A NOTE ON THE SELF-GENERATING CONSECUTIVE-SUM GREEDY SEQUENCE

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Abstract. Let $(a_n)_{n\geq 1}$ be the greedy self-generating sequence with $a_1 = 1$, $a_2 = 2$, and for $k \geq 3$ let a_k be the least integer $> a_{k-1}$ that can be written as a sum of at least two consecutive earlier terms. Writing $b_n := a_n - n$, we prove that $(b_n)_{n\geq 1}$ is nondecreasing and unbounded. In particular, the sequence omits infinitely many positive integers, settling a conjecture recorded in the OEIS entry A005243.

1. Introduction

A *self-generating* process introduced by Hofstadter starts from $1, 2$ and repeatedly adjoins all sums of at least two consecutive previous terms. Equivalently, one may define a strictly increasing sequence $(a_n)_{n\geq 1}$ by $a_1 = 1$, $a_2 = 2$, and for each $k \geq 3$ letting a_k be the least integer $a_k > a_{k-1}$ admitting a representation

$$a_k = \sum_{i=p}^{q} a_i \qquad \text{for some } 1 \leq p \leq q \leq k-1, \quad q-p+1 \geq 2.$$

This is OEIS A005243. In [2, p. 71], [3, p. 83], and [4, E31], Hofstadter asked the following question (although Erdős notes in [2] that the question is originally due to Ulam); it also appears as Problem #423 on Bloom's Erdős Problems website [1].

Problem 1.1. *Let $a_1 = 1$ and $a_2 = 2$, and for $k \geq 3$ choose a_k to be the least integer $> a_{k-1}$ that is the sum of at least two consecutive earlier terms of the sequence. What is the asymptotic behaviour of $(a_n)_{n\geq 1}$?*

Empirically, most integers occur. However, a conjecture recorded since 2006 in the *COMMENTS* section of OEIS A005243 asserts that the sequence has infinitely many *nonmembers.*

Conjecture 1.2. *There are infinitely many positive integers that never appear in the sequence $(a_n)_{n\geq 1}$.*

The main purpose of this note is to prove Conjecture 1.2 by establishing the following theorem.

Theorem 1.3. *With $(a_n)_{n\geq 1}$ as above and $b_n := a_n - n$, the sequence $(b_n)_{n\geq 1}$ is nondecreasing and unbounded. Consequently, the set $\{a_n : n \geq 1\}$ omits infinitely many positive integers.*

Remark 1.4. In particular, Theorem 1.3 yields a qualitative step toward Problem 1.1: unboundedness of b_n rules out eventual linearity $a_n = n + B$ for all sufficiently large n. However, it does not by itself determine the growth rate of b_n; for instance, it remains compatible with $b_n = o(n)$.

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2. Preliminaries

The following result is standard; see [5, Corollary 1].

Lemma 2.1 ([5]). *If a polynomial $P(x) \in \mathbb{Q}[x]$ has at least two simple zeros, then the equation*

$$y^m = P(x), \quad x, y \text{ integers}, \quad |y| > 1,$$

has only finitely many integer solutions m, x, y with $m > 2$, $|y| > 1$ and these solutions can be found effectively.

As a direct consequence , we obtain the following.

Proposition 2.2. *Fix $E \in \mathbb{Z}$. Then the equation*

$$v^2 + v + E = 2^m$$

has only finitely many integer solutions $(v, m) \in \mathbb{Z} \times \mathbb{Z}_{>2}$.

Proof. Apply Lemma 2.1 to $P(v) = v^2 + v + E$ and $y = 2$. The discriminant of P is $1 - 4E \neq 0$ for $E \in \mathbb{Z}$, hence P has two simple zeros, so the lemma applies. □

Lemma 2.3. *A positive integer $N \geq 3$ is a sum of at least two consecutive positive integers if and only if N is not a power of 2.*

Proof. If $N = x + (x+1) + \cdots + (x + \ell - 1)$ with $\ell \geq 2$, then

$$2N = \ell(2x + \ell - 1).$$

If ℓ is odd, then $\ell \mid 2N$ implies $\ell \mid N$, and $\ell > 1$ is an odd divisor of N, so N is not a power of 2. If ℓ is even, then $2x + \ell - 1$ is odd and > 1 and divides $2N$, hence divides N, again forcing N to have an odd divisor > 1, so N is not a power of 2.

Conversely, if N is not a power of 2, then N has an odd divisor $d > 1$. Write $N = dm$. If $m \geq (d+1)/2$, then

$$N = \Big(m - \frac{d-1}{2}\Big) + \Big(m - \frac{d-1}{2} + 1\Big) + \cdots + \Big(m + \frac{d-1}{2}\Big)$$

is a sum of $d \geq 3$ consecutive positive integers. If $m < (d+1)/2$, then $2m \geq 2$ and

$$N = \Big(\frac{d+1}{2} - m\Big) + \Big(\frac{d+1}{2} - m + 1\Big) + \cdots + \Big(\frac{d+1}{2} + m - 1\Big)$$

is a sum of $2m$ consecutive positive integers. □

3. Proof of Theorem 1.3

First, we show that b_n is nondecreasing.

Lemma 3.1. *The sequence (b_n) is nondecreasing. In particular, since $b_1 = 0$, we have $b_n \geq 0$ for all n. If (b_n) is bounded above, then there exist integers $n_0 \geq 1$ and $B \geq 0$ such that*

$$a_n = n + B \qquad \forall\, n \geq n_0.$$

Proof. Because (a_n) is strictly increasing in integers, $a_{n+1} \geq a_n + 1$ for all n. Hence

$$b_{n+1} - b_n = (a_{n+1} - a_n) - 1 \geq 0,$$

so (b_n) is nondecreasing. Since $b_1 = a_1 - 1 = 0$, monotonicity gives $b_n \geq 0$.

If (b_n) is bounded above, then a nondecreasing integer sequence must be eventually constant: there exist n_0 and an integer B such that $b_n = B$ for all $n \geq n_0$, i.e. $a_n = n + B$ for all $n \geq n_0$. Finally $B \geq 0$ because $B = b_n \geq 0$ for all n. □

Assume for contradiction that (b_n) is bounded above, and fix n_0, B as in Lemma 3.1. Set

$$T := n_0 + B.$$

Define the finite set

$$\mathcal{C} := \Big\{ \sum_{k=p}^{n_0-1} a_k :\ 1 \le p \le n_0 \Big\},$$

where the case $p = n_0$ gives the empty sum $0 \in \mathcal{C}$.

Lemma 3.2. *Assume $a_n = n + B$ for all $n \ge n_0$. Let $t = a_n$ with $n \ge n_0$ and suppose $t > S_{n_0-1}$ (where $S_m = \sum_{i=1}^m a_i$). Then any representation*

$$t = \sum_{k=p}^{q} a_k \qquad (1 \le p \le q \le n-1, \quad q-p+1 \ge 2)$$

falls into exactly one of the following two types:

(A) *$p \ge n_0$, and then*

$$t = \sum_{m=u}^{v} m \quad \textit{for some integers } u \ge T,\ v \ge u+1.$$

(B) *$p < n_0 \le q$, and then*

$$t = C + \sum_{m=T}^{v} m \quad \textit{for some } C \in \mathcal{C} \textit{ and some integer } v \ge T.$$

Moreover, the case $q < n_0$ is impossible whenever $t > S_{n_0-1}$.

Proof. Let $t = \sum_{k=p}^{q} a_k$ with $q-p+1 \ge 2$ and $q \le n-1$. If $q < n_0$, then the block lies entirely in $\{1, \dots, n_0 - 1\}$, hence

$$t = \sum_{k=p}^{q} a_k \le \sum_{k=1}^{n_0-1} a_k = S_{n_0-1},$$

contradicting $t > S_{n_0-1}$. Thus $q < n_0$ is impossible, so $q \ge n_0$.

If $p \ge n_0$, then using $a_k = k + B$ for $k \ge n_0$ we obtain

$$t = \sum_{k=p}^{q} (k+B) = \sum_{k=p}^{q} k + (q-p+1)B.$$

Set $u := p + B$ and $v := q + B$. Then $u \ge n_0 + B = T$, and since $q - p + 1 \ge 2$ we have $v - u + 1 = q - p + 1 \ge 2$, i.e. $v \ge u + 1$. Also

$$\sum_{k=p}^{q} k + (q-p+1)B = \sum_{m=u}^{v} m,$$

giving type (A).

If instead $p < n_0$, then $q \ge n_0$ as shown above, and we split the block:

$$t = \sum_{k=p}^{n_0-1} a_k \ + \sum_{k=n_0}^{q} a_k.$$

The first part equals some $C \in \mathcal{C}$ by definition. For the second part,

$$\sum_{k=n_0}^{q} a_k = \sum_{k=n_0}^{q} (k+B) = \sum_{m=T}^{q+B} m.$$

Setting $v := q + B$ (so $v \geq T$) yields type (B). These two cases are disjoint as conditions on the indices (either $p \geq n_0$ or $p < n_0$), although the resulting sets of attainable values may overlap. $\square$

Lemma 3.3. *Assume $a_n = n + B$ for all $n \geq n_0$ and write $T = n_0 + B$. Then for every integer r with*

$$2^r \geq \max\{T,\ S_{n_0-1} + 1,\ B + 3\},$$

the number 2^r is a term of the sequence and admits a representation of type (B)*. Consequently, for each such r there exist $C \in \mathcal{C}$ and an integer $v \geq T$ such that*

$$2^{r+1} = v^2 + v + E_C, \qquad E_C := 2C - (T-1)T.$$

Moreover, there exists a fixed integer E for which the equation

$$v^2 + v + E = 2^m$$

has infinitely many integer solutions (v, m).

Proof. Fix r with $2^r \geq T = n_0 + B$. Since the tail is consecutive integers $a_n = n + B$ for $n \geq n_0$, taking $n := 2^r - B$ gives $n \geq n_0$ and

$$a_n = n + B = 2^r,$$

so 2^r is indeed a term of the sequence.

Because $n \geq 3$ for all sufficiently large r, the defining rule of the sequence provides a representation

$$2^r = a_n = \sum_{k=p}^{q} a_k \qquad (1 \leq p \leq q \leq n-1,\quad q - p + 1 \geq 2).$$

Also $2^r > S_{n_0-1}$ by the hypothesis on r, so Lemma 3.2 applies. If the representation were of type (A), then Lemma 3.2(A) would give

$$2^r = \sum_{m=u}^{v} m \quad \text{with } v \geq u + 1,$$

a sum of at least two consecutive positive integers, contradicting Lemma 2.3 since 2^r is a power of 2. Thus the representation must be of type (B), so

$$2^r = C + \sum_{m=T}^{v} m \quad \text{for some } C \in \mathcal{C},\ v \geq T.$$

Using $\sum_{m=T}^{v} m = \frac{v(v+1)}{2} - \frac{(T-1)T}{2}$ and multiplying by 2 yields

$$2^{r+1} = v^2 + v + \big(2C - (T-1)T\big) = v^2 + v + E_C,$$

as claimed.

For the final statement, consider the set of pairs

$$A := \Big\{(r, C) \in \mathbb{Z}_{>1} \times \mathcal{C} :\ \exists\, v \geq T \text{ with } 2^{r+1} = v^2 + v + E_C\Big\}.$$

We have shown: for every sufficiently large r, there exists at least one $C \in \mathcal{C}$ with $(r, C) \in A$. Therefore A is infinite. Since $\mathcal{C}$ is finite, the infinite pigeonhole principle implies that there exists a fixed $C_* \in \mathcal{C}$ such that $(r, C_*) \in A$ for infinitely many r. For each such r, choose one corresponding v; then setting $m := r + 1$ and $E := E_{C_*}$ produces infinitely many solutions $(v, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>2}$ to $v^2 + v + E = 2^m$. $\square$

Now we conclude the main contradiction. By Lemma 3.3, boundedness of (b_n) implies the existence of a fixed integer E for which $v^2 + v + E = 2^m$ has infinitely many integer solutions (v, m). This contradicts Proposition 2.2. Therefore (b_n) is not bounded above, i.e. (b_n) is unbounded.

Finally, if $\{a_n\}$ omitted only finitely many positive integers, then there would exist N_1 such that every integer $t \geq N_1$ equals some a_n. Since (a_n) is strictly increasing, this would force $a_n = n + B$ for all sufficiently large n, i.e. boundedness of b_n, contradicting unboundedness. Hence $\{a_n\}$ omits infinitely many integers. This proves Theorem 1.3. □

References

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