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Stellar Structure Calculation: Report

For my final project in Stellar Structure and Evolution, I created a Python script to calculate the structure of a ZAMS star. These zero-age main sequence stars are a bit less complicated to model because the stars are just becoming what we think of as generic stars. Certain assumptions can be made – such as, energy generation only due to fusion, the application of the ideal gas law everywhere, pressure only due to gas and radiation, and the full ionization of the star. I can ignore more complex parts of stars, including rotation, inhomogeneous composition (as ZAMS stars are just starting to striate), and gravitational contraction (which should be minimal after the onset of hydrogen fusion). All of these things combine to make a one-dimensional stellar structure calculation on a typical laptop feasible.

The heart of stellar structure calculations is the four nonlinear, coupled ordinary differential equations. These are reproduced below in Lagrangian form – pressure, radius, luminosity, and temperature are all functions of mass. These represent, in order, hydrostatic equilibrium, mass conservation, energy generation, and energy transport.

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}$$

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dL_r}{dM_r} = \varepsilon$$

$$\frac{dT}{dM_r} = \frac{d\ln T}{d\ln P}\left(-\frac{GM_r T}{4\pi r^4 P}\right)$$

While some of these variables are relatively straightforward, others depend on a multitude of other factors. First, I chose the stellar mass as 1.4 solar. This is close enough to the mass of our Sun that the assumption of stellar composition (70% hydrogen, 28% helium, and 2% everything else) is reasonable. Additionally, I can use the homology relations for low-mass stars for radius and luminosity.

$$\frac{R}{R_\odot} \approx \left(\frac{M}{M_\odot}\right)^{0.75}$$

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

Similarly, I can assume a constant-density core to determine temperature and pressure.

$$T_c \approx \frac{1}{2}\frac{4}{\mu N_A k_B}\frac{GM}{R}$$

$$P_c \approx \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Temperature and pressure coupled together yield the star's density.

$$\rho = P - \frac{aT^4}{3} \times \frac{\mu}{N_A kT}$$

The mean molecular weight depends on the mass fraction of hydrogen, X.

$$\mu = \frac{4}{3 + 5X} \approx 0.6$$

This is the expected value.

The density and temperature combine to produce an opacity. Opacities are incredibly complicated, so I used the tables compiled from OPAL – the project from Lawrence Livermore National Laboratory.

The opacity determines the gradient of temperature with respect to pressure.

$$\nabla_{rad} = \left(\frac{d \ln T}{d \ln P}\right)_{rad} = \frac{3P\kappa L}{16\pi acT^4 GM}$$

I compare this to the nominal value for complete ionization to determine whether the core is convective (greater) or radiative (equal or less).

$$\nabla_{\mathrm{ad}} = \left(\frac{d \ln T}{d \ln P}\right)_{ad} = \frac{\gamma - 1}{\gamma} \rightarrow \gamma = \frac{5}{3} \rightarrow \nabla_{\mathrm{ad}} = 0.4$$

The last part of these equations concerns the methods of energy generation. For a 1.4 solar mass star, both the proton-proton chain (pp-chain) and Carbon-Nitrogen-Oxygen cycle (CNO cycle) contribute. For the former, I use pp1.

$$\varepsilon_{pp} = 2.57 \times 10^4 \psi f_{11} g_{11} \rho X_1^2 T_9^{-2/3} e^{-\frac{3.381}{T_9^{1/3}}}$$

The gaunt factor is reproduced below. All temperatures with a subscript refer to the temperature divided by that power of ten.

$$g_{11} = 1 + 3.82T_9 + 1.51T_9^2 + 0.144T_9^3 - 0.0114T_9^4$$

I assume a psi of 1, since I am only considering pp1. Additionally, I use weak screening.

$$f_{11} = e^{\frac{E_D}{k_B T}}$$

$$\frac{E_D}{k_B T} = \frac{Z_1 Z_2 e^2}{r_D k_B T} = 5.92 \times 10^{-3} Z_1 Z_2 \left(\frac{\zeta \rho}{T_7^3}\right)^{1/2}$$

Zeta is assumed to be 1, and $Z_1 = Z_2$ = about 1 for pp1.

The procedure is similar for the CNO cycle.

$$\varepsilon_{CNO} = 8.24 \times 10^{25} g_{14,1} X_{CNO} X_1 \rho T_9^{-2/3} e^{-15.231 T_9^{-\frac{1}{3}} - \left(\frac{T_9}{0.8}\right)^2}$$

$$g_{14,1} = 1 - 2.00T_9 + 3.41T_9^2 - 2.43T_9^3$$

$$X_{CNO} = X_c + X_N + X_O \approx 0.7Z$$

Armed with these equations, I determined values for luminosity, pressure, radius, and temperature for starting points just outside the star's core and just inside the star's photosphere. This avoids mathematical issues directly at the core and surface. The internal pressure requires a correction from that at the core, and the temperature depends on whether the interior is radiative or convective.

$$P = P_c - \frac{3G}{8\pi}\left(\frac{4\pi\rho_c}{3}\right)^{\frac{4}{3}} m_c^{2/3}$$

$$T_{conv} = e^{T_c - \left(\frac{\pi}{6}\right)^{\frac{1}{3}} \frac{G\nabla_{ad}}{P_c} \rho_c^{\frac{4}{3}} m_c^{2/3}}$$

$$T_{rad} = \left(T_c^4 - \left(\frac{\kappa_c \varepsilon}{2ac}\right) rho_c^{\frac{4}{3}} \left(\frac{3m_c}{4\pi}\right)^{2/3}\right)^{1/4}$$

Similarly, the star's exterior depends on its surface gravity and effective temperature.

$$g = \frac{GM}{R^2}$$

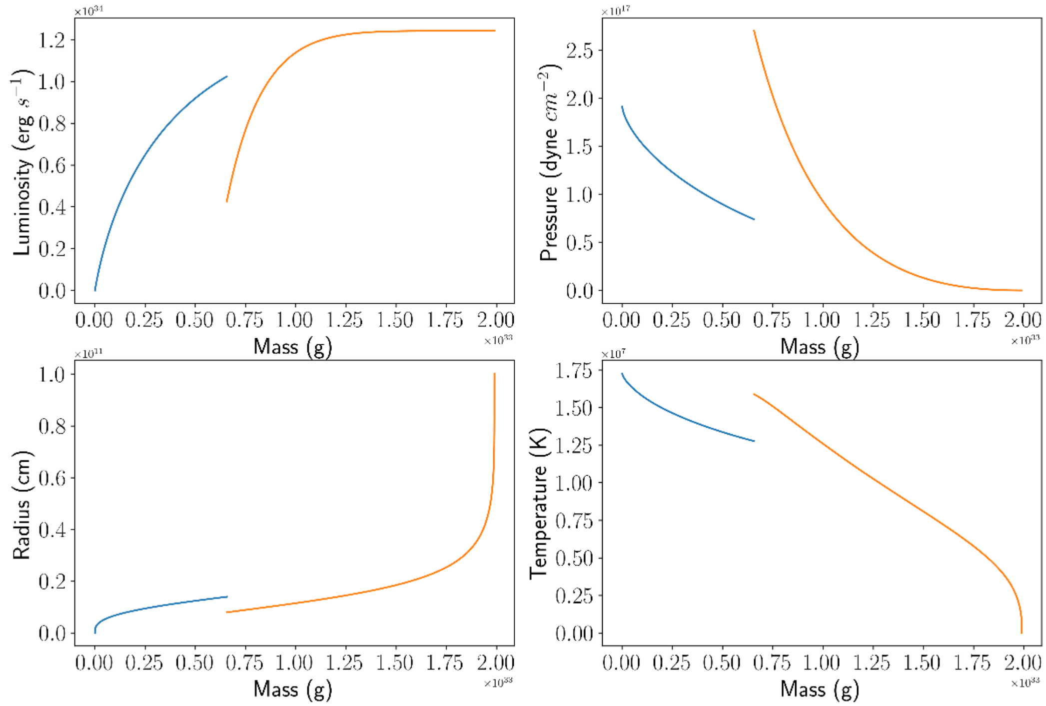
$$T_{eff} = \left(\frac{L}{4\pi\sigma_{SB}R^2}\right)^{1/4}$$

Additionally, the density could be determined by opacity or the equation of state, itself a mixture of ideal gas and radiation. Whichever is lower influences the pressure.

$$P = \frac{2g}{3\kappa}\left(1 + \frac{L\kappa}{4\pi cGM}\right)$$

With the initial conditions set for both inside and outside of the star, I employed the shooting method by solving initial value problems. I solved the coupled differential equations using 4th-order Runge-Kutta from both directions, checking whether they met at a point one-third of the way between the core and the surface. Since more of the mass resides in the stellar core, I used this point as a very rough weighted average. The results of these inside-out and outside-in integrations are presented as four graphs: luminosity, pressure, radius, and temperature as a function of mass.

The luminosity presents the biggest problem, as the two integrations are clearly about to diverge. Pressure also has a jump-discontinuity. This indicates there might be an issue with my energy generation rates (luminosity) and/or density calculations (pressure). I suspect my method of interpolating opacities is not as robust as it should be.



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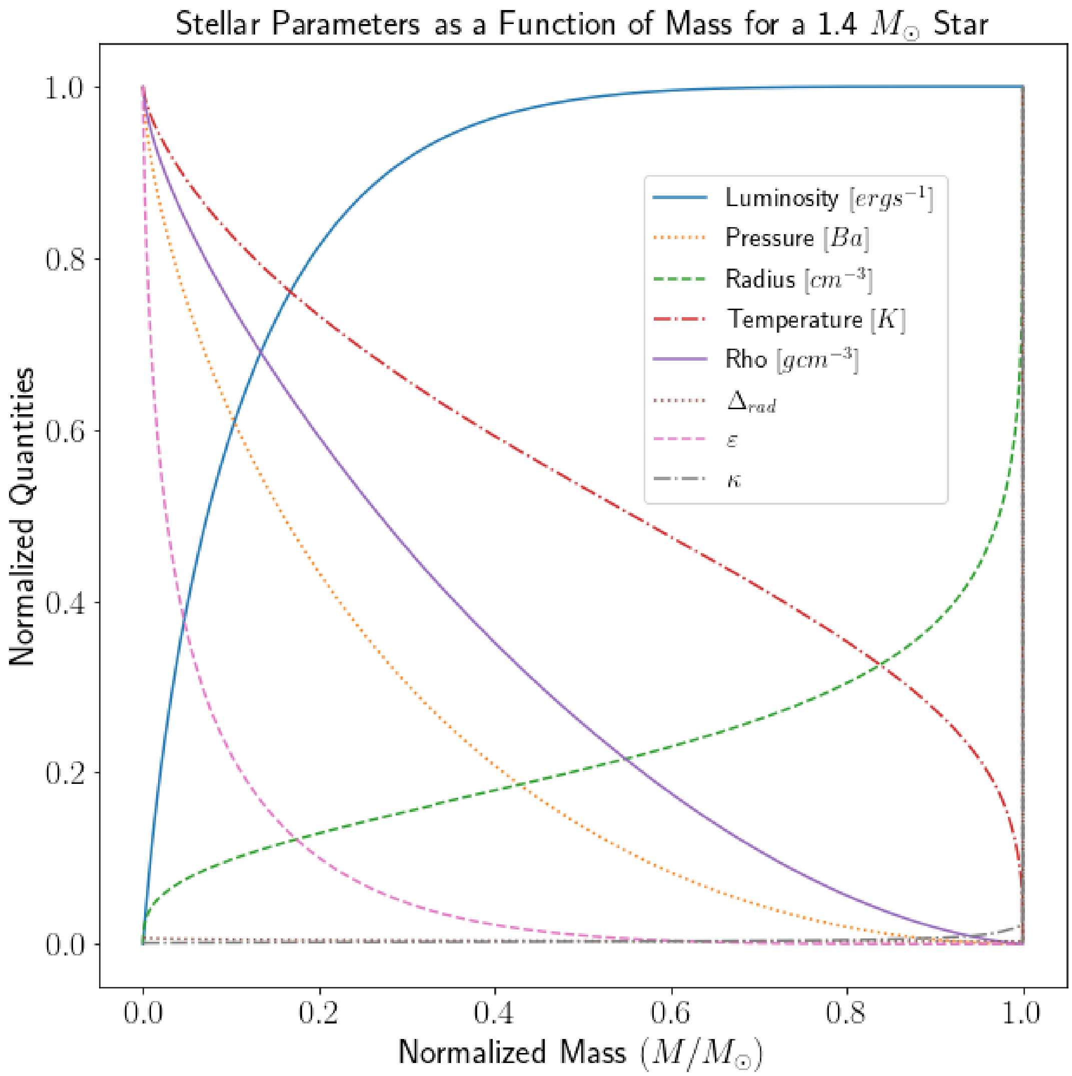
These integrations were performed iteratively, with the residuals from each run minimized using the least-squares method. The model converged in 94 iterations – longer than the process should have taken. I conclude that some of the assumptions were invalid, or there could be a problem with the underlying code. The results are presented in a graph showing the luminosity, pressure, radius, temperature, density, temperature-pressure gradient, energy generation, and opacity as a function of mass.



Comparing my stellar structure code with the same stellar mass run through MESA, I find that the code did a somewhat decent job of reproducing the star. Pressure, radius, and temperature are all within 5% of their MESA values. Luminosity, however, is only within 15%. This is another indication that there may be a problem with the underlying assumptions or coding. These results are summarized in a table.

Variable	**Converged Solution**	**MESA**	**Percent Difference**
Luminosity (erg s^{-1})	1.24e34	1.44e34	14.76%
Pressure (dyne cm^{-2})	1.92e17	1.97e17	3.05%
Radius (cm)	1.00e11	9.75e10	2.91%
Temperature (K)	1.73e7	1.75e7	1.44%

In closing, I would like to thank my fellow scientists for their support and troubleshooting during this project – specifically William Balmer, Stephen Schmidt, Celia Mulcahey, Ezra Sukay, and Patrick McCreery, with extra emotional support from Nicole Crumpler. All documents for this project can be found in my GitHub repo: https://github.com/Quasara/ZAMS-Project