1. Introduction

In [2, Section 1], Erdős proposed the following problem (see also Problem #614 in [1]):

Problem 1.1. *Let $f(n,k)$ be minimal such that there is a graph with n vertices and $f(n,k)$ edges where every set of $k+2$ vertices induces a subgraph with maximum degree at least k. Determine $f(n,k)$.*

Now, we denote $f(n,k)$ for the minimum number of edges of an n-vertex graph G with the property that every $(k+2)$-subset of vertices induces a subgraph whose maximum degree is at least k. Equivalently, if $H = \overline{G}$ denotes the complement of G, then

$$\text{every induced subgraph of } H \text{ on } k+2 \text{ vertices has minimum degree at most } 1. \qquad (\star)$$

Thus $f(n,k) = \binom{n}{2} - \max\{e(H) : H \text{ satisfies } (\star)\}$ and our task is to maximize the number of edges under the constraint $(\star)$.

2. When $k = 1$

We first dispose of the easy base $k = 1$.

Theorem 2.1. *For all $n \geq 2$, we have*

$$f(n,1) = \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. For $k = 1$, property $(\star)$ says that $\overline{G}$ is triangle-free. Hence $e(\overline{G}) \leq \lfloor n^2/4 \rfloor$ by Mantel's theorem, with equality attained by the balanced complete bipartite graph. The formula for $f(n,1)$ follows. $\qquad \square$

3. When $k = 2$

Theorem 3.1. *For all $n \geq 4$, we have*

$$f(n,2) = \binom{n}{2} - \mathrm{ex}(n, C_4).$$

Proof. Let $k = 2$. First assume that H satisfies $(\star)$. If H contains a copy of C_4 on vertex set S with $|S| = 4$, then the induced subgraph $H[S]$ contains that cycle and hence has minimum degree at least 2, contradicting $(\star)$. Thus H is C_4-free.

Conversely, assume that H is C_4-free and let $S \subseteq V(H)$ with $|S| = 4$. If $\delta(H[S]) \geq 2$, then $H[S]$ is a graph on 4 vertices with minimum degree at least 2, hence by Dirac's theorem it is Hamiltonian. In particular, $H[S]$ contains a spanning cycle of length 4, contradicting that H is C_4-free. Therefore $\delta(H[S]) \leq 1$ for every 4-set S, so H satisfies $(\star)$.

Finally,

$$f(n,2) = \binom{n}{2} - \max\{e(H) : H \text{ satisfies } (\star)\} = \binom{n}{2} - \mathrm{ex}(n, C_4),$$

since the admissible H are exactly the C_4-free graphs. $\qquad \square$

4. A general Turán reduction

Definition 4.1. Fix an integer $k \geq 2$ and set $m := k + 2$. We recall that an n-vertex graph H satisfies $(\star)$ if

$$\forall S \subseteq V(H) \text{ with } |S| = m, \qquad \delta\big(H[S]\big) \leq 1.$$

Let $\mathcal{F}_k$ be the (finite) family of all graphs F on exactly m vertices with

$$\delta(F) \geq 2.$$

Let $\mathcal{F}_k^{\min} \subseteq \mathcal{F}_k$ be the subfamily consisting of those $F \in \mathcal{F}_k$ that are *edge-minimal* with $\delta(F) \geq 2$, i.e. $\delta(F) \geq 2$ but for every edge $e \in E(F)$ we have

$$\delta(F - e) \leq 1.$$

Lemma 4.2. *A graph H satisfies $(\star)$ if and only if H is $\mathcal{F}_k$-free as a (not necessarily induced) subgraph.*

Proof. ($\Rightarrow$) Suppose H satisfies $(\star)$. If H contained some $F \in \mathcal{F}_k$ as a subgraph on a vertex set S with $|S| = m$, then in the induced graph $H[S]$ every vertex would have degree at least its degree in F, hence $\delta(H[S]) \geq \delta(F) \geq 2$, contradicting $(\star)$.

($\Leftarrow$) Conversely, suppose H violates $(\star)$. Then there exists $S \subseteq V(H)$ with $|S| = m$ such that $\delta(H[S]) \geq 2$. Taking $F := H[S]$, we have $F \in \mathcal{F}_k$ and F appears as a subgraph of H. $\quad\square$

Lemma 4.3. *A graph H is $\mathcal{F}_k$-free if and only if it is $\mathcal{F}_k^{\min}$-free.*

Proof. The forward implication is trivial since $\mathcal{F}_k^{\min} \subseteq \mathcal{F}_k$.

For the reverse implication, suppose H contains a copy of some $F \in \mathcal{F}_k$ on a fixed vertex set S of size m. Starting from this copy, delete edges (within S) one by one as long as the minimum degree remains at least 2. Since S is finite, this process terminates at a subgraph $F' \subseteq F$ on the same vertex set S such that $\delta(F') \geq 2$ but deleting any further edge would force the minimum degree to drop to at most 1. Thus $F' \in \mathcal{F}_k^{\min}$ and H contains F'. $\quad\square$

Proposition 4.4. *Let $k \geq 2$ be fixed. Then*

$$\max\{e(H) : |V(H)| = n, \ H \text{ satisfies } (\star)\} = \mathrm{ex}\big(n, \mathcal{F}_k^{\min}\big).$$

Proof. By Lemma 4.2, graphs satisfying $(\star)$ are precisely the $\mathcal{F}_k$-free graphs, so the maximum number of edges among them is $\mathrm{ex}(n, \mathcal{F}_k)$ by definition of $\mathrm{ex}(\cdot, \cdot)$. By Lemma 4.3, $\mathcal{F}_k$-freeness is equivalent to $\mathcal{F}_k^{\min}$-freeness, hence $\mathrm{ex}(n, \mathcal{F}_k) = \mathrm{ex}(n, \mathcal{F}_k^{\min})$. $\quad\square$

Let $f(n, k)$ be as in Problem 1.1 and set $m := k + 2$. Passing to the complement $H = \overline{G}$, the defining property for G is equivalent to the condition that H satisfies $(\star)$. Consequently, we obtain the following Turán-type expression for $f(n, k)$.

Theorem 4.5.

$$f(n, k) = \binom{n}{2} - \mathrm{ex}\big(n, \mathcal{F}_k^{\min}\big).$$

For $3 \leq k \leq 6$, we computed the family $\mathcal{F}_k^{\min}$ by an exhaustive Mathematica search; see Figures 1–5.

In particular, let $K_1 \vee 2K_2$ denote the join of a single vertex with two disjoint edges. Our computation in Figure 2 yields

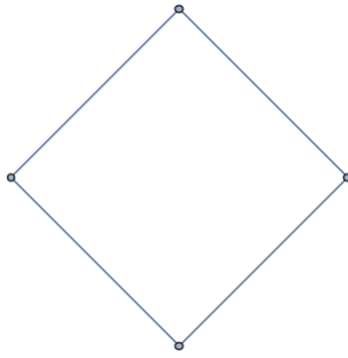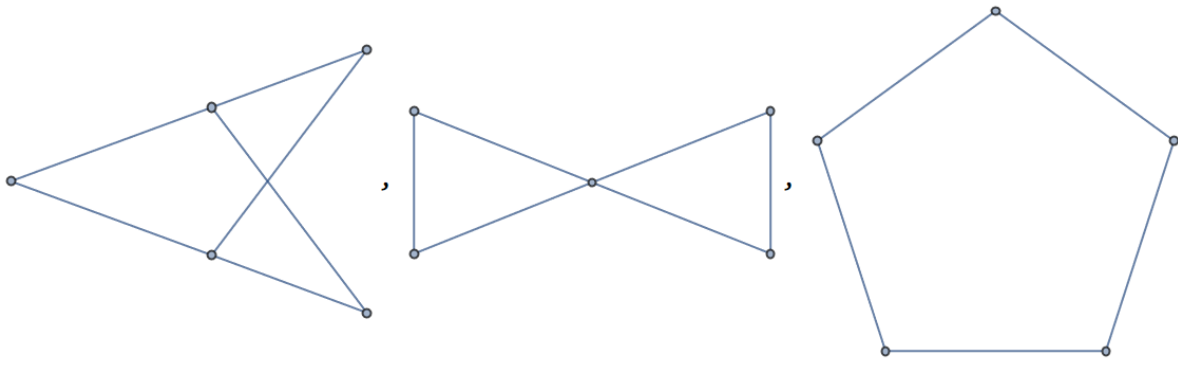$$\mathcal{F}_3^{\min} = \{C_5, \ K_{2,3}, \ K_1 \vee 2K_2\},$$

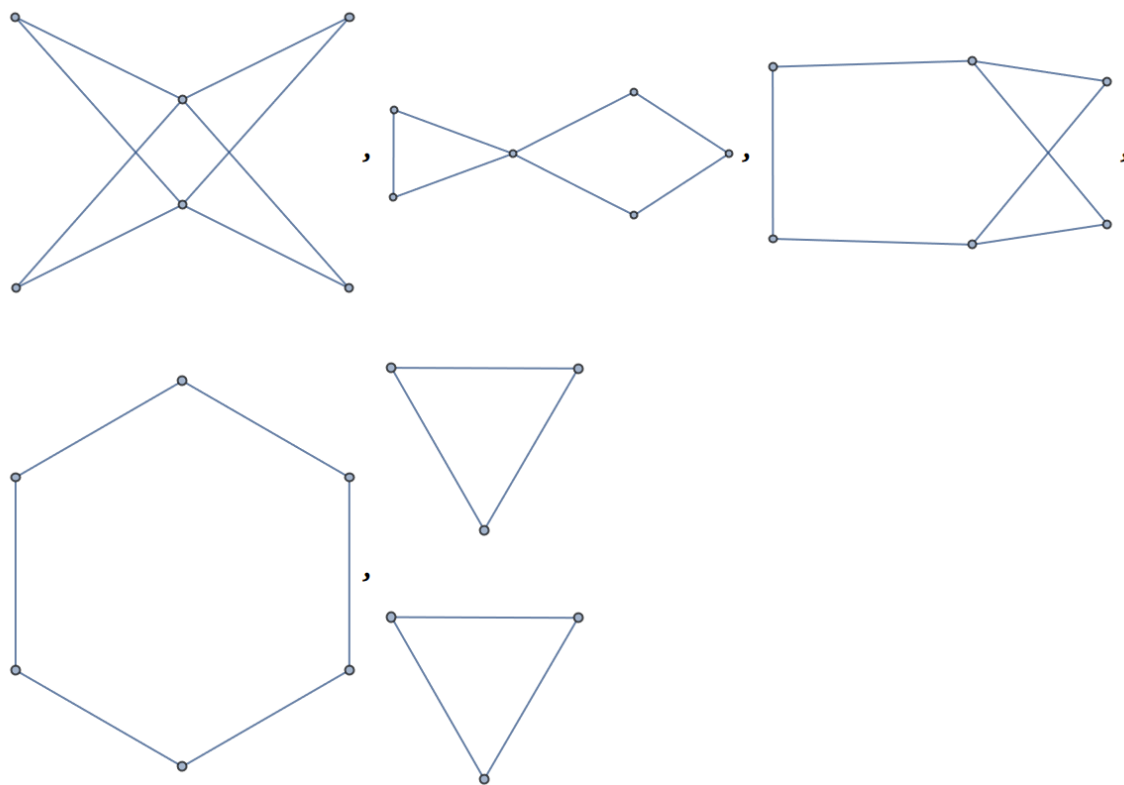and hence we obtain an explicit Turán formulation for $f(n, 3)$.

Corollary 4.6. *For all $n \geq 5$,*

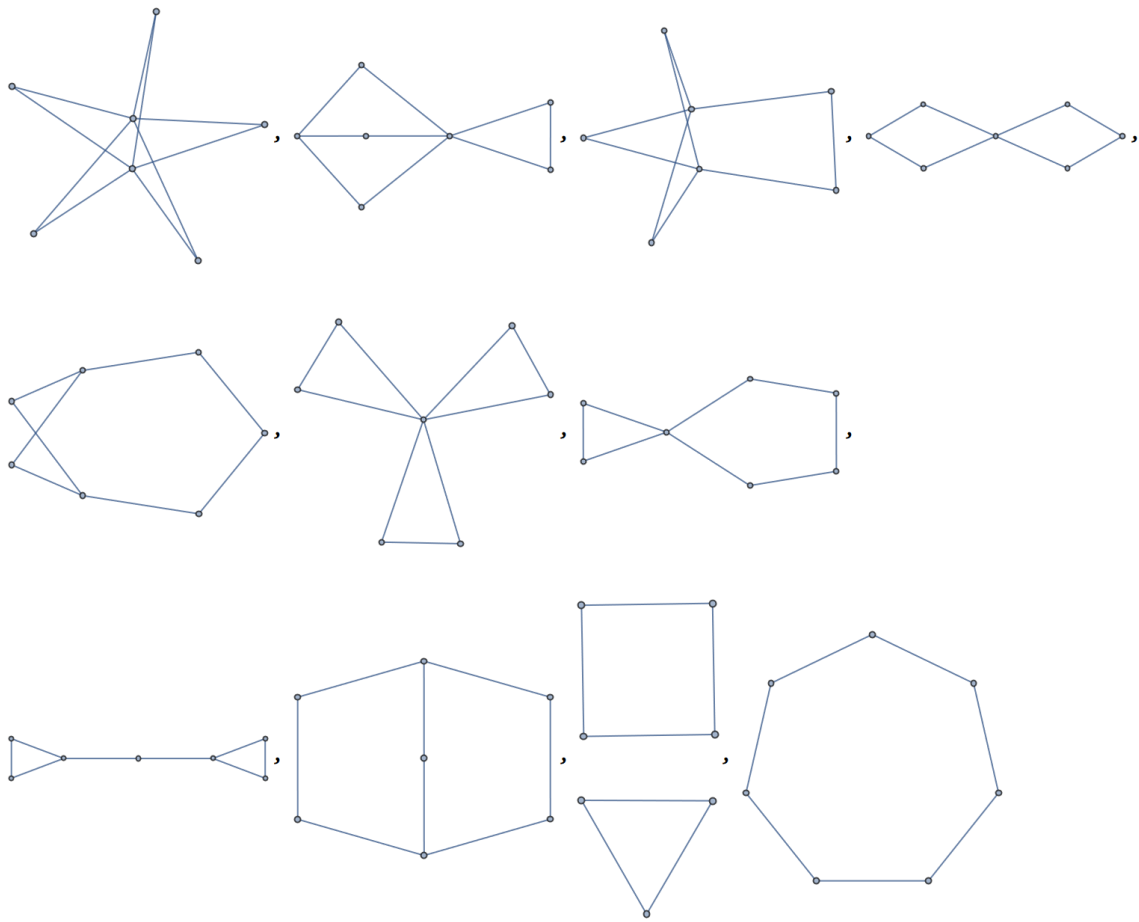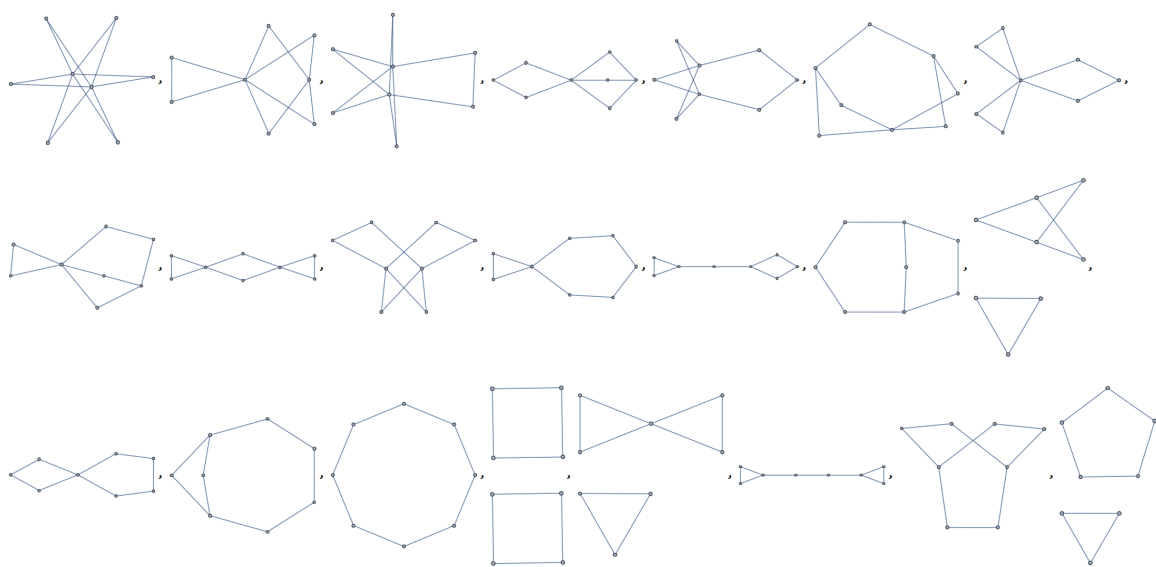$$f(n, 3) = \binom{n}{2} - \mathrm{ex}\big(n, \{C_5, K_{2,3}, K_1 \vee 2K_2\}\big).$$

References

[1] T. F. Bloom, Erdős Problem #614, `https://www.erdosproblems.com/614`, accessed 2025-10-26.

[2] Faudree, R. J. and Rousseau, C. C. and Schelp, R. H., Problems in graph theory from Memphis. The mathematics of Paul Erdős, II (1997), 7–26.

FIGURE 1. $k = 2$

FIGURE 2. $k = 3$

FIGURE 3. $k = 4$

FIGURE 4. $k = 5$

Figure 5. $k = 6$