Contents

I   A First Course on General Relativity

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5 CHAPTER 1
Special Relativity

Part I

A First Course on General Relativity

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Chapter 1

Special Relativity

1.1 Einstein's Original Paper on "Special Relativity"

> ⚠ This section is not from the book, but simply my extra interests on the history of SR

> The original paper is in The Collected Papers of Albert Einstein, Vol.2, page 140, *On the Electrodynamics of Moving Bodies*

Reading the original paper requires the prerequisites of

- Michelson-Morley Experiment

  - An excellent experiment intro

  - What I care most about this experiment is the **way we handle "unsolvable" problems**. Michelson-Morley experiment had led to extensive followups trying to explain what was seen in the experiment. All the *mediocre* conclusion simply said: "Dude, we don't know." Albert Einstein innovated a new era of Physics out of this conflict. **When a problem seems to lead to a dead end, it's time to innovate; it's time to take on the risk and bring the human into a new world of new opportunities!**

- Maxwell's Electrodynamics

> **Does the Electromagnetic Field *physically* exist?**
>
> "There exists a model of the universe which includes a field known as the Electromagnetic Field. This model does a remarkably good job of predicting the observations we make in the world. It does so good at making such predictions that it is often phrased as 'existing in the world"[a]
>
> ───────────────
> [a] https://philosophy.stackexchange.com/a/28010

1.1.1 Reading Notes...

...of the Paper

§1. Definition of Simultaneity

Definition 1.1.1.1: Simultaneity

If an event occurs at (t, x, y, z), then all observers would see this event at (t, x', y', z'), where $x \neq x'$, $y \neq y'$, and $y \neq y'$

This definition is ideal but proves to be inefficient if we are going to look at a series of events happening one after another, according to Einstein, because light takes time to travel. But two clocks can *synchronize* in the following way:

Suppose an event occurs at A and a ray of light leaves from A toward B at t_A; the light is reflected from B towards A at t_B, and arrives back at A at t'_A. The two clocks at A and B satisfies

$$t_B - t_A = t'_A - t_B$$

which means

$$t_B = \frac{t'_A + t_A}{2} \tag{1.1}$$

Imagine a person holding a watch and manages to precisely record t_A and t'_A, they will be able to state with perfect confidence that any other person (or observer) at arbitrary location B sees their event at time t_B, which can be calculated by Eq.1.1, where both t_A and t'_A can be read at that person's hand watch

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Definition 1.1.1.2: Synchronism

Suppose a ray of light leaves from A toward B at "A-time" at t_A, is reflected from B toward A at "B-time" t_B, and arrives back at A at "A-time" t'_A. The two clocks are synchronous by defintion if

$$t_B - t_A = t'_A - t_B \tag{1.2}$$

If follows naturally that

1. If the clock in B is synchronous with the clock in A, then the clock in A is synchronous with the clock in B

2. If the clock in A is synchronous with the clock in B as well as with the clock in C, then the clocks in B and C are also synchronous relative to each other

The **speed of light as a universal constant in empty space is thus**:

$$c = \frac{2\overline{AB}}{t'_A - t_A} \tag{1.3}$$

and there is a BIG assumption: all clocks are at *rest* in a system at *rest*

With that, we have a pretty good mechanism to talk about series of events in a system happening at different

time, because we know how to synchronize them

§2. On the relativity of lengths and times

> **Principle of Relativity**
>
> The laws governing the changes of the state of any physical system do not depend on which one of two coordinate systems in uniform translational motion relative to each other these changes of the state are referred to

> **Principle of the Constancy of the Velocity of Light**
>
> Each ray of light moves in the coordinate system "at rest" with the definite velocity V independent of whether this ray of light is emitted by a body at rest or a body in motion

1.2 Fundamental principles of special relativity theory (SR)

1.2.1 On "Principle of relativity (Galileo)"

Galilean invariance

Newton's laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).[1] The proof has been given by the book on page 2.

1.3 Construction of the coordinates used by another observer

1.3.1 Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose $\mathcal{O}$ and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1, observer $\mathcal{O}$ sees $\bar{\mathcal{O}}$ at position x_1:
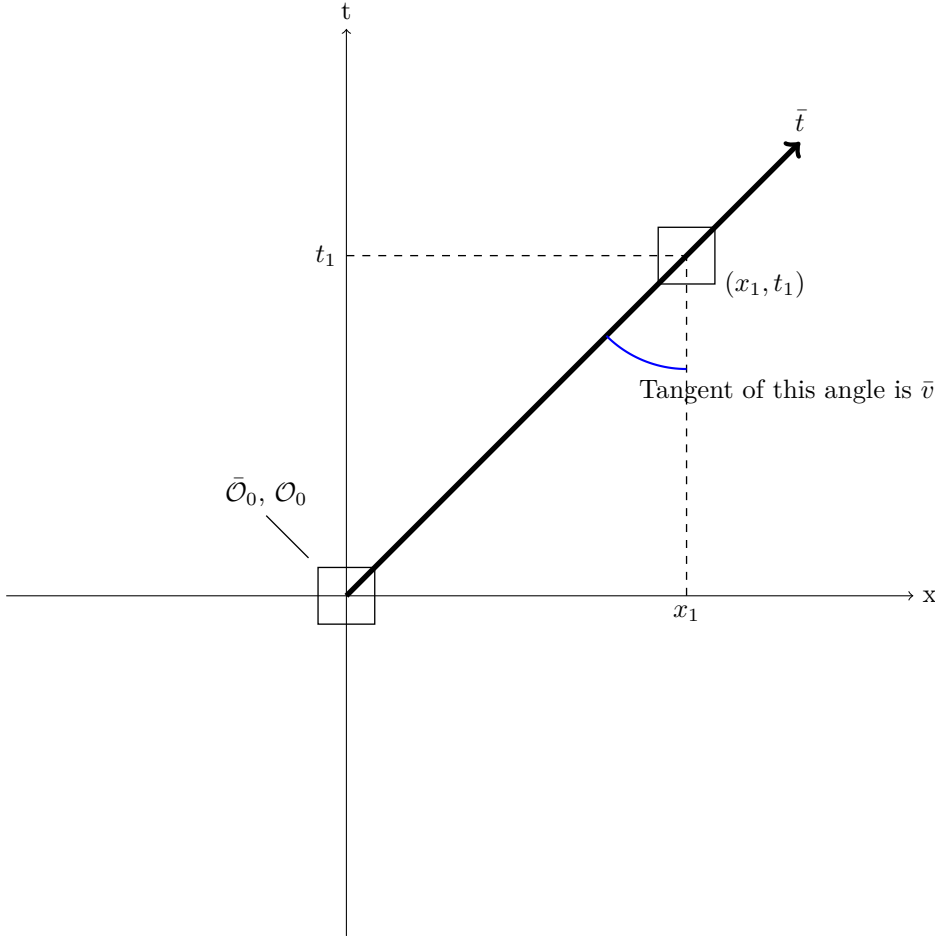
$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themselves at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where "$\bar{t}$ is the locus of events at constant $\bar{x} = 0$", $\bar{t}$ is the straight line that passes the origin and the (x_1, t_1):

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[1]Galilean invariance

1.4 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $\boldsymbol{M}_{\alpha\beta} + \boldsymbol{M}_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $\boldsymbol{M}_{\alpha\beta} = \boldsymbol{M}_{\beta\alpha}$?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha} \right) \left(\Delta x^{\beta} \right)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$\boldsymbol{M}_{\alpha^*\beta^*}\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)$$

$$\boldsymbol{M}_{\beta^*\alpha^*}\left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

Since

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)=\left(\Delta x^{\beta^*}\right)\left(\Delta x^{\alpha^*}\right)$$

We can then group these 2 terms and factor out the product, leaving

$$\left(\Delta x^{\alpha^*}\right)\left(\Delta x^{\beta^*}\right)\left(\boldsymbol{M}_{\alpha^*\beta^*}+\boldsymbol{M}_{\beta^*\alpha^*}\right)$$

The terms of expanded $\Delta\bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix}
\boldsymbol{M}_{00}\Delta x^0\Delta x^0 & \boldsymbol{M}_{01}\Delta x^0\Delta x^1 & \boldsymbol{M}_{02}\Delta x^0\Delta x^2 & \boldsymbol{M}_{03}\Delta x^0\Delta x^3 \\
\boldsymbol{M}_{10}\Delta x^1\Delta x^0 & \boldsymbol{M}_{11}\Delta x^1\Delta x^1 & \boldsymbol{M}_{12}\Delta x^1\Delta x^2 & \boldsymbol{M}_{13}\Delta x^1\Delta x^3 \\
\boldsymbol{M}_{20}\Delta x^2\Delta x^0 & \boldsymbol{M}_{21}\Delta x^2\Delta x^1 & \boldsymbol{M}_{22}\Delta x^2\Delta x^2 & \boldsymbol{M}_{23}\Delta x^2\Delta x^3 \\
\boldsymbol{M}_{30}\Delta x^3\Delta x^0 & \boldsymbol{M}_{31}\Delta x^3\Delta x^1 & \boldsymbol{M}_{32}\Delta x^3\Delta x^2 & \boldsymbol{M}_{33}\Delta x^3\Delta x^3
\end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$\boldsymbol{M}_{\alpha^*\beta^*}=\boldsymbol{M}_{\beta^*\alpha^*}=\frac{\left(\boldsymbol{M}_{\alpha^*\beta^*}+\boldsymbol{M}_{\beta^*\alpha^*}\right)}{2}$$

where $\alpha^*\neq\beta^*$. And since $\boldsymbol{M}_{\alpha\beta}=\boldsymbol{M}_{\beta\alpha}$ if $\alpha=\beta$, we conclude that

$$\boldsymbol{M}_{\alpha\beta}=\boldsymbol{M}_{\beta\alpha}\text{ for all }\alpha\text{ and }\beta$$

Why do we have the 2nd term in equation

$$\Delta\bar{s}^2=\boldsymbol{M}_{00}\left(\Delta r\right)^2+\boxed{2\left(\sum_{i=1}^{3}\boldsymbol{M}_{0i}\Delta x^i\right)\Delta r}+\sum_{i=1}^{3}\sum_{i=1}^{3}\boldsymbol{M}_{ij}\Delta x^i\Delta x^j$$

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.4}$$

$$= \sum_{\alpha=0}^{0} \sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) + \sum_{\alpha=0}^{3} \sum_{\beta=0}^{0} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.5}$$

$$= \sum_{\beta=0}^{3} \boldsymbol{M}_{0\beta} \Delta t \left(\Delta x^{\beta}\right) + \sum_{\alpha=0}^{3} \boldsymbol{M}_{\alpha 0} \left(\Delta x^{\alpha}\right) \Delta t + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.6}$$

$$= \boldsymbol{M}_{00} \left(\Delta t\right)^2 + \sum_{\beta=1}^{3} \boldsymbol{M}_{0\beta} \Delta t \left(\Delta x^{\beta}\right) + \sum_{\alpha=1}^{3} \boldsymbol{M}_{\alpha 0} \left(\Delta x^{\alpha}\right) \Delta t + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.7}$$

$$= \boldsymbol{M}_{00} \left(\Delta t\right)^2 + \boxed{2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x^{i}\right) \right]} + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.8}$$

Why would $\boldsymbol{M}_{0i} = 0$ for $i = 1, 2, 3$ and $\boldsymbol{M}_{ij} = -\boldsymbol{M}_{00}\delta_{ij}$ in Equation 1.8?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes $\Delta x_1 = -\Delta x_2$ to simplify the equation 1.13. This is not sufficient, because what if $\Delta x_1 \neq -\Delta x_2$? This box takes a general approach where we **do not assume any relationship between Δx_1 and Δx_2**

Note that this statement is based on the aforementioned assumption that $\Delta \bar{s}^2 = \Delta s^2 = 0$, which has been proved here. Therefore, by 1.8, we have

$$\Delta \bar{s}^2(\Delta t, \Delta x_1) - \Delta \bar{s}^2(\Delta t, \Delta x_2) \tag{1.9}$$

$$= \boldsymbol{M}_{00} \left(\Delta t\right)^2 + 2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x^{i}\right) \right] + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^{\alpha}\right) \left(\Delta x^{\beta}\right) \tag{1.10}$$

$$= 2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x_1^{i}\right) \right] + \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x_1^{\alpha}\right) \left(\Delta x_1^{\beta}\right) -$$

$$2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x_2^{i}\right) \right] - \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x_2^{\alpha}\right) \left(\Delta x_2^{\beta}\right) \tag{1.11}$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x_1^{\alpha}\right) \left(\Delta x_1^{\beta}\right) - \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x_2^{\alpha}\right) \left(\Delta x_2^{\beta}\right) +$$

$$2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x_1^{i}\right) \right] - 2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x_2^{i}\right) \right] \tag{1.12}$$

$$= \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta} \left[\left(\Delta x_1^{\alpha}\right) \left(\Delta x_1^{\beta}\right) - \left(\Delta x_2^{\alpha}\right) \left(\Delta x_2^{\beta}\right) \right] + 2 \left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \Delta t \left(\Delta x_1^{i} - \Delta x_2^{i}\right) \right] = 0 \tag{1.13}$$

We won't be able to go further unless with some assumed relationships between Δx_1^{i} and Δx_2^{i}. But since we do not assume any relations between them, let's step back and re-think about this problem then and forget about Δx_1^{i} and Δx_2^{i}.

We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta \bar{s}^2$. The (detail) is about Δx_1^i and Δx_2^i but the (goal) is to derive some form of

$$\Delta \bar{s}^2 = f\left(\Delta s^2\right) = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \boldsymbol{M}_{\alpha\beta} \left(\Delta x^\alpha\right) \left(\Delta x^\beta\right)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^{3} (\Delta x^i)^2$$

Let's work on $f\left(\Delta s^2\right)$ directly toward that goal then

Assuming $\Delta \bar{s}^2 = \Delta s^2 = 0$, we have $\Delta t = \pm\Delta x$; plugging it into Eq. 1.8 gives us

$$\Delta \bar{s}^2 = \boldsymbol{M}_{00} \sum_{i=1}^{3} \left(\Delta x^i\right)^2 + 2\left[\sum_{i=1}^{3} \boldsymbol{M}_{0i} \left(\Delta x^i\right)^2\right] + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.14}$$

$$= \left(\boldsymbol{M}_{00} + 2\boldsymbol{M}_{0i}\right)\sum_{i=1}^{3}\left(\Delta x^i\right)^2 + \sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} \boldsymbol{M}_{\alpha\beta}\left(\Delta x^\alpha\right)\left(\Delta x^\beta\right) \tag{1.15}$$

Eq.1.15 seems to suggest a linear relationship between Δs^2 and $\Delta \bar{s}^2$. How do we go about proving it? We now start the formal proof of **Invariance of Interval**

Theorem 1.4.1

Let $n, p \geq 1$ be integers, $d := n + p$ and V a vector space over $\mathbb{R}$ of dimension d. Let h be an indefinite-inner product[a] on V with signature type (n, p)[b]. Suppose g is a symmetric bilinear form on V such that the null set of the associated quadratic form of h is contained in that of g (i.e. suppose that for every $v \in V$, if $h(v, v) = 0$ then $g(v, v) = 0$). Then, there exists a constant $C \in \mathbb{R}$ such that $g = Ch$. Futhermore, if we assume $n \neq p$ and that g also has signature type (n, p), then we have $C > 0$

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[a]Start with page 447 of Introduction to Linear Algebra, 4th Edition for everything you need to know about indefinite-inner product

[b]Read Ch. 6 of Introduction to Linear Algebra, 4th Edition and then Matrix Signature