Homework7

November 8, 2022

[ ]:
```python
import numpy as np

def AdaptiveQuadrature(f, a, b, tol=1e-6):
    """
    Adaptive quadrature for numerical integration.

    Args:
        f (function): function to integrate
        a (float): lower bound of integration
        b (float): upper bound of integration
        tol (_type_, optional): tolerance for stopping criterion. Defaults to␣
    ↪1e-6.

    Returns:
        _type_: _description_
    """
    # Simpson's Rule
    def Simpson(f, a, b):
        h = (b - a) / 2
        return h / 3 * (f(a) + 4 * f(a + h) + f(b))

    # Recursive function
    def Recurse(f, a, b, tol):
        c = (a + b) / 2
        left = Simpson(f, a, c)
        right = Simpson(f, c, b)
        if abs(left + right - Simpson(f, a, b)) < 15 * tol:
            return left + right + (left + right - Simpson(f, a, b)) / 15
        return Recurse(f, a, c, tol / 2) + Recurse(f, c, b, tol / 2)

    return Recurse(f, a, b, tol)
```

0.1 Assignment 7 MATH 4610

0.1.1 Section 4.6

1. Compute the Simpson's rule approximations $S(a,b)$, $S(a,\frac{(a+b)}{2})$ and $S(\frac{(a+b)}{2},b)$ for the following integral, and verify the estimate given in the approximation formula.

$$\int_0^1 x^2 e^{-x}\,dx$$

$$a = x_0 = 0, b = x_2 = 1, h = \frac{b-a}{2}$$

$$\int_0^1 x^2 e^{-x}\,dx \approx S(a,b), S(a,a+\frac{h}{2}) + S(a+\frac{h}{2},b) - \frac{1}{16}(\frac{h^5}{90})f^{(4)}(\xi)$$

$$S(a,b) = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$

$$\implies \frac{1}{6}(f(0) + 4f(0.5) + f(1))$$

$$\implies \frac{1}{6}(0 + 4(0.151632664928) + 0.367879441171)$$

$$\implies S(a,b) = 0.162401683481$$

$$S(a,a+\frac{h}{2}) = \frac{1}{12}(f(0) + 4f(0.25) + f(0.5))$$

$$\implies \frac{1}{12}(0 + 4(0.048675048942) + 0.151632664928)$$

$$\implies S(a,a+\frac{h}{2}) = 0.0288610717247$$

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$$S(a+\frac{h}{2},b) = \frac{1}{12}(f(0.5) + 4f(0.75) + f(1))$$

$$\implies \frac{1}{12}(0.151632664928 + 4(0.265706185917) + 0.367879441171)$$

$$\implies S(a+\frac{h}{2},b) = 0.131861404147$$

$\Longrightarrow$
$\int_0^1 x^2 e^{-x}\, dx \approx$
$S(a,b) =$
0.162401683481
$\Longrightarrow$
$\int_0^1 x^2 e^{-x}\, dx \approx$
$S(a, a + \frac{h}{2}) + S(a + \frac{h}{2}, b) =$
$= 0.0288610717247 + 0.131861404147 =$
0.160722475872
$\frac{1}{16}(\frac{h^5}{90}) f^{(4)}(\xi)$
the max value on the inverval [0,1] is at 1, so $\xi = 1$
$\frac{d}{dx}(f(x)) = 2e^{-x}x - e^{-x}x^2$
$\frac{d^2}{dx^2}(f(x)) = e^{-x}x^2 - 4e^{-x}x + 2e^{-x}$
$\frac{d^3}{dx^3}(f(x)) = -e^{-x}x^2 + 6e^{-x}x - 6e^{-x}$
$\frac{d^4}{dx^4}(f(x)) = e^{-x}x^2 - 8e^{-x}x + 12e^{-x}$
$\frac{d^4}{dx^4}(f(1)) = 1.83939720586$
$h = 0.5$
$\frac{1}{16}\frac{0.5^5}{90}(1.83939720586) = 0.0000399174740854$
$0.160722475872 - 0.0000399174740854 = 0.160682558398$

Using the approximation formula,

$$|S(a,b) - (S(a,a+\frac{h}{2}) + S(a+\frac{h}{2},b))| < 15\epsilon$$

$$|0.162401683481 - (0.160722475872)| < 0.025188114135$$

and

$$|\int_0^1 f(x)dx - (S(a,a+\frac{h}{2}) + S(a+\frac{h}{2},b))| < 0.000119681729212$$

2. Use Adaptive quadrature to find approximations to within 10^{-3} for the integrals in Exercise 1.

[ ]:
```
def f(x):
    return x**2*np.exp(-x)
AdaptiveQuadrature(f, 0, 1, tol=1e-3)
```

[ ]: 0.16061052869798964