

## RCWA Formulation

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Reference: EMPossible

- Formulation of Rigorous Coupled-Wave Analysis (RCWA)  
<https://empossible.net/wp-content/uploads/2019/08/Lecture-7a-RCWA-Formulation.pdf>
- R. C. Rumpf, "IMPROVED FORMULATION OF SCATTERING MATRICES FOR SEMI-ANALYTICAL METHODS THAT IS CONSISTENT WITH CONVENTION," PIER B, vol. 35, pp. 241–261, 2011, doi: 10.2528/PIERB11083107.
- Lecture 19 (CEM) -- Formulation of Rigorous Coupled-Wave Analysis  
[https://www.youtube.com/watch?v=LEWTvwrYxii&t=1s&ab\\_channel=EMPossible](https://www.youtube.com/watch?v=LEWTvwrYxii&t=1s&ab_channel=EMPossible)

## Maxwell's equation

Let's start from the most fundamental equation, there are four:

|                                |   |
|--------------------------------|---|
| Gauss's law                    | $\nabla \cdot D = \rho$                               |
| Gauss's law for magnetism      | $\nabla \cdot B = 0$                                  |
| Maxwell-Faraday                | $\nabla \times E = -\frac{\partial B}{\partial t}$    |
| Ampere with Maxwell's addition | $\nabla \times H = J + \frac{\partial D}{\partial t}$ |

Assume there is no source:  $\rho = 0, J = 0$  and under linear material  $D = \epsilon E, B = \mu H$

We have:

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t}$$

Assume  $E = E_0 e^{-i\omega t}, H = H_0 e^{-i\omega t} \rightarrow \frac{\partial E}{\partial t} = (-i\omega)E, \frac{\partial H}{\partial t} = (-i\omega)H$

$$\nabla \times E = -\mu(-i\omega)H$$

$$\nabla \times H = \epsilon(-i\omega)E$$

We want to normalize the equation so that:  $\nabla \times E = -\mu(-i\omega)H = k_0 \mu_r \bar{H}, k_0 = \frac{2\pi}{\lambda_0}$

Therefore,  $-\mu(-i\omega)H = k_0 \mu_r \bar{H} \rightarrow \bar{H} = \frac{i\mu\omega}{k_0 \mu_r} H = i c \mu_0 H = i \sqrt{\frac{\mu_0}{\epsilon_0}} H = i \eta_0 H$

After normalization of  $\bar{H} = i\eta_0 H \rightarrow H = -i\sqrt{\frac{\epsilon_0}{\mu_0}}\bar{H}$

$$\nabla \times E = -\mu(-i\omega)H = \mu i\omega \left( -i\sqrt{\frac{\epsilon_0}{\mu_0}}\bar{H} \right) = \mu_r \omega \sqrt{\mu_0 \epsilon_0} \bar{H} = \mu_r \omega \frac{k_0}{\omega} \bar{H} = k_0 \mu_r \bar{H}$$

$$\nabla \times \left( -i\sqrt{\frac{\epsilon_0}{\mu_0}}\bar{H} \right) = \epsilon(-i\omega)E \rightarrow \nabla \times \bar{H} = \left( i\sqrt{\frac{\mu_0}{\epsilon_0}} \right) \epsilon(-i\omega)E = \epsilon_r \omega \sqrt{\mu_0 \epsilon_0} E = k_0 \epsilon_r E$$

So we got the normalized maxwell equation:

$$\nabla \times E = k_0 \mu_r \bar{H}$$

$$\nabla \times \bar{H} = k_0 \epsilon_r E$$

$$k_0 = \frac{2\pi}{\lambda_0}, \quad \bar{H} = i\eta_0 H$$

Expanding the curl operation:

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$\nabla \times \bar{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{H}_x & \bar{H}_y & \bar{H}_z \end{vmatrix} = \left( \frac{\partial \bar{H}_z}{\partial y} - \frac{\partial \bar{H}_y}{\partial z}, \quad \frac{\partial \bar{H}_x}{\partial z} - \frac{\partial \bar{H}_z}{\partial x}, \quad \frac{\partial \bar{H}_y}{\partial x} - \frac{\partial \bar{H}_x}{\partial y} \right)$$

We have six equation

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \bar{H}_x, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \mu_r \bar{H}_y, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0 \mu_r \bar{H}_z$$

$$\frac{\partial \bar{H}_z}{\partial y} - \frac{\partial \bar{H}_y}{\partial z} = k_0 \epsilon_r E_x, \quad \frac{\partial \bar{H}_x}{\partial z} - \frac{\partial \bar{H}_z}{\partial x} = k_0 \epsilon_r E_y, \quad \frac{\partial \bar{H}_y}{\partial x} - \frac{\partial \bar{H}_x}{\partial y} = k_0 \epsilon_r E_z$$

## Field Expansion

$$E(z) = \sum_{mn} E_{mn}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\bar{H}(z) = \sum_{mn} \bar{H}_{mn}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\varepsilon_r(x, y) = \sum_{mn} \varepsilon_{r_{mn}} \cdot \exp(i(G_x m x + G_y n y))$$

$$\mu_r(x, y) = \sum_{mn} \mu_{r_{mn}} \cdot \exp(i(G_x m x + G_y n y))$$

$$r = (x, y), \quad k_{mn} = k_0 + (G_x m, G_y n), \quad G_x = \frac{2\pi}{\Lambda_x}, \quad G_y = \frac{2\pi}{\Lambda_y}$$

$E_{mn}(z), \bar{H}_{mn}(z), \varepsilon_{r_{mn}}, \mu_{r_{mn}}$  is the amplitude coefficient, complex number

Using  $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \bar{H}_x$  as a example,

$$\frac{\partial E_z}{\partial y} = \sum_{mn} E_{z_{mn}}(z) \cdot \frac{\partial}{\partial y} \exp(ik_{mn} \cdot r) = \sum_{mn} \mathbf{k}_{y_{mn}} \cdot E_{z_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\frac{\partial E_y}{\partial z} = \sum_{mn} \frac{\mathbf{d}}{\mathbf{dz}} E_{y_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\mu_r \bar{H}_x = \sum_{mn} \mu_{r_{mn}} \cdot \exp(i(G_x m x + G_y n y)) \sum_{mn} \bar{H}_{x_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{m''n''}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp(i(G_x m' x + G_y n' y)) \cdot \exp(ik_{mn} \cdot r)$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{m''n''}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp(i(k_0 \cdot r + G_x(m'' + m')x + G_y(n'' + n')y))$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{mn}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp(ik_{m''+m',n''+n'} \cdot r)$$

Let  $m = m'' + m', m'' = m = m'$

$$= \sum_{mn} \sum_{m'n'} \mathbf{\mu}_{r_{m-m',n-n'}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp(ik_{mn} \cdot r)$$

Combine above to get:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \bar{H}_x$$

$$\sum_{mn} \left( k_{y_{mn}} \cdot E_{z_{mn}}(z) - \frac{d}{dz} E_{y_{mn}}(z) = k_0 \sum_{m'n'} \mu_{r_{m-m'n-n'}} \cdot \bar{H}_{x_{m'n'}}(z) \right) \cdot \exp(ik_{mn} \cdot r)$$

For each m, n:

$$k_{y_{mn}} \cdot E_{z_{mn}}(z) - \frac{d}{dz} E_{y_{mn}}(z) = k_0 \sum_{m'n'} \mu_{r_{m-m'n-n'}} \cdot \bar{H}_{x_{m'n'}}(z)$$

Write in matrix form:

$$K_y E_z - \frac{d}{dz} E_y = k_0 \mu_r \bar{H}_x$$

Where  $E_z$  and  $\bar{H}_x$  are vector,  $K_y$  is diagonal matrix:

$$E_z = \begin{bmatrix} E_{z1} \\ \vdots \\ E_{zA} \end{bmatrix}, \quad \bar{H}_x = \begin{bmatrix} \bar{H}_{x1} \\ \vdots \\ \bar{H}_{xA} \end{bmatrix}, \quad K_y = \begin{bmatrix} k_{y1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_{yA} \end{bmatrix}$$

$A$  is number of modes, for each term  $E_{z_a}$  will corresponding to certain mode  $E_{z_{mn}}$

Which means,  $E_{z_a}$  is the flattened  $E_{z_{mn}}$ ,  $\bar{H}_{x_a}$  is the flattened  $\bar{H}_{x_{mn}}$ ,  $k_{y_a}$  is the flattened  $k_{y_{mn}}$ , and so on.

And  $\mu_r$  becomes a convolution matrix:

$$\mu_r = \begin{bmatrix} \mu_{r11} & \cdots & \mu_{r1b} & \cdots & \mu_{r1B} \\ \vdots & \ddots & & & \vdots \\ \mu_{ra1} & & \mu_{rab} & & \mu_{raB} \\ \vdots & & & \ddots & \vdots \\ \mu_{rA1} & \cdots & \mu_{rAb} & \cdots & \mu_{rAB} \end{bmatrix}$$

$$\mu_{rab} = \mu_{r_{m-m'n-n'}}$$

$A = B$  is number of modes, where index  $b \rightarrow m'n'$ ,  $a \rightarrow mn$

## Compare to DFT algorithm:

Our expansion is:

$$a(x) = \sum_m A_m \cdot \exp(iGmx)$$

DFT definition is:

$$a(x') = \frac{1}{n} \sum_{m=0}^{n-1} A'_m \cdot \exp\left(i2\pi \frac{m}{n} x'\right)$$

Where  $n$  is number of sample point,  $x'$  is coordinate in index order:  $x' = \frac{n}{\Lambda} x$

Therefore:

$$a(x) = \frac{1}{n} \sum_{m=0}^{n-1} A'_m \cdot \exp\left(i2\pi \frac{m}{\Lambda} x\right) = \frac{1}{n} \sum_{m=0}^{n-1} A'_m \cdot \exp(iGmx)$$

By comparison, the term we need  $A_m$ , is the DFT term  $A'_m$  divid by number of sample point  $n$ :

$$A_m = \frac{1}{n} A'_m$$

Also, By DFT Difinition:

$$A'_m = \sum_{m=0}^{n-1} a(x') \exp\left(-i2\pi \frac{m}{n} x'\right)$$

$$\exp\left(-i2\pi \frac{m}{n} x'\right) = \exp\left(-i2\pi \left(\frac{m}{n} x' - N\right)\right) = \exp\left(-i2\pi \frac{m}{n} \left(x' - \frac{n}{m} N\right)\right)$$

$N$  can be any integer, let  $N = m$

$$\exp\left(-i2\pi \frac{m}{n} x'\right) = \exp\left(-i2\pi \frac{m}{n} (x' - n)\right)$$

$$A'_m = A'_{m-n} = A'_{mod(m,n)}$$

Therefore,

$$A_m = \frac{1}{n} A'_{mod(m,n)}$$

Mod  $m$  by  $n$  then it's safe to access any coefficient even if  $m$  is out of bounds.

Original six equations:

$$\begin{aligned}\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= k_0 \mu_r \bar{H}_x, & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= k_0 \mu_r \bar{H}_y, & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= k_0 \mu_r \bar{H}_z \\ \frac{\partial \bar{H}_z}{\partial y} - \frac{\partial \bar{H}_y}{\partial z} &= k_0 \varepsilon_r E_x, & \frac{\partial \bar{H}_x}{\partial z} - \frac{\partial \bar{H}_z}{\partial x} &= k_0 \varepsilon_r E_y, & \frac{\partial \bar{H}_y}{\partial x} - \frac{\partial \bar{H}_x}{\partial y} &= k_0 \varepsilon_r E_z\end{aligned}$$

In matrix form:

$$\begin{aligned}K_y E_z - \frac{dE_y}{dz} &= k_0 \mu_r \bar{H}_x, & \frac{dE_x}{dz} - K_x E_z &= k_0 \mu_r \bar{H}_y, & K_x E_y - K_y E_x &= k_0 \mu_r \bar{H}_z \\ K_y \bar{H}_z - \frac{d\bar{H}_y}{dz} &= k_0 \varepsilon_r E_x, & \frac{d\bar{H}_x}{dz} - K_x \bar{H}_z &= k_0 \varepsilon_r E_y, & K_x \bar{H}_y - K_y \bar{H}_x &= k_0 \varepsilon_r E_z\end{aligned}$$

Normalize by  $k_0$ :

$$\begin{aligned}\bar{K} &= \frac{K}{k_0} \\ \bar{z} &= k_0 z \\ d\bar{z} &= k_0 dz \\ \bar{K}_y E_z - \frac{dE_y}{d\bar{z}} &= \mu_r \bar{H}_x, & \frac{dE_x}{d\bar{z}} - \bar{K}_x E_z &= \mu_r \bar{H}_y, & \bar{K}_x E_y - \bar{K}_y E_x &= \mu_r \bar{H}_z \\ \bar{K}_y \bar{H}_z - \frac{d\bar{H}_y}{d\bar{z}} &= \varepsilon_r E_x, & \frac{d\bar{H}_x}{d\bar{z}} - \bar{K}_x \bar{H}_z &= \varepsilon_r E_y, & \bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x &= \varepsilon_r E_z\end{aligned}$$

Solve for  $\bar{H}_z$  and  $E_z$ :

$$\begin{aligned}\bar{H}_z &= \mu_r^{-1} (\bar{K}_x E_y - \bar{K}_y E_x) \\ E_z &= \varepsilon_r^{-1} (\bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x)\end{aligned}$$

Substitute back to the equation:

$$\begin{aligned}\bar{K}_y \varepsilon_r^{-1} (\bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x) - \frac{dE_y}{d\bar{z}} &= \mu_r \bar{H}_x \\ \frac{dE_x}{d\bar{z}} - \bar{K}_x \varepsilon_r^{-1} (\bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x) &= \mu_r \bar{H}_y\end{aligned}$$

We get:

$$\begin{aligned}\frac{dE_y}{d\bar{z}} &= -\mu_r \bar{H}_x - \bar{K}_y \varepsilon_r^{-1} (\bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x) = (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r) \bar{H}_x - (\bar{K}_y \varepsilon_r^{-1} \bar{K}_x) \bar{H}_y \\ \frac{dE_x}{d\bar{z}} &= \mu_r \bar{H}_y - \bar{K}_x \varepsilon_r^{-1} (\bar{K}_x \bar{H}_y - \bar{K}_y \bar{H}_x) = (\bar{K}_x \varepsilon_r^{-1} \bar{K}_y) \bar{H}_x + (\mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x) \bar{H}_y\end{aligned}$$

## Matrix Equation

In matrix form:

$$\frac{d}{d\bar{z}} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \bar{K}_x \varepsilon_r^{-1} \bar{K}_y & \mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \\ \bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r & -\bar{K}_y \varepsilon_r^{-1} \bar{K}_x \end{bmatrix} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = P \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix}$$

Similarly:

$$\frac{d}{d\bar{z}} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = \begin{bmatrix} \bar{K}_x \mu_r^{-1} \bar{K}_y & \varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x \\ \bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r & -\bar{K}_y \mu_r^{-1} \bar{K}_x \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = Q \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

We have:

$$P = \begin{bmatrix} \bar{K}_x \varepsilon_r^{-1} \bar{K}_y & \mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \\ \bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r & -\bar{K}_y \varepsilon_r^{-1} \bar{K}_x \end{bmatrix}, \quad Q = \begin{bmatrix} \bar{K}_x \mu_r^{-1} \bar{K}_y & \varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x \\ \bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r & -\bar{K}_y \mu_r^{-1} \bar{K}_x \end{bmatrix}$$

And differential equation:

$$\frac{d}{d\bar{z}} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = P \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix}, \quad \frac{d}{d\bar{z}} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = Q \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Solving:

$$\frac{d^2}{d\bar{z}^2} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = P \frac{d}{d\bar{z}} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = PQ \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \Omega^2 \begin{bmatrix} E_x \\ E_y \end{bmatrix}, \quad \Omega^2 = PQ$$

General Solution:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = e^{\Omega \bar{z}} c = W e^{\lambda \bar{z}} W^{-1} c, \quad \Omega = W \lambda W^{-1}, \quad \Omega^2 = W \lambda^2 W^{-1}$$

Where  $\lambda^2$  is the eigen value matrix of  $\Omega^2$ ,  $W$  is the eigen vector matrix of  $\Omega^2$ ,  $c$  is coefficient.

Further separate the solution into forward and backward wave:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = W e^{\lambda \bar{z}} c^+ + W e^{-\lambda \bar{z}} c^-$$

And assume similar solution for magnetic field:

$$\begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = V e^{\lambda \bar{z}} c^+ - V e^{-\lambda \bar{z}} c^-$$

Recall  $\frac{d}{d\bar{z}} \begin{bmatrix} \bar{H}_x \\ \bar{H}_y \end{bmatrix} = Q \begin{bmatrix} E_x \\ E_y \end{bmatrix}$ :

$$V \lambda e^{\lambda \bar{z}} c^+ + V \lambda e^{-\lambda \bar{z}} c^- = Q W e^{\lambda \bar{z}} c^+ + Q W e^{-\lambda \bar{z}} c^-$$

$$V \lambda = Q W$$

$$V = Q W \lambda^{-1}$$

We are able to fomulate the solution:

$$\psi(\bar{z}) = \begin{bmatrix} E_x \\ E_y \\ \bar{H}_x \\ \bar{H}_y \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} e^{\lambda \bar{z}} & 0 \\ 0 & e^{-\lambda \bar{z}} \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

And start to match the boundary condition:

Assume three layer, and their eigen modes are:

|                   | Layer 1   | Layer N   | Layer 2   |
|-------------------|---|---|---|
| Eigen modes       | $\begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix}$ | $\begin{bmatrix} W & W \\ V & -V \end{bmatrix}$ | $\begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix}$ |
| Field coefficient | $\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$          | $\begin{bmatrix} c^+ \\ c^- \end{bmatrix}$      | $\begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$          |
| Thickness         | 0   | $L$   | 0   |

Between Layer 1 and Layer N,  $\bar{z} = 0$

$$\begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

Between Layer N and Layer 2 ,  $\bar{z} = k_0 z = k_0 L$

$$\begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} c^+ \\ c^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} \begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix}^{-1} \begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} \begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

The inverse of the matrix:

$$\begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} = \frac{\begin{bmatrix} -V & -W \\ -V & W \end{bmatrix}}{-2VW} = \frac{1}{2} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix}, \quad \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix}^{-1} = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix}$$

Verify:

$$\frac{1}{2} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W & W \\ V & -V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Therefore:

$$\begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$



## Scattering matrix

Simplify:

$$\begin{bmatrix} W^{-1}W_1 + V^{-1}V_1 & W^{-1}W_1 - V^{-1}V_1 \\ W^{-1}W_1 - V^{-1}V_1 & W^{-1}W_1 + V^{-1}V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} \\ = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix} \begin{bmatrix} W^{-1}W_2 + V^{-1}V_2 & W^{-1}W_2 - V^{-1}V_2 \\ W^{-1}W_2 - V^{-1}V_2 & W^{-1}W_2 + V^{-1}V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

Let

$$A_i = W^{-1}W_i + V^{-1}V_i$$

$$B_i = W^{-1}W_i - V^{-1}V_i$$

$$X = e^{\lambda k_0 L}$$

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} X^{-1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

Write into form of scattering matrix:

$$\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}$$

$$\text{Let } \begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$$

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} 1 \\ S_{11} \end{bmatrix} = \begin{bmatrix} X^{-1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} S_{21} \\ 0 \end{bmatrix}$$

|  |   |
|--|---|
| 1. From the matrix<br>$A_1 + B_1 S_{11} = X^{-1} A_2 S_{21}$<br>$B_1 + A_1 S_{11} = X B_2 S_{21}$  | 2. Solve for S21 and S11<br>$S_{21} = A_2^{-1} X (A_1 + B_1 S_{11})$<br>$S_{11} = A_1^{-1} (X B_2 S_{21} - B_1)$  |
| 3. Substitute back into 1<br>$A_1 + B_1 A_1^{-1} (X B_2 S_{21} - B_1) = X^{-1} A_2 S_{21}$<br>$B_1 + A_1 S_{11} = X B_2 A_2^{-1} X (A_1 + B_1 S_{11})$ | 4. Rearrange, brings out S21 and S11<br>$(X^{-1} A_2 - B_1 A_1^{-1} X B_2) S_{21} = A_1 - B_1 A_1^{-1} B_1$<br>$(A_1 - X B_2 A_2^{-1} X B_1) S_{11} = X B_2 A_2^{-1} X A_1 - B_1$ |

We have:

$$S_{11} = (A_1 - X B_2 A_2^{-1} X B_1)^{-1} (X B_2 A_2^{-1} X A_1 - B_1)$$

$$S_{21} = (A_2 - X B_1 A_1^{-1} X B_2)^{-1} X (A_1 - B_1 A_1^{-1} B_1)$$

Similarly:

$$S_{22} = (A_2 - X B_1 A_1^{-1} X B_2)^{-1} (X B_1 A_1^{-1} X A_2 - B_2)$$

$$S_{12} = (A_1 - X B_1 A_1^{-1} X B_2)^{-1} X (A_2 - B_2 A_2^{-1} B_2)$$

Summary:

$$A_i = W^{-1}W_i + V^{-1}V_i$$

$$B_i = W^{-1}W_i - V^{-1}V_i$$

$$X = e^{\lambda k_0 L}$$

$$S_{11} = (A_1 - XB_2A_2^{-1}XB_1)^{-1}(XB_2A_2^{-1}XA_1 - B_1)$$

$$S_{12} = (A_1 - XB_1A_1^{-1}XB_2)^{-1}X(A_2 - B_2A_2^{-1}B_2)$$

$$S_{21} = (A_2 - XB_1A_1^{-1}XB_2)^{-1}X(A_1 - B_1A_1^{-1}B_1)$$

$$S_{22} = (A_2 - XB_1A_1^{-1}XB_2)^{-1}(XB_1A_1^{-1}XA_2 - B_2)$$

When  $A_1 = A_2 = A$ ,  $B_1 = B_2 = B$ :

$$S_{11} = (A - XBA^{-1}XB)^{-1}(XBA^{-1}XA - B)$$

$$S_{12} = (A - XBA^{-1}XB)^{-1}X(A - BA^{-1}B)$$

$$S_{21} = S_{12}$$

$$S_{22} = S_{11}$$

Reflection side, when  $W = W_1$ ,  $V = V_1$ ,  $X = 1 \rightarrow A_1 = 1 + 1 = 2$ ,  $B_1 = 1 - 1 = 0$

$$S_{11} = (2 - 0)^{-1}(BA^{-1}2 - 0) = BA^{-1}$$

$$S_{12} = (2 - 0)^{-1}(A - BA^{-1}B) = 0.5(A - BA^{-1}B)$$

$$S_{21} = (A - 0)^{-1}(2 - 0) = 2A^{-1}$$

$$S_{22} = (A - 0)^{-1}(0 - B) = -A^{-1}B$$

Transmission side, when  $W = W_2$ ,  $V = V_2$ ,  $X = 1 \rightarrow A_2 = 1 + 1 = 2$ ,  $B_2 = 1 - 1 = 0$

$$S_{11} = (A - 0)^{-1}(0 - B) = -A^{-1}B$$

$$S_{12} = (A - 0)^{-1}(2 - 0) = 2A^{-1}$$

$$S_{21} = (2 - 0)^{-1}(A - BA^{-1}B) = 0.5(A - BA^{-1}B)$$

$$S_{22} = (2 - 0)^{-1}(BA^{-1}2 - 0) = BA^{-1}$$

## Scattering matrix operation

Assume 3 layer connected by 2 scattering matrix A and B

| →  | →  | →  | →  | →  |
|--|--|--|--|--|
| $\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$   | $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ | $\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$ | $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ | $\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$ |
| ←  | ←  | ←  | ←  | ←  |
| $\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}, \quad \begin{bmatrix} c_2^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_3^- \end{bmatrix}$ |  |  |  |  |

Combine into one scattering matrix:

$$\begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix}$$

Case 1:  $\begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$

$$\begin{bmatrix} S_{11} \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 \\ c_2^- \end{bmatrix}, \quad \begin{bmatrix} c_2^- \\ S_{21} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ 0 \end{bmatrix}$$

|   |  |
|---|--|
| 1. From matrix A<br>$S_{11} = A_{11} + A_{12}c_2^- \dots (A1)$<br>$c_2^+ = A_{21} + A_{22}c_2^- \dots (A2)$ | 2. From matrix B<br>$c_2^- = B_{11}c_2^+ \dots (B1)$<br>$S_{21} = B_{21}c_2^+ \dots (B2)$                    |
| 3. Substitute A2 and B1:<br>$c_2^+ = A_{21} + A_{22}B_{11}c_2^+$<br>$c_2^- = B_{11}(A_{21} + A_{22}c_2^-)$  | 4. Solve for C2:<br>$c_2^+ = (I - A_{22}B_{11})^{-1}A_{21}$<br>$c_2^- = (I - B_{11}A_{22})^{-1}B_{11}A_{21}$ |

$$S_{11} = A_{11} + A_{12}(I - B_{11}A_{22})^{-1}B_{11}A_{21}$$

$$S_{21} = B_{21}(I - A_{22}B_{11})^{-1}A_{21}$$

Case 2:  $\begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$

$$\begin{bmatrix} S_{12} \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ c_2^- \end{bmatrix}, \quad \begin{bmatrix} c_2^- \\ S_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ 1 \end{bmatrix}$$

|  |  |
|--|--|
| 1. From matrix A<br>$S_{12} = A_{12}c_2^- \dots (A1)$<br>$c_2^+ = A_{22}c_2^- \dots (A2)$                  | 2. From matrix B<br>$c_2^- = B_{11}c_2^+ + B_{12} \dots (B1)$<br>$S_{22} = B_{21}c_2^+ + B_{22} \dots (B2)$  |
| 3. Substitute A2 and B1:<br>$c_2^+ = A_{22}(B_{21}c_2^+ + B_{12})$<br>$c_2^- = B_{11}A_{22}c_2^- + B_{12}$ | 4. Solve for C2:<br>$c_2^+ = (I - A_{22}B_{11})^{-1}A_{22}B_{12}$<br>$c_2^- = (I - B_{11}A_{22})^{-1}B_{12}$ |

$$S_{12} = A_{12}(I - B_{11}A_{22})^{-1}B_{12}$$

$$S_{22} = B_{21}(I - A_{22}B_{11})^{-1}A_{22}B_{12} + B_{22}$$

Summary:

$$\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}, \quad \begin{bmatrix} c_2^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_3^- \end{bmatrix}, \quad \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix}$$

$$S_{11} = A_{11} + A_{12}(I - B_{11}A_{22})^{-1}B_{11}A_{21}$$

$$S_{12} = A_{12}(I - B_{11}A_{22})^{-1}B_{12}$$

$$S_{21} = B_{21}(I - A_{22}B_{11})^{-1}A_{21}$$

$$S_{22} = B_{21}(I - A_{22}B_{11})^{-1}A_{22}B_{12} + B_{22}$$

Define as  $A \otimes B = S$ , called the Redheffer Star Product.

Final solution

$$\begin{bmatrix} C_R \\ C_T \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} C_I \\ 0 \end{bmatrix}, \quad C_I = W_R^{-1}E_i, \quad E_R = W_R C_R, \quad E_T = W_T C_T$$

$$E_R = W_R S_{11} W_R^{-1} E_i$$

$$E_T = W_T S_{21} W_R^{-1} E_i$$

Finding z component:

$$\vec{K} \cdot E = 0 \rightarrow k_x E_x + k_y E_y + k_z E_z = 0$$

$$E_z = -\frac{k_x E_x + k_y E_y}{k_z}, \quad H_z = -\frac{k_x H_x + k_y H_y}{k_z}$$

Power:

$$|E| = E_x^2 + E_y^2 + E_z^2$$

$$P = |E \times H| = \frac{|E|^2}{\eta} = |E|^2 \sqrt{\frac{\epsilon}{\mu}}$$

Power Ratio

$$\frac{P_2 \cos \theta_2}{P_1 \cos \theta_1} = \frac{|E_2|^2}{|E_1|^2} \frac{\sqrt{\frac{\epsilon_2}{\mu_2}} \cos \theta_2}{\sqrt{\frac{\epsilon_1}{\mu_1}} \cos \theta_1} = \frac{|E_2|^2 \mu_{r2}^{-1} n_2 \cos \theta_2}{|E_1|^2 \mu_{r1}^{-1} n_1 \cos \theta_1} = \frac{|E_2|^2 \mu_{r2}^{-1} \operatorname{Re}(k_{z2})}{|E_1|^2 \mu_{r1}^{-1} \operatorname{Re}(k_{z1})}$$

### Special Case: No Magnetic

$$\begin{aligned}
 \Omega^2 &= PQ = \begin{bmatrix} \bar{K}_x \varepsilon_r^{-1} \bar{K}_y & I - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \\ \bar{K}_y \varepsilon_r^{-1} \bar{K}_y - I & -\bar{K}_y \varepsilon_r^{-1} \bar{K}_x \end{bmatrix} \begin{bmatrix} \bar{K}_x \bar{K}_y & \varepsilon_r - \bar{K}_x^2 \\ \bar{K}_y^2 - \varepsilon_r & -\bar{K}_y \bar{K}_x \end{bmatrix} \\
 \Omega_{11}^2 &= (\bar{K}_x \varepsilon_r^{-1} \bar{K}_y)(\bar{K}_x \bar{K}_y) + (I - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x)(\bar{K}_y^2 - \varepsilon_r) \\
 &= \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \bar{K}_y^2 + \bar{K}_y^2 - \varepsilon_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \bar{K}_y^2 + \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \varepsilon_r = \bar{K}_y^2 + (\bar{K}_x \varepsilon_r^{-1} \bar{K}_x - I) \varepsilon_r \\
 \Omega_{12}^2 &= (\bar{K}_x \varepsilon_r^{-1} \bar{K}_y)(\varepsilon_r - \bar{K}_x^2) + (I - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x)(-\bar{K}_y \bar{K}_x) \\
 &= \bar{K}_x \varepsilon_r^{-1} \bar{K}_y \varepsilon_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x^2 \bar{K}_y - \bar{K}_x \bar{K}_y + \bar{K}_x \varepsilon_r^{-1} \bar{K}_x^2 \bar{K}_y = \bar{K}_x \varepsilon_r^{-1} \bar{K}_y \varepsilon_r - \bar{K}_x \bar{K}_y \\
 \Omega_{21}^2 &= (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - I)(\bar{K}_x \bar{K}_y) + (-\bar{K}_y \varepsilon_r^{-1} \bar{K}_x)(\bar{K}_y^2 - \varepsilon_r) \\
 &= \bar{K}_y \varepsilon_r^{-1} \bar{K}_x \bar{K}_y^2 - \bar{K}_x \bar{K}_y - \bar{K}_y \varepsilon_r^{-1} \bar{K}_x \bar{K}_y^2 + \bar{K}_y \varepsilon_r^{-1} \bar{K}_x \varepsilon_r = \bar{K}_y \varepsilon_r^{-1} \bar{K}_x \varepsilon_r - \bar{K}_x \bar{K}_y \\
 \Omega_{22}^2 &= (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - I)(\varepsilon_r - \bar{K}_x^2) + (-\bar{K}_y \varepsilon_r^{-1} \bar{K}_x)(-\bar{K}_y \bar{K}_x) \\
 &= \bar{K}_y \varepsilon_r^{-1} \bar{K}_y \varepsilon_r - \bar{K}_y \varepsilon_r^{-1} \bar{K}_x^2 \bar{K}_y - \varepsilon_r + \bar{K}_x^2 + \bar{K}_y \varepsilon_r^{-1} \bar{K}_x^2 \bar{K}_y = \bar{K}_x^2 + (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - I) \varepsilon_r \\
 \Omega^2 &= \begin{bmatrix} \bar{K}_y^2 + (\bar{K}_x \varepsilon_r^{-1} \bar{K}_x - I) \varepsilon_r & \bar{K}_x \varepsilon_r^{-1} \bar{K}_y \varepsilon_r - \bar{K}_x \bar{K}_y \\ \bar{K}_y \varepsilon_r^{-1} \bar{K}_x \varepsilon_r - \bar{K}_x \bar{K}_y & \bar{K}_x^2 + (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - I) \varepsilon_r \end{bmatrix}
 \end{aligned}$$

### Special Case: Homogeneous Layer

$$\begin{aligned}
 \Omega^2 &= PQ = \begin{bmatrix} \bar{K}_x \varepsilon_r^{-1} \bar{K}_y & \mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x \\ \bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r & -\bar{K}_y \varepsilon_r^{-1} \bar{K}_x \end{bmatrix} \begin{bmatrix} \bar{K}_x \mu_r^{-1} \bar{K}_y & \varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x \\ \bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r & -\bar{K}_y \mu_r^{-1} \bar{K}_x \end{bmatrix} \\
 \Omega_{11}^2 &= (\bar{K}_x \varepsilon_r^{-1} \bar{K}_y)(\bar{K}_x \mu_r^{-1} \bar{K}_y) + (\mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x)(\bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r) \\
 &= \bar{K}_x^2 \bar{K}_y^2 \varepsilon_r^{-1} \mu_r^{-1} + \bar{K}_y^2 - \mu_r \varepsilon_r - \bar{K}_x^2 \bar{K}_y^2 \varepsilon_r^{-1} \mu_r^{-1} + \bar{K}_x^2 = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r \\
 \Omega_{12}^2 &= (\bar{K}_x \varepsilon_r^{-1} \bar{K}_y)(\varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x) + (\mu_r - \bar{K}_x \varepsilon_r^{-1} \bar{K}_x)(-\bar{K}_y \mu_r^{-1} \bar{K}_x) \\
 &= \bar{K}_x \bar{K}_y - \bar{K}_x^3 \bar{K}_y^1 \varepsilon_r^{-1} \mu_r^{-1} - \bar{K}_x \bar{K}_y + \bar{K}_x^3 \bar{K}_y^1 \varepsilon_r^{-1} \mu_r^{-1} = 0 \\
 \Omega_{21}^2 &= (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r)(\bar{K}_x \mu_r^{-1} \bar{K}_y) + (-\bar{K}_y \varepsilon_r^{-1} \bar{K}_x)(\bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r) \\
 &= \bar{K}_x^1 \bar{K}_y^3 \varepsilon_r^{-1} \mu_r^{-1} - \bar{K}_x \bar{K}_y - \bar{K}_x^1 \bar{K}_y^3 \varepsilon_r^{-1} \mu_r^{-1} + \bar{K}_x \bar{K}_y = 0 \\
 \Omega_{22}^2 &= (\bar{K}_y \varepsilon_r^{-1} \bar{K}_y - \mu_r)(\varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x) + (-\bar{K}_y \varepsilon_r^{-1} \bar{K}_x)(-\bar{K}_y \mu_r^{-1} \bar{K}_x) \\
 &= \bar{K}_y^2 - \bar{K}_x^2 \bar{K}_y^2 \varepsilon_r^{-1} \mu_r^{-1} - \mu_r \varepsilon_r + \bar{K}_x^2 + \bar{K}_x^2 \bar{K}_y^2 \varepsilon_r^{-1} \mu_r^{-1} = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r \\
 \Omega^2 &= \begin{bmatrix} \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r & 0 \\ 0 & \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r \end{bmatrix}
 \end{aligned}$$

Since it is a diagonal matrix, it's already diagonalized.

$$\Omega^2 = \begin{bmatrix} \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r & 0 \\ 0 & \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r \end{bmatrix} = \lambda^2, \quad W = W^{-1} = I$$

Interesting fact:

$$\lambda^2 = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r \varepsilon_r, \quad \lambda = \sqrt{\bar{K}_x^2 + \bar{K}_y^2 - n^2} = i \sqrt{n^2 - \bar{K}_x^2 - \bar{K}_y^2} = i \bar{K}_z$$

Recall, solution is:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = W e^{\lambda \bar{z}} c^+ + W e^{-\lambda \bar{z}} c^- = e^{i K_z z} c^+ + e^{-i K_z z} c^-$$

Note that relation between E and H is  $\frac{\bar{K}}{|\bar{K}|} \times E = \eta H = i \frac{\eta}{\eta_0} \bar{H} = i \sqrt{\frac{\mu_r}{\varepsilon_r}} \bar{H}$ ,  $|\bar{K}| = n = \sqrt{\mu_r \varepsilon_r}$

$$\frac{1}{\sqrt{\mu_r \varepsilon_r}} \bar{K} \times E = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \bar{K}_x & \bar{K}_y & \bar{K}_z \\ E_x & E_y & E_z \end{bmatrix} = \frac{1}{\sqrt{\mu_r \varepsilon_r}} (\bar{K}_y E_z - \bar{K}_z E_y, \bar{K}_z E_x - \bar{K}_x E_z, \bar{K}_x E_y - \bar{K}_y E_x)$$

$$E_z = -\frac{k_x E_x + k_y E_y}{k_z}$$

$$\bar{H} = -i \sqrt{\frac{\mu_r}{\varepsilon_r}} \frac{1}{\sqrt{\mu_r \varepsilon_r}} \left( -\bar{K}_y \frac{\bar{K}_x E_x + \bar{K}_y E_y}{\bar{K}_z} - \bar{K}_z E_y, \bar{K}_z E_x + \bar{K}_x \frac{\bar{K}_x E_x + \bar{K}_y E_y}{\bar{K}_z}, \bar{K}_x E_y - \bar{K}_y E_x \right)$$

$$\begin{aligned} V &= -i \mu_r^{-1} \begin{bmatrix} \frac{\bar{K}_y \bar{K}_x}{\bar{K}_z} & \frac{\bar{K}_y^2}{\bar{K}_z} - \bar{K}_z \\ -\bar{K}_z - \frac{\bar{K}_x^2}{\bar{K}_z} & -\frac{\bar{K}_x \bar{K}_y}{\bar{K}_z} \end{bmatrix} = -i \mu_r^{-1} \begin{bmatrix} \bar{K}_y \bar{K}_x & \bar{K}_y^2 + \bar{K}_z^2 \\ -\bar{K}_z^2 - \bar{K}_x^2 & -\bar{K}_x \bar{K}_y \end{bmatrix} \begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix} \\ &= -i \mu_r^{-1} \begin{bmatrix} \bar{K}_y \bar{K}_x & n^2 - \bar{K}_x^2 \\ \bar{K}_y^2 - n^2 & -\bar{K}_x \bar{K}_y \end{bmatrix} \begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix} \end{aligned}$$

Also, for magnetic, the other solution also match the result:

$$\begin{aligned} V &= Q W \lambda^{-1} = Q \lambda^{-1} \\ &= \begin{bmatrix} \bar{K}_x \mu_r^{-1} \bar{K}_y & \varepsilon_r - \bar{K}_x \mu_r^{-1} \bar{K}_x \\ \bar{K}_y \mu_r^{-1} \bar{K}_y - \varepsilon_r & -\bar{K}_y \mu_r^{-1} \bar{K}_x \end{bmatrix} \begin{bmatrix} i \bar{K}_z & 0 \\ 0 & i \bar{K}_z \end{bmatrix}^{-1} = -i \mu_r^{-1} \begin{bmatrix} \bar{K}_x \bar{K}_y & n^2 - \bar{K}_x^2 \\ \bar{K}_y^2 - n^2 & -\bar{K}_x \bar{K}_y \end{bmatrix} \begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix} \end{aligned}$$

Free space:

$$V = -i \begin{bmatrix} \bar{K}_x \bar{K}_y & I - \bar{K}_x^2 \\ \bar{K}_y^2 - I & -\bar{K}_y \bar{K}_x \end{bmatrix} \begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix}$$