Corrigé du TD N°1 - Exercices 11 à 13

Exercice 11 :

1) $\frac{1+z}{1-z} \in i\mathbb{R} \iff \frac{1+z}{1-z} + \overline{\left(\frac{1+z}{1-z}\right)} = 0 \iff \frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}} = 0$

$$\iff \frac{(1+z)(1-\bar{z}) + (1+\bar{z})(1-z)}{(1-z)(1-\bar{z})} = 0$$

$$\iff (1+z)(1-\bar{z}) + (1+\bar{z})(1-z) = 0$$

$$\iff (1-\bar{z}+z-z\bar{z}) + (1-z+\bar{z}-\bar{z}z) = 0$$

$$\iff 2 - 2z\bar{z} = 0$$

$$\iff z\bar{z} = 1$$

$$\iff |z|^2 = 1$$

$$\iff |z| = 1 \quad (\text{car } |z| \geq 0)$$

2) a) $Z \in \mathbb{R} \iff Z - \bar{Z} = 0 \iff \frac{z+2i}{z-2i} - \overline{\left(\frac{z+2i}{z-2i}\right)} = 0$

$$\iff \frac{z+2i}{z-2i} - \frac{\bar{z}-2i}{\bar{z}+2i} = 0$$

$$\iff \frac{(z+2i)(\bar{z}+2i) - (\bar{z}-2i)(z-2i)}{(z-2i)(\bar{z}+2i)} = 0$$

$$\iff (z+2i)(\bar{z}+2i) - (\bar{z}-2i)(z-2i) = 0$$

$$\iff (z\bar{z} + 2iz + 2i\bar{z} - 4) - (\bar{z}z - 2i\bar{z} - 2iz - 4) = 0$$

$$\iff 4iz + 4i\bar{z} = 0$$

$$\iff 4i(\bar{z} + \bar{z}) = 0$$

$$\iff z + \bar{z} = 0$$

$$\iff z \in i\mathbb{R}$$

$$\iff M(z) \in (Oy)$$

b) $Z \in i\mathbb{R} \iff Z + \bar{Z} = 0 \iff \frac{z+2i}{z-2i} + \overline{\left(\frac{z+2i}{z-2i}\right)} = 0$

$$\iff \frac{z+2i}{z-2i} + \frac{\bar{z}-2i}{\bar{z}+2i} = 0$$

$$\iff \frac{(z+2i)(\bar{z}+2i) + (\bar{z}-2i)(z-2i)}{(z-2i)(\bar{z}+2i)} = 0$$

$$\iff (z+2i)(\bar{z}+2i) + (\bar{z}-2i)(z-2i) = 0$$

$$\iff (z\bar{z} + 2iz + 2i\bar{z} - 4) + (\bar{z}z - 2i\bar{z} - 2iz - 4) = 0$$

$$\iff 2z\bar{z} - 8 = 0$$

$$\iff z\bar{z} - 4 = 0$$

$$\iff |z|^2 = 4$$

$$\iff |z| = 2 \quad (\text{car } |z| \geq 0)$$

$$\iff OM = 2$$

$\iff$ [M est sur le cercle de centre O et de rayon 2]

$\iff$ [M est sur le cercle de diamètre [AB]]

c) $|Z|=1 \iff |Z|^2 = 1 \iff Z\bar{Z} = 1$

$$\iff \left(\frac{z+2i}{z-2i}\right) \times \overline{\left(\frac{z+2i}{z-2i}\right)} = 1$$

$$\iff \frac{z+2i}{z-2i} \times \frac{\bar{z}-2i}{\bar{z}+2i} = 1$$

$$\iff \frac{(z+2i)(\bar{z}-2i)}{(z-2i)(\bar{z}+2i)} = 1$$

$$\iff (z+2i)(\bar{z}-2i) = (z-2i)(\bar{z}+2i)$$

$$\iff z\bar{z} - 2iz + 2i\bar{z} + 4 = z\bar{z} + 2iz - 2i\bar{z} + 4$$

$$\iff -4iz + 4i\bar{z} = 0$$

$$\iff (-4i)(z-\bar{z}) = 0$$

$$\iff z - \bar{z} = 0$$

$$\iff z \in \mathbb{R}$$

$$\iff M(z) \in (Ox)$$

d) $|Z| = 3 \iff |Z|^2 = 9 \iff Z\bar{Z} = 9$

$$\iff \left(\frac{z+2i}{z-2i}\right) \times \overline{\left(\frac{z+2i}{z-2i}\right)} = 9$$

$$\iff \frac{z+2i}{z-2i} \times \frac{\bar{z}-2i}{\bar{z}+2i} = 9$$

$$\iff (z+2i)(\bar{z}-2i) = 9(z-2i)(\bar{z}+2i)$$

$$\iff z\bar{z} - 2iz + 2i\bar{z} + 4 = 9(z\bar{z} + 2iz - 2i\bar{z} + 4)$$

$$\iff -8z\bar{z} - 20iz + 20i\bar{z} - 32 = 0$$

$$\iff 2z\bar{z} + 5iz - 5i\bar{z} + 8 = 0 \qquad (z = x+iy)$$

$$\iff 2(x^2+y^2) + 5i(x+iy) - 5i(x-iy) + 8 = 0$$

$$\iff 2(x^2+y^2) - 10y + 8 = 0$$

$$\iff x^2 + y^2 - 5y + 4 = 0$$

$$\iff x^2 + \left(y-\frac{5}{2}\right)^2 - \frac{25}{4} + 4 = 0$$

$$\iff x^2 + \left(y-\frac{5}{2}\right)^2 - \frac{9}{4} = 0$$

$$\iff x^2 + \left(y-\frac{5}{2}\right)^2 = \frac{9}{4}$$

$$\iff \sqrt{x^2 + \left(y-\frac{5}{2}\right)^2} = \frac{3}{2}$$

$$\iff \Omega M = 3/2 \qquad (\text{où } \Omega(0, 5/2))$$

$\iff$ [M(x, y) est sur le cercle de centre $\Omega(0, 5/2)$ et de rayon $\frac{3}{2}$]

Exercice 12 :

1) Calcul du discriminant Δ :

$$\Delta = (-(1+3i))^2 - 4(3i-4) = (1+3i)^2 - 4(3i-4)$$
$$= 1 + 6i - 9 - 12i + 16$$
$$= 8 - 6i \neq 0$$

racines carrées de Δ

$$\delta^2 = \Delta \iff \begin{cases} x^2 - y^2 = 8 & (1) \\ x^2 + y^2 = |8-6i| = 10 & (2) \\ 2xy = -6 & (3) \end{cases} \iff \begin{cases} 2x^2 = 18 & (2)+(1) \\ 2y^2 = 2 & (2)-(1) \\ xy = -3 & (3) \end{cases}$$

$$\iff \begin{cases} x^2 = 9 \\ y^2 = 1 \\ xy = -3 < 0 \end{cases} \iff \begin{cases} x = -3 \text{ et } y = 1 \\ \text{ou} \\ x = 3 \text{ et } y = -1 \end{cases} \iff \delta = \pm(3-i)$$

Solutions de l'équation :

$$z = \frac{(1+3i)-(3-i)}{2} \quad \text{ou} \quad z = \frac{(1+3i)+(3-i)}{2}$$

$$= -1+2i \qquad\qquad = 2+i$$

conclusion :

$$\mathcal{S} = \{-1+2i \,;\, 2+i\}$$

2) Posons $z_1 = 2+i$ et $z_2 = -1+2i$. Soient $M_1(z_1)$ et $M_2(z_2)$

- On a : $\begin{cases} OM_1 = |z_1 - 0| = |z_1| = |2+i| = \sqrt{5} \\ OM_2 = |z_2 - 0| = |z_2| = |-1+2i| = \sqrt{5} \end{cases}$

Donc $OM_1 = OM_2$. Par conséquent :

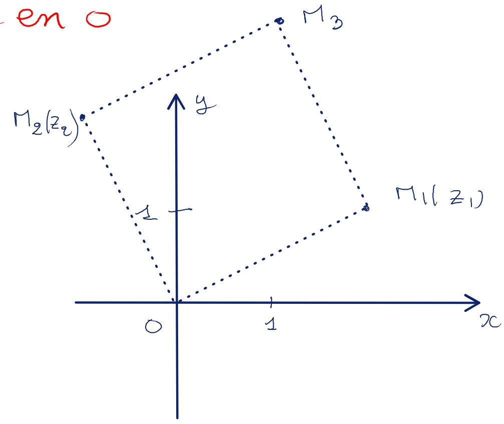
le triangle (OM_1M_2) est isocèle

- on a : $M_1M_2 = |z_2 - z_1| = |(-1+2i)-(2+i)| = |-3+i| = \sqrt{10}$

Donc : $M_1O^2 + OM_2^2 = (\sqrt{5})^2 + (\sqrt{5})^2 = 5+5 = 10 = (\sqrt{10})^2 = M_1M_2^2$

Par conséquent, d'après la réciproque du th de Pythagore :

le Triangle (OM_1M_2) est rectangle en O



3) On doit avoir $\overrightarrow{OM_3} = \overrightarrow{OM_1} + \overrightarrow{OM_2}$

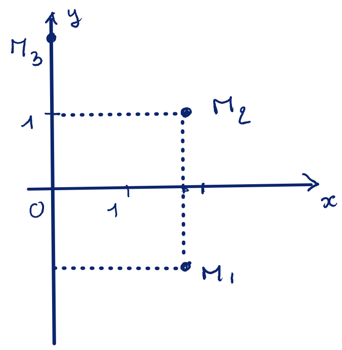
Or : $\overrightarrow{OM_1}(z_1)$, $\overrightarrow{OM_2}(z_2)$ et $\overrightarrow{OM_3}(z_3)$

Donc $z_3 = z_1 + z_2$

$= (2+i) + (-1+2i)$

Donc : $z_3 = 1+3i$

Exercice 13 :



1) $\begin{cases} OM_1 = |z_1 - 0| = |z_1| = |\sqrt{3} - i| = 2 \\ OM_2 = |z_2 - 0| = |z_2| = |\sqrt{3} + i| = 2 \\ OM_3 = |z_3 - i| = |z_3| = |2i| = 2 \end{cases}$

On a $OM_1 = OM_2 = OM_3$.

Donc M_1, M_2 et M_3 sont sur le cercle de centre O et de rayon 2

2) $z_2 - z_1 = 2i$ et $z_2 - z_3 = \sqrt{3} - i$

- Or $\overrightarrow{M_1M_2}\ (\underbrace{z_2 - z_1}_{2i})$ et $\overrightarrow{OM_3}\ (\underbrace{z_3}_{2i})$; donc $\overrightarrow{M_1M_2} = \overrightarrow{OM_3}$

Par conséquent $(OM_1M_2M_3)$ est un parallélogramme.

- $OM_1 = |z_1| = 2$ et $M_1M_2 = |z_2 - z_1| = |2i| = 2$.

Donc $OM_1 = M_1M_2$

On en déduit que le parallélogramme $(OM_1M_2M_3)$ a 2 côtés consécutifs de même longueur.

Par conséquent : $(OM_1M_2M_3)$ est un losange.