Ultrafilters and Ultralimits

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Filters

Given a set X, we define a filter on X to be a collection of subsets $F \subset \mathcal{P}X$ such that:

- $B \supseteq A \in \mathcal{F} \implies B \in \mathcal{F}$ (F is closed upwards)
- $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$ ($\mathcal{F}$ is closed under finite intersection)
- $\emptyset \notin \mathcal{F}$ ($\mathcal{F}$ is nontrivial)

A filter gives some notion of *sufficiently large* subsets of X.

Generating Filters

The upwards closure of a family $\mathcal{A}$ of subsets of X is the family of supersets of of elements of $\mathcal{A}$. A family of sets $\mathcal{B}$ is a filter base on X if its upwards closure is a filter on X.

The following are equivalent:

- $\mathcal{B}$ is a filter base,
- $\mathcal{B}$ is nonempty, does not contain $\emptyset$, and is closed under finite intersection.

Bases for ultrafilters are of particular interest.

Ultrafilters

An ultrafilter on X is a maximal filter $\mathcal{F}$ on X, that is, no more elements can be added to $\mathcal{F}$ or it will become $\mathcal{P}X$ and therefore not a filter. A particularly useful equivalent definition is as a filter that decides between every subset and its complement.

Ultrafilters are usually distinguished between principal and non-principal (or free) ultrafilters. A principal ultrafilter is one with a minimal element, equivalently it is generated by a single element (called the principal element).

An important result (when assuming AC) is the following:

Any filter is contained in an ultrafilter.

Applications of (Ultra)filters

Filters are not just useful in abstract set theory! They are especially useful in topology:

- Filters and topologies are closely related structures,
- Continuity can be characterised in terms of filters,
- Compactness can be characterised in terms of ultrafilters (NB: this),
- and much more!

And, of course, we get ultralimits.

Gromov-Hausdorff Metric

The Hausdorff metric between two subsets is the supremum distance from a point in one subset to the other subset. It is a definition of how far apart two subsets are from one another.

The Gromov-Hausdorff metric extends this to compare how close two compact metric spaces are to being isometric. We define $d_{\mathrm{GH}}(X, Y)$ to be the infimum of $d_{\mathrm{H}}(f(X), g(Y))$ for all compact metric spaces M and isometries $f : X \to M, g : Y \to M$. This induces a topology on the set of compact metric spaces modulo isometry, this is the Gromov-Hausdorff space.

Limits w.r.t. Ultrafilters

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Let (X, d) be a metric space, $\langle x_n \rangle_{n\in\omega}$ a sequence of points in X, and $x \in X$ a point. Then x is the $\mathcal{U}$-limit of $\langle x_n \rangle_{n\in\omega}$, written $\mathcal{U}\text{-lim}\langle x_n \rangle$ or $\mathcal{U}\text{-lim}_n\, x_n$, if for any nbhd N of x, the set $\{i : x_i \in N\} \subseteq \omega$ is in $\mathcal{U}$.

$\mathcal{U}$-limits are unique, and when a conventional limit is defined, it is equal to the $\mathcal{U}$-limit. So why do we care about this?

If (X, d) is compact, then every $\mathcal{U}$-limit in X converges.

This might seem familiar...

Ultralimits of Metric Spaces

We want to generalise the Gromov-Hausdorff convergence to wider classes of spaces. Suppose we have a sequence $\langle X_n \rangle_{n \in \omega}$ with metrics $d_{X_n} : X_n \times X_n \to [0, \infty]$ of (not necessarily compact) metric spaces. For a non-principal ultrafilter $\mathcal{U}$ we define the ultralimit $X_{\mathcal{U}}$ (sometimes written X_∞, $\mathcal{U}\text{-}\lim_n X_n$) as follows:

Let Seq be the space of sequences $\langle x_n \rangle_{n \in \omega}$ wth $x_n \in X_n$. Given $x = \langle x_n \rangle_{n \in \omega}, y = \langle y_n \rangle_{n \in \omega} \in \mathrm{Seq}$ we define their distance as

$$d_{\mathcal{U}}(x, y) = \mathcal{U}\text{-}\lim_n d_{X_n}(x_n, y_n)$$

Now if we identify all points with zero distance we get a metric space, this is $X_{\mathcal{U}}$. If we restrict to points having finite distance from some other fixed point, then we get a based (or pointed) ultralimit, notated $X^0_{\mathcal{U}}$.

Ultralimits Extending GH-Convergence

Does this work for our purposes?

Let each X_n compact, and converges in the Gromov-Hausdorff topology to a compact metric space X. Then for every non-principal ultrafilter $\mathcal{U}$, X is isometric to $X_{\mathcal{U}}$.

Now we can use this stronger notion of convergence to talk about any metric space, and know that for our already established results they still hold without needing to worry about the ultrafilter.

Properties of Ultralimits

- A based ultralimit of metric spaces is complete,
- A based ultralimit of geodesic metric spaces is geodesic,
- A based ultralimit of $\mathrm{Cat}(\kappa)$ spaces where $\kappa \leq 0$ is $\mathrm{Cat}(\kappa)$,
- A based ultralimit of $\mathrm{Cat}(\kappa_n)$ spaces where $\kappa_n \to -\infty$ is a $\mathbb{R}$-tree.

References

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[2] Misha Kapovich and Bernhard Leeb. "On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds". In: *Geometric & Functional Analysis GAFA* 5 (1995), pp. 582–603.