

Best-Response Learning in Budgeted α -Fair Kelly Mechanisms

Cleque Marlain Mboulou-Moutoubi¹, Younes Ben Mazziane¹
Francesco De Pellegrini¹ and Eitan Altman^{1,2*}

¹LIA, Avignon university, Avignon, France

²INRIA, Sophia Antipolis, France.

Abstract. The Kelly mechanism is a proportional allocation auction widely adopted in decentralized resource allocation systems to share an infinitely divisible resource among competing agents. We analyze the sequential game it induces when agents have α -fair utilities and behave strategically. Our main result proves that synchronous best-response updates drive bids to the unique Nash equilibrium at a linear rate for $\alpha \in \{0, 1, 2\}$. Extensive simulations reveal that best-response dynamics reach equilibrium significantly faster than previously proposed no-regret learning algorithms.

Keywords: decentralized resource allocation, auctions, game theory, Kelly mechanism, α -fair allocation.

1 Introduction

Decentralized resource allocation problems arise in edge computing, cloud platforms, and communication networks [1–3]. Kelly [4] proposed a proportional allocation scheme in which each agent receives a share of the resource proportional to their bid. In that seminal work, the author made the explicit example of proportional fairness type of utilities. Later on, many extensions of the Kelly mechanism adopted the α -fair utility framework [5–7]. This class of utility functions is widely studied as they capture a range of trade-offs between fairness and efficiency: from efficiency-maximizing behavior ($\alpha = 0$), to proportional fairness ($\alpha = 1$), and approaching max-min fairness as $\alpha \rightarrow \infty$. α -fair utilities of the type studied in this work have since become a standard in modeling user preferences in network economics and decentralized optimization [7, 8].

Johari and Tsitsiklis [9] proved that the game induced by the Kelly allocation mechanism admits a unique Nash equilibrium when utilities are smooth and concave, and that the social welfare at this equilibrium is at least 3/4 of the optimal. These guarantees, however, presuppose that agents can play the NE, which requires complete knowledge of every player’s utility by all players. In this work we part from this setting and we assume instead that each agent knows

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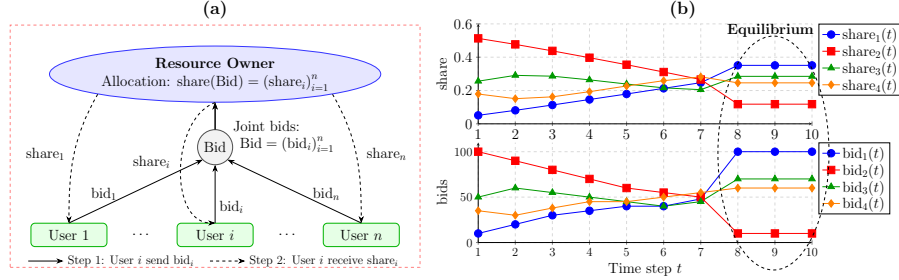


Fig. 1: From the static to the Sequential Kelly mechanism.

only their own utility. But, they can adapt their bid over repeated, synchronous rounds using limited feedback, e.g., the aggregate bid across all players.

Generally, in repeated games, the behavior of selfish agents is captured according to two main models, namely *best-response dynamics* (BRD) or *no-regret* learning algorithms. In best-response dynamics (BRD), each player exploits priors on opponents' actions and determines the best response action to be played at the next step accordingly [10]. No-regret learning algorithms are state of the art solutions for online convex optimization problems [11].

Mertikopoulos et al. [12] showed that Dual Averaging (DA) updates—a class of no-regret learning algorithms—converge to an NE whenever the underlying static game is *variationally stable*. Unfortunately, verifying variational stability is hard; it is known only for special cases of the Kelly game with linear utilities [13] or logarithmic utilities [14]. A further drawback of DA-based bidding schemes for the Kelly game is the lack of provable convergence *rates*. If DA converges slowly and the horizon is limited, agents may not reach the NE, undermining the social welfare guarantees that hold only at equilibrium.

Contributions. We show that if agents follow a simple synchronous best-response rule that uses only the previous round's aggregate bid, the system converges to the unique NE in *linear* time, provided that minimum-bid constraints are satisfied and agents utilities are of the α -fair form in the allocated resource for $\alpha \in \{0, 1, 2\}$. To our knowledge, this is the first linear-time convergence guarantee for the repeated Kelly game. Extensive simulations confirm that best-response dynamics reach equilibrium substantially faster than prior DA-based algorithms [13, 14], preserving the welfare guarantees even in short horizons.

2 Problem formulation

We consider a decentralized setting in which a unit-sized, divisible resource must be allocated among n strategic agents. These agents interact repeatedly in T rounds, each submitting a bid to request a share of the resource (see Fig. 1).

At each round, the resource owner assigns a share of the resource according to the Kelly mechanism. This mechanism extends the classical Kelly network op-

timization framework [4] by incorporating general pricing and utility functions, as in [15]. It allows for general α -fair utilities and non-linear bid-to-payment mappings.

In the following, we define the corresponding game \mathcal{G}_α , namely the sequential Kelly mechanism with α -fair utility under budget constraints. We detail the players' action set, i.e., their bids, the allocation rule, i.e., the fraction of resources the resource owner assigns to each player given the bid vector, and the players' resulting utility. Each agent aims to maximize their own utility by signaling their request to the resource owner, subject to a budget constraint.

Bidding and Allocation.

In the basic formulation of the Kelly mechanism [9], each user i submits a bid z_i to the resource owner; we assume $z_i > 0$. Let's denote by $\mathbf{z} = (z_1, \dots, z_n)$ the profile of bids across all agents.

Given a strategy profile \mathbf{z} and a system reservation parameter $\delta \geq 0$, the mechanism allocates to each agent i a fraction of the unit resource as follows:

$$x_i(\mathbf{z}) = \begin{cases} \frac{z_i}{\sum_{j=1}^n z_j + \delta}, & \text{if } z_i > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The corresponding payment function is $p_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ defined as $p_i(z_i)$, and the budget constraint imposes $p_i(z_i) \leq c_i$, where $c_i > 0$ is agent i 's private budget. Finally, each agent chooses their bid from the constrained strategy space:

$$\mathcal{R}_i = \{z_i \geq 0 \mid \epsilon_i \leq p_i(z_i) \leq c_i\}, \quad (2)$$

The strategy space of the game \mathcal{G}_α is thus $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$.

Utility Function. From their assigned resource fraction $x_i = x_i(\mathbf{z})$, agent i obtains α -fair valuation function $V_i(\cdot)$

$$V_i(\mathbf{z}) = \begin{cases} \frac{x_i(\mathbf{z})^{1-\alpha}}{1-\alpha}, & \text{if } \alpha \neq 1, \\ \log(x_i(\mathbf{z})), & \text{if } \alpha = 1, \end{cases} \quad (3)$$

Also, each agent is assigned a valuation weight or scaling factor $a_i > 0$ which reflects the worth assigned by agent i for each unit of received resource. In the rest of the paper, we consider a linear cost model $p_i(z_i) = \lambda z_i$, where λ denotes the resource price determined by the resource owner. Consequently, their total payoff writes

$$\varphi_i(\mathbf{z}) = a_i V_i(\mathbf{z}) - \lambda z_i, \quad \forall i \in [n] \quad (4)$$

Let $\mathbf{z}_{-i} = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$ be the action of all players except i .

Definition 1 (Nash Equilibrium). A strategy profile $\mathbf{z}^* = (z_1^*, z_2^*, \dots, z_n^*) \in \mathcal{R}$ is a Nash Equilibrium (NE) if, for every player $i \in [n]$,

$$\varphi_i(z_i^*, \mathbf{z}_{-i}^*) \geq \varphi_i(z_i, \mathbf{z}_{-i}^*) \quad \forall z_i \in \mathcal{R}_i \quad (5)$$

We observe that the function φ_i is concave in its i -th component and twice continuously differentiable on $\mathbb{R}_{>0}^n$, and the actions set \mathcal{R} is non empty, closed, bounded, and convex. Thus, existence of a Nash Equilibrium (NE) of the game \mathcal{G}_α follows by [16, Thm. 1]. The minimum bid can be seen as a way to impose a reservation $\delta = \sum \epsilon_i$ and enforce minimal participation from all agents.

The next section presents the Best Response Dynamics (BRD), a simple action update rule that leads to equilibrium, with theoretical convergence guarantees in \mathcal{G}_α .

3 Main Results

We study a system where each agent updates their strategy according to a Best-Response Dynamic (BRD) based on the latest actions of the other players. Formally, in a game with n players and strategy profile $\mathbf{z} \in \mathcal{R}$, the best response $\text{BR}_i : \mathbb{R}_+^{n-1} \mapsto \mathbb{R}_+$ of agent i is defined as:

$$\text{BR}_i(\mathbf{z}_{-i}) \in \arg \max_{z_i \in \mathcal{R}_i} \phi_i(z_i, \mathbf{z}_{-i}) \quad (6)$$

In \mathcal{G}_α , the utility function (4) of each agent depends only on their own bid and the aggregate bid of the others, making it an *aggregative game*. Consequently, the best-response update simplifies and depends only on the aggregate bid of the other agents, defined as $s_{-i} = \sum_{j \neq i} z_j + \delta$, so that $\text{BR}_i(\mathbf{z}_{-i}) \triangleq \text{BR}_i(s_{-i})$. Since \mathcal{G}_α is also a concave game, the best response is unique [17]:

$$\text{BR}_i(s_{-i}) = \arg \max_{z_i \in \mathcal{R}_i} \varphi_i(z_i, \mathbf{z}_{-i}). \quad (7)$$

We study the Synchronous Best Response Dynamic (SBRD), in which all agents simultaneously update their actions based on their best responses. The pseudocode is reported in Alg. 1.

Algorithm 1 Synchronous Best Response Dynamic (SBRD)

- 1: **Input:** Initial bids $\{z_i(0)\}_{i=1}^n$, number of iterations T .
 - 2: **Output:** Bid trajectory $\{\mathbf{z}(t)\}_{t=1}^T$.
 - 3: **for** $t = 1$ to T **do**
 - 4: **for each** bidder $i \in [n]$ **simultaneously do**
 - 5: $s_{-i}(t-1) \leftarrow \sum_{j \neq i} z_j(t-1) + \delta$
 - 6: $z_i(t) \leftarrow \text{BR}_i(s_{-i}(t-1))$ ▷ See Eq. 9
 - 7: **end for**
 - 8: **end for**
-

Now, we analyze the convergence of SBRD(1) in the setting of \mathcal{G}_α . The best response $\text{BR}_i(s_{-i})$ of agent i , given the aggregate bid s_{-i} , is obtained by solving the first-order optimality condition $\frac{\partial \varphi_i(z_i, \mathbf{z}_{-i})}{\partial z_i} = 0$, and projecting the

solution $\widetilde{\text{BR}}_i(s_{-i})$ onto the feasible set \mathcal{R}_i : $\text{BR}_i(s_{-i}) = \Pi_{\mathcal{R}_i}(\widetilde{\text{BR}}_i(s_{-i}))$, where $\Pi_{\mathcal{R}_i}(x) = \min \left\{ \max \left\{ x, \frac{\epsilon_i}{\lambda} \right\}, \frac{c_i}{\lambda} \right\}$. For $\alpha = 0$ the calculation is straightforward. For $\alpha > 0$, this condition yields a nonlinear equation that depends of the fairness parameter α and can be reformulated as:

$$\phi_{s_{-i}}(z_i, \alpha) = \underbrace{\left(\frac{z_i}{z_i + s_{-i}} \right)^\alpha}_{\phi_{s_{-i}}^1(z_i, \alpha)} - \underbrace{\frac{a_i s_{-i}}{\lambda(z_i + s_{-i})^2}}_{\phi_{s_{-i}}^2(z_i)} = 0 \quad (8)$$

Where the solution $\widetilde{\text{BR}}_i(s_{-i})$ is given by the intersection of the functions $\phi_{s_{-i}}^1(\cdot, \alpha)$ and $\phi_{s_{-i}}^2(\cdot)$. However, solving this nonlinear equation for general $\alpha > 0$ is analytically intractable. Therefore, we derive explicit solutions for three key and representative cases $\alpha = 0, 1$, and 2 corresponding respectively to *efficiency maximization*, *proportional fairness*, and *minimum potential delay fairness* [18], and analyze how the solutions depend on agents' valuations and budgets. We identify further conditions to ensure convergence of SBRDs to a unique NE.

Lemma 1 (Best Response Operator). *Consider the game \mathcal{G}_α in the strategy space \mathcal{R} . The best response of player i to the strategy profile of the others, denoted $\text{BR}_i(s_{-i})$, is defined as*

$$\text{BR}_i(s_{-i}) = \Pi_{\mathcal{R}_i}(\widetilde{\text{BR}}_i(s_{-i})) \quad (9)$$

where $\widetilde{\text{BR}}_i(s_{-i})$ depends on the fairness parameter α . In particular, it holds

$$\begin{aligned} i. \quad \widetilde{\text{BR}}_i(s_{-i}) &= \sqrt{\frac{a_i s_{-i}}{\lambda}} - s_{-i} \quad \text{if } \alpha = 0; & iii. \quad \widetilde{\text{BR}}_i(s_{-i}) &= \sqrt{\frac{a_i s_{-i}}{\lambda}} \quad \text{if } \alpha = 2 \\ ii. \quad \widetilde{\text{BR}}_i(s_{-i}) &= \frac{-s_{-i} + \sqrt{s_{-i}^2 + \frac{4a_i s_{-i}}{\lambda}}}{2} \quad \text{if } \alpha = 1 \end{aligned}$$

The explicit form of $\widetilde{\text{BR}}_i(s_{-i})$ is obtained by solving Equation (8) given α . Here we state the convergence properties of the SBRD in \mathcal{G}_α , under the representative values of α :

Theorem 1 (SBRD Convergence). *In the game \mathcal{G}_α , the SBRD converges to the unique Nash equilibrium \mathbf{z}^* of the Kelly mechanism if the minimum bid ϵ satisfies:*

1. For $\alpha = 0$ and $n = 2$ if : $\epsilon > \frac{\max(a_1, a_2)}{16\lambda} - \delta$
2. For $\alpha = 1$ if : $\epsilon > \frac{1}{\lambda\sqrt{n(n-1)}} (n - 2\sqrt{n} + 1) a_i^{\max} - \frac{\delta}{n-1}$
3. for $\alpha = 2$ if : $\epsilon > \frac{1}{4\lambda(n-1)} \left(\max_j \sum_{i \neq j} \sqrt{a_i} \right)^2 - \frac{\delta}{n-1}$

Moreover, the SBRD converges linearly to the Nash equilibrium of game \mathcal{G}_α .

The idea for the proof, reported in Appendix 6 (see Proof of Theorem 1), is that the best-response operator 9 is in fact a contraction. More precisely, we

prove that it is a contraction under the $\|\cdot\|_\infty$ norm for $\alpha = 0$ and $\alpha = 1$, whereas it is a contraction under $\|\cdot\|_1$ norm for $\alpha = 0$ and $\alpha = 2$. This is sufficient to grant convergence based on Banach's Fixed Point Theorem. Incidentally, this also grants the uniqueness of the Nash equilibrium, i.e., of the fixed point. The contraction argument used in the proof also brings the speed of convergence of the best-response dynamics, namely: **Remarks:** (a) Theorem 1 requires $\epsilon = o(1/n)$ for $\alpha = 1$. In practice, as shown in the next section, ϵ can be set to a small value for a small number of players without hindering the convergence of the SBRD. (b) Classical results on the convergence of the BRD rely on the existence of a potential [19]. However, for the Kelly mechanism, it can be shown that a potential does exist if and only if $\alpha = 1$ (we omit here the proof for the space's sake). Nevertheless, in our framework, it is immediate to see that the *unilateral* best-response dynamics also converge, under the same assumptions.

4 Numerical simulations

In this section we compare convergence properties of the SBRD with two established decentralized learning algorithms adapted to the Kelly mechanism. The comparison is performed for the three reference fairness regimes, namely $\alpha = 0, 1, 2$. The two algorithms are the following ones.

- (i) **Dual Averaging with Quadratic Regularizer (DAQ)** [14, 20], each player i maintains a *cumulative discounted gradient* of their utility $g_i(t)$, define as: $g_i(t) = \sum_{k=0}^{t-1} \eta_k \partial_i \varphi_i(z_i(k))$, with η_k being the step size at time k . The bid $z_i(t)$ is updated at time t by projecting $g_i(t)$ using $\Pi_{\mathcal{R}_i}$;
- (ii) **Exponential Learning (XL)** [13], where each player i uses $g_i(t)$ to update their bid via a sigmoid mapping projected on action space: $z_i(t) = \Pi_{\mathcal{R}_i}(c_i \sigma(g_i(t)))$ with $\sigma(x) = \frac{1}{1+\exp(-x)}$ and c_i is the budget.

Convergence to the unique Nash equilibrium in both DAQ and XL is ensured under DSC condition [16, Thm. 2], already proved for the case $\alpha = 1$ for DAQ (see [14, Lma. 2]) and for $\alpha = 0$ for XL [13, Thm. 2]. The step size η_t is often selected such that $\sum_{t=1}^{\infty} \eta_t = +\infty$ and $\sum_{t=1}^{\infty} \eta_t^2 < +\infty$. However, we use a fixed step-size η_t to speed up the convergence of those algorithms [13, 14, 20].

Convergence is measured by tracking the best response residual $\|\text{BR}(\mathbf{z}(t)) - \mathbf{z}(t)\|$, that is the distance of the current multi-strategy vector from the NE. In all tests, convergence is attained for a best response residual of 10^{-5} .

The experiments compare the convergence trajectories of the three methods, focusing on how the fairness parameter α and the update strategy affect convergence speed and stability in a game with heterogeneous players.

The game consists of $n \geq 2$ agents, $i = 1, 2, \dots, n$ with utility function as in (4), where $a_i = 100 \cdot i^{-\gamma}$ and a fixed price $\lambda = 1$. We note that increasing γ introduces larger heterogeneity among players; $\gamma = 0$ represents homogeneity. Each agent has a limited budget $c_i = 4 \times 10^3$. The minimum bid is set to $\epsilon_i = 10^{-3}$ and $z_i(0) = 1$ is the initial bid. The reservation parameter in the allocation function is $\delta = 0.1$.

Proportional Fairness ($\alpha = 1$): Figs. 2a, 2b, 2c. In the proportional fairness regime, SBRD consistently achieves the fastest convergence, driving the system to the NE with less than **10 iterations**. XL is slower and convergence occurs in the order of 500 steps. We observe that its convergence speed is not affected by the players' utility heterogeneity, i.e., different values of γ . In contrast, DAQ requires more than 1500 iterations to converge.

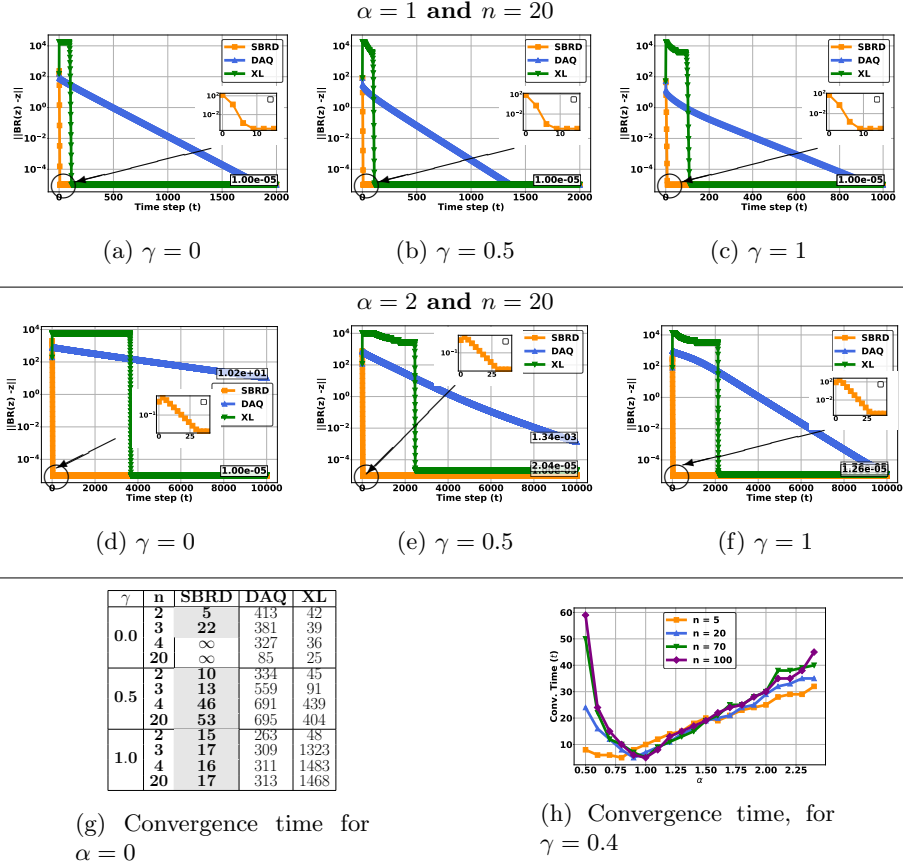
Minimum Potential Delay Fairness ($\alpha = 2$): Figs. 2d, 2e, 2f. As in the previous case, in this fairness regime SBRD shows a similar convergence speed, for all tested values of γ . Here, it converges in **less than 30 iterations**. In contrast, XL converges in more than 2000 steps and DAQ beyond 10^5 steps (even for decreasing stepsize). We observe that, as for $\alpha = 1$, both XL and DAQ appear to converge faster for larger values of γ , i.e., for more pronounced heterogeneity in the players' payoff.

Efficiency Maximization ($\alpha = 0$): Fig. 2g. In the efficiency-maximizing regime, the convergence of SBRD, DAQ, and XL is guaranteed for $n = 2$. Again, SBRD reaches the NE significantly faster than DAQ and XL. When $n > 2$, the convergence of SBRD is not theoretically established, but we can further evaluate it empirically. Fig. 2g shows that SBRD maintains consistent performance when agents are heterogeneous (i.e., $\gamma > 0$), for the tested size of the game $n = 2, 3, 4, 20$. However, as we could somehow expect, SBRD fails to converge in highly homogeneous environments with $n \geq 4$, confirming some limitations in its applicability in this regime in a homogeneous setting.

General case ($\alpha > 0$): Fig. 2h. We want to test the performance of SBRD for general values of $\alpha > 0$, where the BR can be evaluated numerically using a simple bisection search. Fig. 2h shows the SBRD time to convergence, corresponding to a value 10^{-5} for the BR residual. Here, $c_i = 3 \cdot 10^4$ and $\gamma = 0.4$. For $\alpha \geq 1/2$, the SBRD converges rapidly across all game sizes. It appears to have a minimum with convergence time decreasing from $\alpha = 0.5$ to 1, and increasing in a linear fashion beyond.

5 Conclusion

We have explored the decentralized proportional resource allocation, commonly known as the Kelly mechanism, within the context of α -fair utility functions, and subject to budget limitations. Under mild assumptions on the minimum bid, we demonstrated linear convergence of the synchronous best-response dynamics for several cases of interest. We evaluated its performance against dual averaging algorithms. Our findings indicate that, within its stability region, SBRD converges at a markedly faster rate when compared with no-regret algorithms that often require a large number of iterations to achieve equilibrium. In particular, in our numerical results we could confirm the convergence of SBRD for $\alpha > 1/2$. In that region, in sight of its speed of convergence, SBRD appears the natural option to obtain a lightweight sequential scheme based on the Kelly mechanisms under α -fair utilities.

Fig. 2: SBRD Convergence as a function of γ for different values of α .

In future works, we plan to extend our analysis in order to define analytically the region of convergence of SBRD with respect to fairness parameter α . Another interesting direction entails the presence of multiple coupled resources for which players need to bid. This would require to study a multi-dimensional version of the Kelly mechanism and the consequent extension of the SBRD to this case.

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6 Appendix: Proof of Theorem 1

Proof. In order for the best response operator to be a contraction on \mathcal{R} we need to identify $0 < q < 1$ such that

$$\|\text{BR}(\mathbf{z}_1) - \text{BR}(\mathbf{z}_2)\| \leq q \|\mathbf{z}_1 - \mathbf{z}_2\|$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{R}$.

From the generalized mean value theorem (see [21][Cor. 3.2]), it holds

$$\|\text{BR}(\mathbf{z}_1) - \text{BR}(\mathbf{z}_2)\| \leq \sup_{\mathbf{z} \in \mathcal{R}} \|\mathcal{J}_{\text{BR}}(\mathbf{z})\| \|\mathbf{z}_1 - \mathbf{z}_2\|$$

so that the BR operator is a contraction if $q := \sup_{\mathbf{z} \in \mathcal{R}} \|\mathcal{J}_{\text{BR}}(\mathbf{z})\| < 1$. Note that vector and operator norms need to be chosen consistently.

Note that BR converges linearly, with convergence rate determined by $q = \|\mathcal{J}_{\text{BR}}(\mathbf{z})\|$ [22, Thm. 4.2.1]. We thus focus on proving that the norm of the Jacobian matrix is strictly smaller than 1.

Since \mathcal{G}_α is an aggregative game 7, the Jacobian satisfies $(\mathcal{J}_{\text{BR}})_{i,j}(\mathbf{z}) = (\mathcal{J}_{\text{BR}})_{i,j}(s_{-i})$. Let $\mathbf{S}_0 = (s_0)_{i=1}^n$ where $s_0 = (n-1)\epsilon + \delta$ the smallest aggregated bid.

1. **Case $\alpha = 1$:** Use $\|\cdot\|_\infty$ norm. The Jacobian entries are: $(\mathcal{J}_{\text{BR}})_{i,j}(s_{-i}) = \begin{cases} \frac{-1}{2} + \frac{s_{-i} + \frac{2a_i}{\lambda}}{2\sqrt{s_{-i}^2 + \frac{4a_i s_{-i}}{\lambda}}} \geq 0, & j \neq i, \\ 0, & j = i. \end{cases}$, $(\mathcal{J}_{\text{BR}})_{i,j}(\cdot)$ is decreasing,
Hence, $\|\mathcal{J}_{\text{BR}}(\mathbf{S}_0)\|_\infty = \sup_{\mathbf{z} \in \mathcal{R}} \|\mathcal{J}_{\text{BR}}(\mathbf{z})\|_\infty = \max_i \sum_j (\mathcal{J}_{\text{BR}})_{i,j}(s_0)$.
Imposing $\|\mathcal{J}_{\text{BR}}(\mathbf{S}_0)\|_\infty < 1$ leads to a quadratic inequality,
 $\lambda^2 s_0^2 + 4a_i^{\max} \lambda s_0 - \frac{(a_i^{\max})^2 (n-1)^2}{n} > 0$, satisfied for: $s_0 > (n - 2\sqrt{n} + 1) \frac{a_i^{\max}}{\lambda\sqrt{n}}$.

2. **Case $\alpha = 2$:** Use $\|\cdot\|_1$ norm. The Jacobian entries are:
 $(\mathcal{J}_{\text{BR}})_{i,j}(s_{-i}) = \begin{cases} \frac{1}{2} \sqrt{\frac{a_i}{\lambda s_{-i}}} (> 0), & j \neq i, \\ 0, & j = i. \end{cases}$, $(\mathcal{J}_{\text{BR}})_{i,j}(\cdot)$ is also decreasing .
The contraction condition becomes: $\|\mathcal{J}_{\text{BR}}(\mathbf{S}_0)\|_1 = \max_j \sum_{i \neq j} \frac{1}{2} \sqrt{\frac{a_i}{\lambda s_0}} < 1$,
implying: $s_0 > \frac{1}{4\lambda} \left(\max_j \sum_{i \neq j} \sqrt{a_i} \right)^2$.

3. **Case $\alpha = 0$:** Use $\|\cdot\|_\infty$ norm. The Jacobian satisfies:

$$(\mathcal{J}_{\text{BR}})_{i,j}(s_{-i}) = \begin{cases} -1 + \frac{1}{2} \sqrt{\frac{a_i}{\lambda s_{-i}}}, & j \neq i, \\ 0, & j = i, \end{cases}$$

leading to: $\|\mathcal{J}_{\text{BR}}(\mathbf{z})\|_\infty = (n-1) \max_i \left| -1 + \frac{1}{2} \sqrt{\frac{a_i}{\lambda s_{-i}}} \right| < 1$. Solving the inequality yields: $\frac{a_i}{s_{-i}} < 16\lambda$, $\forall i \in \{1, 2\}$. Setting a_i to be $\max(a_1, a_2)$ and s_{-i} to s_0 leads to constraints on ϵ that depend on δ and $\max(a_1, a_2)$.

This finishes the proof. \square