Fintech 545 Project Week04

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Problem 1:

1. For the Classical Brownian Motion, the price at time t is given by:

$$P_t = P_{t-1} + r_t$$

Since r_t has a mean of 0, the expected value of P_t is simply:

$$E[P_t] = E[P_{t-1} + r_t] = E[P_{t-1}] + E[r_t] = P_{t-1}$$

And for the variance of P_t :

Since $P_{t\text{-}1}$ is a constant, whose value can be known from the previous time step, thus with $Var[P_t]$=0 and $Cov(P_{t-1}, r_t)$=0:

$$Var[P_t]=Var[r_t]=\sigma^2$$

$$std[P_t] = \sigma$$

Therefore, the standard deviation of P_t is just σ.

2. For Arithmetic Return System, the price at time t is :

$$P_t = P_{t-1} \times (1 + r_t)$$

Therefore, as r_t has a mean of 0,

$$E[P_t] = E[P_{t-1} \times (1 + r_t)] = P_{t-1} \times E[1 + r_t] = P_{t-1}$$

And for the variance,

$$Var[P_t] = Var[P_{t-1} \times (1 + r_t)] = {P_{t-1}}^2 \times Var(r_t) = {P_{t-1}}^2 \times \sigma^2$$

Therefore, the standard deviation of P_t is $P_{t-1} \times \sigma$.

3. For the Log Return or Geometric Brownian Motion, the price is:

$$P_t = P_{t-1} \times e^{r_t}$$

For e^{r_t}, which is a log-normally distributed variable, as $r_t \sim N(0, \sigma^2)$,

$$E[e^{r_t}] = e^{\frac{\sigma^2}{2}}$$

Thus,

$$E[P_t] = P_{t-1} \times e^{\frac{\sigma^2}{2}}$$

And for the varaince,

$$Var[P_t] = {P_{t-1}}^2 \times Var(e^{r_t})$$

For the log-normally distributed variable, its variance should be:

$$Var[e^{r_t}] = (e^{\sigma^2} - 1) \times e^{\sigma^2}$$

Therefore,

$$Var[P_t] = {P_{t-1}}^2 \times (e^{\sigma^2} - 1) \times e^{\sigma^2}$$

And the standard deviation of P_t is $P_{t-1} \times e^{\frac{\sigma^2}{2}} \times \sqrt{e^{\sigma^2} - 1}$.

For the simulation, I choose to proceed with a commonly used value, and suppose that, $\sigma = 1\%$, $P_t = 100$, and I will repeat this process for 10000 times, the estimated results are like below:

1. Classical Brownian Motion:

$$E[P_t] = 99.99816, \text{ Std Dev} = 0.09876$$

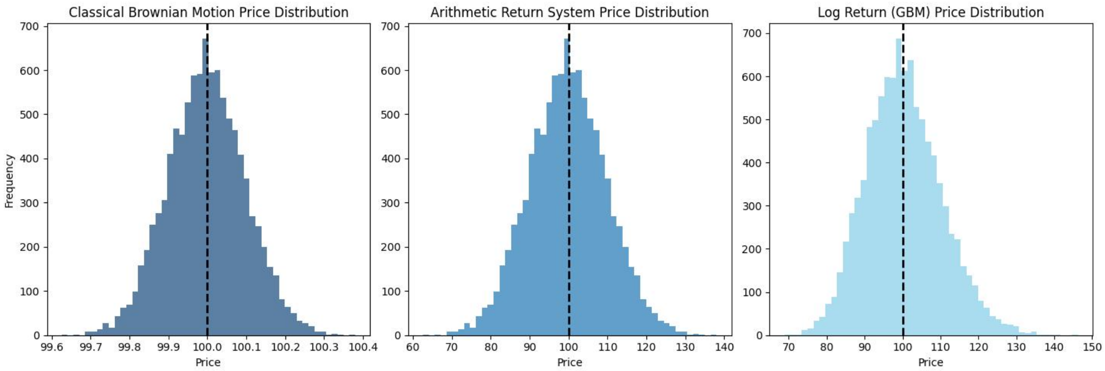
2. Arithmetic Return System:

$$E[P_t] = 100.10988, \text{ Std Dev} = 9.93066$$

3. Log Return (GBM):

$$E[P_t] = 100.44098, \text{ Std Dev} = 10.02366$$

And the figure for all these simulations are like:



While, with the formulas above, the expected results are like below:

1. Classical Brownian Motion:

$$E[P_t] = 100, \text{ Std Dev} = 0.1$$

2. Arithmetic Return System:

$$E[P_t] = 100, \text{ Std Dev} = 10$$

3. Log Return or Geometric Brownian Motion:

$$E[P_t] = 100.5013, \text{ Std Dev} = 10.0753$$

From all the simulations, we can see that the mean price for the Classical Brownian Motion model is closest to the initial price, which is expected since the returns are centered around zero and just add a small fluctuation around the starting price.

For the Arithmetic and Log Return models, the means are lower, and the standard deviations are significantly higher. This is due to the compounding effect present in these models, where returns are applied multiplicatively. The compounding effect, especially when considering log-normal distributions, tends to produce a wider spread of outcomes, which is reflected in the larger standard deviation.

The log return model often results in a log-normally distributed price at the end, which has a different mean and standard deviation than the arithmetic return model due to the exponential function applied to the returns.

Problem 2:

For the arithmetic returns, I made another csv file and add it to the file of project.

For the second part of this question, I firstly extract the return of META from the new data form, and in order to make sure it has a zero mean, I center the returns and shift the distribution of returns.

Then, I will use different required ways to calculate the VaRs:

Method-1: Normal distribution

Firstly, I will calculate the standard deviation of the centered returns, with std and the

Z-score corresponding to the 95% confidence level, and as VaR is the loss threshold, I used a negative number.

$$\mathrm{VaR} = |\mathrm{Z}_{\mathrm{score}}| \times \mathrm{std}$$

And the value of VaR using normal distribution is 5.43%, which means there is a 95% confidence level that the loss on 1 share of META will not exceed 5.43% in one day.

Method-2: Normal distribution with an Exponentially Weighted variance

In the EWMA model, the variance for each time perod is adjusted based on the decay factor λ and the returns, the formula is:

$$\sigma_t{}^2 = \lambda \times \sigma_{t-1}{}^2 + (1 - \lambda) r_t^2$$

And the value of VaR using normal distribution with EWMA model is 3.01%, which means there is a 95% CI the loss on 1 share of META will not exceed 3.01% in one day.

Method-3: MLE fitted T-distribution

For T-distribution, the VaR is calculated by the formula below, where F is the CDF of the T-distribution, and α is the CI, which is 0.95:

$$\mathrm{VaR} = -\mathrm{F}^{-1} \times (1 - \alpha)$$

With MLE I firstly fit my data with T-distribution, so that I can use the above formula to calculate my VaR, which is 4.31%, meaning that within 95% CI, the loss on 1 share of META will not exceed 4.31% in one day.

Method-4: Fitted AR(1) model

The AR(1) model is represented as:

$$r_t = \phi_0 + \phi_1 \times r_{t-1} + \epsilon_t$$

In this formula, the error term is normally distributed with mean 0 and variance σ^2. Then, I will use the parameters of the fitted model to estimate the distribution of the forecasted returns to get VaR value.

And the result is that the value of VaR usingfitted AR(1) model is 5.43%, which means that within a 95% confidence level that the loss on 1 share of META will not exceed 5.43% in one day.

Method-5: The Historic Simulation

Historical Simulation for VaR is just about using the actual historical returns to directly estimate VaR, firstly, sort the data from the smallest to the largest to form the empirical distribution of returns, and then VaR is just the return at the 5th percentile of the sorted returns list, as 5% of returns will be worse than this value, which is just 3.95%.

Problem 3:

For this question, I used the daily returns from Q2 with Arithmetic method:

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Then, I need to get the exponentially weighted covariance matrix with $\lambda = 0.94$, based on the math formula, as well as that the expected return on all stocks is 0, the covariance between two assets a and b should be:

$$Cov_{ab}(t) = \lambda \times Cov_{ab}(t-1) + (1-\lambda) \times r_{a,t} \times r_{b,t}$$

To get each portfolio variance, I need to use this covariance matrix as well as the portfolio weights matrix, suppose we have stock i in portfolio j and the element $W_{i,j}$ is the weight of stock i in portfolio j, therefore, for each portfolio, the portfolio varianace is:

$$\sigma_j{}^2 = W_j^T \cdot \sum \cdot W_j$$

With W_j having the vector of weights for portfolio j, $\sum$ is the covaraince matrix and is the transpose of W_j. Then I am going to calculate VaR with the confidence level α which is:

$$VaR_{\alpha,j} =- Z_\alpha \cdot \sigma_j$$

Commonly, people choose 95% as the confidence lever, which means $\alpha = 95\%$:

The VaR of portfolio 'A' is: \$15426.9682

The VaR of portfolio 'B' is: \$8082.5724

The VaR of portfolio 'C' is: \$18163.2916

The VaR of all portfolios is: \$38941.3757

This method uses the EWMA model, which assumes the returns are multivariate normally distributed, which may not always be the case, so it's important to consider the limitations of the VaR metric.

To test the accuracy of this result, I choose to use a historic simulation to do this again, and I suppose the sample size is 10000, the results are like below:

The VaR of portfolio A is \$18320.2590

The VaR of portfolio B is \$11340.1686

The VaR of portfolio C is \$23807.7645

The VaR for all portfolios is \$51701.8287

I choose to use such model because this method uses actual historical returns to simulate possible future returns. If recent market conditions have been more volatile or there have been extreme market movements in the observed period, it will reflect this in a higher VaR value.

And the model change does not effect the results very much, however, the value of all VaR increase under historic simulation, which means there might be some extreme market movements and it should be considered as a signal of market.

And the comparision figure of these two methods is: