MAO102 - S1A3
16/09/2020

Feuille de TD N°1

Ex 11 :

h) $$\frac{(n!)^2}{(2n)!} = \frac{(\cancel{1\times 2\times 3\times \cdots \times n})\times((n-1)!\times \cancel{n})}{\cancel{1\times 2\times 3\times \cdots\times n}\times(n+1)\times\cdots\times(2\cancel{n})}$$

$$= \frac{(n-1)!}{\underbrace{(n+1)(n+2)\cdots(2n-1)}\times 2}$$

$$= \frac{(n-1)!}{2\times \prod_{k=n+1}^{2n-1} k}$$

i) $$\frac{n!\,(n+1)!}{(n-1)!\,(n+3)!} = \frac{(\cancel{(n-1)!}\times n)\times \cancel{(n+1)!}}{\cancel{(n-1)!}\,(\cancel{(n+1)!}\times(n+2)\times(n+3))} = \frac{n}{(n+2)\times(n+3)}$$

Ex 12

a) $$\prod_{k=1}^{n}\left(\frac{k-1}{k+1}\right) = \overbrace{\underbrace{\frac{1-1}{1+1}}_{k=1}}^{\frac{0}{2}=0}\times\underbrace{\frac{2-1}{2+1}}_{k=2}\times\underbrace{\frac{3-1}{3+1}}_{k=3}\times\cdots\cdots\times\underbrace{\frac{n-1}{n+1}}_{k=n} = 0$$

b) $$\prod_{k=2}^{n}\frac{k-1}{k+1} = \underbrace{\frac{2-1}{2+1}}_{k=2}\times\underbrace{\frac{3-1}{3+1}}_{k=3}\times\cdots\cdots\times\underbrace{\frac{n-1}{n+1}}_{k=n}$$

$$= \frac{1\times 2\times\cancel{3}\times\cancel{\cdot}\times\cancel{\cdot}\cdots\times(\cancel{n-1})}{\cancel{3}\times\cancel{4}\times\cancel{5}\times\cdots\times(\cancel{n-1})\times n\times(n+1)}$$

$$= \frac{1\times 2}{n\times(n+1)} = \frac{2}{n(n+1)}$$

$$\prod_{k=2}^{n}\frac{k-1}{k+1} = \frac{\prod_{k=2}^{n}(k-1)}{\prod_{k=2}^{n}(k+1)}$$

$$= \frac{\prod_{j=1}^{n-1} j}{\prod_{j=3}^{n+1} j}$$

$\begin{cases} j = k-1 \\ k=2 \to j = 2-1 = 1 \\ k = n \to j = n-1 \end{cases}$ $\begin{cases} j = k+1 \\ k = 2 \to j = 3 \\ k = n \to j = n+1 \end{cases}$

$$\prod_{k=?}^{?}(a_k b_k) = \prod_{k=?}^{?} a_k \times \prod_{k=?}^{?} b_k$$

$$\prod_{k=.}^{.}\frac{a_k}{b_k} = \frac{\prod_{k=.}^{.} a_k}{\prod_{k=.}^{.} b_k}$$

$$\begin{cases} \prod_{k=1}^{n} a\,a_k = a^n \prod_{k=1}^{n} a_k \\ \sum_{k=1}^{n} a\,a_k = a\sum_{k=1}^{n} a_k \end{cases}$$

$$= \frac{1 \times 2 \times \cancel{\prod_{j=3}^{n-1} j}}{\cancel{\prod_{j=3}^{n-1} j} \times n \times (n+1)} = \frac{2}{n(n+1)}$$

c) $$\prod_{k=1}^{n} \frac{k(k+1)}{k+2} = \frac{\prod_{k=1}^{n} k \times \prod_{k=1}^{n} (k+1)}{\prod_{k=1}^{n} (k+2)}$$

$$= \frac{\overbrace{\prod_{k=1}^{n} k}^{n!} \times \prod_{j=2}^{n+1} j}{\prod_{j=3}^{n+2} j}$$

$$\begin{cases} j = k+1 \\ k = 1 \rightarrow j = 2 \\ k = n \rightarrow j = n+1 \end{cases}$$

$$\begin{cases} j = k+2 \\ k = 1 \rightarrow j = 3 \\ k = n \rightarrow j = n+2 \end{cases}$$

$$= \frac{n! \times \left(2 \times \cancel{\prod_{j=3}^{n+1} j}\right)}{\left(\cancel{\prod_{j=3}^{n+1} j} \times (n+2)\right)}$$

$$= \frac{2 \times n!}{n+2}$$

$$\prod_{j=3}^{n+2} j = \underbrace{3 \times 4 \times 5 \times \cdots\cdots (n+1)}_{} \times (n+2)$$
$$= \prod_{j=3}^{n+1} j \times (n+2)$$

d) $$\prod_{k=3}^{n} \frac{k^2-1}{k^2-4} = \prod_{k=3}^{n} \frac{(k-1)(k+1)}{(k-2)(k+2)}$$

$$\begin{cases} k^2 - 1 = (k-1)(k+1) \\ k^2 - 4 = (k-2)(k+2) \end{cases}$$

$$= \frac{\prod_{k=3}^{n} (k-1) \times \prod_{k=3}^{n} (k+1)}{\prod_{k=3}^{n} (k-2) \times \prod_{k=3}^{n} (k+2)}$$

$$= \frac{\prod_{j=2}^{n-1} j \times \prod_{j=4}^{n+1} j}{\prod_{j=1}^{n-2} j \times \prod_{j=5}^{n+2} j}$$

$j = k-1 \begin{cases} k=3 \rightarrow j=2 \\ k=n \rightarrow j=n-1 \end{cases}$

$j = k+1 \begin{cases} k=3 \rightarrow j=4 \\ k=n \rightarrow j=n+1 \end{cases}$

$j = k-2 \begin{cases} k=3 \rightarrow j=1 \\ k=n \rightarrow j=n-2 \end{cases}$

$j = k+2 \begin{cases} k=3 \rightarrow j=5 \\ k=n \rightarrow j=n+2 \end{cases}$

$$= \frac{\left(\cancel{\prod_{j=2}^{n-2} j} \times (n-1)\right) \times \left(4 \times \cancel{\prod_{j=5}^{n+1} j}\right)}{\left(1 \times \cancel{\prod_{j=2}^{n-2} j}\right) \times \left(\cancel{\prod_{j=5}^{n+1} j} \times (n+2)\right)}$$

$$= \frac{4\,(n-1)}{n+2}$$

Exo

1) $x, x', y, y' \in \mathbb{R}$

$\left.\begin{array}{l} x \leq y \\ \text{et} \\ x' \leq y' \end{array}\right\} \Longrightarrow x + x' \leq y + y'$ VRAIE

$x + x' \leq y + y' \Longrightarrow \begin{cases} x \leq y \\ \text{et} \\ x' \leq y' \end{cases}$ FAUSSE

Justification :

Contre-exemple : $\begin{cases} x = 0 \\ x' = 1 \\ y = 2 \\ y' = 0 \end{cases}$

on a : $\overbrace{x + x'}^{1} \leq \overbrace{y + y'}^{2}$

et on n'a pas $\begin{cases} \overbrace{x \leq y}^{0 \leq 2} \\ \text{et} \\ \underbrace{x' \leq y'}_{1 \leq 0 \;\rightarrow\; \text{Fausse}} \end{cases}$

2.) $\left.\begin{array}{l} 0 \leq x \leq y \\ \text{et} \\ 0 \leq x' \leq y' \end{array}\right\} \Longrightarrow xx' \leq yy'$ Vraie (cours)

$\left.\begin{array}{l} x \leq y \\ x' \leq y' \end{array}\right\} \Longrightarrow xx' \leq yy'$ Fausse

Contre-exemple : $\begin{cases} x = -1 \\ x' = -2 \\ y = 0 \\ y' = 1 \end{cases}$

Pour ces valeurs choisies, on a : $\begin{cases} \overbrace{x \leq y}^{-1 \leq 0 \text{ vraie}} \\ \text{et} \\ \underbrace{x' \leq y'}_{-2 \leq 1 \text{ vraie}} \end{cases}$

mais on n'a pas $\underbrace{xx' \leq yy'}_{2 \leq 0 \;\rightarrow\; \text{Fausse}}$

3.) a) $a, b \in]0, +\infty[$

Mq : $a > b \Longleftrightarrow \frac{a}{b} > 1$

$\Longleftrightarrow$: $\begin{cases} \Longrightarrow \\ \Longleftarrow \end{cases}$

si et seulement si
équivalent

- Mq $a > b \Longrightarrow \frac{a}{b} > 1$:

(H) Supposons que $a > b$

(D) On a $0 < b < a$ et $\frac{1}{b} > 0$; donc

$b \times \frac{1}{b} < a \times \frac{1}{b}$; c'est à dire $1 < \frac{a}{b}$

(C) Donc $\frac{a}{b} > 1$

- Mq $\frac{a}{b} > 1 \Longrightarrow a > b$

(H) Supposons que $\frac{a}{b} > 1$

(D) On a $\frac{a}{b} > 1$ et $b > 0$; donc

$\left(\frac{a}{b} \times b\right) > (1 \times b)$; c'est à dire : $a > b$

(C) Donc $a > b$

b) $a \neq 0$ et $b \neq 0$

$$\frac{a}{b} > 1 \iff a > b$$ Fausse (q^d on ne suppose pas $a > 0$ et $b > 0$)

Contre-exemple : ($\iff$ est vrai lorsque les 2 membres de $\iff$ sont tous les 2 vrais, ou bien tous les 2 faux

[$2 < 1 \iff \pi = 4$] ← Enoncé vrai

[$2 < 3 \iff \pi \neq 4$] ← //

[$2 < 1 \iff \pi \neq 4$] ← Enoncé faux

$\begin{cases} a = 1 \\ b = -2 \end{cases}$ on a : $\begin{cases} \overbrace{\frac{a}{b}}^{-1/2} > 1 \text{ est fausse.} \\ \underbrace{a > b}_{1 > -2} \text{ est vraie} \end{cases}$

c) $\boxed{a, b < 0}$

- $\underbrace{\frac{a}{b} > 1}_{H} \implies \underbrace{\frac{a}{b} \times b < 1 \times b}_{\text{car } b < 0}$ $\underbrace{(a < b)}_{C}$

on vient de montrer que $\frac{a}{b} > 1 \implies a < b$

- $\underbrace{a < b}_{H} \implies a \times \frac{1}{b} \underset{\text{car } \frac{1}{b} < 0}{>} b \times \frac{1}{b}$ $\underbrace{\left(\frac{a}{b} > 1\right)}_{C}$

on a aussi : $a < b \implies \frac{a}{b} > 1$

$$\boxed{\text{Si } a, b < 0 \text{, on a : } a < b \iff \frac{a}{b} > 1}$$

4°) $a, b \in \mathbb{R}$ et $b \neq 0$:

$\left(\frac{a}{b}\right) \times (ab) = a^2 \geqslant 0$

Si $a = 0$: $\frac{a}{b} = 0$ et $ab = 0$

Si $a \neq 0$: alors $a^2 > 0$

donc $\left(\frac{a}{b}\right) \times (ab) > 0$

donc : $\begin{cases} \frac{a}{b} > 0 \text{ et } ab > 0 \\ \text{ou} \\ \frac{a}{b} < 0 \text{ et } ab < 0 \end{cases}$

Ex 13 (uniquement les réponses)

1°) • $\mathcal{D}_a = \mathbb{R}$

• $a(x) = \frac{1}{2}\left(e^{2x} + e^{x}\right)\left(e^{x} + 1\right)$

2°) $f'(x) = \dfrac{-2e^{3x} + 3e^{2x} + 3}{e^{x}\left(e^{2x}+3\right)^2}$

3°) • $\mathcal{D}_a = \mathbb{R}$ et $a(x) = x^2 + 4$

• $\mathcal{D}_b = \mathbb{R}^*$ et $b(x) = \sqrt{x^2+1}$

4°) $S_n = 1 + 1/2 - \frac{1}{n+1} - \frac{1}{n+2}$

5°) (i) $p_1 = 7$ (ii) $p_n = n(n+2)$

(iii) $p'_n = \dfrac{n+2}{2(n+1)}$.