RCWA Formulation

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Reference: EMPossible

- Formulation of Rigorous Coupled-Wave Analysis (RCWA) https://empossible.net/wp-content/uploads/2019/08/Lecture-7a-RCWA-Formulation.pdf
- R. C. Rumpf, "IMPROVED FORMULATION OF SCATTERING MATRICES FOR SEMI-ANALYTICAL METHODS THAT IS CONSISTENT WITH CONVENTION," PIER B, vol. 35, pp. 241–261, 2011, doi: 10.2528/PIERB11083107.
- Lecture 19 (CEM) -- Formulation of Rigorous Coupled-Wave Analysis https://www.youtube.com/watch?v=LEWTvwrYxiI&t=1s&ab_channel=EMPossible

Maxwell's equation

Let's start with the most fundamental four equations:

Gauss's law	$\nabla \cdot D = \rho$
Gauss's law for magnetism	$\nabla \cdot B = 0$
Maxwell-Faraday	$\nabla \times E = -\frac{\partial B}{\partial t}$
Ampere with Maxwell's addition	$\nabla \times H = J + \frac{\partial D}{\partial t}$

Assume there is no source: $\rho = 0, J = 0$ and under linear material $D = \varepsilon E, B = \mu H$

We have:

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t}$$

Assume $E = E_0 e^{-i\omega t}, H = H_0 e^{-i\omega t} \rightarrow \frac{\partial E}{\partial t} = (-i\omega)E, \frac{\partial H}{\partial t} = (-i\omega)H$

$$\nabla \times E = -\mu(-i\omega)H$$

$$\nabla \times H = \varepsilon(-i\omega)E$$

We want to normalize the equation so that: $\nabla \times E = -\mu(-i\omega)H = k_0 \mu_r \bar{H}, k_0 = \frac{2\pi}{\lambda_0}$

Therefore, $-\mu(-i\omega)H = k_0 \mu_r \bar{H} \rightarrow \bar{H} = \frac{i\mu\omega}{k_0 \mu_r} H = ic\mu_0 H = i\sqrt{\frac{\mu_0}{\varepsilon_0}} H = i\eta_0 H$

After normalization of $\overline{H} = i\eta_0 H \rightarrow H = -i\sqrt{\frac{\varepsilon_0}{\mu_0}}\overline{H}$

$$\nabla \times E = -\mu(-i\omega)H = \mu i\omega\left(-i\sqrt{\frac{\varepsilon_0}{\mu_0}}\overline{H}\right) = \mu_r\omega\sqrt{\mu_0\epsilon_0}\overline{H} = \mu_r\omega\frac{k_0}{\omega}\overline{H} = k_0\mu_r\overline{H}$$

$$\nabla \times \left(-i\sqrt{\frac{\varepsilon_0}{\mu_0}}\overline{H}\right) = \varepsilon(-i\omega)E \rightarrow \nabla \times \overline{H} = \left(i\sqrt{\frac{\mu_0}{\varepsilon_0}}\right)\varepsilon(-i\omega)E = \varepsilon_r\omega\sqrt{\mu_0\epsilon_0}E = k_0\varepsilon_r E$$

So we got the normalized Maxwell equation:

$$\nabla \times E = k_0\mu_r\overline{H}$$

$$\nabla \times \overline{H} = k_0\varepsilon_r E$$

$$k_0 = \frac{2\pi}{\lambda_0}, \qquad \overline{H} = i\eta_0 H$$

Expanding the curl operation:

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\nabla \times \overline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \overline{H}_x & \overline{H}_y & \overline{H}_z \end{vmatrix} = \left(\frac{\partial \overline{H}_z}{\partial y} - \frac{\partial \overline{H}_y}{\partial z}, \qquad \frac{\partial \overline{H}_x}{\partial z} - \frac{\partial \overline{H}_z}{\partial x}, \qquad \frac{\partial \overline{H}_y}{\partial x} - \frac{\partial \overline{H}_x}{\partial y}\right)$$

We have six equation.

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0\mu_r\overline{H}_x, \qquad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0\mu_r\overline{H}_y, \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = k_0\mu_r\overline{H}_z$$

$$\frac{\partial \overline{H}_z}{\partial y} - \frac{\partial \overline{H}_y}{\partial z} = k_0\varepsilon_r E_x, \qquad \frac{\partial \overline{H}_x}{\partial z} - \frac{\partial \overline{H}_z}{\partial x} = k_0\varepsilon_r E_y, \qquad \frac{\partial \overline{H}_y}{\partial x} - \frac{\partial \overline{H}_x}{\partial y} = k_0\varepsilon_r E_z$$

Field Expansion

$$E(z) = \sum_{mn} E_{mn}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\bar{H}(z) = \sum_{mn} \bar{H}_{mn}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\varepsilon_r(x,y) = \sum_{mn} \varepsilon_{r_{mn}} \cdot \exp\big(i(G_x m x + G_y n y)\big)$$

$$\mu_r(x,y) = \sum_{mn} \mu_{r_{mn}} \cdot \exp\big(i(G_x m x + G_y n y)\big)$$

$$r = (x,y), \qquad k_{mn} = k_0 + (G_x m, G_y n), \qquad G_x = \frac{2\pi}{\Lambda_x}, \qquad G_y = \frac{2\pi}{\Lambda_y}$$

$E_{mn}(z), \bar{H}_{mn}(z), \varepsilon_{r_{mn}}, \mu_{r_{mn}}$ is the amplitude coefficient, complex number

Using $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \bar{H}_x$ as an example,

$$\frac{\partial E_z}{\partial y} = \sum_{mn} E_{z_{mn}}(z) \cdot \frac{\partial}{\partial y} \exp(ik_{mn} \cdot r) = \sum_{mn} \boldsymbol{k_{y_{mn}}} \cdot E_{z_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\frac{\partial E_y}{\partial z} = \sum_{mn} \boldsymbol{\frac{d}{dz}} E_{y_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$\mu_r \bar{H}_x = \sum_{mn} \mu_{r_{mn}} \cdot \exp\Big(i\big(G_x m x + G_y n y\big)\Big) \sum_{mn} \bar{H}_{x_{mn}}(z) \cdot \exp(ik_{mn} \cdot r)$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{m''n''}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp\Big(i\big(G_x m' x + G_y n' y\big)\Big) \cdot \exp(ik_{mn} \cdot r)$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{m''n''}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp\Big(i\big(k_0 \cdot r + G_x(m'' + m')x + G_y(n'' + n')y\big)\Big)$$

$$= \sum_{m''n''} \sum_{m'n'} \mu_{r_{mn}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp\big(ik_{m''+m',n''+n'} \cdot r\big)$$

Let $m = m'' + m', m'' = m = m'$

$$= \sum_{mn} \sum_{m'n'} \boldsymbol{\mu_{r_{m-m'n-n'}}} \cdot \bar{H}_{x_{m'n'}}(z) \cdot \exp(ik_{mn} \cdot r)$$

Combine the above to get:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \mu_r \overline{H}_x$$

$$\sum_{mn} \left(k_{y_{mn}} \cdot E_{z_{mn}}(z) - \frac{d}{dz} E_{y_{mn}}(z) = k_0 \sum_{m'n'} \mu_{r_{m-m'n-n'}} \cdot \overline{H}_{x_{m'n'}}(z) \right) \cdot \exp(i k_{mn} \cdot r)$$

For each m, n:

$$k_{y_{mn}} \cdot E_{z_{mn}}(z) - \frac{d}{dz} E_{y_{mn}}(z) = k_0 \sum_{m'n'} \mu_{r_{m-m'n-n'}} \cdot \overline{H}_{x_{m'n'}}(z)$$

Write in matrix form:

$$K_y E_z - \frac{d}{dz} E_y = k_0 \mu_r \overline{H}_x$$

WhereE_z and $\overline{H}_x$ are vector, K_y is the diagonal matrix:

$$E_z = \begin{bmatrix} E_{Z_1} \\ \vdots \\ E_{Z_A} \end{bmatrix}, \qquad \overline{H}_x = \begin{bmatrix} \overline{H}_{x_1} \\ \vdots \\ \overline{H}_{x_A} \end{bmatrix}, \qquad K_y = \begin{bmatrix} k_{y_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_{y_A} \end{bmatrix}$$

A is the number of modes for each term E_{Z_a} will correspond to a certain mode $E_{Z_{mn}}$

Which means, E_{Z_a} is the flattened $E_{Z_{mn}}$, $\overline{H}_{x_a}$ is the flattened $\overline{H}_{x_{mn}}$, $k_{y_a}$is the flattened$k_{y_{mn}}$, and so on.

And μ_r becomes a convolution matrix:

$$\mu_r = \begin{bmatrix} \mu_{r_{11}} & \cdots & \mu_{r_{1b}} & \cdots & \mu_{r_{1B}} \\ \vdots & \ddots & & & \vdots \\ \mu_{r_{a1}} & & \mu_{r_{ab}} & & \mu_{r_{aB}} \\ \vdots & & & \ddots & \vdots \\ \mu_{r_{A1}} & \cdots & \mu_{r_{Ab}} & \cdots & \mu_{r_{AB}} \end{bmatrix}$$

$$\mu_{r_{ab}} = \mu_{r_{m-m'n-n'}}$$

$A = B$ is the number of modes, where index $b \to m'n'$, $a \to mn$

Compare to the DFT algorithm:

Our expansion is:

$$a(x) = \sum_m A_m \cdot \exp(iGmx)$$

DFT definition is:

$$a(x') = \frac{1}{n}\sum_{m=0}^{n-1} A'_m \cdot \exp\left(i2\pi\frac{m}{n}x'\right)$$

Where n is the number of sample points, x' is the coordinate in the index scale: $x' = \frac{n}{\Lambda}x$

Therefore:

$$a(x) = \frac{1}{n}\sum_{m=0}^{n-1} A'_m \cdot \exp\left(i2\pi\frac{m}{\Lambda}x\right) = \frac{1}{n}\sum_{m=0}^{n-1} A'_m \cdot \exp(iGmx)$$

By comparison, the term we need A_m, is the DFT term A'_m divide by the number of sample points n:

$$A_m = \frac{1}{n}A'_m$$

Also, By DFT Definition:

$$A'_m = \sum_{m=0}^{n-1} a(x')\exp\left(-i2\pi\frac{m}{n}x'\right)$$

$$\exp\left(-i2\pi\frac{m}{n}x'\right) = \exp\left(-i2\pi\left(\frac{m}{n}\,x' - N\right)\right) = \exp\left(-i2\pi\frac{m}{n}\left(\,x' - \frac{n}{m}N\right)\right)$$

N can be any integer; let $N = m$

$$\exp\left(-i2\pi\frac{m}{n}x'\right) = \exp\left(-i2\pi\frac{m}{n}(\,x' - n)\right)$$

$$A'_m = A'_{m-n} = A'_{mod(m,n)}$$

Therefore,

$$A_m = \frac{1}{n}A'_{mod(m,n)}$$

Mod m by n, it's safe to access any coefficient even if m is out of bounds.

Original six equations:

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=k_0\mu_r\overline{H}_x,\qquad \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=k_0\mu_r\overline{H}_y,\qquad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=k_0\mu_r\overline{H}_z$$

$$\frac{\partial \overline{H}_z}{\partial y}-\frac{\partial \overline{H}_y}{\partial z}=k_0\varepsilon_r E_x,\qquad \frac{\partial \overline{H}_x}{\partial z}-\frac{\partial \overline{H}_z}{\partial x}=k_0\varepsilon_r E_y,\qquad \frac{\partial \overline{H}_y}{\partial x}-\frac{\partial \overline{H}_x}{\partial y}=k_0\varepsilon_r E_z$$

In matrix form:

$$K_yE_z-\frac{dE_y}{dz}=k_0\mu_r\overline{H}_x,\qquad \frac{dE_x}{dz}-K_xE_z=k_0\mu_r\overline{H}_y,\qquad K_xE_y-K_yE_x=\ k_0\mu_r\overline{H}_z$$

$$K_y\overline{H}_z-\frac{d\overline{H}_y}{dz}=k_0\varepsilon_rE_x,\qquad \frac{d\overline{H}_x}{dz}-K_x\overline{H}_z=k_0\varepsilon_rE_y,\qquad K_x\overline{H}_y-K_y\overline{H}_x=\ k_0\varepsilon_rE_z$$

Normalize by k_0:

$$\overline{K}=\frac{K}{k_0}$$

$$\bar{z}=k_0z$$

$$d\bar{z}=k_0dz$$

$$\overline{K}_yE_z-\frac{dE_y}{d\bar{z}}=\mu_r\overline{H}_x,\qquad \frac{dE_x}{d\bar{z}}-\overline{K}_xE_z=\mu_r\overline{H}_y,\qquad \overline{K}_xE_y-\overline{K}_yE_x=\mu_r\overline{H}_z$$

$$\overline{K}_y\overline{H}_z-\frac{d\overline{H}_y}{d\bar{z}}=\varepsilon_rE_x,\qquad \frac{d\overline{H}_x}{d\bar{z}}-\overline{K}_x\overline{H}_z=\varepsilon_rE_y,\qquad \overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x=\varepsilon_rE_z$$

Solve for $\overline{H}_z$ and E_z:

$$\overline{H}_z=\mu_r^{-1}(\overline{K}_xE_y-\overline{K}_yE_x)$$

$$E_z=\varepsilon_r^{-1}\left(\overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x\right)$$

Substitute back to the equation:

$$\overline{K}_y\varepsilon_r^{-1}\left(\overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x\right)-\frac{dE_y}{d\bar{z}}=\mu_r\overline{H}_x$$

$$\frac{dE_x}{d\bar{z}}-\overline{K}_x\varepsilon_r^{-1}\left(\overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x\right)=\mu_r\overline{H}_y$$

We get:

$$\frac{dE_y}{d\bar{z}}=-\mu_r\overline{H}_x-\overline{K}_y\varepsilon_r^{-1}\left(\overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x\right)=\left(\overline{K}_y\varepsilon_r^{-1}\overline{K}_y-\mu_r\right)\overline{H}_x-\left(\overline{K}_y\varepsilon_r^{-1}\overline{K}_x\right)\overline{H}_y$$

$$\frac{dE_x}{d\bar{z}}=\mu_r\overline{H}_y-\overline{K}_x\varepsilon_r^{-1}\left(\overline{K}_x\overline{H}_y-\overline{K}_y\overline{H}_x\right)=\left(\overline{K}_x\varepsilon_r^{-1}\overline{K}_y\right)\overline{H}_x+(\mu_r-\overline{K}_x\varepsilon_r^{-1}\overline{K}_x)\overline{H}_y$$

Matrix Equation

In matrix form:

$$\frac{d}{d\bar{z}}\begin{bmatrix}E_x\\E_y\end{bmatrix}=\begin{bmatrix}\bar{K}_x\varepsilon_r^{-1}\bar{K}_y & \mu_r-\bar{K}_x\varepsilon_r^{-1}\bar{K}_x\\ \bar{K}_y\varepsilon_r^{-1}\bar{K}_y-\mu_r & -\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\end{bmatrix}\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=P\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}$$

Similarly:

$$\frac{d}{d\bar{z}}\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=\begin{bmatrix}\bar{K}_x\mu_r^{-1}\bar{K}_y & \varepsilon_r-\bar{K}_x\mu_r^{-1}\bar{K}_x\\ \bar{K}_y\mu_r^{-1}\bar{K}_y-\varepsilon_r & -\bar{K}_y\mu_r^{-1}\bar{K}_x\end{bmatrix}\begin{bmatrix}E_x\\E_y\end{bmatrix}=Q\begin{bmatrix}E_x\\E_y\end{bmatrix}$$

We have:

$$P=\begin{bmatrix}\bar{K}_x\varepsilon_r^{-1}\bar{K}_y & \mu_r-\bar{K}_x\varepsilon_r^{-1}\bar{K}_x\\ \bar{K}_y\varepsilon_r^{-1}\bar{K}_y-\mu_r & -\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\end{bmatrix},\qquad Q=\begin{bmatrix}\bar{K}_x\mu_r^{-1}\bar{K}_y & \varepsilon_r-\bar{K}_x\mu_r^{-1}\bar{K}_x\\ \bar{K}_y\mu_r^{-1}\bar{K}_y-\varepsilon_r & -\bar{K}_y\mu_r^{-1}\bar{K}_x\end{bmatrix}$$

And differential equation:

$$\frac{d}{d\bar{z}}\begin{bmatrix}E_x\\E_y\end{bmatrix}=P\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix},\qquad \frac{d}{d\bar{z}}\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=Q\begin{bmatrix}E_x\\E_y\end{bmatrix}$$

Solving:

$$\frac{d^2}{d\bar{z}^2}\begin{bmatrix}E_x\\E_y\end{bmatrix}=P\frac{d}{d\bar{z}}\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=PQ\begin{bmatrix}E_x\\E_y\end{bmatrix}=\Omega^2\begin{bmatrix}E_x\\E_y\end{bmatrix},\qquad \Omega^2=PQ$$

General Solution:

$$\begin{bmatrix}E_x\\E_y\end{bmatrix}=e^{\Omega\bar{z}}c=We^{\lambda z}W^{-1}c,\qquad \Omega=W\lambda W^{-1},\qquad \Omega^2=W\lambda^2W^{-1}$$

Where λ^2 is the eigenvalue matrix of Ω^2, W is the eigenvector matrix of Ω^2 , c is the coefficient.

Further, separate the solution into forward and backward waves:

$$\begin{bmatrix}E_x\\E_y\end{bmatrix}=We^{\lambda\bar{z}}c^{+}+We^{-\lambda\bar{z}}c^{-}$$

And assume a similar solution for magnetic field:

$$\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=Ve^{\lambda\bar{z}}c^{+}-Ve^{-\lambda\bar{z}}c^{-}$$

Recall $\frac{d}{d\bar{z}}\begin{bmatrix}\bar{H}_x\\ \bar{H}_y\end{bmatrix}=Q\begin{bmatrix}E_x\\E_y\end{bmatrix}$:

$$V\lambda e^{\lambda\bar{z}}c^{+}+V\lambda e^{-\lambda\bar{z}}c^{-}=QWe^{\lambda\bar{z}}c^{+}+QWe^{-\lambda\bar{z}}c^{-}$$

$$V\lambda=QW$$

$$V=QW\lambda^{-1}$$

Boundary Condition

We can formulate the solution:

$$\psi(\bar{z}) = \begin{bmatrix} E_x \\ E_y \\ \bar{H}_x \\ \bar{H}_y \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} e^{\lambda \bar{z}} & 0 \\ 0 & e^{-\lambda \bar{z}} \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

And start to match the boundary condition:

Assume a structure with three layers, and their eigenmodes are:

	Layer 1	Layer N	Layer 2
Eigen modes	$\begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix}$	$\begin{bmatrix} W & W \\ V & -V \end{bmatrix}$	$\begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix}$
Field coefficient	$\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$	$\begin{bmatrix} c^+ \\ c^- \end{bmatrix}$	$\begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$
Thickness	0	L	0

Between Layer 1 and Layer N, $\bar{z} = 0$

$$\begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

Between Layer N and Layer 2, $\bar{z} = k_0 z = k_0 L$

$$\begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} c^+ \\ c^- \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} \begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix}^{-1} \begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} \begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

The inverse of the matrix:

$$\begin{bmatrix} W & W \\ V & -V \end{bmatrix}^{-1} = \frac{\begin{bmatrix} -V & -W \\ -V & W \end{bmatrix}}{-2VW} = \frac{1}{2} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix}, \qquad \begin{bmatrix} e^{\lambda k_0 L} & 0 \\ 0 & e^{-\lambda k_0 L} \end{bmatrix}^{-1} = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix}$$

Verify:

$$\frac{1}{2} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W & W \\ V & -V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$$

Therefore:

$$\begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W_1 & W_1 \\ V_1 & -V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix} \begin{bmatrix} W^{-1} & V^{-1} \\ W^{-1} & -V^{-1} \end{bmatrix} \begin{bmatrix} W_2 & W_2 \\ V_2 & -V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

Scattering matrix

Simplify:

$$\begin{bmatrix} W^{-1}W_1 + V^{-1}V_1 & W^{-1}W_1 - V^{-1}V_1 \\ W^{-1}W_1 - V^{-1}V_1 & W^{-1}W_1 + V^{-1}V_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} e^{-\lambda k_0 L} & 0 \\ 0 & e^{\lambda k_0 L} \end{bmatrix} \begin{bmatrix} W^{-1}W_2 + V^{-1}V_2 & W^{-1}W_2 - V^{-1}V_2 \\ W^{-1}W_2 - V^{-1}V_2 & W^{-1}W_2 + V^{-1}V_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

Let

$$A_i = W^{-1}W_i + V^{-1}V_i$$

$$B_i = W^{-1}W_i - V^{-1}V_i$$

$$X = e^{\lambda k_0 L}$$

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix} = \begin{bmatrix} X^{-1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_2^- \end{bmatrix}$$

Write into the form of a scattering matrix:

$$\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}$$

Let $\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} 1 \\ S_{11} \end{bmatrix} = \begin{bmatrix} X^{-1} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} S_{21} \\ 0 \end{bmatrix}$$

1. From the matrix $A_1 + B_1 S_{11} = X^{-1} A_2 S_{21}$ $B_1 + A_1 S_{11} = X B_2 S_{21}$	2. Solve for S21 and S11 $S_{21} = A_2^{-1}\, X(A_1 + B_1 S_{11})$ $S_{11} = A_1^{-1}(X B_2 S_{21} - B_1)$
3. Substitute back into 1 $A_1 + B_1 A_1^{-1}(X B_2 S_{21} - B_1) = X^{-1} A_2 S_{21}$ $B_1 + A_1 S_{11} = X B_2 A_2^{-1}\, X(A_1 + B_1 S_{11})$	4. Rearrange, bringing out S21 and S11 $(X^{-1}A_2 - B_1 A_1^{-1} X B_2) S_{21} = A_1 - B_1 A_1^{-1} B_1$ $(A_1 - X B_2 A_2^{-1}\, X B_1) S_{11} = X B_2 A_2^{-1}\, X A_1 - B_1$

We have:

$$S_{11} = (A_1 - X B_2 A_2^{-1}\, X B_1)^{-1} (X B_2 A_2^{-1}\, X A_1 - B_1)$$

$$S_{21} = (A_2 - X B_1 A_1^{-1} X B_2)^{-1} X (A_1 - B_1 A_1^{-1} B_1)$$

Similarly:

$$S_{22} = (A_2 - X B_1 A_1^{-1}\, X B_2)^{-1} (X B_1 A_1^{-1}\, X A_2 - B_2)$$

$$S_{12} = (A_1 - X B_1 A_1^{-1} X B_2)^{-1} X (A_2 - B_2 A_2^{-1} B_2)$$

Summary:

$$A_i = W^{-1}W_i + V^{-1}V_i$$

$$B_i = W^{-1}W_i - V^{-1}V_i$$

$$X = e^{\lambda k_0 L}$$

$$S_{11} = (A_1 - XB_2A_2^{-1}\,XB_1)^{-1}(XB_2A_2^{-1}\,XA_1 - B_1)$$

$$S_{12} = (A_1 - XB_1A_1^{-1}XB_2)^{-1}X(A_2 - B_2A_2^{-1}B_2)$$

$$S_{21} = (A_2 - XB_1A_1^{-1}XB_2)^{-1}X(A_1 - B_1A_1^{-1}B_1)$$

$$S_{22} = (A_2 - XB_1A_1^{-1}\,XB_2)^{-1}(XB_1A_1^{-1}\,XA_2 - B_2)$$

When $A_1 = A_2 = A$, $B_1 = B_2 = B$:

$$S_{11} = (A - XBA^{-1}\,XB)^{-1}(XBA^{-1}\,XA - B)$$

$$S_{12} = (A - XBA^{-1}XB)^{-1}X(A - BA^{-1}B)$$

$$S_{21} = S_{12}$$

$$S_{22} = S_{11}$$

Reflection side, when $W = W_1, V = V_1, X = 1 \rightarrow A_1 = 1 + 1 = 2, B_1 = 1 - 1 = 0$

$$S_{11} = (2 - 0)^{-1}(BA^{-1}\,2 - 0) = BA^{-1}$$

$$S_{12} = (2 - 0)^{-1}(A - BA^{-1}B) = 0.5(A - BA^{-1}B)$$

$$S_{21} = (A - 0)^{-1}(2 - 0) = 2A^{-1}$$

$$S_{22} = (A - 0)^{-1}(0 - B) = -A^{-1}B$$

Transmision side, when $W = W_2, V = V_2, X = 1 \rightarrow A_2 = 1 + 1 = 2, B_2 = 1 - 1 = 0$

$$S_{11} = (A - 0)^{-1}(0 - B) = -A^{-1}B$$

$$S_{12} = (A - 0)^{-1}(2 - 0) = 2A^{-1}$$

$$S_{21} = (2 - 0)^{-1}(A - BA^{-1}B) = 0.5(A - BA^{-1}B)$$

$$S_{22} = (2 - 0)^{-1}(BA^{-1}\,2 - 0) = BA^{-1}$$

Scattering matrix operation

Assume a structure with three layers connected by two scattering matrices, A and B

$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
$\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$	$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$	$\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$	$\begin{bmatrix} c_1^+ \\ c_1^- \end{bmatrix}$
$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$

$$\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}, \qquad \begin{bmatrix} c_2^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_3^- \end{bmatrix}$$

Combine into one scattering matrix:

$$\begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix}$$

Case 1: $\begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$

$$\begin{bmatrix} S_{11} \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 \\ c_2^- \end{bmatrix}, \qquad \begin{bmatrix} c_2^- \\ S_{21} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ 0 \end{bmatrix}$$

1. From matrix A $S_{11} = A_{11} + A_{12}c_2^- \ldots (A1)$ $c_2^+ = A_{21} + A_{22}c_2^- \ldots (A2)$	2. From matrix B $c_2^- = B_{11}c_2^+ \ldots (B1)$ $S_{21} = B_{21}c_2^+ \ldots (B2)$
3. Substitute A2 and B1: $c_2^+ = A_{21} + A_{22}B_{11}c_2^+$ $c_2^- = B_{11}(A_{21} + A_{22}c_2^-)$	4. Solve for C2: $c_2^+ = (I - A_{22}B_{11})^{-1}A_{21}$ $c_2^- = (I - B_{11}A_{22})^{-1}B_{11}A_{21}$

$$S_{11} = A_{11} + A_{12}(I - B_{11}A_{22})^{-1}B_{11}A_{21}$$

$$S_{21} = B_{21}(I - A_{22}B_{11})^{-1}A_{21}$$

Case 2: $\begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$

$$\begin{bmatrix} S_{12} \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ c_2^- \end{bmatrix}, \qquad \begin{bmatrix} c_2^- \\ S_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ 1 \end{bmatrix}$$

1. From matrix A $S_{12} = A_{12}c_2^- \ldots (A1)$ $c_2^+ = A_{22}c_2^- \ldots (A2)$	2. From matrix B $c_2^- = B_{11}c_2^+ + B_{12} \ldots (B1)$ $S_{22} = B_{21}c_2^+ + B_{22} \ldots (B2)$
3. Substitute A2 and B1: $c_2^+ = A_{22}(B_{21}c_2^+ + B_{12})$ $c_2^- = B_{11}A_{22}c_2^- + B_{12}$	4. Solve for C2: $c_2^+ = (I - A_{22}B_{11})^{-1}A_{22}B_{12}$ $c_2^- = (I - B_{11}A_{22})^{-1}B_{12}$

$$S_{12} = A_{12}(I - B_{11}A_{22})^{-1}B_{12}$$

$$S_{22} = B_{21}(I - A_{22}B_{11})^{-1}A_{22}B_{12} + B_{22}$$

Summary:

$$\begin{bmatrix} c_1^- \\ c_2^+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_2^- \end{bmatrix}, \qquad \begin{bmatrix} c_2^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} c_2^+ \\ c_3^- \end{bmatrix}, \qquad \begin{bmatrix} c_1^- \\ c_3^+ \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} c_1^+ \\ c_3^- \end{bmatrix}$$

$$S_{11} = A_{11} + A_{12}(I - B_{11}A_{22})^{-1}B_{11}A_{21}$$

$$S_{12} = A_{12}(I - B_{11}A_{22})^{-1}B_{12}$$

$$S_{21} = B_{21}(I - A_{22}B_{11})^{-1}A_{21}$$

$$S_{22} = B_{21}(I - A_{22}B_{11})^{-1}A_{22}B_{12} + B_{22}$$

Define as $A \otimes B = S$, called the Redheffer Star Product.

Final solution

$$\begin{bmatrix} C_R \\ C_T \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} C_I \\ 0 \end{bmatrix}, \qquad C_I = W_R^{-1}E_i, \qquad E_R = W_R C_R, \qquad E_T = W_T C_T$$

$$E_R = W_R S_{11} W_R^{-1} E_i$$

$$E_T = W_T S_{21} W_R^{-1} E_i$$

Finding z component:

$$\bar{K} \cdot E = 0 \rightarrow k_x E_x + k_y E_y + k_z E_Z = 0$$

$$E_Z = -\frac{k_x E_x + k_y E_y}{k_z}, \qquad H_Z = -\frac{k_x H_x + k_y H_y}{k_z}$$

Power:

$$|E| = E_x^2 + E_y^2 + E_Z^2$$

$$P = |E \times H| = \frac{|E|^2}{\eta} = |E|^2 \sqrt{\frac{\varepsilon}{\mu}}$$

Power Ratio

$$\frac{P_2 \cos\theta_2}{P_1 \cos\theta_1} = \frac{|E_2|^2}{|E_1|^2} \frac{\sqrt{\frac{\varepsilon_2}{\mu_2}} \cos\theta_2}{\sqrt{\frac{\varepsilon_1}{\mu_1}} \cos\theta_1} = \frac{|E_2|^2}{|E_1|^2} \frac{{\mu_r}_2^{-1}}{{\mu_r}_1^{-1}} \frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} = \frac{|E_2|^2}{|E_1|^2} \frac{{\mu_r}_2^{-1}}{{\mu_r}_1^{-1}} \frac{Re({k_z}_2)}{Re({k_z}_1)}$$

Special Case: No Magnetic

$$\Omega^2 = PQ = \begin{bmatrix} \bar{K}_x\varepsilon_r^{-1}\bar{K}_y & I - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x \\ \bar{K}_y\varepsilon_r^{-1}\bar{K}_y - I & -\bar{K}_y\varepsilon_r^{-1}\bar{K}_x \end{bmatrix}\begin{bmatrix} \bar{K}_x\bar{K}_y & \varepsilon_r - \bar{K}_x^2 \\ \bar{K}_y^2 - \varepsilon_r & -\bar{K}_y\bar{K}_x \end{bmatrix}$$

$$\Omega_{11}^2 = \left(\bar{K}_x\varepsilon_r^{-1}\bar{K}_y\right)\left(\bar{K}_x\bar{K}_y\right) + (I - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x)\left(\bar{K}_y^2 - \varepsilon_r\right)$$

$$= \bar{K}_x\varepsilon_r^{-1}\bar{K}_x\bar{K}_y^2 + \bar{K}_y^2 - \varepsilon_r - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x\bar{K}_y^2 + \bar{K}_x\varepsilon_r^{-1}\bar{K}_x\varepsilon_r = \bar{K}_y^2 + (\bar{K}_x\varepsilon_r^{-1}\bar{K}_x - I)\varepsilon_r$$

$$\Omega_{12}^2 = \left(\bar{K}_x\varepsilon_r^{-1}\bar{K}_y\right)(\varepsilon_r - \bar{K}_x^2) + (I - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x)\left(-\bar{K}_y\bar{K}_x\right)$$

$$= \bar{K}_x\varepsilon_r^{-1}\bar{K}_y\varepsilon_r - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x^2\bar{K}_y - \bar{K}_x\bar{K}_y + \bar{K}_x\varepsilon_r^{-1}\bar{K}_x^2\bar{K}_y = \bar{K}_x\varepsilon_r^{-1}\bar{K}_y\varepsilon_r - \bar{K}_x\bar{K}_y$$

$$\Omega_{21}^2 = \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - I\right)\left(\bar{K}_x\bar{K}_y\right) + \left(-\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\right)\left(\bar{K}_y^2 - \varepsilon_r\right)$$

$$= \bar{K}_y\varepsilon_r^{-1}\bar{K}_x\bar{K}_y^2 - \bar{K}_x\bar{K}_y - \bar{K}_y\varepsilon_r^{-1}\bar{K}_x\bar{K}_y^2 + \bar{K}_y\varepsilon_r^{-1}\bar{K}_x\varepsilon_r = \bar{K}_y\varepsilon_r^{-1}\bar{K}_x\varepsilon_r - \bar{K}_x\bar{K}_y$$

$$\Omega_{22}^2 = \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - I\right)(\varepsilon_r - \bar{K}_x^2) + \left(-\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\right)\left(-\bar{K}_y\bar{K}_x\right)$$

$$= \bar{K}_y\varepsilon_r^{-1}\bar{K}_y\varepsilon_r - \bar{K}_y\varepsilon_r^{-1}\bar{K}_x^2\bar{K}_y - \varepsilon_r + \bar{K}_x^2 + \bar{K}_y\varepsilon_r^{-1}\bar{K}_x^2\bar{K}_y = \bar{K}_x^2 + \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - I\right)\varepsilon_r$$

$$\Omega^2 = \begin{bmatrix} \bar{K}_y^2 + (\bar{K}_x\varepsilon_r^{-1}\bar{K}_x - I)\varepsilon_r & \bar{K}_x\varepsilon_r^{-1}\bar{K}_y\varepsilon_r - \bar{K}_x\bar{K}_y \\ \bar{K}_y\varepsilon_r^{-1}\bar{K}_x\varepsilon_r - \bar{K}_x\bar{K}_y & \bar{K}_x^2 + \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - I\right)\varepsilon_r \end{bmatrix}$$

Special Case: Homogeneous Layer

$$\Omega^2 = PQ = \begin{bmatrix} \bar{K}_x\varepsilon_r^{-1}\bar{K}_y & \mu_r - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x \\ \bar{K}_y\varepsilon_r^{-1}\bar{K}_y - \mu_r & -\bar{K}_y\varepsilon_r^{-1}\bar{K}_x \end{bmatrix}\begin{bmatrix} \bar{K}_x\mu_r^{-1}\bar{K}_y & \varepsilon_r - \bar{K}_x\mu_r^{-1}\bar{K}_x \\ \bar{K}_y\mu_r^{-1}\bar{K}_y - \varepsilon_r & -\bar{K}_y\mu_r^{-1}\bar{K}_x \end{bmatrix}$$

$$\Omega_{11}^2 = \left(\bar{K}_x\varepsilon_r^{-1}\bar{K}_y\right)\left(\bar{K}_x\mu_r^{-1}\bar{K}_y\right) + (\mu_r - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x)\left(\bar{K}_y\mu_r^{-1}\bar{K}_y - \varepsilon_r\right)$$

$$= \bar{K}_x^2\bar{K}_y^2\varepsilon_r^{-1}\mu_r^{-1} + \bar{K}_y^2 - \mu_r\varepsilon_r - \bar{K}_x^2\bar{K}_y^2\varepsilon_r^{-1}\mu_r^{-1} + \bar{K}_x^2 = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r$$

$$\Omega_{12}^2 = \left(\bar{K}_x\varepsilon_r^{-1}\bar{K}_y\right)(\varepsilon_r - \bar{K}_x\mu_r^{-1}\bar{K}_x) + (\mu_r - \bar{K}_x\varepsilon_r^{-1}\bar{K}_x)\left(-\bar{K}_y\mu_r^{-1}\bar{K}_x\right)$$

$$= \bar{K}_x\bar{K}_y - \bar{K}_x^3\bar{K}_y^1\varepsilon_r^{-1}\mu_r^{-1} - \bar{K}_x\bar{K}_y + \bar{K}_x^3\bar{K}_y^1\varepsilon_r^{-1}\mu_r^{-1} = 0$$

$$\Omega_{21}^2 = \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - \mu_r\right)\left(\bar{K}_x\mu_r^{-1}\bar{K}_y\right) + \left(-\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\right)\left(\bar{K}_y\mu_r^{-1}\bar{K}_y - \varepsilon_r\right)$$

$$= \bar{K}_x^1\bar{K}_y^3\varepsilon_r^{-1}\mu_r^{-1} - \bar{K}_x\bar{K}_y - \bar{K}_x^1\bar{K}_y^3\varepsilon_r^{-1}\mu_r^{-1} + \bar{K}_x\bar{K}_y = 0$$

$$\Omega_{22}^2 = \left(\bar{K}_y\varepsilon_r^{-1}\bar{K}_y - \mu_r\right)(\varepsilon_r - \bar{K}_x\mu_r^{-1}\bar{K}_x) + \left(-\bar{K}_y\varepsilon_r^{-1}\bar{K}_x\right)\left(-\bar{K}_y\mu_r^{-1}\bar{K}_x\right)$$

$$= \bar{K}_y^2 - \bar{K}_x^2\bar{K}_y^2\varepsilon_r^{-1}\mu_r^{-1} - \mu_r\varepsilon_r + \bar{K}_x^2 + \bar{K}_x^2\bar{K}_y^2\varepsilon_r^{-1}\mu_r^{-1} = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r$$

$$\Omega^2 = \begin{bmatrix} \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r & 0 \\ 0 & \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r \end{bmatrix}$$

Since it is a diagonal matrix, it's already diagonalized.

$$\Omega^2 = \begin{bmatrix} \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r & 0 \\ 0 & \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r \end{bmatrix} = \lambda^2, \qquad W = W^{-1} = I$$

Interesting fact:

$$\lambda^2 = \bar{K}_x^2 + \bar{K}_y^2 - \mu_r\varepsilon_r, \qquad \lambda = \sqrt{\bar{K}_x^2 + \bar{K}_y^2 - n^2} = i\sqrt{n^2 - \bar{K}_x^2 - \bar{K}_y^2} = i\bar{K}_z$$

Recall solution for the electric field is:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = We^{\lambda\bar{z}}c^+ + We^{-\lambda\bar{z}}c^- = e^{iK_z z}c^+ + e^{-iK_z z}c^-$$

Note that the relation between E and H is $\frac{\bar{K}}{|\bar{K}|} \times E = \eta H = i\frac{\eta}{\eta_0}\bar{H} = i\sqrt{\frac{\mu_r}{\varepsilon_r}}\bar{H}$, $|\bar{K}| = n = \sqrt{\mu_r\varepsilon_r}$

$$\frac{1}{\sqrt{\mu_r\varepsilon_r}}\bar{K} \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \bar{K}_x & \bar{K}_y & \bar{K}_z \\ E_x & E_y & E_z \end{vmatrix} = \frac{1}{\sqrt{\mu_r\varepsilon_r}}\left(\bar{K}_y E_z - \bar{K}_z E_y, \bar{K}_z E_x - \bar{K}_x E_z, \bar{K}_x E_y - \bar{K}_y E_x\right)$$

$$E_Z = -\frac{k_x E_x + k_y E_y}{k_z}$$

$$\bar{H} = -i\sqrt{\frac{\mu_r}{\varepsilon_r}}\frac{1}{\sqrt{\mu_r\varepsilon_r}}\left(-\bar{K}_y\frac{\bar{K}_x E_x + \bar{K}_y E_y}{\bar{K}_z} - \bar{K}_z E_y, \bar{K}_z E_x + \bar{K}_x\frac{\bar{K}_x E_x + \bar{K}_y E_y}{\bar{K}_z}, \bar{K}_x E_y - \bar{K}_y E_x\right)$$

$$V = -i\mu_r^{-1}\begin{bmatrix} \dfrac{\bar{K}_y\bar{K}_x}{\bar{K}_z} & \dfrac{\bar{K}_y^2}{\bar{K}_z} - \bar{K}_z \\ -\bar{K}_z - \dfrac{\bar{K}_x^2}{\bar{K}_z} & -\dfrac{\bar{K}_x\bar{K}_y}{\bar{K}_z} \end{bmatrix} = -i\mu_r^{-1}\begin{bmatrix} \bar{K}_y\bar{K}_x & \bar{K}_y^2 + \bar{K}_z^2 \\ -\bar{K}_z^2 - \bar{K}_x^2 & -\bar{K}_x\bar{K}_y \end{bmatrix}\begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix}$$

$$= -i\mu_r^{-1}\begin{bmatrix} \bar{K}_y\bar{K}_x & n^2 - \bar{K}_x^2 \\ \bar{K}_y^2 - n^2 & -\bar{K}_x\bar{K}_y \end{bmatrix}\begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix}$$

Also, for magnetic, the other solution matches the result given above:

$$V = QW\lambda^{-1} = Q\lambda^{-1}$$

$$= \begin{bmatrix} \bar{K}_x\mu_r^{-1}\bar{K}_y & \varepsilon_r - \bar{K}_x\mu_r^{-1}\bar{K}_x \\ \bar{K}_y\mu_r^{-1}\bar{K}_y - \varepsilon_r & -\bar{K}_y\mu_r^{-1}\bar{K}_x \end{bmatrix}\begin{bmatrix} i\bar{K}_z & 0 \\ 0 & i\bar{K}_z \end{bmatrix}^{-1} = -i\mu_r^{-1}\begin{bmatrix} \bar{K}_x\bar{K}_y & n^2 - \bar{K}_x^2 \\ \bar{K}_y^2 - n^2 & -\bar{K}_y\bar{K}_x \end{bmatrix}\begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix}$$

In free space:

$$V = -i\begin{bmatrix} \bar{K}_x\bar{K}_y & I - \bar{K}_x^2 \\ \bar{K}_y^2 - I & -\bar{K}_y\bar{K}_x \end{bmatrix}\begin{bmatrix} \bar{K}_z^{-1} & 0 \\ 0 & \bar{K}_z^{-1} \end{bmatrix}$$