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LETTER

Statistical physics of the Schelling model of segregation

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Abstract. We investigate the static and dynamic properties of a celebrated model of social segregation, providing a complete explanation of the mechanisms leading to segregation both in one- and two-dimensional systems. Standard statistical physics methods shed light on the rich phenomenology of this simple model, exhibiting static phase transitions typical of kinetic constrained models, non-trivial coarsening like in driven-particle systems and percolation-related phenomena.

Keywords: coarsening processes (theory), critical phenomena of socio-economic systems, stochastic processes

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1. Introduction

Individuals with similar ideas, habits or preferences tend to create cliques and cluster in communities, which results in segregation at the scale of the society. A first theoretical analysis of segregation phenomena in social environments was performed by Schelling, who defined a model in which agents divided into two species move on a checkerboard according to a given utility function [1]. Within this simple model, Schelling was able to show that segregation occurs even when individuals have a very mild preference for neighbors of their own type, as long as they are allowed to move in order to satisfy their preference. Interest for Schelling's results has grown recently among social scientists [2, 3], mathematicians [4] and statistical physicists [5]. In particular, [2] suggested relations with models of binary mixtures in physics. Indeed, the way in which segregation takes place in the Schelling model resembles the coarsening processes governing phase separation kinetics [6, 7]. However, apart from a qualitative picture, [2] does not provide any quantitative measure of coarsening from which the scaling behavior of the segregation process could be inferred. On the other hand, looking at the non-equilibrium process of such a model, several questions spontaneously arise: Does the system fully segregate? What are the properties of the stationary state? If coarsening takes place in this system, does it fall into one of the known universality classes? In simpler terms, we want to understand to what distance segregation extends and how long the process will take.

In this letter, we shed light on the static and dynamical properties of the Schelling model in one and two dimensions, applying methods taken from different fields of statistical physics, such as (vacancy mediated) coarsening dynamics [7], diffusion–annihilation particle systems [9], and kinetically constrained systems [10, 11]. The model is the same as that investigated in [2]: individuals are initially distributed at random on a line ($d = 1$) or a square lattice ($d = 2$) of $N = L^d$ sites. The occupation of each site i is described by a spin variable σ_i, taking values $\sigma_i = 0$ if the site is empty and $\sigma_i = \pm 1$ if the site is occupied by an individual of type ± 1. Let $\rho_0 = N_0/N$ and $\rho_\pm = N_\pm/N$ be the densities of vacancies and of occupied sites, of either type, respectively. The general principle governing the dynamics is that an individual can tolerate at most a fraction $f = f^*$ of neighbors being different from him or her. Following the tradition in economics, we consider the Moore neighborhood, corresponding to the d-dimensional hypercube surrounding a site. For $d = 1$, the Moore neighborhood contains only the two nearest neighbors of a site, whereas in two dimensions it contains eight sites, i.e. four

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nearest neighbors and four next-nearest neighbors. If $f \leq f^*$ the individual is *happy* or has utility 1; otherwise he or she feels *unhappy* (utility 0). Unless otherwise stated we consider $f^* = 1/2$, as in the original Schelling model. Two types of dynamics are possible: *constrained* dynamics (called the 'solid model' in [2]), where only unhappy individuals are allowed to move, as long as they are able to find a vacancy where they can be happy; and *unconstrained* dynamics (called the 'liquid model' in [2]) where agents are allowed to move to vacancies as long as their situation does not get worse. In both cases, following [2], we assume an infinite range dynamics, i.e. individuals can move to any suitable vacancy, irrespective of distance. This spin–vacancy exchange dynamics conserves the magnetization only globally, and not locally. Natural quantities for characterizing the state of the system are the densities of unhappy sites $u(t)$ and of interfaces $n(t)$, defined as the fraction of neighboring spins of opposite type. We stress that, while it might be appealing to introduce a notion of energy, e.g. the number of unhappy individuals, this might be confusing as the dynamics is not derived from an energy functional, as in physics. Individuals move solely in order to maximize their utility, with no regard for the welfare of the fellow neighbors. For example, even if a displacement is beneficial to the mover, this might make some neighbors unhappy with the composition of their new neighborhood. So a move may cause an *increase* of the number of unhappy individuals. In the following, we focus on the static properties of the constrained model and the dynamical properties of the unconstrained one, showing that statistical physics allows one to understand many aspects of the rich phenomenology of the model. We shall use the 1D case to uncover the main properties of the segregation process and show that this provides key insight on the behavior of the 2D case.

2. The constrained model

Let us first discuss the case where only 'condensation' moves, which increase the utility of the moving agent, are allowed. At each time step, an individual is drawn at random and swapped with a randomly chosen vacancy if his or her utility increases. If there is no such vacancy, the agent is not displaced. This dynamics converges in a finite time to frozen states, called (myopic) Nash equilibria (NE), where no agent can find a better location in terms of utility given the location chosen by the others [3, 4]. In these states segregation extends up to a finite length.

Starting from a random initial condition, numerical simulations reveal the existence of a continuous transition in the properties of the final blocked configurations (figure 1). For $m = \rho_+ - \rho_- \neq 0$, we observe that the fraction of unhappy individuals (u_∞) vanishes for large enough vacancy density ($\rho_0 > \rho_0^*(m)$), while below the threshold a non-zero fraction ($u_\infty > 0$) of unhappy individuals of the minority type remains. The threshold decreases as the magnetization is reduced and the transition disappears for $m \to 0$ (figure 1, inset)[3].

It is interesting to compare these results with a static approach [12] which describes the organization of NE in the space of all configurations. In order to do this, we consider the ensemble of all NE *with equal a priori weight*. The *partition function* for this ensemble is $\mathcal{Z}_L(h, \mu, \beta) = \sum_{\sigma \in \mathcal{C}} \mathrm{e}^{hM(\sigma)+\mu N(\sigma)+\beta U(\sigma)}$ where $\mathcal{C}$ is the set of all blocked

[3] As pointed out in [13], it is possible to derive the existence of the phase transition from a simple mean field model, where unhappy individuals are displaced one after another.

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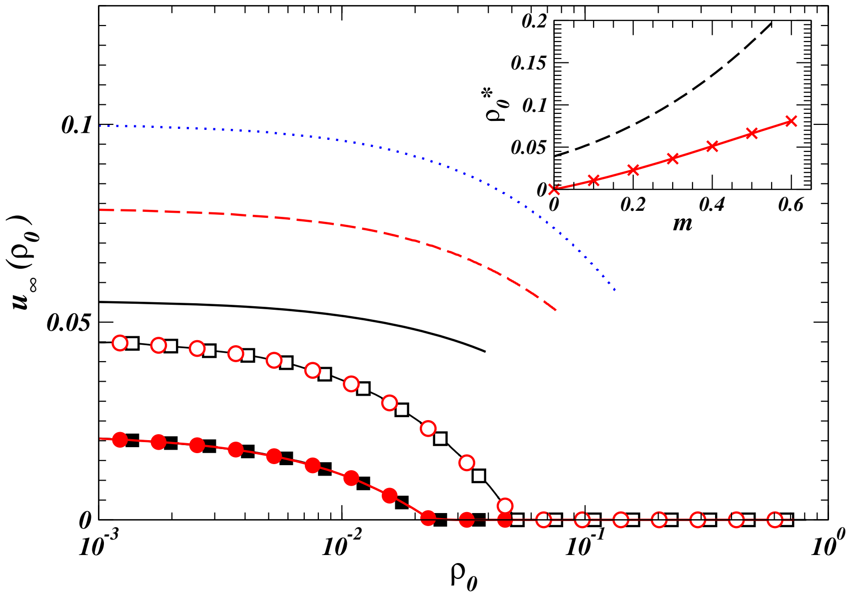

Figure 1. Main: density of unhappy individuals u_∞ in the blocked configurations of the constrained 1D model as a function of the density of vacancies ρ_0 and for different values of magnetization $m > 0$. Results from the transfer matrix approach are shown for $m = 0$ (solid black line), $m = 0.2$ (dashed red line), and $m = 0.4$ (dotted blue line). Lines with symbols are the results of simulations for $m = 0.2$ (full symbols), $m = 0.4$ (open symbols), and sizes $L = 10^6$, 10^7. Inset: phase diagram of the model with the transition line resulting from simulations (solid line) and from the static approach (dashed).

configurations $\sigma = (\sigma_1, \ldots, \sigma_L)$ of size L, with $\sigma_i = 0, \pm$ (with periodic boundary conditions), $M(\sigma) = \sum_i \sigma_i$ is the magnetization, $N(\sigma) = \sum_i \sigma_i^2$ is the number of individuals and $U(\sigma)$ is the number of unhappy individuals. On the basis of standard saddle point arguments, once we have fixed h, μ, β, the partition function is dominated by configurations characterized by values of M, N, U that can be easily determined using a Legendre transform. The number of NE with such a set of values for M, N, U is provided by the coefficient of the term dominating the partition sum. For instance, when $\beta = 0$, the coefficient of the term proportional to $v^{L(\rho_+ - \rho_-)} w^{L(\rho_+ + \rho_-)}$ in $\mathcal{Z}_L(\ln v, \ln w, 0)$ is the number of NE for a chain with $L\rho_+$ individuals of one type and $L\rho_-$ individuals of the other type. $\mathcal{Z}_L$ can be computed using the transfer matrix technique, following the same approach as in [11]. In brief, one can write $\mathcal{Z}_L = \mathrm{Tr}\, \mathcal{T}^L$, where the transfer matrix $\mathcal{T}$ relates the statistical weights of configurations of length $\ell + 1$ to those of configurations of length ℓ. In order to build configurations without constraints, $\mathcal{T}$ must have elements $T^{(\sigma_{\ell-1}, \sigma_\ell), (\sigma_\ell, \sigma_{\ell+1})}$, i.e. $\mathcal{T}$ is a 9×9 matrix. It is important to realize that NE can be of three types. In other words, the partition function has the form

$$\mathcal{Z}_L(h, \mu, \beta) = \mathrm{Tr}\, \mathcal{T}_0^L + \mathrm{Tr}\, \mathcal{T}_+^L + \mathrm{Tr}\, \mathcal{T}_-^L, \tag{1}$$

where $\mathcal{T}_0$ generates all configurations with no unhappy spins, while $\mathcal{T}_+$ ($\mathcal{T}_-$) generates those with unhappy spins of type $+$ ($-$) but no corresponding vacancies, which could accommodate them.

For $L \gg 1$, each term in equation (1) is dominated by the largest eigenvalue $\lambda_q(h, \mu, \beta)$ of the corresponding matrix $\mathcal{T}_q$ $(q = 0, \pm)$. Hence $\log \mathcal{Z}_L/L \cong \max_q \log \lambda_q(h, \mu, \beta)$. The number of NE with $m = \rho_+ - \rho_-$ and a density ρ_0 of vacancies is given by $\mathrm{e}^{LS(m,\rho_0;\beta=0)}$ where the entropy $S(m, \rho_0; \beta) = \max_{h,\mu,q}[\log \lambda_q(h, \mu, \beta) - hm - \mu(1 - \rho_0)]$ is obtained from λ_q via a Legendre transform. This allows us to access the statistical properties of NE depending on the various parameters of the problem. In particular, it is interesting to look at the behavior of the static density $u_\infty = \partial_\beta S(\beta = 0)$ of unhappy individuals as a function of ρ_0. For $\rho_0 > \rho_0^*(m)$ the solution is dominated by the configurations with no unhappy site $(u_\infty = 0, \lambda_0 > \lambda_\pm)$ but the converse is true for small density of vacancies $(\rho_0 < \rho_0^*(m))$, where the leading contribution to $\mathcal{Z}_L$ comes from the term $\mathcal{T}_+^L$ (i.e. $\lambda_+ > \lambda_0$) for $m \geq 0$. Indeed the density $u_\infty(\rho_0)$ of unhappy sites features a jump at ρ_0^* (figure 1). Such a first-order phase transition is typical of cases where the partition sum can be separated into different components as in equation (1). Notice that, for $m = 0$, the symmetry between the two types of agents is spontaneously broken for $\rho_0 < \rho_0^*(0)$: a randomly drawn NE typically has a fraction $u_\infty > 0$ of unhappy individuals of one type, all individuals of the other type being happy. The results of the static approach qualitatively agree with simulations, with two differences. In simulations: (i) the transition is continuous rather than discontinuous and (ii) when $m = 0$ no spontaneous symmetry breaking is observed [14]. This discrepancy is due to the difference in sampling of blocked configurations in the two cases. The static approach weights all blocked configurations with the same weight. In simulations instead each blocked configuration is weighted by the overlap of their basin of attraction and the distribution of initial conditions. The absence of symmetry breaking in the dynamics is a consequence of this: for $m = 0$, typical initial conditions have equal densities of unhappy individuals of the two types. Such a symmetry is preserved by the dynamics, so also the final blocked configuration is expected to share this property, i.e. $u_\infty = 0$. The same qualitative picture applies for $d = 2$, where also the static calculation can only be carried out using numerical simulations [14].

3. The unconstrained model

In the unconstrained model, at each time step a randomly chosen agent is relocated into a randomly chosen vacancy (long range diffusion) when his or her utility does not decrease. No blocked configurations exist since individuals can always be relocated. For this reason, the system can enter a stationary regime in which the densities of unhappy individuals $u(t)$ remain constant on average. We are thus interested in understanding the asymptotic properties of the dynamics, and characterizing the evolution to the stationary state. We study these properties for $d = 1, 2$ as a function of the density of vacancies ρ_0, assuming $m = 0$. For $d = 1$, for any ρ_0, $u(t)$ has an initial decay followed by a stationary state with asymptotic value $u_\infty(\rho_0)$. The initial decay is exponential for large densities of vacancies $(\rho_0 \to 1)$, while it becomes a power law in the limit of vanishing ρ_0. The density $n(t)$ of interfaces between domains of unlike spins exhibits a similar behavior. Figure 2 shows that for $\rho_0 \ll 1$ the interface density follows the scaling law $n(t) \sim \rho_0 \psi(\rho_0 t^{1/2})$ with $\psi(x) \sim x^{-1}$ for $x \to 0$ and $\psi(x) \sim$ const for $x \to \infty$. A similar power law behavior $(u(t) \sim t^{-3/2})$ is found for $u(t)$ in this limit, but $u(\rho_0, t)$ does not satisfy any scaling law. The behavior of the stationary densities as functions of ρ_0 is reported in the inset of figure 2. Both u_∞ and n_∞ have a maximum at some value $0 < \rho_0^* < 1$, then vanish in the limits $\rho_0 \to 0, 1$.

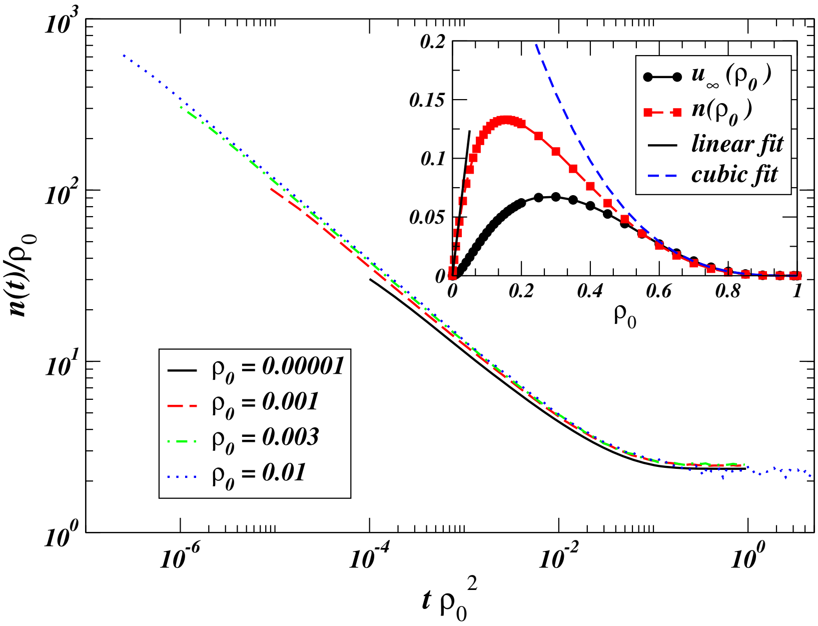

Figure 2. Scaling behavior of the density of interfaces $n(t)$ observed in the unconstrained model, in the limit of low density of vacancies ρ_0 for a 1D system of size $L = 10^6$ and zero magnetization $m = 0$. The inset shows the behavior of the stationary value of the densities of unhappy spins and interfaces as a function of ρ_0.

In particular, for $\rho_0 \to 1$, $u_\infty(\rho_0)$ vanishes as $(1-\rho_0)^3$, while in the opposite limit $n_\infty \sim \rho_0$ and $u_\infty \sim \rho_0^2$. An explanation for the observed behavior of u_∞ for $\rho_0 \approx 1$ can be given in terms of a phenomenological dynamics for $u(t)$. This takes the form $\dot{u} = -au + b$ where a and b describe processes which annihilate or create unhappy sites, respectively. For $\rho_0 \approx 1$ an unhappy individual with high probability becomes happy, once displaced; hence $a \simeq 1$. Unhappy individuals are created in processes such as $\cdots + - - \cdots \to \cdots + -0 \cdots$ where a displaced site leaves one neighbor in an uncomfortable neighborhood. For $\rho_0 \simeq 1$, the leading contribution is given by initial configurations of three occupied sites, whose probability is $4[(1-\rho_0)/2]^3$, where the factor 4 accounts for the degeneracy of the possible types. Hence $u(t)$ decays exponentially to $u_\infty = b/a \simeq (1-\rho_0)^3/2$, which agrees very well with numerical simulations.

A more elaborate analysis is however necessary in order to explain the observed phenomenology in the limit $\rho_0 \to 0$. It is convenient to think in terms of interfaces between clusters of opposite spin values. These can be thought of as particles, like in diffusion-limited annihilation processes. For two adjacent clusters of opposite spins, the leading process is the diffusion of interfaces, e.g. $++--- \to ++0-- \to +++--$. The other relevant processes are the creation and annihilation of clusters of size 1, i.e. unhappy spins. The annihilation rate of unhappy spins is obviously $\propto n^2$, whereas the creation of interfaces involves 'vacancy mediated' processes of the type $++--- \to ++-0-$. Here the displacement of an individual makes one neighbor unhappy, and the latter can in turn move in the bulk of a cluster of individuals of the other species. In the stationary state, the creation rate is proportional to the probability of finding an empty site close

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to an interface, which is $\rho_0 n$. The balance condition $\rho_0 n \approx n^2$ implies that the stationary interface density is $n_\infty \propto \rho_0$ and hence that the creation rate is proportional to ρ_0^2.

The solution to a problem of diffusion-limited annihilation with input of particle pairs is well known [9] and it leads to the following scaling form for the density of interfaces: $n(\rho_0, t) \sim \rho_0^{1/\delta}\psi(\rho_0 t^{1/\Delta})$, with $\psi(x) \to const$ for $x \to \infty$ and $\psi(x) \to x^{-1/\delta}$ for $x \to 0$. The injection rate ρ_0^2 implies $\delta = 1$ and $\Delta = 2$, in perfect agreement with numerical simulations. At $\rho_0 = 0^+$ (single-vacancy limit), the dynamics is purely diffusive; thus $n(\rho_0 = 0^+, t) \sim t^{-1/z}$, with $z = \delta\Delta = 2$. In other words, the $\rho_0 \to 0$ limit can be considered as a critical point for the one-dimensional unconstrained Schelling model. In this respect, the Schelling model for $\rho_0 > 0$ produces a dynamics with the same universal features of finite temperature coarsening dynamics as in Ising-like models with non-conserved order parameter [9].

Unlike $n(\rho_0, t)$, the density of unhappy individuals does not follow a scaling form. Indeed one can write $u(\rho_0, t) = -\partial n(\rho_0 = 0, t)/\partial t + c\rho_0 n$, with $c > 0$ a constant. The first term is the probability of having a cluster of size 1 in the bulk of larger domains, whereas the second arises from the creation of unhappy individuals close to interfaces (see above). For early times ($t \ll \rho_0^{-1}$, as suggested by dimensional analysis) the first term dominates; hence $u(\rho_0, t) \sim t^{-3/2}$, whereas $u(\rho_0, t)$ converges to $n_\infty^2(\rho_0)$ for $t \gg \rho_0^{-1}$.

We now turn to discussing the two-dimensional unconstrained dynamics that was already studied in [2], where it was suggested that some coarsening phenomenon should take place. We remark that, in the dynamics, voids are expected to be randomly distributed in the system. If ρ_0 is too large, a cluster of void sites starts percolating throughout the system, preventing the growth of the clusters of spins [5] beyond a finite size. In a $d = 2$ square lattice with Moore neighborhood, the percolation transition takes place at $\rho_0 \simeq 0.407$ [5]. Since here voids can move we expect this value to be only a lower bound for the real transition, that takes place at some $0.45 < \hat{\rho}_0 < 0.5$. For $\rho_0 \geq \hat{\rho}_0$ we find that clusters grow up to a finite size depending on ρ_0 but independent of the system size N [14]. In contrast, below the void percolation transition, we observe convergence to a (quasi)ordered state, with two domains spanning the whole system. A clear coarsening process with the typical length of clusters growing as $t^{1/z}$ with $z = 2$ (see figure 3(a)) is observed for small ρ_0. Even if the magnetization is globally conserved, the value $z = 2$ correctly describes a coarsening process with non-conserved order parameter in agreement with the model C dynamics [15] and with renormalization group results for domain growth scaling in the presence of long range diffusion [8].

Finally, it is interesting to discuss the behavior of the unconstrained model as a function of the maximal fraction f^* of unlike neighbors which individuals tolerate (so far equal to 1/2). For 1D, for any $0 < f^* < 1/2$, the same analysis as was carried out above [14] reveals a picture qualitatively similar to the $f^* = 1/2$ case. The key difference is that each unhappy individual has at least one unhappy neighbor. In the unconstrained case this causes the creation rate to vanish and, as a consequence, the system coarsens until it reaches configurations where vacancies localize at the interfaces between extended domains of individuals of the same type. For $d = 2$ figure 3(b) shows that, for $\rho_0 \to 0$, the coarsening process is robust against the variation of f^* over a wide range around 1/2, as also found in [2]. Segregation takes place even if individuals are satisfied with as many as five of their eight neighbors of different type. If instead individuals are so tolerant that they are happy with six unlike neighbors out of eight, then no coarsening

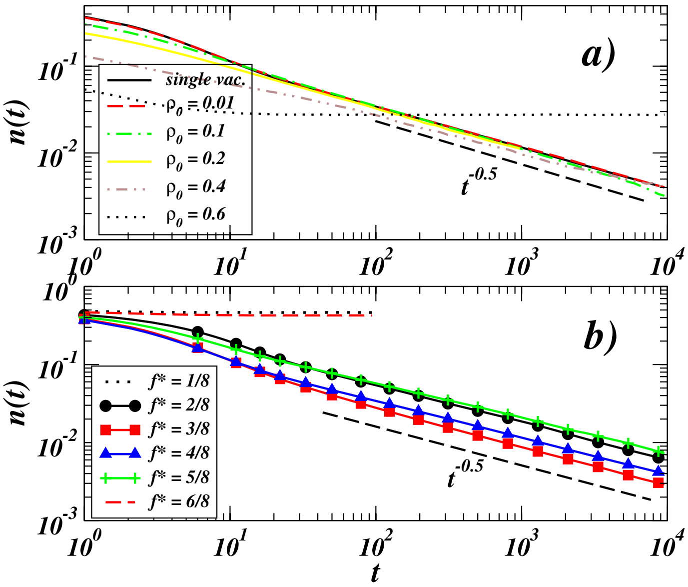

Figure 3. Density of interfaces $n(t)$ in a 2D system of size $L = 10^3$ with unconstrained dynamics: (a) $f^* = 1/2$, the coarsening process is present for $\rho_0 < \hat{\rho}_0$; (b) for some values of the threshold $f^* \neq 1/2$ the system does not order even in the limit $\rho_0 \to 0$.

and no segregation takes place and the system remains in a dynamic disordered state. No coarsening takes place also when individuals are extremely intolerant (see figure 3(b)). This is somewhat remarkable, because a fully segregated state would indeed be optimal, but it cannot be reached dynamically, as the system remains trapped in a disordered state.

4. Conclusions

In summary, we have investigated both analytically and numerically the static and dynamic properties of the Schelling model of segregation in one- and two-dimensional systems. The constrained version of the model presents non-trivial static properties characterized by the existence of a transition with symmetry breaking, whereas the unconstrained dynamics exhibits coarsening typical of systems with non-conserved order parameter. A further sharp transition takes place as the tolerance threshold (f^*) of individuals gets either very large or very small, with the system being trapped in disordered dynamical states. Many possible directions for future research can be identified. Among the most interesting are the consideration of a local range dynamics (i.e. unhappy individuals move to the closest available vacancy) and of a larger 'vision' [16] (i.e. the utility depends on a larger number of neighbors).

References

[1] Schelling T C, 1971 *J. Math. Sociol.* **1** 143
Schelling T C, 1978 *Micromotives and Macrobehavior* (New York: Norton)
[2] Vinković D and Kirman A, 2006 *Proc. Natl Am. Soc.* **103** 19261
[3] Pancs R and Vriend N, 2007 *J. Publ. Econ.* **91** 1

[4] Pollicott M and Weiss H, 2001 *Adv. Appl. Math.* **27** 17
Gerhold S, Glebsky L, Schneider C, Weiss H and Zimmermann B, 2008 *Commun. Nonlinear Sci. Numer. Simul.* **13** 2236
[5] Stauffer D and Solomon S, 2007 *Preprint* physics/0701051
[6] Gunton J D, san Miguel M and Sahni P S, 1983 *Phase Transitions and Critical Phenomena* ed C Domb and J L Lebowitz (New York: Academic)
[7] Bray A J, 2002 *Adv. Phys.* **51** 481
[8] Bray A J, 1989 *Phys. Rev. Lett.* **62** 2841
Bray A J, 1990 *Phys. Rev.* B **41** 6724
[9] Rácz Z, 1985 *Phys. Rev. Lett.* **55** 1707
Lushnikov A A, 1987 *Phys. Lett.* A **120** 135
Masser T O and ben-Avraham D, 2001 *Phys. Rev.* E **63** 066108
[10] Ritort F and Sollich P, 2002 *Adv. Phys.* **52** 219
[11] De Smedt G, Godrèche C and Luck J M, 2003 *Eur. Phys. J.* B **32** 215
De Smedt G, Godrèche C and Luck J M, 2002 *Eur. Phys. J.* B **27** 363
[12] Edwards S F, 1994 *Granular Matter: An Interdisciplinary Approach* ed A Mehta (New York: Springer)
[13] Sobkowicz P, 2007 *Preprint* 0712.3027
[14] Dall'Asta L, Castellano C and Marsili M, 2008 in preparation
[15] Sen P, 1999 *J. Phys. A: Math. Gen.* **32** 1623
[16] Laurie A J and Jaggi N K, 2002 *Solid State Phys.* **45** 183

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