

ANOTHER IMPROVED LOWER BOUND FOR EVEN n

For $z_1, \dots, z_n \in \mathbb{C}$ define

$$\Delta(z_1, \dots, z_n) := \prod_{i \neq j} |z_i - z_j|.$$

Let

$$\Delta_n^* := \sup \left\{ \Delta(z_1, \dots, z_n) : \max_{i,j} |z_i - z_j| \leq 2 \right\}.$$

Theorem 0.1. *Along even integers $n \rightarrow \infty$ one has*

$$\liminf_{\substack{n \rightarrow \infty \\ n \text{ even}}} \frac{\Delta_n^*}{n^n} \geq \exp \left(\frac{7}{24} \zeta(3) - \frac{\pi^4}{864} \right) \approx 1.26853.$$

0.1. **The triangular wave.** Let $\text{tri} : \mathbb{R} \rightarrow \mathbb{R}$ be the 2π -periodic function defined by

$$\text{tri}(x) := 1 - \frac{2}{\pi} \arccos(\cos x).$$

Equivalently, tri is the 2π -periodic extension of the piecewise linear function on $[-\pi, \pi]$ given by

$$\text{tri}(x) = 1 - \frac{2}{\pi} |x| \quad (x \in [-\pi, \pi]).$$

Define $g(\theta) := \text{tri}(3\theta)$. For $x, y \in \mathbb{R}$ define the circular distance

$$d(x, y) := \min_{k \in \mathbb{Z}} |x - y - 2\pi k| \in [0, \pi].$$

Lemma 0.2. *The function $g(\theta) := \text{tri}(3\theta)$ satisfies:*

- (1) $|g(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$.
- (2) g is even: $g(-\theta) = g(\theta)$.
- (3) g is π -antiperiodic: $g(\theta + \pi) = -g(\theta)$.
- (4) g is Lipschitz on the circle: for all $\theta, \phi \in \mathbb{R}$,

$$|g(\theta) - g(\phi)| \leq L d(\theta, \phi), \quad L := \frac{6}{\pi}.$$

Proof. Items (1)–(3) are immediate from the definition of tri .

For (4), note that tri is the 2π -periodic extension of the piecewise linear function $1 - \frac{2}{\pi} |x|$ on $[-\pi, \pi]$, hence tri is globally Lipschitz on \mathbb{R} with constant $2/\pi$. Therefore $g(\theta) = \text{tri}(3\theta)$ is globally Lipschitz on \mathbb{R} (with respect to the usual distance) with constant $3 \cdot (2/\pi) = 6/\pi$:

$$|g(u) - g(v)| \leq \frac{6}{\pi} |u - v| \quad (u, v \in \mathbb{R}).$$

Now fix $\theta, \phi \in \mathbb{R}$ and choose $k \in \mathbb{Z}$ such that $|\theta - (\phi + 2\pi k)| = d(\theta, \phi)$. Using 2π -periodicity of g ,

$$|g(\theta) - g(\phi)| = |g(\theta) - g(\phi + 2\pi k)| \leq \frac{6}{\pi} |\theta - (\phi + 2\pi k)| = \frac{6}{\pi} d(\theta, \phi). \quad \square$$

0.2. **The configuration for even n .** Fix an even integer $n = 2m$. Let

$$\theta_k := \frac{2\pi k}{n} \quad (k = 0, 1, \dots, n-1), \quad \zeta_k := e^{i\theta_k}.$$

Define the parameter

$$t_n := \frac{\pi^2}{12n} \left(1 - \frac{1}{n}\right),$$

and the points

$$z_k := (1 + t_n g(\theta_k)) \zeta_k, \quad k = 0, 1, \dots, n-1.$$

0.3. Diameter bound.

Lemma 0.3. *There exists N_0 such that for all even $n = 2m \geq N_0$ the configuration $\{z_k\}_{k=0}^{n-1}$ in Section 0.2 satisfies*

$$\max_{i,j} |z_i - z_j| \leq 2.$$

Moreover, for every i we have $|z_i - z_{i+m}| = 2$.

Proof. Write $r_k := 1 + t_n g(\theta_k)$, so $z_k = r_k e^{i\theta_k}$.

Antipodal pairs. Using $\zeta_{k+m} = -\zeta_k$ and Lemma 0.2(3), $g(\theta_{k+m}) = -g(\theta_k)$, we get

$$z_{k+m} = (1 - t_n g(\theta_k)) e^{i(\theta_k + \pi)} = -(1 - t_n g(\theta_k)) e^{i\theta_k},$$

hence $z_k - z_{k+m} = 2e^{i\theta_k}$ and therefore $|z_k - z_{k+m}| = 2$.

General pairs. Fix $i \neq j$ and set $\beta := d(\theta_i, \theta_j) \in (0, \pi)$. Then

$$|z_i - z_j|^2 = r_i^2 + r_j^2 - 2r_i r_j \cos \beta.$$

Case 1: $\beta \leq \pi/2$. Then $\cos \beta \geq 0$, hence $|z_i - z_j|^2 \leq r_i^2 + r_j^2 \leq 2(1 + t_n)^2$. Since $t_n \leq \pi^2/24 < \sqrt{2} - 1$, we have $2(1 + t_n)^2 < 4$, so $|z_i - z_j| < 2$.

Case 2: $\beta \in (\pi/2, \pi)$. Write $\beta = \pi - \alpha$ where $\alpha \in (0, \pi/2)$. Then $\cos \beta = -\cos \alpha$ and

$$|z_i - z_j|^2 = r_i^2 + r_j^2 + 2r_i r_j \cos \alpha.$$

We may write

$$\theta_j \equiv \theta_i + \pi - \varepsilon\alpha \pmod{2\pi} \quad \text{for some } \varepsilon \in \{\pm 1\},$$

with $\alpha = \frac{2\pi\ell}{n}$ for some integer $\ell \in \{1, 2, \dots, \lfloor (n-1)/4 \rfloor\}$. By Lemma 0.2(3),

$$g(\theta_j) = g(\theta_i + \pi - \varepsilon\alpha) = -g(\theta_i - \varepsilon\alpha).$$

Set $a := g(\theta_i)$, $b := g(\theta_i - \varepsilon\alpha)$. Then $r_i = 1 + t_n a$ and $r_j = 1 - t_n b$. A direct expansion gives

$$|z_i - z_j|^2 - 4 = -4 \sin^2 \frac{\alpha}{2} + 2t_n(a-b)(1+\cos\alpha) + t_n^2 \left[(a-b)^2 + 2(1-\cos\alpha)ab \right]. \quad (0.1)$$

We bound each term from above. By Lemma 0.2(4), $|a-b| \leq L\alpha$ with $L = 6/\pi$. Also $|ab| \leq 1$ and $1 - \cos \alpha \leq \alpha^2/2$, hence

$$(a-b)^2 + 2(1-\cos\alpha)ab \leq (L\alpha)^2 + \alpha^2 = (L^2 + 1)\alpha^2.$$

Moreover, for all $\alpha \geq 0$ one has $-4 \sin^2 \frac{\alpha}{2} \leq -\alpha^2 + \frac{\alpha^4}{12}$. Finally $1 + \cos \alpha \leq 2$, so $2t_n(a-b)(1+\cos\alpha) \leq 4t_n L\alpha$. Putting these bounds into (0.1) gives

$$|z_i - z_j|^2 - 4 \leq -\alpha^2 + \frac{\alpha^4}{12} + 4t_n L\alpha + (L^2 + 1)t_n^2 \alpha^2. \quad (0.2)$$

Now substitute $\alpha = \frac{2\pi\ell}{n}$ and $t_n = \frac{\pi^2}{12n}(1 - \frac{1}{n})$. We now show that the right-hand side of (0.2) is ≤ 0 for all even $n \geq 8$. Since $t_n \leq \pi^2/(12n)$ and $L^2 + 1 = (\pi^2 + 36)/\pi^2$, we obtain

$$4t_n L\alpha = \frac{4\pi^2\ell}{n^2} \left(1 - \frac{1}{n}\right), \quad \frac{\alpha^4}{12} = \frac{4}{3} \frac{\pi^4\ell^4}{n^4}$$

and

$$(L^2 + 1)t_n^2 \alpha^2 \leq \frac{\pi^2 + 36}{\pi^2} \cdot \frac{\pi^4}{144n^2} \cdot \left(\frac{2\pi\ell}{n}\right)^2 = \frac{\pi^4(\pi^2 + 36)}{36} \frac{\ell^2}{n^4}.$$

Moreover,

$$-\alpha^2 + 4t_n L\alpha = -\left(\frac{2\pi\ell}{n}\right)^2 + \frac{4\pi^2\ell}{n^2} \left(1 - \frac{1}{n}\right) = -\frac{4\pi^2}{n^2} \left(\ell(\ell - 1) + \frac{\ell}{n}\right).$$

Therefore (0.2) implies the explicit bound

$$|z_i - z_j|^2 - 4 \leq -\frac{4\pi^2}{n^2} \left(\ell(\ell - 1) + \frac{\ell}{n}\right) + \frac{\pi^4}{n^4} \left(\frac{4}{3}\ell^4 + \frac{\pi^2 + 36}{36}\ell^2\right) =: E_{n,\ell}. \quad (0.3)$$

We will check $E_{n,\ell} \leq 0$ by cases.

Case 1: $\ell = 1$. Then (0.3) gives

$$E_{n,1} \leq -\frac{4\pi^2}{n^3} + \frac{\pi^4}{n^4} \left(\frac{7}{3} + \frac{\pi^2}{36}\right).$$

Thus $E_{n,1} \leq 0$ follows as soon as $n \geq \frac{\pi^2}{4} \left(\frac{7}{3} + \frac{\pi^2}{36}\right) \approx 6.43372$. Hence $E_{n,1} \leq 0$ for all $n \geq 8$.

Case 2: $\ell \geq 2$. Since $\ell(\ell - 1) \geq \ell^2/2$ for $\ell \geq 2$, (0.3) yields

$$E_{n,\ell} \leq -\frac{2\pi^2\ell^2}{n^2} + \frac{\pi^4}{n^4} \left(\frac{4}{3}\ell^4 + \frac{\pi^2 + 36}{36}\ell^2\right) = \frac{\ell^2}{n^4} \left(-2\pi^2n^2 + \pi^4 \left(\frac{4}{3}\ell^2 + \frac{\pi^2 + 36}{36}\right)\right).$$

Using $\ell \leq n/4$ we have

$$E_{n,\ell} \leq \frac{\ell^2}{n^4} \left(n^2 \left(-2\pi^2 + \frac{\pi^4}{12}\right) + \frac{\pi^4(\pi^2 + 36)}{36}\right).$$

For $n \geq 8$ we have $n^2(2 - \pi^2/12) \geq \pi^2(\pi^2 + 36)/36$, so indeed $E_{n,\ell} \leq 0$ for all $\ell \geq 2$ whenever $n \geq 8$.

Combining the two cases, we have $E_{n,\ell} \leq 0$ for every $1 \leq \ell \leq n/4$ once $n \geq 8$. Hence (0.3) implies $|z_i - z_j|^2 - 4 \leq 0$, i.e. $|z_i - z_j| \leq 2$. \square

0.4. Factorization of Δ and a uniform bound on ρ_{ij} . For $i \neq j$ define

$$v_k := g(\theta_k)\zeta_k, \quad \rho_{ij} := \frac{v_i - v_j}{\zeta_i - \zeta_j}.$$

Then $z_i - z_j = (\zeta_i - \zeta_j)(1 + t_n \rho_{ij})$, and therefore

$$\frac{\Delta(z_0, \dots, z_{n-1})}{\prod_{i \neq j} |\zeta_i - \zeta_j|} = \prod_{i \neq j} |1 + t_n \rho_{ij}|. \quad (0.4)$$

Lemma 0.4. For $\zeta_k = e^{2\pi ik/n}$ one has $\prod_{i \neq j} |\zeta_i - \zeta_j| = n^n$.

Proof. Let $p(z) = z^n - 1 = \prod_{j=0}^{n-1} (z - \zeta_j)$. For each root ζ_i , $p'(\zeta_i) = n\zeta_i^{n-1} = \prod_{j \neq i} (\zeta_i - \zeta_j)$. Taking absolute values and then the product over i gives

$$\prod_i \prod_{j \neq i} |\zeta_i - \zeta_j| = \prod_i |p'(\zeta_i)| = \prod_i n = n^n.$$

\square

Combining Lemma 0.4 with (0.4) yields

$$\frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = \prod_{i \neq j} |1 + t_n \rho_{ij}|. \quad (0.5)$$

Lemma 0.5. *For all $i \neq j$, we have $|\rho_{ij}| \leq 4$.*

Proof. Write $g_i := g(\theta_i)$. Then $v_i - v_j = g_i(\zeta_i - \zeta_j) + (g_i - g_j)\zeta_j$, hence

$$\rho_{ij} = g_i + (g_i - g_j) \frac{\zeta_j}{\zeta_i - \zeta_j}.$$

Let $\delta_{ij} := d(\theta_i, \theta_j) \in (0, \pi]$. Since g is 2π -periodic, $|g_i - g_j| \leq L\delta_{ij}$ and $|\zeta_i - \zeta_j| = 2 \sin(\delta_{ij}/2)$. Therefore

$$|\rho_{ij}| \leq |g_i| + \frac{|g_i - g_j|}{|\zeta_i - \zeta_j|} \leq 1 + \frac{L\delta_{ij}}{2|\sin(\delta_{ij}/2)|}.$$

For $0 < x \leq \pi$, concavity of \sin on $[0, \pi/2]$ gives $\sin(x/2) \geq \frac{x}{\pi}$, hence $|\rho_{ij}| \leq 1 + \frac{L\pi}{2} = 1 + \frac{6}{\pi} \cdot \frac{\pi}{2} = 4$. \square

0.5. Second-order expansion of $\log(\Delta/n^n)$. Taking logarithms on both sides of (0.5), we obtain

$$\log \frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = \sum_{i \neq j} \log |1 + t_n \rho_{ij}|. \quad (0.6)$$

Let $\Re z$ denote the real part of a complex number z .

Lemma 0.6. *Assume $|u| \leq 1/2$. Then*

$$\log |1 + u| = \Re \left(u - \frac{u^2}{2} \right) + R(u), \quad |R(u)| \leq \frac{2}{3} |u|^3.$$

Proof. For $|u| < 1$ we have $\log(1+u) = \sum_{k \geq 1} (-1)^{k+1} u^k / k$. Taking real parts gives $\log |1+u| = \Re \log(1+u)$. The tail satisfies

$$\left| \sum_{k \geq 3} \frac{(-1)^{k+1}}{k} u^k \right| \leq \sum_{k \geq 3} \frac{|u|^k}{k} \leq \frac{1}{3} \sum_{k \geq 3} |u|^k = \frac{|u|^3}{3(1-|u|)} \leq \frac{2}{3} |u|^3. \quad \square$$

Lemma 0.7. *Let $n = 2m$ be even. Then $\sum_{i \neq j} \rho_{ij} = 0$.*

Proof. We have $\zeta_{k+m} = -\zeta_k$. By Lemma 0.2(3), $g(\theta_{k+m}) = -g(\theta_k)$, hence

$$v_{k+m} = g(\theta_{k+m})\zeta_{k+m} = (-g(\theta_k))(-\zeta_k) = v_k.$$

Thus for any $i \neq j$,

$$\rho_{i+m, j+m} = \frac{v_{i+m} - v_{j+m}}{\zeta_{i+m} - \zeta_{j+m}} = \frac{v_i - v_j}{-(\zeta_i - \zeta_j)} = -\rho_{ij}.$$

The map $(i, j) \mapsto (i+m, j+m)$ is a bijection on ordered pairs $i \neq j$, so the sum cancels. \square

Lemma 0.8. *Along even $n \rightarrow \infty$,*

$$\log \frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = -\frac{t_n^2}{2} \sum_{i \neq j} \Re(\rho_{ij}^2) + o(1).$$

Proof. For all sufficiently large n , Lemma 0.5 and $t_n \leq \pi^2/(12n)$ imply $|t_n \rho_{ij}| \leq 1/2$. Apply Lemma 0.6 to (0.6) termwise:

$$\log \frac{\Delta}{n^n} = \sum_{i \neq j} \Re \left(t_n \rho_{ij} - \frac{t_n^2}{2} \rho_{ij}^2 \right) + \sum_{i \neq j} R(t_n \rho_{ij}).$$

By Lemma 0.7, the linear term vanishes. For the remainder, using $|R(u)| \leq \frac{2}{3}|u|^3$ and $|\rho_{ij}| \leq M$,

$$\left| \sum_{i \neq j} R(t_n \rho_{ij}) \right| \leq \frac{2}{3} n(n-1) (t_n M)^3 = O(n^2 t_n^3) = O(1/n) \rightarrow 0. \quad \square$$

0.6. **A Riemann-sum limit for $\sum \Re(\rho_{ij}^2)$.** Define

$$\xi(x) := e^{ix}, \quad f(x) := g(x)e^{ix}, \quad \rho(x, y) := \begin{cases} \frac{f(x) - f(y)}{\xi(x) - \xi(y)}, & d(x, y) > 0, \\ 0, & d(x, y) = 0, \end{cases} \quad (0.7)$$

and set

$$F(x, y) := \Re(\rho(x, y)^2) \quad (x, y \in [0, 2\pi]).$$

For $i \neq j$ we have $d(\theta_i, \theta_j) > 0$, hence $\rho(\theta_i, \theta_j) = \rho_{ij}$ and $F(\theta_i, \theta_j) = \Re(\rho_{ij}^2)$.

Lemma 0.9. *The function F is bounded on $[0, 2\pi]^2$ and is continuous at every point (x, y) with $d(x, y) > 0$. In particular, F is Riemann integrable on $[0, 2\pi]^2$, and*

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} F(\theta_i, \theta_j) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F(x, y) dx dy.$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i \neq j} \Re(\rho_{ij}^2) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \Re(\rho(x, y)^2) dx dy =: J.$$

Proof. Boundedness follows from the same estimate as in Lemma 0.5 (with $\theta_i - \theta_j$ replaced by $x - y$). Continuity holds for $d(x, y) > 0$ since the denominator is nonzero. Let

$$D := \{(x, y) \in [0, 2\pi]^2 : d(x, y) = 0\} = \{(x, y) \in [0, 2\pi]^2 : e^{ix} = e^{iy}\}.$$

On $[0, 2\pi]^2$ one has $D = \{x = y\} \cup \{(0, 2\pi), (2\pi, 0)\}$, hence D has Lebesgue measure 0 and F is Riemann integrable. Let $h := 2\pi/n$ and $\theta_i := ih$ for $i = 0, 1, \dots, n-1$. Consider the uniform partition of $[0, 2\pi]^2$ into n^2 squares of side h and choose the sample point (θ_i, θ_j) in the (i, j) th square. Since F is Riemann integrable, these Riemann sums converge:

$$\lim_{n \rightarrow \infty} h^2 \sum_{i,j=0}^{n-1} F(\theta_i, \theta_j) = \int_0^{2\pi} \int_0^{2\pi} F(x, y) dx dy.$$

Dividing by $(2\pi)^2$ gives

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i,j=0}^{n-1} F(\theta_i, \theta_j) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F(x, y) dx dy.$$

Finally, since $F(\theta_i, \theta_i) = 0$ for all i by definition, we have $\sum_{i,j} F(\theta_i, \theta_j) = \sum_{i \neq j} F(\theta_i, \theta_j)$, and for $i \neq j$, $F(\theta_i, \theta_j) = \Re(\rho_{ij}^2)$. This yields the stated consequence and the definition of J . \square

0.7. **Computing J via Fejér approximation and Fourier orthogonality.**

0.7.1. Fourier series of the triangular wave.

Lemma 0.10. *The 2π -periodic function tri has the Fourier expansion*

$$\text{tri}(x) = \sum_{\substack{k \geq 1 \\ k \text{ odd}}} \frac{8}{\pi^2 k^2} \cos(kx),$$

with absolute (hence uniform) convergence. Consequently,

$$g(\theta) = \text{tri}(3\theta) = \sum_{\substack{r \geq 1 \\ r \text{ odd}}} \frac{8}{\pi^2 r^2} \cos(3r\theta).$$

Proof. Since tri is even and has mean 0, only cosine coefficients appear:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{tri}(x) \cos(kx) dx = \frac{2}{\pi} \int_0^\pi \left(1 - \frac{2x}{\pi}\right) \cos(kx) dx.$$

The first term integrates to 0. For the second term, integration by parts gives

$$\int_0^\pi x \cos(kx) dx = \frac{(-1)^k - 1}{k^2},$$

hence

$$a_k = \frac{4}{\pi^2} \cdot \frac{1 - (-1)^k}{k^2} = \begin{cases} \frac{8}{\pi^2 k^2}, & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

Absolute convergence follows from $\sum_{k \text{ odd}} 1/k^2 < \infty$. \square

0.7.2. A kernel orthogonality identity. For integers k define, for $x \neq y$,

$$R_k(x, y) := \frac{e^{ikx} - e^{iky}}{e^{ix} - e^{iy}}.$$

Lemma 0.11. *For integers k, ℓ one has*

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R_k(x, y) R_\ell(x, y) dx dy = \begin{cases} 1 - |k - \ell|, & k + \ell = 2, \\ 0, & k + \ell \neq 2. \end{cases}$$

Proof. Let $z = e^{ix}$ and $w = e^{iy}$. If $k \geq 1$ then

$$R_k(x, y) = \frac{z^k - w^k}{z - w} = \sum_{m=0}^{k-1} z^{k-1-m} w^m = \sum_{m=0}^{k-1} e^{i((k-1-m)x+my)}.$$

If $k \leq 0$ then, writing $k = -p$ with $p \geq 0$,

$$R_{-p}(x, y) = \frac{z^{-p} - w^{-p}}{z - w} = \frac{w^p - z^p}{z^p w^p (z - w)} = -\frac{z^p - w^p}{z^p w^p (z - w)} = -\sum_{m=0}^{p-1} z^{-(m+1)} w^{-(p-m)}.$$

In all cases, $R_k R_\ell$ is a finite sum of exponentials $e^{i(ax+by)}$. Using

$$\frac{1}{2\pi} \int_0^{2\pi} e^{iax} dx = \begin{cases} 1, & a = 0, \\ 0, & a \neq 0, \end{cases}$$

one checks that a nonzero contribution can occur only when the total x -frequency and y -frequency both vanish, which forces $k + \ell = 2$. When $k + \ell = 2$, the number of surviving terms is $1 - |k - \ell|$, and each contributes 1. \square

0.7.3. *Fejér approximation to justify passage to the Fourier side.* Let K_N be the Fejér kernel

$$K_N(t) := \frac{1}{N+1} \left(\frac{\sin((N+1)t/2)}{\sin(t/2)} \right)^2 \geq 0, \quad \frac{1}{2\pi} \int_0^{2\pi} K_N(t) dt = 1.$$

Define the *Fejér mean* of a 2π -periodic continuous function f by

$$f^{(N)}(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x-t) K_N(t) dt.$$

Then $f^{(N)}$ is a trigonometric polynomial of degree $\leq N$ and $f^{(N)} \rightarrow f$ uniformly as $N \rightarrow \infty$.

Lemma 0.12. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be 2π -periodic and Lipschitz. Write*

$$\text{Lip}(f) := \sup_{d(x,y)>0} \frac{|f(x) - f(y)|}{d(x,y)}.$$

then $\text{Lip}(f^{(N)}) \leq \text{Lip}(f)$ for all N . Moreover, for all $x, y \in \mathbb{R}$ with $d(x, y) > 0$,

$$\left| \frac{f^{(N)}(x) - f^{(N)}(y)}{e^{ix} - e^{iy}} \right| \leq \frac{\pi}{2} \text{Lip}(f),$$

uniformly in N .

Proof. Since K_N is a probability kernel, and $d(x-t, y-t) = d(x, y)$ for all t ,

$$\begin{aligned} |f^{(N)}(x) - f^{(N)}(y)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} (f(x-t) - f(y-t)) K_N(t) dt \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(x-t) - f(y-t)| K_N(t) dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \text{Lip}(f) d(x-t, y-t) K_N(t) dt \\ &= \text{Lip}(f) d(x, y). \end{aligned}$$

Thus $\text{Lip}(f^{(N)}) \leq \text{Lip}(f)$.

If $d(x, y) > 0$, then $|e^{ix} - e^{iy}| = 2 \sin(d(x, y)/2) \geq 2d(x, y)/\pi$, and hence

$$\left| \frac{f^{(N)}(x) - f^{(N)}(y)}{e^{ix} - e^{iy}} \right| \leq \text{Lip}(f) \frac{d(x, y)}{2 \sin(d(x, y)/2)} \leq \frac{\pi}{2} \text{Lip}(f). \quad \square$$

Lemma 0.13. *Let $\rho(x, y) = \frac{f(x)-f(y)}{e^{ix}-e^{iy}}$ as in (0.7) and define*

$$B(f) := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho(x, y)^2 dx dy.$$

Then $B(f)$ is real and equals

$$B(f) = \sum_{k \in \mathbb{Z}} (1 - |k - 1|) a_k a_{2-k},$$

where $f(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}$ is the Fourier series of f .

Proof. Let $f^{(N)}$ be Fejér means. Define

$$\rho_N(x, y) := \frac{f^{(N)}(x) - f^{(N)}(y)}{e^{ix} - e^{iy}}.$$

By uniform convergence $f^{(N)} \rightarrow f$, we have $\rho_N(x, y) \rightarrow \rho(x, y)$ for every $x \neq y$. For our specific $f(x) = g(x)e^{ix}$, Lemma 0.2 implies $|g| \leq 1$ and $\text{Lip}(g) \leq L$. Hence, for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq |g(x) - g(y)| + |g(y)| |e^{ix} - e^{iy}| \leq L d(x, y) + 2 \sin(d(x, y)/2) \leq (L+1) d(x, y),$$

so $\text{Lip}(f) \leq L+1$. By Lemma 0.12, $|\rho_N(x, y)| \leq \frac{\pi}{2} \text{Lip}(f)$ uniformly in N and (x, y) . Hence dominated convergence applies and

$$B(f) = \lim_{N \rightarrow \infty} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho_N(x, y)^2 dx dy.$$

Now $f^{(N)}$ is a trigonometric polynomial, say $f^{(N)}(x) = \sum_{|k| \leq N} a_k^{(N)} e^{ikx}$. Then for $x \neq y$,

$$\rho_N(x, y) = \sum_{|k| \leq N} a_k^{(N)} R_k(x, y),$$

a finite sum. Therefore,

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho_N(x, y)^2 dx dy = \sum_{|k|, |\ell| \leq N} a_k^{(N)} a_\ell^{(N)} \cdot \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R_k R_\ell dx dy.$$

By Lemma 0.11, only terms with $k + \ell = 2$ survive, giving

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho_N(x, y)^2 dx dy = \sum_{k \in \mathbb{Z}} (1 - |k - 1|) a_k^{(N)} a_{2-k}^{(N)}.$$

For Fejér means, $a_k^{(N)} = (1 - |k|/(N+1))a_k$ for $|k| \leq N$ and 0 otherwise, hence $a_k^{(N)} \rightarrow a_k$ for each fixed k . In our application the resulting infinite series is absolutely convergent (indeed it will reduce to $\sum_{r \text{ odd}} O(1/r^3)$), so we may pass $N \rightarrow \infty$ termwise. This yields the desired series for $B(f)$. Since the series is real (coefficients are real in our case), $B(f) \in \mathbb{R}$. \square

Finally, since $B(f)$ is real,

$$J = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \Re(\rho^2) = \Re B(f) = B(f).$$

0.7.4. *Evaluating the series for our f .* By Lemma 0.10,

$$g(\theta) = \sum_{\substack{r \geq 1 \\ r \text{ odd}}} \frac{8}{\pi^2 r^2} \cos(3r\theta).$$

Hence

$$f(\theta) = g(\theta) e^{i\theta} = \sum_{\substack{r \geq 1 \\ r \text{ odd}}} \frac{4}{\pi^2 r^2} \left(e^{i(1+3r)\theta} + e^{i(1-3r)\theta} \right).$$

Therefore the Fourier coefficients of f satisfy

$$a_{1+3r} = a_{1-3r} = \frac{4}{\pi^2 r^2} \quad (r \geq 1, r \text{ odd}), \quad a_k = 0 \text{ otherwise.}$$

In Lemma 0.13, only pairs $(k, 2-k)$ contribute. The only nonzero pairings are $k = 1+3r$ with $2-k = 1-3r$ (and vice versa), and for such k , $1-|k-1|=1-3r$. Thus

$$J = \sum_{r \text{ odd} \geq 1} 2(1-3r) \left(\frac{4}{\pi^2 r^2} \right)^2 = \frac{32}{\pi^4} \sum_{r \text{ odd} \geq 1} \frac{1-3r}{r^4}.$$

Using

$$\sum_{r \text{ odd}} \frac{1}{r^4} = \left(1 - \frac{1}{2^4}\right) \zeta(4) = \frac{15}{16} \cdot \frac{\pi^4}{90} = \frac{\pi^4}{96}, \quad \sum_{r \text{ odd}} \frac{1}{r^3} = \left(1 - \frac{1}{2^3}\right) \zeta(3) = \frac{7}{8} \zeta(3),$$

we obtain

$$J = \frac{32}{\pi^4} \left(\frac{\pi^4}{96} - 3 \cdot \frac{7}{8} \zeta(3) \right) = \frac{1}{3} - \frac{84 \zeta(3)}{\pi^4}.$$

In particular $J < 0$.

0.8. Asymptotic value of Δ/n^n . By Lemma 0.8 and Lemma 0.9,

$$\log \frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = -\frac{t_n^2}{2} \left(n^2 J + o(n^2) \right) + o(1) = -\frac{(nt_n)^2}{2} J + o(1).$$

Since $nt_n \rightarrow \pi^2/12$,

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} \log \frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = -\frac{1}{2} \left(\frac{\pi^2}{12} \right)^2 \left(\frac{1}{3} - \frac{84 \zeta(3)}{\pi^4} \right) = \frac{7}{24} \zeta(3) - \frac{\pi^4}{864}.$$

Exponentiating,

$$\lim_{\substack{n \rightarrow \infty \\ n \text{ even}}} \frac{\Delta(z_0, \dots, z_{n-1})}{n^n} = \exp \left(\frac{7}{24} \zeta(3) - \frac{\pi^4}{864} \right).$$

By Lemma 0.3, for all sufficiently large even n the configuration is feasible (diameter ≤ 2), hence $\Delta_n^* \geq \Delta(z_0, \dots, z_{n-1})$ and therefore

$$\liminf_{\substack{n \rightarrow \infty \\ n \text{ even}}} \frac{\Delta_n^*}{n^n} \geq \exp \left(\frac{7}{24} \zeta(3) - \frac{\pi^4}{864} \right).$$