

Part I

A First Course on General Relativity

By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole

A person being able to master GR has the best advantage in analyzing business and technical problems right, which is why I study GR and Physics in general (Sorry, Einstein)

Chapter 1

Special Relativity

1.1 Einstein's Original Paper on "Special Relativity"



This section is not from the book, but simply my extra interests on the history of SR

The original paper is in [The Collected Papers of Albert Einstein, Vol.2](#), page 140, *On the Electrodynamics of Moving Bodies*

1.1.1 Michelson-Morley experiment

- [An excellent experiment intro](#)
- What I care most about this experiment is the **way we handle "unsolvable" problems**. Michelson-Morley experiment had led to extensive followups trying to explain what was seen in the experiment. All the *mediocre* conclusion simply said: "Dude, we don't know." Albert Einstein innovated a new era of Physics out of this conflict. **When a problem seems to lead to a dead end, it's time to innovate; it's time to take on the risk and bring the human into a new world of new opportunities!**

1.2 On "Principle of relativity (Galileo)"

1.2.1 Galilean invariance

[Newton's laws of motion](#) hold in all frames related to one another by a [Galilean transformation](#). In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).¹ The proof has been given by the book on page 2.

1.3 1.5 - Construction of the coordinates used by another observer

1.3.1 Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose \mathcal{O} and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1 , observer \mathcal{O} sees $\bar{\mathcal{O}}$ at position x_1 :

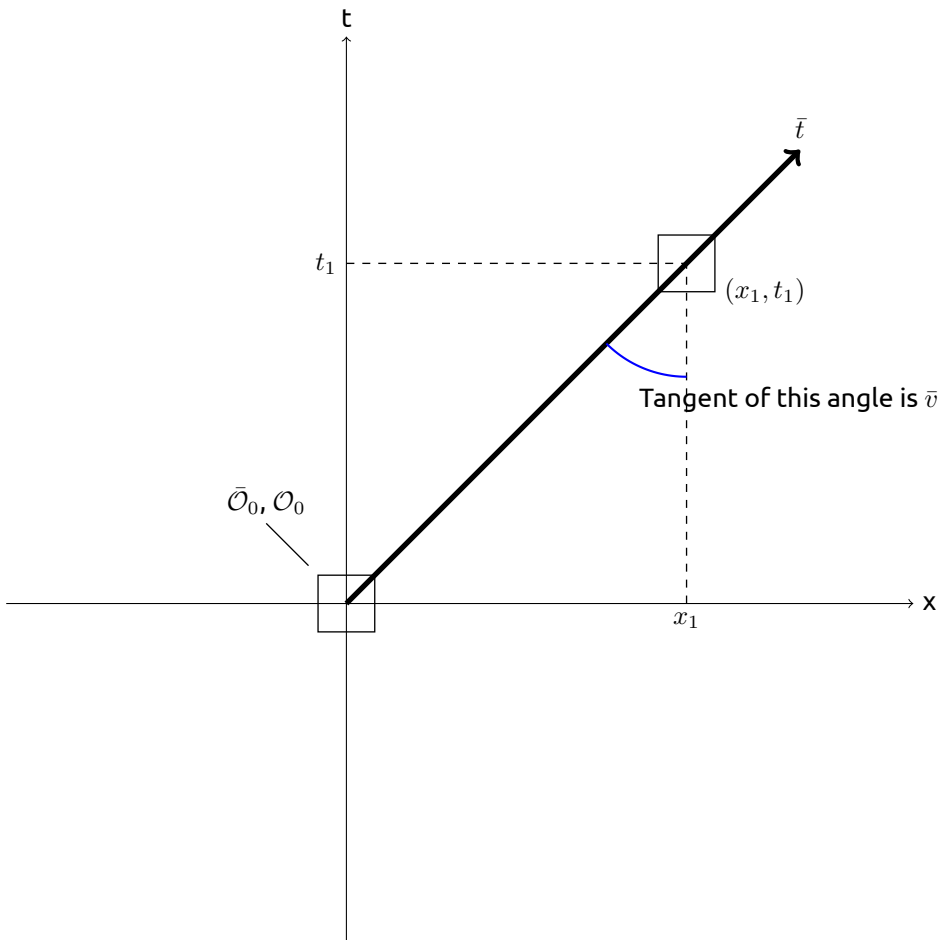
¹[Galilean invariance](#)

$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themselves at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where " \bar{t} is the locus of events at constant $\bar{x} = 0$ ", \bar{t} is the straight line that passes the origin and the (x_1, t_1) :



1.4 1.6 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $M_{\alpha\beta} + M_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $M_{\alpha\beta} = M_{\beta\alpha}$?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$M_{\alpha^*\beta^*} (\Delta x^{\alpha^*}) (\Delta x^{\beta^*})$$

$$M_{\beta^*\alpha^*} (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

Since

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) = (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

We can then group these 2 terms and factor out the product, leaving

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) (M_{\alpha^*\beta^*} + M_{\beta^*\alpha^*})$$

The terms of expanded $\Delta \bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix} M_{00}\Delta x^0\Delta x^0 & M_{01}\Delta x^0\Delta x^1 & M_{02}\Delta x^0\Delta x^2 & M_{03}\Delta x^0\Delta x^3 \\ M_{10}\Delta x^1\Delta x^0 & M_{11}\Delta x^1\Delta x^1 & M_{12}\Delta x^1\Delta x^2 & M_{13}\Delta x^1\Delta x^3 \\ M_{20}\Delta x^2\Delta x^0 & M_{21}\Delta x^2\Delta x^1 & M_{22}\Delta x^2\Delta x^2 & M_{23}\Delta x^2\Delta x^3 \\ M_{30}\Delta x^3\Delta x^0 & M_{31}\Delta x^3\Delta x^1 & M_{32}\Delta x^3\Delta x^2 & M_{33}\Delta x^3\Delta x^3 \end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$M_{\alpha^*\beta^*} = M_{\beta^*\alpha^*} = \frac{(M_{\alpha^*\beta^*} + M_{\beta^*\alpha^*})}{2}$$

where $\alpha^* \neq \beta^*$. And since $M_{\alpha\beta} = M_{\beta\alpha}$ if $\alpha = \beta$, we conclude that

$$M_{\alpha\beta} = M_{\beta\alpha} \text{ for all } \alpha \text{ and } \beta$$

Why do we have the 2nd term in equation

$$\Delta \bar{s}^2 = M_{00} (\Delta r)^2 + 2 \left(\sum_{i=1}^3 M_{0i} \Delta x^i \right) \Delta r + \sum_{i=1}^3 \sum_{j=1}^3 M_{ij} \Delta x^i \Delta x^j$$

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.1)$$

$$= \sum_{\alpha=0}^0 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=0}^3 \sum_{\beta=0}^0 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.2)$$

$$= \sum_{\beta=0}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=0}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.3)$$

$$= M_{00} (\Delta t)^2 + \sum_{\beta=1}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=1}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.4)$$

$$= M_{00} (\Delta t)^2 + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.5)$$

Why would $M_{0i} = 0$ for $i = 1, 2, 3$ and $M_{ij} = -M_{00}\delta_{ij}$ in Equation 1.5?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes $\Delta x_1 = -\Delta x_2$ to simplify the equation 1.10. This is not sufficient, because what if $\Delta x_1 \neq -\Delta x_2$? This box takes a general approach where we **do not assume any relationship between Δx_1 and Δx_2**

Note that this statement is based on the aforementioned assumption that $\Delta \bar{s}^2 = \Delta s^2 = 0$, which has been proved [here](#). Therefore, by 1.5, we have

$$\Delta \bar{s}^2(\Delta t, \Delta x_1) - \Delta \bar{s}^2(\Delta t, \Delta x_2) \quad (1.6)$$

$$= M_{00} (\Delta t)^2 + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.7)$$

$$= 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) -$$

$$2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) \quad (1.8)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) +$$

$$2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] - 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] \quad (1.9)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} \left[(\Delta x_1^\alpha) (\Delta x_1^\beta) - (\Delta x_2^\alpha) (\Delta x_2^\beta) \right] + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i - \Delta x_2^i) \right] = 0 \quad (1.10)$$

We won't be able to go further unless with some assumed relationships between Δx_1^i and Δx_2^i . But since **we do not assume any relations between them**, let's step back and re-think about this problem then and forget about Δx_1^i and Δx_2^i .

We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta \bar{s}^2$. The **detail** is about Δx_1^i and Δx_2^i but the **goal** is to derive some form of

$$\Delta \bar{s}^2 = f(\Delta s^2) = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^3 (\Delta x^i)^2$$

Let's work on $f(\Delta s^2)$ directly toward that goal then

Assuming $\Delta \bar{s}^2 = \Delta s^2 = 0$, we have $\Delta t = \pm \Delta x$; plugging it into Eq. 1.5 gives us

$$\Delta \bar{s}^2 = M_{00} \sum_{i=1}^3 (\Delta x^i)^2 + 2 \left[\sum_{i=1}^3 M_{0i} (\Delta x^i)^2 \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.11)$$

$$= (M_{00} + 2M_{0i}) \sum_{i=1}^3 (\Delta x^i)^2 + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.12)$$

Eq. 1.12 seems to suggest a linear relationship between Δs^2 and $\Delta \bar{s}^2$. How do we go about proving it? We now start the **formal proof of Invariance of Interval**

Theorem 1.4.1

Let $n, p \geq 1$ be integers, $d := n+p$ and V a vector space over \mathbb{R} of dimension d . Let h be an indefinite-inner product^a on V with signature type (n, p) ^b. Suppose g is a symmetric bilinear form on V such that the null set of the associated quadratic form of h is contained in that of g (i.e. suppose that for every $v \in V$, if $h(v, v) = 0$ then $g(v, v) = 0$). Then, there exists a constant $C \in \mathbb{R}$ such that $g = Ch$. Furthermore, if we assume $n \neq p$ and that g also has signature type (n, p) , then we have $C > 0$

^aStart with page 447 of [Introduction to Linear Algebra, 4th Edition](#) for everything you need to know about indefinite-inner product

^bRead Ch. 6 of [Introduction to Linear Algebra, 4th Edition](#) and then [Matrix Signature](#)