COUNTEREXAMPLES FOR PROBLEM #1045

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1. Introduction

On Bloom's Erdős Problems website [1], Problem #1045 asks the following:

Problem 1.1. *Let $z_1, \dots, z_n \in \mathbb{C}$ with $|z_i - z_j| \le 2$ for all i, j, and*

$$\Delta(z_1, \dots, z_n) = \prod_{i \neq j} |z_i - z_j|.$$

Is Δ maximised by taking the z_i to be the vertices of a regular n–gon?

In this short note we show that the answer is negative in general. We construct explicit counterexamples for $n = 4$ and $n = 6$ where the regular n–gon fails to maximise Δ. For $n = 5$ our computations support the regular pentagon as a maximiser, suggesting that the problem may hold for odd n, while it fails at least for some even values of n.

2. An explicit counterexample for $n = 4$

Take

$$z_1 = (0, 0), \quad z_2 = (\sqrt{3}, 1), \quad z_3 = (\sqrt{3}, -1), \quad z_4 = (2, 0).$$

Pairwise distances (unordered):

$$|z_1 z_2| = |z_1 z_3| = |z_1 z_4| = |z_2 z_3| = 2, \qquad |z_2 z_4| = |z_3 z_4| = 2\sqrt{2 - \sqrt{3}}.$$

Therefore

$$\prod_{i<j} |z_i - z_j| = 2^4 \cdot \left(2\sqrt{2 - \sqrt{3}}\right)^2 = 16 \cdot 4(2 - \sqrt{3}) = 64(2 - \sqrt{3}),$$

and hence

$$\Delta_{\text{ours}} = \left[64(2 - \sqrt{3})\right]^2 = 4096(7 - 4\sqrt{3}) \approx 294.0796.$$

For comparison, consider the regular quadrilateral (square) with radius $R = 1$, given by the four vertices $(\pm 1, 0), (0, \pm 1)$. Its distances are four edges of length $\sqrt{2}$ and two diagonals of length 2. Thus

$$\prod_{i<j} d = (\sqrt{2})^4 \cdot 2^2 = 16, \qquad \Delta_{\square} = 16^2 = 256.$$

Hence,

$$\Delta_{\text{ours}} \approx 294.08 \ > \ \Delta_{\square} = 256,$$

so the square is not a maximiser when $n = 4$.

3. A computer-verified counterexample for $n = 6$

Consider the six-point set

$$\begin{aligned} z_1 &= (-1.00000000,\ 0.00000000), & z_2 &= (\ \ 1.00000000,\ 0.00000000), \\ z_3 &= (\ \ 0.61214607,\ -1.18363214), & z_4 &= (-0.83657305,\ -0.79183296), \\ z_5 &= (-0.10715844,\ \ \ 0.64741216), & z_6 &= (-0.28640287,\ -1.34450602). \end{aligned} \tag{3.1}$$

Direct computation shows that the maximal pairwise distance among the six points is exactly 2 (numerically within 10^{-9}). The product of unordered distances satisfies

$$\prod_{1 \le i < j \le 6} |z_i - z_j| \ \approx\ 218.970266 \quad \Longrightarrow \quad \boxed{\Delta\ \approx\ 47920.81584}.$$

For comparison, the regular hexagon (circumradius 1) has $\prod_{i<j} |z_i - z_j| = 6^3 = 216$ and hence

$$\Delta(\text{regular hexagon}) = 6^6 = 46656.$$

Therefore

$$\Delta\big(\text{configuration (3.1)}\big) - \Delta(\text{regular hexagon})\ \approx\ 1264.81584\ >\ 0.$$

4. Remarks

For $n = 5$ we performed extensive random sampling and local optimisation. We consistently observed

$$\Delta(\text{any tested configuration})\ <\ \Delta(\text{regular pentagon with longest chord } = 2)\ \approx\ 8525.57169,$$

with the best non-regular samples around 7585.03. This numerical evidence suggests that, at least for $n = 5$, the regular pentagon remains optimal. It is therefore plausible that for odd n the original statement of Problem #1045 might still hold, although we do not make any definitive claim.

Acknowledgements

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References

[1] T. F. Bloom, Erdős Problem #1045, `https://www.erdosproblems.com/1045`, accessed 2025-10-03

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