Abstract

This report investigates the accuracy of three finite difference schemes for the ordinary differential equation $u' = -au$ with the aid of numerical experiments. Numerical artifacts are in particular demonstrated.

Contents

1 Mathematical problem

We address the initial-value problem

$$u'(t) = -au(t), \quad t \in (0, T], \tag{1}$$
$$u(0) = I, \tag{2}$$

where a, I, and T are prescribed parameters, and $u(t)$ is the unknown function to be estimated. This mathematical model is relevant for physical phenomena featuring exponential decay in time.

2 Numerical solution method

We introduce a mesh in time with points $0 = t_0 < t_1 \cdots < t_N = T$. For simplicity, we assume constant spacing Δt between the mesh points: $\Delta t = t_n - t_{n-1}$, $n = 1, \ldots, N$. Let u^n be the numerical approximation to the exact solution at t_n.

The θ-rule is used to solve (**??**) numerically:

$$u^{n+1} = \frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t}u^n,$$

for $n = 0, 1, \ldots, N-1$. This scheme corresponds to

- The Forward Euler scheme when $\theta = 0$
- The Backward Euler scheme when $\theta = 1$
- The Crank-Nicolson scheme when $\theta = 1/2$

3 Implementation

The numerical method is implemented in a Python function:

```
def solver(I, a, T, dt, theta):
    """Solve u'=-a*u, u(0)=I, for t in (0,T] with steps of dt."""
    dt = float(dt)           # avoid integer division
```

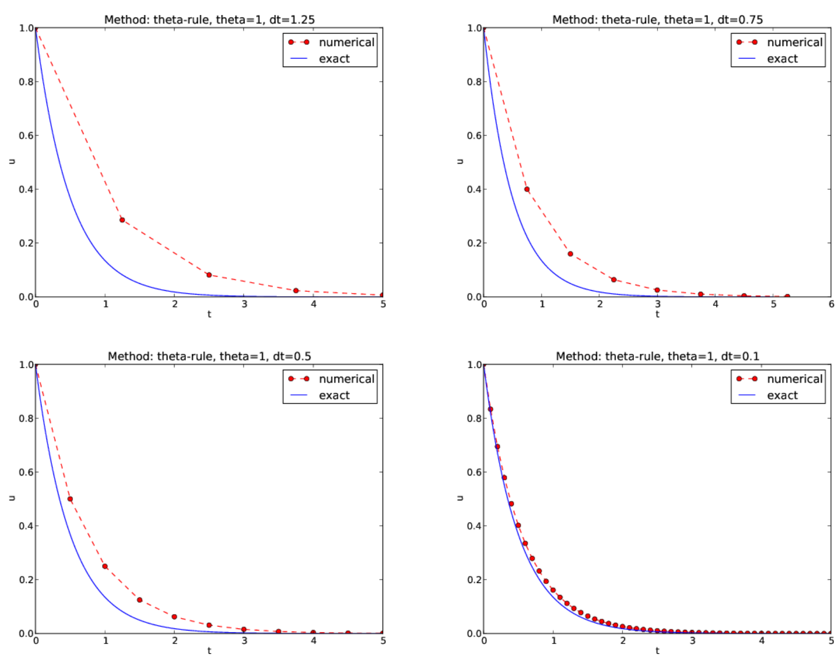
```
N = int(round(T/dt))      # no of time intervals
T = N*dt                  # adjust T to fit time step dt
u = zeros(N+1)            # array of u[n] values
t = linspace(0, T, N+1)   # time mesh

u[0] = I                  # assign initial condition
for n in range(0, N):     # n=0,1,...,N-1
    u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
return u, t
```

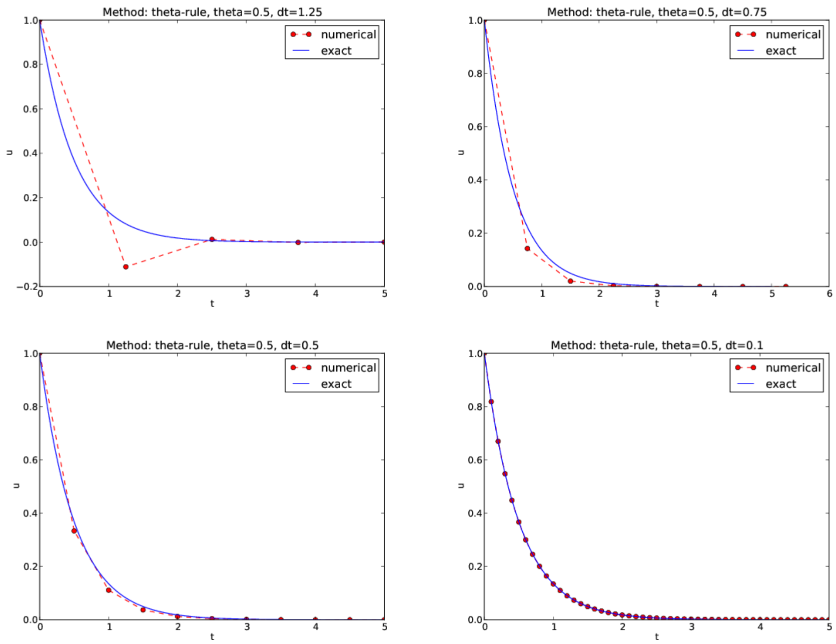
4 Numerical experiments

We define a set of numerical experiments where I, a, and T are fixed, while Δt and θ are varied. In particular, $I = 1$, $a = 2$, $\Delta t = 1.25, 0.75, 0.5, 0.1$.

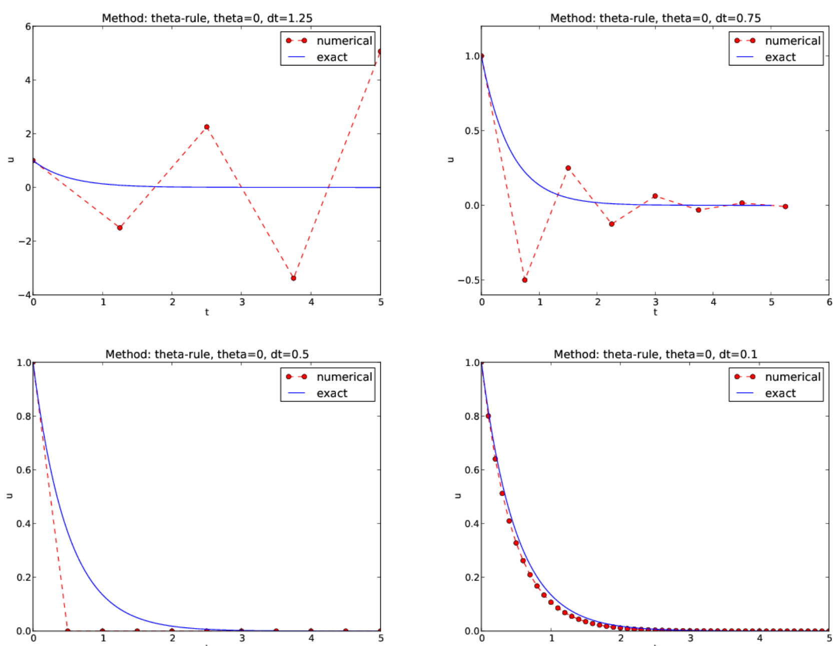
4.1 The Backward Euler method



4.2 The Crank-Nicolson method



4.3 The Forward Euler method



4.4 Error vs Δt

How E varies with Δt for $\theta = 0, 0.5, 1$ is shown in Figure **??**.

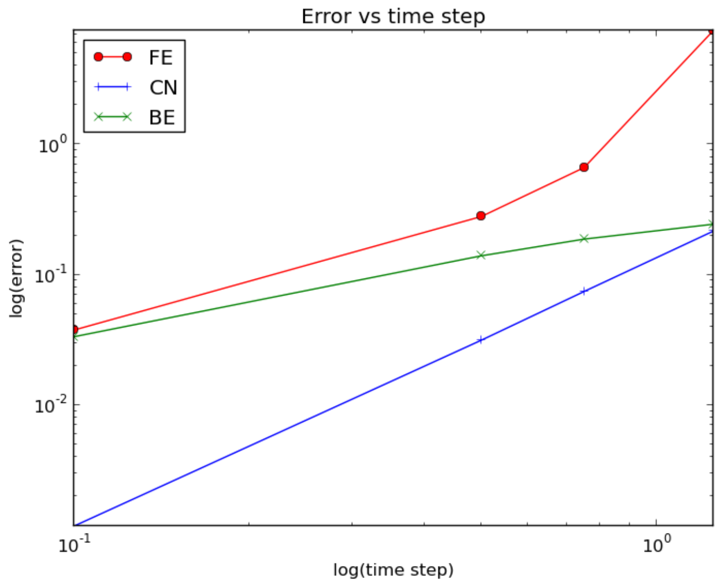


Figure 1: Error versus time step.