Part I

A First Course on General Relativity

By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole

A person being able to master GR has the best advantage in analyzing business and technical problems right, which is why I study GR and Physics in general (Sorry, Einstein)

Chapter 1

Special Relativity

1.1 Einstein's Original Paper on "Special Relativity"

This section is not from the book, but simply my extra interests on the history of SR

The original paper is in The Collected Papers of Albert Einstein, Vol.2, page 140, *On the Electrodynamics of Moving Bodies*

Reading the original paper requires the prerequisites of

- Michelson-Morley Experiment
  - An excellent experiment intro
  - What I care most about this experiment is the **way we handle "unsolvable" problems**. Michelson-Morley experiment had led to extensive followups trying to explain what was seen in the experiment. All the *mediocre* conclusion simply said: "Dude, we don't know." Albert Einstein innovated a new era of Physics out of this conflict. **When a problem seems to lead to a dead end, it's time to innovate; it's time to take on the risk and bring the human into a new world of new opportunities!**
- Maxwell's Electrodynamics

Does the Electromagnetic Field *physically* exist?

"There exists a model of the universe which includes a field known as the Electromagnetic Field. This model does a remarkably good job of predicting the observations we make in the world. It does so good at making such predictions that it is often phrased as 'existing in the world"[a]

[a] https://philosophy.stackexchange.com/a/28010

Reading Notes of Paper:

Definition of "Simultaneity"

If an event occurs at (t, x, y, z), then all observers would see this event at (t, x', y', z'), where $x \neq x'$, $y \neq y'$, and $y \neq y'$

This definition is ideal but proves to be inefficient if we are going to look at a series of events happening one after another, according to Einstein, because light takes time to travel. But two clocks can *synchronize* in the following way:

Suppose an event occurs at A and a ray of light leaves from A toward B at t_A; the light is reflected from B towards A at t_B, and arrives back at A at t'_A. The two clocks at A and B satisfies

$$t_B - t_A = t'_A - t_B$$

which means

$$t_B = \frac{t'_A + t_A}{2} \tag{1.1}$$

Imagine a person holding a watch and manages to precisely record t_A and t'_A, they will be able to state with perfect confidence that any other person (or observer) at arbitrary location B sees their event at time t_B, which can be calculated by Eq.1.1, where both t_A and t'_A can be read at that person's hand watch

Definition of "Synchronism"

Suppose a ray of light leaves from A toward B at "A-time" at t_A, is reflected from B toward A at "B-time" t_B, and arrives back at A at "A-time" t'_A. The two clocks are *synchronous* by defintion if

$$t_B - t_A = t'_A - t_B \tag{1.2}$$

If follows naturally that

1. If the clock in B is synchronous with the clock in A, then the clock in A is synchronous with the clock in B
2. If the clock in A is synchronous with the clock in B as well as with the clock in C, then the clocks in B and C are also synchronous relative to each other

The **speed of light as a universal constant in empty space is thus**:

$$c = \frac{2\overline{AB}}{t'_A - t_A} \tag{1.3}$$

1.2 Fundamental principles of special relativity theory (SR)

1.2.1 On "Principle of relativity (Galileo)"

Galilean invariance

Newton's laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).[1] The proof has been given by the book on page 2.

[1] Galilean invariance

1.3 Construction of the coordinates used by another observer

1.3.1 Why would the tangent of the angle is the speed in Fig. 1.2?

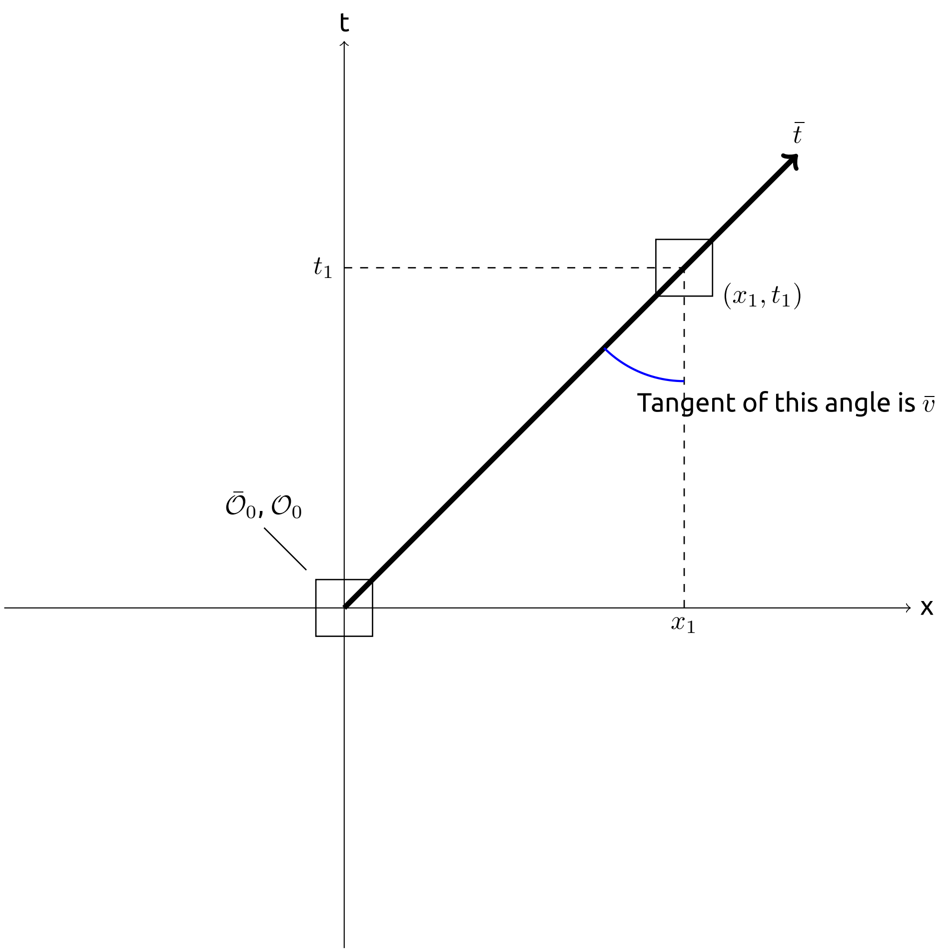
Suppose $\mathcal{O}$ and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1, observer $\mathcal{O}$ sees $\bar{\mathcal{O}}$ at position x_1:

$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themself at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where "$\bar{t}$ is the locus of events at constant $\bar{x} = 0$", $\bar{t}$ is the straight line that passes the origin and the (x_1, t_1):



1.4 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $\boldsymbol{M}_{\alpha\beta} + \boldsymbol{M}_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$?

$$\Delta\bar{s}^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^\alpha\right)\left(\Delta x^\beta\right)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$\boldsymbol{M}_{\alpha^*\beta^*} \left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)$$
$$\boldsymbol{M}_{\beta^*\alpha^*} \left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

Since

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right) = \left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

We can then group these 2 terms and factor out the product, leaving

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)\left(\boldsymbol{M}_{\alpha^*\beta^*} + \boldsymbol{M}_{\beta^*\alpha^*}\right)$$

The terms of expanded $\Delta\bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix}
\boldsymbol{M}_{00}\Delta x^0\Delta x^0 & \boldsymbol{M}_{01}\Delta x^0\Delta x^1 & \boldsymbol{M}_{02}\Delta x^0\Delta x^2 & \boldsymbol{M}_{03}\Delta x^0\Delta x^3 \\
\boldsymbol{M}_{10}\Delta x^1\Delta x^0 & \boldsymbol{M}_{11}\Delta x^1\Delta x^1 & \boldsymbol{M}_{12}\Delta x^1\Delta x^2 & \boldsymbol{M}_{13}\Delta x^1\Delta x^3 \\
\boldsymbol{M}_{20}\Delta x^2\Delta x^0 & \boldsymbol{M}_{21}\Delta x^2\Delta x^1 & \boldsymbol{M}_{22}\Delta x^2\Delta x^2 & \boldsymbol{M}_{23}\Delta x^2\Delta x^3 \\
\boldsymbol{M}_{30}\Delta x^3\Delta x^0 & \boldsymbol{M}_{31}\Delta x^3\Delta x^1 & \boldsymbol{M}_{32}\Delta x^3\Delta x^2 & \boldsymbol{M}_{33}\Delta x^3\Delta x^3
\end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$\boldsymbol{M}_{\alpha^*\beta^*} = \boldsymbol{M}_{\beta^*\alpha^*} = \frac{(\boldsymbol{M}_{\alpha^*\beta^*} + \boldsymbol{M}_{\beta^*\alpha^*})}{2}$$

where $\alpha^* \neq \beta^*$. And since $\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$ if $\alpha = \beta$, we conclude that

$\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$ for all α and β

Why do we have the 2nd term in equation

$$\Delta\bar{s}^2 = \boldsymbol{M}_{00}(\Delta r)^2 + \boxed{2\left(\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta x^i\right)\Delta r} + \sum_{i=1}^{3}\sum_{i=1}^{3}\boldsymbol{M}_{ij}\Delta x^i\Delta x^j$$

$$\Delta\bar{s}^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.4}$$

$$= \sum_{\alpha=0}^{0}\sum_{\beta=0}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) + \sum_{\alpha=0}^{3}\sum_{\beta=0}^{0}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.5}$$

$$= \sum_{\beta=0}^{3}\boldsymbol{M}_{0\beta}\Delta t(\Delta x^\beta) + \sum_{\alpha=0}^{3}\boldsymbol{M}_{\alpha 0}(\Delta x^\alpha)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.6}$$

$$= \boldsymbol{M}_{00}(\Delta t)^2 + \sum_{\beta=1}^{3}\boldsymbol{M}_{0\beta}\Delta t(\Delta x^\beta) + \sum_{\alpha=1}^{3}\boldsymbol{M}_{\alpha 0}(\Delta x^\alpha)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.7}$$

$$= \boldsymbol{M}_{00}(\Delta t)^2 + \boxed{2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t(\Delta x^i)\right]} + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}(\Delta x^\alpha)(\Delta x^\beta) \tag{1.8}$$

Why would $\boldsymbol{M}_{0i} = 0$ for $i = 1, 2, 3$ and $\boldsymbol{M}_{ij} = -\boldsymbol{M}_{00}\delta_{ij}$ in Equation 1.8?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes $\Delta x_1 = -\Delta x_2$ to simplify the equation 1.13. This is not sufficient, because what if $\Delta x_1 \neq -\Delta x_2$? This box takes a general approach where we **do not assume any relationship between Δx_1 and Δx_2**

Note that this statement is based on the aforementioned assumption that $\Delta\bar{s}^2 = \Delta s^2 = 0$, which has been proved here. Therefore, by 1.8, we have

$$\Delta\bar{s}^2(\Delta t, \Delta x_1) - \Delta\bar{s}^2(\Delta t, \Delta x_2) \tag{1.9}$$

$$= \boldsymbol{M}_{00}(\Delta t)^2 + 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.10}$$

$$= 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) -$$

$$2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) \tag{1.11}$$

$$= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) +$$

$$2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] - 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] \tag{1.12}$$

$$= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left[\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right)\right] + 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i - \Delta x_2^i\right)\right] = 0 \tag{1.13}$$

We won't be able to go further unless with some assumed relationships between Δx_1^i and Δx_2^i. But since we do not assume any relations between them, let's step back and re-think about this problem then and forget about Δx_1^i and Δx_2^i.

We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta\bar{s}^2$. The detail is about Δx_1^i and Δx_2^i but the goal is to derive some form of

$$\Delta\bar{s}^2 = f\left(\Delta s^2\right) = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^{3}(\Delta x^i)^2$$

Let's work on $f\left(\Delta s^2\right)$ directly toward that goal then

Assuming $\Delta\bar{s}^2 = \Delta s^2 = 0$, we have $\Delta t = \pm\Delta x$; plugging it into Eq. 1.8 gives us

$$\Delta\bar{s}^2 = \boldsymbol{M}_{00}\sum_{i=1}^{3}\left(\Delta x^i\right)^2 + 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\left(\Delta x^i\right)^2\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.14}$$

$$= \left(\boldsymbol{M}_{00} + 2\boldsymbol{M}_{0i}\right)\sum_{i=1}^{3}\left(\Delta x^i\right)^2 + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.15}$$

Eq.1.15 seems to suggest a linear relationship between Δs^2 and $\Delta\bar{s}^2$. How do we go about proving it? We now start the formal proof of **Invariance of Interval**

Theorem 1.4.1

Let $n, p \geq 1$ be integers, $d := n+p$ and V a vector space over $\mathbb{R}$ of dimension d. Let h be an indefinite-inner product[a] on V with signature type (n, p)[b]. Suppose g is a symmetric bilinear form on V such that the null set of the associated quadratic form of h is contained in that of g (i.e. suppose that for every $v \in V$, if $h(v, v) = 0$ then $g(v, v) = 0$). Then, there exists a constant $C \in \mathbb{R}$ such that $g = Ch$. Futhermore, if we assume $n \neq p$ and that g also has signature type (n, p), then we have $C > 0$

[a] Start with page 447 of Introduction to Linear Algebra, 4th Edition for everything you need to know about indefinite-inner product

[b] Read Ch. 6 of Introduction to Linear Algebra, 4th Edition and then Matrix Signature