

## Part I

# A First Course on General Relativity

**By studying General Relativity, I spotted my weakness of being too submerged in details and not being able to solve problems systematically and as a whole**

***A person being able to master GR has the best advantage in analyzing business and technical problems right, which is why I study GR and Physics in general (Sorry, Einstein)***



# Chapter 1

## Special Relativity

### On “Principle of relativity (Galileo)”

#### Galilean invariance

Newton's laws of motion hold in all frames related to one another by a Galilean transformation. In other words, all frames related to one another by such a transformation are inertial (meaning, Newton's equation of motion is valid in these frames).<sup>1</sup> The proof has been given by the book on page 2.

### 1.5 - Construction of the coordinates used by another observer

#### Why would the tangent of the angle is the speed in Fig. 1.2?

Suppose  $\mathcal{O}$  and  $\bar{\mathcal{O}}$  both start out at the same position where  $\bar{\mathcal{O}}$  moves along the  $x$  at some speed. After  $t_1$ , observer  $\mathcal{O}$  sees  $\bar{\mathcal{O}}$  at position  $x_1$ :

$$\bar{\mathcal{O}}_1 = (x_1, t_1)$$

Observer  $\bar{\mathcal{O}}$ , however, still sees themselves at  $x = 0$ :

$$\bar{\mathcal{O}}_1 = (0, t_1)$$

By definition where “ $\bar{t}$  is the locus of events at constant  $\bar{x} = 0$ ”,  $\bar{t}$  is the straight line that passes the origin and the  $(x_1, t_1)$ :

---

<sup>1</sup>Galilean invariance



## 1.6 Invariance of the interval

Why  $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (\Delta t)^2 = 0$  for two events in the same light beam?

Let's say, in a simplified 1D case, event  $\mathcal{E} = (x_0, t_0)$  and  $\mathcal{P} = (x_1, t_1)$ .

$$(\Delta x)^2 - (\Delta t)^2 = (x_1 - x_0)^2 - (t_1 - t_0)^2$$

Since the speed of light is 1,

$$(x_1 - x_0)^2 - (t_1 - t_0)^2 = (x_1 - x_0)^2 - (t_1 \times 1 - t_0 \times 1)^2 = (x_1 - x_0)^2 - (x_1 - x_0)^2 = 0$$

Why does the equation contains only  $M_{\alpha\beta} + M_{\beta\alpha}$  terms when  $\alpha \neq \beta$ , which guarantees  $M_{\alpha\beta} = M_{\beta\alpha}$ ?

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

Before spending too much time on expanding the equation, we can pick up a pair of indices of  $(\alpha, \beta) = (\alpha^*, \beta^*)$  where  $\alpha^* \neq \beta^*$ . Then we would definitely have the following 2 terms in the expansion:

$$M_{\alpha^* \beta^*} (\Delta x^{\alpha^*}) (\Delta x^{\beta^*})$$

$$M_{\beta^* \alpha^*} (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

Since

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) = (\Delta x^{\beta^*}) (\Delta x^{\alpha^*})$$

We can then group these 2 terms and factor out the product, leaving

$$(\Delta x^{\alpha^*}) (\Delta x^{\beta^*}) (M_{\alpha^* \beta^*} + M_{\beta^* \alpha^*})$$

The terms of expanded  $\Delta \bar{s}^2$  can be expressed in a matrix of

$$\begin{bmatrix} M_{00} \Delta x^0 \Delta x^0 & M_{01} \Delta x^0 \Delta x^1 & M_{02} \Delta x^0 \Delta x^2 & M_{03} \Delta x^0 \Delta x^3 \\ M_{10} \Delta x^1 \Delta x^0 & M_{11} \Delta x^1 \Delta x^1 & M_{12} \Delta x^1 \Delta x^2 & M_{13} \Delta x^1 \Delta x^3 \\ M_{20} \Delta x^2 \Delta x^0 & M_{21} \Delta x^2 \Delta x^1 & M_{22} \Delta x^2 \Delta x^2 & M_{23} \Delta x^2 \Delta x^3 \\ M_{30} \Delta x^3 \Delta x^0 & M_{31} \Delta x^3 \Delta x^1 & M_{32} \Delta x^3 \Delta x^2 & M_{33} \Delta x^3 \Delta x^3 \end{bmatrix}$$

Because the off-diagonal terms always appear in pairs above, we could effectively replace them with their mean value:

$$M_{\alpha^* \beta^*} = M_{\beta^* \alpha^*} = \frac{(M_{\alpha^* \beta^*} + M_{\beta^* \alpha^*})}{2}$$

where  $\alpha^* \neq \beta^*$ . And since  $M_{\alpha\beta} = M_{\beta\alpha}$  if  $\alpha = \beta$ , we conclude that

$$M_{\alpha\beta} = M_{\beta\alpha} \text{ for all } \alpha \text{ and } \beta$$

Why do we have the 2nd term in equation

$$\Delta \bar{s}^2 = M_{00} (\Delta r)^2 + 2 \left( \sum_{i=1}^3 M_{0i} \Delta x^i \right) \Delta r + \sum_{i=1}^3 \sum_{j=1}^3 M_{ij} \Delta x^i \Delta x^j$$

$$\Delta \bar{s}^2 = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.1)$$

$$= \sum_{\alpha=0}^0 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=0}^3 \sum_{\beta=0}^0 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.2)$$

$$= \sum_{\beta=0}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=0}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.3)$$

$$= M_{00} (\Delta t)^2 + \sum_{\beta=1}^3 M_{0\beta} \Delta t (\Delta x^\beta) + \sum_{\alpha=1}^3 M_{\alpha 0} (\Delta x^\alpha) \Delta t + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.4)$$

$$= M_{00} (\Delta t)^2 + 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.5)$$

Why would  $M_{0i} = 0$  for  $i = 1, 2, 3$  and  $M_{ij} = -M_{00}\delta_{ij}$  in Equation 1.5?

The answer is: **not necessarily**. We are probably looking at a wrong problem.

The solution to exercise 1.8 takes  $\Delta x_1 = -\Delta x_2$  to simplify the equation 1.10. This is not sufficient! What if  $\Delta x_1 \neq -\Delta x_2$ ? This box takes a general approach where we **do not assume any relationship between  $\Delta x_1$  and  $\Delta x_2$**

Note that this statement is based on the aforementioned assumption that  $\Delta \bar{s}^2 = \Delta s^2 = 0$ , which has been proved [here](#). Therefore, by 1.5, we have

$$\Delta \bar{s}^2(\Delta t, \Delta x_1) - \Delta \bar{s}^2(\Delta t, \Delta x_2) \quad (1.6)$$

$$= M_{00} (\Delta t)^2 + 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.7)$$

$$= 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) - 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) \quad (1.8)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_1^\alpha) (\Delta x_1^\beta) - \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x_2^\alpha) (\Delta x_2^\beta) + 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i) \right] - 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x_2^i) \right] \quad (1.9)$$

$$= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} \left[ (\Delta x_1^\alpha) (\Delta x_1^\beta) - (\Delta x_2^\alpha) (\Delta x_2^\beta) \right] + 2 \left[ \sum_{i=1}^3 M_{0i} \Delta t (\Delta x_1^i - \Delta x_2^i) \right] = 0 \quad (1.10)$$

We won't be able to go further unless with some assumed relationships between  $\Delta x_1^i$  and  $\Delta x_2^i$ . But since [we do not assume any relations between them](#), let's step back and re-think about this

problem then and forget about  $\Delta x_1^i$  and  $\Delta x_2^i$ .

We go through all these for the proof of invariance of the interval. This is to work out a relation between  $\Delta s^2$  and  $\Delta \bar{s}^2$ . The **detail** is about  $\Delta x_1^i$  and  $\Delta x_2^i$  but the **goal** is to derive some form of

$$\Delta \bar{s}^2 = f(\Delta s^2) = \sum_{\alpha=0}^3 \sum_{\beta=0}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta)$$

where

$$\Delta s^2 = -(\Delta t)^2 + \sum_{i=1}^3 (\Delta x^i)^2$$

*Let's work on  $f(\Delta s^2)$  directly toward that goal then*

Assuming  $\Delta \bar{s}^2 = \Delta s^2 = 0$ , we have  $\Delta t = \pm \Delta x$ ; plugging it into Eq. 1.5 gives us

$$\Delta \bar{s}^2 = M_{00} \sum_{i=1}^3 (\Delta x^i)^2 + 2 \left[ \sum_{i=1}^3 M_{0i} (\Delta x^i)^2 \right] + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.11)$$

$$= (M_{00} + 2M_{0i}) \sum_{i=1}^3 (\Delta x^i)^2 + \sum_{\alpha=1}^3 \sum_{\beta=1}^3 M_{\alpha\beta} (\Delta x^\alpha) (\Delta x^\beta) \quad (1.12)$$

Eq. 1.12 seems to suggest a linear relationship between  $\Delta s^2$  and  $\Delta \bar{s}^2$ . How do we go about proving it? We now start the **formal proof of Invariance of Interval**

### Theorem 1.0.1

Let  $n, p \geq 1$  be integers,  $d := n+p$  and  $V$  a vector space over  $\mathbb{R}$  of dimension  $d$ . Let  $h$  be an indefinite-inner product<sup>a</sup> on  $V$  with signature type  $(n, p)$ . Suppose  $g$  is a symmetric bilinear form on  $V$  such that the null set of the associated quadratic form of  $h$  is contained in that of  $g$  (i.e. suppose that for every  $v \in V$ , if  $h(v, v) = 0$  then  $g(v, v) = 0$ ). Then, there exists a constant  $C \in \mathbb{R}$  such that  $g = Ch$ . Furthermore, if we assume  $n \neq p$  and that  $g$  also has signature type  $(n, p)$ , then we have  $C > 0$

<sup>a</sup>Start with page 447 of [Introduction to Linear Algebra, 4th Edition](#) for everything you need to know about indefinite-inner product