A NOTE ON ERDŐS PROBLEM #835

JIE MA AND QUANYU TANG

1. Introduction

Erdős proposed the following problem (see also Problem #835 in [1]):

Problem 1.1. *Does there exist a $k > 2$ such that the k-sized subsets of $\{1, \dots, 2k\}$ can be coloured with $k+1$ colours such that for every $A \subset \{1, \dots, 2k\}$ with $|A| = k+1$ all $k+1$ colours appear among the k-sized subsets of A?*

We recall the definition of Johnson graph.

Definition 1.2. For $n \in \mathbb{N}$ write $[n] = \{1, \dots, n\}$. The Johnson graph $J(n, k)$ has vertex set $\binom{[n]}{k}$; two k-sets are adjacent iff they intersect in exactly $k-1$ elements.

It is easy to see that Problem 1.1 is equivalent to the following problem:

Problem 1.3. *Does there exists $k > 2$ such that the chromatic number of the Johnson graph $J(2k, k)$ is $k+1$?*

For a graph G let $\alpha(G)$ denote its independence number and $\chi(G)$ its chromatic number. We remark that $k+1 \leq \chi(J(2k, k)) \leq 2k$ is a known result.

2. A necessary condition

Proposition 2.1. *Assume $\chi(J(2k, k)) = k+1$. Then for every integer t with $1 \leq t \leq k$ one must have*

$$t \mid \binom{k+t}{t-1}.$$

Proof. We may assume $k \geq 2$, as the case $k = 1$ is trivial. Fix an integer t with $1 \leq t \leq k$. The case $t = 1$ is also trivial since $1 \mid \binom{k+1}{0} = 1$. Hence, in what follows we assume $2 \leq t \leq k$.

Let $\mathcal{F} \subset \binom{[2k]}{k}$ be any maximal independent set in $J(2k, k)$. For a fixed $(k-t)$-subset $\lambda \subset [2k]$, define

$$\mathcal{F}_\lambda := \{\, S \in \mathcal{F} : \ \lambda \subset S \,\}.$$

Write $U := [2k] \setminus \lambda$; then $|U| = k+t$. Every $S \in \mathcal{F}_\lambda$ can be written uniquely as

$$S = \lambda \cup T \qquad \text{with} \quad T \in \binom{U}{t}.$$

We will compare $\mathcal{F}_\lambda$ to a packing of complete hypergraphs inside the complete $(t-1)$-uniform hypergraph on U.

Claim 1. *Let $\mathcal{T}_\lambda := \{\, T \in \binom{U}{t} : \ \lambda \cup T \in \mathcal{F}_\lambda \,\}$. Then the family*

$$\{\, K_t^{(t-1)}[T] \ : \ T \in \mathcal{T}_\lambda \,\}$$

is an edge-disjoint packing of copies of the complete $(t-1)$-uniform hypergraph $K_t^{(t-1)}$ inside the complete $(t-1)$-uniform hypergraph $K_{k+t}^{(t-1)}$ on vertex set U. In particular,

$$|\mathcal{F}_\lambda| \ \leq \ \nu_{t-1}(k+t),$$

where $\nu_{t-1}(n)$ denotes the maximum number of pairwise edge-disjoint copies of $K_t^{(t-1)}$ in $K_n^{(t-1)}$.

Proof of Claim 1. Take distinct $T, T' \in \mathcal{T}_\lambda$. Suppose for contradiction that $K_t^{(t-1)}[T]$ and $K_t^{(t-1)}[T']$ share an edge in $K_{k+t}^{(t-1)}$; equivalently, T and T' share a common $(t-1)$-subset $R \subset U$. Then for the corresponding k-sets $S = \lambda \cup T$ and $S' = \lambda \cup T'$ we have

$$|S \cap S'| = |\lambda| + |T \cap T'| = (k-t) + (t-1) = k-1,$$

so S and S' are adjacent in $J(2k,k)$, contradicting the independence of $\mathcal{F}$. Hence all the $(t-1)$-edges used by these copies are pairwise disjoint, proving the claim. $\square$

We now double-count the pairs (S,λ) with $S \in \mathcal{F}$, $\lambda \subset S$ and $|\lambda| = k-t$. From the S-side, each $S \in \mathcal{F}$ contains $\binom{k}{k-t} = \binom{k}{t}$ choices of λ, hence

$$\sum_{\lambda} |\mathcal{F}_\lambda| \; = \; \sum_{S \in \mathcal{F}} \binom{k}{t} \; = \; |\mathcal{F}| \binom{k}{t}. \tag{2.1}$$

From the λ-side, there are $\binom{2k}{k-t}$ possible λ, and by Claim 1 each contributes at most $\nu_{t-1}(k+t)$, so

$$\sum_{\lambda} |\mathcal{F}_\lambda| \; \le \; \binom{2k}{k-t} \nu_{t-1}(k+t). \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$|\mathcal{F}| \binom{k}{t} \; \le \; \binom{2k}{k-t} \nu_{t-1}(k+t). \tag{2.3}$$

Claim 2. *For all $n \ge t$,*

$$\nu_{t-1}(n) \; \le \; \left\lfloor \frac{1}{t} \binom{n}{t-1} \right\rfloor.$$

Proof of Claim 2. The hypergraph $K_n^{(t-1)}$ has exactly $\binom{n}{t-1}$ edges. Each copy of $K_t^{(t-1)}$ uses $\binom{t}{t-1} = t$ distinct $(t-1)$-edges. If these copies are edge-disjoint, M copies use at least tM edges in total, hence $tM \le \binom{n}{t-1}$ and $M \le \lfloor \binom{n}{t-1}/t \rfloor$. $\square$

Assume now that $\chi(J(2k,k)) = k+1$. Since $\chi(G)\alpha(G) \ge |V(G)|$, we know that

$$|\mathcal{F}| \; \ge \; \frac{1}{k+1} \binom{2k}{k}. \tag{2.4}$$

Substituting (2.4) into (2.3) and rearranging yields

$$\nu_{t-1}(k+t) \; \ge \; \frac{\binom{2k}{k}}{k+1} \cdot \frac{\binom{k}{t}}{\binom{2k}{k-t}}. \tag{2.5}$$

Now we compute directly:

$$\frac{\binom{2k}{k}}{k+1} \cdot \frac{\binom{k}{t}}{\binom{2k}{k-t}} = \frac{(2k)!}{(k+1)\,k!\,k!} \cdot \frac{k!}{t!\,(k-t)!} \cdot \frac{(k-t)!\,(k+t)!}{(2k)!} = \frac{1}{t} \binom{k+t}{t-1}.$$

Thus we obtain the lower bound

$$\nu_{t-1}(k+t) \; \ge \; \frac{1}{t} \binom{k+t}{t-1}.$$

Together with the upper bound in Claim 2 (and the integrality of $\nu_{t-1}(k+t)$) this forces

$$\nu_{t-1}(k+t) \;=\; \frac{1}{t}\binom{k+t}{t-1} \qquad \text{and hence} \qquad t \;\Big|\; \binom{k+t}{t-1}.$$

Since t was an arbitrary integer with $2 \le t \le k$, the stated divisibility holds for all $t = 2, 3, \dots, k$. As noted at the start, the case $t = 1$ is trivial, which completes the proof. □

As a first corollary of Proposition 2.1, we completely resolve Problem 1.3 for all integers $k > 2$ such that $k+1$ is not prime.

Theorem 2.2. *If $k > 2$ and $k+1$ is not prime, then $\chi(J(2k,k)) \ge k+2$.*

Proof. Let p be a prime divisor of $k+1$. Since $k+1$ is composite, we may choose $p \le (k+1)/2 \le k$, so the divisibility necessity in Proposition 2.1 with $t = p$ gives

$$p \;\Big|\; \binom{k+p}{p-1}.$$

Write $k = pq + (p-1)$ (because $k \equiv -1 \pmod p$). By Lucas' theorem,

$$\binom{k+p}{p-1} \equiv \binom{p-1}{p-1} \cdot \binom{q+1}{0} \equiv 1 \pmod p,$$

a contradiction. Therefore $\chi(J(2k,k)) \ne k+1$. □

We now point out that, for even $k > 2$, the conditions in Proposition 2.1 alone

$$\Big(\; \forall\, t \in [1,k], \quad t \mid \binom{k+t}{t-1} \;\Big)$$

cannot rule out the case $k+1$ prime (equivalently, $k = p-1$ with p odd prime), because by the following Lemma 2.3 all these divisibilities do hold.

Lemma 2.3. *Let $k \ge 2$ and suppose $k+1 = p$ is prime. Then for every $1 \le t \le k$,*

$$t \;\Big|\; \binom{k+t}{t-1}.$$

Proof. Write $k = p-1$ with p prime and fix t with $1 \le t \le k = p-1$. We use the standard identity

$$\binom{p-1+t}{t-1} \;=\; \frac{t}{p}\binom{p-1+t}{t}, \tag{2.6}$$

It therefore suffices to prove that $p \mid \binom{p-1+t}{t}$. Note that

$$\binom{p-1+t}{t} = \frac{(p-1+t)!}{t!\,(p-1)!} = \frac{p(p+1)\cdots(p+t-1)}{t!}.$$

The numerator contains a factor p, whereas the denominator $t!$ is not divisible by p because $t \le p-1$. Hence p divides $\binom{p-1+t}{t}$. Combining this with (2.6) shows that $t \mid \binom{p-1+t}{t-1}$, i.e.,

$$t \;\Big|\; \binom{k+t}{t-1}.$$

This holds for every $1 \le t \le k$, completing the proof. □

References

[1] T. F. Bloom, Erdős Problem #835, https://www.erdosproblems.com/835, accessed 2025-10-26.

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China. Research supported by National Key Re-search and Development Program of China 2023YFA1010201 and National Natural Science Foundation of China grant 12125106.
Email address: jiema@ustc.edu.cn

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: tang_quanyu@163.com