1 MLE estimation for GSD

Denote by $(\hat{\psi}, \hat{\rho})$ the MLE for GSD.

1.1 Step 1

If there exist $k \in \{1, ..., 4\}$ such that $n_k + n_{k+1} = n$ then

$$(\hat{\psi}, \hat{\rho}) = \left(\frac{kn_k + (k+1)n_{k+1}}{n}, 1 \right)$$

1.2 Step 2

If $n_1 + n_5 = n$ then

$$(\hat{\psi}, \hat{\rho}) = \left(\frac{n_1 + 5n_5}{n}, 0 \right)$$

1.3 Step 3

If both conditions in Step 1 and Step 2 are not fulfilled then denote

$$V_{\min}(\psi) = (\lceil \psi \rceil - \psi)(\psi - \lfloor \psi \rfloor),$$

$$V_{\max}(\psi) = (\psi - 1)(5 - \psi),$$

$$C(\psi) := \frac{3}{4} \frac{V_{\max}(\psi)}{V_{\max}(\psi) - V_{\min}(\psi)},$$

Consider Beta-binomial distribution i.e.,

$$P(X = k) = \frac{\mathcal{B}(\alpha + k, \beta + 4 - k)}{\mathcal{B}(\alpha, \beta)},$$

where $X \in \{0, ..., 4\}$. In this case $n_1, ..., n_5$ denote the number of observations $0, ..., 4$ respectively (answer 1 is treated like 0 etc.) Denote

$$l_{BB}(\alpha, \beta) = -n \sum_{i=0}^{3} \log(\alpha + \beta + i) + \sum_{k=0}^{4} n_{k+1} \left[\sum_{i=0}^{k-1} \log(\alpha + i) + \sum_{i=0}^{4-k-1} \log(\beta + i) \right],$$

where $\sum_{i=0}^{-1} \log(\alpha + i) = 0$ and $\sum_{i=0}^{-1} \log(\beta + i) = 0$.

Denote $\hat{\alpha}$, $\hat{\beta}$ the MLE estimators of α and β calculated numerically i.e.,

$$(\hat{\alpha}, \hat{\beta}) = \underset{(\alpha, \beta) \in (0, \infty) \times (0, \infty)}{\arg\max} l_{BB}(\alpha, \beta),$$

by using Newton-Raphson method and following starting points (see [Biometrics])

$$(\alpha_s, \beta_s) = \left(\frac{n\hat{\mu}_1 - \hat{\mu}_2}{n\left(\frac{\hat{\mu}_2}{\hat{\mu}_1} - \hat{\mu}_1 - 1\right) + \hat{\mu}_1}, \frac{(n - \hat{\mu}_1)(n - \frac{\hat{\mu}_2}{\hat{\mu}_1})}{n\left(\frac{\hat{\mu}_2}{\hat{\mu}_1} - \hat{\mu}_1 - 1\right) + \hat{\mu}_1} \right),$$

where

$$\hat{\mu}_1 = \frac{n_2 + 2n_3 + 3n_4 + 4n_5}{n},$$

$$\hat{\mu}_2 = \frac{n_2 + 4n_3 + 9n_4 + 16n_5}{n}.$$

We have

$$\hat{\psi}_{BB} = \frac{5\hat{\alpha} + \hat{\beta}}{\hat{\alpha} + \hat{\beta}}$$

$$\hat{\rho}_{BB} = \frac{4C(\hat{\psi}_{BB})\hat{\alpha}}{4\hat{\alpha} + \hat{\psi}_{BB} - 1}$$

Now denote

$$a(k,\psi) = [1 - |k - \psi|]_+$$

$$b(k,\psi) = \binom{4}{k-1} \left(\frac{\psi - 1}{4}\right)^{k-1} \left(\frac{5 - \psi}{4}\right)^{5-k}$$

$$u(k,\psi) = a(k,\psi) - b(k,\psi)$$

$$v(k,\psi) = b(k,\psi) - a(k,\psi)C(\psi)$$

for $k = 1, ..., 5$ and

$$x(l,\psi) = -nu(l,\psi)u(l+1,\psi)$$

$$y(l,\psi) = u(l,\psi)u(l+1,\psi)(n_l + n_{l+1}) - n(u(l,\psi)v(l+1,\psi) + u(l+1,\psi)v(l,\psi))$$

$$z(l,\psi) = n_l u(l,\psi)v(l+1,\psi) + n_{l+1}u(l+1,\psi)v(l,\psi) - (n - n_l - n_{l+1})v(l,\psi)v(l+1,\psi)$$

$$\Delta(l,\psi) = [y(l,\psi)]^2 - 4x(l,\psi)z(l,\psi)$$

for $l = 1, ..., 4$. Denote

$$l_{mix}(\psi,\rho) = -n\log(1 - C(\psi)) + \sum_{k=1}^{5} n_k \log[a(k,\psi)(\rho - C(\psi)) + b(k,\psi)(1 - \rho)],$$

$$r_1(\psi) = \frac{-y(\lfloor\psi\rfloor,\psi) - \sqrt{\Delta(\lfloor\psi\rfloor,\psi)}}{2x(\lfloor\psi\rfloor,\psi)},$$

$$r_2(\psi) = \frac{n_{\lfloor\psi\rfloor}(1 - b(\lfloor\psi\rfloor,\lfloor\psi\rfloor)) - (n - n_{\lfloor\psi\rfloor})(b(\lfloor\psi\rfloor,\lfloor\psi\rfloor) - \frac{3}{4})}{n(1 - b(\lfloor\psi\rfloor,\lfloor\psi\rfloor))}$$

and

$$\rho_{mix}(\psi) = \begin{cases} r_1(\psi), \text{ if } \psi \notin \mathbb{N},\ \Delta(\lfloor\psi\rfloor,\psi) > 0,\ r_1(\psi) \in (C(\psi), 1) \\ \\ r_2(\psi), \text{ if } \psi \in \{2,3,4\},\ r_2(\psi) \in (C(\psi), 1) \\ \\ C(\psi) \quad \text{in other cases} \end{cases}$$

Let

$$(\hat{\psi}_{mix}, \hat{\rho}_{mix}) = \underset{(\psi,\rho)\in[1,5]\times[C(\psi),1]}{\arg\max} l_{mix}(\psi,\rho)$$

Then

$$\hat{\psi}_{mix} = \underset{\psi\in[1,5]}{\arg\max}\ l_{mix}(\psi, \rho_{mix}(\psi))$$

$$\hat{\rho}_{mix} = \rho_{mix}(\hat{\psi}_{mix}),$$

which can be calculated numerically using starting point

$$\psi_s = \frac{n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5}{n}.$$

Finally

$$(\hat{\psi}, \hat{\rho}) = \begin{cases} (\hat{\psi}_{BB}, \hat{\rho}_{BB}) & \text{if} \quad l_{BB}(\hat{\alpha}, \hat{\beta}) > l_{mix}(\hat{\psi}_{mix}, \hat{\rho}_{mix}) \\ (\hat{\psi}_{mix}, \hat{\rho}_{mix}) & \text{in other case} \end{cases}$$

Notice that $(\hat{\psi}, \hat{\rho})$ belongs to the interior of the set $[1, 5] \times [0, 1]$ if and only if both conditions in Step 1 and Step 2 are not fulfilled. This is very convenient since in that case we do not have to check the edge. On the other hand if the condition in Step 1 or in Step 2 is fulfilled we now that $(\hat{\psi}, \hat{\rho})$ belongs to the edge of the set $[1, 5] \times [0, 1]$ and we know immediately the solution.