OPERATIONS ON POLYHEDRA

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Operations on polyhedra to produce other polyhedra date back as far as Kepler. Conway defined a set of operations that could be performed on the Platonic solids to obtain the Archimedean and Catalan solids, and others added operators after him. Initially there was not much theory supporting operations on polyhedra, and the set of named operators on polyhedra was somewhat of a zoo. A 2017 paper by Brinkmann et. al. provides a framework. [1]

This text is an continuation of [1] to find ways to quantify, analyze, and expand these operators. In particular, it focuses on operators on that can be described in terms of a linear operator on the counts of vertices, edges, and faces. These linear operators can be used to examine the composition and decomposition of operations on polyhedra. Such operators do not constitute all possible operations on polyhedra, or even all those that can be represented by available software, but they are an interesting subset of those operators with many nice aesthetic and geometric qualities.

The goal of this text is to explore two main topics:

- Classification. The inflation factor introduced in [1] is a good measure of the complexity of certain operators. Are there more invariants that can be used to classify operators?
- Relationships between operators. How can operators be composed? How can an operator be decomposed into other operators?

1. Definitions

An *abstract polytope* P is a ranked poset satisfying the properties below. Elements of rank 0 may be called vertices; of rank 1, *edges*; of rank 2, faces. (In most literature on abstract polytopes all elements of the poset are called (abstract) faces, but since this work is primarily concerned with polyhedra, it would be confusing.) The order relation $<$ on the poset represents incidence between elements: for example, a face is incident on an edge, which is incident on a vertex.[13]

(1) P contains a least element $\perp$ with rank -1 and a greatest element $\top$ with rank n. (n is the rank of the abstract polytope.)
(2) Each flag of P has the same length and includes the least and greatest elements. (A flag is a series of incident elements from the least element to the greatest)
(3) P is strongly connected: any flag can be changed into any other flag by changing one element of the flag at a time.
(4) P satisfies the diamond property: if an element A has rank $k-1$, and an element C has rank $k+1$, there are exactly 2 elements B of rank k such that $A < B < C$.

Occasionally a poset structure that violates one of these properties may be useful; these will be discussed as they appear. The skeleton of a polytope is the (possibly multi-) graph formed by its vertices and edges. The dual of an abstract polytope is simply the abstract polytope with its order and rank reversed. (This corresponds to the usual geometric notion of dual.)

An element a of a poset with least element $\perp$ (and therefore an abstract polytope) is an *atom* if $\perp < a$ and there is no x such that $\perp < x < a$. A poset is *atomistic* if every element can be described unambiguously as a unique set of atoms.

The *f-vector* of a polytope is the vector of counts of elements of each rank. Normally the greatest and least elements are omitted (since they're always 1), and the vector has the form $(a_0, \dots, a_{d-1})$ where a_i is the count of elements of rank i. The *extended* f-vector includes those elements: $(a_{-1} = 1, a_0, \dots, a_{d-1}, a_d = 1)$. The vector is *left/right half-extended* if only one of those endpoints is included. As such, the (extended) f-vector

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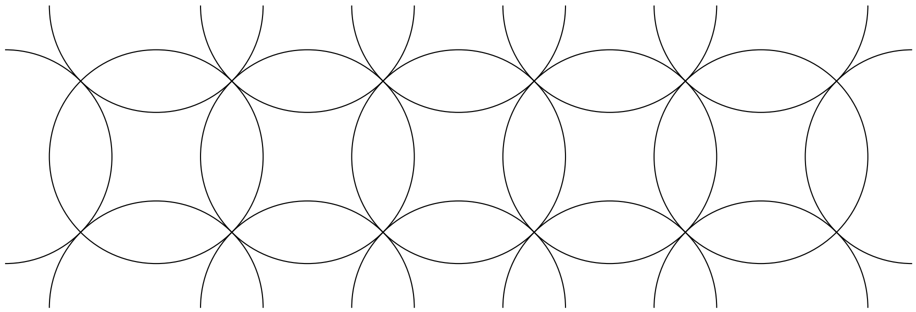

FIGURE 1. A tiling of the plane created by overlapping circles, consisting of digons and concave quadrilaterals with curved edges.

of the dual of a polytope is the (extended) f-vector of the original polytope, reversed. The f-polynomial is $F(x) = \sum_{i=-1}^{d} a_i x^{i+1}$. The *Euler characteristic* of a polytope is the alternating sum of the elements of the f-vector: $\chi = a_0 - a_1 + a_2 \ldots$.

A *realization* of an abstract polytope is an abstract polytope P mapped into a topological space, usually, but not necessarily, $\mathbb{R}^n$. Some realizations may be called *faithful*, which is not consistently defined, but usually means that the polytope does not have self-intersections, repeated points, digons, etc.

An *abstract polyhedron* P is an abstract polytope of rank 3. This generalizes things like convex polyhedra, tilings of the plane, and spherical polyhedra. Some of the properties above have more explicit consequences for polyhedra: [9]

- Each edge is incident to 2 distinct vertices. This is the usual definition of edge, but here it is a consequence of earlier definitions.
- Each edge is incident to 2 distinct faces. This excludes things like space-filling honeycomb structures (where more than 2 faces may meet at an edge)[1] and partial tilings of the plane or partial polyhedra (where fewer than 2 faces may meet at an edge).
- The faces may be self-intersecting, but may not contain interior holes.

The *degree of a vertex* is the number of edges to which it is incident. The *degree of a face* is the number of edges which are incident to it, i.e. its number of sides. (An edge of a polyhedron is always incident to 2 faces and 2 vertices, so defining degree for edges is not useful here, although it is for higher polytopes.) Abstract polyhedra permit faces of degree 2, called *digons*. Digons are disallowed by many definitions of polyhedra, but appear in spherical polyhedra and regular maps (for example). Digons occur naturally in the study of operations on polyhedra. The count of vertices, edges, and faces is be denoted v, e, f. The Euler characteristic is $\chi = v - e + f$. The count of vertices or faces of a certain degree is denoted v_i and f_i where i is the degree, such that $\sum_i v_i = v$ and $\sum_i f_i = f$.

The term *polyhedron* here will mean "realization of an abstract polyhedron." Unless otherwise specified, polyhedra here do not require distinct vertices, convex faces, straight edges, non-intersecting faces, flat faces, or other things that may be required of other classes of polyhedra. This might seem odd if you're used to standard polyhedra, but examples appear in the world. Lu and Steinhardt [12] assert that the girih patterns in Islamic art are based on a particular set of tiles, one of which is a concave polygon. For a more mundane example, Figure 1, with digons and concave quadrilaterals, appears on the author's bedspread.

We won't make much use of abstract polyhedra directly, but we need them as a theoretical underpinning. A structure that resembles a polyhedron but does not satisfy the axioms of an abstract polytope is here called a quasipolyhedron. Examples of quasipolyhedra include the monogonal dihedron (f-vector $[1, 1, 2]$), its dual the digonal hosohedron ($[2, 1, 1]$), and the monohedron ($[1, 0, 1]$), all spherical "polyhedra" that violate the diamond property.

An *orientable* polyhedron is one where each face's vertices may be assigned a cyclic order such that each edge, say with vertices a and b, has an order of a to b in one face and the reverse order b to a in the other. [7] A polyhedron that is not orientable is *non-orientable*. All polyhedra with an odd Euler characteristic

[1] Honeycombs may be valid polytopes of rank 4, however.

are non-orientable, e.g. the hemicuboctahedron. That said, all non-orientable polyhedra have an orientable double cover that is connected. (An orientable polyhedron also has an orientable double over, consisting of two disjoint copies of the original polyhedron.)

An *achiral polyhedron* is one that has mirror symmetry: a *chiral polyhedron* is one that does not. Note that the particular handedness of a chiral polyhedron is a quality of the realization, not the underlying abstract polyhedron. Also note that chirality depends on the space the polyhedron is embedded in: polyhedra that are chiral in $\mathbb{R}^3$ are not in $\mathbb{R}^4$.

An *acoptic polyhedron* is (loosely) a polyhedron that does not self-intersect. [8] Its faces are simple polygons with straight edges that do not self-intersect (although they may be concave). A *convex polyhedron* is an acoptic polyhedron that is convex: any line between points on the surface of the polyhedron is contained in the interior of the polyhedron. By Steinitz's theorem, the skeleton of every convex polyhedron is a 3-vertex-connected planar graph. Furthermore, all convex polyhedra have a realization such that each edge is tangent to the unit sphere, each face is flat, and the centroid of the vertices lies at the origin.[15] This is called the *canonical realization*.

An *operator on polyhedra* is simply a map $x : \mathcal{P} \to \mathcal{P}$, where $\mathcal{P}$ is the set of polyhedra. We'll often apply operators to a restricted set of polyhedra, e.g. convex polyhedra, orientable polyhedra, or polyhedra with triangular faces. Sometimes we will look at maps satisfying $x : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are different subsets of polyhedra: we will still call these operators on polyhedra (instead of transformations). We'll use a calligraphic typeface for sets of polyhedra or quasipolyhedra. The set of polyhedra without digons or degree-2 vertices is here denoted $\mathcal{P}_3$, and the set of polyhedra with 3-vertex-connected skeletons is $\mathcal{P}_{3v}$.

Some of these sets have additional structure, such as a specific coloring of the vertices or faces. One fundamental distinction is between orientable and non-orientable polyhedra. If a polyhedron is orientable, then there are two choices of orientation to apply to it. We'll call the orientable polyhedra, plus a choice of orientation, the *oriented polyhedra*, and denote them $\mathcal{P}^o$. Other subsets of polyhedra that are orientable will share this superscript o notation, so e.g. $\mathcal{P}_3^o$ is the set of orientable polyhedra without digons or degree-2 vertices.

The set of polyhedra with even-sided faces is denoted $\mathcal{E}$. If also orientable, $(\mathcal{E}^o)$, the vertices of the polyhedron can be 2-colored. The particular choice of coloring (out of the two possible colorings) is also part of the additional structure of $\mathcal{E}^o$, in addition to the orientation.

$\mathcal{W}$ is here defined as the set of polyhedra with only triangular faces, such that the vertices of the polyhedron can be 3-colored. The faces of a polyhedron in $\mathcal{W}^o$ can also be 2-colored. Such polyhedra are important to Antiprism's implementation of many operations.

2. Notable operations

This section isn't a complete survey of operations on polyhedra. Some operations have had ambiguous or changing definitions in the past, so going through all that history would just muddy the waters. Instead, this section summarizes some useful operators that motivate the discussion herein or will be applied.

2.1. Operations defined in Coxeter's Regular Polytopes.

In Coxeter's classic text *Regular Polytopes*[4], he defines a handful of operations on regular polytopes:

- The dual is defined in the usual way, compatible with how it was defined earlier in this text.
- Various forms of truncation are defined. The simple truncation is described by analogy with the polygonal case: cut off each corner of a polygon in such a way that the new polygon has vertices at the midpoint of each original edge. Other forms of truncation that retain part of the original edge are mentioned but not described. It is also commented that for regular polyhedra, the truncation is equivalent to the intersection of the (canonical realization of) the original polyhedra and its dual.
- An operation, partial truncation or alternation, is defined on regular polytopes with even-degree faces. Alternating vertices are cut off or retained. This results in digons for degree-4 faces, which are treated as edges.
- The snub operation (different than Conway's snub) is equivalent to simple truncation followed by alternation.

Coxeter's construction of simplexes can be cast in terms of abstract polyhedra. Simply take the poset direct product of a number of copies of the unique abstract polyhedron of rank 1 (a single point). The product of two points is an edge: of three, a triangle, of four, a tetrahedron, and so on. This can be thought of as a binary operation on simplexes, so e.g. the product of two edges is a tetrahedron. (Also, suggestively, the f-polynomial of a direct product of posets is the product of the f-polynomial of each poset.)

2.2. **Conway's operations.** Conway described a set of operations on polyhedra and a notation for describing those operations, with the intent of creating a systematic naming scheme for polyhedra.[2] He used a prefix notation, where the rightmost element is a polyhedral seed and operators apply from right to left. Each operator is assigned a letter and a name. So, for instance, dO is the dual of a regular octahedron (i.e. a cube), and taO is a truncated ambo octahedron, more commonly known as the truncated cuboctahedron. Conway's original set of operations is denoted with the letters $abdegjkmost$: a full list of their descriptions will be given in Appendix B. These operations are sufficient to create all of the Archimedean and Catalan solids from the Platonic solids. Others have defined more operations. [10][11][14] In particular, Hart [10] defined r, which reverses the chirality of a polyhedra. The term "Conway operator" is actually a little ambiguous: it may refer to Conway's original set of operations, or an expanded set, or the general idea of notation polyhedra by alphabetical prefixes.

Aside from g, s, j, and a, Conway's operations preserve the symmetry of the resultant polyhedra. g and s are achiral, so do not preserve mirror symmetry. j and a actually increase the symmetry: in fact, the other 4 Platonic solids can be expressed in terms of the tetrahedron T using j and a.

Some of Conway's operations can be expressed in terms of other operations: $e = aa, o = jj, m = kj$, and $b = ta$, as well as the ones that are dual to each other.

Some Conway operators have an indexed form that indicates that only certain faces or vertices are operated on. For instance, k_i applies kis to faces with i sides, and t_i truncates vertices of degree i. Operations like this do not in general preserve the symmetry of the seed polyhedron.

2.3. **Goldberg-Coxeter operations.** The Goldberg-Coxeter operation (GC operation) was defined by Deza and Dutour [5], based on the Goldberg polyhedra, the viral capsid structure defined by Coxeter [3], the geodesic domes of Buckminster Fuller, and similar structures. Essentially it amounts to replacing the faces of a polyhedra with a section from a grid of triangular or square faces, termed the "master polygon" by Deza and Dutour. Here the operation on triangle-faced polyhedra will be denoted $\Delta_{n,m}$, and on quadrilateral-faced polyhedra $\square_{n,m}$, where n and m are integers, $n > 0$, and $m \geq 0$.

$\Delta_{n,m}$ can be described using the triangular lattice over the Eisenstein integers. It is useful for this operation to parameterize the Eisenstein integers as $x = n+mu$ where $u = \frac{1}{2}(1+i\sqrt{3}) = e^{\pi i/3}$, for reasons to be explained later. The master polygon is the section of the grid contained in the triangle $0, x, xu$. It may also be useful to identify an edge-centered section of the grid, given by $0, x(2-u)/3, x, x(1+u)/3$.

$\square_{n,m}$ can be described using the triangular lattice over the Gaussian integers, $x = n + mi$ where $i = \sqrt{-1}$. The master polygon is contained in $0, x, (x+i), xi$, and the edge-centered section is $0, x(1-i)/2, x, x(1+i)/2$.

Each operator has an invariant g, equivalent to $g = |x|^2$. For $\Delta_{n,m}$, $g = n^2+m^2$: for $\square_{n,m}$, $g = n^2+nm+m^2$. This can be used to calculate the f-vector of the resulting polyhedron based on the f-vector of the seed polyhedron. (The actual formula will be shown later in a more general form.)

Two elements of the Eisenstein integers x and y are associates if $y = u^k x$ for some integer k. Similarly, two elements of the Gaussian integers are associates if $y = i^k x$ for some k. The associated element with $n > 0$ and $m \geq 0$ is the normal form. (We use the alternate definition for Eisenstein integers so that the same definition for normal form applies to both operators. This is also the traditional definition used by Goldberg polyhedra, geodesic domes, viral capsids, etc.) Iff $n + mu$ is associated with $(a + bu)(c + du)$, then $\Delta_{a,b}\Delta_{c,d} = \Delta_{n,m}$, and similarly for $\square_{a,b}$.

Another consequence of the relationship between these operators and the Eisenstein and Gausssian integers is that these operators are commutative and associative: $\Delta_{a,b}\Delta_{c,d} = \Delta_{c,d}\Delta_{a,b}$, and similarly for $\square_{a,b}$. Furthermore, the Eisenstein and Gaussian integers are Euclidean domains, which means elements of the domains can be factored uniquely (if not irreducible), and there is a straightforward way to do so using an extension of the Euclidean algorithm. (The invariant g is the Euclidean function in the Euclidean algorithm.) Therefore, it's straightforward to reduce $\Delta_{n,m}$ and $\square_{n,m}$ into other operators, in and the irreducible operators correspond to Eisenstein or Gaussian primes.

These operators are divided into three classes.

- Class I: $m = 0$, achiral
- Class II: $n = m$, achiral
- Class III: All others, chiral, requires orientable polyhedron.

All Class II operators can be reduced as $\Delta_{n,n} = \Delta_{1,1}\Delta_{n,0}$, and possibly further (and similiarly for $\square_{n,n}$). In fact, $\Delta_{n,m}$ is reducible if $n \equiv m \mod 3$ (as $\Delta_{1,1}$ and some other $\Delta_{c,d}$), and $\square_{n,m}$ is reducible if $n \equiv m \mod 2$ (as $\square_{1,1}$ and some other $\square_{c,d}$).

When applied to the triangular tiling of the plane, $\Delta_{n,m}$ produces a (possibly scaled and rotated) triangular tiling of the plane, regardless of subscript. The same holds for $\square_{n,m}$ and the quadrilateral tiling of the plane.

2.4. **Operations in the software package Antiprism.** The software package Antiprism [14] includes a number of applications that perform operations on polyhedra. (Among other things, it contains an implementation of Conway operations in `conway`.) One caveat: The file format used by antiprism, OFF, consists of a list of vertex positions and a list of faces that references the list of vertices. This is not a fully faithful representation of a polyhedra, as it does not contain explicit incidence information. Any polyhedron whose abstract polyhedron is not atomistic cannot be represented faithfully in Antiprism: e.g. polyhedra with overlapping faces or digons. Digons are referred to as explicit edges by antiprism.

Of particular interest is the application `wythoff`, in which a notation for operations on polyhedra is introduced. Despite the name, the new notation is much more flexible than Wythoff notation. `wythoff` automatically applies Conway's meta operation to produce a polyhedron in $\mathcal{W}^o$. The meta operation retains the original vertices and adds a vertex at the center of each edge and each face. Respectively, these are labeled V, E, and F. If a polyhedron is in $\mathcal{W}^o$ it can be used directly: the labeling of vertices uses the same letters VEF. This is somewhat confusing since E and F no longer relate to edges or faces, so alternately the vertices can be labeled ABC. If applied to a non-orientable polyhedron, `wythoff` instead uses the orientable double cover of the polyhedron.

For example, this is the string that it uses to implement Conway's kis operation: `[F, V] 0_1v1v, 1E`. The extended Wythoff notation comprises two parts. The first part, in brackets, defines points on each triangular face using barycentric coordinates. Each point is specified as `aVbEcF`, where `a`, `b`, and `c` are numbers. If any of those are 1, the number may be left off: if 0, the component may be left off. So in the above, it defines two points, more explicitly as `0V0E1F` and `1V0E0F`: a point at vertices labeled F, and a point at vertices labeled V.

The second part defines faces as paths between these points. A `+`, `-`, or `*` at the start of a path denotes which triangle to start with: if none, then `+` is assumed. An underscore indicates remaining on the same triangle. A lowercase `v`, `e`, or `f` indicates that the path crosses the edge opposite of the vertex labelled V, E, or F. An uppercase `V`, `E`, or `F` indicates a rotation by two triangles about the vertex labeled V, E, or F: explicitly, these are shorthands for `ef`, `fv`, and `ve` respectively. The first path starts at point 0 on + triangles, moves to point 1 on the same triangle, then moves over the edge opposite V to point 1 on that triangle. It then moves back over the edge and completes the path at 0 on the original triangle. (The second produces an explicit edge, and is needed so that the operation produces a polyhedron when applied directly to a polyhedron in $\mathcal{W}^o$.)

This notation is capable of representing many operations on polyhedra, including all of Conway's operators. (In fact the current version of `conway` uses `wythoff` to implement the operators.) Notation to create the regular hemihedra from the Platonic solids exists, as does notation to create hollowed-out, da Vinci-style renderings of polyhedra. It can be applied to partial tilings of the plane or partial polyhedra, although it more or less skips the edges that are adjacent to less than 2 faces. It may also produce quasipolyhedra: in fact, care should be taken to make sure that the notation produces a polyhedron in all cases, not just when applied with the meta operation.

3. Smoothing

Digons can be present in a faithful realization of a polyhedron, such as in Figure 1, or spherical hosohedra. However, many definitions or subsets of polyhedra exclude digons. (For instance, Grünbaum excludes digons in [9].) Degree-2 vertices are also problematic, being dual to digons. These elements can be eliminated in a

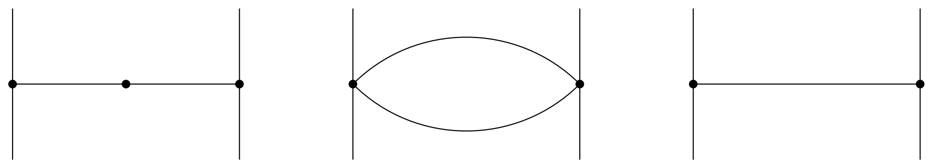

FIGURE 2. Depiction of smoothing operator \$. a) a degree-2 vertex b) digon c) smoothed result when applied to either.

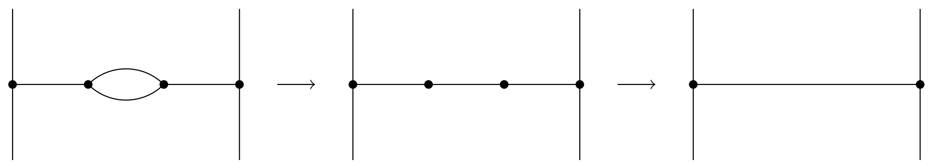

FIGURE 3. A multi-step smoothing series.

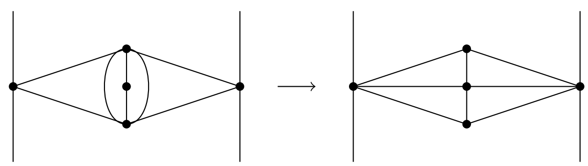

FIGURE 4. A more complicated degree-2 feature and its surroundings, and a particular way it admits a smoothing that preserves the count of elements.

systematic manner. We define a smoothing operator, \$, that transforms digons into single edges and handles degree-2 vertices by removing that vertex and merging the vertex's incident edges, as depicted in figure 2. In graph-theoretic terms, degree-2 vertices are transformed by contracting one of the edges it connects to, and digons are transformed by collapsing their edges into one, which is the equivalent act on the dual graph. [6] Antiprism effectively smooths digons automatically by treating them as "explicit edges".

Smoothing a single digon removes 1 face and 1 edge from the polyhedron. Smoothing a single degree-2 vertex removes 1 vertex and 1 edge. If f_p is the f-vector of a polyhedron, then $f_{\$p} \leq f_p$, where the inequality holds pairwise.

Conceptual complications arise when a polyhedron contains multiple digons or 2-vertices incident to one another. There is some choice in which element to start on, and degree-2 elements may be adjacent to one another. A single smoothing step may create other degree-2 elements, as depicted in Figure 3. It can be shown that with repeated reduction of single elements, the polyhedron eventually reaches a state where it has no degree-2 features. We choose to define \$ so that it produces a polyhedron where all degree-2 features have been removed. There may also be other ways to smooth a particular configuration involving a degree-2 feature that may have other benefits, e.g. Figure 4, but \$ is the minimum-effort way to remove the degree-2 features.

In special circumstances \$ may produce quasipolyhedra. For instance, the result for any spherical hosohedron or spherical dihedron is the digonal hosohedron or monogonal dihedron, respecetively. It may be helpful to define the operator such that it returns a proper polyhedra, stopping before it produces a quasipolyhedron. This ambiguity will not affect the rest of this text.

With some odd exceptions (those in the last paragraph), the smoothing operator \$ produces polyhedra where the minimum vertex and face degree is 3. Since the original polyhedron was required to be connected, this also means that the polyhedron has a 3-edge-connected skeleton. This is necessary but not sufficient in order to have a 3-vertex-connected skeleton (which, if the graph is also planar, makes it the skeleton of a convex polyhedron). One feature preventing a 3-edge-connected skeleton from being a 3-vertex-connected skeleton can be seen on the left of Figure 5. In general, if a 3-edge-connected graph has any subgraphs that can be reduced to a digon or degree-2 vertex by edge contraction, those are responsible for the graph not being a 3-vertex-connected graph. So, we introduce an operator that does that edge contraction and then removes

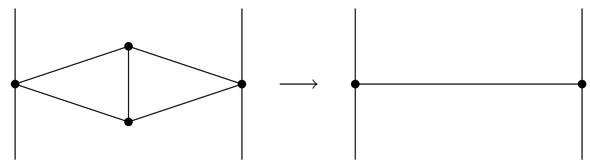

FIGURE 5. Depiction of pound operator £.

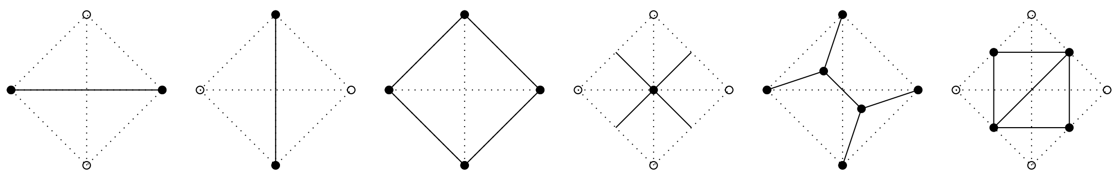

FIGURE 6. Edge-centered diagram of EROs (LSPs and LOPSPs). From right to left: Seed (S), dual (d), join (j), ambo (a), gyro (g), snub (s).

the resulting degree-2 feature, and call it *pound*, denoted £. (The name is by analogy with pounding out the dents in a metal object to make it convex, and using a currency symbol makes it similar to $.) That said, polyhedra with such features and operators that produce them are interesting in ways we'll get to later.

4. Edge-replacement operations

From this point, the discussion will exist in an uncomfortable middle ground. We will usually only be concerned with the combinatorial or topological structure of the polyhedron, but we still need to make reference to the ambient space. For instance, a *face center* as used here need not be at any exact center of the face, or even really on the face, but the most intuitive way to talk about it is to put it at some sort of center on the face. When we talk about equality of operators, we mean it in the topological and combinatorial sense.

In [1], two sets of operations on polyhedra are defined: local symmetry-preserving operations (LSP), on $\mathcal{P}$, and local operations that preserve orientation-preserving symmetries (LOPSP), on $\mathcal{P}^o$. *Symmetry-preserving* means that the symmetry group of the seed polyhedron is the same as or a subgroup of the symmetry group of the result polyhedron. All of Conway's original operators are LSPs, except for gyro (g) and snub (s), which are LOPSPs. If the seed polyhedron's faces have an orientation, such that the seed already doesn't have mirror symmetry, then LOPSP are symmetry-preserving in the same sense as LSPs. r is not an LOPSP, since it reverses the orientation of an oriented polyhedron, but rxr is an LOPSP if x is. If x is an LSP, then r is equivalent to S: $rx = xr = rxr = x$, at least in combinatorial terms.

For LSPs and LOPSPs, the ratio of edges in the result polyhedron to the seed polyhedron is an invariant of the operator, and is always a positive integer. This is termed the *inflation rate* by [1], and denoted g (not to be confused with the operator). It can also be termed the *edge multiplier*.

Both LSPs and LOPSPs can be depicted in an edge-centered diagram like in Figure 6. The idea is quadrilaterals can be identified on a polyhedron that contain each edge and have vertices going from one vertex of that edge, the face center to one side of the edge, the other vertex of the edge, and the face center to the other side of the edge. Then, that quadrilateral can be replaced with these edge-centered diagrams. The original vertices of the seed polyhedron are the point of the quadrilateral on the left and right of the diagram, while the original face centers are those on the top and bottom. Vertices are a filled dot, while face centers that are retained as a face are an open dot. Because LSPs and LOPSPs can be depicted like this, and since [1] didn't introduce a term for it, we call the two sets of operators together *edge-replacement operators*, or *ERO*s.

Edge-centered diagrams show four chambers, two to each side of the seed edge. For LOPSPs, the chambers of the top 2 chambers are a one-half-turn rotation of the bottom two. For LSPs, the chambers are all reflections of each other over the dotted lines. There is some redundancy built into these diagrams. We might think back to our explanation of `wythoff` and how it generally starts with a meta operation m, and note that the m's edge-centered diagram would create a line over each dotted line, and wonder if we can create these operators as a composition of m and some other operator from $\mathcal{W}$ to $\mathcal{P}$ or $\mathcal{W}^o$ to $\mathcal{P}^o$.

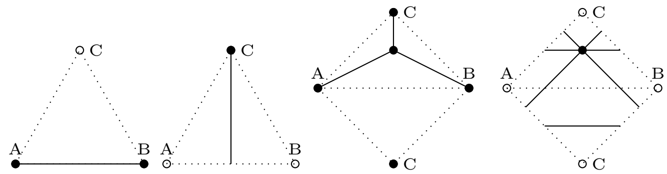

FIGURE 7. Face diagram of some operators.

The last statement is true, in a more complicated form. Figure 7 depicts face diagrams for operators that can be used to produce the operators from Figure 6. The first thing to note is that an edge cannot pass through the corners of the face diagram without a vertex at that corner. If it could, there could be edges connecting 3 or more vertices, which is forbidden. The second is that we can permute the vertices of these face diagrams to produce new operators. Here we introduce an operator, @, to represent that permutation. $@_{ABC}$ maps A, B, and C to V, E and F in the seed polyhedron produced by m. (In general, @ acts on the vertex coloring of polyhedra in $\mathcal{W}$.) The third is that the m operator always produces a vertex of degree 4 at the center of each edge.

Let x be the operator in the first face diagram from Figure 7. $x@m$ produces different operators, depending on the subscript to @. ACB and BCA produce the join operator j. ABC and BAC produce something that is almost, but not quite the identity operator S, while CAB and CBA almost produce the dual operator d. The degree 4 vertex at the edge center has had its order halved, to a degree 2 vertex. Applying the smoothing operator \$ will remove that vertex. So it makes sense, if degree-2 elements are undesirable, to create operators as $\$x@m$.

If y is operator in the second face diagram from Figure 7, then $\$x@m$ is d for ABC and BAC, S for CAB and CBA, and a for ACB and BCA. Note that we have two different expressions for S and d now. For the expression in terms of x, degree-2 vertices were removed, and in terms of y, digons were removed. At most six operators can arise from the different permutations of an operator: the aforementioned have some the same due to symmetry.

For LOPSPs, the situation is a little more complicated by keeping track of orientations. See the last two face diagrams in Figure 7, which correspond to g and s. Starting with A and travelling counterclockwise in each triangle, the first triangle is ABC, while the second is ACB. Thus, the upper triangle is applied to even-permutation faces from the seed in $\mathcal{W}^o$, while the lower triangle is applied to odd-permutation faces. (Also, note that for these operators, all even permutations result in the same operator for these face diagrams, and all odd permutations result in the chiral pair of that operator.) There is some flexibility in where vertices can be placed in the face diagram while representing a (topologically) equivalent operator: they can be moved onto a seed edge, or to the other face. Thus, some of the operations in this text may not have exactly the same appearance as in other publications.

Given what we've discussed so far, it is easily shown that all EROs can be expressed as $y = \$x@m$ for some x, where y operates on $\mathcal{P}_3$. (If the seed polyhedron contains degree-2 elements, those will also be reduced by the \$ term, so to exclude that case we restrict it to polyhedra without such elements. It's possible that there is a definition of \$ or grouping of operators that would allow for extension to all polyhedra.) If the edge multiplier g for y is even, there is an x such that $y = x@m$, with no smoothing step needed. If g is odd, then there are at least two x satisfying $y = \$x@m$, one which will have a digon removed and one which will have a degree-2 vertex removed. (There may be further relations where more than one digon or degree-2 vertex is removed.) Relatedly, if g is even it either has a vertex or a face at the center of its edge-centered diagram: if odd, it has an edge there.

The aforementioned also makes it very easy to describe yd given y. Remember that the dual operator interchanges vertices and faces. Therefore, md produces a polyhedra with V and F in opposite places. In a formula, $\$xmd = \$x@_{CBA}d$. Thus, the edge-centered diagram for yd is that for y, rotated one-quarter-turn.

Borrowing an idea from ring theory, we refer to d (dual), S (seed, identity) as the units of the EROs, and operators that are related by d are called associates. (r is a little odd and will be left out.) At most 4 distinct operators can be associated with each other, corresponding to x, xd, dx, and dxd. Conway's operators are associated as so:

- $j = jd, a = dj = djd$
- $k, t = dkd$ (as well as $n = kd$ and $z = dk$)
- $o = od, e = do = dod$
- $g, s = dgd$ (as well as $rgr = gd$ and $rsr = sd$)
- $m = md, b = dm = dmd$

Appendix B lists all named operators with their associates.

We'll call operations on polyhedra that share their operator on $\mathcal{W}$ or the dual of $\mathcal{W}$ as a *cohort*. If operators are associated, they share a cohort. There are at most 12 distinct operators in a cohort. Appendix C lists all named operators and their cohort.

As a final note, EROs allow us to extend the Goldberg-Coxeter operations $\square_{n,m}$ and $\Delta_{n,m}$ to faces with any number of sides. We simply have to identify the edge-centered diagram for EROs. The vertices of this quadrilateral were given in the earlier section on GC operations. These operators retain the same commutation and factoring properties in this extension.

5. Alternating EROs

Recall that $m = kj$, so $y = \$x@m = \$x@kj$. $m : \mathcal{P} \to \mathcal{W}$, as mentioned earlier, and $j : \mathcal{P} \to \mathcal{E}$, since join produces only quadrilateral faces. As well, when k's seed is in $\mathcal{E}$, its result is in $\mathcal{W}$: this is true in general, not just for the quadrilaterals j produces. Combining this information leads us to operators of the form $\$x@k : \mathcal{E} \to \mathcal{P}$. We call these *Vertex-Alternating EROs*, or *VAEROs*. They have edge-centered diagrams like EROs, but without the vertical center line. The leftmost corner represents one color of vertex, and the rightmost represents another, while the top and bottom still represent face centers. Notice that the orientation of the diagram matters: if you have a diagram from $-$ to $+$, flip it one half turn to get the diagram from $+$ to $-$. The similiar case for dual polyhedra would be called *Face-Alternating EROs*, or *FAEROs*. Both taken together are *Alternating EROs* or *AEROs*.

Coxeter's alternation operation h is an VAERO. Let x be the first face diagram from Figure 7, then $\$x@_{ACB}k = h$. The inner component $x@_{ACB}k$ creates degree-2 vertices where the seed polyhedron had quadrilateral faces. Like with EROs, there is another x that would create digons instead. While the inner component has an inflation rate $g = 1$, this is not necessarily true for h itself, since faces with 4 sides don't exist in any fixed ratio for an arbitrary polyhedron in $\mathcal{E}$.

The join operator j has a similar relation with duals as m. j creates vertices at the vertices and faces of the seed polyhedron, which implies a 2-coloring of the vertices. In jd, the same vertices are created, but their correspondence to the vertices and faces of the seed is reversed. We can expand the @ notation to apply to 2-colored vertices as well: $jd = @j$. Since there are only two possible colorings, we don't need a subscript, and $@^2 = S$. Furthermore, we can use the relation with m from earlier: $md = @_{CBA}m = kjd = k@j$, so we define $k@ = @_{CBA}k$.

This idea leads one to a possible edge-centered diagram method to construct operators like t_n, truncation of vertices of degree n. Create three edge-centered diagrams, one for edges between vertices of degree not equal to n, one for edges between vertices of degree n, and one for edges with one vertex of degree n and one that is not. (Respectively, the first and second would just be the diagrams for S and t, and the last would look like t or a on the left side and S on the right.) Then use these diagrams to replace their respective edges. A similar method could be used to handle partial polyhedra: define an edge-centered diagram for edges where 2 faces meet, and for edges adjacent to only 1 face.

6. Other operations

Technically, LSPs and LOPSPs as defined in [1] are operators on $\mathcal{P}_{3v}$. That excludes any operator that creates digons or degree-2 vertices, as well as any operator that creates the kind of features that $\pounds$ removes, such as the Lozenge operator. It also excludes r, as we mentioned earlier.

Some operators that do not preserve the topology of the seed polyhedron can be described as EROs. `leonardo` in Antiprism creates attractive hollowed-out polyhedra from seed polyhedra. The particular operation of `leonardo` can't be performed in `wythoff`, but a similar one can with the input string `[V, VF] 0_1v1_0v, 1v1f, 1V`. We call this the *Hollow* operation. The resulting polyhedron is acoptic if the seed has positive curvature everywhere. If the seed had Euler characteristic 2 (genus 0), the result has Euler characteristic $4-2f$ (genus $(f-1)$). One could also create operators that add arbitrary numbers of holes per edge. (Operators that add cross-caps, e.g. based on a self-intersecting polyhedron with Euler characteristic 1 such as the tetrahemihexahedron, may be possible. Such operators probably have more theoretical uses than aesthetic or practical ones, and their result would be hard to realize faithfully.)

It can help to understand operators like the Hollow operation by considering two disjoint copies of the seed polyhedra that are then stitched together. We define an operator, *n-copy*, as the operator that creates n disjoint copies of the seed. That said, the result of this operator is not a polyhedron (it's multiple polyhedra), and it is ambiguous how to represent it as a poset (specifically the maximal and minimal elements).

If we want to get even more abstract, we can think of these more general topological operators as having the structure of a $\mathbb{N}$-module over a semiring, where addition is the disjoint union of polyhedra, multiplication is composition of operators, and a coefficient from $\mathbb{N} = 0, 1, \dots$ represents creating disjoint copies of a polyhedron. Then, the module has an obvious homomorphism with the 3x3 matrices over $\mathbb{N}$.

7. Invariants of EROs

We already mentioned the inflation factor for EROs. [1] Expanding on that, many operations act on the f-vector as a linear operator. Where x is the operator, then the linear operator can be described with a matrix:

$$M_x = \begin{bmatrix} a & b & c \\ d & g & h \\ a' & b' & c' \end{bmatrix} \tag{1}$$

If x is an ERO, then $d = h = 0$. If x perserves the Euler characteristic, then the vector $(1, -1, 1)$ is a left eigenvector of M_x with eigenvalue 1. (Explicitly, $a + a' = d + 1$, $c + c' = h + 1$, and $b + b' + 1 = g$).

Some operators can also be expressed as an infinite linear operator L_x on the values v_i, e, and f_i: vertices of degree i, edges, and faces with i sides, respectively. In particular, EROs take this form:

$$\begin{aligned} E &= ge \\ V_i &= av_{i/k} + eb_i + cf_{i/\ell} \\ F_i &= a'v_{i/k} + eb'_i + c'f_{i/\ell} \end{aligned} \tag{2}$$

v_i, e, and f_i are the input to the operator and V_i, E, and F_i are the result. a, a', c, and c' are either 0 or 1 if the Euler characteristic is preserved. g is a positive integer, all b_i and b'_i are nonnegative integers, and k and ℓ are positive integers. $\sum b_i = b$ and $\sum b'_i = b'$. The subscripted values like $v_{i/k}$ should be interpreted as 0 if i/k is not an integer.

L_x and M_x for an ERO can be determined by counting elements off the edge-centered diagram. Step by step:

- Seed vertices are either retained or converted into faces centered on that vertex. (Other options are precluded by symmetry). Let $a = 1$ if the seed vertices are retained, and 0 otherwise. Also, the degree of the vertex or face is either the same as the seed vertex, or a multiple of it; let k be that multiple.
- Seed face centers are either retained (possibly of in a smaller face) or converted into vertices. (Again, other options are precluded by symmetry). Let $c = 0$ if the seed faces are retained, and 1 otherwise. Let ℓ serve a similar role as k above: the degree of the vertex or face corresponding to the seed face center is k times the degree of the seed vertex.
- Except for the faces or vertices corresponding to the seed vertices and face centers, the added elements are in proportion to to the number of edges in the seed. g is the count of added edges (the edge multiplier or inflation rate), b_i is the number of vertices of degree i added, and b'_i is the number of faces of degree i added.

Count elements lying on or crossing the outer edge of the chamber structure as half. It may help to draw an adjacent chamber, particularly when determining the number of sides on a face.

Applying the handshake lemma to the skeleton graph of the polyhedron and its dual gives relations between the values for EROs:

$$\begin{aligned} 2g &= 2ak + 2c\ell + \sum ib_i \\ 2g &= 2a'k + 2c'\ell + \sum ib'_i \end{aligned} \tag{3}$$

For Euler-characteristic preserving operations, these relations can be manipulated into the form

$$2k + 2\ell - 4 = \sum (4-i)(b_i + b'_i), \tag{4}$$

which is interesting because it eliminates g, a and c, and because it suggests that features with degree 5 or more exist in balance with features of degree 3 (triangles and degree-3 vertices), and that in some sense degree 4 features come "for free".

With these relations, and the assumption that there are no degree 2 features and therefore $i \geq 3$, a series of inequalities can be derived for EROs:

$$\begin{aligned} g + 1 \leq 2a + 3b + 2c &\leq 2g \\ 2k + 2\ell &\leq g + 3 \\ 0 \leq 2k + 2\ell - 4 &\leq b_3 + b'_3 \end{aligned} \tag{5}$$

Note that these inequalities are only necessary, not sufficient. For instance, $g = 4, a = 1, c = 0, b_4 = 1, b'_3 = 2, k = 2, \ell = 1$ satisfies the relations, but doesn't appear to correspond to any ERO. Furthermore, the Lozenge operator satisfies the relations.

It can be demonstrated that $M_{xy} = M_x M_y$ and $L_{xy} = L_x L_y$. The expansion factor g for the operator xy is the product of the g invariants for operators x and y. It can also be seen that a, a', c, c' form their own linear system, a submatrix of M_x: let $\Lambda_x = \begin{bmatrix} a & c \\ a' & c' \end{bmatrix}$, then $\Lambda_{xy} = \Lambda_x \Lambda_y$. Λ_x represents the effect of the operator on the seed faces and vertices. By cofactor expansion, $\det(M_x) = g \det(\Lambda_x)$. Λ_x has a determinant of $-1, 0$, or 1. (In fact, Λ_x has two eigenvalues, one of which is always 1, and one of which may be -1, 0, or 1. M_x has three eigenvalues: two it shares with Λ_x, and one is g.) The dual operator has $\det(M_d) = \det(\Lambda_d) = -1$, and it is easy to see that of the four possible Λ_x, the two with determinant 0 are related by the dual operator, and the two with determinant ± 1 are related by the dual operator. With that motivation, we define the *Type* of the operator as the absolute value of the determinant of Λ_x.

For EROs, the parity of the invariants g and b also describe the center of the chamber structure. In particular, an ERO with both g and b odd is not possible.

- g even, b even: A face with even degree lies at the center.
- g even, b odd: A vertex with even degree lies at the center.
- g odd, b even: An edge crosses the center.
- g odd, b odd: Excluded by symmetry.

EROs form a monoid (a group without inverses, or a semigroup with identity). We have derived here a series of monoid homomorphisms, as $x \to L_x \to M_x \to (g, \Lambda_x)$. None of these homomorphisms are injections: there are certain L_x or M_x that correspond to more than one EROs. Examples for M_x are easy to come by: where $n = kd$, $M_k = M_n$. For an example where the operators are not related by duality, $M_l = M_p$. For L_x, $L_{prp} = L_{pp}$ but prp is not the same as pp (one's chiral, one's not). For the Waffle operator, $W \neq Wd$, but $L_W = L_{Wd}$. A general counterexample would be operators with sufficiently large g based on $\square_{n,m}$, with a single square face (not touching the seed vertices or face centers) divided into two triangles: the counts of vertices of each degree, faces of each degree, and edges would be the same no matter which faces was chosen, but the operators would be different. With this construction, it is possible (with a sufficiently large g) to create arbitrarily large sets of operators with the same invariants.

Unfortunately, no invariant for chirality has been discovered so far. The relationship with operators and chirality is complicated: two chiral operators may produce another chiral operator (e.g. p^2, g^2) or an achiral operator $prpr$. Further confusing things are LOPSPs where r and d interact. Some EROs have $xd = x$, while some others have $xd = rxr$. g and B are examples of the latter. Some have $x = dxd$, like p, but none with $rxr = dxd$ have been observed or proven/disproven to exist.

8. Invariants of other operators

Some VAEROs have a 4x3 matrix form from v^+, v^-, e, f to v, e, f, where v^+, v^- are the counts of vertices of one color or the other. The matrix takes this form:

$$M_x = \begin{bmatrix} a^+ & a^- & b & c \\ 0 & 0 & g & 0 \\ a'^+ & a'^- & b' & c' \end{bmatrix} \tag{6}$$

where a^+, a^-, a'^+, and a'^- are either 0 or 1. This matrix may be expressed in condensed form as a standard 3x3 matrix on v, e, f if $a^+ = a^-$ and $a'^+ = a'^-$. VAEROs may also have a linear operator form like so, where k^+ and k^- are positive integers and ℓ takes values in $\mathbb{N}/2 = \{1/2, 1, 3/2, 2, ...\}$:

$$\begin{aligned} E &= ge \\ V_i &= a^+ v^+_{i/k^+} + a^- v^-_{i/k^-} + eb_i + cf_{i/\ell} \\ F_i &= a'^+ v^+_{i/k^+} + a'^- v^-_{i/k^-} + eb'_i + c' f_{i/\ell} \end{aligned} \tag{7}$$

Operators from $\mathcal{W}$ to $\mathcal{P}$ may have a 5x3 matrix form from $v^{(1)}, v^{(2)}, v^{(3)}, e, f$ to v, e, f, where $v^{(1)}, v^{(2)}, v^{(3)}$ are the counts of 3-colored vertices. However, since all the faces of $\mathcal{W}$ are triangles, the edges and faces are in a constant ratio: $2e = 3f$. Therefore M_x is not unique unless other information is brought in, e.g. if the operator can be applied to some larger set of polyhedra that contains $\mathcal{W}$. For the sake of simplicity, we'll constrain the matrix to a form similar to EROs and AEROs unless we have a good reason not to for a particular operator.

$$M_x = \begin{bmatrix} a^{(A)} & a^{(B)} & a^{(C)} & b & c \\ 0 & 0 & 0 & g & 0 \\ a'^{(A)} & a'^{(B)} & a'^{(C)} & b' & c' \end{bmatrix} \tag{8}$$

The zeros in the second row are a consequence of the fact that $x@m$ is an ERO when x is an operator from $\mathcal{W}$ to $\mathcal{P}$. All values can be rational numbers, even negative. Again, if $a^{(A)} = a^{(B)} = a^{(C)}$ and $a'^{(A)} = a'^{(B)} = a'^{(C)}$ the matrix may be presented in a condensed 3x3 form. L_x has a form like that for AEROs, modified similarly.

$M_@$ can be expressed as a 4x4 permutation matrix (for $\mathcal{E}$) or a 5x5 permutation matrix (for $\mathcal{W}$). m and j, which produce such polyhedra, have these expanded forms:

$$\begin{gathered} M_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, j : \mathcal{P} \to \mathcal{E} \\ M_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & 4 & 0 \end{bmatrix}, m : \mathcal{P} \to \mathcal{W} \end{gathered} \tag{9}$$

These operators may also have a defined inflation factor g. An analogous Λ_x can be defined, though it is not uniquely defined for operators from $\mathcal{W}$. (Type, being based on determinants, is not defined unless the matrix can be condensed, and is also not uniquely defined for operators from $\mathcal{W}$.)

9. Reducibility and irreducibility

An operator that cannot be expressed in terms of EROs aside from d, S, and r is *irreducible over the EROs.* For instance, k (Kis) and j (Join) are irreducible in terms of EROs, but m (Meta) is not (it is equal to kj). A polyhedron that cannot be expressed in terms of another polyhedron and one or more EROs other than the units S and d is an irreducible polyhedron. Out of all of the Platonic, Archimedean, and Catalan solids, the only irreducible one is the tetrahedron.

The relations defined earlier can be used to help reduce an operator, with some caveats. The above representations do not give us a completely reliable way to decompose an arbitrary operator into a sequence of operators, although it does suggest a (trial-and-error filled) heuristic to reduce an operator into two operators by starting at the bottom of the homomorphism chain and going up.

- Determine the g of the two operators from the factors of the g of the operator to be factored.
- Determine Λ of the two operators.
- Determine b, b' for the two operators.
- Determine k, ℓ, b_i, b_i'. for the two operators.
- Figure out if the representations you've produced actually correspond to an ERO.

Which set of operators you're trying to reduce an operator over is important. It may be possible to reduce a given ERO into an AERO and j, for instance. Be careful of naively reducing based on M_x, as the matrices for operators outside of the EROs may pop up. Also keep in mind that a LSP may reduce into two LOPSPs which are inapplicable to non-orientable surfaces, although this is more of a technical issue.

Some facts relating to decomposition can be derived from what we have so far. (Unless otherwise specified, the operators in question are over the EROs.)

- If a polyhedron has a prime number of edges, it is irreducible.
- Operators where g is a prime number are irreducible.
- If $x = xd$ or $rxr = xd$, x has type 0.
- If $x = dxd$ or $rxr = dxd$, x has type 1, g is odd, $b = b'$, and b and b' are even.
- If an ERO has type 1, its decomposition cannot contain any EROs of type 0. Correspondingly, if an ERO has type 0, its decomposition must contain at least one type 0 ERO.
- There are no type 1 EROs with $g = 2$, so therefore type 1 EROs with $g = 2p$, where p is prime, are irreducible in terms of EROs. (However, it may be reducible into an AERO or an operator on $\mathcal{W}$.)
- $\square_{n,m}$ that correspond to the Gaussian primes, and $\Delta_{n,m}$ that correspond to the Eisenstein primes, are irreducible. (Proof below.) As a consequence of this, there are an infinite number of irreducible EROs.

Proof of the last statement: A Gaussian integer $a + bi$ is prime if its square norm $a^2 + b^2$ is prime or the square of a prime. In the first case, that prime has the form $p = 4k + 1$; in the latter, $p = 4k + 3$. Remember that the squared norm of the integer is just the inflation factor g for the corresponding operator. If g is prime, the operator is irreducible. If g is the square of a prime, the operator $\square_{n,m}$ is type 1, specifically, $\det(\Lambda_{\square_{n,m}}) = 1$. Suppose the operator can be decomposed into $\square_{n,m} = xy$, where x and y both have inflation factor $g' = \sqrt{g}$. Without loss of generality, assume $\det(\Lambda_x) = \det(\Lambda_y) = 1$. Their matrix forms are:

$$\mathbf{M}_x\mathbf{M}_y = \begin{bmatrix} 1 & b & 0 \\ 0 & g' & 0 \\ 0 & b' & 1 \end{bmatrix} \begin{bmatrix} 1 & B & 0 \\ 0 & g' & 0 \\ 0 & B' & 1 \end{bmatrix} = \begin{bmatrix} 1 & B + bg' & 0 \\ 0 & g & 0 \\ 0 & B' + b'g' & 1 \end{bmatrix} = \mathbf{M}_{\square_{n,m}} = \begin{bmatrix} 1 & (T-1)/2 & 0 \\ 0 & T & 0 \\ 0 & (T-1)/2 & 1 \end{bmatrix} \tag{10}$$

therefore, $B + bg' = B' + b'g'$. It can be demonstrated using the ERO invariant inequalities from earlier that the only solution to this that could correspond to an actual ERO is $b = b'$ and $B = B'$. $g' = p = 4k + 3$, so b, b', B, B' must all be odd. As mentioned earlier, there are no EROs with both b and g odd, so we have a contradiction, and $\square_{n,m}$ is irreducible.

The proof for $\Delta_{n,m}$ is analogous. An Eisenstein integer $a + bu$, $u = \exp(\pi i/3)$, is prime if its square norm $a^2 + ab + b^2$ is prime or the square of a prime. The prior (except for $(1+u)$, which we corresponds to the ERO n which we already know is irreducible) have the form $p = 3k + 1$; the latter, $p = 3k + 2$. When the prime is of the latter form, the ERO is type 1 with $\det(\Lambda_{\Delta_{n,m}}) = 1$ and its matrix form is:

$$\mathbf{M}_{\Delta_{n,m}} = \begin{bmatrix} 1 & (T-1)/3 & 0 \\ 0 & T & 0 \\ 0 & 2(T-1)/3 & 1 \end{bmatrix}. \tag{11}$$

Define x and y as before: then $2(B + bg') = B' + b'g'$. Using the inequalities to exclude other choices, $B' = 2B$ and $b' = 2b$. $g = 3k + 2$, but $g = b + b' + 1 = 3b + 1$: there is no simultaneous integer solution to both equations, so we have a contradiction, and $\Delta_{n,m}$ is irreducible.

10. $\mathcal{P}$, $\mathcal{P}_3$, and $\mathcal{P}_{3v}$

There are some polyhedra or operators that have non-unique reductions. However, all known examples involve degree-2 features or operators that do not produce polyhedra in $\mathcal{P}_{3v}$. For example, $kD_4 = O = aT$, where D_4 is the 4-dihedron, and $jx = lj$, where x is the Lozenge operator on $\mathcal{P}_3$ (not $\mathcal{P}_{3v}$).

In [1], they ask if all LSP operations which increase the number of symmetries of a polyhedron can be written as ambo or some composition of operators involving ambo. ($j = da$, so this question could be asked in terms of join as well.) There are counterexamples, but again they involve degree-2 features. As a general example, $m_n D_k = B_{k(n+1)}$ with $n > 0$ and $k \geq 2$, where D_j is the j-dihedron, B_j is the bipyramid formed by gluing 2 n-pyramids together, and both D_j and B_j have j-dihedral symmetry.

The question now is: do any of the things we've just talked about in $\mathcal{P}$ or $\mathcal{P}_3$ also occur in $\mathcal{P}_{3v}$? If no, what is it about $\mathcal{P}_{3v}$ that disallows them from occuring? We already know that $\mathcal{P}_{3v}$ is special because of Steinitz's theorem, but it's not clear how that would lead to the proofs we would need perform here.

11. Lower polytopes

Examining operations on the lower polytopes may help us form analogies. There is only one abstract polytope of each rank -1, 0, and 1: the null polytope, a single vertex, and a single edge. Therefore, given a rank $k \leq 1$, there is only one operator on that set of abstract polytopes, which is the identity operator.

The polygons are the polytopes of rank 2. A polygon has an equal number of vertices and edges, and a (finite) abstract polygon is uniquely determined by its count of edges (or vertices). (There's also the apeirogon, with an infinite number of vertices and edges, which is also unique.) Therefore, abstract polygons are self-dual. An abstract polygon may have 2 or more vertices. Therefore, the operators on abstract polygons correspond to the operators on the integers 2 or greater. (Monogons violate the diamond property, so they're not abstract polygons. If we were to include those, the operators would correspond to operators on the integers 1 and greater. This analysis is basically the same whether they're included or not.)

In the set of abstract polygons, the analog to EROs is the operators that multiply the number of edges by some integer $k \geq 1$. The symmetry group of an n-edged polygon is, by definition, the dihedral group D_n, and D_n is a subgroup of D_{kn}, so like the EROs these operators also preserve or increase the symmetry of the polygon. Furthermore, the operator can be reduced into a series of operators as per the integer factorization of k, and these operators commute with each other.

One can also make analogy with the alternating EROs by considering polygons with an even number of sides. Essentially, these are the extension of the last paragraph to $k = j + \frac{1}{2}$, $j \geq 0$. (Including monogons may help here: for the operator where $k = 1/2$, a digon would map to a monogon.)

12. Conclusion

With L_x, M_x, Λ_x, and the constituent parts of those, the invariants defined on EROs has been expanded significantly. We have demonstrated their use in the categorization and decomposition of EROs. Some open questions persist:

- Are there irreducible EROs other than j that produce only polyhedra in $\mathcal{E}$?
- Are there EROs other than m_{2k+1} that produce only polyhedra in $\mathcal{W}$?
- Are there any LOPSPs such that $rxr = dxd$? (They would have to be type 1 operators.)
- Are there other conditions that can be added to the invariants for L_x to make the set of conditions sufficient as well as necessary?
- Can a useful invariant related to the chirality of an operator be defined?

- What other invariants need to be added to fully characterize EROs and related operators?
- Is the decomposition of an ERO on $\mathcal{P}_{3v}$ in terms of other EROs unique (up to associates)?

Following this, there are two related ideas to pursue in the future.

First, to explore operations on abstract polyhedra (and polytopes) without reference to the realization. The way they have been described in this text, there is a dependence on the underlying space. Removing that dependence would make the theory clearer. Furthermore, some of these operations may be valid on posets that do not satisfy all the axioms of an abstract polytope. For example, the dual operation is defined for all posets. Finding the most general restriction on the poset for a certain operation may help us understand how to deal with quasipolyhedra.

Second, to explore operations on general polytopes. Here we've explored operations on polyhedra. Coxeter defined his operations on regular polytopes. A general theory of operations on polytopes would be the next logical step. These operations need not necessarily be between polytopes of the same rank: consider the embedding of a higher polytope in 3-space, or producing the skeleton of a polytope.

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Appendix A. `wythoff` strings for new operators in this text

- Opposite-Lace L_{-1}: `[V, E2F] 1F, 1e1_0e, 0_1f1f, 1E`
- Ethel E: `[V, VE, VF] 0_1_2e1e, 2F, 1_2v2f`
- Waffle W: `[V, E, F, V2E, VF] 0_4_3f4f, 2_4_3v3_4v, 3E`
- Bowtie B: `[V, E, F, VE, EF] 1_3_4, 0_3_4_2e4_1_3e`

- Lozenge: `[V, EF] 0_1F, 1_0f1f, 1E`
- Hollow: `[V, VF] 0_1v1_0v, 1v1f, 1V`

Appendix B. Table of operator associates

Items marked with a dagger (†) indicate an operator that may produce non-convex polyhedra from convex polyhedra. Items marked with a double dagger (‡) indicate an operator that does not preserve topology: it may produce polyhedra with holes, or disjoint polyhedra. The origin of the operators is indicated like so; `c`: Conway's original set [2]; `h`: George Hart [11][10]; `a`: Antiprism extensions[14]; `g`: Goldberg-Coxeter, as per the section earlier; `n`: new in this text. $, £, and @ are not included in the list below. r is even though it's not itself an ERO. Where not specified, k and ℓ are 1, and b_i and b'_i are 0.

Table 1: Operators with linear represenations, organized by associates.

M_x	x	xd	dx	dxd	k, ℓ, b_i, b'_i	Notes
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	Reflect: r	$rd = dr$	$rd = dr$	r		chiral, `h`
$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	Seed: $S = \square_{1,0} = \Delta_{1,0}$	Dual: d	d	S		d: `c`, S: `a`
$\begin{bmatrix}1&0&1\\0&2&0\\0&1&0\end{bmatrix}$	Join: $j = \square_{1,1}$	j (@j)	Ambo: a	a (@a)	$b'_4 = 1$	`c`
$\begin{bmatrix}1&0&1\\0&3&0\\0&2&0\end{bmatrix}$	Kis: k	Needle: $n = \Delta_{1,1}$	Zip: z	Truncate: t	$k = 2$, $b'_3 = 2$	k, t: `c`. n, z: `a`.
$\begin{bmatrix}1&1&1\\0&4&0\\0&2&0\end{bmatrix}$	Ortho: $o = jj = \square_{2,0}$	o	Expand: $e = aa$	e		`c`
$\begin{bmatrix}1&2&1\\0&5&0\\0&2&0\end{bmatrix}$	Gyro: g	$gd = rgr$	$sd = rsr$	Snub: s	$b_3 = 2$, $b'_5 = 2$	chiral, `c`
$\begin{bmatrix}1&1&1\\0&6&0\\0&4&0\end{bmatrix}$	Meta: $m = kj$	m (@m)	Bevel: $b = ta$	b (@b)		`c`
$\begin{bmatrix}1&2&0\\0&4&0\\0&1&1\end{bmatrix}$	Chamfer: c	$cd = du$	$dc = ud$	Subdivide: $u = \Delta_{2,0}$	$b_3 = 2$, $b'_6 = 1$	`a`
$\begin{bmatrix}1&2&0\\0&5&0\\0&2&1\end{bmatrix}$	Propeller: $p = \square_{2,1}$	$dp = pd$	$pd = dp$	p	$b_4 = 2$, $b'_4 = 2$	chiral, `h`
$\begin{bmatrix}1&2&0\\0&5&0\\0&2&1\end{bmatrix}$	Loft: l	ld	dl	dld	$k = 2$, $b_3 = 2$, $b'_4 = 2$	`a`
$\begin{bmatrix}1&3&0\\0&6&0\\0&2&1\end{bmatrix}$	Quinto: q	qd	dq	dqd	$b_3 = 2$, $b_4 = 1$, $b'_5 = 2$	`a`
$\begin{bmatrix}1&2&0\\0&6&0\\0&3&1\end{bmatrix}$	Joined-Lace: L_0	L_0d	dL_0	dL_0d	$k = 2$, $b_4 = 2$, $b'_3 = 2$, $b'_4 = 1$	`a`

Table 1: Operators with linear represenations, organized by associates.

M_x	x	xd	dx	dxd	k, ℓ, b_i, b'_i	Notes
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	Lace: L	Ld	dL	dLd	$k=3$, $b_4=2$, $b'_3=4$	`a`
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	Opposite-Lace: L_{-1}	$L_{-1}d$	dL_{-1}	$dL_{-1}d$	$k=2$, $b_5=2$, $b'_3=4$	`n`
$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	Medial: M	Md	dM	dMd		`a`
$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	Stake: K	Kd	dK	dKd	$k=3$, $b_3=2$, $b'_3=2$, $b'_4=2$	`a`
$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 7 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	Whirl: w	wd	dw	$dwd = \Delta_{2,1}$	$b_3=4$, $b'_6=2$	chiral, `a`
$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	Ethel: E	Ed	dE	dEd	$b_3=2$, $b_4=2$, $b'_4=2$, $b'_6=1$	`n`
$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 8 & 0 \\ 0 & 5 & 0 \end{bmatrix}$	Join-kis-kis[2]: J	Jd	dJ	dJd	$k=3$, $\ell=2$, $b_3=2$, $b'_3=4$, $b'_4=1$	`a`
$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	Waffle: W	Wd	dW	dWd	$b_3=2$, $b_4=2$, $b'_4=2$, $b'_5=2$	`n`
$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 10 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	Bowtie: B	Bd	dB	dBd	$b_3=4$, $b_4=1$, $b'_3=2$, $b'_7=2$	chiral, `n`
$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 10 & 0 \\ 0 & 6 & 0 \end{bmatrix}$	Cross: X	Xd	dX	dXd	$k=2$, $b_4=2$, $b_6=1$, $b'_3=4$, $b'_4=2$	`a`
$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+3 & 0 \\ 0 & 2n+2 & 0 \end{bmatrix}$	m_n	m_nd	b_n	b_nd	$k=2$, $\ell=n+1$, $b_4=n$, $b'_3=2n+2$	$n \geq 0$. $m_1 = m$. `a`
$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+1 & 0 \\ 0 & 2n & 0 \end{bmatrix}$	M_n	M_nd	dM_n	dM_nd	$\ell=n$, $b_4=n$, $b'_3=2n-2$, $b'_4=2$	`a`, $n \geq 1$

[2]Antiprism calls this "joined-medial".

Table 1: Operators with linear represenations, organized by associates.

M_x	x	xd	dx	dxd	k, ℓ, b_i, b'_i	Notes
$\begin{bmatrix} 1 & \frac{T}{2}-1 & 1 \\ 0 & T & 0 \\ 0 & \frac{T}{2} & 0 \end{bmatrix}$	$\square_{n,m}$	$\square_{n,m}$	$d\square_{n,m}$	$d\square_{n,m}$	$b_4 = b'_4 = b$	$n \equiv m \mod 2$, $T = n^2 + m^2$. `g`, `a`[3]
$\begin{bmatrix} 1 & \frac{T-1}{2} & 0 \\ 0 & T & 0 \\ 0 & \frac{T-1}{2} & 1 \end{bmatrix}$	$\square_{n,m}$	$d\square_{n,m}$	$d\square_{n,m}$	$\square_{n,m}$	$b_4 = b'_4 = b$	$n \not\equiv m \mod 2$, $T = n^2 + m^2$. `g`
$\begin{bmatrix} 1 & \frac{T}{3}-1 & 1 \\ 0 & T & 0 \\ 0 & \frac{2T}{3} & 0 \end{bmatrix}$	$\Delta_{n,m}$	$\Delta_{n,m}d$	$d\Delta_{n,m}$	$d\Delta_{n,m}d$	$b_6 = b$, $b'_3 = b'$	$n \equiv m \mod 3$, $T = n^2 + nm + m^2$. `g`
$\begin{bmatrix} 1 & \frac{T-1}{3} & 0 \\ 0 & T & 0 \\ 0 & 2\frac{T-1}{3} & 1 \end{bmatrix}$	$\Delta_{n,m}$	$\Delta_{n,m}d$	$d\Delta_{n,m}$	$d\Delta_{n,m}d$	$b_6 = b$, $b'_3 = b'$	$n \not\equiv m \mod 3$, $T = n^2 + nm + m^2$. `g`
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	Lozenge				$k = 2$, $\ell = 2$, $b_3 = 2$, $b'_3 = 2$	`n`†
$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 1 & 3 & 0 \end{bmatrix}$	Hollow				$k = 2$, $b_5 = 2$, $b'_4 = 3$	`n`‡
$\begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{bmatrix}$	n-Copy	d (n-Copy) = (n-Copy) d	d (n-Copy) = (n-Copy) d	n-Copy		`n`, ‡for $n > 1$

Appendix C. Cohorts of LSPs

With the exception of some named operators, if x is shown, xd is not, since its information can be easily determined from the information for x. An arbitrary named member of the cohort is chosen to label the cohort, e.g. the a-cohort.

Appendix D. Cohorts of LOPSPs

In the following tables, permutations in parentheses, such as (CBA), indicate a permutation that produces the chiral pair of the depicted operator. For some operators, the edge-centered diagram has been redrawn for clarity in a topologically-equivalent manner. (Typically this means vertices were moved to an edge of the diagram.)

[3] Antiprism implements $\square$, but only where $b = 0$: it calls it o_n and numbers it differently.

Table 2. a-cohort of LSPs

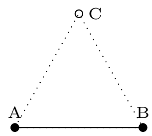
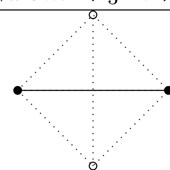
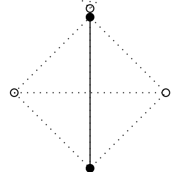
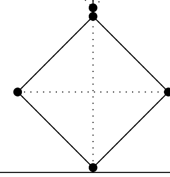
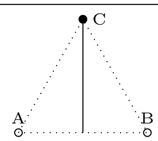
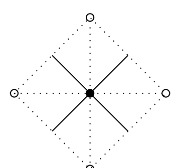
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
 $\begin{bmatrix} 1 & 1 & 0 & -2/3 & 1 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	ABC, BAC		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
	CAB, CBA		$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
	ACB, BCA		$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$x@m = j$
 $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 \\ 1 & 1 & 0 & -2/3 & 1 \end{bmatrix}$	ABC, BAC	d	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
	CAB, CBA	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
	ACB, BCA		$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$x@m = a$

TABLE 3. o-cohort of LSPs

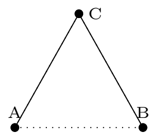
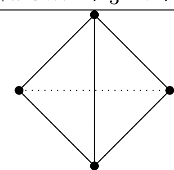
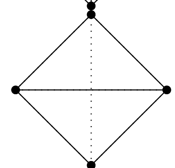
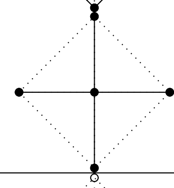
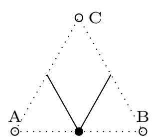
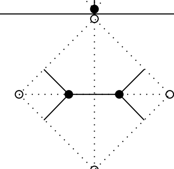
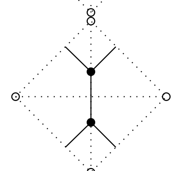
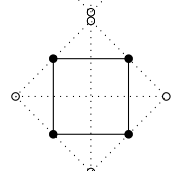
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC, BAC		$\begin{bmatrix}1 & 0 & 1\\0 & 3 & 0\\0 & 2 & 0\end{bmatrix}$	$\$x@m = n$
$\begin{bmatrix}1 & -2/3 & 1\\0 & 2/3 & 0\\0 & 1/3 & 0\end{bmatrix}$	ACB, BCA		$\begin{bmatrix}1 & 0 & 1\\0 & 3 & 0\\0 & 2 & 0\end{bmatrix}$	$\$x@m = k$
	CAB, CBA		$\begin{bmatrix}1 & 1 & 1\\0 & 4 & 0\\0 & 2 & 0\end{bmatrix}$	$x@m = o$
	ACB, BCA		$\begin{bmatrix}0 & 2 & 0\\0 & 3 & 0\\1 & 0 & 1\end{bmatrix}$	$\$x@m = t$
$\begin{bmatrix}0 & 1/3 & 0\\0 & 2/3 & 0\\1 & -2/3 & 1\end{bmatrix}$	ABC, BAC		$\begin{bmatrix}0 & 2 & 0\\0 & 3 & 0\\1 & 0 & 1\end{bmatrix}$	$\$x@m = z$
	CAB, CBA		$\begin{bmatrix}0 & 2 & 0\\0 & 4 & 0\\1 & 1 & 1\end{bmatrix}$	$x@m = e$

TABLE 4. u-cohort of LSPs

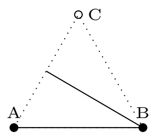
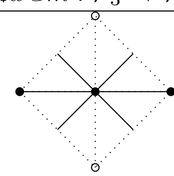
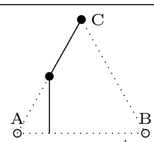
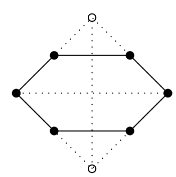
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
 $\begin{bmatrix} 1 & 1 & 0 & -2/3 & 1 \\ 0 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$	ABC		$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$x@m = u$
	ACB	n	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = n$
	BAC	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
 $\begin{bmatrix} 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 2/3 & 0 \\ 1 & 1 & 0 & -2/3 & 1 \end{bmatrix}$	ACB	t	$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\$x@m = t$
	CAB	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
	CBA		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$x@m = c$

Table 5. l-cohort of LSPs

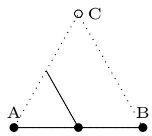
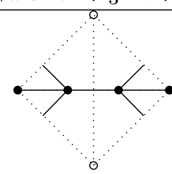
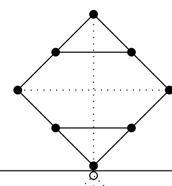
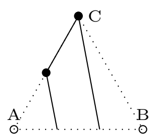
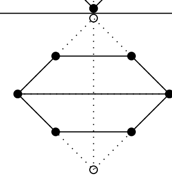
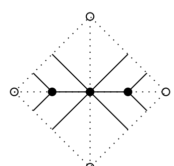
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
 $\begin{bmatrix} 1 & 1 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}$	ABC		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$\$x@m = dld$
	BAC	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
	BCA		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 0 \\ 0 & 3 & 0 \end{bmatrix}$	$x@m$
 $\begin{bmatrix} 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1/3 & 1 \end{bmatrix}$	CBA		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$\$x@m = l$
	CAB	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
	ACB		$\begin{bmatrix} 0 & 3 & 0 \\ 0 & 6 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$x@m$

TABLE 6. L_0-cohort of LSPs

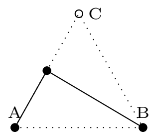
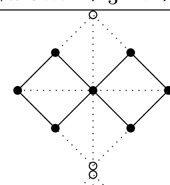
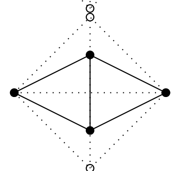
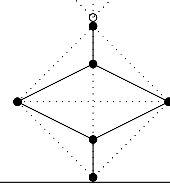
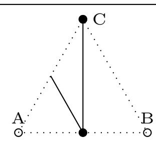
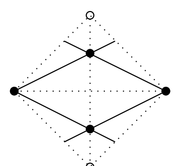
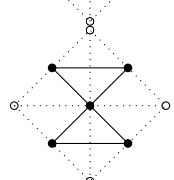
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix}1&3&0\\0&6&0\\0&2&1\end{bmatrix}$	$x@m = dL_0d$
$\begin{bmatrix}1&1&0&-1/3&1\\0&0&0&1&0\\0&0&1&1/3&0\end{bmatrix}$	BAC		$\begin{bmatrix}1&2&0\\0&5&0\\0&2&1\end{bmatrix}$	$\$x@m$ = Lozenge †
	BCA		$\begin{bmatrix}1&2&1\\0&6&0\\0&3&0\end{bmatrix}$	$x@m = jk$
	CAB	Lozenge	$\begin{bmatrix}1&2&0\\0&5&0\\0&2&1\end{bmatrix}$	$\$x@m$ = Lozenge †
$\begin{bmatrix}0&0&1&1/3&0\\0&0&0&1&0\\1&1&0&-1/3&1\end{bmatrix}$	CBA		$\begin{bmatrix}1&2&0\\0&6&0\\0&3&1\end{bmatrix}$	$x@m = L_0$
	BCA		$\begin{bmatrix}0&3&0\\0&6&0\\1&2&1\end{bmatrix}$	$x@m = djk$

TABLE 7. q-cohort of LSPs

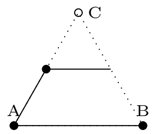
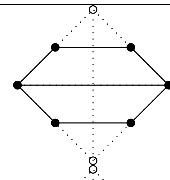
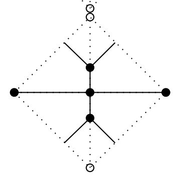
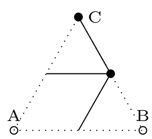
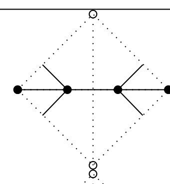
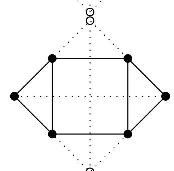
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix}1 & 2 & 0\\ 0 & 5 & 0\\ 0 & 2 & 1\end{bmatrix}$	$\$x@m = l$
$\begin{bmatrix}1 & 1 & 0 & -1/3 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 1/3 & 0\end{bmatrix}$	BAC		$\begin{bmatrix}1 & 3 & 0\\ 0 & 6 & 0\\ 0 & 2 & 1\end{bmatrix}$	$x@m = q$
	ACB	k	$\begin{bmatrix}1 & 0 & 1\\ 0 & 3 & 0\\ 0 & 2 & 0\end{bmatrix}$	$\$x@m = k$
	CBA		$\begin{bmatrix}1 & 2 & 0\\ 0 & 5 & 0\\ 0 & 2 & 1\end{bmatrix}$	$\$x@m = dld$
$\begin{bmatrix}0 & 0 & 1 & 1/3 & 0\\ 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & -1/3 & 1\end{bmatrix}$	CAB		$\begin{bmatrix}1 & 2 & 0\\ 0 & 6 & 0\\ 0 & 3 & 1\end{bmatrix}$	$x@m = dqd$
	BCA	t	$\begin{bmatrix}0 & 2 & 0\\ 0 & 3 & 0\\ 1 & 0 & 1\end{bmatrix}$	$\$x@m = t$

TABLE 8. m-cohort of LSPs

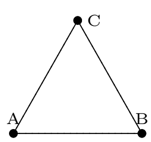
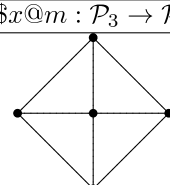
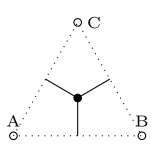
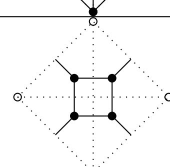
$x : \mathcal{W} \to \mathcal{P}$	M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$	All		$\begin{bmatrix}1 & 1 & 1\\ 0 & 6 & 0\\ 0 & 4 & 0\end{bmatrix}$	$x = S$, $xm = m = kj$
	$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$	All		$\begin{bmatrix}0 & 4 & 0\\ 0 & 6 & 0\\ 1 & 1 & 1\end{bmatrix}$	$x = d$, $xm = b = ta$

Table 9. K-cohort of LSPs

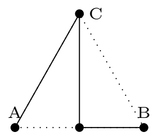
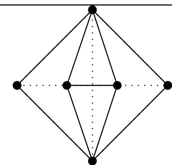
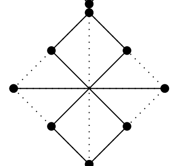
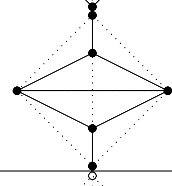
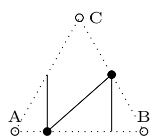
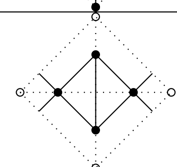
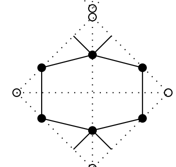
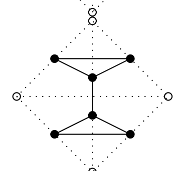
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix}1&2&1\\0&7&0\\0&4&0\end{bmatrix}$	$\$x@m$ †
$\begin{bmatrix}1&-1/3&1\\0&4/3&0\\0&2/3&0\end{bmatrix}$	ACB		$\begin{bmatrix}1&3&1\\0&8&0\\0&4&0\end{bmatrix}$	$x@m$
	CAB		$\begin{bmatrix}1&2&1\\0&7&0\\0&4&0\end{bmatrix}$	$\$x@m = K$
	ABC		$\begin{bmatrix}1&2&1\\0&7&0\\0&4&0\end{bmatrix}$	$\$x@m$ †
$\begin{bmatrix}0&2/3&0\\0&4/3&0\\1&-1/3&1\end{bmatrix}$	ACB		$\begin{bmatrix}0&4&0\\0&8&0\\1&3&1\end{bmatrix}$	$x@m$
	CAB		$\begin{bmatrix}0&4&0\\0&7&0\\1&2&1\end{bmatrix}$	$\$x@m = dK$

TABLE 10. M-cohort of LSPs

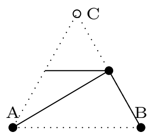
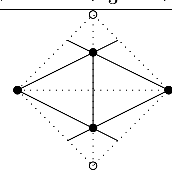
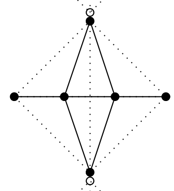
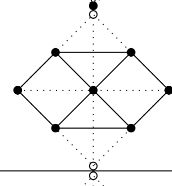
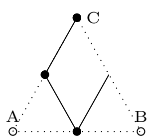
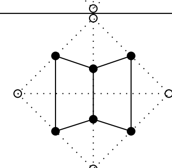
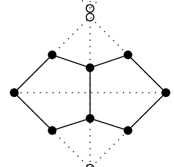
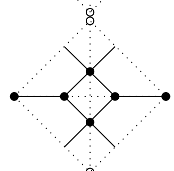
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	$\$x@m = L_{-1}$
$\begin{bmatrix} 1 & 1 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 4/3 & 0 \\ 0 & 0 & 1 & 2/3 & 0 \end{bmatrix}$	BCA		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m = M$
	BAC		$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	$x@m$
	BCA		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 7 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$\$x@m = dM$
$\begin{bmatrix} 0 & 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 4/3 & 0 \\ 1 & 1 & 0 & -1/3 & 1 \end{bmatrix}$	CBA		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 7 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$\$x@m = dL_{-1}d$
	CAB		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$x@m =$

TABLE 11. E-cohort of LSPs

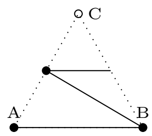
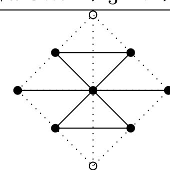
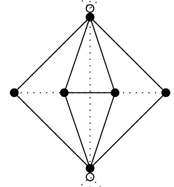
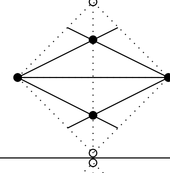
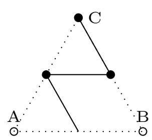
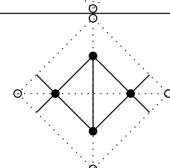
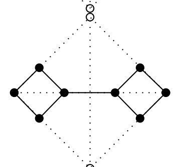
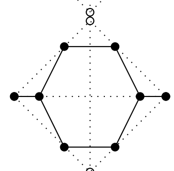
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	$x@m = dEd$
$\begin{bmatrix} 1 & 1 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 4/3 & 0 \\ 0 & 0 & 1 & 2/3 & 0 \end{bmatrix}$	ACB		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m$ †
	BAC		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$\$x@m = L$
	ACB		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m$ †
$\begin{bmatrix} 0 & 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 4/3 & 0 \\ 1 & 1 & 0 & -1/3 & 1 \end{bmatrix}$	CAB		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 7 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$\$x@m = dLd$
	CBA		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$x@m = E$

Table 12. J-cohort of LSPs

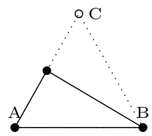
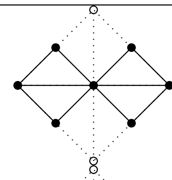
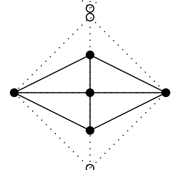
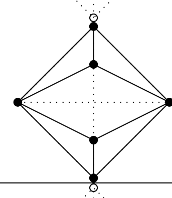
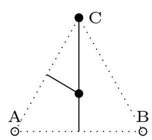
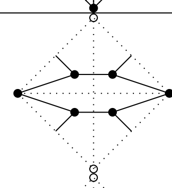
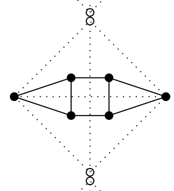
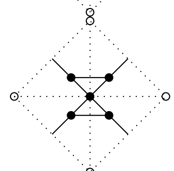
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	$x@m$
$\begin{bmatrix} 1 & 1 & 0 & -1/3 & 1 \\ 0 & 0 & 0 & 4/3 & 0 \\ 0 & 0 & 1 & 2/3 & 0 \end{bmatrix}$	BAC		$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 4 & 1 \end{bmatrix}$	$x@m$ †
	BCA		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 8 & 0 \\ 0 & 5 & 0 \end{bmatrix}$	$x@m = J$
	CBA		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$x@m$
$\begin{bmatrix} 0 & 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 4/3 & 0 \\ 1 & 1 & 0 & -1/3 & 1 \end{bmatrix}$	CAB		$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 3 & 1 \end{bmatrix}$	$x@m$ †
	BCA		$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 8 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$x@m = dJ$

Table 13. W-cohort of LSPs

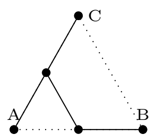
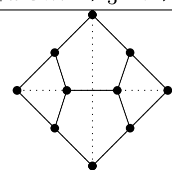
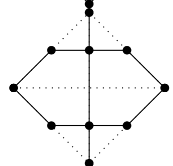
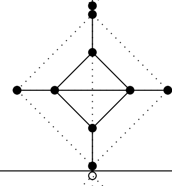
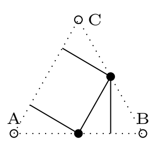
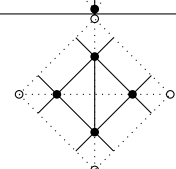
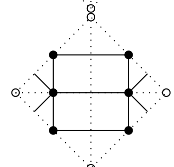
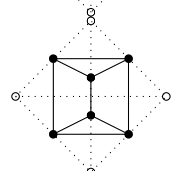
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
 $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 5/3 & 0 \\ 0 & 2/3 & 0 \end{bmatrix}$	ABC		$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m = W$
	BCA		$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m$
	CAB		$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m$
 $\begin{bmatrix} 0 & 2/3 & 0 \\ 0 & 5/3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	ABC		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 9 & 0 \\ 1 & 4 & 1 \end{bmatrix}$	$\$x@m = dW$
	BCA		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 9 & 0 \\ 1 & 4 & 1 \end{bmatrix}$	$\$x@m$
	CAB		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 9 & 0 \\ 1 & 4 & 1 \end{bmatrix}$	$\$x@m$

TABLE 14. M_n-cohort of LSPs, $n = 2k, k \geq 1$

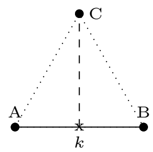
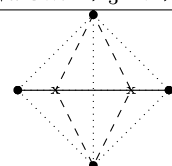
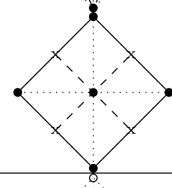
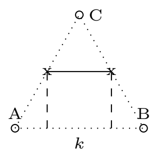
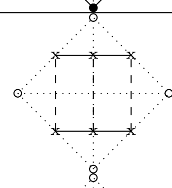
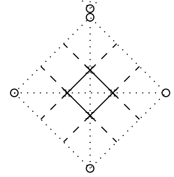
$x : \mathcal{W} \to \mathcal{P},\ M_x$	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC BAC		$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+1 & 0 \\ 0 & 2n & 0 \end{bmatrix}$	$\$x@m = M_n$
$\begin{bmatrix} 1 & \frac{k-2}{3} & 1 \\ 0 & k+\frac{1}{3} & 0 \\ 0 & \frac{2}{3}k & 0 \end{bmatrix}$	BCA ACB		$\begin{bmatrix} 1 & n+1 & 1 \\ 0 & 3n+2 & 0 \\ 0 & 2n & 0 \end{bmatrix}$	$x@m$
	ABC BAC		$\begin{bmatrix} 0 & 2n & 0 \\ 0 & 3n+1 & 0 \\ 1 & n & 1 \end{bmatrix}$	$\$x@m$
$\begin{bmatrix} 0 & \frac{2}{3}k & 0 \\ 0 & k+\frac{1}{3} & 0 \\ 1 & \frac{k-2}{3} & 1 \end{bmatrix}$	BCA ACB		$\begin{bmatrix} 0 & 2n & 0 \\ 0 & 3n+2 & 0 \\ 1 & n+1 & 1 \end{bmatrix}$	$x@m$

Table 15. M_n/m_n-cohort of LSPs

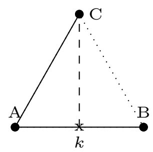
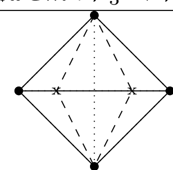
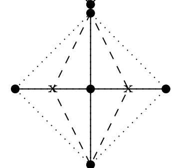
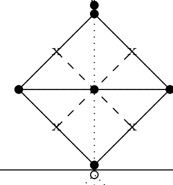
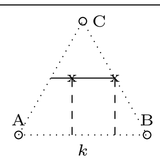
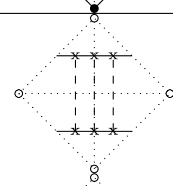
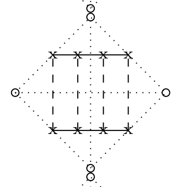
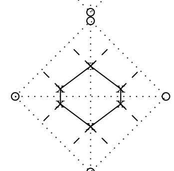
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC		$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+3 & 0 \\ 0 & 2n+2 & 0 \end{bmatrix}$	$n = 2k$, $k \geq 0$, $\$x@m = m_n$
$\begin{bmatrix} 1 & \frac{2k-5}{6} & 1 \\ 0 & k+\frac{1}{6} & 0 \\ 0 & \frac{2}{3}k & 0 \end{bmatrix}$	BAC		$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+1 & 0 \\ 0 & 2n & 0 \end{bmatrix}$	$n = 2k+1$, $k \geq 0$, $x@m = M_n$
	ACB		$\begin{bmatrix} 1 & n & 1 \\ 0 & 3n+1 & 0 \\ 0 & 2n & 0 \end{bmatrix}$	$n = 2k$, $k \geq 1$, $x@m$
	ABC		$\begin{bmatrix} 0 & 2n+2 & 0 \\ 0 & 3n+3 & 0 \\ 1 & n & 1 \end{bmatrix}$	$n = 2k$, $k \geq 0$, $\$x@m = b_n$
$\begin{bmatrix} 0 & \frac{2}{3}k & 0 \\ 0 & k+\frac{1}{6} & 0 \\ 1 & \frac{2k-5}{6} & 1 \end{bmatrix}$	BAC		$\begin{bmatrix} 0 & 2n & 0 \\ 0 & 3n+1 & 0 \\ 1 & n & 1 \end{bmatrix}$	$n = 2k+1$, $k \geq 0$, $x@m$
	ACB		$\begin{bmatrix} 0 & 2n & 0 \\ 0 & 3n+1 & 0 \\ 1 & n & 1 \end{bmatrix}$	$n = 2k$, $k \geq 1$, $x@m$

TABLE 16. m_n-cohort of LSPs, $n = 2k+1, k \geq 0$

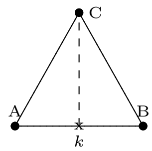
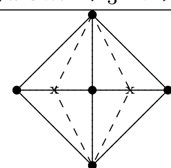
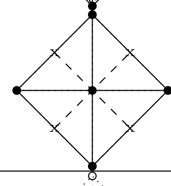
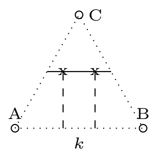
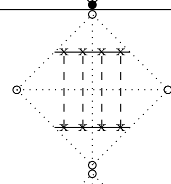
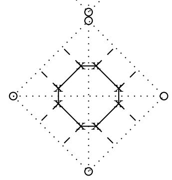
$x : \mathcal{W} \to \mathcal{P}$, M_x	@	$\$x@m : \mathcal{P}_3 \to \mathcal{P}_3$	$M_{\$x@m}$	Note
	ABC BAC		$\begin{bmatrix}1 & n & 1\\0 & 3n+3 & 0\\0 & 2n+2 & 0\end{bmatrix}$	$x@m = m_n$
$\begin{bmatrix}1 & \frac{2k-5}{6} & 1\\0 & k+\frac{1}{2} & 0\\0 & \frac{2k+1}{3} & 0\end{bmatrix}$	BCA ACB		$\begin{bmatrix}1 & n & 1\\0 & 3n+3 & 0\\0 & 2n+2 & 0\end{bmatrix}$	$x@m$
	ABC BAC		$\begin{bmatrix}0 & 2n+2 & 0\\0 & 3n+3 & 0\\1 & n & 1\end{bmatrix}$	$x@m = b_n$
$\begin{bmatrix}0 & \frac{2k+1}{3} & 0\\0 & k+\frac{1}{2} & 0\\1 & \frac{2k-5}{6} & 1\end{bmatrix}$	BCA ACB		$\begin{bmatrix}0 & 2n+2 & 0\\0 & 3n+3 & 0\\1 & n & 1\end{bmatrix}$	$x@m$

TABLE 17. p-cohort of LOPSPs

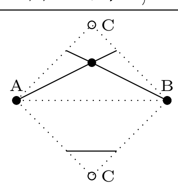
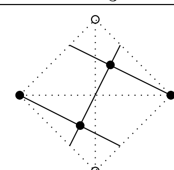
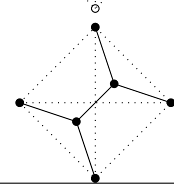
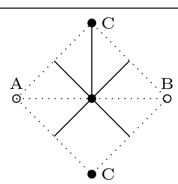
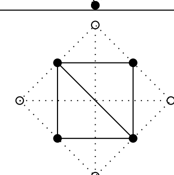
$x : \mathcal{W}^o \to \mathcal{P}^o$, M_x	@	$\$x@m : \mathcal{P}^o_3 \to \mathcal{P}^o_3$	$M_{\$x@m}$	Note
	BAC (ABC)		$\begin{bmatrix}1 & 2 & 0\\0 & 5 & 0\\0 & 2 & 1\end{bmatrix}$	$\$x@m = p$
$\begin{bmatrix}1 & 1 & 0 & -1/3 & 1\\0 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 1/3 & 0\end{bmatrix}$	ACB (BCA)		$\begin{bmatrix}1 & 2 & 1\\0 & 5 & 0\\0 & 2 & 0\end{bmatrix}$	$\$x@m = g$
	ACB (BCA)		$\begin{bmatrix}0 & 2 & 0\\0 & 5 & 0\\1 & 2 & 1\end{bmatrix}$	$\$x@m = rsr$
$\begin{bmatrix}0 & 0 & 1 & 1/3 & 0\\0 & 0 & 0 & 1 & 0\\1 & 1 & 0 & -1/3 & 1\end{bmatrix}$	BAC (ABC)	p	$\begin{bmatrix}1 & 2 & 0\\0 & 5 & 0\\0 & 2 & 1\end{bmatrix}$	$\$x@m = p$

Table 18. g-cohort of LOPSPs

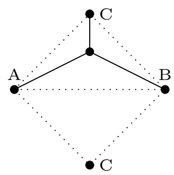
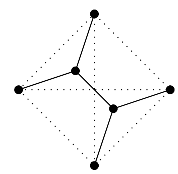
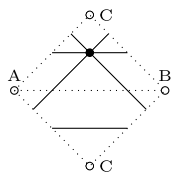
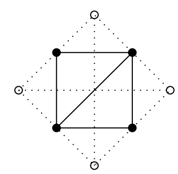
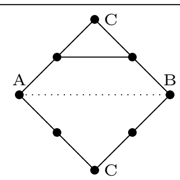
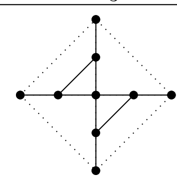
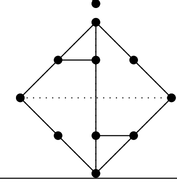
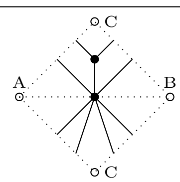
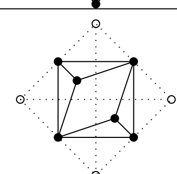
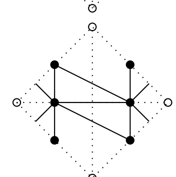
$x : \mathcal{W}^o \to \mathcal{P}^o$	M_x	@	$\$x@m : \mathcal{P}^o_3 \to \mathcal{P}^o_3$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & -1/3 & 1 \\ 0 & 1 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$	ABC BCA CAB (CBA) (BAC) (ACB)		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = rgr$
	$\begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1 & 0 \\ 1 & -1/3 & 1 \end{bmatrix}$	ABC BCA CAB (CBA) (BAC) (ACB)		$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$\$x@m = s$

Table 19. B-cohort of LOPSPs

$x : \mathcal{W}^o \to \mathcal{P}^o$, M_x	@	$\$x@m : \mathcal{P}^o_3 \to \mathcal{P}^o_3$	$M_{\$x@m}$	Note
	BCA (ACB)		$\begin{bmatrix} 1 & 5 & 1 \\ 0 & 10 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$x@m = B$
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 5/3 & 0 \\ 0 & 2/3 & 0 \end{bmatrix}$	ABC (BAC)		$\begin{bmatrix} 1 & 4 & 1 \\ 0 & 9 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$\$x@m$
	BCA (ACB)		$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 10 & 0 \\ 1 & 4 & 1 \end{bmatrix}$	$x@m = dB$
$\begin{bmatrix} 0 & 2/3 & 0 \\ 0 & 5/3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	ABC (BAC)		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 9 & 0 \\ 1 & 4 & 1 \end{bmatrix}$	$\$x@m$