Introduction to Probability

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- *Biology*: Genetics is deeply intertwined with probability; both in the inheritance of genes and in modelling random mutations.
- *Computer Sciences*: Randomized algorithms make random choices while they are run, and in many important applications they are simpler and more efficient than any currently known deterministic alternatives. Probability also plays an essential role in studying the performance of algorithms, and in machine learning and artificial intelligence.

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- *Political Sciences*: In recent years, political sciences has become more and more quantitative and statistical. For example, Nate Silver’s successes in predicting election results, such as in the 2008 and 2012 U. S. presidential elections, were achieved using probability models to make sense of polls and to drive simulations

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- *Medicine*: The development of randomized clinical trials, in which patients are randomly assigned to receive treatment or placebo, has transformed medical research in recent years. As the biostatistian David Harringto remarked, "Some have conjectured that it could be the most significant advance in scientific medicine in the twentieth century$\cdots$. In one of the delightful ironies of modern science, the randomized trial adjusts for both observed and unobserved heterogeneity in a controlled experiment by introducing chance variation into the study design".

Review of Set Theoretical Notation

A set is a collection of objects we want to study. Each object in a set is called an *element*. If A is a set and a is an element in A, then we denote $a \in A$. If B is another set in which all elements of B are also elements of the set A, then B is called a *subset* of A. $B \subseteq A$

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Sets can be specified in a variety of ways. If S contains a finite number of elements, say $x_1, x_2, ..., x_n$, we write it as a list of the elements, in braces:

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If S contains infinitely many elements $x_1, x_2, \cdots$ which can be enumerated in a list we write

$$S = \{x_1, x_2, \cdots\}. = \{x_1, x_2, \cdots x_n, \cdots\}$$

We say that S is *countably infinite* or *denumerable*. For example the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

is denumerable

Some operations on sets

- Union of sets: if $A_1, \cdots, A_n$ are sets, then the union of the sets is $A_1 \cup A_2 \cup \cdots \cup A_n = \{a | a \in A_i \text{ for some } i\}$.

Venn Diagram

$n = 2$

$A_1 \cup A_2$

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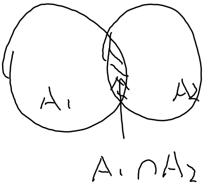

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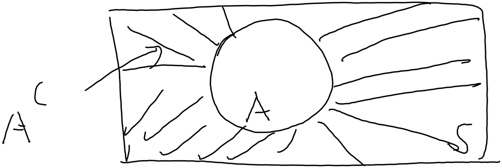

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- If A and B are two sets such that $A \bigcap B = \phi$, then A and B are called *mutually exclusive or disjoint*.
- The cardinality of a finite set: The cardinality of a set S is the number of the elements in the set S and denoted by $|S|$.

The Algebra of Sets

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De Morgan's Laws

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c, \qquad \left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c.$$

Sample Spaces

Definition 1 (Sample Spaces and Events)

The *Sample space* S of an experiment is the set of all possible outcomes of the experiment. An event A is a subset of the sample space S, and we say that A *occurred* if the actual outcome is in A

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Example 1

Pick a card from a deck of 52 cards. The sample space S is the set of all 52 cards.

- A: card is an ace.
- B: card has a black suit.
- D: card is a diamond.
- H: card is a heart.

Then $H = \{$ Ace of hearts, Two of hearts,$\cdots$, King of hearts $\}$. What is $A \cap H$? How about $A \cap B$?

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Let A be an event for an experiment with a finite sample space Ω. The probability of A is

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Example 2

Compute the probability of A in example 1.

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2. (Additivity) If A and B are two disjoint events, then the probability of their union satisfies

$$P(A \cup B) = P(A) + P(B).$$

   More generally, if the sample space has an finite number of elements and $A_1, A_2, \cdots, A_n$ is a sequence of disjoint events, then the probability of their union satisfies

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3. (Normalization) The probability of the entire sample space Ω is 1, i.e., $P(\Omega) = 1$.

Probability Models

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Example: The above example 1 is a discrete model.

2. **Continuous Models.**

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Example 4

A wheel of fortune is continuously calibrated from 0 to 1, so the possible outcomes of an experiment consisting of a single spin are the numbers in the interval $\Omega = [0, 1]$. Assume a fair wheel,it is appropriate to consider all outcomes equally likely, but what is the probability of the event consisting of a single event?

Properties of Probability Laws

Properties of probability laws

Consider the probability laws. Let A and B, C be events.
(a) If $A \subset B$, then $P(A) \leq P(B)$.
(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
(c) $P(A \cup B) \leq P(A) + P(B)$.
(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$.