Part I

A First Course on General Relativity

By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole

A person being able to master GR has the best advantage in analyzing business and tech problems right, which is why I study GR and Physics in general (Sorry, Einstein)

Chapter 1

Special Relativity

On “Principle of relativity (Galileo)”

Galilean invariance

Newton’s laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton’s equation of motion is valid in these frames).[1] The proof has been given by the book on page 2.

1.5 - Construction of the coordinates used by another observer

Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose $\mathcal{O}$ and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1, observer $\mathcal{O}$ sees $\bar{\mathcal{O}}$ at position x_1:

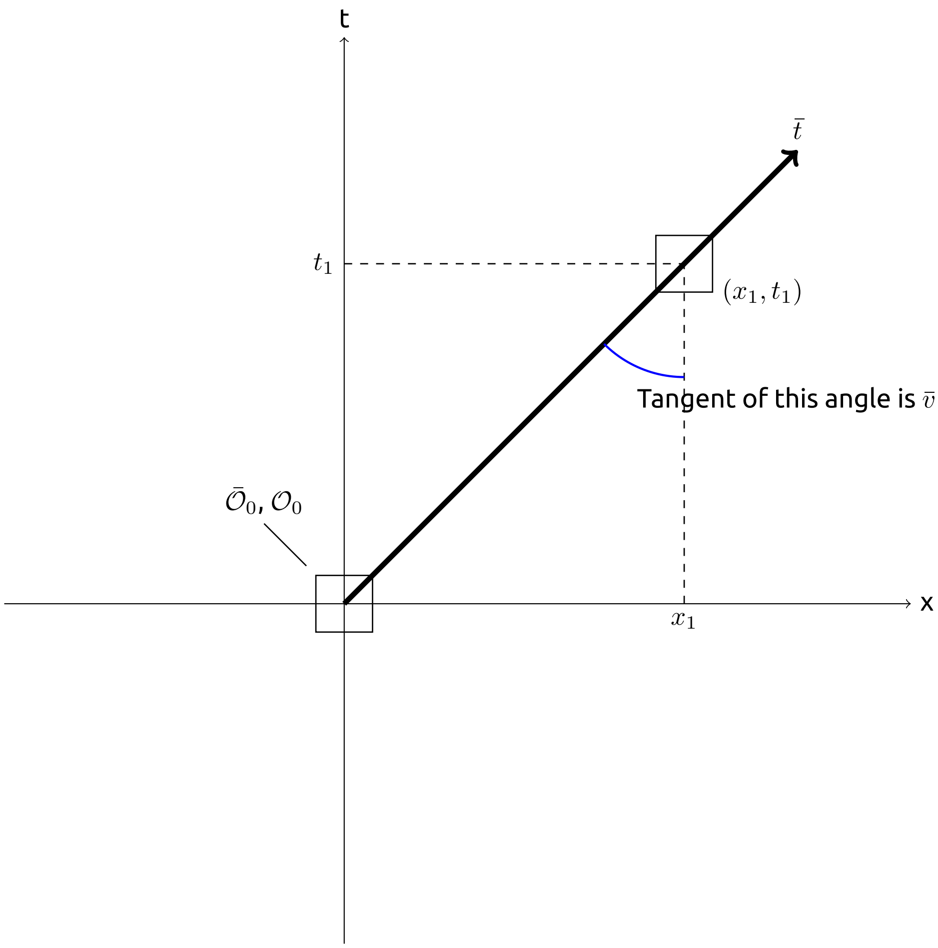
$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themself at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where “$\bar{t}$ is the locus of events at constant $\bar{x} = 0$”, $\bar{t}$ is the straight line that passes the origin and the (x_1, t_1):

[1] Galilean invariance



1.6 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $M_{\alpha\beta} + M_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $M_{\alpha\beta} = M_{\beta\alpha}$?

$$\Delta\bar{s}^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} M_{\alpha\beta} \left(\Delta x^\alpha\right)\left(\Delta x^\beta\right)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$M_{\alpha^*\beta^*}\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)$$

$$M_{\beta^*\alpha^*}\left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

Since

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)=\left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

We can then group these 2 terms and factor out the product, leaving

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)\left(M_{\alpha^*\beta^*}+M_{\beta^*\alpha^*}\right)$$

The terms of expanded $\Delta\bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix}
M_{00}\Delta x^0\Delta x^0 & M_{01}\Delta x^0\Delta x^1 & M_{02}\Delta x^0\Delta x^2 & M_{03}\Delta x^0\Delta x^3 \\
M_{10}\Delta x^1\Delta x^0 & M_{11}\Delta x^1\Delta x^1 & M_{12}\Delta x^1\Delta x^2 & M_{13}\Delta x^1\Delta x^3 \\
M_{20}\Delta x^2\Delta x^0 & M_{21}\Delta x^2\Delta x^1 & M_{22}\Delta x^2\Delta x^2 & M_{23}\Delta x^2\Delta x^3 \\
M_{30}\Delta x^3\Delta x^0 & M_{31}\Delta x^3\Delta x^1 & M_{32}\Delta x^3\Delta x^2 & M_{33}\Delta x^3\Delta x^3
\end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$M_{\alpha^*\beta^*}=M_{\beta^*\alpha^*}=\frac{\left(M_{\alpha^*\beta^*}+M_{\beta^*\alpha^*}\right)}{2}$$

where $\alpha^*\neq\beta^*$. And since $M_{\alpha\beta}=M_{\beta\alpha}$ if $\alpha=\beta$, we conclude that

$M_{\alpha\beta}=M_{\beta\alpha}$ for all α and β

Why do we have the 2nd term in equation

$$\Delta\bar{s}^2=M_{00}\left(\Delta r\right)^2+\boxed{2\left(\sum_{i=1}^{3}M_{0i}\Delta x^i\right)\Delta r}+\sum_{i=1}^{3}\sum_{i=1}^{3}M_{ij}\Delta x^i\Delta x^j$$

$$\Delta\bar{s}^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.1}$$

$$= \sum_{\alpha=0}^{0}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) + \sum_{\alpha=0}^{3}\sum_{\beta=0}^{0} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.2}$$

$$= \sum_{\beta=0}^{3} \boldsymbol{M}_{0\beta}\Delta t\left(\Delta x^\beta\right) + \sum_{\alpha=0}^{3} \boldsymbol{M}_{\alpha 0}\left(\Delta x^\alpha\right)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.3}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + \sum_{\beta=1}^{3} \boldsymbol{M}_{0\beta}\Delta t\left(\Delta x^\beta\right) + \sum_{\alpha=1}^{3} \boldsymbol{M}_{\alpha 0}\left(\Delta x^\alpha\right)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.4}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + \boxed{2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x^i\right)\right]} + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.5}$$

Why would $M_{0i} = 0$ for $i = 1, 2, 3$ and $M_{ij} = -M_{00}\delta_{ij}$ in Equation 1.5?

Note that this statement is based on the aforementioned assumption that $\Delta\bar{s}^2 = \Delta s^2 = 0$, which has been proved here. Therefore, by 1.5, we have

$$\Delta\bar{s}^2(\Delta t, \Delta x_1) - \Delta\bar{s}^2(\Delta t, \Delta x_2) \tag{1.6}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.7}$$

$$= 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) \tag{1.8}$$

$$= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] - 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] \tag{1.9}$$

$$= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left[\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right)\right] + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i - \Delta x_2^i\right)\right] \tag{1.10}$$

We won't be able to go further unless with some assumed relationships between Δx_1^i and Δx_2^i. Let's step back and re-think about this problem then.
We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta\bar{s}^2$. So the details is about Δx_1^i and Δx_2^i but the goal is to derive some form of $\Delta\bar{s}^2 = f\left(\Delta s^2\right)$. Let work on 1.10 directly toward that goal then