ANOTHER IMPROVED LOWER BOUND FOR EVEN n

For $z_1, \dots, z_n \in \mathbb{C}$ define

$$\Delta(z_1, \dots, z_n) := \prod_{i \neq j} |z_i - z_j|.$$

Let

$$\Delta_n^* := \sup\Big\{\Delta(z_1, \dots, z_n) : \max_{i,j} |z_i - z_j| \leq 2\Big\}.$$

Theorem 0.1. *Along even integers $n \to \infty$ one has*

$$\liminf_{\substack{n \to \infty \\ n \text{ even}}} \frac{\Delta_n^*}{n^n} \geq \exp\left(\frac{7}{24}\zeta(3) - \frac{\pi^4}{864}\right) > 1.26852.$$

In fact, there is an explicit family of configurations (with diameter 2) for which the ratio $\Delta(z_1, \dots, z_n)/n^n$ converges to the constant on the right-hand side.

0.1. **The triangular wave.** Let $\operatorname{tri} : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function defined by

$$\operatorname{tri}(x) := 1 - \frac{2}{\pi} \arccos(\cos x).$$

Equivalently, tri is the 2π-periodic extension of the piecewise linear function on $[-\pi, \pi]$ given by

$$\operatorname{tri}(x) = 1 - \frac{2}{\pi}|x| \qquad (x \in [-\pi, \pi]).$$

Define $g(\theta) := \operatorname{tri}(3\theta)$.

Lemma 0.2. *The function g satisfies:*

(1) $|g(\theta)| \leq 1$ for all θ.
(2) g is even: $g(-\theta) = g(\theta)$.
(3) g is π-antiperiodic: $g(\theta + \pi) = -g(\theta)$.
(4) g is Lipschitz with constant $L := 6/\pi$, i.e. $|g(\theta) - g(\phi)| \leq L|\theta - \phi|$.

Proof. Items (1) and (2) are immediate from the definition on $[-\pi, \pi]$. For (3), since $\operatorname{tri}(x+\pi) = -\operatorname{tri}(x)$ and $3(\theta+\pi) = 3\theta+3\pi \equiv 3\theta+\pi \pmod{2\pi}$, we get $g(\theta+\pi) = \operatorname{tri}(3\theta+3\pi) = \operatorname{tri}(3\theta+\pi) = -\operatorname{tri}(3\theta) = -g(\theta)$. For (4), tri has slope $\pm 2/\pi$ on each linear piece, hence is $(2/\pi)$-Lipschitz. Composing with 3θ multiplies the Lipschitz constant by 3, giving $L = 6/\pi$. □

0.2. **The configuration for even n.** Fix an even integer $n = 2m$. Let

$$\theta_k := \frac{2\pi k}{n} \qquad (k = 0, 1, \dots, n-1), \qquad \zeta_k := e^{i\theta_k}.$$

Define the (small) parameter

$$t_n := \frac{\pi^2}{12n}\left(1 - \frac{1}{n}\right),$$

and the points

$$z_k := (1 + t_n g(\theta_k))\,\zeta_k, \qquad k = 0, 1, \dots, n-1.$$

0.3. **Diameter bound: $|z_i - z_j| \le 2$ for large even n.**

Lemma 0.3. *There exists N_0 such that for all even $n \ge N_0$ the configuration $\{z_k\}_{k=0}^{n-1}$ satisfies*

$$\max_{i,j} |z_i - z_j| \le 2.$$

Moreover, for every i we have $|z_i - z_{i+m}| = 2$.

Proof. Write $r_k := 1 + t_n g(\theta_k)$, so $z_k = r_k e^{i\theta_k}$.

Antipodal pairs. Using $\zeta_{k+m} = -\zeta_k$ and Lemma 0.2(3), $g(\theta_{k+m}) = -g(\theta_k)$, we get

$$z_{k+m} = (1 - t_n g(\theta_k))\, e^{i(\theta_k+\pi)} = -(1 - t_n g(\theta_k)) e^{i\theta_k},$$

hence $z_k - z_{k+m} = 2e^{i\theta_k}$ and therefore $|z_k - z_{k+m}| = 2$.

General pairs. Fix $i \neq j$ and set $\beta := |\theta_i - \theta_j| \in (0, \pi]$ (take the smaller arc). Then

$$|z_i - z_j|^2 = r_i^2 + r_j^2 - 2r_i r_j \cos\beta.$$

Case 1: $\beta \le \pi/2$. Then $\cos\beta \ge 0$, hence $|z_i - z_j|^2 \le r_i^2 + r_j^2 \le 2(1+t_n)^2$. Since $t_n \le \pi^2/24 < \sqrt{2} - 1$, we have $2(1+t_n)^2 < 4$, so $|z_i - z_j| < 2$.

Case 2: $\beta \in (\pi/2, \pi)$. Write $\beta = \pi - \alpha$ where $\alpha \in (0, \pi/2)$. Then $\cos\beta = -\cos\alpha$ and

$$|z_i - z_j|^2 = r_i^2 + r_j^2 + 2r_i r_j \cos\alpha.$$

We may write

$$\theta_j \equiv \theta_i + \pi - \varepsilon\alpha \pmod{2\pi} \quad \text{for some } \varepsilon \in \{\pm 1\},$$

with $\alpha = \dfrac{2\pi\ell}{n}$ for some integer $\ell \in \{1, 2, \ldots, \lfloor n/4 \rfloor - 1\}$. By Lemma 0.2(3),

$$g(\theta_j) = g(\theta_i + \pi - \varepsilon\alpha) = -g(\theta_i - \varepsilon\alpha).$$

Set

$$a := g(\theta_i), \qquad b := g(\theta_i - \varepsilon\alpha).$$

Then $r_i = 1 + t_n a$ and $r_j = 1 - t_n b$. A direct expansion gives

$$|z_i - z_j|^2 - 4 = -4\sin^2\frac{\alpha}{2} + 2t_n(a-b)(1+\cos\alpha) + t_n^2\Big[(a-b)^2 + 2(1-\cos\alpha)\, ab\Big]. \tag{0.1}$$

We bound each term from above. By Lemma 0.2(4), $|a - b| \le L\alpha$ with $L = 6/\pi$. Also $|ab| \le 1$ and $1 - \cos\alpha \le \alpha^2/2$, hence

$$(a-b)^2 + 2(1-\cos\alpha)ab \le (L\alpha)^2 + \alpha^2 = (L^2+1)\alpha^2.$$

Moreover, for all $\alpha \ge 0$ one has $\sin x \ge x - x^3/6$; applying this with $x = \alpha/2$ yields $\sin^2(\alpha/2) \ge \alpha^2/4 - \alpha^4/48$, hence

$$-4\sin^2\frac{\alpha}{2} \le -\alpha^2 + \frac{\alpha^4}{12}.$$

Finally $1 + \cos\alpha \le 2$, so $2t_n(a-b)(1+\cos\alpha) \le 4t_n L\alpha$. Putting these bounds into (0.1) gives

$$|z_i - z_j|^2 - 4 \le -\alpha^2 + \frac{\alpha^4}{12} + 4t_n L\alpha + (L^2+1)t_n^2\alpha^2. \tag{0.2}$$

Now substitute $\alpha = \frac{2\pi\ell}{n}$ and $t_n = \frac{\pi^2}{12n}(1 - \frac{1}{n})$. We now show that the right-hand side of (0.2) is ≤ 0 for all even $n \ge 8$. Since $t_n \le \pi^2/(12n)$ and $L^2 + 1 = (\pi^2+36)/\pi^2$, we obtain

$$4t_n L\alpha = 4 \cdot \frac{\pi^2}{12n}\Big(1 - \frac{1}{n}\Big) \cdot \frac{6}{\pi} \cdot \frac{2\pi\ell}{n} = \frac{4\pi^2\ell}{n^2}\Big(1 - \frac{1}{n}\Big),$$

$$\frac{\alpha^4}{12} = \frac{1}{12}\Big(\frac{2\pi\ell}{n}\Big)^4 = \frac{4}{3}\frac{\pi^4\ell^4}{n^4},$$

and

$$(L^2+1)t_n^2\alpha^2 \le \frac{\pi^2+36}{\pi^2}\cdot\frac{\pi^4}{144n^2}\cdot\Big(\frac{2\pi\ell}{n}\Big)^2 = \frac{\pi^4(\pi^2+36)}{36}\cdot\frac{\ell^2}{n^4}.$$

Moreover,

$$-\alpha^2+4t_nL\alpha = -\Big(\frac{2\pi\ell}{n}\Big)^2+\frac{4\pi^2\ell}{n^2}\Big(1-\frac{1}{n}\Big) = -\frac{4\pi^2}{n^2}\Big(\ell(\ell-1)+\frac{\ell}{n}\Big).$$

Therefore (0.2) implies the explicit bound

$$|z_i-z_j|^2-4 \le -\frac{4\pi^2}{n^2}\Big(\ell(\ell-1)+\frac{\ell}{n}\Big)+\frac{\pi^4}{n^4}\Big(\frac{4}{3}\ell^4+\frac{\pi^2+36}{36}\ell^2\Big) =: E_{n,\ell}. \tag{0.3}$$

We will check $E_{n,\ell}\le 0$ by cases.

Case 1: $\ell=1$. Then (0.3) gives

$$E_{n,1}\le -\frac{4\pi^2}{n^3}+\frac{\pi^4}{n^4}\Big(\frac{7}{3}+\frac{\pi^2}{36}\Big).$$

Thus $E_{n,1}\le 0$ follows as soon as $n\ge \frac{\pi^2}{4}\left(\frac{7}{3}+\frac{\pi^2}{36}\right)\approx 6.43372$. Hence $E_{n,1}\le 0$ for all $n\ge 8$.

Case 2: $\ell\ge 2$. Since $\ell(\ell-1)\ge \ell^2/2$ for $\ell\ge 2$, (0.3) yields

$$E_{n,\ell}\le -\frac{2\pi^2\ell^2}{n^2}+\frac{\pi^4}{n^4}\Big(\frac{4}{3}\ell^4+\frac{\pi^2+36}{36}\ell^2\Big) = \frac{\ell^2}{n^4}\Big(-2\pi^2n^2+\pi^4\Big(\frac{4}{3}\ell^2+\frac{\pi^2+36}{36}\Big)\Big).$$

Using $\ell\le n/4$ we have

$$E_{n,\ell}\le \frac{\ell^2}{n^4}\Big(n^2\Big(-2\pi^2+\frac{\pi^4}{12}\Big)+\frac{\pi^4(\pi^2+36)}{36}\Big).$$

For $n\ge 8$ we have

$$n^2\Big(2-\frac{\pi^2}{12}\Big) \;\ge\; \frac{\pi^2(\pi^2+36)}{36},$$

so indeed $E_{n,\ell}\le 0$ for all $\ell\ge 2$ whenever $n\ge 8$.

Combining the two cases, we have $E_{n,\ell}\le 0$ for every $1\le\ell\le n/4$ once $n\ge 8$. Hence (0.3) implies $|z_i-z_j|^2-4\le 0$, i.e. $|z_i-z_j|\le 2$, for all near–antipodal pairs. □

0.4. **Factorization of Δ and a uniform bound on ρ_{ij}.** For $i\ne j$ define

$$v_k := g(\theta_k)\zeta_k, \qquad \rho_{ij} := \frac{v_i-v_j}{\zeta_i-\zeta_j}.$$

Then

$$z_i-z_j=(\zeta_i-\zeta_j)(1+t_n\rho_{ij}),$$

and therefore

$$\frac{\Delta(z_0,\dots,z_{n-1})}{\prod_{i\ne j}|\zeta_i-\zeta_j|} = \prod_{i\ne j}|1+t_n\rho_{ij}|. \tag{0.4}$$

Lemma 0.4. *For $\zeta_k=e^{2\pi ik/n}$ one has*

$$\prod_{i\ne j}|\zeta_i-\zeta_j| = n^n.$$

Proof. Let $p(z) = z^n - 1 = \prod_{j=0}^{n-1}(z - \zeta_j)$. For each root ζ_i,

$$p'(\zeta_i) = n\zeta_i^{n-1} = \prod_{j \neq i}(\zeta_i - \zeta_j).$$

Taking absolute values and then the product over i gives

$$\prod_i \prod_{j \neq i} |\zeta_i - \zeta_j| = \prod_i |p'(\zeta_i)| = \prod_i n = n^n. \qquad \square$$

Combining Lemma 0.4 with (0.4) yields

$$\frac{\Delta(z_0, \ldots, z_{n-1})}{n^n} = \prod_{i \neq j} |1 + t_n \rho_{ij}|.$$

Lemma 0.5. *There is a constant M independent of n such that $|\rho_{ij}| \leq M$ for all $i \neq j$. In fact one may take $M = 4$.*

Proof. Write $g_i := g(\theta_i)$. Then

$$v_i - v_j = g_i(\zeta_i - \zeta_j) + (g_i - g_j)\zeta_j,$$

hence

$$\rho_{ij} = g_i + (g_i - g_j)\frac{\zeta_j}{\zeta_i - \zeta_j}.$$

Therefore

$$|\rho_{ij}| \leq |g_i| + \frac{|g_i - g_j|}{|\zeta_i - \zeta_j|} \leq 1 + \frac{L|\theta_i - \theta_j|}{2|\sin((\theta_i - \theta_j)/2)|}.$$

For $0 < x \leq \pi$, concavity of sin on $[0, \pi/2]$ gives $\sin(x/2) \geq \frac{x}{\pi}$, hence

$$\frac{x}{2\sin(x/2)} \leq \frac{\pi}{2}.$$

Thus $|\rho_{ij}| \leq 1 + \frac{L\pi}{2} = 1 + \frac{6}{\pi} \cdot \frac{\pi}{2} = 4$. $\square$

0.5. Second-order expansion of $\log(\Delta/n^n)$.

Taking logs,

$$\log \frac{\Delta(z_0, \ldots, z_{n-1})}{n^n} = \sum_{i \neq j} \log|1 + t_n \rho_{ij}|. \tag{0.5}$$

Lemma 0.6. *Assume $|u| \leq 1/2$. Then*

$$\log|1 + u| = \Re\left(u - \frac{u^2}{2}\right) + R(u), \qquad |R(u)| \leq \frac{2}{3}|u|^3.$$

Proof. For $|u| < 1$ we have $\log(1 + u) = \sum_{k \geq 1}(-1)^{k+1}u^k/k$. Taking real parts gives $\log|1 + u| = \Re \log(1 + u)$. The tail satisfies

$$\left|\sum_{k \geq 3} \frac{(-1)^{k+1}}{k} u^k\right| \leq \sum_{k \geq 3} \frac{|u|^k}{k} \leq \frac{1}{3} \sum_{k \geq 3} |u|^k = \frac{|u|^3}{3(1 - |u|)} \leq \frac{2}{3}|u|^3. \qquad \square$$

Lemma 0.7. *Let $n = 2m$ be even. Then $\sum_{i \neq j} \rho_{ij} = 0$.*

Proof. We have $\zeta_{k+m} = -\zeta_k$. By Lemma 0.2(3), $g(\theta_{k+m}) = -g(\theta_k)$, hence

$$v_{k+m} = g(\theta_{k+m})\zeta_{k+m} = (-g(\theta_k))(-\zeta_k) = v_k.$$

Thus for any $i \neq j$,

$$\rho_{i+m,\, j+m} = \frac{v_{i+m} - v_{j+m}}{\zeta_{i+m} - \zeta_{j+m}} = \frac{v_i - v_j}{-(\zeta_i - \zeta_j)} = -\rho_{ij}.$$

The map $(i, j) \mapsto (i + m, j + m)$ is a bijection on ordered pairs $i \neq j$, so the sum cancels. $\square$

Lemma 0.8. *Along even $n \to \infty$,*

$$\log \frac{\Delta(z_0, \ldots, z_{n-1})}{n^n} = -\frac{t_n^2}{2} \sum_{i \neq j} \Re(\rho_{ij}^2) + o(1).$$

Proof. For all sufficiently large n, Lemma 0.5 and $t_n \leq \pi^2/(12n)$ imply $|t_n\rho_{ij}| \leq 1/2$. Apply Lemma 0.6 to (0.5) termwise:

$$\log \frac{\Delta}{n^n} = \sum_{i \neq j} \Re\left(t_n\rho_{ij} - \frac{t_n^2}{2}\rho_{ij}^2\right) + \sum_{i \neq j} R(t_n\rho_{ij}).$$

By Lemma 0.7, the linear term vanishes. For the remainder, using $|R(u)| \leq \frac{2}{3}|u|^3$ and $|\rho_{ij}| \leq M$,

$$\left| \sum_{i \neq j} R(t_n\rho_{ij}) \right| \leq \frac{2}{3}\, n(n-1)\, (t_n M)^3 = O(n^2 t_n^3) = O(1/n) \to 0. \qquad \square$$

0.6. **A Riemann-sum limit for $\sum \Re(\rho_{ij}^2)$.** Define for $x, y \in [0, 2\pi]$,

$$\xi(x) := e^{ix}, \qquad f(x) := g(x)e^{ix}, \qquad \rho(x, y) := \frac{f(x) - f(y)}{\xi(x) - \xi(y)} \quad (x \neq y),$$

and $F(x, y) := \Re(\rho(x, y)^2)$ for $x \neq y$, with $F(x, x) := 0$. Then $\rho(\theta_i, \theta_j) = \rho_{ij}$.

Lemma 0.9. *The function F is bounded on $[0, 2\pi]^2$ and is continuous off the diagonal $\{x = y\}$. Hence F is Riemann integrable and*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} F(\theta_i, \theta_j) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F(x, y)\, dx\, dy.$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i \neq j} \Re(\rho_{ij}^2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \Re(\rho(x, y)^2)\, dx\, dy =: J.$$

Proof. Boundedness follows from the same estimate as in Lemma 0.5 (with $\theta_i - \theta_j$ replaced by $x - y$). Continuity holds for $x \neq y$ since the denominator is nonzero. The diagonal has measure 0, hence F is Riemann integrable. For Riemann integrable functions, uniform-grid Riemann sums converge to the integral. Finally, adding/removing the diagonal terms changes the sum by at most $O(n)$, hence $o(n^2)$ after normalization. $\square$

0.7. **Computing J via Fejér approximation and Fourier orthogonality.**

0.7.1. *Fourier series of the triangular wave.*

Lemma 0.10. *The 2π-periodic function* tri *has the Fourier expansion*

$$\mathrm{tri}(x) = \sum_{\substack{k\geq 1\\ k \text{ odd}}} \frac{8}{\pi^2 k^2}\cos(kx),$$

with absolute (hence uniform) convergence. Consequently,

$$g(\theta) = \mathrm{tri}(3\theta) = \sum_{\substack{r\geq 1\\ r \text{ odd}}} \frac{8}{\pi^2 r^2}\cos(3r\theta).$$

Proof. Since tri is even and has mean 0, only cosine coefficients appear:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{tri}(x)\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos(kx)\,dx.$$

The first term integrates to 0. For the second term, integration by parts gives

$$\int_0^{\pi} x\cos(kx)\,dx = \frac{(-1)^k - 1}{k^2},$$

hence

$$a_k = \frac{4}{\pi^2}\cdot\frac{1-(-1)^k}{k^2} = \begin{cases} \frac{8}{\pi^2 k^2}, & k \text{ odd},\\ 0, & k \text{ even}.\end{cases}$$

Absolute convergence follows from $\sum_{k \text{ odd}} 1/k^2 < \infty$. □

0.7.2. *A kernel orthogonality identity.* For integers k define, for $x \neq y$,

$$R_k(x,y) := \frac{e^{ikx} - e^{iky}}{e^{ix} - e^{iy}}.$$

Lemma 0.11. *For integers k, ℓ one has*

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R_k(x,y)\,R_\ell(x,y)\,dx\,dy = \begin{cases} 1-|k-1|, & k+\ell = 2,\\ 0, & k+\ell \neq 2.\end{cases}$$

Proof. Let $z = e^{ix}$ and $w = e^{iy}$. If $k \geq 1$ then

$$R_k(x,y) = \frac{z^k - w^k}{z-w} = \sum_{m=0}^{k-1} z^{k-1-m}w^m = \sum_{m=0}^{k-1} e^{i((k-1-m)x+my)}.$$

If $k \leq 0$ then, writing $k = -p$ with $p \geq 0$,

$$R_{-p}(x,y) = \frac{z^{-p} - w^{-p}}{z-w} = \frac{w^p - z^p}{z^p w^p (z-w)} = -\frac{z^p - w^p}{z^p w^p (z-w)} = -\sum_{m=0}^{p-1} z^{-(m+1)} w^{-(p-m)}.$$

In all cases, $R_k R_\ell$ is a finite sum of exponentials $e^{i(ax+by)}$. Using

$$\frac{1}{2\pi}\int_0^{2\pi} e^{iax}\,dx = \begin{cases} 1, & a = 0,\\ 0, & a \neq 0,\end{cases}$$

one checks that a nonzero contribution can occur only when the total x-frequency and y-frequency both vanish, which forces $k+\ell = 2$. When $k+\ell = 2$, the number of surviving terms is $1-|k-1|$, and each contributes 1. □

0.7.3. *Fejér approximation to justify passage to the Fourier side.* Let K_N be the *Fejér kernel*

$$K_N(t) := \frac{1}{N+1}\left(\frac{\sin((N+1)t/2)}{\sin(t/2)}\right)^2 \geq 0, \qquad \frac{1}{2\pi}\int_0^{2\pi} K_N(t)\,dt = 1.$$

Define the *Fejér mean* of a 2π-periodic continuous f by

$$f^{(N)}(x) := \frac{1}{2\pi}\int_0^{2\pi} f(x-t)\,K_N(t)\,dt.$$

Then $f^{(N)}$ is a trigonometric polynomial of degree $\leq N$ and $f^{(N)} \to f$ uniformly as $N \to \infty$.

Lemma 0.12. *If f is Lipschitz with constant* $\mathrm{Lip}(f)$*, then* $\mathrm{Lip}(f^{(N)}) \leq \mathrm{Lip}(f)$ *for all N. Moreover, for all $x \neq y$,*

$$\left|\frac{f^{(N)}(x) - f^{(N)}(y)}{e^{ix} - e^{iy}}\right| \leq \frac{\pi}{2}\,\mathrm{Lip}(f),$$

uniformly in N.

Proof. Since K_N is a probability kernel,

$$\begin{aligned}|f^{(N)}(x) - f^{(N)}(y)| &= \left|\frac{1}{2\pi}\int_0^{2\pi}(f(x-t) - f(y-t))K_N(t)\,dt\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Lip}(f)\,|x-y|\,K_N(t)\,dt = \mathrm{Lip}(f)\,|x-y|.\end{aligned}$$

Thus $\mathrm{Lip}(f^{(N)}) \leq \mathrm{Lip}(f)$. Also $|e^{ix} - e^{iy}| = 2|\sin((x-y)/2)|$ and for $0 < |x-y| \leq \pi$ we have $\sin(|x-y|/2) \geq |x-y|/\pi$, hence $|x-y|/|e^{ix} - e^{iy}| \leq \pi/2$. $\square$

For our specific $f(x) = g(x)e^{ix}$, Lemma 0.2 implies $|g| \leq 1$ and $\mathrm{Lip}(g) \leq L$. Hence

$$|f(x) - f(y)| \leq |g(x) - g(y)| + |g(y)|\,|e^{ix} - e^{iy}| \leq L|x-y| + |x-y| = (L+1)|x-y|,$$

so $\mathrm{Lip}(f) \leq L+1$.

Lemma 0.13. *Let* $\rho(x,y) = \frac{f(x)-f(y)}{e^{ix}-e^{iy}}$ *as above and define*

$$B(f) := \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\rho(x,y)^2\,dx\,dy.$$

Then $B(f)$ is real and equals

$$B(f) = \sum_{k\in\mathbb{Z}}(1 - |k-1|)\,a_k\,a_{2-k},$$

where $f(x) = \sum_{k\in\mathbb{Z}} a_k e^{ikx}$ *is the Fourier series of f.*

Proof. Let $f^{(N)}$ be Fejér means. Define

$$\rho_N(x,y) := \frac{f^{(N)}(x) - f^{(N)}(y)}{e^{ix} - e^{iy}}.$$

By uniform convergence $f^{(N)} \to f$, we have $\rho_N(x,y) \to \rho(x,y)$ for every $x \neq y$. By Lemma 0.12, $|\rho_N(x,y)| \leq \frac{\pi}{2}\,\mathrm{Lip}(f)$ uniformly in N and (x,y). Hence dominated convergence applies and

$$B(f) = \lim_{N\to\infty}\frac{1}{4\pi^2}\iint \rho_N(x,y)^2\,dx\,dy.$$

Now $f^{(N)}$ is a trigonometric polynomial, say $f^{(N)}(x) = \sum_{|k|\le N} a_k^{(N)} e^{ikx}$. Then for $x \neq y$,

$$\rho_N(x,y) = \sum_{|k|\le N} a_k^{(N)}\, R_k(x,y),$$

a finite sum. Therefore,

$$\frac{1}{4\pi^2}\iint \rho_N(x,y)^2\,dx\,dy = \sum_{|k|,|\ell|\le N} a_k^{(N)} a_\ell^{(N)} \cdot \frac{1}{4\pi^2}\iint R_k R_\ell\,dx\,dy.$$

By Lemma 0.11, only terms with $k+\ell = 2$ survive, giving

$$\frac{1}{4\pi^2}\iint \rho_N(x,y)^2\,dx\,dy = \sum_{k\in\mathbb{Z}} (1-|k-1|) a_k^{(N)} a_{2-k}^{(N)}.$$

For Fejér means, $a_k^{(N)} = (1-|k|/(N+1))a_k$ for $|k|\le N$ and 0 otherwise, hence $a_k^{(N)} \to a_k$ for each fixed k. In our application the resulting infinite series is absolutely convergent (indeed it will reduce to $\sum_{r \text{ odd}} O(1/r^3)$), so we may pass $N\to\infty$ termwise. This yields the desired series for $B(f)$. Since the series is real (coefficients are real in our case), $B(f) \in \mathbb{R}$. $\square$

Finally, since $B(f)$ is real,

$$J = \frac{1}{4\pi^2}\iint \Re(\rho^2) = \Re B(f) = B(f).$$

0.7.4. *Evaluating the series for our f.* By Lemma 0.10,

$$g(\theta) = \sum_{\substack{r\ge 1\\ r \text{ odd}}} \frac{8}{\pi^2 r^2}\cos(3r\theta).$$

Hence

$$f(\theta) = g(\theta)e^{i\theta} = \sum_{\substack{r\ge 1\\ r \text{ odd}}} \frac{4}{\pi^2 r^2}\left(e^{i(1+3r)\theta} + e^{i(1-3r)\theta}\right).$$

Therefore the Fourier coefficients of f satisfy

$$a_{1+3r} = a_{1-3r} = \frac{4}{\pi^2 r^2} \quad (r\ge 1,\ r \text{ odd}), \qquad a_k = 0 \text{ otherwise.}$$

In Lemma 0.13, only pairs $(k, 2-k)$ contribute. The only nonzero pairings are $k = 1+3r$ with $2-k = 1-3r$ (and vice versa), and for such k, $1-|k-1| = 1-3r$. Thus

$$J = \sum_{r \text{ odd}\ge 1} 2(1-3r)\left(\frac{4}{\pi^2 r^2}\right)^2 = \frac{32}{\pi^4}\sum_{r \text{ odd}\ge 1}\frac{1-3r}{r^4}.$$

Using

$$\sum_{r \text{ odd}} \frac{1}{r^4} = \left(1-\frac{1}{2^4}\right)\zeta(4) = \frac{15}{16}\cdot\frac{\pi^4}{90} = \frac{\pi^4}{96}, \qquad \sum_{r \text{ odd}} \frac{1}{r^3} = \left(1-\frac{1}{2^3}\right)\zeta(3) = \frac{7}{8}\zeta(3),$$

we obtain

$$J = \frac{32}{\pi^4}\left(\frac{\pi^4}{96} - 3\cdot\frac{7}{8}\zeta(3)\right) = \frac{1}{3} - \frac{84\,\zeta(3)}{\pi^4}.$$

In particular $J < 0$.

0.8. **Asymptotic value of Δ/n^n.** By Lemma 0.8 and Lemma 0.9,

$$\log \frac{\Delta(z_0,\ldots,z_{n-1})}{n^n} = -\frac{t_n^2}{2}\Big(n^2 J + o(n^2)\Big) + o(1) = -\frac{(nt_n)^2}{2} J + o(1).$$

Since $nt_n \to \pi^2/12$,

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \log \frac{\Delta(z_0,\ldots,z_{n-1})}{n^n} = -\frac{1}{2}\left(\frac{\pi^2}{12}\right)^2 \left(\frac{1}{3} - \frac{84\zeta(3)}{\pi^4}\right) = \frac{7}{24}\zeta(3) - \frac{\pi^4}{864}.$$

Exponentiating,

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} \frac{\Delta(z_0,\ldots,z_{n-1})}{n^n} = \exp\left(\frac{7}{24}\zeta(3) - \frac{\pi^4}{864}\right).$$

By Lemma 0.3, for all sufficiently large even n the configuration is feasible (diameter ≤ 2), hence $\Delta_n^* \ge \Delta(z_0,\ldots,z_{n-1})$ and therefore

$$\liminf_{\substack{n\to\infty \\ n \text{ even}}} \frac{\Delta_n^*}{n^n} \ge \exp\left(\frac{7}{24}\zeta(3) - \frac{\pi^4}{864}\right).$$