第二部分　定理定义公式的英文表达

Part2　English Expression for Theorem, Definition and Formula

第一章　函数与极限

Chapter 1　Function and Limit

1.1　映射与函数(Mapping and Function)

一、集合(Set)

二、映射(Mapping)

映射概念(The Concept of Mapping)

设 X，Y 是两个非空集合，如果存在一个法则 f，使得对 X 中每个元素 x，按法则 f，在 Y 中有唯一确定的元素 y 与之对应，则称 f 为从 X 到 Y 的映射，记作 f: $X \to Y$。

Let X, Y be two nonempty sets, if there exists a rule f that associates a unique element y of Y to every element x in X, then f is called a mapping from X to Y, denoted by $f: X \to Y$.

三、函数(Function)

函数概念(The Concept of Function)

设数集 $D \subset R$，则称映射 $f: D \to R$ 为定义在 D 上的函数，通

常简记为 $y=f(x)$，$x\in D$，其中 x 称为自变量，y 称为因变量，D 称为定义域，记作 $D_f=D$。

Let the number set $D\subset R$, then the mapping $f: D\to R$ is called a function defined on D, usually denoted by $y=f(x)$, $x\in D$, where x is called an independent variable, y is called a dependent variable, D is called a domain, denoted by $D_f=D$.

1.2 数列的极限(Limit of the Sequence of Number)

一、数列极限的定义(Definition of the Limit of Sequence of Number)

设$\{x_n\}$为一数列，如果存在常数 a，对于任意给定的正数 ε (不论它多么小)，总存在正整数 N，使得当 $n>N$ 时，不等式 $|x_n-a|<\varepsilon$ 都成立，那么就称常数 a 是数列$\{x_n\}$的极限，或者称数列$\{x_n\}$收敛于 a，记为

$$\lim_{n\to\infty} x_n=a \text{ 或 } x_n\to a(n\to\infty)。$$

Let $\{x_n\}$ be a sequence of number, If there exists a constant a, such that for any given positive number ε, there exists a positive integer N, such that for every $n>N$, $|x_n-a|<\varepsilon$, then the constant a is called the limit of the sequence $\{x_n\}$, or we call that the sequence $\{x_n\}$ converges to a, denoted by

$$\lim_{n\to\infty} x_n=a \text{ or } x_n\to a(n\to\infty).$$

二、收敛数列的性质(Properties of Convergent Sequence)

定理 1 (极限的唯一性) 如果数列$\{x_n\}$收敛，那么它的极限唯一。

Theorem 1 (Uniqueness of Limit) If the sequence $\{x_n\}$ is convergent, then its limit is unique.

定理 2 (收敛数列的有界性) 如果数列$\{x_n\}$收敛，那么数列$\{x_n\}$必定有界。

Theorem 2 (Boundedness of a Convergent Sequence) If the sequence $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.

定理3 如果 $\lim\limits_{n\to\infty} x_n=a$ 且 $a>0$(或 $a<0$)，那么存在正整数 $N>0$，当 $n>N$ 时，都有 $x_n>0$ (或 $x_n<0$)。

Theorem 3 If $\lim\limits_{n\to\infty} x_n=a$, and $a>0$(or $a<0$), then there exists a positive integer N, such that $x_n>0$(or $x_n<0$), for every $n>N$.

定理 4 (收敛数列与其子数列间的关系) 如果数列 $\{x_n\}$ 收敛于 a，那么它的任一子数列也收敛，且极限也是 a。

Theorem 4(Relation of a Convergent Sequence between its Subsequence) If the sequence $\{x_n\}$ converges to a, then any of its subsequence is also convergent, and the limit is also a.

1.3 函数的极限(Limit of Function)

一、函数极限的定义 (Definition of Limit of Function)

1. 自变量趋于有限值时函数的极限(Definition of Limit of Function (as x Tends to a Real Number x_0))

定义 1 设函数 $f(x)$在某个包含 x_0 的邻域内有定义(可能在 x_0 无定义)，如果存在常数 A，对于任意给定的正数 ε，总存在正数 δ，使得当 x 满足不等式 $0<|x-x_0|<\delta$ 时，对应的函数值 $f(x)$都满足不等式 $|f(x)-A|<\varepsilon$，那么常数 A 就叫做函数 $f(x)$当 $x\to x_0$ 时的极限，记作 $\lim\limits_{x\to x_0} f(x)=A$ 或 $f(x)\to A$ (当 $x\to x_0$)。

Definition 1 Let $f(x)$ be a function defined on some neighborhood of x_0, maybe not at x_0 itself. If there exists a constant A, such that for any given positive number ε, there exists a positive number δ, such that if $0<|x-x_0|<\delta$, then $|f(x)-A|<\varepsilon$. Then the constant A is called the limit of $f(x)$ as x approaches x_0. Denoted by $\lim\limits_{x\to x_0} f(x)=A$ or $f(x)\to A(as\ x\to x_0)$.

2. 自变量趋于无穷大时函数的极限(Definition of Limit of Function (as x Tends to Infinity))

定义 2 设函数$f(x)$当$|x|$大于某一正数时有定义。如果存在常数A,对于任意给定的正数ε,总存在正数X,使得当x满足不等式$|x|>X$时,对应的函数值$f(x)$都满足不等式$|f(x)-A|<\varepsilon$,那么常数A就叫做函数$f(x)$当$x\to\infty$时的极限,记作$\lim\limits_{x\to\infty} f(x)=A$或$f(x)\to A(当x\to\infty)$。

Definition 2 Let $f(x)$ be a function which is defined if $|x|$ is greater than some positive number. If there exists a constant A, such that for any given positive number ε, there is a positive number X, such that if $|x|>\delta$, then $|f(x)-A|<\varepsilon$. Then the constant A is called the limit of $f(x)$ as x approaches ∞. Denoted by

$$\lim_{x\to\infty} f(x)=A \quad \text{or} \quad f(x)\to A(\text{as } x\to\infty).$$

二、函数极限的性质 (Properties of the Limit of Function)

定理 1(函数极限的唯一性) 如果$\lim\limits_{x\to x_0} f(x)$存在,那么这极限唯一。

Theorem 1 (Uniqueness of Limit of Function) If $\lim\limits_{x\to x_0} f(x)$ exists, then its limit is unique.

定理 2(函数极限的局部有界性) 如果$\lim\limits_{x\to x_0} f(x)=A$,那么存在常数$M>0$,使得当$0<|x-x_0|<\delta$时,有$|f(x)|\leqslant M$。

Theorem2 (Locally Boundedness of the Limit of Function) If $\lim\limits_{x\to x_0} f(x)=A$, then there exists $M>0$, such that $0<|x-x_0|<\delta$ implies $|f(x)|\leqslant M$.

定理 3 如果$\lim\limits_{x\to x_0} f(x)=A$且$A>0$(或$A<0$),那么存在常数$\delta>0$,使得当$0<|x-x_0|<\delta$时,有$f(x)>0$(或$f(x)<0$)。

Theorem 3 If $\lim\limits_{x\to x_0} f(x)=A$, and $A>0$ (or $A<0$), then there exists a constant $\delta>0$, such that whenever $0<|x-x_0|<\delta$, then $f(x)>0$ (or $f(x)<0$).

定理 4 (函数极限与数列极限间的关系) 如果极限$\lim\limits_{x\to x_0} f(x)$存在,$\{x_n\}$为函数$f(x)$的定义域内任一收敛于$x_0$的数列,且满足:$x_n\neq x_0$ $(n\in N^+)$,那么相应的函数值数列$\{f(x_n)\}$必收敛,且$\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{x\to x_0} f(x)$。

Theorem 4 (Relations between Limit of Function and Limit of Sequence) If $\lim\limits_{x\to x_0} f(x)$ exists, $\{x_n\}$ is any sequence in the domain of $f(x)$ that converges to x_0, and satisfies: $x_n\neq x_0$ $(n\in N^+)$, then the corresponding sequence of function value $\{f(x_n)\}$ must be convergent, and $\lim\limits_{n\to\infty} f(x_n)=\lim\limits_{x\to x_0} f(x)$.

1.4 无穷小与无穷大 (Infinitesimal and Infinity)

定义 1 如果函数$f(x)$当$x\to x_0$(或$x\to\infty$)时的极限为零,那

么称函数$f(x)$为当$x \to x_0$(或$x \to \infty$)时的无穷小。

Definiton 1 If the limit of function $f(x)$ is zero as $x \to x_0$ (or $x \to \infty$), then $f(x)$ is called an infinitesimal as $x \to x_0$(or $x \to \infty$).

定义 2 设函数$f(x)$在x_0的某一去心邻域内有定义（或$|x|$大于某一正数时有定义），如果对于任意给定的正数M，总存在正数δ (或正数X), 只要x适合不等式$0<|x-x_0|<\delta$ (或$|x|>X$)，对应的函数值$f(x)$总满足不等式$|f(x)|>M$, 则称函数$f(x)$为当$x \to x_0$(或$x \to \infty$)时的无穷大。

Definition 2 Let $f(x)$ be a function defined on some neighborhood of x_0, maybe not at x_0 itself (or is defined for $|x|$ greater than some positive number), if for any given $M>0$, there exists $\delta>0$ (or $X>0$), such that whenever $0<|x-x_0|<\delta$ (or $|x|>X$), then $|f(x)|>M$. Then $f(x)$ is called an infinity as $x \to x_0$ (or $x \to \infty$).

1.5 极限运算法则(Operation Rule of Limit)

定理 1 有限个无穷小的和也是无穷小。

Theorem 1 The sum of finite number of infinitesimal is an infinitesimal.

定理 2 有界函数与无穷小的乘积是无穷小。

Theorem 2 The product of a bounded function and an infinitesimal is an infinitesimal.

推论 1 常数与无穷小的乘积是无穷小。

Corollary 1 The product of a constant and an infinitesimal is an infinitesimal.

推论 2 有限个无穷小的乘积是无穷小。

Corollary 2 The product of finite number of infinitesimal is an infinitesimal.

定理 6(复合函数的极限运算法则) 设函数$y=f[g(x)]$是由函数$y=f(u)$与函数$u=g(x)$复合而成，$f[g(x)]$在点x_0的某去心邻域内有定义，若$\lim\limits_{x \to x_0} g(x)=u_0$，$\lim\limits_{u \to u_0} f(u)=A$，且存在$\delta_0>0$，当$x \in \mathring{U}(x_0,\delta_0)$时，有$g(x) \neq u_0$，则$\lim\limits_{x \to x_0} f[g(x)]=\lim\limits_{u \to u_0} f(u)=A$。

Theorem 6 (Operation Rule of Limits of Composite Functions) Suppose the function $y=f[g(x)]$ is the composition of $y=f(u)$ and $u=g(x)$, and $f[g(x)]$ is defined on some neighborhood of x_0 (except possibly at x_0).If $\lim\limits_{x \to x_0} g(x)=u_0$, $\lim\limits_{u \to u_0} f(u)=A$, and there exists $\delta_0>0$, for $x \in \mathring{U}(x_0,\delta_0)$, we have $g(x) \neq u_0$, then

$$\lim_{x \to x_0} f[g(x)]=\lim_{u \to u_0} f(u)=A.$$

1.6 极限存在准则 两个重要极限(Rule for the Existence of Limits Two Important Limits)

准则 I 如果数列$\{x_n\}$、$\{y_n\}$及$\{z_n\}$满足下列条件：

(1) $y_n \leqslant x_n \leqslant z_n$ $(n=1,2,3,\cdots)$,

(2) $\lim\limits_{n \to \infty} y_n=a$, $\lim\limits_{n \to \infty} z_n=a$,

那么数列$\{x_n\}$的极限存在，且$\lim\limits_{n \to \infty} x_n=a$。

Rule I Suppose the sequences $\{x_n\}$、$\{y_n\}$ and $\{z_n\}$ satisfy:

(1) $y_n \leqslant x_n \leqslant z_n$ $(n=1,2,3,\cdots)$,

(2) $\lim\limits_{n \to \infty} y_n=a$, $\lim\limits_{n \to \infty} z_n=a$,

Then the limit of the sequence $\{x_n\}$ exists, and $\lim\limits_{n \to \infty} x_n=a$.

准则Ⅰ' 如果

(1) 当 $x\in \overset{o}{U}(x_0,r)$(或$|x|>M$)时，$g(x)\leqslant f(x)\leqslant h(x)$

(2) $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} g(x)=A$, $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} h(x)=A$,

那么 $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} f(x)$ 存在，且等于A。

Rule Ⅰ' Suppose

(1) for $x\in \overset{o}{U}(x_0,r)$ (or $|x|>M$)，$g(x)\leqslant f(x)\leqslant h(x)$

(2) $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} g(x)=A$, $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} h(x)=A$,

then $\lim\limits_{\substack{x\to x_0\\(x\to\infty)}} f(x)$ exists，and equals A.

准则Ⅱ 单调有界数列必有极限。

Rule Ⅱ Every bounded monotonic sequence has limit.

准则Ⅱ' 设函数$f(x)$在点x_0的某个左邻域内单调并且有界，则$f(x)$在x_0的左极限$f(x_0^-)$必定存在。

Rule Ⅱ' Suppose the function $f(x)$ is bounded monotonic on some left neighborhood of x_0, then the left–hand limit $f(x_0^-)$ of $f(x)$ exists.

1.7 无穷小的比较 (The Comparison of Infinitesimal)

定义

如果$\lim\dfrac{\beta}{\alpha}=0$，就说$\beta$是比$\alpha$高阶的无穷小，记作$\beta$=o($\alpha$)；

如果$\lim\dfrac{\beta}{\alpha}=\infty$，就说$\beta$是比$\alpha$低阶的无穷小；

如果$\lim\dfrac{\beta}{\alpha}=c\neq 0$，就说$\beta$与$\alpha$是同阶无穷小；

如果$\lim\dfrac{\beta}{\alpha^k}=c\neq 0,k>0$，就说$\beta$是关于$\alpha$的$k$阶无穷小；

如果$\lim\dfrac{\beta}{\alpha}=1$，就说$\beta$与$\alpha$是等价无穷小，记作$\alpha\sim\beta$；

Definition

If $\lim\dfrac{\beta}{\alpha}=0$, then β is called a higher order infinitesimal of α, denoted by β=o(α);

If $\lim\dfrac{\beta}{\alpha}=\infty$, then β is called a lower order infinitesimal of α,

If $\lim\dfrac{\beta}{\alpha}=c\neq 0$, then β is called an infinitesimal of the same order as α;

If $\lim\dfrac{\beta}{\alpha^k}=c\neq 0,k>0$, then β is called an infinitesimal of the kth order as α;

If $\lim\dfrac{\beta}{\alpha}=1$, then β is called an equivalent infinitesimal of α, denoted by $\alpha\sim\beta$;

定理1 β与α是等价无穷小的充分必要条件为$\beta=\alpha+o(\alpha)$。

Theorem 1 β and α are equivalent infinitesimals if and only if $\beta=\alpha+o(\alpha)$.

定理2 设$\alpha\sim\alpha'$, $\beta\sim\beta'$，且$\lim\dfrac{\beta'}{\alpha'}$存在，则$\lim\dfrac{\beta}{\alpha}=\lim\dfrac{\beta'}{\alpha'}$。

Theorem 2 Let $\alpha \sim \alpha'$, $\beta \sim \beta'$, and $\lim \frac{\beta'}{\alpha'}$ exists, then $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$.

1.8 函数的连续性与间断点 (Continuity of Function and Discontinuity Points)

函数的连续性 (Continuity of Function)

定义 设函数 $y=f(x)$在点 x_0 的某一邻域内有定义，如果 $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} f(x_0 + \Delta x) - f(x_0) = 0$，那么就称函数 $y=f(x)$在点 x_0 连续。

Definition Suppose the function $y=f(x)$ is defined on some neighborhood of x_0, if $\lim\limits_{\Delta x \to 0} \Delta y = \lim\limits_{\Delta x \to 0} f(x_0 + \Delta x) - f(x_0) = 0$, then we call that $y=f(x)$ is continuous at the point x_0.

1.9 连续函数的运算与初等函数的连续性(Operation of Continuous Functions and Continuity of Elementary Functions)

一、连续函数的和、差、积、商的连续性 (Continuity of the Sum, Difference, Product and Quotient of Continuous Functions)

定理 1 设函数 $f(x)$和 $g(x)$在点 x_0 连续，则它们的和（差）$f \pm g$，积 fg 及商 f/g (当 $g(x_0) \neq 0$ 时)都在点 x_0 连续。

Theorem 1 Let the function $f(x)$ and $g(x)$ are continuous at x_0, then their sum (difference) $f \pm g$, product fg and quotient f/g (provided $g(x_0) \neq 0$) are continuous at x_0.

二、反函数与复合函数的连续性 (Continuity of the Inverse Function and Composite Function)

定理 2 如果函数 $y=f(x)$ 在区间 I_x 上单调增加 (或单调减少) 且连续，那么它的反函数 $x=f^{-1}(y)$也在对应的区间 $I_y=\{y \mid y=f(x), x \in I_x\}$上单调增加 (或单调减少)且连续。

Theorem 2 If the function $y=f(x)$ is continuous and increasing (or decreasing) on the interval I_x, then its inverse function $x=f^{-1}(y)$ is also continuous and increasing (or decreasing) on the corresponding interval $I_y=\{y \mid y=f(x), x \in I_x\}$.

定理 3 设函数 $y=f[g(x)]$是由函数 $y=f(u)$与函数 $u=g(x)$复合而成，$\overset{o}{U}(x_0) \subset D_{f \circ g}$。若 $\lim\limits_{x \to x_0} g(x) = u_0$，而函数 $y=f(u)$在 $u=u_0$ 连续，则

$$\lim_{x \to x_0} f[g(x)] = \lim_{u \to u_0} f(u) = f(u_0)\text{。}$$

Theorem 3 Suppose the function $y=f[g(x)]$ is the composition of $y=f(u)$ and $u=g(x)$, $\overset{o}{U}(x_0) \subset D_{f \circ g}$. If $\lim\limits_{x \to x_0} g(x) = u_0$, and $y=f(u)$ is continuous at $u=u_0$, then $\lim\limits_{x \to x_0} f[g(x)] = \lim\limits_{u \to u_0} f(u) = f(u_0)$.

定理 4 设函数 $y=f[g(x)]$是由函数 $y=f(u)$与函数 $u=g(x)$复合而成，$\overset{o}{U}(x_0) \subset D_{f \circ g}$。若函数 $u=g(x)$ 在 $x=x_0$ 连续，且 $g(x_0)=u_0$，而函数 $y=f(u)$ 在 $u=u_0$ 连续，则复合函数 $y=f[g(x)]$ 在 $x=x_0$ 也连续。

Theorem 4 Suppose the function $y=f[g(x)]$ is the composition of $y=f(u)$ and $u=g(x)$, $\overset{o}{U}(x_0)\subset D_{f\circ g}$. If $u=g(x)$ is continuous at $x=x_0$, $g(x_0)=u_0$, and $y=f(u)$ is continuous at $u=u_0$, then the composite function $y=f[g(x)]$ is also continuous at $x=x_0$.

三、初等函数的连续性 (Continuity of the Elementary Functions)

基本初等函数在它们的定义域内都是连续的。

Basic elementary functions are continuous on their domain.

一切初等函数在其定义区间内都是连续的。

Every elementary function is continuous on its well-defined intervals.

1.10 闭区间上连续函数的性质 (Properties of Continuous Functions on a Closed Interval)

一、有界性与最大值最小值定理 (Boundedness and Max-min Theorem)

定理1（有界性与最大值最小值定理） 在闭区间上连续的函数在该区间上有界且一定能取得它的最大值和最小值。

Theorem 1 (Boundedness and max-min theorem) If $f(x)$ is a continuous function on a closed interval, then $f(x)$ has an absolute maximum and an absolute minimum on the interval. In particular, $f(x)$ must be bounded on the interval.

二、零点定理与介值定理 (The Intermediate Value Theorem)

定理 2 设函数 $f(x)$在闭区间$[a, b]$上连续，且 $f(a)$与 $f(b)$异号(即 $f(a)\cdot f(b)<0$)，那么在开区间(a, b)内至少有一点 ξ 使 $f(\xi)=0$。

Theorem 2 Let $f(x)$ be a continuous function on the interval $[a, b]$ and $f(a)\cdot f(b)<0$, then there is a $\xi\in(a, b)$, such that $f(\xi)=0$.

定理 3 (介值定理) 设函数 $f(x)$在闭区间$[a, b]$上连续，且在这区间的端点取不同的函数值 $f(a)=A$ 及 $f(b)=B$，那么，对于 A 与 B 之间的任意一个数 C，在开区间(a, b)内至少有一点 ξ 使 $f(\xi)=C$ $(a<\xi<b)$。

Theorem 3 (The Intermediate Value Theorem) Let $f(x)$ be a continuous function on the interval $[a, b]$ and let C be a number between $f(a)=A$ and $f(b)=B$, then there is at least one number $\xi\in(a, b)$, such that $f(\xi)=C$ $(a<\xi<b)$.

推论 在闭区间上连续的函数必取得介于最大值 M 与最小值 m 之间的任何值。

Corollary Let $f(x)$ be a continuous function on a closed interval, then $f(x)$ can obtain any number between its absolute maximum M and its absolute minimum m.

▶第二章 导数与微分 Chapter 2 Derivative and Differential

2.1 导数概念(The Concept of Derivative)

一、引例(Introduction)

1. 直线运动的速度 velocity of rectilinear motion
2. 切线问题 tangent line problem

二、导数的定义(Definition of Derivative)

函数在一点处的导数与导函数 (The Derivative at a Point and the Derivative as a Function)

定义 设函数 $y=f(x)$在点 x_0 的某个邻域内有定义，当自变量 x 在 x_0 处取得增量Δx (点 $x_0+\Delta x$ 仍在该邻域内)时，相应地函数 y 取得增量$\Delta y=f(x_0+\Delta x)-f(x_0)$; 如果$\Delta y$ 与Δx 之比当$\Delta x\to 0$ 时的极限存在，则称函数 $y=f(x)$在点 x_0 处可导，并称这个极限为函数 $y=f(x)$在点 x_0 处的导数，记为 $f'(x_0)$，即

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x},$$

也可记作 $y'|_{x=x_0}$, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$ 或 $\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0}$ 。

Definition Let $f(x)$ be a well–defined function on some neighborhood of x_0, if the value of the independent variable x changes from x_0 to $x_0+\Delta x$, the corresponding change in the dependent variable, y, will be $\Delta y=f(x_0+\Delta x)-f(x_0)$; if the limit of the ratio $\Delta y/\Delta x$ exists as $\Delta x\to 0$, then we say that $f(x)$ is differentiable at $x=x_0$, i.e.

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x},$$

and we call this limit the derivative of $f(x)$ at $x=x_0$, denoted by $f'(x_0)$ or $y'|_{x=x_0}$, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0}$, or $\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=x_0}$.

三、导数的几何意义(The Geometric Meaning of Derivatives)

四、函数可导性与连续性的关系(Continuity and Derivability of Functions)

2.2 函数的求导法则 (Rules for Finding Derivatives)

一、函数的和、差、积、商的求导法则(Rules for Finding Derivatives of the Sum, Difference, Product, Quotient of Two Functions)

定理 1 如果函数 $u=u(x)$及 $v=v(x)$都在点 x 具有导数，那么它们的和、差、积、商（除分母为零的点外）都在点 x 具有导数，且

(1) $[u(x)\pm v(x)]'=u'(x)\pm v'(x)$;

(2) $[u(x)\,v(x)]'=u'(x)\,v(x)+u(x)v'(x)$;

(3) $[u(x)/v(x)]'=[u'(x)\,v(x)-u(x)v'(x)]/v^2(x)\quad (v(x)\neq 0)$。

Theorem 1 If the function $u=u(x)$ and $v=v(x)$ are differentiable at x, then the sum, difference, product and quotient $(v(x)\neq 0)$ of the two funcitons are differentiable at x, and

(1) $[u(x)\pm v(x)]'=u'(x)\pm v'(x)$;

(2) $[u(x)\,v(x)]'=u'(x)\,v(x)+u(x)v'(x)$;

(3) $[u(x)/v(x)]'=[u'(x)\,v(x)-u(x)v'(x)]/v^2(x)\ (v(x)\neq 0)$.

二、反函数的求导法则(Rule for Derivative of Inverse Functions)

定理 2 如果函数 $x=f(y)$在区间 I_y 内单调、可导且 $f'(y)\neq 0$, 则它的反函数 $y=f^{-1}(x)$ 在区间 $I_x=\{x\mid x=f(y), y\in I_y\}$内也可导，且

$$[f^{-1}(x)]'=1/f'(y),\quad \mathrm{d}y/\mathrm{d}x=1/(\mathrm{d}x/\mathrm{d}y)。$$

Theorem 2 If the function $x=f(y)$ is monotonic , differentiable on the interval I_y and $f'(y)\neq 0$, then its inverse function $y=f^{-1}(x)$ is differentiable on the interval $I_x=\{x \mid x=f(y), y\in I_y\}$ and $[f^{-1}(x)]'=1/f'(y)$，$dy/dx=1/(dx/dy)$.

三、复合函数的求导法则—链式法则 (Rule for Derivative of Composite Functions—Chain Rule)

定理 3 如果 $u=g(x)$在点 x 可导，而 $y=f(u)$在点 $u=g(x)$可导，则复合函数 $y=f[g(x)]$在点 x 可导，且其导数为

$$\frac{dy}{dx}=f'(u)\cdot g'(x) \quad 或 \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Theorem 3 Let $y=f(u)$ and $u=g(x)$. If $g(x)$ is differentiable at x and $f(u)$ is differentiable at $u=g(x)$, then the composite function $y=f[g(x)]$ is differentiable at x and $\frac{dy}{dx}= f'(u)\cdot g'(x)$ or $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

四、基本求导法则与导数公式 (Basic Rules for Finding Derivatives and the Derivative Formulas)

常数和基本初等函数的导数公式 (The Derivative Formulas of the Constant Function and the Basic Elementary Functions)

2.3 高阶导数(Higher-order Derivatives)

2.4 隐函数及由参数方程所确定的函数的导数 相关变化率(Derivatives of Implicit Functions and Functions Determined by Parametric Equation and Correlative Change Rate)

一、隐函数的导数(Derivatives of Implicit Functions)

二、由参数方程所确定的函数的导数 (Derivatives of Functions Determined by Parametric Equation)

三、相关变化率(Correlative Change Rate)

2.5 函数的微分 (Differential of a Function)

一、微分的定义(Definition of differential)

定义 设函数 $y=f(x)$在某区间内有定义，x_0及 $x_0+\Delta x$ 在这区间内，如果函数的增量 $\Delta y=f(x_0+\Delta x)-f(x_0)$ 可表示为 $\Delta y=A\Delta x+o(\Delta x)$，其中 A 是不依赖于Δx 的常数，那么称函数 $y=f(x)$在点 x_0是可微的，而 $A\Delta x$ 叫做函数 $y=f(x)$在点 x_0相应于自变量增量Δx 的微分，记作 dy，即 $dy=A\Delta x$。

Definition Let $y=f(x)$ be a function defined on some interval I, $x_0\in I$, $x_0+\Delta x\in I$, if the increment of the dependent variable $\Delta y=f(x_0+\Delta x)-f(x_0)$ can be expressed as $\Delta y=A\Delta x+o(\Delta x)$, where A is a constant which is independent of Δx, then we say that $f(x)$ is differentiable at x_0, and $A\Delta x$ is called the differential of $y=f(x)$ at x_0 corresponding to the increment Δx of the independent variable, denoted by dy, i.e. $dy=A\Delta x$.

二、微分的几何意义(The Geometric Meaning of Differential)

三、基本初等函数的微分公式与微分运算法则 (The Differential Formulas of the Basic Elementary Functions and the Rules for Differentiation)

1. 基本初等函数的微分公式(The Differential Formulas of the Basic Elementary Functions)

2. 函数的和、差、积、商的微分法则(The Differentiation Rules of the Sum, Difference, Product, Quotient of Functions)

3. 复合函数的微分法则(The Differentiation Rule of

Composite Functions)

四、微分在近似计算中的应用(Application of Differential in Approximation)

1. 函数的近似计算(Approximation of Functions)

2. 误差估计(Error Estimate)

▶第三章 微分中值定理与导数的应用
Chapter 3 Mean Value Theorem of Differentials and the Application of Derivatives

3.1 微分中值定理 (The Mean Value Theorem)

一、罗尔定理(Rolle's Theorem)

费马引理（Fermat Lemma）

设函数 $f(x)$在点 x_0 的某邻域 $U(x_0)$内有定义，并且在 x_0 处可导，如果对任意的 $x\in U(x_0)$，有 $f(x)\leqslant f(x_0)$ (或 $f(x)\geqslant f(x_0)$)，那么 $f'(x_0)=0$。

Let $f(x)$ be defined on the open interval $(x_0-\delta, x_0+\delta)$ for some δ. If $f(x)$ is differentiable at x_0, and for any x in $(x_0-\delta, x_0+\delta)$, $f(x)\leqslant f(x_0)$ (or $f(x)\geqslant f(x_0)$), then $f'(x_0)=0$.

驻点、奇异点和临界点

(1) 如果函数在 c 点的导数 $f'(c)=0$，则称 c 点为驻点；

(2) 如果c是区间$I=[a, b]$的内点，且函数在c点的导数 $f'(c)$ 不存在，则称 c 点为奇异点；

(3) 函数的定义域内的驻点、奇异点和端点统称为函数的临界点。

Stationary Point, Singular Point, and Critical Point

(1) If c is a point at which $f'(c)=0$, we call c a stationary point;

(2) If c is an interior point of $I=[a, b]$ where $f'(c)$ fails to exist, we call c a singular point;

(3) Any point of the three types, including stationary point, singular point and end point, in the domain of a function is called a critical point of $f(x)$.

罗尔定理(Rolle's Theorem)

如果函数 $f(x)$满足：

(1) 在闭区间$[a, b]$上连续；

(2) 在开区间(a, b)内可导；

(3) 在区间端点处的函数值相等，即 $f(a)=f(b)$，那么在(a, b)内至少有一点ξ（$a<\xi<b$），使得 $f'(\xi)=0$。

Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a)=f(b)$, then there is at least one number ξ in (a, b) such that $f'(\xi)=0$.

二、拉格朗日中值定理(Lagrange's Mean Value Theorem)

如果函数 $f(x)$满足：

(1) 在闭区间$[a, b]$上连续；

(2) 在开区间(a, b)内可导，

那么在(a, b)内至少有一点ξ（$a<\xi<b$），使得

$$f(b)-f(a)=f'(\xi)(b-a)。$$

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one number ξ in (a, b) where

$$\frac{f(b)-f(a)}{b-a}=f'(\xi),$$

or equivalently, where $f(b)-f(a)=f'(\xi)(b-a)$.

定理 如果函数$f(x)$在区间 I 上的导数恒为 0，则$f(x)$在区间I上是一个常数。

Theorem If $f(x)$ is defined on an interval I with $f'(x)=0$ for all x in I, then there is a constant C such that $f(x) = C$ for all x in I.

三、柯西中值定理 (Cauchy's Mean Value Theorem)

如果函数$f(x)$及$F(x)$满足:

(1) 在闭区间$[a, b]$上连续;

(2) 在开区间(a, b)内可导;

(3) 对任一$x\in(a, b)$，$F'(x)\neq 0$，

那么在(a, b)内至少有一点ξ，使得

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)}。$$

Let the functions $f(x)$ and $F(x)$ be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). If $F'(x)\neq 0$ for all x in (a, b), then there is at least one number ξ in (a, b) such that

$$\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(\xi)}{F'(\xi)}.$$

3.2 洛必达法则(L'Hopital's Rule)

未定式(Indeterminate Form)

如果当$x\to a$ (或$x\to\infty$)时，两个函数都趋于零或都趋于无穷大，那么极限$\lim\limits_{\substack{x\to a\\(x\to\infty)}}\frac{f'(x)}{F'(x)}$可能存在，也可能不存在。通常把这种极限叫作未定式，并分别简记为 0/0 或 ∞/∞。

If the limits of $f(x)$ and $F(x)$ are both 0 or infinity as x approaches a (or ∞), then $\lim\limits_{\substack{x\to a\\(x\to\infty)}}\frac{f'(x)}{F'(x)}$ exists or does not exist, which is called the indeterminate form and represented by 0/0 or ∞/∞.

定理 1 未定式 0/0，$x\to a$ (Indeterminate Form 0/0 as x Approaches a)

设

(1) 当$x\to a$时，函数$f(x)$和$F(x)$都趋于零;

(2) 在点a的某去心邻域内$f'(x)$和$F'(x)$都存在且$F'(x)\neq 0$;

(3) $\lim\limits_{x\to a}\frac{f'(x)}{F'(x)}$存在（或为无穷大），

那么

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)}。$$

定理 2 未定式 0/0，$x\to\infty$ (Indeterminate Form 0/0 as x Approaches ∞)

设

(1) 当$x\to\infty$时，函数$f(x)$和$F(x)$都趋于零;

(2) 当$|x|>N$时，$f'(x)$和$F'(x)$都存在且$F'(x)\neq 0$;

(3) $\lim\limits_{x\to a}\frac{f'(x)}{F'(x)}$存在（或为无穷大），

那么

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)}$$

L'Hôpital's Rule for Forms of Type 0/0

Suppose that $\lim_{x\to a}f(x)=\lim_{x\to a}F(x)=0$. If $\lim_{x\to a}\frac{f'(x)}{F'(x)}$ exists in either the finite or infinite sense (i.e. , if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)},$$

here a may stand for any of the symbols a, a^-, a^+, $-\infty$ or $+\infty$.

未定式∞/∞情形(Indeterminate Form ∞/∞ as x Approaches a)

设

(1) 当 $x\to a$ 时，函数 $f(x)$和 $F(x)$都趋于∞;

(2) 在点 a 的某去心邻域内 $f'(x)$和 $F'(x)$都存在且 $F'(x)\neq 0$;

(3) $\lim_{x\to a}\frac{f'(x)}{F'(x)}$存在（或为无穷大），

那么

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)}。$$

L'Hôpital's Rule for Forms of Type ∞/∞

Suppose that $\lim_{x\to a}f(x)=\lim_{x\to a}F(x)=\infty$. If $\lim_{x\to a}\frac{f'(x)}{F'(x)}$ exists in either the finite or infinite sense, then

$$\lim_{x\to a}\frac{f(x)}{F(x)}=\lim_{x\to a}\frac{f'(x)}{F'(x)},$$

here a may stand for any of the symbols a, a^-, a^+, $-\infty$ or $+\infty$.

3.3 泰勒公式 (Taylor's Formula)

泰勒中值定理 (Taylor's Mean Value Theorem)

如果函数$f(x)$在含有x_0的某个开区间(a, b)内具有直到$n+1$阶导数，则对任意 $x\in(a, b)$，有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x), \tag{3}$$

其中

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \tag{4}$$

这里ξ是 x 与 x_0之间的某个值。

公式(3)称为$f(x)$按 $(x-x_0)$ 的幂展开的带有拉格朗日型余项的 n 阶泰勒公式，$R_n(x)$的表达式(4)称为拉格朗日型余项。

Let $f(x)$ be a function whose $(n+1)st$ derivative $f^{(n+1)}(x)$ exists for each x in an open interval (a, b) containing x_0. Then for each x in (a, b),

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x), \tag{3}$$

where the remainder (or error) $R_n(x)$ is given by the formula

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}, \tag{4}$$

and ξ is some point between x and x_0.

The equation (3) is called the Taylor's Formula with Lagrange Remainder term of order n based at x_0, and (4) is called the Lagrange Remainder term.

佩亚诺型余项 (Peano Remainder Term)

n 阶泰勒公式也可以写成

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o[(x-x_0)^n],$$

上式称为 $f(x)$ 按 $(x-x_0)$ 的幂展开的带有佩亚诺型余项 n 阶泰勒公式，$o[(x-x_0)^n]$ 称为佩亚诺型余项。

Taylor's Formula of order n can be written in the following:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o[(x-x_0)^n],$$

which is called the Taylor's Formula with Peano Remainder term of order n based at x_0, and $o[(x-x_0)^n]$ is called the Peano Remainder term.

麦克劳林公式 (Maclaurin's Formula)

当 $x_0=0$ 时，带有拉格朗日型余项的 n 阶泰勒公式简化为带有拉格朗日型余项的 n 阶麦克劳林公式，形式如下：

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}\quad(0<\theta<1)。$$

When $x_0=0$, the Taylor's Formula with Lagrange Remainder term of order n simplifies the Maclaurin's Formula with Lagrange Remainder term of order n, which is in the form of the following:

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1},\quad(0<\theta<1).$$

3.4 函数的单调性和曲线的凹凸性 (Monotonicity of Functions and Concavity of Curves)

一、函数单调性的判别法(Test for Increasing or Decreasing Functions)

定理 1 单调性定理 (Monotonicity Theorem)

设函数 $f(x)$ 在 $[a, b]$ 上连续，在 (a, b) 内可导。

(1) 如果在 (a, b) 内 $f'(x)>0$，那么函数 $y=f(x)$ 在 $[a, b]$ 上单调增加；

(2) 如果在 (a, b) 内 $f'(x)<0$，那么函数 $y=f(x)$ 在 $[a, b]$ 上单调减少。

Let $f(x)$ be continuous on an interval $[a, b]$ and differentiable at every interior point of $[a, b]$.

(1) If $f'(x)>0$ for all x interior to $[a, b]$, then $f(x)$ is increasing on $[a, b]$;

(2) If $f'(x)<0$ for all x interior to $[a, b]$, then $f(x)$ is decreasing on $[a, b]$.

寻找函数单调区间的方法(Guidelines for Finding Intervals on which a Function is Increasing or Decreasing)

设函数$f(x)$在(a, b)上连续。寻找函数的单调区间可按下面的步骤：

(1) 找出函数$f(x)$的临界点，并根据这些点决定验证区间；

(2) 在每个区间取一个点确定$f'(x)$的符号；

(3) 利用函数的单调性定理判定函数在每个区间上是单调增还是单调减。

Let $f(x)$ be continuous on the interval (a, b). To find the open intervals on which $f(x)$ is increasing or decreasing, we suggest the following steps:

(1) Locate the critical numbers of $f(x)$ in (a, b), and use the numbers to determine test intervals;

(2) Determine the sign of $f'(x)$ at one value in each of the test intervals;

(3) Use the monotonicity theorem to decide whether $f(x)$ is increasing or decreasing on each intervals.

二、曲线的凹凸性与拐点(Concavity and Inflection Point)

定义 设$f(x)$在区间I上连续，如果对I上任意两点x_1, x_2，恒有

$$f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2},$$

那么称$f(x)$在I上的图形是(向上)凹的(或凹弧)；如果恒有

$$f\left(\frac{x_1+x_2}{2}\right)>\frac{f(x_1)+f(x_2)}{2},$$

那么称$f(x)$在I上的图形是(向上)凸的(或凸弧)。

Definition Let $f(x)$ be continuous on an interval I. For any two points x_1 and x_2 in interval I, if

$$f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2},$$

we say that $f(x)$ (as well as its graph) is concave up on I; if

$$f\left(\frac{x_1+x_2}{2}\right)>\frac{f(x_1)+f(x_2)}{2},$$

we say that $f(x)$ (as well as its graph) is concave down on I.

定理 2 凹凸性定理(Concavity Theorem)

如果$f(x)$在$[a, b]$上连续，在(a, b)内具有一阶和二阶导数，那么

(1) 若在(a, b)内$f''(x)>0$，则$f(x)$在$[a, b]$上的图形是凹的；

(2) 若在(a, b)内$f''(x)<0$，则$f(x)$在$[a, b]$上的图形是凸的。

Let $f(x)$ be twice differentiable on the open interval (a, b). Then

(1) If $f''(x)>0$ for all x in (a, b), then $f(x)$ is concave up on (a, b);

(2) If $f''(x)<0$ for all x in (a, b), then $f(x)$ is concave down on (a, b).

拐点 (Points of Inflection)

设$y=f(x)$在区间I上连续，x_0是I的内点。称$(x_0, f(x_0))$是曲线$y=f(x)$的一个拐点，如果$f(x)$在x_0点的一侧是凹的，而在x_0点的另一侧是凸的。

Let $f(x)$ be continuous at x_0. We call $(x_0, f(x_0))$ an inflection point of the graph of $f(x)$ if $f(x)$ is concave up on one side of x_0 and concave down on the other side.

3.5 函数的极值与最大最小值 (Extrema, Maxima and Minima of Functions)

一、函数极值及其求法(Extrema and the Guidelines for Finding Extrema)

定义 极值 (Definition of Extrema)

设函数 $f(x)$在点 x_0 的某邻域 $U(x_0)$内有定义，如果对于去心邻域 $\mathring{U}(x_0)$内的任一点 x，有

$$f(x)<f(x_0)\quad (\text{或} f(x)>f(x_0)),$$

那么称 $f(x_0)$是函数 $f(x)$的一个极大值(或极小值)。 函数的极大值和极小值统称为极值。

Let $f(x)$ be defined on $U(x_0)$ which contains the point x_0.

(1) $f(x_0)$ is the maximum of $f(x)$ if $f(x_0)>f(x)$ for all x in $\mathring{U}(x_0)$;

(2) $f(x_0)$ is the minimum of $f(x)$ if $f(x_0)<f(x)$ or all x in $\mathring{U}(x_0)$;

The minimum and maximum of a function on an interval are called the extreme values, or extrema, of the function on the interval.

定理 1 (必要条件) 设函数 $f(x)$在 x_0 处可导，且在 x_0 处取得极值，那么 $f'(x_0)=0$。

Theorem 1 (Necessary Condition) If $f(x)$ is differentiable at x_0, and $f(x_0)$ is a local extreme value of $f(x)$, then $f'(x_0)=0$.

定理 2 (第一充分条件) (The First Sufficient Condition)

设函数 $f(x)$在 x_0 处连续，且在 x_0 的某去心邻域$\mathring{U}(x_0,\delta)$ 内可导，

(1) 若 $x\in(x_0-\delta,\ x_0)$时，$f'(x)>0$，而 $x\in(x_0, x_0+\delta)$ 时，$f'(x)<0$，

则 $f(x)$在 x_0 处取得极大值；

(2) 若 $x\in(x_0-\delta, x_0)$时，$f'(x)<0$，而 $x\in(x_0, x_0+\delta)$ 时，$f'(x)>0$，则 $f(x)$在 x_0 处取得极小值；

(3) 若 $x\in\mathring{U}(x_0,\delta)$ 时，$f'(x)$的符号不变，则 $f(x_0)$不是极值。

Let $f(x)$ be continuous on an open interval $(x_0-\delta,\ x_0+\delta)$ that contains a critical point x_0;

(1) If $f'(x)>0$ for all x in $(x_0-\delta, x_0)$ and $f'(x)<0$ for all x in $(x_0, x_0+\delta)$, then $f(x_0)$ is a local maximum value of $f(x)$;

(2) If $f'(x)<0$ for all x in $(x_0-\delta, x_0)$ and $f'(x)>0$ for all x in $(x_0, x_0+\delta)$, then $f(x_0)$ is a local minimum value of $f(x)$;

(3) If $f'(x)$ has the same sign on both sides of x_0, then $f(x_0)$ is not a local extreme value of $f(x)$.

寻找闭区间上函数极值的方法(Guidelines for Finding Extrema on a Closed Interval)

如果函数 $f(x)$在区间$[a, b]$上连续，可按下列步骤求 $f(x)$在该区间内的极值点和相应的极值：

(1) 求出导数 $f'(x)$；

(2) 求出 $f(x)$的全部驻点和不可导点；

(3) 考察 $f'(x)$的符号在每个驻点或不可导点的左、右临近的情形，以确定该点是否为极值点；如果是极值点，进一步确定是极大值点还是极小值点；

(4) 求出每个极值点的函数值，就得到函数 $f(x)$的全部极值。

To find the extrema of a continuous function $f(x)$on a closed interval $[a, b]$, we suggest the following steps:

(1) Find the critical numbers of $f(x)$;

(2) Evaluate $f(x)$ at each critical number in (a, b);

(3) Evaluate $f(x)$ at each end point of $[a, b]$;

(4) The least of these values is the minimum, and the greatest is the maximum.

定理 3 (第二充分条件) (The Second Sufficient Condition)

设函数$f(x)$在 x_0 处具有二阶导数且 $f'(x_0)=0, f''(x_0)\neq 0$，那么

(1) 当 $f''(x_0)<0$ 时，那么 $f(x)$在 x_0 处取得极大值；

(2) 当 $f''(x_0)>0$ 时，那么 $f(x)$在 x_0 处取得极小值。

Let $f(x)$ be twice differentiable at x_0 and $f'(x_0)=0$, $f''(x_0)\neq 0$, then

(1) If $f''(x_0)<0$, $f(x_0)$ is a local maximum value of $f(x)$;

(2) If $f''(x_0)>0$, $f(x_0)$ is a local minimum value of $f(x)$.

二、最大最小值问题(Max-Min Problems)

求函数$f(x)$在$[a, b]$上的最大值和最小值的方法：

步骤 1 画出问题的图形，并用合适的变量来刻画其中主要的量；

步骤 2 设要对量 $Q(x)$求极值，用上面所给出的变量对 $Q(x)$公式化；

步骤 3 利用题目的条件消去其它变量，将 $Q(x)$表示成单个变量的函数，比如 $Q(x)$；

步骤 4 求出 x 的所有可能的值，通常是一个区间(a, b)；

步骤 5 寻找 $Q(x)$在(a, b)内的所有临界点（端点，驻点和奇异点）。通常，主要的临界点是驻点，即，使得 $dQ/dx=0$ 的点；

步骤 6 计算 $Q(x)$在所有临界点的取值，利用本章的定理去判断哪个临界点是最大值(或最小值)。

A Summary of the Method in Max-min Problems

Step 1 Draw a picture for the problem and assign appropriate variables to the key quantities;

Step 2 Write a formula for the quantity $Q(x)$ to be maximized (minimized) in terms of those variables;

Step 3 Use the condition of the problem to eliminate all but one of these variables, and thereby express $Q(x)$ as a function of a single variable, such as $Q(x)$;

Step 4 Determine the set of possible values for x, usually an interval (a, b);

Step 5 Find the critical points (end points, stationary points, singular points) of $Q(x)$ in (a, b). Frequently, the key critical points are the stationary points where $dQ/dx=0$;

Step 6 Compute the value of $Q(x)$ at the critical points, and use the theory of this chapter to decide which critical point gives the maximum (minimum).

3.6 函数图形的描绘(Graphing Functions)

函数图形描绘的方法 (Method of Graphing Functions)

步骤 1 预分析

(a) 判定函数的定义域和值域，去掉平面区域中不在其中的点；

(b) 验证函数是否关于 y 轴或原点对称（即函数是否为奇函数或偶函数)；

(c) 求出截距；

步骤 2 微积分学的分析

(a) 利用函数的一阶导数找出图形的临界点，判断图形的增

减性；

(b) 判断哪些临界点是最大值点或最小值点；

(c) 利用函数的二阶导数判断图形的凹凸性，找出拐点；

(d) 找出函数的水平、铅直渐近线以及其它变化趋势；

步骤 3 绘出一些点（包括所有的临界点和拐点）；

步骤 4 勾画出图形。

Step 1 Precalculus analysis

(a) Check the domain and range of the function to see if any regions of the plane are excluded;

(b) Test for symmetry with respect to the y-axis and the origin(Is the function even or odd?).

(c) Find the intercepts.

Step 2 Calculus analysis

(a) Use the first derivative to find the critical points and to find out where the graph is increasing and decreasing.

(b) Test the critical points for local maxima and minima.

(c) Use the second derivative to find out where the graph is concave upward and concave downward and to locate inflection points.

(d) Find the asymptotes.

Step 3 Plot a few points (including all critical points and inflection points).

Step 4 Sketch the graph.

3.7 曲率(Curvature)

一、弧微分(Arc Differential)

设函数 $f(x)$在区间(a, b)内具有连续导数，且曲线 $y=f(x)$上

点$(x, f(x))$的弧长为 $s(x)$，则弧长的微分是 $ds=\sqrt{1+y'^2}dx$。

Let $f(x)$ have continuous derivative on the open interval (a, b), and $s(x)$ is the function of arc length at point $(x, f(x))$, then

$$ds=\sqrt{1+y'^2}dx,$$

which is called the arc differential of the curve.

二、曲率及其计算公式(Curvature and its Computation Formula)

曲率(Curvature)

设曲线 C(平面曲线或空间曲线)是光滑的，在曲线 C 上选定一点 M_0 作为度量弧 s 的基点。设曲线上点 $M(x, y)$对应弧 s，在 M 点处切线的倾角为α，曲线上另外一点 M' 对应于弧 $s+\Delta s$，在点 M'处切线的倾角为$\alpha+\Delta\alpha$，则弧段 MM'的**平均曲率**为 $\bar{K}=|\frac{\Delta\alpha}{\Delta s}|$，而曲线 C 在点 M 处的**曲率**为 $K=\lim\limits_{\Delta s\to 0}|\frac{\Delta\alpha}{\Delta s}|$。

Let C be a smooth curve (in plane or in space). α is the angle between the tangent line of C at point M and x-axis, and $\alpha+\Delta\alpha$ is the angle between the tangent line of C at point $M'$and x-axis. The arc length from M to M' is Δs. Then **the average curvature** of MM' is $\bar{K}=\left|\frac{\Delta\alpha}{\Delta s}\right|$, and the **curvature** of the curve C at M is $K=\lim\limits_{\Delta s\to 0}|\frac{\Delta\alpha}{\Delta s}|$.

设平面曲线的参数方程为$\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$，则曲率 K 为

$$K=\frac{|\varphi'(t)\psi''(t)-\varphi''(t)\psi'(t)|}{[\varphi'^2(t)+\psi'^2(t)]^{3/2}}。$$

Consider a plane curve with parametric equation $\begin{cases}x=\varphi(t)\\ y=\psi(t)\end{cases}$.

Then the curvature K is given by

$$K=\frac{|\varphi'(t)\psi''(t)-\varphi''(t)\psi'(t)|}{[\varphi'^2(t)+\psi'^2(t)]^{3/2}}.$$

三、曲率圆和曲率半径(Circle of Curvature and the Radius of Curvature)

设曲线 $y=f(x)$在点 $M(x, y)$处的曲率为 K ($K\neq 0$)。在点 M 处的曲线的法线上，在凹的一侧取一点 D，使得$|DM|=\frac{1}{K}=\rho$，以 D 为圆心，半径为 ρ 的圆叫做曲线在点 M 处的**曲率圆**，曲率圆的圆心 D 叫做曲线在点 M 处的**曲率中心**，曲率圆的半径ρ叫做曲线在点 M 处的**曲率半径**。

Let $M(x, y)$ be a point on a curve $y=f(x)$ where the curvature $K\neq 0$. Consider the circle that is tangent to the curve at M and has the same curvature there. Its center will lie on the concave side of the curve. This circle is called the circle of curvature (or osculating circle). Its radius $\rho=\frac{1}{K}$ is the radius of curvature, and its center is the center of curvature.

四、曲率中心的计算公式 渐屈线与渐伸线(Computation Formula of the Center of Curvature, Evolute and Involute)

设已知曲线的方程是 $y=f(x)$，且其二阶导数在点 x 不为 0，则曲线在对应点 $M(x, y)$的曲率中心 $D(\alpha, \beta)$的坐标为

$$\begin{cases}\alpha=x-\dfrac{y'(1+y'^2)}{y''}\\ \beta=y+\dfrac{1+y'^2}{y''}\end{cases}。$$

Consider the plane curve with equation $y=f(x)$, and $y''(x)\neq 0$. Let $D(\alpha, \beta)$ be the center of Curvature at $M(x, y)$. Then

$$\begin{cases}\alpha=x-\dfrac{y'(1+y'^2)}{y''}\\ \beta=y+\dfrac{1+y'^2}{y''}\end{cases}.$$

当点$(x, f(x))$沿曲线 C 移动时，相应的曲率中心 D 的轨迹曲线 G 称为曲线 C 的**渐屈线**，而曲线 C 称为曲线 G 的**渐伸线**。

Denote the center of Curvature at $(x, f(x))$ by D. When the point $(x, f(x))$ moves along the plane curve C, the trace curve of the point D is called the **evolute** of the curve C, denoted by G, and C is called the **involute** of the curve G.

3.8 方程的近似解(Solving Equation Numerically)

一、二分法(Bisection Method)

二、切线法(Tangent Line Method)

▶第四章　不定积分 Chapter 4　Indefinite Integrals

4.1　不定积分的概念与性质(The Concept and Properties of Indefinite Integrals)

一、原函数和不定积分的概念 (Antiderivatives and Indefinite Integrals)

定义 1　如果在区间 I 上，可导函数 $F(x)$导函数为 $f(x)$，即对任一 $x\in I$，都有

$$F'(x)=f(x)\ \text{或}\ \mathrm{d}F(x)=f(x)\mathrm{d}x,$$

那么函数 $F(x)$就称为 $f(x)$ (或 $f(x)\mathrm{d}x$)在区间 I 上的原函数。

Definition 1　A function $F(x)$ is called an antiderivative of the function $f(x)$ if for every x in the domain of $f(x)$，$F'(x)=f(x)$ or $\mathrm{d}F(x)=f(x)\mathrm{d}x$.

原函数存在定理(Existence Theorem of Antiderivatives)

如果函数 $f(x)$在区间 I 上连续，那么在区间 I 上存在可导函数 $F(x)$，使对任一 $x\in I$ 都有 $F'(x)=f(x)$。简单的说连续函数一定有原函数。

Let $f(x)$ be continuous on the interval I, then there exists differentiable function $F(x)$ on I, such that for any x in I, $F'(x)=f(x)$. In words, the continuous functions must have antiderivatives.

原函数的表示(Representation of Antiderivatives)

若 $F(x)$是 $f(x)$在区间 I 上的一个原函数，则 $G(x)$是 $f(x)$ 在区间 I 上的原函数当且仅当存在常数 C，对所有的 $x\in I$，有 $G(x)=F(x)+C$。

If $F(x)$ is an antiderivative of $f(x)$ on an interval I, then $G(x)$ is an antiderivative of $f(x)$ on the interval I if and only if $G(x)$ is of the form

$$G(x)=F(x)+C,\ \text{for all } x \text{ in } I,$$

where C is a constant.

定义 2　不定积分(Indefinite Integrals)

在区间 I 上，函数 $f(x)$的带有任意常数项的原函数称为 $f(x)$ (或 $f(x)\mathrm{d}x$)在区间 I 上的不定积分，记作 $\int f(x)dx$。其中 $\int$ 被称为积分号，$f(x)$称为被积函数，$f(x)\mathrm{d}x$ 被称为被积表达式，x 称为积分变量。

函数 $f(x)$的原函数的图形称为 $f(x)$的积分曲线。

Definition 2　We shall often use the term indefinite integral in place of antiderivative, that is, the antiderivative of $f(x)$ with any constant is called the indefinite integral of $f(x)$ (or $f(x)\mathrm{d}x$) in I, denoted by $\int f(x)\mathrm{d}x$. Where $\int$ is called the integral sign, $f(x)$ is called the integrand, and x is the integral variable.

The graph of the antiderivative of the function $f(x)$ is called the integral curve of $f(x)$.

二、基本积分表(Basic Integration Tables)

三、不定积分的性质(The Properties of Indefinite Integrals)

性质 1　不定积分是线性算子(Indefinite Integral is a Linear Operator)

设 $f(x)$和 $g(x)$都具有原函数(可积)，k 为常数，则

(1) $\int kf(x)\mathrm{d}x=k\int f(x)\mathrm{d}x$;

(2) $\int[f(x)+g(x)]\mathrm{d}x=\int f(x)\mathrm{d}x+\int g(x)\mathrm{d}x$，相应地，

(3) $\int[f(x)-g(x)]\mathrm{d}x=\int f(x)\mathrm{d}x-\int g(x)\mathrm{d}x$。

Property 1 Let $f(x)$ and $g(x)$ have antiderivatives (indefinite integral) and let k be a constant. Then

(1) $\int kf(x)\mathrm{d}x=k\int f(x)\mathrm{d}x$;

(2) $\int[f(x)+g(x)]\mathrm{d}x=\int f(x)\mathrm{d}x+\int g(x)\mathrm{d}x$, and consequently,

(3) $\int[f(x)-g(x)]\mathrm{d}x=\int f(x)\mathrm{d}x-\int g(x)\mathrm{d}x$.

4.2 换元积分法(Substitution Rule for Indefinite Integrals)

一、第一类换元法(First Substitution Rule for Indefinite Integrals)

定理 1 设函数 $u=\varphi(x)$可微，且 $F(u)$是函数 $f(u)$的一个原函数，则有换元公式

$$\int f(\varphi(x))\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}=F(u)+C=F(\varphi(x))+C。$$

Theorem 1 Let $\varphi(x)$ be a differentiable function and suppose that $F(u)$ is an antiderivative of $f(u)$. Then, if $u=\varphi(x)$,

$$\int f(\varphi(x))\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}=F(u)+C=F(\varphi(x))+C.$$

二、第二类换元法(Second Substitution Rule for Indefinite Integrals)

定理 2 设 $x=\varphi(t)$是单调的、可导的函数，并且$\varphi'(t)\neq0$，又设$f[\varphi(t)]\varphi'(t)$具有原函数，则有换元公式

$$\int f(x)\mathrm{d}x=[\int f(\varphi(t))\varphi'(t)\mathrm{d}t]_{t=\varphi^{-1}(x)},$$

其中 $t=\varphi^{-1}(x)$是 $x=\varphi(t)$的反函数。

Theorem 2 Let $x=\varphi(t)$ be a monotonic and differentiable function, and $\varphi'(t)\neq0$. The function $f[\varphi(t)]\varphi'(t)$ has an antiderivative, then

$$\int f(x)\mathrm{d}x=[\int f(\varphi(t))\varphi'(t)\mathrm{d}t]_{t=\varphi^{-1}(x)}.$$

Where $t=\varphi^{-1}(x)$ is the inverse function of $x=\varphi(t)$.

4.3 分部积分法 (Integration by Parts)

分部积分法(Integration by Parts)

设函数 $u=u(x)$和 $v=v(x)$ 具有连续导数，那么两个函数乘积的导数公式为

$$D_x[u(x)v(x)]=u(x)v'(x)+u'(x)v(x)$$

移项得

$$u(x)v'(x)=D_x[u(x)v(x)]-u'(x)v(x)$$

对这个等式两边求不定积分，得

$$\int u(x)v'(x)\mathrm{d}x=u(x)v(x)-\int u'(x)v(x)\mathrm{d}x$$

称上面的公式为**分部积分公式**；

因为 $\mathrm{d}v=v'(x)\mathrm{d}x$，$\mathrm{d}u=u'(x)\mathrm{d}x$，则上面的等式可以写成

$$\int u\mathrm{d}v=uv-\int v\mathrm{d}u。$$

Let $u=u(x)$ and $v=v(x)$ Then

$$D_x[u(x)v(x)]=u(x)v'(x)+u'(x)v(x),$$

or

$$u(x)v'(x)= D_x[u(x)v(x)]-u'(x)\, v(x).$$

By integrating both members of this equation, we obtain

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\mathrm{d}x\,,$$

Since $\mathrm{d}v=v'(x)\mathrm{d}x$ and $\mathrm{d}u=u'(x)\mathrm{d}x$, the preceding equation is usually written symbolically as follows:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u\,,$$

which is the **integration by parts.**

4.4 有理函数的积分(Integration of Rational Functions)

一、有理函数的积分(Integration of Rational Functions)

有理函数是指由两个多项式的商所表示的函数，即具有如下形式的函数：

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}\,,$$

其中 n 和 m 都是非负整数；a_0，a_1，…，a_n 及 b_0，b_1，…，b_m 都是实数，并且 $a_0\neq 0$，$b_0\neq 0$。

当有理函数的分子多项式的次数 n 小于其分母多项式的次数 m，即 $n<m$ 时，称该有理函数为**真分式**；而当 $n\geqslant m$ 时，称该有理函数为**假分式**。

A **rational function** is by definition the quotient of two polynomial functions，which has the following form:

$$\frac{P(x)}{Q(x)}=\frac{a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_{m-1}x+b_m}\,,$$

where n and m are nonnegative integer, the a's and b's are real numbers, and $a_0\neq 0$, $b_0\neq 0$.

A rational function $f(x)$ is a **proper fraction** if the degree of the numerator is less than that of the denominator, that is, $n<m$. While $n\geqslant m$, the rational function is an **improper fraction.**

有理函数 $\frac{P(x)}{Q(x)}$ 的分解 (The Decomposition of the Rational Functions)

可以按照下面的步骤将有理函数 $f(x)=\frac{P(x)}{Q(x)}$ 进行分解：

步骤 1 如果 $f(x)$ 是假分数，即 $P(x)$ 的次数不小于 $Q(x)$ 的次数，用 $Q(x)$ 去除 $P(x)$，得到

$$f(x) = \text{一个多项式} + \frac{N(x)}{D(x)};$$

步骤 2 将 $D(x)$ 在实数范围内分解为一次因式和二次质因式的乘积。利用代数基本定理，这种分解在理论上总是可行的；

步骤 3 $D(x)$ 如果有因式 $(ax+b)^k$，那么分解后有下面 k 个部分分式之和：

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k};$$

步骤 4 $D(x)$ 如果有因式 $(ax^2+bx+c)^m$，那么分解后有下面 m 个部分分式之和：

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_mx+C_m}{(ax^2+bx+c)^m};$$

步骤 5 令 $\frac{N(x)}{D(x)}$ 等于**步骤** 3 和 4 中所得到部分分式和相

加，待定常数的个数等于分母 $D(x)$的次数；

步骤 6　在**步骤** 5 所得到的等式的两边同乘以 $D(x)$后求解其中的未知常数。有两种求解方法：(1)利用等式两端 x 的系数和常数项都相等；或(2)代入特殊的 x 的值，从而求出待定常数。

To decompose a rational function $f(x)=\dfrac{P(x)}{Q(x)}$ into partial fractions, proceed as follows:

Step 1　If $f(x)$ is improper, that is, if $P(x)$ is of degree at least that of $Q(x)$, divide $P(x)$ by $Q(x)$, obtaining

$$f(x)=\text{a polynomial}+\frac{N(x)}{D(x)};$$

Step 2　Factor $D(x)$ into a product of the linear and irreducible quadratic fractions with real coefficients.By a theorem of algebra, this is always (theoretically) possible;

Step 3　For each factor of the form $(ax+b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k};$$

Step 4　For each factor of the form $(ax^2+bx+c)^m$, expect the decomposition to have the terms

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_mx+C_m}{(ax^2+bx+c)^m};$$

Step 5　Set $\dfrac{N(x)}{D(x)}$ equal to the sum of the terms found in steps 3 and 4. The number of constants to be determined should

equal the degree of the denominator, $D(x)$;

Step 6　Multiply both sides of the equation found in step 5 by $D(x)$ and solve for the unknown constants. This can be done by either of two methods: (1) Equate coefficients of like-degree terms or (2) assign convenient value to the variable x.

第五章　定积分
Chapter 5　Definite Integrals

5.1　定积分的概念和性质 (Concept of Definite Integral and its Properties)

一、定积分问题举例 (Examples of Definite Integral)

设 $y=f(x)$在区间$[a, b]$上非负、连续，由 $x=a$，$x=b$，$y=0$ 以及曲线 $y=f(x)$所围成的图形称为**曲边梯形**，其中曲线弧称为曲边。

Let $f(x)$ be continuous and nonnegative on the closed interval $[a, b]$. Then the region bounded by the graph of $f(x)$, the x–axis, the vertical lines $x=a$，and $x=b$ is called the **trapezoid with curved edge.**

黎曼和的定义 (Definition of Riemann Sum)

设 $f(x)$是定义在闭区间$[a, b]$上的函数，Δ是$[a, b]$的任意一个分割，

$$a=x_0<x_1<\cdots<x_{n-1}<x_n=b,$$

其中Δx_i 是第 i 个小区间的长度，c_i 是第 i 个小区间的任意一点，那么和

$$\sum_{i=1}^{n} f(c_i)\Delta x_i\,,\ x_{i-1} \leqslant c_i \leqslant x_i$$

称为**黎曼和**。

Let $f(x)$ be defined on the closed interval $[a, b]$, and let Δ be an arbitrary partition of $[a, b]$, $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$, where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i\,,\ x_{i-1} \leqslant c_i \leqslant x_i$$

is called a **Riemann sum for the partition Δ.**

二、定积分的定义(Definition of Definite Integral)

定义　定积分(Definite Integral)

设函数 $f(x)$ 在区间 $[a, b]$ 上有界，在 $[a, b]$ 中任意插入若干个分点 $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$，把区间 $[a, b]$ 分成 n 个小区间：

$$[x_0, x_1],\ [x_1, x_2],\ \cdots,\ [x_{n-1}, x_n],$$

各个小区间的长度依次为 $\Delta x_1= x_1-x_0$，$\Delta x_2= x_2-x_1$，$\cdots$，$\Delta x_n= x_n-x_{n-1}$。在每个小区间 $[x_{i-1}, x_i]$ 上任取一点 ξ_i $(x_{i-1}\leqslant\xi_i\leqslant x_i)$，作函数 $f(\xi_i)$ 与小区间长度 Δx_i 的乘积 $f(\xi_i)\,\Delta x_i$ $(i=1，2，\cdots，n)$，并作出和

$$S=\sum_{i=1}^{n} f(\xi_i)\Delta x_i\,。$$

记 $|P|=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$，如果不论对 $[a, b]$ 怎样分法，也不论在小区间 $[x_{i-1}, x_i]$ 上点 ξ_i 怎样取法，只要当 $|P|\to 0$ 时，和 S 总趋于确定的极限 I，这时我们称这个极限 I 为函数 $f(x)$ 在区间 $[a, b]$ 上的定积分(简称积分)，记作 $\int_a^b f(x)\mathrm{d}x$，即

$$\int_a^b f(x)\mathrm{d}x = I = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i\,,$$

其中 $f(x)$ 叫做**被积函数**，$f(x)\mathrm{d}x$ 叫做**被积表达式**，x 叫做**积分变量**，a 叫做**积分下限**，b 叫做**积分上限**，$[a, b]$ 叫做**积分区间**。

Let $f(x)$ be a function that is defined on the closed interval $[a, b]$. Consider a partition P of the interval $[a, b]$ into n subinterval (not necessarily of equal length) by means of points $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$, and let $\Delta x_i= x_i-x_{i-1}$. On each subinterval $[x_{i-1}, x_i]$, pick an arbitrary point ξ_i (which may be an end point); we call it a sample point for the ith subinterval. We call the sum $S=\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ a Riemann sum for $f(x)$ corresponding to the partition P.

If $\lim\limits_{|P|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ exists, we say $f(x)$ is integrable on $[a, b]$, where $|P|=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$. Moreover, $\int_a^b f(x)\mathrm{d}x$, called definite integral (or Riemann Integral) of $f(x)$ from a to b, is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i\,.$$

The equality $\lim\limits_{|P|\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i = L$ means that, corresponding to each $\varepsilon>0$, there is a $\delta>0$ such that $\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - L\right|<\varepsilon$ for all Riemann sums $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ for $f(x)$ on $[a, b]$ for which the norm $|P|$ of the associated partition is less than δ.

In the symbol $\int_a^b f(x)\mathrm{d}x$, a is called the **lower limit of integral**, b **the upper limit of integral**, and $[a, b]$ the **integral** interval.

定理 1 可积性定理 (Integrability Theorem)

设$f(x)$在区间$[a, b]$上连续，则$f(x)$在$[a, b]$上可积。

Theorem 1 If a function $f(x)$ is continuous on the closed interval $[a, b]$, it is integrable on $[a, b]$.

定理 2 可积性定理(Integrability Theorem)

设 $f(x)$在区间$[a, b]$上有界，且只有有限个间断点，则 $f(x)$在$[a, b]$可积。

Theorem 2 If $f(x)$ is bounded on $[a, b]$ and if it is continuous there except at a finite number of points, then $f(x)$ is integrable on $[a, b]$.

三、定积分的性质 (Properties of Definite Integrals)

两个特殊的定积分

(1) 如果$f(x)$在 $x=a$ 点有定义，则$\int_a^a f(x)\mathrm{d}x=0$;

(2) 如果$f(x)$在$[a, b]$上可积，则$\int_b^a f(x)\mathrm{d}x=-\int_a^b f(x)\mathrm{d}x$。

Two Special Definite Integrals

(1) If $f(x)$ is defined at $x=a$. Then $\int_a^a f(x)\mathrm{d}x=0$;

(2) If $f(x)$ is integrable on $[a, b]$. Then $\int_b^a f(x)\mathrm{d}x=-\int_a^b f(x)\mathrm{d}x$.

定积分的线性性 (Linearity of the Definite Integral)

设函数$f(x)$和 $g(x)$在$[a, b]$上都可积，k 是常数，则 $kf(x)$和$f(x)+g(x)$都可积，并且

(1) $\int_a^b kf(x)\mathrm{d}x=k\int_a^b f(x)\mathrm{d}x$;

(2) $\int_a^b [f(x)+g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$; and consequently,

(3) $\int_a^b [f(x)-g(x)]\mathrm{d}x=\int_a^b f(x)\mathrm{d}x-\int_a^b g(x)\mathrm{d}x$.

Suppose that $f(x)$ and $g(x)$ are integrable on $[a, b]$ and k is a constant. Then $kf(x)$ and $f(x)+g(x)$ are integrable, and

(1) $\int_a^b kf(x)\mathrm{d}x=k\int_a^b f(x)\mathrm{d}x$;

(2) $\int_a^b [f(x)+g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$; and consequently,

(3) $\int_a^b [f(x)-g(x)]\mathrm{d}x=\int_a^b f(x)\mathrm{d}x-\int_a^b g(x)\mathrm{d}x$.

性质 3 定积分对于积分区间的可加性 (Interval Additive Property of Definite Integrals)

设$f(x)$在区间上可积，且 a，b 和 c 都是区间内的点，则不论 a, b 和 c 的相对位置如何，都有$\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$。

Property 3 If $f(x)$ is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^c f(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x+\int_b^c f(x)\mathrm{d}x$$

no matter what the order of a, b, and c.

性质 4 如果在区间$[a, b]$上$f(x)\equiv 1$，则$\int_a^b 1\mathrm{d}x=\int_a^b \mathrm{d}x= b-a$。

Property 4 If $f(x) \equiv 1$ for every x in $[a, b]$, then $\int_a^b 1\mathrm{d}x=\int_a^b \mathrm{d}x=b-a$.

性质5 如果在区间$[a, b]$上$f(x)\geqslant 0$，则$\int_a^b f(x)\mathrm{d}x \geqslant 0\ (a<b)$。

Property 5 If $f(x)$ is integrable and nonnegative on the closed interval $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x \geqslant 0 \quad (a<b).$$

推论1、2 定积分的可比性（Comparison Property for Definite Integrals)

如果在区间$[a, b]$上，$f(x)\leqslant g(x)$，则

$$\int_a^b f(x)\mathrm{d}x \leqslant \int_a^b g(x)\mathrm{d}x,$$

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\mathrm{d}x.$$

用通俗明了的话说，就是定积分保持不等号。

Corollary 1，2 If $f(x)$ and $g(x)$ are integrable on the closed interval $[a, b]$, and $f(x)\leqslant g(x)$ for all x in $[a, b]$. Then

$$\int_a^b f(x)\mathrm{d}x \leqslant \int_a^b g(x)\mathrm{d}x,$$

and

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\mathrm{d}x.$$

In informal but descriptive language, we say that the definite integral preserves inequalities.

性质 6 积分的有界性(Boundedness Property for Definite Integrals)

如果$f(x)$在$[a, b]$上连续，且对任意的$x\in[a, b]$，都有$m\leqslant f(x)\leqslant M$，则

$$m(b-a)\leqslant \int_a^b f(x)\mathrm{d}x \leqslant M(b-a)。$$

Property 6 If $f(x)$ is continuous on $[a, b]$ and $m\leqslant f(x)\leqslant M$ for all x in $[a, b]$. Then

$$m(b-a) \leqslant \int_a^b f(x)\mathrm{d}x \leqslant M(b-a).$$

性质7 积分中值定理 (Mean Value Theorem for Definite Integrals)

如果函数$f(x)$在闭区间$[a, b]$上连续，则在积分区间$[a, b]$上至少存在一点ξ，使下式成立

$$\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a),$$

且

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

称为函数$f(x)$在区间$[a, b]$上的平均值。

Property 7 If $f(x)$ is continuous on $[a, b]$, there is at least one number ξ between a and b such that

$$\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a),$$

and

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

is called **the average value of $f(x)$** on $[a, b]$.

5.2 微积分基本定理(Fundamental Theorem of Calculus)

一、积分上限的函数及其导数(Accumulation Function and Its Derivative)

定理 1 微积分基本定理（Fundamental Theorem of

Calculus)

如果函数 $f(x)$在区间$[a, b]$上连续，则积分上限函数$\phi(x)=\int_a^x f(t)\mathrm{d}t$在$[a, b]$上可导，并且它的导数是

$$\phi'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t=f(x) \quad (a\leqslant x\leqslant b)。$$

Theorem 1 Let $f(x)$ be continuous on the closed interval $[a, b]$ and let x be a (variable) point in (a, b). Then $\phi(x)=\int_a^x f(t)\mathrm{d}t$ is differentiable on $[a, b]$, and $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t=f(x) \quad (a\leqslant x\leqslant b)$.

定理 2 原函数存在定理 (The Existence Theorem of Antiderivative)

如果函数$f(x)$在区间$[a, b]$上连续，则函数$\phi(x)=\int_a^x f(t)\mathrm{d}t$就是$f(x)$在$[a, b]$上的一个原函数。

Theorem 2 If $f(x)$ is continuous on the closed interval $[a, b]$, then $\phi(x)=\int_a^x f(t)\mathrm{d}t$ is an antiderivative of $f(x)$ on $[a, b]$.

二、牛顿—莱布尼茨公式(Newton–Leibniz Formula)

定理 3 微积分第一基本定理 (First Fundamental Theorem of Calculus)

如果函数 $F(x)$是连续函数$f(x)$在区间$[a, b]$上的一个原函数，则

$$\int_a^b f(x)\mathrm{d}x=F(b)-F(a)$$

称上面的公式为**牛顿–莱布尼茨公式。**

Theorem 3 Let $f(x)$ be continuous (hence integrable) on $[a, b]$, and let $F(x)$ be any antiderivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)\mathrm{d}x=F(b)-F(a),$$

which is called the **Newton–Leibniz Formula.**

5.3 定积分的换元法和分部积分法(Integration by Substitution and Definite Integrals by Parts)

一、定积分的换元法(Substitution Rule for Definite Integrals)

定理 定积分的换元法 (Substitution Rule for Definite Integrals)

假设函数$f(x)$在区间$[a, b]$上连续，函数 $x=\phi(t)$满足条件

(1) $\phi(\alpha)=a, \phi(\beta)=b$;

(2) $\phi(t)$在$[\alpha, \beta]$(或$[\beta, \alpha]$)上具有连续导数，且其值域$R_\phi\subset[a, b]$，则有

$$\int_a^b f(x)\mathrm{d}x=\int_\alpha^\beta f[\phi(t)]\phi'(t)\mathrm{d}t,$$

上面的公式叫做定积分的换元公式。

Theorem Let $\phi(t)$ have a continuous derivative on $[\alpha, \beta]$(or $[\beta, \alpha]$) , and let $f(x)$be continuous on $[a, b]$. If $\phi(\alpha)=a$, $\phi(\beta)=b$ and the range of $\phi(x)$ is a subset of $[a, b]$. Then

$$\int_a^b f(x)\mathrm{d}x=\int_\alpha^\beta f[\phi(t)]\phi'(t)\mathrm{d}t,$$

which is called the substitution rule for definite integrals.

二、定积分的分部积分法(Definite Integration by Parts)

根据不定积分的分部积分法，有

$$\int_a^b u(x)v'(x)\mathrm{d}x=\left[\int u(x)v'(x)\mathrm{d}x\right]_a^b$$

$$=[u(x)v(x)-\int u'(x)v(x)\mathrm{d}x]_a^b$$

$$=[u(x)v(x)]_a^b-\int_a^b v(x)u'(x)\mathrm{d}x$$

简写为

$$\int_a^b uv'\mathrm{d}x=[u\,v]_a^b-\int_a^b vu'\mathrm{d}x$$

或

$$\int_a^b u\mathrm{d}v=[u\,v]_a^b-\int v\mathrm{d}u\text{。}$$

According to the indefinite integration by parts,

$$\int_a^b u(x)v'(x)\mathrm{d}x=\left[\int u(x)v'(x)\mathrm{d}x\right]_a^b$$

$$=\left[u(x)v(x)-\int u'(x)v(x)\mathrm{d}x\right]_a^b$$

$$=\left[u(x)v(x)\right]_a^b-\int_a^b v(x)u'(x)\mathrm{d}x$$

For simplicity,

$$\int_a^b uv'\mathrm{d}x=[uv]_a^b-\int_a^b vu'\mathrm{d}x,$$

or

$$\int_a^b u\mathrm{d}v=[uv]_a^b-\int_a^b v\mathrm{d}u.$$

5.4 反常积分(Improper Integrals)

一、无穷限的反常积分(Improper Integrals with Infinite Limits of Integration)

定义 1 设函数$f(x)$在区间$[a,+\infty)$上连续，取$t>a$，如果极限$\lim\limits_{t\to+\infty}\int_a^t f(x)\mathrm{d}x$存在且为有限值，则此极限为函数$f(x)$在无穷区间$[a,+\infty)$上的**反常积分**，记作$\int_a^{+\infty}f(x)\mathrm{d}x$，即

$$\int_a^{+\infty}f(x)\mathrm{d}x=\lim_{t\to+\infty}\int_a^t f(x)\mathrm{d}x\text{。}$$

这时也称反常积分$\int_a^{+\infty}f(x)\mathrm{d}x$**收敛**；如果上述极限不存在，函数$f(x)$在无穷区间$[a,+\infty)$上的反常积分就没有意义，习惯上称为反常积分$\int_a^{+\infty}f(x)\mathrm{d}x$**发散**。

Let $f(x)$ be continuous on $[a,+\infty)$, and $t>a$. If the limit $\lim\limits_{t\to+\infty}\int_a^t f(x)\mathrm{d}x$ exists and have finite value, the value is **the improper integral of $f(x)$** on $[a,+\infty)$, which is denoted by $\int_a^{+\infty}f(x)\mathrm{d}x$, that is,

$$\int_a^{+\infty}f(x)\mathrm{d}x=\lim_{t\to+\infty}\int_a^t f(x)\mathrm{d}x,$$

We say that the corresponding improper integral **converges.** Otherwise, the integral is said to **diverge**.

设函数$f(x)$在区间$(-\infty,b]$上连续，取$t<b$，如果极限$\lim\limits_{t\to-\infty}\int_t^b f(x)\mathrm{d}x$存在且为有限值，则此极限为函数$f(x)$在**无穷区间**$(-\infty,b]$上的**反常积分**，记作$\int_{-\infty}^b f(x)\mathrm{d}x$，即

$$\int_{-\infty}^b f(x)\mathrm{d}x=\lim_{t\to-\infty}\int_t^b f(x)\mathrm{d}x,$$

这时也称反常积分$\int_{-\infty}^b f(x)\mathrm{d}x$收敛；如果上述极限不存在，就称反常积分$\int_{-\infty}^b f(x)\mathrm{d}x$发散。

Let $f(x)$ be continuous on $(-\infty,b]$, and $t<b$. If the limit $\lim\limits_{t\to-\infty}\int_t^b f(x)\mathrm{d}x$ exists and have finite value, the value is **the**

improper integral of $f(x)$ on $(-\infty, b]$, which is denoted by $\int_{-\infty}^{b} f(x)\mathrm{d}x$, that is ,

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = \lim_{t\to-\infty}\int_{t}^{b} f(x)\mathrm{d}x,$$

We say that the corresponding improper integral converges. Otherwise, the integral is said to diverge.

定义 设函数 $f(x)$在区间$(-\infty, +\infty)$上连续，如果反常积分 $\int_{-\infty}^{0} f(x)\mathrm{d}x$和 $\int_{0}^{+\infty} f(x)\mathrm{d}x$都收敛，则称上述反常积分之和为函数 $f(x)$在无穷区间$(-\infty, +\infty)$上的反常积分，记作$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$，即

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = \int_{-\infty}^{0} f(x)\mathrm{d}x + \int_{0}^{+\infty} f(x)\mathrm{d}x$$

$$= \lim_{t\to-\infty}\int_{t}^{0} f(x)\mathrm{d}x + \lim_{t\to+\infty}\int_{0}^{t} f(x)\mathrm{d}x$$

这时也称反常积分 $\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$ 收敛；否则就称反常积分 $\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$ 发散。

Let $f(x)$ be continuous on $(-\infty, +\infty)$. If both $\int_{-\infty}^{0} f(x)\mathrm{d}x$ and $\int_{0}^{+\infty} f(x)\mathrm{d}x$ converge, then $\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$ is said to converge and have value

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = \int_{-\infty}^{0} f(x)\mathrm{d}x + \int_{0}^{+\infty} f(x)\mathrm{d}x$$

$$= \lim_{t\to-\infty}\int_{t}^{0} f(x)\mathrm{d}x + \lim_{t\to+\infty}\int_{0}^{t} f(x)\mathrm{d}x,$$

Otherwise, $\int_{-\infty}^{+\infty} f(x)\mathrm{d}x$ diverges.

二、无界函数的反常积分(Improper Integrals of Infinite Integrands)

定义 无界函数的反常积分(Improper Integrals of Infinite Integrands)

设函数$f(x)$在半开半闭区间$[a, b)$上连续，且 $\lim_{x\to b^-}|f(x)|=\infty$，则

$$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to b^-}\int_{a}^{t} f(x)\mathrm{d}x,$$

如果等式右边的极限存在且为有限值，此时称反常积分收敛，否则称反常积分发散。

Definition Let $f(x)$ be continuous on the half–open interval $[a, b)$ and suppose that $\lim_{x\to b^-}|f(x)|=\infty$. Then

$$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to b^-}\int_{a}^{t} f(x)\mathrm{d}x$$

provided that this limit exists and is finite, in which case we say that the integral converge. Otherwise, we say that the integral diverges.

无界函数的反常积分(Improper Integrals of Infinite Integrands)

定义 设函数 $f(x)$在半开半闭区间$(a, b]$上连续，且 $\lim_{x\to a^+}|f(x)|=\infty$，则

$$\int_{a}^{b} f(x)\mathrm{d}x = \lim_{t\to a^+}\int_{t}^{b} f(x)\mathrm{d}x,$$

如果等式右边的极限存在且为有限值，此时称反常积分收敛，否则称反常积分发散。

Definition Let $f(x)$ be continuous on the half–open interval $(a, b]$ and suppose that $\lim\limits_{x\to a^+}|f(x)|=\infty$. Then

$$\int_a^b f(x)\mathrm{d}x=\lim_{t\to a^+}\int_t^b f(x)\mathrm{d}x,$$

provided that this limit exists and is finite, in which case we say that the integral converge. Otherwise, we say that the integral diverges.

积分函数在内点极限为∞的反常积分 (Integrands That are Infinite at an Interior Point)

设函数在 $f(x)$在$[a, b]$上除点 c $(a<c<b)$外连续，且$\lim\limits_{x\to c}|f(x)|=\infty$，则定义

$$\int_a^b f(x)\mathrm{d}x=\int_a^c f(x)\mathrm{d}x+\int_c^b f(x)\mathrm{d}x$$

$$=\lim_{t\to c^-}\int_a^t f(x)\mathrm{d}x+\lim_{t\to c^+}\int_t^b f(x)\mathrm{d}x$$

如果等式右边的两个反常积分都收敛，否则称反常积分$\int_a^b f(x)\mathrm{d}x$发散。

Let $f(x)$ be continuous on $[a, b]$ except at a number c, where $a<c<b$, and suppose that $\lim\limits_{x\to c}|f(x)|=\infty$. Then we define

$$\int_a^b f(x)\mathrm{d}x=\int_a^c f(x)\mathrm{d}x+\int_c^b f(x)\mathrm{d}x$$

$$=\lim_{t\to c^-}\int_a^t f(x)\mathrm{d}x+\lim_{t\to c^+}\int_t^b f(x)\mathrm{d}x,$$

provided both integrals on the right converge. Otherwise, we say that $\int_a^b f(x)\mathrm{d}x$ diverges.

一类特殊的反常积分(A Special Type of Improper Integral)

$$\int_1^{\infty}\frac{\mathrm{d}x}{x^p}=\begin{cases}\dfrac{1}{p-1}, & \text{if } p>1\\ \text{diverges}, & \text{if } p\leqslant 1\end{cases}.$$

▶第六章 定积分的应用 Chapter 6 Applications of the Definite Integrals

6.1 定积分的元素法(The Element Method of Definite Integrals)

分割、近似、积分方法:

步骤 1 画图

步骤 2 将其分割成小薄片。

步骤 3 用矩形面积近似代替小薄片面积。

步骤 4 将所有小薄片面积的近似值相加。

步骤 5 设所有小薄片宽度的极限趋于零，得出定积分。

Slice, approximate, integrate method:

Step 1 Sketch the region;

Step 2 Slice it into thin pieces (strips);

Step 3 Approximate the area of each thin piece as if it were a rectangle;

Step 4 Add up the approximations to the areas of the pieces;

Step 5 Take the limit as the width of the pieces approaches zero, thus getting a definite integral.

6.2 定积分在几何学上的应用(Application of the Definite Integrals to Geometry)

一、平面图形的面积(The Area of a Plane Figure)

直角坐标的情形(The Case of Rectangular Coordinates)

曲边梯形的面积(The Area of the Trapezoid with Curved Edge)

设 $y=f(x)$在区间$[a, b]$上非负、连续，由曲线 $y=f(x)$，直线 $x=a$，$x=b$ 与 x 轴所围成的曲边梯形的面积 A 是定积分 $A=\int_a^b f(x)\mathrm{d}x$。

Let $f(x)$ be continuous and nonnegative on the closed interval $[a, b]$. Then the area of the region bounded by the graph of $f(x)$, the x-axis, and the vertical lines $x=a$, and $x=b$ is given by $A=\int_a^b f(x)\mathrm{d}x$.

极坐标情形(The Case of Polar Coordinates)

如果$\varphi(\theta)$是$[\alpha, \beta]$上的非负、连续函数，则由图形$\rho=\varphi(\theta)$，射线$\theta=\alpha$ 和$\theta=\beta$所围成的图形的面积 $A=\int_\alpha^\beta \frac{1}{2}[\varphi(\theta)]^2\mathrm{d}\theta$。

If $\varphi(\theta)$ is continuous and nonnegative on the interval$[\alpha,\beta]$. Then the area of the region bounded by the graph of $\rho=\varphi(\theta)$ between the radial lines $\theta=\alpha$ and $\theta=\beta$ is given by

$$A=\int_\alpha^\beta \frac{1}{2}[\varphi(\theta)]^2\mathrm{d}\theta.$$

二、体积(Volume)

旋转体就是由一个平面图形绕这平面内的一条直线旋转一周而成的立体，这直线叫做**旋转轴**。

If a region in the plane is revolved about a line, the resulting solid is called **a solid of revolution,** and the line is called **the axis of revolution.**

旋转体的体积 (Volume of Revolution Solids)

由曲线 $x=\varphi(y)$，直线 $y=c$，$y=d$ 与 y 轴所围成的曲边梯形，绕 y 轴旋转一周而成的旋转体的体积为 $V=\int_c^d \pi[\varphi(y)]^2\mathrm{d}y$。

If the solid of revolution is generated by the revolution of the trapezoid with curved edge bounded by $x=\varphi(y)$，$y=c$，$y=d$ around y-axis. Then the volume of the above solid is given by $V=\int_c^d \pi[\varphi(y)]^2\mathrm{d}y$.

平行截面面积为已知的立体的体积(The Volume of the Solid with Known Cross Sections)

设立体在过点 $x=a$，$x=b$ 且垂直于 x 轴的两个平面之间，以 $A(x)$表示过点 x 且垂直于 x 轴的截面的面积。假定 $A(x)$为 x 的已知的连续函数。则旋转体的体积为

$$V=\int_a^b A(x)\mathrm{d}x。$$

For cross sections of area $A(x)$, taken perpendicular to the x-axis, then the volume of the solid is $V=\int_a^b A(x)\mathrm{d}x$.

三、平面曲线的弧长(Arc Length of the Plane Curve)

称平面曲线是光滑的，如果曲线可以由参数方程 $x=f(t)$,

$y=g(t)$，$a\leqslant t\leqslant b$，决定，其中$f'(t)$和$g'(t)$在$[a, b]$上连续，且$f'(t)$和$g'(t)$在(a, b)上不同时为0。

A plane curve is smooth if it is determined by a pair of parametric equations $x=f(t)$，$y=g(t)$, $a\leqslant t\leqslant b$, where $f'(t)$ and $g'(t)$ exist and are continuous on $[a, b]$, and $f'(t)$ and $g'(t)$ are not simultaneously zero on (a, b).

定理 光滑曲线弧是可求长的。

Theorem The arc length of a plane curve which is smooth can be calculated.

平面曲线的弧长(Arc Length of a Plane Curve)

设平面曲线由参数方程 $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$，$\alpha\leqslant t\leqslant\beta$，给出，其中$\varphi'(t)$和$\psi'(t)$在区间$[\alpha, \beta]$上连续。则平面曲线的弧长公式为$s=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}\mathrm{d}t$；

如果曲线由$y=f(x)$, $a\leqslant x\leqslant b$给出，其中$f'(x)$在区间$[a, b]$上连续，则

$$s=\int_a^b\sqrt{1+y'^2}\mathrm{d}x\text{；}$$

如果曲线是由极坐标方程$\rho=\rho(\theta)$,$\alpha\leqslant\theta\leqslant\beta$所给出的，其中$\rho'(\theta)$在$[\alpha, \beta]$上连续，则$s=\int_\alpha^\beta\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\mathrm{d}\theta$。

Let a plane curve be determined by a pair of parametric equations

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases},\ \alpha\leqslant t\leqslant\beta,$$

where $\varphi'(t)$ and $\psi'(t)$ are continuous on the given interval $[\alpha, \beta]$. As t increase from α to β, the point (x, y) traces out a curve in the plane. Then the arc length is defined as

$$s=\int_\alpha^\beta\sqrt{\varphi'^2(t)+\psi'^2(t)}\mathrm{d}t\,.$$

If the curve is given by $y=f(x)$, $a\leqslant x\leqslant b$, where $f'(x)$ is continuous on $[a, b]$, then the arc length of $f(x)$ is given by $s=\int_a^b\sqrt{1+y'^2}\mathrm{d}x$.

If the plane curve is given by $\rho=\rho(\theta)$, $\alpha\leqslant\theta\leqslant\beta$, where $\rho'(\theta)$ is continuous on $[\alpha, \beta]$, then

$$s=\int_\alpha^\beta\sqrt{\rho^2(\theta)+\rho'^2(\theta)}\mathrm{d}\theta\,.$$

6.3 定积分在物理学上的应用(Application of the Definite Integrals to Physics)

变力沿直线所作的功(Work Done by Variable Force)

如果一物体在作直线运动的过程中有一持续变化的力$F(x)$作用在该物体上，且这个力的方向与运动的方向一致，则物体从$x=a$移动到$x=b$，该力对物体所做的功W为$W=\int_a^b F(x)\mathrm{d}x$。

If an object is moved along a straight line by a continuously varying force $F(x)$, then the work W done by the force as the object moved from $x=a$ to $x=b$ is given by

$$W=\int_a^b F(x)\mathrm{d}x\,.$$

▶第七章　空间解析几何与向量代数
Chapter 7　Space Analytic Geometry and Vector Algebra

7.1　向量及其线性运算 (Vector and Its Linear Operation)

一、向量概念(Definition of Vectors)

客观世界中有这样一些量，它们既有大小，又有方向，这一类量叫做向量。通常用有向线段来表示，有向线段的长度表示向量的大小，有向线段的方向表示向量的方向；箭头有两端，分别称为尾部和头部。

如果两个向量的大小相等，且方向相同，则称这两个向量相等。

Many quantities that occur in science require both magnitude and a direction for complete specification. We call such quantities **vectors** and represent them by arrows. The length of the arrow represents the magnitude, or length, of the vector; its direction is the direction of the vector. Arrows have two ends. There is initial point, called the tail, and the terminal point, called head.

Two vectors are considered to be equivalent if they have the same magnitude and direction.

二、向量的线性运算(Linear Operations of Vectors)

向量的加减法(The Sum of the Vectors)

向量的加法的三角形法则(The Triangle Law of Vector Addition)

设有两个向量 $\boldsymbol{a}$ 与 $\boldsymbol{b}$，任取一点 A，做 $\overline{AB}=\boldsymbol{a}$，再以 B 为起点，作 $\overline{BC}=\boldsymbol{b}$，连接 $\overline{AC}$，那么向量 $\overline{AC}=\boldsymbol{c}$ 称为向量 $\boldsymbol{a}$ 与 $\boldsymbol{b}$ 的和，记作 $\boldsymbol{a}+\boldsymbol{b}$，即 $\boldsymbol{c}=\boldsymbol{a}+\boldsymbol{b}$，上述作出两向量之和的方法叫做**向量相加的三角形法则**。

To find the sum, or resultant, of $\boldsymbol{a}$ and $\boldsymbol{b}$, move $\boldsymbol{b}$ without changing its magnitude or direction until its tail coincides with the head of $\boldsymbol{a}$. Then $\boldsymbol{a}+\boldsymbol{b}$ is the vector connecting the tail of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$, this method is called the Triangle Law.

向量的加法的平行四边形法则(The Parallelogram Law of Vector Addition)

设有两个向量 $\boldsymbol{a}$ 与 $\boldsymbol{b}$，当 $\boldsymbol{a}$ 与 $\boldsymbol{b}$ 不平行时作 $\overline{AB}=\boldsymbol{a}$，$\overline{AD}=\boldsymbol{b}$，以 AB、AD 为边作一平行四边形 $ABCD$，连接对角线 AC，显然向量 $\overline{AC}$ 即等于向量 $\boldsymbol{a}$ 与 $\boldsymbol{b}$ 的和。上述作出两向量之和的方法叫做向量相加的**平行四边形法则**。

As an alternative way to find $\boldsymbol{a}+\boldsymbol{b}$, move $\boldsymbol{b}$ so that its tail coincides with that of $\boldsymbol{a}$. Then $\boldsymbol{a}+\boldsymbol{b}$ is the vector with this common tail and coinciding with the diagonal of the parallelogram that has $\boldsymbol{a}$ and $\boldsymbol{b}$ as sides. This method is called the Parallelogram Law.

向量的加法符合交换律和结合律，即

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$$

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$$

Vector addition is commutative and associative，that is

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a},$$

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}).$$

设 $\boldsymbol{a}$ 为一向量，与 $\boldsymbol{a}$ 的模相同而方向相反的向量叫做 $\boldsymbol{a}$ 的

负向量，记作$-\boldsymbol{a}$。且$\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$，其中$\boldsymbol{0}$是零向量，它的模等于零，起点和终点重合均为原点，它的方向可以看做是任意的。

Let $\boldsymbol{a}$ be a vector, and $-\boldsymbol{a}$ has the same length as $\boldsymbol{a}$, but opposite direction. It is called the negative of $\boldsymbol{a}$, and $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}$, where $\boldsymbol{0}$ is a zero vector whose initial point and terminate point lie at the origin.

向量与数的乘法(Scalar Multiple of Vectors)

设$\boldsymbol{a}$为一向量，λ是实数。如果$\lambda>0$，则向量$\lambda\boldsymbol{a}$与$\boldsymbol{a}$方向相同，但$\lambda\boldsymbol{a}$的模是$\boldsymbol{a}$的模的λ倍；如果$\lambda<0$，则向量$\lambda\boldsymbol{a}$与$\boldsymbol{a}$方向相反，且$\lambda\boldsymbol{a}$的模是$\boldsymbol{a}$的模的λ倍。通常称$\lambda\boldsymbol{a}$是$\boldsymbol{a}$的一个数乘，它满足结合律和分配律，即

$$\lambda(\mu\boldsymbol{a})=\mu\lambda\boldsymbol{a}=(\lambda\mu)\boldsymbol{a},$$
$$(\lambda+\mu)\boldsymbol{a}=\lambda\boldsymbol{a}+\mu\boldsymbol{a},$$
$$\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}。$$

Let $\boldsymbol{a}$ be a vector, and λ a real number. If $\lambda>0$, then $\lambda\boldsymbol{a}$ is a vector with the same direction as $\boldsymbol{a}$ but λ times as long; otherwise $\lambda\boldsymbol{a}$ is a vector with the opposite direction as $\boldsymbol{a}$ but λ times as long. In general, $\lambda\boldsymbol{a}$ is called a scalar multiple of $\boldsymbol{a}$, which satisfies associate law and distributive law, that is,

$$\lambda(\mu\boldsymbol{a})=\mu\lambda\boldsymbol{a}=(\lambda\mu)\boldsymbol{a},$$
$$(\lambda+\mu)\boldsymbol{a}=\lambda\boldsymbol{a}+\mu\boldsymbol{a},$$
$$\lambda(\boldsymbol{a}+\boldsymbol{b})=\lambda\boldsymbol{a}+\lambda\boldsymbol{b}.$$

定理　设向量$\boldsymbol{a}\neq\boldsymbol{0}$，那么，向量$\boldsymbol{b}$平行于$\boldsymbol{a}$的充分必要条件是：存在实数$\lambda$，使$\boldsymbol{b}=\lambda\boldsymbol{a}$。

Theorem　If $\boldsymbol{a}\neq\boldsymbol{0}$, then vector $\boldsymbol{b}$ is parallel with $\boldsymbol{a}$ if and only if there is a real number λ such that $\boldsymbol{b}=\lambda\boldsymbol{a}$.

空间中，起点在P点，终点在Q点的有向线段为$\overrightarrow{PQ}$，记$\|\overrightarrow{PQ}\|$为有向线段的大小，大小和方向相同的有向线段称为等价的。与给定的有向线段等价的所有有向线段的集合称为空间中的向量，记为$\boldsymbol{v}=\overrightarrow{PQ}$。

The directed line segment $\overrightarrow{PQ}$ has initial point P and terminal point Q, and we denote its length by $\|\overrightarrow{PQ}\|$. Two directed line segments that have the same length and direction are called equivalent. We call the set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ a vector in the space and write $\boldsymbol{v}=\overrightarrow{PQ}$.

三、空间直角坐标系(The Cartesian Coordinates System in Three-space)

空间中三条相互垂直的坐标线，它们零点相互重合，称为原点，记为O。坐标轴遵循右手法则，即以右手握住z-轴，当右手的四个手指从正向x-轴以$\frac{\pi}{2}$角度转向正向y-轴时，大拇指的指向就是z-轴的正向。

Consider three mutually perpendicular coordinate lines (the x-, y-, and z-axes) with their zero points at a common point O, called the origin. The three axes form the right-handed system, that is, if the fingers of the right hand are curled so that they curve from the positive x-axis toward the positive y-axis, the thumb points in the direction of the positive z-axis.

三个坐标轴决定三个坐标面，yOz面，xOz面和xOy面，这些坐标面将空间分成八个卦限。空间中的每个点P都对应一个有序的三元组(x, y, z)，x，y，和z分别表示P点到三个坐标

面的有向距离。

The three axes determine three planes, the yz-, xz- and xy-plane, which divide space into eight octants. To each point P in space corresponds an ordered triple of numbers (x, y, z), its Cartesian coordinates, which measure its directed distances from the three planes.

空间中，起点在原点，终点在(a_x, a_y, a_z)的向量的坐标分解式为 $\boldsymbol{a} = (a_x, a_y, a_z) = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$，其中 $\boldsymbol{i} =(1, 0, 0)$，$\boldsymbol{j} =(0, 1, 0)$ 和 $\boldsymbol{k} =(0, 0,1)$是指向坐标轴正方向的标准单位向量，称为基向量。

In space, the component form of a vector whose initial point is the origin and whose terminal point is (a_x, a_y, a_z) is $\boldsymbol{a} = (a_x, a_y, a_z) = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$, where $\boldsymbol{i} =(1, 0, 0)$, $\boldsymbol{j} =(0, 1, 0)$, $\boldsymbol{k} =(0, 0,1)$ are the standard unit vectors, called basis vectors, in the directions of the three positive coordinate axes.

四、利用坐标作向量的线性运算(Vector Operations by Components of Vectors)

向量 $\boldsymbol{a} =(a_x, a_y, a_z)$，$\boldsymbol{b} = (b_x, b_y, b_z)$，$k$ 为一个数量，则：

(1) 向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的和向量为 $\boldsymbol{c} = (a_x+ b_x, a_y+ b_y, a_z+ b_z)$;

(2) 向量 $\boldsymbol{a}$ 和数量 k 的乘积为 $k\boldsymbol{a} = (ka_x, ka_y, ka_z)$;

(3) 向量 $\boldsymbol{a}$ 的负向量为$-\boldsymbol{a} = (-a_x, -a_y, -a_z)$;

(4) 向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的差向量为 $\boldsymbol{c} = (a_x-b_x, a_y-b_y, a_z-b_z)$。

For vectors $\boldsymbol{a}=(a_x, a_y, a_z)$, and $\boldsymbol{b}=(b_x, b_y, b_z)$ and scalar k, we define the following operation:

(1) The vector sum of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector $\boldsymbol{c} = (a_x+ b_x, a_y+ b_y, a_z+ b_z)$;

(2) The scalar multiple of $\boldsymbol{a}$ and k is $k\boldsymbol{a} = (ka_x, ka_y, ka_z)$;

(3) The nonnegative of $\boldsymbol{a}$ is $-\boldsymbol{a} = (-a_x, -a_y, -a_z)$;

(4) The difference of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector $\boldsymbol{c} = (a_x-b_x, a_y-b_y, a_z-b_z)$.

五、向量的模、方向角、投影(Magnitude of Vectors, Direction Angles and Projections)

空间两点的距离公式 (The Distance Formula between Two Points in Space)

设 $A(x_1, y_1, z_1)$和 $B(x_2, y_2, z_2)$ 是空间内两点，则 A 和 B 之间的距离等于向量$\overline{AB}$的模，即

$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\text{。}$$

Consider two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in three-space ,then the distance between A and B is the magnitude of the vector $\overrightarrow{AB}$, which is given by

$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

方向角和方向余弦(Direction Angles and Direction cosines)

非零向量 $\boldsymbol{a}$ 和单位向量 $\boldsymbol{i}, \boldsymbol{j}$ 和 $\boldsymbol{k}$ 之间的非负的最小的角称为方向角，分别记为α, β和γ。如果 $\boldsymbol{r} = (x, y, z)$，则向量 $\boldsymbol{r}$ 的方向余弦 $\cos\alpha$，$\cos\beta$ 和 $\cos\gamma$分别为：$\cos\alpha = \frac{x}{|\boldsymbol{r}|}$，$\cos\beta = \frac{y}{|\boldsymbol{r}|}$，$\cos\gamma = \frac{z}{|\boldsymbol{r}|}$，因此，

$$(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{x}{|\boldsymbol{r}|}, \frac{y}{|\boldsymbol{r}|}, \frac{z}{|\boldsymbol{r}|}\right) = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}\text{。}$$

The (smallest nonnegative) angles between a nonzero vector $\boldsymbol{a}$ and the basis vectors $\boldsymbol{i},\boldsymbol{j}$ and $\boldsymbol{k}$ are the direction angles of $\boldsymbol{a}$; they are designed by α, β and γ, respectively. It is generally more convenient to work with the direction cosines $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. If $\boldsymbol{r}=(x, y, z)$, then $\cos\alpha=\frac{x}{|\boldsymbol{r}|}$, $\cos\beta=\frac{y}{|\boldsymbol{r}|}$, $\cos\gamma=\frac{z}{|\boldsymbol{r}|}$, hence

$$(\cos\alpha, \cos\beta, \cos\gamma)=(\frac{x}{|\boldsymbol{r}|},\frac{y}{|\boldsymbol{r}|},\frac{z}{|\boldsymbol{r}|})=\frac{\boldsymbol{r}}{|\boldsymbol{r}|}.$$

向量在轴上的投影(The Projection of Vector on Axis)

向量的投影的性质(Properties of Projection)

(1) $\mathrm{Pr}\,j_u(\boldsymbol{a})=|\boldsymbol{a}|\cos\varphi$，其中$\varphi$是向量 $\boldsymbol{a}$ 与 $\boldsymbol{u}$ 轴的夹角；

(2) $\mathrm{Pr}\,j_u(\boldsymbol{a}+\boldsymbol{b})=\mathrm{Pr}\,j_u\boldsymbol{a}+\mathrm{Pr}\,j_u\boldsymbol{b}$；

(3) $\mathrm{Pr}\,j_u(\lambda\boldsymbol{a})=\lambda\mathrm{Pr}\,j_u\boldsymbol{a}$，其中 λ 是一数量。

Properties of projection:

(1) $\mathrm{Pr}\,j_u(\boldsymbol{a})=|\boldsymbol{a}|\cos\varphi$, where φ is the angle between vector $\boldsymbol{a}$ and *u-axis*;

(2) $\mathrm{Pr}\,j_u(\boldsymbol{a}+\boldsymbol{b})=\mathrm{Pr}\,j_u\boldsymbol{a}+\mathrm{Pr}\,j_u\boldsymbol{b}$;

(3) $\mathrm{Pr}\,j_u(\lambda\boldsymbol{a})=\lambda\mathrm{Pr}\,j_u\boldsymbol{a}$, where λ is a scalar.

7.2 数量积 向量积(Dot Product and Cross Product)

一、两向量的数量积(Dot Product of Two Vectors)

向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的数量积定义为：$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$，其中$\theta$是向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的夹角。从数量积的定义知：

(1) $\boldsymbol{a}\cdot\boldsymbol{a}=|\boldsymbol{a}|^2$；

(2) 对于两个非零向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$，向量 $\boldsymbol{a}\perp\boldsymbol{b}$ 的充分必要条件 $\boldsymbol{a}\cdot\boldsymbol{b}=0$。此时称向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 相互垂直，或称 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 是正交的。

The dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by $\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. According to the definition of the dot product, we know:

(1) $\boldsymbol{a}\cdot\boldsymbol{a}=|\boldsymbol{a}|^2$;

(2) If $\boldsymbol{a}\neq\boldsymbol{0}$ and $\boldsymbol{b}\neq\boldsymbol{0}$, then $\boldsymbol{a}\perp\boldsymbol{b}$ if and only if $\boldsymbol{a}\cdot\boldsymbol{b}=0$. In this case, we say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular or orthogonal.

数量积的性质(The Properties of the Dot Product)

设 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 为三个向量，λ 为数，则有：

(1) $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$;

(2) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$;

(3) $\lambda(\boldsymbol{a}\cdot\boldsymbol{b})=(\lambda\boldsymbol{a})\cdot\boldsymbol{b}=\boldsymbol{a}\cdot(\lambda\boldsymbol{b})$;

(4) $\boldsymbol{0}\cdot\boldsymbol{a}=\boldsymbol{0}$.

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors and λ is a scalar, then these properties hold:

(1) $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$;

(2) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$;

(3) $\lambda(\boldsymbol{a}\cdot\boldsymbol{b})=(\lambda\boldsymbol{a})\cdot\boldsymbol{b}=\boldsymbol{a}\cdot(\lambda\boldsymbol{b})$;

(4) $\boldsymbol{0}\cdot\boldsymbol{a}=\boldsymbol{0}$.

二、两向量的向量积(The Cross Product of Two Vectors)

设 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 是空间中的两个向量，θ是向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 之间的夹角。则 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 的向量积可以如下定义：

(1) $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$，即 $\boldsymbol{a}\times\boldsymbol{b}$ 的模等于 $\boldsymbol{a}$ 的模，$\boldsymbol{b}$ 的模以及它们夹角正弦的乘积；

(2) $\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{b}(\boldsymbol{a}\times\boldsymbol{b})=0$，即 $\boldsymbol{a}\times\boldsymbol{b}$ 垂直于 $\boldsymbol{a}$ 和 $\boldsymbol{b}$ 所在的平面，且 $\boldsymbol{a}$，$\boldsymbol{b}$ 和 $\boldsymbol{a}\times\boldsymbol{b}$ 符合右手法则。

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors in three-space and θ be the angle between them. Then the cross product $\boldsymbol{a}\times\boldsymbol{b}$ of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined by

(1) $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$; that is, the may nitude of $\boldsymbol{a}\times\boldsymbol{b}$ is the multiplication of that of $\boldsymbol{a}$,$\boldsymbol{b}$ and $\sin\theta$;

(2) $\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b})=0$, that is, $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ form a right handed triple.

向量积的性质(The Properties of the Cross Product)

向量积具有如下性质:

The properties of the cross product:

(1) $\boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}, \boldsymbol{a}\times\boldsymbol{0}= \boldsymbol{0}, \boldsymbol{0}\times\boldsymbol{a} = \boldsymbol{0}$;

(2) 向量 $\boldsymbol{a}/\!/\boldsymbol{b}$ 的充分必要条件是（$\boldsymbol{a}\times\boldsymbol{b}$）=0，即向量 $\boldsymbol{a}$，$\boldsymbol{b}$ 在空间中平行的充分必要条件为 $\boldsymbol{a}\times\boldsymbol{b}=0$。

$\boldsymbol{a}/\!/\boldsymbol{b}$ if and only if $(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{0}$, that is, two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in three-space are parallel if and only if $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{0}$;

(3) $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$, 称为反交换律; which is called the anticommutative law;

(4) $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$ 分配律 distributive law;

(5) $\lambda(\boldsymbol{a}\times\boldsymbol{b}) = (\lambda\boldsymbol{a})\times\boldsymbol{b} = \boldsymbol{a}\times(\lambda\boldsymbol{b})$, 其中$\lambda$为数 where λ is a scalar.

三、向量的混合积(The Triple Scalar Product)

对空间向量 $\boldsymbol{a}$, $\boldsymbol{b}$ 和 $\boldsymbol{c}$，向量 $\boldsymbol{a}$ 和 $\boldsymbol{b}\times\boldsymbol{c}$ 的数量积 $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ 称为混合积。

For vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in space, the dot product of $\boldsymbol{a}$ and $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is called the triple scalar product.

向量 $\boldsymbol{a}=(a_x, a_y, a_z)$, $\boldsymbol{b}=(b_x, b_y, b_z)$和 $\boldsymbol{c}=(c_x, c_y, c_z)$的混合积表示为:

For $\boldsymbol{a} = (a_x, a_y, a_z)$, $\boldsymbol{b} = (b_x, b_y, b_z)$ and $\boldsymbol{c} = (c_x, c_y, c_z)$, the triple scalar product is given by

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ b_x & c_y & c_z \end{vmatrix}.$$

以向量 $\boldsymbol{a}$, $\boldsymbol{b}$ 和 $\boldsymbol{c}$ 为棱的平行六面体的体积为 $V=\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$。

The volume V of a parallelepiped with vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as adjacent sides is given by

$$V=\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}).$$

7.3 曲面及其方程 (Surface and Its Equation)

一、曲面方程的概念(Equation of Surface)

三维空间中方程的图形一般为曲面。如果曲面 S 和方程

$$F(x, y, z) = 0 \tag{1}$$

满足

(1) 曲面 S 上的任一点的坐标都满足方程(1);

(2) 不在曲面 S 上的点的坐标都不满足方程(1);

那么，方程(1)就叫做曲面 **S 的方程**，而曲面 S 叫做方程(1)的图形。

The graph of an equation in three-space is normally a surface. If the surface S and the equation

$$F(x, y, z) = 0 \tag{1}$$

Satisfy

(1) any point at the surface makes the equation (1) an equality;

(2) any point that is not at the surface can not make equation (1) an equality;

then equation (1) is called **the equation of the surface S**, and S is called the surface of the equation (1).

二、旋转曲面 (Surfaces of Revolution)

一个连续函数的图形绕一条直线旋转所得到的曲面为**旋转曲面**。

If the graph of a continuous function is revolved about a line, the resulting surface is called **a surface of revolution.**

三、柱面(Cylindrical Surfaces)

一般地，平行于定直线并沿定曲线 C 移动的直线 L 形成的轨迹叫做**柱面**，定曲线 C 叫做柱面的**准线**，动直线 L 叫做柱面的**母线**。

Let C be a plane curve in a plane and L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a **cylinder**. C is called the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are called rulings.

四、二次曲面(Quadric Surface)

三元二次方程 $F(x, y, z) = 0$ 所表示的曲面称为二次曲面。平面称为一次曲面。下面为几种二次曲面的标准方程：

If a surface is the graph in three-space of an equation of second degree, it is called a quadric surface. Some general types of quadric surface will be showed:

椭圆锥面：Elliptic cone: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=z^2$;

椭球面：Ellipsoid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$;

单叶双曲面：Hyperboloid of one Sheet: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$;

双叶双曲面：Hyperboloid of two Sheets: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=-1$;

椭圆抛物面：Elliptic Paraboloid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=z$;

双曲抛物面：Hyperbolic Paraboloid: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=z$;

椭圆柱面：Elliptic Cylinder: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$;

双曲柱面：Hyperbolic Cylinder: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$;

抛物柱面：Paraboloid Cylinder: $x^2=ay$.

平面与曲面的交线称为曲面在平面上的截痕。

The intersection of a surface with a plane is called the trace of the surface in the plane.

7.4 空间曲线及其方程 (The Curve in Three-space and Its Equation)

一、空间曲线的一般方程(General Form Equation of Space Curve)

空间曲线可以看作两个曲面的交线。设 $F(x, y, z) = 0$ 和 $G(x, y, z) = 0$ 是两个曲面的方程，它们的交线 C 可以用方程组

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \tag{1}$$

来表示。方程组(1)叫做曲线 C 的方程。

The space curve is the intersection curve of two space surfaces. Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$, then the intersection curve C of such two space surfaces is given by

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \tag{1}$$

二、空间曲线的参数方程(The Parametric Equations of Curve in Three-space)

空间曲线可由三个参数方程$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$决定，称为空间曲线的参数方程。

A space curve is determined by a triple of parametric equations:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

which is called the parametric equations of curve in three-space.

三、空间曲线在坐标面上的投影(Projecting Cylinder of Space Curve onto the Coordinate Planes)

设空间曲线 C 的一般方程为 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$，在上面的方程中消去变量 z 后所得到的方程为 $H(x, y)=0$，称上面的方程所表示的柱面为曲线关于 xOy 面的投影柱面；投影柱面与 xOy 面的交线叫做空间曲线 C 在 xOy 面上的投影曲线，或简称投影，其方程为

$$\begin{cases} H(x,y)=0 \\ z=0 \end{cases}$$

Consider a space curve C with parametric equations $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$. Eliminating variable z, we obtain $H(x, y) =0$. The

cylinder determined by the above equation is called the projecting cylinder of space curve C onto xOy plane. The intersection of cylinder and xOy plane is called the projection of C onto xOy plane, which is given by

$$\begin{cases} H(x,y)=0 \\ z=0 \end{cases}.$$

7.5 平面及其方程(Plane in Space and Its Equation)

一、平面的点法式方程(Point-norm Form Equations of a Plane)

如果一非零向量 $\boldsymbol{n}$ 垂直于一平面，这向量 $\boldsymbol{n}$ 就叫做该平面的法线向量。

A vector n is called the normal vector of a plane if n is prependicular to the plane.

过点(x_0, y_0, z_0)，法向量为 $n=(A, B, C)$ 的平面的点法式方程为

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0。$$

The plane containing the point (x_0, y_0, z_0) and having a normal vector $n=(A, B, C)$, can be represented, in point-norm form, by the equation:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

二、平面的一般方程(General Form Equations of a Plane)

过点(x_0, y_0, z_0) ，法向量为 $n=(A, B, C)$ 一般方程为

$$Ax + By + Cz+D= 0。$$

The plane containing the point (x_0, y_0, z_0) and having a normal vector $n=(A, B, C)$, can be represented, in general form, by the equation:

$$Ax + By + Cz+D= 0.$$

三、两平面的交角(The Angle between Two Planes)

两平面的法线向量的夹角(通常指锐角)称为两平面的夹角。

设两平面的法线向量依次为 $\boldsymbol{n}_1=(A_1, B_1, C_1)$ 和 $\boldsymbol{n}_2=(A_2, B_2, C_2)$，两平面的夹角为$\theta$，则

$$\cos\theta=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}。$$

两平面垂直如果 $\boldsymbol{n}_1\cdot\boldsymbol{n}_2=0$，即 $A_1A_2+B_1B_2+C_1C_2=0$;

两平面平行如果 $\boldsymbol{n}_1=\lambda\boldsymbol{n}_2$，其中$\lambda$是一数量，即 $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$。

If vectors $\boldsymbol{n}_1=(A_1, B_1, C_1)$ and $\boldsymbol{n}_2=(A_2, B_2, C_2)$ are normal to two intersecting planes, then the angles θ between the normal vectors is equal to the angle between the two plane and is given by

$$\cos\theta=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}.$$

Two planes with normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are perpendicular if $\boldsymbol{n}_1\cdot\boldsymbol{n}_2=0$, i.e. $A_1A_2+B_1B_2+C_1C_2=0$; and parallel if $\boldsymbol{n}_1=\lambda\boldsymbol{n}_2$, where λ is a scalar, that is $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$.

四、点到平面的距离(Distance from a Point to a Plane)

平面外一点 $P(x_0, y_0, z_0)$到平面 $Ax + By + Cz+D= 0$ 的距离

为

$$D=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}。$$

The distance from a point $P(x_0, y_0, z_0)$ (not on the plane) to a plane $Ax + By + Cz+D= 0$ is given by

$$D=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}.$$

7.6 空间直线及其方程(Lines in Space and Their Equations)

一、空间直线的一般方程(General Equation of a Line in Space)

空间直线 L 可以看作是两个平面的交线。如果两个相交平面的方程分别为 $A_1x + B_1y + C_1z+D_1 = 0$ 和 $A_2x + B_2y + C_2z+D_2 = 0$，那么直线 L 的方程为

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}\tag{1}$$

方程组(1)叫做空间直线的一般方程。

The space line is defined by the intersection of two planes. If $A_1x + B_1y + C_1z+D_1 = 0$ and $A_2x + B_2y + C_2z+D_2 = 0$ are the equations of two planes respectively. Then the line L is given by

$$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\A_2x+B_2y+C_2z+D_2=0\end{cases}\tag{1}$$

which is called the general equation of space line.

二、空间直线的方向向量(Direction vectors for the Lines in Space)

如果一个非零向量 $\boldsymbol{v}=(m, n, p)$平行于空间一条已知直线，这个向量 $\boldsymbol{v}$ 就叫做这条直线的方向向量，m，n 和 p 叫做这条直线的方向数，它们不是唯一的，对任意的非零常数 k，km，kn 和 kp 仍然是这条直线的方向数。

If the nonzero vector $\boldsymbol{v} = (m, n, p)$ parallel to a line in three-space, then $\boldsymbol{v}$ is called the direction vector for the line and the numbers m, n and p are called the direction numbers for the line; they are not unique, any nonzero constant multiples k km, kn and kp are also direction numbers.

三、空间直线的对称式方程与参数方程(Symmetric Equations and the Parametric Equations of the Line in Space)

1. 空间直线的对称式方程(The Symmetric Equations of the Lines in Space)

过(x_0, y_0, z_0)点，且方向数为 m，n 和 p 的空间直线的对称式方程为

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$$

The symmetric equations for the line through (x_0, y_0, z_0) with direction numbers m, n and p is defined as

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}.$$

2. 空间直线的参数方程(The Parametric Equations of the Lines in Space)

在空间直线的对称式方程中取 $\dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}=t$，

得空间直线的参数方程 $\begin{cases} x=x_0+mt \\ y=y_0+nt \\ z=z_0+pt \end{cases}$

Let $\dfrac{x-x_0}{m}=\dfrac{y-y_0}{n}=\dfrac{z-z_0}{p}=t$, we obtain $\begin{cases} x=x_0+mt \\ y=y_0+nt \\ z=z_0+pt \end{cases}$, which is called the parametric equations of the line through (x_0, y_0, z_0).

四、两直线的夹角(The Angle between Two Lines)

两直线的方向向量的夹角 φ (通常指锐角)叫做两直线的夹角。设直线 L_1 和 L_2 的方向向量分别是 $s_1=(m_1, n_1, p_1)$, $s_2=(m_2, n_2, p_2)$，则 L_1 和 L_2 的夹角可由

$$\cos\varphi=\frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\cdot\sqrt{m_2^2+n_2^2+p_2^2}}$$

确定，且

(1) 两直线 L_1 和 L_2 相互垂直的充分必要条件是 $m_1m_2+n_1n_2+p_1p_2=0$；

(2) 两直线 L_1 和 L_2 相互平行的充分必要条件是 $\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}=\dfrac{p_1}{p_2}$。

The angel φ between two space lines is defined by the angle between the direction vectors of such two lines. If the direction vectors of space lines L_1 and L_2 are $s_1=(m_1, n_1, p_1)$ and $s_2=(m_2, n_2, p_2)$ respectively. Then

$$\cos\varphi=\frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\cdot\sqrt{m_2^2+n_2^2+p_2^2}}.$$

Two space lines L_1 and L_2 are

(1) perpendicular if and only if $m_1 m_2+n_1n_2+p_1 p_2=0$;

(2) parallel if and only if $\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}=\dfrac{p_1}{p_2}$.

五、直线与平面的夹角(The Angle between Space Line and Plane)

当直线与平面不垂直时，直线和它在平面上的投影直线的夹角φ $(0\leqslant\varphi\leqslant\dfrac{\pi}{2})$称为直线与平面的交角。

设直线的方向向量为(m, n, p)，平面的法向量为$\boldsymbol{n}=(A, B, C)$，直线与平面的交角为φ，则

$$\sin\varphi=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{m^2+n^2+p^2}}。$$

(1) 直线与平面垂直等价于$\dfrac{A}{m}=\dfrac{B}{n}=\dfrac{C}{p}$；

(2) 直线与平面平行或直线在平面上等价于$Am+Bn+Cp=0$。

If space line and plane is not perpendicular, the angle between space line and plane is defined by that of the space line and its projection onto the plane.

Let the direction vector of space line be (m, n, p), and vectors $\boldsymbol{n}=(A, B, C)$ be normal to the plane. The angle between space line and plane is φ, then

$$\sin\varphi=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{m^2+n^2+p^2}}.$$

Space line and plane are

(1) perpendicular if and only if $\dfrac{A}{m}=\dfrac{B}{n}=\dfrac{C}{p}$;

(2) parallel if and only if $\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}=\dfrac{p_1}{p_2}$.

通过定直线的所有平面的全体称为平面束。设直线 L 的方程为$\begin{cases}A_1x+B_1y+C_1z=0\\A_2x+B_2y+C_2z=0\end{cases}$，则通过直线 L 的平面束方程为

$$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0。$$

The set of all planes through the fixed line is called the pencil of planes. Consider the line with parametric equation

$$\begin{cases}A_1x+B_1y+C_1z=0\\A_2x+B_2y+C_2z=0\end{cases}.$$

Then the pencil of planes is given by

$$A_1x+B_1y+C_1z+D_1+\lambda(A_2x+B_2y+C_2z+D_2)=0.$$

▶第八章　多元函数微分法及其应用
Chapter 8　Differentiation of Functions of Several Variables and Its Application

8.1　多元函数的基本概念(The Basic Concepts of Functions of Several Variables)

定义 1　设 D 是 R^2 的一个非空子集，称映射 $f: D\to R$ 为定义在 D 上的二元函数，通常记为 $z=f(x, y), (x, y)\in D$，或 $z=f(P)$，$P\in D$。其中点集 D 称为该函数的定义域，x、y 称为自变量，z

称为因变量。

Definition 1　Let D be a nonempty subset of R^2, we call the mapping f: $D \to R$ the function of two variables defined on, usually denoted by $z=f(x, y) \in D$, or $z=f(P)$, $P \in D$. The set D is called the domain of the function. We call x and y the independent variables and z the dependent variable.

定义 2　设二元函数$f(P)=f(x, y)$的定义域为 D，$P_0(x_0, y_0)$是 D 的聚点。如果存在常数 A，对于任意给定的正数ε，总存在正数δ，使得当点 $P(x,y) \in D \cap \overset{0}{U}(P_0,\delta)$ 时，都有 $|f(P)-A|=|f(x,y)-A|<\varepsilon$ 成立，那么就称常数 A 为函数$f(x, y)$当$(x, y) \to (x_0, y_0)$时的极限，记作

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=A \text{ 或 } f(x,y)\to A((x,y)\to(x_0,y_0)),$$

也记作 $\lim\limits_{P\to P_0} f(P)=A$ 或 $f(P)\to A(P\to P_0)$。

Definition 2　Let D be the domain of the function $f(P)=f(x, y)$ of two variables, P_0（x_0, y_0）be a point of accumulation of D. If there exists a constant A, such that, for each $\varepsilon>0$ there is a corresponding $\delta>0$ such that $|f(P)-A|=|f(x,y)-A|<\varepsilon$, provided that $P(x,y)\in D\cap \overset{0}{U}(P_0,\delta)$, then we call the constant A the limit of $f(x, y)$ as $(x, y) \to (x_0, y_0)$.

定义 3　设二元函数$f(P)=f(x, y)$的定义域为 D，$P_0(x_0,y_0)$是 D 的聚点，且 $P_0\in D$。如果 $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$，则称函数$f(x, y)$在点 $P_0(x_0,y_0)$连续。

Definition 3　Let D be the domain of the function $f(P)=f(x, y)$ of two variables, $P_0(x_0,y_0)$ be a point of accumulation of D and $P_0\in D$. If

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0),$$

then we say that $f(x, y)$ is continuous at the point $P_0(x_0,y_0)$.

定义 4　设二元函数$f(P)=f(x, y)$的定义域为 D，$P_0(x_0,y_0)$是 D 的聚点。如果函数$f(x, y)$在点 $P_0(x_0,y_0)$不连续，则称 $P_0(x_0,y_0)$为函数$f(x, y)$的间断点。

Definition 4　Let D be the domain of the function $f(P)=f(x, y)$ of two variables, $P_0(x_0,y_0)$ be a point of accumulation of D. If $f(x, y)$ is not continuous at $P_0(x_0,y_0)$, then we say that $P_0(x_0,y_0)$ is a discontinuity point of $f(x, y)$.

性质 1 (有界性与最大值最小值定理)　在有界闭区域 D 上的多元连续函数 f，必定在 D 上有界且一定能取得它的最大值和最小值。

Property 1　(Boundedness and max-min theorem) Let f be a continuous function of several variables on a closed region D, then f has an absolute maximum and an absolute minimum on the region. In particular, f must be bounded on the region.

性质 2 (介值定理)　在有界闭区域 D 上的多元连续函数 f，必定在 D 上取得界于最大值 M 与最小值 m 之间的任何值。

Property 2 (Intermediate Value Theorem)　Let f be a continuous function of several variables on a closed region D, then f can obtain any number between its absolute maximum M and its absolute minimum m.

8.2 偏导数(Partial Derivative)

定义 设函数$f(x, y)$是关于x, y的二元函数，如果y固定为常数$y=y_0$，则$f(x, y_0)$是关于x的一元函数。它在$x=x_0$的导数叫做$f(x, y)$在关于x的偏导数，记作$f_x(x_0, y_0)$。即

$$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

类似地，$f(x, y)$在(x_0, y_0)关于y的偏导数，记作$f_y(x_0, y_0)$，由$f_y(x_0,y_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$所给出。

Definition Suppose that $f(x, y)$ is a function of two variables x and y. If y is held constant, say $y=y_0$, then $f(x, y_0)$ is a function of the single variable x. Its derivative at $x=x_0$ is called the partial derivative of $f(x, y)$ with respect to x at $x=x_0$ and is denoted by $f_x(x_0, y_0)$. Thus

$$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}.$$

Similarly, the partial derivative of $f(x, y)$ with respect to y at (x_0, y_0), and is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}.$$

定理 若f_{xy}和f_{yx}在开集S连续，则对于S的每一点有$f_{xy}=f_{yx}$。

Theorem If f_{xy} and f_{yx} are continuous on an open set S, then $f_{xy}=f_{yx}$ at each point of S.

8.3 全微分(Total Differential)

定义 如果函数$z=f(x, y)$在点(x, y)的全增量

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$$

可表示为$\Delta z=A(x,y)\Delta x+B(x,y)\Delta y+o(\rho)$，其中$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称函数$z=f(x, y)$在点$(x,y)$可微分，而$A(x,y)\Delta x+B(x,y)\Delta y$称为函数$z=f(x, y)$在点$(x,y)$的全微分，记作$dz$，即

$$dz=A(x,y)\Delta x+B(x,y)\Delta y\text{。}$$

Definition If the total increment of the function $z=f(x, y)$ at (x, y) can be expressed as $\Delta z=A(x,y)\Delta x+B(x,y)\Delta y+o(\rho)$, where $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$, then $f(x, y)$ is said to be differentiable at (x, y). $A(x,y)\Delta x+B(x,y)\Delta y$ is called the total differential of $z=f(x, y)$ at (x, y), denoted by dz, i.e. $dz=A(x,y)\Delta x+B(x,y)\Delta y$.

定理 1(必要条件) 如果函数$z=f(x, y)$在点(x, y)可微分，则该函数在点(x, y)的偏导数$\partial z/\partial x,\partial z/\partial y$必定存在，且函数$z=f(x, y)$在点$(x, y)$的全微分为$dz=\dfrac{\partial z}{\partial x}\Delta x+\dfrac{\partial z}{\partial y}\Delta y$。

Theorem1 (Necessary Condition) Suppose the function $z=f(x, y)$ is differentiable at (x, y), then its partial derivative $\partial z/\partial x,\partial z/\partial y$ at (x, y) exist, and the total differential of $z=f(x, y)$ at (x, y) is given by $dz=\dfrac{\partial z}{\partial x}\Delta x+\dfrac{\partial z}{\partial y}\Delta y$.

定理 2(充分条件) 如果函数$z=f(x, y)$在点(x, y)的偏导数$\partial z/\partial x,\partial z/\partial y$连续，则函数在该点可微分。

Theorem 2(Sufficient Condition) If $z=f(x, y)$has continuous partial derivatives $\partial z/\partial x, \partial z/\partial y$ at (x, y), then $f(x, y)$ is differentiable at (x, y).

8.4 链式法则(The Chain Rule)

定理 1 设函数$x=x(t)$和$y=y(t)$在点t可微，$z=f(x, y)$在点$(x(t), y(t))$可微。则 $z=f(x(t), y(t))$ 在点 t 可微且$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$。

Theorem 1 Let $x=x(t)$ and $y=y(t)$ be differentiable at t, and let $z=f(x, y)$ be differentiable at $(x(t), y(t))$.Then $z=f(x(t),y(t))$ is differentiable at t and $\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$.

定理 2 设 $x=x(s, t)$, $y=y(s, t)$在点(s, t)具有一阶偏导数，且$z=f(x, y)$在点$(x(s,t), y(s,t))$可微，则复合函数 $z=f(x(s,t), y(s,t))$在点(s, t)的一阶偏导数存在，且有

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s},\quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}。$$

Theorem 2 Let $x=x(s, t)$ and $y=y(s, t)$ have first partial derivatives at (s, t), and let $z=f(x, y)$ be differentiable at $(x(s,t), y(s,t))$. Then $z=f(x(s,t), y(s,t))$ has first partial derivatives at (s,t),and

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s},\quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

8.5 隐函数的求导公式(Derivative Formula for Implicit Functions)

假设方程 $F(x, y)=0$ 确定了 y 为 x 的隐函数，比如 $y=g(x)$但

是 $g(x)$的显式表达式很难或者不可能求出来。我们依然可以求出dy/dx。应用链式法则，方程 $F(x, y)=0$ 两边对 x 求导。我们得到$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$，解得$\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}$。

Suppose that $F(x, y)=0$ defines y implicitly as a function of x, for example, $y=g(x)$ but that the function $g(x)$ is difficult or impossible to determine. We can still find dy/dx. Let's differentiate both sides of $F(x, y)=0$ with respect to x using the Chain Rule. We obtain $\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$. Solving for $\frac{dy}{dx}=-\frac{\partial F/\partial x}{\partial F/\partial y}$.

假设 z 是由方程 $F(x, y, z)=0$ 所确定的 x 和 y 的隐函数，两边关于 x 求偏导数，保持 y 不变，则有$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$。注意到$\partial y/\partial x=0$,即可解得$\partial z/\partial x$,得到下面的第一个公式。类似地,固定 x,对 y 求偏导数即得第二个公式。

$$\frac{\partial z}{\partial x}=-\frac{\partial F/\partial x}{\partial F/\partial z},\quad \frac{\partial z}{\partial y}=-\frac{\partial F/\partial y}{\partial F/\partial z}\quad。$$

If z is an implicit function of x and y defined by the equation $F(x, y, z)=0$, then differentiation of both sides with respect to x, holding y fixed, yields $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$. If we solve for $\partial z/\partial x$ and note that $\partial y/\partial x=0$, we get the first of the formulas below. A similar calculation holding x fixed and differentiating with respect to y produces the second formula

$$\frac{\partial z}{\partial x}=-\frac{\partial F/\partial x}{\partial F/\partial z},\quad \frac{\partial z}{\partial y}=-\frac{\partial F/\partial y}{\partial F/\partial z}.$$

8.6 多元函数微分学的几何应用(Geometric Applications of Differentiation of Functions of Several Variables)

假设曲线 L 由参数方程 $x=x(t)$, $y=y(t)$, $z=z(t)$ $(\alpha\leqslant t\leqslant\beta)$ 所给出,且 x, y, z 在区间 $[\alpha,\beta]$ 上可微。假设点 $M(x_0,y_0,z_0)$ 对应到参数值 t_0，则 L 在点 M 的切线方程为 $\dfrac{x-x_0}{x'(t_0)}=\dfrac{y-y_0}{y'(t_0)}=\dfrac{z-z_0}{z'(t_0)}$。

Assume that the curve L is given by the parametric equations $x=x(t)$, $y=y(t)$, $z=z(t)$ $(\alpha\leqslant t\leqslant\beta)$, and functions x, y, z are differentiable on the interval $[\alpha,\beta]$. Assume also that point $M(x_0,y_0,z_0)$ corresponds the value t_0 of parameter t, then the equation of the tangent line to the curve L at point M is given by

$$\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)}.$$

假设 $M(x_0,y_0,z_0)$ 是方程 $F(x, y, z)=0$ 所确定的曲面 S 上的一点，F 的偏导数 F_x, F_y, F_z 在该点连续且不同时为零, 则曲面 S 在点 M 的**切平面的方程**是

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+$$

$$F_z(x_0,y_0,z_0)(z-z_0)=0,$$

曲面 S 在点 M 的**法线方程**是

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}。$$

Suppose that $M(x_0,y_0,z_0)$ is a point on a surface S with equation $F(x, y, z)=0$ and that the partial derivatives F_x, F_y and F_z are continuous and not all equal to zero at M, then the **tangent plane to surface S at point M** is given by

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+$$

$$F_z(x_0,y_0,z_0)(z-z_0)=0$$

and the **normal line to surface S at point M** is given by

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}.$$

8.7 方向导数与梯度(Directional Derivatives and Gradients)

定理 如果函数 $f(x, y)$在点 $P_0(x_0,y_0)$ 可微分，那么函数在该点沿任一方向 l 的方向导数存在，且有

$$\left.\frac{\partial f}{\partial l}\right|_{(x_0,y_0)}=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta,$$

其中 $\cos\alpha,\cos\beta$ 是方向 l 的方向余弦。

Theorem Let $f(x, y)$ be differentiable at $P_0(x_0,y_0)$. Then $f(x, y)$ has a directional derivative at $P_0(x_0,y_0)$ in any direction l and

$$\left.\frac{\partial f}{\partial l}\right|_{(x_0,y_0)}=f_x(x_0,y_0)\cos\alpha+f_y(x_0,y_0)\cos\beta,$$

where $\cos\alpha$, $\cos\beta$ is the direction cosines of the direction l.

设 $f(x,y,z)$是定义在三维空间 R^3 的区域 D 上的标量场，$P_0(x_0,y_0,z_0)\in D$，则向量

$$f_x(x_0,y_0,z_0)i+f_y(x_0,y_0,z_0)j+f_z(x_0,y_0,z_0)k$$

叫作标量场 f 在 P_0 点的**梯度**，记作 grad$f$。

Let $f(x,y,z)$, a scalar field, be defined on a domain D in R^3, $P_0(x_0,y_0,z_0)\in D$, then the vector $f_x(x_0,y_0,z_0)i+f_y(x_0,y_0,z_0)j+f_z(x_0,y_0,z_0)k$ is called the gradient at P_0 of the scalar field f, denoted by grad$f$.

8.8 多元函数的极值(Extreme Value of Functions of Several Variables)

定义 设函数 $f(x, y)$的定义域为 S, (x_0, y_0)为 S 中一点。

(1) 若对 S 中的任何点(x, y)，有$f(x_0, y_0)>f(x, y)$，则称$f(x_0, y_0)$是 f 在 S 上的**最大值**；

(2) 若对 S 中的任何点(x, y)，有$f(x_0, y_0)<f(x, y)$，则称$f(x_0, y_0)$是 f 在 S 上的**最小值**；

(3) f 的最大值或最小值统称为**最值**。

在(1)和(2)中，如果不等式只需要在 $N\cap S$ 上成立，其中 N 是(x_0, y_0)的某个邻域，我们就得到极大值和极小值的定义。f 的极大值或极小值统称为**极值**。

Definition Let $f(x, y)$ be a function with domain S, and let (x_0, y_0) be a point in S.

(1) $f(x_0, y_0)$ is a **global maximum value** of f on S if $f(x_0, y_0)> f(x, y)$ for all (x, y) in S;

(2) $f(x_0, y_0)$ is a **global minimum value** of f on S if $f(x_0, y_0) < f(x, y)$ for all (x, y) in S;

(3) $f(x_0, y_0)$ is a **global extreme value** of f on S if $f(x_0, y_0)$ is either a global maximum value or a global minimum value.

We obtain definitions for **local maximum value** and **local minimum value** if in (1) and (2) we require only that the inequalities hold on $N\cap S$, where N is some neighborhood of (x_0, y_0). $f(x_0, y_0)$ is a **local extreme value** of f on S if $f(x_0, y_0)$ is either a local maximum value or a local minimum value.

定理 1(必要条件) 设函数 $z=f(x, y)$在点(x_0, y_0)具有偏导数，且在点(x_0, y_0)处有极值，则有 $f_x(x_0, y_0)=0$，$f_y(x_0, y_0)=0$。

Theorem1 (Necessary Condition) If $z=f(x, y)$ has partial derivatives at (x_0, y_0), and has an extreme value at (x_0, y_0), then we have $f_x(x_0, y_0)=0, f_y(x_0, y_0)=0$.

定理 2 (充分条件) 假设函数 $f(x, y)$在点(x_0, y_0)的某个邻域内具有二阶连续偏导数，且 $f_x(x_0, y_0)=0, f_y(x_0, y_0)=0$ 设

$$D=D(x_0,y_0)=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}^{\ 2}(x_0,y_0)。$$

则有

(i) 若 $D>0$ 且 $f_{xx}(x_0, y_0)<0$，$f(x_0, y_0)$是极大值；

(ii) 若 $D>0$ 且 $f_{xx}(x_0, y_0)>0$，$f(x_0, y_0)$是极小值；

(iii) 若 $D<0$, $f(x_0, y_0)$不是极值；

(iv) 若 $D=0$，不确定。

Theorem2 (Sufficient Condition) Suppose that $f(x, y)$ has continuous second partial derivatives in a neighborhood of (x_0, y_0) and that $f_x(x_0, y_0)=0, f_y(x_0, y_0)=0$. Let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

Then

(i) if $D>0$ and $f_{xx}(x_0, y_0)<0$, $f(x_0, y_0)$ is a local maximum value;

(ii) if $D>0$ and $f_{xx}(x_0, y_0)>0$, is a local minimum value;

(iii) if $D<0$, $f(x_0, y_0)$ is not an extreme value;

(iv) if $D=0$, the test is inconclusive.

拉格朗日乘数法

要求函数$f(x, y)$在限制条件 $g(x, y)=0$ 下的极值,只要解方程组

$$\begin{cases} f_x(x,y) + \lambda g_x(x,y) = 0, \\ f_y(x,y) + \lambda g_y(x,y) = 0, \\ g(x,y) = 0。 \end{cases}$$

得到 x, y 和λ的值,这样得到的(x, y)就是函数在附加条件 $g(x, y)=0$ 下的可能极值点,参数λ称为拉格朗日乘子。

Lagrange's Method

To maximize or minimize $f(x, y)$ subject to the constraint $g(x, y)=0$ solve the system of equations

$$\begin{cases} f_x(x,y) + \lambda g_x(x,y) = 0, \\ f_y(x,y) + \lambda g_y(x,y) = 0, \\ g(x,y) = 0. \end{cases}$$

for x, y and λ. Each such point (x, y) is critical point for the constrained extremum problem, and the corresponding λ is called a Lagrange multiplier.

▶第九章　重积分 Chapter 9　Multiple Integrals

9.1　二重积分的概念与性质(The Concept of Double Integrals and Its Properties)

一、二重积分的概念(Double Integrals)

定义（二重积分的定义） 设 D 是 xy 平面的有界闭区域，f 是定义在 D 上的函数。将 D 任意分成 n 个小区域σ_i,它们的面积用$\Delta\sigma_i$（$i=1,2,\cdots,n$）表示。在每个σ_i（$i=1,2,\cdots,n$）上任取一点(ξ_i,η_i)，并作和$\sum\limits_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$。假设存在一个确定的数 I，满足：任给$\varepsilon>0$，存在$\delta>0$，使得当各小区域σ_i的直径中的最大值λ小于δ 时，就有

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i - I\right| < \varepsilon,$$

不管区域 D 的分法如何，(ξ_i,η_i)的取法如何。这样就称 f 在 D 上可积，I 称为 f 在 D 上的二重积分，记作$\iint\limits_D f(x,y)\mathrm{d}\sigma = I$，或

$$\iint_D f(x,y)\mathrm{d}\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i。$$

Definition (The Double Integral) Let D be a bounded closed region in the xy plane and f a function defined on D. Partition D arbitrarily into n subregions σ_i, whose area is denoted by $\Delta\sigma_i$（$i=1,2,\cdots,n$）Choose arbitrarily a point (ξ_i,η_i) in σ_i

($i=1,2,\cdots,n$) and then form the sum $\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i$. Suppose that there exists a fixed number I such that for any $\varepsilon>0$, there exists a $\delta>0$ such that if the length λ of the longest diameter of those subregions σ_i in a partition of D is less than δ, then

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i - I\right| < \varepsilon,$$

no matter how the partition is and how those points (ξ_i,η_i) are chosen from σ_i ($i=1,2,\cdots,n$) Then f is said to be integrable over D and I is the double integral of f over D, written $\iint\limits_D f(x,y)\mathrm{d}\sigma = I$, or

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i.$$

二、二重积分的性质 (Properties of Double Integrals)

性质 1 两个函数和(或差)的二重积分等于它们二重积分的和(或差)，即

$$\iint\limits_D (f(x,y)\pm g(x,y))\mathrm{d}\sigma = \iint\limits_D f(x,y)\mathrm{d}\sigma \pm \iint\limits_D g(x,y)\mathrm{d}\sigma。$$

Property 1 The double integral of the sum(or difference) of two functions is equal to the sum(or difference) of their double integrals, that is

$$\iint\limits_D (f(x,y)\pm g(x,y))\mathrm{d}\sigma = \iint\limits_D f(x,y)\mathrm{d}\sigma \pm \iint\limits_D g(x,y)\mathrm{d}\sigma.$$

性质 2 被积函数前面的常数因子可以提到积分号前面,即 $\iint\limits_D kf(x,y)\mathrm{d}\sigma = k\iint\limits_D f(x,y)\mathrm{d}\sigma$, 若 k 为常数。

Property 2 The constant factor in the integrand function can be taken out of the double integral, that is $\iint\limits_D kf(x,y)\mathrm{d}\sigma = k\iint\limits_D f(x,y)\mathrm{d}\sigma$, if k is a constant.

性质 3 二重积分关于积分区域具有可加性，即如果 D 被分成两个区域 D_1 和 D_2，$D_1\cap D_2$ 的面积为 0，则有

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \iint\limits_{D_1} f(x,y)\mathrm{d}\sigma + \iint\limits_{D_2} f(x,y)\mathrm{d}\sigma。$$

Property 3 The double integral is additive with respect to the integration region, that is, if D is divided into two regions D_1 and D_2 and the area of $D_1\cap D_2$ is 0, then

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \iint\limits_{D_1} f(x,y)\mathrm{d}\sigma + \iint\limits_{D_2} f(x,y)\mathrm{d}\sigma.$$

性质 4 若对任意 $(x,y)\in D$,有 $f(x,y)\geqslant 0$, 则 $\iint\limits_D f(x,y)\mathrm{d}\sigma \geqslant 0$。

Property 4 If $f(x, y) \geqslant 0$ for every $(x, y)\in D$, then $\iint\limits_D f(x,y)\mathrm{d}\sigma \geqslant 0$.

性质 5 若对任意 $(x, y)\in D$，有 $f(x, y)\leqslant g(x, y)$，则

$$\iint\limits_D f(x,y)\mathrm{d}\sigma \leqslant \iint\limits_D g(x,y)\mathrm{d}\sigma。$$

Property 5 If $f(x, y)\leqslant g(x, y)$ for every $(x, y)\in D$, then

$$\iint\limits_D f(x,y)\mathrm{d}\sigma \leqslant \iint\limits_D g(x,y)\mathrm{d}\sigma.$$

性质 6 假设 M 和 m 分别是函数 f 在 D 上的最大值和最小值，则

$$m\sigma \leqslant \iint_D f(x,y)\mathrm{d}\sigma \leqslant M\sigma,$$

其中σ是区域D的面积。

Property 6 Suppose that M and m are respectively the maximum and minimum values of function f on D, then $m\sigma \leqslant \iint_D f(x,y)\mathrm{d}\sigma \leqslant M\sigma$, where σ is the area of D.

性质 7 (二重积分的中值定理) 若$f(x,y)$在闭区域D上连续，则在D上至少存在一点(ξ,η)使得$\iint_D f(x,y)\mathrm{d}\sigma = f(\xi,\eta)\sigma$，其中$\sigma$是区域$D$的面积。

Property 7 (The Mean Value Theorem for Double Integral) If $f(x, y)$ is continuous on the closed region D, then there exists at least a point (ξ,η) in D such that $\iint_D f(x,y)\mathrm{d}\sigma = f(\xi,\eta)\sigma$, where σ is the area of D.

9.2 二重积分的计算法(Evaluation of Double Integrals)

一、直角坐标的二重积分(Double Integrals in Rectangular Coordinates)

定理 设$f(x,y)$在xy平面上的有界闭区域D上连续，

(1) 若区域D由$a\leqslant x\leqslant b$和$g_1(x)\leqslant y\leqslant g_2(x)$所给出，其中$g_1(x)$，$g_2(x)$是$[a,b]$上的连续函数，则$\iint_D f(x,y)\mathrm{d}\sigma = \int_a^b \mathrm{d}x\int_{g_1(x)}^{g_2(x)} f(x,y)\mathrm{d}y$。

(2) 若区域D由$c\leqslant y\leqslant d$和$h_1(y)\leqslant x\leqslant h_2(y)$所给出，其中$h_1(y)$，$h_2(y)$是$[c,d]$上的连续函数，则$\iint_D f(x,y)\mathrm{d}\sigma = \int_c^d \mathrm{d}y\int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x$。

Theorem Let $f(x, y)$ be continuous on a bounded closed region D in the xy plane.

(1) If D is given by $a\leqslant x\leqslant b$ and $g_1(x)\leqslant y\leqslant g_2(x)$, where $g_1(x)$, $g_2(x)$ are continuous functions of x on $[a, b]$, then

$$\iint_D f(x,y)\mathrm{d}\sigma = \int_a^b \mathrm{d}x\int_{g_1(x)}^{g_2(x)} f(x,y)\mathrm{d}y.$$

(2) If D is given by $c\leqslant y\leqslant d$ and $h_1(y)\leqslant x\leqslant h_2(y)$, where $h_1(y)$, $h_2(y)$ are continuous functions of y on $[c, d]$, then

$$\iint_D f(x,y)\mathrm{d}\sigma = \int_c^d \mathrm{d}y\int_{h_1(y)}^{h_2(y)} f(x,y)\mathrm{d}x.$$

二、极坐标下的二重积分(Double Integrals in Polar Coordinates)

如果积分区域D由极坐标形式$D=\{(r,\theta)|\alpha\leqslant\theta\leqslant\beta, h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$给出，则

$$\iint_D f(x,y)\mathrm{d}\sigma = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta。$$

If the region D is given by $D=\{(r,\theta)|\alpha\leqslant\theta\leqslant\beta, h_1(\theta)\leqslant r\leqslant h_2(\theta)\}$ in polar coordinates, then

$$\iint_D f(x,y)\mathrm{d}\sigma = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\mathrm{d}r\mathrm{d}\theta.$$

9.3 三重积分(Triple Integrals)

一、三重积分的概念 (Triple Integrals)

定义 设f是定义在空间有界闭区域Ω上的三元函数。如果存在一个确定的数A，满足：任给$\varepsilon>0$，存在$\delta>0$，对Ω的任意一个分法，设小区域为v_i，它们的体积用Δv_i表示，在每个v_i

$(i=1,2,\cdots,n)$中任取一点(ξ_i,η_i,ζ_i)，当小区域的最大直径λ小于δ时，不等式$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i - A\right| < \varepsilon$成立，则$A$称为$f$在$D$上的三重积分，记作$\iiint_\Omega f(x,y,z)\mathrm{d}v = A$，或

$$\iiint_\Omega f(x,y,z)\mathrm{d}v = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i = A。$$

Definition Suppose that f is a continuous function of three variables defined on a bounded closed region Ω in space. If there is a number A such that for any $\varepsilon>0$, there exists a $\delta>0$ such that, for any partition to Ω with subregions v_i, whose volume are denoted by Δv_i, and any points (ξ_i,η_i,ζ_i) chosen arbitrarily from v_i $(i=1,2,\ldots,n)$respectively, the inequality

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i - A\right| < \varepsilon$$

holds true whenever the diameter of the largest subregion is less than δ, then A is said to be the **triple integral** of function f over the region Ω, written $\iiint_\Omega f(x,y,z)\mathrm{d}v = A$, or

$$\iiint_\Omega f(x,y,z)\mathrm{d}v = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta v_i = A.$$

二、三重积分的计算(Evaluation of Triple Integrals)

1. 利用直角坐标(Rectangular Coordinates) 计算三重积分

设f是定义在空间有界闭区域Ω上的连续的三元函数，Ω由

$$\{(x,y,z)\mid a\leqslant x\leqslant b, y_1(x)\leqslant y\leqslant y_2(x), z_1(x,y)\leqslant z\leqslant z_2(x,y)\}$$

给出，则$\iiint_\Omega f(x,y,z)\mathrm{d}v = \int_a^b \mathrm{d}x\int_{y_1(x)}^{y_2(x)}\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$。

Suppose that f is a continuous function of three variables defined on a bounded closed region Ω in space given by

$$\{(x,y,z)\mid a\leqslant x\leqslant b, y_1(x)\leqslant y\leqslant y_2(x), z_1(x,y)\leqslant z\leqslant z_2(x,y)\}.$$

Then $\iiint_\Omega f(x,y,z)\mathrm{d}v = \int_a^b \mathrm{d}x\int_{y_1(x)}^{y_2(x)}\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$.

2. 利用柱面坐标(Cylindrical Coordinates)计算三重积分

当空间立体区域Ω有一个对称轴时，计算Ω上的三重积分通常使用柱坐标比较容易。

When a solid region Ω in three-space has an axis of symmetry, the evaluation of triple integrals over Ω is often facilitated by using cylindrical coordinates.

柱坐标和笛卡尔(直角)坐标之间的关系为

$$x=r\cos\theta, y=r\sin\theta, z=z。$$

Cylindrical and Cartesian (rectangular) coordinates are related by the equations $x=r\cos\theta, y=r\sin\theta, z=z$.

设Ω是一个z型立体区域，它在xy平面的投影是r型的。如果f在Ω上连续，则

$$\iiint_\Omega f(x,y,z)\mathrm{d}v = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta。$$

Let Ω be a z-sample solid and suppose that its projection in the xy-plane is r-sample. If f is continuous on Ω, then

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta .$$

3. 利用球面坐标(Spherical Coordinates)计算三重积分

当空间立体区域Ω关于某个点对称时,计算Ω上的三重积分通常使用球坐标比较容易。

When a solid region Ω in three-space is symmetric with respect to a point, the evaluation of triple integrals over Ω is often facilitated by using spherical coordinates.

球坐标和笛卡尔(直角)坐标之间的关系为:

$$x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta, z=\rho\cos\phi 。$$

The equations $x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi$ relate spherical coordinates and Cartesian (rectangular) coordinates.

球坐标中的体积元素由 $\mathrm{d}v=\rho^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$ 给出，因此有

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \iiint\limits_{\Omega} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi 。$$

The volume element in spherical coordinates is given by $\mathrm{d}v=\rho^2\sin\phi\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$, hence we have

$$\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}v = \iiint\limits_{\Omega} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi .$$

4. 重积分中的变量代换 (Change of Variables in Multiple Integrals)

假设 $x=m(u,v)$, $y=n(u,v)$建立了旧变量 x, y 与新变量 u,v

的关系。定义一个函数 $J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$，叫做**雅可比行列式**。

这样，在适当的限制条件下，就有

$$\iint f(x,y)\mathrm{d}x\mathrm{d}y = \iint f[m(u,v),n(u,v)]\,|J(u,v)|\,\mathrm{d}u\mathrm{d}v 。$$

Suppose that $x=m(u,v)$, $y=n(u,v)$relate the old variables x and y to new variables u and v. Define a function $J(u,v)$, called the **Jacobian,** by $J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$. Then, under suitable conditions on the functions m and n and with appropriate limits on the integral signs,

$$\iint f(x,y)\mathrm{d}x\mathrm{d}y = \iint f[m(u,v),n(u,v)]\,|J(u,v)|\,\mathrm{d}u\mathrm{d}v .$$

对于三元情形，其中 $x=m(u,v,w)$, $y=n(u,v,w)$, $z=p(u,v,w)$，则雅克比行列式由式子

$$J(u,v,w)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w}\\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{vmatrix}$$

给出，同样在变换后的积分会出现因子 $|J(u,v,w)|$。

In the three-variable case, where $x=m(u,v,w)$, $y=n(u,v,w)$, $z=p(u,v,w)$; the Jacobian $J(u,v,w)$ is defined by

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Again, $|J(u,v,w)|$ is the extra factor that appears in the transformed integral.

9.4 重积分的应用(Applications of Multiple Integrals)

设曲面 S 由方程 $z=f(x,y)$ 给出，D 为曲面 S 在 xy 平面上的投影区域，函数 f 在 D 上具有一阶连续偏导数 f_x 和 f_y。则 S 的表面积为

$$A=\iint_D \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}\sigma \text{。}$$

Suppose the surface S is defined by the equation $z=f(x, y)$. D is the projection region of S in the xy-plane. Assume that f has continuous first partial derivatives f_x and f_y on D. Then the furface area of S is given by

$$A=\iint_D \sqrt{1+f_x^2(x,y)+f_y^2(x,y)}\,\mathrm{d}\sigma .$$

考虑一平面薄片，占有 xy 平面上的区域 D，其密度函数为 $\mu(x, y)$。则该薄片的**质心**的坐标为

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_D x\mu(x,y)\mathrm{d}\sigma}{\iint_D \mu(x,y)\mathrm{d}\sigma},\quad \bar{y}=\frac{M_x}{m}=\frac{\iint_D y\mu(x,y)\mathrm{d}\sigma}{\iint_D \mu(x,y)\mathrm{d}\sigma}\text{。}$$

该薄片关于 x 轴和 y 轴的**转动惯量**为

$$I_x=\iint_D y^2\mu(x,y)\mathrm{d}\sigma, I_y=\iint_D x^2\mu(x,y)\mathrm{d}\sigma \text{。}$$

Consider now a lamina of variable density $\mu(x, y)$ covering a region D in the xy-plane. Then the **center of mass** of the lamina is given by

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_D x\mu(x,y)\mathrm{d}\sigma}{\iint_D \mu(x,y)\mathrm{d}\sigma},\quad \bar{y}=\frac{M_x}{m}=\frac{\iint_D y\mu(x,y)\mathrm{d}\sigma}{\iint_D \mu(x,y)\mathrm{d}\sigma}.$$

The **moments of inertia** of the lamina about the x- and y-axes are given by

$$I_x=\iint_D y^2\mu(x,y)\mathrm{d}\sigma,\quad I_y=\iint_D x^2\mu(x,y)\mathrm{d}\sigma .$$

▶第十章　曲线积分与曲面积分
Chapter 10　Line(Curve)Integrals and Surface Integrals

10.1 对弧长的曲线积分(Line Integrals with Respect to Arc Length)

对弧长的曲线积分的概念和性质(The Concept and Properties of Line Integrals with Respect to Arc Length)

设 C 是一光滑的平面曲线，即如果 C 的参数方程由 $x=x(t)$, $y=y(t)$（$a\leqslant t\leqslant b$）给出。其中 x' 和 y' 都是连续的，且在区间(a, b)上它们不同时为 0。规定 C 的正向对应到 t 增加的方向。假设 C 取正向，且当 t 从 a 变到 b 时，C 无重点。这样 C 的起点为

$A=(x(a),y(a))$，终点为 $B=(x(b),y(b))$。对参数区间$[a,b]$插入分点 $a=t_0<t_1<t_2<\cdots<t_n=b$ 得到分法 P。这个分法导致了将曲线 C 分成了 n 个小弧段 $P_{i-1}P_i$，其中点 P_i 对应到 t_i。用 Δs_i 表示弧 $P_{i-1}P_i$ 的长度，λ表示小弧段的最大长度。最后,在小弧段选择点 $Q_i(\bar{x}_i,\bar{y}_i)$。现在考虑黎曼和 $\sum_{i=1}^{n} f(\bar{x}_i,\bar{y}_i)\Delta s_i$。如果当 $\lambda\to 0$ 时，该黎曼和有极限，该极限就叫做f沿曲线 C 从 A 到 B 对弧长的曲线积分。即

$$\int_C f(x,y)\mathrm{d}s=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\bar{x}_i,\bar{y}_i)\Delta s_i 。$$

Let C be a smooth plane curve; that is, let C be given parametrically by $x=x(t)$, $y=y(t)$, $a\leqslant t\leqslant b$, where x' and y' are continuous and not simultaneously zero on (a,b). We say that C is positively oriented if its direction corresponds to increasing values of t. We suppose that C is positively oriented and that C is traced only once as t varies from a to b. Thus, C has initial point $A=(x(a),y(a))$ and terminal point $B=(x(b),y(b))$. Consider the partition P of the parameter interval $[a,b]$ obtained by inserting the points $a=t_0<t_1<t_2<\ldots<t_n=b$. This partition of $[a,b]$ results in a division of the curve C into n subarcs $P_{i-1}P_i$ in which the point P_i corresponds to t_i. Let Δs_i denote the length of the arc $P_{i-1}P_i$, and let λ be the longest length of the subarcs. Finally, choose a sample point $Q_i(\bar{x}_i,\bar{y}_i)$ on the subarc $P_{i-1}P_i$. Now consider the Riemann sum $\sum_{i=1}^{n} f(\bar{x}_i,\bar{y}_i)\Delta s_i$. If this Riemann sum has a limit as $\lambda\to 0$. This limit is called the **line integral of f along C from A to B with respect to arc length**; that is

$$\int_C f(x,y)\mathrm{d}s=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\bar{x}_i,\bar{y}_i)\Delta s_i .$$

如果 C 是分段光滑的，即 C 是由几段光滑曲线 $C_1, C_2,\cdots,C_k$ 连接起来的。我们规定 C 上的积分为各段上的积分之和。

If C is piecewise smooth, that is, consists of several smooth curves C_1, C_2, $\cdots$, C_k joined together. We simply define the integral over C to be the sum of the integrals over the individual curves.

10.2 对坐标的曲线积分(Line Integrals with Respect to Coordinate Variables)

对坐标的曲线积分的概念与性质(The Concept and Properties of Line Integrals with Respect to Coordinate Variables)

定义 设 $P(x,y)$和 $Q(x,y)$是连续函数，$y=\varphi(x)$是一光滑曲线，x 从 a 变到 b，则对应的关于坐标的曲线积分或第二类曲线积分为

$$\int_C P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=\int_a^b[P(x,\varphi(x))+\varphi'(x)Q(x,\varphi(x))]\mathrm{d}x 。$$

If $P(x,y)$ and $Q(x,y)$ are continuous functions and $y=\varphi(x)$ is a smooth curve C which runs from a to b as x varies, then the corresponding **line integral with respect to coordinate variables** or **line integral of the second type** is expressed as follow:

$$\int_C P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=\int_a^b[P(x,\varphi(x))+\varphi'(x)Q(x,\varphi(x))]\mathrm{d}x .$$

对于更一般的情形当曲线 C 由参数方程 $x=\varphi(t)$, $y=\psi(t)$给出，t 从 a 变到 b，我们有

$$\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y =$$

$$\int_a^b [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]\mathrm{d}t \text{。}$$

In the more general case when the curve C is represented parametrically, $x=\varphi(t)$, $y=\psi(t)$ where t varies from a to b, we have

$$\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y =$$

$$\int_a^b [P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)]\mathrm{d}t .$$

对于空间曲线上的第二类曲线积分也有类似的公式。当积分路径改变方向时,第二类曲线积分改变符号。

Similar formulae hold for a line integral of the second type taken over a space curve. A line integral of the second type changes sign when the direction of the integration path is reversed.

10.3 格林公式及其应用(Green's Formula and Its Applications)

一、格林公式(Green's Formula)

定理 1 假设 C 是一个分段光滑的简单闭曲线,围成平面区域 D。设 $P(x, y)$, $Q(x, y)$在 D 上有连续的偏导数。则

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint_C P\mathrm{d}x + Q\mathrm{d}y \text{。}$$

Theorem 1 Suppose C is a piecewise smooth simple closed curve bounding the domain D in the plane. Suppose $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on D. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint_C P\mathrm{d}x + Q\mathrm{d}y .$$

格林公式建立了曲线积分和二重积分的关系。这是微积分基本定理在高维情形的推广：它将 D 上 P、Q 某种形式的导数的积分用 D 的边界上 P、Q 的积分表示出来。我们可以使用格林公式将一种类型的积分转化为另外一种类型的积分来计算。

Green's formula establishes a relation between a line integral and a double integral. This result is a higher-dimentional analogue of the **Fundamental Theorem of Calculus**: it expresses the integral over D of some kind of derivative of P and Q in terms of an integral of P and Q over the boundary of D. We can use Green's foumula to replace the computation of one kind of integral by the computation of the other kind of integral.

二、平面上曲线积分与路径无关的条件(Conditions for Path Independence of a Line Integral)

定理 2 曲线积分与路径无关的充要条件

设 D 是一个单连通域，$P(x, y)$，$Q(x, y)$在 D 内具有一阶连续偏导数，则曲线积分 $\int_C P\mathrm{d}x+Q\mathrm{d}y$ 在 D 内与路径无关的充分必要条件是 $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$，对所有的$(x, y)\in D$ 均成立。

Theorem2 (Necessary and Sufficient Conditions for Path Independence of a Line Integral) Let D be a simply-connected region. $P(x, y)$, $Q(x, y)$ have continuous first partial derivatives on

D. Then the line integral $\int_C P\mathrm{d}x + Q\mathrm{d}y$ is independent of path if and only if $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$ for all (x, y) in D.

三、二元函数的全微分求积(Integration by the Total Differential of Functions of two Variables)

定理 3 设 D 是一个单连通域，$P(x, y)$, $Q(x, y)$在 D 内具有一阶连续偏导数，则 $P(x, y)\mathrm{d}x+Q(x, y)\mathrm{d}y$ 在 D 内为某一函数 $u(x,y)$ 的全微分的充分必要条件是

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x},$$

对所有的$(x,y)\in D$ 均成立。

Theorem 3 Let D be a simply-connected region. $P(x, y)$, $Q(x, y)$ have continuous first partial derivatives on D. Then $P(x, y)\mathrm{d}x+Q(x, y)\mathrm{d}y$ is a total differential of some function $u=u(x, y)$, i.e., $P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=\mathrm{d}u(x,y)$, if and only if $\dfrac{\partial P(x,y)}{\partial y} = \dfrac{\partial Q(x,y)}{\partial x}$ for all (x, y) in D.

10.4 对面积的曲面积分(Surface Integrals with Respect to Area)

一、对面积的曲面积分的概念与性质(Concept and Properties of Surface Integrals with Respect to Area)

定义 设Σ是三维空间中的光滑曲面，$f(x,y,z)$在Σ上有界。将Σ分成 n 部分 $\sum_1, \sum_2, \cdots, \sum_n$ 。每一部分的面积记为 $\Delta S_1, \cdots, \Delta S_n$ 。在$\sum_i$ 上任取一点(ξ_i, η_i, ζ_i)，作黎曼和$\sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta S_i$。设$\lambda$表示所有$\Sigma_i$ 的直径的最大值。如果$\lambda \to 0$时，黎曼和的极限存在，则称此极限为$f(x,y,z)$在曲面Σ上对面积的曲面积分或第一类曲面积分，记作

$$\iint_\Sigma f(x,y,z)\mathrm{d}S = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta S_i 。$$

Definition Let Σ be a smooth surface in three-space, and let $f(x,y,z)$ be bounded on Σ. Subdivide Σ into n parts Σ_1, Σ_2, $\cdots$, Σ_n, with areas $\Delta S_1, \cdots, \Delta S_n$, and form the Riemann sum $\sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta S_i$, where (ξ_i, η_i, ζ_i) is an arbitrary point on Σ_i. Let λ be the longest diameter of those Σ_i. If the Riemann sum has a limit as $\lambda\to 0$. This limit is called the **surface integral with respect to area** (or the **surface integral of the first type**) of $f(x,y,z)$ over Σ and is denoted by

$$\iint_\Sigma f(x,y,z)\mathrm{d}S = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i)\Delta S_i .$$

二、对面积的曲面积分的计算法(Computation of Surface Integrals with Respect to Area)

定理 设曲面的方程为$z=z(x,y)$，它在xy平面的投影为R。如果$z(x,y)$在R上有一阶连续偏导数，$f(x, y, z)$在Σ上连续，则

$$\iint_\Sigma f(x,y,z)\mathrm{d}S = \iint_R f(x,y,z(x,y))\sqrt{1+z_x^{\,2}(x,y)+z_y^{\,2}(x,y)}\mathrm{d}x\mathrm{d}y 。$$

Theorem Let Σ be a surface with equation $z=z(x,y)$ and let R be the projection on the xy-plane. If $z(x, y)$ has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on Σ, then

$$\iint_{\Sigma} f(x,y,z)\mathrm{d}S = \iint_{R} f(x,y,z(x,y))\sqrt{1+z_x^{\,2}(x,y)+z_y^{\,2}(x,y)}\mathrm{d}x\mathrm{d}y\,.$$

10.5 对坐标的曲面积分(Surface Integrals with Respect to Coordinate Variables)

一、对坐标的曲面积分的概念和性质(Concept and Properties of Surface Integrals with Respect to Coordinate Variables)

定义 设Σ是三维空间中的有向光滑曲面，$f(x,y,z)$在Σ上有界。将Σ分成n部分Σ_1，Σ_2，$\cdots\Sigma_n$，每一部分的面积记为$\Delta S_1,\cdots,\Delta S_n$。$\Delta S_i$在$xoy$平面的投影为$(\Delta S_i)_{xy}$。在$\Sigma_i$上任取一点$(\xi_i,\eta_i,\zeta_i)$，作黎曼和$\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}$。设λ表示所有$\Sigma_i$的直径的最大值。如果$\lambda\to 0$时，黎曼和的极限存在，则称此极限为$f(x, y, z)$在曲面Σ上对坐标的曲面积分或第二类曲面积分，记作

$$\iint_{\Sigma} f(x,y,z)\mathrm{d}x\mathrm{d}y = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}\text{。}$$

Definition Let Σ be a smooth directed surface in 3-space, and let $f(x, y, z)$ be bounded on Σ. Subdivide Σ into n parts $\Sigma_1, \cdots, \Sigma_n$ with areas $\Delta S_1,\cdots,\Delta S_n$. Let $(\Delta S_i)_{xy}$ be the projection on the xoy-plane of ΔS_i. Form the Riemann sum $\sum\limits_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}$, where (ξ_i,η_i,ζ_i) is an arbitrary point on Σ_i. Let λ be the longest diameter of those Σ_i. If the Riemann sum has a limit as $\lambda\to 0$. This limit is called the **surface integral with respect to coordinate variables** (or the **surface integral of the second type**) of $f(x, y, z)$ over Σ and is denoted by

$$\iint_{\Sigma} f(x,y,z)\mathrm{d}x\mathrm{d}y = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)(\Delta S_i)_{xy}\,.$$

二、对坐标的曲面积分的计算法(Computation of Integrals with Respect to Coordinate Variables)

设Σ是由方程$z=z(x, y)$所给出的曲面上侧，Σ在xoy面上的投影为D_{xy}，函数$z=z$（x，y）在D_{xy}上具有一阶连续偏导数，R（x，y，z）在Σ上连续，则

$$\iint_{\Sigma} R(x,y,z)\mathrm{d}x\mathrm{d}y = \iint_{D_{xy}} R[x,y,z(x,y)]\mathrm{d}x\mathrm{d}y\text{。}$$

Let Σ be a surface given by the equation $z=z(x,y)$, and suppose that Σ is oriented upward. Let D_{xy} be the projection of Σ on the xoy-plane. The function $z=z(x, y)$ has continuous first partial derivatives on D_{xy}. $R(x, y, z)$ is continuous on Σ. Then

$$\iint_{\Sigma} R(x,y,z)\,\mathrm{d}x\mathrm{d}y = \iint_{D_{xy}} R[x,y,z(x,y)]\,\mathrm{d}x\mathrm{d}y.$$

三、两类曲面积分之间的联系(The Relation between the two types of Surface Integrals)

定理 设有向曲面Σ的方程为$z=z(x,y)$，它在xoy平面的投影为R。如果$z(x,y)$在R上有一阶连续偏导数，$f(x,y,z)$在Σ上连续，则

$$\iint_{\Sigma} f(x,y,z)\mathrm{d}x\mathrm{d}y = \pm\iint_{R} f[x,y,z(x,y)]\mathrm{d}x\mathrm{d}y\text{。}$$

Theorem Let Σ be a directed surface with equation $z=z(x, y)$ and let R be the projection on the xoy-plane.If $z(x, y)$has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on Σ, then

$$\iint_{\Sigma} f(x,y,z)\mathrm{d}x\mathrm{d}y = \pm\iint_{R} f[x,y,z(x,y)]\mathrm{d}x\mathrm{d}y .$$

两类曲面积分由下式建立联系：

$$\iint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S ,$$

其中 $\cos\alpha$、$\cos\beta$、$\cos\gamma$ 是有向曲面 Σ 在点 (x, y, z) 处的法向量的方向余弦。

The two types of surface integrals are related by the formula

$$\iint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S ,$$

Where $\cos\alpha$、$\cos\beta$ and $\cos\gamma$ is the direction cosines of the normal vector of the directed surface Σ at the point (x, y, z).

10.6 高斯公式 通量与散度(Gauss's Formula Flux and Divergence)

高斯公式(Gauss's Formula)

定理(高斯定理) 设Ω是空间闭区域，其边界曲面Σ是分片光滑的，Σ取外侧。函数 $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$在Ω上具有一阶连续偏导数，则有

$$\iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \oiint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y 。$$

Theorem (Gauss's Theorem) Let Ω be a closed region in three-space with boundary surface Σ oriented outword. Suppose that Σ is piecewise smooth. $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ have continuous first partial derivatives on Ω, then

$$\iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \oiint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y .$$

设 $\boldsymbol{A}(x,y,z)= P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+ R(x,y,z)\boldsymbol{k}$ 是一向量场，P, Q, R 有一阶连续偏导数。Σ 是场内的一有向曲面，$\boldsymbol{n}$ 是 Σ 在点(x,y,z)处的单位法向量，则 $\iint_{\Sigma} A\cdot n\mathrm{d}S$ 叫做向量场 $\boldsymbol{A}$ 通过曲面 Σ 的**通量**，而 $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ 叫做向量场 $\boldsymbol{A}$ 的**散度**，记作 $\mathrm{div}\,\boldsymbol{A}$。

Let $\boldsymbol{A}(x,y,z)= P(x,y,z)\boldsymbol{i}+ Q(x,y,z)\boldsymbol{j}+ R(x,y,z)\boldsymbol{k}$ be a vector field, where P, Q, R have continuous first partial derivatives. Σ is a directed surface. $\boldsymbol{n}$ is the unit normal vector of Σ at (x,y,z), then $\iint_{\Sigma} A\cdot n\mathrm{d}S$ is called the **flux** of the vector field $\boldsymbol{A}$ across Σ, and $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is called the **divergence** of the vector field $\boldsymbol{A}$, denoted by $\mathrm{div}\,\boldsymbol{A}$.

10.7 斯托克斯公式 环流量与旋度 (Stokes's Formula Circulation and Rotation)

斯托克斯公式 (Stokes's Formula)

定理(斯托克斯定理) 设 Γ 为分段光滑的空间有向闭曲线，Σ 是以 Γ 为边界的分片光滑的有向曲面，Γ 的正向与 Σ 的侧符合右手规则(即如果右手大拇指表示曲面 Σ 的方向，则弯曲四指表示曲线 Γ 的方向)，函数 $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ 在曲面 Σ 上(连同边界 Γ) 上具有一阶连续偏导数，则有下述公式成

立：

$$\iint_{\Sigma}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\mathrm{d}y\mathrm{d}z+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\mathrm{d}z\mathrm{d}x+ (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\mathrm{d}x\mathrm{d}y=\oint_{\Gamma}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z。$$

Theorem (Stokes's Theorem) Let Γ be a piecewise smooth oriented curve in three-space, Σ be a piecewise smooth oriented surface with boundary curve Γ. The surface Σ and the curve Γ are related by a **right-hand relationship,** i.e., if the fingers of the right hand are cupped in the direction of Γ, then the thumb points in the direction of orientation of Σ. $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ have continuous first partial derivatives on Σ and Γ, then

$$\iint_{\Sigma}(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\mathrm{d}y\mathrm{d}z+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\mathrm{d}z\mathrm{d}x+ (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\mathrm{d}x\mathrm{d}y=\oint_{\Gamma}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z.$$

斯托克斯公式是格林公式在三维空间中的推广。它建立了有向曲面Σ上的曲面积分与它的边界曲线Γ上的曲线积分的关系。

Stokes' formula is a generalization of Green's formula to three-space. It gives a relationship between a surface integral over an oriented surface Σ and a line integral along a simple closed curve Γ.

设 $\boldsymbol{A}(x,y,z)= P(x,y,z)\boldsymbol{i}+ Q(x,y,z)\boldsymbol{j}+ R(x,y,z)\boldsymbol{k}$ 是一向量场，则向量 $(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})$ 叫做向量场 $\boldsymbol{A}$ 的**旋度**，记作

rot $\boldsymbol{A}$。沿有向闭曲线Γ的曲线积分 $\oint_{\Gamma}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z$ 叫做向量场 $\boldsymbol{A}$ 沿有向闭曲线Γ的**环流量**，记作 **curl** $\boldsymbol{A}$。

Let $\boldsymbol{A}(x,y,z)=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$ be a vector field, then the vector $(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})$ is called the **rotation** of the vector field $\boldsymbol{A}$, denoted by **rot** $\boldsymbol{A}$. The line integral $\oint_{\Gamma}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z$ along the closed oriented curve Γ is called the **circulation** of the vector field $\boldsymbol{A}$ along the curve Γ , denoted by **curl** $\boldsymbol{A}$.

第十一章　无穷级数
Chapter 11　Infinite Series

11.1 常数项级数的概念和性质(The Concept and Properties of the Constant Series)

一、常数项级数的概念 (The Concept of Constant Series)

一般的，如果给定一个数列 $u_1, u_2, u_3,\cdots$，u_n，$\cdots$，则由这个数列构成的表达式

$$u_1+u_2+u_3+\cdots+u_n+\cdots \tag{1}$$

叫做(常数项)无穷级数，简称(常数项)级数，记为 $\sum_{n=1}^{\infty}u_n$，即

$$\sum_{n=1}^{\infty}u_n=u_1+u_2+u_3+\cdots+u_n+\cdots$$

其中第 n 项 u_n 叫做级数的一般项。$s_n= u_1+u_2+u_3+\cdots+u_n$ 称为级数(1)的部分和。

Generally, for the given sequence u_1, u_2, u_3, $\cdots$, u_n, $\cdots$, the formula

$$u_1+u_2+u_3+\cdots+u_n+\cdots$$

is called the infinite series of the constant term, denoted by $\sum_{n=1}^{\infty}u_n$, that is

$$\sum_{n=1}^{\infty}u_n=u_1+u_2+u_3+\cdots+u_n+\cdots,$$

Where the nth term u_n is said to be the general term of the series, moreover, the nth partical sum of the series is given by $s_n=u_1+u_2+u_3+\cdots+u_n$.

定义 如果级数$\sum_{n=1}^{\infty}u_n$的部分和数列$\{s_n\}$有极限 s，即$\lim_{n\to\infty}s_n=s$，则称无穷级数$\sum_{n=1}^{\infty}u_n$收敛，这时极限s叫做这级数的和，并写成$s=u_1+u_2+u_3+\cdots+u_n+\cdots$；如果$\{s_n\}$没有极限，则称无穷级数$\sum_{n=1}^{\infty}u_n$发散。

Definition The infinite series $\sum_{n=1}^{\infty}u_n$ converges and has sum s if the sequence of partial sums $\{s_n\}$ converges to s, that is $\lim_{n\to\infty}s_n=s$. If $\{s_n\}$ diverges, then the series $\sum_{n=1}^{\infty}u_n$ diverges. A divergent series has no sum.

等比级数(Geometric Series)

无穷级数$\sum_{n=0}^{\infty}aq^n=a+aq+aq^2+\cdots+aq^n+\cdots$叫做等比级数(又称为几何级数)，其中$a\neq0$，$q$叫做级数的公比。

A series of the form $\sum_{n=0}^{\infty}aq^n=a+aq+aq^2+\cdots+aq^n+\cdots$, where $a\neq0$, is called a geometric series, and q is called the common ratio of the series.

二、收敛级数的性质(Properties of Convergent Series)

性质 1 如果级数$\sum_{n=1}^{\infty}u_n$收敛于和s，则级数$\sum_{n=1}^{\infty}ku_n$也收敛，且其和为ks。

Property 1 If $\sum_{n=1}^{\infty}u_n$ converges to s, then $\sum_{n=1}^{\infty}ku_n$ also converges, and

$$\sum_{n=1}^{\infty}ku_n=k\sum_{n=1}^{\infty}u_n=ks.$$

性质 2 如果级数$\sum_{n=1}^{\infty}u_n$、$\sum_{n=1}^{\infty}v_n$分别收敛于s和δ，则级数$\sum_{n=1}^{\infty}(u_n\pm v_n)$也收敛，且其和为$s\pm\delta$.

Property 2 If $\sum_{n=1}^{\infty}u_n$, $\sum_{n=1}^{\infty}v_n$ converge to s and δ respectively, then $\sum_{n=1}^{\infty}(u_n\pm v_n)$ also converges, and $\sum_{n=1}^{\infty}(u_n\pm v_n)=s\pm\delta$.

性质 3 在级数中去掉、加上或改变有限项，不会改变级数的收敛性。

Property 3 Deleting, adding and altering the finite terms of the infinite series keep the convergence of the series.

性质 4 如果级数$\sum_{n=1}^{\infty}u_n$收敛，则对这级数的项任意加括号后所成的级数

$(u_1+u_2+u_3+\cdots+u_{n1})+\cdots+(u_{n1+1}+u_{n1+2}+u_{n1+3}+\cdots+u_{n1+nk})+\cdots$
仍然收敛，且其和不变。

Property 4　The terms of a convergent series can be grouped in any way (provided that the order of the terms is maintained). And the new series $(u_1+u_2+u_3+\cdots+u_{n1})+\cdots+\ (u_{n1+1}+u_{n1+2}+u_{n1+3}+\text{n}+u_{n1+nk})+\text{n}$ will converge with the same sum as the original series.

性质 5 (级数收敛的必要条件)　如果级数 $\sum_{n=1}^{\infty}u_n$ 收敛，则它的一般项 u_n 趋于零，即

$$\lim_{n\to\infty}u_n=0 。$$

Property 5 (Necessary Condition for Convergence)　If the series $\sum_{n=1}^{\infty}u_n$ converges，then $\lim_{n\to\infty}u_n=0$.

三、柯西审敛原理(Cauchy Convergence Criteria)

定理(柯西审敛原理)　级数 $\sum_{n=1}^{\infty}u_n$ 收敛的充分必要条件为：对于任意给定的正数ε，总存在自然数 N，使得当 $n>N$ 时，对于任意的自然数 p，都有

$$|u_{n+1}+u_{n+2}+u_{n+3}+\cdots+u_{n+p}|<\varepsilon$$

成立。

Theorem(Cauchy Convergence Criteria)　The series $\sum_{n=1}^{\infty}u_n$ converges if and only if for any $\varepsilon>0$, there is a natural number N, such that for any $n>N$, and any natural number p, the inequality

$$|u_{n+1}+u_{n+2}+u_{n+3}+\cdots+u_{n+p}|<\varepsilon$$

holds.

11.2　常数项级数的审敛法(Test for Convergence of the Constant Series)

一、正项级数及其审敛法(Test for the Convergence of Positive Series)

定理 1　正项级数 $\sum_{n=1}^{\infty}u_n$ 收敛的充分必要条件是：它的部分和数列{ s_n }有上界。

Theorem 1 (Bounded Sum Test)　A series $\sum_{n=1}^{\infty}u_n$ of nonnegative terms converges if and only if its partial sums $\{s_n\}$are bounded above.

定理 2 (比较审敛法)　设 $\sum_{n=1}^{\infty}u_n$ 和 $\sum_{n=1}^{\infty}v_n$ 都是正项级数，且 $u_n\leqslant v_n(n=1,2,\cdots)$。若级数 $\sum_{n=1}^{\infty}v_n$ 收敛，则级数 $\sum_{n=1}^{\infty}u_n$ 收敛；反之，若级数 $\sum_{n=1}^{\infty}u_n$ 发散，则级数 $\sum_{n=1}^{\infty}v_n$ 发散。

Theorem 2 (Ordinary Comparison Test)　Suppose that $\sum u_n$ and $\sum v_n$ are positive series，and $u_n\leqslant v_n(n=1,2,\cdots)$.If the series $\sum_{n=1}^{\infty}v_n$ converges, so does $\sum_{n=1}^{\infty}u_n$; if the series $\sum_{n=1}^{\infty}u_n$ diverges, so does the series $\sum_{n=1}^{\infty}v_n$.

推论 设$\sum_{n=1}^{\infty}u_n$和$\sum_{n=1}^{\infty}v_n$都是正项级数，如果级数$\sum_{n=1}^{\infty}v_n$收敛，且存在自然数N，使当$n\geqslant N$时，有$u_n\leqslant kv_n(k>0)$成立，则级数$\sum_{n=1}^{\infty}u_n$收敛；如果级数$\sum_{n=1}^{\infty}v_n$发散，且当$n\geqslant N$时，有$u_n\geqslant kv_n(k>0)$成立，则级数$\sum_{n=1}^{\infty}u_n$发散。

Corollary Suppose that $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ are positive series, if the series $\sum_{n=1}^{\infty}v_n$ converges, and there is a natural number N, such that for any $n\geqslant N$, $u_n\leqslant kv_n\,(k>0)$ holds, then the series $\sum_{n=1}^{\infty}u_n$ also converges; If $\sum_{n=1}^{\infty}v_n$ diverges, and there is a natural number N, such that for any $n\geqslant N$, $u_n\geqslant kv_n\,(k>0)$ holds, then $\sum_{n=1}^{\infty}u_n$ also diverges.

***p*-级数审敛法 (*p*-series Test)**

称级数

$$\sum_{n=0}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$$

为p-级数，且当$p>1$时，p-级数收敛；当$p\leqslant 1$时，p-级数发散。

The series

$$\sum_{n=0}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$$

where p is a constant, is called a p-series. Show each of the following:

The p-series converges if $p>1$; and the p-series diverges if $p\leqslant 1$.

定理 3 (比较审敛法的极限形式) 设$\sum_{n=1}^{\infty}u_n$和$\sum_{n=1}^{\infty}v_n$都是正项级数，

(1) 如果$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l\ (0\leqslant l<+\infty)$，且级数$\sum_{n=1}^{\infty}v_n$收敛，则级数$\sum_{n=1}^{\infty}u_n$收敛；

(2) 如果$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l\ (l>0)$，或$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=+\infty$，且级数$\sum_{n=1}^{\infty}v_n$发散，则级数$\sum_{n=1}^{\infty}u_n$发散。

Theorem 3 (Limit Comparison Test) Suppose that $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ are positive series. Then

(1) If $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l\ (0\leqslant l<+\infty)$, and the series $\sum_{n=1}^{\infty}v_n$ converges, so does $\sum_{n=1}^{\infty}u_n$;

(2) If $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l\ (l>0)$, or $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=+\infty$, and the series $\sum_{n=1}^{\infty}v_n$ diverges, then the series $\sum_{n=1}^{\infty}u_n$ also diverges.

定理 4 (比值审敛法，达朗贝尔(D'Alembert)判别法) 设$\sum_{n=1}^{\infty}u_n$为正项级数，如果$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\rho$，则当$\rho<1$时级数收敛；$\rho>1$（或$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$）时级数发散；$\rho=1$时级数可能收敛也可能发散。

Theorem 4 (Ratio Test，D'Alembert Test) Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms and suppose that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\rho$, then if $\rho<1$, the series converges; If $\rho>1$ (or $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\infty$), the series diverges; If $\rho=1$, the test is inconclusive.

定理 5 (根值审敛法，柯西审敛法) 设$\sum_{n=1}^{\infty} u_n$为正项级数，如果

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\rho,$$

则当 $\rho<1$ 时级数收敛；$\rho>1$(或$\lim_{n\to\infty}\sqrt[n]{u_n}=\infty$)时级数发散；$\rho=1$时级数可能收敛也可能发散。

Theorem 5 (Cauchy Test) Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms and suppose that $\lim_{n\to\infty}\sqrt[n]{u_n}=\rho$. Then if $\rho<1$, the series converges; If $\rho>1$ (or $\lim_{n\to\infty}\sqrt[n]{u_n}=\infty$), the series diverges; If $\rho=1$, the test is inconclusive.

定理 6 (极限审敛法) 设$\sum_{n=1}^{\infty} u_n$为正项级数，

(1) 如果$\lim_{n\to\infty} nu_n=l>0$(或$\lim_{n\to\infty} nu_n=+\infty$)，则级数$\sum_{n=1}^{\infty} u_n$发散；

(2) 如果$p>1$，而$\lim_{n\to\infty} n^p u_n=l\ (0\leqslant l<+\infty)$，则级数$\sum_{n=1}^{\infty} u_n$收敛。

Theorem 6 (Limit Test) Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms. Then

(1) If $\lim_{n\to\infty} nu_n=l>0$ (or $\lim_{n\to\infty} nu_n=+\infty$), the series $\sum_{n=1}^{\infty} u_n$ diverges;

(2) If $p>1$, and $\lim_{n\to\infty} n^p u_n=l\ (0\leqslant l<+\infty)$, the series $\sum_{n=1}^{\infty} u_n$ converges.

二、交错级数及其审敛法(Alternating Series and The tests for Convergence)

交错级数 (The Alternating Series) 形如

$$u_1-u_2+u_3-u_4+\cdots$$

的级数称为交错级数，其中 u_1，u_2，…都是正数。

The series of the form

$$u_1-u_2+u_3-u_4+\cdots$$

is called the alternating series, where $u_n>0$ for all n.

定理 7 (莱布尼茨定理) 如果交错级数$\sum_{n=1}^{\infty}(-1)^n u_n$满足条件：

(1) $u_n\geqslant u_{n+1}\ (n=1,2,\cdots)$;

(2) $\lim_{n\to\infty} u_n=0$。

则级数收敛，且其和 $s\leqslant u_1$，其余项 r_n 的绝对值$|r_n|\leqslant u_{n+1}$。

Theorem 7 (Leibniz Theorem)

If the alternating series $\sum_{n=1}^{\infty}(-1)^n u_n$ satisfy:

(1) $u_n \geqslant u_{n+1}$ $(n=1, 2,\cdots)$;

(2) $\lim_{n\to\infty} u_n = 0$.

then the series converges. Moreover, $s \leqslant u_1$, and the error r_n made by using the sum s_n of the first n terms to approximate the sum s of the series is not more than u_{n+1}, that is, $|r_n| \leqslant u_{n+1}$.

三、绝对收敛与条件收敛(Absolute Convergence and Conditional Convergence)

绝对收敛(Absolute Convergence) 如果级数$\sum_{n=1}^{\infty}|u_n|$收敛；称级数$\sum_{n=1}^{\infty}u_n$绝对收敛，如果级数$\sum_{n=1}^{\infty}u_n$收敛，而$\sum_{n=1}^{\infty}|u_n|$发散，则称级数$\sum_{n=1}^{\infty}u_n$条件收敛。

A series $\sum_{n=1}^{\infty}u_n$ is said to be converges absolutely if the series $\sum_{n=1}^{\infty}|u_n|$ converges; the series $\sum_{n=1}^{\infty}u_n$ is called conditionally convergent, if the series $\sum_{n=1}^{\infty}u_n$ converges but $\sum_{n=1}^{\infty}|u_n|$ diverges.

定理 8 (绝对收敛判别法) 如果级数$\sum_{n=1}^{\infty}u_n$绝对收敛，则级数$\sum_{n=1}^{\infty}u_n$必定收敛。

Theorem 8 (Absolute Convergence Test) If the series $\sum_{n=1}^{\infty}u_n$ converges absolutely, the series $\sum_{n=1}^{\infty}u_n$ must be convergent.

定理 9 (可交换性定理) 绝对收敛级数经改变项的位置后构成的级数也收敛，且与原级数有相同的和（即绝对收敛级数具有可交换性）。

Theorem 9 (Rearrangement Theorem) The terms of an absolutely convergent series can be rearranged without affecting either the convergence or the sum of the series.

定理 10 (绝对收敛级数的乘法) 设级数$\sum_{n=1}^{\infty}u_n$和$\sum_{n=1}^{\infty}v_n$都绝对收敛，其和分别为 s 和 σ，则它们的柯西乘积

$$u_1v_1+(u_1v_2+u_2v_1)+\cdots+(u_1v_n+u_2v_{n-1}\cdots+u_nv_1)+\cdots$$

也是绝对收敛的，且其和为$s\cdot\sigma$。

Theorem 10 (Multiplication of Absolute Convergent series) Let the series $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ be absolutely converge to s and σ respectively, then the Cauchy product of them which is denoted by

$$u_1v_1+(u_1v_2+u_2v_1)+\cdots+(u_1v_n+u_2v_{n-1}\cdots+u_nv_1)+\cdots$$

converges to $s\cdot\sigma$.

11.3 幂级数(Power Series)

一、函数项级数的概念(The Series of Functions)

如果给定一个定义在区间 I 上的函数列

$$u_1(x), u_2(x), u_3(x),\cdots, u_n(x),\cdots,$$

则由这函数列构成的表达式

$$u_1(x)+u_2(x)+u_3(x)+\cdots+u_n(x)+\cdots+ \qquad (1)$$

称为定义在区间 I 上的(函数项)无穷级数，简称(函数项)级数。

The formula

$$u_1(x)+u_2(x)+u_3(x)+\cdots+u_n(x)+\cdots+ \quad (1)$$

is called **the infinite series of functions** on the interval I, where

$$u_1(x), u_2(x), u_3(x),\cdots, u_n(x),\cdots,$$

are all functions defined on the interval I.

对于每一个确定的 $x_0\in I$，若级数

$$u_1(x_0)+u_2(x_0)+u_3(x_0)+\cdots+u_n(x_0)+\cdots+$$

收敛，则称 x_0 为函数项级数(1)的收敛点。函数项级数的所有收敛点的全体称为它的收敛域。

For the series of functions (1), the point x_0 is called the convergent point if

$$u_1(x_0)+u_2(x_0)+u_3(x_0)+\cdots+u_n(x_0)+\cdots+$$

is convergent. We call the set on which a power series converges its convergence set.

二、幂级数及其收敛性 (Power Series and Its Convergence)

定理 1 (阿贝尔定理) 如果级数 $\sum_{n=0}^{\infty}a_nx^n$ 当 $x=x_0$ $(x_0\neq 0)$时收敛，则适合不等式$|x|<|x_0|$的一切 x 使这幂级数绝对收敛；反之，如果级数 $\sum_{n=0}^{\infty}a_nx^n$ 当 $x=x_0$ 时发散，则适合不等式$|x|>|x_0|$的一切 x 使这幂级数发散。

Theorem 1 (Abel Theorem) If the power series $\sum_{n=0}^{\infty}a_nx^n$ is convergent at $x=x_0$ $(x_0\neq 0)$, then for any x that $|x|<|x_0|$, the power series $\sum_{n=0}^{\infty}a_nx^n$ absolutely converges at point x; if the power series $\sum_{n=0}^{\infty}a_nx^n$ is divergent at $x=x_0$ $(x_0\neq 0)$, then for any x that $|x|>|x_0|$, the power series $\sum_{n=0}^{\infty}a_nx^n$ diverges at point x.

推论 如果幂级数 $\sum_{n=0}^{\infty}a_nx^n$ 不是仅在 $x=0$ 一点收敛，也不是在整个数轴上都收敛，则必有一个确定的正数 R 存在，使得

(1) 当$|x|<R$ 时，幂级数绝对收敛；

(2) 当$|x|>R$ 时，幂级数发散；

(3) 当 $x=R$ 与 $x=-R$ 时，幂级数可能收敛也可能发散。

正数 R 通常叫做幂级数的收敛半径，开区间$(-R, R)$叫做幂级数的收敛区间。

Corollary If the convergence set for a power series $\sum_{n=0}^{\infty}a_nx^n$ is not $\{x=0\}$nor the real line, then there exists a positive number, such that

(1) if $|x|<R$, $\sum_{n=0}^{\infty}a_nx^n$ is absolutely convergent

(2) if $|x|>R$, $\sum_{n=0}^{\infty}a_nx^n$ is divergent

(3) if $x=R$ and $x=-R$, $\sum_{n=0}^{\infty}a_nx^n$ may convergence or divergence.

The positive number R is called the convergent radius of power series, and $(-R, R)$ is said to be the convergent interval of power series.

In (1), (2), and (3), the series is said to have radius of convergence 0, R, and ∞, respectively.

定理 2 如果

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho ,$$

其中 a_n, a_{n+1} 是幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的相邻两项的系数，则这幂级数的收敛半径为

$$R = \begin{cases} \dfrac{1}{\rho} & \rho \neq 0 \\ +\infty & \rho = 0 \\ 0 & \rho = +\infty \end{cases}$$

Theorem 2 If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

where a_n, a_{n+1} is the coefficients of power series $\sum_{n=0}^{\infty} a_n x^n$ .The radius of convergence is given by

$$R = \begin{cases} \dfrac{1}{\rho} & \rho \neq 0 \\ +\infty & \rho = 0 \\ 0 & \rho = +\infty \end{cases}$$

三、幂级数的运算(Operations on Power Series)

性质 1 幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $s(x)$在其收敛域 I 上连续。

Property 1 The sum function of a power series $\sum_{n=0}^{\infty} a_n x^n$ is continuous on the interior of its convergence set.

性质 2 (逐项积分) 幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $s(x)$在其收敛域 I 上可积，并有逐项积分公式，$\int_0^x s(x)\mathrm{d}x = \int_0^x \left[\sum_{n=0}^{\infty} a_n x^n\right]\mathrm{d}x = \sum_{n=0}^{\infty}\int_0^x a_n x^n \mathrm{d}x = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$，逐项积分后所得到的幂级数和原级数有相同的收敛半径。

Property 2 (Term-by-Term Integration) Suppose that $s(x)$ is the sum of a power series on interval I; that is,

$$s(x)= \sum_{n=0}^{\infty} a_n x^n .$$

Then, if x is interior to I,

$$\int_0^x s(x)\,\mathrm{d}x = \int_0^x \left[\sum_{n=0}^{\infty} a_n x^n\right]\mathrm{d}x = \sum_{n=0}^{\infty}\int_0^x a_n x^n\,\mathrm{d}x = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1} ,$$

and the radius of convergence of the integrated series is the same as for the orginal series.

性质 3 (逐项求导) 幂级数 $\sum_{n=0}^{\infty} a_n x^n$ 的和函数 $s(x)$在其收敛区间$(-R, R)$内可导，且有逐项求导公式

$$s'(x)=\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty}(a_n x^n)' = \sum_{n=0}^{\infty} n a_n x^{n-1} \ (|x|< R)$$

逐项求导后所得到的幂级数和原级数有相同的收敛半径。

Property 3 (Term-by-Term Differentiation) Suppose that $s(x)$ is the sum of a power series on interval $(-R, R)$; that is,

$$s(x)=\sum_{n=0}^{\infty}a_n x^n.$$

Then, if x is interior to I,

$$s'(x)=(\sum_{n=0}^{\infty}a_n x^n)'=\sum_{n=0}^{\infty}(a_n x^n)'=\sum_{n=0}^{\infty}na_n x^{n-1}\ (|x|<R),$$

and the radius of convergence of the differentiated series is the same as for the orginal series.

11.4 函数展开成幂级数(Represent the Function as Power Series)

一、泰勒级数(Taylor Series)

定理 设函数 $f(x)$在点 x_0 的某一邻域 $U(x_0)$内具有各阶导数，则$f(x)$在该邻域内能展开成泰勒级数的充分必要条件是$f(x)$的泰勒公式中的余项 $R_n(x)$当 $n\to\infty$时的极限为零，即

$$\lim_{n\to\infty}R_n(x)=0。\quad (x\in U(x_0))$$

Theorem Let $f(x)$ be a function with derivatives of all orders in a neighborhood $U(x_0)$. Then $f(x)$ can be represented as Taylor series if and only if

$$\lim_{n\to\infty}R_n(x)=0,\ (x\in U(x_0))$$

where $R_n(x)$ is the remainder in Taylor formula.

二、函数展开成幂级数(Represent the Function as Power Series)

11.5 函数的幂级数展开式的应用(The Application of the Power Series representation of a Function)

一、近似计算(Approximate Calculation)

二、欧拉公式(Euler's Formula)

11.6 函数项级数的一致收敛性及一致收敛级数的基本性质(The Unanimous Convergence of the Series of Functions and Its Properties)

一、函数项级数的一致收敛性(The Unanimous Convergence of the Series of Functions)

定义 设有函数项级数$\sum_{n=1}^{\infty}u_n(x)$。如果对于任意给定的正数$\varepsilon$，都存在着一个只依赖于$\varepsilon$ 的自然数 N，使得当 $n>N$ 时，对区间 I 上的一切 x，都有不等式

$$|r_n(x)|=|s(x)-s_n(x)|<\varepsilon$$

成立，则称函数项级数$\sum_{n=1}^{\infty}u_n(x)$在区间 I 上一致收敛于 $s(x)$，也称函数序列$\{s_n(x)\}$在区间 I 上一致收敛于 $s(x)$。

Definition The series $\sum_{n=1}^{\infty}u_n(x)$ is said to be unanimous convergent on the interval I, if for any $\varepsilon>0$, there is a natural number N only dependent on ε, such that for any $n>N$, and any x in I, we have

$$|r_n(x)|=|s(x)-s_n(x)|<\varepsilon.$$

定理 1 (魏尔斯特拉斯判别法) 如果函数项级数$\sum_{n=1}^{\infty}u_n(x)$在区间 I 上满足条件：

(1) $|u_n(x)|\leqslant a_n\ (n=1,2,\cdots)$;

(2) 正项级数$\sum_{n=1}^{\infty}a_n$ 收敛，

则函数项级数$\sum_{n=1}^{\infty}u_n(x)$在区间 I 上一致收敛。

Theorem 1 (Weierstrass Test) Suppose that $\sum_{n=1}^{\infty}u_n(x)$ satisfies:

(1) $|u_n(x)| \leqslant a_n$ on the interval I for all n;

(2) The positive series $\sum_{n=1}^{\infty}a_n$ is covergent on the interval I.

Then $\sum_{n=1}^{\infty}u_n(x)$ is unanimous convergent on the interval I.

二、一致收敛级数的基本性质(Properties of the Unanimous Convergent Series)

定理 1 如果级数$\sum_{n=1}^{\infty}u_n(x)$的各项 u_n (x)在区间$[a, b]$上连续，且$\sum_{n=1}^{\infty}u_n(x)$在区间$[a, b]$上一致收敛于 $s(x)$，则 $s(x)$ 在$[a, b]$上也连续。

Theorem 1 If the general term u_n (x) of the series $\sum_{n=1}^{\infty}u_n(x)$ is continuous on $[a, b]$, and $\sum_{n=1}^{\infty}u_n(x)$ is unanimous convergent with the sum function $s(x)$ on $[a, b]$. Then $s(x)$ is also continuous on $[a, b]$.

定理 2 如果级数$\sum_{n=1}^{\infty}u_n(x)$的各项 u_n (x)在区间$[a, b]$上连续，且$\sum_{n=1}^{\infty}u_n(x)$在区间$[a, b]$上一致收敛于 $s(x)$，则级数$\sum_{n=1}^{\infty}u_n(x)$在$[a, b]$上可以逐项积分，即

$$\int_{x_0}^{x}s(x)\mathrm{d}x=\int_{x_0}^{x}u_1(x)\mathrm{d}x+\int_{x_0}^{x}u_2(x)\mathrm{d}x+\cdots+\int_{x_0}^{x}u_n(x)\mathrm{d}x+\cdots,$$

其中 $a\leqslant x_0< x\leqslant b$，并且上式右端的级数在$[a, b]$上也一致收敛。

Theorem 2 If the general term u_n (x) of the series $\sum_{n=1}^{\infty}u_n(x)$ is continuous on $[a, b]$, and $\sum_{n=1}^{\infty}u_n(x)$ is unanimous convergent with the sum function $s(x)$. Then

$$\int_{x_0}^{x}s(x)\mathrm{d}x=\int_{x_0}^{x}u_1(x)\mathrm{d}x+\int_{x_0}^{x}u_2(x)\mathrm{d}x+\cdots+\int_{x_0}^{x}u_n(x)\mathrm{d}x+\cdots,$$

where $a\leqslant x_0< x\leqslant b$, and the series on the right is also unanimous convergent on $[a, b]$.

定理 3 如果级数$\sum_{n=1}^{\infty}u_n(x)$在区间$[a, b]$上收敛于 $s(x)$，它的各项 u_n (x)都具有连续导数 u'_n (x)，并且级数$\sum_{n=1}^{\infty}u'_n(x)$在$[a, b]$上一致收敛，则级数$\sum_{n=1}^{\infty}u_n(x)$在$[a, b]$上也一致收敛，且可逐项求导。

Theorem 3 Let $\sum_{n=1}^{\infty}u_n(x)$ be a series with the sum functions $s(x)$ on the interval $[a, b]$. If the derivative u'_n (x) of the general tems u_n (x) is continuous, and $\sum_{n=1}^{\infty}u'_n(x)$ is unanimous convergent on $[a, b]$. Then $\sum_{n=1}^{\infty}u_n(x)$ is also unanimous convergent on $[a, b]$, and can be differentiated term by term.

定理 4 如果幂级数$\sum_{n=0}^{\infty}a_nx^n$的收敛半径为 $R>0$，则此级数

在$(-R, R)$内的任一闭区间$[a, b]$上一致收敛。

Theorem 4 Let the radius of the convergence set of the power series $\sum_{n=0}^{\infty} a_n x^n$ be $R>0$, then $\sum_{n=0}^{\infty} a_n x^n$ is unanimous convergent on any interval $[a, b] \subset (-R, R)$.

定理 5 如果幂级数$\sum_{n=0}^{\infty} a_n x^n$的收敛半径为$R>0$，则其和函数$s(x)$在$(-R, R)$内可导，且有逐项求导公式

$$s'(x)=(\sum_{n=0}^{\infty} a_n x^n)'=\sum_{n=0}^{\infty}(a_n x^n)'=\sum_{n=1}^{\infty} n a_n x^{n-1} \ (|x|<R),$$

逐项求导后所得到的幂级数和原级数有相同的收敛半径。

Theorem 5 Let the radius of the convergence set of the power series $\sum_{n=0}^{\infty} a_n x^n$ be $R>0$, then the sum function $s(x)$ is derivable on the interval $(-R, R)$, and

$$s'(x)=(\sum_{n=0}^{\infty} a_n x^n)'=\sum_{n=0}^{\infty}(a_n x^n)'=\sum_{n=1}^{\infty} n a_n x^{n-1} \ (|x|<R),$$

and the radius of convergence of the differentiated series is the same as for the orginal series.

11.7 傅里叶级数(Fourier Series)

一、三角级数 三角函数系的正交性(Series of Trigonometric, Function Orthogonal Properties of the Trigonometric Function System)

二、函数展开成傅里叶级数(Fourier Series of the Function)

定理（收敛定理，狄利克雷充分条件） 设$f(x)$是周期为2π的周期函数，如果它满足：

(1) 在一个周期内连续或只有有限个第一类间断点；

(2) 在一个周期内至多只有有限个极值点；

则$f(x)$的傅里叶级数收敛，并且

当x是$f(x)$的连续点时，级数收敛于$f(x)$；

当x是$f(x)$的间断点时，级数收敛于

$$\frac{1}{2}[f(x^-)+f(x^+)]。$$

Theorem (Convergent Theorem, Dirichlet Condition)

Let $f(x)$ be a function with the period of 2π, and it satisfies:

(1) $f(x)$ has no or finite discontinuity points in one period;

(2) There are finite points where $f(x)$ attains its extreme value;

Then the Fourier series of $f(x)$ is convergent, and the sum function is $f(x)$ if $f(x)$ is continuous at x; If x is the discontinuity point of $f(x)$, the sum function is

$$\frac{1}{2}[f(x^-)+f(x^+)].$$

三、正弦级数和余弦级数(The Series of Sine Function and Cosine Function)

11.8 一般周期函数的傅里叶级数(Fourier Series of Periodic Functions)

周期为 $2l$ 的周期函数的傅里叶级数(Fourier Series of Function with period $2l$)

定理 设周期为$2l$的周期函数$f(x)$满足收敛定理的条件，则它的傅立叶级数展开式为

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l}+b_n \sin\frac{n\pi x}{l}) \quad (x \in C)$$

其中

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \quad (n=0,1,2,\cdots)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \quad (n=1,2,3,\ \cdots)$$

$$C=\{x|\ f(x)=\frac{1}{2}[f(x^-)+f(x^+)]\}$$

当 $f(x)$ 为奇函数时

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \quad (x\in C)$$

其中

$$b_n=\frac{2}{l}\int_{0}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \quad (n=1,2,3,\ \cdots)$$

当 $f(x)$ 为偶函数时

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \quad (x\in C)$$

其中

$$a_n=\frac{2}{l}\int_{0}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \quad (n=0,1,2,3,\ \cdots)$$

Theorem Let $f(x)$ be a function of period $2l$. If $f(x)$ satisfies the convergent theorem, the Fourier series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}) \quad (x\in C)$$

where

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \quad (n=0,1,2,\cdots)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \quad (n=1,2,3,\ \cdots)$$

$$C=\{x|\ f(x)=\frac{1}{2}[f(x^-)+f(x^+)]\}.$$

While $f(x)$ is an odd function,

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \quad (x\in C)$$

where

$$b_n=\frac{2}{l}\int_{0}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x \quad (n=1,2,3,\ \cdots)$$

while $f(x)$ is an even function

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l} \quad (x\in C)$$

where

$$a_n=\frac{2}{l}\int_{0}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x \quad (n=0,1,2,3,\ \cdots)$$

▶第十二章　微分方程
Chapter 12　Differential Equation

12.1　微分方程的基本概念(The Concept of Differential Equation)

在一个方程中，如果未知量是一个函数，且方程中含有这个函数的导数，则称此方程为**微分方程**。

形如

$$F(x, y, y^{(1)}, y^{(2)}, \cdots, y^{(n)})=0$$

的方程称为 ***n* 阶常微分方程**。

Any equation in which the unknown is a function and that involves derivatives (or differentials) of this unknown function is called a **differential equation.**

An equation of the form

$$F(x, y, y^{(1)}, y^{(2)}, \cdots, y^{(n)})=0$$

in which $y^{(k)}$ denotes the kth derivatives of y with respect to x, is called **an ordinary differential equation of order n.**

如果把函数 $y=\varphi(x)$代入方程能使方程在一个区间上成为恒等式，则函数 $y=\varphi(x)$叫做微分方程在这个区间上的解。

when $y=\varphi(x)$ is substituted for y in the differential equation, the resulting equation is an identity for all x in some interval, then $y=\varphi(x)$ is called a solution of the differential equation.

12.2 可分离变量的微分方程(Separable Differential Equation)

如果一个一阶微分方程能写成 $g(y)dy=f(x)dx$ 的形式，即能把微分方程写成一端只含 y 的函数和 dy，另一端只含 x 的函数和 dx，则原方程称为可分离变量的微分方程。

The first-order separable differential equations are those involving just the first derivative of the unknown function and are such that the variables can be separated, one on each side of the equation as the following:

$$g(y)dy=f(x)dx.$$

12.3 齐次方程(Homogeneous Equation)

一、齐次方程(Homogeneous Equation)

如果一阶微分方程

$$\frac{dy}{dx}=f(x,y)$$

中的函数 $f(x, y)$能够写成 $\frac{y}{x}$ 的函数，即 $f(x, y)=\varphi(\frac{y}{x})$，则称这方程为齐次方程。

The first-order differential equation

$$\frac{dy}{dx}=f(x,y)$$

is called a homogeneous equation if the function $f(x, y)$ can be denoted in the form of $f(x, y)=\varphi(\frac{y}{x})$.

二、可化为齐次的方程(The Differential Equation which can be Turned to Homogeneous Equation)

12.4 一阶线性微分方程(Linear Differential Equation of the First Order)

一、线性微分方程(Linear Differential Equation)

方程

$$\frac{dy}{dx}+P(x)y=Q(x)$$

叫做一阶线性微分方程。如果 $Q(x)\equiv0$，则称此方程为齐次的；如果 $Q(x)$不恒等于零，则称此方程为非齐次的。

The differential equation

$$\frac{dy}{dx}+P(x)y=Q(x)$$

is called the first-order linear differential equation. The equation is said to be homogeneous if $Q(x)\equiv0$, and non-homogeneous otherwise.

二、伯努利方程(Bernoulli Equation)

方程

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

叫做伯努利方程

The equation

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

is called the Bernoulli equation.

12.5 全微分方程(Total Differential Equation)

一个一阶微分方程写成

$$P(x, y)dx + Q(x, y)dy=0$$

形式后，如果它的左端恰好是某一个函数 $u=u(x, y)$的全微分：

$$du(x, y)= P(x, y)dx + Q(x, y)dy$$

则称此方程为**全微分方程**。

如果一个微分方程不是全微分方程，但有一个适当的函数 $\mu=\mu(x, y)$ 使得方程 $P(x, y)dx + Q(x, y)dy=0$ 在乘上 $\mu(x, y)$后所得到的方程

$$\mu(x, y)P(x, y)dx + \mu(x, y)Q(x, y)dy=0$$

是全微分方程，则函数 $u=u(x, y)$叫做**积分因子**。

A differential equation is called **a total differential equation** if it can be written in the form of

$$P(x, y)dx + Q(x, y)dy=0,$$

and there exists a function $u=u(x, y)$ such that

$$du(x, y)= P(x, y)dx + Q(x, y)dy.$$

If the differential equation is not a total differential equation, and there is a function $\mu=\mu(x, y)$ such that

$$\mu(x, y)P(x, y)dx + \mu(x, y)Q(x, y)dy=0$$

is a total differential equation.Then the function $\mu=\mu(x, y)$ is called **an integrating factor**.

12.6 可降阶的高阶微分方程(Higher-order Differential Equation Turned to Lower-order Differential Equation)

一、$y^{(n)}=f(x)$型的微分方程(The Differential Equation of the Form $y^{(n)}=f(x)$)

二、$y''=f(x, y')$型的微分方程(The Differential Equation of the Form $y''=f(x, y')$)

三、$y''=f(y, y')$型的微分方程(The Differential Equation of the Form $y''=f(y, y')$)

12.7 高阶线性微分方程(Linear Differential Equation of Higher Order)

一、二阶线性微分方程举例(The Examples of the Second-order Linear Differential Equation)

形如

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+k^2x=0$$

的微分方程称为自由振动的微分方程。

The differential equation in the form of

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+k^2x=0$$

is called **the differential equation of free vibration.**

形如

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+k^2x=h\sin pt$$

的微分方程称为强迫振动的微分方程。

The differential equation of the form

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+k^2x=h\sin pt$$

is called **the differential equation of forced oscillation.**

二、线性微分方程的解的结构(The Structure of the Solution of the Linear Differential Equation)

定理 1 如果函数 $y_1(x)$与 $y_2(x)$是方程

$$y''+P(x)y'+Q(x)y=0$$

的两个解，则

$$y=C_1y_1(x)+C_2y_2(x)$$

也是方程的解，其中 C_1，C_2是任意常数。

Theorem 1 If $y_1(x)$ and $y_2(x)$ are solutions of the second-order differential equation

$$y''+P(x)y'+Q(x)y=0.$$

Then $y=C_1y_1(x)+C_2y_2(x)$ is also the solution of this equation, where C_1 and C_2 are any constants.

定理 2 如果 $y_1(x)$与 $y_2(x)$是方程

$$y''+P(x)y'+Q(x)y=0$$

的两个线性无关的特解，则

$$y=C_1y_1(x)+C_2y_2(x)\quad (C_1，C_2\text{是任意常数})$$

是此方程的通解。

Theorem 2 If $y_1(x)$ and $y_2(x)$ are linear independent particular solutions of the second-order differential equation

$$y''+P(x)y'+Q(x)y=0.$$

Then $y=C_1y_1(x)+C_2y_2(x)$ is general solution of this equation.

推论 如果 $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$是 n 阶齐次线性方程

$$y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0,$$

的 n 个线性无关的解，则此方程的通解为

$$y=C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x),$$

其中 C_1，C_2，$\cdots$，C_n为任意常数。

Corollary If $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ are linear independent solutions of the nth-order homogeneous linear differential equation

$$y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0.$$

Then the general solution of this equation is

$$y=C_1y_1(x)+C_2y_2(x)+\cdots+C_ny_n(x),$$

where C_1, C_2, $\cdots$, C_n are any constants.

定理 3 设 $y^*(x)$是二阶非齐次线性方程

$$y''+P(x)y'+Q(x)y=f(x) \tag{5}$$

的一个特解。$Y(x)$是与此方程对应的齐次方程 $y''+P(x)y'+Q(x)y=0$ 的通解，则

$$y=Y(x)+y^*(x)$$

是二阶非齐次线性方程的通解。

Theorem 3 If $y^*(x)$ is any particular solution to the second-order nonhomogeneous equation

$$y''+P(x)\,y'+Q(x)y=f(x), \tag{5}$$

and $Y(x)$ is the general solution to the corresponding homogeneous equation. Then

$$y=Y(x)+y^*(x)$$

is the general solution of the equation $y''+P(x)\,y'+Q(x)y=f(x)$.

定理 4 设非齐次线性方程(5)的右端 $f(x)$是几个函数之和，如

$$y''+P(x)\,y'+Q(x)y=f_1(x)+f_2(x),$$

而 $y_1^*(x)$和 $y_2^*(x)$分别是方程

$$y''+P(x)\,y'+Q(x)y=f_1(x),$$

和

$$y''+P(x)\,y'+Q(x)y=f_2(x)$$

的特解，则 $y_1^*(x)+y_2^*(x)$就是原方程的一个特解。

Theorem 4 If the right side of the equation (5) is the sum of several functions, e. g.

$$y''+P(x)\,y'+Q(x)y=f_1(x)+f_2(x),$$

and if $y_1^*(x)$, $y_2^*(x)$ are particular solutions of the equation

$$y''+P(x)\,y'+Q(x)y=f_1(x),$$

and

$$y''+P(x)\,y'+Q(x)y=f_2(x).$$

respectively. Then $y_1^*(x)+y_2^*(x)$ is a particular solution of the original equation.

三、常数变易法(Method of Variation of Constant)

12.8 常系数齐次线性微分方程(Homogeneous Linear Differential Equation with Constant Coefficient)

在二阶齐次线性微分方程

$$y''+P(x)\,y'+Q(x)y=0 \tag{1}$$

中，如果 y，y'的系数 $P(x)$、$Q(x)$均为常数，即

$$y''+py'+qy=0 \tag{2}$$

其中 p、q 是常数，则称方程(2)为二阶常系数齐次线性微分方程。否则称方程为二阶变系数**齐次线性微分方程**。

If $P(x)$、$Q(x)$ in the second-order homogeneous linear differential equation

$$y''+P(x)\,y'+Q(x)y=0 \tag{1}$$

are both constant, that is,

$$y''+py'+qy=0 \tag{2}$$

where p、q are constants. Then equation (2) is called the homogeneous linear differential equation with constant coefficient; otherwise, it is called **the second-order homogeneous linear differential equation with variable coefficient.**

***n* 阶常系数齐次线性微分方程**的一般形式为

$$y^{(n)}+p_1y^{(n-1)}+\cdots+p_{n-1}\,y'+p_n\,y=0$$

其中 $p_1, p_2, \cdots, p_n$ 都是常数。

The differential equation in the form of

$$y^{(n)}+p_1y^{(n-1)}+\cdots+p_{n-1}\,y'+p_n\,y=0$$

is called **the *n*th-order homogeneous linear equation with constant coefficient**, where $p_1, p_2, \cdots, p_n$ are all constants.

12.9 常系数非齐次线性微分方程(Nonhomogeneous Linear Differential Equation with Constant Coefficient)

二阶常系数非齐次线性微分方程的一般形式为

$$y''+py'+qy=f(x)$$

其中 p，q 是常数。

The general form of the second-order nonhomogeneous linear differential equation is

$$y''+py'+qy=f(x)$$

where p，q are constants.

12.10 欧拉方程(Euler Equation)

形如

$$x^n y^{(n)}+p_1 x^{n-1} y^{(n-1)}+\cdots+p_{n-1} x y'+p_n y=f(x)$$

的方程(其中 $p_1, p_2, \cdots, p_n$ 为常数)，叫做欧拉方程。

The differential equation of the form

$$x^n y^{(n)}+p_1 x^{n-1} y^{(n-1)}+\cdots+p_{n-1} x y'+p_n y=f(x)$$

where $p_1, p_2, \cdots, p_n$ are all constants, is called Euler Equation.

12.11 微分方程的幂级数解法(Power Series Solution to Differential Equation)

定理 如果方程

$$y''+P(x)y'+Q(x)y=0 \tag{3}$$

中的系数 $P(x)$与 $Q(x)$可在 $-R<x<R$ 内展开为 x 的幂级数，那么在 $-R<x<R$ 内，方程(3)必有形如

$$y=\sum_{n=0}^{\infty} a_n x^n$$

的解。

Theorem If the coefficient function $P(x)$ and $Q(x)$ in the differential equation (3) can be represented as power series in x on the interval $-R<x<R$. Then there must be solution of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

to the differential equation.