Corrigé du TD N° 1 - Exercices 14 à 19

1.) • $\Delta = (4+3i)^2 - 4(1+5i) = (16+24i-9)-(4+20i) = 3+4i$

• $\delta^2 = \Delta \iff \begin{cases} x^2 - y^2 = 3 \\ x^2 + y^2 = |3+4i| = 5 \\ 2xy = 4 \end{cases} \iff \begin{cases} 2x^2 = 8 \\ 2y^2 = 2 \\ xy = 2 \end{cases} \iff \begin{cases} x^2 = 4 \\ y^2 = 1 \\ xy = 2 \end{cases}$

$\iff \begin{cases} x = \pm 2 \\ y = \pm 1 \\ xy = 2 \end{cases} \iff \delta = \pm(2+i)$

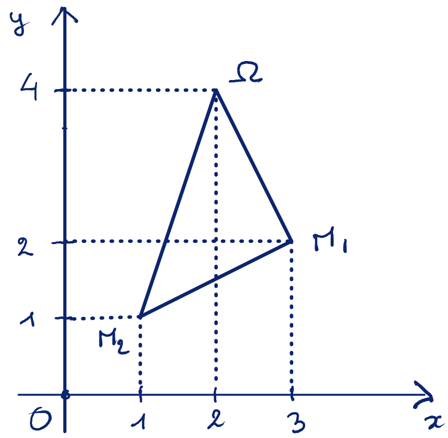
• $z = \dfrac{(4+3i)-(2+i)}{2} = 1+i$ ou $z = \dfrac{(4+3i)+(2+i)}{2} = 3+2i$

Donc $z_1 = 3+2i$ et $z_2 = 1+i$

2.) • $z_1 + z_2 = (3+2i) + (1+i) = 4+3i$

• $z_1 z_2 = (3+2i)(1+i) = 3+3i+2i-2 = 1+5i$

3) a)



b) • $\begin{cases} M_1\Omega = |\omega - z_1| = |-1+2i| = \sqrt{5} \\ M_1M_2 = |z_2 - z_1| = |-2-i| = \sqrt{5} \end{cases}$

Donc le triangle $(M_1M_2\Omega)$ est isocèle (puisque $M_1\Omega = M_1M_2$)

• $\Omega M_2 = |z_2 - \omega| = |-1-3i| = \sqrt{10}$

Donc $\underbrace{\Omega M_1^2}_{5} + \underbrace{M_1M_2^2}_{5} = \underbrace{\Omega M_2^2}_{10}$

Donc d'après la réciproque du th de Pythagore,

le triangle $(M_1M_2\Omega)$ est rectangle en M_1

Exercice 15 :

1°) • $a = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$; donc $a = e^{2i\pi/3}$

• $b = \frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$; donc $b = e^{i\pi/6}$

2°) a) $$2\cos\left(\frac{\theta-\theta'}{2}\right)e^{i\frac{\theta+\theta'}{2}} = 2\,\frac{e^{i\frac{\theta-\theta'}{2}} + e^{-i\frac{\theta-\theta'}{2}}}{2} \times e^{i\frac{\theta+\theta'}{2}}$$
$$= \left(e^{i\frac{\theta-\theta'}{2}} + e^{-i\frac{\theta-\theta'}{2}}\right)e^{i\frac{\theta+\theta'}{2}}$$
$$= e^{i\left(\frac{\theta-\theta'}{2}+\frac{\theta+\theta'}{2}\right)} + e^{i\left(\frac{\theta'-\theta}{2}+\frac{\theta+\theta'}{2}\right)}$$
$$= e^{i\theta} + e^{i\theta'}$$

Donc $e^{i\theta} + e^{i\theta'} = 2\cos\left(\frac{\theta-\theta'}{2}\right)e^{i\frac{\theta+\theta'}{2}}$

b) $$a+b = e^{2i\pi/3} + e^{i\pi/6} = 2\cos\left(\frac{\frac{2\pi}{3}-\frac{\pi}{6}}{2}\right)e^{i\frac{2\pi/3+\pi/6}{2}}$$ (on applique a) avec $\theta = \frac{2\pi}{3}$ et $\theta' = \pi/6$)
$$= 2\underbrace{\cos\left(\frac{\pi}{4}\right)}_{\sqrt{2}/2}e^{i\frac{5\pi}{12}}$$ donc $a+b = \sqrt{2}\,e^{\frac{5i\pi}{12}}$

(on ne peut pas appliquer la formule du a) car c'est $e^{i\theta} - e^{i\theta'}$ et pas $e^{i\theta} + e^{i\theta'}$)

$$a-b = e^{2i\pi/3} - e^{i\pi/6} = e^{\frac{2i\pi}{3}} + e^{i\pi}e^{i\pi/6}$$ (car $e^{i\pi} = -1$)
$$= e^{2i\pi/3} + e^{7i\pi/6}$$
$$= 2\cos\left(\frac{2\pi/3 - 7\pi/6}{2}\right)e^{i\frac{2\pi/3+7\pi/6}{2}}$$
$$= 2\underbrace{\cos\left(-\frac{\pi}{4}\right)}_{\sqrt{2}/2}e^{\frac{11i\pi}{12}}$$ donc $a-b = \sqrt{2}\,e^{\frac{11i\pi}{12}}$

3°) • $\Delta = (1-i\sqrt{3})^2 - 4\times(-(1+i\sqrt{3})) = (1-2i\sqrt{3}-3) + (4+4i\sqrt{3})$
$= 2 + 2i\sqrt{3}$

• $$\delta^2 = \Delta \iff \begin{cases} x^2 - y^2 = 2 \\ x^2+y^2 = |2+2i\sqrt{3}| = 2|1+i\sqrt{3}| = 2\times 2 = 4 \\ 2xy = 2\sqrt{3} \end{cases} \iff \begin{cases} x^2 = 3 \\ y^2 = 1 \\ xy = \sqrt{3} \end{cases}$$
$$\iff \begin{cases} x = \sqrt{3} \text{ et } y = 1 \\ \text{ou} \\ x = -\sqrt{3} \text{ et } y = -1 \end{cases} \iff \delta = \pm(\sqrt{3}+i)$$

$$\bullet \begin{cases} z = \dfrac{-(1-i\sqrt{3})-(\sqrt{3}+i)}{2} = \dfrac{-1-\sqrt{3}}{2} + i\,\dfrac{\sqrt{3}-1}{2} \\ \text{ou} \\ z = \dfrac{-(1-i\sqrt{3})+(\sqrt{3}+i)}{2} = \dfrac{-1+\sqrt{3}}{2} + i\,\dfrac{\sqrt{3}+1}{2} \end{cases}$$

Donc : $z_1 = \dfrac{-1+\sqrt{3}}{2} + i\,\dfrac{\sqrt{3}+1}{2}$ et $z_2 = \dfrac{-1-\sqrt{3}}{2} + i\,\dfrac{\sqrt{3}-1}{2}$

b) $z_1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ donc $z_1 = a+b$

$z_2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ donc $z_2 = a-b$

c) Donc d'après 2)b) : $z_1 = \sqrt{2}\, e^{\frac{5i\pi}{12}}$ et $z_2 = \sqrt{2}\, e^{\frac{11i\pi}{12}}$

Exercice 16 :

1.) $\bullet\ Z = \dfrac{-3+i}{1-2i} = \dfrac{(-3+i)(1+2i)}{1^2+2^2} = \dfrac{-3-6i+i-2}{5} = -1-i$

$$\bullet \begin{cases} |Z| = |-1-i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \\ \cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \text{ et } \sin\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}\ ; \text{ donc } \theta = -\frac{3\pi}{4} \end{cases}$$

Donc $Z = -1-i = \sqrt{2}\, e^{-\frac{3i\pi}{4}}$

2.) $\widehat{(\overrightarrow{OA}, \overrightarrow{OB})} = \arg\left(\dfrac{z_B - z_O}{z_A - z_O}\right)\ [2\pi]$ (Formule de cours)

$= \arg\left(\dfrac{z_B}{z_A}\right)\ [2\pi]$

$= \arg(Z) = -\dfrac{3\pi}{4}\ [2\pi]$

Exercice 17 :

1.) a) $z^2 = 35+12i \iff \begin{cases} x^2-y^2 = 35 \\ x^2+y^2 = |35+12i| = \sqrt{35^2+12^2} = \sqrt{\underbrace{1225+144}_{1369}} = 37 \\ 2xy = 12 \end{cases}$

$(z = x+iy)$

$\iff \begin{cases} 2x^2 = 72 \\ 2y^2 = 2 \\ xy = 6 \end{cases} \iff \begin{cases} x^2 = 36 \\ y^2 = 1 \\ xy = 6 \end{cases} \iff \begin{cases} x = \pm 6 \\ y = \pm 1 \\ xy = 6 \end{cases}$

$\iff z = \pm(6+i)$

2.) $\Delta = (2-3i)^2 - 4i(-6+10i) = 4-12i-9+24i+40 = 35+12i$

Donc d'après a) : $\delta = \pm(6+i)$

Donc $z = \dfrac{-(2-3i)-(6+i)}{2i}$ ou $z = \dfrac{-(2-3i)+(6+i)}{2i}$

$= \dfrac{-8+2i}{2i}$ $\qquad = \dfrac{4+4i}{2i}$

$= \dfrac{-4+i}{i}$ $\qquad = \dfrac{2+2i}{i}$

$= (-4+i)(-i)$ $\qquad = (2+2i)(-i)$

$= 1+4i$ $\qquad = 2-2i$

Donc $z_1 = 2-2i$ et $z_2 = 1+4i$

c) $\begin{cases} M_1M_2 = |z_2 - z_1| = |-1-6i| = \sqrt{37} \\ \Omega M_1 = |z_1 - \omega| = |12+2i| = 2\sqrt{37} \\ \Omega M_2 = |z_2 - \omega| = |11+8i| = \sqrt{185} \end{cases}$

- On a $M_1M_2^2 + \Omega M_1^2 = 37 + 4 \times 37 = 185$

  Donc $M_1M_2^2 + \Omega M_1^2 = \Omega M_2^2$.

  Par conséquent, d'après la réciproque du th de Pythagore

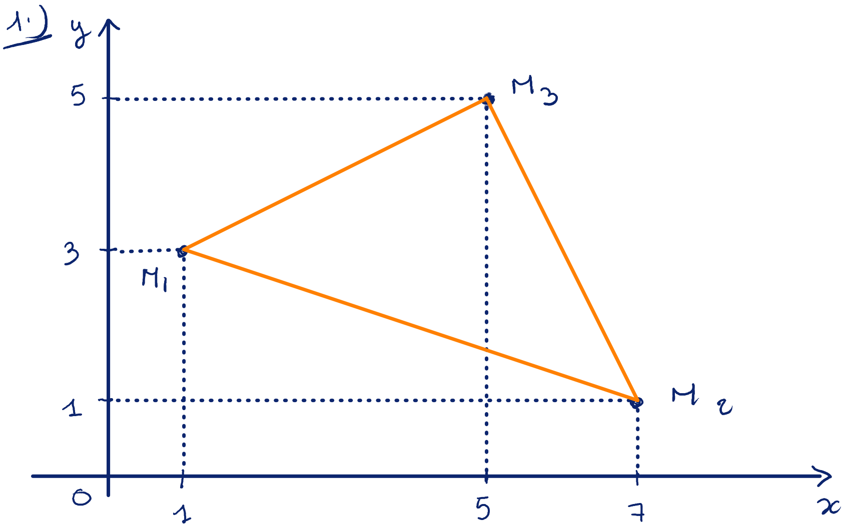
  le triangle $(\Omega M_1 M_2)$ est rectangle en M_1

- $M_1M_2 \neq \Omega M_1$ et $M_1M_2 \neq \Omega M_2$ et $\Omega M_1 \neq \Omega M_2$.

  Donc le triangle $(\Omega M_1 M_2)$ n'est pas isocèle.

Exercice 18 :

1.)



$\begin{cases} M_2M_3 = |z_3 - z_2| = |-2+4i| = \sqrt{20} \\ M_1M_3 = |z_3 - z_1| = |4+2i| = \sqrt{20} \end{cases}$

Donc $M_2M_3 = M_1M_3$

Donc le triangle $(M_1M_2M_3)$ est isocèle

2.) $Z = \dfrac{(7+i)-(5+5i)}{(1+3i)-(5+5i)} = \dfrac{2-4i}{-4-2i} = \dfrac{i(-2i-4)}{-4-2i} = i = \cos\frac{\pi}{2} + i\sin\pi/2$

Donc $Z = e^{i\pi/2}$

3) $(\overrightarrow{M_3M_1}, \overrightarrow{M_3M_2}) = \arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) [2\pi]$ (Formule du cours)

$$= \arg Z \quad [2\pi]$$

$$= \frac{\pi}{2} \quad [2\pi]$$

Donc $\overrightarrow{M_3M_1} \perp \overrightarrow{M_3M_2}$; par conséquent, le triangle $(M_1M_2M_3)$ est rectangle en M_3

Exercice 19 :

Multiplication du numérateur et dénominateur par i

- $$z_1 = \frac{2}{\sin\left(\frac{3\pi}{13}\right) - i\cos\left(\frac{3\pi}{13}\right)} = \frac{i \times 2}{i \times \left(\sin\left(\frac{3\pi}{13}\right) - i\cos\left(\frac{3\pi}{13}\right)\right)}$$

Développement du dénominateur

$$= \frac{2i}{i\sin\left(\frac{3\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right)}$$

$$= \frac{2e^{i\pi/2}}{e^{\frac{3i\pi}{13}}}$$ (On reconnaît des formes exponentielles)

$$= 2e^{i\left(\frac{\pi}{2} - \frac{3\pi}{13}\right)}$$

$$= 2e^{\frac{7i\pi}{26}}$$

Factorisation par i

- $$z_2 = \sin\left(\frac{\pi}{9}\right) + i\cos\left(\frac{\pi}{9}\right) = i\left(\frac{1}{i}\sin\left(\frac{\pi}{9}\right) + \cos\left(\pi/9\right)\right)$$

$$= i\left(-i\sin\left(\frac{\pi}{9}\right) + \cos\left(\frac{\pi}{9}\right)\right)$$ (car $\frac{1}{i} = -i$)

$$= i\left(\cos\left(\frac{\pi}{9}\right) - i\sin\left(\frac{\pi}{9}\right)\right)$$

$$= e^{i\pi/2}\, e^{-i\pi/9}$$ (car $i = e^{i\pi/2}$ et $\cos\frac{\pi}{9} - i\sin\pi/9 = e^{-i\pi/9}$)

$$= e^{i\left(\pi/2 - \pi/9\right)}$$

$$= e^{\frac{7i\pi}{18}}$$