A NOTE ON ERDŐS PROBLEM #835

JIE MA AND QUANYU TANG

1. Introduction

In [2, p. 283], Erdős proposed the following problem (see also Problem #835 in [1]):

Problem 1.1. *Does there exist a $k > 2$ such that the k-sized subsets of $\{1, \dots, 2k\}$ can be coloured with $k+1$ colours such that for every $A \subset \{1, \dots, 2k\}$ with $|A| = k+1$ all $k+1$ colours appear among the k-sized subsets of A?*

We recall the definition of Johnson graph.

Definition 1.2. For $n \in \mathbb{N}$ write $[n] = \{1, \dots, n\}$. The Johnson graph $J(n, k)$ has vertex set $\binom{[n]}{k}$; two k-sets are adjacent iff they intersect in exactly $k-1$ elements.

It is easy to see that Problem 1.1 is equivalent to the following problem:

Problem 1.3. *Does there exists $k > 2$ such that the chromatic number of the Johnson graph $J(2k, k)$ is $k+1$?*

For a graph G let $\alpha(G)$ denote its independence number and $\chi(G)$ its chromatic number. We remark that $k+1 \leq \chi(J(2k, k)) \leq 2k$ is a known result.

2. A necessary condition

Proposition 2.1. *Assume $\chi(J(2k, k)) = k+1$. Then for every integer t with $1 \leq t \leq k$ one must have*

$$t \mid \binom{k+t}{t-1}.$$

Proof. We may assume $k \geq 2$, as the case $k = 1$ is trivial. Fix an integer t with $1 \leq t \leq k$. The case $t = 1$ is also trivial since $1 \mid \binom{k+1}{0} = 1$. Hence, in what follows we assume $2 \leq t \leq k$.

Let $\mathcal{F} \subset \binom{[2k]}{k}$ be any maximal independent set in $J(2k, k)$. For a fixed $(k-t)$-subset $\lambda \subset [2k]$, define

$$\mathcal{F}_\lambda := \{ S \in \mathcal{F} : \lambda \subset S \}.$$

Write $U := [2k] \setminus \lambda$; then $|U| = k+t$. Every $S \in \mathcal{F}_\lambda$ can be written uniquely as

$$S = \lambda \cup T \qquad \text{with} \quad T \in \binom{U}{t}.$$

We will compare $\mathcal{F}_\lambda$ to a packing of complete hypergraphs inside the complete $(t-1)$-uniform hypergraph on U.

Claim 1. *Let $\mathcal{T}_\lambda := \{ T \in \binom{U}{t} : \lambda \cup T \in \mathcal{F}_\lambda \}$. Then the family*

$$\{ K_t^{(t-1)}[T] \ : \ T \in \mathcal{T}_\lambda \}$$

is an edge-disjoint packing of copies of the complete $(t-1)$-uniform hypergraph $K_t^{(t-1)}$ inside the complete $(t-1)$-uniform hypergraph $K_{k+t}^{(t-1)}$ on vertex set U. In particular,

$$|\mathcal{F}_\lambda| \ \leq \ \nu_{t-1}(k+t),$$

where $\nu_{t-1}(n)$ denotes the maximum number of pairwise edge-disjoint copies of $K_t^{(t-1)}$ in $K_n^{(t-1)}$.

Proof of Claim 1. Take distinct $T, T' \in \mathcal{T}_\lambda$. Suppose for contradiction that $K_t^{(t-1)}[T]$ and $K_t^{(t-1)}[T']$ share an edge in $K_{k+t}^{(t-1)}$; equivalently, T and T' share a common $(t-1)$-subset $R \subset U$. Then for the corresponding k-sets $S = \lambda \cup T$ and $S' = \lambda \cup T'$ we have

$$|S \cap S'| = |\lambda| + |T \cap T'| = (k-t) + (t-1) = k-1,$$

so S and S' are adjacent in $J(2k,k)$, contradicting the independence of $\mathcal{F}$. Hence all the $(t-1)$-edges used by these copies are pairwise disjoint, proving the claim. □

We now double-count the pairs (S, λ) with $S \in \mathcal{F}$, $\lambda \subset S$ and $|\lambda| = k-t$. From the S-side, each $S \in \mathcal{F}$ contains $\binom{k}{k-t} = \binom{k}{t}$ choices of λ, hence

$$\sum_{\lambda} |\mathcal{F}_\lambda| \; = \; \sum_{S \in \mathcal{F}} \binom{k}{t} \; = \; |\mathcal{F}| \binom{k}{t}. \tag{2.1}$$

From the λ-side, there are $\binom{2k}{k-t}$ possible λ, and by Claim 1 each contributes at most $\nu_{t-1}(k+t)$, so

$$\sum_{\lambda} |\mathcal{F}_\lambda| \; \leq \; \binom{2k}{k-t} \nu_{t-1}(k+t). \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$|\mathcal{F}| \binom{k}{t} \; \leq \; \binom{2k}{k-t} \nu_{t-1}(k+t). \tag{2.3}$$

Claim 2. *For all $n \geq t$,*

$$\nu_{t-1}(n) \; \leq \; \left\lfloor \frac{1}{t} \binom{n}{t-1} \right\rfloor.$$

Proof of Claim 2. The hypergraph $K_n^{(t-1)}$ has exactly $\binom{n}{t-1}$ edges. Each copy of $K_t^{(t-1)}$ uses $\binom{t}{t-1} = t$ distinct $(t-1)$-edges. If these copies are edge-disjoint, M copies use at least tM edges in total, hence $tM \leq \binom{n}{t-1}$ and $M \leq \lfloor \binom{n}{t-1}/t \rfloor$. □

Assume now that $\chi(J(2k,k)) = k+1$. Since $\chi(G)\alpha(G) \geq |V(G)|$, we know that

$$|\mathcal{F}| \; \geq \; \frac{1}{k+1} \binom{2k}{k}. \tag{2.4}$$

Substituting (2.4) into (2.3) and rearranging yields

$$\nu_{t-1}(k+t) \; \geq \; \frac{\binom{2k}{k}}{k+1} \cdot \frac{\binom{k}{t}}{\binom{2k}{k-t}}. \tag{2.5}$$

Now we compute directly:

$$\frac{\binom{2k}{k}}{k+1} \cdot \frac{\binom{k}{t}}{\binom{2k}{k-t}} = \frac{(2k)!}{(k+1)\, k!\, k!} \cdot \frac{k!}{t!\, (k-t)!} \cdot \frac{(k-t)!\, (k+t)!}{(2k)!} = \frac{1}{t} \binom{k+t}{t-1}.$$

Thus we obtain the lower bound

$$\nu_{t-1}(k+t) \; \geq \; \frac{1}{t} \binom{k+t}{t-1}.$$

Together with the upper bound in Claim 2 (and the integrality of $\nu_{t-1}(k+t)$) this forces

$$\nu_{t-1}(k+t) \;=\; \frac{1}{t}\binom{k+t}{t-1} \qquad \text{and hence} \qquad t \;\Big|\; \binom{k+t}{t-1}.$$

Since t was an arbitrary integer with $2 \le t \le k$, the stated divisibility holds for all $t = 2, 3, \ldots, k$. As noted at the start, the case $t = 1$ is trivial, which completes the proof. □

As a corollary of Proposition 2.1, we resolve Problem 1.3 for all integers $k > 2$ such that $k+1$ is not prime.

Theorem 2.2. *If $k > 2$ and $k+1$ is not prime, then $\chi(J(2k,k)) \ge k+2$.*

Proof. Let p be a prime divisor of $k+1$. Since $k+1$ is composite, we may choose $p \le (k+1)/2 \le k$, so the divisibility necessity in Proposition 2.1 with $t = p$ gives

$$p \;\Big|\; \binom{k+p}{p-1}.$$

Write $k = pq + (p-1)$ (because $k \equiv -1 \pmod p$). By Lucas' theorem,

$$\binom{k+p}{p-1} \equiv \binom{p-1}{p-1} \cdot \binom{q+1}{0} \equiv 1 \pmod p,$$

a contradiction. Therefore $\chi(J(2k,k)) \ne k+1$. □

We now point out that, for even $k > 2$, the conditions in Proposition 2.1 alone

$$\Big(\; \forall\, t \in [1,k], \quad t \mid \binom{k+t}{t-1} \;\Big)$$

cannot rule out the case $k+1$ prime (equivalently, $k = p-1$ with p odd prime), because by the following Lemma 2.3 all these divisibilities do hold.

Lemma 2.3. *Let $k \ge 2$ and suppose $k+1 = p$ is prime. Then for every $1 \le t \le k$,*

$$t \;\Big|\; \binom{k+t}{t-1}.$$

Proof. Write $k = p-1$ with p prime and fix t with $1 \le t \le k = p-1$. We use the standard identity

$$\binom{p-1+t}{t-1} \;=\; \frac{t}{p}\binom{p-1+t}{t}, \tag{2.6}$$

It therefore suffices to prove that $p \mid \binom{p-1+t}{t}$. Note that

$$\binom{p-1+t}{t} = \frac{(p-1+t)!}{t!\,(p-1)!} = \frac{p(p+1)\cdots(p+t-1)}{t!}.$$

The numerator contains a factor p, whereas the denominator $t!$ is not divisible by p because $t \le p-1$. Hence p divides $\binom{p-1+t}{t}$. Combining this with (2.6) shows that $t \mid \binom{p-1+t}{t-1}$, i.e.,

$$t \;\Big|\; \binom{k+t}{t-1}.$$

This holds for every $1 \le t \le k$, completing the proof. □

References

[1] T. F. Bloom, Erdős Problem #835, https://www.erdosproblems.com/835, accessed 2025-10-26.
[2] P. Erdős, Unsolved Problems. (1974), 278–297.

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China. Research supported by National Key Re-search and Development Program of China 2023YFA1010201 and National Natural Science Foundation of China grant 12125106.
Email address: jiema@ustc.edu.cn

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: tang_quanyu@163.com