Corrigé du TD N°1 - Exercices 6 à 10

Exercice 6 :

- $\cos(3\theta)$? On utilise la formule de Moivre et la formule du binôme.

$$\cos(3\theta) + i\sin(3\theta) = e^{3i\theta} = (e^{i\theta})^3 = \underbrace{(\cos\theta + i\sin\theta)^3}_{(a+b)^3}$$

$(a+b)^3$ (où $a = \cos\theta$ et $b = i\sin\theta$)

D'après la formule du binôme :

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Triangle de Pascal

1
1 1
1 2 1
1 3 3 1

Donc :

$$\cos(3\theta) + i\sin(3\theta) = (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$$

Or : $\begin{cases} (i\sin\theta)^2 = i^2\sin^2\theta = -\sin^2\theta \\ (i\sin\theta)^3 = i^3\sin^3\theta = -i\sin^3\theta \end{cases}$. Donc :

$$\cos(3\theta) + i\sin\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Donc, par identification de la partie réelle :

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$

(et par identification de la partie imaginaire ;
$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$)

- $\sin(7\theta)$?

$$\cos(7\theta) + i\sin(7\theta) = \underbrace{(\cos\theta + i\sin\theta)^7}_{=(a+b)^7}$$

$(a+b)^7$ où $a = \cos\theta$ et $b = i\sin\theta$

Le Triangle de Pascal est :

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Donc :

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

On en déduit que :

$$\cos(7\theta) + i\sin(7\theta) = (\cos\theta)^7 + 7(\cos\theta)^6(i\sin\theta) + 21(\cos\theta)^5(i\sin\theta)^2$$
$$+ 35(\cos\theta)^4(i\sin\theta)^3 + 35(\cos\theta)^3(i\sin\theta)^4$$
$$+ 21(\cos\theta)^2(i\sin\theta)^5 + 7\cos\theta\,(i\sin\theta)^6 + (i\sin\theta)^7$$

or $\begin{cases} (i\sin\theta)^2 = i^2\sin^2\theta = -\sin^2\theta \\ (i\sin\theta)^3 = i^3\sin^3\theta = -i\sin^3\theta \\ (i\sin\theta)^4 = i^4\sin^4\theta = \sin^4\theta \\ (i\sin\theta)^5 = i^5\sin^5\theta = i\sin^5\theta \\ (i\sin\theta)^6 = i^6\sin^6\theta = -\sin^6\theta \\ (i\sin\theta)^7 = i^7\sin^7\theta = -i\sin^7\theta \end{cases}$

Donc :

$$\cos(7\theta) + i\sin(7\theta) = \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

$$= \left(\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\right) + i\left(7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta\right)$$

Donc :

$$\sin(7\theta) = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

(et $\cos(7\theta) = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$)

Pour continuer à s'entraîner :

- $\cos(4\theta)$ et $\sin(4\theta)$?

$$\cos(4\theta) + i\sin(4\theta) = e^{4i\theta} = \left(e^{i\theta}\right)^4 = \underbrace{(\cos\theta + i\sin\theta)^4}_{(a+b)^4 \text{ où } \begin{cases} a = \cos\theta \\ b = i\sin\theta \end{cases}}$$

Or $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

(Voir triangle de Pascal ci-dessus)

Donc :

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta)^4 + 4(\cos\theta)^3(i\sin\theta) + 6(\cos\theta)^2(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$$

or $\begin{cases} (i\sin\theta)^2 = i^2\sin^2\theta = -\sin^2\theta \\ (i\sin\theta)^3 = i^3\sin^3\theta = -i\sin^3\theta \\ (i\sin\theta)^4 = i^4\sin^4\theta = \sin^4\theta \end{cases}$

D'où :

$$\cos(4\theta) + i\sin(4\theta) = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

$$= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right)$$
$$+ i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$$

Donc :
$$\begin{cases} \cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ \text{et} \\ \sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \end{cases}$$

- $\cos(6\theta)$? $\sin(6\theta)$?

$$\cos(6\theta) + i\sin(6\theta) = (\cos\theta + i\sin\theta)^6$$
$$= (\cos\theta)^6 + 6(\cos\theta)^5(i\sin\theta) + 15(\cos\theta)^4(i\sin\theta)^2$$
$$+ 20(\cos\theta)^3(i\sin\theta)^3 + 15(\cos\theta)^2(i\sin\theta)^4$$
$$+ 6(\cos\theta)(i\sin\theta)^5 + (i\sin\theta)^6$$
$$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta$$
$$- 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta$$
$$+ 6i\cos\theta\sin^5\theta - \sin^6\theta$$
$$= \left(\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta\right)$$
$$+ i\left(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta\right)$$

D'où :
$$\begin{cases} \cos(6\theta) = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \\ \text{et} \\ \sin(6\theta) = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta \end{cases}$$

Exercice 7 :

- $\sin^3 x$?

D'après une des deux formules d'Euler : $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Donc :
$$\sin^3 x = (\sin x)^3 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{1}{(2i)^3}\left(e^{ix} - e^{-ix}\right)^3$$

or $(2i)^3 = 2^3 i^3 = -2^3 i$ et :

$(e^{ix} - e^{-ix})^3 = (a-b)^3$ où $a = e^{ix}$ et $b = e^{-ix}$

On a : $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. Donc

$$(e^{ix}-e^{-ix})^3 = (e^{ix})^3 - 3(e^{ix})^2(e^{-ix}) + 3(e^{ix})(e^{-ix})^2 - (e^{-ix})^3$$
$$= e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}$$
$$= e^{3ix} - 3e^{2ix-ix} + 3e^{ix-2ix} - e^{-3ix}$$
$$= e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}$$

Donc :

$$\sin^3 x = -\frac{1}{2^3 i}\left(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}\right)$$
$$= -\frac{1}{2^2} \times \frac{1}{2i}\left(e^{3ix} - e^{-3ix} - 3(e^{ix} - e^{-ix})\right)$$
$$= -\frac{1}{2^2}\left(\frac{e^{3ix}-e^{-3ix}}{2i} - 3\,\frac{e^{ix}-e^{-ix}}{2i}\right)$$
$$= -\frac{1}{4}\left(\sin(3x) - 3\sin x\right)$$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$$

- $\cos^5 x = ?$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \quad \text{(Formule d'Euler)}$$

Donc : $\cos^5 x = (\cos x)^5 = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^5 = \frac{1}{2^5}\left(e^{ix}+e^{-ix}\right)^5$

$= \frac{1}{2^5}(a+b)^5$ où $a = e^{ix}$ et $b = e^{-ix}$

Or : $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Donc :

$$(e^{ix}+e^{-ix})^5 = (e^{ix})^5 + 5(e^{ix})^4e^{-ix} + 10(e^{ix})^3(e^{-ix})^2 + 10(e^{ix})^2(e^{-ix})^3 + 5e^{ix}(e^{-ix})^4 + (e^{-ix})^5$$
$$= e^{5ix} + 5e^{4ix}e^{-ix} + 10e^{3ix}e^{-2ix} + 10e^{2ix}e^{-3ix} + 5e^{ix}e^{-4ix} + e^{-5ix}$$
$$= e^{5ix} + 5e^{4ix-ix} + 10e^{3ix-2ix} + 10e^{2ix-3ix} + 5e^{ix-4ix} + e^{-5ix}$$
$$= e^{5ix} + 5e^{3ix} + 10e^{ix} + 10e^{-ix} + 5e^{-3ix} + e^{-5ix}$$

Donc :

$$\cos^5 x = \frac{1}{2^5}\left(e^{5ix} + 5e^{3ix} + 10e^{ix} + 10e^{-ix} + 5e^{-3ix} + e^{-5ix}\right)$$
$$= \frac{1}{2^4} \times \frac{1}{2}\left(e^{5ix} + e^{-5ix} + 5(e^{3ix}+e^{-3ix}) + 10(e^{ix}+e^{-ix})\right)$$

$$= \frac{1}{2^4}\left(\frac{e^{5ix}+e^{-5ix}}{2} + 5\,\frac{e^{3ix}+e^{-3ix}}{2} + 10\,\frac{e^{ix}+e^{-ix}}{2}\right)$$

$$\cos^5 x = \frac{1}{16}\left(\cos(5x) + 5\cos(3x) + 10\cos x\right)$$

- $\sin(5x)\cos(6x)$?

$$\sin(5x)\cos(6x) = \frac{e^{5ix}-e^{-5ix}}{2i} \times \frac{e^{6ix}+e^{-6ix}}{2}$$

$$= \frac{e^{5ix}e^{6ix}+e^{5ix}e^{-6ix} - e^{-5ix}e^{6ix} - e^{-5ix}e^{-6ix}}{2i\times 2}$$

$$= \frac{e^{11ix}+e^{-ix}-e^{ix}-e^{-11ix}}{2\times 2i}$$

$$= \frac{1}{2}\left(\underbrace{\frac{e^{11ix}-e^{-11ix}}{2i}}_{\sin(11x)} - \underbrace{\frac{e^{ix}-e^{-ix}}{2i}}_{\sin x}\right)$$

$$\sin(5x)\cos(6x) = \frac{1}{2}\sin(11x) - \frac{1}{2}\sin x$$

$\cos x \sin^4 x$?

$$\cos x\sin^4 x = \frac{e^{ix}+e^{-ix}}{2} \times \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4$$

$$= \frac{e^{ix}+e^{-ix}}{2} \times \frac{1}{(2i)^4}\times\left(e^{ix}-e^{-ix}\right)^4$$

$$= \frac{e^{ix}+e^{-ix}}{2} \times \frac{1}{2^4}\left((e^{ix})^4 - 4(e^{ix})^3e^{-ix} + 6(e^{ix})^2(e^{-ix})^2 - 4(e^{ix})(e^{-ix})^3 + (e^{-ix})^4\right)$$

$$= \frac{1}{2^5}\left(e^{ix}+e^{-ix}\right)\left(e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}\right)$$

$$= \frac{1}{2^5}\left(e^{5ix} - 4e^{3ix} + 6e^{ix} - 4e^{-ix} + e^{-3ix} + e^{3ix} - 4e^{ix} + 6e^{-ix} - 4e^{-3ix} + e^{-5ix}\right)$$

$$= \frac{1}{2^5}\left(e^{5ix}+e^{-5ix} - 3(e^{3ix}+e^{-3ix}) + 2(e^{ix}+e^{-ix})\right)$$

$$= \frac{1}{2^4}\left(\frac{e^{5ix}+e^{-5ix}}{2} - 3\,\frac{e^{3ix}+e^{-3ix}}{2} + 2\,\frac{e^{ix}+e^{-ix}}{2}\right)$$

c'est à dire :

$$\cos x\sin^4 x = \frac{1}{16}\left(\cos(5x) - 3\cos(3x) + 2\cos x\right)$$

Pour continuer à s'entraîner :

- $\sin^4 x$?

$$\sin^4 x = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4$$

$$= \frac{1}{(2i)^4}\left(e^{ix}-e^{-ix}\right)^4$$

$$= \frac{1}{2^4}\left((e^{ix})^4 - 4(e^{ix})^3 e^{-ix} + 6(e^{ix})^2(e^{-ix})^2 - 4e^{ix}(e^{-ix})^3 + (e^{-ix})^4\right)$$

$$= \frac{1}{2^4}\left(e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}\right)$$

$$= \frac{1}{2^3}\left(\frac{e^{4ix}+e^{-4ix}}{2} - 4\,\frac{e^{2ix}+e^{-2ix}}{2} + \frac{6}{2}\right)$$

$$\sin^4 x = \frac{1}{8}\left(\cos(4x) - 4\cos(2x) + 3\right)$$

• $\cos^6 x$?

$$\cos^6 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^6$$

$$= \frac{1}{2^6}\left(e^{6ix} + 6e^{5ix}e^{-ix} + 15e^{4ix}e^{-2ix} + 20e^{3ix}e^{-3ix} + 15e^{2ix}e^{-4ix} + 6e^{ix}e^{-5ix} + e^{-6ix}\right)$$

$$= \frac{1}{2^6}\left(e^{6ix} + 6e^{4ix} + 15e^{2ix} + 20 + 15e^{-2ix} + 6e^{-4ix} + e^{-6ix}\right)$$

$$= \frac{1}{2^5}\left(\frac{e^{6ix}+e^{-6ix}}{2} + 6\,\frac{e^{4ix}+e^{-4ix}}{2} + 15\,\frac{e^{2ix}+e^{-2ix}}{2} + \frac{20}{2}\right)$$

$$\cos^4 x = \frac{1}{32}\left(\cos(6x) + 6\cos(4x) + 15\cos(2x) + 10\right)$$

• $\cos^2 x \sin^3 x$? (On va commencer par utiliser une formule de duplication afin de simplifier un peu les calculs)

$$\cos^2 x \sin^3 x = (\cos x \sin x)^2 \sin x$$

$$= \left(\frac{1}{2}\sin(2x)\right)^2 \sin x$$

$$= \frac{1}{2^2}\sin^2(2x)\sin x$$

$$= \frac{1}{4}\left(\frac{e^{2ix}-e^{-2ix}}{2i}\right)^2 \times \frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{1}{4} \times \frac{1}{(2i)^3}\left(e^{4ix} - 2 + e^{-4ix}\right)\left(e^{ix}-e^{-ix}\right)$$

$$= -\frac{1}{32i}\left(e^{5ix} - e^{3ix} - 2e^{ix} + 2e^{-ix} + e^{-3ix} - e^{-5ix}\right)$$

$$= -\frac{1}{16}\left(\frac{e^{5ix}-e^{-5ix}}{2i} - 2\,\frac{e^{ix}-e^{-ix}}{2i} - \frac{e^{3ix}-e^{-3ix}}{2i}\right)$$

$$= -\frac{1}{16}\left(\sin(5x) - 2\sin x - \sin(3x)\right)$$

C'est à dire :

$$\cos^2 x \sin^3 x = \frac{1}{8}\sin x + \frac{1}{16}\sin(3x) - \frac{1}{16}\sin(5x)$$

Exercice 8 :

- Posons $C_n = \sum_{k=0}^{n} \cos(kx)$ et $S_n = \sum_{k=0}^{n} \sin(kx)$

→ si $x = 0\ [2\pi]$: $C_n = \sum_{k=0}^{n} 1$ et $S_n = \sum_{k=0}^{n} 0$

c'est à dire : $C_n = n+1$ et $S_n = 0$

→ si $x \neq 0\ [2\pi]$:

$$
\begin{aligned}
C_n + i S_n &= \sum_{k=0}^{n} \cos(kx) + i \sum_{k=0}^{n} \sin(kx) \\
&= \sum_{k=0}^{n} \left(\cos(kx) + i\sin(kx)\right) \\
&= \sum_{k=0}^{n} e^{ikx} \\
&= \sum_{k=0}^{n} \left(e^{ix}\right)^k
\end{aligned}
$$

Posons $q = e^{ix}$. Alors $q \neq 1$ car $x \neq 0\ [2\pi]$. Donc

$$
\begin{aligned}
\sum_{k=0}^{n} \left(e^{ix}\right)^k = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} &= \frac{1-e^{i(n+1)x}}{1-e^{ix}} \\
&= \frac{e^{i\frac{n+1}{2}x}\left(e^{-i\frac{n+1}{2}x} - e^{i\frac{n+1}{2}x}\right)}{e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}\right)} \\
&= e^{i\frac{n}{2}x} \times \frac{-2\sin\left(\frac{n+1}{2}x\right)}{-2\sin\left(\frac{x}{2}\right)} \\
&= e^{i\frac{n}{2}x} \frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \\
&= \left(\cos\left(\frac{n}{2}x\right) + i\sin\left(\frac{n}{2}x\right)\right) \frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}
\end{aligned}
$$

c'est à dire :

$$C_n + i S_n = \frac{\cos\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)} + i \frac{\sin\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}$$

Donc :
$$
\begin{cases}
C_n = \dfrac{\cos\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)} \\[2ex]
S_n = \dfrac{\sin\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}
\end{cases}
$$

- On en déduit :

$$\sum_{k=0}^{n} \cos(x+ky) = \sum_{k=0}^{n} \left(\cos x \cos(ky) - \sin x \sin(ky)\right)$$

$$= \cos x \cdot \sum_{k=0}^{n} \cos(ky) - \sin x \cdot \sum_{k=0}^{n} \sin(ky)$$

Si $y = 0 \ [2\pi]$: $\sum_{k=0}^{n} \cos(ky) = n+1$ et $\sum_{k=0}^{n} \sin(ky) = 0$

Donc $\sum_{k=0}^{n} \cos(x+ky) = (n+1)\cos x$

Si $y \neq 0 \ [2\pi]$

$$\begin{cases} \sum_{k=0}^{n} \cos(ky) = \dfrac{\cos\left(\frac{n}{2}y\right)\sin\left(\frac{n+1}{2}y\right)}{\sin\left(\frac{y}{2}\right)} \\ \sum_{k=0}^{n} \sin(ky) = \dfrac{\sin\left(\frac{n}{2}y\right)\sin\left(\frac{n+1}{2}y\right)}{\sin\left(\frac{y}{2}\right)} \end{cases}$$

Donc

$$\sum_{k=0}^{n} \cos(x+ky) = \cos x \cdot \frac{\cos\left(\frac{n}{2}y\right)\sin\left(\frac{n+1}{2}y\right)}{\sin\left(\frac{y}{2}\right)} - \sin x \cdot \frac{\sin\left(\frac{n}{2}y\right)\sin\left(\frac{n+1}{2}y\right)}{\sin\left(\frac{y}{2}\right)}$$

$$= \frac{\sin\left(\frac{n+1}{2}y\right)}{\sin\left(\frac{y}{2}\right)} \cdot \underbrace{\left(\cos x \cos\left(\frac{n}{2}y\right) - \sin x \sin\left(\frac{n}{2}y\right)\right)}_{\cos\left(x+\frac{n}{2}y\right)}$$

$$\sum_{k=0}^{n} \cos(x+ky) = \frac{\sin\left(\frac{n+1}{2}y\right)\cos\left(x+\frac{n}{2}y\right)}{\sin\left(\frac{y}{2}\right)}$$

Exercice 9 :

1) les racines carrées complexes de :

- 7 sont $-\sqrt{7}$ et $\sqrt{7}$

$\left\{\begin{array}{l} \bullet\ -\sqrt{7} \text{ et } \sqrt{7} \text{ sont deux racines carrées complexe de } 7 \text{ puisque } (-\sqrt{7})^2 = 7 \text{ et } (\sqrt{7})^2 = 7 \\ \bullet\ 7 \text{ admet deux racines carrées complexes} \end{array}\right.$

- -3 sont $-\sqrt{3}i$ et $\sqrt{3}i$ car

$\left\{\begin{array}{l} \bullet\ -i\sqrt{3} \text{ et } i\sqrt{3} \text{ sont deux racines carrées complexes puisque } (-i\sqrt{3})^2 = (-i)^2(\sqrt{3})^2 = -3 \\ \quad \text{et } (i\sqrt{3})^2 = i^2(\sqrt{3})^2 = -3 \\ \bullet\ -3 \text{ admet deux racines carrées complexes} \end{array}\right.$

- $3i$ sont ??

On cherche $u = x + iy \in \mathbb{C}$ tel que $u^2 = 3i$.

On a : $u^2 = 3i \iff \begin{cases} x^2 - y^2 = 0 & (1) \\ x^2 + y^2 = |3i| = 3 & (2) \\ 2xy = 3 & (3) \end{cases} \iff \begin{cases} x^2 - y^2 = 0 & \\ 2x^2 = 3 & (1)+(2) \\ 2y^2 = 3 & (2)-(1) \\ 2xy = 3 & \end{cases}$

$\iff \begin{cases} x^2 = \frac{3}{2} \\ y^2 = \frac{3}{2} \\ xy = \frac{3}{2} > 0 \end{cases} \iff \begin{cases} x = \pm\sqrt{\frac{3}{2}} \\ y = \pm\sqrt{\frac{3}{2}} \\ xy > 0 \end{cases}$

$\iff \begin{cases} x = \sqrt{\frac{3}{2}} \text{ et } y = \sqrt{\frac{3}{2}} \\ \text{ou} \\ x = -\sqrt{\frac{3}{2}} \text{ et } y = -\sqrt{\frac{3}{2}} \end{cases}$

$\iff \left[u = \sqrt{\frac{3}{2}} + i\sqrt{\frac{3}{2}} \text{ ou } u = -\sqrt{\frac{3}{2}} - i\sqrt{\frac{3}{2}} \right]$

les deux racines carrées de $3i$ sont $\pm\left(\sqrt{\frac{3}{2}} + i\sqrt{\frac{3}{2}}\right)$

- $-5i$ sont ??

On cherche $u = x + iy$ tel que $u^2 = -5i$

On a : $u^2 = -5i \iff \begin{cases} x^2 - y^2 = 0 & (1) \\ x^2 + y^2 = |-5i| = 5 & (2) \\ 2xy = -5 & (3) \end{cases} \iff \begin{cases} 2x^2 = 5 & (1)+(2) \\ 2y^2 = 5 & (2)-(1) \\ xy = -\frac{5}{2} < 0 & \end{cases}$

$\iff \begin{cases} x = \pm\sqrt{\frac{5}{2}} \text{ et } y = \pm\sqrt{\frac{5}{2}} \\ xy < 0 \end{cases}$

$\iff \begin{cases} x = \sqrt{\frac{5}{2}} \text{ et } y = -\sqrt{\frac{5}{2}} \\ \text{ou} \\ x = -\sqrt{\frac{5}{2}} \text{ et } y = \sqrt{\frac{5}{2}} \end{cases}$

$\iff \left[u = \sqrt{\frac{5}{2}} - i\sqrt{\frac{5}{2}} \text{ ou } u = -\sqrt{\frac{5}{2}} + i\sqrt{\frac{5}{2}}\right]$

Donc les racines carrées complexes de $-5i$ sont : $\pm\left(\sqrt{\frac{5}{2}} - i\sqrt{\frac{5}{2}}\right)$

• $8-6i$ sont ??

$$u^2 = 8-6i \iff \begin{cases} x^2-y^2=8 & (1) \\ x^2+y^2 = |8-6i| = \sqrt{8^2+(-6)^2} = \sqrt{100} = 10 & (2) \\ 2xy = -6 & (3) \end{cases}$$

$$\iff \begin{cases} 2x^2 = 18 & (1)+(2) \\ 2y^2 = 2 & (2)-(1) \\ xy = -3 < 0 \end{cases} \iff \begin{cases} x^2 = 9 \\ y^2 = 1 \\ xy < 0 \end{cases}$$

$$\iff \begin{cases} x = \pm 3 \\ y = \pm 1 \\ xy < 0 \end{cases} \iff \begin{cases} x = 3 \text{ et } y = -1 \\ \quad \text{ou} \\ x = -3 \text{ et } y = 1 \end{cases}$$

$$\iff [u = 3-i \text{ ou } u = -3+i]$$

Les racines carrées complexes de $8-6i$ sont : $\pm(3-i)$

• $-3-4i$ sont ??

$$u^2 = -3-4i \iff \begin{cases} x^2-y^2=-3 & (1) \\ x^2+y^2 = |-3-4i| = \sqrt{3^2+4^2} = 5 & (2) \\ 2xy = -4 & (3) \end{cases}$$

$$\iff \begin{cases} 2x^2 = 2 & (1)+(2) \\ 2y^2 = 8 & (2)-(1) \\ xy = -2 < 0 & (3) \end{cases} \iff \begin{cases} x = \pm 1 \\ y = \pm 2 \\ xy < 0 \end{cases}$$

$$\iff [(x=-1 \text{ et } y=2) \text{ ou } (x=1 \text{ et } y=-2)]$$

$$\iff [u = -1+2i \text{ ou } u = 1-2i]$$

Les racines carrées complexes de $-3-4i$ sont : $\pm(1-2i)$

• $40i+9$ sont ??

$$u^2 = 40i+9 \iff \begin{cases} x^2-y^2 = 9 & (1) \\ x^2+y^2 = |40i+9| = \sqrt{40^2+9^2} = \sqrt{1681} = 41 & (2) \\ 2xy = 40 & (3) \end{cases}$$

$$\iff \begin{cases} 2x^2 = 50 & (1)+(2) \\ 2y^2 = 32 & (2)-(1) \\ xy = 20 & (3) \end{cases}$$

$$\iff [x = \pm 5 \text{ et } y = \pm 4 \text{ et } xy > 0]$$

$$\iff [(x=5 \text{ et } y=4) \text{ ou } (x=-5 \text{ et } y=-4)]$$

$\Leftrightarrow [u = 5+4i \text{ ou } u = -5-4i]$

Les racines carrées complexes de $-3-4i$ sont : $\pm(5+4i)$

2) $z^4 = -119+120i \Leftrightarrow \begin{cases} Z = z^2 \\ Z^2 = -119+120i \end{cases}$

Résolvons $Z^2 = -119+120i$

$Z^2 = -119+120i \Leftrightarrow \begin{cases} X^2 - Y^2 = -119 & (1) \\ X^2+Y^2 = |-119+120i| = \sqrt{119^2+120^2} = 169 & (2) \\ 2XY = 120 & (3) \end{cases}$

$(Z = X+iY)$

$\Leftrightarrow \begin{cases} 2X^2 = 50 & (1)+(2) \\ 2Y^2 = 288 & (2)-(1) \\ XY = 60 & (3) \end{cases} \Leftrightarrow \begin{cases} X^2 = 25 \\ Y^2 = 144 \\ XY > 0 \end{cases}$

$\Leftrightarrow \begin{cases} X = \pm 5 \text{ et } Y = \pm 12 \\ XY > 0 \end{cases}$

$\Leftrightarrow [Z = 5+12i \text{ ou } Z = -5-12i]$

Donc $Z^4 = -119+120i \Leftrightarrow \begin{cases} z^2 = 5+12i & (E_1) \\ \text{ou} \\ z^2 = -5-12i & (E_2) \end{cases}$

Or $(E_1) \Leftrightarrow z^2 = 5+12i \Leftrightarrow \begin{cases} x^2-y^2 = 5 & (1) \\ x^2+y^2 = |5+12i| = 13 & (2) \\ xy = 6 & (3) \end{cases}$

$(z = x+iy)$

$\Leftrightarrow \begin{cases} x^2 = 9 & (1)+(2) \\ y^2 = 4 & (2)-(1) \\ xy > 0 & (3) \end{cases}$

$\Leftrightarrow [(x = 3 \text{ et } y = 2) \text{ ou } (x = -3 \text{ et } y = -2)]$

$\Leftrightarrow z = \pm(3+2i)$

et $(E_2) \Leftrightarrow z^2 = -5-12i \Leftrightarrow -z^2 = 5+12i$

$\Leftrightarrow (iz)^2 = 5+12i$

$\Leftrightarrow iz$ est solution de (E_1)

$\Leftrightarrow iz = \pm(3+2i)$

$\Leftrightarrow z = \pm \dfrac{3+2i}{i}$

$\Leftrightarrow z = \pm((3+2i)\cdot(-i))$

$$\Longleftrightarrow \quad z = \pm(-3i+2)$$
$$\Longleftrightarrow \quad z = \pm(2-3i)$$

Donc $z^4 = -119+120i \Longleftrightarrow \left[z = \pm(3+2i) \text{ ou } z = \pm(2-3i) \right]$

Exercice 10 :

- $z^2 + (1-4i)z + i - 5 = 0$

→ On calcule Δ :

$$\Delta = (1-4i)^2 - 4\times 1 \times (i-5) = 1 - 8i - 16 - 4i + 20 = 5-12i \neq 0$$

→ on calcule les racines carrées δ de Δ $\quad (\delta = x+iy)$

$$\delta^2 = \Delta \Longleftrightarrow \delta^2 = 5-12i \Longleftrightarrow \begin{cases} x^2-y^2 = 5 \\ x^2+y^2 = |5-12i| = 13 \\ 2xy = -12 \end{cases} \Longleftrightarrow \begin{cases} 2x^2 = 18 \\ 2y^2 = 8 \\ xy = -6 \end{cases}$$

$$\Longleftrightarrow \left[x = \pm 3 \text{ et } y = \pm 2 \text{ et } xy < 0 \right]$$

$$\Longleftrightarrow \delta = \pm(3-2i)$$

→ On calcule les solutions de l'équation

$$z = \frac{-(1-4i)+(3-2i)}{2} \quad \text{ou} \quad z = \frac{-(1-4i)-(3-2i)}{2}$$
$$= 1+i \qquad\qquad = -2+3i$$

→ On conclut :

$$\mathcal{S} = \{1+i, -2+3i\}$$

- $-iz^2 + 2(1+i)z + 5(2+i) = 0$

→ Calcul du discriminant Δ

$$\Delta = (2(1+i))^2 - 4\times(-i)\times(5(2+i))$$
$$= 4(1+i)^2 + 20i(2+i)$$
$$= 4(\underbrace{1+2i-1}_{2i}) + 40i - 20$$
$$= 48i - 20 \neq 0$$

→ Calcul des racines carrées complexes de Δ

$$\delta^2 = \Delta \Longleftrightarrow \begin{cases} x^2-y^2 = -20 & (1) \\ x^2+y^2 = |20+48i| = 4|5+12i| = 4\underbrace{\sqrt{25+144}}_{13} = 52 & (2) \\ 2xy = 48 & (3) \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2x^2 = 32 & (1)+(2) \\ 2y^2 = 72 & (2)-(1) \\ xy = 24 & (3) \end{cases} \Longleftrightarrow \begin{cases} x = \pm 4 \\ y = \pm 6 \\ xy > 0 \end{cases} \Longleftrightarrow \delta = \pm(4+6i)$$

→ Calcul des solutions de l'équation.

$$z = \frac{-2(1+i)+(4+6i)}{2\times(-i)} \quad \text{ou} \quad z = \frac{-2(1+i)-(4+6i)}{2\times(-i)}$$

$$= \frac{2+4i}{2\times(-i)} \qquad\qquad = \frac{-6-8i}{2\times(-i)}$$

$$= \frac{1+2i}{-i} \qquad\qquad = \frac{3+4i}{i}$$

$$= (1+2i)\times i \qquad\qquad = (3+4i)(-i)$$

$$= -2+i \qquad\qquad = 4-3i$$

→ Conclusion : $S = \{-2+i,\ 4-3i\}$

• $(1-i)z^2+(3+6i)z-7+i=0$

→ $\Delta = (3+6i)^2 - 4(1-i)(-7+i)$

$$= 9+36i-36-4(\underbrace{-7+i+7i+1}_{-6+8i})$$

$$= -27+36i+24-32i$$

$$= -3+4i$$

→ $\delta^2 = \Delta \iff \begin{cases} x^2-y^2=-3 \\ x^2+y^2=|-3+4i|=5 \\ 2xy=4 \end{cases} \iff \begin{cases} x^2=1 \\ y^2=4 \\ xy=2>0 \end{cases} \iff \begin{cases} x=\pm 1 \\ y=\pm 2 \\ xy>0 \end{cases}$

$$\iff \delta = \pm(1+2i)$$

→ $z = \dfrac{-(3+6i)+(1+2i)}{2(1-i)}$ ou $z = \dfrac{-(3+6i)-(1+2i)}{2(1-i)}$

$$= \frac{-2-4i}{2(1-i)} \qquad\qquad = \frac{-4-8i}{2(1-i)}$$

$$= \frac{-1-2i}{1-i} \qquad\qquad = \frac{-2-4i}{1-i}$$

$$= (-1-2i)\,\frac{1+i}{2} \qquad\qquad = (-2-4i)\,\frac{1+i}{2}$$

$$= \frac{1}{2}(-1-i-2i+2) \qquad\qquad = \frac{1}{2}(-2-2i-4i+4)$$

$$= \frac{1}{2}-\frac{3}{2}i \qquad\qquad = 1-3i$$

→ $S = \left\{\frac{1}{2}-\frac{3}{2}i,\ 1-3i\right\}$

• $z^4+2z^2+4=0 \iff \begin{cases} Z=z^2 \\ Z^2+2Z+4=0 \end{cases}$

$\rightarrow$ Solutions de $Z^2 + 2Z + 4 = 0$

$\Delta = 2^2 - 4 \times 4 = -12$; donc $\delta = \pm i\sqrt{12} = \pm 2i\sqrt{3}$

Donc $\begin{cases} Z = \dfrac{-2 + 2i\sqrt{3}}{2} = -1 + i\sqrt{3} \\ \text{ou} \\ Z = \dfrac{-2 - 2i\sqrt{3}}{2} = -1 - i\sqrt{3} \end{cases}$

On en déduit que :

$$z^4 + 2z^2 + 4 = 0 \iff \begin{cases} z^2 = -1 + i\sqrt{3} \quad (E_1) \\ \text{ou} \\ z^2 = -1 - i\sqrt{3} \quad (E_2) \end{cases}$$

On a ;

$(E_1) \iff z^2 = -1 + i\sqrt{3}$ $(z = x + iy)$ $\iff \begin{cases} x^2 - y^2 = -1 \\ x^2 + y^2 = |-1 + i\sqrt{3}| = 2 \\ 2xy = \sqrt{3} \end{cases} \iff \begin{cases} x^2 = \frac{1}{2} \\ y^2 = 3/2 \\ xy = \frac{\sqrt{3}}{2} \end{cases}$

$\iff \left[x = \pm \frac{1}{\sqrt{2}} \text{ et } y = \pm\sqrt{\frac{3}{2}} \text{ et } xy > 0\right]$

$\iff z = \pm\left(\frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\right)$

$(E_2) \iff z^2 = -1 - i\sqrt{3} \iff \overline{z^2} = \overline{-1 - i\sqrt{3}}$

$\iff \overline{z}^2 = -1 + i\sqrt{3}$

$\iff \overline{z}$ solution de (E_1)

$\iff \overline{z} = \pm\left(\frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\right)$

$\iff z = \overline{\pm\left(\frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\right)}$

$\iff z = \pm\left(\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}\right)$

Donc $\mathcal{S} = \left\{ \pm\left(\frac{1}{\sqrt{2}} + i\sqrt{\frac{3}{2}}\right), \pm\left(\frac{1}{\sqrt{2}} - i\sqrt{\frac{3}{2}}\right) \right\}$

- $z^4 - (5 - 14i)z - 2(5i + 12) = 0 \iff \begin{cases} Z = z^2 \\ Z^2 - (5 - 14i)Z - 2(5i + 12) = 0 \end{cases}$

$\rightarrow$ Solutions de $Z^2 - (5 - 14i)Z - 2(5i + 12) = 0$

$\Delta = \left(-(5 - 14i)\right)^2 - 4 \times \left(-2(5i + 12)\right)$

$= (5 - 14i)^2 + 8(5i + 12)$

$= 25 - 140i - 196 + 40i + 96$

$= -75 - 100i$

$$\delta^2 = \Delta \iff \begin{cases} x^2 - y^2 = -75 \\ x^2 + y^2 = |-75 - 100i| = 25\,\overbrace{|-3-4i|}^{5} = 125 \\ 2xy = -100 \end{cases}$$

$$\iff \begin{cases} x^2 = 25 \\ y^2 = 100 \\ xy = -50 < 0 \end{cases} \iff \delta = \pm(5 - 10i)$$

$$Z = \frac{(5-14i) + (5-10i)}{2} \quad \text{ou} \quad Z = \frac{(5-14i) - (5-10i)}{2}$$

$$= 5 - 12i \qquad\qquad = -2i$$

On en déduit que :

$$z^4 - (5-14i)z - 2(5i+12) = 0 \iff \begin{cases} z^2 = 5 - 12i \quad (E_1) \\ \text{ou} \\ z^2 = -2i \quad (E_2) \end{cases}$$

On a :

$$(E_1) \iff z^2 = 5 - 12i \iff \begin{cases} x^2 - y^2 = 5 \\ x^2 + y^2 = |5 - 12i| = \sqrt{169} = 13 \\ 2xy = -12 \end{cases}$$

$$\iff \begin{cases} x^2 = 9 \\ y^2 = 4 \\ xy = -6 \end{cases} \iff \begin{cases} x = \pm 3 \\ y = \pm 2 \\ xy < 0 \end{cases} \iff z = \pm(3 - 2i)$$

$$(E_2) \iff z^2 = -2i \iff \begin{cases} x^2 - y^2 = 0 \\ x^2 + y^2 = |-2i| = 2 \\ 2xy = -2 \end{cases} \iff \begin{cases} x^2 = 1 \\ y^2 = 1 \\ xy = -1 \end{cases} \iff \begin{cases} x = \pm 1 \\ y = \pm 1 \\ xy < 0 \end{cases}$$

$$\iff z = \pm(1 - i)$$

Conclusion : $S = \{\pm(3 - 2i), \pm(1 - i)\}$