Mutual Exclusions Insertion Formalisation

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Based on the definitions of mutual exclusions presented in the paper, one can now add to G the mutually exclusive tasks not already belonging to it. The tasks not already belonging to G are $\overline{V} = \bigcup\limits_{T \in S_T} \{v \in \texttt{tasks}(T) \mid v \notin G\}$. To perform the insertion, we must consider the set of mutually exclusive tasks of each task $t \in \overline{V}$, that is M_t. Let us then consider two possible cases: either (i) the set only contains tasks that do not belong to G, i.e., $M_t \cap V = \emptyset$, or (ii) the set contains at least one task belonging to G, i.e., $M_t \cap V \neq \emptyset$.

For case (i), the solution is rather simple: t and each task of M_t are added to the graph, without any connection, as initial nodes. By doing so, $\mathcal{P}_G^0$ now contains $|M_t| + 1$ new paths, containing each a single task of M_t, or t. By construction, there is no path of G containing both t and a task of M_t. Thus, they are mutually exclusive, as desired.

For case (ii), the solution is slightly more complex. Let us break M_t into two sets: the set of tasks already belonging to G called $\tilde{M_t}$, and the set of tasks not belonging to G called $\overline{M_t}$. We have that $\tilde{M_t} \cup \overline{M_t} = M_t$. A simple—yet naive—way of inserting t and the tasks belonging to $\overline{M_t}$ into G would be to do just as in case (i), that is, adding them to G without any connection, as initial nodes. However, unlike in case (i), t is, by definition, constrained with regards to some tasks of the graph (the $\tilde{M_t}$). Consequently, inserting these tasks as performed in case (i) would create many unspecified mutual exclusions. To avoid this, the proposed method consists in connecting t and the tasks belonging to $\overline{M_t}$ to a particular node of G while preserving the existing mutual exclusions and limiting the number of unspecified mutual exclusions. This particular node is one of the *closest inevitable common ancestors* of the tasks belonging to $\tilde{M_t}$, and of the mutually exclusive tasks of the tasks of $\overline{M_t}$ already belonging to G.

Definition 1 (Closest Inevitable Common Ancestors)**.** *Let* $G = (V, E, \Sigma)$ *be a BPMN process.* $\forall v_1, ..., v_n \in V$*, the* closest inevitable common ancestors *of* $(v_1, ..., v_n)$ *are all the nodes* $v_C \in V$ *such that:*

- $\forall i \in [1...n]$, $\forall \widehat{p} \in \widehat{\mathcal{P}}_G^0$, $v_i \in \widehat{p} \Rightarrow (v_C \in \widehat{p} \wedge \texttt{index}(v_C) < \texttt{index}(v_i))$ *(inevitability, commonality, ancestrality);*
- $\forall p_{v_C} = (v_C, v_b, ..., v_m) \in \mathcal{P}_G(v_C)$, $\nexists j \in [b...m]$ *such that* v_j *is a common ancestor of* $(v_1, ..., v_n)$ *(closeness).*

Among the eventual multiple closest inevitable common ancestors, one of them is selected, and t and the tasks belonging to $\overline{M_t}$ are inserted to G as children of this ancestor. As desired, task t is now mutually exclusive of the tasks of M_t.

Proposition 1 (Validity of the Closest Inevitable Common Ancestors)**.** *Let* $G = (V, E, \Sigma)$ *be a BPMN process, let* M_t *be the set of mutually exclusive tasks of a task* $t \in V$[1]*, let* $\tilde{M}_t = M_t \cap V$*, let* $\overline{M}_t = M_t \setminus \tilde{M}_t$*, and let* V_C *be the set of closest inevitable common ancestors of the tasks belonging to* $\tilde{M}_t \cup \bigcup\limits_{\overline{t} \in \overline{M_t}} \bigcup\limits_{\substack{m \in \texttt{mutex}(\overline{t}) \\ m \in G}}$*. We state that inserting* t *and the tasks* $\overline{M}_t$ *into* G *as children of any*

[1]One could take $t \notin V$ without changing the validity of the statement.

$v_C \in V_C$ *make* t *weakly mutually exclusive of the tasks* M_t.

Proof. Let $G = (V,\ E,\ \Sigma)$ be a BPMN process, let M_t be the set of mutually exclusive tasks of a task $t \in V$[2], let $\tilde{M}_t = M_t \cap V$, let $\overline{M}_t = M_t \setminus \tilde{M}_t$, and let V_C be the set of closest inevitable common ancestors of the tasks belonging to $\tilde{M}_t \cup \bigcup\limits_{\bar{t} \in \overline{M_t}} \bigcup\limits_{\substack{m \in \texttt{mutex}(\bar{t}) \\ m \in G}}$. We will show that, $\forall v_C \in V_C$, adding t and the tasks of $\overline{M}_t$ as children of v_c make t weakly mutually exclusive of the tasks belonging to M_t.

Adding t and the tasks $\overline{M}_t$ as children of v_C creates a BPMN process $G' = (V', E', \Sigma')$, where:

— $V' = V \ \cup\ \overline{M}_t \ \cup\ \{t\}$;
— $E' = E \ \cup\ \bigcup\limits_{\bar{t} \in \overline{M_t}} \{v_C \rightarrow \bar{t}\} \ \cup\ \{v_C \rightarrow t\}$;
— $\Sigma' = \Sigma \ \cup\ \bigcup\limits_{\bar{t} \in \overline{M_t}} \{\sigma(\bar{t})\} \ \cup\ \{\sigma(t)\}$.

Consequently, we have that

$$\widehat{\mathcal{P}}^0_{G'} = \widehat{\mathcal{P}}^0_G \cup \bigcup_{\bar{t} \in \overline{M_t}} \bigcup_{p \in \widehat{\mathcal{P}}^0_G} \{(p[: v_C], \bar{t}) \mid v_C \in p\} \ \cup \bigcup_{p \in \widehat{\mathcal{P}}^0_G} \{(p[: v_C], t) \mid v_C \in p\}$$

By construction, there is no $p \in \widehat{\mathcal{P}}^0_{G'}$ containing both t and a task $\bar{t} \in \overline{M_t}$. Moreover, by definition of v_C, we have that $\forall \tilde{t} \in \tilde{M}_t,\ \forall p \in \widehat{\mathcal{P}}^0_{G'},\ \tilde{t} \in p \Rightarrow p = (v_1, ..., v_C, ..., \tilde{t}, ..., v_z)$. Thus, by construction of G', there is no $p \in \widehat{\mathcal{P}}^0_{G'}$ containing both t and a task $\tilde{t} \in \tilde{M}_t$. Consequently, t is weakly mutually exclusive of all $\tilde{t} \in \tilde{M}_t$, and of all $\bar{t} \in \overline{M_t}$, which corresponds to all the tasks of M_t. ☐

Remark 1. *It is worth mentioning that considering the closest inevitable common ancestor of (the barbarian expression)* $\tilde{M}_t \cup \bigcup\limits_{\bar{t} \in \overline{M_t}} \bigcup\limits_{\substack{m \in \texttt{mutex}(\bar{t}) \\ m \in G}}$ *is mandatory in order to preserve the original mutual exclusions. Indeed, adding the tasks of* $\overline{M_t}$ *as children of the closest inevitable common ancestor of* $\tilde{M}_t$ *only could potentially prevent a task* $\bar{t} \in \overline{M_t}$ *from being mutually exclusive of one of its mutually exclusive tasks, in the case where such a task is a predecessor of the closest inevitable common ancestor of* $\tilde{M}_t$.

[2] One could take $t \notin V$ without changing the validity of the statement.