# MATH 315 Miniproject 1: Sollow-Swan ODE Numerical Analysis Using Euler's Method

Ethan Thomas and Paul Quidu

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### Description of Chosen ODE and Parameters

The Solow-Swan model uses the following differential equation to describe capital accumulation per effective worker in a closed economy:

$$k'(t) = s\, f\big(k(t)\big) - (\delta + n + g)\, k(t)$$

Where:
- $k(t)$ = capital per effective worker at time $t$ (tracks how much capital there is for each unit of "effective" labor in the economy)
- $s$ = savings rate: The fraction of output that is saved and invested instead of consumed
- $f(k(t))$ = output per effective worker given $k(t)$: Specifies the amount of output produced per effective worker given capital per effective worker; often a Cobb-Douglas form such as $f(k) = k^{\alpha}$, with $0 < \alpha < 1$
- $\delta$ = depreciation rate
- $n$ = population growth rate
- $g$ = technology growth rate

For the purpose of our study we will analyse the Solow-Swan ODE for Canada. These parameter values are based on 2024 "Statistics Canada" data, $\alpha$ and $\delta$ are based on average for developed countries.

- $\alpha = 0.31$
- $\delta = 5\%$
- $s = 25.2\%$
- $n = 3\%$
- $g = 0.85\%$

**Numerical Approximation Using Euler's Method**

Euler's method approximates solutions to ordinary differential equations using the iterative formula:

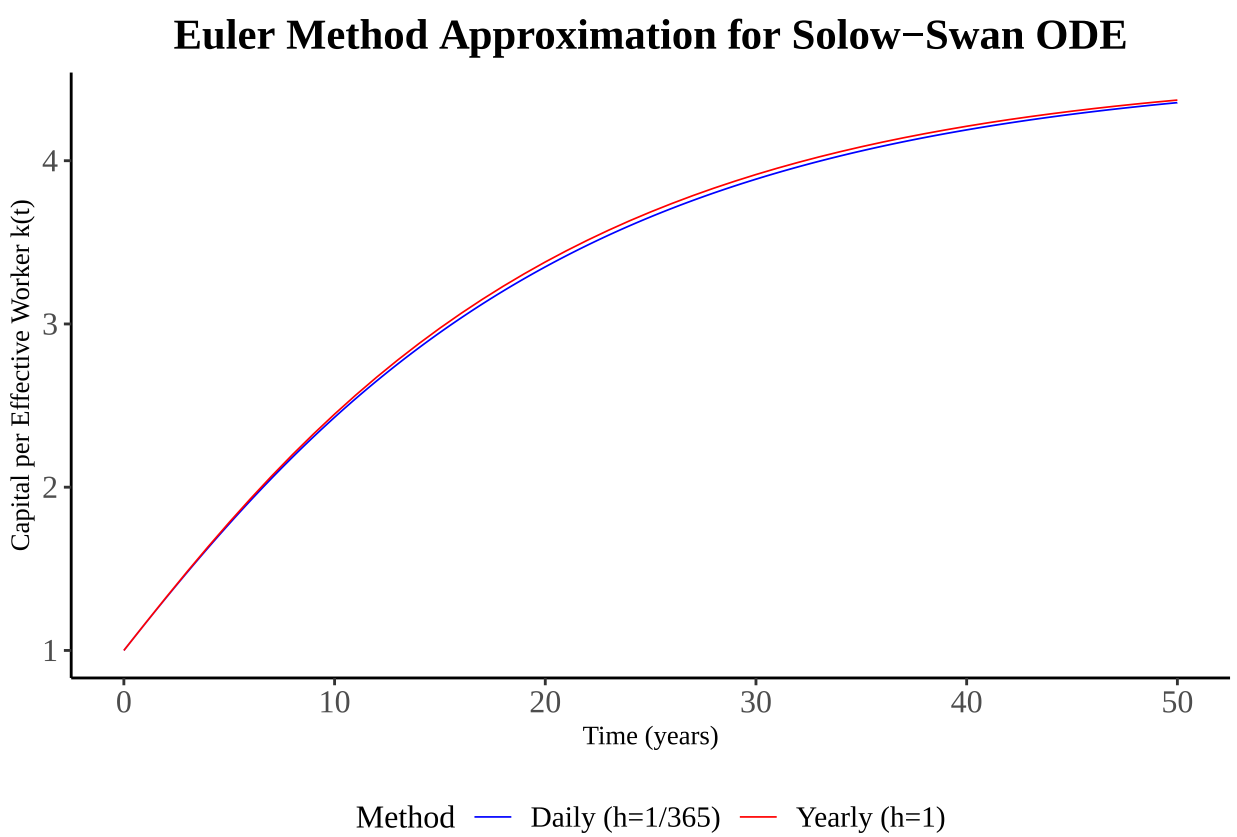
$$k_{n+1} = k_n + h\, f(t_n, k_n)$$

where $h$ is the step size, $k_n$ is the approximation of $k(t_n)$, and $f(t,k)$ is the right-hand side of the ODE.

- We chose a step size of $h = 1$ **(yearly)** to capture the dynamics on a rough scale and compare with a more precise simulation.

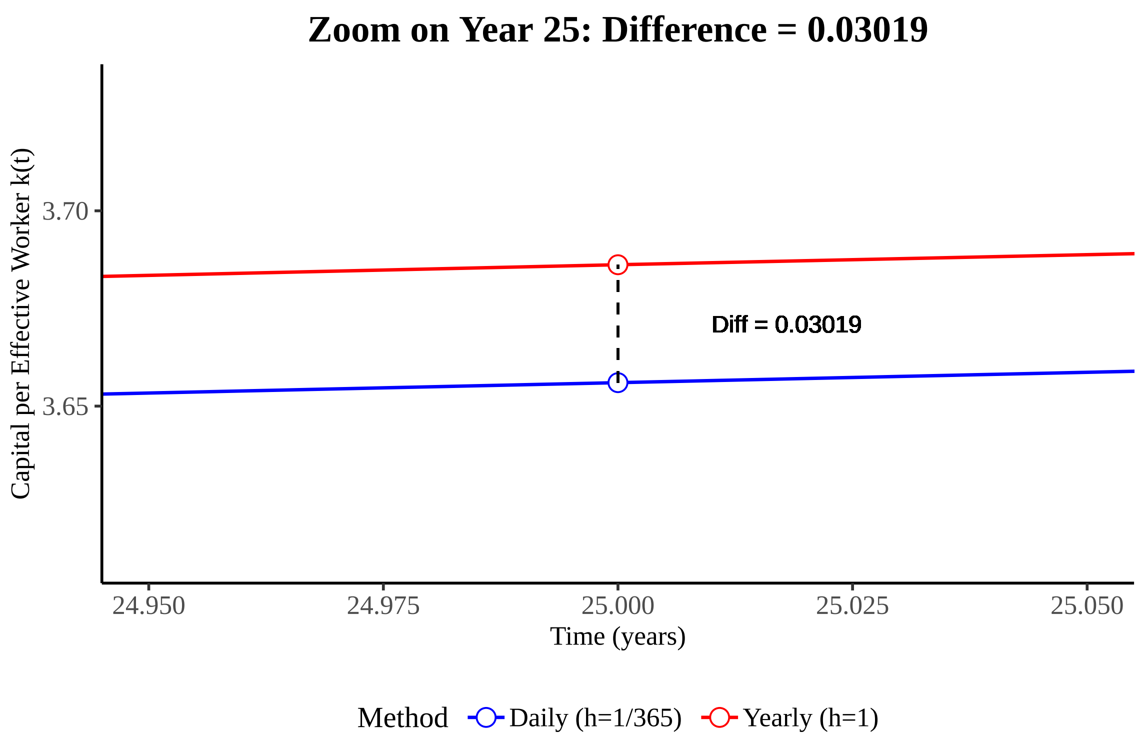
- We also chose a step size of $h = \frac{1}{365}$ **(daily)** to approximate the continuous dynamics of the model much more closely.

- The **initial condition** was set as $k(0) = 1$ (starting point (0,1)), representing an arbitrary normalized unit of capital per effective worker at the beginning of the simulation.




As we can see in the first plot, the difference between the yearly step ($h = 1$) and the daily step ($h = 1/365$) is very small. This is because the Solow-Swan ODE is relatively smooth over time, and even a yearly step captures the overall trend of capital accumulation with good accuracy.

To better illustrate the subtle differences between the two approximations, we next zoom in around the 25th year, where we can clearly observe how the step size affects the Euler method's numerical solution.




**R function for Euler's method**

```
f <- function(k) { s * k^alpha - delta_n_g * k }

# Step sizes
steps <- list(
  "Yearly (h=1)" = 1,
  "Daily (h=1/365)" = 1/365
)

sim_data <- data.frame(time = numeric(), k = numeric(), Method = character())

for (name in names(steps)) {
  h <- steps[[name]]
  n_steps <- n_years / h
  t_vals <- numeric(n_steps + 1)
  k_vals <- numeric(n_steps + 1)
  t_vals[1] <- t0
  k_vals[1] <- k0

  for (i in 1:n_steps) {
    k_vals[i+1] <- k_vals[i] + h * f(k_vals[i])
    t_vals[i+1] <- t_vals[i] + h
  }

  sim_data <- rbind(sim_data,
                    data.frame(time = t_vals, k = k_vals, Method = name))
}
```

**Steady-State Analysis and Dynamics**

The Solow-Swan model has a **steady-state (equilibrium) point** where capital per effective worker does not change, i.e., $k'(t) = 0$.
From the ODE:

$$k'(t) = s\,k^{\alpha} - (\delta + n + g)k = 0$$

Solving for $k^*$ gives the **stable steady point**:

$$k^* = \left( \frac{s}{\delta + n + g} \right)^{\frac{1}{1-\alpha}}$$

This represents the long-term level of capital per effective worker that the economy converges to.

**Stability check of $k^*$**

Let

$$F(k) = sk^{\alpha} - (\delta + n + g)k,$$

so the steady state $k^*$ satisfies

$$s(k^*)^{\alpha} = (\delta + n + g)k^* \quad \Longrightarrow \quad s(k^*)^{\alpha-1} = \delta + n + g.$$

Differentiate:

$$F'(k) = s\alpha k^{\alpha-1} - (\delta + n + g).$$

Evaluate at $k^*$ and use the steady-state identity:

$$F'(k^*) = s\alpha(k^*)^{\alpha-1} - (\delta + n + g) = \alpha(\delta + n + g) - (\delta + n + g) = (\alpha - 1)(\delta + n + g).$$

Because $0 < \alpha < 1$ and $\delta + n + g > 0$, we have $F'(k^*) < 0$. Hence $k^*$ is a stable steady point.

Solving for $k^*$ gives the **stable steady point**:

$$k^* = \left( \frac{s}{\delta + n + g} \right)^{\frac{1}{1-\alpha}}$$

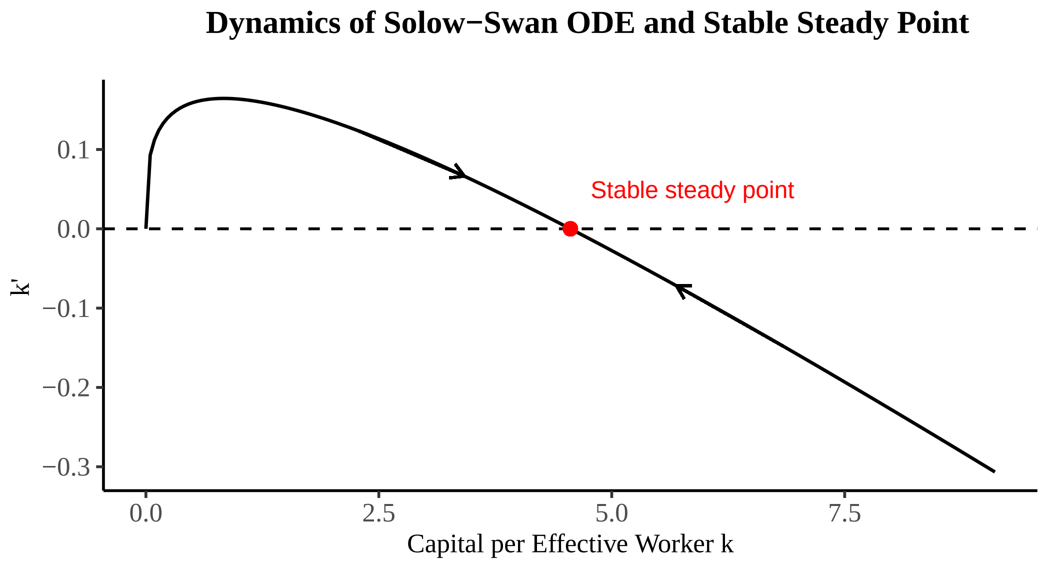
This represents the long-term level of capital per effective worker that the economy converges to.

Using the Canadian parameter values ($s = 0.252$, $\delta + n + g = 0.0885$, $\alpha = 0.31$), we compute

$$k^* = \left( \frac{0.252}{0.0885} \right)^{\frac{1}{1-0.31}} \approx 4.56.$$

This means that, in the long run, the economy converges to about **4.56 units of capital per effective worker**.
At this point, investment exactly offsets depreciation, population growth, and technological progress, so $k(t)$ remains constant.

Below, we illustrate the dynamics of $k(t)$ around the steady-state. The arrows indicate whether $k(t)$ increases or decreases, showing the **stability** of the steady point.




From the dynamics plot, we can see that the steady state $k^*$ is **stable**.

- **When $k < k^*$:** The curve lies above the $x$-axis, so $k'(t) > 0$. This means capital per effective worker increases over time, pushing the system back toward $k^*$.
- **When $k > k^*$:** The curve lies below the $x$-axis, so $k'(t) < 0$. In this case, capital per effective worker decreases, again pulling the system back toward $k^*$.

Thus, no matter whether the economy starts with "too little" or "too much" capital per effective worker, the dynamics naturally drive it back to the steady state.

Economically, this means the Solow-Swan model predicts **long-run convergence**: different initial conditions for $k(0)$ do not matter, since the economy eventually settles at the same steady-state capital stock per effective worker, determined solely by parameters $s$, $\alpha$, $\delta$, $n$, and $g$.