Part I

A First Course on General Relativity

By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole

A person being able to master GR has the best advantage in analyzing business and technical problems right, which is why I study GR and Physics in general (Sorry, Einstein)

Chapter 1

Special Relativity

On "Principle of relativity (Galileo)"

Galilean invariance

Newton's laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).[1] The proof has been given by the book on page 2.

1.5 - Construction of the coordinates used by another observer

Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose $\mathcal{O}$ and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1, observer $\mathcal{O}$ sees $\bar{\mathcal{O}}$ at position x_1:

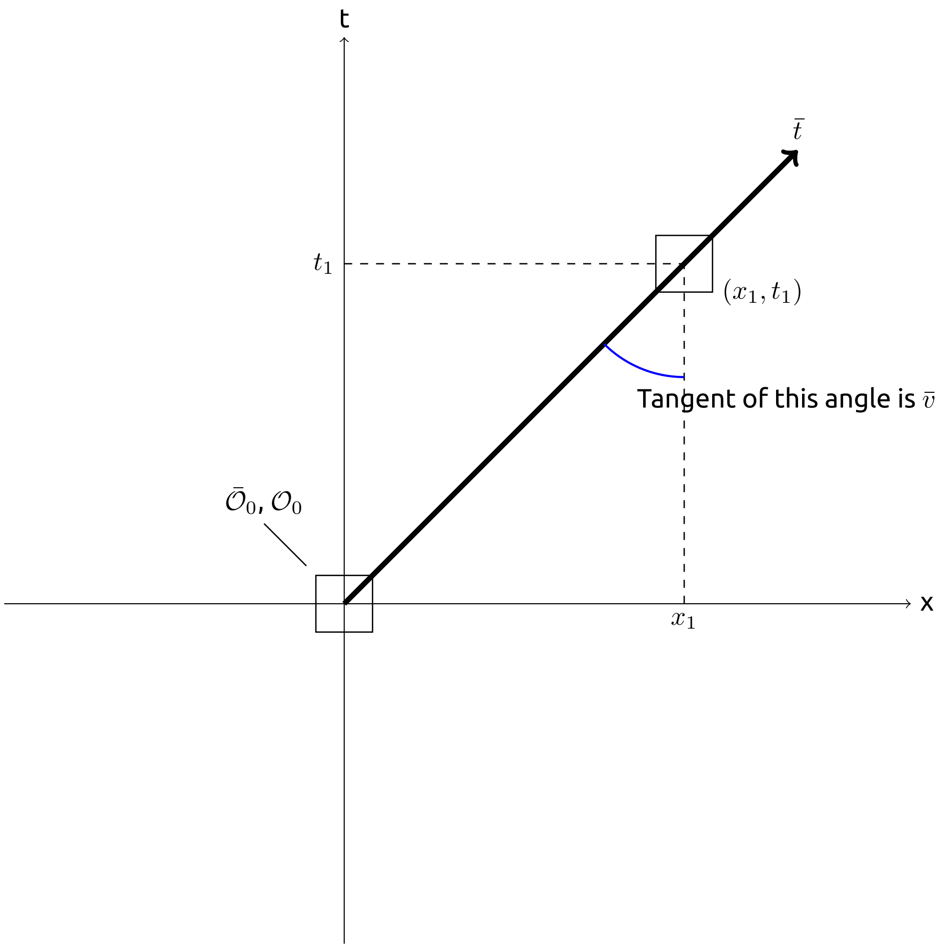
$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themself at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where "$\bar{t}$ is the locus of events at constant $\bar{x} = 0$", $\bar{t}$ is the straight line that passes the origin and the (x_1, t_1):

[1] Galilean invariance



1.6 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $M_{\alpha\beta} + M_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $M_{\alpha\beta} = M_{\beta\alpha}$?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} M_{\alpha\beta} \left(\Delta x^\alpha\right) \left(\Delta x^\beta\right)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$\boldsymbol{M}_{\alpha^*\beta^*}\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)$$

$$\boldsymbol{M}_{\beta^*\alpha^*}\left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

Since

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right) = \left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

We can then group these 2 terms and factor out the product, leaving

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)\left(\boldsymbol{M}_{\alpha^*\beta^*} + \boldsymbol{M}_{\beta^*\alpha^*}\right)$$

The terms of expanded $\Delta\bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix} \boldsymbol{M}_{00}\Delta x^0\Delta x^0 & \boldsymbol{M}_{01}\Delta x^0\Delta x^1 & \boldsymbol{M}_{02}\Delta x^0\Delta x^2 & \boldsymbol{M}_{03}\Delta x^0\Delta x^3 \\ \boldsymbol{M}_{10}\Delta x^1\Delta x^0 & \boldsymbol{M}_{11}\Delta x^1\Delta x^1 & \boldsymbol{M}_{12}\Delta x^1\Delta x^2 & \boldsymbol{M}_{13}\Delta x^1\Delta x^3 \\ \boldsymbol{M}_{20}\Delta x^2\Delta x^0 & \boldsymbol{M}_{21}\Delta x^2\Delta x^1 & \boldsymbol{M}_{22}\Delta x^2\Delta x^2 & \boldsymbol{M}_{23}\Delta x^2\Delta x^3 \\ \boldsymbol{M}_{30}\Delta x^3\Delta x^0 & \boldsymbol{M}_{31}\Delta x^3\Delta x^1 & \boldsymbol{M}_{32}\Delta x^3\Delta x^2 & \boldsymbol{M}_{33}\Delta x^3\Delta x^3 \end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$\boldsymbol{M}_{\alpha^*\beta^*} = \boldsymbol{M}_{\beta^*\alpha^*} = \frac{\left(\boldsymbol{M}_{\alpha^*\beta^*} + \boldsymbol{M}_{\beta^*\alpha^*}\right)}{2}$$

where $\alpha^* \neq \beta^*$. And since $\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$ if $\alpha = \beta$, we conclude that

$\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$ for all α and β

Why do we have the 2nd term in equation

$$\Delta\bar{s}^2 = \boldsymbol{M}_{00}\left(\Delta r\right)^2 + \boxed{2\left(\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta x^i\right)\Delta r} + \sum_{i=1}^{3}\sum_{i=1}^{3}\boldsymbol{M}_{ij}\Delta x^i\Delta x^j$$

$$\Delta\bar{s}^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.1}$$

$$= \sum_{\alpha=0}^{0}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) + \sum_{\alpha=0}^{3}\sum_{\beta=0}^{0} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.2}$$

$$= \sum_{\beta=0}^{3} \boldsymbol{M}_{0\beta}\Delta t\left(\Delta x^\beta\right) + \sum_{\alpha=0}^{3} \boldsymbol{M}_{\alpha 0}\left(\Delta x^\alpha\right)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.3}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + \sum_{\beta=1}^{3} \boldsymbol{M}_{0\beta}\Delta t\left(\Delta x^\beta\right) + \sum_{\alpha=1}^{3} \boldsymbol{M}_{\alpha 0}\left(\Delta x^\alpha\right)\Delta t + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.4}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + \boxed{2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x^i\right)\right]} + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.5}$$

Why would $\boldsymbol{M}_{0i} = 0$ for $i = 1, 2, 3$ and $\boldsymbol{M}_{ij} = -\boldsymbol{M}_{00}\delta_{ij}$ in Equation 1.5?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes $\Delta x_1 = -\Delta x_2$ to simplify the equation 1.10. This is not sufficient! What if $\Delta x_1 \neq -\Delta x_2$? This box takes a general approach where we **do not assume any relationship between** Δx_1 **and** Δx_2

Note that this statement is based on the aforementioned assumption that $\Delta\bar{s}^2 = \Delta s^2 = 0$, which has been proved here. Therefore, by 1.5, we have

$$\Delta\bar{s}^2(\Delta t, \Delta x_1) - \Delta\bar{s}^2(\Delta t, \Delta x_2) \tag{1.6}$$

$$= \boldsymbol{M}_{00}\left(\Delta t\right)^2 + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.7}$$

$$\begin{aligned} &= 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \\ &\quad 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) \end{aligned} \tag{1.8}$$

$$\begin{aligned} &= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right) + \\ &\quad 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i\right)\right] - 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_2^i\right)\right] \end{aligned} \tag{1.9}$$

$$= \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left[\left(\Delta x_1^\alpha\right)\left(\Delta x_1^\beta\right) - \left(\Delta x_2^\alpha\right)\left(\Delta x_2^\beta\right)\right] + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i}\Delta t\left(\Delta x_1^i - \Delta x_2^i\right)\right] = 0 \tag{1.10}$$

We won't be able to go further unless with some assumed relationships between Δx_1^i and Δx_2^i. But since we do not assume any relations between them, let's step back and re-think about this problem then and forget about Δx_1^i and Δx_2^i.

We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta \bar{s}^2$. The detail is about Δx_1^i and Δx_2^i but the goal is to derive some form of

$$\Delta \bar{s}^2 = f\left(\Delta s^2\right) = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^{3}(\Delta x^i)^2$$

Let's work on $f\left(\Delta s^2\right)$ directly toward that goal then

Assuming $\Delta \bar{s}^2 = \Delta s^2 = 0$, we have $\Delta t = \pm \Delta x$; plugging it into Eq. 1.5 gives us

$$\Delta \bar{s}^2 = \boldsymbol{M}_{00}\sum_{i=1}^{3}\left(\Delta x^i\right)^2 + 2\left[\sum_{i=1}^{3}\boldsymbol{M}_{0i}\left(\Delta x^i\right)^2\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.11}$$

$$= \left(\boldsymbol{M}_{00} + 2\boldsymbol{M}_{0i}\right)\sum_{i=1}^{3}\left(\Delta x^i\right)^2 + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}\boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.12}$$

Eq.1.12 seems to suggest a linear relationship between $\Delta \bar{s}^2$ and $\Delta \bar{s}^2$. How do we go about proving it? We now start the formal proof of **Invariance of Interval**

Theorem 1.0.1

Let $n, p \geq 1$ be integers, $d := n + p$ and V a vector space over $\mathbb{R}$ of dimension d. Let h be an indefinite-inner product on V with signature type (n, p). Suppose g is a symmetric bilinear form on V such that the null set of the associated quadratic form of h is contained in that of g (i.e. suppose that for every $v \in V$, if $h(v, v) = 0$ then $g(v, v) = 0$). Then, there exists a constant $C \in \mathbb{R}$ such that $g = Ch$. Futhermore, if we assume $n \neq p$ and that g also has signature type (n, p), then we have $C > 0$