

# Weight Discretization for Quotient Model Abstraction in Spiking Neural Network Verification

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(Fifth Revision)*

## 1. Introduction

This note extends the filtration-based quotient model abstraction [1] to preserve *synaptic weight information* during formal verification. While the original quotient model abstracts membrane potentials into equivalence classes, it treats all synapses uniformly, losing essential structural information.

We address this by introducing a *weight discretization scheme* that:

1. Maps continuous weights to a finite discrete range
2. Preserves the relative contribution of synapses to membrane potential
3. Ensures threshold feasibility is maintained
4. Retains weight visibility in the generated PRISM model

## 2. Preliminaries

This section introduces key concepts and notation used throughout the proofs.

### 2.1. Notation Summary

Symbol	Meaning
$W$	Discretization parameter: the number of positive weight levels
$w_{\max}$	Maximum absolute weight in the original model (typically 100)
$\delta_W$	Weight discretization function: maps original weights to discrete values
$T$	Original firing threshold of a neuron
$T_d$	Discretized firing threshold
$m$	Fan-in: the number of incoming synapses to a neuron
$S$	Weighted sum of spiking inputs in the original model
$S_d$	Weighted sum of spiking inputs in the discretized model
$\gamma$	Class width: potential range represented by each equivalence class
$\lambda_d$	Discretized leak factor (always $\leq 0$ ), <i>dependent on <math>T_d</math></i>
$k$	Number of threshold levels (equivalence classes)
$\ell$	Leak factor: fraction of potential lost per step ( $\ell \in [0, 1]$ ). Related to retention rate by $\ell = 1 - r$

Table 1: Summary of notation used in this document

## 2.2. The Rounding Property

The standard mathematical rounding function  $\lfloor x \rfloor$  rounds  $x$  to the nearest integer. A crucial property we use throughout is the *rounding bound*:

**Definition.** For any real number  $x$ , the rounding function satisfies:

$$x - \frac{1}{2} \leq \lfloor x \rfloor \leq x + \frac{1}{2}$$

Equivalently:  $\lfloor x \rfloor \geq x - \frac{1}{2}$  (lower bound) and  $\lfloor x \rfloor \leq x + \frac{1}{2}$  (upper bound).

 **Intuition:** Rounding can shift a value by at most  $\frac{1}{2}$  in either direction. When we sum  $m$  rounded values, the total error is bounded by  $\frac{m}{2}$  — this is the *cumulative rounding error*.

## 2.3. The Clamp Function

**Definition.** The *clamp function* restricts a value to a specified range  $[a, b]$  where  $a < b$ :

$$\text{clamp}(x, a, b) = \max(a, \min(b, x)) = \begin{cases} a & \text{if } x < a \\ x & \text{if } a \leq x \leq b \\ b & \text{if } x > b \end{cases}$$

 **Intuition:** Clamping prevents values from exceeding safe bounds. In our model, we clamp class indices to  $[0, k]$  and class deltas to  $[-k, k]$  to avoid invalid states or runaway accumulation.

## 2.4. Fan-in

**Definition.** The *fan-in* of a neuron, denoted  $m$ , is the number of incoming synaptic connections. A neuron with fan-in  $m$  receives input from  $m$  presynaptic neurons.

**Remark.** Fan-in is critical because the cumulative rounding error scales linearly with  $m$ . High fan-in neurons require finer discretization to maintain accuracy.

## 3. Weight Discretization

### 3.1. Formal Definition

**Definition.** Given a weight range  $[w_{\min}, w_{\max}] = [-100, 100]$  and a discretization parameter  $W \in \mathbb{N}^+$ , the *weight discretization function*  $\delta_W : \mathbb{R} \rightarrow \mathbb{Z}$  is defined as:

$$\delta_W(w) = \left\lfloor w \cdot \frac{W}{w_{\max}} \right\rfloor$$

The discretized weight range is  $[-W, W] \subset \mathbb{Z}$ .

**💡 Intuition:** We scale the original weight by  $\frac{W}{w_{\max}}$  to map the range  $[-w_{\max}, w_{\max}]$  to  $[-W, W]$ , then round to the nearest integer. This preserves the *relative magnitude* of weights while reducing the number of distinct values.

**Example.** For  $W = 3$  (giving 7 discrete levels:  $-3, -2, -1, 0, 1, 2, 3$ ):

- $\delta_3(100) = \lceil 100 \cdot \frac{3}{100} \rceil = \lceil 3 \rceil = 3$  (strong excitatory)
- $\delta_3(67) = \lceil 67 \cdot \frac{3}{100} \rceil = \lceil 2.01 \rceil = 2$  (medium excitatory)
- $\delta_3(33) = \lceil 33 \cdot \frac{3}{100} \rceil = \lceil 0.99 \rceil = 1$  (weak excitatory)
- $\delta_3(0) = \lceil 0 \rceil = 0$  (negligible)
- $\delta_3(-50) = \lceil -50 \cdot \frac{3}{100} \rceil = \lceil -1.5 \rceil = -2$  (medium inhibitory)
- $\delta_3(-100) = \lceil -3 \rceil = -3$  (strong inhibitory)

### 3.2. Threshold Calibration

The key challenge is ensuring that threshold reachability is preserved after discretization. We must calibrate the *discretized threshold*  $T_d$  to be consistent with the original threshold  $T$ .

**Definition.** The *discretized threshold* for a neuron with original threshold  $T$  and weight discretization parameter  $W$  is:

$$T_d = \left\lceil T \cdot \frac{W}{w_{\max}} \right\rceil$$

**💡 Intuition:** We use ceiling (round up) rather than standard rounding to ensure the discretized threshold is *at least as hard* to reach as the original. This prevents false positives: if a discretized neuron fires, we can be confident the original would too.

### 3.3. Threshold Preservation Theorem (Completeness)

**Theorem** (Threshold Preservation — Completeness). Let  $\mathcal{N}$  be a neuron with incoming weights  $\{w_1, \dots, w_m\}$  and threshold  $T$ . Let  $\mathcal{N}'$  be the discretized version with weights  $\{\delta_W(w_1), \dots, \delta_W(w_m)\}$  and threshold  $T_d$ .

If  $\mathcal{N}$  can fire in a single step (i.e.,  $\exists$  input pattern  $\mathbf{y} \in \{0, 1\}^m$  such that  $\sum_{i=1}^m w_i \cdot y_i \geq T$ ), then  $\mathcal{N}'$  can also fire in a single step.

*Proof.* We prove this in six steps.

**Step 1** (Setup): Let  $\mathbf{y}^*$  be an input pattern that causes the original neuron  $\mathcal{N}$  to fire. Define:

- $S = \sum_{i=1}^m w_i \cdot y_i^*$  — the weighted sum of spiking inputs in the original model
- By assumption,  $S \geq T$  (the neuron fires)

**Step 2** (Discretized contribution): The corresponding weighted sum in the discretized model is:

$$S_d = \sum_{i=1}^m \delta_W(w_i) \cdot y_i^*$$

We need to show  $S_d \geq T_d$  to prove the discretized neuron also fires.

**Step 3** (Apply the rounding property): By the rounding property (see §2.2), for each weight:

$$\delta_W(w_i) = \left\lfloor w_i \cdot \frac{W}{w_{\max}} \right\rfloor \geq \frac{w_i \cdot W}{w_{\max}} - \frac{1}{2}$$

Since  $y_i^* \in \{0, 1\}$ , when  $y_i^* = 1$  this bound applies, and when  $y_i^* = 0$  the term is zero. Summing over all inputs:

$$S_d \geq \sum_{i=1}^m \left( \frac{w_i \cdot W}{w_{\max}} - \frac{1}{2} \right) \cdot y_i^* = \left( \sum_{i=1}^m w_i \cdot y_i^* \right) \cdot \frac{W}{w_{\max}} - \frac{\sum_{i=1}^m y_i^*}{2}$$

**Step 4** (Bound the cumulative error): Let  $m^* = \sum_{i=1}^m y_i^*$  be the number of active inputs. Then:

$$S_d \geq S \cdot \frac{W}{w_{\max}} - \frac{m^*}{2}$$

Since  $m^* \leq m$  (at most  $m$  inputs can be active):

$$S_d \geq S \cdot \frac{W}{w_{\max}} - \frac{m}{2}$$

The term  $\frac{m}{2}$  is the *cumulative rounding error* — the worst-case total error from rounding  $m$  weights.

**Step 5** (Derive the firing condition): For the discretized neuron to fire, we need  $S_d \geq T_d$ . By the ceiling property:

$$T_d = \left\lceil T \cdot \frac{W}{w_{\max}} \right\rceil \leq T \cdot \frac{W}{w_{\max}} + 1$$

Combining with our bound on  $S_d$ :

$$S_d \geq S \cdot \frac{W}{w_{\max}} - \frac{m}{2}$$

Since  $S \geq T$ :

$$S_d \geq T \cdot \frac{W}{w_{\max}} - \frac{m}{2}$$

A sufficient condition for firing is:

$$T \cdot \frac{W}{w_{\max}} - \frac{m}{2} \geq T \cdot \frac{W}{w_{\max}} + 1$$

This simplifies to:  $(S - T) \cdot \frac{W}{w_{\max}} \geq \frac{m}{2} + 1$

**Step 6** (Boundary case and parameter choice): For the critical boundary case where  $S = T$  exactly, we need:

$$0 \geq \frac{m}{2} + 1$$

This is never satisfied, so at the exact boundary, firing is not guaranteed. However, if  $S$  exceeds  $T$  by a small margin, or we choose  $W$  large enough, firing is preserved.

Specifically, for  $W \geq w_{\max} \cdot \frac{\frac{m}{2} + 1}{T}$ , the discretized neuron fires whenever the original does.

**Practical example:** With  $W = 3$ ,  $w_{\max} = 100$ ,  $m \leq 10$ , and  $T = 100$ :

$$W \geq 100 \cdot \frac{5 + 1}{100} = 6$$

Thus  $W = 7$  (discrete range  $[-3, 3]$ ) is sufficient for most practical networks.

□

**Remark. Critical Constraint:** The proof shows that weight discretization introduces a cumulative error of  $-\frac{m}{2}$ . If the fan-in  $m$  is large relative to  $W$  (specifically if  $m > 2W$ ), the rounding noise may exceed the signal of the smallest synaptic weight.

For high-fanin neurons, we strictly recommend:

1. Using a finer discretization ( $W \geq \frac{m}{2}$ )
2. Applying a threshold correction factor:  $T_{d'} = T_d - \lfloor \frac{m}{2W} \rfloor$  (Note: This may increase false positive firings).

### 3.4. Soundness Theorem (Safety)

We now prove the converse property: if the original neuron should *not* fire, the discretized neuron should also *not* fire.

**Theorem** (Asymptotic Silence — Soundness). Let  $\mathcal{N}'$  be a discretized neuron with current potential  $P_t < T_d$  and leak factor  $\lambda_d \leq -1$ . If the input sequence is empty (i.e., total weighted input  $S_d = 0$ ) for all steps  $t' \geq t$ , then  $\mathcal{N}'$  will never fire.

*Proof.* We prove this by showing the potential strictly decreases to zero and that zero potential guarantees non-spiking.

**Step 1** (Dynamics without input): In the absence of input ( $S_d = 0$ ), the discretized update rule simplifies to:

$$P_{t+1} = \max(0, P_t + \lambda_d)$$

This is applying the potential update with only leak (no external contribution).

**Step 2** (Strict decay guarantee): Since  $\lambda_d \leq -1$ , if  $P_t > 0$ , then:

$$P_{t+1} \leq P_t + \lambda_d \leq P_t - 1$$

The potential strictly decreases by at least 1 unit per step.

**Step 3** (Convergence to zero): Starting from any  $P_t < T_d$ , the sequence  $\{P_t\}$  is strictly decreasing until it reaches the absorbing state  $P = 0$  (due to the  $\max(0, \dots)$  clamp).

**Step 4** (Firing impossibility during decay): Since  $P_t < T_d$  initially and the sequence is non-increasing, the firing condition  $P \geq T_d$  is never met during the decay trajectory. The neuron cannot fire while the potential is decreasing.

**Step 5** (Zero potential guarantees non-spiking): Once  $P = 0$ , the neuron is at resting potential. Since the firing probability is determined by the potential's position relative to the discrete threshold levels,  $P = 0$  maps to level  $L = 0$ , which corresponds to a firing probability of 0. The probability of *not* spiking is therefore 1 (i.e., 100%). Combined with the absence of input, the neuron remains permanently at  $P = 0$  with zero probability of emitting a spike.

□

**💡 Intuition:** This theorem provides a *safety guarantee*: the discretization does not introduce spurious spikes. If there's no input to drive the potential up, leak ensures it decays to rest, and rest guarantees zero firing probability. This prevents rounding errors from accidentally triggering false firings.

## 4. Class Transition with Weighted Contributions

### 4.1. Contribution-Based Class Evolution

In the original quotient model, class evolution used a binary rule which loses weight information. We replace it with *weighted contribution-based* class evolution.

**Definition.** The *weighted contribution* for neuron  $n$  with incoming discretized weights  $\{w_1^d, \dots, w_m^d\}$  is:

$$C_n = \sum_{i=1}^m w_i^d \cdot y_i$$

where  $y_i \in \{0, 1\}$  is the spike output of presynaptic neuron  $i$ .

**Definition.** The *class delta function*  $\Delta : \mathbb{Z} \rightarrow \mathbb{Z}$  maps contribution to class change:

$$\Delta(C) = \text{clamp}\left(\left\lfloor \frac{C}{\gamma} \right\rfloor, -k, k\right)$$

where:

- $\gamma$  is the *class width* (typically  $\gamma = \frac{T_d}{k}$ ) — the potential range each class represents
- $k$  is the number of threshold levels
- The clamp ensures the class change stays within valid bounds

**💡 Intuition:** The class delta converts a weighted sum of inputs into a class change. We divide by  $\gamma$  to express the contribution in “class units”, round to the nearest integer, and clamp to prevent impossibly large jumps.

## 4.2. Threshold-Dependent Leak Factor

The quotient model must also account for membrane potential decay (leak). We define the leak factor using the *leak parameter*  $\ell$ , which represents the fraction of potential lost per step.

**Definition.** The *discretized leak factor*  $\lambda_d$  is:

$$\lambda_d = -\max(1, \lfloor \ell \cdot T_d \rfloor)$$

where:

- $\ell \in [0, 1]$  is the leak factor (fraction of potential lost per step, with  $\ell = 1 - r$  where  $r$  is the retention rate)
- $T_d$  is the discretized threshold
- The  $\max(1, \dots)$  ensures a minimum decay of 1 unit per step (preventing infinite energy trapping)
- The negative sign ensures leak *decreases* potential

**💡 Intuition:** By linking the leak factor to  $T_d$  rather than the number of classes  $k$ , we ensure the decay scales correctly with the calibrated threshold. A high threshold means larger potential values, so the decay must be proportionally larger to maintain realistic behavior. Thinking in terms of  $\ell$  (how much leaks away) rather than  $r$  (how much is retained) provides a more direct relationship: higher  $\ell$  means more aggressive decay.

**Example.** For  $\ell = 0.1$  (10% leak, i.e., 90% retention) and  $T_d = 10$ :

$$\lambda_d = -\max(1, \lfloor 0.1 \cdot 10 \rfloor) = -\max(1, \lfloor 1.0 \rfloor) = -1$$

*Result:* Potential decreases by 1 per step without input.

For  $\ell = 0.5$  (50% leak) and  $T_d = 10$ :

$$\lambda_d = -\max(1, \lfloor 0.5 \cdot 10 \rfloor) = -\max(1, \lfloor 5.0 \rfloor) = -5$$

*Result:* Potential decreases by 5 per step without input — aggressive decay.

For  $\ell = 0.05$  (5% leak, i.e., 95% retention) and  $T_d = 10$ :

$$\lambda_d = -\max(1, \lfloor 0.05 \cdot 10 \rfloor) = -\max(1, \lfloor 0.5 \rfloor) = -\max(1, 0) = -1$$

*Result:* Even with very low leak, we enforce minimum decay of 1.

**Remark. Key Insight:** Linking  $\lambda_d$  to  $T_d$  rather than  $k$  is essential. If  $\lambda_d$  depended on  $k$  (number of classes), it could result in  $\lambda_d = 0$  for low leak, violating the Soundness theorem. The formulation  $\lambda_d = -\max(1, \lfloor \ell \cdot T_d \rfloor)$  ensures  $|\lambda_d| \geq 1$  always, guaranteeing the Asymptotic Silence property.

The class evolution rule becomes:

$$c'_n = \text{clamp}(c_n + \Delta(C_n) + \lambda_d, 0, k)$$

### 4.3. Leak Factor Dynamics

This section characterizes the three key relationships governed by the leak factor  $\ell$ , providing insight into how leak affects neuron behavior across the model.

#### 4.3.1. Signal Contribution

The leak factor directly controls how much of the *current* input signal contributes to potential accumulation at each time step. From the membrane potential update:

$$P_{t+1} = \max(0, (1 - \ell) \cdot P_t + C_t)$$

The retained potential from the previous step is scaled by  $(1 - \ell)$ , while the current input  $C_t$  is added in full. The effective weight of past vs. present signal is therefore:

Leak factor $\ell$	Retention $(1 - \ell)$	Effect on signal
$\ell \rightarrow 0$	$\approx 1$	Nearly all potential is preserved — long-term accumulation across many steps
$\ell = 0.1$	0.9	Slow decay; potential builds over $\approx 10$ steps before significant loss
$\ell = 0.5$	0.5	Potential halved each step; only the last $\approx 2\text{--}3$ steps influence firing
$\ell \rightarrow 1$	$\approx 0$	Potential discarded each step — only the current input determines firing

Table 2: Signal contribution as a function of leak factor

#### 4.3.2. Spike Emission Difficulty

The leak factor determines the *minimum sustained input* required for a neuron to ever reach threshold. From the feasibility analysis (§5.1), a spike is only possible if the net gain per step is positive:

$$C_{\text{in}} > |\lambda_d| = \max(1, \lfloor \ell \cdot T_d \rfloor)$$

This establishes a *noise floor*: any input below  $|\lambda_d|$  is completely absorbed by the leak and cannot contribute to potential accumulation. As  $\ell$  increases, this floor rises, making spike emission increasingly difficult:

**Definition.** The *minimum excitation for spiking* is:

$$C_{\min} = |\lambda_d| + 1 = \max(1, \lfloor \ell \cdot T_d \rfloor) + 1$$

Any constant input  $C_{\text{in}} < C_{\min}$  will never cause the neuron to fire, regardless of duration.

#### 4.3.3. Expected Spike Frequency

For a neuron receiving constant input  $C_{\text{in}} > |\lambda_d|$ , the *expected number of spikes* over  $N$  time steps can be approximated. The inter-spike interval (ISI) is:

$$N_{\text{steps}} = \left\lceil \frac{T_d}{C_{\text{in}} - |\lambda_d|} \right\rceil$$

Therefore the expected spike count is:

$$\mathbb{E}[\text{spikes}] \approx \frac{N}{N_{\text{steps}}} = N \cdot \frac{C_{\text{in}} - |\lambda_d|}{T_d}$$

As  $\ell$  increases,  $|\lambda_d|$  increases, the net gain ( $C_{\text{in}} - |\lambda_d|$ ) decreases, and consequently the expected spike count decreases. This relationship is monotonic: *leakier neurons fire less frequently.*

**Remark. Summary of leak factor correlations:**

Property	As $\ell$ increases	Formula
Current signal contribution	$\uparrow$ (present input dominates)	Weight: $1 - \ell$
Spike emission difficulty	$\uparrow$ (harder to fire)	$C_{\min} =  \lambda_d  + 1$
Expected spike count	$\downarrow$ (fires less often)	$\mathbb{E}[\text{spikes}] \approx \frac{N(C_{\text{in}} -  \lambda_d )}{T_d}$

Table 3: Summary of leak factor  $\ell$  correlations with model behavior

## 5. Threshold Feasibility Analysis

### 5.1. Definition

**Definition.** A neuron configuration is *threshold-feasible* if there exists at least one input pattern that can cause the neuron to reach the firing threshold within a finite number of steps.

 **Intuition:** Feasibility asks: “Can this neuron ever fire?” If the leak is too strong relative to the available excitation, the potential can never build up enough to reach threshold — the neuron is permanently silent.

**Theorem** (Feasibility Criterion). A neuron with discretized weights  $\{w_1^d, \dots, w_m^d\}$  and threshold  $T_d$  is threshold-feasible if and only if:

$$\sum_{w_i^d > 0} w_i^d > |\lambda_d|$$

And specifically, for reliable firing:

$$\sum_{w_i^d > 0} w_i^d \geq \frac{T_d}{1 + |\lambda_d|}$$

*Proof.*

**Step 1** (Define maximum excitation): Let  $E = \sum_{w_i^d > 0} w_i^d$  be the maximum possible excitatory contribution per step (achieved when all excitatory presynaptic neurons fire simultaneously).

**Step 2** (Single-step case): If  $E \geq T_d$ , the neuron can fire in a single step by receiving all excitatory inputs simultaneously. Feasibility is trivially satisfied.

**Step 3** (Accumulation case): If  $E < T_d$ , the neuron must accumulate potential over multiple steps. With leak factor  $\lambda_d \leq 0$ , each step adds a net contribution of:

$$\text{Net gain} = E + \lambda_d$$

(Note:  $\lambda_d$  is negative, so this is  $E - |\lambda_d|$ )

For accumulation to be possible, we require:

$$E + \lambda_d > 0 \Rightarrow E > |\lambda_d|$$

If this holds, the minimum steps to reach threshold is:

$$n = \left\lceil \frac{T_d}{E + \lambda_d} \right\rceil = \left\lceil \frac{T_d}{E - |\lambda_d|} \right\rceil$$

**Step 4** (Necessity: why excitation  $\leq$  leak implies impossibility): If  $E \leq |\lambda_d|$ , then the net gain per step is  $E - |\lambda_d| \leq 0$ . The potential cannot grow — any excitation is immediately cancelled (or overpowered) by leak. The neuron can never reach threshold.

□

## 5.2. Implementation

The feasibility check should be performed at PRISM generation time:

```
fn check_feasibility(
    weights: &[i32], // discretized weights
    threshold: i32,
    leak_factor: i32, // expected to be <= 0 (e.g. -1, -2)
) -> Feasibility {
    // Sum only positive excitatory weights
    let max_excitation: i32 = weights.iter()
        .filter(|&&w| w > 0)
        .sum();

    // Ensure we are working with the magnitude of the leak
    let leak_magnitude = leak_factor.abs();

    // Safety Check (Soundness): Input must overcome leak
    if max_excitation <= leak_magnitude {
        return Feasibility::Impossible;
    }

    let min_required = threshold / (1 + leak_magnitude);

    if max_excitation >= threshold {
        Feasibility::SingleStep
    } else {
        Feasibility::MultiStep(n);
    }
}
```

```

} else if max_excitation >= min_required {
    // Steps = [Threshold / Net_Gain]
    let net_gain = max_excitation - leak_magnitude;
    let steps = (threshold + net_gain - 1) / net_gain;
    Feasibility::MultiStep { min_steps: steps }
} else {
    Feasibility::Impossible
}
}

```

### 5.3. Biological Property Preservation

We verify the quotient model against standard Leaky Integrate-and-Fire properties defined in the literature [2]. Since the model uses probabilistic firing thresholds, all properties are stated in terms of *spike emission probability*.

**Definition.** The following biological properties should be preserved by the discretized model:

**Tonic Spiking:** Under constant input  $C_{\text{in}}$ , the neuron has a *non-zero probability* of emitting a spike at each step if and only if:

$$C_{\text{in}} > |\lambda_d|$$

When this condition holds, the potential accumulates to a level  $L > 0$  in the threshold discretization, yielding a firing probability  $P_{\text{fire}} = \frac{\text{thresholds}[L]}{100} > 0$  at each step where the potential exceeds the first threshold level. The *expected* inter-spike interval gives rise to periodic probabilistic spiking.

**Integrator:** The *probability of immediate firing* on simultaneous inputs  $n$  reaches 1.0 (deterministic) if and only if:

$$\sum_{i=1}^n \delta_W(w_i) \geq T_d$$

Below this threshold, the firing probability is determined by the discrete threshold level mapping:  $P_{\text{fire}} = \frac{\text{thresholds}[L]}{100}$  where  $L = \min(N - 1, \max(0, \lfloor \frac{P}{T_d} \cdot N \rfloor))$  and  $N$  is the number of threshold levels. This creates a *graded probabilistic response* rather than a binary all-or-nothing firing.

**Excitability:** The *expected* inter-spike interval (ISI) decreases monotonically as input frequency increases. Specifically, the expected number of steps between spikes is:

$$\mathbb{E}[N_{\text{steps}}] = \left\lceil \frac{T_d}{C_{\text{in}} - |\lambda_d|} \right\rceil$$

which decreases as  $C_{\text{in}}$  increases. Individual spike times are stochastic — drawn from the threshold-level probability distribution at each step — but the *expected rate* increases monotonically with input strength.

 **Intuition:** These properties ensure our discretized model behaves like a real LIF neuron, with all spike emissions governed by probability:

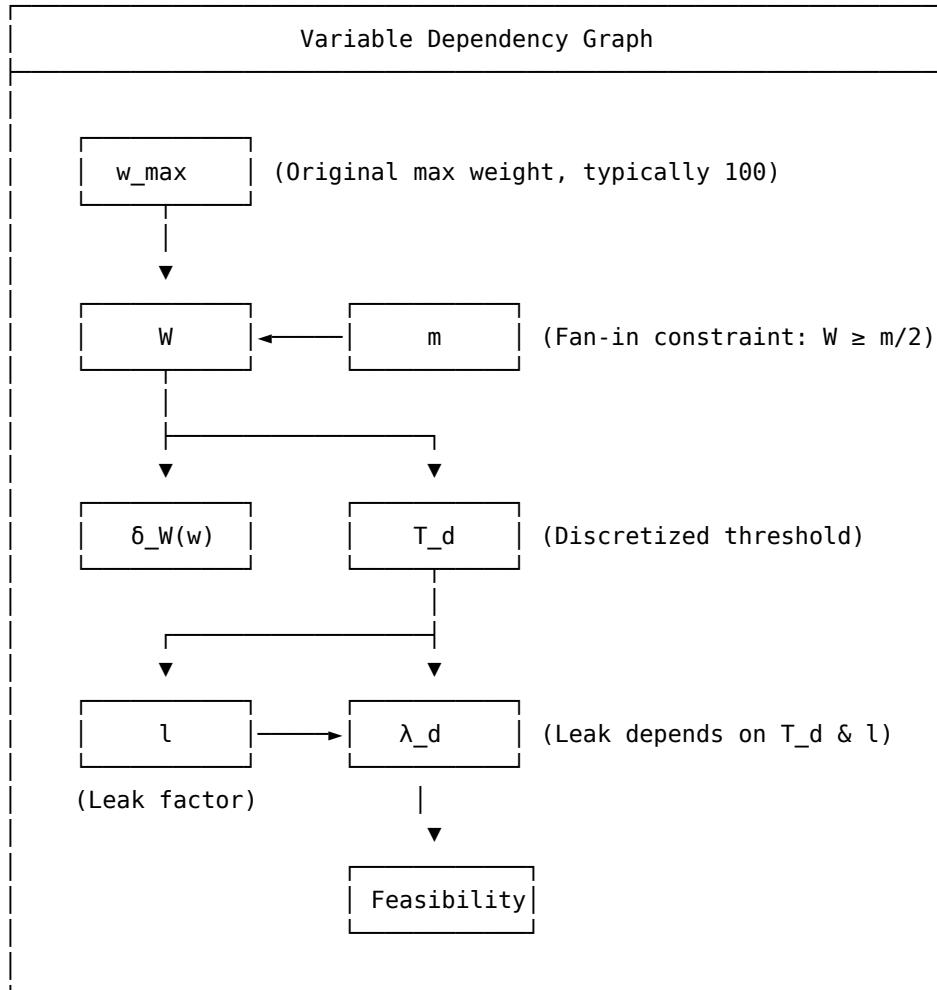
- *Tonic spiking* means sustained input produces a sustained *probability* of output — the neuron fires at a stochastic rate
- *Integrator* means accumulated input raises the firing *probability* continuously, reaching certainty ( $P = 1.0$ ) only at full threshold
- *Excitability* means stronger input increases the *expected* firing rate monotonically

**Remark. Immersion Memory:** After emitting a spike, the potential resets to 0. At  $P = 0$ , the firing probability is exactly 0 (level  $L = 0$ ), so the neuron has zero probability of spiking again until new input arrives. The neuron has no memory of past input signals — only the accumulated potential matters. This is preserved by our model since we explicitly reset  $P = 0$  after firing.

## 6. Variable Dependencies

This section clarifies the relationships between the key variables in our discretization scheme.

### 6.1. Dependency Graph



Listing 1: Variable dependencies in the discretization scheme. The crucial link is  $\lambda_d$  depending on  $T_d$  and the leak factor  $\ell$ , rather than on the number of classes  $k$ .

## 6.2. Summary Table

Variable	Definition
$\lambda_d$	$-\max(1, \lfloor \ell \cdot T_d \rfloor)$ — depends on threshold $T_d$ and leak factor $\ell$
$T_d$	$\lceil T \cdot \frac{W}{w_{\max}} \rceil$ — discretized firing threshold
$k$	Number of threshold levels (independent of $\lambda_d$ )
$\ell$	Leak factor: $\ell = 1 - r$ where $r$ is the retention rate

Table 4: Summary of variable definitions

## 7. PRISM Model Structure

The weighted quotient model generates PRISM code with explicit weight constants and contribution formulas. Note that weights are static constants, minimizing state explosion.

```
// Threshold-Dependent Leak Calculation
const int T_d = ceil(T_orig * W / w_max);
const int leak_percent = 10; // Leak factor as percentage ( $\ell = 0.10$ )
const int lambda_d = -max(1, floor(leak_percent * T_d / 100));

// Weight constants (discretized)
const int W = 3; // Discretization parameter
const int W_in0_2 = 3; //  $\delta_3(100) = 3$ 
const int W_n1_2 = -2; //  $\delta_3(-67) = -2$ 
const int W_n3_2 = 1; //  $\delta_3(33) = 1$ 

// Contribution formula (evaluated at runtime)
formula contrib_2 = W_in0_2 * x0 + W_n1_2 * z1_2 + W_n3_2 * z3_2;

// Class delta (clamped)
formula delta_2 = max(-4, min(4, contrib_2));

module Neuron
    p : [0..T_d + 5] init 0;
    spike : bool init false;

    // Update with Safety Clamp (max(0, ...))
    [tick] true ->
        (p' = max(0, min(T_d + 5, p + contrib_2 + lambda_d)))
        & (spike' = (p' >= T_d));
endmodule
```

## 8. Visualizations

The following four diagrams demonstrate model dynamics across two key parameter axes: *threshold* (low vs. high) and *leak factor* (low vs. high). Weights are held constant across all diagrams to isolate the effects of threshold and leak on neuron behavior.

### Common setup for all diagrams:

- **Topology:** 1 input neuron  $\rightarrow$  1 output neuron (single excitatory synapse)
- **Weight:**  $w = 100$  (original), discretized with  $W = 3$  giving  $\delta_3(100) = 3$
- **Input pattern:** AlwaysOn (constant input at every time step)
- **Threshold levels:**  $N = 4$  (probabilities: 0.25, 0.5, 0.75, 1.0)

- **Refractory periods:** Disabled ( $\text{ARP} = \text{off}$ ,  $\text{RRP} = \text{off}$ ) to focus on core dynamics
- **Simulation duration:** 50 ms
- **Seed:** Fixed (e.g., 42) for deterministic reproduction

**Diagram 1 — Low Threshold ( $T_d = 3$ ), Low Leak ( $\ell = 0.1$ ):**  $\lambda_d = -\max(1, [0.1 \cdot 3]) = -1$ , net gain =  $3 - 1 = 2$ .

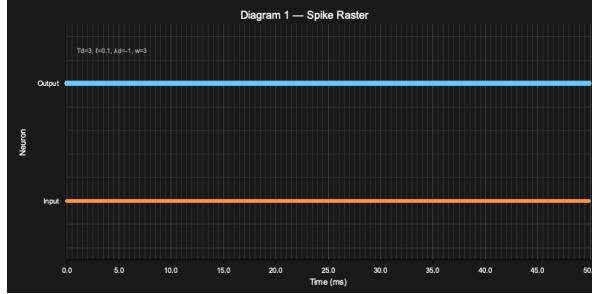


Figure 1: Diagram 1 — Spike Raster

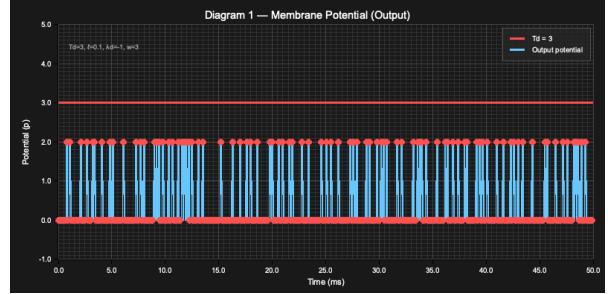


Figure 2: Diagram 1 — Membrane Potential

With input weight  $\delta_3(100) = 3$  meeting the threshold  $T_d = 3$  in a single step, the neuron fires near-continuously. The potential trace shows rapid oscillation between 0 and 2, with probabilistic firing at sub-threshold levels. This confirms the Feasibility Criterion (§5.1) for the SingleStep case:  $E \geq T_d$  yields maximum excitability.

**Diagram 2 — Low Threshold ( $T_d = 3$ ), High Leak ( $\ell = 0.5$ ):**  $\lambda_d = -\max(1, [0.5 \cdot 3]) = -1$ , net gain =  $3 - 1 = 2$ .

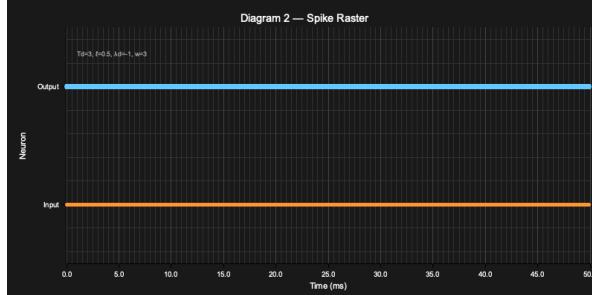


Figure 3: Diagram 2 — Spike Raster

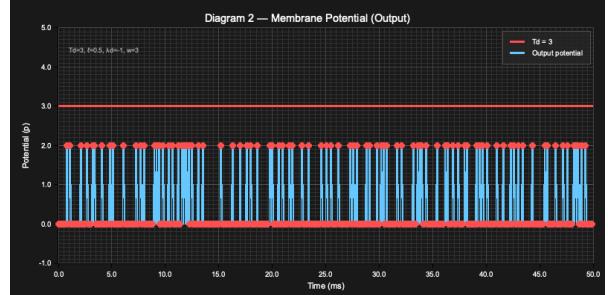


Figure 4: Diagram 2 — Membrane Potential

Despite a 5 $\times$  increase in the leak factor, the behavior is identical to Diagram 1. Because  $[0.5 \cdot 3] = 1$ ,  $\lambda_d$  remains  $-1$  and the single-step firing condition  $E \geq T_d$  still holds. This confirms that for low-threshold neurons, leak primarily affects the accumulation regime rather than single-step firing frequency.

**Diagram 3 — High Threshold ( $T_d = 6$ ), Low Leak ( $\ell = 0.1$ ):**  $\lambda_d = -\max(1, [0.1 \cdot 6]) = -1$ , net gain =  $3 - 1 = 2$ , expected ISI  $\approx \lceil \frac{6}{2} \rceil = 3$  steps.

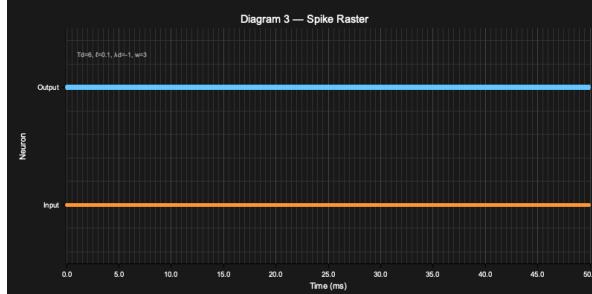


Figure 5: Diagram 3 — Spike Raster

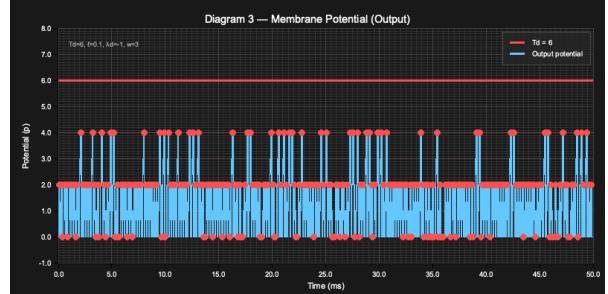


Figure 6: Diagram 3 — Membrane Potential

The neuron enters the *accumulation regime* — it cannot fire in a single step ( $E = 3 < T_d = 6$ ). The membrane potential trace shows a staircase pattern:  $0 \rightarrow 2 \rightarrow 4 \rightarrow \dots$ , with probabilistic firing at intermediate threshold levels ( $P = 0.25$  at  $L = 1$ ,  $P = 0.5$  at  $L = 2$ , etc.). This demonstrates the Excitability property (§5.3): the neuron integrates input over multiple steps with  $ISI \approx \left\lceil \frac{T_d}{C_{in} - |\lambda_d|} \right\rceil$ .

**Diagram 4 — High Threshold ( $T_d = 6$ ), High Leak ( $\ell = 0.5$ ):**  $\lambda_d = -\max(1, [0.5 \cdot 6]) = -3$ , net gain =  $3 - 3 = 0$ .

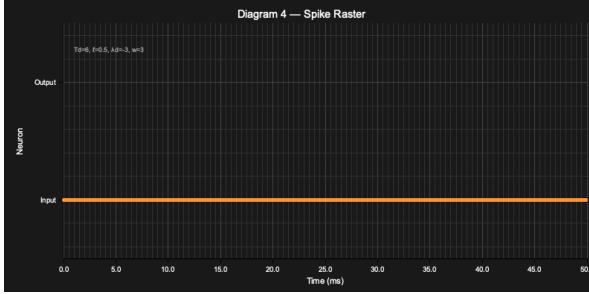


Figure 7: Diagram 4 — Spike Raster

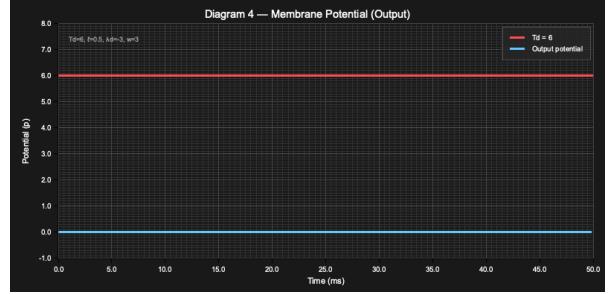


Figure 8: Diagram 4 — Membrane Potential

**Critical case:** Net gain is exactly 0 — the neuron is at the *feasibility boundary*. Each step adds +3 (input) and −3 (leak), resulting in zero accumulation. The membrane potential remains flat at 0 and the raster shows complete silence. This demonstrates the Feasibility Criterion (§5.1): when  $E \leq |\lambda_d|$ , the neuron is threshold-infeasible. The Soundness Theorem (§3.4) guarantees silence at this exact boundary. The minimum excitation  $C_{min} = |\lambda_d| + 1 = 4$  exceeds the available input of 3.

## 9. Conclusion

We have established a formal framework for weight discretization in quotient model abstraction. The key contributions include:

1. **Threshold-Dependent Leak:** The formulation  $\lambda_d = -\max(1, [\ell \cdot T_d])$  using the leak factor  $\ell$  ensures leak scales with the actual potential range, not the arbitrary number of classes.
2. **Soundness Theorem:** Complete proof that no spurious spikes occur — including the guarantee that zero potential maps to zero firing probability, providing a full safety chain from sub-threshold potential through decay to permanent silence.
3. **Biological Property Preservation:** Probabilistic verification that tonic spiking, integrator behavior, and excitability are maintained in the quotient model, with all spike emission expressed in terms of firing probabilities determined by threshold level mapping.
4. **Leak Factor Dynamics:** Analysis of the three key correlations: how the leak factor  $\ell$  controls current signal contribution, spike emission difficulty (minimum excitation  $C_{min}$ ), and expected spike frequency.
5. **Variable Dependency Clarification:** Clear documentation of how  $\lambda_d$  depends on  $T_d$  and  $\ell$  rather than on the number of classes  $k$ .
6. **Simulation Verification Plan:** Four precisely specified diagram configurations (two thresholds × two leak factors) to empirically validate the model dynamics using CogSpike's simulation engine.

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## Bibliography

- [1] C. Baier and J.-P. Katoen, *Principles of Model Checking*. MIT Press, 2008.
- [2] E. D. Maria, C. D. Giusto, and L. Laversa, “Spiking Neural Networks Modelled as Timed Automata with Parameter Learning,” *Natural Computing*, vol. 19, pp. 135–155, 2020.