1. Introduction

In [3, p. 40], Erdős and Graham posed the following question, which also appears as Problem #311 on Bloom's Erdős Problems website [1].

Problem 1.1. *Let $\delta(N)$ be the minimal value of $|1-\sum_{n\in A}\frac{1}{n}|$ as A ranges over all subsets of $\{1,\dots,N\}$ which contain no S such that $\sum_{n\in S}\frac{1}{n}=1$? Is it*

$$e^{-(c+o(1))N}$$

for some constant $c\in(0,1)$?

It is trivially at least $1/\mathrm{lcm}(1,\dots,N)$.

In this note, we use a representation lemma of Liu–Sawhney [2] to obtain the following nontrivial upper bound: there exists $c_0>0$ such that for all sufficiently large N,

$$\delta(N)\le \exp\left(-c_0\frac{N}{(\log N)^3(\log\log N)^4}\right).$$

2. A nontrivial upper bound for $\delta(N)$ via Liu–Sawhney

2.1. **Definition.** Throughout, log denotes the natural logarithm. For a positive integer N, define

$$\delta(N):=\min\left\{\left|1-\sum_{n\in A}\frac{1}{n}\right|:\ A\subseteq\{1,\dots,N\}\text{ and } A \text{ is admissible}\right\},$$

where we say that A is *admissible* if it contains no subset $S\subseteq A$ with $\sum_{n\in S}\frac{1}{n}=1$. Note that any A with $\sum_{n\in A}\frac{1}{n}<1$ is automatically admissible, since then $\sum_{n\in S}\frac{1}{n}<1$ for every $S\subseteq A$.

We say that a positive integer n is *S-smooth* if every prime power $q|n$ satisfies $q\le S$.

2.2. **Two standard inputs.**

Lemma 2.1 (Liu–Sawhney [2, Lemma 4.1])**.** *There exist absolute constants $c_{\mathrm{LS}}>0$ and $N_{\mathrm{LS}}\in\mathbb{N}$ such that the following holds for all $N\ge N_{\mathrm{LS}}$. Let*

$$S:=c_{\mathrm{LS}}\frac{N}{(\log N)^3(\log\log N)^4}.$$

*If t is S–*smooth*, and s is an integer with*

$$\frac{t}{3}\le s\le t,$$

then there exists a subset $A\subset[N/16,N]\cap\mathbb{N}$ such that

$$\sum_{n\in A}\frac{1}{n}=\frac{s}{t}.$$

Lemma 2.2. *Let*

$$\psi(x):=\sum_{p^k\le x}\log p$$

be Chebyshev's function. Then for every real $x\ge 1$,

$$\log\mathrm{lcm}(1,2,\dots,\lfloor x\rfloor)=\psi(x). \tag{2.1}$$

Moreover, by the prime number theorem (equivalently $\psi(x)\sim x$), there exist absolute constants $x_0\ge 2$ and $c_\psi>0$ such that for all $x\ge x_0$,

$$\psi(x)\ge c_\psi x. \tag{2.2}$$

2.3. **Main theorem.**

Theorem 2.3. *There exist absolute constants $c_0 > 0$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,*

$$\delta(N) \ \leq \ \exp\left(-c_0 \, \frac{N}{(\log N)^3 (\log\log N)^4}\right).$$

Proof. Let $N \geq N_0$ be sufficiently large, where N_0 will be chosen at the end. Define

$$S := c_{\mathrm{LS}} \frac{N}{(\log N)^3 (\log\log N)^4}, \qquad S_0 := \lfloor S \rfloor, \qquad t := \mathrm{lcm}(1, 2, \ldots, S_0), \qquad s := t - 1.$$

Since every prime power divisor of t is at most $S_0 \leq S$, the integer t is S–smooth. Also, once $S_0 \geq 2$ we have $t \geq 2$, hence $t/3 \leq t - 1 = s \leq t$.

Applying Lemma 2.1 to these parameters (s, t), we obtain a set

$$A \subset [N/16, N] \cap \mathbb{N} \subseteq \{1, 2, \ldots, N\}$$

such that

$$\sum_{n \in A} \frac{1}{n} = \frac{s}{t} = 1 - \frac{1}{t}.$$

In particular $\sum_{n \in A} \frac{1}{n} < 1$, so A is admissible for the definition of $\delta(N)$. Therefore

$$\delta(N) \ \leq \ \left| 1 - \sum_{n \in A} \frac{1}{n} \right| = \frac{1}{t}. \tag{2.3}$$

It remains to lower bound t. By (2.1) and (2.2), provided $S_0 \geq x_0$ we have

$$\log t = \log \mathrm{lcm}(1, 2, \ldots, S_0) = \psi(S_0) \geq c_\psi S_0.$$

If additionally $S_0 \geq 2$, then $S_0 \geq S/2$, hence

$$\log t \ \geq \ c_\psi S_0 \ \geq \ \frac{c_\psi}{2} S.$$

Consequently,

$$\frac{1}{t} \ \leq \ \exp\left(-\frac{c_\psi}{2} S\right) = \exp\left(-\frac{c_\psi c_{\mathrm{LS}}}{2} \, \frac{N}{(\log N)^3 (\log\log N)^4}\right).$$

Combining this with (2.3) gives the claimed bound with $c_0 := \frac{c_\psi c_{\mathrm{LS}}}{2}$.

Finally, choose N_0 large enough so that simultaneously: (i) $N \geq N_{\mathrm{LS}}$ (to apply Lemma 2.1), (ii) $S_0 = \lfloor S \rfloor \geq x_0$ (to apply (2.2)), and (iii) $S_0 \geq 2$ (so that $t \geq 2$ and $S_0 \geq S/2$). This completes the proof. $\square$

References

[1] T. F. Bloom, Erdős Problem #311, `https://www.erdosproblems.com/311`, accessed 2026-01-12.

[2] Y. P. Liu, M. Sawhney, On further questions regarding unit fractions. arXiv preprint arXiv:2404.07113. `https://arxiv.org/abs/2404.07113v1`

[3] Erdős, P. and Graham, R., Old and new problems and results in combinatorial number theory. Monographies de L'Enseignement Mathematique (1980).