Feuille de TD n°1.

Ex7.

3) • Mq : $\forall x \in \mathbb{R}^{*+}$

$$\frac{1}{x(x+1)(x+3)} = \frac{1/6}{x} - \frac{1/2}{x+1} + \frac{1/2}{x+2} - \frac{1/6}{x+3}$$

$$= \frac{1/6(x+1)(x+2)(x+3) - 1/2\, x(x+2)(x+3) + 1/2\, x(x+1)(x+3) - 1/6\, x(x+1)(x+2)}{x(x+1)(x+2)(x+3)}$$

$$= \frac{1/6\left[(x+1)(x+3)(x+2) - x(x+1)(x+2)\right] + 1/2\left[x(x+1)(x+3) - x(x+2)(x+3)\right]}{x(x+1)(x+2)(x+3)}$$

$$= \frac{1/6\left[(x+1)(x+2)\left((x+3)-x\right)\right] + 1/2\left[x(x+3)\left((x+1)-(x+2)\right)\right]}{x(x+1)(x+2)(x+3)}$$

$$= \frac{1/2(x+1)(x+2) - 1/2\, x(x+3)}{x(x+1)(x+2)(x+3)}$$

$$= \frac{1}{2}\,\frac{(x^2+2x+x+2) - (x^2+3x)}{x(x+1)(x+2)(x+3)}$$

$$= \frac{1}{x(x+1)(x+2)(x+3)}$$

• $$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \sum_{k=1}^{n}\left(\frac{1/6}{k} - \frac{1/2}{k+1} + \frac{1/2}{k+2} - \frac{1/6}{k+3}\right)$$

$$= \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+2} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k+3}$$

$$= \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2}\sum_{l=2}^{n+1}\frac{1}{l} + \frac{1}{2}\sum_{m=3}^{n+2}\frac{1}{m} - \frac{1}{6}\sum_{p=4}^{n+3}\frac{1}{p}$$

$$\begin{cases} l = k+1 \\ k=1 \to l=2 \\ k=n \to l=n+1 \end{cases} \quad \begin{cases} m = k+2 \\ k=1 \to m=3 \\ k=n \to m=n+2 \end{cases} \quad \begin{cases} p = k+3 \\ p=4 \\ p=n+3 \end{cases}$$

$$= \frac{1}{6}\left(1 + \frac{1}{2} + \frac{1}{3} + \sum_{k=4}^{n}\frac{1}{k}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \sum_{l=4}^{n}\frac{1}{l} + \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{3} + \sum_{m=4}^{n}\frac{1}{m} + \frac{1}{n+1} + \frac{1}{n+2}\right) - \frac{1}{6}\left(\sum_{p=4}^{n}\frac{1}{p} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{1}{6}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{3}+\frac{1}{n+1}+\frac{1}{n+2}\right)$$

$$-\frac{1}{6}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$$

$$= \frac{1}{18} - \frac{1}{6}\,\frac{1}{n+1} + \frac{1}{3}\,\frac{1}{n+2} - \frac{1}{6}\,\frac{1}{n+3}$$

2ème méthode

$$\sum_{k=1}^{n}\left(\frac{1/6}{k} - \frac{1/2}{k+1} + \frac{1/2}{k+2} - \frac{1/6}{k+3}\right)$$

$$= \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+2} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k+3}$$

$$= \frac{1}{6}\left(1+\frac{1}{2}+\frac{1}{3}+\cancel{\frac{1}{4}}+\cancel{\dots}+\cancel{\frac{1}{n}}\right) - \frac{1}{2}\left(\frac{1}{2}+\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\dots+\cancel{\frac{1}{n+1}}\right) + \frac{1}{2}$$

$$\left(\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\cancel{\dots}+\frac{1}{n+2}\right) - \frac{1}{6}\left(\cancel{\frac{1}{4}}+\cancel{\frac{1}{5}}+\cancel{\dots}+\frac{1}{n+3}\right)$$

$$= \frac{1}{6}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{n+2} - \frac{1}{6}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right).$$

Ex 8

$$S_n = \sum_{k=0}^{n} q^k$$

<u>si $q = 1$</u> : $q^k = 1^k = 1$.

1) $S_n = \sum_{k=0}^{n} 1 = n+1$.

2) si $q \neq 1$

a) $qS_n = q\sum_{k=0}^{n} q^k$

$$= \sum_{k=0}^{n} q^{k+1} \qquad j = k+1 \begin{cases} k=0 \to j=1 \\ k=n \to j=n+1 \end{cases}$$

$$= \sum_{j=1}^{n+1} q^j$$

b) $qS_n - S_n = \sum_{j=1}^{n+1} q^j - \sum_{k=0}^{n} q^k$

$$= \left(\cancel{q}+\cancel{q^2}+\dots+\cancel{q^n}+q^{n+1}\right) - \left(1+\cancel{q}+\cancel{q^2}+\dots+\cancel{q^n}\right).$$

$$= q^{n+1} - 1.$$

c) $(q-1)\, S_n = q^{n+1} - 1$

Donc comme $q-1 \neq 0$

$$S_n = \frac{q^{n+1}-1}{q-1} = \frac{-(q^{n+1}-1)}{-(q-1)}$$

$$= \frac{1-q^{n+1}}{1-q}.$$

Ex 9.

• $\displaystyle\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$S_n = \sum_{k=0}^{n} k^3$$

1) $\displaystyle S_{n+1} = \sum_{k=0}^{n+1} k^3 = \sum_{k=1}^{n+1} k^3 = \sum_{j=0}^{n} (j+1)^3 = \sum_{k=0}^{n} (k+1)^3.$

$$\left\{\begin{array}{l} j = k-1 \Leftrightarrow k = j+1 \\ k = 1 \rightarrow j = 0 \\ k = n+1 \rightarrow j = n. \end{array}\right\}$$

2) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

Donc :

$$(k+1)^3 = k^3 + 3k^2 * 1 + 3k * 1^2 + 1^3$$

$$= k^3 + 3k^2 + 3k + 1.$$

$$S_{n+1} = \sum_{k=0}^{n} (k+1)^3$$

$$= \sum_{k=0}^{n} \left(k^3 + 3k^2 + 3k + 1\right).$$

3) $(n+1)^3 = S_{n+1} - S_n$

$$S_{n+1} = \sum_{k=0}^{n+1} k^3 = \underbrace{\sum_{k=0}^{n} k^3}_{S_n} + (n+1)^3.$$

$S_{n+1} = S_n + (n+1)^3$ donc $(n+1)^3 = S_{n+1} - S_n$.

$$S_{n+1} - S_n = 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + (n+1)$$

$$S_{n+1} - S_n = \sum_{k=0}^{n} (k^3 + 3k^2 + 3k + 1) - \sum_{k=0}^{n} k^3$$

$$= \cancel{\sum_{k=0}^{n} k^3} + 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 - \cancel{\sum_{k=0}^{n} k^3}$$

$$= 3\sum_{k=0}^{n} k^2 + 3\sum_{k=0}^{n} k + (n+1).$$

4) $(n+1)^3 = 3\sum_{k=0}^{n} k^2 + 3\underbrace{\sum_{k=0}^{n} k}_{\frac{n(n+1)}{2}} + (n+1)$

$$3\sum_{k=0}^{n} k^2 = \frac{1}{3}\left[(n+1)^3 - 3\,\frac{n(n+1)}{2} - (n+1)\right]$$

$$= \frac{1}{3}(n+1)\left[n^2 + 2n + 1 - \frac{3n}{2} - 1\right].$$

$$= \frac{1}{3}(n+1)\left(n^2 + \frac{1}{2}n\right)$$

$$= \frac{1}{3}(n+1)\,\frac{2n^2+n}{2}$$

$$= \frac{1}{6}(n+1)\,n\,(2n+1)$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

Faire Exercice 10.

Ex 11.

1) $1! = 1$

$2! = 2$

$3! = 6$

$4! = 3! \times 4 = 24$

$5! = 4! \times 5 = 120$

$6! = 5! \times 6 = 720.$

2) a) $\dfrac{7!}{4!} = \dfrac{4! \times 5 \times 6 \times 7}{4!} = 210.$

b) $\dfrac{6!}{2!\,4!} = \dfrac{4! \times 5 \times 6}{2 \times 4!} = 15.$

c) $\dfrac{5!\,7!}{(6!)^2} = \dfrac{5! \times 6! \times 7}{(5! \times 6) \times 6!} = \dfrac{7}{6}.$

d) $\dfrac{4!\,5!}{6!\,3!} = \dfrac{(\cancel{3!} \times 4) \times \cancel{5!}}{(\cancel{5!} \times 6) \times \cancel{3!}} = \dfrac{2}{3}.$

e) $\dfrac{(3!)^2}{36} = \dfrac{6^2}{36} = \dfrac{36}{36} = 1.$

f) $(3!)! = 6! = 720.$

g) $\frac{(2n)!}{2 \times n!} = \frac{(2n)(2n-1)(2n-2) \cdots \times (n+1)}{2} = (2n-1)(2n-2) \times \cdots \times (n+1) \times n.$

Finir Ex 11 et faire Ex 12.