Cardinal Characteristics and Computability

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The Continuum Hypothesis

Cantor proposed the Continuum Hypothesis (CH) in 1878:

> There is no set whose cardinality is strictly between that of the integers and the real numbers.

That is: $\aleph_1 = 2^{\aleph_0}$.

Gödel proved in 1940 that CH is consistent with the axioms of ZFC (that it doesn't lead to contradictions). In 1963, Cohen completed the independence proof, developing the method of forcing, showing that $\neg$CH is also consistent with ZFC.

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Mod Finite Relations

When we work with infinite objects, sometimes the usual comparisons (e.g. subset, function comparison) between these are too strong. Sometimes more interesting results and richer structures can be obtained with weaker relations.

We consider the following mod finite relations:

- $f \leq^* g \iff f(x) \leq g(x)$ for all but finitely many x
- $A \subseteq^* B \iff x \in A \to x \in B$ for all but finitely many x

We say that g *eventually dominates* f, and that A is *almost contained* in B. When one of these relations holds in both directions we can use $=^*$.

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Cardinal Characteristics of the Continuum

We like to make our results as general as possible. Consider the following statement which assumes a set is countable:

> For any **countable** set of sequences of natural numbers, there exists a sequence of natural numbers which eventually dominates each sequence in the set.

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The (Un)Bounding Number $\mathfrak{b}$

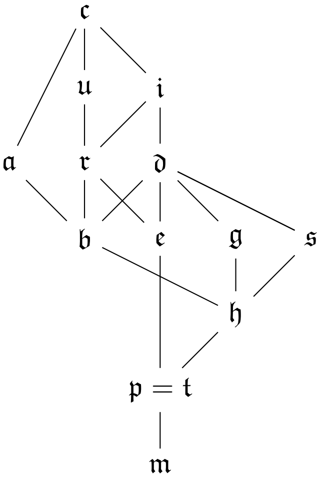
For any ~~countable~~ set of sequences of natural numbers **with cardinality less than** $\mathfrak{b}$, there exists a sequence of natural numbers which eventually dominates each sequence in the set.

We define the bounding number $\mathfrak{b}$ to be the minimum size of a set of sequences of natural numbers, such that there is no sequence that eventually dominates each sequence in the set:

$$\mathfrak{b} = \min \left| \{ F \subseteq {}^{\omega}\omega : g \in {}^{\omega}\omega \to \exists f \in F (g \not\geq^* f) \} \right|$$

Such a family can be interpreted as an unbounded set in the natural number sequences ordered by eventual domination... so perhaps it should be referred to as the *un*bounding number.

Hasse Diagram of Combinatorial Characteristics



This diagram[1] illustrates provable (in ZFC) relations between many characteristics which arise from combinatorial properties.

1 [1]Andreas Blass, [3]Malliaris and Shelah

Computability Theory

Computability theory works to describe the kinds of objects we can construct recursively.

- A function is called computable if it can be defined recursively from some very basic functions.
- A set is called computable if its characteristic function is a computable function.

In some sense, an object is computable if it only contains a finite amount of information, even though it may be an infinite object in a classical sense.

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Analogy of Set Theory

In a lot of ways, computability acts like countability. In many settings we have very analogues of set theoretic concepts in the setting of computability (cardinal characteristics, ordinals, large cardinals).

Whereas the witnesses of these cardinal characteristics are uncountable families, in their computability analogues they are uncomputable:

We say a set A is *high* if there is a function f that can be computed using the additional information from A (we write $f \leq_T A$) such that $g \leq^* f$ for any computable function g. Note that A can not be computable.

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Characteristics and Mass Problems

We call the collection of high sets the *mass problem* High. It is analagous the the bounding number $\mathfrak{b}$, though in the setting of computability (restricting to computable functions of natural numbers).

Various other cardinal characteristics have analogous mass problems, which are often simple adaptations of their set theoretic counterparts.[2] Proving relations between these often follows a very different method than in the set theoretic context, though a lot of similar results hold.

My current interest is in investigating the computability theoretic analogue of a characteristic of maximal ideal independent families. As of the past few weeks we have some new results adapted in this context.

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References

[1] Andreas Blass. "Combinatorial Cardinal Characteristics of the Continuum". In: *Handbook of Set Theory*. Springer Netherlands, 2010, pp. 395–489.

[2] Steffen Lempp, Joseph S. Miller, André Nies, and Mariya I. Soskova. "Maximal Towers and Ultrafilter Bases in Computability Theory". In: *The Journal of Symbolic Logic* 88.3 (2023), pp. 1170–1190.

[3] Maryanthe Malliaris and Saharon Shelah. "Cofinality spectrum theorems in model theory, set theory, and general topology". In: *J. Amer. Math. Soc.* 29.1 (2016), pp. 237–297.