Counting Pseudo Progressions

Jay Cummings, Quin Darcy, Natalie Hobson, Drew Horton,
Keith Rhodewalt, Morgan Throckmorton, Ry Ulmer-Strack

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Abstract

An *m-psuedo progression* is an increasing list of numbers for which there are at most m distinct differences between consecutive terms. This object generalizes the notion of an arithmetic progression. In this paper, we give two counts for the number of k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$. We also provide computer-generated tables of values which agree with both counts and graphs that display the growth rates of these functions. Finally, we present a generating function which counts k-term progressions in $\{1, 2, \ldots, n\}$ whose differences are all distinct, and we discuss further directions in Ramsey theory.

1 Introduction and Motivation

Arithmetic progressions have been well-studied. Their existence within partitions of $\mathbb{Z}$ is a central theme of Ramsey theory. The existence of long arithmetic progressions within the primes was famously solved by Green and Tao [5]. Landman and Robertson recently asked how the theory changes when instead of searching for arithmetic progressions, one searches for a specific generalization of an arithmetic progression, called an m-pseudo progression; in Section 6 we provide more details on this connection (see also [6]). Motivated by this question, we sought to better understand m-psuedo progressions by counting them. In this paper we provide two explicit methods and formulas to count these objects, which we now define.

1.1 Defining m-Pseudo Progressions

A k-term arithmetic progression is a list of numbers, $a_1, a_2, \ldots, a_k$, for which there exists some $d \in \mathbb{Z}^+$ where $a_{i+1} - a_i = d$ for all i. We now generalize this definition to include progressions with a greater number of differences between consecutive terms.

Definition 1.1. A (k-term) *m-pseudo progression* is a list

$$a_1, a_2, \ldots, a_k$$

of increasing integers from $\mathbb{Z}^+$ for which there exists a set $\{d_1, d_2, \ldots, d_m\}$, where $a_{i+1} - a_i \in \{d_1, d_2 \ldots, d_m\}$ for all i. If the m is not specified, it is simply called a *progression*.

For a given progression and any difference d_i, we denote $||d_i||$ for the number of times d_i appears as a difference in the progression.

Notice that if $m = 1$, we get the definition of an arithmetic progression. Furthermore, every m-pseudo progression is an ℓ-pseudo progression for $\ell \geq m$, because the definition of an ℓ-pseudo progression requires *at most* ℓ differences.

The list $2, 5, 8, 13, 16, 21, 24$ is a 7-term 2-pseudo progression since there are at most two common differences:

$$\begin{aligned} 5 - 2 &= 3 \\ 8 - 5 &= 3 \\ 13 - 8 &= 5 \\ 16 - 13 &= 3 \\ 21 - 16 &= 5 \\ 24 - 21 &= 3. \end{aligned}$$

Therefore, by letting $\{d_1, d_2\} = \{3, 5\}$, it is indeed true that $a_{i+1} - a_i \in \{d_1, d_2\}$ for all i.

In this paper, we provide two separate counts of how many k-term m-pseudo progressions there are in $\{1, 2, \ldots, n\}$, and we discuss in detail some special cases.

We note that there are other generalizations of arithmetic progressions. Psuedo progressions are in fact a generalization of what are called *generalized arithmetic progressions*, sometimes also called *d-dimensional arithmetic progressions* or *quasi-progressions*, (see for example [2, 3]), which are increasing sequences $a_1, a_2, \ldots, a_k$ for which there is some d such that $a_{i+1} - a_i \in \{1, \ldots, d\}$ for all i. These progressions demand that all the differences are "close" to each other, while a pseudo progression simply restricts the number of such differences.

2 Counting 2-Pseudo Progressions

2.1 Arithmetic Progressions

We begin with a discussion of the solved problem of counting arithmetic progressions, which we first formally provide. It is beneficial to see this approach, as two of our main theorems generalize the ideas highlighted in this proof.

Theorem 2.1. *There are*

$$F_1 = n \cdot \left\lfloor \frac{n}{k-1} \right\rfloor - (k-1) \cdot \binom{\lfloor \frac{n}{k-1} \rfloor + 1}{2}$$

k-term arithmetic progressions in $\{1, 2, \ldots, n\}$.

Proof. Fix a $d \in \mathbb{Z}^+$. Notice that if you know that a k-term arithmetic progression has common difference d, and you know the first element, then you have determined the entire progression. The possible arithmetic progressions with a common difference of d are

$$\begin{aligned} &1, 1 + d, 1 + 2d, \ldots, 1 + (k-1)d \\ &2, 2 + d, 2 + 2d, \ldots, 2 + (k-1)d \\ &\qquad\qquad \vdots \\ &n - (k-1) \cdot d, n - (k-1) \cdot d + d, n - (k-1) \cdot d + 2d, \ldots, n \end{aligned}$$

That is, there are $n - (k-1) \cdot d$ such progressions in $\{1, 2, \ldots, n\}$.

Notice that the largest possible d is $\lfloor n/(k-1) \rfloor$, and as for any integer $m > \lfloor n/(k-1) \rfloor$, the first possible arithmetic progression is

$$1, 1+m, 1+2m, \ldots, 1+(k-1)m,$$

making its last element $1 + (k-1)m > n$. Adding up the number of progressions for all possible d and using the combinatorial identity $\sum_{d=1}^{L} d = \binom{L+1}{2}$, it follows that

$$\begin{aligned}
\sum_{d=1}^{\lfloor \frac{n}{k-1} \rfloor} (n - (k-1) \cdot d) &= n \cdot \left\lfloor \frac{n}{k-1} \right\rfloor - (k-1) \cdot \sum_{d=1}^{\lfloor \frac{n}{k-1} \rfloor} d \\
&= n \cdot \left\lfloor \frac{n}{k-1} \right\rfloor - (k-1) \cdot \binom{\lfloor \frac{n}{k-1} \rfloor + 1}{2}. \qquad \square
\end{aligned}$$

Remark 2.2. There are a few elements of this proof which will be important later. First, the maximum common difference of a k-term arithmetic progression in $\{1, 2, \ldots, n\}$ is $\lfloor \frac{n-1}{k-1} \rfloor$. Second, the number of k-term arithmetic progressions in $\{1, 2, \ldots, n\}$ with common difference d is $n - d(k-1)$. Both of these will be generalized in this paper.

2.2 2-Pseudo Progressions

In this section, we provide two counts for the number of k-term 2-pseudo progressions in $\{1, 2, \ldots, n\}$. First, some notation.

Notation 2.3. Suppose

$$a_1, a_2, \ldots, a_k$$

is a k-term 2-pseudo progression. We will use a and b to refer to possible differences of this progression; that is, $a_{i+1} - a_i \in \{a, b\}$ for all i. Recall, we will use $||a||$ to refer to the number of differences of size a and $||b||$ to refer to the number of differences of size b.

For instance, in Example 1.1 we have $a = 3$ with $||a|| = 4$, and $b = 5$ with $||b|| = 2$.

We will also wish to refer to different orderings of the differences a and b. For example, in Example 1.1 the differences are in the order $3, 3, 5, 3, 5, 3$. We call this a difference pattern.

Definition 2.4. Given an m-pseudo progression

$$a_1, a_2, \ldots, a_k,$$

the list

$$a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}$$

is called the progression's *difference pattern.*

Lemma 2.5. *Fix a set of differences $\{d_1, d_2, \ldots, d_m\}$ and numbers $||d_1||, ||d_2||, \ldots, ||d_m||$. The number of k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$ with a fixed difference pattern is independent of the difference pattern you choose.*

Proof. Fix a set of differences $\{d_1, d_2, \ldots, d_m\}$ and their multiplicities $||d_1||, ||d_2||, \ldots, ||d_m||$, as given in the lemma. Next, fix two difference patterns D_1 and D_2 using the d_i from this set. Let S be the collection of all k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$ with difference pattern D_1, and let T be the collection of all k-term m-pseudo progressions with difference pattern D_2. We will show a bijection between the elements in S and T.

Given an arbitrary $s \in S$, say s starts at p_0. Note that there is only one progression starting at s whose difference pattern matches D_1. Let f be the function that maps s to the progression $t \in T$ which starts at p_0, which is likewise unique since its difference pattern is again specified. Moreover, both s and t must end at $p_0 + d_1||d_1|| + d_2||d_2|| + \cdots + d_m||d_m||$, so if one is in $\{1, 2, \ldots, n\}$, then the other is too. Because a starting point uniquely determines the progression, f is invertible and hence a bijection. This shows there are the same number of elements in S and T, completing the proof. $\square$

Once you have fixed your set of differences and their multiplicities, it will not be difficult to determine how many difference patterns match those criteria. Thus, Lemma 2.5 will be beneficial in that it allows one to focus on a special class of difference patterns, such as ones in which all instances of one of the differences occur before any instance of the second difference. The following lemma pushes this further, by fixing the multiplicities but relaxing the differences themselves.

Lemma 2.6. *Fix some $s_0, t_0 \in \{0, 1, 2, \ldots\}$ where $s_0 + t_0 = k - 1$, and consider any two lists, L_1 and L_2, each consisting of s_0 copies of the variable a and t_0 copies of the variable b, in some order. Let D_1 be the collection of all k-term 2-pseudo progressions in $\{1, 2, \ldots, n\}$ whose difference pattern matches L_1 (note that a and b can differ between progressions within D_1, provided the order of the differences matches L_1), and let D_2 be the collection of all k-term 2-pseudo progressions in $\{1, 2, \ldots, n\}$ whose difference pattern matches L_2 (for appropriate substitutions of a and b). Then, $|D_1| = |D_2|$.*

Proof. Consider a k-term 2-pseudo progression P_1 from D_1, and suppose this progression starts at i. Then, there exist a and b for which the progression makes s_0 jumps of size a and t_0 jumps of size b (in the order matching L_1), and the last term of the progression is, therefore, $i + s_0a + t_0b$. Since all the progressions in D_2 also have s_0 copies of one difference and t_0 copies of a second, and the differences are allowed to be anything, if you simply reorder the differences in P_1 to match the ordering of L_2, you get a new 2-pseudo progression P_2 which begins at i and ends at $i + s_0a + t_0b$ and which is now in D_2.

That is, by permuting the order of which you make your jumps of size a and b, you necessarily get a new progression from the same beginning point to the same ending point. So if one of these progressions is in $\{1, 2, \ldots, n\}$, then the other is too. And since this procedure is clearly invertible, we have a bijection between D_1 and D_2. $\square$

2.3 Recursive Count

In this section we present our first count of 2-pseudo progressions. By Lemma 2.6, the number of m-pseudo progressions is independent on the difference pattern. Therefore, we define a simple difference pattern for which the progressions will be simpler to count, and then later scale up this count to include all m-pseudo progressions.

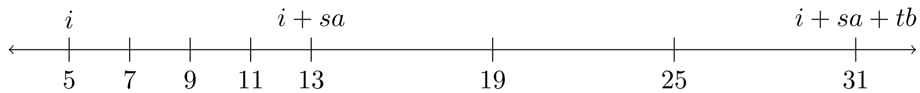
Definition 2.7. Call a 2-pseudo progression *s-t-simple* if its difference pattern is of the form

$$\underbrace{a, a, \ldots, a}_{s \text{ terms}}, \underbrace{b, b, \ldots, b}_{t \text{ terms}},$$

where s, t, a and b are positive integers, and let $S(n, s, t)$ be the number of s-t-simple 2-pseudo progressions in $\{1, 2, \ldots, n\}$.

Note that if $a = b$, then the 2-pseudo progression is in fact an arithmetic progression and is s-t-simple whenever $s + t = k - 1$, where k is the length of the progression.

To better visualize the general case, suppose we wanted to create a 4-3-simple 2-pseudo progression that started at $i = 5 \in \{1, 2, \ldots, n\}$ and where $a = 2$ and $b = 6$. Since our 2-pseudo progression is of the form a, a, a, a, b, b, b, it would look like the following:



Here, $i + sa = 13$ and $i + sa + tb = 31$. It is also important to note that given any k-term 2-pseudo progression, $s + t = k - 1$. This is because s and t count the number of differences between terms, so the total number of those differences will always be one less than the number of terms in the progression. For instance, in the example above we have $k = 8$, $s = 4$ and $t = 3$, so we see that $4 + 3 = 8 - 1$. In general, we see that s and t are dependent on k.

Proposition 2.8.

$$S(n, s, t) = \sum_{a=1}^{\left\lfloor \frac{n-1-t}{s} \right\rfloor} \sum_{i=1}^{n-sa-t} \left\lfloor \frac{n - (i + sa)}{t} \right\rfloor.$$

Proof. Given an s-t-simple 2-pseudo progression, let us assume that the progression starts at $i \in \{1, 2, \ldots, n\}$. Since the first s common differences are of size a, the $(s+1)^{\text{st}}$ term of the pseudo progression will be $i + sa$. After $i + sa$, the terms in the 2-pseudo progression have the characteristic that $a_{j+1} - a_j = b$ and since the pseudo progression proceeds with t common differences of size b, the 2-pseudo progression will end at $i + sa + tb$.

Since s and t are dependent on k, instead of fixing a k we choose to fix s and t, and consider the cases for which i, a, and b vary. Without loss of generality, by using the inequality $i + sa + tb \leq n$ we can see that $b \leq \dfrac{n - (i + sa)}{t}$. We know that b must be a positive integer, and so by using the floor function we have

$$1 \leq b \leq \left\lfloor \frac{n - (i + sa)}{t} \right\rfloor.$$

Therefore, given any valid selection of a and i, the above gives the possible values of b. That is,

$$S(n, s, t) = \sum_a \sum_i \left\lfloor \frac{n - (i + sa)}{t} \right\rfloor,$$

where the sums are over all the valid values of a and of i. Thus, our focus turns to determining these valid values. We begin with the range of i. We know that $i + sa + tb \leq n$, and so we have that $i \leq n - sa - tb$. And since i is at its greatest when b is at its smallest (when $b = 1$), we have

$$1 \leq i \leq n - sa - t.$$

Finally, for a fixed i, consider the valid values of a. In order to find the greatest possible value that a can be, note that a is at its greatest when both b is at its smallest (when $b = 1$) and the sequence starts at the earliest index (when $i = 1$). Thus, by again using the inequality $i+sa+tb \leq n$ we have that $a \leq \dfrac{n-i-tb}{s}$. And in the case that $b = 1$ and $i = 1$, we have

$$1 \leq a \leq \left\lfloor \frac{n-1-t}{s} \right\rfloor.$$

From this, we obtain our final count,

$$S(n,s,t) = \sum_{a=1}^{\lfloor \frac{n-1-t}{s} \rfloor} \sum_{i=1}^{n-sa-t} \left\lfloor \frac{n-(i+sa)}{t} \right\rfloor,$$

which completes the proof. $\square$

Note, though, that there are many other progressions with s common differences of size a and t common differences of size b. For example,

$$\underbrace{a, a, \ldots, a}_{s-2 \text{ terms}}, \underbrace{b, b, \ldots, b}_{t-2 \text{ terms}}, a, b, b, a.$$

We now count these other forms.

Corollary 2.9. Let F be a list of s copies of a and t copies of b, in some order. Let D be the collection of difference patterns which, for some substitution of a and b, match F. Then, the number of 2-pseudo progressions in $\{1, 2, \ldots, n\}$ with a difference pattern in D is equal to $S(n, s, t)$.

Proof. This follows immediately from Lemma 2.6. $\square$

Definition 2.10. Let $F_m(n, k)$ be the number of m-pseudo progressions. When the context is clear, we will write simply F_m.

Note that $F_1(n, k)$ is the number of k-term arithmetic progressions in $\{1, 2, \ldots, n\}$, which we counted in Section 2.1.

Theorem 2.11. *Fix n and k. Let $a_1, a_2, \ldots, a_k$ be a 2-pseudo progression where $a_{i+1} - a_i \in \{a, b\}$ for all i. Let s be defined as the number of elements such that $a_{i+1} - a_i = a$ and t be defined as the number of elements such that $a_{j+1} - a_j = b$. Then,*

$$F_2 = F_1 + \frac{1}{2}\binom{k-1}{\frac{k-1}{2}}\left[S\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) - F_1\right] + \sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s}\left[S(n,s,t) - F_1\right].$$

Proof. Recall that s is the number copies of a and t is the number of copies of b, while a and b can be any integers. Since one of these integers will occur at least as many times as the other, we may assume without loss of generality that $s \leq t$. First, assume $s < t$.

By Proposition 2.8, $S(n, s, t)$ counts the number of progressions of the form

$$\underbrace{a, a, \ldots, a}_{s \text{ terms}}, \underbrace{b, b, \ldots, b}_{t \text{ terms}}.$$

By Corollary 2.9, we know that given any form with s copies of a and t copies of b, there are $S(n,s,t)$ progressions of that form. Moreover, one can see that there are $\binom{s+t}{s}$ such forms—the $s+t$ terms in the form are each either an a or a b, and is determined by choosing which s of these terms are an a. Now, $S(n,s,t)$ includes all the arithmetic progressions (in the case where $a=b$). Therefore,

$$S(n,s,t) - F_1$$

counts the number of progressions of any fixed form containing s copies and a and t copies of b, which contains *exactly* two distinct common differences. There are then

$$\binom{s+t}{s}[S(n,s,t) - F_1]$$

2-pseudo progressions, excluding the arithmetic progressions. Note that $s+t=k-1$, and so in our current case where $s<t$ (i.e., $s \le t-1$), we know that $2s+1 \le k-1$, implying that $s \le \left\lfloor \frac{k-2}{2} \right\rfloor$. And so, in the case where $s<t$, the total number of 2-psuedo progressions which are not arithmetic progressions is

$$\sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s} [S(n,s,t) - F_1].$$

Therefore, among all the cases in which $s<t$, the total number of 2-pseudo progressions is

$$F_1 + \sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s} [S(n,s,t) - F_1].$$

The last case to consider is when $s=t$; note that this is only possible if k is odd. And since $s+t=k-1$, we have $s=t=\frac{k-1}{2}$. Just like above, given any form using $\frac{k-1}{2}$ copies of a and $\frac{k-1}{2}$ copies of b, there are $S(n, \frac{k-1}{2}, \frac{k-1}{2})$ progressions of this form. And so there are

$$S\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) - F_1$$

progressions that have exactly two distinct common differences. The only difference is in the next step. Note that if we simply multiply by $\binom{s+t}{s}$, we will be over-counting by a factor of 2. Indeed, since $s=t$, any progression comes about in two ways: once when the copies of a are counted by s and the copies of b are counted by t, and once when the copies of a are counted by t and the copies of b are counted by s. Thus, the count in the $s=t$ case is

$$\frac{1}{2}\binom{k-1}{\frac{k-1}{2}} \left[S\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) - F_1 \right].$$

Note that the binomial here evaluates to 0 in the event that k is even, and so including this term in the even case is consistent. This gives us our final answer:

$$F_2 = F_1 + \frac{1}{2}\binom{k-1}{\frac{k-1}{2}} \left[S\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) - F_1 \right] + \sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s} [S(n,s,t) - F_1].$$

$\square$

Since there are $F_\ell(n,k)$ sequences where you are allowed up to ℓ differences, and $F_{\ell-1}(n,k)$ sequences where you are allowed up to $\ell-1$ differences, there are

$$F_\ell(n,k) - F_{\ell-1}(n,k)$$

with exactly ℓ differences.

If we let $\tilde{F}_\ell(n,k)$ denote the number of ℓ-pseudo progressions with *exactly* ℓ distinct differences, and $\tilde{S}(n,s,t)$ likewise to be the number of s-t-simple progressions with exactly two differences, then $\tilde{S}(n,s,t) = S(n,s,t) - F_1(n,k)$ and the Theorem 2.11 has the reduced form

$$\tilde{F}_2(n,k) = \frac{1}{2}\binom{k-1}{\frac{k-1}{2}}\tilde{S}\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) + \sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s}\tilde{S}(n,s,t).$$

Moreover, the above two equations imply

$$F_2(n,k) - F_1(n,k) = \frac{1}{2}\binom{k-1}{\frac{k-1}{2}}\tilde{S}\left(n, \frac{k-1}{2}, \frac{k-1}{2}\right) + \sum_{s=1}^{\lfloor \frac{k-2}{2} \rfloor} \binom{s+t}{s}\tilde{S}(n,s,t),$$

which is another form of Theorem 2.11.

2.4 Iterative Count

As in the previous sections, we use a and b to denote the two differences in a 2-pseudo progression. In this section, we will insist that $a < b$, which in particular prohibits $a = b$. We will often refer to a k-term 2-pseudo progression in $\{1, 2, \ldots, n\}$ as just a 'progression' if the context is clear.

Remark 2.12. Recall, for a k-term 2-pseudo progression,

$$||a|| + ||b|| = k - 1.$$

That is, the total number of differences of size a and b is equal to the total number of differences in the progression, $k-1$. If we are considering k-term progressions, then we only need to know either $||a||$ or $||b||$ and the other will follow.

First, given a fixed number of two differences, we determine the maximum value these differences can be.

Lemma 2.13. *For a k-term 2-pseudo progression with a fixed number of differences, say $||a||$ and $||b||$ (without the sizes of the differences a and b being determined), the largest possible value for a is,*

$$a_{max} = \left\lfloor \frac{(n-1) - ||b||}{k-1} \right\rfloor.$$

Proof. We begin by noting that similar to Remark 2.2, the maximum possible difference between the first and last terms of a 2-pseudo progression is $n-1$. Since we are assuming $a < b$, if we want to find the largest possible value for a, we can assume $a = b - 1$. Thus, we must distribute the difference of $n-1$ into $k-1$ groups ($||a||$ groups of size a, and $||b||$ groups of size $a+1$). To do this, we subtract $||b||$ from $n-1$ in order to account for the $||b||$ groups of one larger value than a. The maximum possible value for the difference a is the result in the lemma. $\square$

Similarly, with a fixed difference of size a and number of differences, say $||a||$ and $||b||$, we can determine the maximum value of the difference b in a k-term 2-pseudo progression in $\{1, 2, \ldots, n\}$.

Lemma 2.14. *Suppose a k-term 2-pseudo progression has difference a and some number of differences, say $||a||$ and $||b||$. Then, the maximum possible value for b is*

$$\bar{b_a} = \begin{cases} 0 & ||b|| = 0, \\ \left\lfloor \frac{(n-1)-a\cdot||a||}{||b||} \right\rfloor & ||b|| \neq 0. \end{cases}$$

Proof. Similar to our argument in Lemma 2.13, the largest possible value between the first and final terms of a progression in $\{1, 2, \ldots, n\}$ is $n-1$. Thus, to determine the maximum possible b that can create a progression, we must divide $n-1$ into $k-1$ groups ($||a||$ groups of size a and $||b||$ groups of size b). In order to account for the $||a||$ differences of size a, we must subtract off the product $a \cdot ||a||$, and divide the remaining $(n-1) - a \cdot ||a||$ into $||b||$ groups. Thus, we have our resulting maximum above. The equality is also guaranteed, since a 2-pseudo progression with these metrics which begins at 1 will end at $1 + a \cdot ||a|| + \bar{b_a} \cdot ||b|| \leq n$. □

Proposition 2.15. *Given a fixed a, b, $||a||$ and $||b||$, the number of k-term 2-pseudo progressions in a set $\{1, 2, \ldots, n\}$ with a fixed difference pattern is*

$$n - a \cdot ||a|| - b \cdot ||b||.$$

Proof. Fix the integers n, k, a, b, $||a||$ and $||b||$, and a difference pattern. Similar to the argument in the proof of Theorem 2.1, we proceed by first considering a progression with inital term 1. Since our differences are of size a and b and there are $||a||$ $a's$, and $||b||$ $b's$, we have that the final term in the progression will be $1 + a \cdot ||a|| + b \cdot ||b||$. In general, if a progression with these parameters has initial term p_0, then the final term will be $p_0 + a \cdot ||a|| + b \cdot ||b||$. Such a progression is valid in $\{1, 2, \ldots, n\}$ if this final term $p_0 + a \cdot ||a|| + b \cdot ||b|| \leq n$. Such will be the case when

$$p_0 \leq n - a \cdot ||a|| - b \cdot ||b||.$$

Thus, the total number of valid 2-pseudo progressions with these parameters is equal to the largest p_0 such that the above inequality is true. And so the result of the proposition follows. □

Proposition 2.16. *The total number of k-term 2-pseudo progressions in $\{1, 2, \ldots, n\}$ can be counted using the following formula:*

$$F_1 + \sum_{||a||=1}^{k-1} \sum_{a=1}^{a_{max}} \sum_{b=a+1}^{\bar{b_a}} \left[\binom{k-1}{||a||} (n - a \cdot ||a|| - b \cdot ||b||) \right].$$

Proof. A formula for F_1 is given in Theorem 2.1, and it counts the number of arithmetic progressions in $\{1, 2, \ldots, n\}$; we now turn to count the number of 2-pseudo progressions which contain two distinct differences.

As a consequence of Proposition 2.15, the total number of k-term 2-pseudo progressions in $\{1, 2, \ldots, n\}$ with a given difference pattern and fixed values of $a, b, ||a||, ||b|| \in \mathbb{Z}^+$ is given by $(n - a \cdot ||a|| - b \cdot ||b||)$. Furthermore, by Lemma 2.5, this value does not depend on the difference pattern chosen. Thus, given a fixed a, b, and $||a||$ (which implies the value of $||b||$), we count the total number of progressions with these parameters by scaling the count in Proposition 2.15 by the total number of difference patterns that could occur with $||a||$ $a's$ and $||b||$ $b's$. In particular, there are $\binom{k-1}{||a||}$ such ways, by choosing which of the total $k-1$ skips to place the $||a||$ skips of size a.

Now we must determine all possible values of a and b that are possible. We assume $0 < a < b$, so we have $a = 1$ being the smallest possible value for a. Thus, $b = a + 1$ is the smallest possible value for b given a value for a. This gives us the bounds for the inner two sums.

Finally, we iterate this process through all possible positive values of $||a||$ and $||b||$, by summing up all valid progressions for each value of $||a||$, with $1 \leq ||a|| \leq k-1$. Since $||a|| + ||b|| = k-1$, iterating through all possible values of $||a||$ will indeed iterate all valid positive pairs of $||a||$ and $||b||$. $\square$

3 Counting m-Pseudo Progressions

We now generalize Lemma 2.6, which will be used to generalize Corollary 2.9

Lemma 3.1. *Fix some $d_0, d_1, \ldots, d_m \in \{0, 1, 2, \ldots\}$ where $\sum_{i=1}^{m} d_i = k-1$, and consider any two lists, L_1 and L_2, each consisting of $||d_i||$ copies of the variable d_i for each i, in some order. Let D_1 be the collection of all k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$ whose difference pattern matches L_1 (for appropriate substitutions of $d_1, d_2, \ldots, d_m$), and let D_2 be the collection of all k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$ whose difference pattern matches L_2 (for appropriate substitutions of $d_1, d_2, \ldots, d_m$). Then, $|D_1| = |D_2|$.*

Proof. We will show a bijective correspondence between k-term m-pseudo progressions with the same number of differences, $||d_i||$ for all i, regardless of the ordering of the differences. Let S and T be the set of m-pseudo progressions with distinct difference patterns containing differences $d_1, d_2, \ldots, d_m$ such that the number of differences $||d_i||$ is the same in each difference pattern for all i.

Given an arbitrary $s \in S$, if s starts at, say, p_0, then it will end at $p_0 + d_1 \cdot ||d_1|| + d_2 \cdot ||d_2|| + \cdots + d_m \cdot ||d_m||$. Let f be the function that maps s to the m-pseudo progression $t \in T$ starting at p_0. Note that this is well-defined, as pairing the starting term of an m-pseudo progression with that progression's difference pattern uniquely determines the progression. Any $t \in T$ is the image of the m-pseudo progression in S starting at the initial term of t. Therefore, the number of progressions with a fixed difference pattern is independent of the ordering of the differences in the difference pattern. $\square$

3.1 Recursive Count

In this section we generalize the ideas of Section 2.3 to count k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$.

Definition 3.2. A progression to be s_1,s_2,…,s_m-simple if it is of the form

$$\underbrace{a_1, a_1, \ldots, a_1}_{s_1 \text{ terms}}, \underbrace{a_2, a_2, \ldots, a_2}_{s_2 \text{ terms}}, \ldots, \underbrace{a_m, a_m, \ldots, a_m}_{s_m \text{ terms}},$$

where $s_i, a_j \in \mathbb{Z}^+$ for all i, j. Let $S(n, s_1, \ldots, s_m)$ be the number of s_1,s_2,…,s_m-simple progressions in $\{1, 2, \ldots, n\}$.

By the same reasoning as in Section 2.3, there is a bijection between the set of progressions of this form and the set of progressions of any permuted form, and there are $\binom{k-1}{s_1 \ldots s_m}$ such permuted forms. And

$$S(n, s_1, \ldots, s_{m+1}) = \sum_{i=1}^{n} \left\lfloor \frac{i}{s_{m+1}} \right\rfloor \cdot S(n-i, s_1, \ldots, s_m),$$

since an s_1,s_2,...,s_{m+1}-simple progression in $\{1, 2, \ldots, n\}$ is simply an s_1,s_2,...,s_m-simple progression in $[n-i]$, for some i, followed immediately by s_{m+1} more terms. And within the remaining i numbers, starting with the first, there are $\left\lfloor \frac{i}{s_{m+1}} \right\rfloor$ possible common differences that will keep the entire s_1,s_2,...,s_{m+1}-simple progression in $\{1, 2, \ldots, n\}$.

So in this way we have recursively found the number of s_1,s_2,...,s_m-simple progressions for an arbitrary m. And so

$$S(n, s_1, \ldots, s_{m+1}) - F_m(n)$$

counts the number of s_1,s_2,...,s_m-simple progressions with *exactly* $m+1$ distinct common differences. And if $s_1 < s_2 < \cdots < s_{m+1}$, then by Lemma 3.1,

$$\binom{k-1}{s_1 \ldots s_{m+1}} [S(n, s_1, \ldots, s_{m+1}) - F_m(n)]$$

counts the number of all progressions which are of a form which is a permutation of the s_1,s_2,...,s_m-simple form.

Otherwise, if we only assume $s_1 \leq s_2 \leq \cdots \leq s_{m+1}$, then just like with 2-pseudo progressions, we may over count. Indeed, if in the multiset $\{s_1, s_2, \ldots, s_{m+1}\}$ we have, say, 7 appearing 3 times (say, $s_3 = s_4 = s_5 = 7$), then this alone causes the above expression to over count by a factor of $3! = 6$ (just like in the 2-pseudo progressions, when $s = t$ implied that we over counted by a factor of $2! = 2$). To see this, note that each such progression will have been counted once when the copies of a_3 were counted by s_3, the copies of a_4 were counted by s_4, and the copies of a_5 were counted by s_5; but will also have been counted once for each of the other 3! pairings between these a_is and s_js. This reasoning gives the answer of

$$\frac{1}{n_1! n_2! \cdots n_\ell!} \binom{k-1}{s_1 \ldots s_{m+1}} [S(n, s_1, \ldots, s_{m+1}) - F_m(n)]$$

where n_i is the multiplicity of the i^{th} distinct number in $s_1, s_2, \ldots, s_{m+1}$, and therefore ℓ is the size of $\{s_1, s_2, \ldots, s_{m+1}\}$ as a set (i.e., removing multiplicities). Note that our answer in the 2-pseudo progression case is a special case of this. Adding up all the possibilities, and adding back in the progressions with at most m distinct common differences, gives you the final count:

$$F_{m+1}(n,k) = F_m(n) + \sum_{s_1 \leq \cdots \leq s_{m+1}} \frac{1}{n_1! n_2! \cdots n_\ell!} \binom{k-1}{s_1 \ldots s_{m+1}} [S(n, s_1, \ldots, s_{m+1}) - F_m(n)].$$

This also gives a particularly nice formula for $\tilde{F}_{m+1}(n,k)$, which you recall is the number of k-term progressions in $\{1, 2, \ldots, n\}$ with *exactly* $m+1$ common differences. The above reduces to the following:

$$\tilde{F}_{m+1}(n,k) = \sum_{s_1 \leq \cdots \leq s_{m+1}} \frac{1}{n_1! n_2! \cdots n_\ell!} \binom{k-1}{s_1 \ldots s_{m+1}} \tilde{S}(n, s_1, \ldots, s_{m+1}).$$

3.2 Iterative Count

In this section we generalize the ideas of Section 2.4 to count k-term m-pseudo progressions in $\{1, 2, \ldots, n\}$.

Remark 3.3. As before, the differences in an m-pseudo progression will be denoted d_i for $1 \leq i \leq m$. In this section, we assume

$$0 < d_1 < d_2 < \cdots < d_m.$$

Proposition 3.4. *For any i such that $0 < i \leq m$, and fixed list of positive integers $||d_1||, ||d_2||, \ldots, ||d_m||$ such that $\sum_{j=1}^{m} ||d_j|| = k - 1$, and fixed multiplicities $0 < d_1 < d_2 < \cdots < d_{i-1}$, the maximum value of the difference d_i of a k-term m-pseudo progression with m distinct differences is,*

$$\bar{d_i} = \left\lfloor \frac{(n-1) - \left(\sum_{j=1}^{i-1} d_j \cdot ||d_j|| + \sum_{j=i+1}^{m} ((j-i) \cdot ||d_j||) \right)}{k - 1 - \sum_{j=1}^{i-1} \cdot ||d_j||} \right\rfloor .$$

Proof. Assume the setup of Proposition 3.4. We have a fixed list of numbers $||d_1||, ||d_2||, \ldots, ||d_m||$ such that $\sum_{j=1}^{m} ||d_j|| = k - 1$, a difference pattern, and a fixed size of differences $d_1, d_2, \ldots, d_{i-1}$.

Similar to the proof for Lemmas 2.14 and 2.13, the largest difference between the initial and final term of a progression in $\{1, 2, \ldots, n\}$ is $n - 1$. In other words, for a progression that starts at 1, the last term in the progression is, $1 + \sum_{i=1}^{m} d_i \cdot ||d_i||$. If we want to determine a difference that is as *large* as possible, we can assume that $1 + \sum_{i=1}^{m} d_i \cdot ||d_i||$ is as large as possible. Thus, we assume $1 + \sum_{i=1}^{m} d_i \cdot ||d_i|| = n$ (and we consider issues of whether this value is an integer later). Thus, to determine the maximum value for d_i we must remove $d_j \cdot ||d_j||$ from $n - 1$ for each known difference $d_1, d_2, \ldots, d_{i-1}$.

Similar to the proof for Lemma 2.13, since we want to determine the largest possible value for d_i, we will assume for each $j > i$ that d_j is as small as possible while still maintaining the inequality from Remark 3.3. That is, we will assume for each $j > i$ that $d_j = d_i + (j - i)$. For example, $d_{i+1} = d_i + 1$.

However, since we are determining the value of d_i, we will account for each d_j of size $d_i + (j - i)$ by removing $(j - i) \cdot ||d_j||$ from $n - 1$ and distributing the remaining value equally between the remaining $k - 1 - \sum_{j=1}^{i-1} \cdot ||d_j||$ possible skips.

The floor of this expression gives the largest possible value for the difference of size d_i.

□

Proposition 3.5. *Given a fixed $d_1, d_2, \ldots, d_m, ||d_1||, ||d_2||, \ldots ||d_m||$ such that $\sum_{i=1}^{m} ||d_i|| = k - 1$, and difference pattern, the number of k-term m-pseudo progressions in a set $\{1, 2, \ldots, n\}$ with a fixed difference pattern is*

$$n - \sum_{j=1}^{m} d_j ||d_j||.$$

Proof. This result follows the same reasoning as Proposition 2.15.

□

Proposition 3.6. *Given an n, k and m, the total number of k-term pseudo progressions in $\{1, 2, \dots, n\}$ with exactly m distinct differences is*

$$\sum_{||d_1||=1}^{k-1} \sum_{||d_2||=1}^{k-2} \cdots \sum_{||d_{m-1}||=1}^{k-(m-1)} \sum_{d_1=1}^{\bar{d_1}} \sum_{d_2=d_1+1}^{\bar{d_2}} \cdots \sum_{d_m=d_{m-1}+1}^{\bar{d_m}} \binom{k-1}{||d_1||, ||d_2||, \dots, ||d_m||} \left(n - \sum_{j=1}^{m} d_j ||d_j||\right).$$

This can be written more succinctly as

$$\sum_{\mathbb{D}} \binom{k-1}{||d_1||, ||d_2||, \dots, ||d_m||} \left(n - \sum_{j=1}^{m} d_j \cdot ||d_j||\right),$$

where $\mathbb{D} = \{(d_1, \dots, d_m, ||d_1||, \dots, ||d_m||)$ such that $||d_i|| \neq 0$, $\sum_{i=1}^{m} ||d_i|| = k-1$, $1 \leq d_i \leq \bar{d_i}$, and $d_i < d_j$ whenever $i < j\}$.

In order to compute the total number of k-term m-pseudo progressions (that is, progressions with up to m distinct differences), sum the above over all m from 1 to m.

Proof. We can determine the total number of m-pseudo progressions in $\{1, 2, \dots, n\}$ with a given set of fixed positive integers $d_1, d_2, \dots, d_m, ||d_1||, ||d_2||, \dots, ||d_m||$ such that $\sum_{i=1}^{m} ||d_i|| = k-1$ and a fixed difference pattern from Proposition 3.5. By Lemma 2.5, we can scale this count by the number of possible difference patterns to determine the number of m-pseudo progressions with fixed parameters $d_1, d_2, \dots, d_m, ||d_1||, ||d_2||, \dots, ||d_m||$. The number of such difference patterns is the number of ways to choose where the $||d_i||$ differences of size d_i occur for each i from 1 to m. That is, the multinomial coefficient

$$\binom{k-1}{||d_1||, ||d_2||, \dots, ||d_m||}.$$

To determine the allowable collections of numbers $d_1, d_2, \dots, d_m, ||d_1||, ||d_2||, \dots, ||d_m||$, we continue with similar reasoning as in Proposition 2.16. That is, we iterate over all possible values of $||d_i||$ from 1 to $k-1-(m-1)$ (in order to ensure no $||d_i|| = 0$) such that $\sum_{i=1}^{m} ||d_i|| = k-1$. In order to maintain the inequality from Remark 3.3 and the maximum in Proposition 3.4, we iterate over the values of d_i from $d_{i-1}+1$ to $\bar{d_i}$. All such valid lists of differences $d_1, d_2, \dots, d_m$ and amounts $||d_1||, ||d_2||, \dots, ||d_m||$ can be represented by the set $\mathbb{D}$. $\square$

3.3 Reinterpreting combinatorial identities

Observe that if a $(k-1)$-pseudo progression has j numbers in $[v]$ (which can occur in $\binom{v}{j}$ ways), then the other $k-j$ numbers in the pseudo progression can be anywhere in the set $\{v+1, v+2, \dots, n\}$, which has size $n-v$. The number of ways to complete this is $F_{k-1-j}(n-v, k-j)$. Thus,

$$F_{k-1}(n, k) = \sum_{j=0}^{k} \binom{v}{j} F_{k-1-j}(n-v, k-j).$$

Recalling that $F_{k-1}(n, k) = \binom{n}{k}$, this gives a new proof of the Chu-Vandermonde identity:

$$\binom{n}{k} = \sum_{j=0}^{k} \binom{v}{j} \binom{n-v}{k-j}.$$

4 Generating Functions

An m-pseudo progression places a limit of m on the number of distinct differences within such a progression. In this section, we go to the opposite extreme and ask what happens if we demand all of the differences be distinct. Indeed, below we find the generating function which counts the number of k-term pseudo progressions in $\{1, 2, \ldots, n\}$ where all of the $k-1$ differences are distinct.

We will be discussing m-pseudo progressions without needing to refer to any particular m ($m = k-1$ would suffice, except that we do not wish to allow fewer than $k-1$ distinct differences). Therefore we will continue to refer to these as pseudo progressions without mentioning any m.

It is well known that

$$\frac{x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}$$

is the generating function for integer partitions with $k-1$ distinct parts. That is, the coefficient of x^t in this generating function gives the number of partitions of t into $k-1$ distinct parts: $t = p_1 + p_2 + \cdots + p_{k-1}$, where each p_i is a positive integer and $p_1 < p_2 < \cdots < p_{k-1}$.

Definition 4.1. Fix a k and n. For $t < n$, let $c(t)$ be the number of partitions of t into $k-1$ distinct parts.

Lemma 4.2. *There are*

$$\sum_{t=\frac{k(k-1)}{2}}^{n-1} (k-1)! \cdot (n-t) \cdot c(t)$$

k-term pseudo progressions in $\{1, 2, \ldots, n\}$ with distinct common differences.

Proof. Given a partition of t into $k-1$ distinct parts, note that we can create a pseudo progression in $\{1, 2, \ldots, n\}$ which starts at 1, ends at $t+1$, and whose common differences are distinct. Namely, if the partition is $t = p_1 + p_2 + \cdots + p_{k-1}$, then the pseudo progression is

$$1\ ,\ 1+p_1\ ,\ 1+p_1+p_2\ ,\ \ldots\ ,\ 1+t.$$

Also, observe that because $p_1 < p_2 < \cdots < p_{k-1}$, we in fact can find $(k-1)!$ pseudo progressions which start at 1 and end at $t+1$ by simply considering all possible permutations of $\{p_1, \ldots, p_{k-1}\}$, and adding in the p_i in the order determined by the permutation.

Moreover, all k-term pseudo progressions with distinct common differences that start at 1 and end at $t+1$ can be realized in this way. To see this, simply take such a pseudo progression, $1 = a_1, a_2, \ldots, a_k = t+1$, and observe the $k-1$ distinct common differences,

$$a_2 - a_1\ , a_3 - a_2\ ,\ \ldots\ ,\ a_k - a_{k-1}.$$

The sum of these common differences telescopes, so their sum can be seen as $a_k - a_1 = (t+1) - 1 = t$. And being distinct, once they are reordered in increasing order they do indeed form a partition of t with $k-1$ distinct parts.

So there is in fact a total of $(k-1)! \cdot c(t)$ k-term pseudo progressions in $\{1, 2, \ldots, n\}$ which start at 1, end at $t+1$, and have distinct common differences. To obtain a count for all such progressions, we simply need to multiply by the number of possible starting points. The progressions could begin at 1 and end at $t+1$, begin at 2 and end at $t+2$, ..., begin at $n-t$ and end at $t+(n-t)$. In total, there are $n-t$ ways that we can "shift" these progressions which start at 1 into progression that start at higher values. Thus, by multiplying by $n-t$ we get the total number of $(n-t)\cdot(k-1)!\cdot c(t)$ k-term pseudo progressions in $\{1,2,\ldots,n\}$ with distinct common differences.

Finally, we must sum over all possible values of t. The smallest t corresponds to the smallest value which can be partitioned into $k-1$ distinct parts, which is

$$1+2+3+\cdots+(k-1)=\frac{k(k-1)}{2}.$$

The largest possible t is $n-1$, since this corresponds to a progression which starts at 1 and ends at $t+1=n$. Thus, by summing over these possible vales of t, we get our final count:

$$\sum_{t=\frac{k(k-1)}{2}}^{n-1}(k-1)!\cdot(n-t)\cdot c(t).$$

□

We now use this to find the generating function for the number of k-term pseudo progressions in $\{1,2,\ldots,n\}$ with distinct common differences.

Theorem 4.3. *The number of k-term pseudo progressions in $\{1,2,\ldots,n\}$ with distinct common differences is the coefficient on x^n in the generating function*

$$\frac{(k-1)!x^{1+k(k-1)/2}}{(1-x)^3(1-x^2)(1-x^3)\cdots(1-x^{k-1})}.$$

Proof. Recall that the generating function for the number of integer partitions with distinct parts is

$$\frac{x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}.$$

That is, the coefficient of x^t in this generating function gives $c(t)$. By scaling, the coefficient of x^n in

$$\frac{x^{n-t+k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}$$

now gives $c(t)$. Thus, by Lemma 4.2, since there are $\sum_{t=\frac{k(k-1)}{2}}^{n-1}(k-1)!\cdot(n-t)\cdot c(t)$ k-term pseudo progressions in $\{1,2,\ldots,n\}$ with distinct common differences, this value is given by the coefficient of x^n in

$$\sum_{t=\frac{k(k-1)}{2}}^{n-1}\frac{(k-1)!(n-t)x^{n-t+k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}=\frac{(k-1)!x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}\sum_{t=\frac{k(k-1)}{2}}^{n-1}(n-t)x^{n-t}.$$

By substituting i for $n-t$, which reverses the order of summation, the above is equivalent to

$$\frac{(k-1)!x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}\sum_{i=1}^{n-\frac{k(k-1)}{2}}ix^i.$$

Notice that the coefficient on x^n here is the same as in

$$\frac{(k-1)!x^{k(k-1)/2}}{(1-x)(1-x^2)\cdots(1-x^{k-1})}\sum_{i=1}^{\infty} ix^i,$$

and so this new expression also has the property that the coefficient on x^n gives the number of k-term pseudo progressions in $\{1,2,\ldots,n\}$ with distinct common differences. Since $\sum_{i=1}^{\infty} ix^i$ has generating function $\frac{x}{(x-1)^2}$, this is equivalent to

$$\frac{(k-1)!x^{1+k(k-1)/2}}{(1-x)^3(1-x^2)(1-x^3)\cdots(1-x^{k-1})},$$

as desired. □

5 Symmetries

We have observed (see Section 7) that for certain small values of k, the number of k-term m-pseudo progressions in $\{1,2,\ldots,n\}$ is equal to the number of $(n-k)$-term m-pseudo progressions in $\{1,2,\ldots,n\}$. Indeed, the relationship seems to be related to the compliment. Consider a k-term m-pseudo progression and let K be the subset of $\{1,2,\ldots,n\}$ consisting of the elements of the progression. Then, the set $K^c = \{1,2,\ldots,n\} \setminus K$ corresponds to an $(n-k)$-term progression.

Note that the K^c progression will include a difference of 1 whenever there are two adjacent numbers in $\{1,2,\ldots,n\}$ which are not in K (for $k < \frac{n}{2} - 1$, this is guaranteed). The K^c progression will include a difference of 2 whenever the K progression had a term $i \in \{2,3,\ldots,n-1\}$ for which $i-1$ and $i+1$ are not in K (for most sets K of small size, such an i will exist). For the K^c progression to have a difference of $d > 1$, the K progression would have to include $d-1$ consecutive terms.

Since terms from the K progression have to be used to create differences in the K^c progressions, $|K|$ creates a bound on how many differences the K^c can have. Indeed, by this reasoning, it is impossible for the K^c progression to have more than m differences if

$$|K| < 1 + \sum_{d=2}^{m+1}(d-1) = 1 + \binom{m+1}{2}.$$

In Section 7 you will see these symmetries in our tables of values, which also show other interesting behavior. For example, the number of 4-term 3-pseudo progressions in $\{1,2,\ldots,14\}$ is equal to the number of 10-term 3-pseudo progressions in $\{1,2,\ldots,14\}$.

6 Further Directions

We were motivated to study this problem because of a problem in Ramsey theory, and it is in this direction that we plan to move to next.

Consider the positive integers $\mathbb{Z}^+ = \{1,2,3,4,\ldots\}$. An r-coloring of these integers is produced by assigning each of these integers one of r colors. The question is whether *every* r-coloring of $\mathbb{Z}^+$ contains a *k-term monochromatic arithmetic progression*. Such a progression is a collection of integers $a, a+d, a+2d, \ldots, a+(k-1)d$ which are all assigned the same color. Here, d is called the *common difference*. The seminal van der Waerden theorem [7] says that given any k and r, there exists some N such that every r-coloring of $\{1, 2, 3, \ldots, N\}$ contains a k-term monochromatic arithmetic progression; the smallest such N is denoted $w(k, r)$. For example, $w(3, 2) = 9$. That is, every 2-coloring of $\{1, 2, 3, \ldots, 9\}$ contains a 3-term monochromatic arithmetic progression, and furthermore it is not true that every such coloring of $\{1, 2, 3, \ldots, 8\}$ does. For example, here is a 2-coloring that avoids such a progression:

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8.$$

Much work has been done to try to bound $w(k, r)$. The best upper bound is that $w(k, r) \leq 2^{2^{r^{2^{2^{k+9}}}}}$, and is due to Tim Gowers.

Brown, Graham and Landman [1] investigated what happens when you restrict the allowable set of arithmetic progressions. In particular, if $D \subseteq \mathbb{Z}^+$ is a set of allowable common differences, they asked whether there must still exist an N for which every r-coloring of $\{1, 2, 3, \ldots, N\}$ contains a monochromatic arithmetic progression whose common difference is in D. That is, their research focused on a subset of the collection of arithmetic progressions. It seems natural then to ask what happens when you instead consider a superset of this collection.

Landman and Robertson recently asked about generalizations of van der Waerden's theoerm to m-pseudo progressions. Now that m-pseudo progressions are better understood through their count, we aim to determine the smallest values of N for which every r-coloring of $\{1, 2, \ldots, N\}$ contains a monochromatic m-pseudo progression.

7 Tables of Values and Graphs

Below are tables of values for the number of k-term m-pseudo progressions, as well as graphs to visualize these values.

2-pseudo progressions table

$k\backslash n$	5	6	7	8	9	10	11	12	13	14	15	16
1	5	6	7	8	9	10	11	12	13	14	15	16
2	10	15	21	28	36	45	55	66	78	91	105	120
3	10	20	35	56	84	120	165	220	286	364	455	560
4	5	15	29	52	84	126	180	249	331	431	549	686
5	1	6	21	44	78	120	186	264	363	478	627	792
6	0	1	7	28	64	120	182	274	386	533	715	918
7	0	0	1	8	36	90	180	282	426	582	795	1060
8	0	0	0	1	9	45	123	264	433	672	919	1236
9	0	0	0	0	1	10	55	164	379	658	1057	1472
10	0	0	0	0	0	1	11	66	214	533	987	1654
11	0	0	0	0	0	0	1	12	78	274	735	1458
12	0	0	0	0	0	0	0	1	13	91	345	995
13	0	0	0	0	0	0	0	0	1	14	105	428
14	0	0	0	0	0	0	0	0	0	1	15	120
15	0	0	0	0	0	0	0	0	0	0	1	16
16	0	0	0	0	0	0	0	0	0	0	0	1

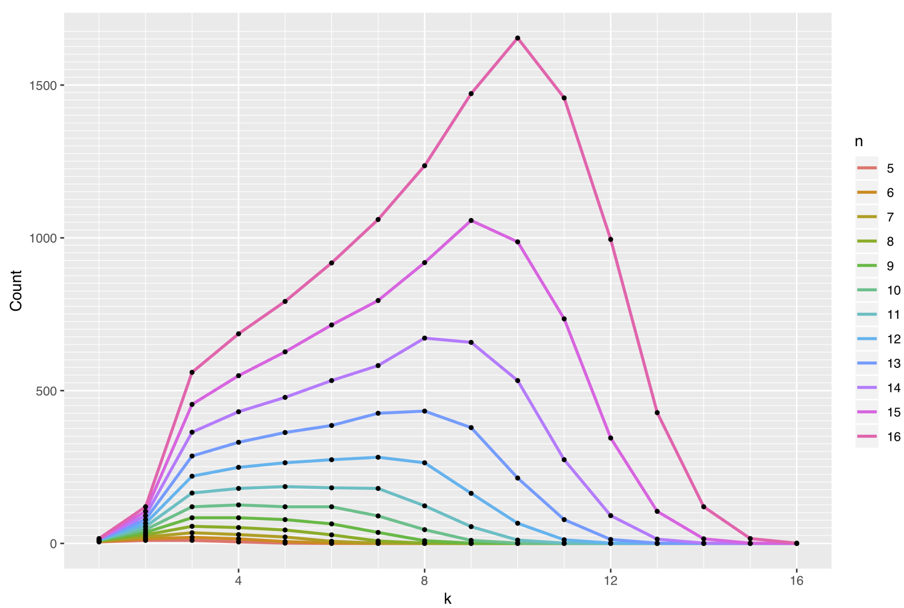
3-pseudo progressions table

$k\backslash n$	5	6	7	8	9	10	11	12	13	14	15	16
1	5	6	7	8	9	10	11	12	13	14	15	16
2	10	15	21	28	36	45	55	66	78	91	105	120
3	10	20	35	56	84	120	165	220	286	364	455	560
4	5	15	35	70	126	210	330	495	715	1001	1365	1820
5	1	6	21	56	126	252	438	720	1119	1666	2379	3312
6	0	1	7	28	84	210	462	864	1476	2343	3505	5128
7	0	0	1	8	36	120	330	792	1596	2892	4755	7240
8	0	0	0	1	9	45	165	495	1287	2793	5385	9300
9	0	0	0	0	1	10	55	220	715	2002	4669	9592
10	0	0	0	0	0	1	11	66	286	1001	3003	7504
11	0	0	0	0	0	0	1	12	78	364	1365	4368
12	0	0	0	0	0	0	0	1	13	91	455	1820
13	0	0	0	0	0	0	0	0	1	14	105	560
14	0	0	0	0	0	0	0	0	0	1	15	120
15	0	0	0	0	0	0	0	0	0	0	1	16
16	0	0	0	0	0	0	0	0	0	0	0	1

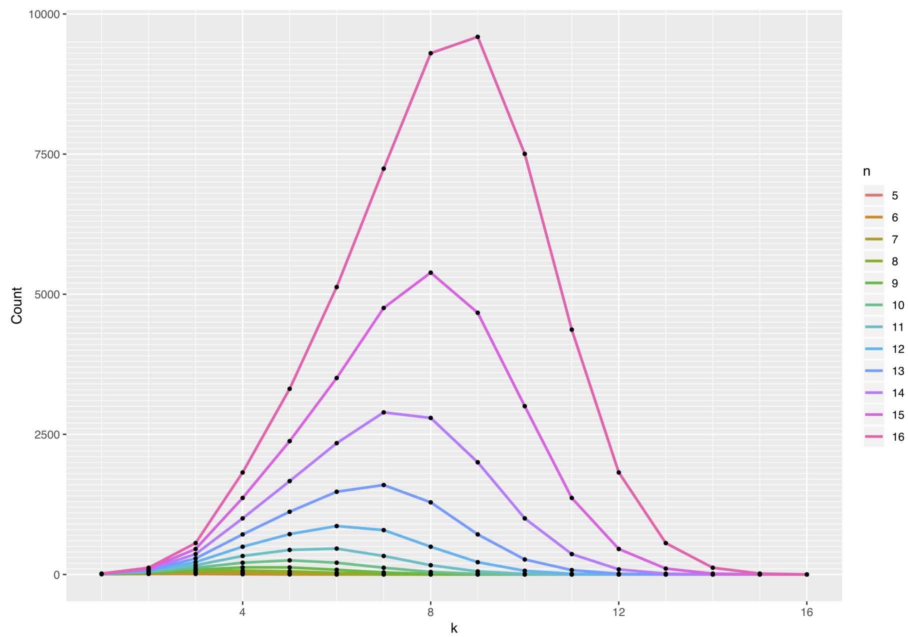
4-pseudo progressions table

$k\backslash n$	5	6	7	8	9	10	11	12	13	14	15	16
1	5	6	7	8	9	10	11	12	13	14	15	16
2	10	15	21	28	36	45	55	66	78	91	105	120
3	10	20	35	56	84	120	165	220	286	364	455	560
4	5	15	35	70	126	210	330	495	715	1001	1365	1820
5	1	6	21	56	126	252	462	792	1287	1666	2379	4368
6	0	1	7	28	84	210	462	924	1716	3003	5005	7888
7	0	0	1	8	36	120	330	792	1716	3432	6435	11440
8	0	0	0	1	9	45	165	495	1287	3003	5385	12870
9	0	0	0	0	1	10	55	220	715	1666	5005	11440
10	0	0	0	0	0	1	11	66	286	1001	3003	8008
11	0	0	0	0	0	0	1	12	78	364	1365	4368
12	0	0	0	0	0	0	0	1	13	91	455	1820
13	0	0	0	0	0	0	0	0	1	14	105	560
14	0	0	0	0	0	0	0	0	0	1	15	120
15	0	0	0	0	0	0	0	0	0	0	1	16
16	0	0	0	0	0	0	0	0	0	0	0	1

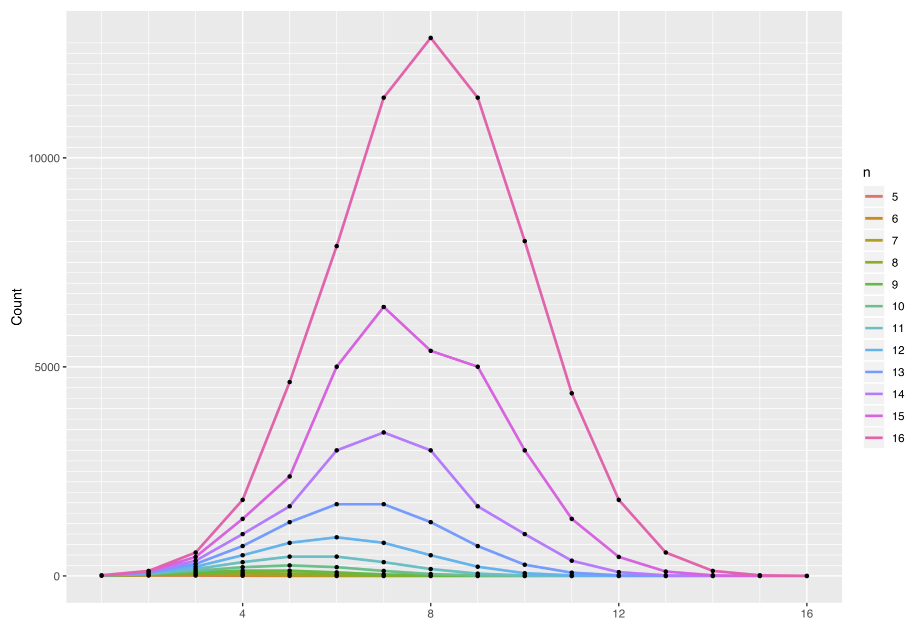
2-pseudo progressions graph



3-pseudo progressions graph



4-pseudo progressions graph



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