

## **Part I**

# **A First Course on General Relativity**

**By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole**

***A person being able to master GR has the best advantage in analyzing business and tech problems right, which is why I study GR and Physics in general (Sorry, Einstein)***



# Chapter 1

## Special Relativity

### On “Principle of relativity (Galileo)”

#### Galilean invariance

Newton’s laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton’s equation of motion is valid in these frames).<sup>1</sup> The proof has been given by the book on page 2.

### 1.5 - Construction of the coordinates used by another observer

#### Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose  $\mathcal{O}$  and  $\bar{\mathcal{O}}$  both start out at the same position where  $\bar{\mathcal{O}}$  moves along the  $x$  at some speed. After  $t_1$ , observer  $\mathcal{O}$  sees  $\bar{\mathcal{O}}$  at position  $x_1$ :

$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer  $\bar{\mathcal{O}}$ , however, still sees themselves at  $x = 0$ :

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where “ $\bar{t}$  is the locus of events at constant  $\bar{x} = 0$ ”,  $\bar{t}$  is the straight line that passes the origin and the  $(x_1, t_1)$ :

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<sup>1</sup>Galilean invariance



## 1.6 Invariance of the interval

Why does the equation contains only  $M_{\alpha\beta} + M_{\beta\alpha}$  terms when  $\alpha \neq \beta$ , which guarantees  $M_{\alpha\beta} = M_{\beta\alpha}$ ?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of  $(\alpha, \beta) = (\alpha^*, \beta^*)$  where  $\alpha^* \neq \beta^*$ . Then we would definitely have the following 2 terms in the expansion:

$$M_{\alpha^*\beta^*} (\Delta x^{\alpha^*}) (\Delta x^{\beta^*})$$

$$M_{\beta^*\alpha^*} (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

Since

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) = (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

We can then group these 2 terms and factor out the product, leaving

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)\left(M_{\alpha^*\beta^*}+M_{\beta^*\alpha^*}\right)$$

The terms of expanded  $\Delta \bar{s}^2$  can be expressed in a matrix of

$$\begin{bmatrix} M_{00}\Delta x^0\Delta x^0 & M_{01}\Delta x^0\Delta x^1 & M_{02}\Delta x^0\Delta x^2 & M_{03}\Delta x^0\Delta x^3 \\ M_{10}\Delta x^1\Delta x^0 & M_{11}\Delta x^1\Delta x^1 & M_{12}\Delta x^1\Delta x^2 & M_{13}\Delta x^1\Delta x^3 \\ M_{20}\Delta x^2\Delta x^0 & M_{21}\Delta x^2\Delta x^1 & M_{22}\Delta x^2\Delta x^2 & M_{23}\Delta x^2\Delta x^3 \\ M_{30}\Delta x^3\Delta x^0 & M_{31}\Delta x^3\Delta x^1 & M_{32}\Delta x^3\Delta x^2 & M_{33}\Delta x^3\Delta x^3 \end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$M_{\alpha^*\beta^*} = M_{\beta^*\alpha^*} = \frac{(M_{\alpha^*\beta^*} + M_{\beta^*\alpha^*})}{2}$$

where  $\alpha^* \neq \beta^*$ . And since  $M_{\alpha\beta} = M_{\beta\alpha}$  if  $\alpha = \beta$ , we conclude that

$$M_{\alpha\beta} = M_{\beta\alpha} \text{ for all } \alpha \text{ and } \beta$$

Why do we have the 2nd term in equation

$$\Delta \bar{s}^2 = M_{00}(\Delta r)^2 + 2\left(\sum_{i=1}^3 M_{0i}\Delta x^i\right)\Delta r + \sum_{i=1}^3 \sum_{j=1}^3 M_{ij}\Delta x^i\Delta x^j$$

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.1)$$

$$= \sum_{\alpha=0}^0 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=0}^3 \sum_{\beta=0}^0 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.2)$$

$$= \sum_{\beta=0}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=0}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.3)$$

$$= M_{00} (\Delta t)^2 + \sum_{\beta=1}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=1}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.4)$$

$$= M_{00} (\Delta t)^2 + 2\left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i)\right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.5)$$

**1.6 - Why  $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$  for two events in the same light beam?**

Let's say, in a simplified 1D case, event  $\mathcal{E} = (x_0, t_0)$  and  $\mathcal{P} = (x_1, t_1)$ .

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$