OHAYON
Quentin

Question 6 - Theorical question:

Let A be a matrix of finite size. If A is real, then it admits a decomposition into a real singular values real singular values $A = U \Sigma V^*$ with U and V matrices with real coefficients.

A has a size: $n \times m$

A^*A of size m ~~*~~ is a square matrix of dimension n.

The adjoint matrix of A^*A is written: $(A^*A)^* = (A^*(A^*)^*) = A^*A$

The matrix A^*A is hermitian because it is equal to its assistant.

A^*A is real because its coefficients are the products of coefficients of real A

When we ~~do~~ an eigenvalue decomposition of A^*A, we obtain the diagonal matrix D which contains n eigenvalues. They are 0 to m different eigenvalues λ_m, $m \in \mathbb{N}$, $m \leq n$

We obtain the matrix of singular values Σ containing singular values real and positive

$A = U \Sigma V^*$

The matrix V^* is composed of the eigenvectors of A^*A

To determine them, we pose $(A^*A) x_n = \lambda_m x_n$ where x_n are eigenvectors. Thus we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda_m \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} (a_{11} - \lambda_m) x_1 & a_{12} x_2 & \cdots & a_{1n} x_n \\ a_{21} x_1 & (a_{22} - \lambda_m) x_2 & \cdots & a_{2n} x_n \\ \vdots & & \ddots & \vdots \\ a_{n1} x_1 & a_{n2} x_2 & \cdots & (a_{nn} - \lambda_m) x_n \end{pmatrix} = 0$$

The following system must be solved

$$\begin{cases} (a_{11}-\lambda_m)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda_m)x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots (a_{nn}-\lambda_m)x_n = 0 \end{cases}$$

Where the coefficients a_{ij} and the eigenvalues λ_m are positive values.
There are therefore solutions for these systems.
Solving the system as a function of x_1 and posing $x_1 = 1$
we obtain the eigenvectors for the value λ_m with necessarily
positive coefficient
So we have positive eigenvectors which form V^* which is
consequently also positive. Its adjoint V has real coefficients
U is defined as:

$$A = U \Sigma V^*$$
$$AV = U \Sigma V^* V$$
$$A V \Sigma^{-1} = U \Sigma \Sigma^{-1}$$
$$A V \Sigma^{-1} = U$$

Now it is a matrix product of three matrices with real coefficients,
the matrix U has real coefficients.
So we have shown that if A has real coefficients the vectors U
and V have real coefficients too.