Ramsey-like theorems for the Schreier barrier

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Introduction

Theorems as problems

Many theorems (of ordinary mathematics) can be seen as problems involving instances and solutions:

- For every field, there exists an algebraic closure.
- For infinite binary tree, there exists an infinite branch (Weak König's lemma)
- For every bounded sequence in $\mathbb{R}^n$, there exists a converging subsequence (Bolzano-Weierstrass theorem).

The proof of these theorems often give a "procedure" to build the solution given the instance.
We can then wonder about the effectiveness of such constructions.

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Computational content of theorems

Given $X \subseteq \mathbb{N}$ an instance of a problem P, can we compute a solution $Y \subseteq \mathbb{N}$ from X ?

In a lot of cases the answer will be no:

- There exists computable consistent theories that have no computable models (e.g. Tennenbaum's theorem)
- There exists computable commutative rings with no computable maximal ideals.

Such theorems will therefore not be provable in constructive mathematics.

But we can push the computational analysis further and obtain more precise proof-theoretic results.

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Reverse Mathematics

To prove that an axiom was needed for a theorem, we can try to prove the axiom from the theorem:

- Modulo ZF, Zorn's lemma implies the axiom of choice.
- Modulo absolute geometry, the Pythagorean theorem implies the Parallel postulate.

Reverse mathematics does the same, but modulo constructive mathematics.

Framework

We will use the formalism of second-order arithmetic.

All the models considered will be ω-models, i.e. models $(\mathbb{N}, S, +, \times, 0)$ with $(\mathbb{N}, +, \times, 0)$ the standard integers, and $S \subseteq 2^{\mathbb{N}}$ the second-order part.

Base theory RCA_0

Base theory (corresponding to computable mathematics) RCA_0:

- Robinson's arithmetic Q
- Δ^0_1-comprehension (The computable sets exists)
- Σ^0_1-induction (Every set of finite cardinality is bounded)

The computable sets form a minimal ω-model of RCA_0.

The "Big Five"

Modulo RCA_0, most theorems of ordinary mathematics are equivalent to one the following theories (from weakest to strongest):

$$
\begin{array}{cl}
\Pi^1_1 - \mathsf{CA}_0 & : \text{ impredicativism} \\
\downarrow & \\
\mathsf{ATR}_0 & : \text{ transfinite recursion} \\
\downarrow & \\
\mathsf{ACA}_0 & : \text{ second-order PA} \\
\downarrow & \\
\mathsf{WKL}_0 & : \text{ compactness arguments} \\
\downarrow & \\
\mathsf{RCA}_0 & : \text{ constructive mathematics}
\end{array}
$$

ACA_0

Definition (ACA_0)

ACA_0 is RCA_0 plus the comprehension axiom for every arithmetical formula.

This is equivalent to comprehension for Σ^0_1 formula, or to the existence of the Turing jump of any set.
The arithmetical sets form a minimal model ω-model of ACA_0.

For a problem P, if there exists an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$ not containing the halting set K, then $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{ACA}_0$.

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Cone avoidance

Definition (Cone avoidance)

A problem P admits *cone avoidance* if for every set Z, every non-Z-computable set C, every Z-computable instance X of P has a solution Y that does not compute C.

Proposition

If P *admits cone avoidance, then for every non computable set* C, *there exists a model of* $\mathsf{RCA}_0 + \mathsf{P}$ *that does not contains* C.
Hence $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{ACA}_0$.

Strong cone avoidance

Definition (strong Cone avoidance)

A problem P admits *strong cone avoidance* if for every set Z, every non-Z-computable set C, every ~~Z-computable~~ instance X of P has a solution Y that does not compute C.

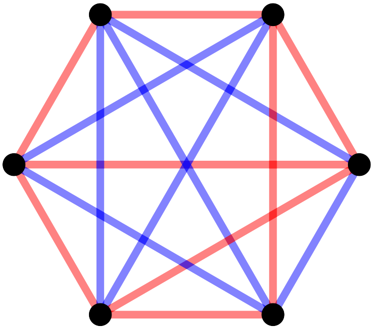
Cone avoidance witness the computable weakness of a problem P.
Strong cone avoidance witness its combinatorial weakness.

The infinite Ramsey's theorem for pairs and two colors escape the structure phenomenon of Ramsey's theory.

$$\begin{tikzcd}
\Pi^1_1 - \mathsf{CA}_0 \arrow[d] & \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] \arrow[dr] & \\
\mathsf{WKL}_0 \arrow[d] & \mathsf{RT}^2_2 \arrow[dl] \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

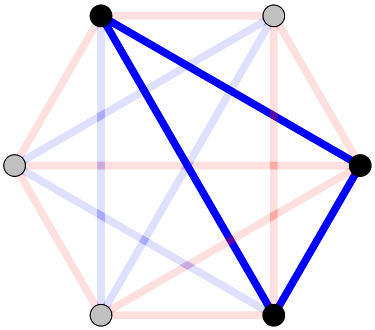
Ramsey Theory

Finite Ramsey's theorem



For every 2-coloring of the edges of K_6

Finite Ramsey's theorem



There exists some monochromatic copy of K_3

Infinite Ramsey's theorem

Let $[X]^n$ be the set of all subsets of X of cardinality n.

Definition (Ramsey's theorem)

RT^n_k is the statement: "For every coloring $f : [\mathbb{N}]^n \to k$, there is an infinite set $H \subseteq \mathbb{N}$ such that $|f([H]^n)| = 1$".

RT^1_k is the infinite pigeonhole principle. It is computably true (albeit not by a uniform procedure, we can't computably guess which color to choose) and $\mathsf{RCA}_0 \vdash \mathsf{RT}^1_k$ for every $k \in \mathbb{N}$.

Facts

- RT^1_2 admits strong cone avoidance, i.e., for every set A and every non computable set Z, there exists some subset $H \subseteq A$ or $H \subseteq \overline{A}$ such that $H \not\geq_T Z$. (Dzhafarov/Jockusch)
- There exists a $\emptyset'$-computable instance of RT^2_2, every solution of which computes $\emptyset'$ (Hence RT^2_2 does not admits strong cone avoidance)
- RT^2_2 admits cone avoidance. (Dzhafarov/Jockusch)
- For every $n \geq 3$, there exists a computable instance of RT^n_2, every solution of which computes $\emptyset^{(n-2)}$. Hence $\mathsf{RCA}_0 + \mathsf{RT}^n_2 \vdash \mathsf{ACA}_0$ (Hirschfeldt/ Jockusch).

Ramsey-like theorems

What about weakenings of Ramsey's theorems ?

Definition (Thin set theorem)

Let TS^n be the statement: "for every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there exists some infinite set $H \subseteq \mathbb{N}$ such that $f([H]^n) \neq \mathbb{N}$."

Definition (Free set theorem)

Let FS^n be the statement: "for every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$, there exists some infinite set $H \subseteq \mathbb{N}$, such that $f(s) \notin H \setminus s$ for every $s \in [H]^n$."

Definition (Rainbow Ramsey theorem)

Let RRT^n_k be the statement: "for every coloring $f : [\mathbb{N}]^n \to \mathbb{N}$ such that $|f^{-1}(\ell)| \leq k$ for every $\ell \in \mathbb{N}$, there exists some infinite $H \subseteq \mathbb{N}$ such that f is injective on $[H]^n$."

$$
\begin{tikzcd}
 & \mathsf{RT}^n_k \arrow[d] & \\
 & \mathsf{FS}^n \arrow[dl] \arrow[dr] & \\
\mathsf{RRT}^n_k & & \mathsf{TS}^n
\end{tikzcd}
$$

Theorem (Wang)

For every n, k, TS^n, FS^n and RRT^n_k admits the strong cone avoidance property.

Schreier barrier

Wrong generalizations

Let $[H]^{<\mathbb{N}}$ be the set of all finite subsets of H.
Let $\mathsf{RT}^{<\mathbb{N}}$ be the statement that for every coloring $f : [\mathbb{N}]^{<\mathbb{N}} \to 2$ there exists an infinite set H such that $|f([H]^{<\mathbb{N}})| = 1$.

Proposition

$\mathsf{RT}^{<\mathbb{N}}$ *is false. (Take f such that $f(s) = 0$ if $|s|$ even and $f(s) = 1$ otherwise.)*

Similarly, the generalizations $\mathsf{TS}^{<\mathbb{N}}$, $\mathsf{FS}^{<\mathbb{N}}$ and $\mathsf{RRT}^{<\mathbb{N}}$ are false. (For $\mathsf{TS}^{<\mathbb{N}}$, take $f(s) = |s|$)

Schreier Barrier

Definition (exactly ω-largeness)

A finite set $s \subseteq \mathbb{N}$ is *exactly ω-large* if $|s| = \min s + 1$.

We write $[H]^{!\omega}$ for the set of all the exactly ω-large subsets of H.

Definition

Let $\mathsf{RT}^{!\omega}$ be the statement: "for every coloring $f : [\mathbb{N}]^{!\omega} \to 2$ there exists an infinite set H such that $|f([H]^{!\omega})| = 1$."

Define $\mathsf{TS}^{!\omega}$, $\mathsf{FS}^{!\omega}$ and $\mathsf{RRT}^{!\omega}$ similarly.

Clopen Ramsey theorem

We write $[H]^{\mathbb{N}}$ for the set of all the infinite subsets of H.

- For every infinite set $X = \{x_0, \dots\}$, there exists a unique prefix $\sigma \prec X$ that is exactly ω-large: $x_0, \dots, x_{\min X}$.
- Therefore, for every coloring $f : [\mathbb{N}]^{!\omega} \to 2$, we can consider the coloring $g : [\mathbb{N}]^{\mathbb{N}} \to 2$ that send an infinite set X to $f(\sigma)$ for σ its exactly ω-large prefix.
- g is a particular instance of the Clopen Ramsey theorem, which is equivalent to ATR_0 (Friedman/McAloon/Simpson)
- $\mathsf{RT}^{!\omega}$ can also be seen as the first step of a hierarchy of statements (due to Nash-Williams) generalizing RT^n, and concerning coloring of barriers (sets of finite sets having the property that no elements is included in another).

Results

Theorem (Carlucci/Gjetaj/L./Levy Patey)

$\mathsf{TS}^{!\omega}$ *and* $\mathsf{FS}^{!\omega}$ *proves the existence of* $\emptyset^{(\omega)}$, *the* ω*-th jump.*
Hence, they imply (and are strictly stronger than) ACA_0.

Sketch.

It is sufficient to show the statement for $\mathsf{TS}^{!\omega}$ as any free set will be thin.

The statement "for every $f : [\mathbb{N}]^n \to k$, there is an infinite $H \subseteq \mathbb{N}$ such that $|f([H])^n| \neq k$" doesn't have the cone avoidance property when $k < C_n$ (where C_n is the n-th Catalan number) (Cholak/Patey).

We can get a uniformly computable family of computable coloring $f_n : [\mathbb{N}]^n \to n$ such that every infinite f_n-thin set computes $\emptyset'$.

We can combine all those instances and obtain some coloring $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ by taking $f(x_0, \ldots, x_{x_0}) = f_{x_0}(x_1, \ldots, x_{x_0})$.

Any f-thin set $H = \{x_0, \ldots\}$ will be f_{x_i}-thin for every i and therefore computes $\emptyset'$.

We can go further than that and computes $\emptyset^{(\omega)}$ by transforming $\emptyset^{(k)}$-computable colorings $f : [\mathbb{N}]^n \to \mathbb{N}$ into computable coloring $f : [\mathbb{N}]^{n+k} \to \mathbb{N}$ using Δ_k approximations. $\square$

Theorem (Carlucci/Gjetaj/L./Levy Patey)

$\mathsf{RRT}^{!\omega}$ *have the strong cone avoidance property.*

Proof Sketch

Let C be a non-computable set, and let $f : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be a coloring such that no color appear more than 2 times.

- For every $s, t \in [\mathbb{N}]^{!\omega}$, if $s \neq t$ then $s \not\subseteq t$ (if $\min s \geq \min t$, then $|s| \geq |t|$).
- Let $g : [\mathbb{N}]^{!\omega} \to \mathbb{N}$ be the partial f'-computable coloring defined by $g(s) = \min(t \setminus s)$ if there exists some $t \neq s$ such that $f(t) = f(s)$ and $s <_{lex} t$.
- g satisfy that either $g(s) > \min s$ or $g(s)$ undefined for every s. We say that g is progressive.
- If H is such that for every $s \in [H]^{!\omega}$, $g(s) \notin H$, then for every $s, t \in [H]^{!\omega}$, $f(s) \neq f(t)$ otherwise we would have $g(s) = \min(t \setminus s) \in H$.
  Hence H is an f-rainbow.

g can be seen as blocking some elements that would prevent a set H from being an f-rainbow.
It is an instance of the free set theorem, but the added assumption that $g(s) > \min s$ allow us to have the strong cone avoidance.

- For every $s \subseteq \mathbb{N}$ such that $|s| \leq \min s$ (i.e. not ω-large), s induce a coloring $f(s, \cdot) : [\mathbb{N}]^{\min s - |s| + 1} \to \mathbb{N}$
- This is a coloring of tuple of a fixed length.
- We can then use the thin set theorem and the free set theorem for tuple of a fixed size.

The set H is then constructed by forcing.

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