

Part I

A First Course on General Relativity

By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole

A person being able to master GR has the best advantage in analyzing business and technical problems right, which is why I study GR and Physics in general (Sorry, Einstein)

Chapter 1

Special Relativity

Einstein's Original Paper on "General Relativity"

The original paper is in [The Collected Papers of Albert Einstein, Vol.2](#), page 140, *On the Electrodynamics of Moving Bodies*

On "Principle of relativity (Galileo)"

Galilean invariance

[Newton's laws of motion](#) hold in all frames related to one another by a [Galilean transformation](#). In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).¹ The proof has been given by the book on page 2.

1.5 - Construction of the coordinates used by another observer

Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose \mathcal{O} and $\bar{\mathcal{O}}$ both start out at the same position where $\bar{\mathcal{O}}$ moves along the x at some speed. After t_1 , observer \mathcal{O} sees $\bar{\mathcal{O}}$ at position x_1 :

$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer $\bar{\mathcal{O}}$, however, still sees themselves at $x = 0$:

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where " \bar{t} is the locus of events at constant $\bar{x} = 0$ ", \bar{t} is the straight line that passes the origin and the (x_1, t_1) :

¹Galilean invariance



1.6 Invariance of the interval

Why $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$ for two events in the same light beam?

Let's say, in a simplified 1D case, event $\mathcal{E} = (x_0, t_0)$ and $\mathcal{P} = (x_1, t_1)$.

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only $M_{\alpha\beta} + M_{\beta\alpha}$ terms when $\alpha \neq \beta$, which guarantees $M_{\alpha\beta} = M_{\beta\alpha}$?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of $(\alpha, \beta) = (\alpha^*, \beta^*)$ where $\alpha^* \neq \beta^*$. Then we would definitely have the following 2 terms in the expansion:

$$M_{\alpha^* \beta^*} (\Delta x^{\alpha^*}) (\Delta x^{\beta^*})$$

$$M_{\beta^* \alpha^*} (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

Since

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) = (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

We can then group these 2 terms and factor out the product, leaving

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) (M_{\alpha^* \beta^*} + M_{\beta^* \alpha^*})$$

The terms of expanded $\Delta \bar{s}^2$ can be expressed in a matrix of

$$\begin{bmatrix} M_{00} \Delta x^0 \Delta x^0 & M_{01} \Delta x^0 \Delta x^1 & M_{02} \Delta x^0 \Delta x^2 & M_{03} \Delta x^0 \Delta x^3 \\ M_{10} \Delta x^1 \Delta x^0 & M_{11} \Delta x^1 \Delta x^1 & M_{12} \Delta x^1 \Delta x^2 & M_{13} \Delta x^1 \Delta x^3 \\ M_{20} \Delta x^2 \Delta x^0 & M_{21} \Delta x^2 \Delta x^1 & M_{22} \Delta x^2 \Delta x^2 & M_{23} \Delta x^2 \Delta x^3 \\ M_{30} \Delta x^3 \Delta x^0 & M_{31} \Delta x^3 \Delta x^1 & M_{32} \Delta x^3 \Delta x^2 & M_{33} \Delta x^3 \Delta x^3 \end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$M_{\alpha^* \beta^*} = M_{\beta^* \alpha^*} = \frac{(M_{\alpha^* \beta^*} + M_{\beta^* \alpha^*})}{2}$$

where $\alpha^* \neq \beta^*$. And since $M_{\alpha\beta} = M_{\beta\alpha}$ if $\alpha = \beta$, we conclude that

$$M_{\alpha\beta} = M_{\beta\alpha} \text{ for all } \alpha \text{ and } \beta$$

Why do we have the 2nd term in equation

$$\Delta \bar{s}^2 = M_{00} (\Delta r)^2 + 2 \left(\sum_{i=1}^3 M_{0i} \Delta x^i \right) \Delta r + \sum_{i=1}^3 \sum_{j=1}^3 M_{ij} \Delta x^i \Delta x^j$$

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.1)$$

$$= \sum_{\alpha=0}^0 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=0}^3 \sum_{\beta=0}^0 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.2)$$

$$= \sum_{\beta=0}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=0}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.3)$$

$$= M_{00} (\Delta t)^2 + \sum_{\beta=1}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=1}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.4)$$

$$= M_{00} (\Delta t)^2 + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.5)$$

Why would $M_{0i} = 0$ for $i = 1, 2, 3$ and $M_{ij} = -M_{00}\delta_{ij}$ in Equation 1.5?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes $\Delta x_1 = -\Delta x_2$ to simplify the equation 1.10. This is not sufficient, because what if $\Delta x_1 \neq -\Delta x_2$? This box takes a general approach where we **do not assume any relationship between Δx_1 and Δx_2**

Note that this statement is based on the aforementioned assumption that $\Delta \bar{s}^2 = \Delta s^2 = 0$, which has been proved [here](#). Therefore, by 1.5, we have

$$\Delta \bar{s}^2(\Delta t, \Delta x_1) - \Delta \bar{s}^2(\Delta t, \Delta x_2) \quad (1.6)$$

$$= M_{00} (\Delta t)^2 + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.7)$$

$$= 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) - 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) \quad (1.8)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] - 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] \quad (1.9)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} \left[(\Delta x_1^\alpha) (\Delta x_1^\beta) - (\Delta x_2^\alpha) (\Delta x_2^\beta) \right] + 2 \left[\sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i - \Delta x_2^i) \right] = 0 \quad (1.10)$$

We won't be able to go further unless with some assumed relationships between Δx_1^i and Δx_2^i . But since [we do not assume any relations between them](#), let's step back and re-think about this

problem then and forget about Δx_1^i and Δx_2^i .

We go through all these for the proof of invariance of the interval. This is to work out a relation between Δs^2 and $\Delta \bar{s}^2$. The **detail** is about Δx_1^i and Δx_2^i but the **goal** is to derive some form of

$$\Delta \bar{s}^2 = f(\Delta s^2) = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^3 (\Delta x^i)^2$$

Let's work on $f(\Delta s^2)$ directly toward that goal then

Assuming $\Delta \bar{s}^2 = \Delta s^2 = 0$, we have $\Delta t = \pm \Delta x$; plugging it into Eq. 1.5 gives us

$$\Delta \bar{s}^2 = M_{00} \sum_{i=1}^3 (\Delta x^i)^2 + 2 \left[\sum_{i=1}^3 M_{0i} (\Delta x^i)^2 \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.11)$$

$$= (M_{00} + 2M_{0i}) \sum_{i=1}^3 (\Delta x^i)^2 + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.12)$$

Eq. 1.12 seems to suggest a linear relationship between Δs^2 and $\Delta \bar{s}^2$. How do we go about proving it? We now start the **formal proof of Invariance of Interval**

Theorem 1.0.1

Let $n, p \geq 1$ be integers, $d := n+p$ and V a vector space over \mathbb{R} of dimension d . Let h be an indefinite-inner product^a on V with signature type (n, p) ^b. Suppose g is a symmetric bilinear form on V such that the null set of the associated quadratic form of h is contained in that of g (i.e. suppose that for every $v \in V$, if $h(v, v) = 0$ then $g(v, v) = 0$). Then, there exists a constant $C \in \mathbb{R}$ such that $g = Ch$. Furthermore, if we assume $n \neq p$ and that g also has signature type (n, p) , then we have $C > 0$

^aStart with page 447 of [Introduction to Linear Algebra, 4th Edition](#) for everything you need to know about indefinite-inner product

^bRead Ch. 6 of [Introduction to Linear Algebra, 4th Edition](#) and then [Matrix Signature](#)