MAPS BETWEEN EUCLIDEAN AND SPHERICAL POLYGONS

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Yadda yadda[2]

In the geodesic dome world, geometry is usually synthetic rather than analytic. That is, geometric constructions are usually expressed in terms of step-by-step geometric constructions rather than in equations and numeric values. See, for instance, [3]. Implementing a synthetic geometric construction on a computer can be a pain; analytic specifications are more amenable to programming idioms.

1. Definitions

1.1. **Spherical geometry.** There are a number of ways to numerically specify points on a sphere. By far the most common is by latitude and longitude, which appear on every modern map of the Earth and probably some other planets. Latitude and longitude are familiar and convenient, but performing extended geometry calculations using latitude and longitude is a complicated task.

Doing spherical geometry using a 3-component unit vector is more convenient in a number of ways: the equations are often simpler, and there are no singularities at the poles. A unit sphere can be defined as the set of all unit vectors in 3-space; i.e., vectors $\mathbf{v} = [v_x, v_y, v_z]$ such that the vector norm $\|\mathbf{v}\| = 1$. Unit vectors are often denoted using a hat: $\hat{\mathbf{v}}$. We'll often find ourselves normalizing vectors, so we may supress the denominator with an ellipsis like so:

$$\hat{\mathbf{v}} = \frac{\mathbf{some} + \mathbf{really} + \mathbf{long} + \mathbf{statement}}{\|\ldots\|} \tag{1}$$

If a vector is an intermediate step to a normalized vector, we may call it pre-normalized and denote it $\mathbf{v}^*$, such that $\hat{\mathbf{v}} = \frac{\mathbf{v}^*}{\|\mathbf{v}^*\|}$

The shortest distance between two points in Euclidean space is a straight line. On the sphere, the shortest distance is an arc of the great circle between those points. The great circle is the intersection of the sphere and a plane passing through the origin. A plane through the origin can be specified as $\hat{\mathbf{n}} \cdot \mathbf{v} = 0$, where $\hat{\mathbf{n}}$ is a unit vector normal to the plane; this vector $\hat{\mathbf{n}}$ can be used to specify a great circle. Given two points $\hat{\mathbf{v}}_\mathbf{1}, \hat{\mathbf{v}}_\mathbf{2}$ on the sphere, the $\hat{\mathbf{n}}$ of the great circle between those two points is (up to normalization) their cross product:

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2}{\|\ldots\|} \tag{2}$$

Small circles are the intersection of the sphere with a plane not through the origin. All planes may be specified in Hessian normal form as $\hat{\mathbf{n}} \cdot \mathbf{v} = r$, where r is the minimum distance between the plane and the origin. The intersection of the plane with the sphere is determined by r as so:

- $r = 0$: Great circle
- $r \in (0, 1)$: Small circle
- $r = 1$: Point
- $r > 1$: None

The following subsections give some formulas for measurements and constructions in Euclidean space and on the sphere. Since a sphere is locally Euclidean, the spherical formulas approach the Euclidean formulas when the measures are small. Note that some spherical formulas require normalized vectors, denoted by the hat $\hat{\mathbf{v}}$.

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1.1.1. *Distance.* In Euclidean space, the distance between two vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by the usual metric. On the sphere, distance is the central angle θ, which is given by many equivalent forms: the most numerically stable one is the one using arctan given below.

$$\begin{aligned} \eta &= \|\mathbf{v}_1 - \mathbf{v}_2\| \\ \theta &= \arctan\left(\frac{\|\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2\|}{\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2}\right) \end{aligned} \tag{3}$$

1.1.2. *Angle.* The angle on a surface at the vertex v_1 between v_2 and v_3 is a standard expression for Euclidean space. For spheres, it is the dihedral angle between the planes defined from v_1 to v_2 and from v_1 to v_3.

$$\cos\phi_1 = \hat{\mathbf{c}}_{12} \cdot \hat{\mathbf{c}}_{13} \tag{4}$$

where $\hat{\mathbf{c}}_{ij} = \frac{\mathbf{v}_i - \mathbf{v}_j}{\|\ldots\|}$ for Euclidean space and $\hat{\mathbf{c}}_{ij} = \frac{\hat{\mathbf{v}}_i \times \hat{\mathbf{v}}_j}{\|\ldots\|}$ for spheres. (This formula also has equivalent forms using sin and tan.)

1.1.3. *Area.* The area of an arbitrary polygon with vertices v_i is given by the shoelace formula. n is the number of vertices in the polygon and $i = 0 \ldots n-1$ is an index for each vertex. i should be treated as if it's mod n, so that it loops around. This formula will give a result for skew polygons, but the areas of skew polygons are not well-defined.

$$A = \frac{1}{2}\|\sum \mathbf{v}_i \times \mathbf{v}_{i+1}\| \tag{5}$$

The area of a spherical triangle is the solid angle Ω, and given by a more elaborate formula. [4][1]

$$\tan(\Omega/2) = \frac{|\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 \times \hat{\mathbf{v}}_3|}{1 + \hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 + \hat{\mathbf{v}}_2 \cdot \hat{\mathbf{v}}_3 + \hat{\mathbf{v}}_3 \cdot \hat{\mathbf{v}}_1} \tag{6}$$

1.1.4. *Means.* When $n = 2$ this formula gives the midpoint between $\mathbf{v}_1$ and $\mathbf{v}_2$. When $n = 3$ it gives the centroid of the triangle with vertices $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

$$\begin{aligned} \mathbf{v}_\mu &= \frac{\sum \mathbf{v}_i}{n} \\ \hat{\mathbf{v}}_\mu &= \frac{\sum \hat{\mathbf{v}}_i}{\|\ldots\|} \end{aligned} \tag{7}$$

1.1.5. *Interpolation.* Interpolation in Euclidean space is standard linear interpolation. On the sphere, interpolation is given by spherical linear interpolation, or slerp.

$$\begin{aligned} \mathrm{Lerp}(\mathbf{v_1}, \mathbf{v_2}; t) &= (1-t)\mathbf{v_1} + t\mathbf{v_2} \\ \mathrm{Slerp}(\hat{\mathbf{v}}_\mathbf{1}, \hat{\mathbf{v}}_\mathbf{2}; t) &= \frac{\sin((1-t)w)}{\sin(w)}\hat{\mathbf{v}}_\mathbf{1} + \frac{\sin(tw)}{\sin(w)}\hat{\mathbf{v}}_\mathbf{2}, w = \arccos \hat{\mathbf{v}}_\mathbf{1} \cdot \hat{\mathbf{v}}_\mathbf{2} \end{aligned} \tag{8}$$

1.1.6. *Face normal.* For the purposes of this text, we define the normal to a (Euclidean) polygon as so, where n is the number of vertices in the polygon and $i = 0 \ldots n-1$ is an index for each vertex:

$$\hat{\mathbf{n}} = \frac{\sum_i^{n-1} \mathbf{v}_i \times \mathbf{v}_{i+1}}{\|\ldots\|} \tag{9}$$

i should be treated as if it's mod n, so that it loops around. (Note the similarity to the shoelace formula.) This definition allows for a somewhat sensible extension to skew polygons: the normal points in a generally reasonable direction for skew polygons. The normal will be outward-facing if the points are ordered counterclockwise, and inward-facing if the points are ordered clockwise.

1.1.7. *Skewness.* There's no standard measure of polygon skewness, so this text uses an ad-hoc measure that seems to work well. This program measures the skewness of a polygon with 4 or more vertices by this method: Let $\mathbf{x}_i = \mathbf{v}_i - \bar{\mathbf{v}}$, where $\bar{\mathbf{v}}$ is the (Euclidean) average of the points. Calculate the SVD decomposition of the matrix that has $\mathbf{x}_i$ as rows (or columns). We only need the singular values: since we're in 3d space, there will be 3 singular values. The *skewness* is the smallest singular value divided by the sum of the other two singular values. If the polygon is flat, the skewness is 0.

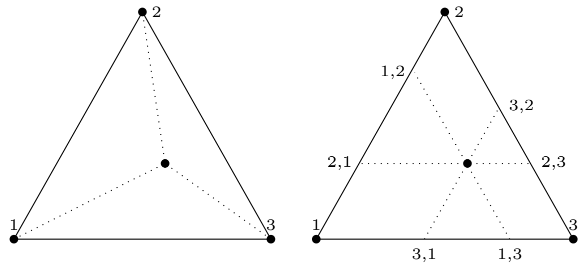

FIGURE 1. Barycentric coordinates. Left: Area opposite of each vertex. Right: Intersection of lines parallel with triangle edges.

1.2. **Triangle and quadrilateral coordinates.** This text will use barycentric coordinates to express Euclidean triangles. Barycentric coordinates are real numbers $\beta_1, \beta_2, \beta_3$ such that $\sum_{i=1}^{3} \beta_i = 1$. Given a triangle with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the corresponding vertex is given by $\mathbf{v} = \sum_{i=1}^{3} \beta_i \mathbf{v}_i$. Given $\mathbf{v}$ and $\mathbf{v}_i$, β_i can be found by e.g. solving the linear system of $\beta_1 + \beta_2 + \beta_3 = 1$ and $\mathbf{v} = \sum_{i=1}^{3} \beta_i \mathbf{v}_i$. β_i are all positive on the interior of the triangle. If a point lies on an edge opposite vertex i, then β_i is zero. (If it lies beyond the edge, then $\beta_i < 0$.)

There are two geometric interpretations of barycentric coordinates that will be useful, as depicted in Figure 1. One is that β_i is the area of the smaller triangle opposite v_i divided by the area of the large triangle. The other is that if a line is placed passing through v parallel to the edge opposite vertex i, it will be at β_i of the distance between the edge and its opposite vertex, with $\beta_i = 0$ being on the edge itself. Let $v_{i,j}$ be the point where the line for i meets the line between vertices i and j: then the vertex lies $\frac{\beta_j}{1-\beta i}$ of the distance from $v_{i,j}$ to $v_{i,j+1}$.

Generalized barycentric coordinates are defined similarly, but the requirement that $\sum_{i=1}^{3} \beta_i = 1$ is dropped. For instance, generalized barycentric coordinates on the unit sphere replace it with a requirement that $\| \sum_{i=1}^{3} \beta_i \mathbf{v}_i \| = 1$. $\sum_{i=1}^{3} \beta_i$ would be > 1 on the interior of the triangle, $= 1$ on the edges, and < 1 on the exterior.

Quadrilaterals are instead specified by what we'll call *uv coordinates* where u and v are $\in [0, 1]$. Given a quadrilateral with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, the transformation is:

$$\mathbf{v} = \mathbf{v}_1 + (\mathbf{v}_2 - \mathbf{v}_1)u + (\mathbf{v}_4 - \mathbf{v}_1)v + (\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4)uv \tag{10}$$

Unlike triangles, quadrilaterals may have points that do not share a common plane: they may be skew quadrilaterals. If the quadrilateral is a skew quadrilateral, u and v smoothly parameterize a surface over that skew quadrilateral.

2. Maps between Euclidean and spherical polygons

2.1. **Conformal.** Schwarz triangle maps

2.2. **Gnomonic.** The gnomonic projection was known to the ancient Greeks, and is the simplest of the transformations listed here. It has the nice property that all lines in Euclidean space are transformed into great circles on the sphere: that is, geodesics stay geodesics, and polygons stay polygons. This is in fact the motivation for the name "geodesic dome": Fuller used this projection to project triangles on the sphere. This is referred to as Method 1 in geodesic dome terminology. The main downside is that the transformation causes shapes near the corners to appear bunched up; this is particularly bad for larger faces e.g. on the tetrahedron.

In general, the gnomonic projection is defined as:

- To sphere: $\hat{\mathbf{v}} = \frac{\mathbf{p}}{\|\mathbf{p}\|}$
- From sphere: $\mathbf{p} = \frac{r}{\hat{\mathbf{n}} \cdot \hat{\mathbf{v}}} \hat{\mathbf{v}}$

where $\mathbf{p}$ is a point on a plane given in Hessian normal form by $\hat{\mathbf{n}} \cdot \mathbf{p} = r$. Projection from Euclidean space to the sphere is literally just normalizing the vector. For the Goldberg-Coxeter operation, this amounts to just normalizing the vectors produced by the coordinate form. For triangles:

$$\mathbf{v}^* = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 \tag{11}$$

For quadrilaterals:

$$\mathbf{v}^* = \mathbf{v}_1 + (\mathbf{v}_2 - \mathbf{v}_1)x + (\mathbf{v}_4 - \mathbf{v}_1)y + (\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4)xy \tag{12}$$

where β_i are (planar) barycentric coordinates and x, y are x-y quadrilateral coordinates.

The triangular case can be thought of in terms of generalized barycentric coordinates. If the generalized coordinates are β'_i, then $\beta'_i = \frac{\beta_1}{\|\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2+\beta_3\mathbf{v}_3\|}$.

2.3. **Spherical areal.** This method applies to triangles only. Instead we look to the relation between barycentric coordinates and area; we treat β_i as the proportion of spherical area in the triangle that is opposite the vertex $\hat{\mathbf{v}}_i$. Let Ω be the spherical area (solid angle) of the spherical triangle and $\Omega_i = \beta_i\Omega$ be the area of the smaller triangle opposite vertex $\hat{\mathbf{v}}_i$.

The formula to find $\hat{\mathbf{v}}$ given β_i more complicated, although it's also derived from the formula for solid angle given earlier.

$$\begin{aligned}
\mathbf{G}\hat{\mathbf{v}} &= \mathbf{h} \\
\mathbf{G} &= \begin{bmatrix}\mathbf{g}_1 & \mathbf{g}_2 & \mathbf{g}_3\end{bmatrix} \\
\mathbf{h} &= \begin{bmatrix}h_1 & h_2 & h_3\end{bmatrix}^T \\
\mathbf{g}_i &= (1+\cos\Omega_i)\,\mathbf{v}_{i-1}\times\mathbf{v}_{i+1} - \sin\Omega_i\,(\mathbf{v}_{i-1}+\mathbf{v}_{i+1}) \\
h_i &= \sin\Omega_i\,(1+\mathbf{v}_{i-1}\cdot\mathbf{v}_{i+1})
\end{aligned} \tag{13}$$

The subscripts loop around: 0 should be interpreted as 3, and 4 should be interpreted as 1. To clarify, $\mathbf{G}$ is the 3x3 matrix where the ith column is $\mathbf{g}_i$, and $\mathbf{h}$ is the column vector where the ith element is h_i. The vector $\hat{\mathbf{v}}$ can be solved for using standard matrix methods.

2.4. **Great Circle.** Method 2 in geodesic dome circles, at least how Antiprism implements it

2.5. **Double Slerp.**

2.6. **Naive Slerp.** This method shares with the gnomonic method an analytic form for the transformation from Euclidean space to the sphere. It has two downsides: there is no analytic form for the reverse transformation, and it can only be used on equilateral faces.

The Naive Slerp method on a triangular face resembles a naive extension of spherical linear interpolation (Slerp) to barycentric coordinates, thus the name. The Naive Slerp methods reduce to slerp on the edges of the face.

Let $\cos(w) = \mathbf{v}_i \cdot \mathbf{v}_{i+1}$ for all i. (As usual, the subscripts loop around.) For triangles:

$$\mathbf{v}^* = \sum_{i=1}^{3} \frac{\sin(w\beta_i)}{\sin(w)}\mathbf{v}_i \tag{14}$$

For quadrilaterals:

$$\begin{aligned}
\mathbf{v}^* &= \sum_{i=1}^{4} \frac{\sin(w\gamma_i)}{\sin(w)}\mathbf{v}_i \\
\gamma_1 &= (1-x)(1-y) \\
\gamma_2 &= x(1-y) \\
\gamma_3 &= xy \\
\gamma_4 &= (1-x)y
\end{aligned} \tag{15}$$

or

$$
\begin{aligned}
\mathbf{v}^* &= \sum_{i=1}^{4} \frac{s_i}{\sin^2(w)} \mathbf{v}_i \\
s_1 &= \sin(w(1-x))\sin(w(1-y)) \\
s_2 &= \sin(wx)\sin(w(1-y)) \\
s_3 &= \sin(wx)\sin(wy) \\
s_4 &= \sin(w(1-x))\sin(wy)
\end{aligned} \tag{16}
$$

Because the projected edges already lie on the sphere, there is a lot of freedom in how to adjust $\mathbf{v}^*$ to lie on the sphere. The easiest is just to centrally project the vertices, that is, to normalize $\mathbf{v}^*$ like we have been. Another option is to perform a parallel projection along the face normal. (See appendix for a formula for the "normal" to a skew face.) We need the parallel distance p from the vertex to the sphere surface in the direction of the face normal $\hat{\mathbf{n}}$, such that $\hat{\mathbf{v}} = \mathbf{v}^* + p\hat{\mathbf{n}}$. p is given by:

$$p = -\mathbf{v}^* \cdot \hat{\mathbf{n}} + \sqrt{1 + \mathbf{v}^* \cdot \hat{\mathbf{n}} - \mathbf{v}^* \cdot \mathbf{v}^*} \tag{17}$$

p can also be approximated as $\widetilde{p} = 1 - \|\mathbf{v}^*\| \leq p$, which is somewhat fewer operations and doesn't require calculation of the face normal. Technically, you can project in almost any direction, not just that of the face normal, but most other choices don't produce anything remotely symmetric.

Sometimes the best polyhedron comes from a compromise of the central and parallel projections. Choose a constant k, typically between 0 and 1, then:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}^* + kp\mathbf{c}}{\|\cdots\|} \tag{18}$$

p may be replaced by $\widetilde{p}$. If our goal is to optimize a measurement of the polyhedron, we can do a 1-variable optimization on k, which is more tractable than the multivariate optimization of the location of every vertex.

3. Maps between polygons and disks in the Euclidean plane

3.1. **Conformal.** Schwarz-Christoffel mapping

3.2. **Naive Slerp.**

References

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