Experiments with Schemes for Exponential Decay

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Abstract

This report investigates the accuracy of three finite difference schemes for the ordinary differential equation $u' = -au$ with the aid of numerical experiments. Numerical artifacts are in particular demonstrated.

Contents

1 Mathematical problem

We address the initial-value problem

$$u'(t) = -au(t), \quad t \in (0, T], \tag{1}$$
$$u(0) = I, \tag{2}$$

where a, I, and T are prescribed parameters, and $u(t)$ is the unknown function to be estimated. This mathematical model is relevant for physical phenomena featuring exponential decay in time.

2 Numerical solution method

We introduce a mesh in time with points $0 = t_0 < t_1 \cdots < t_N = T$. For simplicity, we assume constant spacing Δt between the mesh points: $\Delta t = t_n - t_{n-1}$, $n = 1, \ldots, N$. Let u^n be the numerical approximation to the exact solution at t_n.

The θ-rule is used to solve (1) numerically:

$$u^{n+1} = \frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t} u^n,$$

for $n = 0, 1, \ldots, N-1$. This scheme corresponds to

- The Forward Euler[1] scheme when $\theta = 0$
- The Backward Euler[2] scheme when $\theta = 1$
- The Crank-Nicolson[3] scheme when $\theta = 1/2$

3 Implementation

The numerical method is implemented in a Python function `solver` (found in the `decay_mod`[4] module):

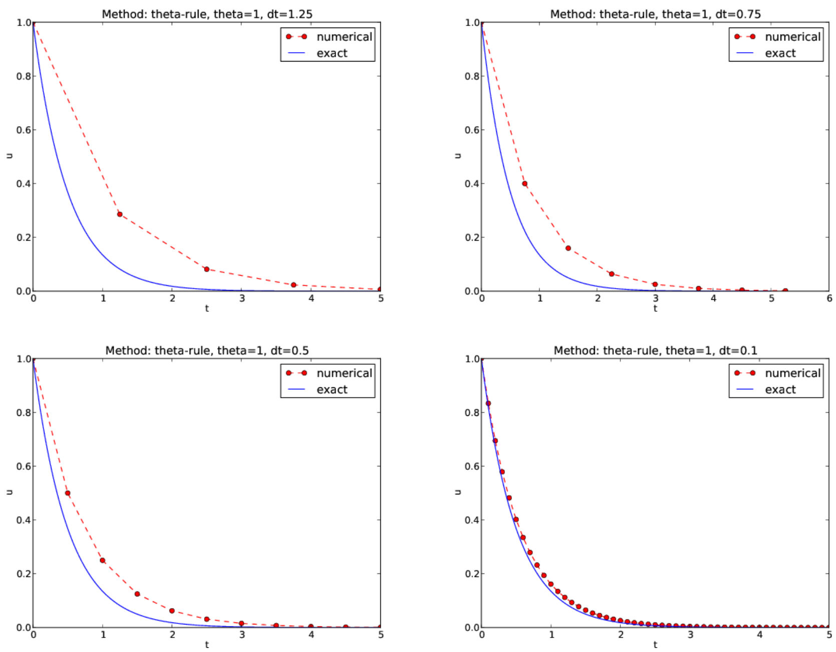
```
def solver(I, a, T, dt, theta):
    """Solve u'=-a*u, u(0)=I, for t in (0,T] with steps of dt."""
    dt = float(dt)            # avoid integer division
    N = int(round(T/dt))      # no of time intervals
    T = N*dt                  # adjust T to fit time step dt
    u = zeros(N+1)            # array of u[n] values
    t = linspace(0, T, N+1)   # time mesh

    u[0] = I                  # assign initial condition
    for n in range(0, N):     # n=0,1,...,N-1
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]
    return u, t
```

4 Numerical experiments

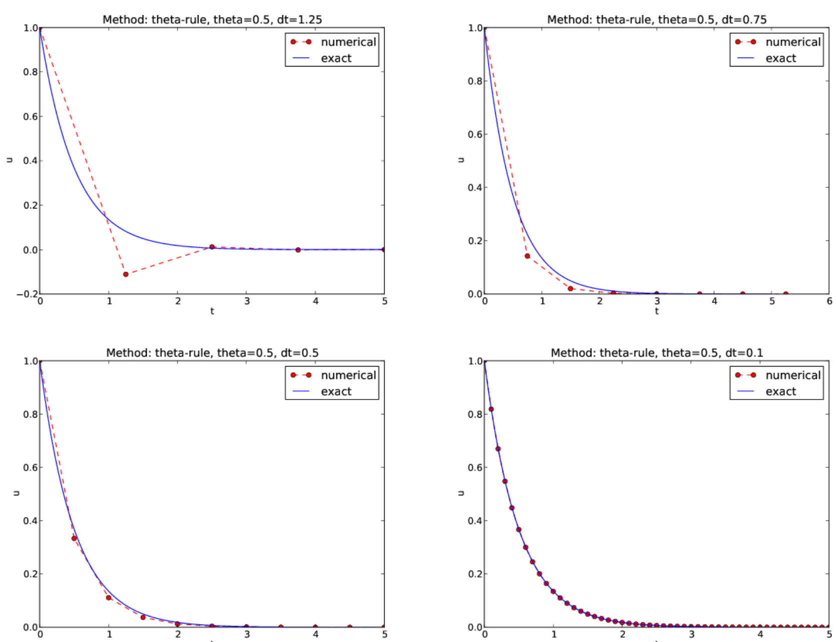
We define a set of numerical experiments where I, a, and T are fixed, while Δt and θ are varied. In particular, $I = 1$, $a = 2$, $\Delta t = 1.25, 0.75, 0.5, 0.1$.

[1] http://en.wikipedia.org/wiki/Forward_Euler_method
[2] http://en.wikipedia.org/wiki/Backward_Euler_method
[3] http://en.wikipedia.org/wiki/Crank-Nicolson
[4] https://github.com/hplgit/INF5620/blob/gh-pages/src/decay/experiments/dc_mod.py

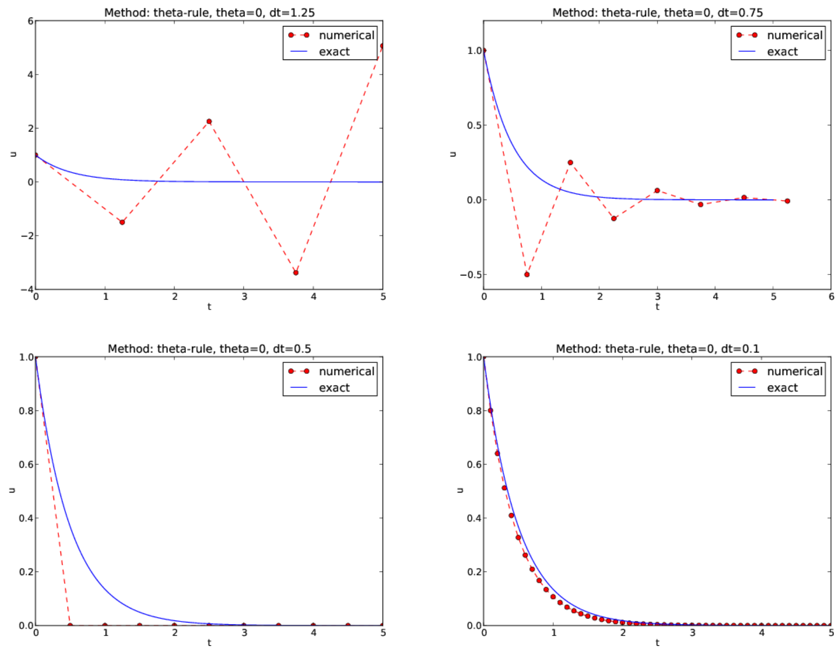
4.1 The Backward Euler method



4.2 The Crank-Nicolson method



4.3 The Forward Euler method



4.4 Error vs Δt

How E varies with Δt for $\theta = 0, 0.5, 1$ is shown in Figure 1.

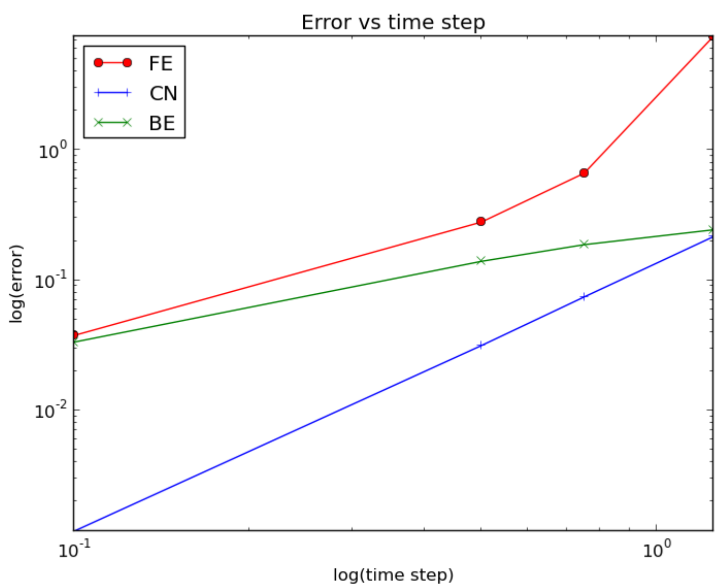


Figure 1: Error versus time step.