The motivations for this line of inquiry are:

- Classification. The inflation factor introduced in Brinkmann et al. [5] is a good measure of the complexity of certain operators. Are there more invariants that can be used to classify operators?
- Relationships between operators. How can operators be composed? How can an operator be decomposed into other operators?

1 Definitions

An abstract polytope P is a ranked poset satisfying the properties below. Elements of rank 0 may be called vertices; of rank 1, edges; of rank 2, faces. (In most literature on abstract polytopes all elements of the poset are called (abstract) faces, but since this work is primarily concerned with polyhedra, it would be confusing.) The order relation $<$ on the poset represents incidence between elements: for example, a face is incident on an edge, which is incident on a vertex. [3]

1. P contains a least element with rank -1 and a greatest element with rank n. (n is the rank of the abstract polytope.)
2. Each flag of P has the same length and includes the least and greatest elements. (A flag is a series of incident elements from the least element to the greatest)
3. P is strongly connected: any flag can be changed into any other flag by changing one element of the flag at a time.
4. P satisfies the diamond property: if an element A has rank $k-1$, and an element C has rank $k+1$, there are exactly 2 elements B of rank k such that $A < B < C$.

Occasionally a structure that violates one of these properties may be useful; these will be discussed as they appear. The skeleton of a polytope is the (possibly multi-) graph formed by its vertices and edges. The dual of an abstract polytope is simply the abstract polytope with its order and rank reversed. (This corresponds to the usual geometric notion of dual.)

The f-vector of a polytope is the vector of counts of elements of each rank. Normally the greatest and least elements are omitted (since they're always 1), and the vector has the form $(a_0, \ldots, a_{d-1})$ where a_i is the count of elements of rank i. The extended f-vector includes those elements: $(a_{-1} = 1, a_0, \ldots, a_{d-1}, a_d = 1)$. The vector is half-extended if only one of those endpoints is included. As such, the (extended) f-vector of the dual of a polytope is the (extended) f-vector of the original polytope, reversed. The f-polynomial is $F(x) = \sum_{i=-1}^{d} a_i x^{i+1}$. The Euler characteristic of a polytope is the alternating sum of the elements of the f-vector: $\chi = a_0 - a_1 + a_2 \ldots$.

A realization of an abstract polytope is an abstract polytope P mapped into a topological space, usually, but not necessarily, $\mathbb{R}^n$. Some realizations may be called faithful, which is not consistently defined, but usually means that the polytope does not have self-intersections, repeated points, digons, etc.

An abstract polyhedron P is an abstract polytope of rank 3. This generalizes things like convex polyhedra, tilings of the plane, and spherical polyhedra. Some of the properties above have more explicit consequences for polyhedra: [1]

- Each edge is incident to 2 distinct vertices. This is the usual definition of edge, but here it is a consequence of earlier definitions.
- Each edge is incident to 2 distinct faces. This excludes things like space-filling honeycomb structures (where more than 2 faces may meet at an edge)[1] and partial tilings of the plane (where fewer than 2 faces may meet at an edge).
- The faces may be self-intersecting, but may not contain interior holes.

The degree of a vertex is the number of edges to which it is incident. The degree of a face is the number of edges which are incident to it, i.e. its number of sides. (An edge of a polyhedron is always incident to 2 faces and 2 vertices, so defining degree for edges is not useful here, although it is for higher polytopes.) Abstract polyhedra permit faces of degree 2, called digons. Digons are disallowed by many definitions of polyhedra, but appear in spherical polyhedra (for example). Digons occur naturally in the study of operations on polyhedra. The count of vertices, edges, and faces is be denoted v, e, f. The Euler characteristic is $\chi = v - e + f$. The count of vertices or faces of a certain degree is denoted v_i and f_i where i is the degree, such that $\sum_i v_i = v$ and $\sum_i f_i = f$.

The term polyhedron here will mean "realization of an abstract polyhedron." Unless otherwise specified, polyhedra here do not require distinct vertices, convex faces, straight edges, non-intersecting faces, flat faces, or other things that may be required of other classes of polyhedra. This might seem odd if you're used to standard polyhedra, but examples appear naturally: see for example Figure 1, which is a common motif in quilting. We won't make much use of abstract polyhedra directly, but we need them as a theoretical underpinning. A structure that resembles a polyhedron but does not satisfy the axioms of an abstract polytope is here called a quasipolyhedron.

An achiral polyhedron is one that has mirror symmetry: a chiral polyhedron is one that does not. Note that the particular handedness of a chiral polyhedron is a quality of the realization, not the underlying abstract structure. An acoptic polyhedron is (loosely) a polyhedron that does not self-intersect. [2] Its faces are simple polygons with straight edges that do not self-intersect (although they may be concave). A convex polyhedron is an acoptic polyhedron that is convex: any line between points on the surface of the polyhedron is contained in the interior of the polyhedron. By Steinitz's theorem, the skeleton of every convex

[1] Honeycombs may be valid polytopes of rank 4, however.

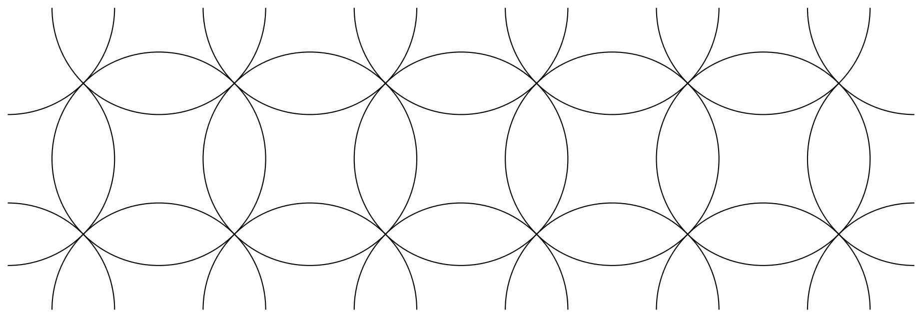


Figure 1: A tiling of the plane created by overlapping circles, consisting of digons and concave quadrilaterals with curved edges.

polyhedron is a 3-vertex-connected planar graph. Furthermore, all convex polyhedra have a realization such that each edge is tangent to the unit sphere, each face is flat, and the centroid of the vertices lies at the origin. This is called the canonical realization.

An operator on polyhedra is simply a map $x : \mathcal{P} \to \mathcal{P}$, where $\mathcal{P}$ is the set of polyhedra. Operators which break the mirror symmetry of a polyhedron are called chiral: ones that preserve it are achiral. We'll often apply operators to a restricted set of polyhedra, e.g. convex polyhedra, or polyhedra with triangular faces. Sometimes we will look at maps satisfying $x : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are different subsets of polyhedra: we will still call these operators on polyhedra (instead of transformations). We'll use a calligraphic typeface for sets of polyhedra or quasipolyhedra.

2 Notable operations

This section isn't a complete survey of operations on polyhedra. The concept dates back to Kepler, and some operations have had ambiguous or changing definitions in the past, so going through all that history would just muddy the waters. Instead, this section summarizes some useful operators that motivate the discussion herein or will be applied.

2.1 Operations defined in Coxeter's Regular Polytopes

In Coxeter's classic text [10], he defines a handful of operations on regular polytopes:

- The dual is defined in the usual way, compatible with how it was defined earlier in this text.
- Various forms of truncation are defined. The simple truncation is described by analogy with the polygonal case: cut off each corner of a polygon in such a way that the new polygon has vertices at the midpoint of

each original edge. Other forms of truncation that retain part of the original edge are mentioned but not described. It is also commented that for regular polyhedra, the truncation is equivalent to the intersection of the (canonical realization of) the original polyhedra and its dual.

- An operation, partial truncation or alternation, is defined on regular polytopes with even-degree faces. Alternating vertices are cut off or retained. This results in digons for degree-4 faces, which are treated as edges.
- The snub operation (different than Conway's snub) is equivalent to simple truncation followed by alternation.

Coxeter's construction of simplexes can be cast in terms of abstract polyhedra. Simply take the poset direct product of a number of copies of the unique abstract polyhedron of rank 1 (a single point). The product of two points is an edge: of three, a triangle, of four, a tetrahedron, and so on. This can be thought of as a binary operation on simplexes, so e.g. the product of two edges is a tetrahedron. (Also, suggestively, the f-polynomial of a direct product of posets is the product of the f-polynomial of each poset.)

2.2 Conway's operations

Conway described a set of operations on polyhedra and a notation for describing those operations, with the intent of creating a systematic naming scheme for polyhedra. He used a prefix notation, where the rightmost element is a polyhedral seed and operators apply from right to left. Each operator is assigned a letter and a name. So, for instance, dO is the dual of a regular octahedron (i.e. a cube), and taO is a truncated ambo octahedron, more commonly known as the truncated cuboctahedron. Conway's original set of operations is denoted with the letters $abdegjkmost$: a full list of their descriptions will be given later. These operations are sufficient to create all of the Archimedean and Catalan solids from the Platonic solids. Others have defined more operations. [12][11][4] In particular, Hart [12] defined r, which reverses the chirality of a polyhedra.

Aside from g, s, j, and a, Conway's operations preserve the symmetry of the resultant polyhedra. g and s are achiral, so do not preserve mirror symmetry. j and a actually increase the symmetry: in fact, it is possible to express the other 4 Platonic solids in terms of the tetrahedron T using j and a.

Some of Conway's operations can be expressed in terms of other operations: $e = aa, o = jj, m = kj$, and $b = ta$. Borrowing an idea from ring theory, we refer to d (dual) and S (seed, identity) as the units of the EROs, and operators that are related by d are called associates. At most 4 operators can be associated with each other, corresponding to x, xd, dx, and dxd. Conway's operators are associated as so:

- $j = jd, a = dj = djd$
- $k, t = dkd$ (as well as $n = kd$ and $z = dk$)

- $o = od, e = do = dod$
- $g, s = dgd$ (as well as $rgr = gd$ and $rsr = sd$)
- $m = md, b = dm = dmd$

Some Conway operators have an indexed form that indicates that only certain faces or vertices are operated on. For instance, k_i applies kis to faces with i sides, and t_i truncates vertices of degree i. Operations like this do not in general preserve the symmetry of the seed polyhedron.

2.3 Goldberg-Coxeter operations

The Goldberg-Coxeter operation (GC operation) was defined by Deza and Dutour [6], based on the Goldberg polyhedra, the viral capsid structure defined by Coxeter [9], the geodesic domes of Buckminster Fuller, and similar structures. Essentially it amounts to replacing the faces of a polyhedra with a section from a grid of triangular or square faces. Here the operation on triangle-faced polyhedra will be denoted $\Delta_{a,b}$, and on quadrilateral-faced polyhedra $\square_{a,b}$, where a and b are integers, $a > 0$, and $b \geq 0$.

$\Delta_{a,b}$ can be described using the triangular lattice over the Eisenstein integers. It is useful for this operation to parameterize the Eisenstein integers as $x = a+bu$ where $u = \frac{1}{2}(1+i\sqrt{3}) = e^{\pi i/3}$, for reasons to be explained later. Take the section of the grid inside the triangle with vertices $0, x(2-u)/3, x, x(1+u)/3$.

$\square_{a,b}$ can be described using the triangular lattice over the Gaussian integers, $x = a + bi$ where $i = \sqrt{-1}$. Take the section of the grid inside the square with vertices $0, x(1-i)/2, x, x(1+i)/2$.

Each operator has an invariant g, equivalent to $g = |x|^2$. For $\Delta_{a,b}$, $g = a^2 + b^2$: for $\square_{a,b}$, $g = a^2 + ab + b^2$. This can be used to calculate the count of elements in the resulting polyhedron based on the count of elements in the original polyhedron. (The actual formula will be shown later in a more general form.)

Two elements of the Eisenstein integers x and y are associates if $y = u^n x$ for some n. Similarly, two elements of the Gaussian integers are associates if $y = i^n x$ for some n. The associated element with $a > 0$ and $b \geq 0$ is the normal form. (We use the alternate definition for Eisenstein integers so that the same definition for normal form applies to both operators. This is also the traditional definition used by Goldberg polyhedra, geodesic domes, viral capsids, etc.) Iff $e + fu$ is associated with $(a + bu)(c + du)$, then $\Delta_{a,b}\Delta_{c,d} = \Delta_{e,f}$, and similarly for $\square_{a,b}$.

Another consequence of the relationship between these operators and the Eisenstein and Gausssian integers is that these operators are commutative and associative: $\Delta_{a,b}\Delta_{c,d} = \Delta_{c,d}\Delta_{a,b}$, and similarly for $\square_{a,b}$. Furthermore, the Eisenstein and Gaussian integers are Euclidean domains, which means elements of the domains can be factored uniquely (if not irreducible), and there is a straightforward way to do so using an extension of the Euclidean algorithm. (The invariant g is the Euclidean function in the Euclidean algorithm.)

These operators are divided into three classes.

- Class I: $b = 0$, achiral
- Class II: $a = b$, achiral
- Class III: All others, chiral

All Class II operators can be reduced as $\Delta_{a,a} = \Delta_{1,1}\Delta_{a,0}$ (and possibly further).

2.4 Operations in the software package Antiprism

The software package Antiprism [4] includes a number of applications that perform operations on polyhedra. (Among other things, it contains an implementation of Conway operations.) One caveat: The file format used by antiprism, OFF, consists of a list of vertex positions and a list of faces that references the list of vertices. This is not a fully faithful representation of a polyhedra, as it does not contain explicit incidence information. Problems show up for overlapping faces and digons. Digons are referred to as explicit edges by antiprism.

Of particular interest is the application `wythoff`, in which a notation for operations on polyhedra is introduced. Despite the name, the new notation is much more flexible than Wythoff notation. `wythoff` requires a polyhedron with only triangular faces, where the vertices of the polyhedron can be 3-colored. As a consequence, the vertices must have even order, and the faces are 2-colorable. The colored faces are labeled $+$ or $-$. We'll call this set of polyhedra $\mathcal{W}$. `wythoff` automatically applies Conway's meta operation to produce a polyhedron in $\mathcal{W}$. The meta operation retains the original vertices and adds a vertex at the center of each edge and each face. Respectively, these are labeled V, E, and F. If a polyhedron is in $\mathcal{W}$ it can be used directly: the labeling of vertices uses the same letters VEF. This is somewhat confusing since E and F no longer relate to edges or faces, so alternately the vertices can be labeled ABC.

For example, this is the string that it uses to implement Conway's kis operation: `[F, V] 0_1v1v, 1E`. The extended Wythoff notation comprises two parts. The first part, in brackets, defines points on each triangular face using barycentric coordinates. Each point is specified as `aVbEcF`, where `a`, `b`, and `c` are numbers. If any of those are 1, the number may be left off: if 0, the component may be left off. So in the above, it defines two points, more explicitly as `0V0E1F` and `1V0E0F`: a point at vertices labeled F, and a point at vertices labeled V.

The second part defines faces as paths between these points. A `+`, `-`, or `*` at the start of a path denotes which triangle to start with: if none, then `+` is assumed. An underscore indicates remaining on the same triangle. A lowercase `v`, `e`, or `f` indicates that the path crosses the edge opposite of the vertex labelled V, E, or F. An uppercase `V`, `E`, or `F` indicates a rotation by two triangles about the vertex labeled V, E, or F: explicitly, these are shorthands for `ef`, `fv`, and `ve` respectively. The first path starts at point 0 on $+$ triangles, moves to point 1 on the same triangle, then moves over the edge opposite V to point 1 on that triangle. It then moves back over the edge and completes the path at 0 on the

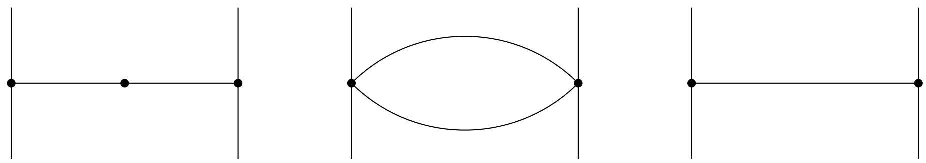


Figure 2: Depiction of smoothing operator. a) a degree-2 vertex b) digon c) smoothed result when applied to either.

original triangle. (The second produces an explicit edge, and is needed so that the operation produces a polyhedron when applied directly to a polyhedron in $\mathcal{W}$.)

This notation is capable of representing many operations on polyhedra, including all of Conway's operators. Notation to create the regular hemihedra from the Platonic solids exists, as does notation to create hollowed-out, da Vinci-style renderings of polyhedra. It is also capable of producing quasipolyhedra.

3 Smoothing

Digons can be present in a faithful realization of a polyhedron, such as in Figure 1, or spherical hosohedra. However, many subsets of polyhedra exclude digons. Degree-2 vertices are also problematic, being dual to digons. These elements can be eliminated in a systematic manner. We define a smoothing operator, \$, that transforms digons into single edges and handles degree-2 vertices by removing that vertex and merging the vertex's incident edges, as depicted in figure 2. This is similar to the topic of homeomorphism in graph theory. Antiprism effectively smooths digons automatically by treating them as "explicit edges".

Smoothing a single digon removes 1 face and 1 edge from the polyhedron. Smoothing a single degree-2 vertex removes 1 vertex and 1 edge. If f_p is the f-vector of a polyhedron, then $f_{\$p} \leq f_p$, where the inequality holds pairwise.

Conceptual complications arise when a polyhedron contains multiple digons or 2-vertices incident to one another. There is some choice in which element to start on, and degree-2 elements may be adjacent to one another. A single smoothing step may create other degree-2 elements, as depicted in Figure 3. It can be shown that with repeated reduction of single elements, the polyhedron eventually reaches a state where it has no degree-2 features. We choose to define \$ so that it produces a polyhedron where all degree-2 features have been removed.

Note that in special circumstances this operator may produce quasipolyhedra. For instance, any spherical hosohedron is reduced to a spherical quasipolyhedron with two vertices, one edge, and one face, and any spherical dihedron is reduced to a spherical quasipolyhedron with one vertex, one edge, and two faces. (These violate the diamond property, and therefore are not polyhedra in the sense defined here.)

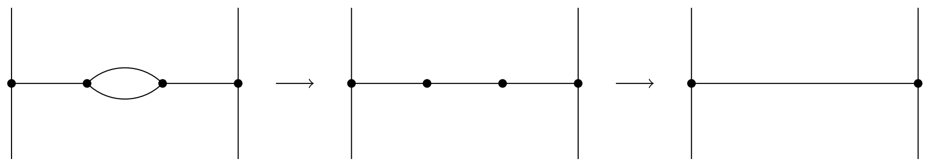


Figure 3: A multi-step smoothing series.

4 Compositions of operations on $\mathcal{W}$ and operations producing $\mathcal{W}$

5 Invariants of operators

Brinkmann et al. [5] prove that in a class of operators they call local symmetry-preserving operations (LSP) and local operations that preserve orientation-preserving symmetries (LOPSP), each operator has an invariant they call g, or the inflation factor. g is the ratio of the number of edges in the result polyhedron to that in the seed polyhedron. All of Conway's operations are LSPs or LOPSPs. For shorthand, we'll refer to LSPs and LOPSPs together as edge-replacement operations (EROs).

Expanding on that, many operations act on the f-vector as a linear operator. Where x is the operator, then the linear operator can be described with a matrix:

$$M_x = \begin{bmatrix} a & b & c \\ d & g & h \\ a' & b' & c' \end{bmatrix} \tag{1}$$

If x is an ERO, then $d = h = 0$. If x perserves the Euler characteristic, then the vector $< 1, -1, 1 >$ is a left eigenvector of M_x with eigenvalue 1. (Explicitly, $a + a' = d + 1$, $c + c' = h + 1$, and $b + b' + 1 = g$).

Some operators can also be expressed as an infinite linear operator L_x on the values v_i, e, and f_i. In particular, EROs take this form:

$$\begin{aligned} E &= ge \\ V_i &= av_{i/k} + eb_i + cf_{i/\ell} \\ F_i &= a'v_{i/k} + eb'_i + c'f_{i/\ell} \end{aligned} \tag{2}$$

v_i, e, and f_i are the input to the operator and V_i, E, and F_i are the result. a, a', c, and c' are either 0 or 1 if the Euler characteristic is preserved. g is a positive integer, all b_i and b'_i are nonnegative integers, and k and ℓ are positive integers. The subscripted values like $v_{i/k}$ should be interpreted as 0 if i/k is not an integer.

Applying the handshake lemma to the skeleton graph of the polyhedron gives relations between the values for EROs:

$$\begin{aligned} 2g &= 2ak + 2c\ell + \sum ib_i \\ 2g &= 2a'k + 2c'\ell + \sum ib'_i \end{aligned} \tag{3}$$

For Euler-characteristic preserving operations, these relations can be manipulated into the form

$$2k + 2\ell - 4 = \sum(4 - i)(b_i + b'_i), \tag{4}$$

which is interesting because it eliminates g, a and c, and because it suggests that features with degree 5 or more exist in balance with features of degree 3 (triangles and degree-3 vertices), and that in some sense degree 4 features come "for free".

With these relations, and the assumption that there are no degree 2 features and therefore $i \geq 3$, a series of inequalities can be derived for EROs:

$$g + 1 \leq 2a + 3b + 2c \leq 2g2k + 2\ell \leq g + 30 \leq 2k + 2\ell - 4 \leq b_3 + b'_3 \tag{5}$$

Note that these inequalities are only necessary, not sufficient.

6 Operator diagrams

While antiprism's extended Wythoff notation can express EROs, it's not necessarily obvious what the notation string does without executing it, or what the composition of two operators would be. Here we introduce a way of diagramming these operators to complement the extended Wythoff notation.

The action of an ERO on the vertices of degree i, edges, and faces with i sides can be described with an infinite linear operator L_x, as mentioned earlier. This operator can be determined by counting elements off the chamber structure. Step by step:

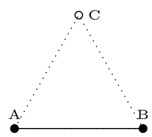
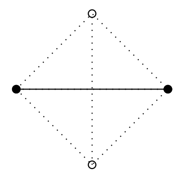
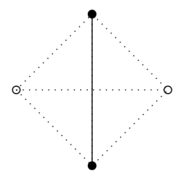
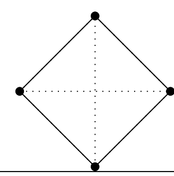
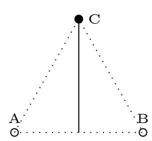
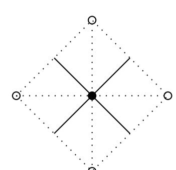
* Seed vertices are either retained or converted into faces centered on that vertex. (Other options are precluded by symmetry). Let $a = 1$ if the seed vertices are retained, and 0 otherwise. Also, the degree of the vertex or face is either the same as the seed vertex, or a multiple of it; let k be that multiple. * Seed face centers are either retained (possibly of in a smaller face) or converted into vertices. (Again, other options are precluded by symmetry). Let $c = 0$ if the seed faces are retained, and 1 otherwise. Let ℓ serve a similar role as k above: the degree of the vertex or face corresponding to the seed face center is k times the degree of the seed vertex. * Except for the faces or vertices corresponding to the seed vertices and face centers, the added elements are in proportion to to the number of edges in the seed. g is the count of added edges (the edge multiplier or inflation rate), b_i is the number of vertices of degree i added, and b'_i is the number of faces of degree i added.

Count elements lying on or crossing the outer edge of the chamber structure as half. It may help to draw an adjacent chamber, particularly when determining the number of sides on a face.

Extension of GCs to all faces

7 Table of operators

Table 1: a-cohort of operators on polyhedra

$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & 1 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	ABC, BAC		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
		CAB, CBA		$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
		ACB, BCA		$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$x@m = j$
	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & ? & ? \end{bmatrix}$	ABC, BAC	d	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
		CAB, CBA	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
		ACB, BCA		$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$x@m = a$

8 Future work

There are two related directions to take this work in the future.

First, to explore operations on general polytopes. Here we've explored operations on polyhedra. Coxeter defined his operations on regular polytopes. A general theory of operations on polytopes would be the next logical step. These operations need not necessarily be between polytopes of the same rank: consider the embedding of a higher polytope in 3-space, or producing the skeleton of a polytope.

Second, to explore operations on abstract polyhedra (and polytopes) without reference to the realization. The way they have been described in this text, there is a dependence on the underlying space. Removing that dependence would probably make the theory clearer. Furthermore, some of these operations may be valid on posets that do not satisfy the axioms of an abstract polytope. For example, the dual operation is defined for all posets. Finding the most general restriction on the poset for a certain operation may help us understand how to deal with quasipolyhedra.

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[12] Hart, G. W. Conway Notation for Polyhedra. http://www.georgehart.com/virtual-polyhedra/conway_notation.html

Table 2: o-cohort of operators on polyhedra

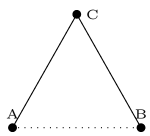
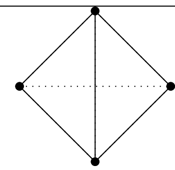
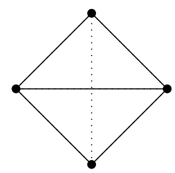
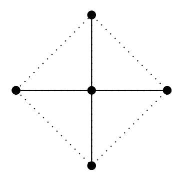
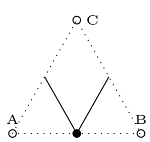
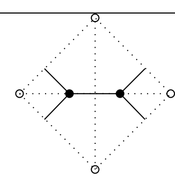
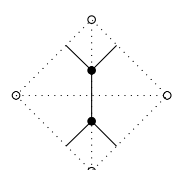
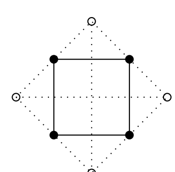
$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & ? & ? \end{bmatrix}$	ABC, BAC		$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$x@m = n$
		ACB, BCA		$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = k$
		CAB, CBA		$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$x@m = o$
	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & ? & ? \end{bmatrix}$	ACB, BCA		$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\$x@m = t$
		ABC, BAC		$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\$x@m = z$
		CAB, CBA		$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$x@m = e$

Table 3: u-cohort of operators on polyhedra

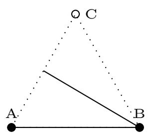
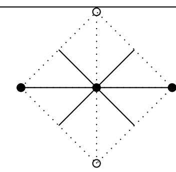
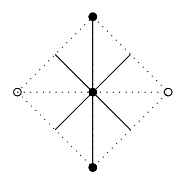
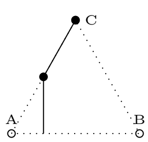
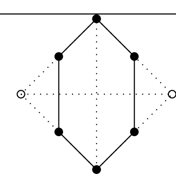
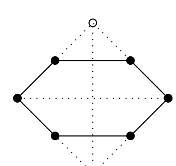
$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & 1/2 \end{bmatrix}$	ABC		$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 1 \end{bmatrix}$	$x@m = u$
		ACB	n	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = n$
		BAC	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
		BCA	k	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = k$
		CAB	d	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
		CBA		$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$	$x@m = ud$
	$\begin{bmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 3/2 \\ 1 & 1 & 0 & 0 & 1/2 \end{bmatrix}$	ABC		$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$x@m = cd$
		ACB	t	$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\$x@m = t$
		BAC	d	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\$x@m = d$
		BCA	z	$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$\$x@m = z$
		CAB	S	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\$x@m = S$
		ABC		$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$x@m = c$

Table 4: p-cohort of operators on polyhedra

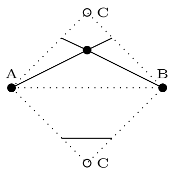
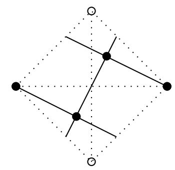
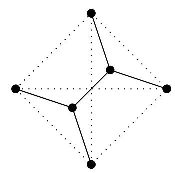
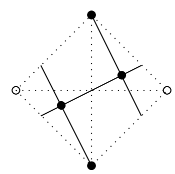
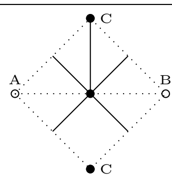
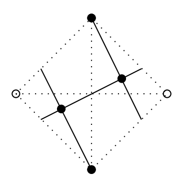
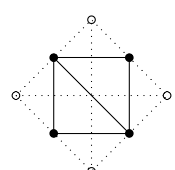
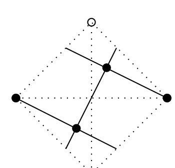
$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix}1 & 1 & 0 & ? & ?\\0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\end{bmatrix}$	BAC		$\begin{bmatrix}1 & 2 & 0\\0 & 5 & 0\\0 & 2 & 1\end{bmatrix}$	$\$x@m = p$
		ACB		$\begin{bmatrix}1 & 2 & 1\\0 & 5 & 0\\0 & 2 & 0\end{bmatrix}$	$\$x@m = rgr$
		CBA		$\begin{bmatrix}0 & 2 & 1\\0 & 5 & 0\\1 & 2 & 0\end{bmatrix}$	$\$x@m = dp = pd$
	$\begin{bmatrix}0 & 0 & 1 & ? & ?\\0 & 0 & 0 & 0 & 0\\1 & 1 & 0 & 0 & 0\end{bmatrix}$	BAC		$\begin{bmatrix}0 & 2 & 1\\0 & 5 & 0\\1 & 2 & 0\end{bmatrix}$	$\$x@m = dp = pd$
		ACB		$\begin{bmatrix}0 & 2 & 0\\0 & 5 & 0\\1 & 2 & 1\end{bmatrix}$	$\$x@m = rsr$
		BCA		$\begin{bmatrix}1 & 2 & 0\\0 & 5 & 0\\0 & 2 & 1\end{bmatrix}$	$\$x@m = p$

Table 5: g-cohort of operators on polyhedra

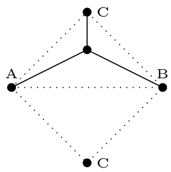
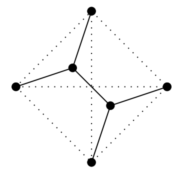
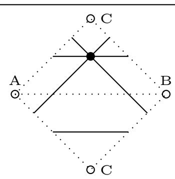
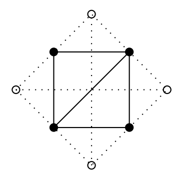
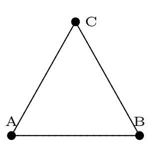
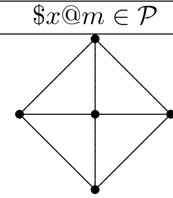
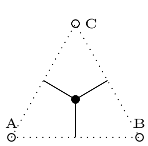
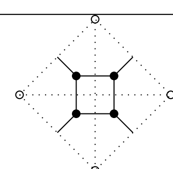
$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	ABC BCA CAB		$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$\$x@m = g$
		CBA BAC ACB	mirror	”	$\$x@m = rgr$
	$\begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 3/2 \\ 1 & -1 & 2 \end{bmatrix}$	ABC BCA CAB		$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 5 & 0 \\ 1 & 2 & 1 \end{bmatrix}$	$\$x@m = s$
		CBA BAC ACB	mirror	”	$\$x@m = rsr$

Table 6: m-cohort of operators on polyhedra

$x \in \mathcal{W}$	M_x	@	$\$x@m \in \mathcal{P}$	$M_{\$x@m}$	Note
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	All		$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 0 \\ 0 & 4 & 0 \end{bmatrix}$	$x = S\ xm = m$
	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	All		$\begin{bmatrix} 0 & 4 & 0 \\ 0 & 6 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$x = d\ xm = b$