AN IMPROVED LOWER BOUND ON ERDŐS PROBLEM #962

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1. Introduction

In [2, p. 96], Erdős proposed the following problem, which also appears as Problem #962 on Bloom's Erdős Problems website [1].

Problem 1.1. *Let $k(n)$ be the maximal integer k such that there exists $m \le n$ such that each of the integers*

$$m+1, \dots, m+k$$

are divisible by at least one prime $> k$. Estimate $k(n)$.

Erdős [2] wrote it is 'not hard to prove' that

$$k(n) \gg_\epsilon \exp\left((\log n)^{1/2-\epsilon}\right)$$

and it 'seems likely' that $k(n) = o(n^\epsilon)$, but had no non-trivial upper bound for $k(n)$.

It is not clear what he meant by a non-trivial bound for this problem, but Tao in the comments section of [1] has given a simple argument proving $k(n) \le (1+o(1))n^{1/2}$.

In this short note we prove a new lower bound for $k(n)$.

Theorem 1.2. *As $n \to \infty$, we have*

$$k(n) \ge \exp\left(\left(\tfrac{1}{\sqrt{2}} - o(1)\right)\sqrt{\log n \, \log\log n}\right).$$

2. Main result

Theorem 2.1. *For every fixed constant c with $0 < c < 1/\sqrt{2}$ there exists $n_0(c)$ such that for all integers $n \ge n_0(c)$,*

$$k(n) \ \ge \ \left\lfloor \exp\left(c\sqrt{\log n \, \log\log n}\right) \right\rfloor.$$

Proof. For an integer $t \ge 1$, let $P^+(t)$ denote the largest prime divisor of t (with the convention $P^+(1) = 1$). For real numbers $x \ge 1$ and $y \ge 2$, define

$$\Psi(x,y) \ := \ \#\{\, t \le x : \ P^+(t) \le y \,\},$$

the counting function of y–smooth integers up to x.

1. A pigeonhole lemma: Fix an integer $y \ge 2$ and set $Q := \lfloor n/y \rfloor$. Partition the interval $\{1, 2, \dots, Qy\}$ into Q disjoint blocks of length y:

$$B_j := \{jy+1, \dots, (j+1)y\} \qquad (j = 0, 1, \dots, Q-1).$$

If every block B_j contains at least one y–smooth integer, then $\Psi(n,y) \ge Q$. Hence, if $\Psi(n,y) < Q$, there exists some j such that B_j contains no y–smooth integer, i.e. $P^+(t) > y$ for all $t \in B_j$. Taking $m := jy$, we have $m \le Qy \le n$ and

$$P^+(m+i) > y \qquad (1 \le i \le y).$$

In particular, each $m+i$ is divisible by a prime $> y$.

Thus, to prove the theorem it suffices to find y with $\Psi(n,y) < \lfloor n/y \rfloor$.

2. Choosing y and estimating $\Psi(n, y)$: Fix $c \in (0, 1/\sqrt{2})$ and set

$$y := \Big\lfloor \exp\big(c\sqrt{\log n\ \log\log n}\big)\Big\rfloor, \qquad u := \frac{\log n}{\log y}.$$

Then $y \to \infty$ and $u \to \infty$ as $n \to \infty$, and moreover $\log y \sim c\sqrt{\log n\ \log\log n}$, hence

$$u = \frac{\log n}{\log y} = \frac{1}{c}\sqrt{\frac{\log n}{\log\log n}}\,(1+o(1)), \qquad \log u = \frac{1}{2}\log\log n + O(\log\log\log n),$$

so

$$u\log u = \Big(\tfrac{1}{2c} + o(1)\Big)\sqrt{\log n\ \log\log n}. \tag{2.1}$$

A classical estimate of de Bruijn, with Hildebrand's extension of the uniformity range, gives

$$\Psi(n, y) = n\,\rho(u)\Big(1 + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big), \tag{2.2}$$

holds for $y > \exp\big((\log\log n)^{5/3+\varepsilon}\big)$, where $\rho(u)$ is the Dickman–de Bruijn function; see [3, (1.8)–(1.10)]. In our choice of parameters, $\log(u+1)/\log y \to 0$, so

$$\Psi(n, y) = n\,\rho(u)\,(1 + o(1)).$$

Moreover, $\rho(u)$ decays as

$$\rho(u) = u^{-u+o(u)} = \exp\big(-(1+o(1))u\log u\big) \qquad (u \to \infty), \tag{2.3}$$

see [3, (1.6)]. Combining (2.1) and (2.3) yields

$$\Psi(n, y) \le n\exp\Big(-\Big(\tfrac{1}{2c} + o(1)\Big)\sqrt{\log n\ \log\log n}\Big).$$

3. Comparison with n/y: Since $\log y = c\sqrt{\log n\ \log\log n} + O(1)$, we have

$$\frac{n}{y} = n\exp\big(-c\sqrt{\log n\ \log\log n} + O(1)\big).$$

Because $c < 1/\sqrt{2}$, we have $\frac{1}{2c} - c > 0$; choose a constant $\delta > 0$ with $\delta < \frac{1}{2c} - c$. Then for all sufficiently large n, the previous bounds imply

$$\Psi(n, y) \ \le\ \frac{n}{y}\,\exp\big(-\delta\sqrt{\log n\ \log\log n}\big) \ <\ \frac{1}{2}\cdot\frac{n}{y} \ \le\ \Big\lfloor\frac{n}{y}\Big\rfloor.$$

Therefore $\Psi(n, y) < \lfloor n/y\rfloor$, and by Step 1 there exists $m \le n$ such that each of $m+1, \dots, m+y$ has largest prime factor $> y$, hence is divisible by a prime $> y$. This shows $k(n) \ge y$. □

References

[1] T. F. Bloom, Erdős Problem #962, `https://www.erdosproblems.com/962`, accessed 2025-12-28.
[2] Erdős, P., Extremal problems in number theory. *Proc. Sympos. Pure Math.*, Vol. VIII (1965), 181–189.
[3] A. Granville, *Smooth numbers: computational number theory and beyond*, in *Algorithmic Number Theory: Lattices, Number Fields, Curves and Cryptography* (MSRI Publications, Vol. 44), J. P. Buhler and P. Stevenhagen (eds.), Cambridge University Press, 2008, pp. 267–324. `doi:10.1017/9781139049801.010`.

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