperfunctions [microfunctions] with holomorphic parameters). Since x^n is zero as a microfunction,

$$\begin{aligned} \lim_{\lambda\to n} \frac{(x+i0)^\lambda}{\lambda - n} &= \lim_{\lambda\to n} \frac{(x+i0)^\lambda - x^n}{\lambda - n} = \frac{d}{d\lambda}(x+i0)^\lambda\Big|_{\lambda=n} \\ &= (x+i0)^\lambda \log(x+i0)\big|_{\lambda=n} \\ &= (x+i0)^n \log(x+i0) = x^n \log(x+i0). \end{aligned}$$

This is the boundary value of $z^n \log z$ from the upper half-plane, and it cannot be analytically continued beyond $z = 0$. Therefore, $x^n \log(x + i0)$ is not zero as a microfunction. That is, $(x + i0)^\lambda$ has zeros of order one at $\lambda = n$, non-negative integers. On the other hand, the gamma function $\Gamma(-\lambda)$ is never zero and has poles of order one at $\lambda = n$, non-negative integers. Therefore, for any λ, $\Gamma(-\lambda)(x + i0)^\lambda$ is not zero as a microfunction. Consequently,

$$\text{S.S. } \Gamma(-\lambda)(x+i0)^\lambda = \{(0, i\,dx\infty)\}$$

holds for an arbitrary λ. The value of $\Gamma(-\lambda)(x + i0)^\lambda$ at $\lambda = n \in \mathbf{Z}_+$, non-negative integers, can be computed as follows (note that, since $\Gamma(-\lambda)(x + i0)^\lambda$ is holomorphic in λ, the restriction $\lambda = n$ makes sense):

$$\begin{aligned} \Gamma(-\lambda)(x+i0)^\lambda\big|_{\lambda=n} &= \frac{(x+i0)^\lambda}{\lambda - n}\bigg|_{\lambda=n} (\lambda - n)\Gamma(-\lambda)\big|_{\lambda=n} \\ &= \frac{(-1)^{n-1}}{n!} x^n \log(x+i0). \end{aligned}$$

Example 2.4.2. Define

$$x_+^\lambda = \begin{cases} x^\lambda & \text{for } x > 0 \\ 0 & \text{for } x < 0, \end{cases} \tag{2.4.1}$$

where Re $\lambda > 0$. Then this is a well-defined continuous function. Therefore it is well defined as a hyperfunction, since distributions are hyperfunctions. For a proof, see the lecture notes by Komatsu mentioned in the Introduction.) We will express x_+^λ explicitly as boundary values of homomorphic functions. When λ is not an integer, define

$$x_+^\lambda = \frac{1}{e^{-\pi i\lambda} - e^{\pi i\lambda}}\{e^{-\pi i\lambda}(x+i0)^\lambda - e^{\pi i\lambda}(x-i0)^\lambda\}$$

$$= \frac{1}{-2i\sin\pi\lambda}\{e^{-\pi i\lambda}(x+i0)^\lambda - e^{\pi i\lambda}(x-i0)^\lambda\}. \tag{2.4.2}$$

Note that the right-hand side is well defined provided $\sin\pi\lambda \neq 0$ by Example 2.4.1. If Re $\lambda > 0$, the hyperfunction defined as (2.4.2) coincides with the continuous function defined in (2.4.1).

Proof. If $x > 0$, then $(x+i0)^\lambda = (x-i0)^\lambda = x^\lambda$, which is the case in (2.4.1). If $x < 0$, then $(x+i0)^\lambda = |x|^\lambda e^{\pi i\lambda}$ and $(x-i0)^\lambda = |x|^\lambda e^{-\pi i\lambda}$. This implies $e^{-\pi i\lambda}(x+i0) - e^{\pi i\lambda}(x-i0) = |x|^\lambda - |x|^\lambda = 0$. Therefore we can define the hyperfunction x_+^λ by (2.4.2). The hyperfunction defined as (2.4.2) has the following properties:

(1) for $\lambda = 0, 1, 2, \ldots$, it is well defined

and

(2) it has a pole of order one at $\lambda = -1, -2, \ldots$.

Suppose $\lambda = n \geqq 0$; then one has

$$e^{-\pi in}(x+i0)^n - e^{\pi in}(x-i0)^n = (-1)^n(x^n - x^n) = 0.$$

On the other hand, $1/(\sin\pi\lambda)$ has only a pole of order one at $\lambda = n$; i.e. (2.4.2) is well defined at $\lambda = n$. (Strictly speaking, we need the theory found in §6 of Chapter III to justify this claim. However, the following calculation guarantees it here.) Let $\lambda = -n < 0$; then

$$(e^{-\pi i\lambda}(x+i0)^\lambda - e^{\pi i\lambda}(x-i0)^\lambda)|_{\lambda=-n} = (-1)^n\left\{\frac{1}{(x+i0)^n} - \frac{1}{(x-i0)^n}\right\} \neq 0,$$

which implies that x_+^λ has a pole of order one.

For each $\lambda = 0, 1, \ldots$, as we have seen, x_+^λ defines a hyperfunction. The hyperfunction corresponding to $\lambda = 0$ is called the Heaviside function, denoted by $Y(x)$; i.e.

$$Y(x) = x_+^\lambda|_{\lambda=0} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

(The symbol $\theta(x)$ is also used to denote the Heaviside function.)

Furthermore,

$$Y(x) = \frac{-2\pi i + \log(x + i0) - \log(x - i0)}{-2\pi i},$$

where the branch of log is taken as $\log(1 + i0) = \log(1 - i0) = 0$. We have

$$x_+^\lambda = \frac{e^{-\pi i\lambda}(x + i0)^\lambda - e^{\pi i\lambda}(x - i0)^\lambda}{-2i \sin \pi\lambda},$$

and

$$\left.\frac{e^{-\pi i\lambda}[(x + i0)^\lambda - 1]}{-2i \sin \pi\lambda}\right|_{\lambda=0} = \frac{e^{-\pi i\lambda}\lambda}{-2i \sin \pi\lambda} \cdot \left.\frac{[(x + i0)^\lambda - 1]}{\lambda}\right|_{\lambda=0}$$
$$= \frac{1}{-2\pi i} \log(x + i0).$$

The other term can be rewritten similarly. Therefore, as a hyperfunction,

$$x_+^\lambda|_{\lambda=0} = \frac{-2\pi i + \log(x + i0) - \log(x - i0)}{-2\pi i}.$$

Similarly, we obtain

$$Y(x) = \frac{\log(-x + i0) - \log(-x - i0)}{2\pi i}.$$

If $x > 0$, then $\log(-x + i0) = \log x + \pi i$ and $\log(-x - i0) = \log x - \pi i$; i.e. the right-hand side of the above equation is 1. If $x < 0$, $\log(-x + i0) = \log(-x - i0) = \log(-x)$. Then the right-hand side is zero. Note also that

$$x_+^\lambda|_{\lambda=n} = x^n \cdot x_+^{\lambda-n}|_{\lambda=n} = x^n Y(x).$$

First we consider $x_+^\lambda/(\Gamma(1 + \lambda))$ in order to study the structure of x_+^λ for $\lambda = -1, -2, \ldots$. By the functional equation $\Gamma(1 + \lambda)\Gamma(-\lambda) = \pi/(-\sin \pi\lambda)$,

$$\left.\frac{x_+^\lambda}{\Gamma(1 + \lambda)}\right|_{\lambda=-n} = \left.\frac{e^{-\pi i\lambda}(x + i0)^\lambda - e^{\pi i\lambda}(x - i0)^\lambda}{-2i \sin \pi\lambda \cdot \Gamma(1 + \lambda)}\right|_{\lambda=-n}$$
$$= \frac{\Gamma(-\lambda)}{2\pi i} \{e^{-\pi i\lambda}(x + i0)^\lambda - e^{\pi i\lambda}(x - i0)^\lambda\}|_{\lambda=-n}$$
$$= \frac{(-1)^n(n - 1)!}{2\pi i} \left\{\frac{1}{(x + i0)^n} - \frac{1}{(x - i0)^n}\right\} = D_x^{n-1}\delta(x),$$

where the hyperfunction $\delta(x)$ is defined as $\delta(x) = (1/2\pi i)(1/(x - i0) - 1/(x + i0))$ and is called the Dirac δ-function.

Proof. Since $D_x x_+^\lambda = \lambda x_+^{\lambda-1}$,

$$D_x \frac{x_+^\lambda}{\Gamma(1+\lambda)} = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}.$$

On the other hand, we have

$$\delta(x) = \left.\frac{x_+^\lambda}{\Gamma(1+\lambda)}\right|_{\lambda=-1}$$

by definition. Therefore,

$$D_x\delta(x) = \left.\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right|_{\lambda=-1} = \left.\frac{x_+^\lambda}{\Gamma(1+\lambda)}\right|_{\lambda=-2}.$$

Repeating the above, we obtain

$$D_x^{n-1}\delta(x) = \left.\frac{x_+^\lambda}{\Gamma(1+\lambda)}\right|_{\lambda=-n}.$$

Remark 1. We have supp $\delta(x) = \{x \in \mathbf{R} \,|\, x = 0\}$. That is, the δ-function is not a function in Dirichlet's sense (i.e. for each x there is a corresponding value). Dirac's introduction of δ-function in quantum mechanics, where the δ-function was used very effectively, gave an impetus to the theory of distributions and led to hyperfunction theory.

Remark 2. Let $\mathscr{D}$ be the sheaf of differential operators with coefficients in holomorphic functions. Then $P \in \mathscr{D}$ defines a homomorphism: $\mathcal{O}_X \to \mathcal{O}_X$. Therefore there is induced a homomorphism

$$\mathscr{B}_M = \mathscr{H}_M^n(\mathcal{O}_X) \otimes \omega_M \xrightarrow{P} \mathscr{H}_M^n(\mathcal{O}_X) \otimes \omega_M.$$

In particular, a holomorphic function acts on $\mathscr{B}$ by the multiplication; e.g. $x\delta(x) = 0$. In general, for $u = \sum_j b(\varphi_j) \in \mathscr{B}$, $Pu = \sum_j b(P\varphi_j)$.

We summarize what we have discussed as the following definition.

Definition 2.4.1.

(1) $x_+^\lambda = \dfrac{1}{-2i \sin \pi\lambda}\{e^{-\pi i\lambda}(x+i0)^\lambda - e^{\pi i\lambda}(x-i0)^\lambda\}$.

(2) *The Heaviside function* $Y(x) = x_+^\lambda|_{\lambda=0} = \dfrac{\log(-x+i0) - \log(-x-i0)}{2\pi i}$.

(3) *The Dirac δ-function of one variable* $\delta(x) = \dfrac{1}{2\pi i}\left(\dfrac{1}{x-i0} - \dfrac{1}{x+i0}\right)$.

(4) *The Dirac δ-function of several variables*

$$\delta(x) = \delta(x_1, \ldots, x_n) = \delta(x_1) \cdots \delta(x_n)$$
$$= \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + i\epsilon_1 0) \cdots (x_n + i\epsilon_n 0)},$$

where $1/((x_1 + i\epsilon_1 0) \cdots (x_n + i\epsilon_n 0))$ *is the boundary value of* $1/(z_1 \cdots z_n)$ *from* $\epsilon_i \operatorname{Im} z_i > 0$, $1 \leqq i \leqq n$.

Remark. For a real analytic function $a(x)$, we have $a(x)\delta(x) = a(0)\delta(x)$. That is, $x_1\delta(x) = \cdots = x_n\delta(x) = 0$ holds. *Proof*:

$$x_1\delta(x) = \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0)}$$
$$= \frac{1}{(2\pi i)^n} \left[\sum_{\epsilon_2, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_2 \cdots \epsilon_n}{(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0)} \right.$$
$$\left. - \sum_{\epsilon_2, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_2 \cdots \epsilon_n}{(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0)} \right] = 0.$$

Proposition 2.4.1. *Let $\delta(x)$ be the δ-function on $\mathbf{R}^n$, and let $GL(n, \mathbf{R})$ be the general linear group of degree n over $\mathbf{R}$ acting on $\mathbf{R}^n$. Then we have the following:*

(1) $\delta(gx) = \dfrac{1}{|\det g|}\, \delta(x)$, *where $x \in \mathbf{R}^n$ and $g \in GL(n, \mathbf{R})$.*

(2) S.S. $\delta(x) = \pi^{-1}(0) = \{(0, \sqrt{-1}\langle \xi, dx \rangle \infty) \mid \xi \in \mathbf{R}^n - \{0\}\}$, *where we often use a representative $\xi \in \mathbf{R}^n - \{0\}$ for a point in S^{n-1}.*

(3) $$\delta(x) = \frac{1}{(2\pi i)^n} \sum_{k=0}^{n} \frac{|\xi_0 \wedge \cdots \wedge \xi_{k-1} \wedge \xi_{k+1} \wedge \cdots \wedge \xi_n|}{(\langle x, \xi_0 \rangle + \sqrt{-1}0) \cdots (\langle x, \xi_{k-1} \rangle + \sqrt{-1}0)}$$
$$\times \frac{1}{(\langle x, \xi_{k+1} \rangle + \sqrt{-1}0) \cdots (\langle x, \xi_n \rangle + \sqrt{-1}0)},$$

where $\xi_0 + \cdots + \xi_n = 0$ such that any n vectors of those $\xi_0, \ldots, \xi_n$ are linearly independent, and $|\xi_1 \wedge \cdots \wedge \xi_n| \underset{\text{def}}{=} |\det(\xi_1, \ldots, \xi_n)|$.

Remark 1. From (1) above, $\delta(x)|dx_1 \wedge \cdots \wedge dx_n|$ is invariant under the action of $GL(n, \mathbf{R})$. Furthermore, if F is a real analytic map from a neighborhood of 0 to a neighborhood of 0 such that $F(0) = 0$ and $dF(0)$ is nondegenerate, one has

$$\delta(F(x)) = \frac{1}{|\det dF(0)|}\, \delta(x).$$

Generally, for a real analytic isomorphism $f: M_1 \to M_2$ there is induced a morphism $\mathscr{B}_{M_2} \to \mathscr{B}_{M_1}$ as follows: for $u(x_2) = \sum_j b(\varphi_j) \in \mathscr{B}_{M_2}$ there corresponds $u(f(x_1)) = \sum_j b(\varphi_j(f(x_1)) \in \mathscr{B}_{M_1}$. See Theorem 3.1.7.

Remark 2. Since $\delta(x)$ is a hyperfunction of n variables, the general theory in Chapter I guarantees that it can be expressed as a sum of boundary values of $(n+1)$ holomorphic functions. The above Proposition 2.4.1 (3) realizes this statement concretely.

Proof of Proposition 2.4.1. Since $GL(n, \mathbf{R})$ is generated by diagonal matrices and matrices of the type

$$\begin{pmatrix} 1 & & \\ & \cdot\lambda & \\ & \ddots & \\ & & 1 \end{pmatrix},$$

it is sufficient to prove (1) for these matrices. First one has

$$\delta(a_1x_1, \ldots, a_nx_n) = \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(a_1x_1 + i\epsilon_1 0) \cdots (a_nx_n + i\epsilon_n 0)},$$

where $(\epsilon_1 \cdots \epsilon_n)/((a_1x_1 + i\epsilon_1 0) \cdots (a_nx_n + i\epsilon_n 0))$ is the boundary value of the holomorphic function $1/((a_1z_1) \cdots (a_nz_n))$ restricted to $\{\epsilon_i \operatorname{Im} a_iz_i > 0, 1 \leqq i \leqq n\}$, i.e. the boundary value from $(\epsilon_i \operatorname{sgn} a_i) \operatorname{Im} z_i > 0$, $1 \leqq i \leqq n$, of $(1/(a_1 \cdots a_n)) \cdot (1/(z_1 \cdots z_n))$. Then

$$\frac{1}{(a_1x_1 + i\epsilon_1 0) \cdots (a_nx_n + i\epsilon_n 0)}$$
$$= \frac{1}{a_1 \cdots a_n} \cdot \frac{1}{(x_1 + i\epsilon_1 \operatorname{sgn} a_1 0) \cdots (x_n + i\epsilon_n \operatorname{sgn} a_n 0)}.$$

Let $\epsilon_i' = \epsilon_i \operatorname{sgn} a_i$. Consequently, one obtains

$$\delta(a_1x_1, \ldots, a_nx_n)$$
$$= \frac{1}{(-2\pi i)^n} \cdot \frac{(\operatorname{sgn} a_1) \cdots (\operatorname{sgn} a_n)}{a_1 \cdots a_n} \sum_{\epsilon_1', \ldots, \epsilon_n' = \pm 1} \frac{\epsilon_1' \cdots \epsilon_n'}{(x_1 + i\epsilon_1' 0) \cdots (x_n + i\epsilon_n' 0)}$$
$$= \left| \frac{1}{a_1 \cdots a_n} \right| \delta(x).$$

Next, when

$$g = \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix},$$

one has

$$\delta(g(x)) = \delta(x_1 + x_2, x_2, \ldots, x_n)$$
$$= \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + x_2 + i\epsilon_1 0)(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0)}.$$

Then

$$\delta(x_1 + x_2, x_2, \ldots, x_n) - \delta(x_1, \ldots, x_n)$$
$$= \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \epsilon_1 \cdots \epsilon_n \times \left\{ \frac{1}{(x_1 + x_2 + i\epsilon_1 0)(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0)} - \frac{1}{(x_1 + i\epsilon_1 0) \cdots (x_n + i\epsilon_n 0)} \right\}.$$

where, for each $\epsilon_1, \ldots, \epsilon_n$, $1/((x_1 + x_2 + i\epsilon_1 0)(x_2 + i\epsilon_2 0) \cdots (x_n + i\epsilon_n 0))$ is the boundary value from $\{\epsilon_1 \operatorname{Im}(z_1 + z_2) > 0$ and $\epsilon_i \operatorname{Im} z_i > 0$ for $i = 2, \ldots, n\}$, and $1/((x_1 + i\epsilon_1 0) \cdots (x_n + i\epsilon_n 0))$ is the boundary value from $\{\epsilon_i \operatorname{Im} z_i > 0$ if $i = 1, 2, \ldots, n\}$. Therefore, the difference of the above boundary values is the boundary value of

$$\frac{1}{(z_1 + z_2)z_2 \cdots z_n} - \frac{1}{z_1 \cdots z_n} = \frac{-1}{(z_1 + z_2)z_2 \cdots z_n}$$

from their intersection $\{\epsilon_1 \operatorname{Im}(z_1 + z_2) > 0$ and $\epsilon_i \operatorname{Im} z_i > 0$ for $i = 1, 2, \ldots, n\}$. If $-1/((x_1 + x_2 + i\epsilon_1 0)(x_1 + i\epsilon_1 0)(x_3 + i\epsilon_3 0) \cdots (x_n + i\epsilon_n 0))$ denotes the boundary value from $\{\epsilon_1 \operatorname{Im}(z_1 + z_2) > 0, \epsilon_1 \operatorname{Im} z_1 > 0$ and $\epsilon_i \operatorname{Im} z_i > 0$ for $i = 3, 4, \ldots, n\}$, we obtain

$$\delta(x_1 + x_2, x_2, \ldots, x_n) - \delta(x_1, \ldots, x_n)$$
$$= \frac{1}{(-2\pi i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + x_2 + i\epsilon_1 0)(x_1 + i\epsilon_1 0)(x_3 + i\epsilon_3 0) \cdots (x_n + i\epsilon_n 0)}$$
$$= 0$$

The general case where

$$g = \begin{pmatrix} 1 & & \lambda \\ & \ddots & \\ & & 1 \end{pmatrix}$$

can be done similarly.

We will prove (2). Since $x_1\delta(x) = \cdots = x_n\delta(x) = 0$, $\operatorname{supp} \delta(x) = \{0\}$. Therefore, since $\delta(x)$ cannot be analytic,

$$\emptyset \neq \text{S.S. } \delta(x) \subset \pi^{-1}(0) = \{(0, \sqrt{-1}\langle \xi, dx\rangle \infty) \mid \xi \in \mathbf{R}^n - \{0\}\}.$$

Hence there exists $\xi \in \mathbf{R}^n - \{0\}$ so that $(0, i\langle \xi, dx\rangle\infty) \in$ S.S.$\delta(x)$. Then, for this $\xi \in \mathbf{R}^n - \{0\}$, $(0, i\langle \xi, d(gx)\rangle\infty) = (0, i\langle {}^t g\xi, dx\rangle\infty)$ is contained in S.S. $\delta(gx)$. Since (1) implies S.S. $\delta(gx) =$ S.S. $\delta(x)$, $(0, i\langle {}^t g\xi, dx\rangle\infty)$ is contained in S.S. $\delta(x)$. On the other hand, $GL(n, \mathbf{R})$ acts transitively on $\mathbf{R}^n - \{0\}$. Hence we obtain S.S. $\delta(x) = \pi^{-1}(0)$.

Lastly we will prove (3) for the case $n = 2$, and we leave it to the reader to prove the general case. For the case $n = 2$, one can let $\xi_0 = (1, 0)$, $\xi_1 = (0, 1)$ and $\xi_2 = (-1, -1)$ without loss of generality. Then the right-hand side of (3) becomes

$$\frac{1}{(-2\pi i)^2}\left\{\frac{1}{(x_2 + i0)(-x_1 - x_2 + i0)} + \frac{1}{(x_1 + i0)(-x_1 - x_2 + i0)} + \frac{1}{(x_1 + i0)(x_2 + i0)}\right\}.$$

By definition,

$$\delta(x) = \frac{1}{(-2\pi i)^2}\left\{\frac{1}{(x_1 + i0)(x_2 + i0)} - \frac{1}{(x_1 + i0)(x_2 - i0)} - \frac{1}{(x - i0)(x_2 + i0)} + \frac{1}{(x_1 - i0)(x_2 - i0)}\right\}.$$

Therefore,

$$\begin{aligned}
&(-2\pi i)^2(\text{the right-hand side of (3)} - \delta(x)) \\
&= \left\{\frac{1}{(x_1 - i0)(x_2 + i0)} - \frac{1}{(x_1 + x_2 - i0)(x_2 + i0)}\right\} \\
&\quad + \left\{\frac{1}{(x_1 + i0)(x_2 - i0)} - \frac{1}{(x_1 + x_2 - i0)(x_1 + i0)}\right\} - \frac{1}{(x_1 - i0)(x_2 - i0)} \\
&= \frac{1}{(x_1 - i0)(x_1 + x_2 - i0)} + \frac{1}{(x_2 - i0)(x_1 + x_2 - i0)} - \frac{1}{(x_1 - i0)(x_2 - i0)} \\
&= 0.
\end{aligned}$$

Remark. One should keep in mind that the summation of two holomorphic functions can be performed only on the intersection of the domain of definition of each function. Although this is an obvious fact, forgetting it can lead to careless mistakes in such calculations as the one above.

The following proposition is often useful to consider explicit examples.

Proposition 2.4.2. *Let M be a real analytic manifold, and let X be a complexification of M. Let $f(x)$ be a real-valued real analytic function on M. Suppose that $df(x_0) \neq 0$ for $x_0 \in M$ and $\langle r_0, df(x_0)\rangle > 0$ for $r_0 \in \sqrt{-1}SM$; then $\{z \in X \mid \operatorname{Im} f(z) > 0\} \cup \sqrt{-1}SM$ is a neighborhood of $x_0 + ir_0 0$.*

Proof. It is sufficient to prove that $\operatorname{Im} f(z) > 0$ holds for $z = x + itv$ such that $|x - x_0| \ll 1$, $|v - v_0| \ll 1$, and $0 < t \ll 1$. Since $f(x + itv) = f(x) + it\langle v, df(x)\rangle + O(t^2)$, one obtains $\operatorname{Im} f(x + itv) = t\langle v, df(x)\rangle + O(t^2) > 0$ for a sufficiently small t.

As an example relating to Proposition 2.4.2, we consider the following.

Example 2.4.3. Let $u(x) = ((x_1 + i0)^2 - x_2^2 - \cdots - x_n^2)^\lambda$, and let $z_j = x_j + \sqrt{-1}y_j$. First we will show that $y_1^2 - y_2^2 - \cdots - y_n^2 > 0$ implies $z_1^2 - z_2^2 - \cdots - z_n^2 \neq 0$. Since $z_1^2 - z_2^2 - \cdots - z_n^2 = [(x_1^2 - x_2^2 - \cdots - x_n^2) - (y_1^2 - y_2^2 - \cdots - y_n^2)] + 2i(x_1 y_1 - x_2 y_2 - \cdots - x_n y_n)$, if $y_1^2 - y_2^2 - \cdots - y_n^2 > 0$ and $z_1^2 - z_2^2 - \cdots - z_n^2 = 0$ hold, then $(x_1^2 - x_2^2 - \cdots - x_n^2) > 0$ and $(x_1 y_1 - x_2 y_2 - \cdots - x_n y_n) = 0$ must hold. This would imply $|x_1 y_1| = |x_2 y_2 + \cdots + x_n y_n| \leqq \sqrt{x_2^2 + \cdots + x_n^2} \cdot \sqrt{y_2^2 + \cdots + y_n^2} < |x_1| \cdot |y_1|$. Therefore we have obtained $z_1^2 - z_2^2 - \cdots - z_n^2 \neq 0$. Let $D = \{(z_1, \ldots, z_n) \in \mathbf{C}^n | y_1 > 0 \text{ and } y_1^2 - y_2^2 - \cdots - y_n^2 > 0, \text{ where } z_j = x_j + \sqrt{-1}y_j\}$, and let $U = \{x + iv0 \in \sqrt{-1}S\mathbf{R}^n | v_1 > 0 \text{ and } v_1^2 - v_2^2 - \cdots - v_n^2 > 0, \text{ where } v = (v_1, \ldots, v_n)\}$. First note $\sqrt{-1}S\mathbf{R}^n \cong \mathbf{R}^n \times \sqrt{-1}S^{n-1}$. Then $D \cup \mathbf{R}^n \times \sqrt{-1}S^{n-1}$ is a neighborhood of U. Since $u(z) = (z_1^2 - z_2^2 - \cdots - z_n^2)^\lambda \in \mathcal{O}_{\mathbf{C}^n}(D)$, as we saw above, $u(x)$ is a well-defined hyperfunction on $\mathbf{R}^n$ by Theorem 2.3.5, and its singularity spectrum S.S. u is contained in U°. That is, S.S. $u \subset \{(x, i\langle \xi, dx\rangle\infty) | \xi_1^2 - \xi_2^2 - \cdots - \xi_n^2 \geqq 0 \text{ and } \xi_1 > 0\}$. On the other hand, if $(x_1^2 - x_2^2 - \cdots - x_n^2) \neq 0$ holds, then u is real-analytic, which implies S.S. $u \subset \{x_1^2 - x_2^2 - \cdots - x_n^2 = 0\}$. Suppose $x_1^2 - x_2^2 - \cdots - x_n^2 = 0$; then $x = (x_1, \ldots, x_n) \neq 0$ holds if and only if $x_1 \neq 0$. Therefore, we must consider the cases $x_1 > 0$, $x_1 = 0$, and $x_1 < 0$.

If $x_1 > 0$, $x_1^2 - x_2^2 - \cdots - x_n^2 = 0$, $y_1^2 - y_2^2 - \cdots - y_n^2 > 0$, and $y_1 > 0$, then we have $\operatorname{Im}(z_1^2 - z_2^2 - \cdots - z_n^2) > 0$. Hence, if we let $f(z) = z_1^2 - z_2^2 - \cdots - z_n^2$, then $u(z) = f(z)^\lambda$ can be continued in a neighborhood of $x_1 > 0$ to the domain $\{z | \operatorname{Im} f(z) > 0\}$, containing D. Then Proposition 2.4.2 implies that $\{z | \operatorname{Im} f(z) > 0\}$ is a conoidal neighborhood of $U = \{x + iv0 \in \sqrt{-1}S\mathbf{R}^n | x_1 > 0 \text{ and } \langle v, df(x)\rangle > 0\}$. Hence the singularity spectrum of the hyperfunction determined by the boundary values of $u(z)$ is contained in $U^\circ = \{(x, idf(x)\infty) | x_1 > 0\}$. That is, if $x_1 > 0$, then we have S.S. $u \subset \{(x, i(x_1, -x_2, \ldots, -x_n)\infty | x_1^2 - x_2^2 - \cdots - x_n^2 = 0\}$. Similarly, if $x_1 < 0$, then S.S. $u \subset \{(x, -i(x_1, -x_2, \ldots, -x_n)\infty | x_1^2 - x_2^2 - \cdots - x_n^2 = 0\}$. Consequently, we obtain

$$\text{S.S. } u \subset \left\{x, i\left(1, -\frac{x_2}{x_1}, \ldots, -\frac{x_n}{x_1}\right)\infty \,\middle|\, x \neq 0 \text{ and } x_1^2 - x_2^2 - \cdots - x_n^2 = 0\right\}$$
$$\cup \{(0, i\xi\infty) | \xi_1^2 - \xi_2^2 - \cdots - \xi_n^2 \geqq 0 \text{ and } \xi_1 > 0\}.$$

Note that Proposition 2.4.2 does not apply to this example at the origin as $df(0) = 0$. Hence, a priori, there was no guarantee that $\{z | \operatorname{Im} f(z) > 0\}$

determines a conoidal neighborhood, although it was verified to be the case for this example by a concrete calculation. Incidentally, let us calculate the value of $u = f^\lambda$ for $f \neq 0$. This computation will be useful in applications. If $x_1 > 0$, by taking the branch such that $u(1, 0, \dots, 0) = (f(1, 0, \dots, 0))^\lambda = 1$,

$$u(x) = (x_1^2 - x_2^2 - \cdots - x_n^2 + i0)^\lambda$$
$$= \begin{cases} (x_1^2 - x_2^2 - \cdots - x_n^2)^\lambda, & x_1 > \sqrt{x_2^2 + \cdots + x_n^2} \\ e^{\pi i\lambda}(x_2^2 + \cdots + x_n^2 - x_1^2)^\lambda, & x_1^2 < x_2^2 + \cdots + x_n^2 \end{cases},$$

where $(x_1^2 - x_2^2 - \cdots - x_n^2 + i0)^\lambda$ denotes the boundary value of $u(z) = (z_1^2 - z_2^2 - \cdots - z_n^2)^\lambda$ from $\operatorname{Im} f(z) = \operatorname{Im}(z_1^2 - z_2^2 - \cdots - z_n^2) > 0$. Similarly, for the case $x_1 < 0$,

$$u(x) = e^{\pi i\lambda}(x_2^2 + \cdots + x_n^2 - x_1^2 + i0)^\lambda$$
$$= \begin{cases} e^{\pi i\lambda}(x_1^2 - x_2^2 - \cdots - x_n^2)^\lambda, & x_1^2 < x_2^2 + \cdots + x_n^2 \\ e^{2\pi i\lambda}(x_2^2 + \cdots + x_n^2 - x_1^2)^\lambda, & x_1 < -\sqrt{x_2^2 + \cdots + x_n^2} \end{cases}$$

holds.

Let us next introduce the notion of positive type. This notion is quite important for the theory of partial differential equations and other applications. See §7 of Chapter III.

Definition 2.4.2. *Let M be a real analytic manifold, and let X be a complexification of M. A function $f(x)$ on X is said to be of positive type if* $\operatorname{Re} f(x) = 0$ *for $x \in M$ implies* $\operatorname{Im} f(x) \geqq 0$.

Proposition 2.4.3. *Let M be a real analytic manifold, and let X be a complexification of M. Let $f(x)$ be a complex-valued real analytic function of positive type such that for $x_0 \in M$ $f(x_0) = 0$ and $df(x_0)$ is a non-zero real vector. Then $\{f^{-1}(D_\epsilon) - M\} \cup \sqrt{-1}SM$ is a neighborhood of $x_0 + iv_0 0$, where $\langle v_0, df(x_0)\rangle > 0$; ϵ is an arbitrary positive real number; and $D_\epsilon = \{\tau \in \mathbf{C} \mid \operatorname{Im} \tau + \epsilon|\operatorname{Re} \tau| > 0\}$.*

Note. Recall the Weierstrass preparation theorem: let $f(z_1, z_2, \dots, z_n, w)$ be a holomorphic function in a neighborhood of the origin such that $f(0) = 0$ and $f(0, w) = a_s w^s + a_{s+1} w^{s+1} + \cdots$, where $a_s \neq 0$. Then there exist holomorphic functions h and g_k, $k = 1, 2, \dots, s$, such that $g_k(0) = 0$ and $h(0) \neq 0$, satisfying the equation

$$f = h(z_1, \dots, z_n, w)\{w^s + g_1(z_1, \dots, z_n)w^{s-1} + \cdots + g_s(z_1, \dots, z_n)\}.$$

See, for example, Hitotumatu [1] for the proof.

Proof of Proposition 2.4.3. One may assume $x_0 = 0$, since the assertion is invariant under coordinate transformations. By the assumption,

$d\,\mathrm{Re}\, f(x) \neq 0$ in a neighborhood of the origin. Therefore one may let $\mathrm{Re}\, f(x) = x_1$. Then, by the Weierstrass preparation theorem, $f(x) = h(x)(x_1 - g(x'))$ holds for some h and g such that $g(0) = 0$ and $h(0) \neq 0$, where $x' = (x_2, x_3, \ldots, x_n)$.

Notice $dx_1(0) = df(0) = h(0)(dx_1(0) - dg(x'))$. From this, one obtains $h(0) = 1$ and $dg(0) = 0$. Since $x_1 = \mathrm{Re}\, f = x_1 \,\mathrm{Re}\, h - \mathrm{Re}(gh)$, $\mathrm{Re}\, f = 0$ implies $\mathrm{Re}(gh) = 0$. Furthermore, since $\mathrm{Im}\, f = x_1 \,\mathrm{Im}\, h - \mathrm{Im}(gh)$, $\mathrm{Re}\, f = 0$ implies $\mathrm{Im}\, f = -\mathrm{Im}(gh)$. Therefore, since f is of positive type, one has $h(0, x')g(x') = -\sqrt{-1}\varphi(x')$, where $\varphi(0) = 0$ and $\varphi(x') \geqq 0$.

We will prove this proposition for $\tilde{f}(x) = x_1 - g(x')$. Since $f(x + iy) = h(x + iy)\tilde{f}(x + iy)$ and $h(0) = 1$, the proof for $\tilde{f}(x)$ would imply that Proposition 2.4.3 holds sufficiently near the origin for f.

After a suitable coordinate transformation, one can assume $v_0 = \partial/\partial x_1$. In fact, for $v_0 = \partial/\partial x_1 + a_2(\partial/\partial x_2) + \cdots + a_n(\partial/\partial x_n)$, let the coordinate transformation be $x_1' = x_1, x_2' = x_2 - a_2 x_1, \ldots, x_n' = x_n - a_n x_1$. Then $v_0(x_1') = 1$ and $v_0(x_j') = 0$ for $j \neq 1$ hold. That is, with the new coordinates $v_0 = \partial/\partial x_1'$.

Then one obtains

$$\begin{aligned}
\mathrm{Im}\, \tilde{f}(x + iy) &+ \epsilon|\mathrm{Re}\, \tilde{f}(x + iy)| \\
&= y_1 - \mathrm{Im}\, g(x' + iy') + \epsilon|x_1 - \mathrm{Re}\, g(x' + iy')| \\
&\geqq y_1 - \mathrm{Im}\, g(x' + iy') + \epsilon\{|x_1| - |\mathrm{Re}\, g(x' + iy')|\} \\
&\geqq y_1 - \{\mathrm{Im}\, g(x' + iy') + \epsilon|\mathrm{Re}\, g(x' + iy')|\}.
\end{aligned}$$

Notice that, since $v_0 = \partial/\partial x_1$, $|v - v_0| \ll 1$ can be rephrased as $0 < |y'|/y_1 \ll 1$.

One has $g(x') = -i\varphi(x)\psi(x')$, where $\psi(x') = h(0, x')^{-1}$ and $\psi(0) = 1$, and $\varphi(x' + iy') = \varphi(x') + i\langle y', d_{x'}, \varphi(x')\rangle + O(|y'|^2) = \varphi(x') + O(|y'|)$. Therefore,

$$\begin{aligned}
-(\mathrm{Im}\, g(x' + iy') &+ \epsilon|\mathrm{Re}\, g(x' + iy')|) \\
&= O(|y'|) + (\mathrm{Im}(i\varphi(x')\psi(x' + iy')) - \epsilon|\mathrm{Re}(i\varphi(x')\psi(x' + iy'))|.
\end{aligned}$$

If, since $\varphi(x') \geqq 0$ and $\psi(0) = 1$, one lets $x' + iy'$ be sufficiently near 0, then for an arbitrary $\epsilon > 0$ one obtains

$$\mathrm{Im}(i\varphi(x')\psi(x' + iy')) - \epsilon|\mathrm{Re}(i\varphi(x')\psi(x' + iy'))| > 0.$$

Consequently, $\mathrm{Im}\;\; \tilde{f}(x + iy) + \epsilon|\mathrm{Re}\;\; \tilde{f}(x + iy)| \geqq y_1 + O(|y'|)$. If $0 < |y'|/y_1 \ll 1$, then $y_1 + O(|y'|) > 0$. Hence, one finally obtains $\mathrm{Im}\, f(x + itv) + \epsilon|\mathrm{Re}\, f(x + itv)| > 0$ for $|x| \ll 1$, $|v - v_0| \ll 1$, and $0 < t \ll 1$.

We will consider the hyperfunction $(x_1 + i(x_2^2 + \cdots + x_n^2) + i0)^\lambda$, where λ is neither zero nor a positive integer, as an application of Proposition 2.4.3. This hyperfunction has its singularity spectrum equaling one point, which is quite important not only from a theoretical point of view but also for applications. Theoretically, this fact and the flabbiness of the sheaf of microfunctions (see §7, Chapter III) are opposite sides of the same coin. As for applications, it is related to the (non-)solvability of linear differential equations; see Theorem 4.3.8.

Example 2.4.4. Let $f(x) = x_1 + i(x_2^2 + \cdots + x_n^2)$. Then $f(x)$ is of positive type on $\mathbf{R}^n$ such that $f(0) = 0$ and $df(0) = (1, 0, \ldots, 0)$. Therefore Proposition 2.4.3 can be applied to the function $f(x)$. That is, $\{z \in \mathbf{C}^n - \mathbf{R}^n \mid \operatorname{Im} f(x) + \epsilon|\operatorname{Re} f(z)| > 0\}$ is a conoidal neighborhood of $\{0 + iv0 \mid \langle v, idf(0)\rangle > 0\}$. Hence the singularity spectrum of $(f(x) + i0)^\lambda$ is contained in $\{(x, i\xi\infty) \mid \xi = (1, 0, \ldots, 0)\}$. On the other hand, $f(x) = 0$ if and only if $x = 0$; i.e. $(f(x) + i0)^\lambda$ is real-analytic everywhere but at the origin. These imply

$$\mathrm{S.S.}(f(x) + i0)^\lambda \subset \{(x, i\xi\infty) \mid x = 0 \text{ and } \xi = (1, 0, \ldots, 0)\}.$$

For $\lambda \neq 0, 1, 2, \ldots$, $(f(x) + i0)^\lambda$ is not real-analytic at the origin. One can now conclude

$$\mathrm{S.S.}(f(x) + i0)^\lambda = \{(x, i\xi0) \mid x = 0 \text{ and } \xi = (1, 0, \ldots, 0)\}.$$

Exercise. Sketch the domain $\{(z_1, z_2) \in \mathbf{C}^2 \mid \operatorname{Im}(z_1 + iz_2^2) > 0\}$. Then notice that one cannot treat $(x_1 + ix_2^2 - i0)^\lambda$ in the natural way, as above, by considering the boundary value of a holomorphic function.

We will consider Fourier series, which provide explicit examples of hyperfunctions.

Proposition 2.4.4. *For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n$ let $u(x) = \sum_{\alpha \in Z^n} a_\alpha e^{2\pi i\langle x, \alpha\rangle}$, where $x \in \mathbf{R}^n$. Suppose that for an arbitrary $\epsilon > 0$ there exists a constant C_ϵ such that $|a_\alpha| \leqq C_\epsilon e^{\epsilon|\alpha|}$ holds for each α, where $|\alpha| = \sum_{j=1}^n |\alpha_j|$. Then $u(x)$ is a well-defined hyperfunction.*

Remark. If $|a_\alpha| \leqq C|\alpha|^N$ for some $N > 0$ and $C > 0$, then $u(x)$ is a distribution.

Proof. Let G_j, $j = 1, 2, \ldots, N$, be a closed convex cone containing no lines in $\mathbf{R}^n$ or having $a \in \mathbf{R}^n$ as its vertex, and let Γ_j satisfy $\Gamma_j \subset G_j$ and $\mathbf{Z}^n = \bigcup_{j=1}^N \Gamma_j$. Then write $u(z)$ as $u(z) = \sum_{j=1}^N \sum_{\alpha \in \Gamma_j} a_\alpha e^{2\pi i\langle z, \alpha\rangle}$. We will first show that $\sum_{\alpha \in \Gamma_j} a_\alpha e^{2\pi i\langle Z, \alpha\rangle}$, $z \in \mathbf{R}^n \times \sqrt{-1}G_j^\circ$, is holomorphic for each j, where G_j° is the polar set of G_j.

Lemma. *Let G be a closed cone without containing a line, and let U be an open cone such that $\bar{U} \subset G^\circ \cup \{0\}$ holds. Then there exists $\delta > 0$ such that*

$$|\langle x, y\rangle| \geq \delta|x||y| \qquad \text{for } x \in U \text{ and } y \in G.$$

Proof. Let $S = \{x \in \mathbf{R}^n \mid |x| = 1\}$. The map $(S \cap \bar{U}) \times (S \cap G) \to \mathbf{R}$, defined by $(x, y) \mapsto \langle x, y\rangle$, has the compact image. Therefore the image is closed, and $\langle x, y\rangle > 0$ by the definition of the polar set. Hence one obtains $|\langle x, y\rangle| \geqq \delta|x|\,|y|$, since U and G are cones.

Proof of Proposition 2.4.4. Let U be an arbitrary open cone such that $U \subset G_j^\circ$. By the lemma, there exists $\delta > 0$ such that $|\langle \operatorname{Im} z, \alpha\rangle| \geqq \delta|\operatorname{Im} z|\,|\alpha|$ for $\operatorname{Im} z \in U$. Therefore one has

$$\left|\sum_{\alpha \in \Gamma_j} a_\alpha e^{2\pi i\langle z,\alpha\rangle}\right| \leqq \sum_{\alpha \in \Gamma_j} C_\epsilon e^{\epsilon|\alpha|} e^{-2\pi\langle \operatorname{Im} z,\alpha\rangle} \leqq C_\epsilon \sum_{\alpha \in \Gamma_j} e^{\epsilon|\alpha| - 2\pi\delta|\operatorname{Im} z|\,|\alpha|}.$$

Note that for a sufficiently small $\epsilon > 0$ one has $e^{\epsilon|\alpha| - 2\pi\delta|\operatorname{Im} z|} < 1$. Hence $\sum_{\alpha \in \Gamma_j} a_\alpha e^{2\pi i\langle z,\alpha\rangle}$ is uniformly convergent on $\mathbf{R}^n \times \sqrt{-1}U$; i.e. it is a holomorphic function. There is defined a hyperfunction

$$u_j(x) = \sum_{\alpha \in \Gamma_j} a_\alpha e^{2\pi i\langle x,\alpha\rangle}$$

as the boundary value. Then $u(x) = \sum_{j=1}^{N} u_j(x)$ is a hyperfunction. The proof of Proposition 2.4.5 shows that the above definition is independent of the choice of partition of $\mathbf{Z}^n$.

Example 2.4.5 (*Poisson's Summation Formula*). Note that

$$\sum_{n=-\infty}^{\infty} e^{2\pi inx} = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

holds.

Proof. Write $u(x) = \sum_{n=0}^{\infty} e^{2\pi inx} + \sum_{n=1}^{\infty} e^{-2\pi inx}$.

Since

$$\sum_{n=0}^{\infty} e^{2\pi inz} = \frac{1}{1 - e^{2\pi iz}} \qquad \text{for } \operatorname{Im} z > 0,$$

it is holomorphic. Therefore the first term equals $1/(1 - e^{2\pi i(x+i0)})$. If $\operatorname{Im} z < 0$, then

$$\sum_{n=1}^{\infty} e^{-2\pi inz} = \frac{e^{-2\pi iz}}{1 - e^{-2\pi iz}} = -\frac{1}{1 - e^{2\pi iz}}.$$

The second term equals $-1/(1 - e^{2\pi i(x-i0)})$. Consequently, one has

$$u(x) = \frac{1}{1 - e^{2\pi i(x+i0)}} - \frac{1}{1 - e^{2\pi i(x-i0)}}.$$

Notice that in a complex neighborhood of $(-1, 1) \subset \mathbf{R}$,

$$\frac{1}{1 - e^{2\pi i z}} = \frac{1}{-2\pi i z} + \text{(a holomorphic function)}$$

holds. Therefore, on $(-1, 1)$

$$u(x) = \frac{1}{-2\pi i}\left(\frac{1}{x + i0} - \frac{1}{x - i0}\right) = \delta(x).$$

On the other hand, $u(x + n) = u(x)$ holds for $n \in \mathbf{Z}$. One now concludes $u(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$.

We will consider a simple case from the theory of Fourier transformation, i.e. changing $\alpha \in \mathbf{Z}^n$ to the continuum $y \in \mathbf{R}^n$.

Proposition 2.4.5. *Let $f(y)$ be a locally Lebesgue integrable function on $\mathbf{R}^n$ such that for an arbitrary $\epsilon > 0$ there exists C_ϵ with the property $|f(y)| \leqq C_\epsilon e^{\epsilon|y|}$ almost everywhere. Then the Fourier transform of $f(y)$, $u(x) = \int_{\mathbf{R}^n} f(y) e^{2\pi i\langle x, y\rangle}\, dy$, is a well-defined hyperfunction.*

Proof. Let $\mathbf{R}^n = \Gamma_1 \cup \cdots \cup \Gamma_N$, where each Γ_j is a closed convex cone which contains no lines, such that for $j \neq k$ the measure of $\Gamma_j \cap \Gamma_k$ is zero. Then it is plain that Γ_i° is a non-empty open set. Furthermore, $\Gamma_j^\circ = \{x \mid \varphi_{\Gamma_j}(x) > 0\}$ holds, where $\varphi_{\Gamma_j}(x) = \inf_{\substack{y \in \Gamma_i \\ |y| = 1}} \langle x, y\rangle$. We now show that $u_{\Gamma_i}(z) \underset{\text{def}}{=} \int_{\Gamma_i} f(y) e^{2\pi i\langle z, y\rangle}\, dy$ is holomorphic in z for $\operatorname{Im} z \in \Gamma_i^\circ$. Since $\operatorname{Im}\langle z, y\rangle = \langle \operatorname{Im} z, y\rangle > 0$ holds for $\operatorname{Im} z \in \Gamma_i^\circ$ and $y \in \Gamma_i$, we have the inequality $\operatorname{Im}\langle z, y\rangle \geqq |y|\varphi_{\Gamma_i}(\operatorname{Im} z)$. Therefore we get

$$|e^{2\pi i\langle z, y\rangle}| = e^{-2\pi \operatorname{Im}\langle z, y\rangle} \leqq \exp(-2\pi|y|\varphi_{\Gamma_i}(\operatorname{Im} z)) \qquad \text{for } y \in \Gamma_i.$$

Consequently,

$$|u_{\Gamma_i}(z)| \leqq \int_{\Gamma_i} |f(y)|\,|e^{2\pi i\langle z, y\rangle}|\, dy \leqq \int_{\Gamma_i} C_\epsilon \exp(-|y|(2\pi\varphi_{\Gamma_i}(\operatorname{Im} z) - \epsilon))\, dy.$$

Let $\alpha = 2\pi\varphi_{\Gamma_i}(\operatorname{Im} z) - \epsilon > 0$ for a sufficiently small ϵ. Then

$$|u_{\Gamma_i}(z)| \leqq C_\epsilon \int_{\mathbf{R}^n} e^{-\alpha|y|}\, dy = C_\epsilon \sigma_{n-1}\alpha^{-n}\Gamma(n) < +\infty,$$

from which we conclude that $u_{\Gamma_i}(z)$, $\operatorname{Im} z \in \Gamma_i$, is a holomorphic function in z, where σ_{n-1} denotes the area of sphere S^{n-1}.

Hence we can now consider the hyperfunction

$$u(x) = \sum_{i=1}^{N} u_{\Gamma_i}(x + \sqrt{-1}\Gamma_i^\circ 0)$$

induced from the above $u_{\Gamma_i} \in \mathcal{O}(\mathbf{R}^n \times \sqrt{-1}\Gamma_i^\circ)$, where $u_{\Gamma_i}(x + \sqrt{-1}\Gamma_i^\circ 0)$ denotes the boundary value from $\mathbf{R}^n \times \sqrt{-1}\Gamma_i$. Next we will prove that $u(x)$ is independently determined from the choice of partition $\mathbf{R}^n = \Gamma_1 \cup \cdots \cup \Gamma_N$. Let $\mathbf{R}^n = \Gamma_1 \cup \cdots \cup \Gamma_N = \Gamma'_1 \cup \cdots \cup \Gamma'_M$ be two partitions of $\mathbf{R}^n$. Then we can have the partition $\mathbf{R}^n = \Gamma''_1 \cup \cdots \cup \Gamma''_L$, where Γ''_k, $1 \leqq k \leqq L$, chosen from among non-zero measure sets $\Gamma''_{ij} = \Gamma_i \cap \Gamma'_j$, $1 \leqq i \leqq N$ and $1 \leqq j \leqq M$. Thus one can assume that one partition, $\Gamma'_1 \cup \cdots \cup \Gamma'_M$, is a refinement of the other, $\Gamma_1 \cup \cdots \cup \Gamma_N$. For each i, let $\Gamma_i = \Gamma'_{i_1} \cup \cdots \cup \Gamma'_{i_k}$. Abbreviate $u_{\Gamma_i}(z)$ and $u_{\Gamma'_j}(z)$ as $u_i(z)$ and $u'_j(z)$ respectively. Note that $\Gamma_i \supset \Gamma'_{i_j}$, $1 \leqq j \leqq k$, implies $\Gamma_i^\circ \subset \Gamma'^\circ_{i_1} \cap \cdots \cap \Gamma'^\circ_{i_k}$. Therefore, $\operatorname{Im} z \in \Gamma_i^\circ$ implies $\operatorname{Im} z \in \Gamma'^\circ_{i_j}$. That is, if $u_i(z)$ can be defined, then $u'_{i_j}(z)$ exists. Since $\Gamma_i = \Gamma'_{i_1} \cup \cdots \cup \Gamma'_{i_k}$, one has $u_i(z) = u'_{i_1}(z) + \cdots + u'_{i_k}(z)$. Hence $u_i(x + \sqrt{-1}\Gamma_i 0) = u'_{i_1}(x + \sqrt{-1}\Gamma_i 0) + \cdots + u'_{i_k}(x + \sqrt{-1}\Gamma_i 0)$ holds. We conclude that $u(x)$ is well defined.

Example 2.4.6. The function $f(y) \equiv 1$ clearly satisfies the condition of Proposition 2.4.5. Therefore $u(x) = \int_{\mathbf{R}} e^{2\pi ixy}\, dy$ is a hyperfunction. This hyperfunction is the δ-function; i.e.

$$\delta(x) = \int_{\mathbf{R}} e^{2\pi ixy}\, dy.$$

Proof. Let $\operatorname{Im} z > 0$. Then

$$\int_0^\infty e^{2\pi izy}\, dy = \frac{e^{2\pi izy}}{2\pi iz}\bigg|_0^\infty = -\frac{1}{2\pi iz}.$$

Hence one obtains

$$\int_0^\infty e^{2\pi ixy}\, dy = -\frac{1}{2\pi i(x + i0)}.$$

Similarly, one has

$$\int_{-\infty}^0 e^{2\pi ixy}\, dy = \frac{1}{2\pi i(x - i0)}.$$

Consequently,

$$\int_{\mathbf{R}} e^{2\pi ixy}\, dy = \int_0^\infty e^{2\pi ixy}\, dy + \int_{-\infty}^0 e^{2\pi ixy}\, dy = \delta(x).$$

Remark. We quote the following fact from the theory of Fourier transformations. Let D^n denote the compactification of $\mathbf{R}^n$ by adding S^{n-1} at infinity. There exists a flabby sheaf $\mathcal{Q}$ (this sheaf has nothing to do with the sheaf $\mathcal{Q}$ in Definition 2.1.3) such that $\mathcal{Q}|_{\mathbf{R}^n} = \mathscr{B}_{\mathbf{R}^n}$, and $\mathcal{Q}(D^n)$ and $\mathcal{Q}(D^{n*})$ correspond, via the Fourier transformation and the inverse Fourier transformation respectively, where D^{n*} denotes the dual of D^n and where it can be identified with D^n. See Kawai [1].

Let us now examine the singularity spectrum of the hyperfunction $u(x) = \int_{\mathbf{R}^n} f(y)e^{2\pi i\langle x,y\rangle}\, dy$, supposing some additional information on supp f.

Definition 2.4.3. *For a subset G of $\mathbf{R}^n$, define $G\infty = \{y \in \mathbf{R}^n - \{0\} \mid$ for an arbitrary $N > 0$ and $\epsilon > 0$, $G \cap \{y' \mid |y'| \geqq N$ and $|y'/|y'| - y/|y|| < \epsilon\} \neq \varnothing\}$.*

For example, let $G_1 = \{(x, y) \mid xy \geqq 1, x > 0\}$, and let $G_2 = \{(x, y) \mid y \geqq x^2\}$ in $\mathbf{R}^2$; then $G_1\infty = \{(x, y) \mid x \geqq 1$ and $y \geqq 1\}$, and $G_2\infty = \{(x, y) \mid x = 0\}$. Generally, $G\infty = \{y \in \mathbf{R}^n - \{0\} \mid$ the intersection of G and an arbitrary open cone containing y is never relatively compact$\}$.

Lemma. *Let G be a subset of $\mathbf{R}^n$. Let $\mathscr{G}$ be the collection of $\{G' \subset \mathbf{R}^n \mid G'$ is a closed cone such that $G' \supset G + a$ for some $a\}$. Then $G\infty = \bigcap_{G' \in \mathscr{G}} G'$ holds.*

Proof. Let $y \in G\infty$, and let U be an open cone containing y. Then one can find a closed cone T and an open cone V so that $y \in V \subset T \subset U$. Therefore, since $T \cap G$ is not relatively compact, $T \cap (G' - a)$ is not relatively compact where $a \in \mathbf{R}^n$, and G' is an arbitrary closed cone such that $G' \supset G + a$. But in general, for closed cones G_1 and G_2 and $a \in \mathbf{R}^n$, $(G + a) \cap G_2$ is relatively compact if and only if $G_1 \cap G_2 = \varnothing$. So one has $T \cap G' \neq \varnothing$ in this case. This implies $U \cap G' \neq \varnothing$. Hence $y \in G'$. This is because, if $y \notin G'$, then there exists an open set Ω such that $y \in \Omega$ and $\Omega \cap G' = \varnothing$ since G' is closed. Let $U = \mathbf{R}^+ \cdot \Omega = \{\lambda x \mid \lambda \in \mathbf{R}^+$ and $x \in \Omega\}$. Then U is an open cone containing y with the property $U \cap G' = \varnothing$. Therefore, one concludes $G\infty \subset \bigcap_{G' \in \mathscr{G}} G'$. Next suppose $y \in \bigcap_{G' \in \mathscr{G}} G'$ and $y \notin G\infty$. Then one can find an open cone U containing y such that $U \cap G$ is contained in a compact set K. Notice that there exists $a \in U$ such that $(U + a) \cap K = \varnothing$ (see Figure 2.4.1). Since one can assume that U is convex, $U + U \subset U$ holds. $(U + a) \cap G \subset U \cap G \subset K$ implies $(U + a) \cap G \subset (U + a) \cap K = \varnothing$. That is, $U \cap (G - a) = \varnothing$. Then $G' = \mathbf{R}^n - U$ is a closed cone such that $G - a \subset G'$ and $y \notin G'$, which is a contradiction. Consequently, $y \in G'$.

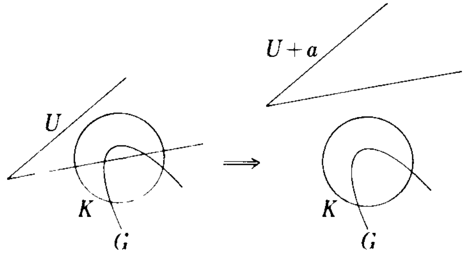

Figure 2.4.1

Proposition 2.4.6. *Keeping the assumptions on $f(x)$ the same as in Proposition 2.4.5, let $G = \operatorname{supp} f$. Then the singularity spectrum of the hyperfunction $u(x) = \int_{\mathbf{R}^n} f(y)e^{2\pi i\langle x,y\rangle}\,dy$ is contained in $\{(x, \sqrt{-1}\xi\infty) \mid \xi \in G\infty\}$.*

Note. The notion of essential support developed by Bros and Iagolnitzer (see the remark following Definition 2.3.3) results from Proposition 2.4.6 being carried to its limit. Probably this is the reason they chose the name "essential support" for their theory. Physically speaking, Proposition 2.4.6 is related to microcausality, but it is macrocausality that has immediate relevance for singularity spectrum theory. With respect to macrocausality, it is quite interesting that physicists were quite independently developing a prototype of the macrocausality notion of the singularity spectrum at around the same time that microfunction theory appeared. See Iagolnitzer and Stapp [1] and the references cited therein.

Proof of Proposition 2.4.6. Let $\xi \notin G\infty$. Then there exists a closed cone G' such that $\xi \notin G'$ and $G + a \subset G'$ for some $a \in \mathbf{R}^n$, from the above lemma. Let G_j be a closed convex cone without containing a line for each $j = 1, \ldots, N$ such that $G' \subset \bigcup_{j=1}^{N} G_j$ and $\xi \notin \bigcup_{j=1}^{N} G_j$. Proposition 2.4.5 implies that $u_j(z) = \int_{G_j - a} f(y)e^{2\pi i\langle z,y\rangle}\,dy$ is a holomorphic function for $z \in \mathbf{R}^n \times \sqrt{-1}G_j^\circ$. Hence S.S. $u_j(x + \sqrt{-1}G_j^\circ 0) \subset \sqrt{-1}G_j$. Since $u(x) = \sum_{j=1}^{N} u_j(x + \sqrt{-1}G_j^\circ 0)$, one has S.S. $u(x) \subset \bigcup_{j=1}^{N} \sqrt{-1}G_j$. Then $(x, \sqrt{-1}\xi\infty) \notin$ S.S. u holds, since $\xi \notin G_j$.

Remark. Note that $\int_{G_j - a} f(y)e^{2\pi i\langle x,y\rangle}\,dy = e^{-\langle x,a\rangle}\int_{G_j} f(y - a)e^{2\pi i\langle x,y\rangle}\,dy$. Therefore the shifting of G_j by a, $G_j - a$, does not affect the convergence.

Example 2.4.7. As an example of Proposition 2.4.6, we consider the hyperfunction $\int_{\mathbf{R}} y_+^\lambda e^{2\pi ixy}\,dy$, Re $\lambda > 0$. Note that we have $G = G\infty = \overline{\mathbf{R}^+}$. By carrying out the integration, we have

$$\int y_+^\lambda e^{2\pi i\langle x,y\rangle}\,dy = \int_0^\infty y^\lambda e^{2\pi ixy}\,dy = \int_0^\infty \left(\frac{t}{-2\pi ix}\right)^\lambda e^{-t}\frac{dt}{(-2\pi ix)}$$

$$= (-2\pi i(x + i0))^{-(\lambda+1)}\Gamma(\lambda + 1) = \frac{\Gamma(\lambda + 1)e^{(\pi/2)i(1+\lambda)}}{(2\pi)^{1+\lambda}(x + i0)^{1+\lambda}}.$$

Then the singularity spectrum is contained in $(0, idx\infty)$.

Note. By the reasoning given in Chapter III, §8, we can verify that the above equality holds for any $\lambda \neq -1, -2, \ldots$.

To conclude this section, we will consider an example which has a different flavor from the previous ones. The equation (2.4.4), below, is important in application.

Example 2.4.8. We will define the hyperfunction $1/(x+\sqrt{-1}y)$ by boundary values. Let

$$\Omega_{\epsilon_1,\epsilon_2}=\{(z,w)\in \mathbf{C}^2 \mid \epsilon_1 \operatorname{Im} z>0 \text{ and } \epsilon_2 \operatorname{Im} w>0\},$$

where $\epsilon_1, \epsilon_2=\pm$. Let $\log z$ denote the logarithm function defined on $\mathbf{C}-\{x\in\mathbf{R} \mid x\leq 0\}$ with the prescription of the branch $\log z|_{z=1}=0$. Then $\log z-\log w+\pi i/2$ has a zero on $z+iw=0$ in $\Omega_{+,+}$. Therefore $\varphi_{+,+}(z,w)=(\log z-\log w+\pi i/2)/(z+\sqrt{-1}w)$ is holomorphic in $\Omega_{+,+}$. Similarly, we obtain that the function $\varphi_{\epsilon_1,\epsilon_2}$, where

$$\varphi_{-,+}=\left(\log z-\log w+\frac{\pi i}{2}\right)\Big/(z+\sqrt{-1}w),$$

$$\varphi_{+,-}=(\log z-\log w-\tfrac{3}{2}\pi i)/(z+\sqrt{-1}w),$$

and

$$\varphi_{-,-}=\left(\log z-\log w+\frac{\pi i}{2}\right)\Big/(z+\sqrt{-1}w),$$

is holomorphic in $\Omega_{\epsilon_1,\epsilon_2}$. Define

$$u(x,y)=\frac{1}{2\pi i}\left(b_{\Omega_{+,+}}(\varphi_{+,+})-b_{\Omega_{+,-}}(\varphi_{+,-})-b_{\Omega_{-,+}}(\varphi_{-,+})+b_{\Omega_{-,-}}(\varphi_{-,-})\right),$$

where $b_{\Omega\epsilon_1,\epsilon_2}(\varphi_{\epsilon_1,\epsilon_2})$ denotes the boundary value of $\varphi_{\epsilon_1,\epsilon_2}$ from $\Omega_{\epsilon_1,\epsilon_2}$. Then

$$\begin{aligned}
2\pi\sqrt{-1}(x+\sqrt{-1}y)u &= b_{\Omega_{+,+}}(\log z-\log w+\pi\sqrt{-1}/2)\\
&\quad -b_{\Omega_{+,-}}(\log z-\log w-3\pi\sqrt{-1}/2)\\
&\quad -b_{\Omega_{-,+}}(\log z-\log w+\pi\sqrt{-1}/2)\\
&\quad +b_{\Omega_{-,-}}(\log z-\log w+\pi\sqrt{-1}/2)\\
&=\{\log(x+\sqrt{-1}0)-\log(y+\sqrt{-1}0)+\pi\sqrt{-1}/2\}\\
&\quad -\{\log(x+\sqrt{-1}0)-\log(y-\sqrt{-1}0)-3\pi\sqrt{-1}/2\}\\
&\quad -\{\log(x-\sqrt{-1}0)-\log(y+\sqrt{-1}0)+\pi\sqrt{-1}/2\}\\
&\quad +\{\log(x-\sqrt{-1}0)-\log(y-\sqrt{-1}0)+\pi\sqrt{-1}/2\}\\
&=2\pi\sqrt{-1}.
\end{aligned}$$

That is,

$$(x+\sqrt{-1}y)u=1 \tag{2.4.3}$$

holds. In particular, $u=1/(x+\sqrt{-1}y)$ for $(x,y)\neq(0,0)$. Next we will prove the following equation which is important for applications:

$$\frac{1}{2}\left(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y}\right)u=\pi\delta(x,y). \tag{2.4.4}$$

First notice that we have

$$\left(\frac{\partial}{\partial z}+\sqrt{-1}\frac{\partial}{\partial w}\right)\left(\frac{1}{z+\sqrt{-1}w}(\log z-\log w+c)\right)$$
$$=\frac{1}{z+\sqrt{-1}w}\left(\frac{1}{z}-\frac{\sqrt{-1}}{w}\right)=-\frac{\sqrt{-1}}{zw}\qquad \text{for } c\in\mathbf{C}.$$

From this we obtain the following:

$$\left(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y}\right)u$$
$$=\frac{1}{2\pi\sqrt{-1}}\left\{\frac{-\sqrt{-1}}{(x+\sqrt{-1}0)(y+\sqrt{-1}0)}+\frac{\sqrt{-1}}{(x+\sqrt{-1}0)(y-\sqrt{-1}0)}\right.$$
$$\left.+\frac{\sqrt{-1}}{(x-\sqrt{-1}0)(y+\sqrt{-1}0)}-\frac{\sqrt{-1}}{(x-\sqrt{-1}0)(y-\sqrt{-1}0)}\right\}$$
$$=(-\sqrt{-1})(2\pi\sqrt{-1})\delta(x,y).$$

Thus, we have verified (2.4.4). We will also give some other differential equations which u satisfies. On the other hand,

$$\left(z\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}+1\right)\varphi_{\epsilon_1,\epsilon_2}=0$$

holds. Hence, we have

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+1\right)u=0. \tag{2.4.5}$$

We also obtain

$$\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}-\sqrt{-1}\right)u=0. \tag{2.4.6}$$

This is because equation (2.4.6) plus equation (2.4.5) $\cdot\sqrt{-1}$ implies

$$\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}-\sqrt{-1}\right)u=\left((y+\sqrt{-1}x)\frac{\partial}{\partial x}-(x-\sqrt{-1}y)\frac{\partial}{\partial y}\right)u$$
$$=(y+\sqrt{-1}x)\left(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y}\right)u$$
$$=2\pi(y+\sqrt{-1}x)\delta(x,y)=0.$$

CHAPTER III

Fundamental Operations

§1. Product, Restriction, and Substitution

We shall devote this section to definitions of the product, restriction, and substitution of hyperfunctions and microfunctions. We will treat Proposition 3.1.1, which follows, as an axiom for the succeeding argument. Note that in SKK [1] these operations are discussed first and that, as a result, the statement of Proposition 3.1.1 (and even the flabbiness of the microfunction sheaf, see §7) is proved. Our treatment in this book is intended to be a more elementary exposition of the material.

Proposition 3.1.1. *Let $V_j, j = 1, 2, \ldots, N < \infty$, be an open set in $\sqrt{-1}S^*M$, and let $V = \bigcup\limits_j V_j$.*

(1) *If* S.S. $f \subset V$ *holds for* $f \in \mathscr{B}_M(M)$, *then for each* $x \in M$ *there exists* $f_j \in \mathscr{B}_M(W)$, *for some neighborhood* W *of* x, *such that* S.S. $f_j \subset V_j \cap \pi^{-1}(W)$ *and* $f = \sum\limits_j f_j$ *on* W.

(2) *Let* $f_j \in \mathscr{B}_M(M)$ *such that* S.S. $f_j \subset V_j$ *and* $\sum f_j = 0$. *Then, for any* x *in* M *there exists* $f_{jk} \in \mathscr{B}_M(W)$, *for some neighborhood* W *of* x, *such that* $f_j = \sum\limits_k f_{jk}$, $f_{jk} = -f_{kj}$ *and* S.S. $f_{jk} \subset V_j \cap V_k \cap \pi^{-1}(W)$.

Proposition 3.1.1 can be rephrased as follows.

Proposition 3.1.1′. *Let $V_j \subset \sqrt{-1}S^*M$, $j = 1, 2, \ldots, N < \infty$, be an open set and let $V = \bigcup\limits_j V_j$. Then for each $p \in \sqrt{-1}S^*M$ there is a neighborhood W with the following properties:*

(1)′ *If* $u \in \mathscr{C}_M(W)$ *satisfies* supp $u \subset V \cap W$, *then* u *can be expressed as* $u = \sum\limits_j u_j$ *for some* $u_j \in \mathscr{C}_M(W)$ *such that* supp $u_j \subset V_j \cap W$.

(2)′ *If* $u_j \in \mathscr{C}_M(W)$ *satisfies* supp $u_j \subset V_j$ *and* $\sum\limits_j u_j = 0$, *then* $u_j = \sum\limits_k u_{jk}$ *for some* $u_{jk} \in \mathscr{C}_M(W)$ *such that* $u_{jk} = -u_{kj}$ *and* supp $u_{jk} \subset V_j \cap V_k \cap W$.

We will define a combined concept of the support of a hyperfunction and that of a microfunction to obtain more systematic results. One may understand why we introduce this concept if one tries to rephrase the following Theorem 3.1.1 without using S.S., defined in Definition 3.1.1. One may also find it more convenient to work with T^*M rather than S^*M when one studies systems of differential equations. See also Definition 3.8.1, where the sheaf $\hat{\mathscr{C}}$, a combined notion of $\mathscr{B}$ and $\mathscr{C}$, is introduced such that $\widehat{\text{S.S.}}$ can be considered as the support of a section of $\hat{\mathscr{C}}$.

Definition 3.1.1. *Let u be a hyperfunction on a real analytic manifold M. Then define*

$$\begin{aligned}\widehat{\text{S.S.}}\, u \underset{\text{def}}{=} &\{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M \mid x \in \operatorname{supp} u \text{ and } \xi = 0\}\\ &\cup \{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M \mid \xi \neq 0 \text{ and } (x, \sqrt{-1}\xi\infty) \in \text{S.S. } u\}.\end{aligned}$$

Remark. $\widehat{\text{S.S.}}\, u$ is closed in T^*M and is invariant under the transform

$$(x, \sqrt{-1}\xi) \mapsto (x, \sqrt{-1}c\xi) \qquad \text{for } c \geqq 0.$$

The reader may find it profitable to read first the examples at the end of this section in order to understand the following statements on products, restrictions, and substitutions of hyperfunctions or microfunctions.

Theorem 3.1.1. *Let M_1 and M_2 be real analytic manifolds, and let $M = M_1 \times M_2$. For hyperfunctions $u_1 = u_1(x_1)$ and $u_2 = u_2(x_2)$, on M_1 and M_2 respectively, one can define canonically the product $u = u(x_1, x_2) \underset{\text{def}}{=} u_1(x_1)u_2(x_2)$ so that*

$$\widehat{\text{S.S.}}\, u \subset \widehat{\text{S.S.}}\, u_1 \times \widehat{\text{S.S.}}\, u_2,$$

where we regard $\sqrt{-1}T^*M = \sqrt{-1}T^*M_1 \times \sqrt{-1}T^*M_2$.

Proof. Let $\sqrt{-1}S^*M_\nu = \bigcup_{j \in I_\nu} V_j^\nu$, $\nu = 1, 2$, where V_j^ν is an open and convex set such that $\tau(V_j^{\nu\circ}) = M_\nu$ and I_ν is a finite set. Then Proposition 3.1.1 can be applied to this situation since the question is local in nature. There exist a neighborhood W_ν and a hyperfunction u_j^ν on W_ν such that $u_\nu = \sum_j u_j^\nu$ and S.S. $u_j^\nu \subset V_j^\nu \cap \pi^{-1}(W_\nu)$. Let $W = W_1 \times W_2$. We will denote $V_j^\nu \cap \pi^{-1}(W_\nu)$ simply by V_j^ν. Let Z_j^ν be a closed convex set such that S.S. $u_j^\nu \subset Z_j^\nu \subset V_j^\nu$. Then $Z_j^{\nu\circ} \supset V_j^{\nu\circ}$ and $Z_j^{\nu\circ}$ is an open set. By Theorem 2.3.5, there is a conoidal neighborhood D_j^ν of $Z_j^{\nu\circ}$ and a holomorphic function $f_j^\nu \in \mathcal{O}(D_j^\nu)$ such that $u_j^\nu = b(f_j^\nu; D_j^\nu)$. Then $f_{j_1}^1(z_1)f_{j_2}^2(z_2) \in \mathcal{O}(D_{j_1}^1 \times D_{j_2}^2)$ holds. Since $D_{j_\nu}^\nu$ is a conoidal neighborhood of $Z_j^{\nu\circ}$, $D_{j_1}^1 \times D_{j_2}^2$ is a conoidal neighborhood of $Z_j^{1\circ} \hat{\times} Z_j^{2\circ} \underset{\text{def}}{=} \{(x_1, x_2) + \sqrt{-1}(\xi_1, \xi_2)0 \mid \xi_1, \xi_2 \neq 0$ and $(x_\nu, \sqrt{-1}\xi_\nu 0) \in Z_j^{\nu\circ}$ for $\nu = 1, 2\}$. Notice that $Z_j^{1\circ} \hat{\times} Z_j^{2\circ}$ is an open convex set such that $\tau(Z_j^{1\circ} \hat{\times} Z_j^{2\circ}) = M$. Hence $b(f_{j_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1}^1 \times D_{j_2}^2)$ is a hyperfunction on M. Define, then, the product of $u_1(x_1)$ and $u_2(x_2)$ as $u_1(x_1)u_2(x_2) = \sum_{j_1 \in I_1} \sum_{j_2 \in I_2} b(f_{j_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1}^1 \times D_{j_2}^2)$. Next we will show that this is well defined. Since one can find a refinement of the covering $\{V_j^\nu\}$, the assertion is independent of the choice of coverings. We will confirm that the assertion does not depend upon the choice of f_j^ν. Suppose $u_\nu = \sum_j b(f_j^\nu; D_j^\nu) = \sum_j b(\tilde{f}_j^\nu; D_j^\nu)$ for $\nu = 1, 2$. If one lets $g_j^\nu = \tilde{f}_j^\nu - f_j^\nu$, one obtains $\sum_j b(g_j^\nu; D_j^\nu) = 0$ and S.S. $g_j^\nu \subset V_j^\nu$. By Proposition 3.1.1, $b(g_j^\nu; D_j^\nu) = \sum_k w_{jk}^\nu$ holds for $w_{jk}^\nu \in \mathscr{B}(W_\nu)$ such that $w_{jk}^\nu = -w_{kj}^\nu$ and S.S. $w_{jk}^\nu \subset V_j^\nu \cap V_k^\nu$. Therefore one can find D_{jk}^ν and $g_{jk}^\nu \in \mathcal{O}(D_{jk}^\nu)$, where $D_{jk}^\nu \supset D_j^\nu \cup D_k^\nu$ and D_{jk}^ν is a conoidal neighborhood of $Z_j^{\nu\circ} \cup Z_k^{\nu\circ}$, so that $w_{jk}^\nu = b(g_{jk}^\nu; D_{jk}^\nu)$.

It is sufficient to prove the case $\tilde{f}_j^2 = f_j^2$. Then,

$$\begin{aligned}
\sum_{j_1,j_2} & b(\tilde{f}_{j_1}^1(z_1)f_{j_2}^2(z_2)) - \sum_{j_1,j_2} b(f_{j_1}^1(z_1)f_{j_2}^2(z_2)) \\
&= \sum_{j_1,j_2} b(g_{j_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1}^1 \times D_{j_2}^2) \\
&= \sum_{j_1,j_2,k_1} b(g_{j_1k_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1}^1 \times D_{j_2}^2) \\
&= \sum_{j_1,j_2,k_1} b(g_{j_1k_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1k_1}^1 \times D_{j_2}^2) \\
&= \sum_{j_1,k_1,j_2} b(g_{j_1k_1}^1(z_1)f_{j_2}^2(z_2) + g_{k_1j_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1k_1}^1 \times D_{j_2}^2) = 0,
\end{aligned}$$

i.e. independent of f_j^ν. We will prove $\widehat{\text{S.S.}}\ u \subset \widehat{\text{S.S.}}\ u_1 \times \widehat{\text{S.S.}}\ u_2$ next. Let S.S. $u_\nu \subset Z_\nu = \bigcup_j Z_{j_\nu}^\nu$, where $Z_{j_\nu}^\nu$ is a closed convex set for $\nu = 1, 2$. Then $D_{j_1}^1 \times D_{j_2}^2$ is a conoidal neighborhood of $Z_{j_1}^{1\circ} \hat{\times} Z_{j_2}^{2\circ}$. Therefore, one has S.S. $(b(f_{j_1}^1(z_1)f_{j_2}^2(z_2); D_{j_1}^1 \times D_{j_2}^2)) \subset (Z_{j_1}^{1\circ} \hat{\times} Z_{j_2}^{2\circ})^\circ$. We will show

$$(Z_{j_1}^{1\circ} \hat{\times} Z_{j_2}^{2\circ})^\circ \subset Z_{j_1}^1 \hat{\times} Z_{j_2}^2 \cup W_1 \times Z_{j_2}^2 \cup Z_{j_1}^1 \times W_2.$$

Let $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty) \in (Z_{j_1}^{1\circ} \hat{\times} Z_{j_2}^{2\circ})$. Then $\langle \xi_1, v_1 \rangle + \langle \xi_2, v_2 \rangle > 0$ holds for $x_1 + \sqrt{-1}v_1 0 \in Z_{j_1}^{1\circ}$ and $x_2 + \sqrt{-1}v_2 0 \in Z_{j_2}^{2\circ}$. This inequality holds for any tv_1 and sv_2, $s > 0$ and $t > 0$, which implies $\langle \xi_1, v_1 \rangle \geqq 0$ and $\langle \xi_2, v_2 \rangle \geqq 0$. Note that if $\xi_1 \neq 0$, then $\langle \xi_1, v_1 \rangle > 0$ (and similarly for ξ_2). In fact, since $Z_{j_1}^\circ$ is an open set, if $\langle \xi_1, v_1 \rangle = 0$, then $v_1 - \epsilon v_1'$ is contained in $Z_{j_1}^\circ$ for a sufficiently small $\epsilon > 0$ and v_1' such that $\langle \xi_1, v_1' \rangle > 0$. This implies $\langle \xi_1, v_1 - \epsilon v_1' \rangle = -\epsilon \langle \xi_1, v_1' \rangle < 0$. Hence for $\xi_1 \neq 0$, $\xi_2 \neq 0$, and an arbitrary $v_\nu \in Z_{j_\nu}^{\nu\circ}$, one has $\langle \xi_\nu, v_\nu \rangle > 0$ for $\nu = 1, 2$; i.e. $(x_\nu, \xi_\nu) \in Z_{j_\nu}^\nu$ for $\nu = 1, 2$. Consequently, one obtains $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty) \in Z_{j_1}^1 \hat{\times} Z_{j_2}^2$. If $\xi_1 = 0$, then $\langle \xi_2, v_2 \rangle > 0$ holds for an arbitrary $v_2 \in Z_{j_2}^\circ$. Thus, $((x_1, x_2), \sqrt{-1}(0, \xi_2)\infty) \in M_1 \times Z_{j_2}^\circ$ holds (similarly for the case $\xi_2 = 0$). Therefore, one has S.S.$(b(f_{j_1}^1(z_1)f_{j_2}^2(z_2))) \subset Z_{j_1}^1 \hat{\times} Z_{j_2}^2 \cup W_1 \times Z_{j_2}^2 \cup Z_{j_1}^1 \times W_2$; i.e. S.S. $u \subset Z_1 \hat{\times} Z_2 \cup W_1 \times Z_2 \supset Z_1 \times W_2$ for an arbitrary point on $M_1 \times M_2$. Hence, S.S. $u \subset Z_1 \hat{\times} Z_2 \cup M_1 \times Z_2 \cup Z_1 \times M_2$

is true for any closed convex sets Z_ν with the property S.S. $u_\nu \subset Z_\nu$, $\nu = 1, 2$. Finally, one obtains S.S. $u \subset$ S.S. $u_1 \hat{\times}$ S.S. $u_2 \cup M_1 \times$ S.S. $u_2 \cup$ S.S. $u_1 \times M_2$. From this we will conclude $\widehat{\text{S.S.}}\, u \subset \widehat{\text{S.S.}}\, u_1 \times \widehat{\text{S.S.}}\, u_2$. Let $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)) \in \widehat{\text{S.S.}}\, u$, $\xi_1 \neq 0$, and $\xi_2 \neq 0$. From the above result, one has $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty) \in$ S.S. $u_1 \hat{\times}$ S.S. u_2. Since $((x_1, x_2), \sqrt{-1}(0, \xi_2)\infty) \in M_1 \times$ S.S. u_2 for $\xi_1 = 0$ and $\xi_2 \neq 0$, one must show $x_1 \in \operatorname{supp} u_1$. But if $x_1 \notin \operatorname{supp} u_1$, then $(x_1, x_2) \notin \operatorname{supp} u$ (whose proof is equivalent to the one for independence of the choice of f_j^ν, as before); and similarly, where $\xi_1 \neq 0$ and $\xi_2 = 0$, we reach a contradiction. If $\xi_1 = \xi_2 = 0$, then $(x_1, x_2) \in \operatorname{supp} u$ implies $x_1 \in \operatorname{supp} u_1$ and $x_2 \in \operatorname{supp} u_2$.

Definition 3.1.2. *Let M_1 and M_2 be real analytic manifolds, and let $M = M_1 \times M_2$. Let $(\sqrt{-1}S^*M)' \underset{\text{def}}{=} \sqrt{-1}S^*M - \sqrt{-1}S^*M_1 \times M_2 - M_1 \times \sqrt{-1}S^*M_2$, and define $p_1 : (\sqrt{-1}S^*M)' \to \sqrt{-1}S^*M_1$ and $p_2 : (\sqrt{-1}S^*M)' \to \sqrt{-1}S^*M_2$ by $p_1((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty)) = (x_1, \sqrt{-1}\xi_1\infty)$ and $p_2((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty)) = (x_2, \sqrt{-1}\xi_2\infty)$ respectively. For an open set Ω_ν in $\sqrt{-1}S^*M_\nu$, $\nu = 1, 2$, let $\Omega_1 \hat{\times} \Omega_2 \underset{\text{def}}{=} \{((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty) \mid \xi_1$ and $\xi_2 \neq 0$ and $(x_\nu, \sqrt{-1}\xi_\nu\infty) \in \Omega_\nu$ for $\nu = 1, 2\}$.*

Remark. We have $\Omega_1 \hat{\times} \Omega_2 = p_1^{-1}\Omega_1 \cap p_2^{-1}\Omega_2$.

Theorem 3.1.2. *There exists a canonical sheaf homomorphism*

$$p_1^{-1}\mathscr{C}_{M_1} \times p_2^{-1}\mathscr{C}_{M_2} \to \mathscr{C}_M|_{(\sqrt{-1}S^*M)'}.$$

Remark. We obtain the product $u_1(x_1)u_2(x_2) \in \mathscr{C}_M(\Omega_1 \hat{\times} \Omega_2)$ of $u_1(x_1) \in \mathscr{C}_{M_1}(\Omega_1)$ and $u_2(x_2) \in \mathscr{C}_{M_2}(\Omega_2)$ by the morphism in Theorem 3.1.2. The construction is as follows. By the sheaf homomorphism $\mathscr{C}_{M_\nu} \to p_{\nu*}p_\nu^{-1}\mathscr{C}_{M_\nu}$, $\nu = 1, 2$, for $u_\nu(x_\nu) \in \mathscr{C}_{M_\nu}(\Omega_\nu)$ we have $u'_\nu(x_\nu) \in p_\nu^{-1}\mathscr{C}_{M_\nu}(p_\nu^{-1}\Omega_\nu)$. Then the restrictions of $u_1(x_1)$ and $u_2(x_2)$ give $(u'_1(x_1), u'_2(x_2)) \in (p_1^{-1}\mathscr{C}_{M_1} \times p_2^{-1}\mathscr{C}_{M_2})(\Omega_1 \hat{\times} \Omega_2)$. Therefore, the image of the morphism in the above theorem is $u_1(x_1)u_2(x_2) \in \mathscr{C}_M(\Omega_1 \hat{\times} \Omega_2)$.

Proof of Theorem 3.1.2. For each $\nu = 1, 2$, let $(x_\nu, \sqrt{-1}\xi_\nu\infty) \in \sqrt{-1}S^*M_\nu$ and let $u_\nu \in \mathscr{C}_{M_\nu,(x_\nu,\sqrt{-1}\xi_\nu\infty)}$. From Proposition 2.1.6, $\pi^{-1}\mathscr{B}_{M_\nu} \xrightarrow{\text{sp}} \mathscr{C}_{M_\nu} \to 0$ is exact. This implies that there are $f_1 \in \mathscr{B}_{M_1,x_1}$ and $f_2 \in \mathscr{B}_{M_2,x_2}$ such that $u_1 = \text{sp}(f_1)$ and $u_2 = \text{sp}(f_2)$. Define, then, $u_1(x_1)u_2(x_2) \underset{\text{def}}{=} \text{sp}(f_1 f_2) \in \mathscr{C}_{M,((x_1,x_2),\sqrt{-1}(\xi_1,\xi_2)\infty)}$. We will show next that this definition is independent of the choice of f_1 and f_2. Suppose $u_1 = \text{sp}(f_1) = \text{sp}(\tilde{f}_1)$ and $u_2 = \text{sp}(f_2) = \text{sp}(\tilde{f}_2)$. Then one has $f_1(x_1)f_2(x_2) - \tilde{f}_1(x_1)\tilde{f}_2(x_2) = f_1(x_1)(f_2(x_2) - \tilde{f}_2(x_2)) + (f_1(x_1) - \tilde{f}_1(x_1))\tilde{f}_2(x_2)$. Since $\text{sp}(f_2 - \tilde{f}_2) = u_2 - u_2 = 0$, one obtains $(x_2, \sqrt{-1}\xi_2) \notin \widehat{\text{S.S.}}(f_2(x_2) - \tilde{f}_2(x_2))$. On the other hand, $\widehat{\text{S.S.}}\, f_1(f_2 - \tilde{f}_2) \subset \widehat{\text{S.S.}}\, f_1 \times$ S.S.$(f_2 - \tilde{f}_2)$ holds by Theorem 3.1.1. Hence

$((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty) \notin \widehat{\mathrm{S.S.}}\, f_1(f_2 - \tilde{f}_2)$ holds; i.e. $\mathrm{sp}\, f_1(f_2 - \tilde{f}_2) = 0$ at $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty)$. In a similar manner, one obtains $\mathrm{sp}(f_1(x_1) - \tilde{f}_1(x_1))\tilde{f}_2(x_2) = 0$ at $((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)\infty)$. Consequently, $\mathrm{sp}\, f_1 f_2 = \mathrm{sp}\, \tilde{f}_1 \tilde{f}_2$ holds.

Remark. We proved Theorem 3.1.2 locally at each stalk. In order to prove the theorem globally, notice first that $\mathscr{C}_{M_1}$ is the sheaf associated to the presheaf $\mathscr{B}_{M_1}(\pi(U))/\mathscr{A}^*_{M_1}(U)$ for an open set U. Then one can complete the proof by constructing the map: $(\mathscr{B}_{M_1}(\pi(U_1))/\mathscr{A}^*_{M_1}(U_1) \times (\mathscr{B}_{M_2}(\pi(U_2))/\mathscr{A}^*_{M_2}(U_2)) \to \mathscr{C}_{M_1 \times M_2}(p_1^{-1}U_1 \cap p_2^{-1}U_2)$ via the spectrum map sp.

Definition 3.1.3. *Let N and M be real analytic manifolds, and let $f: N \to M$ be a real analytic map. For $y \in N$ and $\xi \in T^*_{f(y)}M$, define a map $\hat{\rho}: N \underset{M}{\times} T^*M \to T^*N$ by $\hat{\rho}(y, \xi) = (y, f^*(\xi))$. The kernel of $\hat{\rho}$ is said to be the conormal bundle with supports in N, denoted by $T^*_N M$.*

We denote $(T^*_N M - N)/\mathbf{R}^\times_+$ by $S^*_N M$, regarding $N = \{(y, \xi) \in N \underset{M}{\times} T^*M \mid y \in N$ and $\xi = 0\} \subset T^*_N M$. Notice that $\sqrt{-1}S^*_N M$ is a closed set in $N \underset{M}{\times} \sqrt{-1}S^*M$. Let the maps $\rho = \rho_f: (N \underset{M}{\times} \sqrt{-1}S^*M - \sqrt{-1}S^*_N M) \to \sqrt{-1}S^*N$ and $\varpi = \varpi_f: (N \underset{M}{\times} \sqrt{-1}S^*M - \sqrt{-1}S^*_N M) \to \sqrt{-1}S^*M$ be as follows:

$$\rho((y, \sqrt{-1}\xi\infty)) = (y, \sqrt{-1}f^*(\xi)\infty)$$
$$\varpi((y, \sqrt{-1}\xi\infty)) = (f(y), \sqrt{-1}\xi\infty).$$

Note. In the case when f is an embedding, the above definition of a conormal bundle agrees with the one in Definition 2.1.1.

Remark. The important maps ρ and ϖ, above, will be frequently used hereafter in this book. When N is a submainfold of M, the map ρ is an epimorphism and ϖ is a monomorphism. (Compare with Definition 2.1.1.)

Theorem 3.1.3 (Restriction of a Hperfunction). *Let N be a submanifold of M, and ι denotes the embedding $N \hookrightarrow M$. Let u be a hyperfunction on M, $u \in \mathscr{B}_M(M)$, such that $\mathrm{S.S.}\, u \cap \sqrt{-1}S^*_N M = \varnothing$. Then one can define the restriction of u to N, $u|_N \in \mathscr{B}_N$, such that*

$$\widehat{\mathrm{S.S.}}(u|_N) \subset \hat{\rho}(N \underset{M}{\times} \sqrt{-1}T^*M \cap \widehat{\mathrm{S.S.}}\, u)$$

$$\mathrm{S.S.}(u|_N) \subset \rho(N \underset{M}{\times} \sqrt{-1}S^*M \cap \mathrm{S.S.}\, u).$$

Proof. Let X and Y be complexifications of M and N, respectively, and let Y be a submanifold of X. Let each U_j be an open cone containing no lines such that $\mathrm{S.S.}\, u \subset \bigcup U_j$ and $U_j \cap \sqrt{-1}S^*_N M = \varnothing$. Then one has $[\rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)]^\circ = U_j^\circ \cap \sqrt{-1}SN$, since the left-hand side of this equality is $\{x + \sqrt{-1}v0 \in \sqrt{-1}SN \mid \langle v, \iota^*(\xi)\rangle > 0$ for an arbitrary $(x, \sqrt{-1}\xi\infty) \in U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M\} = \{x + \sqrt{-1}v0 \in \sqrt{-1}SN \mid \langle v, \xi\rangle > 0$ for an arbitrary $(x, \sqrt{-1}\xi\infty) \in U_j\}$, which is the right-hand side. Here we used the fact $\langle v, \iota^*(\xi)\rangle = \langle \iota_*(v), \xi\rangle = \langle v, \xi\rangle$. Furthermore, one has $\tau(U_j^\circ \cap \sqrt{-1}SN) = N$. Here is a proof for this. If there exists x in N such that $x \notin \tau(U_j^\circ \cap \sqrt{-1}SN)$, then $\rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)_x \cap (\sqrt{-1}S^*M)_x = (\sqrt{-1}S^*N)_x$, since $\rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)^\circ \cap (\sqrt{-1}SN)_x = \varnothing$ and $\rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)$ is convex. Therefore, there can be found ξ and ξ' in $\sqrt{-1}T_x^*M - \{0\}$ such that $\iota^*(\xi) \neq 0$ and $\iota^*(\xi') = 0$, satisfying $(x, \sqrt{-1}\xi\infty) \in U_j$ and $(x, -\sqrt{-1}(\xi - 2\xi')\infty) \in U_j$. Note that $\frac{1}{2}\xi + \frac{1}{2}(-\xi + 2\xi') = \xi'$, since U_j is convex, implies $(x, \sqrt{-1}\xi'\infty) \in U_j$. But this contradicts $U_j \cap \sqrt{-1}S_N^*M = \varnothing$, since $(x, \sqrt{-1}\xi'\infty) \in S_N^*M$. Conversely, let $(x, \sqrt{-1}\xi'\infty) \in U_j \cap \sqrt{-1}S_N^*M$. For an arbitrary $\xi \in T_x^*M$, one has $(x, \sqrt{-1}(\epsilon\xi + \xi')\infty) \in U_j$ for a sufficiently small $\epsilon > 0$. Then $\rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M) \cap (\sqrt{-1}S^*M)_x = (\sqrt{-1}S^*N)_x$ holds. Hence, the polar set is an empty set; i.e. $U_j^\circ \cap (\sqrt{-1}SN)_x = \varnothing$.

Lemma. *If D is a conoidal neighborhood of $U \subset \sqrt{-1}SM$, then for a complex neighborhood Y of N, $D \cap Y$ is a conoidal neighborhood of $U \cap \sqrt{-1}SN$.*

Proof. Since the assertion is of a local nature, one may assume $N = \{x_1 = \cdots = x_r = 0\} \subset M \subset \mathbf{R}^n$. If D is a conoidal neighborhood of $0 + \sqrt{-1}(0, v_{r+1}, \ldots, v_n)0 \in \sqrt{-1}SM$, then $x + \sqrt{-1}tv \in D$ for $|x| \ll 1$, $0 < t \ll 1$ and $|v - v_0| \ll 1$. That is, for $|(x_{r+1}, \ldots, x_n)| \ll 1$, $0 < t \ll 1$ and $|(v'_{r+1}, \ldots, v'_n) - (v_{r+1}, \ldots, v_n)| \ll 1$, one obtains $(0, x_{r+1}, \ldots, x_n) + \sqrt{-1}(0, v'_{r+1}, \ldots, v'_n)t \in D \cap Y$, completing the proof of the lemma.

Let $u = \sum_j b_{D_j}(f_j)$ for $u \in \mathscr{B}_M(M)$, where D_j is a conoidal neighborhood of U_j° and $f_j \in \mathcal{O}_X(D_j)$. The above lemma implies that $D_j \cap Y$ is a conoidal neighborhood of $U_j^\circ \cap \sqrt{-1}SN$ and $\tau(U_j^\circ \cap \sqrt{-1}SN) = N$. Since $\sum_j b_{D_j \cap Y}(f_j|_Y)$ defines a hyperfunction on N, one can now let $u|_N = \sum b_{D_j \cap Y}(f_j|_Y)$. On the other hand, one has $U_j^\circ \cap \sqrt{-1}SN = \rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)^\circ$. Then $\mathrm{S.S.}(u|_N) \subset \bigcup_j \rho(U_j \cap N \underset{M}{\times} \sqrt{-1}S^*M)$ holds by Theorem 2.3.4. This implies $\mathrm{S.S.}(u|_N) \subset \rho(N \underset{M}{\times} \sqrt{-1}S^*M \cap \mathrm{S.S.}\, u)$. Therefore, since $\mathrm{supp}(u|_N) \subset N \cap \mathrm{supp}\, u$, $\widehat{\mathrm{S.S.}}(u|_N) \subset \hat{\rho}(N \underset{M}{\times} \sqrt{-1}S^*M \cap \widehat{\mathrm{S.S.}}\, u)$.

Definition 3.1.4. *Let X and Y be topological spaces, let $f: X \to Y$ be a continuous map, and let $\mathscr{F}$ be a sheaf over X. Then we denote $\{s \in \Gamma(X, \mathscr{F}) \mid f|_{\mathrm{supp}\, s}$ is a proper map$\}$ by $\Gamma_{f-pr}(X, \mathscr{F})$. Let $f_!(\mathscr{F})$ be the sheaf over Y associated to the presheaf $\Gamma_{f-pr}(f^{-1}(U), \mathscr{F})$, where U is an open set in Y. The sheaf $R^k f_!(\mathscr{F})$ over Y denotes the sheaf associated to the presheaf $H^k(\Gamma_{f-pr}(f^{-1}(U), \mathscr{L}^\bullet))$, where $0 \to \mathscr{F} \to \mathscr{L}^\bullet$ is a flabby resolution of $\mathscr{F}$ and U is an open set in Y.*

Remark. $R^0 f_!(\mathscr{F}) = f_!(\mathscr{F})$ holds and $R^k f_!(\mathscr{F}) = R^k f_*(\mathscr{F})$ if f itself is a proper map.

Theorem 3.1.4 (Restriction of a Microfunction). *Let N be a submanifold of M; then there exists a sheaf homomorphism*

$$\rho_! \varpi^{-1} \mathscr{C}_M \to \mathscr{C}_N.$$

Proof. Let U be an open set in $\sqrt{-1}S^*N$. For W $(\subset N)$, a neighborhood of $\pi_N(U)$, we let

$$G_1(W, U) = \{s \in \mathscr{B}_M(W) \mid \text{S.S. } s \cap \sqrt{-1}S_N^*M = \varnothing\}$$

$$G_2(W, U) = \{s \in \mathscr{B}_W(W) \mid \text{S.S. } s \cap \rho^{-1}(U) = \varnothing, \text{ S.S. } s \cap \sqrt{-1}S_N^*M = \varnothing\}$$

and let

$$G(W, U) = G_1(W, U)/G_2(W, U).$$

Next we will demonstrate that the sheaf associated to the presheaf $\varinjlim_{W \supset \pi_N(U)} G(W, U)$, for an open set U in $\sqrt{-1}S^*N$, is the sheaf $\rho_! \varpi^{-1}\mathscr{C}_M$. First we will show that a map from $G(W, U)$ to $\Gamma(U, \rho_! \varpi^{-1}\mathscr{C}_M)$ can be defined by the correspondence $s \in \mathscr{B}_M(W)$ to $\mathrm{sp}(s)$. Since $\mathrm{sp}(s) \in \mathscr{C}_M(\pi_M^{-1}(W))$, one may regard $\mathrm{sp}(s) \in \Gamma(\rho^{-1}(U), \varpi^{-1}\mathscr{C}_M)$. Now one needs to show that $\rho|_{\mathrm{supp\,sp}(s) \cap \rho^{-1}(U)}$ is a proper map. Consider the commutative diagrams (3.3.1) and (3.3.2) for $x \in W$.

$$\begin{tikzcd} N \underset{M}{\times} \pi_M^{-1}(W) \arrow[r] \arrow[d] & (\sqrt{-1}S^*M)_x \arrow[d, "\text{proper}"] \\ W \arrow[r] & \{x\} \end{tikzcd} \tag{3.3.1}$$

$$\begin{tikzcd} & \mathrm{supp\,sp}(s) \cap (N \underset{M}{\times} \pi_M^{-1}(W) - \sqrt{-1}S_N^*M) \arrow[dl, "\rho"'] \arrow[dd, "\pi_M"] \\ \pi_N^{-1}(W) \arrow[dr, dashed] & \\ & W \end{tikzcd} \tag{3.3.2}$$

Since the map: $N \underset{M}{\times} \pi_M^{-1}(W) \to W \cap N$ is proper and since supp sp$(s) \cap N \underset{M}{\times} \pi_M^{-1}(W)$ is closed in $N \underset{M}{\times} \pi_M^{-1}(W)$, the map:

$$\operatorname{supp} \operatorname{sp}(s) \cap N \underset{M}{\times} \pi_M^{-1}(W) \to W$$

is proper. Therefore $\operatorname{supp} \operatorname{sp}(s) \cap (N \underset{M}{\times} \pi_M^{-1}(W) - \sqrt{-1}S_N^*M) \to W$ is a proper map, since $\operatorname{supp} \operatorname{sp}(s) \cap \sqrt{-1}S_N^*M = \varnothing$. Consequently, one needs the following lemma to claim that $\rho: \operatorname{supp} \operatorname{sp}(s) \cap (N \underset{M}{\times} \pi_M^{-1}(W) - \sqrt{-1}S_N^*M) \to \pi_N^{-1}(W)$ is a proper map; see (3.3.2).

Lemma 1. *Let X, Y, and Z be topological spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous maps. Assume that Y is a Hausdorff space. Then, if $g \circ f$ is a proper map, f is a proper map.*

Proof. In the commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, "f"] & X \underset{Z}{\times} Y \arrow[r] \arrow[d, "\tilde{f}"] & X \arrow[d, "g \circ f"] \\ Y \arrow[r, equal] & Z \underset{Z}{\times} Y \arrow[r] & Z \end{tikzcd}$$

ι is defined by $\iota(x) = (x, f(y))$, and $\tilde{f}$ is defined by $\tilde{f}(x, y) = ((g \circ f)(x), y) = (g(y), y)$. Then $\tilde{f}$ is a proper map, since the map $g \circ f$ is proper. Hence, it is sufficient to prove that $\iota(X)$ is closed in $X \underset{Z}{\times} Y$. Suppose $(x, y) \in X \underset{Z}{\times} Y$ does not belong to $\iota(X)$. Then $y \neq f(x)$ holds. Since Y is Hausdorff, for some open set U and V, where $U \cap V = \varnothing$, one has $f(x) \in U$ and $y \in V$. For such U and V, the neighborhood $f^{-1}(U) \underset{Z}{\times} V$ of (x, y) does not intersect with X; i.e. X is closed. Therefore, the map $f = \tilde{f}|_X$ is a proper map.

By this lemma, the restriction ρ on $\rho^{-1}(U) \cap \operatorname{supp} \operatorname{sp}(s)$ is also a proper map. Hence $\operatorname{sp}(s) \in \Gamma(U, \rho_! \varpi^{-1}\mathscr{C}_M)$. If S.S. $s \cap \rho^{-1}(U) = \varnothing$ for $s \in \mathscr{B}_M(W)$, then $\operatorname{sp}(s) = 0$ on $\rho^{-1}(U)$ holds. Therefore, the above correspondence is a well-defined map. Conversely, if $\operatorname{sp}(s) = 0$ on $\rho^{-1}(U)$, then $s = 0$ as an element of $G(W, U)$; i.e. the map is monomorphic. It remains to be shown that the map is epimorphic at each stalk. First notice that, for $p \in \sqrt{-1}S^*N$, one has $(\rho_! \varpi^{-1}\mathscr{C}_M)_p = \{s \in \Gamma(\rho^{-1}(p), \varpi^{-1}\mathscr{C}_M) \mid \operatorname{supp} \operatorname{sp}(s) \text{ is compact}\}$ as the corollary of Proposition 2.1.4. Let $x = \pi_N(p)$ in N. Then define a section $\tilde{s}$ of $\mathscr{C}_M$ on $F \underset{\text{def}}{=} \rho^{-1}(p) \cup (\sqrt{-1}S_N^*M)_x$ such that $\tilde{s}|_{\rho^{-1}(p)} = s$ and $\tilde{s}|_{\sqrt{-1}S_N^*M} = 0$; i.e. $\tilde{s} \in \Gamma(F, \mathscr{C}_M|_F)$. Since $\rho^{-1}(p)$ is closed in $(\sqrt{-1}S^*M)_x$ $(\sqrt{-1}S_N^*M)_x$, F is a closed set in $(\sqrt{-1}S^*M)_x = \pi_M^{-1}(x)$.

Lemma 2. *If F is a closed set in $\pi^{-1}(x)$, the restriction map*

$$\Gamma(\pi^{-1}(x), \mathscr{C}_M|_{\pi^{-1}(x)}) \to \Gamma(F, \mathscr{C}_M|_F)$$

is epimorphic.

By Lemma 2, and since the map $\mathscr{B}_x \to \Gamma(\pi^{-1}(x), \mathscr{C}_M|_{\pi^{-1}(x)})$ is an epimorphism, there exists $u \in \mathscr{B}_x$ such that u is mapped onto $\tilde{s} \in \Gamma(F, \mathscr{C}_M|_F)$. From the definition of $\tilde{s}$, $\mathrm{sp}(u) = s$ on $\rho^{-1}(p)$ and S.S. $u \cap \sqrt{-1}s_N^*M = \varnothing$ hold. Hence the sheafication of the presheaf $\left\{U \mapsto \varinjlim_{W \supset \pi_N(U)} G(W, U)\right\}$ is $\rho_! \pi^{-1}\mathscr{C}_M$.

Then the correspondence $s \in \mathscr{B}_M(W)$, where S.S. $s \cap \sqrt{-1}S_N^*M = \varnothing$, to $\mathrm{sp}(s|_N) \in \mathscr{C}_N$ gives a sheaf homomorphism $\rho_! \varpi^{-1}\mathscr{C}_M \to \mathscr{C}_N$.

Proof of Lemma 2. Since $\mathscr{B}_x \to \mathscr{C}_{M,p}$ is an epimorphism for $p \in F$, F has a covering $\{U_j\}$ such that for $s \in \Gamma(F, \mathscr{C}_M|_F)$ $\mathrm{sp}(u_j)|_{U_j \cap F} = s$ for some $u_j \in \mathscr{B}_x$. One can let $F \subset \bigcup_{i=1}^{N} U_i$, since F is compact. We will prove by induction on N the case when $N = 1$ is trivial. Assume the assertion for the case of $N - 1$. Since $F - U_N \subset \bigcup_{i=1}^{N-1} U_i$, there exists $u \in \mathscr{B}_x$ such that $\mathrm{sp}(u) = s$ in a neighborhood $\bar{V}$ of $F - U_N$. Then $F \subset U_N \cup V$; i.e. we need only prove for $N = 2$. If $N = 2$, then S.S. $(u_1 - u_2) \subset \complement\bar{U}_1 \cup \complement\bar{U}_2 \cup \complement F$ holds, where $\complement$ denotes the complement, so that $u_1 - u_2 = w_1 - w_2 + t$ where S.S. $w_1 \subset \complement\bar{U}_1$, S.S. $w_2 \subset \complement\bar{U}_2$, and S.S. $t \subset \complement F$. Then let $u \underset{\mathrm{def}}{=} u_1 - w_1 = u_2 - w_2 + t$. One obtains

$$\mathrm{sp}\, u|_{U_1 \cap F} = \mathrm{sp}\, u_1|_{U_1 \cap F} = s|_{U_1 \cap F}$$

and

$$\mathrm{sp}\, u|_{U_2 \cap F} = \mathrm{sp}\, u_2|_{U_2 \cap F} = s|_{U_2 \cap F};$$

i.e. $\mathrm{sp}(u)|_F = s$, completing the proof of Lemma 2. Therefore, we have proved Theorem 3.1.4.

We will consider the product of hyperfunctions on the same manifold.

Theorem 3.1.5 (Product of Hyperfunctions). *Let $u(x)$ and $v(x)$ be hyperfunctions on a real analytic manifold M such that* S.S. $u \cap (\text{S.S. } v)^a = \varnothing$. *Then the product $u(x)v(x) \in \mathscr{B}(M)$ exists with the following properties:* $\widehat{\text{S.S.}}\,(uv) \subset \{(x, \sqrt{-1}(\xi_1 + \xi_2)) \mid (x, \sqrt{-1}\xi_1) \in \widehat{\text{S.S.}}\, u$ and $(x, \sqrt{-1}\xi_2) \in \widehat{\text{S.S.}}\, v\}$. S.S. $(uv) \subset \{(x, \sqrt{-1}(\theta\xi_1 + (1-\theta)\xi_2)\infty) \mid (x, \sqrt{-1}\xi_1\infty) \in$ S.S. u, $(x, \sqrt{-1}\xi_2\infty) \in$ S.S. v and $0 \leqq \theta \leqq 1\} \cup$ S.S. $u \cup$ S.S. v, *where $a: \sqrt{-1}S^*M \to \sqrt{-1}S^*M$ is the antipodal mapping* (i.e. $a(x, \sqrt{-1}\xi\infty) = (x, -\sqrt{-1}\xi\infty)$) *and where* $(\text{S.S. } v)^a$ *denotes the image of* S.S. v *under the antipodal mapping.*

Proof. First of all, the hyperfunction $u(x_1)v(x_2)$ on $M \times M$ can be defined from Theorem 3.1.1. Let the map $\iota: M \to M \times M$ be defined as $\iota(x) = (x, x) \in M \times M$ so that M may be regarded as a subset of $M \times M$. Then define $u(x)v(x) \underset{\text{def}}{=} u(x_1)v(x_2)|_M$. By Theorem 3.1.3, the restriction on M exists if $\text{S.S.}(u(x_1)v(x_2)) \cap \sqrt{-1}S_M^*(M \times M) = \varnothing$ holds, which will be proved next. Note that, by Theorem 3.1.1, one has

$$\widehat{\text{S.S.}}\, u(x_1)v(x_2) \subset \{((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)) \in \sqrt{-1}T^*(M \times M) | (x_1, \sqrt{-1}\xi_1) \in \widehat{\text{S.S.}}\, u \text{ and } (x_2, \sqrt{-1}\xi_2) \in \widehat{\text{S.S.}}\, v\}.$$

On the other hand, the map $\iota: M \to M \times M$ induces the map

$$\hat{\rho}: M \underset{M \times M}{\times} \sqrt{-1}T^*(M \times M) \to \sqrt{-1}T^*M$$

such that $\sqrt{-1}T_M^*(M \times M) = \operatorname{Ker} \hat{\rho} = \{((x_1, x_2), \sqrt{-1}(\xi_1, \xi_2)) | x_1 = x_2$ and $\xi_1 + \xi_2 = 0\}$, where $\hat{\rho}((x, x), \sqrt{-1}(\xi_1, \xi_2)) = (x, \sqrt{-1}(\xi_1 + \xi_2))$. The hypothesis, $\text{S.S.}\, u \cap (\text{S.S.}\, v)^a = \varnothing$, implies $\widehat{\text{S.S.}}\, u(x_1)v(x_2) \cap \sqrt{-1}T_M^*(M \times M) = \varnothing$. Hence $\text{S.S.}\, u(x_1)v(x_2) \cap \sqrt{-1}S_N^*(M \times M) = \varnothing$ holds. Consequently, $u(x)v(x)$ is defined by Theorem 3.1.1.

Remark 1. Theorem 3.1.5 can be proved more directly as follows. Let u and v be hyperfunctions on M, and let $\text{S.S.}\, u \cap (\text{S.S.}\, v)^a = \varnothing$. Then one can find finitely many properly closed convex sets U_j and V_k so that

$$\text{S.S.}\, u \subset \bigcup_j U_j,$$

$$\text{S.S.}\, v \subset \bigcup_k V_k,$$

and

$$U_j \cap V_k^a = \varnothing.$$

Let D_j be a conoidal neighborhood of U_j°, and let D_k' be a conoidal neighborhood of V_k°. If $\tau(U_j^\circ \cap V_k^\circ) = M$, then $D_j \cap D_k'$ is a conoidal neighborhood of $U_j^\circ \cap V_k^\circ$ and $\varphi_j \cdot \varphi_k' \in \mathcal{O}(D_j \cap D_k')$, where $u = \sum_j b_{D_j}(\varphi_j)$ and $v = \sum_k b_{D_k'}(\varphi_k')$. Therefore, one can define $u \cdot v = \sum_{j,k} b_{D_j \cap D_k'}(\varphi_j \cdot \varphi_k')$ since $b_{D_j \cap D_k'}(\varphi_j \cdot \varphi_k')$ is a hyperfunction on M. If there exists $x \in M$ such that $U_j^\circ \cap U_k^\circ \cap (\sqrt{-1}SM)_x = \varnothing$, then one obtains $(U_j^\circ \cap V_k^\circ) \cap (\sqrt{-1}SM)_x = (U_j \cup V_k)^\circ \cap (\sqrt{-1}SM)_x = \varnothing$; i.e. there is ξ with the properties $(x, \sqrt{-1}\xi\infty) \in U_j \cup V_k$ and $(x, -\sqrt{-1}\xi\infty) \in U_j \cup V_k$. But since U_j and V_k are properly convex, $(x, \sqrt{-1}\xi\infty) \in U_j$ implies $(x, -\sqrt{-1}\xi\infty) \in V_k$; i.e. $(x, \sqrt{-1}\xi\infty) \in U_j \cap V_k^a = \varnothing$, a contradiction. Hence one concludes $\tau(U_j^\circ \cap V_k^\circ) = M$. Therefore, the product $u \cdot v$ exists.

Remark 2. Under the assumption in Theorem 3.1.5, we have the commutativity $u(x)v(x) = v(x)u(x)$, but the associativity $(u_1(x)u_2(x))u_3(x) = u_1(x)(u_2(x)u_3(x))$ does not hold in general even if both sides are well defined. For example, let $u_1(x) = 1/(x + i0)$, $u_2(x) = x$, and $u_3(x) = \delta(x)$; then $(u_1u_2)u_3 = \delta(x)$ and $u_1(u_2u_3) = 0$.

Let S.S. $u_j = Z_j$, $1 \leqq j \leqq 3$, and let $Z_1 \cap Z_2^a = \varnothing$. Then u_1u_2 is defined, and one can also define $Z_1 + Z_2 = \{(x, \sqrt{-1}(\xi_1 + \xi_2)\infty) \mid (x, \sqrt{-1}\xi_1\infty) \in Z_1$ and $(x, \sqrt{-1}\xi_2\infty) \in Z_2\}$. Furthermore, if $(Z_1 + Z_2)^a \cap Z_3 = \varnothing$ holds, then the associativity can hold. In order to see this, let $u_i = \sum_j b_{D_{ij}}(f_{ij})$, where $1 \leqq i \leqq 3$ as in Remark 1. Then $b_{D_{1i} \cap D_{2j} \cap D_{3k}}(f_{1i} \cdot f_{2j} \cdot f_{3k})$ determines a hyperfunction on M so that $(u_1u_2)u_3 = \sum_{i,j,k} b_{D_{1i} \cap D_{2j} \cap D_{3k}}(f_{1i} \cdot f_{2j} \cdot f_{3k}) = u_1(u_2 \cdot u_3)$ may hold.

Remark 3. Note that the product and the restriction defined above are stable under the action of differential operators. This is quite remarkable, as the definition of the product based upon conditions of "quantitative" regularity lacks this property. For example, let $u(x)$ and $v(x)$ be continuous functions on M. Then $u(x)v(x)$ as a continuous function (and therefore as a hyperfunction) can always be defined. However, since for a differential operator $P(D)$, $P(D)u(x)$ need not be a continuous function, the product of $P(D)u(x)$ and $v(x)$ may not exist.

Theorem 3.1.6 (Product of Microfunctions). *Let M be a real analytic manifold, and let Δ_M be the diagonal set of $M \times M$. Then define*

$$N = \Delta_M \underset{M \times M}{\times} (\sqrt{-1}S^*(M \times M)) - \Delta_M \underset{M \times M}{\times} (M \times \sqrt{-1}S^*M)$$
$$- \Delta_M \underset{M \times M}{\times} (\sqrt{-1}S^*_M M \times M) - \sqrt{-1}S^*_M(M \times M).$$

For a point $z = (x, x, \sqrt{-1}(\xi_1, \xi_2)\infty) \in N$, where $\xi_1 \neq 0$, $\xi_2 \neq 0$, and $\xi_1 + \xi_2 \neq 0$, one lets $p_1(z) = (x, \sqrt{-1}\xi_1\infty) \in \sqrt{-1}S^*M$, $p_2(z) = (x, \sqrt{-1}\xi_2\infty) \in \sqrt{-1}S^*M$, and $q(z) = (x, \sqrt{-1}(\xi_1 + \xi_2)\infty) \in \sqrt{-1}S^*M$. Then there exists a sheaf homomorphism

$$q_!(p_1^{-1}\mathscr{C}_M \times p_2^{-1}\mathscr{C}_M) \to \mathscr{C}_M.$$

Proof. Theorem 3.1.2 implies that $p_1^{-1}\mathscr{C}_M \times p_2^{-1}\mathscr{C}_M \to \mathscr{C}_{M \times M}|_N$ exists. On the other hand, one has a homomorphism $q_!(\mathscr{C}_{M \times M}|_N) \to \mathscr{C}_M$ by Theorem 3.1.4. Hence the composite of these homomorphisms gives us what we need.

Lastly, we will discuss substitutions for hyperfunctions and microfunctions. Let N and M be real analytic manifolds, and let $f: N \to M$ be a smooth real analytic map (i.e. $T_yN \xrightarrow{f^*} T_{f(y)}M$ is surjective for each $y \in N$). Then let X and Y be complexifications of M and N, respectively, and let $f: Y \to X$ be an extension of the smooth map $N \to M$. We shall prove that there can be defined a hyperfunction $u(f(x)) = f^*u \in \mathscr{B}_N(N)$ for a given $u \in \mathscr{B}_M(M)$.

Theorem 3.1.7

(1) (Substitution for Hyperfunction). *If a map $f: N \to M$ is smooth, there is induced a sheaf homomorphism*

$$f^*: f^{-1}\mathscr{B}_M \to \mathscr{B}_N.$$

Furthermore, define $\hat{\rho}: N \underset{M}{\times} (\sqrt{-1}T^*M) \to (\sqrt{-1}T^*N$ and $\varpi: N \underset{M}{\times} (\sqrt{-1}T^*M) \to \sqrt{-1}T^*M$ by $\hat{\rho}((y, \sqrt{-1}\xi)) = (y, \sqrt{-1}f^*(\xi))$ and $\varpi((y, \sqrt{-1}\xi)) = (f(y), \sqrt{-1}\xi)$. Then

$$\widehat{\text{S.S.}}(f^*u) = \hat{\rho}\hat{\varpi}^{-1}(\widehat{\text{S.S.}}\, u)$$

and

$$\text{S.S.}(f^*u) = \rho\varpi^{-1}(\text{S.S. } u)$$

hold (see Definition 3.1.3). *Note that, since f is smooth, ρ and $\hat{\rho}$ are monomorphic.*

(2) (Substitution for Microfunction). *There exists a homomorphism*

$$f^*: \varpi^{-1}\mathscr{C}_M \to \mathscr{H}^0_{N \underset{M}{\times} \sqrt{-1}S^*M}(\mathscr{C}_N).$$

Lemma. *Let $D \subset X - M$ be a conoidal neighborhood of $U \subset \sqrt{-1}SM$. Then, if f is smooth, $f^{-1}(D)$ is a conoidal neighborhood of $(f_*)^{-1}U = \{y + \sqrt{-1}v0 \in \sqrt{-1}SN \mid f(y) + \sqrt{-1}f_*v0 \in U\}$.*

Proof of Lemma. The question being local and f being smooth, one may assume $M = \mathbf{R}^n$ and $N = \mathbf{R}^n \times \mathbf{R}^l$. Furthermore, one may let $(x, y) + \sqrt{-1}v0 = 0 + \sqrt{-1}(\xi, \eta)0$. When D is a conoidal neighborhood of $0 + \sqrt{-1}\xi 0$, by definition $x + \sqrt{-1}t\xi' \in D$ for $|x| \ll 1$, $0 < t \ll 1$ and $|\xi' - \xi| \ll 1$. On the other hand, if $(x, y) + \sqrt{-1}t(\xi', \eta') \in \tilde{D}$ holds for some (ξ, η) where $|x| \ll 1$, $|y| \ll 1$, $|\xi' - \xi| \ll 1$, and $|\eta' - \eta| \ll 1$, then $\tilde{D}$ is a conoidal neighborhood of $0 + \sqrt{-1}(\xi, \eta)0$. So if one lets $\tilde{D} = f^{-1}(D)$, the assertion follows from the choice of the coordinate system.

Proof of Theorem 3.1.7. For $u \in \mathscr{B}(M)$, choose finitely many properly closed convex sets U_j so that S.S. $u \subset \bigcup_j U_j$. Let D_j be a conoidal neighborhood of $U_j^\circ \subset \sqrt{-1}SM$, and let $u = \sum_j b_{D_j}(\varphi_j)$. Then $b_{f^{-1}(D_j)}(\varphi_j \circ f)$ is a hyperfunction on N since $f^{-1}(D_j)$ is a conoidal neighborhood of

$((f_*)^{-1}U_j)^\circ$, where $f_*: TN \to N \underset{M}{\times} TM$ is the transposed mapping of $\rho: N \underset{M}{\times} T^*M \to T^*N$. Then define $f^*u = \sum_j b_{f^{-1}(D_j)}(\varphi_j \circ f)$.

Let us study the singularity spectrum of f^*u. We need to show $((f_*)^{-1}U_j)^\circ \subset \rho\varpi^{-1}U_j^\circ$ to claim $\mathrm{S.S.}(f^*u) \subset \rho\varpi^{-1}(\mathrm{S.S.}\, u)$. Let $W = \mathrm{Ker}(f_*: TN \to N \underset{M}{\times} TM)$, and let $(x, \sqrt{-1}\xi\infty) \in ((f_*)^{-1}U_j)^\circ \subset \sqrt{-1}S^*N$. Since $\langle \xi, v\rangle > 0$ holds for an arbitrary v such that $f(x) + \sqrt{-1}f_*(v)0 \in U_j$, one has $\langle \xi, v + aw\rangle = \langle \xi, v\rangle + a\langle \xi, w\rangle > 0$ for any $w \in W$ and $a \in \mathbf{R}^\times$. Hence $\langle \xi, w\rangle = 0$; i.e. $\xi \in W^\perp = \rho(N \underset{M}{\times} T^*M)$. This implies $\xi = \rho(\xi')$ for some $\xi' \in T^*_{f(x)}M$. Therefore one obtains $\langle \xi, v\rangle = \langle \rho(\xi'), v\rangle = \langle \xi', f_*(v)\rangle > 0$ for any v with the condition $f(x) + \sqrt{-1}f_*(v)0 \in U_j$. Since f_* is surjective, one gets $\xi' \in \varpi^{-1}U_j^\circ$, i.e. $(x\sqrt{-1}\xi\infty) \in \rho\varpi^{-1}U_j^\circ$.

$\mathrm{S.S.}(f^*u) \supset \rho\varpi^{-1}(\mathrm{S.S.}\, u)$ will be shown next. One may assume that $N = M \times L$ and that f is a projection on M, owing to the local nature of the assertion. First note that $f^*u|_{M \times \{x_0\}} = u$ for $x_0 \in L$. This can be seen if one represents u as boundary values of a holomorphic function. Then, by Theorem 3.1.3, for $q_{x_0}: (M \times \{x_0\} \underset{M \times L}{\times} \sqrt{-1}T^*(M \times L) \to \sqrt{-1}T^*(M \times \{x_0\})$, one has $\widehat{\mathrm{S.S.}}\, u \subset q_{x_0}(\widehat{\mathrm{S.S.}}\, f^*u)$. By varying x_0 in L, one can consider $L \times \widehat{\mathrm{S.S}}\, u \subset \widehat{\mathrm{S.S.}}(f^*u)$. On the other hand, one may derive $\hat\rho\hat\varpi^{-1}\, \widehat{\mathrm{S.S.}}\, u = L \times \widehat{\mathrm{S.S.}}\, u$ from the definition. Consequently $\hat\rho\hat\varpi^{-1}\widehat{\mathrm{S.S.}}\, u \subset \widehat{\mathrm{S.S.}}(f^*u)$, completing (1) of this theorem.

As for (2), one has the epimorphisms

$$\mathscr{B}_{M,f(y)} \xrightarrow{\mathrm{sp}} \mathscr{C}_{M,(f(y),\sqrt{-1}\xi\infty)}$$

and

$$\mathscr{B}_{N,y} \xrightarrow{\mathrm{sp}} \mathscr{H}^0_{\sqrt{-1}S^*M \underset{M}{\times} N}(\mathscr{C}_N)_{(y,\sqrt{-1}f^*(\xi)\infty)},$$

and the substitution $f^*: \mathscr{B}_{M,f(y)} \to \mathscr{B}_{N,y}$. Hence, one has the following homomorphism:

$$\mathscr{C}_{M,(f(y),\sqrt{-1}\xi\infty)} \to \mathscr{H}^0_{\sqrt{-1}S^*M \underset{M}{\times} N}(\mathscr{C}_N)_{(y,\sqrt{-1}f^*(\xi)\infty)}.$$

One notices that the above homomorphism can be defined independently from the choice of the inverse map of sp.

Remark. Suppose that a real analytic map $f: N \to M$ is smooth and that $\mathscr{X}$ is a vector field in N. If $\mathscr{X}(\varphi \circ f) = 0$ holds for any function φ on M, i.e. $\mathscr{X}$ is tangent to each fibre of f, then we have $\mathscr{X}(f^*u) = 0$ for $u \in \mathscr{B}(M)$. Hence, for $u = \sum_j b(\varphi_j)$ we have

$$\mathscr{X}(f^*u) = \mathscr{X}\left(\sum_j b(\varphi_j \circ f)\right) = \sum_j b\left(\mathscr{X}(\varphi_j \circ f)\right) = 0.$$

We will give some examples of the various operations discussed in this section.

Example 3.1.1. Let $\delta(x)$ denote the δ-function of one variable, and let $f(x_1, \ldots, x_n)$ be a real-valued real analytic function defined in $U \subset \mathbf{R}^n$, satisfying the following condition.

$$\text{If } f(x) = 0, \text{ then } d_x f(x) \neq 0. \tag{3.1.1}$$

Then $f^*\delta(t)$ is well defined as an element of $\mathscr{B}_{\mathbf{R}^n}(U)$. Furthermore, we have the following inclusion:

$$\begin{aligned} \text{S.S. } f^*\delta(t) \subset \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*U \mid f(x) = 0 \text{ and} \\ \xi = c \operatorname{grad}_x f(x), c \in \mathbf{R} - \{0\}\}. \end{aligned} \tag{3.1.2}$$

In order to see (3.1.2), notice that $\delta(t) = 0$ for $t \neq 0$ and that the map $f: f^{-1}(V) \to V$ is smooth for a neighborhood V of the origin. Then, by Theorem 3.1.7, $f^*\delta(t)$ is well defined and (3.1.2) holds. We often denote $f^*\delta(t)$ by $\delta(f(x))$.

Example 3.1.2. Let $f(x)$ be as in Example 3.1.1. Then $f^*(1/(t + \sqrt{-1}0))$ is well defined, as above, and is sometimes denoted by $1/(f(x) + \sqrt{-1}0)$. As before, we have:

$$\begin{aligned} \text{S.S.}\left(\frac{1}{f(x) + \sqrt{-1}0}\right) \subset \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*U \mid f(x) = 0 \text{ and} \\ \xi = c \operatorname{grad}_x f(x), c > 0\}. \end{aligned} \tag{3.1.3}$$

Let W be a sufficiently small Stein neighborhood of U, and let $D = W - \{z \in W \mid \operatorname{Im} f(z) \leqq 0\}$. The boundary value $b_D(1/f(z))$ of $1/f(z)$ from the domain D is the above $1/(f(x) + \sqrt{-1}0)$ from the proof of Theorem 3.1.7; see also Theorem 2.3.4.

Example 3.1.3. Let U be an open set in $\mathbf{R}^n$, and let $H = \{x \in U \mid h(x) = 0\}$ be a non-singular hypersurface; i.e. h is a real-valued real analytic function such that $dh \neq 0$ on H. Let $f(x)$ be as in Example 3.1.1, with an additional condition:

$$\operatorname{grad}_x h \not\parallel \operatorname{grad}_x f \text{ on } H \cap \{f = 0\}.$$

Then the restriction of $\delta(f(x))$ to H, i.e. $\delta(f(x))|_H$, is well defined as an element of $\mathscr{B}_H(H)$ so that $(f|_H)^*\delta(t) \equiv \delta(f(x)|_H)$. Since the sequence

$$0 \to T_H^*U \to T^*U \underset{U}{\times} H \to T^*H \to 0 \tag{3.1.5}$$

is exact, a point on T^*H can be described as $(x, \xi) \in U \times \mathbf{R}^n \sim T^*U$

modulo $\{(x, \xi) | h(x) = 0$ and $\xi = c \operatorname{grad}_x h(x), c \in \mathbf{R}\}$. Then the following (3.1.6) holds:

$$\text{S.S.}(\delta(f(x))|_H) \subset A/B, \tag{3.1.6}$$

where

$$A = \{(x, \sqrt{-1}\xi) \in U \times \sqrt{-1}\mathbf{R}^n | f(x) = h(x) = 0 \text{ and }$$
$$\xi = c_1 \operatorname{grad}_x f(x) + c_2 \operatorname{grad}_x h(x), (c_1, c_2) \in \mathbf{R}^2, c_1 \neq 0\}$$
$$B = \{(x, \sqrt{-1}\xi) \in U \times \sqrt{-1}\mathbf{R}^n | h(x) = 0 \text{ and } \xi = c \operatorname{grad}_x h(x), c \in \mathbf{R}\}.$$

For example, $\delta(x_1 - x_2)|_{\{x_2 = 0\}} = \delta(x_1)$.

Note. In Example 3.1.3, one may let H be a submanifold of $U \subset \mathbf{R}^n$, as well.

Example 3.1.4. Let $f_1(x)$ and $f_2(x)$ be real-valued real analytic functions satisfying the condition (3.1.1). Assume further that the following condition is satisfied:

On $\{f_1(x) = f_2(x) = 0\}$ we have $\operatorname{grad}_x f_1(x)$ and $\operatorname{grad}_x f_2(x)$ being linearly independent. (3.1.7)

Then $\delta(f_1(x))\delta(f_2(x))$ is well defined, and we have the inclusion

$$\begin{aligned}\text{S.S. } &\delta(f_1)\delta(f_2)\\ &\subset \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*U \,|\, f_1(x) = f_2(x) = 0 \text{ and }\\ &\qquad \xi = c_1 \operatorname{grad}_x f_1 + c_2 \operatorname{grad}_x f_2, (c_1, c_2) \in \mathbf{R}^2 - \{0\}\}.\end{aligned} \tag{3.1.8}$$

Example 3.1.5. Let us assume the condition (3.1.9), which is weaker than (3.1.7), on f_1 and f_2.

For arbitrary $\alpha_1 \geqq 0$ and $\alpha_2 \geqq 0$ such that $\alpha_1 + \alpha_2 \neq 0$, $\alpha_1 \operatorname{grad}_x f_1(x) + \alpha_2 \operatorname{grad}_x f_2(x) \neq 0$ on $\{f_1(x) = f_2(x) = 0\}$. (3.1.9)

Then $1/(f_1(x) + \sqrt{-1}0) \cdot 1/(f_2(x) + \sqrt{-1}0)$ is well defined, and

$$\begin{aligned}\widehat{\text{S.S.}}\, &\frac{1}{f_1(x) + \sqrt{-1}0} \cdot \frac{1}{f_2(x) + \sqrt{-1}0}\\ &\subset \{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*U \,|\, \alpha_1 f_1 = 0, \alpha_2 f_2 = 0 \text{ and }\\ &\qquad \xi = \alpha_1 \operatorname{grad}_x f_1(x) + \alpha_2 \operatorname{grad}_x f_2(x), (\alpha_1, \alpha_2) \in \mathbf{R}^2, \alpha_1, \alpha_2 \geqq 0\}.\end{aligned} \tag{3.1.10}$$

For example, $1/(x_1 + \sqrt{-1}0) \cdot 1/(x_1 - x_2^2 + \sqrt{-1}0)$ and $1/(x_1 + \sqrt{-1}0))^2$ are well defined. On the contrary, $\delta(x_1)\delta(x_1 - x_2^2)$ and $\delta(x_1)^2$ are not well defined in the sense of Theorem 3.1.5.

§2. Integration

We shall describe the integrations of a hyperfunction and a microfunction. The next proposition states that "indefinite integrals" exist.

Proposition 3.2.1. *If* $M = \mathbf{R}^n$, *then one has the following* (a), (b), *and* (c):

(a) $D_1 \underset{\text{def}}{=} \partial/\partial x_1 : \mathscr{B}_M \to \mathscr{B}_M$ *and* $D_1 : \mathscr{C}_M \to \mathscr{C}_M$ *are epimorphisms.*

(b) *Let* U_1 *be a connected open set in* $\mathbf{R}$, *let* U_2 *be an open set in* $\mathbf{R}^{n-1}$, *and let* $u \in \mathscr{B}_M(U)$ *where* $U = U_1 \times U_2$. *If* $D_1 u = 0$, *there exists a unique* $v \in \mathscr{B}_{\mathbf{R}^{n-1}}(U_2)$ *such that* $u(x) = v(x')$, *where* $x' = (x_2, \ldots, x_n)$.

(c) $D_1 : \mathscr{C}_M \to \mathscr{C}_M$ *is an isomorphism over a neighborhood of* $(x, \sqrt{-1}\xi\infty)$, *where* $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_1 \neq 0$. *Let* U_1 *be an open interval in* $\mathbf{R}$, *and let* Ω *be an open set in* $\sqrt{-1}S^*\mathbf{R}^{n-1}$. *Then, in a neighborhood of* $(x, \sqrt{-1}\xi\infty)$, $\xi_1 = 0$, *if* $D_1 u = 0$ *for* $u \in \mathscr{C}_{\mathbf{R}^n}(U_1 \times \Omega)$, *there exists* $v \in \mathscr{C}_{\mathbf{R}^{n-1}}(\Omega)$ *such that* $u = v(x')$.

Recall the following well-known lemma, with the interesting proof from Suzuki [1] applied to the convex case.

Lemma. *Define* $F : \mathbf{C}^n \to \mathbf{C}^{n-1}$ by $F(z_1, \ldots, z_n) = (z_2, \ldots, z_n)$. *Let* Ω *be a convex set in* $\mathbf{C}^n$, *and let* $\Omega_1 = F(\Omega)$. *Then,*

(1) $D_1 \mathscr{O}_{\mathbf{C}^n}(\Omega) = \mathscr{O}_{\mathbf{C}^n}(\Omega)$ *and*

(2) *if* $D_1 u = 0$ *for* $u \in \mathscr{O}_{\mathbf{C}^n}(\Omega)$, *there exists* $v \in \mathscr{O}_{\mathbf{C}^{n-1}}(\Omega_1)$ *such that* $u(z) = v(z')$, *where* $z' = F(z)$.

Proof. We will give a proof for (1), since (2) is clearly true. For $f \in \mathscr{O}_{\mathbf{C}^n}(\Omega)$ we will show that $D_1 u = f$ for some $u \in \mathscr{O}_{\mathbf{C}^n}(\Omega)$. Let open convex sets $\{U_j\}$ be a covering of Ω_1, where $\Omega \supset \{z_1 = \alpha_j\} \times U_j$ for $\alpha_j \in \mathbf{C}$. For $f(z) \in \mathscr{O}_{\mathbf{C}^n}(\Omega)$ define $u_j(z) = \int_{\alpha_j}^{z_1} f(z)\, dz_1$. Then $u_j(z)$ is a holomorphic function defined in $\tilde{U}_j \underset{\text{def}}{=} \Omega \cap F^{-1}(U_j)$. Since $D_1 u_j = f$ on $\tilde{U}_j$, one obtains $D_1(u_j - u_k) = 0$ on $\tilde{U}_j \cap \tilde{U}_k$. Then (2) implies $u_j - u_k \underset{\text{def}}{=} u_{jk} \in \mathscr{O}_{\mathbf{C}^{n-1}}(U_j \cap U_k)$, and $\{u_{jk}\}$ is plainly a cocycle. Since Ω_1 is convex, it is a Stein manifold. Hence $H^1(\Omega_1, \mathscr{O}) = 0$ holds. Further, U_j is also a Stein manifold, since it is convex. Therefore $\{U_j\}$ is a Leray covering. The Leray theorem (Theorem 1.2.1) implies that there exists $u'_j \in \mathscr{O}_{\mathbf{C}^{n-1}}(U_j)$ such that $u_{jk} = u'_j - u'_k$. Let $\tilde{u}_j(z) \underset{\text{def}}{=} u_j(z) - u'_j(F(z)) \in \mathscr{O}_{\mathbf{C}^n}(\tilde{U}_j)$. Then one has $D_1 \tilde{u}_j = f$, and $\tilde{u}_j = \tilde{u}_k$ holds in $\tilde{U}_j \cap \tilde{U}_k$. Therefore, one can let $u \in \mathscr{O}_{\mathbf{C}^n}(\Omega)$ so that $u|_{\tilde{U}_j} = \tilde{u}_j$ holds. Consequently, u is a solution for $D_1 u = f$.

Proof of Proposition 3.2.1. We will prove (a) as follows. Let U be an open convex set in $\mathbf{R}^n$, and let $u \in \mathscr{B}_M(U)$. Then u can be expressed as u

$\sum_j b_{U\times\sqrt{-1}G_j}(\varphi_j)$, where G_j, $1 \leqq j \leqq N$, are properly chosen open convex cones in $\mathbf{R}^n$ and $\varphi_j \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_j)$. The above lemma implies that there exists $\psi_j \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_j)$ such that $D_1\psi_j = \varphi_j$. If one lets $v = \sum_j b(\psi_j)$, then $v \in \mathscr{B}_M(U)$ and $D_1 v = u$. Therefore, $D_1 : \mathscr{B}_M \to \mathscr{B}_M$ is epimorphic. It can be shown similarly that $D_1 : \mathscr{C}_M \to \mathscr{C}_M$ is an epimorphism.

Next we will prove (b). Let us recall a fact from §2 of Chapter I for this case of (b). Let $\xi_j \in \mathbf{R}^n$ for $0 \leqq j \leqq n$ such that the convex hull of ξ_j, $j = 0, 1, \ldots, n$, is a neighborhood of the origin; let $G_j = \{y \in \mathbf{R}^n | \langle y, \xi_k\rangle > 0 \text{ for } k \neq j\}$, and let $G_{jk} = \{y \in \mathbf{R}^n | \langle y, \xi_1\rangle > 0 \text{ for } l \neq j, k\}$. Then one has

$$\mathscr{B}(U) = \bigoplus_j \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_j) / \bigoplus_{j,k}{}' \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_{jk}),$$

where $\bigoplus'$ denotes the alternative sum. Hence $u \in \mathscr{B}(U)$ can be expressed as $u = \sum_j b(\varphi_j)$, $\varphi_j \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_j)$; and if $\sum_j b(\varphi_j) = 0$, then $\varphi_j = \sum_{j,k} \varphi_{jk}$ for $\varphi_{jk} \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_{jk})$ such that $\varphi_{jk} = -\varphi_{kj}$.

We will apply the above result in the following manner. Let U_2 be an open convex set in $\mathbf{R}^{n-1}$, and let $U \underset{\text{def}}{=} (a, b) \times U_2 \subset \mathbf{R}^1 \times \mathbf{R}^{n-1}$. For $u \in \mathscr{B}(U)$, one lets $u = \sum_j b(\varphi_j)$, $\varphi_j \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_j)$. If $D_1 u = \sum_j b(D_1\varphi_j) = 0$, there exists $\psi_{jk} \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_{jk})$, with the property $\psi_{jk} = -\psi_{kj}$, such that $D_1\varphi_j = \sum \psi_{jk}$. Since $U \times \sqrt{-1}G_{jk}$ is convex, $\psi_{jk} = D_1\varphi_{jk}$ for $\varphi_{jk} \in \mathcal{O}_{\mathbf{C}^n}(U \times \sqrt{-1}G_{jk})$ such that $\varphi_{jk} = -\varphi_{kj}$. (For $j = k$, let $\varphi_{jk} = 0$; for $j > k$, one can choose a solution φ_{jk} for $\psi_{jk} = D_1\varphi_{jk}$; and for $j < k$, one can define $\{\varphi_{jk}\}$ such that $\varphi_{jk} = -\varphi_{kj}$.) Then $D_1\varphi_j = \sum_k D_1\varphi_{jk}$ holds. Hence one obtains $D_1\left(\varphi_j - \sum_j \varphi_{jk}\right) = 0$. If one lets $\psi_j = \varphi_j - \sum_j \varphi_{jk}$, then ψ_j does not depend upon z_1. Hence $v = \sum_j b(\psi_j(z'))$, $z' = (z_2, \ldots, z_n)$, is a hyperfunction on $\mathbf{R}^{n-1}$. Since one has $\sum_{j,k} b(\varphi_{jk}) = 0$ on $\mathbf{R}^n$, then $v = \sum_j b(\psi_j) = u$.

We will give a proof of (c) last. In order to prove the first half of the assertion, it is sufficient to show that $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S. } u$, provided that $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S.}(D_1 u)$, i.e. $\text{sp}(D_1 u) = 0$ at $(0, \sqrt{-1}\, dx_1 \infty)$. If $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S.}(D_1 u)$ is true, then $D_1 u = \sum_j b(\psi_j)$, $\psi_j \in \mathcal{O}(U_\epsilon \times \sqrt{-1}\Gamma_j)$, where $U_\epsilon = \{x | |x| < \epsilon\}$ and Γ_j is an open convex cone such that $\Gamma_j \subset \{y \in \mathbf{R}^n | y_1 < 0\}$ (see Theorem 2.3.4). If $\Gamma_{j,\epsilon} = \Gamma_j \cap \{y | |y| < \epsilon\}$, since $U_\epsilon \times \sqrt{-1}\Gamma_{j,\epsilon}$ is convex, then $D_1\varphi_j = \psi_j$ holds for $\varphi_j \in \mathcal{O}_{\mathbf{C}^n}(U_\epsilon \times \sqrt{-1}\Gamma_{j,\epsilon})$ by the above lemma. Then let $v = \sum_j b(\varphi_j)$. One obtains $D_1 u = D_1 v$. Therefore, $D_1(u - v) = 0$. Now (b) implies $u - v = w(x')$, $x' = (x_2, \ldots, x_n)$; i.e. $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S.}(u - v)$. On the other hand, $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S.}\, v$ holds from the definition of v and from Theorem 2.3.5. Consequently, one obtains $(0, \sqrt{-1}\, dx_1 \infty) \notin \text{S.S.}\, u$. We will prove the latter half of (c). Let $u = \text{sp}(f)$ for $f \in \mathscr{B}(\mathbf{R}^n)$. Suppose $(x, \sqrt{-1}\xi\infty) \notin \text{S.S.}(D_1 f)$; then, by Theorem 2.3.5, one can express $D_1 f = \sum_j b(\psi_j)$, where $\psi_j \in \mathcal{O}_{\mathbf{C}^n}(U_\epsilon \times \sqrt{-1}\Gamma_{j,\epsilon})$ and $\Gamma_j \subset \{y \mid \langle y, \xi\rangle < 0\}$. Define $v = \sum_j b(\varphi_j)$ for $\varphi_j \in \mathcal{O}_{\mathbf{C}^n}(U_\epsilon \times \sqrt{-1}\Gamma_{j,\epsilon})$ such that $D_1 \varphi_j = \psi_j$. One has $(x, \sqrt{-1}\xi\infty) \notin \text{S.S.}\, v$. Since $D_1 f = D_1 v$, then $D_1 w = 0$ for $w = f - v$. Since in a neighborhood of $(x, \sqrt{-1}\xi\infty)$, $\xi_1 = 0$, $\text{sp}(v) = 0$ holds, one has $\text{sp}(u) = \text{sp}(w)$.

Let N be a real analytic manifold, and let $u(t, x)$ be a hyperfunction on $\mathbf{R} \times N$ such that $\text{supp}\, u(t, x) \subset \{(t, x) \in \mathbf{R} \times N \mid a < t < b\}$. Then there exists $v(t, x)$ such that $D_t v(t, x) = u(t, x)$ by Proposition 3.2.1. (In the case when N is a manifold, Grauert's theorem can be applied to prove the assertions in Proposition 3.2.1.) Then define the integration $\int u(t, x)\, dt = v(b, x) - v(a, x)$. Since $D_t v(t, x) = u(t, x) = 0$ in a neighborhood of a and b, we have $\text{S.S.}\, v \cap \sqrt{-1}S^*_{\{b\}\times N}(\mathbf{R} \times N) = \varnothing$ by the first half of the statement in (c) of Proposition 3.2.1. Notice that $v(t, x)|_{t=b} = v(b, x)$ is well defined by Theorem 3.1.3. Furthermore, we have $v(b', x) = v(b, x)$ for $b' > b$ and $v(a', x) = v(a, x)$ for $a' < a$ from (b) of Proposition 3.2.1. Therefore, $\int u(t, x)\, dt$ does not depend upon the choice of (a, b). Next, it can be seen also from the fact that $\int u(t, x)\, dt$ can be defined as $\tilde{v}(b, x) - \tilde{v}(a, x)$, where $\tilde{v}(t, x)$ is a hyperfunction satisfying $D_t \tilde{v}(t, x) = u(t, x)$ by Proposition 3.2.1(b). Hence $\int u(t, x)\, dt$ can be defined as in the above.

If a hyperfunction $u(t, x) = u(t_1, \ldots, t_m, x)$ on $\mathbf{R}^m \times \mathbf{N}$ satisfies $\text{supp}(u(t, x)) \subset K \times N$ for a relatively compact set K in $\mathbf{R}^m$, then by repeated use of the above argument one can define $\int u(t, x)\, dt = \int(\ldots\int(\ldots(\int u(t, x)\, dt_1)\, dt_2 \ldots)\, dt_m$. Let $u(t)$ be a hyperfunction of one variable, let $\text{supp}\, u(t) = K \subset (a, b)$, and let D be a complex neighborhood of K. One may assume that D contains $[a, b]$, and we denote the restriction of $\tau \in \mathbf{C}$ to $\mathbf{R}$ by $t = \tau|_{\mathbf{R}}$. Then $u(t)$ can be expressed as the boundary values of a holomorphic function $\varphi(\tau)$ in $D - K$. That is, $u(t) = \varphi_+(\tau) - \varphi_-(\tau)$ where $\varphi_\pm(\tau) = \lim\limits_{\text{Im}\,\tau \to \pm 0} \varphi(\tau)$. Let $\psi_+(\tau) = \int_a^\tau \varphi(\tau)\, d\tau$ for $\text{Im}\,\tau > 0$; let $\psi_-(\tau) = \int_a^\tau \varphi(\tau)\, d\tau$ for $\text{Im}\,\tau < 0$; and let $v(t) = \psi_+(t + i0) - \psi_-(t - i0)$. Then $D_t v(t) = \varphi(t + i0) - \varphi(t - i0) = u(t)$ holds. Hence we obtain

$$\int u(t)\, dt = v(b) - v(a) = \int_{a\ \gamma_+}^{b} \varphi(\tau)\, d\tau - \int_{a\ \gamma_-}^{b} \varphi(\tau)\, d\tau = \oint_\gamma \varphi(\tau)\, d\tau,$$

where γ is a path around K as in Figure 3.2.1.

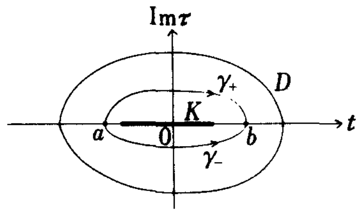

Figure 3.2.1

Example 3.2.1. $\int_{\mathbf{R}} \delta(t)\, dt = 1.$

Proof. Since

$$\delta(t) = \frac{-1}{2\pi i}\left(\frac{1}{t+i0} - \frac{1}{t-i0}\right),$$

the integration around the unit circle of $\varphi(\tau) = -1/2\pi i\tau$ gives

$$\oint_\gamma \frac{d\tau}{\tau} = \int_\pi^0 \frac{ie^{i\theta}}{e^{i\theta}}\, d\theta + \int_0^{-\pi} \frac{ie^{i\theta}}{e^{i\theta}}\, d\theta = [i\theta]_\pi^0 + [i\theta]_0^{-\pi} = -2\pi i.$$

Hence $\oint_\gamma \varphi(\tau)\, d\tau = 1$. One may also prove this more directly, as follows. The derivative of $Y(x) = (-2\pi i + \log(x + i0) - \log(x - i0))/-2\pi i$ with respect to x gives $(d/dx)/Y(x) = -(1/2\pi i)\{1/(x+i0) - 1/(x-i0)\} = \delta(x)$. Therefore

$$\int \delta(x)\, dx = Y(1) - Y(-1) = 1.$$

Note also that x_+ is a primitive function of $Y(x)$.

Example 3.2.2. $\int \delta^{(n)}(x)\, dx = 0$ for $n \geqq 1$, where $\delta^{(n)}(x) = (d^n/dx^n)\,\delta(x)$.

Proof. For $x \neq 0$ one has $\delta^{(n-1)}(x) = 0$ for $n \geqq 1$ and $(d/dx)\delta^{(n-1)}(x) = \delta^{(n)}(x)$.

Example 3.2.3. $\int_{-\infty}^1 x_+^\lambda\, dx = 1/(\lambda + 1)$ for $\lambda \neq -1, -2, \ldots.$

Proof. Recall that x_+^λ is well defined as a hyperfunction for $\lambda \neq -1, -2, \ldots.$ One also has

$$\frac{d}{dx}\left(\frac{1}{\lambda+1}\, x_+^{\lambda+1}\right) = x_+^\lambda.$$

The proof follows from this (see Example 2.4.2).

When a hyperfunction $u(t, x)$ on $\mathbf{R} \times N$ satisfies $\operatorname{supp} u(t, x) \subset \{(t, x) \in \mathbf{R} \times N \mid a < t < b\}$, we define, as before, the definite integral $\int u(t, x)\, dt$

as $v(b, x) - v(a, x)$ via a primitive function $v(t, x)$; i.e. $(\partial/\partial t)v(t, x) = u(t, v)$. For the case $t = (t_1, \ldots, t_m)$, we defined it by repeating the above m times. Thus we obtain a hyperfunction on N through the integration of a hyperfunction on $\mathbf{R} \times N$, satisfying a certain condition on the support. (If $N = \varnothing$, then we get a complex number.) One is naturally led to study the singularity spectrum of $\int u(t, x)\, dt$ when one is provided with the singularity spectrum of $u(t, x)$. We will give an extremely concise answer to this question. The next proposition can be regarded as a modern version of the classical stationary-phase method (see, for example, Lax [1]). It is a geometric interpretation of the smoothing effect of an integral operator. The treatment in Hörmander [3] is closer to the classical one.

Proposition 3.2.2. *Let $u(t, x)$ be a hyperfunction on $\mathbf{R} \times N$ such that $\operatorname{supp} u(t, x) \subset \{(t, x) \in \mathbf{R} \times N \mid a < t < b\}$. If, for $(x_0, \sqrt{-1}\langle \xi_0, dx\rangle\infty) \in \sqrt{-1}S^*N$,*

$$(t, v_0, \sqrt{-1}(0 \cdot dt + \langle \xi_0, dx\rangle)\infty) \notin \text{S.S. } u(t, x)$$

holds for an arbitrary $t \in \mathbf{R}$, then one has

$$(x_0, \sqrt{-1}\langle \xi_0, dx\rangle\infty) \notin \text{S.S.}\left(\int u(t, x)\, dt\right).$$

Remark. It is a quite important fact that such a small set in the singularity spectrum of $u(t, x)$ contributes to the singularity spectrum of $\int u(t, x)\, dt$.

Proof. Let f be the projection $\mathbf{R} \times N \to N$. Since $(\partial/\partial t) \operatorname{sp} v(t, x) = \operatorname{sp} u(t, x) = 0$ holds in a neighborhood of $(t, x_0, \sqrt{-1}\langle \xi_0, dx\rangle\infty)$, by the assumption, then (c) of Proposition 3.2.1 implies that $\operatorname{sp} v(t, x) = f^*w(x)$ for some $w(x) \in \mathscr{C}_{N,(x_0,\sqrt{-1}\xi_0\infty)}$. Consider N as a submanifold of $\mathbf{R} \times N$ by the map $\iota_b : x \mapsto (b, x) \in \mathbf{R} \times N$. Then $f \circ \iota_b = \mathrm{id}_N$ holds, where id_N is the identity map on N. Hence, one obtains $\operatorname{sp} v(b, x) = \operatorname{sp} v(t, x)|_{t=b} = w(x)$; and $\operatorname{sp} v(a, x) = w(x)$ similarly holds. Therefore $\operatorname{sp}(v(b, x) - v(a, x)) = w(x) - w(x) = 0$.

Next we will consider the integration of a microfunction. Let M be a real analytic manifold. By the embedding

$$\mathbf{R} \times \sqrt{-1}S^*M \hookrightarrow \sqrt{-1}S^*(\mathbf{R} \times M),$$

sending $(t, x, \sqrt{-1}\langle \xi, dx\rangle\infty)$ to $((t, x), \sqrt{-1}(0 \cdot dt + \langle \xi, dx\rangle)\infty)$, one can regard $\mathbf{R} \times \sqrt{-1}S^*M$ a subset of $\sqrt{-1}S^*(\mathbf{R} \times M)$. Therefore, $\mathbf{R} \times U$ is a subset of $\sqrt{-1}S^*(\mathbf{R} \times M)$ where U is an open subset of $\sqrt{-1}S^*M$. Let $u(t, x) \in \mathscr{C}_{\mathbf{R}\times M}(\mathbf{R} \times U)$ such that $\operatorname{supp} u(t, x) \subset \{(t, x) \mid a < t < b\}$. Rigorously speaking, $u(t, x)$ is a microfunction on an open subset containing $\mathbf{R} \times U$, since $\mathbf{R} \times U$ is not an open subset of $\sqrt{-1}S^*(\mathbf{R} \times M)$; i.e. $u(t, x) \in \mathscr{C}_{\mathbf{R}\times M}|_{\mathbf{R}\times\sqrt{-1}S^*M}$. Let $v(t, x)$ be a microfunction on $\mathbf{R} \times U$ such that

$(\partial/\partial t)v(t, x) = u(t, x)$. Then define $w(x) = \int u(t, x)\,dt = v(b, x) - v(a, x)$. Consider the diagram

$$
\begin{tikzcd}
\sqrt{-1}S^*(\mathbf{R} \times M) & \Lambda \arrow[l, hook', "\varpi"'] \arrow[d, "\rho"] & \supset (\{b\} \times M) \underset{\mathbf{R}\times M}{\times} (\mathbf{R} \times \sqrt{-1}S^*M) \arrow[dl, "\sim"] \\
& \sqrt{-1}S^*(\{\mathbf{b}\} \times M) &
\end{tikzcd}
$$

where $\Lambda = (\{b\} \times M) \underset{\mathbf{R}\times M}{\times} \sqrt{-1}S^*(\mathbf{R} \times M) - \sqrt{-1}S^*_{\{b\}\times M}(\mathbf{R} \times M)$. Since one has $\varpi^{-1} \operatorname{supp} u \subset (\{b\} \times M) \underset{\mathbf{R}\times M}{\times} (\mathbf{R} \times \sqrt{-1}S^*M)$, the map ρ: $\varpi^{-1} \operatorname{supp} u \to \sqrt{-1}S^*M$ is proper. Then Theorem 3.1.4 implies that $v(t, x)$ has the restriction to $t = b$ (or $t = a$). Therefore $v(b, x)$ and $v(a, x)$ exist. Notice, also, that the definition of $\int u(t, x)\,dt$ does not depend upon the choice of (a, b) and that of v as we showed for the hyperfunction case. Similarly, as before, we define $\int u(t_1, \ldots, t_m, x)\,dt_1 \ldots dt_m = \int(\ldots(\int(\int u(t_1, \ldots, t_m x)\,dt_1)\,dt_2)\ldots)\,dt_m$. Hence we obtain the following theorem.

Theorem 3.2.1

(I) (Integration of Hyperfunction). *Let M and N be real analytic manifolds, and let $f{:}M \times N \to N$ be the natural projection. If $f|_{\operatorname{supp} u}$ is a proper map for a hyperfunction $u(t, x)$ on $M \times N$, then the integration of $u(t, x)$ along the fibre*

$$v(x) = \int_{f^{-1}(x)} u(t, x)\,dt$$

can be defined. Furthermore, one has

$$\text{S.S. } v(x) \subset \pi(\text{S.S. } u \cap M \times \sqrt{-1}S^*N),$$

where π denotes the natural projection from $M \times \sqrt{-1}S^*N$ to $\sqrt{-1}S^*N$. That is, there exists a homomorphism

$$f_!(\mathscr{B}_{M\times N} \otimes v_M) \to \mathscr{B}_N,$$

where $v_M = \Omega^m_M \otimes \omega_M$, Ω^m_M is a sheaf of holomorphic differential forms of degree m on M of dimension m, and ω_M is the orientation sheaf on M.

(II) (Integration of Microfunction). *Let M, N and π be as in* (I), *and let U be an open subset of $\sqrt{-1}S^*N$. If, for $u(t, x) \in \mathscr{C}_{M\times N}(\pi^{-1}(U))$, $\pi|_{\operatorname{supp} u(t,x)}$ is a proper map, then the integration $v(x) = \int_{f^{-1}(x)} u(t, x)\,dt$ is well-defined as a microfunction. Therefore, there exists a homomorphism*

$$\pi_!(\mathscr{C}_{M\times N}|_{M\times\sqrt{-1}S^*N} \otimes v_M) \to \mathscr{C}_N.$$

Proof. We have proved the case when $M = \mathbf{R}^m$. In order to prove the general case, first cover M with coordinate neighborhoods. Then use Proposition 3.1.1 to define the integration in each coordinate neighborhood and paste those together.

Exercise. Find a sufficient condition so that the Fubini theorem $\int dy \int f(x, y)\,dx = \int dx \int f(x, y)\,dy$ may hold. Also find sufficient conditions so that $\int f(x)g(x, y)\,dy = f(x)\int g(x, y)\,dy$ and $\int f(x, y)\,dy)|_{x=a} = \int (f(x, y)|_{x=a})\,dy$ hold.

The following examples are important for applications.

For the δ-function of several variables, we will show $\int_{\mathbf{R}^n} \delta(x)\,dx = 1$. We have

$$\delta(x) = \frac{1}{(-2\pi\sqrt{-1})^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_k}{\prod_{k=1}^{n} (x_k + \sqrt{-1}\epsilon_k 0)}$$

$$= \prod_{k=1}^{n} \frac{1}{(-2\pi\sqrt{-1})} \left(\frac{1}{x_k + \sqrt{-1}0} - \frac{1}{x_k - \sqrt{-1}0} \right) = \delta(x_1)\delta(x_2) \cdots \delta(x_n).$$

Therefore, $\int \delta(x)\,dx = \prod_{i=1}^{n} (\int \delta(x_i)\,dx_i) = 1$ holds, as $\int \delta(x_i)\,dx_i = 1$ was shown before.

We will prove the plane-wave decomposition formula of the δ-function, which is not only of theoretical importance but is also important for applications. This plane-wave decomposition formula is essentially done in John [1] and [2], in which the fundamental solution for elliptic partial differential equations is elegantly constructed, based on this formula. Apparently the plane-wave decomposition formula was crucially important for Sato when he constructed the microfunction theory. We will interpret John's work in the microfunction theoretic framework; see Theorem 3.4.3. Leray [1] is an attempt to apply John's theory to partial differential equations of hyperbolic type. We will give our construction of the fundamental solution for hyperbolic differential equations in §6 of this chapter.

Define an $(n - 1)$-form $\omega(\xi)$ on $\mathbf{R}^n$ as

$$\omega(\xi) = \sum_{i=1}^{n} (-1)^{i-1} \xi_i \, d\xi_1 \wedge \cdots \wedge d\xi_{i-1} \wedge d\xi_{i+1} \wedge \cdots \wedge d\xi_n.$$

Denote $\langle x, \xi \rangle = \sum_{i=1}^{n} x_i \xi_i$ for $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. Since $(t + \sqrt{-1}0)^{-n}$ is well-defined as a hyperfunction of one variable, we have the hyperfunction $(\langle x, \xi \rangle + \sqrt{-1}0)^{-n}$, where t is replaced by $\langle x, \xi \rangle$; see Example 3.1.2. Then the $(n - 1)$-form $\omega(\xi)(\langle x, \xi \rangle + \sqrt{-1}0)^{-n}$ is invariant under the multiplication of ξ by a positive real number. Hence, it can be regarded as an $(n-1)$-form on $S^{n-1} = (\mathbf{R}^n - \{0\})/\mathbf{R}_+^\times$.

Lemma. *To be more precise, let N and M be real analytic manifolds of dimension n and $(n+1)$ respectively, let $f: M \to N$ be a smooth real analytic epimorphism, and let each fibre $f^{-1}(y)$, $y \in N$, be connected. Let v be a vector field on M, such that v is tangent to each fibre, and that v never vanishes on M. If an n-form η on M with hyperfunctions as coefficients satisfies $\iota_v\eta = 0$ and $L_v\eta = 0$, where ι_v denotes the interior product and L_v denotes the Lie derivative, then there exists an n-form, η_0 on N, with coefficients in hyperfunctions such that $\eta = f^*(\eta_0)$.*

Proof. Fix a local coordinate system, and let $f:(t, x) \mapsto x$. Then one can express $v = a(t, x)(\partial/\partial t)$, $\eta = \varphi_0\, dx_1 \wedge \cdots \wedge dx_n + dt \wedge \left(\sum_j g_j(t, x)\, dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n\right)$. One has $\iota_v\eta = a(t, x) \sum_j g_j(t, x)\, dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n = 0$ and $a(t, x) \neq 0$ by the hypothesis. Hence $\eta = \varphi_0\, dx_1 \wedge \cdots \wedge dx_n$. On the other hand, one has

$$L_v\eta = a(t, x)\frac{\partial\varphi_0}{\partial t}\, dx_1 \wedge \cdots \wedge dx_n = 0.$$

Therefore, $\partial\varphi_0/\partial t = 0$ holds. Let $\eta_0 = \varphi_0\, dx_1 \wedge \cdots \wedge dx_n$. Then η_0 is an n-form on N such that $\eta = f^*(\eta_0)$.

In the above, if one lets $v = \sum_{i=1}^n \xi_i(\partial/\partial\xi_i)$, one obtains

$$L_v \frac{\omega(\xi)}{(\langle x, \xi\rangle + i0)^n} = 0,$$

since $L_v\omega(\xi) = n\omega(\xi)$. Similarly,

$$\iota_v \frac{\omega(\xi)}{(\langle x, \xi\rangle + i0)^n} = 0$$

holds. Therefore, by the above lemma, $\omega(\xi)/(\langle x, \xi\rangle + i0)^n$ can be regarded as an $(n$-$1)$-form on S^{n-1}. Next we will prove the following celebrated result of John ([1] and [2]) in our framework.

Proposition 3.2.3 (Planc-Wave Decomposition of the δ-Function).

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int_{S^{n-1}} \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n}$$

holds.

Proof. Since $\omega(\xi)$ is a volume element on S^{n-1}, for $n = 1$, in which case $S^0 = \{+1\} \cup \{-1\}$, one has

$$\frac{1}{-2\pi\sqrt{-1}} \int_{\{\pm 1\}} \frac{|\xi|}{(x\xi + \sqrt{-1}0)}$$

$$= \frac{1}{-2\pi\sqrt{-1}} \left\{ \frac{1}{x + \sqrt{-1}0} + \frac{1}{-x + \sqrt{-1}0} \right\}$$

$$= \frac{1}{-2\pi\sqrt{-1}} \left\{ \frac{1}{x + \sqrt{-1}0} - \frac{1}{x - \sqrt{-1}0} \right\} = \delta(x).$$

The general case will be proved, after the following lemma, by the method of integration along fibres.

Lemma (Feynman). *Let* $a = (a_1, \ldots, a_n) \in \mathbf{C}^n$ *such that* $\langle a, \xi \rangle \neq 0$ *holds for an arbitrary* $\xi = (\xi_1, \ldots, \xi_n) \neq 0$, $\xi_i \geqq 0$ for $i = 1, 2, \ldots, n$. *Then*

$$\frac{1}{a_1 \cdots a_n} = (n-1)! \int_{\xi_1 \geqq 0, \ldots, \xi_n \geqq 0} \frac{\omega(\xi)}{\langle a, \xi \rangle^n}$$

holds.

Proof. We will prove this lemma in the case $\operatorname{Im} a_j < 0$ for $1 \leqq j \leqq n$. The general case can be proved by the analytic continuation. Let $\operatorname{Im} a_k < 0$ and then

$$\frac{1}{a_k} = \sqrt{-1} \int_0^\infty e^{-\sqrt{-1}a_k x_k}\, dx_k$$

holds. Therefore one obtains

$$\frac{1}{a_1 \cdots a_n} = (\sqrt{-1})^n \int_{x_1 \geqq 0, \ldots, x_n \geqq 0} e^{-\sqrt{-1}\langle a, x \rangle}\, dx.$$

Let $x = t\xi$ for $t \in \mathbf{R}$ and $\xi \in S^{n-1}$. Then $dx = t^{n-1}\, dt\omega(\xi)$ holds. Consequently, one obtains

$$\frac{1}{a_1 \cdots a_n} = (\sqrt{-1})^n \int_{\substack{\xi_1 \geqq 0, \ldots, \xi_n \geqq 0 \\ \xi \in S^{n-1}}} \omega(\xi) \int_0^\infty t^{n-1} e^{-\sqrt{-1}\langle a, \xi \rangle t}\, dt$$

$$= (n-1)! \int_{\substack{\xi_1 \geqq 0, \ldots, \xi_n \geqq 0 \\ \xi \in S^{n-1}}} \frac{\omega(\xi)}{\langle a, \xi \rangle^n}.$$

Note. The above formula was used ingeniously by Feynman for the study of Feynman integrals (see §3 of this chapter).

We now return to the proof of Proposition 3.2.3. One has

$$\int \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n} = \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \int_{\epsilon_i \xi_i \geqq 0} \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n}.$$

Then, by the following Proposition 3.2.4, one can first integrate with x being complex and then take the boundary value. That is

$$\begin{aligned}\int_{\epsilon_i \xi_i \geqq 0} \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n} &= b_{\{\mathrm{Im}\langle z, \xi\rangle > 0\}}\left(\int_{\epsilon_i \xi_i \geqq 0} \frac{\omega(\xi)}{\langle z, \xi\rangle^n}\right) \\ &= b\left(\int_{\eta_1 \geqq 0, \ldots, \eta_n \geqq 0} \frac{\omega(\eta)}{\langle z, \xi\eta\rangle^n}\right).\end{aligned}$$

The above lemma implies that

$$\begin{aligned}&\frac{1}{(n-1)!} b\left(\frac{1}{(\epsilon_1 z_1) \cdots (\epsilon_n z_n)}\right) \\ &\qquad = \frac{1}{(n-1)!} \cdot \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + \sqrt{-1}\epsilon_1 0) \cdots (x_n + \sqrt{-1}\epsilon_n 0)}.\end{aligned}$$

Therefore one obtains

$$\begin{aligned}&\int \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n} \\ &\qquad = \frac{1}{(n-1)!} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + \sqrt{-1}\epsilon_1 0) \cdots (x_n + \sqrt{-1}\epsilon_n 0)} \\ &\qquad = \frac{(-2\pi\sqrt{-1})^n}{(n-1)!} \delta(x).\end{aligned}$$

Proposition 3.2.4. *Let M and N be real analytic manifolds, and let $M^{\mathbf{C}}$ and $N^{\mathbf{C}}$ be complexifications of M and N respectively. Further assume that N is compact. Suppose that $N = \bigcup_j K_j$ and $v(N) = \sum_j v(K_j)$ where K_j is a closed subset of N and v denotes the volume. Let U_j be an open subset of $\sqrt{-1}SM$ such that $\tau(U_j) = M$, and let $D_j \subset M^{\mathbf{C}} \times N^{\mathbf{C}}$ be a conoidal neighborhood of $U_j \times K_j$, regarded as a subset of $\sqrt{-1}S(M \times N)$ via the map $\sqrt{-1}SM \times N \hookrightarrow \sqrt{-1}S(M \times N)$. If, in a neighborhood of $M \times K_j$, $u(x, t) = b_{D_j}(f_j(z, \tau))$ holds for some $f_j(z, \tau) \in \mathcal{O}(D_j)$, then there exists a conoidal neighborhood V_j of U_j such that $V_j \times K_j \subset D_j$ and*

$$\int u(x, t)\, dt = \sum_j b_{V_j}\left(\int_{K_j} f_j(z, t)\, dt\right)$$

holds.

Proof. It is sufficient to prove the case when $K_j = [a_1, c_1] \times \cdots \times [a_n, c_n] \subset \mathbf{R}^n$. Let $Y_{a,c}(t) = Y(c-t)Y(t-a)$ for $a, c \in \mathbf{R}$, $a < c$. Then one has

$$\int_{a_1}^{c_1} \cdots \int_{a_n}^{c_n} u(x, t)\, dt = \int u(x, t) Y_{a_1, c_1}(t_1) \cdots Y_{a_n, c_n}(t_n)\, dt.$$

In order to prove that the right-hand side is equal to $b(\int_{a_1}^{c_1} \cdots \int_{a_n}^{c_n} f_j(z, t)\, dt)$, it is enough to prove the case when $n = 1$; i.e.

$$\int_a^c u(x, t)\, dt = b\left(\int_a^c f(z, t)\, dt\right).$$

Let $v(x, t) = b(g(z, t))$ where $g(z, t) = \int_a^t f(z, t)\, dt$. Then it is plain that $(\partial/\partial t)v(x, t) = u(x, t)$ holds. Therefore, one has $b(\int_a^c f(z, t)\, dt) = v(x, c) - v(x, a)$. One now needs to show $\int_a^c u(x, t)\, dt = v(x, c) - v(x, a)$.

Lemma. *If hyperfunctions $u(x, t)$ and $v(x, t)$ satisfy $(\partial/\partial t)v(x, t) = u(x, t)$, then one has*

$$\frac{\partial}{\partial t} w(x, t) = u(x, t) Y_{a,c}(t)$$

where

$$w(x, t) = v(x, t) + Y(t - c)\{v(x, c) - v(x, t)\} + Y(a - t)\{v(x, a) - v(x, t)\}.$$

Proof. One can prove this lemma by direct computation using the next proposition.

We can complete the proof of Proposition 3.2.4 by the above lemma. For the w in the lemma above, one has $\int_a^c u(x, t)\, dt = \int u(x, t) Y_{a,c}(t)\, dt = \int (\partial/\partial t) w(x, t)\, dt = \int_{a'}^{c'} (\partial/\partial t) w(x, t)\, dt = w(x, c') - w(x, a')$, where $a' < a < c < c'$. Furthermore, $w(x, c') = v(x, c)$ and $w(x, a') = v(x, a)$ hold. Hence one obtains

$$\int_a^c u(x, t)\, dt = v(x, c) - v(x, a) = b\left(\int_a^c f(x, t)\, dt\right).$$

Proposition 3.2.5. *If a hyperfunction $u(x, t)$ satisfies the condition $(x, 0; \pm\sqrt{-1}\, dt\infty) \notin$ S.S. u, then the restriction $u(x, 0) = u(x, t)|_{t=0}$ and the product $u(x, t)\delta(t)$ are defined and satisfy*

$$u(x, t)\delta(t) = u(x, 0)\delta(t).$$

Note. The restriction $u(x, 0)$ and $u(x, t)\delta(t)$ are well defined from Theorems 3.1.3 and 3.1.5. Hence, it is sufficient to prove $u(x, t)\delta(t) = u(x, 0)\delta(t)$.

Lemma. *If a hyperfunction $u(x, t)$ satisfies the condition*

$$(x, 0; \pm\sqrt{-1}\, dt\infty) \notin \text{S.S. } u(x, t).$$

then the following two statements are equivalent:

(1) $u(x, 0) = 0$.

(2) *There exists a hyperfunction* $v(x, t)$ *such that* $(x, 0; \pm\sqrt{-1}\, dt\infty) \notin$ S.S. $v(x, t)$ *and* $u(x, t) = tv(x, t)$.

Proof. It is plain that (2) implies (1). We will prove that (1) implies (2). Let $u = \sum_j b(f_j(z, \tau))$. Then $f_j(z, 0)$ is well defined by the assumption on S.S. $u(x, t)$. Since $u|_{t=0} = \sum_j b(f_j(z, 0)) = 0$ by (1), then $f_j(z, 0)$ is a coboundary; i.e. $f_j(z, 0) = \sum_k g_{jk}(z)$ where $g_{jk} = -g_{kj}$. Then let $h_j(z, \tau) = f_j(z, \tau) - \sum_k g_{jk}(z)$. One obtains $u = \sum_j b(f_j) = \sum_j b(f_j - \sum_k g_{jk}) = \sum_j b(h_j)$. Since $h_j(z, \tau)$ is holomorphic and $h_j(z, 0) = 0$, one can express $h_j(z, \tau) = \tau\varphi_j(z, \tau)$. Therefore, if one lets $v = \sum_j b(\varphi_j(z, \tau))$, then $u(x, t) = tv(x, t)$.

From this lemma, $u(x, t) - u(x, 0) = tv(x, t)$ holds. Then, $u(x, t)\delta(t) - u(x, 0)\delta(t) = (tv(x, t))\delta(t) = (v(x, t)t)\delta(t)$ is obtained. The associative law $(v(x, t)t)\delta(t) = u(x, t) \cdot (t\delta(t)) = 0$ holds by the assumption

$$(x, 0; \pm\sqrt{-1}\, dt\infty) \notin \text{S.S. } v(x, t).$$

The following proposition on the δ-function is important for applications.

Proposition 3.2.6. *Let f be a real analytic map from a neighborhood of the origin in* **R** *to a neighborhood of the origin in* **R** *such that* $f(0) = 0$ *and* $f'(0) \neq 0$. *Then one has*

$$\delta(f(x)) = \frac{1}{|f'(0)|}\,\delta(x)$$

in a neighborhood of the origin. Generally, if f is a real analytic map from a neighborhood of the origin in $\mathbf{R}^n$ *to a neighborhood of the origin in* $\mathbf{R}^n$ *such that* $f(0) = 0$ *and* $\det df(0) \neq 0$, *then*

$$\delta(f(x)) = \frac{1}{|\det df(0)|}\,\delta(x)$$

holds.

Proof. Here we will prove the case when $n = 1$, and it is recommended that the reader prove the case in several variables. First consider the case where $f(x) = x\varphi(x)$, $\varphi(x) > 0$. Then one has

$$\delta(f(x)) = \frac{1}{2\pi i}\left(\frac{1}{x\varphi(x) + i0} - \frac{1}{x\varphi(x) - i0}\right) = \frac{1}{\varphi(x)}\,\delta(x) = \frac{1}{\varphi(0)}\,\delta(x).$$

Since $\varphi(0) = |f'(0)|$, then $\delta(f(x)) = (1/|f'(0)|)\delta(x)$ holds. If $\varphi(x) < 0$, Proposition 2.4.1 implies $\delta(f(x)) = \delta(-f(x))$. Then, use $-f(x)$ instead to complete the proof.

Remark. This proposition suggests that coordinate-transformationwise it is $\delta(x)\,dx$ rather than $\delta(x)$ which is defined intrinsically.

Example 3.2.4. $\delta(x^2 - 1) = \frac{1}{2}\delta(x - 1) + \frac{1}{2}\delta(x + 1)$ holds.

Let $X = x - 1$, and let $F(X) = X(2 + X)$ in a neighborhood of $x = 1$. Then we have

$$\delta(x^2 - 1) = \delta(F(X)) = \frac{1}{|F'(0)|}\,\delta(X) = \frac{1}{2}\,\delta(x - 1).$$

We also obtain $\delta(x^2 - 1) = \frac{1}{2}\delta(x + 1)$ in a neighborhood of $x = -1$. The next example can be proved in a similar manner.

Example 3.2.5. $\delta(\cos\theta) = \delta(\theta + \pi/2) + \delta(\theta - \pi/2)$ holds for $-\pi \leqq \theta \leqq \pi$.

We will give useful examples of integrals using the above obtained results.

Example 3.2.6. We have $\int_{\mathbf{R}} \delta(t - x^2)\,dx = t_+^{-1/2}$.

Proof. It is convenient to employ differential equations for this type of computation. We have $x\delta'(x) + \delta(x) = 0$ by differentiating $x\delta(x) = 0$. Hence we have $(t(\partial/\partial t) + \frac{1}{2} + \frac{1}{2}(\partial/\partial x)x)\delta(t - x^2) = t\delta'(t - x^2) + \frac{1}{2}\delta(t - x^2) + \frac{1}{2}\{\delta(t - x^2) + x(-2x)\delta'(t - x^2)\} = (t - x^2)\delta'(t - x^2) + \delta(t - x^2) = 0$. Let $u(t) = \int\delta(t - x^2)\,dx$. Then $(t(\partial/\partial t) + \frac{1}{2})u(t) = \int(t(\partial/\partial t) + \frac{1}{2})\delta(t - x^2)\,dx = \int(-\frac{1}{2}(\partial/\partial x)(x\delta(t - x^2)))\,dx = -\frac{1}{2}\{b\delta(t - x^2) - a\delta(t - a^2)\} = 0$, where, for t in the equations, a and b are chosen so that $t < a^2, b^2$. If $t < 0$, then clearly $u(t) = 0$ holds. Therefore, $u(t) = ct_+^{-1/2}$ for some constant c. Then $c = u(1) = \int\delta(1 - x^2)\,dx = \int(\frac{1}{2}\delta(x - 1) + \frac{1}{2}\delta(x + 1))\,dx = 1$ holds. Consequently, $u(t) = t_+^{-1/2}$.

Example 3.2.7. $\int_{-\pi}^{\pi} d\theta/(\cos\theta + \sqrt{-1}0) = -2\pi\sqrt{-1}$.

Proof. One can prove easily that $A = \int_{-\pi}^{\pi} d\theta/(\cos\theta + i0)$ exists and is finite. By transforming the variable θ to $\theta + \pi$, we have

$$A = \int_{-\pi}^{\pi} \frac{d\theta}{-\cos\theta + i0}.$$

Hence

$$2A = \int_{-\pi}^{\pi}\left(\frac{1}{\cos\theta + i0} + \frac{1}{-\cos\theta + i0}\right)d\theta = \int_{-\pi}^{\pi}(-2\pi i)\delta(\cos\theta)\,d\theta$$

$$= \int_{-\pi}^{\pi}(-2\pi i)\left(\delta\left(\theta + \frac{\pi}{2}\right) + \delta\left(\theta - \frac{\pi}{2}\right)\right)d\theta = \quad 4\pi i$$

holds by the previous example. We conclude

$$A = \int_{-\pi}^{\pi} \frac{d\theta}{(\cos\theta + i0)} = -2\pi i.$$

Example 3.2.8. Let $q(x)$ be a non-degenerate quadratic form, and denote the signature by sgn q. Then we have

$$\int_{S^{n-1}} \frac{\omega(\xi)}{(q(\xi)+\sqrt{-1}0)^{n/2}} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\sqrt{\det(q+\sqrt{-1}0)}}$$

$$= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\exp\left(-\frac{n\sqrt{-1}}{4}(n - \operatorname{sgn} q)\right)}{\sqrt{|\det q|}}.$$

Remark. Recall that $q(x)$ is said to be a quadratic form on $\mathbf{R}^n$ if there exists a real symmetric matrix $A = (a_{ij})$ of order n such that $q(x) = \langle Ax, x\rangle = \sum_{i,j=1}^{n} a_{ij}x_i x_j$. If $\det A \neq 0$, then $q(x)$ is said to be non-degenerate and $\det q$ is defined by $\det A$. All the eigenvalues of A are real, and non-zero if q is non-degenerate. The difference between the number of positive eigenvalues and the number of negative eigenvalues is called the signature of q. (Sometimes the pair of the numbers of the positive and the negative eigenvalues is called the signature of q.) As in Proposition 3.2.3, we consider $\omega(\xi)/(q(\xi)+\sqrt{-1}0)^{n/2}$ as an $(n-1)$-form on S^{n-1}.

Proof. We first consider the case when $q(x)$ is a positive definite form, i.e. sgn $q = n$. Then $q = \langle Ax, x\rangle$ such that $A = {}^tTT$ with T being invertible. Hence one has the following:

$$\int \frac{\omega(\xi)}{(\langle A\xi,\xi\rangle+\sqrt{-1}0)^{n/2}} = \int \frac{\omega(\xi)}{(\langle T\xi, T\xi\rangle + \sqrt{-1}0)^{n/2}}$$

$$= \int \frac{(\det T)^{-1}\omega(\eta)}{(\langle\eta,\eta\rangle+\sqrt{-1}0)^{n/2}}$$

$$= \frac{1}{\sqrt{\det q}} \int_{\xi\in S^{n-1}} \frac{\omega(\xi)}{(\langle\xi,\xi\rangle)^{n/2}}$$

$$= \frac{1}{\sqrt{\det q}} \int_{\xi\in S^{n-1}} \omega(\xi).$$

Notice in the above that $\langle\xi,\xi\rangle = 1$ for $\xi \in S^{n-1}$ holds, and that one then obtains $(\langle \xi, \xi \rangle + \sqrt{-1}0)^{-n/2} = \langle \xi, \xi \rangle^{-n/2}$. Since

$$\int_{\xi \in S^{n-1}} \omega(\xi) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

holds, consequently one obtains

$$\int_{S^{n-1}} \frac{\omega(\xi)}{(q(\xi) + \sqrt{-1}0)^{n/2}} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\sqrt{\det q}}$$

for $\operatorname{sgn} q = n$.

For the general case, we consider each entry of A to be a variable for $q(x) = \langle Ax, x \rangle$. Then $1/(\langle Ax, x \rangle + \sqrt{-1}0)^{n/2}$ can be considered as the boundary value from the domain where $\operatorname{Im} A$ is positive definite; more precisely, $\operatorname{Im}\langle A\xi, \xi \rangle \geqq 0$ regarding ξ as a complex number. If $\operatorname{Re} A$ is positive definite, one has

$$\int \frac{\omega(\xi)}{(q(\xi))^{n/2}} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\sqrt{\det q}}.$$

Therefore, one can continue this equation from the domain where both $\operatorname{Re} A$ and $\operatorname{Im} A$ are positive-definite to the one where $\operatorname{Im} A$ is positive-definite. Hence, one obtains the equation

$$\int \frac{\omega(\xi)}{(\langle A\xi, \xi \rangle + \sqrt{-1}0)^{n/2}} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\sqrt{\det(A + \sqrt{-1}0)}}$$

of hyperfunctions. We will compute $\sqrt{\det(A + \sqrt{-1}0)}$ next.

One may let

$$A = T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} {}^tT, \qquad \operatorname{Im} \lambda_i > 0 \qquad \text{for } 1 \leqq i \leqq n$$

for a positive definite $\operatorname{Im} A$. This implies

$$\sqrt{\det(A(\lambda) + \sqrt{-1}0)} = |\det T| \sqrt{\lambda_1 + \sqrt{-1}0} \cdots \sqrt{\lambda_n + \sqrt{-1}0}.$$

Taking the branch $\sqrt{1 + \sqrt{-1}0} = 1$, one has $\sqrt{-1 + \sqrt{-1}0} = \exp(\pi\sqrt{-1}/2)$. Since the number of negative eigenvalues is $(n - \operatorname{sgn} q)/2$, one obtains $\sqrt{\det(A + \sqrt{-1}0)} = \sqrt{|\det A|} \cdot \exp(\pi\sqrt{-1}(n - \operatorname{sgn} q)/4)$, completing the proof.

Example 3.2.9. $\int_{\mathbf{R}^n} (x_1^2 + \cdots + x_n^2 - t)_-^\lambda \, dx_1 \cdots dx_n = \dfrac{\pi^{n/2}\Gamma(\lambda+1)}{\Gamma(\lambda + n/2 + 1)} t_+^{\lambda + n/2}$.

Proof. As in Example 3.2.6, let $u(t)$ be the left-hand side of this equation. Then $(t(\partial/\partial t) - \lambda - n/2)u(t) = 0$ holds. It is also plain that $u(t) = 0$ for $t < 0$. Therefore, one can express $u(t) = ct_+^{\lambda + n/2}$. Then one obtains

$$c = u(1) = \int_{\mathbf{R}^n} (1 - x_1^2 - \cdots - x_n^2)_+^\lambda \, dx = \int (1 - r^2)_+^\lambda r^{n-1} \, dr\omega(\xi)$$

$$= \int_0^1 (1 - r^2)^\lambda r^{n-1} \, dr \int_{S^{n-1}} \omega(\xi) = \frac{\pi^{n/2}\Gamma(\lambda+1)}{\Gamma\left(\lambda + \frac{n}{2} + 1\right)}.$$

§3. Analyticity of Feynman Integrals

Before discussing the various aspects of the usefulness of microfunction theory for linear partial differential equations, we will study the analyticity of Feynman integrals as an application with a different flavor. Since the results in this section will not be used later on in this book, one may proceed to the next section. As we will explain, Feynman integrals are integrals of hyperfunctions corresponding to Feynman diagrams, which are quite important for the study of the quantum field theory. Even though we will go neither into the background details nor into the applications of algebraic analysis to theoretical physics, now being developed chiefly by Sato, we will give an elementary example here. Those interested are advised to read Eden et al. [1] and Nakanishi [1], and to see *Publ. RIMS, Kyoto Univ. 12*, suppl. (1977) for applications of hyperfunction theory to theoretical physics. Incidentally, it might be worth mentioning that the dispersion relation (see Vladimirov [1]) in quantum field theory had some influence on Sato when he constructed the hyperfunction theory, and that the edge-of-the-wedge theorem (see Morimoto [1]) was apparently within Sato's subconcious when he constructed the microfunction theory. It is also worthy of note that physicists independently found a notion pretty similar to that of microfunctions; see Iagolnitzer and Stapp [1]. It might be said that the naturalness of the theory of hyperfunctions is demonstrating itself.

Definition 3.3.1. *A Feynman diagram D consists of finitely many points, called vertices, $V_1, \ldots, V_{n'}$; finitely many one-dimensional segments, called internal lines, $L_1, \ldots, L_N$; and finitely many half-lines, called external lines, $L_1^e, \ldots, L_n^e$. Each pair of end points W_l^+ and W_l^- of L_l, $l = 1, 2, \ldots, N$, are V_i and V_j for some i and j, $1 \leq i, j \leq n'$. We also assume $W_l^+ \neq W_l^-$ in the following discussion. This vertex of L_r^e, $r = 1, \ldots, n$, coincides with some V_j, $j = 1, \ldots, n'$. Each external line L_r^e associates a real four-dimensional* *vector $p_r = (p_{r,0}, p_{r,1}, p_{r,2}, p_{r,3})$, and each internal line L_l associates a positive constant $m_l^2 \gneq 0$. External and internal lines are oriented, and $\rightarrow$ denotes the orientation. Then the incidence number $[j:l]$ is defined as follows: if V_j is the initial point of L_l, then let $[j:l] = -1$; if V_j is the terminal point of L_l, then $[j, l] = +1$; and if the vertex V_j is neither the initial point nor the terminal point, then the incidence number $[j:l] = 0$. The incidence number $[j, r]$ for V_j and L_r^e is defined in a similar manner.*

An example of a Feynman diagram D is given in Figure 3.3.1, below. We have the incidence numbers as follows in this example: $[1:1] = [1:2] = [1:3] = -1$; $[2:1] = [2, 2] = [2:3] = +1$; $[3:1] = [3:3] = 0$; and so on.

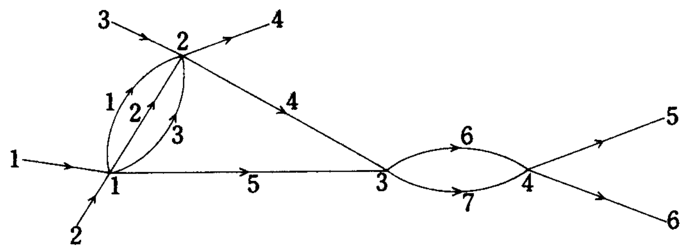

Figure 3.3.1

Remark 1. For the sake of simplicity, we will consider only Feynman diagrams that are connected.

Remark 2. There are some important cases where $m_l^2 = 0$; however, for simplicity, we restrict ourselves to the case $m_l^2 > 0$.

Remark 3. A Feynman diagram expresses (diagrammatically) the interaction of elementary particles, where m_l represents the mass of an elementary particle.

Definition 3.3.2. *Let D be a Feynman diagram. The Feynman integral $F_D(p)$ is formally defined by the integral*

$$F_D(p) = F_D(p_1, \ldots, p_n)$$
$$\underset{\text{def}}{=} \int \frac{\prod_{j=1}^{n'} \delta^4\left(\sum_{r=1}^{n} [j:r]p_r + \sum_{l=1}^{N} [j:l]k_l\right)}{\prod_{l=1}^{N} (k_l^2 - m_l^2 + \sqrt{-1}0)} \prod_{l=1}^{N} d^4k_l, \tag{3.3.1}$$

where $k_l^2 = k_{l,0}^2 - \sum_{v=1}^{3} k_{l,v}^2$. (The square of a four-dimensional vector is defined in this way in this section.)

Note that the above definition is quite formal and that Feynman integrals are divergent integrals in general. The renormalization theory, or self-consistent subtraction theory, gives them a definite meaning consistently. For details, see Nakanishi [1]. We will consider the case where the integral exists, in the microfunction theoretic sense, following Sato [2].

We will assume that for any j there is r such that $[j:r] \neq 0$; i.e. each vertex has at least one external line attached. Then, by replacing $\sum_r [j:r]p_r$ by $-p_j$ in the definition of the integral, one can assume that there is a unique external line leaving each vertex, i.e. identifying j and r. An example of the Feynman diagram under consideration is shown in Figure 3.3.2. Then we have

$$\text{S.S. } \delta^4(p) = \{(p; \sqrt{-1}u\infty) \in \sqrt{-1}S^*\mathbf{R}^4 \mid p = 0, u \neq 0\}$$

$$\text{S.S.}\left(\frac{1}{k^2 - m^2 + \sqrt{-1}0}\right)$$
$$= \{(k; \sqrt{-1}v\infty) \in \sqrt{-1}S^*\mathbf{R}^4 \mid k^2 = m^2, v = ck\,(c > 0)\}.$$

Note. Here the assumption $m^2 \neq 0$ is used. We also identify $\text{grad}_k\, k^2$ with $2k$ via the Minkowsky metric. When one regards $2k_1$ as a cotangent vector, it may be proper to denote it by $\text{grad}_{k_1} k_1^2$ in order to be rigorous. We will follow the notation that physicists use.

Therefore, Theorem 3.1.3 implies that the integrand of (3.3.1) is well defined, and the singularity spectrum is contained in the following set Λ:

$$\Lambda = \left\{(p, k; \sqrt{-1}(u, v)\infty \in \sqrt{-1}S^*\mathbf{R}^{4(n+N)} \,\middle|\, -\sum_{j=1}^{n} [j:l]u_j + \alpha_l k_l = v_l,\right.$$
$$\alpha_l(k_l^2 - m_l^2) = 0 \text{ for } \alpha_l \geqq 0,\ l = 1, \ldots, N \text{ and}$$
$$\left. -p_j + \sum_{l=1}^{N} [j:k]k_l = 0,\ j = 1, \ldots, n\right\}. \tag{3.3.2}$$

We wish to locate the singularity spectrum of $F_D(p)$ by applying Theorem 3.2.1 to the integral (3.3.1). But generally the condition of the theorem

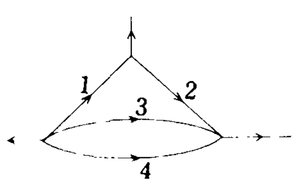

Figure 3.3.2

is not satisfied. Hence, we will look for a point where (3.3.1) determines a microfunction and where it is zero or not zero.

Let

$$\mathscr{L} \underset{\text{def}}{=} \{(p; \sqrt{-1}u\infty) \in \sqrt{-1}S^*\mathbf{R}^{4n} \mid \text{there exist } \alpha_l \text{ and } k_l,\ l = 1, \ldots, N$$

such that

$$\left.\begin{array}{lll} (3.3.3\text{a}) & p_j = \sum_{l=1}^{N} [j:l]k_l, & j = 1, \ldots, n. \\ (3.3.3\text{b}) & \sum_{j=1}^{n} [j:l]u_j = \alpha_l k_l, & l = 1, \ldots, N. \\ (3.3.3\text{c}) & \alpha_l(k_l^2 - m_l^2) = 0, & l = 1, \ldots, N. \\ (3.3.3\text{d}) & \alpha_l \geqq 0, & l = 1, \ldots, N. \end{array}\right\} \qquad (3.3.3)$$

hold$\}$. Furthermore, let

$$\mathscr{L}_0 = \{(p; \sqrt{-1}u\infty) \in \sqrt{-1}S^*\mathbf{R}^{4n} \mid (p; \sqrt{-1}u\infty) \in \mathscr{L}$$
$$\text{and all the } \alpha_1 \text{ are strictly positive}\}.$$

For $(p; \sqrt{-1}u\infty) \in \mathscr{L}_0$, the condition of Theorem 3.2.1 (II) is satisfied in a neighborhood of this point $(p; \sqrt{-1}u\infty)$, since (α_l, k_l) in (3.3.3) uniquely determined (p, u). Notice also that if $(p; \sqrt{-1}u\infty) \notin \mathscr{L}$, then $F_D(p)$ as a microfunction is zero in a neighborhood of $(p; \sqrt{-1}u\infty)$. Therefore, we have the following theorem.

Theorem 3.3.1. *For the Feynman diagram considered in this section, $F_D(p)$ determines a microfunction outside $(\mathscr{L} - \mathscr{L}_0)$ and the support is within $\mathscr{L}_0$.*

Remark 1. Note that $\mathscr{L}$ is called the Landau-Nakanishi variety. Landau and Nakanishi independently found the notion (1959) in their study of the Feynman integral.

Remark 2. As an example, consider Figure 3.3.2. Let a four-dimensional vector u_j be assigned to each vertex in Figure 3.3.2. Then (3.3.3b) indicates that, by assigning a four-dimensional vector k_l to each internal line L, the sum $\alpha_1 k_1 + \alpha_2 k_2 - \alpha_3 k_3$ around a loop in D is zero, where α_1, α_2, and α_3 are properly chosen. Note that treatises in physics often use this form, rather than the condition (3.3.3b).

Remark 3. If $(p; u)$ is a solution to (3.3.3), then $(p; u + a)$ is a solution of (3.3.3) for $a \in \mathbf{R}^4$. This indicates diagrammatically that the diagram may be shifted to any place. In a phrase of analysis, $F_D(p)$, if it is well defined, takes the form $\delta^4\left(\sum_{j=1}^{n} p_j\right)f_D(p)$.

Remark 4. The reader may notice that the obstruction to applying Theorem 3.2.1 to $F_D(p)$ is the fact that the domain of integration is not finite. One of the primary objectives of the self-consistent subtraction theory is to deal with this divergence (ultraviolet divergence) originating from the points where k_l is large. Mathematically this is usually done by compactifying the domain of integration. But we will not go into that here. It is known that $F_D(p)$ is well defined as a hyperfunction under a proper self-consistent subtraction and whose singularity spectrum is contained in $\mathscr{L}$.

Exercise. Let $I_D(p)$ be the integral obtained by replacing $1 \Big/ \prod_{l=1}^{N} (k_l^2 - m_l^2 + \sqrt{-1}0)$ by $\delta(k_l^2 - m_l^2)Y(k_{l,0})$ in Definition 3.3.2. Consider how far one can carry on an argument similar to the one above. Note that $I_D(p)$ is also an important function in physics, called the phase space integral, and that, contrary to $F_D(p)$, the product of integrands is not well defined. See Kawai and Stapp [1] on these topics.

§4. Microlocal Operators and the Fundamental Theorem of Sato

Let M be a real analytic manifold, and let X be a complexification of M. Then let $\Delta_M = \{(x, y) \in M \times M \,|\, x = y\}$, and let

$$\Delta^a_{\sqrt{-1}S^*M} = \{(x, y; \sqrt{-1}(\xi, \eta)\infty \,|\, x = y \text{ and } \xi = -\eta\}.$$

If $K(x, y)$ is a hyperfunction on $M \times M$ with the support in Δ_M and S.S. $K(x, y) \subset \Delta^a_{\sqrt{-1}S^*M}$, then the integral operator $\mathscr{K}$ defined by

$$\mathscr{K}(u(y)) = \int K(x, y)u(y)\, dy$$

is a sheaf homomorphism from $\mathscr{B}_M$ to $\mathscr{B}_M$. Then such an integral operator is a natural generalization of a differential operator of finite order. In this section, we will treat the analogous case for $\mathscr{C}_M$. When one is interested in studying the case for $\mathscr{C}_M$, one may expect that the support of the above hyperfunction $K(x, y)$ need not be contained in Δ_M, judging from §1 and §2 of this chapter. For example, let $M = \mathbf{R}^1$. Then

$$u(y) \mapsto \frac{1}{-2\pi\sqrt{-1}} \int \frac{u(y)}{x - y + \sqrt{-1}0}\, dy$$

induces a sheaf homomorphism on the sheaf $\mathscr{C}_{\mathbf{R}}$ of microfunctions at $\{(x, \sqrt{-1}\, dx\infty)\}$; it is actually an identity map on microfunctions at $\{(x, \sqrt{-1}\, dx\infty)\}$, since

$$\delta(x) = \frac{1}{-2\pi\sqrt{-1}}\left(\frac{1}{x + \sqrt{-1}0} - \frac{1}{x - \sqrt{-1}0}\right).$$

Our intent is to define a class (as general as possible) of integral operators, based on the results in §1 and §2 of this chapter, inducing sheaf homomorphisms on $\mathscr{C}_M$. The class to be considered in this section is too general to enjoy algebraic properties. A desired class for algebraic consideration is the one of microdifferential operators defined in §1 of Chapter IV.

We begin with notations for the discussion which follows. Let M and N be real analytic manifolds, and let Δ_N be the diagonal set of N; i.e. $\Delta_N = \{(x, y) \in N \times N \mid x = y\}$. Define L_1, L_2, and L_3 as follows:

$$L_1 = \sqrt{-1}S^*(M \times N) - M \times \sqrt{-1}S^*N - \sqrt{-1}S^*M \times N,$$

$$\begin{aligned} L_2 = (M \times \Delta_N) \underset{M \times N \times N}{\times} & \sqrt{-1}S^*(M \times N \times N) \\ & - \sqrt{-1}S^*_{M \times \Delta_N}(M \times N \times N) - M \times N \times \sqrt{-1}S^*N \\ & - \sqrt{-1}S^*(M \times N) \times N, \end{aligned}$$

and

$$\begin{aligned} L_3 = \sqrt{-1}S^*(M \times N \times N) & - M \times N \times \sqrt{-1}S^*N \\ & - \sqrt{-1}S^*(M \times N) \times N. \end{aligned}$$

Consider the following diagrams:

$$\begin{tikzcd} & \sqrt{-1}S^*(M \times N) - M \times \sqrt{-1}S^*N - \sqrt{-1}S^*M \times N \arrow[dl, "p_1"] \arrow[dr, "p_2^a"] & \\ \sqrt{-1}S^*M & & \sqrt{-1}S^*N \end{tikzcd}$$

Diagram 3.4.1

where $p_1((x, y; \sqrt{-1}(\xi, \eta)\infty)) = (x, \sqrt{-1}\xi\infty)$ and

$$p_2^a((x, y; \sqrt{-1}(\xi, \eta)\infty)) = (y, -\sqrt{-1}\eta\infty).$$

$$\begin{tikzcd} & & & & \sqrt{-1}S^*(M \times N \times N) \\ & L_1 \arrow[r, hook, "\iota'"] \arrow[ddl, "\gamma_1'"'] & L_2 \arrow[r, hook, "j"] \arrow[dd, "\gamma_1"] & L_3 \arrow[ur, hook, "k"] \arrow[r, "\beta_2"] \arrow[dr, "\beta_1"'] & \sqrt{-1}S^*N \\ & & & & \sqrt{-1}S^*(M \times N) \\ \sqrt{-1}S^*M \times N \arrow[rr, hook, "\iota"] \arrow[d, "\gamma_2"] & & \sqrt{-1}S^*(M \times N) \cong \sqrt{-1}S^*(M \times \Delta_N) & & \\ \sqrt{-1}S^*M & & & & \end{tikzcd}$$

Diagram 3.4.2

where ι, ι', j, and k are natural embeddings. That is, for example, $\iota'((x, y; \sqrt{-1}(\xi, \eta)\infty) = (x, y, y; \sqrt{-1}(\xi, \eta, -\eta)\infty)$, and $\gamma_1((x, y, y; \sqrt{-1}(\xi, \eta_1, \eta_2)\infty)) = (x, y; \sqrt{-1}(\xi, \eta_1 + \eta_2)\infty)$, $\gamma_1' = \gamma_1|_{L_1}$; and β_1, β_2, and γ_2 are natural projections.

Under these notations, a main theorem can be phrased as follows.

Theorem 3.4.1. *Let M and N be real analytic manifolds, and let Z be a locally closed subset of $\sqrt{-1}S^*(M \times N)$ such that $Z \cap M \times \sqrt{-1}S^*N = \varnothing$ and $Z \cap \sqrt{-1}S^*M \times N = \varnothing$. Further assume that $p_1(Z)$ is a locally closed subset of $\sqrt{-1}S^*M$. We denote the sheaf of volume elements of N by v_N. Then define an integral operator $\mathscr{K}$ as*

$$\mathscr{K}(u) = \int_N K(x, y)u(y)\, dy$$

for $K(x, y)\, dy \in H_Z^0(\sqrt{-1}S^*(M \times N), \mathscr{C}_{M\times N} \otimes v_N)$, and $u \in \mathscr{C}_N$. Hence one obtains a sheaf homomorphism

$$\mathscr{K}: (p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_N \to \mathscr{C}_M.$$

Proof. We will consider the restriction of a section $w(x, y_1, y_2)$ of the sheaf $\mathscr{C}_{M\times N\times N}$ over $\sqrt{-1}S^*(M \times N \times N)$ to $\{y_1 = y_2\}$. This operation corresponds to giving a sheaf homomorphism $\gamma_{1!}(k \circ j)^{-1}\mathscr{C}_{M\times N\times N} \to \mathscr{C}_{M\times \Delta_N}$ over $\sqrt{-1}S^*(M \times \Delta_N)$. This induces a sheaf homomorphism

$$\iota^{-1}\gamma_{1!}(k \circ j)^{-1}\mathscr{C}_{M\times N\times N} \to \iota^{-1}\mathscr{C}_{M\times\Delta_N}$$

over $\sqrt{-1}S^*M \times N$. On the other hand, one can show that

$$\begin{aligned}\iota^{-1}\gamma_{1!}(k \circ j)^{-1}\mathscr{C}_{M\times N\times N} &= \gamma'_{1!}\iota'^{-1}(k \circ j)^{-1}\mathscr{C}_{M\times N\times N}\\ &= \gamma'_{1!}(k \circ j \circ \iota')^{-1}\mathscr{C}_{M\times N\times N}\end{aligned}$$

holds. One may identify $\mathscr{C}_{M\times\Delta_N} = \mathscr{C}_{M\times N}$. For the sake of clarity, N_1 and N_2 will be used in place of N. Consequently, one obtains the sheaf homomorphism

$$\gamma'_{1!}(k \circ j \circ \iota')^{-1}(\mathscr{C}_{M\times N_1\times N_2} \otimes v_{N_1}) \to \iota^{-1}\mathscr{C}_{M\times N} \otimes v_N$$

over $\sqrt{-1}S^*M \times N$. Therefore, there is induced a sheaf homomorphism

$$(\gamma_2 \circ \gamma_1')_!(k \circ j \circ \iota')^{-1}(\mathscr{C}_{M\times N_1\times N_2} \otimes v_{N_1}) \to \gamma_{2!}\iota^{-1}(\mathscr{C}_{M\times N} \otimes v_N)$$

over $\sqrt{-1}S^*M$. This map is nothing but the map which assigns $w(x, y, y)\, dy$ for $w(x, y_1, y_2)\, dy_1$. On the other hand, the assignment $w(x, y, y,)\, dy$ to the integral $\int_N w(x, y, y)\, dy$ is a sheaf homomorphism $\gamma_{2!}\iota^{-1}(\mathscr{C}_{M\times N} \otimes v_N) \to \mathscr{C}_M$. Hence we have obtained a sheaf homomorphism

$$(\gamma_2 \circ \gamma_1')_!(k \circ j \circ \iota')^{-1}(\mathscr{C}_{M\times N_1\times N_2} \otimes v_{N_1}) \to \mathscr{C}_M$$

over $\sqrt{-1}S^*M$.

Since we are interested in replacing $w(x, y, y_1, y_2)\, dy_1$ by $K(x, y_1) \cdot u(y_2)\, dy_1$, we first consider the product of K and u. The product corresponds to a sheaf homormophism over L_3,

$$\beta_1^{-1}(\mathscr{C}_{M \times N_1} \otimes v_{N_1}) \times \beta_2^{-1}\mathscr{C}_{N_2} \to k^{-1}\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}.$$

Therefore, for a given $K(x, y_1)\, dy_1 \in H^0_Z(M \times N_1, \mathscr{C}_{M \times N_1} \otimes v_{N_1})$ we have a sheaf homomorphism

$$\beta_2^{-1}\mathscr{C}_{N_2} \to \mathscr{H}^0_{\beta_1^{-1}(Z)}(k^{-1}\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}).$$

From these sheaf homomorphisms we obtain

$$\begin{aligned}(\beta_2 \circ j \circ \imath')^{-1}\mathscr{C}_{N_2} &= \imath'^{-1}j^{-1}\beta_2^{-1}\mathscr{C}_{N_2} \\ &\to \imath'^{-1}j^{-1}\mathscr{H}^0_{\beta_1^{-1}(Z)}(k^{-1}\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}) \\ &\to \mathscr{H}^0_{\imath'^{-1}j^{-1}\beta_1^{-1}(Z)}(\imath'^{-1}j^{-1}k^{-1}(\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}) \\ &= \mathscr{H}^0_{(\beta_1 \circ j \circ \imath')^{-1}(Z)}((k \circ j \circ \imath')^{-1}(\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1})).\end{aligned}$$

Note that $p_1 = \gamma_2 \circ \gamma_1'$, $\beta_1 \circ j \circ \imath' = \mathrm{id}$, and that $\beta_2 \circ j \circ \imath' = p_2^a$ hold. Consequently, one obtains sheaf homomorphisms

$$p_{1!}(k \circ j \circ i')^{-1}(\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}) \to \mathscr{C}_M$$

and

$$(p_2^a)^{-1}\mathscr{C}_{N_2} \to \mathscr{H}^0_Z(k \circ j \circ \imath')^{-1}(\mathscr{C}_{M \times N_1 \times N_2} \otimes v_{N_1}).$$

By restricting the latter homomorphism to Z and operating $p_{1!}$ to it, one obtains, combined with the former homomorphism, a sheaf homomorphism

$$(p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_N \to \mathscr{C}_M$$

over $\sqrt{-1}S^*M$. This homomorphism is the one which, for a given $K(x, y)\, dy \in H^0_Z(\sqrt{-1}S^*(M \times N), \mathscr{C}_{M \times N} \otimes v_N)$, assigns $\int K(x, y)u(y)\, dy$ for $u \in \mathscr{C}_N$.

Corollary. *Let M be a real analytic manifold. Then, for*

$$K(x, y)\, dy \in \mathscr{H}^0_{\sqrt{-1}S^*_M(M \times M)}(\mathscr{C}_{M \times M} \otimes v_M)$$

the integral operator $\mathscr{K}(u) = \int K(x, y)u(y)\, dy$ defines a sheaf homomorphism from $\mathscr{C}_M$ to $\mathscr{C}_M$.

Proof. Suppose $M = N$ in Theorem 3.4.1; then one has a sheaf homomorphism $(p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_M \to \mathscr{C}_M$. Note that in this case $p_1|_Z$ and $p_2^a|_Z$ are isomorphisms; in particular, they are proper and $(p_1|_Z) \circ (p_2^a|_Z)^{-1}$ is an identity on $\sqrt{-1}S^*M$. Hence, $(p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_M = \mathscr{C}_M$.

Remark. We sometimes call $K(x, x')\, dx'$ the kernel function of a microlocal operator $\mathscr{K}$. When there is no fear of confusion, the kernel function and the corresponding microlocal operator are identified and denoted with the same notation.

Definition 3.4.1. *$\mathscr{H}^0_{\sqrt{-1}S^*_M(M\times M)}(\mathscr{C}_{M\times M}\otimes v_M)$ is said to be the sheaf of microlocal operators and denoted by $\mathscr{L}_M$.*

Then we have the following corollary from the corollary of Theorem 3.4.1 and the above definition.

Corollary.

(1) *$\mathscr{L}_M$ acts on $\mathscr{C}_M$ as a sheaf homomorphism.*

(2) *Let $K_1(x, y)\,dy$ and $K_2(x, y)\,dy$ be elements of $\mathscr{L}_M$. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be the integral operators determined by K_1 and K_2 respectively. Then $\mathscr{K}_1 \circ \mathscr{K}_2$ is the microlocal operator having $(\int K_1(x, y')K_2(y', y)\,dy')\,dy$ as the kernel function.*

(3) *$\mathscr{L}_M$ is a ring with identity $\delta(x - y)\,dy$, which acts on $\mathscr{C}_M$ as an identity map.*

Definition 3.4.2. *Let $P(x, D_x) = \sum_{|\alpha|\leqq m} a_\alpha(x)D_x^\alpha$ be a linear differential operator of order m. The principal symbol $\sigma(P)(x, \xi)$ is defined as $\sigma(P)(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \in \mathcal{O}_{T^*M}$.*

Note. $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{Z}^+ \sqcup \{0\})^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$.

Remark 1. Since we seldom deal with non-linear differential operators in this book, we will simply call them differential operators.

Remark 2. A differential operator $P(x, D_x)$ is clearly a microlocal operator with the kernel function $P(x, D_x)\delta(x - y)\,dy$.

Remark 3. Even though the lower-order terms of $P(x, D_x)$ are not invariant under a coordinate transformation, the principal symbol is invariant under a coordinate transformation.

The main aim of the rest of this section is to give a proof of Sato's fundamental theorem, which is the microlocalization of John's construction of a fundamental solution for an elliptic differential equation; see John [1] and [2]. First we will recall the Cauchy-Kovalevsky theorem, which is the most fundamental in the theory of differential equations. Note that the Cauchy-Kovalevsky theorem is valid for the case of non-linear differential equations.

Theorem 3.4.2 (Cauchy-Kovalevsky). *Let $\psi(x)$ be a real analytic function defined in a neighborhood of x_0 such that the hypersurface $\{x\,|\,\psi(x) = 0\}$ is non-characteristic at $x = x_0$ with respect to a differential operator $P(x, D_x)$;*

i.e. $\sigma(P)(x_0, d\psi(x_0)) \neq 0$. *Then, for real analytic functions f and φ defined in a neighborhood of $x = x_0$, there exists a unique real analytic function $u(x)$ such that*

$$Pu = f \quad \text{and} \quad u \equiv \varphi \bmod \psi^m.$$

Proof. See, for example, Oshima and Komatsu [1].

We will prove Sato's fundamental theorem as a consequence of the Cauchy-Kovalevsky theorem. This fundamental theorem from Sato is so important and exquisite that it might not be an exaggeration to say that it was this theorem which convinced all analysts of the importance and the profoundness of the theory of microfunctions. Nowadays, the theorem is so fundamental among specialists that proving the theorem does not occur to them; but Sato's theorem was really revolutionary in the theory of differential equations.

Theorem 3.4.3 (Sato). *A linear differential operator of finite order $P(x, D_x)$ is left- and right-invertible in the ring $\mathscr{L}_M$ over $\{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \mid \sigma(P)(x, \sqrt{-1}\xi) \neq 0\}$.*

Remark. We will later (Corollary of Theorem 4.1.6) prove a stronger result which claims that the left and right inverse is a microdifferential operator.

Proof. Since the statement is local, we will consider it in a neighborhood of $\mathring{x} = 0$ of a local coordinate system. Let $\sigma(P)(0, \sqrt{-1}\xi_0) \neq 0$. By Theorem 3.4.2, there is a real analytic function $u(x, \xi, p)$ defined in a neighborhood of $(x, \xi, p) = (0, \xi_0, 0) \in S^*M \times \mathbf{R}$ so that

$$P(x, D_x)u(x, \xi, p) = 1 \quad \text{and} \quad u \equiv 0 \bmod(\langle x, \xi\rangle - p)^m.$$

Note that $\sigma(P)(0, \xi_0) = (-\sqrt{-1})^m\sigma(P)(0, \sqrt{-1}\xi_0) \neq 0$. We sometimes call such a solution $u(x, \xi, p)$ a unitary solution after Leray. The method of obtaining a fundamental solution by superposing unitary solutions is sometimes called the Duhamel principle. (Generally speaking, the term "Duhamel principle" refers to obtaining a solution v of inhomogeneous equation $Pv = f$ by integrating a family of solutions of homogeneous equation $Pu = 0$ with respect to a parameter, assuming suitable initial conditions on parameterized surfaces.) For the solution $u(x, \xi, p)$, let $v(x, \xi, p) = u(x, \xi, p)Y(\langle x, \xi\rangle - p)$, where $Y(t)$ is the Heaviside function.

We need two lemmas.

Lemma 1. *One has*

$$D_1^j(u(x)Y(x_1)) = (D_1^j u(x))Y(x_1) + \sum_{k=1}^{j} D_1^{j-k}u(0, x')\delta^{(k-1)}(x_1)$$

where $x' = (x_2, \ldots, x_n)$.

Proof. It is plainly true for $j = 0, 1$. We will prove it by induction on j. Suppose the equation holds for $j = j_0$. Then

$$\begin{aligned} D_1^{j_0+1}(u(x)Y(x_1)) &= (D_1^{j_0+1}u(x))Y(x_1) + (D_1^{j_0}u(x))\delta(x_1) \\ &\quad + \sum_{k=1}^{j_0} D_1^{j_0-k}u(0, x')\delta^{(k)}(x_1) \\ &= (D_1^{j_0+1}u(x))Y(x_1) + \sum_{k=1}^{j_0+1} D_1^{j_0+1-k}u(0, x')\delta^{(k-1)}(x_1) \end{aligned}$$

holds. Hence, the inductive assumption implies the above equation and completes the proof.

Lemma 2. *Let $\sigma(P)(x, d\psi(x)) \neq 0$. A real analytic function $u(x)$ such that $P(x, D)u(x) = 1$ and $u \equiv 0 \bmod \psi(x)^m$ satisfies*

$$P(x, D_x)(u(x)Y(\psi(x))) = Y(\psi(x)).$$

Proof. One may assume $\psi = x_1$ after a suitable coordinate transformation, since $d\psi \neq 0$. Let $P(x, D) = \sum_{j=0}^{m} A_j(x, D')D_1^j$, where $A_j(x, D') = \sum_{\alpha' \in \mathbf{Z}_+^{n-1}} a_{\alpha'}(x)D'^{\alpha'}$ and $D' = (D_2, \ldots, D_n)$. By Lemma 1, we obtain

$$\begin{aligned} P(x, D)(u(x)Y(x_1)) &= \sum_{j=0}^{m} A_j(x, D')D_1^j(uY(x_1)) \\ &= \sum_{j=0}^{m} A_j(x, D')(D_1^j u(x))Y(x_1) \\ &\quad + \sum_{j=0}^{m}\sum_{k=1}^{j} A_j(x, D')(D_1^{j-k}u(0, x'))\delta^{(k-1)}(x_1). \end{aligned}$$

On the other hand, since $u \equiv 0 \bmod x_1^m$ and $j - k < m$, one concludes that $D_1^{j-k}u(0, x') = 0$. Consequently, we have

$$P(x, D)(u(x)Y(x_1)) = \sum_{j=0}^{m} A_j(x, D')(D_1^j u(x))Y(x_1) = (Pu)Y(x_1) = Y(x_1),$$

since $Pu = 1$. This completes the proof of Lemma 2.

By Lemma 2, $P(x, D)v(x, \xi, p) = Y(\langle x, \xi\rangle - p)$ holds, and the n-time differentiation with respect to p gives

$$P(x, D_x)(\partial/\partial p)^n v(x, \xi, p) = (-1)^n \delta^{(n-1)}(\langle x, \xi\rangle - p).$$

Let $w(x, \xi, p) = (\partial/\partial p)^n v(x, \xi, p)$. We obtain

$$P(x, D_x)w(x, \xi, p) = (-1)^n \delta^{(n-1)}(\langle x, \xi\rangle - p).$$

If one lets $p = \langle x', \xi\rangle$, then one has

$$P(x, D_x)w(x, \xi, \langle x', \xi\rangle) = (-1)^n\delta^{(n-1)}(\langle x - x', \xi\rangle).$$

We have the following lemma.

Lemma 3. *In a neighborhood of* $(0, \sqrt{-1}\xi_0\infty)$, $\delta(x)$ *as a microfunction, i.e.* $\mathrm{sp}\,\delta(x)$, *is equal to*

$$\frac{(-1)^{n-1}}{(-2\pi\sqrt{-1})^n}\int \delta^{(n-1)}(\langle x, \xi\rangle)\chi_U(\xi)\omega(\xi),$$

where U *is a neighborhood of* $\xi_0 \in S^{n-1}$ *and the function* $\chi_U(\xi)$ *takes values* 1 *in* U *and* 0 *outside* $\bar{U}$.

Proof. First note that

$$\delta^{(n-1)}(t) = \frac{(-1)^{n-1}(n-1)!}{(-2\pi\sqrt{-1})}\left(\frac{1}{(t+\sqrt{-1}0)^n} - \frac{1}{(t-\sqrt{-1}0)^n}\right).$$

Then Proposition 3.2.3 and the above expression imply the conclusion of Lemma 3. The details are left for the reader, as an exercise.

We now integrate $P(x, D)w(x, \xi, \langle x', \xi\rangle) = (-1)^n\delta^{(n-1)}(\langle x - x', \xi\rangle)$ with respect to ξ in a neighborhood of ξ_0. Then, by Lemma 3, in a neighborhood of $(0, 0; \sqrt{-1}\langle\xi_0, d(x - x')\rangle)$, one obtains

$$\begin{aligned} P(x, D_x)\int_U w(x, \xi, \langle x', \xi\rangle)\omega(\xi) &= (-1)^n\int_U \delta^{(n-1)}(\langle x - x', \xi\rangle)\omega(\xi) \\ &= (-2\pi\sqrt{-1})^{n-1}\delta(x - x'). \end{aligned}$$

Therefore, if one lets

$$E(x, x') = \frac{1}{(-2\pi\sqrt{-1})^{n-1}}\int_U w(x, \xi, \langle x', \xi\rangle)\omega(\xi),$$

then, in a neighborhood of $(0, 0; \sqrt{-1}\langle\xi_0, d(x - x')\rangle\infty)$, one obtains

$$P(x, D_x)E(x, x') = \delta(x - x')$$

as an equality of microfunctions. Hence, if one can show that the singularity spectrum of $E(x, x')\,dx$ in a sufficiently small neighborhood of $(0, 0; \sqrt{-1}\langle\xi_0, d(x - x')\rangle\infty)$ is contained in $\sqrt{-1}S^*_M(M \times M)$, then $E(x, x')\,dx'$ is an element of $\mathscr{L}_M$. Denoting $E(x, x')\,dx'$ by E, one has $PE = 1$; i.e. E is a right inverse of P in $\mathscr{L}_M$. In fact, we have $PE = (\int P(x, D_x)\delta(x - y)E(y, x')\,dy)\,dx' = P(x, D_x)E(x, x')\,dx' = \delta(x - x')\,dx'$. Next we will consider the singularity spectrum of $E(x, x')dx'$. By the definition of u, $u(x, \xi, p)$ is a real analytic function defined in a neighborhood of $(x, \xi, p) = (0, \xi_0, 0)$. Hence $v(x, \xi, p) = u(x, \xi, p)Y(\langle x, \xi\rangle - p)$ is a hyperfunction defined in a neighborhood of $(x, \xi, p) = (0, \xi_0, 0)$, and its singularity spectrum must be in $\{\langle x, \xi\rangle = p, \pm\sqrt{-1}d(\langle x, \xi\rangle - p)\infty\}$. Therefore, by Theorem 3.2.1, the singularity spectrum of $E(x, x') = \int_U w(x, \xi, \langle x', \xi\rangle)\omega(\xi)$, in a sufficiently small neighborhood of $(0, 0; \sqrt{-1}\langle\xi_0, d(x - x')\rangle\infty)$, is restricted to $\{x = x', \sqrt{-1}\langle\xi, d(x - x')\rangle\infty\}$. Consequently, $E(x, x')\, dx$ is an element of $\mathscr{L}_M$ in a neighborhood of $(0, \sqrt{-1}\langle\xi_0, dx\rangle\infty)$.

Next we will show that there exists a microlocal operator E' such that $E'P = 1$. We will fix a volume element $dx = |dx_1 \wedge \ldots \wedge dx_n|$. For the sake of smoothness in the following argument, we will introduce the notion of a conjugate operator.

Definition 3.4.3. *Let $K(x, x')\, dx'$ be the kernel function of a microlocal operator $\mathscr{K}$. Then the microlocal operator defined by the kernel function $K(x', x)dx'$ is said to be the conjugate operator (or adjoint operator) of $\mathscr{K}$, denoted with $\mathscr{K}^*$.*

Remark 1. If $\mathscr{K}$ is a microlocal operator defined in a neighborhood of $(x_0, \sqrt{-1}\langle\xi_0, dx\rangle\infty)$, then $\mathscr{K}^*$ is a microlocal operator defined in a neighborhood of $(x_0, -\sqrt{-1}\langle\xi_0, dx\rangle\infty)$. That is, the conjugate operation $*$ induces a sheaf homomorphism $\mathscr{L}_M \to \mathscr{L}^a_M$, where a is the antipodal map: $(x, \sqrt{-1}\langle\xi, dx\rangle\infty) \mapsto (x, -\sqrt{-1}\langle\xi, dx\rangle\infty)$. Note that we have $\mathscr{L}^a_M = a^*\mathscr{L}_M (= a^{-1}\mathscr{L}_M)$ by definition.

Remark 2. The notion of a conjugate operator depends upon the choice of a volume element.

Lemma 4.

(1) *Let $\mathscr{K}$, $\mathscr{K}_1$, and $\mathscr{K}_2$ be microlocal operators defined in a neighborhood of $(x_0, \sqrt{-1}\langle\xi_0, dx\rangle\infty)$. Then $(\mathscr{K}_1\mathscr{K}_2)^* = \mathscr{K}_2^*\mathscr{K}_1^*$ and $(\mathscr{K}^*)^* = \mathscr{K}$ hold.*

(2) *For a linear differential operator $P(x, D_x) = \sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$, the conjugate operator $P^*(x, D_x)$ is a linear differential operator given by $\sum_{|\alpha|\le m} (-1)^{|\alpha|}D_x^\alpha(a_\alpha(x)\cdot)$, where $a_\alpha(x)\cdot$ denotes the multiplication by $a_\alpha(x)$.*

Proof. Statement (1) is plain by the definition of a conjugate operator. Let us prove (2). First we will find the conjugate operator of the multiplication operator $a_\alpha(x)\cdot$. Since the corresponding kernel function of $a_\alpha(x)\cdot$ is $a(x)\delta(x' - x)$, the kernel function corresponding to the conjugate operator $(a(x)\cdot)^*$ of $a(x)\cdot$ is $a(x')\delta(x' - x)$ by the definition. Hence, $a(x')\delta(x' - x) = a(x)\delta(x - x')$ holds, which implies $(a(x)\cdot)^* = a(x)\cdot$. Next we will find the conjugate operator of $D_j = \partial/\partial x_j$. The kernel function corresponding to D_j is $(\partial/\partial x_j)\delta(x - x')$. Therefore, the kernel function corresponding to D_j^*

is, by definition, given by $(\partial/\partial x'_j)\delta(x'-x) = (\partial/\partial x'_j)\delta(x'_1 - x_1)\delta(x'_2 - x_2)\cdots \delta(x'_n - x_n) = -(\partial/\partial x_j)\delta(x_1 - x'_1)\delta(x_2 - x'_2)\cdots\delta(x_n - x'_n)$. Hence, one has

$$(D_x^\alpha)^* = (D_1^{\alpha_1}\cdots D_n^{\alpha_n})^* = (-D_n)^{\alpha_n}\cdots(-D_1)^{\alpha_1} = (-1)^{|\alpha|}D_x^\alpha.$$

Consequently, one obtains

$$P^*(x, D_x) = \sum_{|\alpha|\leq m}(a_\alpha(x)D^\alpha)^* = \sum_{|\alpha|\leq m}(D^\alpha)^*(a_\alpha(x)\cdot)^* = \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D(a_\alpha(x)\cdot).$$

Now we return to the proof of Theorem 3.4.3. Recall that if $\sigma(P)(x, \sqrt{-1}\xi) \neq 0$ holds, then there exists a microlocal operator E in a neighborhood of $(x, \sqrt{-1}\langle\xi, dx\rangle\infty)$ such that $PE = 1$. Let P^* be the conjugate operator of P. Then $\sigma(P^*)(x, -\sqrt{-1}\xi) = \sigma(P)(x, \sqrt{-1}\xi) \neq 0$ holds. Hence there exists a microlocal operator E' such that $P^*E' = 1$ holds in a neighborhood of $(x, -\sqrt{-1}\langle\xi, dx\rangle\infty)$. Then the conjugate operator $(E')^*$ of E' is a microlocal operator defined in a neighborhood of $(x, \sqrt{-1}\langle\xi, dx\rangle\infty)$. One has $(E')^*P = (E')^*P^{**} = (P^*E')^* = 1$; i.e. $(E')^*$ is a left inverse of P. Furthermore, $(E')^* = (E')^*PE = (E'^*P)E = E$ holds. Therefore, for $P(x, D_x)$ there exists the left and right inverse E in $\mathscr{L}_M$ such that $PE = EP = 1$.

Remark. The most crucial point in the proof of Theorem 3.4.3 is the construction of $E(x, x')$ such that $P(x, D_x)E(x, x') = \delta(x - x')$. Such a hyperfunction $E(x, x')$ is sometimes called an elementary solution or a fundamental solution for the differential operator $P(x, D_x)$. Since $f(x) = \int f(x')\delta(x - x')\,dx'$ holds for an arbitrary hyperfunction $f(x)$, then $u(x) = \int E(x, x')f(x')\,dx'$ is a solution of $P(x, D_x)u(x) = f(x)$, provided $\int E(x, x')f(x')dx'$ makes sense and the differentiation and integration commute. The construction of a fundamental solution is one of the most effective methods in the local theory of linear differential equations.

Definition 3.4.4. *A linear differential operator $P(x, D_x)$ is said to be an elliptic operator at x_0 if for an arbitrary $\xi \in \mathbf{R}^n - \{0\}$, $\sigma(P)(x_0, \sqrt{-1}\xi) \neq 0$ holds.*

Theorem 3.4.4.

(i) *If hyperfunctions $u(x)$ and $f(x)$ satisfy $P(x, D_x)u(x) = f(x)$, then one has the inclusion*

$$\text{S.S. } u \subset \{(x, \sqrt{-1}\langle\xi, dx\rangle\infty) \in \sqrt{-1}S^*M \,|\, \sigma(P)(x, \sqrt{-1}\xi) = 0\} \cup \text{S.S. } f.$$

Particularly, if $P(x, D_x)$ is elliptic at arbitrary point on M, and if f is a real analytic function, then $u(x)$ is also a real analytic function.

(ii) *If $P(x, D_x)$ is elliptic at x_0, then $P: \mathscr{B}_{x_0} \to \mathscr{B}_{x_0}$ is an epimorphism.*

Note. The latter half of (i), above, asserts that Weyl's lemma holds. Weyl's lemma, in connection to Hilbert's 19th problem (see Hilbert [1]), is one of the most interesting results for applications in harmonic integral theory and for the theory of partial differential equations. Weyl's lemma triggered the development of the general theory of partial differential equations.

Proof of (i): The equation $P(x, D_x)u(x) = f(x)$ implies that $P(x, D_x)\,\mathrm{sp}(u(x)) = \mathrm{sp}(f(x))$. By Theorem 3.4.3, if $\sigma(P)(x, \sqrt{-1}\xi) \neq 0$, then $EP = 1$ for a microlocal operator E. Therefore, $\mathrm{sp}(u) = (EP)\,\mathrm{sp}(u) = E\,\mathrm{sp}(f)$ holds. From this, one obtains

$$\mathrm{S.S.}\, u \subset \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \,|\, \sigma(P)(x, \sqrt{-1}\xi) = 0\} \cup \mathrm{S.S.}\, f.$$

Next we will prove (ii). Theorem 3.4.3 implies that $P(x, D_x)$ has the right inverse E in a neighborhood of each point $(x_0, \sqrt{-1}\xi\infty)$. Since a left inverse also exists, the right inverse is unique. Therefore, the right inverse operator of P exists in a neighborhood of $\pi^{-1}(x_0)$, globally in the fibre direction. Since $\mathscr{B} \xrightarrow{\mathrm{sp}} \pi_*\mathscr{C} \to 0$ is exact, there exists $u \in \mathscr{B}_{x_0}$ such that $E\,\mathrm{sp}(f) = \mathrm{sp}(u)$; i.e. $\mathrm{sp}(Pu - f) = 0$ holds for such a u in a neighborhood of $\pi^{-1}(x_0)$. Hence, by (i), $g \underset{\mathrm{def}}{=} Pu - f$ is a real analytic function in a neighborhood of x_0. On the other hand, there exists a real analytic function v in a neighborhood of x_0 such that $Pv = g$ holds by Theorem 3.4.2. If one lets u be $u - v$, then $Pu = f$ holds in a neighborhood of x_0.

Remark. From (i) of Theorem 3.4.4, a solution $u(x)$ of $P(x, D_x)u(x) = 0$ has the singularity spectrum in $\{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \,|\, \sigma(P)(x, \sqrt{-1}\xi) = 0\}$. Therefore, the study of the structures of solutions for $Pu = 0$ on this variety is our main goal in the theory of linear differential equations. (In general, the set of zeros of $\sigma(P)(x, \xi) = 0$ is called the characteristic variety.) This central problem has long been recognized in the case of equations with constant coefficients (i.e. Ehrenpreis' fundamental principle; see Ehrenpreis [1]). But one had to wait until the advent of microfunction theory to consider the above problem for the case of variable coefficients. (Though some distinguished experts, for example Hörmander [1] and Mizohata [1], had shown the trend implicitly, their virtuoso performances were really too ingenious to be appreciated by everybody.)

§5. The Wave Equation

In this section we will treat the wave equation, the most elementary example of partial differential equations of hyperbolic type, and consider the initial value problem.

Let $P(x, D_x)$ be a linear differential operator of order m, and let $\varphi_\nu(x')$, $0 \leqq \nu \leqq m-1$, be a hyperfunction, where $x' = (x_2, \ldots, x_n)$. The Cauchy problem, or the initial value problem, is to find a hyperfunction $u(x)$ such that

$$\begin{cases} P(x, D_x)u(x) = 0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^{\nu} u\Big|_{x_1=0} = \varphi_\nu(x'), \qquad 0 \leqq \nu \leqq m-1 \end{cases}$$

hold. Note that $\varphi_\nu(x')$, $0 \leqq \nu \leqq m-1$, are called the Cauchy data, or the initial values.

Note. For given hyperfunctions $f(x)$ and $\varphi_\nu(x')$, $\nu = 0, \ldots, m-1$, consider the problem to find $u(x)$ such that

$$\begin{cases} P(x, D_x)u(x) = f(x) \\ \left(\dfrac{\partial}{\partial x_1}\right)^{\nu} u\Big|_{x_1=0} = \varphi_\nu(x'), \qquad 0 \leqq \nu \leqq m-1. \end{cases}$$

This problem, as well, is sometimes called the Cauchy problem. In view of the terminology used for overdetermined systems, we will consider the case $f(x) = 0$ in this book.

The next proposition explains why we take $\{\varphi_\nu(x')\}_{\nu=0}^{m-1}$ as initial values.

Proposition 3.5.1. *Let $P(x, D_x)$ be a differential operator of order m. Assume that the hyperplane $\{x \mid x_1 = 0\}$ is non-characteristic with respect to P; i.e. if $x_1 = 0$, then $\sigma(P)(x, dx_1) \neq 0$. Then, for a hyperfunction $u(x)$ such that $P(x, D_x)u(x) = 0$, the restriction $D_1^\nu u(x)|_{x_1=0}$ is well defined for any $\nu \in \mathbf{Z}^+ \cup \{0\}$, and all the $D_1^\nu u(x)|_{x_1=0}$ are uniquely determined by $D_1^\nu u(x)|_{x_1=0}$, $\nu = 0, \ldots, m-1$.*

Proof. Since $\{x \mid x_1 = 0\}$ is non-characteristic with respect to P, one has, by Theorem 3.4.4, $(x, \pm\sqrt{-1}\, dx_1\infty) \notin \text{S.S. } u(x)$ if $x_1 = 0$. Therefore, if $x_1 = 0$, then $(x, \pm\sqrt{-1}\, dx_1\infty) \notin \text{S.S. } D_1^\nu u(x)$ for an arbitrary ν. Hence, by Theorem 3.1.3, $D_1^\nu u(x)|_{x_1=0}$ is well defined. Next we will prove the uniqueness in the latter half of this proposition. Let

$$P(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha = \sum_{j=0}^{m} A_j(x, D') D_1^j \cdot$$

Then $A_m(x, D') = a_{(m,0,\ldots,0)}(x) \neq 0$ since $\sigma(P)(0, dx_1) = a_{(m,0,\ldots,0)}(0) \neq 0$. Then, $Pu = \sum_{j=0}^{m} A_j(x, D') D_1^j u = 0$ implies

$$a_{(m,0,\ldots,0)}(x) D_1^m u = \sum_{j=0}^{m-1} A_j(x, D') D_1^j u.$$

Hence, $D_1^m u|_{x_1=0} = 1/a_{(m,0,\ldots,0)}(x') \sum_{j=0}^{m-1} A_j(0, x', D')D_1^j u(x)|_{x_1=0}$ holds. That is, $D_1^m u|_{x_1=0}$ is uniquely determined by $D_1^\nu u(x)|_{x_1=0}$, $0 \leqq \nu \leqq m-1$. Similarly, if $D_1^{m+l}u|_{x_1=0}$, $l \geqq 0$, are given, then

$$D_1^{m+l+1}u = -D_1^{l+1}\left(\frac{1}{a_{(m,0,\ldots,0)}(x')}\sum_{j=0}^{m-1} A_j(0, x', D')D_1^j u\right)$$

implies that $D_1^{m+l+1}u|_{x_1=0}$ is determined uniquely.

Remark. As in the first half of the proof, if a hyperfunction $u(x)$ satisfies $(x, \pm\sqrt{-1}\, dx_1\infty) \notin \text{S.S. } u(x)$, then $D_1^\nu u(x)|_{x_1=0}$ is always well defined. Hyperfunctions with this property, i.e. $(x, \pm\sqrt{-1}\, dx_1\infty) \notin \text{S.S. } u(x)$, are said (Sato [1]) to be hyperfunctions containing real holomorphic parameters. (Since Sato [1] appeared long before the microfunction theory, the notation S.S. was not used there, but the concept is equivalent to the one above.) The above notion is essentially different from that of hyperfunctions containing holomorphic parameters in §8 of this chapter. (Not every hyperfunction containing real holomorphic parameters is obtained from the restriction of a hyperfunction containing holomorphic parameters to the real domain. The latter class of hyperfunctions is much more restrictive than the first one.) Because of this, we will not use such terminology in this book. The psychological effect of this term led us to consider whether the unique continuation theorem might hold, with respect to the real holomorphic parameter, when we attempted to grasp Holmgren's uniqueness theorem (Theorem 3.5.1) from the microfunction point of view. This may suggest to the reader the nature of the atmosphere in which Sato worked during the early stages of microfunction theory, when things were foggy for everyone except Sato. The section on "Hyperfunctions Containing Real Holomorphic Parameters" in Sato [1] gave the impression that the notion was not completely exposed: one had to wait for the appearance of microfunction theory to see the full picture.

In this way, the Cauchy problem for hyperfunction solutions has been formulated. Contrary to the case of real analytic functions (Theorem 3.4.2), there is generally no guarantee that a hyperfunction solution exists for the Cauchy problem. In fact, if $P(x, D_x)$ is elliptic, then $u(x)$ and, therefore, $D_1^\nu u|_{x_1=0}$ are all real analytic functions (Theorem 3.4.4 (i)). Hence, the Cauchy data must be analytic functions, which creates the necessity of introducing the notion of hyperbolic equations in the following section. However, if a hyperfunction solution exists for the Cauchy problem, then it is unique. This is Holmgren's theorem (Theorem 3.5.1). We have the following general proposition in light of the microfunction theory.

Proposition 3.5.2. *Let $\varphi(x)$ be a real-valued real analytic function on a real analytic manifold M such that $\varphi(x_0) = 0$ and $d\varphi(x_0) \neq 0$ at $x_0 \in M$. If* *a hyperfunction* $u(x)$ *on* M *satisfies* $\operatorname{supp} u \subset \{x \in M \mid \varphi(x) \geqq 0\}$, *and if either* $(x_0, \sqrt{-1}\, d\varphi(x_0)\infty)$ *or* $(x_0, -\sqrt{-1}\, d\varphi(x_0)\infty)$ *is not contained in* S.S. u, *then* $u = 0$ *in a neighborhood of* x_0.

Proof. Since $d\varphi(x_0) \neq 0$ holds, after a coordinate transformation in a neighborhood of x_0, one may assume $(t, x) \in M = \mathbf{R}_t \times \mathbf{R}_x^n$, $\varphi = \varphi(t, x) = t - x_1^2 - \cdots - x_n^2$, and $x_0 = (t, x) = (0, 0)$. Then one has $\operatorname{supp} u \subset \{(t, x) \mid t \geqq x_1^2 + \cdots + x_n^2\}$. Let us assume $(0, 0; \sqrt{-1}\, dt\infty) \notin$ S.S. $u(t, x)$. For $\xi \in S^{n-1}$, define

$$v(t, \xi, p) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int \frac{u(t, x)}{(p - \langle x, \xi \rangle + \sqrt{-1}0)^n}\, dx.$$

If $t < \epsilon$, then $u(t, x) = 0$ for $|x|^2 > \epsilon$. Thus the above integral makes sense. (We did in fact choose the coordinate transformation so that the integral might make sense; this transformation is called the Holmgren transformation, which Holmgren used in the proof of his celebrated uniqueness theorem.) Next, substitute $p = \langle x, \xi \rangle$ in $v(t, \xi, p)$, and integrate with respect to ξ to obtain

$$\begin{aligned}
&\int_{S^{n-1}} v(t, \xi, \langle x, \xi \rangle)\omega(\xi) \\
&\quad = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int_{S^{n-1}} \omega(\xi) \int \frac{u(t, x')}{(\langle x - x', \xi \rangle + \sqrt{-1}0)^n}\, dx' \\
&\quad = \int u(t, x')\, dx' \left(\frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int_{S^{n-1}} \frac{\omega(\xi)}{(\langle x - x', \xi \rangle + \sqrt{-1}0)^n} \right) \\
&\quad = \int u(t, x')\delta(x - x')\, dx' = u(t, x).
\end{aligned}$$

We will first consider the singularity spectrum of $v(t, \xi, p)$. Recall that $(0, 0; \sqrt{-1}\, dt\infty) \notin$ S.S. $u(t, x)$ is assumed. If $|t| < \epsilon$ (hence $|x^2| < \epsilon$) and $a > M|\eta|$ hold for a sufficiently small $\epsilon > 0$ and a sufficiently large $M > 0$, then one has

$$(t, x; \sqrt{-1}(a\, dt + \langle \eta, dx \rangle)\infty) \notin \text{S.S. } u(t, x).$$

Therefore,

$$\widehat{\text{S.S.}}\, u(t, x) \cap \{|t| < \epsilon\} \subset \{(t, x; \sqrt{-1}(a\, dt + \langle \eta, dx \rangle)) \mid a \leqq M|\eta|\}$$

holds. On the other hand, we have

$$\begin{aligned}
\widehat{\text{S.S.}}(p - \langle x, \xi \rangle + \sqrt{-1}0)^{-n} \subset \{(p, x, \xi; \sqrt{-1}\alpha d(p - \langle x, \xi \rangle)) \mid \alpha \geqq 0 \\
\text{and } \alpha(p - \langle x, \xi \rangle) = 0\}.
\end{aligned}$$

Hence, Theorem 3.1.5 implies

$$\widehat{\text{S.S.}}(u(t,x)(p-\langle x,\xi\rangle+\sqrt{-1}0)^{-n}\cap\{|t|<\epsilon\}$$
$$\subset\{(t,x,\xi,p,\sqrt{-1}(a\,dt+\langle\eta-\alpha\xi,dx\rangle-\alpha\langle x,d\xi\rangle+\alpha\,dp)|a\leqq M|\eta|,$$
$$\alpha\geqq 0 \text{ and } \alpha(p-\langle x,\xi\rangle)=0\}.$$

Then by Theorem 3.2.1, one obtains

$$\widehat{\text{S.S.}}\,v(t,\xi,p)\cap\{|t|<\epsilon\}\subset\{(t,\xi,p;\sqrt{-1}(a\,dt+\alpha\,dp-\alpha\langle x,d\xi\rangle)|\,|x|^2\leqq\epsilon,$$
$$a\leqq M\alpha|\xi|,\alpha\geqq 0 \text{ and } \alpha(p-\langle x,\xi\rangle)=0\}.$$

Since $a\,dt+\alpha\,dp-\alpha\langle x,d\xi\rangle=\alpha(dp-\langle x,d\xi\rangle+M|\xi|\,dt)+(\alpha M|\xi|-a)(-dt)$ holds, and since $\alpha M|\xi|-a\geqq 0$ by the assumption, consequently S.S. $v(t,\xi,p)\cap\{|t|<\epsilon\}$ is contained in the convex hull of $\{(t,\xi,p;\sqrt{-1}(dp-\langle\zeta,d\xi\rangle+M|\xi|\,dt)\infty|\,|\zeta|\leqq\sqrt{\epsilon}\}\cup\{(t,\xi,p;-\sqrt{-1}\,dt\infty)\}$. Note that this convex hull is properly convex. Therefore, from Theorem 2.3.5, there exists a holomorphic function $\psi(t,\xi,p)$ such that $v(t,\xi,p)=b(\psi(t,\xi,p))$. Note that $\psi(t,\xi,p)$ is holomorphic in t for $|t|<\epsilon$. For $t<0$, $u(t,x)=0$ holds; i.e. $v(t,\xi,p)=0$ holds. The exact sequence $0\to\tilde{\mathscr{A}}\xrightarrow{b}\tau^{-1}\mathscr{B}$ implies that one has $\psi(t,\xi,p)=0$ for $t<0$. By the uniqueness of analytic continuation, we have $\psi(t,\xi,p)\equiv 0$. Then $v(t,\xi,p)=0$ holds. Hence we obtain $u(t,x)=\int v(t,\xi,\langle x,\xi\rangle)\omega(\xi)=0$.

Theorem 3.5.1 (Holmgren). *Let $P(x,D_x)$ be a linear differential operator of order m such that $\sigma(P)(0,dx_1)\neq 0$ holds. If a hyperfunction $u(x)$ satisfies $Pu=0$ and $D_1^\nu u|_{x_1=0}=0$, $0\leqq\nu\leqq m-1$, in a neighborhood of the origin, then $u=0$ holds in a neighborhood of $x=0$. That is, a hyperfunction solution for the Cauchy problem, if one exists, is unique.*

Proof. Since $\sigma(P)(0,dx_1)\neq 0$ implies $\sigma(P)(0,\pm\sqrt{-1}\,dx_1)\neq 0$, one has $(0,\pm\sqrt{-1}\,dx_1)\notin$ S.S. u by Theorem 3.4.4. Then $v(x)=u(x)Y(x_1)$ is well defined. Furthermore,

$$P(x,D)v=(P(x,D)u)Y(x_1)+\sum_{j=0}^{m}\sum_{k=1}^{j}A_j(x,D')(D_1^{j-k}u(0,x'))\delta^{(k-1)}(x_1)=0$$

holds (see Lemma 2 in the proof of Theorem 3.4.3). Therefore, $(0,\pm\sqrt{-1}\,dx_1\infty)\notin$ S.S. v by Theorem 3.4.4. On the other hand, one has $\operatorname{supp} v\subset\{x|x_1\geqq 0\}$. Then, in a neighborhood of $x=0$, $v(x)=u(x)Y(x_1)=0$ holds by Proposition 3.5.2. Similarly, one obtains $u(x)Y(-x_1)=0$. Consequently, $u(x)=u(x)(Y(x_1)+Y(-x_1))=0$ holds.

Note. $(d/dx_1)(Y(x_1)+Y(-x_1))=\delta(x_1)-\delta(-x_1)=0$ holds, and $Y(x_1)+Y(-x_1)=1$ for $x_1\neq 0$. Hence, $Y(x_1)+Y(-x_1)=1$ identically holds.

Remark. We will give refined statements, without proofs, of Holmgren's Theorem.

(1) The Split Watermelon Theorem (Morimoto [2]): If a hyperfunction $u(x)$ satisfies $\operatorname{supp} u \subset \{x \mid x_1 \geqq 0\}$, then there exists a closed convex cone G in $\mathbf{R}^{n-1}$ such that $(\widehat{\text{S.S.}\, u}) \cap \{x = 0\} = \{(0, \sqrt{-1}\langle \xi, dx \rangle) \mid \xi' = (\xi_2, \ldots, \xi_n) \in G\}$. ["Split watermelon" is the name of a Japanese game.] The reader is advised to draw a figure for himself to consider why this name is given to the theorem.

(2) Let u be a hyperfunction on a real analytic manifold M. Denote $G = \operatorname{supp} u \subset M$ and $T = \text{S.S.}\, u \subset \sqrt{-1}S^*M$. Let φ be a real-valued real analytic function such that $\varphi(x_0) = 0$ and $d\varphi(x_0) \neq 0$. Then Proposition 3.5.2 can be rephrased: "If $x_0 \in G \subset \{x \mid \varphi(x) \geqq 0\}$ holds, then $(x_0, \pm\sqrt{-1}\, d\varphi(x_0)\infty) \in T$ holds" under the above notation. Furthermore, the following assertion is true (Bony [1]).

Let $\mathscr{I} = \{C^\infty\text{-functions on } \sqrt{-1}T^*M - M, \text{ homogeneous in } \xi, \text{ which are zero on } T\}$. Then let $\tilde{\mathscr{I}}$ be the smallest ideal among those containing $\mathscr{I}$ and with the following property: if f and g belong to $\tilde{\mathscr{I}}$, then the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial x_i} \right) \in \tilde{\mathscr{I}}.$$

Note that for the zero set $\tilde{T}$ of $\tilde{\mathscr{I}}$ we have $\tilde{T} \subset T$. Then, for the pair $(G, \tilde{T})$ the corresponding statement of Proposition 3.5.2 holds. That is, if $x_0 \in G \subset \{x \mid \varphi(x) \geqq 0\}$ holds, then one has $(x_0, \pm\sqrt{-1}\, d\varphi(x_0)\infty) \in \tilde{T}$.

Now we consider a differential operator $P = D_t^2 - D_{x_1}^2 - \cdots - D_{x_n}^2$ on $M = R_t \times R_x^n$, which is called the d'Alambertian (or wave operator). The equation $Pu = 0$ is said to be the wave equation. We will study the structure of solutions of the Cauchy problem, using the wave equation as an example, and consider the problems raised by this example. We abbreviate the Laplacian $D_{x_1}^2 + \cdots + D_{x_2}^2$ as Δ_x or Δ.

By direct computation, we obtain

$$P(x_1^2 + \cdots + x_n^2 - t^2)^\lambda = -4\lambda\left(\lambda + \frac{n-1}{2}\right)(x_1^2 + \cdots + x_n^2 - t^2)^{\lambda - 1}.$$

On the other hand, $\tilde{u}_\pm = (x_1^2 + \cdots + x_n^2 - (t \pm i0)^2)^\lambda$ is well defined as a hyperfunction on $\mathbf{R}_x^n \times \mathbf{R}_t$ by Example 2.4.3. Then we have

$$\begin{aligned} \text{S.S.}\, \tilde{u}_\pm \subset \{((x, t); \sqrt{-1}(\pm x_1/t, \ldots, \pm x_n/t, \mp 1)\infty) \mid t \neq 0, \\ t^2 = x_1^2 + \cdots + x_n^2\} \cup \{((0, 0), \sqrt{-1}(\xi_1, \ldots, \xi_n, \eta)\infty) \mid \\ \eta^2 \geqq \xi_1^2 + \cdots + \xi_n^2 \text{ and } \pm\eta > 0\}. \end{aligned}$$

where the order of the signs $\pm$ is taken respectively. In particular, we will consider the case $\lambda = (1-n)/2$. Let $u_\pm = (x_1^2 + \cdots + x_n^2 - (t \pm i0)^2)^{(1-n)/2}$. Then, for the d'Alambertian P, we have $Pu_\pm = 0$. Hence, by Theorem 3.4.4 (1), (S.S. $u_\pm) \cap \{t = x = 0\} \subset \{(0, 0); \sqrt{-1}(\xi_1, \ldots, \xi_n, \eta)\infty \,|\, \eta^2 = \xi_1^2 + \cdots + \xi_n^2$ and $\pm\eta > 0\}$. (We will show later that actually the equality holds.) Therefore, $u_\pm(t, x)$ can be restricted to $t = 0$. Similarly, we can restrict $(\partial/\partial t)u_\pm(t, x)$ to $t = 0$. For $x \neq 0$, we have $u_+(0, x) - u_-(0, x) = 0$ and

$$(\partial/\partial t)u_\pm(t, x)\big|_{t=0} = \frac{1-n}{2}\, t(x^2 - (t + i0)^2)^{(1-n)/2-1}\big|_{t=0} = 0.$$

That is, the hyperfunctions $u_+(0, x) - u_-(0, x)$ and $(\partial/\partial t)u_\pm(0, x)$ on $\mathbf{R}^n$ have their supports only at the origin $x = 0$. The following fact is known about the structure of hyperfunctions on $\mathbf{R}^n$ whose supports are only at the origin.

Proposition 3.5.3. *Let $u(x)$ be an element of $H^0_{\{0\}}(\mathbf{R}^n, \mathscr{B}_{\mathbf{R}^n})$. Then $u(x)$ can be expressed as $\sum_{\alpha \in (\mathbf{Z}^+ \cup \{0\})^n} a_\alpha \delta^{(\alpha)}(x)$, where $\mathbf{Z}^+ = \{1, 2, \ldots\}$ and $\delta^{(\alpha)}(x) = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}\delta(x)$. The coefficients $a_\alpha \in \mathbf{C}$ satisfy the following estimate: for an arbitrary $\epsilon > 0$, there exists C_ϵ such that*

$$|a_\alpha| \leqq C_\epsilon \frac{\epsilon^{|\alpha|}}{|\alpha|!} \qquad \text{for any } \alpha.$$

Note that a_α can be determined from $u(x)$ as $a_\alpha = (-1)^{|\alpha|}/\alpha! \int x^\alpha u(x)\, dx$.

Proof. Let

$$v(\xi, p) = \frac{(n-1)!}{(-2\pi i)^n} \int \frac{u(x)}{(p - \langle x, \xi\rangle - \sqrt{-1}0)^n}\, dx$$

Then one obtains $u(x) = \int_{S^{n-1}} v(\xi, \langle x, \xi\rangle)\omega(\xi)$ in the same way as in the proof of Proposition 3.5.2. Define

$$v(\zeta, \tau) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int \frac{u(x)}{(\tau - \langle x, \zeta\rangle)^n}\, dx \qquad \text{for } \zeta \in \mathbf{C}^n \text{ and } \tau \in \mathbf{C}.$$

Then v is a holomorphic function defined for $\tau \neq 0$. Note that $u(x) = 0$ for $x \neq 0$. By definition, one has $v(\xi, p) = v(\xi, p + \sqrt{-1}0)$. Since $v(\zeta, \tau)$ is homogeneous in (ζ, τ) of degree $(-n)$, one can write $v(\zeta, \tau) = \sum_{\alpha \in (\mathbf{Z}^+ \cup \{0\})^n} a_\alpha \zeta^\alpha \tau^{-n-|\alpha|}$. In particular for $\tau = 1$, one has $\sum_{\alpha \in (\mathbf{Z}^+ \cup \{0\})^n} a_\alpha \zeta^\alpha$, which is an entire function. Hence the Cauchy-Hadamard theorem for power series implies that for an arbitrary $\epsilon > 0$ there exists C_ϵ such that

$|a_\alpha| \leqq C_\alpha \epsilon^{|\alpha|}$ holds for any $\alpha \in (\mathbf{Z}^+ \cup \{0\})^n$. Since $v(\xi, p) = \sum_\alpha a_\alpha \xi^\alpha (p + \sqrt{-1}0)^{-n-|\alpha|}$, then

$$u(x) = \int_{S^{n-1}} \sum_\alpha a_\alpha \xi^\alpha (\langle x, \xi \rangle + \sqrt{-1}0)^{-n-|\alpha|} \omega(\xi)$$

holds. On the other hand, one has $D_1(\langle x, \xi\rangle)^{-n} = -n\xi_1(\langle x, \xi\rangle)^{-n-1}$. Generally, one has

$$D_x^\alpha(\langle x, \xi\rangle)^{-n} = (-n)(-n-1)\cdots(-n-|\alpha|+1)\xi^\alpha(\langle x, \xi\rangle)^{-n-|\alpha|}.$$

Therefore,

$$\begin{aligned} u(x) &= \int \sum_\alpha \frac{a_\alpha}{(-n)(-n-1)\cdots(-n-|\alpha|+1)} D_x^\alpha(\langle x, \xi\rangle + \sqrt{-1}0)^{-n}\omega(\xi) \\ &= \sum_\alpha C_\alpha D_x^\alpha \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \int \frac{\omega(\xi)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n}, \end{aligned}$$

where

$$C_\alpha = \frac{a_\alpha}{(-n)(-n-1)\cdots(-n-|\alpha|+1)} \cdot \frac{(-2\pi\sqrt{-1})^n}{(n-1)!}.$$

By Feynman's lemma in the proof of Proposition 3.2.3,

$$\frac{1}{(-2\pi\sqrt{-1})^n} \int_{\epsilon_j \xi_j \geqq 0} \frac{\omega(\xi)}{\langle x, \xi\rangle^n} = \frac{\operatorname{sgn}(\epsilon_1 \cdots \epsilon_n)}{(-2\pi\sqrt{-1})^n} \cdot \frac{1}{x_1 \cdots x_n}$$

holds, where $\epsilon_1, \ldots, \epsilon_n = \pm 1$. If one lets

$$f(z) = \frac{1}{(-2\pi\sqrt{-1})^n} \sum_\alpha c_\alpha D_z^\alpha \frac{1}{z_1 \cdots z_n},$$

then one obtains $u(x) = \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \operatorname{sgn}(\epsilon_1 \cdots \epsilon_n) f(x_1 + \sqrt{-1}\epsilon_1 0, \ldots, x_n + \sqrt{-1}\epsilon_n 0) = \sum c_\alpha \delta^{(\alpha)}(x)$. Recall $\delta(x) = 1/(-2\pi\sqrt{-1})^n \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} (\epsilon_1 \cdots \epsilon_n)/((x_1 + \sqrt{-1}\epsilon_1 0) \cdots (x_n + \sqrt{-1}\epsilon_n 0))$. Conversely, if $u(x)$ is expressed as $\sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \operatorname{sgn}(\epsilon_1 \cdots \epsilon_n) f(x_1 + \sqrt{-1}\epsilon_1 0, \ldots, x_n + \sqrt{-1}\epsilon_n 0)$,

then $u(x) = 0$ for $x_1 \neq 0$ holds. This is because

$$f(x_1 + \sqrt{-1}0, x_2 + \sqrt{-1}\epsilon_2 0, \ldots, x_n + \sqrt{-1}\epsilon_n 0) = f(x_1 - \sqrt{-1}0, x_2 + \sqrt{-1}\epsilon_2 0, \ldots, x_n + \sqrt{-1}\epsilon_n 0)$$

holds for $x_1 \neq 0$. Repeated arguments for $x_2, \ldots, x_n$ imply that $u(x) = 0$ for $x \neq 0$; i.e. $u(x)$ has its support only at the origin.

Lastly, we will find a formula for C_α when a hyperfunction having the support only at the origin $u(x)$ is given. For each j, $1 \leqq j \leqq n$, let γ_j be a closed path in the z_j-plane around the origin, oriented counter clockwise. Then, for a hyperfunction $u(x)$ having support only at the origin, one has

$$\int x^\alpha u(x)\,dx = (-1)^n \oint_{\gamma_1} \cdots \oint_{\gamma_n} z^\alpha f(z)\,dz.$$

By definition,

$$f(z) = \frac{1}{(-2\pi\sqrt{-1})^n} \sum_\alpha (-1)^{|\alpha|} \alpha! C_\alpha \frac{1}{z_1^{1+\alpha_1} \cdots z_n^{1+\alpha_n}}$$

holds. Hence, the coefficient of $1/(z_1 \cdots z_n)$ in $z^\alpha f(z)$ is given by $(1/(-2\pi i)^n)(-1)^{|\alpha|}\alpha! C_\alpha$. On the other hand, we have

$$\oint_{\gamma_i} \frac{dz_i}{z_i^n} = \begin{cases} 2\pi i & \text{for } n = 1 \\ 0 & \text{for } n \neq 1. \end{cases}$$

Therefore, $\int x^\alpha u(x)\,dx = (-1)^{|\alpha|}\alpha! C_\alpha$ holds; i.e. $C_\alpha = ((-1)^{|\alpha|}/\alpha!) \int x^\alpha u(x)\,dx$.

Corollary. *Let $u(x)$ be a hyperfunction on $\mathbf{R}^n$ whose support is only at the origin and such that the homogeneous degree is λ; i.e. Euler's formula $\sum_{i=1}^n (x_i D_i - \lambda)u = 0$ holds. Then the following two statements are true:*

(1) *If $\lambda \neq -n, -n-1, -n-2, \ldots$, then $u(x) = 0$.*
(2) *If $\lambda = -n - m$, $m \geqq 0$, then $u(x) = \sum_{|\alpha| = m} a_\alpha \delta^{(\alpha)}(x)$.*

Proof. From Proposition 3.5.3, $u(x)$ can be expressed as $u(x) = \sum_\alpha a_\alpha \delta^{(\alpha)}(x)$. Since $\left(\sum_{i=1}^n x_i D_i\right) \delta^{(\alpha)}(x) = -(n + |\alpha|)\delta^{(\alpha)}(x)$ holds, one has

$$\begin{aligned} 0 = \left(\sum_{i=1}^n x_i D_i - \lambda\right) u &= \sum_\alpha a_\alpha \left(\sum_{i=1}^n x_i D_i - \lambda\right) \delta^{(\alpha)}(x) \\ &= -\sum_\alpha (n + |\alpha| - \lambda) a_\alpha \delta^{(\alpha)}(x). \end{aligned}$$

Hence, $(n + |\alpha| - \lambda)a_\alpha = 0$ holds for any α; i.e. $a_\alpha = 0$ holds for $\lambda \neq n + |\alpha|$, which completes the proof.

We will now return to the wave equation. First we will consider $u_+(0, x) - u_-(0, x)$. Recall that $u_\pm(t, x) = (x_1^2 + \cdots + x_n^2 - (t \pm \sqrt{-1}\,0)^2)^{(1-n)/2}$; then we have $\left(\sum_{i=1}^n x_i D_i + n - 1\right) u_\pm(0, x) = 0$. That is, $u_+(0, x) - u_-(0, x)$ is a hyperfunction of homogeneous degree $(-n + 1)$, whose support is only at the origin. Then the above corollary implies $u_+(0, x) = u_-(0, x)$.

Next, we will study the structure of $((\partial/\partial t)u_\pm(0, x))$. Note first that

$$\begin{aligned} 0 &= D_t\left(tD_t + \sum_{i=1}^{n} x_iD_i + (n-1)\right)u_\pm(t, x) \\ &= \left(tD_t + \sum_{i=1}^{n} x_iD_i + n\right)D_tu_\pm(t, x) \end{aligned}$$

holds. Hence the restriction $t = 0$ provides $\sum_{i=1}^{n} (x_iD_i + n)D_tu_\pm(0, x) = 0$, which implies that $D_tu_\pm(0, x)$ is a homogeneous hyperfunction of degree $(-n)$. Again by the above corollary, $D_tu_\pm(0, x) = C_\pm\delta(x)$ for some constant $C_\pm$. $C_\pm$ can be computed as follows: $C_\pm = \int C_\pm\delta(x)\,dx = \int D_tu_\pm(0, x)\,dx$ holds, and the origin is the only support of $D_tu_\pm(0, x)$. Therefore, we have $C_\pm = \int_{|x|\leqq a} D_tu_\pm(0, x)\,dx$ for $a > 0$. On the other hand, we know

$$\mathrm{S.S.}(D_tu_\pm(t, x)) \subset \{((t, x), \sqrt{-1}(\tau\,dt + \langle\xi, dx\rangle)\infty) | \tau^2 = |\xi|^2\}.$$

Hence one can define the product $D_tu_\pm(t, x)Y(a - |x|)$. Then, the integral

$$\begin{aligned} \psi_\pm(t) &= \int_{|x|\leqq a} D_tu_\pm(t, x)\,dx = \int D_tu_\pm(t, x)\cdot Y(a - |x|)\,dx \\ &= D_t\int u_\pm(t, x)Y(a - |x|)\,dx \end{aligned}$$

is also well defined. (Notice that the integrand as a function of x has the compact support.)

In order to compute $\psi_\pm$, we will first compute $\int_{|x|\leqq a}(x^2 - (t + i0)^2)^\lambda\,dx$. This integral is the boundary value of the integral $\int_{|x|\leqq a}(x^2 - \tau^2)^\lambda\,dx$ from $\operatorname{Im}\tau > 0$, by Proposition 3.2.4; i.e. $\int_{|x|\leqq a}(x^2 - (t + i0)^2)^\lambda\,dx = b(\int_{|x|\leqq a}(x^2 - \tau^2)^\lambda\,dx)$. With the coordinate transformation $x = s\xi$ and $|\xi| = 1$, one has

$$\int_{|x|\leqq a}(x^2 - \tau^2)^\lambda\,dx = c\int_0^a (s - \tau^2)^\lambda s^{n-1}\,ds,$$

where c = the surface area of n-dimensional sphere $= 2\pi^{n/2}/\Gamma(n/2)$. Therefore, we obtain

$$\begin{aligned} \psi(t) &= c\frac{d}{dt}\int_0^a (s^2 - (t + i0)^2)^\lambda s^{n-1}\,ds \\ &= -2\lambda ct\int_0^a (s^2 - (t + i0)^2)^{\lambda-1}s^{n-1}\,ds. \end{aligned}$$

Let $t = \sqrt{-1}t'\,(t' > 0)$. Then

$$\psi(\sqrt{-1}t') = -2\lambda c\sqrt{-1}t' \int_0^a (s^2 + t'^2)^{\lambda-1} s^{n-1}\, ds$$
$$= -2\lambda c\sqrt{-1}t' \int_0^{a/t'} t'^{2\lambda+n-2}(s^2 + 1)^{\lambda-1} s^{n-1}\, ds$$

holds. In particular, for $\lambda = (1 - n)/2$ we have

$$\psi(\sqrt{-1}t') = (n-1)c\sqrt{-1} \int_0^{a/t'} (s^2 + 1)^{-(n+1)/2} s^{n-1}\, ds.$$

Consequently, we obtain

$$C_+ = \lim_{t' \to 0} \psi(\sqrt{-1}t') = (n-1)c\sqrt{-1} \int_0^{\infty} (s^2 + 1)^{-(n+1)/2} s^{n-1}\, ds$$
$$= (n-1)c\sqrt{-1}\,\frac{\sqrt{\pi}\,\Gamma\left(\dfrac{n}{2}\right)}{2\Gamma\left(\dfrac{1+n}{2}\right)} = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\dfrac{n-1}{2}\right)} \cdot \sqrt{-1}.$$

By a similar computation for C_-, we obtain the final form

$$D_t u_\pm(0, x) = \pm\frac{2\pi^{(n+1)/2}\sqrt{-1}}{\Gamma\left(\dfrac{n-1}{2}\right)}\,\delta(x).$$

Note also from the above that we have

$$(\text{S.S. } u_\pm) \cap \{t = x = 0\} = \{(0, 0; \sqrt{-1}(\xi_1, \ldots, \xi_n, \eta)\infty) \,|\, \eta^2 = |\xi|^2,\ \pm\eta > 0\},$$

where the order of the double sign is taken respectively.

We will summarize what we have obtained as follows. Let $P = D_t^2 - \Delta$, and let

$$u_\pm = (x_1^2 + \cdots + x_n^2 - (t \pm \sqrt{-1}0)^2)^{(1-n)/2}.$$

Then $Pu_\pm = 0$ holds, and we also have

$$u_+(0, x) = u_-(0, x) \quad \text{and} \quad D_t u_\pm(0, x) = \pm\frac{2\pi^{(n+1)/2}\sqrt{-1}}{\Gamma\left(\dfrac{n-1}{2}\right)}\,\delta(x).$$

Define a hyperfunction $K_1(t, x)$ as

$$K_1(t, x) = \frac{\Gamma\left(\dfrac{n-1}{2}\right)}{4\sqrt{-1}\pi^{(n+1)/2}} \{(x_1^2 + \cdots + x_n^2 - (t + \sqrt{-1}0)^2)^{(1-n)/2}$$
$$- (x_1^2 + \cdots + x_n^2 - (t - \sqrt{-1}0)^2)^{(1-n)/2}\}.$$

Then we have

$$\begin{cases} (D_t^2 - \Delta)K_1(t, x) = 0 \\ K_1(0, x) = 0 \\ \dfrac{\partial K_1}{\partial t}(0, x) = \delta(x). \end{cases}$$

A similar computation to Example 2.4.3 provides $\operatorname{supp} K_1 \subset \{(t, x) \mid |x|^2 \leqq t^2\}$.

For this hyperfunction $K_1(t, x)$ we have the following theorem.

Theorem 3.5.2. *Let $K_0(t, x) = D_t K_1(t, x)$.*

(1) *If $(D_t^2 - \Delta)u(t, x) = 0$ holds on $\mathbf{R}^{n+1}$, then we have*

$$u(t, x) = \int K_0(t, x - x')u(0, x')\, dx' + \int K_1(t, x - x')\frac{\partial u}{\partial t}(0, x')\, dx'.$$

(2) *Conversely, for arbitrary hyperfunctions $\varphi_0(x)$ and $\varphi_1(x)$ on $\mathbf{R}^n$, one lets*

$$u(t, x) = \int K_0(t, x - x')\varphi_0(x')\, dx' + \int K_1(t, x - x')\varphi_1(x')\, dx'.$$

Then $u(t, x)$ satisfies

$$\begin{cases} (D_t^2 - \Delta)u(t, x) = 0 \\ u(0, x) = \varphi_0(x) \\ \dfrac{\partial u}{\partial t}(0, x) = \varphi_1(x). \end{cases}$$

That is, the Cauchy problem for the wave equation can be solved uniquely.

Proof. We will prove (2) first. As we noted before, we have

$$\begin{cases} (D_t^2 - \Delta)K_1 = 0 \\ K_1(0, x) = 0 \\ \dfrac{\partial K_1}{\partial t}(0, x) = \delta(x). \end{cases}$$

Hence, $(D_t^2 - \Delta)K_0 = D_t(D_t^2 - \Delta)K_1 = 0$ and $K_0(0, x) = \delta(x)$ hold. Furthermore, we also have $(\partial K_0/\partial t)(0, x) = 0$. This is because $D_t K_0 = D_t^2 K_1 = \Delta K_1$ implies $D_t K_0|_{t=0} = \Delta K_1|_{t=0} = \Delta(K_1|_{t=0}) = 0$. Notice that the support of K_1 (therefore of K_0) is restricted to $\{|x|^2 \leq t^2\}$. Then the integral defining $u(t, x)$ is well defined and, furthermore, we obtain the following:

$$(D_t^2 - \Delta)u(t, x) = \int [(D_t^2 - \Delta_x)K_0](t, x - x')\varphi_0(x')\, dx'$$
$$+ \int [(D_t^2 - \Delta_x)K_1](t, x - x')\varphi_1(x')\, dx' = 0,$$
$$u(0, x) = \int K_0(0, x - x')\varphi_0(x')\, dx' + \int K_1(0, x - x')\varphi_1(x')\, dx'$$
$$= \int \delta(x - x')\varphi_0(x')\, dx' = \varphi_0(x)$$

and

$$\frac{\partial u}{\partial t}(0, x') = \int \frac{\partial K_0}{\partial t}(0, x - x')\varphi_0(x')\, dx' + \frac{\partial K_1}{\partial t}(0, x - x')\varphi_1(x')\, dx'$$
$$= \int \delta(x - x')\varphi_1(x')\, dx' = \varphi_1(x),$$

completing the proof for (2).

In order to prove (1), note that one has the following:

$$\int_0^t \left(\int D_{t'}(D_t K_1(t - t', x - x')u(t', x'))\, dx' \right) dt'$$
$$= \int D_t K_1(0, x - x')u(t, x')\, dx' - \int D_t K_1(t, x - x')u(0, x')\, dx'$$
$$= u(t, x) - \int D_t K_1(t, x - x')u(0, x')\, dx'$$
$$= u(t, x) - \int K_0(t, x - x')u(0, x')\, dx'.$$

We also have

$$\int_0^t \left(\int D_{t'}[K_1(t - t', x - x')D_{t'}u(t', x')]\, dx' \right) dt'$$
$$= \int K_1(0, x - x')D_t u(t, x')\, dx' - \int K_1(t, x - x')(D_t u)(0, x')\, dx'$$
$$= -\int K_1(t, x - x')D_t u(0, x')\, dx'.$$

From the above, then, we obtain the following:

$$u(t, x) - \int K_0(t, x - x')u(0, x')\, dx' - \int K_1(t, x - x')(D_t u)(0, x')\, dx'$$
$$= \int_0^t \left(\int D_{t'}[K_1(t - t', x - x')D_{t'}u(t', x') + D_t K_1(t - t', x - x')u(t', x')]\, dx' \right) dt'$$
$$= \int_0^t \left(\int [K_1(t - t', x - x')D_{t'}^2 u(t', x') - (D_t^2 K_1(t - t', x - x')u(t', x'))]\, dx' \right) dt'$$
$$= \int_0^t \left(\int [K_1(t - t', x - x')\Delta_{x'} u(t', x') - (\Delta_{x'} K_1)(t - t', x - x')u(t', x')]\, dx' \right) dt'.$$

Let $\Omega_{t'}$ be a domain with smooth boundary $\partial\Omega_{t'}$ in $\mathbf{R}^n_{x'}$ such that

$$\{x' \in \mathbf{R}^n \mid (t-t')^2 \leqq |x-x'|^2\} \subset \Omega_{t'}.$$

From the condition on the support of K_1, the Stokes theorem can be applied to the above. Then one obtains that the above integral is

$$\int_0^\infty \left[K_1(t-t', x-x') \sum_i \frac{\partial}{\partial x_i} u(t', u') \right.$$
$$\left. - \sum_i \left(\frac{\partial}{\partial x_i} K_1\right)(t-t', x-x') u(t', x')\Big|_{\partial\Omega_{t'}} \right] dt' = 0,$$

which completes the proof of (1).

Remark. The pair of hyperfunctions $(K_0(t, x), K_1(t, x))$, in Theorem 3.5.2, is sometimes called a fundamental solution for the Cauchy problem.

Exercise. In Theorem 3.5.2, we consider the problem in $\mathbf{R}^{n+1}$. If $\varphi_0(x)$ and $\varphi_1(x)$ are given in $\Omega \subset \mathbf{R}^n$, then consider the domain where the hyperfunction $u(t, x)$ defined as

$$u(t, x) = \int K_0(t, x-x')\varphi_0(x')\,dx' + \int K_1(t, x-x')\varphi_1(x')\,dx'$$

is well defined. Generally, a domain in $\{(t, x) \mid t = 0\}$ which determines the solution near (t, x) is said to be a domain of dependence of u at (t, x). Conversely, when $\varphi_0(x)$ and $\varphi_1(x)$ vary in a neighborhood of x_0 in $\{(t, x) \mid t = 0\}$, a domain on which the solution $u(t, z)$ is affected is said to be a domain of influence.

One can show that the support of $K_0(t, x)$ in Theorem 3.5.2 is contained in $\{(t, x) \mid t^2 = |x|^2\}$ for any odd integer strictly greater than 1; otherwise by direct computation, it can be shown that the support is the entirety of $\{(t, x) \mid t^2 \leq |x|^2\}$. (When this implication holds, one says that Huyghens' principle holds for an odd integer $n \geqq 3$, or that diffusion of waves does not occur.) This property—that the support of K_0 varies sensitively for the dimension n, which interested many analysts,—was crystallized as the lacuna theory in Petrowsky [1]. However, as Atiyah, Bott, and Gårding ([1] and [2]) indicated, the study of supports seems to be too specialized a topic in the whole structure of the theory of linear partial differential equations (in fact, Huyghens' principle holds as an extremely exceptional case). In the theory of microfunctions, to one's astonishment, the Huygens principle for microfunctions (sometimes called Huyghen's principle in the wider sense) holds quite generally. We will return to this topic in the next section. In this section, we will discuss how a microfunction solution of the wave equation propagates. First, we will recall the definition of a bicharacteristic strip in the classical theory of partial differential equations.

Definition 3.5.1. *Let $p_m(x, \xi)$ be the principal symbol $\sigma(P)$* of a linear differential operator *$P(x, D_x)$. An integral curve $(x(t), \xi(t))$ of*

$$\frac{dx_1}{\dfrac{\partial p_m}{\partial \xi_1}} = \cdots = \frac{dx_n}{\dfrac{\partial p_m}{\partial \xi_n}} = \frac{d\xi_1}{-\dfrac{\partial p_m}{\partial x_1}} = \cdots = \frac{d\xi_n}{-\dfrac{\partial p_m}{\partial x_n}}$$

with the property $p_m(x(t), \xi(t)) = 0$ is said to be a bicharacteristic strip of the equation $P(x, D_x)u(x) = 0$. The image $\{x(t)\}$ of the projection of a bicharacteristic strip onto the base space is called the bicharacteristic curve.

We obtain the following theorem, in terms of a bicharacteristic strip on the structure of microfunction solutions for the wave equation $(D_t^2 - \Delta_x)u(t, x) = 0$.

Theorem 3.5.3. *Let Ω be an open subset of $\sqrt{-1}S^*\mathbf{R}^{n+1}$. For a microfunction solution u and an arbitrary bicharacteristic strip b,* supp *$u \cap b$ is open and closed in $b \cap \Omega$; i.e.* supp *$u \cap b$ is a union of connected subsets of $b \cap \Omega$.*

Remark. This theorem implies that the microfunction solution of the wave equation propagates only along a bicharacteristic strip. On the other hand, the characteristic manifold $\{(t, x; \sqrt{-1}(\tau, \xi)\infty) \in \sqrt{-1}S^*\mathbf{R}^{n+1} \mid \tau^2 = \xi^2\}$ of the wave equation is covered by an n-parameter family of bicharacteristic strips. Hence, the structure problem, mentioned in the last section, for microfunction solutions of the wave equation is solved.

Proof of Theorem 3.5.3. Let $K(t, x)$ be a solution of

$$\begin{cases} (D_t^2 - \Delta_x)K(t, x) = 0 \\ K(0, x) = 0 \\ \dfrac{\partial K}{\partial t}(0, x) = \delta(x), \end{cases}$$

and let $E(t, x) = K(t, x)Y(t)$, where $Y(t)$ is the Heaviside function. First we need the following lemma.

Lemma.

(i) $(D_t^2 - \Delta_x)E(t, x) = \delta(t, x)$.

(ii) $\operatorname{supp} E \subset \{(t, x) \in \mathbf{R}^{n+1} \mid |x| \leqq t\}$.

(iii) $\text{S.S. } E \subset \{(0, 0; \sqrt{-1}(\tau, \xi)\infty) \in \sqrt{-1}S^*\mathbf{R}^{n+1} \mid (\tau, \xi) \neq 0\}$
$\cup \{(t, x; \sqrt{-1}(\tau, \xi)\infty) \in \sqrt{-1}S^*\mathbf{R}^{n+1} \mid \tau^2 = \xi^2 \neq 0, t > 0, x = -(\xi/\tau)t\}$.

Proof. We will prove (i) first. Since one has

$$D_t E(t, x) = (D_t(K(t, x))Y(t) + K(t, x)\delta(t) = (D_t K)Y(t) + K(0, x)\delta(t)$$
$$= (D_t K)Y(t),$$

then

$$D_t^2 E(t, x) = (D_t^2 K)Y(t) + D_t K(0, x)\delta(t) = (D_t^2 K)Y(t) + \delta(t, x)$$

holds. On the other hand, $\Delta_x E(t, x) = (\Delta_x K)Y(t)$; hence one obtains

$$(D_t^2 - \Delta)E(t, x) = ((D_t^2 - \Delta)K)Y(t) + \delta(t, x) = \delta(t, x).$$

Statement (ii) follows from the facts $\operatorname{supp} K \subset \{(t, x) \in \mathbf{R}^{n+1} \,|\, |x|^2 \leqq t^2\}$ and $\operatorname{supp} Y(t) \subset \{(t, x) \in \mathbf{R}^{n+1} \,|\, t \geqq 0\}$. We will prove (iii). One has S.S. $E. \cap \{t \leqq 0\} \subset \{t = x = 0\}$ by (ii). Since $Y(t) = 1$ for $t > 0$, one has

$$E(t, x) = K(t, x) = \frac{\Gamma\left(\dfrac{n-1}{2}\right)}{4\sqrt{-1}\pi^{(n+1)/2}} \{(x_1^2 + \cdots + x_n^2 - (t + \sqrt{-1}0)^2)^{(1-n)/2}$$
$$- (x_1^2 + \cdots + x_n^2 - (t - \sqrt{-1}0)^2)^{(1-n)/2}\}.$$

By Example 2.4.3, as the boundary value from $\operatorname{Im} t > 0$ and $\operatorname{Im}(x_1^2 + \cdots + x_n^2 - t^2) < 0$, then $(x_1^2 + \cdots + x_n^2 - (t \pm \sqrt{-1}0)^2)^\lambda = (x_1^2 + \cdots + x_n^2 - t^2 \pm \sqrt{-1}0)^\lambda$ holds. Therefore, the singularity spectrum is contained in $\{(x, t; (\xi, \tau)\infty) \,|\, x_1^2 + \cdots + x_n^2 - t^2 = 0, \quad (\xi, \tau) = (2x, 2t)\}$. Hence, there exists a positive number $c > 0$ such that $\xi = cx$ and $\tau = -ct$. Then $x = -(\xi/\tau)t$ holds. On the other hand, one has $(D_t^2 - \Delta_x)E = 0$ $(t > 0)$. Hence, $\tau^2 = \xi^2 \neq 0$ holds by Theorem 3.4.4. This completes the proof.

Now we will prove Theorem 3.5.3, using the above lemma. It is sufficient to prove the following statement:

Let $u(t, x)$ be a microfunction defined in a neighborhood of $(0, 0; \sqrt{-1}(\tau_0, \xi_0)\infty)$, where $\tau_0^2 = \xi_0^2$, such that $(D_t^2 - \Delta)u(t, x) = 0$ holds. If $u(t, x)$ is 0 at $(0, 0; \sqrt{-1}(\tau_0, \xi_0)\infty)$, then $u(t, x)$ is 0 at $(t, -(t/\tau_0)\xi_0, \sqrt{-1}(\tau_0, \xi_0)\infty)$.

Define a microfunction $\tilde{u}$ as $\tilde{u} = u(t, x)Y(t)$. Then it is sufficient to prove $\tilde{u} = 0$. Notice that $(D_t^2 - \Delta_x)\tilde{u}(t, x) = 0$ clearly holds in a neighborhood of $\{(t, x; \sqrt{-1}(\tau, \xi)\infty) \,|\, \tau = \tau_0 \text{ and } \xi = \xi_0\}$. The above lemma (i) implies

$$\tilde{u}(t, x) = \int \delta(t - t', x - x')\tilde{u}(t', x')\, dt'\, dx'$$
$$= \int [(D_{t'}^2 - \Delta_{x'})E(t - t', x - x')]\tilde{u}(t', x')\, dt'\, dx'.$$

Through integration by parts and (ii) and (iii) of our lemma,

$$\tilde{u}(t, x) = \int E(t - t', x - x')(D_{t'}^2 - \Delta_{x'})\tilde{u}(t', x'))\, dt'\, dx' = 0$$

holds in a neighborhood of $\{(t, x; \sqrt{-1}(\tau, \xi)\infty) \in \sqrt{-1}S^*\mathbf{R}^{n+1} | \tau = \tau_0$ and $\xi = \xi_0\}$. Details are left as an exercise for the reader.

§6. Fundamental Solutions for Regularly Hyperbolic Operators

We constructed the inverse in $\mathscr{L}_M$ of a linear differential operator $P(x, D_x)$ under the provision that $\sigma(P)(x, \sqrt{-1}\xi) \neq 0$. In the preceding section, we discussed the solvability of an initial value problem relating to Holmgren's theorem. There we treated the wave operator as an example when one can explicitly construct a fundamental solution for the initial value problem. One can ask, then, how to locate wave operators among non-elliptic operators. In this section, we will show that an operator called a regularly hyperbolic operator has properties similar to those of the wave operator. The material in this section is connected with that in §3 of Chapter IV.

We begin with the definition of a regularly hyperbolic operator.

Definition 3.6.1. *A linear differential operator $P(x, D_x)$ is said to be hyperbolic at the origin in the direction* $(1, 0, \ldots, 0)$ *if*

$$\sigma(P)(x, \sqrt{-1}\xi) = a(x) \prod_{l=1}^{m} (\xi_1 - \lambda_l(x, \xi'))$$

such that $a(0) \neq 0$ and $\lambda_l(x, \xi')$ is real for $(x, \xi') \in \mathbf{R}^n \times \mathbf{R}^{n-1}$. A hyperbolic operator is said to be regularly hyperbolic when $\lambda_l(x, \xi') \neq \lambda_k(x, \xi')$ holds for $l \neq k$ and $\xi' \neq 0$.

Remark. The above definition is the most natural one for the study of hyperfunction solutions; for distribution solutions, however, the notion of hyperbolic operators is usually defined in a more narrow sense. The facts which led to the above definition are, first of all, that it is such an operator, as it has a fundamental solution for the initial value problem (see Gårding [1]), and secondly that in the framework of hyperfunction theory there exists a fundamental solution of the initial value problem for the operator defined as above (Bony and Schapira [1]). We will not treat this problem in this book, since it is rather technical. The micro-local version of a hyperbolic operator, called a micro-hyperbolic operator (Kashiwara and Kawai [3]), is theoretically more important. But it will not be considered since it is a bit beyond this book's scope.

For a regularly hyperbolic operator, we will locally construct a fundamental solution for the initial value problem and study its singularity spectrum. The following, very fundamental theorem of Hamada will be used most effectively to carry out our plan. Basically we will follow Kawai [2] for our treatment of this topic.

Theorem (Hamada). *Let $P(z, D_z) = P(z_1, \ldots, z_n, \partial/\partial z_1, \ldots, \partial/\partial z_n)$ be a linear differential operator with holomorphic coefficients defined in a neighborhood of $z = 0 \in \mathbf{C}^n$. Regarding $\sigma_m(P)(z, \zeta_1, \zeta') = 0$, $\zeta' = (\zeta_2, \ldots, \zeta_n)$, as an equation of ζ_1, assume that all the roots $\lambda_1(z, \zeta'), \ldots, \lambda_m(z, \zeta')$ are distinct in a neighborhood of $(z_1, z'; \zeta') = (0, z_0'; \zeta_0')$, $z' = (z_2, \ldots, z_n)$. Then the following initial value problem (3.6.2) has a solution $u(z, w', \zeta')$, as in (3.6.3). We let $\varphi_k(z, w', \zeta')$ be a solution of the initial value problem*

$$\left.\begin{aligned} &\frac{\partial \varphi_k}{\partial z_1} - \lambda_k(z, \operatorname{grad}_{z'} \varphi_k) = 0 \\ &\varphi_k(z, w', \zeta')\big|_{z_1=0} = \langle z' - w', \zeta' \rangle \end{aligned}\right\} \tag{3.6.1}$$

$$\begin{cases} P(z, D_z)u(z, w', \zeta) = 0 \\ \dfrac{\partial^j}{\partial z_1^j} u(z, w', \zeta')\big|_{z_1=0} = \dfrac{c_j}{\langle z' - w', \zeta' \rangle^{l_j}} \qquad \text{for } j = 0, \ldots, m-1. \end{cases} \tag{3.6.2}$$

where c_j is a constant and l_j is a positive integer.

$$u = \sum_{k=1}^{m} u_k, \tag{3.6.3}$$

where

$$u_k(z, w', \zeta') = \sum_{j \geqq j_k} a_{j_k}(z, w', \zeta')\varphi_k(z, w', \zeta')^j + b_k(z, w', \zeta') \log \varphi_k(z, w', \zeta')$$

and where a_{j_k} and b_k are holomorphic in a neighborhood of $(0, z_0', \zeta_0')$.

Remark 1. Since $\log \varphi_k$ is many-valued in (3.6.2), perhaps one should consider (3.6.3) over a covering space to be precise. In what follows, we choose a branch of $\log \varphi_k$ so that the equation (3.6.3) may have a clear meaning.

Remark 2. It is known that one can have u_k such that $P(z, D_z)u_k = 0$.

Remark 3. It is the classical Hamilton-Jacobi theory which says that there exists a solution φ_k to the equation (3.6.1). Consult Yosida [1], pt. 2, chap. 4, for details. Notice, also, that $\operatorname{grad}_z \varphi_k(z, w', \zeta')|_{z=z(s)} = \zeta(s)$ holds on the bicharacteristic strip $(z(s), \zeta(s))_{|s|<<1}$ of $\zeta_1 - \lambda_k(z, \zeta')$. The Hamilton-Jacobi theory implies that the solution to (3.6.1) is a real-valued real analytic function, under the assumption that $\lambda_k(x, \xi')$ is real for a real (x, ξ'). From the uniqueness of a solution to the initial value problem (3.6.1), φ_k is homogeneous in ζ' of degree one since λ_k is homogeneous in ζ' of degree one (i.e. $\sum_{j=2}^{n} \zeta_j(\partial/\partial\zeta_j)\lambda_k(z, \zeta') = \lambda_k(z, \zeta')$). This is because $\varphi_k(z, w', c\zeta')/c$ $(c > 0)$ and $\varphi_k(z, w', \zeta')$ are both solutions to (3.6.1).

We will not prove the above theorem here. It is achieved by finding a formal solution of the above form and then proving its convergence.

Theorem 3.6.2. *Let a linear differential operator $P(x, D_x)$ be regularly hyperbolic at the origin in the direction of* $(1, 0, \ldots, 0)$. *Then there exists a solution $E(x, y')$ in a neighborhood of $(x, y') = (0, 0)$, $y' = (y_2, \ldots, y_n)$ to the initial value problem* (3.6.4), *and it is unique*:

$$\left.\begin{aligned} &P(x, D_x)E(x, y') = 0 \\ &\frac{\partial^j}{\partial x_1^j} E(x, y')\Big|_{x_1=0} = \delta_{j,m-1}\delta(x' - y') \qquad \text{for } j = 0, \ldots, m-1 \end{aligned}\right\} \qquad (3.6.4)$$

Proof. Considering the plane-wave decomposition of the δ-function in Proposition 3.2.3, we will solve the following initial value problem:

$$\left.\begin{aligned} &P(x, D_x)F(x, y', \xi') = 0 \\ &\frac{\partial^j}{\partial x_1^j} F(x, y', \xi')\Big|_{x_1=0} = \frac{(n-2)\delta_{j,m-1}}{(2\pi\sqrt{-1})^{n-1}(\langle x' - y', \xi'\rangle + \sqrt{-1}0)^{n-1}}, \\ &\qquad\qquad j = 0, \ldots, m-1. \end{aligned}\right\} \qquad (3.65)$$

Since $P(x, D_x)$ can be extended to an operator $P(z, D_z)$, defined in a neighborhood of the origin in $\mathbf{C}^n$, one can apply Theorem 3.6.1 to the case where

$$c_j = \frac{(n-2)!\delta_{j,m-1}}{(2\pi\sqrt{-1})^{n-1}} \quad \text{and} \quad l_j = n - 1.$$

As in Remark 3, $\varphi_k(z, w', \xi')$ is real if (z, w', ξ') is real and $\operatorname{grad}_z \varphi_k \neq 0$ holds for the points $\varphi_k = 0$. Hence, by Theorem 2.3.4 (see also Example 3.1.2), a single-valued holomorphic function in

$$\Omega_k = \{(z, w', \zeta') \in \omega \,|\, \operatorname{Im} \varphi_k(z, w', \zeta') > 0\},$$

where ω is a sufficiently small neighborhood of $(0, 0, \xi_0')$ for $\xi_0' \in \mathbf{R}^{n-1} - \{0\}$, defines a hyperfunction. On the other hand, φ_k^j and $\log \varphi_k$ are single-valued holomorphic functions in Ω_k. Therefore, $u_k(z, w', \zeta')$, which exists by Theorem 3.6.1, is a single-valued holomorphic function in Ω_k for a sufficiently small neighborhood ω. Since $P(z, D_z)u_k = 0$ holds from Remark 2 above, the hyperfunction $F_k(x, y', \xi')$, induced by u_k, satisfies $P(x, D_x)F_k(x, y', \xi') = 0$. Furthermore, one has $\varphi_k(z, w', \zeta')|_{z_1=0} = \langle z' - w', \zeta'\rangle$. Hence the restriction

$$\frac{\partial^j}{\partial z_1^j}\left(\sum_{k=1}^m u_k(z, w', \zeta')\right)\Bigg|_{z_1=0}$$

is a well-defined, single-valued holomorphic function in $\Omega_k \cap \{z_1 = 0\}$ and is equal to

$$\frac{(n-2)!\delta_{j,m-1}}{(2\pi\sqrt{-1})^{n-1}(\langle z' - w', \zeta'\rangle)^{n-1}}.$$

That is, one obtains

$$\frac{\partial^j}{\partial x_1^j}\left(\sum_{k=1}^{m} F_k(x, y', \xi')\right)\Bigg|_{x_1=0} = \frac{(n-2)!\delta_{j,m-1}}{(2\pi\sqrt{-1})^{n-1}(\langle x' - y', \xi'\rangle + \sqrt{-1}0)^{n-1}}.$$

Lastly, we will construct a solution $E(x, y')$ by integrating $F_k(x, y', \xi')$ with respect to ξ' over $S_{\xi'}^{n-1}$. As we noted previously, $\varphi_k(z, w', \zeta')$ is homogeneous in ζ' of degree one. Hence, by Euler's identity, one has

$$\left(\sum_{j=2}^{n} \zeta \frac{\partial}{\partial \zeta_j} - j\right)\varphi_k^j = 0 \quad \text{and} \quad \sum_{j=2}^{n} \zeta_j \frac{\partial}{\partial \zeta_j} \log \varphi_k = 1.$$

Therefore, their boundary values satisfy the same equations. Then $F_k(x, y', \xi')|_{|\xi'|=1}$ is well defined from Proposition 3.5.1. Applying Theorem 3.2.1 to this case,

$$E_k(x, y') = \int_{\xi' \in S^{n-1}} F_k(x, y', \xi')\omega(\xi')$$

is well defined. Let $E(x, y') = \sum_{k=1}^{m} E_k(x, y')$. Then $E(x, y')$ satisfies the required equations. The uniqueness of $E(x, y')$ is obtained from Holmgren's theorem (Theorem 3.5.1).

Next we will study where the singularity spectrum of $E(x, y')$ is located. It will be self-evident how useful the microfunction theory is to the study of the singularity of a hyperfunction.

Theorem 3.6.3. *If $(x, y'; \sqrt{-1}(\eta, \theta')\infty)$ is in* S.S. $E(x, y')$, *then $\theta' = -\xi'$, and (x, η) is on the bicharacteristic strip of $\eta_1 - \lambda_k(x, \eta')$ going through $(0, y'; \lambda_k(0, y', \xi'), \xi')$ for some $\xi' \in \mathbf{R}^{n-1} - \{0\}$.*

Remark. Since we assume $\lambda_k(x, \eta') \neq \lambda_l(x, \eta')$ for $k \neq l$, the union of bicharacteristic strips of $\eta_1 - \lambda_k(x, \eta')$ is the one of P.

Corollary 1. *The hyperfunction $E(x, y')$ is real-analytic outside the set of bicharacteristic curves through y' (i.e. the image of the projection of a bicharacteristic strip on the base space).*

Remark. Such a set in Corollary 1 is called a characteristic conoid of $P(x, D_x)$ with its vertex at y'.

Corollary 2. *The support of $E(x, y')$ is contained in a cone with y' as its vertex; i.e. for the operator P, it has a finite propagation.*

Proof. Since P is regularly hyperbolic, a characteristic conoid with its vertex at y' is contained in some cone, provided $|x_1| \ll 1$ and $|x' - y'| \ll 1$. On the other hand, Holmgren's theorem implies that $E(x, y')$ is 0 in a neighborhood of $\{x | x_1 = 0 \text{ and } x' \neq y'\}$. Hence, by the uniqueness of the continuation of an analytic function, the assertion of Corollary 2 follows, since $E(x, y')$ is real-analytic outside the characteristic conoid.

Proof of Theorem 3.6.3. Denote a cotangent vector at $(x, y', \xi') \in \mathbf{R}^n \times \mathbf{R}^{n-1} \times (\mathbf{R}^{n-1} - \{0\})$ by (η, θ', τ'), where $\eta \in \mathbf{R}^n$ and $\theta', \tau' \in \mathbf{R}^{n-1}$. By the definition of Ω_k and by Theorem 2.3.4 (see, also, Example 3.1.2), one has

$$\begin{aligned}\text{S.S. } F_k(x, y', \xi') \subset \{(x, y', \xi'; \sqrt{-1}(\eta, \theta', \tau')\infty) \,|\, \varphi_k(x, y', \xi') = 0 \text{ and}\\ (\eta, \theta', \tau') = \alpha \operatorname{grad}_{(x,y',\xi')}\varphi_k(x, y', \xi'), (\alpha > 0)\}.\end{aligned} \tag{3.6.6}$$

Furthermore, by Euler's identities, satisfied by $\varphi_k(z, w', \zeta')^j$ and $\log \varphi_k(z, w', \zeta')$ in the proof of Theorem 3.6.2, and with Theorem 3.4.4, one obtains

$$\begin{aligned}\text{S.S. } F_k(x, y', \xi') \subset \Big\{(x, y', \xi'; \sqrt{-1}(\eta, \theta', \tau') \,|\, \varphi_k(x, y', \xi') = 0 \text{ and}\\ (\eta, \theta', \tau') = \alpha \operatorname{grad}_{(x,y',\xi')} \varphi_k(x, y', \xi'), (\alpha > 0)\\ \text{and } \sum_{j=2}^{n} \xi_j \tau_j = 0\Big\}.\end{aligned} \tag{3.6.7}$$

Therefore, Theorem 3.1.4 implies

$$\begin{aligned}\text{S.S.}(F_k(x, y', \xi')|_{S^{n-1}_{\xi'}}) \subset \Big\{(x, y', \xi'; \sqrt{-1}(\eta, \theta', \tau')\infty) \,|\, \varphi_k(x, y', \xi') = 0,\\ (\eta, \theta', \tau' + c\xi') = \alpha \operatorname{grad}_{(x,y',\xi')} \varphi_k(x, y', \xi'),\\ (\alpha > 0 \text{ and } c \in \mathbf{R}) \text{ and } \sum_{j=2}^{n} \xi_j(\tau_j + c\xi_j) = 0\Big\}.\end{aligned} \tag{3.6.8}$$

Generally, one has, for a submanifold M of $\mathbf{R}^n$, the exact sequence

$$0 \to T^*_M\mathbf{R}^n \to T^*\mathbf{R}^n \underset{\mathbf{R}^n}{\times} M \to T^*M \to 0. \tag{3.6.9}$$

In the equation (3.6.8), we used the fact that a point on T^*M is identified with $(x, \xi) \in T^*\mathbf{R}^n \cong \mathbf{R}^n \times \mathbf{R}^n$ modulo elements of $T^*_M\mathbf{R}^n$ (see Example 3.1.3). That is, τ' is actually τ' mod $\operatorname{grad}_{\xi'} \xi'^2$ or τ' mod ξ'. Therefore, one obtains, by Theorem 3.2.1,

$$\text{S.S.} \int_{S^{n-1}_{\xi'}} F_k(x, y', \xi')\omega(\xi') \subset \Lambda_k, \tag{3.6.10}$$

where $\Lambda_k = \{(x, y'; \sqrt{-1}(\eta, \theta')\infty) \mid \varphi_k(x, y', \xi') = 0$ for some $\xi' \in \mathbf{R}^n - \{0\}$, $(\eta, \theta') = \alpha\, \mathrm{grad}_{(x,y',\xi')}\, \varphi_k(x, y', \xi')$, $\alpha > 0$ and $\alpha\, \mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi') = 0\}$.

Note. By the above agreement, the last equation in Λ_k ought to be phrased as "$\alpha\, \mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi') + c'\xi' = 0$ for some $c' \in \mathbf{R}$." But the homogeneity of φ_k in ξ' implies that $0 = \langle \xi', \mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi') + c\xi' \rangle = \varphi_k(x, y', \xi') + c'\xi'^2 = c'\xi'^2 = c'$. If one pays close attention to the construction of F_k in Hamada [1], one notices, without going through the somewhat lengthy argument above, that $F_k(x, y', \xi')$ as a microfunction is homogeneous in ξ' of degree $n - 1$. Hence, one can regard $F_k\omega(\xi')$ as being defined on $S^{n-1}_{\xi'}$. In that case, one may take the normalized (ξ', τ') on $T^*S^{n-1}_{\xi'}$ so that $\langle \xi', \tau' \rangle = 0$ holds (Exercise: explain why). Then $\varphi_k(x, y', \xi') = 0$ implies $\langle \xi', \mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi') \rangle = 0$. Thus, one can obtain (3.6.10) directly from (3.6.6). We often use this kind of computational skill on homogeneous functions defined on a sphere (or on a projective space).

We have obtained an estimate on S.S. $E_k(x, y')$. Next, we shall consider the geometric meaning of Λ_k in (3.6.10). The next lemma gives its meaning.

Lemma. *On a bicharacteristic strip of* $\eta_1 - \lambda_k(x, \eta')$, $\mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi')$ *and* $\mathrm{grad}_{y'}\, \varphi_k(x, y', \xi')$ *are constant. In particular, one has* $\mathrm{grad}_{y'}\, \varphi_k = -\xi'$.

Proof. Let $(x(s), \eta(s))_{|s| << 1}$ be the bicharacteristic strip under consideration. Then the following (3.6.11) holds:

$$
\begin{aligned}
&\frac{d}{ds}\left(\frac{\partial}{\partial \xi_j}\, \varphi_k(x(s), y', \xi')\right) \\
&\qquad = \sum_{l=1}^{n} \frac{\partial^2 \varphi_k}{\partial x_l\, \partial \xi_j} \frac{dx_l}{ds} \\
&\qquad = \frac{\partial}{\partial \xi_j}\, (\lambda_k(x, \mathrm{grad}_{x'}\, \varphi_k(x, y', \xi')))\Big|_{x = x(s)} \\
&\qquad\quad + \sum_{l=2}^{n} \left(\frac{\partial^2 \varphi_k}{\partial x_l\, \partial \xi_j}\left(\left(-\frac{\partial \lambda_k(x, \eta')}{\partial \eta_l}\right)\Big|_{\eta' = \mathrm{grad}_{x'}\varphi}\right)\right)\Big|_{x = x(s)} \\
&\qquad = \sum_{l=2}^{n} \left(\frac{\partial^2 \varphi_k}{\partial x_l\, \partial \xi_j}\left(\frac{\partial \lambda_k}{\partial \eta_l}\Big|_{\eta' = \mathrm{grad}_{x'}\, \varphi}\right)\right)\Big|_{x = x(s)} \\
&\qquad\quad - \sum_{l=2}^{n} \left(\frac{\partial^2 \varphi_k}{\partial x_l\, \partial \xi_j}\left(\frac{\partial \lambda_k}{\partial \eta_l}\Big|_{\eta' = \mathrm{grad}_{x'}\, \varphi}\right)\right)\Big|_{x = x(s)} \\
&\qquad = 0.
\end{aligned} \tag{3.6.11}
$$

Hence, $\mathrm{grad}_{\xi'}\, \varphi_k(x, y', \xi')$ is constant on the bicharacteristic strip, and similarly for the second assertion.

We return to the proof of Theorem 3.6.3. Let $(x, y'; \sqrt{-1}(\eta, \theta')\infty) \in \Lambda_k$. Take a bicharacteristic strip $(x(s), \eta(s))$ of $\eta_1 - \lambda_k(x, \eta')$ such that $(x(0), \eta(0)) = (x, \eta)$. Then p^* denotes the point of intersection with $\{x_1 = 0\}$. By the above lemma, $\operatorname{grad}_{\xi'} \varphi_k = 0$ holds at p^*. The equation (3.6.1) for φ_k implies $\operatorname{grad}_{\xi'} \varphi_k = x' - y'$ for $x_1 = 0$. Hence, one has $x' = y'$ at p^*. Again by (3.6.1), one has $\operatorname{grad}_x \varphi_k = (\lambda_k(0, y', \xi'), \xi')$ at p^*. From Remark 3 on the construction of φ_k, one concludes that (x, η) is on the bicharacteristic strip of $\eta_1 - \lambda_k(x, \eta')$ passing through $(0, y', \lambda_k(0, y, \xi'), \xi')$. The above lemma implies that $\theta' = -\xi'$.

Theorem 3.6.4. *Let a differential operator $P(x, D_x)$ be regularly hyperbolic in the direction of* $(1, 0, \ldots, 0)$ *at the origin. Then the following initial value problem* (3.6.12) *has a unique solution in a neighborhood of the origin, and the singularity spectrum of the solution $u(x)$ is contained in the set of bicharacteristic strips of $P(x, D_x)$ initiating from the singularity spectrums of the initial conditions $\mu_j(x')$:*

$$\left.\begin{aligned} &P(x, D_x)u(x) = 0 \\ &\frac{\partial^j}{\partial x_1^j} u(x)\Big|_{x_1=0} = \mu_j(x'), \qquad u = 0, 1, \ldots, m-1. \end{aligned}\right\} \tag{3.6.12}$$

Proof. By the same argument for Theorem 3.6.3, one can construct $E_j(x, y')$, satisfying

$$\begin{cases} P(x, D_x)E_j(x, y') = 0 \\ \dfrac{\partial^l}{\partial x_1^l} E_j(x, y')\Big|_{x_1=0} = \delta_{l,j}\delta(x' - y'), \qquad (0 \leqq l, j \leqq m-1) \end{cases}$$

and having the same singularity spectrum as $E(x, y')$ in Theorem 3.6.3. By defining $u(x)$ as

$$u(x) = \sum_{j=0}^{m-1} \int E_j(x, y')\mu_j(y')\,dy, \tag{3.6.13}$$

one can show the existence of a solution. The details are left to the reader. (One ought to check carefully that the integral is well defined. Then notice that the domain of the integration is compact. One can use the flabbiness of the hyperfunction sheaf to cut off $\mu_j(y')$ and then show that the operation does not affect the solution $u(x)$ in a sufficiently small neighborhood of the origin. See the problem below.) The uniqueness is nothing but Holmgren's theorem (Theorem 3.5.1). The statement on S.S. $u(x)$ is left to be proved. This should be done during the process of showing the well-definedness of the integral in (3.6.13). The reader is expected to prove it. One should then be careful to check that there is no contribution from the real analytic region of $E(x, y')$ or $\mu(y')$.

Problem. For what kind of domain does the uniqueness of a solution of (3.6.12) hold?

As in Theorems 3.6.3 and 3.6.4, bicharacteristic strips play an important role. One might think that the importance of bicharacteristic strips as carriers of singularities would have been recognized implicitly since the publication of Courant and Hilbert [1]. However, microfunction theory was needed to make possible the clear formulation given here. For example, consider the hyperfunction in Example 2.4.4 as a Cauchy datum. Then one can find a solution u, for a regularly hyperbolic operator P, such that $Pu = 0$ and S.S. u is contained in one bicharacteristic strip (see Kawai [2]). From this, for $P = \partial^2/\partial t^2 - \partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$, we can easily find a solution u of $Pu = 0$ so that u is not analytic outside the cylinder $Z = \{(t, x_1, x_2) \in \mathbf{R}^3 | x_1^2 + x_2^2 \leqq 1\}$ and analytic in the interior of Z. (Hint: For each point on the boundary of Z, there is a bicharacteristic curve going through the point and tangent to Z.) The existence of such a solution, however, seems to have puzzled even such an expert as John in 1960. On finding such a solution, John says: "What is remarkable is that this cylinder is not a characteristic surface for the differential equation. Apparently not *all* types of singularities propagate along characteristic surfaces" (John [3], p. 574). For those of us who have succeeded in grasping the notion of bicharacteristic strips as building blocks for the singularities of a solution, the above fact is a natural consequence. This is a very good example for showing just how profoundly the theory of linear differential equations has advanced over the past twenty years. Furthermore, the notion of a bicharacteristic strip has its origins in contact geometry, and S^*M is the most typical contact manifold (for example, the classical Darboux theorem states that an arbitrary contact manifold is locally isomorphic to an open subset of S^*M as contact manifolds). Hence, it is natural that one should visualize the interplay between the theory of linear differential equations and contact geometry; note, also, that the theory of first order (non-linear) partial differential equations and contact geometry are two sides of the same coin. This visualization will be realized in the next chapter. There we reach a more sophisticated level, where we study not only the structure of solutions but also the structure of equations.

§7. The Flabbiness of the Sheaf of Microfunctions

The sheaf of hyperfunctions has a remarkable property of being a flabby sheaf, which provided a great clue to its usefulness in analysis. As we will show below, it is easy to prove that $\mathscr{B}_M/\mathscr{A}_M$ is a flabby sheaf. The next question is whether the sheaf of microfunctions is flabby or not. Since $\pi_*\mathscr{C}_M \cong \mathscr{B}_M/\mathscr{A}_M$ holds, the flabbiness of $\mathscr{C}_M$ would imply the flabbiness of $\mathscr{B}_M/\mathscr{A}_M$. But the converse does not follow immediately. We will give an affirmative answer to the question.

Theorem 3.7.1. *The sheaf of microfunctions is a flabby sheaf.*

Proof. Let $N = \sqrt{-1}S^*M$, and let $\pi: N \to M$. $\mathscr{B}_N$ and $\mathscr{A}_N$ denote the sheaf of hyperfunctions on N and the sheaf of real analytic functions on N respectively.

Lemma 1. $\mathscr{B}_N/\mathscr{A}_N$ *is a flabby sheaf.*

Proof. By Grauert's theorem (Theorem 1.2.3) one has $H^1(\Omega, \mathscr{A}_N) = 0$ for an (oriented) open subset Ω in N. Hence one has the canonical exact sequence

$$0 \to \mathscr{A}_N \to \mathscr{B}_N \to \mathscr{B}_N/\mathscr{A}_N \to 0.$$

From this, the sequence

$$\Gamma(\Omega, \mathscr{B}_N) \to \Gamma(\Omega, \mathscr{B}_N/\mathscr{A}_N) \to H^1(\Omega, \mathscr{A}_N) = 0$$

is exact. On the other hand, $\Gamma(N, \mathscr{B}_N) \to \Gamma(\Omega, \mathscr{B}_N) \to 0$ is also an exact sequence since $\mathscr{B}_N$ is flabby. From the commutative diagram

$$\begin{tikzcd}
\Gamma(N, \mathscr{B}_N) \arrow[r] \arrow[d] & \Gamma(N, \mathscr{B}_N/\mathscr{A}_N) \arrow[d] & \\
\Gamma(\Omega, \mathscr{B}_N) \arrow[r] \arrow[d] & \Gamma(\Omega, \mathscr{B}_N/\mathscr{A}_N) \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

one concludes that $\Gamma(N, \mathscr{B}_N/\mathscr{A}_N) \to \Gamma(\Omega, \mathscr{B}_N/\mathscr{A}_N)$ is an epimorphism. Hence, the sheaf $\mathscr{B}_N/\mathscr{A}_N$ is flabby.

We plan to derive the flabbiness of $\mathscr{C}_M$ from that of $\mathscr{B}_N/\mathscr{A}_N$. Suppose that there exist sheaf homomorphisms $\Phi: \mathscr{C}_M \to \mathscr{B}_N/\mathscr{A}_N$ and $\Psi: \mathscr{B}_N/\mathscr{A}_N \to \mathscr{C}_M$ such that $\Psi \circ \Phi = 1$ holds. Then $\Phi(u) \in \Gamma(\Omega, \mathscr{B}_N/\mathscr{A}_N)$ for $u \in \mathscr{C}_M(\Omega)$. By Lemma 1, there exists $s \in \Gamma(N, \mathscr{B}_N/\mathscr{A}_N)$ such that $s|_\Omega = \Phi(u)$ holds. Then one has $\Psi(s) \in \Gamma(N, \mathscr{C}_M)$ and $\Psi(s)|_\Omega = \Psi \circ \Phi(u) = u$; i.e. $\mathscr{C}_M$ is flabby. Therefore, we must construct Φ and Ψ such that $\Psi \circ \Phi = 1$. Flabbiness is a local property (see Theorem 1.1.3). Hence it is sufficient to prove for the case where $M = \mathbf{R}^n$ and $N = \sqrt{-1}S^*M \cong \mathbf{R}^n \times S^{n-1}$. Let us recall Theorem 3.4.1. Let $K(x, \xi) \in \mathscr{C}_N$ such that $\operatorname{supp} K(x, \xi) \subset \{(x, \xi; \sqrt{-1}\langle \xi, dx\rangle\infty) \in \sqrt{-1}S^*N \mid x = 0\}$. Then one has $\operatorname{supp} K(x - y, \xi)\, dy \subset Z = \{(x, \xi, y; \sqrt{-1}\langle \xi, d(x - y)\rangle\infty) \mid x = y\}$ for $K(x - y, \xi)\, dy \in \mathscr{C}_{N \times M} \otimes v_M$. From the definitions, for $z = (x, \xi, y; \sqrt{-1}\langle \xi, d(x - y)\rangle\infty) \in Z$, one has $p_1(z) = (x, \xi, \sqrt{-1}\langle \xi, dx\rangle\infty) \in \sqrt{-1}S^*N$ and $p_2^a(z) = (y, \sqrt{-1}\langle \xi, dy\rangle\infty) \in \sqrt{-1}S^*M$. Define a subset G of $\sqrt{-1}S^*N$ by $G = \{(x, \xi; \sqrt{-1}\langle \xi, dx\rangle\infty)\}$. Then $(p_1|_Z)_*(p_2^a|_Z)^{-1}\mathscr{C}_M = (\pi_N|_G)^{-1}\mathscr{C}_M$ holds.

Hence, by Theorem 3.4.1, there exists a sheaf homomorphism

$$(\pi_N|_G)^{-1}\mathscr{C}_M \to \mathscr{H}^0_G(\mathscr{C}_N).$$

That is, one obtains a sheaf homomorphism

$$\mathscr{C}_M \to (\pi_N|_G)_*\mathscr{H}^0_G(\mathscr{C}_N) = \pi_{N*}(\mathscr{H}^0_G(\mathscr{C}_N)) \subset \pi_{N*}(\mathscr{C}_N) = \mathscr{B}_N/\mathscr{A}_N.$$

We denote this homomorphism with Φ, i.e.

$$\Phi: u(x) \mapsto \int K(x-y, \xi)u(y)\,dy.$$

Next, let $T(x, \xi) \in \mathscr{C}_N$ satisfy

$$\operatorname{supp} T(x, \xi) \subset \{(x, \xi; \sqrt{-1}\langle \xi, dx\rangle\infty) | x = 0\}.$$

Then one has $\operatorname{supp} T(x-y, \xi) \subset \{(x, y, \xi; \sqrt{-1}\langle\xi, d(x-y)\rangle\infty) | x = y\} = Z$, and also $p_1(z) = (x; \sqrt{-1}\langle\xi, dx\rangle\infty) \in N = \sqrt{-1}S^*M$, $p_2^a(z) = (y, \xi; \sqrt{-1}\langle\xi, dy\rangle\infty) \in \sqrt{-1}S^*N$ for $z = (x, y, \xi; \sqrt{1}\langle\xi, d(x-y)\rangle\infty)$. Theorem 3.4.1 implies that there exists a sheaf homomorphism

$$(p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_N \to \mathscr{C}_M.$$

Since $(p_1|_Z)_!(p_2^a|_Z)^{-1}\mathscr{C}_N = \pi_{N*}(\mathscr{C}_N|_G) \subset \pi_{N*}(\mathscr{C}_N) = \mathscr{B}_N/\mathscr{A}_N$, consequently a sheaf homomorphism $\Psi: \mathscr{B}_N/\mathscr{A}_N \to \mathscr{C}_M$ has been obtained. That is,

$$\Psi: u(x, \xi) \mapsto \int T(x-y, \xi)u(y, \xi)\,dy\omega(\xi).$$

We will compute $\Psi \circ \Phi$ next:

$$\begin{aligned}\Psi\circ\Phi: u(x) \mapsto \int T(x-x', \xi)\,dx'\omega(\xi) \int K(x'-y, \xi)u(y)\,dy \\ = \int u(y)\,dy \int T(x-x', \xi)K(x'-y, \xi)\,dx'\omega(\xi) \\ = \int u(y)\,dy \int T(x-y-x', \xi)K(x', \xi)\,dx'\omega(\xi).\end{aligned}$$

Therefore, if one can choose T and K such that

$$\int T(x-y-x', \xi)K(x', \xi)\,dx'\omega(\xi) = \delta(x-y),$$

then one obtains $\Psi \circ \Phi = 1$. Hence our proof rests upon the construction of $K(x, \xi)$ and $T(x, \xi)$ in $\mathscr{C}_N$ such that

$$\operatorname{supp} K(x, \xi) \quad \text{and} \quad \operatorname{supp} T(x, \xi) \subset \{(x, \xi; \sqrt{-1}\langle\xi, dx\rangle\infty) | x = 0\} \tag{3.7.1}$$

and

$$\int T(x-y, \xi)K(y, \xi)\,dy\omega(\xi) = \delta(x) \tag{3.7.2}$$

hold. In order to construct K and T, we need several lemmas in connection with the plane-wave decomposition of the δ-function. The following Lemma 2 is of particular interest, as it stands and has theoretical importance.

Lemma 2. *Let $\varphi_j(x, \xi)$ be a real analytic function defined in $\mathbf{R}^n \times \mathbf{R}^n$ for $j = 1, 2, \ldots, n$. Assume that $\Phi(x, \xi) \underset{\text{def}}{=} (\varphi_1(x, \xi), \varphi_2(x, \xi), \ldots, \varphi_n(x, \xi))$*

is homogeneous in ξ of degree one such that $\Phi(0, \xi) = \xi$. Furthermore, we assume that $\langle x, \Phi(x, \xi)\rangle$ is of positive type (see Definition 2.4.2). Then we have

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int \frac{J(x, \xi)}{(\langle x, \Phi(x, \xi)\rangle + \sqrt{-1}0)^n}\, \omega(\xi),$$

where $J(x, \xi) = \det\,(\partial\Phi(x, \xi)/\partial\xi) = \det\,(\partial\varphi_i(x, \xi)/\partial\xi_j)_{1 \leqq i,j \leqq n}$.

Remark. The above equation is obtained formally from the plane-wave decomposition by the change of variables $\zeta = \Phi(x, \xi)$,

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \int \frac{\omega(\zeta)}{(\langle x, \xi\rangle + \sqrt{-1}0)^n}.$$

Since $\Phi(x, \xi)$ is not assumed to be real-valued, we need a proof.

Proof. First note that

$$F(x) = \int \frac{J(x, \xi)}{(\langle x, \Phi(x, \xi)\rangle + \sqrt{-1}0)^n}\, \omega(\xi)$$

is well defined as a hyperfunction since $\langle x, \Phi(x, \xi)\rangle$ is of positive type. Define a multivalued function Φ with

$$\Phi(z) = \int \frac{J(x, \xi)}{\langle z, \Phi(z, \xi)\rangle^n}\, \omega(\xi) \qquad \text{for } z \in \mathbf{C}^n.$$

If one can prove $z_j\Phi(z) = 0$ for $j = 1, \ldots, n$, then $J(x, \xi)/(\langle x, \Phi(x, \xi)\rangle + \sqrt{-1}0)^n$ does make sense by taking the boundary values of $J(z, \xi)/\langle z, \Phi(z, \xi)\rangle^n$ from $\operatorname{Im}\langle z, \Phi(z, \xi)\rangle > 0$, depending on ξ. Then $x_jF(x) = 0$ holds for $j = 1, \ldots, n$; i.e. the support of $F(x)$ is only at the origin. We will prove $z_j\Phi(z) = 0$ for $j = 1, \ldots, n$ first. We will denote the exterior differential operator and the interior product by d and ι respectively. Then, by direct computation,

$$d(f\iota_{\partial/\partial\zeta_j}\omega(\zeta)) = \frac{\partial f}{\partial\zeta_j}\,\omega(\zeta) + (-1)^j\left((n-1)f + \sum_k \zeta_k \frac{\partial f}{\partial\zeta_k}\right) d\zeta_1 \wedge \cdots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \cdots \wedge d\zeta_n$$

(see Matsushima [1]). In particular, if $(1-n)$ is the homogeneous degree of f, then $\partial f/\partial\zeta_j\omega(\zeta) = d(f\iota_{\partial/\partial\zeta_j}\omega(\zeta))$. From this equation, one obtains

$$z_jF(z) = \int \frac{z_jJ(z, \xi)}{\langle z, \Phi\rangle^n}\,\omega(\xi) = \int \frac{z_j}{\langle z, \zeta\rangle^n}\,\omega(\zeta) = \int \frac{\partial}{\partial\zeta_j}\left(\frac{1}{(1-n)\langle z, \zeta\rangle^{n-1}}\right)\omega(\zeta)$$
$$= \int d\left(\frac{1}{(1-n)\langle z, \zeta\rangle^{n-1}}\,\iota_{\partial/\partial\zeta_j}\omega(\zeta)\right) = 0.$$

Therefore, $F(x)$ is a hyperfunction whose support is only at the origin. On the other hand,

$$\left(\sum_{i=1}^{n} x_i D_i\right) F = \sum_{i=1}^{n} (D_i x_i - 1) F = -nF$$

holds. Hence, the homogeneous degree of F is $-n$. From the corollary of Proposition 3.5.3, one concludes $F(x) = c\delta(x)$. Next, we shall determine the constant c. Since sp $F = c$ sp δ, it is sufficient to consider a neighborhood of $(0, \sqrt{-1}d(x_1 + \ldots + x_n)\infty)$ for the computation of c. We let:

$$G_z = \{\zeta = \Phi(tz, \xi) \mid 0 \leqq t \leqq 1, \xi = (\xi_1, \ldots, \xi_n), \xi_1 + \ldots + \xi_n = 1, \\ \xi_i \geqq 0 \text{ for } 1 \leqq i \leqq n\};$$

$$\Gamma_z = \{\zeta = \Phi(z, \xi) \mid \xi_1 + \ldots + \xi_n = 1, \xi_i \geqq 0 \text{ for } 1 \leqq i \leqq n\};$$

$$\Gamma_0 = \{\xi \mid \xi_1 + \ldots + \xi_n = 1, \xi_i \geqq 0 \text{ for } 1 \leqq i \leqq n\};$$

$$G_j = \{\zeta = \Phi(tz, \xi) \mid 0 \leqq t \leqq 1, \xi_1 + \ldots + \xi_n = 1, \xi_j = 0, \text{ and for} \\ \text{an arbitrary } i (\neq j)\ \xi_i \geqq 0\}, j = 1, 2, \ldots, n.$$

Then the boundary ∂G_z of G_z is clearly $\Gamma_z \cup \Gamma_0 \cup G_1 \cup \ldots \cup G_n$. Stokes' theorem implies

$$\int_{\partial G_z} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n} = \int_{G_z} d\, \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n} = 0,$$

since n-forms are 0 on the $(n-1)$-dimensional space. By taking the orientation into consideration, one has,

$$\int_{\Gamma_z} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n} = \int_{\Gamma_0} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n} + \sum_{j=1}^{n} \int_{G_j} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n}.$$

Therefore, in a neighborhood of $(0, \sqrt{-1}\, d(x_1 + \cdots + x_n)\infty)$, one gets

$$F(x) = b\left(\int_{\xi_i \geqq 0,\, 1 \leqq i \leqq n} \frac{J(z, \xi)}{\langle z, \Phi(z, \xi)\rangle^n}\, \omega(\xi)\right) = b\left(\int_{\Gamma_z} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n}\right)$$

$$= b\left(\int_{\Gamma_0} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n} + \sum_{j=1}^{n} b\left(\int_{G_j} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n}\right)\right).$$

Then,

$$b\left(\int_{\Gamma_0} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n}\right) = \int \frac{\omega(\xi)}{(\langle x, \xi \rangle + \sqrt{-1}0)^n} = \frac{(-2\pi\sqrt{-1})^n}{(n-1)!}\, \delta(x)$$

holds. Therefore, if one can prove

$$b\left(\int_{G_j} \frac{\omega(\zeta)}{\langle z, \zeta \rangle^n}\right) = 0 \qquad \text{for all } j$$

as a microfunction, then one obtains

$$F(x) = \frac{(-2\pi\sqrt{-1})^n}{(n-1)!}\,\delta(x),$$

and

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int \frac{J(x,\xi)}{\langle x, \Phi\rangle^n}\,\omega(\xi).$$

If a real analytic function $\varphi(x)$ is of positive type such that $\varphi(0) = 0$ and $d\varphi(0) = \xi$ hold, then by Proposition 2.4.3 one has $\varphi(z) \neq 0$ for a sufficiently small z such that $\langle \operatorname{Im} z, \xi\rangle > \epsilon|\operatorname{Im} z|$. Let us apply this to $\langle z, \Phi(z,\xi)\rangle = \langle z, \zeta\rangle$. Then one has $\langle z, \zeta\rangle \neq 0$ in G_j if $\sum_{k\neq j}(\operatorname{Im} z_k)\xi_k = \sum_k (\operatorname{Im} z_k)\xi_k > \epsilon|\operatorname{Im} z|$ holds. Let $D_j = \{z\,|\operatorname{Im} z_k > \epsilon|\operatorname{Im} z|$ for an arbitrary $k \neq j\}$. Hence, in particular, for $z \in D_j$ and $\zeta \in G_j$ one obtains $\langle z, \zeta\rangle \neq 0$. Therefore, $\int_{G_j}\omega(\zeta)/\langle z,\zeta\rangle^n$ is holomorphic on D_j. Consequently,

$$b\left(\int_{G_j}\frac{\omega(\zeta)}{\langle z,\zeta\rangle^n}\right) = b_{D_j}\left(\int_{G_j}\frac{\omega(\zeta)}{\langle z,\zeta\rangle^n}\right)$$

holds. On the other hand, D_j is a conoidal neighborhood of $U_j = \{(0 + \sqrt{-1}(v_1,\ldots,v_n)0)\,|\,v_k > \epsilon\sqrt{v_1^2 + \cdots + v_n^2}$ for any $k \neq j\}$. By Theorem 2.3.5, one can conclude that the singularity spectrum of $b_{D_j}(\int_{G_j}\omega(\zeta)/\langle z,\zeta\rangle^n)$ is contained in U_j°. But, by definition,

$$U_j^\circ \cap \{x = 0\} = \left\{(0, \sqrt{-1}(\xi_1,\ldots,\xi_n)\infty\,\middle|\,\sum_{i=1}^n \xi_i v_i > 0\right.$$

for $(v_1,\ldots,v_n)$, such that $v_k > \epsilon\sqrt{v_1^2 + \cdots + v_n^2}$ for any $k \neq j\}$. Hence, $(0, \sqrt{-1}(1,\ldots,1)\infty) \notin U_j^\circ$. In fact, for an arbitrary $k \neq j$, one can clearly find $(v_1,\ldots,v_n) \in \mathbf{R}^n$ such that $v_k > \epsilon\sqrt{v_1^2 + \cdots + v_n^2}$ and $\sum_{i=1}^n v_i < 0$ hold for a sufficiently small ϵ. Therefore, one obtains $b(\int_{G_j}\omega(\zeta)/\langle z,\zeta\rangle^n) = 0$ in a neighborhood of $(0, \sqrt{-1}(dx_1 + \cdots + dx_n))$.

Lemma 3. *Let* $\alpha, \beta \in \mathbf{C}$ *such that* $\operatorname{Re}\alpha = \operatorname{Re}\beta = 0$ *and* $\operatorname{Im}\alpha > 0$ *hold. Then one has*

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int_{|\xi|=1}$$

$$\frac{(1-\beta\langle x,\xi\rangle)^{n-1}(1 + (\alpha-\beta)\langle x,\xi\rangle + \alpha\beta(1-\beta\langle x,\xi\rangle))^{n-2}(|x|^2 - \langle x,\xi\rangle^2)}{(\langle x,\xi\rangle + \alpha|x|^2 - \beta\langle x,\xi\rangle^2 + \sqrt{-1}\,0)^n}.$$

In particular, for $\alpha = \beta$,

$$\delta(x) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \cdot \int_{|\xi|=1} \frac{(1-\alpha\langle x, \xi\rangle)^{n-1} + \alpha^2(1-\alpha\langle x, \xi\rangle)^{n-2}(|x|^2 - \langle x, \xi\rangle^2)}{(\langle x, \xi\rangle + \alpha(|x|^2 - \langle x, \xi\rangle^2) + \sqrt{-1}0)^n}\, \omega(\xi)$$

holds.

Note. The right-hand sides of the above equations are not homogeneous in ξ. Hence, our computation has to be done for $|\xi| = 1$.

Proof. Let $\Phi(x, \xi) = \xi + \alpha|\xi|x - (\beta\langle x, \xi\rangle/|\xi|)\xi$. Then Φ satisfies the conditions for Lemma 2. First we will compute $J(x, \xi) = \det(\partial\Phi(x, \xi)/\partial\xi)$. If one lets $\Phi(x, \xi) = (\varphi_1, \ldots, \varphi_n)$, then $\varphi_j(x, \xi) = \xi_j + \alpha|\xi|x_j - (\beta\langle x, \xi\rangle/|\xi|)\cdot\xi_j$. Hence, $J(x, \xi)$ can be computed from $d\varphi_1 \wedge \cdots \wedge d\varphi_n = J(x, \xi)d\xi_1 \wedge \cdots \wedge d\xi_n$. One has $d\varphi_i = (1 - \beta\langle x, \xi\rangle)\, d\xi_i + (\alpha x_i + \beta\langle x, \xi\rangle\xi_i)d|\xi| - \beta\xi_i\langle x, d\xi\rangle$ and $d|\xi| = \sum_{i=1}^n \xi_i\, d\xi_i$ since $|\xi| = 1$. Then, one obtains

$$\begin{aligned}
d\varphi_1 &\wedge \cdots \wedge d\varphi_n \\
&= (1 - \beta\langle x, \xi\rangle)^n\, d\xi_1 \wedge \cdots \wedge d\xi_n + (1 - \beta\langle x, \xi\rangle)^{n-1} \\
&\quad \cdot \left(\sum_{i=1}^n (\alpha x_i + \beta\langle x, \xi\rangle\xi_i)\, d\xi_1 \wedge \cdots \wedge d\overset{\overset{i}{\smile}}{|\xi|} \wedge \cdots \wedge d\xi_n \right) \\
&\quad + (1 - \beta\langle x, \xi\rangle)^{n-1} \\
&\quad \cdot \left(\sum_{i=1}^n (-\beta\xi_i)\, d\xi_1 \wedge \cdots \wedge \overset{\overset{i}{\smile}}{\langle x, d\xi\rangle} \wedge \cdots \wedge d\xi_n \right) \\
&\quad + (1 - \beta\langle x, \xi\rangle)^{n-2} \cdot \\
&\quad \cdot \sum_{i \neq j} (\alpha x_i + \beta\langle x, \xi\rangle\xi_i)(-\beta\xi_j)\, d\xi_1 \wedge \cdots \wedge d\overset{\overset{i}{\smile}}{|\xi|} \wedge \cdots \wedge \overset{\overset{j}{\smile}}{\langle x, d\xi\rangle} \wedge \cdots \wedge d\xi_n \\
&= [(1 - \beta\langle x, \xi\rangle)^n + (1 - \beta\langle x, \xi\rangle)^{n-1}(\alpha + \beta)\langle x, \xi\rangle) \\
&\quad - (1 - \beta\langle x, \xi\rangle)^{n-1}\beta\langle x, \xi\rangle + \alpha\beta(|x|^2 \\
&\quad - \langle x, \xi\rangle^2)(1 - \beta\langle x, \xi\rangle)^{n-2}]\, d\xi_1 \wedge \cdots \wedge d\xi_n \\
&= [(1 - \beta\langle x, \xi\rangle)^{n-1}(1 + (\alpha - \beta)\langle x, \xi\rangle) \\
&\quad + \alpha\beta(1 - \beta\langle x, \xi\rangle)^{n-2}(|x|^2 - \langle x, \xi\rangle^2)]\, d\xi_1 \wedge \cdots \wedge \xi_n.
\end{aligned}$$

Note. The notation $d\overset{\overset{i}{\smile}}{|\xi|}$ means that $d\xi_i$ is replaced by $d|\xi|$.

Lemma 4. *For* $\alpha \in \mathbf{C}$, $\operatorname{Im} \alpha > 0$, *one can express*

$$\delta(x) = \sum_{j,k} c_{jk} \int_{|\xi|=1} \frac{(|x|^2 - \langle x, \xi\rangle^2)^j}{(\langle x, \xi\rangle + \alpha(|x|^2 - \langle x, \xi\rangle^2) + \sqrt{-1}0)^{n+k}}\, \omega(\xi)$$

for some finitely many $c_{jk} \in \mathbf{C}$.

Proof. Let $t = \langle x, \xi\rangle + \alpha(|x|^2 - \langle x, \xi\rangle^2)$, and let $y = |x|^2 - \langle x, \xi\rangle^2$. Then one can verify easily that there exists $c_{jk} \in \mathbf{C}$ such that

$$\frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{(1-\alpha\langle x, \xi\rangle)^{n-1} + \alpha^2(1-\alpha\langle x, \xi\rangle)^{n-2}(|x|^2 - \langle x, \xi\rangle^2)}{(\langle x, \xi\rangle + \alpha(|x|^2 - \langle x, \xi\rangle^2))^n}$$

$$= \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{(1-\alpha t + \alpha^2 y)^{n-1} + \alpha^2 y(1 - \alpha t + \alpha^2 y^{n-2})}{t^n} = \sum_{j,k} c_{jk} \frac{y^j}{t^{n-k}}$$

By Lemma 3, one obtains

$$\delta(x) = \sum_{j,k} c_{jk} \int_{|\xi|=1} \frac{y^j}{(t + \sqrt{-1}0)^{n-k}}\, \omega(\xi)$$

$$= \sum_{j,k} c_{jk} \int_{|\xi|=1} \frac{(|x|^2 - \langle x, \xi\rangle^2)^j}{(\langle x, \xi\rangle + \alpha(|x|^2 - \langle x, \xi^2\rangle) + \sqrt{-1}0)^{n-k}}\, \omega(\xi).$$

Lemma 5. *The following three equations hold:*

(1) $\int (t + \sqrt{-1}0)^\alpha (x - t + \sqrt{-1}0)^\beta \, dt$

$$= 2\pi\sqrt{-1}\, \frac{\Gamma(-\alpha - \beta - 1)}{\Gamma(-\alpha)\Gamma(-\beta)} (x + \sqrt{-1}0)^{\alpha+\beta+1}.$$

(2) $\displaystyle\int (x_1^2 + \cdots + x_n^2 + c)^\lambda \, dx_1 \cdots dx_n = \pi^{n/2} \frac{\Gamma\left(-\lambda - \frac{n}{2}\right)}{\Gamma(-\lambda)} c^{\lambda + n/2}, c > 0.$

(3) $\int (\langle x - y, \xi\rangle + \alpha(|x - y|^2 - \langle x - y, \xi\rangle^2))^\lambda (\langle y, \xi\rangle$
$+ \beta(|y|^2 - \langle y, \xi\rangle^2))^\mu \, dy$

$$= -2\pi^{(n+1)/2}\sqrt{-1}\, \frac{\Gamma\left(-\lambda - \mu - \frac{n+1}{2}\right)}{\Gamma(-\lambda)\Gamma(-\mu)} (\alpha + \beta)^{-(n-1)/2}$$

$$\cdot \left(\langle x, \xi\rangle + \frac{\alpha\beta}{\alpha + \beta}(|x|^2 - \langle x, \xi\rangle^2)\right)^{\lambda + \mu + (n+1)/2},$$

where $\operatorname{Im}\alpha$, $\operatorname{Im}\beta > 0$.

Proof. First note that $u(x) = \int (t + \sqrt{-1}0)^\alpha (x - t + \sqrt{-1}0)^\beta \, dt$ is well defined as the integration of a microfunction. Then

$$\operatorname{supp} u(x) = \{0, \sqrt{-1}\, dx\infty\}$$

holds. On the other hand, it is clear that $u(cx) = c^{\alpha+\beta+1} u(x)$ holds. Therefore, one obtains $u(x) = c(x + \sqrt{-1}0)^{\alpha+\beta+1}$. Next we will compute c. By

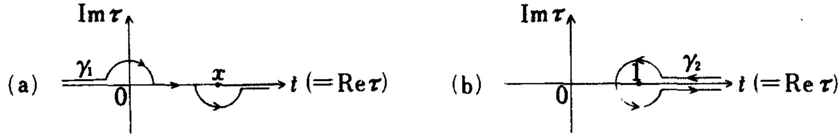

Figure 3.5.1

the definition of integration of a microfunction, one has

$$u(x) = \int_{\gamma_1} \tau^\alpha (x - \tau)^\beta \, d\tau,$$

where γ_1 is as in Figure 3.5.1(a). As an integration of a microfunction, the integral using γ_2 in Figure 3.5.1(b),

$$\int_{\gamma_2} \tau^\alpha (x - \tau)^\beta \, d\tau,$$

is equal to the above integral in the term of γ_1. Hence one obtains

$$\begin{aligned} c &= \int (t + \sqrt{-1}0)^\alpha (1 - t + \sqrt{-1}0)^\beta \, dt = \int_{\gamma_2} \tau^\alpha (1 - \tau)^\beta \, dt \\ &= \int_1^\infty t^\alpha [(t-1)^\beta e^{\pi\sqrt{1}\beta} - (t-1)^\beta e^{-\pi\sqrt{1}\beta}] \, dt \\ &= 2\sqrt{-1} \sin \pi\beta \, B(\beta + 1, -\alpha - \beta - 1) \\ &= 2\sqrt{-1} \sin \pi\beta \frac{\Gamma(\beta + 1)\Gamma(-\alpha - \beta - 1)}{\Gamma(-\alpha)} \\ &= -2\pi\sqrt{-1} \frac{\Gamma(-\alpha - \beta - 1)}{\Gamma(-\alpha)\Gamma(-\beta)}. \end{aligned}$$

Note. Here, $B(p, q)$ is the beta function defined as

$$\int_0^1 x^{p-1}(1 - x)^{q-1} \, dx \left(= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} \right).$$

Even though the above computation is valid, with the proviso that $\int_{\gamma_2} \tau^\alpha (1 - \tau)^\beta \, d\tau$ converges absolutely, the result is still correct for any (α, β) by virtue of the method of analytic continuation. This note will apply to the computations below.

We will prove (2). By the transformation of variables $x_i = r\xi_i$, $(\xi_1^2 + \cdots + \xi_n^2 = 1)$, the left-hand side of (2) equals

$$\int_0^\infty (r^2 + c)^\lambda r^{n-1} \, dr \int_{|\xi| = 1} \omega(\xi).$$

Since $\int_{|\xi|=1}\omega(\xi)$ is the surface area of S^{n-1}, it is $2\pi^{n/2}/\Gamma(n/2)$. On the other hand, one has

$$\int_0^\infty (x+1)^\lambda x^{n/2-1}\,dx = \int_1^\infty x^\lambda (x-1)^{n/2-1}\,dx$$
$$= \int_1^0 \frac{1}{x^\lambda}\cdot\left(\frac{1}{x}-1\right)^{n/2-1}\left(-\frac{dx}{x^2}\right)$$
$$= \int_0^1 x^{-\lambda-n/2-1}(1-x)^{n/2-1}\,dx = B\left(-\lambda-\frac{n}{2},\frac{n}{2}\right).$$
$$= \frac{\Gamma\left(-\lambda-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(-\lambda)}.$$

Since one has

$$\int_0^\infty (r^2+c)^\lambda r^{n-1}\,dr = \tfrac{1}{2}c^{\lambda+n/2}\int_0^\infty (x+1)^\lambda x^{(n-1)/2}\,dx,$$

the left-hand side of (2) is equal to

$$\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{c^{\lambda+n/2}}{2}\cdot\frac{\Gamma\left(-\lambda-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(-\lambda)} = \pi^{n/2}\,\frac{\Gamma\left(-\lambda-\frac{n}{2}\right)}{\Gamma(-\lambda)}\,c^{\lambda+n/2}.$$

Finally, we will prove (3). Since both sides of the equation in (3) are invariant with respect to ξ under the action of the rotation group, it is sufficient to prove the case where $\xi = (1, 0, \ldots, 0)$; i.e.

$$\int (x_1 - y_1 + \alpha((x_2-y_2)^2 + \cdots + (x_n-y_n)^2))^\lambda (y_1 + \beta(y_2^2 + \cdots + y_n^2))^\mu\,dy$$
$$= -2\pi^{(n+1)/2}\sqrt{-1}\,\frac{\Gamma\left(-\lambda-\mu-\frac{n+1}{2}\right)}{\Gamma(-\lambda)\Gamma(-\mu)}(\alpha+\beta)^{-(n-1)/2}$$
$$\cdot\left(x_1 + \frac{\alpha\beta}{\alpha+\beta}(x_2^2 + \cdots + x_n^2)^{\lambda+\mu+(n+1)/2}\right). \qquad (3.7.3)$$

By virtue of (1) for the integration with respect to y_1, the left-hand side of (3.7.3), letting $y_i - \alpha x_i/(\alpha+\beta)$ be y_i, is equal to

$$-2\pi\sqrt{-1}\,\frac{\Gamma(-\lambda-\mu-1)}{\Gamma(-\lambda)\Gamma(-\mu)}\cdot\int\left(x_1 + \frac{\alpha\beta}{\alpha+\beta}(x_2^2 + \cdots + x_n^2)\right.$$
$$\left. + (\alpha+\beta)(y_2^2 + \cdots + y_n^2)\right)^{\lambda+\mu+1} dy_2\cdots dy_n.$$

Then, by (2), this equation can be written as

$$-2\pi\sqrt{-1}\,\frac{\Gamma(-\lambda-\mu-1)}{\Gamma(-\lambda)\Gamma(-\mu)}(\alpha+\beta)^{\lambda+\mu+1}\,\pi^{(n-1)/2}\,\frac{\Gamma\left(-\lambda-\mu-1-\dfrac{n-1}{2}\right)}{\Gamma(-\lambda-\mu-1)}$$

$$\cdot\left\{\frac{x_1}{\alpha+\beta}+\frac{\alpha\beta}{(\alpha+\beta)^2}(x_2^2+\cdots+x_n^2)\right\}^{\lambda+\mu+1+(n-1)/2},$$

which is the right-hand side of (3).

We have finished the necessary preparation. Now we return to the proof of Theorem 3.7.1. Let

$$L_{\lambda,\mu}(\alpha,x,\xi)=\frac{\Gamma(-\lambda)\Gamma(-\mu)}{2\pi^{(n+1)/2}\sqrt{-1}\,\Gamma\left(-\lambda-\mu-\dfrac{n+1}{2}\right)}$$

$$\cdot(\alpha+\sqrt{-1})^{(n-1)/2}(\langle x,\xi\rangle+\alpha(|x|^2-\langle x,\xi\rangle^2))^{\lambda},$$

and let

$$K_\mu(x,\xi)=(\langle x,\xi\rangle+\sqrt{-1}(|x|^2-\langle x,\xi\rangle^2))^{\mu}.$$

Then Lemma 5, (3) implies

$$\int L_{\lambda,\mu}(\alpha,x-y,\xi)K_\mu(y,\xi)\,dy$$

$$=-\left(\langle x,\xi\rangle+\frac{\alpha\sqrt{-1}}{\alpha+\sqrt{-1}}(|x|^2-\langle x,\xi\rangle^2)\right)^{\lambda+\mu+(n+1)/2}.$$

Hence one obtains

$$\left(-(\alpha+\sqrt{-1})^2\frac{d}{d\alpha}\right)^j\int L_{\lambda,\mu}(\alpha,x-y,\xi)K_\mu(y,\xi)\,dy$$

$$=-(|x|^2-\langle x,\xi\rangle^2)^j\left(\langle x,\xi\rangle+\frac{\alpha\sqrt{-1}}{\alpha+\sqrt{-1}}(|x|^2-\langle x,\xi\rangle^2)\right)^{\lambda+\mu+(n+1)/2-j}.$$

On the other hand, by Lemma 4, one has,

$$\delta(x)=\sum_{j,k}c_{j,k}\int_{|\xi|=1}\frac{(|x|^2-\langle x,\xi\rangle^2)^j}{\langle x,\xi\rangle+\dfrac{\sqrt{-1}}{2}(|x|^2-\langle x,\xi\rangle^2+\sqrt{-1}0)^{n-k}}\,\omega(\xi)$$

for some $c_{j,k}$. Then let

$$T_\mu(x,\xi)=-\sum_{j,k}c_{j,k}\left(-(\alpha+\sqrt{-1})^2\frac{d}{d\alpha}\right)^j L_{j+k-\mu-(3/2)n-1,\mu}(\alpha,x,\xi).$$

One obtains

$$\int\left(\int T_\mu(x-y,\xi)K_\mu(y,\xi)\,dy\right)\omega(\xi)=\delta(x).$$

By Proposition 2.4.3 and by definition, S.S. $T_\mu(x, \xi)$ and S.S. $K_\mu(x, \xi)$ are contained in $\{(x, \xi; \sqrt{-1}d\langle x, \xi\rangle\infty) \in \sqrt{-1}S^*N | x = 0\} = \{(x, \xi; \sqrt{-1}\langle \xi, dx\rangle\infty | x = 0\}$. Fixing μ, let $T(x, \xi) = T_\mu(x, \xi)$, and let $K(x, \xi) = K_\mu(x, \xi)$. Then define sheaf homomorphisms $\Phi: \mathscr{C}_M \to \mathscr{B}_N/\mathscr{A}_N$ and $\Psi: \mathscr{B}_N/\mathscr{A}_N \to \mathscr{C}_M$ as

$$\Phi: u(x) \mapsto \int K(x-y, \xi)u(y)\,dy$$

and

$$\Psi: u(x, \xi) \mapsto \int T(x-y, \xi)u(y, \xi)\,dy\omega(\xi).$$

The sheaf homomorphisms Φ and Ψ satisfy the conditions required to claim the flabbiness of the sheaf $\mathscr{C}_M$ of microfunctions.

§8. Appendix

We will supply a few important motions concerning microfunction theory. First a sheaf $\hat{\mathscr{C}}_M$ can be introduced in connection with Definition 3.1.1.

Definition 3.8.1. *Define a sheaf $\hat{\mathscr{C}}_M$ on $\sqrt{-1}T^*M(= T^*_MX)$ such that the stalk of $\hat{\mathscr{C}}_M$ at $(x, \sqrt{-1}\xi)$ is given by*

$$\hat{\mathscr{C}}_{M,(x,\sqrt{-1}\xi)} = \begin{cases} \mathscr{C}_{M,(x,\sqrt{-1}\xi\infty)} & \text{for } \xi \neq 0 \\ \mathscr{B}_{M,x} & \text{for } \xi = 0. \end{cases}$$

Remark 1. The notion $\widehat{\text{S.S.}}\, u$ in Definition 3.1.1 can be interpreted as the support of a section of $\hat{\mathscr{C}}_M$.

Remark 2. By virtue of the sheaf $\hat{\mathscr{C}}_M$, one can conveniently treat the hyperfunction sheaf and the microfunction sheaf simultaneously. The combined version of Theorems 3.1.5 and 3.1.6 is the following theorem.

Theorem 3.8.1. *Define $p_1: \sqrt{-1}T^*(M \times M) \underset{M\times M}{\times} M \to \sqrt{-1}T^*M$ by*

$$p_1((x, \sqrt{-1}\xi_1), (x, \sqrt{-1}\xi_2)) = (x, \sqrt{-1}\xi_1);$$

$p_2: \sqrt{-1}T^*(M \times M) \underset{M\times M}{\times} M \to T^*M$ by

$$p_2((x, \sqrt{-1}\xi_1), (x, \sqrt{-1}\xi_2)) = (x, \sqrt{-1}\xi_2);$$

and $q: \sqrt{-1}T^*(M \times M) \underset{M\times M}{\times} M \to \sqrt{-1}T^*M$ by

$$q((x, \sqrt{-1}\xi_1), (x, \sqrt{-1}\xi_2)) = (x, \sqrt{-1}(\xi_1 + \xi_2)).$$

Then there exists a sheaf homomorphism

$$q_1(p_1^{-1}\hat{\mathscr{C}}_M \times p_2^{-1}\hat{\mathscr{C}}_M) \to \hat{\mathscr{C}}_M.$$

Exercise. Rephrase Theorem 3.2.1 in terms of $\hat{\mathscr{C}}_M$.

The next proposition gives us a link between $\hat{\mathscr{C}}_M$ and $\hat{\mathscr{C}}_{M\times\mathbf{R}}$, which is of theoretical importance.

Proposition 3.8.1. *Let M be a real analytic manifold. Define $s:\mathscr{B}_M \to \mathscr{B}_{M\times\mathbf{R}}$ by $s(u(x)) = u(x)\delta(t)$. Then define a map $s:\hat{\mathscr{C}}_M \to \hat{\mathscr{C}}_{M\times\mathbf{R}}$ in the same manner. The following sequences are exact:*

$$0 \to \mathscr{B}_M \xrightarrow{s} \mathscr{B}_{M\times\mathbf{R}} \xrightarrow{t\cdot} \mathscr{B}_{M\times\mathbf{R}} \to 0. \tag{3.8.1}$$

$$0 \to \hat{\mathscr{C}}_M \xrightarrow{s} \hat{\mathscr{C}}_{M\times\mathbf{R}} \xrightarrow{t\cdot} \hat{\mathscr{C}}_{M\times\mathbf{R}} \to 0. \tag{3.8.2}$$

Here, $t\cdot$, $t \in \mathbf{R}$, means the multiplication by t.

Proof. First we will prove the exactness of (3.8.1).

(i) We will prove that s is monomorphic. Consider the map $u(x)\delta(x) \mapsto \int u(x)\delta(t)\,dt = u(x)$. Then this map is a left inverse of s. Hence s is monomorphic.

(ii) We will prove that $t\cdot$ is epimorphic. Since the question is local in nature, we may assume $M \subset \mathbf{R}^n$ without loss of generality. Then, for $u(x, t) \in \mathscr{B}_{M\times\mathbf{R}}$, there exist finitely many open convex cones Γ_j in $\mathbf{R}^n \times \mathbf{R}$, such that $u = \sum_j b(\varphi_j)$ for $\varphi_j \in \mathcal{O}((M \times \mathbf{R}) \times \sqrt{-1}\Gamma_j \cap \Omega)$, where Ω is a complex neighborhood of $M \times \mathbf{R}$. If necessary, one can arrange Γ_j so that $\Gamma_j \cap (\mathbf{R}^n \times \{0\}) = \varnothing$ holds. Then $\psi_j = \varphi_j/t$ is a holomorphic function in $(M \times \mathbf{R}) \times \sqrt{-1}\Gamma_j) \cap \Omega$. Hence $v(x, t) \in \mathscr{B}_{M\times\mathbf{R}}$ can be defined by $\sum_j b(\psi_j)$. Clearly, $u(x, t) = tv(x, t)$ holds; i.e. $t\cdot$ is an epimorphism.

(iii) We will prove $\operatorname{Ker}(t\cdot) = \operatorname{Im} s$. Since $\operatorname{Im} s \subset \operatorname{Ker}(t\cdot)$ is plainly true, we will give a proof for $\operatorname{Ker}(t\cdot) \subset \operatorname{Im} s$. As shown in §2 of Chapter I, an arbitrary hyperfunction $u(x, t) \in \mathscr{B}_{M\times\mathbf{R}}$ can be expressed as $\sum_\epsilon f_\epsilon(x_1 + \sqrt{-1}\epsilon_1 0, \ldots, x_n + \sqrt{-1}\epsilon_n 0, t + \sqrt{-1}\epsilon_{n+1}0)$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_n, \epsilon_{n+1}) \in \{-1, 1\}^{n+1}$. Let $u(x, t) \in \operatorname{Ker}(t\cdot)$. For the sake of simplicity, rewrite $t = x_{n+1}$. Then $x_{n+1}u(x, x_{n+1}) = 0$ holds. Hence, by using a holomorphic function $\varphi_{\hat{\epsilon}_k}(x, x_{n+1}) \underset{\text{def}}{=} \varphi_{\epsilon_1,\ldots,\epsilon_{k-1},\epsilon_{k+1},\ldots,\epsilon_{n+1}}(x, x_{n+1})$ on $\Omega_k \underset{\text{def}}{=} \{(x, x_{n+1}) \in \Omega \mid \epsilon_j \operatorname{Im} x_j > 0$ for any $j \neq k\}$, one can express $x_{n+1}u(x, x_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} \cdot \varphi_{\hat{\epsilon}_k}(x, x_{n+1})$. Since $x_{n+1} \neq 0$ holds where $\varphi_{\hat{\epsilon}_k}$ is defined for $k \leq n$, then $\varphi_{\hat{\epsilon}_k} \underset{\text{def}}{=} \psi_{\hat{\epsilon}_k}/x_{n+1}$ is holomorphic in Ω_k. If $k = n + 1$, then construct holomorphic functions $\psi_{\hat{\epsilon}_{n+1}}$ and $\xi_{\epsilon_1,\dots,\epsilon_n}$ such that one has $\varphi_{\hat{\epsilon}_{n+1}} = x_{n+1}\psi_{\hat{\epsilon}_{n+1}}(x, x_{n+1}) + \xi_{\epsilon_1,\dots,\epsilon_n}(x)$. (Note that it is not trivial that $\psi_{\hat{\epsilon}_{n+1}}$ and $\xi_{\epsilon_1,\dots,\epsilon_n}$ exist with the above property, since one needs to construct them globally. The reader is expected to verify this point.) Then one obtains

$$\begin{aligned} u(x, x_{n+1}) &= \sum_{\epsilon} f_\epsilon = \sum_{\epsilon}\left(f_\epsilon + \sum_{k=1}^{n+1}(-1)^k\epsilon_k\psi_{\epsilon_k}\right) \\ &= \sum_{\epsilon}\frac{(-1)^{n+1}\epsilon_{n+1}}{x_{n+1}}\xi_{\epsilon_1,\dots,\epsilon_n}(x) \\ &= (-1)^{n+1}\left(\frac{1}{x_{n+1}+\sqrt{-1}0}\right. \\ &\qquad \left.-\frac{1}{x_{n+1}-\sqrt{-1}0}\right)\sum_{\epsilon_1,\dots,\epsilon_n=\pm1}\xi_{\epsilon_1,\dots,\epsilon_n}(x). \end{aligned}$$

Then, if one lets $v(x) = (-1)^{n+1}2\pi\sqrt{-1}\sum_{\epsilon_1,\dots,\epsilon_n=\pm1}\xi_{\epsilon_1,\dots,\epsilon_n}(x)$, then $u(x, x_{n+1}) = v(x)\delta(x_{n+1})$ holds, i.e. $u(x, x_{n+1}) \in \operatorname{Im} s$.

Next we will prove (3.8.2).

(i) We begin with the proof that $t\cdot$ is an epimorphism. Since the sheaf of microfunctions is a flabby sheaf by Theorem 3.7.1, then Theorem 2.3.2 implies that the onto-ness in (3.8.2) is reduced to the onto-ness in (3.8.1).

(ii) Next we will prove that $\operatorname{Ker}(t\cdot) = \operatorname{Im} s$ in (3.8.2). Again, $\operatorname{Im} s \subset \operatorname{Ker}(t\cdot)$ is clearly true; so we will prove $\operatorname{Ker}(t\cdot) \subset \operatorname{Im} s$. Let us first consider this question in a neighborhood of $(x_0, 0; \sqrt{-1}(\langle\xi_0, dx\rangle + dt)\infty)$. The hypothesis $tu = 0$ implies that one can find a conoidal neighborhood Γ_j of $U_j \subset S(M \times \mathbf{R})$, $j = 1, 2, \dots, j_0$, such that $\sqrt{-1}(\langle\xi_0, dx\rangle + dt)\infty \in U_j^\circ$, and one can also find a holomorphic function φ_j in Γ_j so that $tu = \sum_{j=1}^{j_0} b_{\Gamma_j}(\varphi_j)$ holds. Using $\psi_j(x, t)$ and $\xi_j(x)$ such that $\varphi_j(x, t) = t\psi_j(x, t) + \xi_j(x)$, as we noted, one lets

$$\tilde{u}(x, t) = u(x, t) - \sum_{j=1}^{j_0} b_{\Gamma_j}(\psi_j) - \frac{1}{t-\sqrt{-1}0}\sum_{j=1}^{j_0} b_{\Gamma_j}(\xi_j).$$

Clearly, then, $\tilde{u}(x, t) = u(x, t)$ holds in a neighborhood of $(x_0, 0; \sqrt{-1}(\langle\xi_0, dx\rangle + dt)\infty)$, and one also has $t\tilde{u}(x, t) = 0$ since ψ_j and ξ_j are chosen as above. Therefore, by the exactness of (3.8.1), one can find a hyperfunction $v(x)$ such that $\tilde{u}(x, t) = v(x)\delta(t)$ holds. That is, $u = \mathrm{sp}(v(x)\delta(t))$ holds in a neighborhood of that point. Hence, $\mathrm{Ker}\,(t\cdot) \subset \mathrm{Im}\, s$ has been proved. One can prove this in a neighborhood of $(x_0, 0; \sqrt{-1}\langle\xi_0, dx\rangle\infty)$ by using the coordinate transformation $x \mapsto x + at$.

(iii) Lastly, we will prove that s is monomorphic. From the definition of $\hat{\mathscr{C}}_M$, we need to prove two cases.

First, the exactness of the sequence $0 \to \mathscr{B}_{M,x} \xrightarrow{s} \mathscr{C}_{M,(x,0;\sqrt{-1}\,dt\infty)}$ will be proved. We have $\delta(ct) = (1/|c|)\delta(t)$ for $c \neq 0$. If $\mathrm{sp}(u(x)\delta(t)) = 0$ holds in a neighborhood of $(x, 0; \sqrt{-1}(\langle\xi_0, dx\rangle + dt)\infty)$ for some $\xi_0 \neq 0$, then one has $\mathrm{sp}(u(x)\delta(t)) = 0$ for an arbitrary ξ in a neighborhood of $(x, 0; \sqrt{-1}\langle\xi, dx\rangle + dt)\infty)$. On the other hand, $\mathrm{sp}(u(x)\delta(t)) = 0$ holds anywhere by the coordinate transformation $x \mapsto x + at$. Hence, $u(t)\delta(t)$ is a real analytic function whose support is restricted to $\{t = 0\}$. Then one has $u(x)\delta(t) = 0$. The map s, being monomorphic in (3.8.1), implies $u(x) = 0$.

We will prove that

$$0 \to \mathscr{C}_{M,(x;\sqrt{-1}\langle\xi,dx\rangle\infty)} \xrightarrow{s} \mathscr{C}_{M\times\mathbf{R},(x,0;\sqrt{-1}(\langle\xi,dx\rangle+\tau\,dt)\infty)}$$

is an exact sequence. By the coordinate transformation $x \mapsto x + at$, one obtains $u(x)\delta(t) = 0$, including the case where $\tau = 0$. Then

$$u(x) = \int u(x)\delta(t)\,dt$$

implies that s is a monomorphism. The proof of Proposition 3.8.1 is now completed.

As our next topic, we will discuss the concept of a hyperfunction containing holomorphic parameters. By way of introduction, we first recall the notion of a complex conjugate.

Definition 3.8.2. *Let T be a complex manifold. Then a complex manifold $\bar{T}$ is said to be the complex conjugate of T when there exists a homeomorphism $\alpha: T \to \bar{T}$, as topological spaces, such that $f \in \mathcal{O}_{\bar{T}}$ if and only if $\overline{f \circ \alpha} \in \mathcal{O}_T$, where $\overline{f \circ \alpha}(z) = \overline{f(\alpha(z))}$.*

Remarks.

(1) If $\bar{T}$ is the complex conjugate of T, then T is the complex conjugate of $\bar{T}$.

(2) Let $\bar{T}$ be the complex conjugate of T. Then one can embed T into $T \times \bar{T}$ by the map $T \ni \tau \mapsto (\tau, \alpha(\tau)) \in T \times \bar{T}$. Let us denote by T_{real} the case when T is regarded as a real analytic manifold. Then $T \times \bar{T}$ is a complexification of T_{real}. We can identify the tangent space $T(T)$ as a complex manifold with the tangent space $T(T_{\text{real}})$ as a real analytic manifold as follows. Let $X_0 \in T(T)$ and $X \in T(T_{\text{real}})$. Then X_0 and X are identified if and only if $Xf = X_0 f$ holds for an arbitrary $f \in \mathcal{O}_T$.

Next we will describe the above identification via a local coordinate system. Let $(\tau_1, \ldots, \tau_l)$ be local complex coordinates of T, and let $\tau_j = t_j + \sqrt{-1}s_j$, $j = 1, 2, \ldots, l$. Then, for $f \in \mathcal{O}_T$,

$$\frac{\partial}{\partial t_j} f(\tau) = \frac{\partial}{\partial \tau_j} f(\tau) \frac{\partial \tau_j}{\partial t_j} = \frac{\partial}{\partial \tau_j} f(\tau)$$

and

$$\frac{\partial}{\partial s_j} f(\tau) = \frac{\partial}{\partial \tau_j} f(\tau) \frac{\partial \tau_j}{\partial s_j} = \sqrt{-1} \frac{\partial}{\partial \tau_j} f(\tau) \qquad .$$

holds. Hence, the correspondence should be

$$\frac{\partial}{\partial \tau_j} \leftrightarrow \frac{\partial}{\partial t_j}, \qquad \sqrt{-1} \frac{\partial}{\partial \tau_j} \leftrightarrow \frac{\partial}{\partial s_j} \qquad \text{for } j = 1, \ldots, l.$$

One can also identify the dual spaces $T^*(T)$ with $T^*(T_{\text{real}})$ through the above identification. In terms of coordinates, this is given by

$$d\tau_j \leftrightarrow dt_j, \qquad -\sqrt{-1}\, d\tau_j \leftrightarrow ds_j.$$

In what follows, these identifications are assumed.

Let M be a real analytic manifold, and let X and $\bar{X}$ be complexifications of M such that X and $\bar{X}$ are complex conjugates to each other. Assume that the restriction of the homeomorphism $\alpha: X \to \bar{X}$ to M is an identity map on M. For example, if $M = \mathbf{R}^n$ and $X = \bar{X} = \mathbf{C}^n$, then one may let $\alpha(z_1, \ldots, z_n) = (\bar{z}_1, \ldots, \bar{z}_n)$.

The correspondence $\mathcal{O}_{\bar{X}} \ni f(\bar{z}) \mapsto \bar{f}(z) = \overline{f(\bar{z})} \in \mathcal{O}_X$ induces a sheaf homomorphism $\alpha^{-1}\mathcal{O}_{\bar{X}} \to \mathcal{O}_X$. Hence this induces a sheaf homomorphism $\alpha^{-1}\mathcal{H}^n_M(\mathcal{O}_{\bar{X}}) \to \mathcal{H}^n_M(\mathcal{O}_X)$, which is an isomorphism. Following these preparations, we begin our discussion.

Definition 3.8.3. *For a hyperfunction* $u(x) = \sum_j b(f_j(z)) \in \mathcal{B}_M$, $\overline{u(x)} = \sum_j b(\bar{f}_j(\alpha(z))) \in \mathcal{B}_M$ *is called the complex conjugate of* $u(x)$. *A hyperfunction* $u(x)$ *is said to be a real-valued hyperfunction if* $u(x) = \overline{u(x)}$ *holds.*

Example 3.8.1. δ-function is a real-valued hyperfunction. Since

$$\delta(x_1, \ldots, x_n) = \frac{1}{(-2\pi\sqrt{-1})^n} \sum_\epsilon \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 + \sqrt{-1}\epsilon_1 0) \cdots (x_n + \sqrt{-1}\epsilon_n 0)},$$

we have by definition

$$\overline{\delta(x_1, \ldots, x_n)} = \frac{1}{(2\pi\sqrt{-1})^n} \sum_\epsilon \frac{\epsilon_1 \cdots \epsilon_n}{(x_1 - \sqrt{-1}\epsilon_1 0) \cdots (x_n - \sqrt{-1}\epsilon_n 0)}.$$

Replace ϵ_i with $-\epsilon_i$ in the above. Then we obtain

$$\overline{\delta(x_1, \ldots, x_n)} = \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\epsilon} \frac{(-1)^n \epsilon_1 \cdots \epsilon_n}{(x_1 + \sqrt{-1}\epsilon_1 0) \cdots (x_n + \sqrt{-1}\epsilon_n 0)}$$
$$= \delta(x_1, \ldots, x_n).$$

Example 3.8.2. For $\lambda \in \mathbf{C}$, $\overline{(x + i0)^\lambda} = (x - \sqrt{-1}0)^{\bar{\lambda}}$ holds.

If $\bar{X}$ is the complex conjugate of X, then $\alpha: X \to \bar{X}$ naturally induces the antipodal maps a on $\sqrt{-1}SM$ and $\sqrt{-1}S^*M$. It is also plain that we have

$$\text{S.S. } \bar{u} = (\text{S.S. } u)^a = \{(x, -\sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \mid (x, \sqrt{-1}\xi\infty) \in \text{S.S. } u\}.$$

One can also define the complex conjugate of a microfunction as in the above.

Definition 3.8.4. *Let M be an n-dimensional real analytic manifold, and let T be an l-dimensional complex manifold. Let $u(x, \tau, \bar{\tau})$ be a hyperfunction (or a microfunction) on $M \times T$, regarded as an $(n + 2l)$-dimensional real analytic manifold. Then u is said to contain τ as holomorphic parameters, or u depends upon τ holomorphically, if $u(x, \tau, \bar{\tau})$ satisfies the Cauchy-Riemann differential equations $\partial u/\partial \bar{\tau}_j = 0$ for $j = 1, \ldots, l$. In that case, we denote $u = u(x, \tau)$, since $\partial u/\partial \bar{\tau} = 0$.*

Remark. We have $\sqrt{-1}S^*(M \times T) = \{(x, \tau; \sqrt{-1}(\langle \xi, dx\rangle + \langle \zeta, d\tau\rangle + \langle \bar{\zeta}, d\bar{\tau}\rangle)\infty\}$ and $T \times \sqrt{-1}S^*M = \{(x, \tau; \sqrt{-1}\langle \xi, dx\rangle\infty)\}$. Hence, we can regard $T \times \sqrt{-1}S^*M \subset \sqrt{-1}S^*(M \times T)$. Since the principal symbol of the differential operator $\partial/\partial\bar{\tau}_j$ is $\bar{\zeta}_j$, from Theorem 3.4.4, we have S.S. $u \subset \{\bar{\zeta}_1 = \cdots = \bar{\zeta}_l = 0\} \subset T \times \sqrt{-1}S^*M$.

We will consider a hyperfunction containing τ as holomorphic parameters. Even though we will restrict our discussion to the case $T \subset \mathbf{C}$, the general case $T \subset \mathbf{C}^l$ can be carried out equally well.

Proposition 3.8.2. *Let $T \subset \mathbf{C}$, and let $u(x, \tau)$ be a hyperfunction on $M \times T$ containing a holomorphic parameter. If X and $T \times \bar{T}$ are complexifications of M and T, respectively, then one can choose an open subset D_j of X and an open subset U_j of T such that* $u(x, \tau) = \sum_j b_{D_j \times U_j}(f_j(z, \tau))$ holds.

Proof. First let $u(x, \tau) = \sum_j b(f_j(z, \tau, \bar{\tau}))$. Then one obtains

$$\frac{\partial u}{\partial \bar{\tau}} = \sum_j b\left(\frac{\partial}{\partial \bar{\tau}} f_j(z, \tau, \bar{\tau})\right) = 0.$$

That is, $(\partial/\partial\bar{\tau})f_j$ is a coboundary. Hence, one can express

$$\frac{\partial}{\partial \bar{\tau}} f_j = \sum_k g_{jk}, \qquad \text{where } g_{jk} = -g_{kj}.$$

As in the proof of the Lemma following Proposition 3.2.1, there exists $h_{jk}(z, \tau, \bar{\tau}), j > k$, such that $g_{jk} = (\partial/\partial\bar{\tau})h_{jk}$ holds. For the cases $j < k$ and $j = k$, let h_{jk} be $h_{jk} = -h_{kj}$ and $h_{jk} = 0$ respectively. Then define $F_j(z, \tau, \bar{\tau}) = f_j - \sum_k h_{jk}$. One has $(\partial/\partial\bar{\tau}\ F_j(z, \tau, \bar{\tau}) = 0$. Therefore, one has $F_j(z, \tau, \bar{\tau}) = F_j(z, \tau)$. Furthermore, $\sum_j b(F_j(z, \tau)) = \sum_j b\left(f_j - \sum_k h_{jk}\right) = \sum_j b(f_j) = u(x, \tau)$ holds.

Proposition 3.8.3. *Let a hyperfunction $u(x, \tau)$ depend holomorphically upon τ. If $u(x, \tau_0) = 0$ holds, then there exists a unique hyperfunction $v(x, \tau)$ holomorphically dependent on τ such that $u(x, \tau) = (\tau - \tau_0)v(x, \tau)$ holds.*

Proof. One can let $\tau_0 = 0 \in \mathbf{C}$ without loss of generality. Express $u(x, \tau) = \sum_j b(f_j(z, \tau))$, where $f_j(z, \tau)$ is as in the previous proposition. Then $u(x, 0) = \sum_j b(f_j(z, 0)) = 0$ holds. On the other hand, there is a holomorphic function $h_j(z, \tau)$ so that $f_j(z, \tau) - f_j(z, 0) = \tau h_j(z, \tau)$ holds. Hence, one obtains $u(x, \tau) = \sum_j b(f_j(z, \tau) - f_j(z, 0)) = \tau \sum_j b(h_j(z, \tau))$; i.e. $v(x, \tau) = \sum_j b(h_j(z, \tau))$. Next, we will prove the uniqueness part. Suppose $\tau \cdot u(x, \tau) = 0$. Then $u(x, \tau) = 0$ for $\tau \neq 0$. Hence, $\operatorname{supp} u(x, \tau) \subset \{\operatorname{Re} \tau \geqq 0\}$ holds. One has $d \operatorname{Re} \tau((x, 0)) \neq 0$ and $((x, 0); \sqrt{-1}d \operatorname{Re} \tau(x, 0)\infty) \notin \text{S.S. } u(x, \tau)$. Consequently, by Proposition 3.5.2, $u(x, \tau) = 0$ holds.

Proposition 3.8.4 (Cauchy's Integral Theorem for a Holomorphic Parameter). *Let γ be a closed Jordan curve $\subset T(\subset \mathbf{C})$. Then, for a hyperfunction $u(x, \tau)$ containing a holomorphic parameter τ, $\int_\gamma u(x, \tau)\, d\tau = 0$ holds.*

Note. The above integration $\int_\gamma u(x, \tau)\, d\tau$ means: one first restricts $u(x, \tau)$ on γ and performs the line integral $\oint_\gamma d\tau$.

Proof. Using $f_j(z, \tau)$ in Proposition 3.8.2, express $u(x, \tau) = \sum_j b(f_j(z, \tau))$. Then the proof can be completed by Cauchy's integral theorem for a holomorphic function.

Proposition 3.8.5 (Cauchy's Integral Formula for a Holomorphic Parameter). *Let $\gamma \subset T(\subset \mathbf{C})$ be a closed Jordan curve. Then one has $u(x, \tau) = (1/2\pi\sqrt{-1})\oint_\gamma u(x, \zeta)(1/\zeta - \tau)\, d\zeta$ for a hyperfunction containing a holomorphic parameter τ, where the closed curve γ is taken in the counter-clockwise direction once around τ.*

Proof. From Proposition 3.8.3, there exists a hyperfunction v depending holomorphically upon τ such that

$$u(x, \tau + w) - u(x, w) = \tau v(x, w, \bar{w}, \tau)$$

holds. Taking the derivative of this equation with respect to $\bar{w}$ gives us $0 = \tau(\partial/\partial\bar{w})v(x, w, \bar{w}, \tau)$. By the uniqueness assertion in Proposition 3.8.3, one has $\partial v/\partial\bar{w} = 0$. Hence, $v = v(x, w, \tau)$ holds. Therefore one obtains

$$u(x, \zeta) - u(x, \tau) = u(x, (\zeta - \tau) + \tau) - u(x, \tau) = (\zeta - \tau)v(x, \zeta - \tau, \tau).$$

Consequently, Proposition 3.8.4 and the above imply

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\oint u(x, \zeta)\frac{1}{\zeta - \tau}\,d\zeta &= u(x, \tau) + \frac{1}{2\pi\sqrt{-1}}\oint v(x, \zeta - \tau, \tau)\,d\zeta \\ &= u(x, \tau).\end{aligned}$$

Our next aim is to prove the uniqueness of the analytic continuation with respect to a holomorphic parameter of a hyperfunction (and a microfunction). For this purpose, we will prove Proposition 3.8.6, a local version of Bochner's theorem on the envelope of holomorphy of a tube domain. This proposition has many applications.

Proposition 3.8.6 (Komatsu [2]; SKK [1], Chap. I, §3.1). *For* $0 < \epsilon < 1$, *define* G_ϵ *and* F_ϵ *as follows:*

$$\begin{aligned}G_\epsilon \underset{\text{def}}{=} \{&(x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2) \in \mathbf{C}^2 \,|\, 0 \leqq y_1, 0 \leqq y_2, y_1 + y_2 < 1 \\ &\text{and } \epsilon(x_1^2 + x_2^2) + (y_1 + y_2) - \epsilon(y_1^2 + y_2^2) < 1 - \epsilon\}\end{aligned}$$

and

$$F_\epsilon \underset{\text{def}}{=} G_\epsilon \cap \{y_1 = 0 \text{ or } y_2 = 0\}.$$

Furthermore, let U' *be an open set containing* F_ϵ, *and let*

$$U = \{(x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2) \in U' \,|\, y_1 < 0 \text{ or } y_2 < 0\}.$$

Then, for any open subset $W \subset \mathbf{C}^n$,

$$\mathcal{O}_{\mathbf{C}^2 \times W}((U \cup G_\epsilon) \times W) \to \mathcal{O}_{\mathbf{C}^2 \times W}(U \times W)$$

is an epimorphism.

Proof. From the long exact sequence of relative cohomology groups, it is sufficient to prove $H^1_{G_\epsilon \times W}(\mathbf{C}^2 \times W, \mathcal{O}) = 0$. Let $\varphi(z_1, z_2) = z_1 + z_2 + \sqrt{-1}\epsilon(z_1^2 + z_1^2)$. Then $G_\epsilon = \{(z_1, z_2) \,|\, \operatorname{Im} z_1 \geqq 0, \operatorname{Im} z_2 \geqq 0, \operatorname{Im}(z_1 + z_2) < 1 \text{ and } \operatorname{Im}\varphi < 1 - \epsilon\}$ holds. If one defines

$$K_\epsilon^1 = \{(z_1, z_2) \in \mathbf{C}^2 \,|\, \operatorname{Im} z_1 \geqq 0, \operatorname{Im} z_2 \geqq 0, \operatorname{Im}(z_1 + z_2) \leqq 1, \operatorname{Im}\varphi \leqq 1 - \epsilon\}$$

$$K_\epsilon^2 = \{(z_1, z_2) \in \mathbf{C}^2 \,|\, \operatorname{Im} z_1 \geqq 0, \operatorname{Im} z_2 \geqq 0, \operatorname{Im}(z_1 + z_2) \leqq 1, \operatorname{Im}\varphi = 1 - \epsilon\},$$

then one has $G_\epsilon = K_\epsilon^1 - K_\epsilon^2$. Clearly, $G_\epsilon \subset K_\epsilon^1 - K_\epsilon^2$ holds. One needs to prove $K_\epsilon^1 - K_\epsilon^2 \subset G_\epsilon$. That is, it suffices to prove, "$\operatorname{Im} z_1 > 0$, $\operatorname{Im} z_2 > 0$, and $\operatorname{Im}\varphi < 1-\epsilon$ imply $\operatorname{Im}(z_1+z_2) \neq 1$." One has $\operatorname{Im}(z_1+z_2) - \epsilon(\operatorname{Im}(z_1+z_2))^2 \leqq \operatorname{Im}\varphi < 1-\epsilon$ for $\operatorname{Im} z_1 \geqq 0$, $\operatorname{Im} z_2 \geqq 0$, and $\operatorname{Im}\varphi < 1-\epsilon$. Hence, if $\operatorname{Im}(z_1+z_2) = 1$, it would then imply $1-\epsilon < 1-\epsilon$, a contradiction.

Since K_ϵ^1 and K_ϵ^2 are compact analytic polyhedrons, one obtains $H^1_{G_\epsilon \times W}(\mathbf{C}^2 \times W, \mathcal{O}) = 0$ by Proposition 2.2.2.

Proposition 3.8.7. *Let* $0 < \epsilon < \frac{1}{2}$. *Define*

$$F_1 = \left\{(x_1+\sqrt{-1}y_1, x_2+\sqrt{-1}y_2) \in \mathbf{C}^2 \,|\, y_1 = 0, 0 \leqq y_2 < 1,\right.$$

$$\left. x_1^2 + x_2^2 < 4\left(\frac{1-\epsilon}{\epsilon}\right)\right\};$$

$$F_2 = \left\{x_1+\sqrt{-1}y_1, x_2+\sqrt{-1}y_2) \in \mathbf{C}^2 \,|\, y_2 = 0, 0 \leqq y_1 < 1,\right.$$

$$\left. x_1^2 + x_2^2 < 4\left(\frac{1-\epsilon}{\epsilon}\right)\right\};$$

$$G_1 = \left\{(x_1+\sqrt{-1}y_1, x_2+\sqrt{-1}y_2) \in \mathbf{C}^2 \,|\, y_1 \geqq 0, y_2 \geqq 0,\right.$$

$$\left. y_2 + (1-2\epsilon)(y_1 - 1) < 0, x_1^2 + x_2^2 < \frac{1-\epsilon}{\epsilon}\right\};$$

$$G_2 = \left\{(x_1+\sqrt{-1}y_1, x_2+\sqrt{-1}y_2) \in \mathbf{C}^2 \,|\, y_1 \geqq 0, y_2 \geqq 0,\right.$$

$$\left. y_1 + (1-2\epsilon)(y_2 - 1) < 0, x_1^2 + x_2^2 < \frac{1-\epsilon}{\epsilon}\right\}.$$

Then let $F = F_1 \cup F_2$, and let $\mathrm{G} = \mathrm{G}_1 \cup \mathrm{G}_2$. *For a neighborhood* U' *of* F, *define* $U = U' \cap \{y_1 < 0$ or $y_2 < 0\}$. *Then the map*

$$\mathcal{O}_{\mathbf{C}^2 \times W}((U \cup G) \times W) \to \mathcal{O}_{\mathbf{C}^2 \times W}(U \times W)$$

is an epimorphism for an arbitrary open subset $W \subset \mathbf{C}^N$.

Proof. For $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^2$, define $G_{\epsilon,\alpha}$, $F_{\epsilon,\alpha}$, and $\tilde{G}_{\epsilon,\alpha}$ as follows: $G_{\epsilon,\alpha} = \{(z_1, z_2) \in \mathbf{C}^2 \,|\, y_1 \geqq 0,\ \ y_2 \geqq 0,\ \ y_1 + y_2 < 1$ and $\epsilon((x_1-\alpha_1)^2 + (x_2-\alpha_2)^2) + (y_1+y_2) - \epsilon(y_1^2+y_2^2) < 1-\epsilon\}$; $F_{\epsilon,\alpha} = G_{\epsilon,\alpha} \cap \{y_1 = 0$ or $y_2 = 0\}$; and $\tilde{G}_{\epsilon,\alpha} = G_{\epsilon,\alpha} \cap \{x_1 = \alpha_1, x_2 = \alpha_2\}$. Then $F_{\epsilon,\alpha} \subset F$ for $\alpha_1^2 + \alpha_2^2 < (1-\epsilon)/\epsilon$. Therefore, U' is a neighborhood of $F_{\epsilon,\alpha}$ if U' is a neighborhood of F. Then Proposition 3.8.6 implies that a holomorphic function on U can be extended to one over $U \cup G_{\epsilon,\alpha}$; in particular, over

$U \cup \tilde{G}_{\epsilon,\alpha}$. Let $\tilde{G} \underset{\text{def}}{=} \bigcup_{\alpha_1^2+\alpha_2^2<(1-\epsilon)/\epsilon} \tilde{G}_{\epsilon,\alpha}$. Then $\tilde{G} = \{(z_1, z_2) \in \mathbf{C}^2 | x_1^2 + x_2^2 < (1-\epsilon)/\epsilon,\ y_1 \geqq 0,\ y_2 \geqq 0,\ (y_1+y_2) - \epsilon(y_1^2+y_2^2) < 1-\epsilon\}$. Since a holomorphic function in U can be extended to a holomorphic function in $U \cup \tilde{G}$, and since $G \subset \tilde{G}$, particularly, it can be extended to one in $U \cup G$. This means that

$$\mathcal{O}_{\mathbf{C}^2 \times W}((U \cup G) \times W) \to \mathcal{O}_{\mathbf{C}^2 \times W}(U \times W)$$

is an epimorphism.

The following is an easy corollary from the proposition, which is quite important for applications.

Corollary 1. *A holomorphic function in a neighborhood of* $\{z = (z_1, \ldots, z_n) \in \mathbf{C}^n | |\operatorname{Re} z_j| < a$ *for* $j = 1, 2, \ldots, n$, *and* $0 < \operatorname{Im} z_1 < b$, $\operatorname{Im} z_2 = \cdots = \operatorname{Im} z_n = 0\}$ *can be extended to a holomorphic function in*

$$\left\{z \in \mathbf{C}^n \middle| |\operatorname{Re} z_j| < \epsilon, \frac{1}{\epsilon}\sqrt{\sum_{j=2}^{n} |\operatorname{Im} z_j|^2} < \operatorname{Im} z_1 < \epsilon\right\} \qquad \text{for some } \epsilon > 0.$$

Remark. The above Corollary 1 indicates that $f(z)$ determines an element of $\tilde{\mathscr{A}}$ in a neighborhood of $0 + \sqrt{-1}(\partial/\partial x_1)0$. Hence, from this corollary, when one considers a boundary value of a holomorphic function, one may ignore the higher-order infinitesimals as approaching reals from imaginaries. This seems to be a hidden reason why the microfunction sheaf can be defined on such a geometrically simple structured manifold as $\sqrt{-1}S^*M$, i.e. without using such a notion as a jet of higher order. Of course, this explanation is a kind of afterthought. Sato regarded this fact (Corollary 1) as being obvious when he constructed the theory of microfunctions—the sort of clear intuition that only a genius has!

The following corollary of Corollary 1 is also quite useful.

Corollary 2. *Let M be an open subset of* $\mathbf{R}^n$, *and let ξ_0 be a non-zero n-dimensional real vector. If, for a real analytic function $f(x)$ on M and x_0 on M, one has* $f(x + \sqrt{-1}t\xi_0) \neq 0$ *for* $0 < t \ll 1$, $|x - x_0| \ll 1$, *then* $\{z \in \mathbf{C}^n - \mathbf{R}^n | f(z) \neq 0\}$ *is a conoidal neighborhood of* $x_0 + \sqrt{-1}\xi_0 0$.

By these results, we will prove a proposition on the propagation of analyticity.

Proposition 3.8.8. *Let D be a connected open subset of* $\mathbf{C}$, *and let Ω be an open subset containing* $\{0\} \times D$ *in* $\mathbf{R}^n \times \mathbf{C}$. *Suppose $u(x, \tau)$ is a hyperfunction defined on Ω containing a holomorphic parameter τ such that* S.S. $u \cap \{x = 0\} \subset \Omega \times \sqrt{-1}\Gamma$ *for a properly convex closed set Γ in* S^{n-1}, *If $u(x, \tau)$ is a real analytic function in a neighborhood of $x=0$ and $\tau=\tau_0 \in D$, then $u(x, \tau)$ is real-analytic in a neighborhood of $\{0\} \times D$.*

Proof. Since Γ is a properly convex closed set, one can assume, without loss of generality, that $\Gamma \subset V \underset{\text{def}}{=} \{(\xi_1, \ldots, \xi_n) \in S^{n-1} | \xi_1 > |\xi_2| + \cdots + |\xi_n|\}$ after a certain coordinate transformation. One has $\{\tau||\text{Re}\,\tau| \leqq \alpha$ and $|\text{Im}\,\tau| \leqq \alpha\} \times \{x \in \mathbf{R}^n||x| \leqq \epsilon\} \subset \Omega$ for some $\alpha(<1)$ and ϵ, since $D \times \{0\} \subset \Omega$. Hence, this allows one to assume $\bar{D} \times \{x \in \mathbf{R}^n||x| \leqq 1\} \subset \Omega$ to prove the assertion. Since S.S. $u \cap \{x = 0\} \subset \Omega \times \sqrt{-1}V$, Theorem 2.3.4 implies that one can express u as $b_{\tilde{D}}(f(z, \tau))$, where $f(z, \tau)$ is a holomorphic function in $\tilde{D} = \{(z, \tau) \in \mathbf{C}^n \times \mathbf{C} | \tau \in D, |z| \ll 1$ and $\text{Im}\, z_1 > |\text{Im}\, z_2|, \ldots, |\text{Im}\, z_n|\}$. Then we have the following lemma.

Lemma 1. *Let $u(x, \tau) = b(f(z, \tau))$ be defined for $|\tau| < 1$ and be real-analytic in $\text{Im}\,\tau > 0$. Then, for some $\epsilon > 0$, $u(x, \tau)$ is a real analytic function in $|\tau| < \epsilon$.*

Proof. First we will prove Proposition 3.8.8 using Lemma 1. In order to prove Proposition 3.8.8, it is sufficient to prove the statement

(A) If $b(f(z, \tau))$ is defined for $|\tau| < 1$ and is real-analytic in a neighborhood of $\tau = 0$, then $b(f(z, \tau))$ is real-analytic in $|\tau| < 1$.
It suffices to prove the following (B) to claim (A).

(B) If $b(f(z, \tau))$ is real-analytic in $|\tau| < a(<1)$, then there exists $\epsilon > 0$ such that $b(f(z, \tau)$ is real-analytic in $|\tau| < a + \epsilon$.

We will derive (B) from Lemma 1. Let $|\tau_0| = a$. Then $b(f(z, \tau))$ is real-analytic in $|\tau| < a$. From Lemma 1, one may assume, after a suitable linear fractional transformation, that $b(f(z, \tau))$ is real-analytic in $U_{\tau_0} = \{\tau||\tau - \tau_0| < \epsilon_{\tau_0}\}$ for some $\epsilon_{\tau_0} > 0$. Since $\{\tau||\tau| = a\}$ is compact, one has finitely many τ_i, $i = 1, \ldots, l$, such that $\{\tau||\tau| = a\} \subset \bigcup_{i=1}^{l} U_{\tau_i}$ holds. Hence, for $\epsilon = \min_i \epsilon_{\tau_i}$, $b(f(z, \tau))$ is real-analytic in $|\tau| < a + \epsilon$. Therefore, it is reduced to prove Lemma 1.

One can let $f(z, \tau)$ be a holomorphic function defined in the domain where $|\text{Re}\,\tau| < 1$, $|\text{Im}\,\tau| < 1$, $|\text{Re}\, z_j| < \epsilon$, $|\text{Im}\, z_j| < \epsilon$ $(1 \leqq j \leqq n)$ and where $\text{Im}\, z_1 > |\text{Im}\, z_2|, \ldots, |\text{Im}\, z_n|$. Since the boundary value $b(f(z, \tau))$ is real-analytic for $\text{Im}\,\tau > 0$, then $f(z, \tau)$ remains to be holomorphic where $\text{Im}\, z_1 \to 0$ (hence $|\text{Im}\, z_j| \to 0$, $j = 2, \ldots, n$). Therefore, for $\text{Im}\,\tau > \delta > 0$, $f(z, \tau)$ is holomorphic so long as $\text{Im}\, z_1 > -\epsilon'$, $|\text{Im}\, z_j| < \epsilon'$ for some $\epsilon' > 0$, $j = 2, \ldots, n$. Then Proposition 3.8.7 implies that, for sufficiently small δ and $\epsilon > 0$, there exists $\epsilon'' > 0$ such that $f(z, \tau)$ is holomorphic for $|\text{Im}\,\tau| < \epsilon''$ and $|\text{Im}\, z_j| < \epsilon''$, $j = 1, \ldots, n$. Therefore, particularly $f(z, \tau)$ is real-analytic where $\text{Im}\, z = 0$ and $|\text{Im}\,\tau| < \epsilon''$. That is, $u(x, \tau) = b(f(z, \tau))$ is real-analytic in $|\tau| < \epsilon''$, completing the proof of Lemma 1. Hence Proposition 3.8.8 has been proved.

We finally obtain the next theorem.

Theorem 3.8.1.

(1) *Let* $u(x, \tau)$ *be a microfunction containing a holomorphic parameter* τ. *For a connected open subset* D *in* $\mathbf{C}$, *let* $u(x, \tau)$ *be defined on* $\tilde{D}_{(x_0,\xi_0)} = \{(x_0, \tau; \sqrt{-1}\langle\xi_0, dx\rangle\infty) | \tau \in D \subset \mathbf{C}\}$. *If* $u(x, \tau) = 0$ *holds in a neighborhood of* $(x_0, \tau_0; \sqrt{-1}\langle\xi_0, dx\rangle\infty)$, *then* $u(x, \tau) = 0$ *in a neighborhood of* $\tilde{D}_{(x_0,\xi_0)}$.

(2) (Uniqueness of Analytic Continuation): *Let* $u(x, \tau)$ *be a hyperfunction containing a holomorphic parameter* τ, *which is defined in a neighborhood of* $\{x_0\} \times D$. *If one has* $u(x, \tau) = 0$ *in a neighborhood of* (x_0, τ_0), $\tau_0 \in D$, *then* $u(x, \tau) = 0$ *in a neighborhood of* $\{x_0\} \times D$.

Proof. The assertion (2) is immediate from Proposition 3.8.8 and from the uniqueness of analytic continuation of a holomorphic function. We will prove (1). Since $u(x, \tau)$ is defined in a neighborhood of $\{(x_0, \sqrt{-1}\xi_0\infty)\} \times D$, there exists an open set V containing $(x_0, \sqrt{-1}\xi_0\infty)$ such that $u(x, \tau)$ is defined in $V \times D$. Furthermore, one may assume that $u = 0$ holds in a neighborhood of $V \times \{\tau_0\}$. For the sake of simplicity, let $V = U \times \sqrt{-1}W$, where U and W are convex. Note that the sheaf $\mathscr{L}_M$ of microlocal operators on $\sqrt{-1}S^*M$ is also flabby, because of the flabbiness of the microfunction sheaf. From this, one concludes that there exists a microlocal operator K such that $K = 1$ and $\operatorname{supp} K \subset V$ in a neighborhood of $(x_0, \sqrt{-1}\xi_0\infty)$. For this K, let $v = Ku$. Then, from the exactness of $\mathscr{B} \xrightarrow{\mathrm{sp}} \pi_*\mathscr{C} \to 0$, one has a hyperfunction g such that $v = \mathrm{sp}(g)$. Then one has $\mathrm{sp}(\partial g/\partial\bar{\tau}) = \partial v/\partial\bar{\tau} = 0$. Hence, $h = \partial g/\partial\bar{\tau}$ is real-analytic in $U \times D$. Let L be a compact convex set contained in U, and let L' be an arbitrary compact convex subset of D. Then it is well known in the theory of partial differential equations with constant coefficients (e.g. Ehrenpreis [1]) that there exists a real analytic function g_0 such that $\partial g_0/\partial\bar{\tau} = h$ holds on $L \times L'$. Define $v = \mathrm{sp}(g - g_0)$. Then, if L is chosen to contain x_0, then by definition $g - g_0$ is real-analytic in a neighborhood of (x_0, τ_0). Since $g - g_0$ has τ as a holomorphic parameter inside $L \times L'$, by Proposition 3.8.8, $g - g_0$ is real-analytic in a neighborhood of $\{x_0\} \times$ (inside L). That is, $u = v = \mathrm{sp}(g - g_0) = 0$ holds in a neighborhood of $\{(x_0, \sqrt{-1}\xi_0)\} \times$ (inside L) for an arbitrary compact convex set L in D. This completes the proof.

CHAPTER IV

Microdifferential Operators

§1. Definition of the Microdifferential Operator and Its Fundamental Properties

The class of microlocal operators, introduced in §4 of Chapter III, is too wide for practical manipulation. For example, in practice it is a very difficult task to compute the composition of two microlocal operators. In this chapter, we will define a microdifferential operator, as a special case of a microlocal operator, and will consider its properties.

The class of microdifferential operators is located between the class of microlocal operators and the class of differential operators. Its appearance has already been shown in Theorem 3.4.3 (Sato's fundamental theorem). According to this theorem, if the principal symbol $p_m(x, \xi)$ of a linear differential operator $P(x, D)$ does not vanish at $(x_0, \sqrt{-1}\xi_0)$, then there exists a microlocal operator as the inverse element. A main topic of this chapter is to define a class among microlocal operators where the inverse exists and where algebraic manipulations can be performed.

Let X be an open subset of $\mathbf{R}^n$, and let $P(x, D) = \sum a_\alpha(x)D^\alpha$ be a differential operator with coefficients in analytic functions, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, as in Definition 3.4.2. We will begin with the computation of the kernel function $K(x, x')\,dx'$ of $P(x, D)$ regarded as a microlocal operator. As we noted in §4 of Chapter III, this is nothing but $(P(x, D_x)\delta(x - x'))\,dx'$. On the other hand, the plane-wave expansion formula (Proposition 3.2.3),

$$\delta(x - x') = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int \frac{\omega(\xi)}{(\langle x - x', \xi\rangle + \sqrt{-1}0)^n}$$

gives, by taking the derivative D_{x_j} of both sides,

$$D_{x_j}\delta(x - x') = \frac{1}{(-2\pi\sqrt{-1})^n}\int \frac{-n!\xi_j}{(\langle x - x', \xi\rangle + \sqrt{-1}0)^{n+1}}\,\omega(\xi).$$

Further differentiation implies

$$D_x^\alpha \delta(x - x') = \frac{1}{(-2\pi\sqrt{-1})^n} \int \frac{(-1)^{|\alpha|}\xi^\alpha(n + |\alpha| - 1)!}{(\langle x - x', \xi\rangle + \sqrt{-1}0)^{n+|\alpha|}}\, \omega(\xi).$$

Hence, we obtain

$$\begin{aligned} K(x, x') &= P(x, D_x)\delta(x - x') \\ &= \frac{1}{(-2\pi\sqrt{-1})^n} \int \sum_\alpha \frac{(-1)^{|\alpha|}a_\alpha(x)\xi^\alpha(n + |\alpha| - 1)!}{(\langle x - x', \xi\rangle + \sqrt{-1}0)^{n+|\alpha|}}\, \omega(\xi) \\ &= \frac{1}{(-2\pi\sqrt{-1})^n} \int \sum_j \frac{(-1)^j(j + n - 1)!p_j(x, \xi)}{(\langle x - x', \xi\rangle + \sqrt{-1}0)^{n+j}}\, \omega(\xi), \end{aligned}$$

where $p_j(x, \xi) = \sum_{|\alpha|=j} a_\alpha(x)\xi^\alpha$. Here $p_j(x, \xi)$ is a homogeneous polynomial in ξ of degree j. A microdifferential operator corresponds to a generalization of the polynomial $p_j(x, \xi)$ to a holomorphic function $p_j(x, \xi)$.

Let us introduce the following function for convenience:

$$\Phi_\lambda(\tau) = \frac{\Gamma(\lambda)}{(-\tau)^\lambda} \qquad \text{for } \tau \in \mathbf{C} - \{\tau \in \mathbf{R} \mid \tau \geqq 0\}. \tag{4.1.1}$$

Since $(-\tau)^\lambda$ is a multivalued function, we choose a branch such that $(-1)^\lambda = 1$ holds for $\tau = -1$. Furthermore, $\Phi_\lambda(\tau)$ cannot be defined at $\lambda = 0, -1, -2, \ldots$ as in the equation (4.1.1), since the gamma-function $\Gamma(\lambda)$ has a pole of order one at $\lambda = 0, -1, -2, \ldots$. But the function

$$(\lambda + m)\Phi_\lambda(\tau) = (\lambda + m)\Gamma(\lambda)/(-\tau)^\lambda = \frac{\Gamma(\lambda + m + 1)}{\lambda(\lambda + 1)\ldots(\lambda + m - 1)(-\tau)^\lambda}$$

is holomorphic in a neighborhood of $\lambda = -m$ and has the value $\tau^m/m!$ at $\lambda = -m$. Hence, $\Phi_\lambda(\tau) - \tau^m/(\lambda + m)m!$ is holomorphic in a neighborhood of $\lambda = -m$. Since our concern is the singularity with respect to τ, we may take $\tau^m/(\lambda + m)m!$ for $\Phi_\lambda(\tau)$. Then let the value at $\lambda = -m$ be $\Phi_{-m}(\tau)$. When we carry out the computation, we have

$$\Phi_{-m}(\tau) = \frac{-1}{m!}\tau^m\left(\log(-\tau) + \left(\gamma - 1 - \frac{1}{2} - \cdots - \frac{1}{m}\right)\right) \qquad \text{for } m = 0, 1, 2, \ldots,$$

where $\gamma = 0.57721\ldots$ is the Euler constant. We have

$$\frac{\partial}{\partial\tau}\Phi_\lambda(\tau) = \Phi_{\lambda+1}(\tau).$$

Since $\Gamma(\lambda) = \int_0^\infty e^{-t}t^{\lambda-1}\,dt$, we have $\Phi_\lambda(\tau) = \int_0^\infty e^{t\tau}t^{\lambda-1}\,dt$ for $\operatorname{Re}\tau < 0$ and $\operatorname{Re}\lambda > 0$.

We can rewrite $K(x, x')$ in terms of $\Phi_\lambda(\tau)$, as follows:

$$\begin{aligned}K(x, x') &= P(x, D_x)\delta(x - x')\\ &= \frac{1}{(2\pi)^n}\int \sum_j p_j(x, \sqrt{-1}\xi)\Phi_{n+j}(\sqrt{-1}(\langle x - x', \xi\rangle + \sqrt{-1}0))\omega(\xi).\end{aligned}$$

Note that $\Phi_\lambda(\sqrt{-1}\tau)$ is defined for $\operatorname{Im}\tau > 0$. We may omit $+\sqrt{-1}0$ when it is obvious.

Proposition 4.1.1. *Let $\{a_j(z)\}_{j\in Z}$ be a sequence of holomorphic functions defined in $\Omega \subset \mathbf{C}^n$, which satisfies the conditions* (4.1.2) *and* (4.1.3) *as follows:*

For an arbitrary compact subset K of Ω, there exists a positive real number R_K such that

$$\sup_{z\in K} |a_j(z)| \leqq (-j)! R_K^{-j} \qquad \text{for } j < 0$$

holds. (4.1.2)

Let K be an arbitrary compact subset of Ω, and let ϵ be an arbitrary positive real number. Then there exists $C_{\epsilon,K}$ such that

$$\sup_{z\in K} |a_j(z)| \leqq \frac{1}{j!} C_{\epsilon,K}\epsilon^j \qquad \text{for } j \geqq 0$$

holds. (4.1.3)

Then, the following (1) *and* (2) *hold:*

(1) *The series $\sum_{j=-\infty}^{\infty} a_j(z)\Phi_{\lambda+j}(\tau)$, $z \in \Omega$, is uniformly absolutely convergent in wider sense for $\{\tau \in \mathbf{C} \mid 0 < |\tau| \ll 1,\ 0 \neq \tau\}$ and is multivalued for $\lambda \in \mathbf{C} - \mathbf{Z}$. Furthermore, it has a pole of order one at $\lambda = 0, \pm 1, \pm 2, \ldots$.*

(2) *The boundary value of $\sum_j a_j(z)\Phi_{\lambda+j}(\sqrt{-1}\tau)$ from $\{\operatorname{Im}\tau > 0\}$ defines a microfunction containing $\lambda \in \mathbf{C}$ as a holomorphic parameter.*

Proof. One has

$$\begin{aligned}f_1(z, \tau) \underset{\text{def}}{=} \sum_{j=0}^{\infty} a_j(z)\Phi_{\lambda+j}(\tau) &= (-\tau)^{-\lambda} \sum_{j=0}^{\infty} a_j(z)\Gamma(\lambda + j)(-\tau)^{-j}\\ &= \Gamma(\lambda)(-\tau)^{-\lambda} \sum_{j=0}^{\infty} \lambda(\lambda + 1)\cdots(\lambda + j - 1)a_j(z)(-\tau)^{-j}.\end{aligned}$$

Since $|\lambda(\lambda + 1)\cdots(\lambda + j - 1)| \leqq |\lambda|(1 + |\lambda|)\cdots(j - 1 + |\lambda|) \leqq (j - 1)!|\lambda|(1 + |\lambda|)\cdots(1 + |\lambda|/(j-1))$ and $1 + x < e^x$ $(x \geqq 0)$ hold, the last term is less than or equal to $(j-1)!|\lambda|e^{|\lambda|(1+\cdots+1/(j-1))}$. Furthermore, one has $1+\ldots+1/(j-1) \leqq \int_1^{j-1} dx/x = 1+\log(j-1)$. Consequently,

$$|\lambda(\lambda+1)\cdots(\lambda+j-1)| \leqq (j-1)!|\lambda|e^{|\lambda|}(j-1)^{|\lambda|} \tag{4.1.4}$$

holds. Hence, one obtains

$$|\lambda(\lambda+1)\cdots(\lambda+j-1)a_j(z)| < |\lambda|(j-1)^{|\lambda|-1}e^{|\lambda|}C_{\epsilon,K}\epsilon^j$$

for $z \in K$. Therefore, $((-\tau)^\lambda/\Gamma(\lambda))f_1(z,\tau)$ is holomorphic for $z \in \operatorname{Int} K$, the interior of K, and $\epsilon < |\tau|$. Since ϵ and K are arbitrary, consequently $(-\tau)^\lambda f_1(z,\tau)/\Gamma(\lambda)$ is holomorphic for $z \in \Omega$ and $\tau \in \mathbf{C} - \{0\}$. Next we will consider

$$f_2(z,\tau) = \sum_{j<0} a_j(z)\Phi_{\lambda+j}(\tau) = \sum_{j>0} \Gamma(\lambda-j)a_{-j}(-\tau)^{j-\lambda}.$$

This can be rewritten as

$$f_2(z,\tau) = \sum_{0<j<m} \Gamma(\lambda-j)a_{-j}(z)(-\tau)^{j-\lambda} + \sum_{j\geqq m} \Gamma(\lambda-j)a_{-j}(z)(-\tau)^{j-\lambda},$$

where m is a sufficiently large integer. The first term on the right-hand side is a multivalued meromorphic function for $\lambda \in \mathbf{C}$ and $0 < |\tau|$, having a pole at $\lambda \in \mathbf{Z}$. Hence, it is sufficient to prove the assertion for the second term. We have

$$\begin{aligned}\sum_{j\geqq m} \Gamma(\lambda-j)a_{-j}(z)(-\tau)^{j-\lambda} &= \Gamma(\lambda-m)(-\tau)^{-\lambda} \sum_{j\geqq m} \frac{a_{-j}(z)}{(\lambda-m-1)\cdots(\lambda-j)}(-\tau)^j \\ &= \Gamma(\lambda-m)(-\tau)^{-\lambda} \sum_{j\geqq m} \frac{(-1)^m a_{-j}(z)}{(1+m-\lambda)\cdots(j-\lambda)}\tau^j.\end{aligned}$$

For $|\lambda| \leqq m$, $|(1+m-\lambda)\cdots(j-\lambda)| \geqq (1+m-|\lambda|)\cdots(j-|\lambda|) \geqq (j-m)!$ holds. Hence, for $z \in K$ one obtains

$$\left|\frac{a_{-j}(z)}{(1+m-\lambda)\cdots(j-\lambda)}\right| \leqq \frac{j!R_K^j}{(j-m)!}.$$

That is, the second term converges for $z \in \operatorname{Int} K$ and $0 < |\tau| < 1/R_K$.

Remark. Conversely, if $\sum_j a_j(z)\Phi_{\lambda+j}(\tau)$ is uniformly absolutely convergent in wider sense for $0 < |\tau| \ll 1$, then $\{a_j(z)\}$ satisfies the conditions (4.1.2) and (4.1.3).

By virtue of Proposition 4.1.1, we can introduce a special class of micro-local operators. Take $\mathbf{R}^n$ for a real analytic manifold M, and take $\mathbf{C}^n$ for a complexification X of M. Denote the coordinates of the tangent bundle

$T^*X \cong \mathbf{C}^n \times \mathbf{C}^n$ by $(z, \zeta) = (z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$. Let λ be a complex number, and let Ω be an open subset of T^*X.

Definition 4.1.1. *Denote by $\mathscr{E}^\infty_{(\lambda)}(\Omega)$ the totality of sequences $\{p_{\lambda+j}(z, \zeta)\}_{j \in \mathbf{Z}}$ of holomorphic functions defined in Ω satisfying the following conditions* (4.1.5) *and* (4.1.6):

$p_{\lambda+j}(z, \zeta)$ *is a holomorphic function defined in Ω and homogeneous in ζ of degree $\lambda + j$; i.e.*

$$\sum_{i=1}^{n} \zeta_i(\partial/\partial\zeta_i)p_{\lambda+j}(z, \zeta) = (\lambda + j)p_{\lambda+j}(z, \zeta) \text{ holds.} \tag{4.1.5}$$

$p_{\lambda+j}(z, \zeta)$ *satisfies the growth conditions as follows:*

(4.1.6a) *For an arbitrary compact subset K in Ω, there exists a positive C_K such that*

$$|p_{\lambda+j}(z, \zeta)| \leqq C_K^{-j}(-j)!$$

holds for $(z, \zeta) \in K$ *and* $j < 0$.

(4.1.6b) *For an arbitrary compact subset K in Ω and for an arbitrary $\epsilon > 0$, there exists a positive $C_{K,\epsilon}$ such that*

$$|p_{\lambda+j}(z, \zeta)| < \frac{1}{j!} C_{K,\epsilon}\epsilon^j$$

holds for $(z, \zeta) \in K$ *and* $j \geqq 0$. (4.1.6)

Let

$$P(z, D) = \sum_j p_{\lambda+j}(z, D).$$

Then $P(z, D)$ is called a microdifferential operator defined in Ω. Since the presheaf $\{\mathscr{E}^\infty_{(\lambda)}(\Omega)\}$ is clearly a sheaf, we denote the sheaf by $\mathscr{E}^\infty_{(\lambda)}$. $\mathscr{E}(\lambda)$ denotes the sheaf of operators $P = \sum_j p_{\lambda+j}(z, D)$ with the property $p_{\lambda+j} = 0$ for $j > 0$. Furthermore, define $\mathscr{E}_{(\lambda)} = \bigcup_{j \in \mathbf{Z}} \mathscr{E}(\lambda + j)$. We also denote $\mathscr{E}^\infty_{(\lambda)}$ and $\mathscr{E}_{(\lambda)}$ for $\lambda = 0$ by $\mathscr{E}^\infty$ and $\mathscr{E}$ respectively. Let $\lambda = \operatorname{ord} P$ for $P \in \mathscr{E}(\lambda)$ and then λ is said to be the order of P. An element of $\mathscr{E}_{(\lambda)}$ is called a microdifferential operator of finite order. An element of $\mathscr{E}(\lambda)$ is called a microdifferential operator of order at most λ. An element of $\mathscr{E}^\infty_{(\lambda)}$, which is not in $\mathscr{E}_{(\lambda)}$, is said to be a microdifferential operator of infinite order.

Remark 1. $\cosh(\sqrt{D_1}) \underset{\text{def}}{=} \sum_{n=0}^{\infty} (D_1^n/(2n)!)$ is one of the simplest examples of microdifferential operators of infinite order (in fact, this is a differential operator). Even though the notion of microdifferential operators of infinite order is profoundly related to the structures of holomorphic functions over the field of complex numbers and is a very useful tool in analysis, it is quite difficult to manipulate. Hence we seldom use it in this book. We also mention that a microdifferential operator of infinite order acts on the sheaf of microfunctions as a sheaf homomorphism, as it will be shown below. But it does not act as a sheaf homomorphism on analogues of microfunctions constructed from distributions.

In the above, $P(z, D)$ is called a microdifferential "operator"; in fact, $P(z, D)$ determines a microlocal operator, as will be shown below. Let Ω be invariant under the actions by $\mathbf{R}_+^\times$; i.e. for $(z, \zeta) \in \Omega$ and $c > 0$ one has $(z, c\zeta) \in \Omega$. Then we will prove that $P(z, D)$ determines a microlocal operator on $R(\Omega) = \{(x\sqrt{-1}\xi\infty) \in \sqrt{-1}S^*\mathbf{R}^n \,|\, (x, \sqrt{-1}\xi) \in \Omega\}$. From Proposition 4.1.1, the boundary value of

$$\sum_j p_{\lambda+j}(z, \sqrt{-1}\zeta)\Phi_{\lambda+j+n}(\sqrt{-1}\tau)$$

from $\operatorname{Im} \tau > 0$ defines a hyperfunction. Let $\tau = \langle x - y, \zeta\rangle$, and let $K(x, y, \zeta)$ be the boundary value of $\sum_j p_{\lambda+j}(x, \sqrt{-1}\zeta)\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x - y, \zeta\rangle))$ from $\operatorname{Im}(\langle x - y, \zeta\rangle) > 0$. Then $K(x, y, \zeta)$ is a hyperfunction defined in an open set U containing $\{(x, y, \zeta) \,|\, x = y \text{ and } (x, \sqrt{-1}\zeta) \in \Omega\}$. Furthermore, it is obtained from the holomorphic function defined in $\operatorname{Im}\langle x - y, \zeta\rangle > 0$ and is holomorphic where $\langle x - y, \zeta\rangle \neq 0$. Hence,

$$\begin{aligned} \text{S.S. } K(x, y, \zeta) \subset G = \{&(x, y, \zeta; \sqrt{-1}(\xi, \eta, \rho)\infty \,|\, (x, y, \zeta) \in U, \langle x - y, \zeta\rangle = 0 \\ &\text{and } \xi = -\eta = k\zeta, \rho = k(x - y) \text{ for some } k > 0\} \end{aligned}$$

holds. Then let

$$K(x, y) = \int K(x, y, \zeta)\omega(\zeta).$$

If a point in G satisfies $\rho = 0$, then $x = y$ holds since $k > 0$. Hence,

$$\begin{aligned} Z \underset{\text{def}}{=} {} & G \cap \{(x, y, \zeta; \sqrt{-1}(\xi, \eta, \rho)\infty) \,|\, \rho = 0\} \\ = {} & \{(x, y, \zeta; \sqrt{-1}(\xi, \eta, \rho)\infty) \,|\, (x, \sqrt{-1}\zeta) \in \Omega, \\ & x = y, \xi = -\eta = k\zeta, (k > 0), \rho = 0\} \end{aligned}$$

holds. Therefore, the projection from Z to the $(x, y, \sqrt{-1}(\xi, \eta)\infty)$-space is a proper map, and its image is contained in

$$\{(x, y; \sqrt{-1}(\xi, \eta)\infty) \,|\, \xi = -\eta, x = y, (x, \sqrt{-1}\xi) \in \Omega\}.$$

Consequently, $K(x, y)$ is a microfunction defined in $\{(x, y; \sqrt{-1}(\xi, \eta)\infty) \,|\, (x, \sqrt{-1}\xi) \in R(\Omega)\}$ with its support contained in $\{x = y, \xi = -\eta\}$. Hence, $K(x, y)\,dx$ defines a microlocal operator defined over $R(\Omega)$.

Remark 2. The flabbiness of the microfunction sheaf implies that the sheaf $\mathscr{L}$ of microlocal operators is also flabby. Contrary to these flabby sheaves, the sheaf $\mathscr{E}$ of microdifferential operators is a "rigid" sheaf, having uniqueness of continuation. This indicates how great the difference is between these two notions.

In order to establish various formulas in connection with microdifferential operators, we will begin with the next formula.

Proposition 4.1.2.

$$\int_{S^{n-1}} \xi_{1+}^{\lambda_1-1} \cdots \xi_{n+}^{\lambda_n-1} \Phi_{\lambda_1+\cdots+\lambda_n}(\sqrt{-1}(\langle \xi, x\rangle + \sqrt{-1}0))\omega(\xi)$$
$$= \Phi_{\lambda_1}(\sqrt{-1}(x_1 + \sqrt{-1}0)) \cdots \Phi_{\lambda_n}(\sqrt{-1}(x_n + \sqrt{-1}0)). \quad (4.1.5')$$

Proof. It is sufficient to prove the above formula for the case where x_j is the complex number $x_j + \sqrt{-1}\epsilon_j$ for $\epsilon_j > 0$. If $\operatorname{Im} x_j > 0$ and $\xi_j \geqq 0$, then one has

$$\Phi_{\lambda_1+\cdots+\lambda_n}(\sqrt{-1}(\langle \xi, x\rangle + \sqrt{-1}0)) = \int_0^\infty e^{\sqrt{-1}t\langle \xi, x\rangle} t^{\lambda_1+\cdots+\lambda_n-1}\, dt.$$

Hence, the left-hand side of (4.1.5′) becomes

$$\int_{S^{n-1}} \omega(\xi)\xi_{1+}^{\lambda_1-1} \cdots \xi_{n+}^{\lambda_n-1} \int_0^\infty e^{\sqrt{-1}t\langle \xi, x\rangle} t^{\lambda_1+\cdots+\lambda_n-1}\, dt,$$

which can be rewritten as follows:

$$\begin{aligned}
\int_0^\infty t^{n-1}\, dt \int_{S^{n-1}} \omega(\xi)(t\xi_1)_+^{\lambda_1-1} \cdots (t\xi_n)_+^{\lambda_n-1} e^{\sqrt{-1}\langle t\xi, x\rangle}& \\
&= \int_{\mathbf{R}^n} d\xi\, \xi_{n+}^{\lambda_1-1} \cdots \xi_{n+}^{\lambda_n-1} e^{\sqrt{-1}\langle \xi, x\rangle} \\
&= \prod_{j=1}^n \int_0^\infty \xi_j^{\lambda_j-1} e^{\sqrt{-1}\xi_j x_j}\, d\xi_j \\
&= \prod_{j=1}^n \Phi_{\lambda_j}(\sqrt{-1}x_j) \\
&= \text{the right-hand side of (4.1.5′)}.
\end{aligned}$$

Using this result, we will give proofs to the formulas (4.1.8) and (4.1.9) of Radon transforms. (Consult Gel'fand, Graev and Vilenkin [1] for the geometric meanings of these equations.) For an arbitrary complex number λ, define a function $\delta_\lambda(x, y)$ on $(\mathbf{R}^n - \{0\}) \times (\mathbf{R}^n - \{0\})$ by

$$\delta_\lambda(x, y) = |x_j|^\lambda \prod_{k \neq j} \delta\left(y_k - \frac{y_j}{x_j} x_k\right)(\pm y_j)_+^{-\lambda-1} \qquad \text{for } \pm x_j > 0.$$

For $\epsilon_j x_j > 0$ and $\epsilon_k x_k > 0$, $(\epsilon_j, \epsilon_k = \pm 1)$,

$$|x_j|^\lambda \prod_{l \neq j} \delta\left(y_l - \frac{y_j}{x_j} x_l\right)(\epsilon_j y_j)_+^{-\lambda-1} = |x_k|^\lambda \prod_{l \neq k} \delta\left(y_l - \frac{y_k}{x_k} x_l\right)(\epsilon_k y_k)_+^{-\lambda-1}$$

holds. Hence $\delta_\lambda(x, y)$ is well defined as a hyperfunction on $(\mathbf{R}^n - \{0\}) \times (\mathbf{R}^n - \{0\})$. From the definition, the homogeneous degrees of $\delta_\lambda(x, y)$ in x and y are λ and $-\lambda - n$ respectively. One has

$$\operatorname{supp} \delta_\lambda(x, y) \subset \{(x, y) | x = ty, t > 0\}$$

and

$$\delta_\lambda(x, y) = \delta_{-n-\lambda}(y, x) = \delta_\lambda(-x, -y).$$

The following lemma is obvious from the definition.

Lemma 1. *Let $u(x)$ be a hyperfunction on $\mathbf{R}^n - \{0\}$ of homogeneous degree λ. Then one has*

$$u(x) = \int \delta_\lambda(x, y) u(y) \omega(y).$$

Lemma 2.

$$\frac{1}{(2\pi)^n} \int_{S^{n-1}} \Phi_{-\lambda}(\sqrt{-1}(\langle x, \xi \rangle + \sqrt{-1}0)) \cdot \Phi_{\lambda+n}(-\sqrt{-1}(\langle y, \xi \rangle - \sqrt{-1}0))\omega(\xi) = \delta_\lambda(x, y). \quad (4.1.6')$$

Proof. One obtains from Proposition 4.1.2.

$$\Phi_{-\lambda}(\sqrt{-1}(\langle x, \xi \rangle + \sqrt{-1}0))\Phi_{\lambda+n}(-\sqrt{-1}(\langle y, \xi \rangle - \sqrt{-1}0)) = \int t_+^{-\lambda-1} s_+^{\lambda+n-1} \Phi_n(\sqrt{-1}(\langle tx - sy, \xi \rangle + \sqrt{-1}0))\omega(t, s).$$

Hence, the left-hand side of (4.1.6′) becomes

$$\frac{1}{(2\pi)^n} \int t_+^{-\lambda-1} s_+^{\lambda+n-1} \omega(t, s) \int \Phi_n(\sqrt{-1}(\langle tx - sy, \xi \rangle + \sqrt{-1}0)\omega(\xi) = \int t_+^{-\lambda-1} s_+^{\lambda+n-1} \delta(tx - sy)\omega(t, s).$$

If, for example, $x_1 > 0$, then

$$\int t_+^{-\lambda-1} s_+^{\lambda+n-1} \delta(tx - sy)\omega(t, s) = \int t_+^{-\lambda-1} \delta(tx - y)\, dt = \left(\frac{y_1}{x_1}\right)_+^{-\lambda-1} \prod_{j \neq 1} \delta\left(\frac{y_1}{x_1} x_j - y_j\right) \frac{1}{x_1},$$

which is equal to $\delta_\lambda(x, y)$.

One easily has the following lemma from the definition.

Lemma 3. *For $g \in GL(n, \mathbf{R})$, one has*

$$\delta_\lambda(gx, gy) = |\det g|^{-1}\delta_\lambda(x, y). \tag{4.1.7}$$

Proposition 4.1.3. *There is a one-to-one correspondence between homogeneous functions $u(x)$ on $\mathbf{R}^n - \{0\}$ of degree λ and homogeneous functions $v(x)$ on $\mathbf{R}^n - \{0\}$ of degree $-n-\lambda$, given by*

$$v(\xi) = \frac{1}{(2\pi)^{n/2}} \int u(x)\Phi_{\lambda+n}(\sqrt{-1}(\langle x, \xi\rangle + \sqrt{-1}0))\omega(x) \tag{4.1.8}$$

and

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int v(\xi)\Phi_{-\lambda}(-\sqrt{-1}(\langle x, \xi\rangle - \sqrt{-1}0)\omega(\xi). \tag{4.1.9}$$

Proof. When $u(x)$ and $v(\xi)$ are given as in (4.1.8), one has

$$\begin{aligned}
&\frac{1}{(2\pi)^{n/2}} \int v(\xi)\Phi_{-\lambda}(-\sqrt{-1}(\langle x, \xi\rangle - \sqrt{-1}0))\omega(\xi) \\
&\qquad = \frac{1}{(2\pi)^n} \int u(y)\omega(y) \int \varphi_{\lambda+n}(\sqrt{-1}\langle y, \xi\rangle + \sqrt{-1}0) \\
&\qquad\quad \cdot \Phi_{-\lambda}(-\sqrt{-1}(\langle x, \xi\rangle - \sqrt{-1}0))\omega(\xi) \\
&\qquad = \int u(y)\delta_\lambda(x, y)\omega(y) = u(x).
\end{aligned}$$

A similar computation can be done when $u(x)$ and $v(\xi)$ are given as in (4.1.9).

Proposition 4.1.4. *Let $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$. Then, for an integer $\mu \geqq n$,*

$$\begin{aligned}
&P(x, D_x)\Phi_\mu(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)) \\
&\qquad = \sum_j p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{\lambda+\mu+j}(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0))
\end{aligned} \tag{4.1.10}$$

holds.

Proof. Let us first consider the case when $\mu = n$. Let

$$K(x, \xi, p) = \frac{1}{(2\pi)^n} \sum p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{n+\lambda+j}(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)).$$

Then, the left-hand side of (4.1.10) equals

$$\begin{aligned}
&\int K(x, \eta, -\langle y, \eta\rangle)\Phi_n(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\omega(\eta)\, dy \\
&\qquad = \int K(x, \eta, t)\, dt \int \delta(t + \langle y, \eta\rangle)\Phi_n(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\omega(\eta)\, dy.
\end{aligned}$$

On the other hand, one has

$$\delta(t + \langle y, \eta\rangle) = \frac{1}{2\pi}\,\Phi_1(-\sqrt{-1}(t + \langle y, \eta\rangle - \sqrt{-1}0)).$$

Hence,

$$\begin{aligned}
&\int \delta(t + \langle y, \eta\rangle)\Phi_n(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\omega(\eta)\,dy \\
&\qquad = \frac{1}{2\pi}\int \Phi_1(-\sqrt{-1}(y_0 t + \langle y, \eta\rangle - \sqrt{-1}0))\cdot\Phi_n(\sqrt{-1}(\langle y, \xi\rangle \\
&\qquad\qquad + y_0 p + \sqrt{-1}0))\omega(y_0, y)\omega(\eta) \\
&\qquad = (2\pi)^n \delta_n((p, \xi), (t, \eta)).
\end{aligned}$$

Consequently, the left-hand side of (4.1.10) is

$$(2\pi)^n \int K(x, \eta, t)\delta_n((p, \xi), (t, \eta))\omega(t, \eta) = (2\pi)^n K(x, \xi, p).$$

For the general μ, first note

$$\begin{aligned}
&\Phi_\mu(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)) \\
&\quad = \frac{1}{2\pi}\int \Phi_n(\sqrt{-1}(\langle x, \xi\rangle + t + \sqrt{-1}0))\Phi_{\mu-n+1}(\sqrt{-1}(p - t + \sqrt{-1}0))\,dt.
\end{aligned}$$

Therefore, one obtains the following:

$$\begin{aligned}
&P(x, D_x)\Phi_\mu(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)) \\
&\quad = \frac{1}{2\pi}\int \Phi_{\mu-n+1}(\sqrt{-1}(p - t + \sqrt{-1}0)) \\
&\qquad\quad \cdot\left(\sum_j p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{n+\lambda+j}(\sqrt{-1}(\langle x, \xi\rangle + t + \sqrt{-1}0)\right)dt \\
&\quad = \left(\frac{D_p}{\sqrt{-1}}\right)^{\mu-n}\sum_j p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{n+\lambda+j}(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)).
\end{aligned}$$

Hence the next lemma completes the proof.

Lemma 1. *Let $a_{jk}(t, z)$, $j, k \in \mathbf{Z}$, be a holomorphic function defined in a neighborhood U of $t = 0$ and $z = 0$ such that $\sum_{j,k} a_{jk}(t, z)\Phi_{\lambda+j}(p)\Phi_{\mu+k}(q)$ is uniformly absolutely convergent in a wider sense for $(t, z) \in U$ and $0 < |p|$, $|q| \ll 1$. Let $u(t, s, x)$ be the boundary value of*

$$\sum_{j,k} a_{jk}(t, x)\Phi_{\lambda+j}(\sqrt{-1}(t - s))\,\Phi_{\mu+k}(\sqrt{-1}s)$$

from $\operatorname{Im}(t-s)$ *and* $\operatorname{Im} s>0$. *Then, as microfunctions, one has*

$$\int u(t, s, x)\, ds = 2\pi \sum_{j,k} a_{j,k}(t, x)\Phi_{\lambda+\mu+j+k-1}(\sqrt{-1}t)$$

in a neighborhood of $\sqrt{-1}\, dt$.

Proof. Let $a>0$ be sufficiently small and let

$$I_a(t, x) = \int_{\gamma_a} u(t, s, x)\, ds,$$

where γ_a is a path from $-a$ to a, contained in $\operatorname{Im} s \geqq 0$ and not going through the origin. Then $\int u(t, s, x)\, ds$ is the boundary value of $I_a(t, x)$. On the other hand,

$$I_a(t, x) = \sum_{j,k} a_{jk}(t, x) \int_{\gamma_a} \Phi_{\lambda+j}(\sqrt{-1}(t-s))\Phi_{\mu+k}(\sqrt{-1}s)\, ds$$

holds. We need the following lemma to compute the right-hand side of the above equation.

Lemma 2.

$$\begin{aligned}
&\int_{-a}^{b} \Phi_\lambda(\sqrt{-1}(t-s+\sqrt{-1}0))\Phi_\mu(\sqrt{-1}(s+\sqrt{-1}0))\, ds \\
&\qquad - 2\pi\Phi_{\lambda+\mu-1}(\sqrt{-1}(t+\sqrt{-1}0)) \\
&\quad = -\Gamma(\mu)\sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{n!(\lambda+\mu+n-1)} \cdot \left(e^{(\pi i/2)(\mu-\lambda)} b^{1-\lambda-\mu}\left(\frac{t}{b}\right)^n \right. \\
&\qquad \left. + (-1)^n e^{(\pi i/2)(\lambda-\mu)} a^{1-\lambda-\mu}\left(\frac{t}{a}\right)^n \right).
\end{aligned} \tag{4.1.11}$$

Proof. By virtue of analytic continuation with respect to λ and μ, it is sufficient to consider the case where $\operatorname{Re}\lambda \gg 0$ and $\operatorname{Re}\mu \gg 0$. In that case, one has

$$\begin{aligned}
2\pi\Phi_{\lambda+\mu-1}(\sqrt{-1}(t+\sqrt{-1}0)) = \int_{-\infty}^{\infty} &\Phi_\lambda(\sqrt{-1}(t-s+\sqrt{-1}0)) \\
&\cdot \Phi_\mu(\sqrt{-1}(s+\sqrt{-1}0))\, ds.
\end{aligned}$$

Hence, the left-hand side of (4.1.11) can be written as

$$\left(\int_{-a}^{-\infty} - \int_{b}^{\infty}\right) \Phi_\lambda(\sqrt{-1}(t-s+\sqrt{-1}0))\Phi_\mu(\sqrt{-1}(s+\sqrt{-1}0))\, ds.$$

Here, note

$$\int_b^\infty \Phi_\lambda(\sqrt{-1}(t-s+\sqrt{-1}0))\Phi_\mu(\sqrt{-1}(s+\sqrt{-1}0))\,ds$$

$$= e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}\Gamma(\lambda)\Gamma(\mu)\int_b^\infty (s-t)^{-\lambda}s^{-\mu}\,ds$$

$$= e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}\Gamma(\lambda)\Gamma(\mu)\int_b^\infty s^{-\lambda-\mu}\left(1-\frac{t}{s}\right)^{-\lambda}ds$$

$$= e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}\Gamma(\lambda)\Gamma(\mu)\int_b^\infty \left(\sum_{n=0}^\infty \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{n!}t^n s^{-\lambda-\mu-n}\right)ds$$

$$= e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}\Gamma(\lambda)\Gamma(\mu)\sum_{n=0}^\infty \frac{\lambda(\lambda+1)\cdots(\lambda+n-1)}{n!(\lambda+\mu+n-1)}b^{-\lambda-\mu-n+1}t^n.$$

The term involving $\int_{-\infty}^{a}$ can be done similarly. This completes the proof for Lemma 2.

Now we return to the proof of Lemma 1. Lemma 2 implies

$$\begin{aligned} I_a(t,x) &= 2\pi\sum_{j,k} a_{jk}(t,x)\Phi_{\lambda+\mu+j+k-1}(\sqrt{-1}t) \\ &\quad - \sum_{\substack{j,k,n\\ n\geqq 0}} \frac{\Gamma(\lambda+n+j)\Gamma(\mu+k)}{n!(\lambda+\mu+j+k+n-1)}\left(e^{(\pi\sqrt{-1}/2)(\mu+k-\lambda-j)}\right. \\ &\quad \left. +(-1)^n e^{(\pi\sqrt{-1}/2)(\lambda+j-\mu-k)}\right)\cdot a_{jk}(t,x)a^{1-\lambda-\mu-j-k}\left(\frac{t}{a}\right)^n. \end{aligned} \tag{4.1.12}$$

It remains to be proved that the second-term sum of the above converges absolutely for $0<|a|\ll 1$ and $|t/a|\ll 1$. The second term can be written as

$$\int da \sum_{\substack{j,k,n\\ n\geqq 0}} \frac{\Gamma(\lambda+n+j)e^{(\pi\sqrt{-1}/2)(k-j)}\left(e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}+(-1)^{n+j+k}e^{(\pi\sqrt{-1}/2)(\lambda-\mu)}\right)}{n!\Gamma(\lambda+j)}$$
$$\cdot a_{jk}(t,x)\Phi_{\lambda+j}(-a)\Phi_{\mu+k}(-a)(t/a)^n. \tag{4.1.13}$$

For $c \geqq \frac{1}{2}\left|e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}\right|, \frac{1}{2}\left|e^{(\pi\sqrt{-1}/2)(\lambda-\mu)}\right|$, one has

$$\sum_{n=0}^\infty \left|\frac{\Gamma(\lambda+n+j)e^{(\pi\sqrt{-1}/2)(k-j)}\left(e^{(\pi\sqrt{-1}/2)(\mu-\lambda)}+(-1)^{n+j+k}e^{(\pi\sqrt{-1}/2)(\lambda-\mu)}\right)}{n!\Gamma(\lambda+j)}\left(\frac{t}{a}\right)^n\right|$$
$$\leqq c\sum_{n=0}^\infty \frac{(|\lambda|+|j|+n-1)\cdots(|\lambda|+|j|)}{n!}\left|\frac{t}{a}\right|^n = c\left(1-\left|\frac{t}{a}\right|\right)^{-|\lambda|-|j|}.$$

On the other hand,

$$\sum_{j,k}|a_{jk}(t,x)|\,|\Phi_{\lambda+j}(-a)|\,|\Phi_{\mu+k}(-a)|\left(1-\left|\frac{t}{a}\right|\right)^{-|j|}$$

converges absolutely. Therefore, the integrand of (4.1.13) is absolutely convergent, which completes the proof of Lemma 1.

By Lemma 1, one can easily complete the proof of Proposition 4.1.4. Let $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$ be a microdifferential operator for $x \in \mathbf{R}^n$, and let $K(x, x')\,dx'$ be the kernel function. For a coordinate system y of $\mathbf{R}^m$, consider

$$K(x, x')\delta(y - y')\,dx'\,dy'.$$

This determines the kernel function of a microlocal operator on $\mathbf{R}^{n+m}$. Furthermore, this is a microdifferential operator, which is equal to $\sum_j r_{\lambda+j}(x, y, D_x, D_y)$, where $r_{r+j}(x, y, \xi, \eta) = p_{\lambda+j}(x, \xi)$. In fact,

$$K(x, x')\delta(y - y') = P(x, D_x)\delta(x - x')\delta(y - y')$$

holds, and Proposition 4.1.4 implies

$$\begin{aligned} &P(x, D_x)\Phi_{n+m}(\sqrt{-1}(\langle x - x', \xi\rangle + \langle y - y', \eta\rangle + \sqrt{-1}0) \\ &= \sum_j p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{n+m+\lambda+j}(\sqrt{-1}(\langle x - x', \xi\rangle + \langle y - y', \eta\rangle + \sqrt{-1}0)) \\ &= \sum_j r_{\lambda+j}(x, y, \sqrt{-1}\xi, \sqrt{-1}\eta)\Phi_{n+m+\lambda+j}(\sqrt{-1}(\langle x - x', \xi\rangle \\ &\quad + \langle y - y', \eta\rangle + \sqrt{-1}0)). \end{aligned}$$

Hence, one obtains

$$\begin{aligned} &R(x, y, D_x, D_y)\delta(x - x')\delta(y - y') \\ &\qquad = \frac{1}{(2\pi)^{n+m}} \int p(x, D_x)\Phi_{n+m}(\sqrt{-1}(\langle x - x', \xi\rangle + \langle y - y', \eta\rangle \\ &\qquad\quad + \sqrt{-1}0))\omega(\xi, \eta) \\ &\qquad = P(x, D_x)\delta(x - x', y - y'). \end{aligned}$$

Henceforth, one regards a microdifferential operator $P(x, D_x)$ on $\mathbf{R}^n$ as one on $\mathbf{R}^{n+m}$.

Proposition 4.1.5. *Let $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$ be a microdifferential operator, let $\{a_j(x)\}_{j \in \mathbf{Z}}$ be a sequence which satisfies the conditions* (4.1.2) *and* (4.1.3), *and let*

$$u(x, \xi, p) = \sum_j a_j(x)\Phi_{\mu+j}(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)).$$

Then the sequence $\{b_l(x, \xi)\}_{l \in \mathbf{Z}}$, where

$$b_l(x, \xi) = \sum_{l = j+k-|\alpha|} \frac{1}{\alpha!}\left(\frac{1}{\sqrt{-1}} D_\xi\right)^\alpha p_{\lambda+j}(x, \sqrt{-1}\xi)(D_x^\alpha a_k(x)),$$

satisfies the conditions (4.1.2) *and* (4.1.3), *and then*

$$P(x, D_x)u(x, \xi, p) = \sum_l b_l(x, \xi)\Phi_{\lambda+\mu+l}(\sqrt{-1}(\langle x, \xi\rangle + p + \sqrt{-1}0)) \quad (4.1.14)$$

holds.

Proof.

$$P(x, D_x)u(x, \xi, p)$$
$$= \frac{1}{(2\pi)^n}\int\sum_{j,k} p_{\lambda+j}(x, \sqrt{-1}\eta)\Phi_{\lambda+n+j}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))$$
$$\cdot a_k(y)\Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\omega(\eta)\,dy$$

holds. The Taylor expansion $a_k(y) = \sum_\alpha (1/\alpha!)(y-x)^\alpha D_x^\alpha a_k(x)$ and

$$(y-x)^\alpha\Phi_{\lambda+n+j}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))$$
$$= (\sqrt{-1}D_\eta)^\alpha\Phi_{\lambda+n+j-|\alpha|}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))$$

provide one with

$$P(x, D_x)u(x, \xi, p)$$
$$= \frac{1}{(2\pi)^n}\int\sum_{j,k,\alpha}\frac{1}{\alpha!} p_{\lambda+j}(x, \sqrt{-1}\eta)(D_x^\alpha a_k(x))\Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))$$
$$\cdot(\sqrt{-1}D_\eta)^\alpha\Phi_{\lambda+n+j-|\alpha|}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))\omega(\eta)\,dy. \quad (4.1.15)$$

When one performs a termwise integration by parts, one obtains

$$P(x, D_x)u(x, \xi, p)$$
$$= \frac{1}{(2\pi)^n}\int\Bigg(\sum_{j,k,\alpha}\frac{1}{\alpha!}(-\sqrt{-1}D_\eta)^\alpha p_{\lambda+j+|\alpha|}(x, \sqrt{-1}\eta)$$
$$\cdot(D_x^\alpha a_k(x))\Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))$$
$$\cdot\Phi_{\lambda+n+j}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))\Bigg)\omega(\eta)\,dy.$$

Therefore, for

$$c_{jk}(x, \eta) \underset{\text{def}}{=} \sum\frac{1}{\alpha!}(-\sqrt{-1}D_\eta)^\alpha p_{\lambda+j+|\alpha|}(x, \sqrt{-1}\eta)D_x^\alpha a_k(x),$$

one has

$$P(x, D_x)u(x, \xi, p) = \frac{1}{(2\pi)^n}\int\sum_{j,k} c_{jk}(x, \eta)\Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0)$$
$$\cdot\Phi_{\lambda+n+j}(\sqrt{-1}(\langle x-y, \eta\rangle + \sqrt{-1}0))\omega(\eta)\,dy.$$

If one lets

$$f(x, \eta, t, s) \underset{\text{def}}{=} \sum_{j,k} c_{jk}(x, \eta)\Phi_{\mu+k}(\sqrt{-1}(t+\sqrt{-1}0))\Phi_{\lambda+1+j}(\sqrt{-1}(s+\sqrt{-1}0)),$$

then one has the following:

$$P(x, D_x)u(x, \xi, p) = \frac{1}{(2\pi)^n}\int_s^{n-1} f(x, \eta, \langle y, \xi\rangle + p, \langle x - y, \eta\rangle)\omega(\eta)\, dy$$

$$= \frac{(\sqrt{-1})^{n-1}}{(2\pi)^n}\int f(x, \eta, t + p, \langle x, \eta\rangle + s)\delta(t - \langle y, \xi\rangle)$$

$$\cdot\, \delta^{(n-1)}(s + \langle y, \eta\rangle)\omega(\eta)\, dy\, dt\, ds.$$

Let us compute

$$\int \delta(t - \langle y, \xi\rangle)\delta^{(n-1)}(s + \langle y, \eta\rangle)\, dy.$$

The above integral is microlocally (i.e. locally on the cotangent bundle) equal to

$$\varphi(t, s, \xi, \eta) \underset{\text{def}}{=} \frac{(-\sqrt{-1})^{n-1}}{(2\pi)^2}\int \Phi_1(\sqrt{-1}(t - \langle y, \xi\rangle + \sqrt{-1}0))$$

$$\cdot\, \Phi_n(\sqrt{-1}(s + \langle y, \eta\rangle + \sqrt{-1}0))\, dy.$$

Then one obtains

$$\varphi(t, s, \xi, \eta) = \frac{(-\sqrt{-1})^{n-1}}{(2\pi)^2}\int \Phi_1(\sqrt{-1}(y_0 t + \langle y, -\xi\rangle + \sqrt{-1}0))$$

$$\cdot\, \Phi_n(\sqrt{-1}(y_0 s + \langle y, \eta\rangle + \sqrt{-1}0))\omega(\tilde{y})$$

$$= (2\pi)^{-2}(2\pi)^{n+1}(-\sqrt{-1})^{n-1}\delta_{-1}((-t, \xi), (s, \eta)),$$

where $\tilde{y} = (y_0, y)$. Therefore,

$$\frac{(\sqrt{-1})^{n-1}}{(2\pi)^n}\int f(x, \eta, t+p, \langle x, \eta\rangle+s)\delta(t-\langle y, \xi\rangle)\delta^{(n-1)}(s+\langle y, \eta\rangle)\omega(\eta)\, dy\, ds$$

$$= \frac{1}{2\pi}\int f(x, \eta, t + p, \langle x, \eta\rangle + s)\delta_{-1}((-t, \xi), (s, \eta))\omega(\eta)\, ds$$

$$= \frac{1}{2\pi}\int f(x, \eta, t + p, \langle x, \eta\rangle + s)\delta_{-1}((-t, \xi), (s, \eta))\omega(\eta, s)$$

$$= \frac{1}{2\pi} f(x, \xi, t + p, \langle x, \xi\rangle \quad t)$$

holds. Consequently, one has

$$P(x, D_x)u(x, \xi, p) = \frac{1}{2\pi}\int f(x, \xi, t + p, \langle x, \xi\rangle - t)\, dt.$$

Hence, Lemma 1, in the proof of Proposition 4.1.4, implies the equation (4.1.14). We need to prove that (4.1.15) can be integrated by parts. It is sufficient to show that there exist hyperfunctions G_ν, $\nu = 1, \dots, n$, such that

$$\begin{aligned}
&\left\{\sum_{j,k,\alpha} \frac{1}{\alpha!} p_{\lambda+j}(x, \sqrt{-1}\eta)(D_x^\alpha a_k(x))\Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\right.\\
&\qquad\left.\cdot(\sqrt{-1}D_\eta)^\alpha \Phi_{\lambda+n+j-|\alpha|}(\sqrt{-1}(\langle x - y, \eta\rangle + \sqrt{-1}0))\right\}\\
&\qquad - \left\{\sum_{j,k,\alpha} \frac{1}{\alpha!}(-\sqrt{-1}D_\eta)^\alpha p_{\lambda+j}(x, \sqrt{-1}\eta)(D_x^\alpha a_k(x))\right.\\
&\qquad\cdot \Phi_{\mu+k}(\sqrt{-1}(\langle y, \xi\rangle + p + \sqrt{-1}0))\\
&\qquad\left.\cdot \Phi_{\lambda+n+j-|\alpha|}(\sqrt{-1}(\langle x - y, \eta\rangle + \sqrt{-1}0)\right\}\\
&= \sum_{\nu=1}^{n} \frac{1}{\sqrt{-1}}\frac{\partial}{\partial \eta_\nu} G_\nu
\end{aligned}$$

holds. This is because the above equation implies

$$\int \frac{\partial}{\partial \eta_\nu} G_\nu \omega(\eta) = 0.$$

The construction of G_ν is left for the reader. (A formal infinite series expression of G_ν can be found from the following lemma. Then the convergence of the series must be checked).

Lemma. *Let $P(\xi)$ be a homogeneous polynomial of degree m. Then*

$$\begin{aligned}
&a(x)(P(D_x)b(x)) - (P(-D_x)a(x))b(x)\\
&\qquad = \sum_j \frac{\partial}{\partial x_j}\left(\sum_{|\alpha|<m} \frac{(-1)^\alpha (m - |\alpha| - 1)!|\alpha|!}{m!\alpha!}(D_x^\alpha a)P^{(\alpha+\delta_j)}(D_x)b\right)
\end{aligned}$$

holds, where $P^{(\alpha)}(\xi) = D_\xi^\alpha P$ and $\delta_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$.

Proof. The right-hand side is equal to

$$\begin{aligned}
&\sum_{|\alpha|<m} \frac{(-1)^{|\alpha|}(m - |\alpha| - 1)!|\alpha|!}{m!\alpha!}(D^{\alpha+\delta_j}a)P^{(\alpha+\delta_j)}(D)b\\
&\qquad + \sum_{|\alpha|<m} \frac{(-1)^{|\alpha|}(m - |\alpha| - 1)!|\alpha|!}{m!\alpha!}(D^\alpha a)\left(\sum_j D_j P^{(\alpha+\delta_j)}(D)b\right). \qquad (4.1.15')
\end{aligned}$$

Since the homogeneous degree of $P^{(\alpha)}$ is $m - |\alpha|$,

$$\sum_j D_j P^{(\alpha+\delta_j)}(D) = (m - |\alpha|)P^{(\alpha)}$$

holds. Then the second term of (4.1.15′) becomes

$$\sum_{|\alpha|<m} \frac{(-1)^{|\alpha|}(m - |\alpha|)!|\alpha|!}{m!\alpha!} (D^\alpha a)(P^{(\alpha)}(D)b),$$

and the first term becomes

$$\sum_{0<|\beta|\leqq m} \frac{(-1)^{|\beta|-1}(m - |\beta|)!(|\beta| - 1)!}{m!} \left(\sum_{\beta=\alpha+\delta_j} \frac{1}{\alpha!}\right)(D^\beta a)P^{(\beta)}(D)b.$$

On the other hand, one has

$$\sum_{\beta=\alpha+\delta_j} \frac{1}{\alpha!} = \sum \frac{\beta_j}{\beta!} = \frac{|\beta|}{\beta!}.$$

Hence, the first term of (4.1.15′) is

$$\sum_{0<|\beta|\leqq m} \frac{(-1)^{|\beta|-1}(m - |\beta|)!|\beta|!}{m!\beta!} (D^\beta a)P^{(\beta)}(D)b.$$

Consequently one obtains

$$\sum_{|\alpha|=m} \frac{(-1)^{|\alpha|-1}(m - |\alpha|)!|\alpha|!}{m!\alpha!} (D^\alpha a)P^{(\alpha)}(D)b + aP(D)b$$

$$= aP(D)b - \sum_{|\alpha|=m} \frac{1}{\alpha!} (P^{(\alpha)}(D)(-D)^\alpha a)b = aP(D)b - (P(-D)a)b.$$

We obtain from Proposition 4.1.5 the following fundamental result on the composition of microdifferential operators.

Theorem 4.1.1. *Let $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$ be a microdifferential operator in $\mathscr{E}_{(\lambda)}$, and let $Q(x, D_x) = \sum_k q_{k+\mu}(x, D_x)$ be a microdifferential operator in $\mathscr{E}_{(\mu)}$. Then the composition $R = PQ$ is an element of $\mathscr{E}_{(\lambda+\mu)}$. If $R = \sum_l r_{\lambda+\mu+l}(x, D_x)$, then one has*

$$r_{\lambda+\mu+l}(x, \sqrt{-1}\xi) = \sum_{l=j+k-|\alpha|} \frac{1}{\alpha!} \left(\left(\frac{1}{\sqrt{-1}} D_\xi\right)^\alpha p_{\lambda+j}(x, \sqrt{-1}\xi)\right)$$

$$\cdot (D_x^\alpha q_{\mu+k}(x, \sqrt{-1}\xi)). \tag{4.1.16}$$

Proof. Proposition 4.1.5 implies

$$Q(x, D_x)\Phi_n(\sqrt{-1}(\langle x, \xi\rangle + p)$$

$$= \sum_k q_{\mu+k}(x, \sqrt{-1}\xi)\Phi_{\mu+k+n}(\sqrt{-1}(\langle x, \xi\rangle + p)).$$

Applying Proposition 4.1.5 to $PQ\Phi_n(\sqrt{-1}(\langle x, \xi\rangle + p))$, one has

$$
\begin{aligned}
&PQ\Phi_n(\sqrt{-1}(\langle x, \xi\rangle + p)) \\
&\qquad = \sum_l r_{\lambda+\mu+l}(x, \sqrt{-1}\xi)\Phi_{\lambda+\mu+l+n}(\sqrt{-1}(\langle x, \xi\rangle + p))
\end{aligned}
$$

where

$$
\begin{aligned}
&r_{\lambda+\mu+l}(x, \sqrt{-1}\xi) \\
&\qquad = \sum_{l=j+k-|\alpha|} \frac{1}{\alpha!}\left(\frac{1}{\sqrt{-1}} D_\xi\right)^\alpha p_{\lambda+j}(x, \sqrt{-1}\xi)(D_x^\alpha q_{\mu+k}(x, \sqrt{-1}\xi))
\end{aligned}
$$

holds. Hence one obtains

$$
\begin{aligned}
PQ\delta(x-y) &= \frac{1}{(2\pi)^n}\int PQ\Phi_n(\sqrt{-1}(\langle x, \xi\rangle - \langle y, \xi\rangle))\omega(\xi) \\
&= \frac{1}{(2\pi)^n}\int \sum_l r_{\lambda+\mu+l}(x, \sqrt{-1}\xi)\Phi_{\lambda+\mu+l+n}(\sqrt{-1}\langle x-y, \xi\rangle)\omega(\xi),
\end{aligned}
$$

completing the proof.

Remark. By a similar argument (for a fixed volume element $dx = dx_1 \cdots dx_n$), the conjugate operator $P^*(x, D_x) \equiv \sum_k q_{\lambda+k}(x, D_x)$ of $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$, in the sense of Definition 3.4.3, satisfies

$$
q_{\lambda+k}(x, -\sqrt{-1}\xi) = \sum_{k=j-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\frac{1}{\sqrt{-1}} D_\xi\right)^\alpha D_x^\alpha p_{\lambda+j}(x, \sqrt{-1}\xi).
$$

Note that this is a generalization of Lemma 4(2), following Definition 3.4.3, to the case of a microdifferential operator.

Let $\mathcal{O}(\lambda)$ be the sheaf of holomorphic functions on T^*X whose homogeneous degree in the fibre coordinate ζ is λ. Assigning $\sum_j p_{\lambda+j}(x, D_x)$ to $p_\lambda(z, \zeta)$ induces a sheaf homomorphism σ_λ from $\mathscr{E}(\lambda)$ to $\mathcal{O}(\lambda)$. As in the case of a linear differential operator, $\sigma_\lambda(P)$ is said to be the principal symbol of P. Then we obtain the following theorem from Theorem 4.1.1.

Theorem 4.1.2

(i) *If* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *then* $PQ \in \mathscr{E}(\lambda+\mu)$ *and* $\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q)$ *hold.*

(ii) *If* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *then* $[P, Q] \in \mathscr{E}(\lambda+\mu-1)$ *holds, and one has*

$$
\sigma_{\lambda+\mu-1}([P, Q]) = \frac{1}{\sqrt{-1}}\{p_\lambda(x, \sqrt{-1}\xi), q_\mu(x, \sqrt{-1}\xi)\},
$$

which (see §5, *Chapter* III) *is equal to*

$$\frac{1}{\sqrt{-1}}\sum_{j=1}^{n}\left(\frac{\partial}{\partial\xi_j}p_\lambda(x,\sqrt{-1}\xi)\frac{\partial}{\partial x_j}q_\mu(x,\sqrt{-1}\xi)\right.$$
$$\left.-\frac{\partial}{\partial\xi_j}q_\mu(x,\sqrt{-1}\xi)\frac{\partial}{\partial x_j}p_\lambda(x,\sqrt{-1}\xi)\right).$$

Note that $\mathscr{E}(\lambda)$ is a sheaf of non-commutative rings and that $\mathcal{O}(\lambda)$ is a sheaf of commutative rings. In what follows, we will show how to capture the structures of microdifferential equations via information from the commutative object $\mathcal{O}(\lambda)$. Before this investigation, we will prove the invariance of the class of microdifferential operators under a coordinate transformation.

Let $x=(x_1,\ldots,x_n)$ and $\tilde{x}=(\tilde{x}_1,\ldots,\tilde{x}_n)$ be coordinate systems of X, and let $x=\varphi(\tilde{x})$ and $\tilde{x}=\tilde{\varphi}(x)$. Also let $P(x,D_x)=\sum_j P_{\lambda+j}(x,D_x)$ be a microdifferential operator defined in a neighborhood of $(x_0,\sqrt{-1}\xi_0\infty)$, and denote the point corresponding to $(x_0,\sqrt{-1}\xi_0)$ by $(\tilde{x}_0,\sqrt{-1}\tilde{\xi}_0)$ on $\sqrt{-1}T^*X$. Then we have

$$(\tilde{\xi})_j=\sum_k\frac{\partial x_k}{\partial\tilde{x}_j}(\xi)_k.$$

From the definition,

$$P(x,D_x)\delta(x-y)=\frac{1}{(2\pi)^n}\int K(x,y,\xi)\omega(\xi),$$

where $K(x,y,\xi)=\sum_j P_{\lambda+j}(x,\sqrt{-1}\xi)\Phi_{\lambda+j}(\sqrt{-1}(\langle x-y,\xi\rangle+\sqrt{-1}0))$. Denoting $|\det\partial x/\partial\tilde{x}|$ by $j(\tilde{x})$, we have

$$\delta(x-y)=j(\tilde{y})\delta(\tilde{x}-\tilde{y}),$$

where $\tilde{x}=\varphi(x)$ and $\tilde{y}=\varphi(y)$. Then

$$P(x,D_x)\delta(x-y)=P(x,D_x)\delta(\tilde{x}-\tilde{y})j(\tilde{y})$$

holds. Therefore, we obtain

$$\begin{aligned}P(x,D_x)\delta(\tilde{\varphi}(x)-\tilde{\varphi}(y))&=j(\tilde{y})^{-1}P(x,D_x)\delta(x-y)\\&=\frac{1}{(2\pi)^n}\int j(\tilde{\varphi}(y))^{-1}\sum_j p_{\lambda+j}(x,\sqrt{-1}\xi)\\&\qquad\cdot\Phi_{\lambda+j}(\sqrt{-1}(\langle x-y,\xi\rangle)+\sqrt{-1}0)\omega(\xi).\qquad(4.1.17)\end{aligned}$$

On the other hand, we can write

$$\langle x-y,\xi\rangle=\langle\varphi(\tilde{x})-\varphi(\tilde{y}),\tilde{\Theta}(\tilde{x},\tilde{y},\xi)\rangle.$$

where $\tilde{\Theta}(\tilde{x}, \tilde{y}, \xi)$ is an n-vector such that

$$\tilde{\Theta}_k(\tilde{x}, \tilde{y}, \xi) = \sum_j \frac{\partial x_k}{\partial \tilde{x}_j} \xi_j.$$

Hence, one can solve $\tilde{\xi} = \tilde{\Theta}(\tilde{x}, \tilde{y}, \xi)$ for ξ, i.e. $\xi = \Theta(\tilde{x}, \tilde{y}, \tilde{\xi})$. Then one has

$$\Theta_j(\tilde{x}, \tilde{y}, \tilde{\xi}) = \sum_k \frac{\partial \tilde{x}_j}{\partial x_k} \tilde{\xi}_k.$$

Since

$$\omega(\tilde{\xi}) = \left|\frac{\partial \tilde{\xi}}{\partial \xi}\right| \omega(\xi)$$

holds for the coordinate transformation $\tilde{\xi} = \tilde{\Theta}(\tilde{x}, \tilde{y}, \xi)$, one obtains from (4.1.17)

$$P(x, D_x)\delta(\tilde{x} - \tilde{y}) = \frac{1}{(2\pi)^n} \int j(\tilde{y})^{-1} \Big(\sum_j p_{\lambda+j}(x, \sqrt{-1}\xi) \cdot \Phi_{\lambda+j}(\sqrt{-1}(\langle \tilde{x} - \tilde{y}, \tilde{\xi}\rangle + \sqrt{-1}0)) \Big) |\partial\tilde{\xi}/\partial\xi|^{-1} \omega(\tilde{\xi}), \tag{4.1.18}$$

where $\xi = \Theta(\tilde{x}, \tilde{y}, \tilde{\xi})$ and $x = \varphi(\tilde{x})$.

Let

$$r_{\lambda+j}(\tilde{x}, \tilde{y}, \sqrt{-1}\tilde{\xi}) = j(\tilde{y})^{-1} |\partial\tilde{\xi}/\partial\xi|^{-1} p_{\lambda+j}(x, \sqrt{-1}\xi).$$

Then one obtains

$$P(x, D_x)\delta(\tilde{x} - \tilde{y}) = \frac{1}{(2\pi)^n} \int \Big(\sum_j r_{\lambda+j}(\tilde{x}, \tilde{y}, \sqrt{-1}\tilde{\xi}) \Phi_{\lambda+j}(\sqrt{-1}\langle \tilde{x} - \tilde{y}, \tilde{\xi}\rangle + \sqrt{-1}0) \Big) \omega(\tilde{\xi}).$$

If one can show that the above integration can be written as

$$\frac{1}{(2\pi)^n} \int \Big(\sum_j \tilde{p}_{\lambda+j}(\tilde{x}, \sqrt{-1}\tilde{\xi}) \Phi_{\lambda+j}(\sqrt{-1}(\langle \tilde{x} - \tilde{y}, \tilde{\xi}\rangle) \Big) \omega(\tilde{\xi}),$$

then it follows that $P(x, D_x)$ becomes $\sum_j \tilde{p}_{\lambda+j}(\tilde{x}, D_{\tilde{x}})$ by the coordinate transformation. Notice that

$$r_{\lambda+j}(\tilde{x}, \tilde{x}, \sqrt{-1}\tilde{\xi}) = p_{\lambda+j}(x, \sqrt{-1}\xi)$$

holds.

Theorem 4.1.3. *For $j \in \mathbf{Z}$, let $a_{\lambda+j}(x, y, \sqrt{-1}\xi)$ satisfy the growth conditions* (4.1.2) *and* (4.1.3), *and let*

$$p_{\lambda+j}(x, \sqrt{-1}\xi) = \sum_{j=k-|\alpha|} \frac{1}{\alpha!} D_y^\alpha \left(\frac{1}{\sqrt{-1}} D_\xi\right)^\alpha a_{\lambda+k}(x, y, \sqrt{-1}\xi)\big|_{x=y}. \quad (4.1.19)$$

Then one has (i) *and* (ii):

(i) $$\int \sum_j a_{\lambda+j}(x, y, \sqrt{-1}\xi)\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\omega(\xi)$$
$$= \int \sum_j p_{\lambda+j}(x, \sqrt{-1}\xi)\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\omega(\xi).$$

(ii) *Furthermore, if $a_{\lambda+j} = 0$ for $j > 0$, then $p_\lambda(x, \sqrt{-1}\xi) = a_\lambda(x, x, \sqrt{-1}\xi)$.*

Proof. First let $c^\alpha_{\lambda+j}(x, \sqrt{-1}\xi) = D^\alpha_y a_{\lambda+j}(x, y, \sqrt{-1}\xi)|_{x=y}$, and write $a_{\lambda+j}(x, y, \sqrt{-1}\xi)$ as the Taylor expansion

$$a_{\lambda+j}(x, y, \sqrt{-1}\xi) = \sum_\alpha \frac{1}{\alpha!}(y-x)^\alpha c^\alpha_{\lambda+j}(x, \sqrt{-1}\xi).$$

Then one obtains

$$\int\left(\sum_j a_{\lambda+j}(x, y, \sqrt{-1}\xi)\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\right)\omega(\xi)$$
$$= \int\left(\sum_{j,\alpha} \frac{1}{\alpha!} c^\alpha_{\lambda+j}(x, \sqrt{-1}\xi)(y-x)^\alpha\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\right)\omega(\xi)$$
$$= \int\left(\sum_{j,\alpha} \frac{1}{\alpha!} c^\alpha_{\lambda+j}(x, \sqrt{-1}\xi)(\sqrt{-1}D_\xi)^\alpha\right.$$
$$\left.\cdot\,\Phi_{\lambda+j+n-|\alpha|}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\right)\omega(\xi)$$
$$= \int\left(\sum_{j,\alpha} \frac{1}{\alpha!} c^\alpha_{\lambda+j+|\alpha|}(x, \sqrt{-1}\xi)(\sqrt{-1}D_\xi)^\alpha\right.$$
$$\left.\cdot\,\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0))\right)\omega(\xi).$$

Through the integral by parts with respect to ξ, as in the proof of Theorem 4.1.1, the above integral is equal to

$$\int\sum_\alpha \frac{1}{\alpha!}((-\sqrt{-1}D_\xi)^\alpha c^\alpha_{\lambda+j+|\alpha|}(x, \sqrt{-1}\xi)$$
$$\cdot\,\Phi_{\lambda+j+n}(\sqrt{-1}(\langle x-y, \xi\rangle + \sqrt{-1}0)))\omega(\xi).$$

Hence, the class of microdifferential operators is invariant under a coordinate transformation, which completes the proof.

Suppose $P(x, D_x) = \sum_j p_{\lambda+j}(x, D_x)$ belongs to $\mathscr{E}(\lambda)$; i.e. $p_{\lambda+j} = 0$ for $j > 0$. Then the operator $\tilde{p}(\tilde{x}, D_x)$, obtained from $P(x, D_x)$ by a coordinate transformation $x = \varphi(\tilde{x})$, is an element of $\mathscr{E}(\lambda)$ by Theorem 4.1.3 (ii). Furthermore,

$$\tilde{p}_\lambda(\tilde{x}, \sqrt{-1}\tilde{\xi}) = r_\lambda(\tilde{x}, \tilde{x}, \sqrt{-1}\tilde{\xi}) = p_\lambda(x, \sqrt{-1}\xi)$$

holds. This implies a sheaf homomorphism

$$\sigma_\lambda : \mathscr{E}(\lambda) \to \mathcal{O}(\lambda)$$

defined by $P = \sum_{j \leqq 0} p_{\lambda+j} \mapsto p_\lambda$ is invariant under a coordinate transformation.

We will describe the fundamental properties of a microdifferential operator, which are needed for our main goal in this treatise—the structure theory of microdifferential equations. In the analysis of microdifferential operators, after obtaining a formal solution $\sum_j p_{\lambda+j}(x, D_x)$, it is always a troublesome task to ensure that each homogeneous part $p_{\lambda+j}(x, \xi)$ satisfies the growth conditions (4.1.6a) and (4.1.6b). A relatively convenient method introduced in Boutet de Monvel and Krée [1] is that of a formal norm $N_l^\omega(P; t)$. We will assume $\lambda = 0$ for the sake of simplicity in the following discussion.

Definition 4.1.2. *Let $(x^0, \sqrt{-1}\xi^0\infty)$ be a point on $\sqrt{-1}S^*M$, let ω be a complex neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$, and let* dim $M = n$. *Suppose that $p_j(z, \zeta)$ is a holomorphic function in ω for a microdifferential operator $P(x, D_x) = \sum_{j=-\infty}^{l} p_j(x, D_x)$, where the homogeneous degree of $p_j(x, \xi)$ in ξ is j. Then the formal norm $N_l^\omega(P; t)$ of $P(x, D_x)$ in ω is a formal sum with respect to t, defined as*

$$\sum_{k,\alpha,\beta} \frac{2(2n)^{-k}k!}{(|\alpha| + k)!(|\beta| + k)!} \sup_\omega |D_z^\alpha D_\zeta^\beta p_{l-k}(z, \zeta)| t^{2k+|\alpha+\beta|}. \tag{4.1.20}$$

Proposition 4.1.6. *If $N_l^\omega(P; \epsilon) < \infty$ holds for any $\epsilon > 0$, then the growth condition* (4.1.6a) *is satisfied. If $\{p_j(z, \zeta)\}_{-\infty < j \leqq l}$ satisfies the condition* (4.1.6a), *then $N_l^{\omega'}(P; \epsilon) < \infty$ for some $\omega' \subset \omega$ and $\epsilon > 0$.*

The proof is almost obvious. In order to prove the latter half of the proposition, use $\sup_\omega |p_j|$ to estimate the derivatives of p_j, via Cauchy's integral formula.

Remark. When there is no fear of confusion, we simply write $N_l(P; t)$ without the superscript ω.

The usefulness of the formal norm is mainly due to the following theorem.

Theorem 4.1.5 (Boutet de Monvel and Krée [1]). *Let*

$$P_1 = \sum_{j=-\infty}^{l_1} p_j^{(1)} \quad \text{and} \quad P_2 = \sum_{j=-\infty}^{l_2} p_j^{(2)},$$

where $p_j^{(1)}$ and $p_j^{(2)}$ are defined in ω. Then

$$N_{l_1+l_2}^{\omega}(P_1 P_2; t) \ll N_{l_1}^{\omega}(P_1; t) N_{l_2}^{\omega}(P_2; t) \tag{4.1.21}$$

holds, where for formal power series $A(t)$ and $B(t)$ the notation $A(t) \ll B(t)$ means that $B(t)$ is a majorant series of $A(t)$.

For a proof, see Lemma 1.2 in Boutet de Monvel and Krée [1].

The next theorem and its corollaries will show how powerful Theorem 4.1.5 is.

Theorem 4.1.6. *Let $f(s) = \sum_{s=0}^{\infty} f_k s^k$ be a holomorphic function defined in a neighborhood of $s = 0$. If a microdifferential operator $P(x, D_x)$ of order at most zero is defined in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty) \in \sqrt{-1}S^*M$, and if $P(x, D_x)$ satisfies*

$$\sigma_0(P)(x^0, \sqrt{-1}\xi^0) = 0, \tag{4.1.22}$$

then $f(P) = \sum_{k=0}^{\infty} f_k P^k$ is a microdifferential operator of order at most zero defined in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$.

Proof. From Theorem 4.1.5, one has

$$N_0^{\omega}(f(P); t) = \sum_{k=0}^{\infty} f_k N_0^{\omega}(P^k; t) \ll \sum_{k=0}^{\infty} f_k N_0^{\omega}(P; t)^k.$$

Then (4.1.22) implies that, for any $\delta > 0$, there can be found ω and $\epsilon > 0$ such that $N_0^{\omega}(P; \epsilon) < \delta$. Hence, from Proposition 4.1.6, the proof follows.

Corollary. *Let $P(x, D_x)$ be a microdifferential operator of finite order. Since the principal symbol $\sigma_m(P)(x, \sqrt{-1}\xi)$ does not vanish in $\Omega \subset \sqrt{-1}S^*M$, there exists a microdifferential operator E defined in Ω such that*

$$PE = EP = I \tag{4.1.23}$$

holds.

Proof. It is sufficient to prove the existence of E at each point in Ω, since such an E satisfying (4.1.23) is also unique locally. Let $q_{-m}(x, \sqrt{-1}\xi) = p_m(x, \sqrt{-1}\xi)^{-1}$, and let

$$R(x, D_x) = I - P(x, D_x)q_{-m}(x, D_x). \tag{4.1.24}$$

Then $R(x, D_x)$ satisfies the condition of Theorem 4.1.6. Hence, $\sum_{k=0}^{\infty} R(x, D_x)^k$ determines locally a microdifferential operator $S(x, D_x)$. Hence, $(I - R)S = I$ holds by the definition of S. This implies that $q_{-m}(x, D_x)S(x, D_x)$ is a right inverse of P. In a similar manner, one can locally obtain a left inverse of P as a microdifferential operator. Since the associative law holds for microdifferential operators, the left and right inverses coincide. Then let $q_{-m}S = E$.

Remark. The following assertion can be proved in the same manner.

Let $P(x, D_x) = (P_{ij}(x, D_x))_{1 \leqq i,j \leqq N}$ be an $N \times N$ matrix of microdifferential operators, and let $m_j = \max_{1 \leqq i \leqq N} \operatorname{ord} P_{ij}$. If $\det(\sigma_{m_j}(P_{ij}))$ does not vanish in $\Omega \subset \sqrt{-1}S^*M$, then there exists an $N \times N$ matrix E of microdifferential operators such that

$$PE = EP = I \tag{4.1.25}$$

holds.

Note. The above assertion can be sharpened. But we will not discuss this further, since in what follows we will not need more than what is stated in the above.

Microlocally speaking, the structure of $Pu = 0$ is trivial outside $V_{\mathbf{R}} = \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \,|\, \sigma_m(P)(x, \sqrt{-1}\xi) = 0\}$ by the above corollary. Therefore, our focus is on the structure of a microdifferential equation where $\sigma_m(P) = 0$. As we will show in §3 of this chapter, the structures can be described with extreme clarity, even for overdetermined systems. Schematically speaking, we can treat microdifferential equations as if they were algebraic equations. In order to carry this out, some preparative topics are in order. We begin with the analogues of the Späth theorem and the Weierstrass preparation theorem, which are fundamental for the local theory of holomorphic functions of several complex variables. We will not give proofs for these theorems in this book.

Theorem 4.1.7 (Späth Theorem I for Microdifferential Operators). *Let $P(x, D_x)$ be a microdifferential operator of order m such that*

$$(\sigma_m(P)(x, \sqrt{-1}\xi)/\xi_n^p)|_{(x;\xi)=(0;1,0,\dots,0,\xi_n)} \tag{4.1.26}$$

is a holomorphic function of ξ_n and can never be zero in a neighborhood of $\xi_n = 0$. Then, for an arbitrary microdifferential operator $S(x, D_x)$, there *can be found microdifferential operators $Q(x, D_x)$ and $R(x, D_x)$ so that the following* (4.1.27) *and division theorem* (4.1.28) *hold in a neighborhood ω of $(x; \sqrt{-1}\xi) = (0; \sqrt{-1}(1, 0, \ldots, 0))$:*

$$S(x, D_x) = Q(x, D_x)P(x, D_x) + R(x, D_x). \tag{4.1.27}$$

$\underbrace{[x_n, [\ldots, [x_n}_{p}, R] \ldots] = 0$; *i.e. $R(x, D_x)$ can be written as*

$$\sum_{j=0}^{p-1} R^{(j)}(x, D')D_n^j, \text{ where } D' = (D_1, \ldots, D_{n-1}). \tag{4.1.28}$$

Note also that Q and R are determined uniquely, and that Q and R are of finite order when S is of finite order.

Theorem 4.1.8 (Späth Theorem II for Microdifferential Operator). *Let $P(x, D_x)$ satisfy* (4.1.29), *instead of* (4.1.26):

$$(\sigma_m(P)(x, \sqrt{-1}\xi)/x_1^p)|_{(x;\xi)=(x_1,0;\xi^\circ)} \text{ is a holomorphic function of } x_1 \text{ and can never be zero in a neighborhood of } x_1 = 0. \tag{4.1.29}$$

Then, for an arbitrary microdifferential operator $S(x, D_x)$, one has a unique presentation of division algorithm as follows:

$$S(x, D_x) = Q(x, D_x)P(x, D_x) + R(x, D_x). \tag{4.1.30}$$

$\underbrace{[D_1, [\ldots [D_1}_{p}, R] \ldots] = 0$; *i.e. $R(x, D_x)$ can be written as*

$$\sum_{j=0}^{p-1} x_1^j R^{(j)}(x', D), \text{ where } x' = (x_2, \ldots, x_n). \tag{4.1.31}$$

See SKK [1], Chap. II, §2.2, for proofs.

As in the theory of several complex variables, a Weierstrass-type division theorem is obtained as a corollary of Theorem 4.1.7.

Theorem 4.1.9. *Let $P(x, D_x)$ be a microdifferential operator of order m satisfying the condition* (4.1.26). *Then $P(x, D_x)$ can be decomposed, in a neighborhood ω of $(x, \sqrt{-1}\xi) = (0; \sqrt{-1}(1, 0, \ldots, 0))$, as*

$$P(x, D_x) = Q(x, D_x)W(x, D_x) \text{ uniquely, where } Q(x, D_x) \text{ is invertible in } \omega \text{ and } W \text{ has the following form:} \tag{4.1.32}$$

$$W(x, D_x) = D_n^p + \sum_{j=0}^{p-1} W^{(j)}(x, D')D_n^j,\ D' = (D_1, \ldots, D_{n-1}), \text{ and the order of } W^{(j)}(x, D') \text{ is at most } p-j \text{ such that } \sigma_{p-j}(W^{(j)})(0, \sqrt{-1}(1, 0, \ldots, 0)) = 0 \text{ holds.} \tag{4.1.33}$$

As with the Späth theorem for the local theory of holomorphic functions of several variables, the above theorems provide a fundamental method to normalize microdifferential operators, as one pleases, in the local study of microdifferential equations.

§2. Quantized Contact Transformation for Microdifferential Operators

We have shown in the previous section that the notion of a microdifferential operator is obtained by the microlocalization of a differential operator on $\sqrt{-1}S^*M$. On the other hand, we recognized (in §6 of Chapter III) a bicharacteristic strip as a "carrier" of singularities of solutions of a differential equation, where a glimpse of the interplay contact geometry and differential equations was observed. In this section, we will carry out this program (i.e. "contact transformations of microdifferential operators"). It was probably Maslov [1] in which an attempt was first made to formulate the above idea. But only after Egorov [1] did mathematicians begin to approach this problem—Maslov [1] was not well known outside the USSR and was somewhat lacking in mathematical rigor (perhaps the book was written for physicists). Nowadays this method is crucial for the study of linear differential equations.

Let us begin with some basic notions from contact geometry. Let a complex manifold X be complex-analytic. In the case when X is a real manifold, then T^*X and other notions should be considered over real numbers $\mathbf{R}$; i.e. considering the actions by $\mathbf{R}_+^\times = \{t \in \mathbf{R} \mid t \gneqq 0$, instead of the actions by $\mathbf{C}^\times = \mathbf{C} - \{0\}$ for the complex category.

Definition 4.2.1. *Let T^*X be the cotangent bundle of a $(2n-1)$-dimensional manifold X. If a 1-dimensional sub-bundle L^* of T^*X satisfies the following condition* (4.2.1), *then (X, L) is said to be a contact structure on X, where L^* denotes the dual bundle of L:*

For a nowhere-vanishing local cross-section ω of L^*, $\omega \wedge (d\omega)^{n-1}$ is non-zero anywhere. (4.2.1)

Note. Since $(f\omega) \wedge (d(f\omega))^{n-1} = f^n\omega \wedge (d\omega)^{n-1}$ holds, the above condition is independent of the choice of ω.

Definition 4.2.2. *When ω satisfies* (4.2.1), *ω is called a canonical 1-form.*

Let us denote $L^* - \{\text{zero sections}\}$ by $\hat{X}$. Then $\mathbf{C}^\times = \mathbf{C} - \{0\}$ acts on $\hat{X}$ as a $\mathbf{C}^\times$-principal bundle over X. One can canonically define a 1-form θ on $\hat{X}$ by $s^*(\theta) = \omega$ for a cross-section s of $\hat{X}$. Then, $(d\theta)^n$ is nowhere zero. $\hat{X}$ is called the symplectic manifold associated with X, and θ is said to be a homogeneous canonical 1-form. Let $f(z)$ be a function defined on $\hat{X}$ (i.e. defined in a neighborhood in $\hat{X}$) satisfying $f(az) = a^m f(z)$ for $a \in \mathbf{C}^\times$ and $z \in \hat{X}$. Then, $f(z)$ is said to be a homogeneous function of degree m, in which case f is said also to be a homogeneous function of degree m on X.

The sheaf of homogeneous functions of degree m on X is isomorphic to $L^{\otimes m}$.

The above definitions are more explicit in the case where X is the cotangent projective bundle P^*Y of an n-dimensional manifold Y; then $\hat{X} = T^*Y - Y$. The classical theorem of Darboux states that a canonical 1-form ω can be written as $\omega = dx_n - p_1\,dx_1 - \cdots - p_{n-1}\,dx_{n-1}$ for a local coordinate system $(x_1, \ldots, x_n, p_1, \ldots, p_{n-1})$, where $(x_1, \ldots, x_n, p_1, \ldots, p_{n-1})$ are said to be canonical coordinates. Note also that in the above case a coordinate system $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$ of $\hat{X}$ can be taken to satisfy $p_j = -\eta_j/\eta_n$ for $j = 1, \ldots, n-1$ and $\theta = \eta_n\omega = \eta_1\,dx_1 + \cdots + \eta_n\,dx_n$, where $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$ are called canonical homogeneous coordinates.

The Poisson bracket, introduced in §5, Chapter III, can be defined intrinsically through a contact structure.

Definition 4.2.3. *The Poisson bracket $\{f, g\}$ of functions f and g on a symplectic manifold $\hat{X}$ of dimension $2n$ is defined as follows:*

$$n\,df \wedge dg \wedge (d\theta)^{n-1} = \{f, g\}(d\theta)^n. \tag{4.2.2}$$

Exercise. Let $(x_1, \ldots, x_n, \eta_1, \ldots, \eta_n)$ be canonical homogeneous coordinates. Then show that the Poisson bracket $\{f, g\}$ can be written as

$$\sum_{j=1}^{n}\left(\frac{\partial f}{\partial \eta_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \eta_j}\right). \tag{4.2.2'}$$

The Poisson bracket $\{f, g\}$ satisfies the following relations, as one can easily see from (4.2.2′).

Theorem 4.2.1.

$$\{f, g\} = -\{g, f\}. \tag{4.2.3}$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \tag{4.2.4}$$

If the homogeneous degree of f is l, and one of g is m, then the homogeneous degree of $\{f, g\}$ is $l + m - 1$. (4.2.5)

Connecting with the Poisson bracket, the Hamiltonian vector field

$$H_f = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \eta_j}\right) \tag{4.2.6}$$

is also important. A bicharacteristic strip, introduced in Definition 3.5.1, can be rephrased as the integral curve of H_{p_m}, which is important for the recognition of the notion of a bicharacteristic strip as a concept in contact geometry. Notice also that

$$H_{\{f,g\}} = [H_f, H_g]\ (\underset{\text{def}}{=} H_f H_g - H_g H_f) \tag{4.2.7}$$

holds from the definitions.

To summarize the above, a canonical 1-form ω determines a contact structure on X and, furthermore, a homogeneous canonical 1-form θ_X on $\hat{X}$ is naturally defined by ω. Conversely, if a $\mathbf{C}^\times$-principal bundle S is given on X such that a 1-form θ satisfies the following conditions (4.2.8) and (4.2.9), then we will show that a contact structure can be defined on X:

$$(d\theta)^n \text{ does not vanish anywhere on } S. \tag{4.2.8}$$

The 1-form θ is homogeneous; i.e. for $(c, x) \in \mathbf{C}^\times \times S, \theta(cx) = c^r\theta(x)$ holds for some integer r. (4.2.9)

For an arbitrary cross-section s of S, one can take $s^*\theta$ for a canonical 1-form ω. If $r = 1$, then S and $\hat{X}$ coincide. For the general r, $\hat{X} = S^{\otimes r}$ holds. That is, for a canonical homogeneous 1-form θ_X on $\hat{X}$, there exists a map F from the principal bundle S to the principal bundle $\hat{X}$ such that $F(cx) = c^r F(x)$ and $\theta = F^*\theta_X$ hold. We will give an example of this construction of a contact structure.

Example 4.2.1. Let V be a symplectic linear space of dimension $2n$; i.e. there is given a skew-symmetric non-degenerate quadratic form $E(v_1, v_2)$ on a linear space V. Define $X = P(V) = (V - \{0\})/\mathbf{C}^\times$. Then define a 1-form θ on V as $E(v, dv)/2$. Since E is non-degenerate, $(d\theta)^n$ is non-degenerate. For $(c, v) \in \mathbf{C}^\times \times (V - \{0\})$, $\theta(cv) = c^2\theta(v)$ holds. Hence, $P(V)$ has a contact structure induced from θ.

Let us recall some of the basic notions pertaining to a contact structure.

Definition 4.2.4. *Let X and Y be contact manifolds of the same dimension, and let φ be a map from X to Y. If, for an arbitrary canonical 1-form ω_Y on Y, $\varphi^*\omega_Y$ is a canonical 1-form on X, then φ is said to be a contact transformation.*

Example 4.2.2. Let M and N be open subsets of $\mathbf{C}^n$. Consider $\Omega(x, y)$, $x \in M$, $y \in N$, a holomorphic function defined on $M \times N$. Suppose that $\Omega(x, y)$ satisfies the following conditions (4.2.10) and (4.2.11):

The hypersurface $H = \{(x, y) \in M \times N \mid \Omega(x, y) = 0\}$ is non-singular; i.e. $\operatorname{grad}_{(x,y)}\Omega(x, y) \neq 0$ holds on H. (4.2.10)

$$\text{On } H, \det \begin{bmatrix} 0 & \dfrac{\partial\Omega}{\partial y_1} & \cdots & \dfrac{\partial\Omega}{\partial y_n} \\ \dfrac{\partial\Omega}{\partial x_1} & \dfrac{\partial^2\Omega}{\partial y_1\,\partial x_1} & \cdots & \dfrac{\partial^2\Omega}{\partial y_n\,\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\Omega}{\partial x_n} & \dfrac{\partial^2\Omega}{\partial y_1\,\partial x_n} & \cdots & \dfrac{\partial^2\Omega}{\partial y_n\,\partial x_n} \end{bmatrix} \neq 0 \text{ holds.} \tag{4.2.11}$$

Then one can define a local isomorphism from P^*M to P^*N via

$$P_H^*(M \times N) = \{(x, y; \xi, \eta) \in P^*(M \times N) | \Omega(x, y) = 0 \text{ and } (\xi, \eta) = c \operatorname{grad}_{(x,y)} \Omega(x, y) \text{ for } c \neq 0\},$$

as follows. Implicit function theorem implies that the projection $P_H^*(M \times N) \xrightarrow{p} P^*M$ is a local isomorphism. Note that

$$\det\begin{pmatrix} 0 & d_y\Omega \\ d_x\Omega & cd_xd_y\Omega \end{pmatrix} = c^{n-2} \det\begin{pmatrix} 0 & d_y\Omega \\ d_x\Omega & d_xd_y\Omega \end{pmatrix}$$

holds. Similarly, one has a local isomorphism $P_H^*(M \times N) \xrightarrow{q} P^*N$. Hence, p and q induce local isomorphisms $p \circ q^{-1}: P^*N \to P^*M$ and $q \circ p^{-1}: P^*M \to P^*N$. Then, on H, one has $d\Omega = \sum_{j=1}^{n} (\partial\Omega/\partial x_j)\, dx_j + \sum_{j=1}^{n} (\partial\Omega/\partial y_j)\, dy_j \neq 0$. Hence, $p \circ q^{-1}$ and $q \circ p^{-1}$ are clearly contact transformations.

Definition 4.2.5. *The contact transformation obtained as above is called a contact transformation having Ω as a generating function.*

Remark. A classical result states that an arbitrary contact transformation can be obtained by the consecutive use of two contact transformations having generating functions. In this sense, a contact transformation having a generating function is a "generic" contact transformation.

Example 4.2.3. The well-known classical Legendre transformation is a contact transformation where the generating function $\Omega(x, y) = x_n - y_n + \sum_{j=1}^{n-1} x_i y_i$. Then the explicit correspondence between (x, ξ) and (y, η) is given by

$$\left.\begin{aligned} x_j &= -\eta_j\eta_n^{-1} && \text{for } j < n \\ x_n &= \langle y, \eta\rangle\eta_n^{-1} && \\ \xi_j &= y_j\eta_n && \text{for } j < n \\ \xi_n &= \eta_n && \end{aligned}\right\} \tag{4.2.12a}$$

and

$$\left.\begin{aligned} y_j &= \xi_j\xi_n^{-1} && \text{for } j < n \\ y_n &= \langle x, \xi\rangle\xi_n^{-1} && \\ \eta_j &= -x_j\xi_n && \text{for } j < n \\ \eta_n &= \xi_n && \end{aligned}\right\} \tag{4.2.12b}$$

Example 4.2.4. A contact structure was defined on $P(V)$ via a skew-symmetric non-degenerate quadratic form E on V in Example 4.2.1. Let

φ be a linear transformation of V with the property $E(v_1, v_2) = E(\varphi v_1, \varphi v_2)$; i.e. it is a symplectic transformation of (V, E). Then a 1-form θ defined on V is invariant under φ. Hence, φ defines a contact transformation of $X = P(V)$.

Definition 4.2.6. *An analytic subset V of a contact manifold X is said to be involutory if the following condition* (4.2.13) *is satisfied:*

$$\text{If } f|_V = g|_V = 0, \text{ then } \{f, g\}|_V = 0. \tag{4.2.13}$$

Remark 1. It is known that the characteristic variety of an arbitrary system of microdifferential equations is involutory (see SKK [1], Chap. II, Theorem 5.3.2). This is extremely fundamental and important for the study of linear partial differential equations. Although the proof given in SKK [1] is transcendental, an algebraic proof of this theorem is available (see Kashiwara [2]).

Remark 2. An involutory subset V of a $(2n\text{-}1)$-dimensional contact manifold X has a codimension which is always less than or equal to n.

Definition 4.2.7. *An involutory (non-singular) submanifold V of X is said to be regular if $\omega|_V \neq 0$ holds everywhere on V.*

Definition 4.2.8. *An involutory analytic subset V of a* $(2n\text{-}1)$*-dimensional contact manifold X is said to be Lagrangian if the codimension of V in X is n.*

Remark 1. A Lagrangian manifold can never be regular in the sense of Definition 4.2.7. It is known that a submanifold Y of codimension n in X is Lagrangian if and only if $\omega|_Y = 0$ holds. From this fact, one can say that a Lagrangian subset is rather exceptional as an involutory subset. Conversely, one should say, this promises that analysis connected with a Lagrangian set is fruitful. Though many interesting topics are in progress, we will not discuss these here (see *Publ. RIMS, Kyoto Univ.*, vol. 12, suppl. [1977], which may be useful for these topics).

Note. A system of microdifferential equations is said to be a holonomic system or a maximally overdetermined system if its characteristic variety is Lagrangian. One of the most fundamental properties of a holonomic system is the finite dimensionality of the (microfunction) solution space; see Kashiwara [1] and Kashiwara and Kawai [1]. Hence one may hope that holonomic systems will be used effectively as a governing principle of functions of several variables, as ordinary differential equations do in one-variable cases. This viewpoint was advocated by M. Sato as early as 1960, in connection with the question of how to characterize fundamental solutions for the Cauchy problem.

Remark 2. By the Jacobi theory, any regular involutory manifold can be expressed as $p_1 = \cdots = p_r = 0$ for some canonical coordinate system (x, p).

Using these geometric preparations, we will quantize a contact transformation, i.e. "a transformation of a microdifferential operator compatible with a contact transformation." By the remark following Definition 4.2.5, one can restrict oneself to consider the case of a contact transformation having a generating function. For the sake of simplicity and convenience, our treatment is within the real analytic category, rather than the complex analytic one.

Theorem 4.2.2. *Let M and N be real analytic manifolds of dimension n. Assume that a real-valued real analytic function $\Omega(x, y)$ defined on $M \times N$ satisfies the conditions* (4.2.10) *and* (4.2.11). *Then, for an arbitrary microdifferential operator $P(x, D_x)$, a microdifferential operator $Q(y, D_y)$ is uniquely determined such that*

$$\int P(x, D_x)\delta(\Omega(x, y))u(y)\,dy = \int \delta(\Omega(x, y))Q(y, D_y)u(y)\,dy \tag{4.2.14}$$

holds for any microfunction $u(y)$. Conversely, if Q is given, then P is uniquely determined so that (4.2.14) *holds. Furthermore, the order of Q is equal to that of P.*

That is, one has

$$\left.\begin{array}{ll} q \circ p^{-1}\mathscr{E}_M(m) \cong \mathscr{E}_N(m), & q \circ p^{-1}\mathscr{E}_M \cong \mathscr{E}_N \\ p \circ q^{-1}\mathscr{E}_N(m) \cong \mathscr{E}_M(m), & p \circ q^{-1}\mathscr{E}_M \cong \mathscr{E}_N, \end{array}\right\} \tag{4.2.15}$$

and the isomorphism

$$q \circ p^{-1}\mathscr{A}_{\sqrt{-1}S^*M} \cong \mathscr{A}_{\sqrt{-1}S^*_N}, \tag{4.2.16}$$

induced by (4.2.15), *is a contact transformation with the generating function Ω.*

Definition 4.2.8. *The isomorphism in* (4.2.15) *is called a quantized contact transformation (with the generating function Ω).*

Remark. Though we will carry out our proof for the sheaf $\mathscr{E}$ of microdifferential operators of finite order, the proof is valid for the sheaf $\mathscr{E}^\infty$ of microdifferential operators of infinite order.

Proof of Theorem 4.2.2. As p and q are defined via the conormal sphere bundle $S^*_H(M \times N)$ of H, we will prove the isomorphism in (4.2.15) via $\mathscr{E}_{M\times N}(m)\delta(\Omega(x, y))$. That is, one is to prove

$$p^{-1}\mathscr{E}_M(m) \cong \mathscr{E}_{M\times N}(m)\delta(\Omega(x, y)). \tag{4.2.17}$$

One of the motivations that led us to this method comes from the fact S.S. $\delta(\Omega(x, y)) = S^*_H(M \times N)$; actually, the characteristic variety of the holonomic system for $\delta(\Omega)$ is (the complexification of) $S_H^*(M \times N)$. (The concept of a holonomic system is used in this proof, though not explicitly. See the remark following the proof of this theorem.) The condition (4.2.10) implies that one can choose a basis $\mathscr{X}_j$, $1 \leqq j \leqq 2n$, for a vector field defined on $M \times N$, so as to satisfy

$$[\mathscr{X}_j, \mathscr{X}_k] = 0 \qquad (j, k = 1, \ldots, 2n) \tag{4.2.18}$$

and

$$\left.\begin{aligned} &\mathscr{X}_j\Omega = 0 \qquad (j = 1, \ldots, 2n-1) \\ &\mathscr{X}_{2n}\Omega = 1. \end{aligned}\right\} \tag{4.2.19}$$

Define $\mathscr{I} = \{P(x, y, D_x, D_y) \in \mathscr{E}_{M\times N} \mid P(x, y, D_x, D_y)\delta(\Omega(x, y)) = 0\}$. Then $\mathscr{I}$ is generated by $\mathscr{X}_1, \ldots, \mathscr{X}_{2n-1}$ and $(\Omega\mathscr{X}_{2n} - 1)$. By the definition of $\mathscr{I}$, one has

$$\mathscr{E}_{M\times N}\delta(\Omega(x, y)) = \mathscr{E}_{M\times N}/\mathscr{I}. \tag{4.2.20}$$

Notice that the common zeros of the principal symbols $\sigma(\mathscr{X}_j)$ and $\sigma(\Omega\mathscr{X}_{2n} - 1)$ of the generators for $\mathscr{I}$ coincide with $S_H^*(M \times N)$. We will denote a point on $S^*(M \times N)$ by $(x, y; \xi, \eta)$, where ξ and η are cotangent vectors corresponding to x and y respectively.

In order to prove the isomorphism in (4.2.17), it is sufficient to prove that, for an arbitrary $A(x, y, D_x, D_y) \in \mathscr{E}_{M\times N}(m)$, $Q_j \in \mathscr{E}_{M\times N}$ and $\tilde{A}(x, D_x) \in \mathscr{E}_M(m)$ can be chosen so that, determining $\tilde{A}$ uniquely,

$$A = \sum_{j=1}^{2n} Q_jR_j + \tilde{A} \tag{4.2.21}$$

may hold, where $R_j(x, y, D_x, D_y)$ are properly chosen generators for $\mathscr{I}$. Since H is non-singular, i.e. condition (4.2.10), then $S_H^*(M \times N)$ is also non-singular and, furthermore, $S_H^*(M \times N)$ is locally isomorphic to S^*M by (4.2.11). Hence, $S_H^*(M \times N)$ can be expressed as

$$\left.\begin{aligned} &\eta_j = p_j(x, \xi), \qquad j = 1, \ldots, n \\ &y_j = q_j(x, \xi), \qquad j = 1, \ldots, n, \end{aligned}\right\} \tag{4.2.22}$$

where $p_j(x, \xi)$ and $q_j(x, \xi)$ are analytic functions for $j = 1, \ldots, n$, and the homogeneous degree of p_j in ξ is 1 and that of q_j in ξ is 0. Therefore, one can let R_j be a microdifferential operator whose principal part is either $\eta_j - p_j$, for $j = 1, \ldots, n$, or $y_j - p_j$ for $j = n + 1, \ldots, 2n$ so that the decomposition in the form of (4.2.21) is possible for the principal part. We will consider next how to choose the terms of R_j of lower order. Since the question is local on $S_H^*(M \times N)$, one can choose analytic functions $a_{jk}(x, y, \xi, \eta)$ and $b_{jk}(x, y, \xi, \eta)$, where $j = 1, \ldots, n$ and $k = 1, \ldots, 2n$, so that one can have

$$\left.\begin{aligned} \eta_j - p_j &= \sum_{k=1}^{2n-1} a_{jk}\sigma(\mathscr{X}_k) + a_{j2n}\sigma(\Omega\mathscr{X}_{2n}), \qquad j = 1, \ldots, n \\ y_j - q_j &= \sum_{k=1}^{2n-1} b_{jk}\sigma(\mathscr{X}_k) + b_{j2n}\sigma(\Omega\mathscr{X}_{2n}), \qquad j = 1, \ldots, n, \end{aligned}\right\} \tag{4.2.23}$$

where the homogeneous degrees of a_{jk} and b_{jk} in (ξ, η) are 0 and -1 respectively. Furthermore, one may assume

$$\det \begin{bmatrix} a_{1,1}, & \ldots, & a_{1,2n} \\ \cdots & \cdots & \cdots \\ a_{n,1}, & \ldots, & a_{n,2n} \\ b_{1,1}, & \ldots, & b_{1,2n} \\ \cdots & \cdots & \cdots \\ b_{n,1}, & \ldots, & b_{n,2n} \end{bmatrix} \neq 0 \tag{4.2.24}$$

holds on $S_H^*(M \times N)$. Next we will find the generators R_j, $j = 1, \ldots, 2n$, for $\mathscr{I}$, satisfying (4.2.21), in the following form (4.2.25):

$$\left.\begin{aligned} R_j &= \sum_{k=1}^{2n-1} A_{jk}\mathscr{X}_k + A_{j2n}(\Omega\mathscr{X}_{2n} - 1) \qquad \text{for } j = 1, \ldots, n \\ S_j &= \sum_{k=1}^{2n-1} B_{jk}\mathscr{X}_k + B_{j2n}(\Omega\mathscr{X}_{2n} - 1) \qquad \text{for } j = 1, \ldots, n, \end{aligned}\right\} \tag{4.2.25}$$

where $S_j = R_{j-n}$, $j = n + 1, \ldots, 2n$, and A_{jk} and B_{jk} are microdifferential operators whose principal symbols are a_{jk} and b_{jk} respectively. By the remark following the corollary of Theorem 4.1.6, (4.2.24) implies that the $2n \times 2n$-matrix

$$\begin{bmatrix} A_{jk} \\ B_{jk} \end{bmatrix}_{\substack{1 \leqq j \leqq n \\ 1 \leqq k \leqq 2n}}$$

is invertible. Therefore, one obtains

$$\mathscr{I} = \sum_{j=1}^{n} \mathscr{E}_{M \times N} R_j + \sum_{j=1}^{n} \mathscr{E}_{M \times N} S_j \tag{4.2.26}$$

from $\mathscr{I} = \sum_{k=1}^{2n-1} \mathscr{E}_{M \times N}\mathscr{X}_k + \mathscr{E}_{M \times N}(\Omega\mathscr{X}_{2n} - 1)$. When one lets P_j and Q_j satisfy

$$\left.\begin{aligned} R_j(x, y, D_x, D_y) &= D_{y_j} - P_j(x, y, D_x, D_y) \qquad \text{for } j = 1, 2, \ldots, n \\ S_j(x, y, D_x, D_y) &= y_j - Q_j(x, y, D_x, D_y) \qquad \text{for } j = 1, 2, \ldots, n, \end{aligned}\right\} \tag{4.2.27}$$

then we will prove that R_j and S_j can be chosen to satisfy

$$\left.\begin{array}{ll}[y_i, P_j] = 0 & \text{for } i, j = 1, \ldots, n \\ [y_i, Q_j] = 0 & \text{for } i, j = 1, \ldots, n\end{array}\right\} \tag{4.2.28}$$

and

$$\left.\begin{array}{ll}[D_{y_i}, P_j] = 0 & \text{for } i, j = 1, \ldots, n \\ [D_{y_i}, Q_j] = 0 & \text{for } i, j = 1, \ldots, n.\end{array}\right\} \tag{4.2.29}$$

That is, one wishes to find R_j and S_j so that the following equations should hold:

$$\left.\begin{array}{ll}R_j = D_{y_j} - P_j(x, D_x) & \text{for } j = 1, \ldots, n \\ S_j = y_j - Q_j(x, D_y) & \text{for } j = 1, \ldots, n.\end{array}\right\} \tag{4.2.30}$$

If (4.2.30) holds, then Theorems 4.1.7 and 4.1.8 imply that there exists $\tilde{A}(x, D_x)$ satisfying (4.2.21), in which case the order of $\tilde{A}(x, D_x)$ is at most m, provided that the order of $A(x, y, D_x, D_y)$ is at most m.

We will show that one can choose R_j and S_j so that (4.2.28) and (4.2.29) hold. We will do this by induction on i. That is, we will prove the following $(4.2.28)_k$ and $(4.2.29)_k$ by induction on k:

$$\left.\begin{array}{ll}[y_i, P_j] = 0 & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n \\ [y_i, Q_j] = 0 & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n,\end{array}\right\} \tag{4.2.28$_k$}$$

and

$$\left.\begin{array}{ll}[D_{y_i}, P_j] = 0 & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n \\ [D_{y_i}, Q_j] = 0 & \text{for } i = 1, \ldots, k, \text{ and } j = 1, \ldots, n.\end{array}\right\} \tag{4.2.29$_k$}$$

For the case $k = 0$, the assertion is clearly true. Hence, let $k \geqq 1$ and assume that $(4.2.28)_i$ and $(4.2.29)_i$ hold for $i < k$. By the definition, the principal part of R_k is $\eta_k - p_k(x, \xi)$. Therefore, from Theorem 4.1.17, one can find G_j, H_j, $\tilde{P}_j$, $\tilde{Q}_j$ such that

$$\left.\begin{array}{ll}P_j = G_j R_k + \tilde{P}_j & \text{for } j = 1, \ldots, n \\ Q_j = H_j R_k + \tilde{Q}_j & \text{for } j = 1, \ldots, n\end{array}\right\} \tag{4.2.31}$$

and

$$[y_k, \tilde{P}_j] = [y_k, \tilde{Q}_j] = 0 \qquad \text{for } j = 1, \ldots, n \tag{4.2.32}$$

hold. By the inductive assumption,

$$[y_i, P_j] = [y_i, R_k] = [y_i, Q_j] = 0 \qquad \text{for } i < k \text{ and } j = 1, \ldots, n \tag{4.2.33}$$

holds. Then, the uniqueness part of Theorem 4.1.7 implies

$$[y_i, \tilde{P}_j] = [y_i, \tilde{Q}_j] = 0 \qquad \text{for } i \leqq k \text{ and } j = 1, \ldots, n. \tag{4.2.34}$$

From (4.2.31), one has $\sigma_1(P_k) = p_k(x, \xi)$. Hence, one obtains $\sigma_0(G_k) = 0$, $\sigma_1(\tilde{P}_k) = \sigma_1(P_k) = p_k$. Therefore, $(1 + G_k)$ is invertible by the corollary of Theorem 4.1.6. Then one has

$$\mathscr{I} = \sum_{j=1}^{n} \mathscr{E}_{M\times N}(D_{y_j} - \tilde{P}_j) + \sum_{j=1}^{n} \mathscr{E}_{M\times N}(y_j - \tilde{Q}_j). \tag{4.2.35}$$

A similar argument provides us with

$$[D_{y_k}, \tilde{P}_j] = [D_{y_k}, \tilde{Q}_j] = 0 \qquad \text{for } j = 1, \ldots, n. \tag{4.2.36}$$

Consequently, by replacing R_j amd S_j by $D_{y_j} - P_j$ and $y_j - Q_j$ respectively, $(4.2.28)_k$ and $(4.2.29)_k$ hold. Hence, the induction proceeds, concluding the proof of (4.2.30).

Lastly, we will prove that, for a given A, $\tilde{A}(x, D_x)$ in (4.2.21) is uniquely determined. It is sufficient to prove $A = 0$ for $A(x, D_x) \in \mathscr{I}$. Since R_j and S_j generate $\mathscr{I}$, for $A \in \mathscr{I}$ one has

$$A(x, D_x) = \sum_{j=1}^{n} G_jR_j + \sum_{j=1}^{n} H_jS_j \qquad \text{for } G_j, H_j \in \mathscr{E}_{M\times N}. \tag{4.2.37}$$

Again, by induction, we will prove that G_j and H_j can be chosen to satisfy

$$\begin{aligned} &A = \sum_{j=k}^{n} G_jR_j + \sum_{j=1}^{n} H_jS_j \qquad \text{and} \\ &[y_i, G_j] = [y_i, H_j] = 0 \qquad \text{for } i < k. \end{aligned} \tag{4.2.37$_k$}$$

As before, let $\tilde{G}_j$ and $\tilde{H}_j$ be the remainders when G_j and H_j are divided by R_k respectively. Then one may let

$$[y_k, \tilde{G}_j] = [y_k, \tilde{H}_j] = 0 \qquad \text{for } j = 1, \ldots, n. \tag{4.2.38}$$

Again by the uniqueness part of the division theorem,

$$[y_i, \tilde{G}_j] = [y_i, \tilde{H}_j] = 0 \qquad \text{for } i \leqq k \text{ and } j = 1, \ldots, n \tag{4.2.39}$$

holds. Therefore, one obtains the expression

$$A(x, D_x) - \sum_{j=k+1}^{n} \tilde{G}_jR_j - \sum_{j=1}^{n} \tilde{H}_jS_j = T_kR_k, \qquad T_k \in \mathscr{E}_{M\times N}. \tag{4.2.40}$$

Since one may suppose that R_j and S_j are as in (4.2.30), the left-hand side commutes with y_k. Hence the uniqueness of the division implies that $A - \sum_{j=k+1}^{n} \tilde{G}_jR_j - \sum_{j=1}^{n} \tilde{H}_jS_j = 0$. Thus, the induction proceeds. If one repeats the argument for $\sum_{j=1}^{n} \tilde{H}_jS_j$, consequently one obtains $(4.2.37)_n$. From the uniqueness of the division, one has $A = 0$.

Remark. We have obtained the isomorphism in (4.2.15) via the integral transformation (4.2.14) with the kernel function $\delta(\Omega(x, y))$. Note that $Y(\Omega(x, y))$ could have been taken as a kernel function instead of $\delta(\Omega(x, y))$; or, more generally, for a real-valued real analytic function $a(x, y)$, which can never be zero on H, one can use $a(x, y)(\Omega(x, y))_+^\lambda/\Gamma(\lambda + 1)$, $(\lambda \in \mathbb{C})$. Such arbitrariness in the choice of kernel functions indicates that the isomorphism in (4.2.15) is not determined uniquely for a given Ω; that is, an operator transformation cannot be uniquely determined by a geometric transformation. Actually, it is known that the isomorphism is unique up to an inner automorphism by an invertible microdifferential operator of order zero. This is the reason why the terminology "quantized contact transformation" was introduced. We also note that, more generally, we can use any non-degenerate section of a simple holonomic system with its characteristic variety being the conormal bundle of H, instead of $\delta(\Omega(x, y))$ etc., as a kernel function in (4.1.14). It might be better to discuss the problem emphasizing this viewpoint, partly because we can deal with general contact transformations (not necessarily with a kernel function) by that approach, and partly because it is preferable aesthetically. That, however, would require further preparation on the manipulation of holonomic systems; hence we content ourselves here with the above discussion.

Example 4.2.5. We will quantize the Legendre transformation. Let $\Omega = x_n - y_n + \sum_{j=1}^{n-1} x_i y_i$. Then, $(\partial/\partial y_n)^{-1}$ is well defined for $\eta_n \neq 0$ as a microdifferential operator. We have

$$\left(\frac{\partial}{\partial y_n}\right)^{-1} \delta(\Omega) = -Y(\Omega).$$

Hence,

$$x_j \delta(\Omega) = \frac{\delta}{\delta y_j} Y(\Omega) = -\frac{\partial}{\partial y_j}\left(\frac{\partial}{\partial y_n}\right)^{-1} \delta(\Omega) \qquad \text{for } j = 1, \ldots, n-1$$

holds. Since we have

$$x_n \delta(\Omega) = \left(y_n - \sum_{j=1}^{n-1} x_j y_j\right) \delta(\Omega),$$

we obtain the following:

$$x_n \delta(\Omega) = \left(y_n + \sum_{j=1}^{n-1} y_j \frac{\partial}{\partial y_j}\left(\frac{\partial}{\partial y_n}\right)^{-1}\right) \delta(\Omega) = \langle y, D_y \rangle \left(\frac{\partial}{\partial y_n}\right)^{-1} \delta(\Omega).$$

Furthermore,

$$\frac{\partial}{\partial x_j}\delta(\Omega) = y_j\delta'(\Omega) = y_j\left(-\frac{\partial}{\partial y_n}\right)\delta(\Omega) \qquad \text{for } j = 1, \ldots, n-1$$

and

$$\frac{\partial}{\partial x_n}\delta(\Omega) = -\frac{\partial}{\partial y_n}\delta(\Omega)$$

hold. Here, the equation (4.2.14) is equivalent to

$$P(x, D_x)\delta(\Omega(x, y)) = Q^*(y, D_y)\delta(\Omega(x, y)), \tag{4.2.14'}$$

where Q^* denotes the conjugate operator of Q. Consequently, we obtain the correspondence as follows:

$$\left.\begin{aligned}
&x_j = \frac{\partial}{\partial y_j}\left(\frac{\partial}{\partial y_n}\right)^{-1}, && j = 1, \ldots, n-1\\
&x_n = \langle y, D_y\rangle\left(\frac{\partial}{\partial y_n}\right)^{-1} &&\\
&\frac{\partial}{\partial x_j} = y_j\frac{\partial}{\partial y_n}, && j = 1, \ldots, n-1\\
&\frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n}. &&
\end{aligned}\right\} \tag{4.2.41a}$$

The inverse correspondence can be found as follows:

$$\left.\begin{aligned}
&y_j = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_n}\right)^{-1}, && j = 1, \ldots, n-1\\
&y_n = \langle x, D_x\rangle\left(\frac{\partial}{\partial x_n}\right)^{-1} &&\\
&\frac{\partial}{\partial y_j} = -x_j\frac{\partial}{\partial x_n}, && j = 1, \ldots, n-1\\
&\frac{\partial}{\partial y_n} = \frac{\partial}{\partial x_n}. &&
\end{aligned}\right\} \tag{4.2.41b}$$

Exercise. Find the above correspondences when the kernel function $\delta(\Omega)$ in (4.2.14) is replaced by $Y(\Omega)$.

We viewed the quantized contact transformation in Theorem 4.2.2 as a correspondence of microdifferential operators. Next we will show that a quantized contact transformation induces an isomorphism on the sheaf of microfunctions. This guarantees that one may use quantized contact transformations freely as one studies the structures of microfunction solutions of microdifferential equations.

Theorem 4.2.3. *Under the same hypotheses as in Theorem* 4.2.2, *the map* $T:\mathscr{C}_M \to \mathscr{C}_N$, *defined by* $T(u(y)) = \int \delta(\Omega(x, y))u(y)\,dy$, *is an isomorphism.*

Proof. Let

$$I(y', y) \underset{\text{def}}{=} \int \delta(\Omega(x, y'))\delta(\Omega(x, y))\,dx.$$

Then, it is sufficient to prove that I is a kernel function for an invertible microdifferential operator on $\sqrt{-1}S^*M$. By the Weierstrass preparation theorem for a holomorphic function (see Note following Proposition 2.4.3) one may assume that $\Omega(x, y)$ is of the form $x_1 - f(x_2, \ldots, x_n, y)$. We have

$$I(y', y) = \int \delta(f(x_2, \ldots, x_n y) - f_2(x_2, \ldots, x_n, y'))\,dx_2 \cdots dx_n. \tag{4.2.43}$$

From the Taylor expansion, one can find G such that

$$f(x_2, \ldots, x_n, y) - f(x_2, \ldots, x_n, y') = \langle y - y', G(x_2, \ldots, x_n, y, y')\rangle.$$

Hence (4.2.43) becomes

$$\int \delta(\langle y - y', G(x_2, \ldots, x_n, y, y')\rangle)\,dx_2 \cdots dx_n. \tag{4.2.44}$$

Letting $A(y, y', \eta) = |\det(\eta, \partial\eta/\partial x_2, \ldots, \partial\eta/\partial x_n)|^{-1}$, one can rewrite (4.2.44) as

$$\int A(y, y', \eta)\delta(\langle y - y', \eta\rangle)\omega(\eta).$$

Therefore, since $A(y, y', \eta) \neq 0$, Theorem 4.1.3 and the corollary of Theorem 4.1.6 imply that $I(y, y')$ determines an invertible microdifferential operator.

§3. Structures of Systems of Microdifferential Equations

The purpose of this section is to analyze the structure of a general system of microdifferential equations, using quantized contact transformations. In classical mechanics it is an exquisite and fundamental fact that any system can be transformed into an equilibrium system by a canonical transformation (see, for example, Yamanouchi[1]). In terms of analysis, this means that a partial differential equation of the first order can be transformed into a standard form by canonical transformation. The results in this section generalize the above result to the higher order in the linear case. Though we will consider rather restricted systems (see Remarks 4.3.3 and 4.3.4) in this treatise, the essence of our theory will be recognized.

Theorem 4.3.1. *Let $\mathcal{M}$ be an $\mathscr{E}$-Module defined in a neighborhood of* $(x^0, \sqrt{-1}\xi^0\infty) \in \sqrt{-1}S^*M$, *satisfying the conditions* (4.3.1) $\sim$ (4.3.4):

For some left-ideal $\mathscr{I}$ of $\mathscr{E}$, $\mathcal{M} = \mathscr{E}/\mathscr{I}$. (4.3.1)

The zero set $V(J)$ of $J \underset{\text{def}}{=} \bigcup_m \{\sigma_m(P) \mid P \in \mathscr{I} \cap \mathscr{E}(m)\}$ is a non-singular manifold of codimension d in a complex neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$; and $\omega|_{V(J)} \neq 0$ also holds, where ω is the canonical 1-form $\sum_j \xi_j dx_j$. (4.3.2)

The totality of analytic functions, homogeneous in ξ, that vanish on $V(J)$ is J. (4.3.3)

The zero set $V(J)$ is real, i.e. $V(J) = \overline{V(J)}$, where $\overline{V(J)}$ is the complex conjugate of $V(J)$. (4.3.4)

Then, $\mathcal{M}$ can be transformed into the following system $\mathcal{N}$ via a quantized contact transformation

$$\mathcal{N} = \mathscr{E}/(\mathscr{E}D_1 + \cdots + \mathscr{E}D_d). \tag{4.3.5}$$

Remark 4.3.1. Let u be the residue class of $1 \in \mathscr{E}$ at $\mathscr{I}$. Then (4.3.1) may be written as

$$\mathcal{M} = \mathscr{E}u, \qquad \text{where } Pu = 0 \text{ for } P \in \mathscr{I}. \tag{4.3.6}$$

Some readers may consider that (4.3.6) looks more like a system of (micro)-differential equations. However, when one considers a general system (not necessarily with one unknown function) of microdifferential equations, it is more desirable to grasp the system of microdifferential equations as a coherent left $\mathscr{E}$-Module; i.e.

$$0 \leftarrow \mathcal{M} \leftarrow \mathscr{E}^s \leftarrow \mathscr{E}^t$$

is exact, which is more intrinsic. If a system of microdifferential equations is interpreted as an $\mathscr{E}$-Module, then it is clear that an element $\mathscr{H}om_{\mathscr{E}}(\mathcal{M}, \mathscr{C})$ represents a microfunction solution of $\mathcal{M}$ (rigorously speaking, the image of u under an $\mathscr{E}$-homomorphism from $\mathcal{M}$ to $\mathscr{C}$ is a microfunction solution of $\mathcal{M}$). The system $\mathcal{N}$ is sometimes called a de Rham system, or a partial de Rham system.

Remark 4.3.2. Since $\omega|_{V(J)} \neq 0$, then $d \leqq n - 1$ holds in our case.

Before we prove Theorem 4.3.1, a special case of Theorem 4.3.1 will be proved. Note that the proof of Theorem 4.3.1 is essentially reduced to this special case.

Theorem 4.3.2. *Let $P(x, D_x)$ be a microdifferential operator of the first order defined in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty) = (0, \sqrt{-1}(0, 0, \ldots, 1)\infty)$*

such that the principal symbol $\sigma_1(P)$ *is* ξ_1. *Then, in a neighborhood of* $(x^0, \sqrt{-1}\xi^0\infty)$,

$$\mathscr{E}/\mathscr{E}P \cong \mathscr{E}/\mathscr{E}D_1 \tag{4.3.7}$$

holds.

Proof. From Theorem 4.1.9, one can write $P = Q(D_1 - A(x, D'))$, where $D' = (D_2, \dots, D_n)$ and Q is invertible in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$. Hence, one may assume $P = D_1 - A(x, D')$ to prove this theorem. We will find an invertible microdifferential operator $R(x, D')$ such that $R^{-1}PR = D_1$ and $R = \sum_{k=0}^{\infty} R^k(x, D')$, where R^k, $k = 0, \dots,$ satisfy the following (4.3.8) and (4.3.9). When such an R exists, it is clear that (4.3.7) holds:

$$R^0 = 1. \tag{4.3.8}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial x_1} R^k(x, D') = A(x, D')R^{k-1}(x, D') \\ &R^k|_{x_1=0} = 0 \qquad \text{for } k \geqq 1. \end{aligned}\right\} \tag{4.3.9}$$

Here, the left-hand sides $(\partial/\partial x_1)R^k(x, D')$ and $R^k|_{x_1=0}$ of (4.3.9) mean the derivative of R^k with respect to x_1 (regarding the microdifferential operator R^k as depending upon the parameter x_1) and the substitution $x_1 = 0$ respectively. The equation (4.3.9) can be written explicitly as

$$R^k(x, D') = \int_0^{x_1} A(s_k, x', D') \int_0^{s_k} A(s_{k-1}, x', D') \int_0^{s_{k-1}} \cdots$$
$$\int_0^{s_2} A(s_1, x', D') ds_1 \cdots ds_{k-2}\, ds_{k-1}\, ds_k, \tag{4.3.10}$$

where $x' = (x_2, \dots, x_n)$ and where A is regarded as an operator depending on s_j. Denote the domain of integration of (4.3.10) by V_k. Then the volume of V_k is $|x_1|^k/k!$. The order of R^k is at most zero from the expression (4.3.10). Hence, by letting δ, ϵ and $M > 0$ be as

$$\sup_{|s| \leqq \epsilon} N_0(A(s, x', D'); t) \leqq M < \infty \qquad \text{for } 0 < t < \delta, \tag{4.3.11}$$

the following (4.3.12), obtained from Theorem 4.1.5, implies (4.3.13):

$$N_0(R^k(x, D'); t) \ll \int_{V_k} \cdots \int N_0(A(s_k, x', D') \cdots A(s_1, x', D')\, ds_1 \cdots ds_k$$
$$\ll \int_{V_k} \cdots \int \left[\sup_{|s| \leqq |x_1|} N_0(A(s, x', D'); t) \right]^k ds_1 \cdots ds_k. \tag{4.3.12}$$

$$N_0\left(\sum_{k=0}^{\infty} R^k(x, D'); t \right) \leqq \sum_{k=0}^{\infty} \frac{M^k |x_1|^k}{k!} = \exp(Mx_1) < \infty. \tag{4.3.13}$$

Therefore, $R = \sum_{k=0}^{\infty} R^k(x, D')$ is well defined as microdifferential operator. Furthermore, (4.3.8) and the initial condition in (4.3.9) imply $\sigma_0(R)|_{x_1=0} = 1 \neq 0$. Hence, R is invertible by the corollary of Theorem 4.1.6. On the other hand, by definitions (4.3.8) and (4.3.9) of R^k, one obtains

$$(D_1 - A(x, D')) \circ R = R \circ D_1. \tag{4.3.14}$$

Here, $R \circ D_1$ is the composition of microdifferential operators, not in the sense of $(\partial/\partial x_1)R$ in (4.3.9). In fact, we have $D_1 \circ R = \partial R/\partial x_1 + R \circ D_1$ by definition.

Remark 4.3.3. In the case $\sigma(P) = \xi_1^m$, $m \geqq 2$, rather than $\sigma(P) = \xi_1$, one can prove the similar assertion by replacing $\mathscr{E}$ with $\mathscr{E}^\infty$. That is, the introduction of operators of infinite order allows one to treat the terms of lower orders. Even though it is one of the characteristics of hyperfunction theory to be able to treat such an operator, we restrict ourselves to a consideration of operators of finite order. One reason for so doing is that the treatment of operators of infinite order is at present not so "algebro-analytic" (not without reason—an operator of infinite order is transcendental, depending on the topological properties of the field of complex numbers).

We now begin the proof of Theorem 4.3.1. It is easy to see, because of the condition (4.3.3), that $V(J)$ is involutory. This is a special case of the remark following Definition 4.2.6; when one does not have such an algebraic condition as (4.3.3), the proof is non-trivial. Here is a proof. Let $f(x, \xi)|_{V(J)} = g(x, \xi)|_{V(J)} = 0$. Then one can find P and Q in $\mathscr{J}$ such that $\sigma_1(P) = f$ and $\sigma_1(Q) = g$. Note that $[P, Q] = PQ - QP \in \mathscr{J}$, which implies $\sigma_1([P, Q])|_{V(J)} = 0$. Consequently, since $\sigma_1([P, Q]) = \{\sigma_1(P), \sigma_1(Q)\} = \{f, g\}$, one obtains $\{f, g\}|_{V(J)} = 0$; i.e. $V(J)$ is involutory. Hence, conditions (4.3.2) and (4.3.4) imply that, by a quantized contact transformation, one can write $V(J) = \{\xi_1 = \cdots = \xi_d = 0\}$. By (4.3.3), one can choose $P_1, \ldots, P_d$ in $\mathscr{J}$ such that

$$\sigma_1(P_j) = \xi_j \qquad \text{for } j = 1, \ldots, d. \tag{4.3.15}$$

Furthermore, one may assume $P_1 = D_1$ from the proof of Theorem 4.3.2. We will prove by induction on k ($\leqq d$) that we can take

$$P_j = D_j \qquad \text{for } j = 1, \ldots, k \tag*{(4.3.16)$_k$}$$

for suitable generators of $\mathscr{M}$. Let us assume that $(4.3.16)_k$ holds. Then, by $D_j u = 0$ for $j = 1, \ldots, k$, one may assume

$$P_j = P_j(x, D_{k+1}, \ldots, D_n) \qquad \text{for } j = k + 1, \ldots, d. \tag{4.3.17}$$

Theorem 4.1.9 implies that one has

$$P_{k+1}(x, D_{k+1}, \ldots, D_n) \quad R(D_{k+1} + Q_{k+1}) \tag{4.3.18}$$

for some microdifferential operator $Q_{k+1}(x, D_{k+2}, \ldots, D_n)$ of, at most, order 0 and an invertible microdifferential operator $R(x, D_{k+1}, \ldots, D_n)$. The assertion on uniqueness in Theorem 4.1.9 indicates that R and Q_{k+1} take the above forms, i.e. depending on some, not all Dj's. Hence, we may begin with $P_{k+1} = D_{k+1} + Q_{k+1}(x, D_{k+2}, \ldots, D_n)$. Thus, one may assume, using $(D_{k+1} + Q_{k+1})u = 0$,

$$P_j = P_j(x, D_{k+2}, \ldots, D_n) \qquad \text{for } j = k + 2, \ldots, d. \tag{4.3.19}$$

Then, again by virtue of Theorem 4.1.9, one obtains $P_{k+2} = D_{k+2} + Q_{k+2}(x, D_{k+3}, \ldots, D_n)$. The repeated use of this argument implies

$$\left.\begin{aligned} P_j &= D_j \qquad \text{for } j = 1, \ldots, k \\ P_{k+1} &= D_{k+1} + Q_{k+1}(x, D_{d+1}, \ldots, D_n). \end{aligned}\right\} \tag{4.3.20}$$

Since for $j = 1, \ldots, k$ one has $[D_j, P_{k+1}] = D_jP_{k+1} - P_{k+1}D_j \in \mathscr{I}$, then $[D_j, Q_{k+1}] \in \mathscr{I}$ holds for $j = 1, \ldots, k$. Furthermore, we will show that $[D_j, Q_{k+1}] = 0$. Suppose $[D_j, Q_{k+1}] \neq 0$; then let m be the order of $[D_j, Q_{k+1}]$. Since $[D_j, Q_{k+1}] \in \mathscr{I}$, then $\sigma_m([D_j, Q_{k+1}]) = 0$ on $V(J)$. Therefore, if $q_{k+1,m}$ denotes the homogeneous part of degree m of Q_{k+1}, then on $V(J)$

$$\sigma_m([D_j, Q_{k+1}]) = \{\xi_j, q_{k+1,m}\} = \frac{\partial}{\partial x_j} q_{k+1,m} = 0$$

holds. But $q_{k+1,m}$ does not depend on $\xi_1, \ldots, \xi_d$ by (4.3.20). Consequently, $(\partial/\partial x_j)q_{k+1,m} \equiv 0$, which contradicts the assumption $\sigma_m([D_j, Q_{k+1}]) \not\equiv 0$. Therefore we have $[D_j, Q_{k+1}] = 0$ for $j = 1, \ldots, k$. That is, Q_{k+1} depends only upon $(x_{k+1}, \ldots, x_n, D_{d+1}, \ldots, D_n)$. By the proof for Theorem 4.3.2, one can find an invertible microdifferential operator $R(x_{k+1}, \ldots, x_n, D_{k+1}, \ldots, D_n)$ such that

$$R^{-1}P_{k+1}R = D_{k+1} \tag{4.3.21}$$

holds. One has $R^{-1}D_jR = D_j$ for $j = 1, \ldots, k$, since R commutes with D_j, $j = 1, \ldots, k$. Hence, replacing a generator u of $\mathscr{M}$ by Ru, generators P_j for $\mathscr{I}$ can be chosen so as to satisfy $(4.3.16)_{k+1}$. Hence the induction proceeds.

Next, by making use of $(4.3.16)_d$, we will prove that $\mathscr{I}$ is generated by $D_1, \ldots, D_d$. Suppose $D_1, \ldots, D_d$ do not generate $\mathscr{I}$; then there exists $0 \neq R(x, D_{d+1}, \ldots, D_n) \in \mathscr{I}$. By the definition of $V(J)$, one has $\sigma(R)|_{V(J)} = 0$. Since $\sigma(R)$ does not depend upon $\xi_1, \ldots, \xi_d$, this implies $\sigma(R) = 0$, a contradiction. Hence, $\mathscr{I}$ is generated by $D_1, \ldots, D_d$. That is, $\mathscr{M}$ is isomorphic to the $\mathscr{N}$ given in (4.3.5).

Remark 4.3.4. If operators of infinite order are employed, the condition (4.3.3) is not needed. Since a microdifferential operator of infinite order operates on the sheaf of microfunctions as a sheaf homomorphism, the condition (4.3.3) is not needed for the following theorem (Theorem 4.3.3). We will not go into details here.

Once Theorem 4.3.1 is obtained, the properties of the microfunction solutions of a system of microdifferential equations, which are invariant under quantized contact transformations, can be easily found. One of the most fundamental examples is phrased in Theorem 4.3.3, stating the propagation of solutions along a bicharacteristic manifold. First, we will generalize the notion of a bicharacteristic strip given for a single equation (Chapter III, §5, Definition 3.5.1).

Definition 4.3.1. *Suppose that an involutory submanifold V of $\sqrt{-1}S^*M$ satisfies the conditions* (4.3.2) *and* (4.3.4), *such that* $V = \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \mid f_1(x, \xi) = \cdots = f_d(x, \xi) = 0\}$. *The bicharacteristic manifold $b = b_{(x^0, \sqrt{-1}\xi^0\infty)}$, associated with V passing through $(x^0, \sqrt{-1}\xi^0\infty) \in V$, is defined as the integral manifold of dimension d, going through $(x^0, \sqrt{-1}\xi^0\infty)$, of d Hamilton operators H_{f_j} associated with V, where*

$$H_{f_j} = \sum_{l=1}^{n} \left(\frac{\partial f_j}{\partial \xi_l} \frac{\partial}{\partial x_l} - \frac{\partial f_j}{\partial x_l} \frac{\partial}{\partial \xi_l} \right) \quad \text{for } j = 1, \ldots, d. \tag{4.3.22}$$

Remark 4.3.5. Since V is involutory, $\{H_{f_j}\}_{j=1}^d$ satisfy the integrability condition. The definition of b does not depend upon the choice of $\{f_j\}_{j=1}^d$.

Theorem 4.3.3. *Let $\mathcal{M}$ be an $\mathcal{E}$-Module satisfying the conditions in Theorem* 4.3.1. *Then one has the following fact* (4.3.23) *in a neighborhood of* $(x^0, \sqrt{-1}\xi^0\infty)$:

The microfunction solution sheaf $\mathcal{S} \underset{\text{def}}{=} \mathcal{H}om_{\mathcal{E}}(\mathcal{M}, \mathcal{C})$ has its support in V and is locally constant along each bicharacteristic manifold. Furthermore, $\mathcal{S}$ is flabby in the direction transversal to bicharacteristic manifolds. (4.3.23)

Proof. Since the assertion of this theorem is invariant under a quantized contact transformation, it is sufficient to prove the case where $\mathcal{M}$ is of the form $\mathcal{N} = \mathcal{E}/(\mathcal{E}D_1 + \cdots + \mathcal{E}D_d)$. For $\mathcal{N}$, (4.3.23) can be proved in a way similar to the proof of the first lemma in §2 of Chapter III.

Remark 4.3.6. It can also be shown that $\mathcal{M}$ is locally solvable on $\sqrt{-1}S^*M$; i.e., for f_j satisfying an algebraic comparable condition, there exists a microfunction solution u such that $P_j u = f_j$, where P_j's are generators of $\mathcal{I}$. More generally, the higher cohomology groups $\mathcal{E}xt^j_{\mathcal{E}}(\mathcal{M}, \mathcal{C})$, $j \geqq 1$, vanish, but we need further algebraic preparation to prove this.

Remark 4.3.7. Having followed this treatise thus far, one may not have any difficulty understanding Theorem 4.3.3. The statement on "the propagation of singularities along bicharacteristic manifolds" (even for $d = 1$), together with the local solvability problem to be discussed later, has been of central interest in the theory of linear differential equations. One had to wait until the advent of microfunction theory to solve this problem in the real analytic category. Even in the C^∞-category, the solution for the case $d = 1$ had not been established before the 1960s. Furthermore, the description using bicharacteristic curves was not thorough enough to use bicharacteristic strips (see §6 of Chapter III). For further historical information on the era of linear partial differential equations before microfunction theory and quantized contact transformations, consult Hörmander [1]. There one may get an idea of what it was like "B.C. (Before $\mathscr{C}$)." As such a difficult problem becomes obvious, one may see progress in mathematics: it is like a view from a mountain peak!

Remark 4.3.8. Theorem 4.3.1 still holds for a complex domain—in which case, (4.3.4) is not needed. In this sense, the theorem is most fundamental. In this book, we defined the notion of a microdifferential operator on $\sqrt{-1}S^*M$. The reader might expect that compositions, transformations, etc. can also be defined for a complex manifold X and the projective cotangent bundle P^*X. In SKK [1], the theory is developed in that way so as to define an operator on a real manifold by the restriction. This method is more universal but less intuitive. Hence, in this book, we gave a less general definition of a microdifferential operator than the one in SKK [1]. Once $\mathscr{E}$ is defined over P^*X, Theorem 4.3.1 can be rephrased as follows: let $\mathscr{M}$ be an $\mathscr{E}$-Module defined in a neighborhood of $(x^0, \xi^0) \in P^*X$ satisfying (4.3.1), (4.3.25), and (4.3.3) where

$$V(J) \subset P^*X \text{ is a non-singular manifold of codimension } d \text{ and } \omega|_{V(J)} \neq 0. \tag{4.3.25}$$

Then (by a quantized contact transformation) $\mathscr{M}$ is isomorphic to $\mathscr{N} = \mathscr{E}/(\mathscr{E}D_1 + \cdots + \mathscr{E}D_d)$.

Using the theory of microdifferential operators developed over P^*X, one can prove this version in a way similar to that shown here. Note that the only geometric fact used for the proof of Theorem 4.3.1 is the following: if $V(J)$ is real and satisfies (4.3.2), then one obtains $V(J) = \{\xi_1 = \cdots = \xi_d = 0\}$ by a real contact transformation. For a complex domain, needless to say, one needs the condition (4.3.25) only, without the condition of $V(J)$ being real, to obtain the above. The rest of the argument is no different for a real domain or for a complex domain.

The above remark (Remark 4.3.8) indicates that the essential part of Theorem 4.3.1 may be viewed as a result for a complex domain. In fact, the condition (4.3.4), which is needed for the case in a real domain, is satisfied only by very restricted equations (although, we may say, it is not such a restrictive condition for equations actually appearing in applications). Thus, we are naturally led to ask the following question. Theoretically speaking, what are "generic" systems of equations for a real domain? First we must ask how far canonical V can be by a real contact transformation. An answer to this question for a system will be given in (4.3.82). To make the essence of the argument apparent, we will treat the case of a single equation.

Theorem 4.3.4. *Let M be an open neighborhood of $x^0 \in \mathbf{R}^n$, and let $f(x, \xi)$ be a real analytic function defined in a neighborhood of $(x^0, \xi^0) \in T^*M - M$, having the following properties* (4.3.26) $\sim$ (4.3.28):

f is positively homogeneous in ξ of degree $1/2$; i.e. $f(x, c\xi) = c^{1/2}f(x, \xi)$ for $c > 0$. (4.3.26)

$$f(x^0, \xi^0) = 0. \tag{4.3.27}$$

$$\frac{1}{2\sqrt{-1}}\{f, \bar{f}\}(x^0, \xi^0) \gneqq 0. \tag{4.3.28}$$

Then there exists a real-valued real analytic function $\Phi(x, \xi, t, \bar{t})$, defined in a neighborhood of $(x, \xi, t, \bar{t}) = (x^0, \xi^0, 0, 0) \in (T^*M - M) \times \mathbf{C} \times \bar{\mathbf{C}}$ ($\bar{t}$ is the complex conjugate of t), satisfying the following three conditions:

$$\Phi(x^0, \xi^0, 0, 0) \gneqq 0. \tag{4.3.29}$$

$\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi))$ is positively homogeneous in ξ of degree $1/2$. (4.3.30)

$$\frac{1}{2\sqrt{-1}}\{f(x, \xi)\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi)), \bar{f}(x, \xi)\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi))\} = 1. \tag{4.3.31}$$

Proof. We will prove a more general statement (C) than the assertion of this theorem, since the proof of (C) is simpler in notations than the one of the theorem.

(C): Let $F(x, \xi, t, \bar{t})$ be a strictly positive-valued real analytic function defined in a neighborhood of $(x^0, \xi^0, 0, 0)$ such that $F(x, \xi, f(x, \xi), \bar{f}(x, \xi))$ is positively homogeneous in ξ of degree 0. Then, one can find a real-valued real analytic function $\Phi(x, \xi, t, \bar{t})$ such that Φ satisfies

$$\{f(x, \xi)\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi)), \bar{f}(x, \xi)\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi))\}$$
$$\{f(x, \xi), \bar{f}(x, \xi)\}F(x, \xi, f(x, \xi), \bar{f}(x, \xi)), \tag{4.3.32}$$

$\Phi(x^0, \xi^0, 0, 0) > 0$, and $\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi))$ is positively homogeneous in ξ of degree 0.

The assertion of Theorem 4.3.4 is obtained from (C) for the case $F = 2/\{f, \bar{f}\}$.

Proof of (C). For the sake of simplicity, let $\Psi(x, \xi, t, \bar{t}) = (\Phi(x, \xi, t, \bar{t}))^2$. For a function $g(x, \xi)$ on $T^*M - M$, define Θ and $\bar{\Theta}$ as follows:

$$\left.\begin{aligned} \Theta_g &= \frac{\{g, \bar{f}\}}{\{f, \bar{f}\}} \\ \bar{\Theta}_g &= \frac{\{f, g\}}{\{f, \bar{f}\}}. \end{aligned}\right\} \tag{4.3.33}$$

The differential operators Θ and $\bar{\Theta}$ naturally act on $\Psi(x, \xi, t, \bar{t})$. Then notice that (4.3.32) is reduced to finding a solution $\Psi \gneqq 0$ of the following (4.3.34):

$$\begin{aligned} \Psi(x, \xi, t, \bar{t}) &+ \frac{1}{2} t\left(\frac{\partial \Psi(x, \xi, t, \bar{t})}{\partial t} + \Theta\Psi(x, \xi, t, \bar{t})\right) \\ &+ \frac{1}{2}\bar{t}\left(\frac{\partial \bar{\Psi}(x, \xi, t, \bar{t})}{\partial \bar{t}} + \bar{\Theta}\Psi(x, \xi, t, \bar{t})\right) = F(x, \xi, t, \bar{t}). \end{aligned} \tag{4.3.34}$$

Since this equation is degenerate at $t = \bar{t} = 0$, we will show the existence of a solution of (4.3.34), using a singular coordinate transformation at the origin in the $(t, \bar{t})$-space. That is, one is to find a real-valued function Ω satisfying

$$\Omega(\lambda, t, \bar{t})\,(\equiv \Omega(x, \xi; \lambda, t, \bar{t})) = \lambda^2 \Psi(x, \xi, \lambda t, \lambda \bar{t}), \qquad \lambda \in \mathbf{R}. \tag{4.3.35}$$

When the dependency of Ω on (x, ξ) is not crucial in arguments, we abbreviate $\Omega(x, \xi; \lambda, t, \bar{t})$ as $\Omega(\lambda, t, \bar{t})$. By (4.3.35), (4.3.34) can be rewritten in terms of Ω as

$$\frac{\partial}{\partial \lambda}\Omega + \Theta(\bar{t}\Omega) + \bar{\Theta}(t\Omega) = 2\lambda F(x, \xi, \lambda t, \lambda \bar{t}). \tag{4.3.36}$$

If the initial value of Ω on $\{\lambda = 0\}$ is given by 0, then Ω, since $\{\lambda = 0\}$ is non-singular for (4.3.36), is determined uniquely from (4.3.36). Furthermore, the complex conjugate $\bar{\Omega}$ of Ω is also a solution, since F is a real-valued function. By the uniqueness theorem of the Cauchy problem, $\Omega = \bar{\Omega}$ holds; i.e. Ω is a real-valued function. Since Θ and $\bar{\Theta}$ are differential operators with respect to (x, ξ), then (4.3.36) implies

$$\left.\frac{\partial \Omega}{\partial \lambda}\right|_{\lambda = 0} = 0. \tag{4.3.37}$$

The initial condition of Ω and (4.3.37) imply that Ω/λ^2 is holomorphic in a neighborhood of $\{\lambda = 0\}$. If one can show that Ω/λ^2 is an analytic function of $(x, \xi, \lambda t, \lambda \bar{t})$, then one can let $\Psi(x, \xi, \lambda t, \lambda \bar{t}) = \Omega/\lambda^2$. To do so, it is sufficient to show that Ω satisfies the following condition on the homogeneity with respect to $(\lambda, t, \bar{t})$:

$$\Omega(c\lambda, c^{-1}t, c^{-1}\bar{t}) = c^2\Omega(\lambda, t, \bar{t}) \qquad \text{for } c \in \mathbf{R} - \{0\}. \qquad (4.3.38)$$

Let $\mu = c^{-1}\lambda$, $s = ct$, and $\bar{s} = c\bar{t}$; and let $\tilde{\Omega}(\mu, s, \bar{s}) = \Omega(c\mu, c^{-1}s, c^{-1}\bar{s})$. From (4.3.36) one obtains

$$\frac{1}{c}\frac{\partial}{\partial \mu}\tilde{\Omega} + \Theta\left(\frac{\bar{s}}{c}\tilde{\Omega}\right) + \bar{\Theta}\left(\frac{s}{c}\tilde{\Omega}\right) = 2c\mu F(x, \xi, \mu s, \mu \bar{s}). \qquad (4.3.39)$$

Notice that $(\tilde{\Omega}/c^2)(\lambda, t, \bar{t})$ is a solution of (4.3.38), which becomes 0 at $\lambda = 0$. Therefore, by the uniqueness of a solution of (4.3.36), one has

$$\tilde{\Omega}(\lambda, t, \bar{t})/c^2 = \Omega(\lambda, t, \bar{t}). \qquad (4.3.40)$$

Hence, (4.3.40) and the definition of $\tilde{\Omega}$ imply (4.3.38). Then, letting $\Psi = \Omega(1, \lambda t, \lambda \bar{t})$, one obtains a solution Ψ of (4.3.34).

From the coefficients of the Taylor expansions of (4.3.36) with respect to λ, one obtains

$$\left.\frac{\Omega}{\lambda^2}\right|_{\lambda=0} = F(x, \xi, 0, 0) > 0. \qquad (4.3.41)$$

Hence $\Psi(x^0, \xi^0, 0, 0) \gneqq 0$; therefore, one may suppose $\Phi(x^0, \xi^0, 0, 0) > 0$. The homogeneity of $\Phi(x, \xi, f(x, \xi), \bar{f}(x, \xi))$ remains to be proved. Since $f(x, \xi)$ is positively homogeneous in ξ of degree 1/2, it is sufficient to prove

$$\Psi(x, c\xi, c^{1/2}t, c^{1/2}\bar{t}) = \Psi(x, \xi, t, \bar{t}) \qquad \text{for } c > 0. \qquad (4.3.42)$$

Then, from (4.3.35), we need to show

$$\Omega(x, \lambda^2\xi, \lambda, t, \bar{t}) = \lambda^2\Omega(x, \xi, 1, t, \bar{t}) \qquad \text{for } \lambda > 0. \qquad (4.3.43)$$

Therefore, using (4.3.38), it is sufficient to show

$$\Omega(x, \lambda^2\xi, \lambda, t, \bar{t}) = \Omega(x, \xi, \lambda, \lambda^{-1}t, \lambda^{-1}\bar{t}) \qquad \text{for } \lambda > 0. \qquad (4.3.44)$$

By the definition of Θ and $f(x, \xi)$ being positively homogeneous of degree 1/2, (4.3.44) can be proved in the same manner as the proof of (4.3.38). Hence, (C) is proved, which completes the proof of Theorem 4.3.4.

From Theorem 4.3.4, we have the following astonishing result.

Theorem 4.3.5. *Let $P(x, D_x)$ be a microdifferential operator of order m defined in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty) \in \sqrt{-1}S^*M$. Let the principal symbol $\sigma_m(P)(x, \sqrt{-1}\xi) = f(x, \xi)$ satisfy the conditions*

$$f(x^0, \xi^0) = 0 \qquad (4.3.45)$$

and

$$\{f, \bar{f}\}(x^0, \xi^0) \gneqq 0. \tag{4.3.46}$$

Then, the equation $Pu = 0$ can be transformed into the following equation $\mathcal{N}$, defined in a neighborhood of $(y; \sqrt{-1}\eta) = (0; \sqrt{-1}(0, \ldots, 0, 1))$ by an invertible (real) quantized contact transformation:

$$\mathcal{N}:\left(\frac{\partial}{\partial y_1} - \sqrt{-1}y_1\frac{\partial}{\partial y_n}\right)u = 0. \tag{4.3.47}$$

Proof. By considering $(D_1^2 + \cdots + D_n^2)^{-(2m-1)/4}P(x, D_x)$, one may assume that f is positively homogeneous in ξ of degree 1/2. Furthermore, by considering $\Phi(x, -\sqrt{-1}D_x, f(x, -\sqrt{-1}D_x), \bar{f}(x, -\sqrt{-1}D_x))$, Theorem 4.3.4 implies that one may assume that

$$\frac{1}{2\sqrt{-1}}\{f, \bar{f}\} = 1 \tag{4.3.48}$$

holds, where $f(x, -\sqrt{-1}D_x)$ etc. means $\sum_\alpha a_\alpha(x)(-\sqrt{-1}D_x)^\alpha$ etc. for $f(x, \xi) = \sum_\alpha a_\alpha(x)\xi^\alpha$ etc. Therefore, by a real contact transformation, one may have

$$f = y_1\eta_n^{1/2} + \sqrt{-1}\eta_1\eta_n^{-1/2} \tag{4.3.49}$$

in a neighborhood of $(y; \eta) = (0; 0, \ldots, 0, 1)$. Hence, one can assume that the principal symbol of P is $\eta_1 - \sqrt{-1}y_1\eta_n$. Therefore, after a suitable (complex) coordinate transformation, this theorem follows from Theorem 4.3.2.

Notes.

1. $\{f, \bar{f}\} \neq 0$ implies $d_{(x,\xi)}f \not\parallel \omega$; i.e. condition (4.3.2) is automatically satisfied.
2. Since Theorem 4.3.2 is a result for a real domain (see Remark 4.3.8), this argument is not rigorous. It is recommended that the unsatisfied reader prove Theorem 4.3.2 for that case. One needs to check the convergence of the infinite series of operators obtained by a successive approximation. See Theorem 2.1.2 and Remark 1 following Theorem 2.1.2, in Chap. II, §2.1, of SKK [1].

Remark 4.3.9. If (4.3.36) in Theorem 4.3.5 is replaced by the condition $\{f, \bar{f}\}(x^0, \xi^0) \lneqq 0$, then the corresponding canonical equation is $\mathcal{N}:(\partial/\partial y_1 + \sqrt{-1}y_1(\partial/\partial y_n))u = 0$ in a neighborhood of $(y; \sqrt{-1}\eta) = (0; \sqrt{-1}(0, \ldots, 0, 1))$.

Remark 4.3.10. Under the assumptions of Theorem 4.3.5, $V = \{f(x, \xi) = 0\}$ and $\bar{V} = \{\bar{f}(x, \xi) = 0\}$ intersect transversally; and $\omega|_{V \cap \bar{V}}$ defines a contact structure on $V \cap \bar{V}$. When $\operatorname{codim}(V \cap \bar{V}) = 2$ and $V \cap \bar{V}$ has the contact structure from $\omega|_{V \cap \bar{V}}$, then one can ask, as an interesting generalization of the above, "What is a canonical equation corresponding to the equation $Pu = 0$?" It is known (Sato-Kawai-Kashiwara [2]), using an argument similar to that given above, that the canonical equation is given by

$$\left(\frac{\partial}{\partial y_1} \pm \sqrt{-1}\, y_1^k \frac{\partial}{\partial y_n}\right) u = 0. \tag{4.3.50}$$

The nature of the equations of this type was first appreciated in Mizohata [2].

By Theorem 4.3.5, the microlocal study of the solutions of a microdifferential equation is reduced to studying the solutions of a very simple equation, $(\partial/\partial x_1 \pm \sqrt{-1}\, x_1(\partial/\partial x_n))u = 0$. Next, we will study the structure of the solutions of this special equation by direct computation. The reader should notice that one cannot understand the essence of even such a simple equation without microlocal consideration.

Theorem 4.3.6. *Let* $P(x, D_x) = D_1 - \sqrt{-1}\, x_1 D_n$, *and let* $Q(x, D_x) = D_1 + \sqrt{-1}\, x_1 D_n$. *Then there exists a non-zero microlocal operator* $\mathscr{K}$ *defined in a neighborhood of* $(x^0; \sqrt{-1}\xi^0\infty) = (0; \sqrt{-1}(0, 0, \ldots, 0, 1)\infty) \in \sqrt{-1}S^*\mathbf{R}^n$ *such that the sequence*

$$0 \to \mathscr{C} \xrightarrow{Q} \mathscr{C} \xrightarrow{\mathscr{K}} \mathscr{C} \xrightarrow{P} \mathscr{C} \to 0 \tag{4.3.51}$$

is exact.

Hence, in particular, P *is solvable in a neighborhood of* $(x^0; \sqrt{-1}\xi^0\infty)$, Q *is not solvable, and the image of* Q *is characterized as the kernel of* $\mathscr{K}$.

Proof. For computational convenience, let us introduce differential operators R and R' defined at $(x, t) \in \mathbf{R}^{n+1}$, where

$$\left.\begin{aligned} R &= P(x, D_x) + \frac{\sqrt{-1}}{2} D_t \\ R' &= P^*(x', D_{x'}) + \frac{\sqrt{-1}}{2} D_t. \end{aligned}\right\} \tag{4.3.52}$$

Here, $P^*(x', D_{x'}) = -\partial/\partial x_1' + \sqrt{-1}\, x_1' \,\partial/\partial x_n'$, the conjugate operator of $P(x, D_x)$.

Next, define $\varphi(x, x', t)$ as

$$\varphi = x_n - x_n' + (x_1 + x_1')t + \frac{\sqrt{-1}}{4}((x_1 - x_1')^2 + 4t^2)). \tag{4.3.53}$$

Then, by Proposition 2.4.2, $(\varphi + \sqrt{-1}0)^\alpha$ is well defined (see also Example 2.4.3) and, furthermore,

$$\begin{aligned} \text{S.S.}(\varphi + \sqrt{-1}0)^\alpha \subset \{(x, x', t; \sqrt{-1}(\langle \xi, dx\rangle + \langle \xi', dx'\rangle + \langle \tau, dt\rangle)\infty \mid \\ x_1 = x_1', x_n = x_n', t = 0, \\ \xi_1 = \cdots = \xi_{n-1} = \xi_1' = \cdots = \xi_{n-1}' = 0, \\ \xi_n = -\xi_n' = 1, \tau = x_1 + x_1'\} \end{aligned} \tag{4.3.54}$$

holds. Since $R\varphi = 0$ is clearly true, one has

$$R(\varphi + \sqrt{-1}0)^\alpha = 0. \tag{4.3.55}$$

On the other hand, consider the hyperfunction $1/(x + \sqrt{-1}y)$ in Example 2.4.8. Then one has

$$\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)\left(\frac{1}{x + \sqrt{-1}y}\right) = 2\pi\delta(x)\delta(y). \tag{4.3.56}$$

Hence, one obtains

$$\begin{aligned} R\left(\frac{\varphi + \sqrt{-1}0)^\alpha}{t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')}\right) &= R\left(\frac{1}{t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')}\right)(\varphi + \sqrt{-1}0)^\alpha \\ &= 2\pi\sqrt{-1}\delta(x_1 - x_1')\delta(t)(\varphi + \sqrt{-1}0)^\alpha \\ &= 2\pi\sqrt{-1}\delta(x_1 - x_1')\delta(t)(x_n - x_n' + \sqrt{-1}0)^\alpha. \end{aligned} \tag{4.3.57}$$

Note that, by (4.3.54), $(\varphi + \sqrt{-1}0)^\alpha/(t - (\sqrt{-1}/2)(x_1 - x_1'))$ is well defined as a hyperfunction (see Theorem 3.1.5). Similarly, since $R'\varphi = \sqrt{-1}(x_1 - x_1' + 2\sqrt{-1}t)$, one obtains

$$\begin{aligned} &R'\left(\frac{(\varphi + \sqrt{-1}0)^\alpha}{t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')}\right) \\ &= 2\pi\sqrt{-1}\delta(x_1 - x_1')\delta(t)(x_n - x_n' + \sqrt{-1}0)^\alpha - 2\alpha(\varphi + \sqrt{-1}0)^{\alpha-1}. \end{aligned} \tag{4.3.58}$$

Integrating (4.3.57) and (4.3.58) with respect to t gives, respectively,

$$\begin{aligned} &P(x, D_x)\int_{-\infty}^{\infty} \frac{(\varphi + \sqrt{-1}0)^\alpha}{\left(t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')\right)}\,dt \\ &\qquad = 2\pi\sqrt{-1}\delta(x_1 - x_1')(x_n - x_n' + \sqrt{-1}0)^\alpha \end{aligned} \tag{4.3.59}$$

and

$$P^*(x', D_{x'}) \int_{-\infty}^{\infty} \frac{(\varphi + \sqrt{-1}0)^\alpha}{\left(t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')\right)} dt$$

$$= 2\pi\sqrt{-1}\delta(x_1 - x_1')(x_n - x_n' + \sqrt{-1}0)^\alpha - 2\alpha \int_{-\infty}^{\infty} (\varphi + \sqrt{-1}0)^{\alpha - 1}\, dt. \tag{4.3.60}$$

Notice, as an integration of a microfunction,

$$E_\alpha(x, x') = \int_{-\infty}^{\infty} \frac{(\varphi + \sqrt{-1}0)^\alpha}{\left(t - \dfrac{\sqrt{-1}}{2}(x_1 - x_1')\right)} dt \tag{4.3.61}$$

is well defined (see Theorem 3.2.1). For the case $\alpha = -1$, one may use the formula

$$\int_{-\infty}^{\infty} \frac{dt}{(a + \sqrt{-1}t)(c + bt + \sqrt{-1}(a^2 + t^2))}$$

$$= 2\pi i \left\{ \frac{Y(a)}{\sqrt{-1}(c - 2\sqrt{-1}ab)} + \frac{Y(\operatorname{Im} \alpha)}{(a + \sqrt{-1}\alpha)\sqrt{-1}(\alpha - \beta)} + \frac{Y(\operatorname{Im} \beta)}{(a + \sqrt{-1}(\beta - \alpha)} \right\},$$

where

$$\begin{cases} \alpha \\ \beta \end{cases} = \frac{\sqrt{-1}b \pm \sqrt{-1}\sqrt{4a^2 + b^2 - 4\sqrt{-1}c}}{2}.$$

Furthermore, a well-known formula

$$\Gamma(-\alpha) \int_{-\infty}^{\infty} (t^2 + A)^\alpha\, dt = \sqrt{\pi}\,\Gamma(-\alpha - \tfrac{1}{2})A^{\alpha + 1/2}, \qquad \operatorname{Im} A > 0 \tag{4.3.62}$$

provides

$$\int_{-\infty}^{\infty} (\varphi + \sqrt{-1}0)^{\alpha - 1}\, dt$$

$$= e^{-(\pi/4)\sqrt{-1}} \frac{\sqrt{\pi}\,\Gamma(-\alpha + \frac{1}{2})}{\Gamma(-\alpha + 1)} \left(x_n - x_n' + \frac{\sqrt{-1}}{2}(x_1^2 + x_1'^2) + \sqrt{-1}0 \right)^{\alpha - 1/2} \tag{4.3.63}$$

(cf. Example 3.2.8). Let

$$E(x, x') \underset{\text{def}}{=} \frac{1}{4\pi^2} E_{-1}(x, x') \prod_{j=2}^{n-1} \delta(x_j - x_j'). \tag{4.3.64}$$

Then we can summarize what has been obtained as

$$P(x, D_x)E(x, x') = \delta(x - x') \tag{4.3.65}$$

$$P^*(x', D_{x'})E(x, x') = \delta(x - x') - \frac{e^{-(\pi/4)\sqrt{-1}}}{4\pi}\left(x_n - x_n' + \frac{\sqrt{-1}}{2}(x_1^2 + x_1'^2) + \sqrt{-1}0\right)^{-3/2} \prod_{j=2}^{n-1} \delta(x_j - x_j'), \tag{4.3.66}$$

where equalities hold in a neighborhood of

$$(x, x'; \sqrt{-1}(\langle\xi, dx\rangle + \langle\xi', dx\rangle)\infty) = (0, 0; \sqrt{-1}(dx_n - dx_n')\infty) \in \sqrt{-1}S^*(\mathbf{R}^n \times \mathbf{R}^n).$$

If one lets $\mathscr{K}$ and $\mathscr{H}$ be the microlocal operators having $-(e^{-(\pi/4)\sqrt{-1}}/4\pi)\cdot(x_n - x_n' + (\sqrt{-1}/2)(x_1^2 + x_1'^2) + \sqrt{-1}0)^{-3/2} \prod_{j=2}^{n-1} \delta(x_j - x_j')$ and $E(x, x')$ as kernel functions, respectively, then (4.3.65) and (4.3.66) can be phrased in terms of equations of microlocal operators as, in the neighborhood of $(x; \sqrt{-1}\langle\xi, dx\rangle\infty) = (0; \sqrt{-1}\,dx_n\infty)$,

$$P\mathscr{H} = 1 \tag{4.3.65'}$$

and

$$\mathscr{H}P = 1 - \mathscr{K}. \tag{4.3.66'}$$

As for $Q(x, D_x)$, let $R = Q^*(x', D_{x'}) - (\sqrt{-1}D_t/2)$, and let

$$R' = Q(x, D_x) - (\sqrt{-1}D_t/2);$$

then one obtains $R\varphi = 0$ and $R'\varphi = \sqrt{-1}(x_1 - x_1' - 2\sqrt{-1})$. In the same manner as above,

$$Q\mathscr{F} = 1 - \mathscr{K}, \tag{4.3.67}$$

and

$$\mathscr{F}Q = 1 \tag{4.3.68}$$

holds, where $\mathscr{F}$ is the microlocal operator defined in a neighborhood of $(0; \sqrt{-1}\,dx_n\infty)$ with the kernel function

$$\frac{1}{4\pi^2}\left(\int_{-\infty}^{\infty} \frac{1}{\left(t + \frac{\sqrt{-1}}{2}(x_1 - x_1')\right)(\varphi + \sqrt{-1}0)}\,dt\right)\prod_{j=2}^{n-1} \delta(x_j - x_j').$$

Hence, for example, one has $Q = Q(\mathscr{F}Q) = (Q\mathscr{F})Q = (1 - \mathscr{K})Q = Q - \mathscr{K}Q$ from (4.3.67), (4.3.68), and the corollary following Definition 3.4.1, concluding $\mathscr{K}Q = 0$. Suppose $\mathscr{K}f = 0$ holds; then (4.3.67) implies $Q(\mathscr{F}f) =$

$f - \mathcal{K}f = f$. Hence $\operatorname{Ker}\mathcal{K} \subset \operatorname{Im} Q$; therefore, $\operatorname{Ker}\mathcal{K} = \operatorname{Im} Q$ holds. Similarly, one obtains $\operatorname{Im}\mathcal{K} = \operatorname{Ker} P$ from (4.3.65′) and (4.3.66′). From (4.3.68) and (4.3.65′), respectively, Q is a monomorphism and P is an epimorphism. This completes the proof of the exactness of (4.3.51).

We will obtain the following decisive theorem by Theorem 4.3.5 (and Remark 4.3.9) and by the above theorem.

Theorem 4.3.7. *Let $P(x, D_x)$ be a microdifferential operator of order m, which is defined in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty) \in \sqrt{-1}S^*M$, and let $f(x, \xi) = \sigma_m(P)(x, \sqrt{-1}\xi)$. Then one has the following statements:*

(I) *If $f(x^0, \xi^0) = 0$ and $\{f, \bar{f}\}(x^0, \xi^0) \gneqq 0$ hold, then P is epimorphic in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$ and $\operatorname{Ker} P$ is equal to the image of a microlocal operator $\mathcal{K}$.*

(II) *If $f(x^0, \xi^0) = 0$ and $\{f, \bar{f}\}(x^0, \xi^0) \lneqq 0$ hold, then P is monomorphic in a neighborhood of $(x^0, \sqrt{-1}\xi^0\infty)$, but not epimorphic, and the image of P is equal to the kernel of a microlocal operator $\mathcal{K}$. That is, for the equation $Pu = g$ to be solvable, $\mathcal{K}g = 0$ must hold.*

Remark 4.3.11. Since H. Lewy's sensational discovery in 1957 of the equation

$$\left(\frac{1}{2}\left(\frac{\partial}{\partial x_1} + \sqrt{-1}\frac{\partial}{\partial x_2}\right) - (x_1 + \sqrt{-1}x_2)\frac{\partial}{\partial x_3}\right)u = g$$

without (local) solutions (see Lewy [1]), the solvability of a linear partial differential equation has been a central topic. Now, for a "generic" equation, an answer has been obtained in an ideal form, as in the above theorem. We have treated the geometrically "generic" case. We need to consider a system of equations that does not satisfy the condition (4.3.3) in order to treat the truly "generic" case. With operators of infinite order, an argument similar to that above enables one to handle the case (see Remark 4.3.3). Theorem 4.3.7, together with Theorem 4.3.3, has shown just how useful microlocal analysis is for the study of linear partial differential equations. Even though it is possible to adapt the above equation of Lewy for a canonical form, we choose the canonical form in (4.3.47), which clearly suggests a connection with the contact structure and the following theorem. A microdifferential equation, having (4.3.47) as the canonical form, i.e. the equation satisfying (4.3.46) or with the opposite inequality, is said to be the Lewy-Mizohata type.

As we mentioned in Remark 4.3.10, for the characteristic variety V of P, $V \cap \sqrt{-1}S^*M = V \cap \bar{V}$ has a contact structure. If this fact links $\operatorname{Ker} P$ or $\operatorname{Coker} Q$, that would be very interesting. This "expected harmony" beautifully exists, as follows.

Theorem 4.3.8. *Let P, Q, and $\mathscr{K}$ be as in Theorem 4.3.6. Let $M = \mathbf{R}^n$, $N = \{x \in M \mid x_1 = 0\}$, and $Z = \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \mid x_1 = 0,\ \xi_1 = 0 \text{ and } \xi_n > 0\}$. Z is identified with $\{(x', \sqrt{-1}\xi'\infty) \in \sqrt{-1}S^*N \mid \xi_n > 0\}$ via the projection $N \underset{M}{\times} \sqrt{-1}S^*M - \sqrt{-1}S_N^*M \to \sqrt{-1}S^*N$, where $x' = (x_2, \ldots, x_n)$ and $\xi' = (\xi_2, \ldots, \xi_n)$. Then, one can define sheaf homomorphisms Φ and Ψ over $\Omega = \{(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*M \mid \xi_n > 0\}$ such that*

$$\begin{array}{ccc} \Phi: \mathscr{C}_N & \longrightarrow & \mathscr{C}_M \\ \Cup & & \Cup \\ u(x_2, \ldots, x_n) & \mapsto & -\dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{-\infty}^{\infty} \dfrac{u(x_2, x_3, \ldots, x_n')}{\left(x_n - x_n' + \dfrac{\sqrt{-1}}{2}x_1^2 + \sqrt{-1}0\right)}\, dx_n' \end{array} \tag{4.3.69}$$

and

$$\begin{array}{ccc} \Psi: \mathscr{C}_M & \longrightarrow & \mathscr{C}_N \\ \Cup & & \Cup \\ v(x_1, \ldots, x_n) & \longmapsto & \mathscr{K}v|_{x_1=0}; \end{array} \tag{4.3.70}$$

and then

$$\Psi\Phi = 1: \mathscr{C}_N \to \mathscr{C}_N \tag{4.3.71}$$

and

$$\Phi\Psi = \mathscr{K}: \mathscr{C}_M \to \mathscr{C}_M \tag{4.3.72}$$

hold. Therefore, in particular one has an isomorphism

$$\Psi: \mathrm{Ker}_{\mathscr{C}_M} P \xrightarrow{\sim} \mathscr{C}_N. \tag{4.3.73}$$

Similarly, sheaf homomorphisms $\tilde{\Phi}$ and $\tilde{\Psi}$ can be defined such that

$$\begin{array}{ccc} \tilde{\Phi}: \mathscr{C}_N & \longrightarrow & \mathscr{C}_M \\ \Cup & & \Cup \\ u(x_2, \ldots, x_n) & \longmapsto & \mathscr{K}(u(x_2, \ldots, x_n)\delta(x_1)), \end{array} \tag{4.3.74}$$

$$\begin{array}{ccc} \tilde{\Psi}: \mathscr{C}_M & \longrightarrow & \mathscr{C}_N \\ \Cup & & \Cup \\ v(x_1, \ldots, x_n) & \mapsto & -\dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int \dfrac{v(x_1', x_2, x_3, \ldots, x_n')}{\left(x_n - x_n' + \dfrac{\sqrt{-1}}{2}x_1'^2 + \sqrt{-1}0\right)}\, dx_1'\, dx_n', \end{array} \tag{4.3.75}$$

$$\tilde{\Psi}\tilde{\Phi} = 1: \mathscr{C}_N \to \mathscr{C}_N, \tag{4.3.76}$$

and

$$\tilde{\Phi}\tilde{\Psi} = \mathscr{K}:\mathscr{C}_M \to \mathscr{C}_M. \tag{4.3.77}$$

***Furthermore, one has the** isomorphism*

$$\tilde{\Phi}:\mathscr{C}_N \xrightarrow{\sim} \operatorname{Coker}_{\mathscr{C}_M} Q. \tag{4.3.78}$$

Proof. **Since $\mathscr{K}$** is a microlocal operator, it is almost obvious that Φ, Ψ, $\tilde{\Phi}$, $\tilde{\Psi}$ are sheaf homomorphisms. We will prove (4.3.71). By the definitions,

$$\begin{aligned}
&\Phi\Psi(u(x_2, \ldots, x_n)) \\
&= -\frac{e^{-(\pi/4)\sqrt{-1}}}{4\pi} \int \frac{\prod_{j=2}^{n-1} \delta(x_j - x_j')}{\left(x_n - x_n' + \frac{\sqrt{-1}}{2} x_1'^2 + \sqrt{-1}0\right)^{3/2}} \\
&\quad \cdot \int \frac{u(x_2', x_3', \ldots, x_n'')}{(-2\pi\sqrt{-1})\left(x_n' - x_n'' + \frac{\sqrt{-1}}{2} x_1'^2 + \sqrt{-1}0\right)} dx_n'' \prod_{j=1}^{n} dx_j'.
\end{aligned} \tag{4.3.79}$$

On the other hand, by the residue computation and (4.3.62), one obtains

$$\begin{aligned}
&\int \frac{dx_1'\, dx_n'}{\left(x_n - x_n' + \frac{\sqrt{-1}}{2} x_1'^2 + \sqrt{-1}0\right)^{3/2}\left(x_n' - x_n'' + \frac{\sqrt{-1}}{2} x_1'^2 + \sqrt{-1}0\right)} \\
&\qquad = -2\pi\sqrt{-1} \int \frac{dx_1'}{(x_n - x_n'' + \sqrt{-1}x_1'^2 + \sqrt{-1}0)^{3/2}} \\
&\qquad = -2\pi\sqrt{-1}\, \frac{\sqrt{\pi}\Gamma(1)}{e^{(\pi/4)\sqrt{-1}}\Gamma(3/2)} \cdot \frac{1}{(x_n - x_n'' + \sqrt{-1}0)} \\
&\qquad = -4e^{(\pi/4)\sqrt{-1}} \frac{1}{(x_n - x_n'' + \sqrt{-1}0)}.
\end{aligned} \tag{4.3.80}$$

When this is substituted into (4.3.79), one obtains $\Psi\Phi = 1$ on Z. We can prove (4.3.72), (4.3.76), and (4.3.77) in the same manner. Next, we will prove (4.3.73). First, notice that $\mathscr{K}^2 = (\Phi\Psi)(\Phi\Psi) = \Phi(\Psi\Phi)\Psi = \Phi\Psi = \mathscr{K}$ by (4.3.71) and (4.3.72). On the other hand, from the exact sequence (4.3.51), for $v \in \operatorname{Ker} P$ there exists $g \in \mathscr{C}_M$ such that $v = \mathscr{K}g$ holds. Therefore, one obtains $\Psi v = \mathscr{K}v|_{x_1=0} = \mathscr{K}^2 g|_{x_1=0} = \mathscr{K}g|_{x_1=0} = v|_{x_1=0}$, which is the isomorphism in (4.3.73). The isomorphism in (4.3.78) can be seen as follows. By (4.3.67), one has

$$\begin{aligned}
\tilde{\Phi}u &= \delta(x_1)u(x_2, \ldots, x_n) - Q\mathscr{F}(\delta(x_1)u(x_2, \ldots, x_n)) \\
&\equiv \delta(x_1)u(x_2, \ldots, x_n) \bmod Q\mathscr{C}_M.
\end{aligned}$$

The flabbiness of the microfunction sheaf (Theorem 3.7.1), Theorem 4.3.7, and Theorem 4.3.8 imply the following theorem.

Theorem 4.3.9. *Notations being the same as in Theorem* 4.3.7, *sheaves* Ker P *in* (I) *and* Coker P *in* (II) *are flabby over* $V \cap \bar{V}$ $(= V \cap \sqrt{-1}S^*N)$.

We have restricted our arguments to the case of a single equation. Clever use of Theorem 4.3.1 (regarded as a theorem for a complex domain) enables one to extend the result on the Lewy-Mizohata-type equation to the one on overdetermined systems. Theorem 4.3.8 is very useful for the generalization. We will touch this topic only briefly. Consult §2.3, Chap. III of SKK [1] for details.

Definition 4.3.2. *Let M be a real analytic manifold. Let an involutory submanifold V in a complex neighborhood of* $(x^0, \sqrt{-1}\langle\xi^0, dx\rangle\infty) \in \sqrt{-1}S^*M$ *be written as* $\{(x, \sqrt{-1}\langle\xi, dx\rangle\infty \mid p_1(x, \sqrt{-1}\xi) = \cdots = p_d(x, \sqrt{-1}\xi) = 0\}$. *Then the Hermitian matrix*

$$L(x, \xi) = \left(\frac{1}{2\sqrt{-1}}\{p_j(x, \xi), \bar{p}_k(x, \xi)\}\right)_{1 \leq j,k \leq d} \tag{4.3.81}$$

is said to be the "generalized Levi form" of V.

Remark 4.3.12. The number of positive eigenvalues and the number of negative eigenvalues of $L(x, \xi)$ are independent of the choice of defining functions of V. These numbers are also invariant under a real contact transformation. L is called the "generalized Levi form" in an analogy with the terminology used in function theory (e.g. see Hitotumatu [1]). These two notions, however, are closely related (Kashiwara and Kawai [1], Example 3).

Using this notion of a generalized Levi form, the statement on the canonical case of an overdetermined system can be phrased as follows. Let $\mathscr{M}$ be an $\mathscr{E}$-Module defined in a neighborhood of $(x^0, \sqrt{-1}\langle\xi^0, dx\rangle\infty) \in \sqrt{-1}S^*M$ such that $\mathscr{M}$ satisfies (4.3.1), (4.3.2), and (4.3.3). Further, assume that the generalized Levi form of $V(J)$ has p positive eigenvalues and q $(= d - p)$ negative eigenvalues at $(x^0, \sqrt{-1}\xi^0)$. Then, $\mathscr{M}$ can be transformed by a quantized contact transformation into the system of equations $\mathscr{N}_p$ (called the $(p, d-p)$-Lewy-Mizohata system):

$$\mathscr{N}_p: \begin{cases} \left(\dfrac{\partial}{\partial y_j} - \sqrt{-1}y_j \dfrac{\partial}{\partial y_n}\right)u = 0 & \text{for } j = 1, \ldots, p \\ \left(\dfrac{\partial}{\partial y_j} + \sqrt{-1}y_j \dfrac{\partial}{\partial y_n}\right)u = 0 & \text{for } j = p+1, \ldots, p+q = d \end{cases} \tag{4.3.82}$$

in a neighborhood of $(y^0, \sqrt{-1}\langle\eta^0, dy\rangle\infty) = (0; \sqrt{-1}\,dy_n\infty)$.

It is an exercise in homological algebra to study the structure of microfunction solutions of the system of equations $\mathcal{N}_p$, if Theorems 4.3.6 and 4.3.8 are employed.

The system (4.3.82) is nothing but the equations that the system $\tilde{\mathcal{N}}$ over $\mathbf{R}^{n+1}$ satisfies on the hypersurface

$$S_p = \left\{ y \in \mathbf{R}^{n+1} \,\middle|\, y_{n+1} - \tfrac{1}{2}\left(\sum_{j=1}^{p} y_j^2 - \sum_{j=p+1}^{d} y_j^2 \right) = 0 \right\}$$

in $\mathbf{R}^{n+1}$, where

$$\tilde{\mathcal{N}}: \begin{cases} \dfrac{\partial}{\partial y_j} u = 0 & \text{for } j = 1, \ldots, d \ (d \leqq n - 1) \\ \left(\dfrac{\partial}{\partial y_n} + \sqrt{-1}\, \dfrac{\partial}{\partial y_{n+1}} \right) u = 0. \end{cases} \tag{4.3.83}$$

One would expect some links between the solutions of $\mathcal{N}_p$ and the solutions of $\tilde{\mathcal{N}}$, in the sense that one may study the structure of solutions of $\tilde{\mathcal{N}}_p$ from that of $\tilde{\mathcal{N}}$ and vice versa. When the above question is pursued far enough, a question surfaces: "What is the boundary value problem?" A fairly satisfying answer has been obtained in Kashiwara and Kawai [1] and [2], in which the interested reader can find the details.

Let us conclude this book with one final comment. We have obtained the canonical form of each $\mathcal{M}$ in Theorem 4.3.1, providing that the characteristic variety V is real, and in Theorem 4.3.5, providing that $V \cap \bar{V}$ intersects transversally and $V \cap \bar{V}$ has a contact structure, respectively. It is also of theoretical importance to find the canonical form in the rather degenerate case where $V \cap \bar{V}$ is a non-singular involutory submanifold. In this case (assuming $\omega|_{V \cap \bar{V}} \neq 0$) the canonical form is known to be the following partial Cauchy-Riemann system

$$\mathcal{N}: \frac{1}{2}\left(\frac{\partial}{\partial y_{2j-1}} + \sqrt{-1}\, \frac{\partial}{\partial y_{2j}} \right) u = 0 \qquad \text{for } j = 1, \ldots, d, \tag{4.3.84}$$

where $\mathcal{N}$ is considered over a neighborhood of $(y, \sqrt{-1}\langle \eta, dy \rangle \infty) = (0, \sqrt{-1}\, dy_n \infty)$.

Furthermore, a "generic" system $\mathcal{M}$ is isomorphic to the mixture system of the (partial) de Rham system, the Lewy-Mizohata system, and the (partial) Cauchy-Riemann system (SKK [1], Chap. III, §2.4). This result is most fundamental and most exquisite in the local theory of linear partial differential equations.

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