

MATH 315 Miniproject 1: Sollow-Swan ODE

Numerical Analysis Using Euler's Method

Ethan Thomas and Paul Quidu

September 28, 2025

Description of Chosen ODE and Parameters

The Solow-Swan model uses the following differential equation to describe capital accumulation per effective worker in a closed economy:

$$k'(t) = s f(k(t)) - (\delta + n + g) k(t)$$

Where:

- $k(t)$ = capital per effective worker at time t (tracks how much capital there is for each unit of “effective” labor in the economy)
- s = savings rate: The fraction of output that is saved and invested instead of consumed
- $f(k(t))$ = output per effective worker given $k(t)$: Specifies the amount of output produced per effective worker given capital per effective worker; often a Cobb-Douglas form such as $f(k) = k^\alpha$, with $0 < \alpha < 1$
- δ = depreciation rate
- n = population growth rate
- g = technology growth rate

For the purpose of our study we will analyse the Solow-Swan ODE for Canada with the following assumptions from Statistics Canada:

- $\alpha = 0.31$
- $\delta = 5\%$
- $s = 25.2\%$
- $n = 3\%$
- $g = 0.85\%$

Numerical Approximation Using Euler's Method

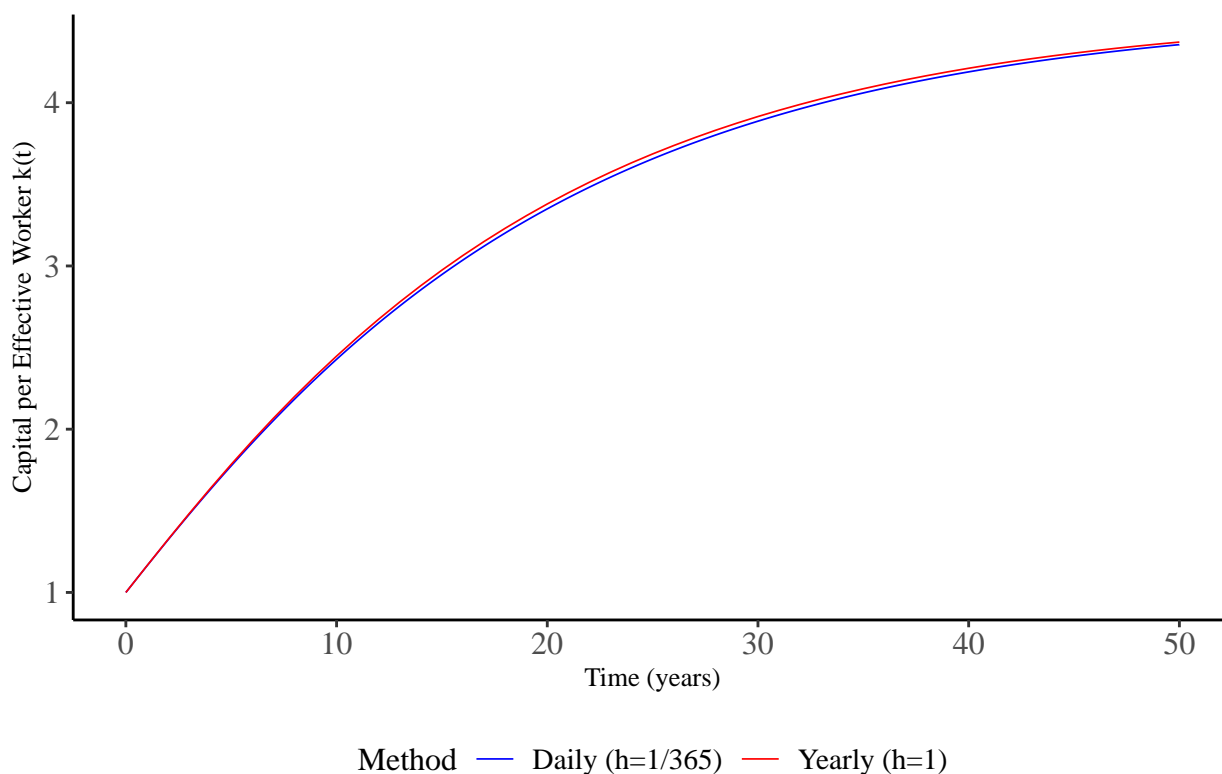
Euler's method approximates solutions to ordinary differential equations using the iterative formula:

$$k_{n+1} = k_n + h f(t_n, k_n)$$

where h is the step size, k_n is the approximation of $k(t_n)$, and $f(t, k)$ is the right-hand side of the ODE.

- We chose a step size of $h = 1$ (**yearly**) to capture the dynamics on a rough scale and compare with a more precise simulation.
- We also chose a step size of $h = \frac{1}{365}$ (**daily**) to approximate the continuous dynamics of the model much more closely.
- The **initial condition** was set as $k(0) = 1$ (starting point (0,1)), representing an arbitrary normalized unit of capital per effective worker at the beginning of the simulation.

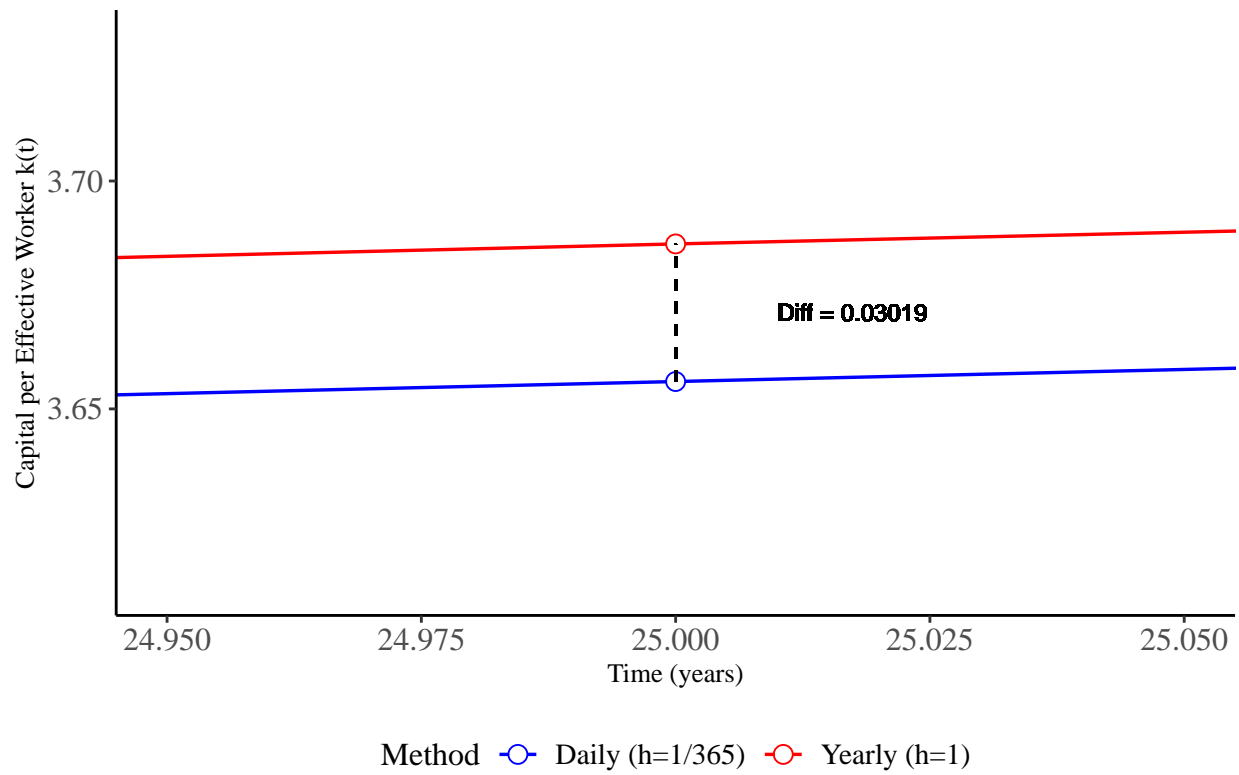
Euler Method Approximation for Solow–Swan ODE



As we can see in the first plot, the difference between the yearly step ($h = 1$) and the daily step ($h = 1/365$) is very small. This is because the Solow-Swan ODE is relatively smooth over time, and even a yearly step captures the overall trend of capital accumulation with good accuracy.

To better illustrate the subtle differences between the two approximations, we next zoom in around the 25th year, where we can clearly observe how the step size affects the Euler method's numerical solution.

Zoom on Year 25: Difference = 0.03019



Steady-State Analysis and Dynamics

The Solow-Swan model has a **steady-state (equilibrium) point** where capital per effective worker does not change, i.e., $k'(t) = 0$.

From the ODE:

$$k'(t) = s k^\alpha - (\delta + n + g)k = 0$$

Solving for k^* gives the **stable fixed point**:

$$k^* = \left(\frac{s}{\delta + n + g} \right)^{\frac{1}{1-\alpha}}$$

This represents the long-term level of capital per effective worker that the economy converges to.

Below, we illustrate the dynamics of $k(t)$ around the steady-state. The arrows indicate whether $k(t)$ increases or decreases, showing the **stability** of the fixed point.

