1 Proofs

In this section, we have all the proofs for the properties of judgment aggregation.

	p	$p \wedge r$	$r \vee s$	$p \wedge q$	t
A_1, A_2, A_3	1	1	1	1	1
A_4	1	0	0	1	1
A_5	1	1	1	1	0
A_6	1	1	1	0	1
A_7, A_8, A_9, A_{10}	0	0	0	0	0
A_{11}	0	0	1	0	1
$Majority$	1	0	1	0	1
$Truth$	1	1	1	1	1

Table 1: Profile

	A_1, A_2, A_3	A_4	A_5	A_6	A_7, A_8, A_9, A_{10}	A_{11}
It1	1	1	1	1	1	1
It2	0.6	0.6	0.4	0.8	0.4	0.8
It3	0.7	0.5	0.5	0.9	0.3	0.7
It4	1	0.6	0.8	0.8	0	0.4

Table 2: Reliability of the agents

	φ_1	$\neg\varphi_1$	φ_2	$\neg\varphi_2$	φ_3	$\neg\varphi_3$	φ_4	$\neg\varphi_4$	φ_5	$\neg\varphi_5$
It1	6	5	5	6	6	5	5	6	6	5
It2	3.6	2.4	3	3	3.8	2.2	2.8	3.2	4	2
It3	4	1.9	3.5	2.4	4.2	1.7	3.1	2.8	4.2	1.7
It4	5.2	0.4	4.6	1	5	.6	4.4	1.2	4.8	0.8

Table 3: Confidence of the formulae

Universal Domain (Dom.). The domain of R is the set of all consistent profiles.

Proposition 1. *$R^{\oplus}$ methods satisfy **Universal Domain**.*

Universal Domain is satisfy by definition for our ICE methods.

The next property is *Anonymity*. We want the judgment aggregation methods to be impartial and not favor any particular agents.

Anonymity (Ano.). For any two profiles $P = (J_1, \ldots, J_n)$ and $P' = (J'_1, \ldots, J'_n)$ in the domain of R which are permutations one another, we have $R(P) = R(P')$.

Proposition 2. *$R^{\oplus}$ methods satisfy **Anonymity**.*

Proof. Anonymity requires agents to be treated in the same way *a priori*, which is the case with our algorithm. The reliability of an agent is evaluated depending on its claims and is not linked to the order given in the profile. In our algorithm, the reliability of the agents is the same with the profile P or P'. Then the outcomes given by the ICE methods are the same: $R(P) = R(P')$.

The next property is *Majority preservation*. It states that the result of a majority voting must be the outcome of the method if the result of the majority voting is consistent and resolute.

Majority preservation (Maj.). If R_P^{maj} is consistent and resolute then
$R(P) = \{R^{maj}(P)\}$

Proposition 3. *$R^{\oplus}$ methods do not satisfy **Majority preservation**.*

Proof. Consider the profile of Table 1, the evaluation of the reliability of agents in Table 2 and the confidence of the formulae in Table 3.

With this example, the consistent majoritarian judgment set is $(\{\varphi_1, \neg\varphi_2, \varphi_3, \neg\varphi_4, \varphi_5\})$ but ICE methods give $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ as the outcome. The score of $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$ maximizes the score for R^{Σ} and $R^{\times}$. R^{lex} orders the vectors of confidence to get $\{5.2, 5, 4.8, 4.6, 4.4\}$ (in a decreasing order) and then selects $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5\}$.

Let us stress that in this case, a method that satisfy majority preservation will not find the truth, contrarily to ours.

ICE methods do not satisfy *Majority preservation* but when $H(P)$ is consistent (the set of consistent judgment sets whose elements have a higher confidence than their negation), the outcome of ICE methods will be the formulae in $H(P)$. We define a new property called *Reliability consistency*. This property ensures that a method selects the most reliable consistent outcomes, if possible.

Reliability consistency (Rel.). If $H(P)$ is not empty, then the outcome of R is $H(P)$, i.e. $R(P) = H(P)$.

Proposition 4. *$R^{\oplus}$ methods satisfy* ***Reliability consistency****.*

Proof. Suppose that $H(P)$ is not empty.

Let $J^R \in H(P)$. By definition of $H(P), \forall\varphi_k \in J^R, c(\varphi_k) \geq c(\neg\varphi_k)$. Suppose that $R^{\oplus}$ does not select J^R. It means that $\exists J \notin J^R$ s.t. $\oplus_{\varphi_k \in J}(c(\varphi_k)) > \oplus_{\varphi'_k \in J^R}(c(\varphi'_k))$. Necessarily, it implies that for at least one k, $c(\varphi_k) > c(\varphi'_k)$, i.e that $c(\neg\varphi'_k) > c(\varphi'_k)$ with $\varphi'_k \in J^R$: contradicts the definition of $H(P)$. So $H(P) \subseteq R^{\oplus}(P)$.

Let $J \in R^{\oplus}(P)$. Suppose that $J \notin H(P)$. As $J \notin H(P)$, there is at least one k s.t. $c(\varphi_k) < c(\neg\varphi_k)$. As $H(P)$ is not empty, there is at least one judgment set $J^R \in H(P)$ s.t. $\neg\varphi_k \in J^R$. We also know that $\forall\varphi_i \in J^R, i \neq k, c(\varphi_i) \geq c(\neg\varphi_i)$. Then $\oplus_{\varphi_i \in J}(c(\varphi_i)) < \oplus_{\varphi'_i \in J^R}(c(\varphi'_i))$: J can not be selected by $R^{\oplus}$. We also obtain a contradiction and $R^{\oplus}(P) \subseteq H(P)$.

Finally, $R^{\oplus}(P) = H(P)$.

Unanimity is the next property. When a formula is in all individual judgment sets, then it must be in all outcomes.

Unanimity (Una.). For any $\varphi_k \in X$, if $J_i(\varphi_k) = x$ with $x \in \{0, 1\}, \forall J_i \in P$, then for every $J \in R(P)$, we have $J(\varphi_k) = x$.

Proposition 5. *$R^{\times}$ and R^{lex} satisfy* ***Unanimity****. R^{Σ} does not satisfy* ***Unanimity****.*

Proof. Consider that φ_k is the formula assigns to every individual judgment set and $\neg\varphi_k$ is the formula that is not supported by the agents i.e. it is not in any judgment sets.

We know that the confidence of φ_k will be the highest because all the agents claim unanimously that this formula is the truth ($c(\varphi_k) = \sum\limits_{a_i \in \mathcal{A}_P(\varphi_k)} r(a_i)$) and the confidence of $\neg\varphi_k$ will be equal to zero ($c(\neg\varphi_k) = \sum\limits_{a_i \in \emptyset} 0$). Also, we know that there is at least one judgment set that contains φ_k because φ_k is in every individual judgment set.

All maximal reliability vectors for lex contain φ_k. R^{lex} will chose its result in these sets. We know that φ_k has the highest confidence and that φ_k is at least in one (consistent) judgment set. So φ_k will be in all the outcomes of R^{lex}.

All the judgment sets which contain $\neg\varphi_k$ obtain a confidence of 0 with $R^{\times}$. There is at least one consistent judgment set, say J, which contains φ_k, because φ_k belongs to the judgment sets of all the agents, which are consistent. As there is unanimity on φ_k, all the agents have a reliability strictly greater than 0. Then all the formulae in J have a confidence strictly greater than 0. As a consequence, any judgment set selected by $R^{\times}$ will contain φ_k.

$R^{\times}$ and R^{lex} satisfy *Unanimity*.

R^{Σ} does not satisfy *Unanimity*. Consider the propositional formulae $\varphi_1 = a$, $\varphi_2 = b$, $\varphi_3 = c$, $\varphi_4 = d$, $\varphi_5 = \neg a$, $\varphi_6 = \neg b$, $\varphi_7 = \neg c$, $\varphi_8 = \neg d$, $\varphi_9 = a \vee b$, $\varphi_{10} = \neg(a \wedge b \wedge c \wedge d)$. We have four agents in the profile P, with the agents claiming
$\{\varphi_1, \varphi_2, \varphi_3, \neg\varphi_4, \neg\varphi_5, \neg\varphi_6, \neg\varphi_7, \varphi_8, \varphi_9, \varphi_{10}\}$,
$\{\varphi_1, \varphi_2, \neg\varphi_3, \varphi_4, \neg\varphi_5, \neg\varphi_6, \varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}$,
$\{\varphi_1, \neg\varphi_2, \varphi_3, \varphi_4, \neg\varphi_5, \varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}$ and
$\{\neg\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \neg\varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}$.

φ_1	$\neg\varphi_1$	φ_2	$\neg\varphi_2$	φ_3	$\neg\varphi_3$	φ_4	$\neg\varphi_4$	φ_5	$\neg\varphi_5$	φ_6	$\neg\varphi_6$	φ_7	$\neg\varphi_7$	φ_8	$\neg\varphi_8$	φ_9	$\neg\varphi_9$	φ_{10}	$\neg\varphi_{10}$
2.4	0.8	2.4	0.8	2.4	0.8	2.4	0.8	0.8	2.4	0.8	2.4	0.8	2.4	0.8	2.4	3.2	0	3.2	0

Table 4: Confidence of the formulae of the profile P

We have the confidence of the formulae in Table 4. The outcomes of $R^{\Sigma}(P)$ are
$\{\ \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \neg\varphi_5, \neg\varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \neg\varphi_{10}\}$,
$\{\varphi_1, \varphi_2, \varphi_3, \neg\varphi_4, \neg\varphi_5, \neg\varphi_6, \neg\varphi_7, \varphi_8, \varphi_9, \varphi_{10}\}$,
$\{\varphi_1, \varphi_2, \neg\varphi_3, \varphi_4, \neg\varphi_5, \neg\varphi_6, \varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}$,
$\{\varphi_1, \neg\varphi_2, \varphi_3, \varphi_4, \neg\varphi_5, \varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}$,
$\{\neg\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \neg\varphi_6, \neg\varphi_7, \neg\varphi_8, \varphi_9, \varphi_{10}\}\ \}$

The five outcomes have a score of 22.4 with Σ. The agents are unanimous on φ_{10} but in one of the outcomes of R^{Σ} we have $\neg\varphi_{10}$. As a consequence, the method R^{Σ} does not satisfy *Unanimity*.

Now we talk about *Systematicity*, a property criticized in the literature. It states that all the formulae with the same number of votes must be treated in the same way. The principal criticism of this property is that it ignores the rest of the profile when a decision has to be made for a formula. This is already a problem in standard judgment aggregation. But it is even worse in this unknown variable reliability framework.

Systematicity (Syst.). For any two profiles $P = (J_1, \dots, J_n)$ and $P' = (J'_1, \dots, J'_n)$ in the domain of R, and any two propositions φ_k and φ_l of X, such that $J_i(\varphi_k) = J'_i(\varphi_l) \forall i$, if $J_P(\varphi_k) = x$ for all $J_P \in R(P)$, then $J'_{P'}(\varphi_l) = x$ for all $J'_{P'} \in R(P')$.

Proposition 6. *$R^{\oplus}$ methods do not satisfy* ***Systematicity****.*

This property does not take into account the other formulae. But the opinion of the agents on all the formulae must be considered to find the right outcome. This is how the reliability of the agents is estimated.

Proof. Consider the profile P and P' from Table 5 and Table 6 with $\varphi_1 = p, \varphi_2 = p \wedge r, \varphi_3 = r \vee s, \varphi_4 = p \wedge q, \varphi_5 = t$. Consider $k = 3$ and $l = 4$, so the agents express the same opinion on φ_3 and φ_4. The agents A_6 and A_7 change their opinion on φ_1 and the agent A_{10} on φ_5.

We have the confidence of the formulae in Table 7 for the profile P and in Table 8 for the profile P'.

	φ_1	φ_2	φ_3	φ_4	φ_5
A_1	1	1	1	1	1
A_2	1	1	1	1	1
A_3	1	1	1	1	0
A_4	1	0	0	1	1
A_5	1	0	0	1	0
A_6	0	0	1	0	0
A_7	0	0	1	0	1
A_8	0	0	0	0	0
A_9	0	0	0	0	0
A_{10}	0	0	0	0	0

Table 5: Profile P

	φ_1	φ_2	φ_3	φ_4	φ_5
A_1	1	1	1	1	1
A_2	1	1	1	1	1
A_3	1	1	1	1	0
A_4	1	0	0	1	1
A_5	1	0	0	1	0
A_6	1	0	1	0	0
A_7	1	0	1	0	1
A_8	0	0	0	0	0
A_9	0	0	0	0	0
A_{10}	0	0	0	0	1

Table 6: Profile P'

φ_1	$\neg\varphi_1$	φ_2	$\neg\varphi_2$	φ_3	$\neg\varphi_3$	φ_4	$\neg\varphi_4$	φ_5	$\neg\varphi_5$
1.2	4.4	0.2	5.4	1.6	4	1.2	4.4	1	4.6

Table 7: Confidence of the formulae of the profile P

φ_1	$\neg\varphi_1$	φ_2	$\neg\varphi_2$	φ_3	$\neg\varphi_3$	φ_4	$\neg\varphi_4$	φ_5	$\neg\varphi_5$
5	0.8	2.2	3.6	3.6	2.2	3.6	2.2	3.6	2.2

Table 8: Confidence of the formulae of the profile P'

With the profile P, the outcome of our methods with Σ, $\times$ and lex is the same with the profile P and P' and it is $R^{\oplus}(P) = \{\neg\varphi_1, \neg\varphi_2, \neg\varphi_3, \neg\varphi_4, \neg\varphi_5\}$. With the profile P', we have $R^{\Sigma}(P') = \{\varphi_1, \neg\varphi_2, \varphi_3, \varphi_4, \varphi_5\}$, which is different on φ_3 and φ_4 despite the fact that the opinions of the agents have not changed. We have $R(P) \neq R(P')$ even if $\forall i, J_i(\varphi_k) = J'_i(\varphi_l)$.

The next property is *Neutrality*. The elements of the agenda must be considered in the same way.

Neutrality (Neut.). If $X = \{\varphi_1, \ldots, \varphi_m\}$ and $X' = \{\varphi'_1, \ldots, \varphi'_m\}$ are two agendas such that there exists a permutation σ over $\{1, \ldots, m\}$ satisfying $\varphi_k \equiv \varphi'_{\sigma(k)}$ for every $k \in \{1, \ldots, m\}$ then for any profiles $P = (J_1, \ldots, J_n)$ on X and $P' = (J'_1, \ldots, J'_n)$ on X' such that for every $i \in \{1, \ldots, n\}$, for every $k \in \{1, \ldots, m\}$, $J_i(\varphi_k) = J'_i(\varphi'_{\sigma(k)})$, we have $R(P) = R(P')$.

Proposition 7. *$R^{\oplus}$ methods satisfy **Neutrality**.*

Proof. A permutation on X will not change the judgments sets in P so the confidence of the formulae will be the same with P' because the confidence is not evaluated depending on the order of the formulae in the agenda. A permutation on X will not change the outcomes of R: $R(P) = R(P')$.

Our methods satisfy *Neutrality*.

The results are summarized in Table 9.

	Dom.	Ano.	Maj.	Rel.	Una.	Syst.	Neut.
R^{Σ}	✓	✓	✗	✓	✗	✗	✓
$R^{\times}$	✓	✓	✗	✓	✓	✗	✓
R^{lex}	✓	✓	✗	✓	✓	✗	✓

Table 9: Properties satisfied by ICE methods