COMPREHENSIVE EXAM
ALGEBRA
Spring 2017

Part I: Group Theory (Do 4 of the following 5 problems)

1. Let G be an Abelian group.

   (a) Prove that if G is an infinite Abelian group, then G is not simple.

   (b) Prove that a finite Abelian group, G, is simple iff $|G| = p$ for some prime p.

2. Let $G = S_7$ and $\alpha = (1\ 2\ 3\ 4)\,(5\ 6\ 7) \in G$ and let $\beta = (2\ 4\ 6\ 7)\,(1\ 3\ 5)$.

   (a) Find $\sigma \in G$ such that $\beta = \sigma\alpha\sigma^{-1}$.

   (b) Prove $C_G(\alpha) = \langle\alpha\rangle$, where $C_G(\alpha)$ denotes the centralizer of α in G.

3. Let G be a group of order $539 = 7^2 \cdot 11$.

   (a) Prove that G is Abelian.

   (b) Give an example from each isomorphism class of groups of order 539.

   (c) For each isomorphism class of groups of order 539, determine (with explanation) the number of elements of order 7.

4. Let G be a group.

   (a) Prove $Z(G) \lhd G$, where $Z(G)$ denotes the center of the group G.

   (b) Prove $\dfrac{G}{Z(G)} \cong \mathrm{Inn}(G)$, where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms on G. (Recall: for each $a \in G$, the function $f_a : G \to G$ defined by $f_a(x) = axa^{-1}$ is called an inner automorphism. $\mathrm{Inn}(G) = \{f_a : a \in G\}$.)

5. Let H and K be subgroups of G and define $HK = \{xy : x \in H \text{ and } y \in K\}$.

   (a) Prove that HK is a subgroup of G if and only if $HK = KH$.

   (b) Prove that if $H \lhd G$, then HK is a subgroup of G.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let R and S be commutative rings with unity, and let $\phi : R \to S$ by a surjective homomorphism.

   (a) Prove that if R is a principal ideal domain, then every ideal in S is principal.

   (b) Prove that if R is a principal ideal domain and $\ker \phi$ is a prime ideal, then S is a principal ideal domain.

2. An element, a, is called **nilpotent** if $a^n = 0$ for some natural number n.

   (a) Let R be a commutative ring with unity, and let $I = \{c \in R : c \text{ is nilpotent }\}$. Prove that I is an ideal in R.

   (b) Let R and I be as in part (a). Prove that R/I does not have any nilpotent elements.

3. Let K be a splitting field for some polynomial over the field F such that $\text{Gal}(K/F) \cong \mathbb{Z}_n$.

   (a) Prove that if $m|n$, then there is exactly one intermediate field, E, such that $[K : E] = m$.

   (b) Prove that if E is an intermediate field (between F and K), then $\text{Gal}(E/F)$ is cyclic.

4. Let K be an algebraic extension field of the field F, and let D be a ring such that $F \subseteq D \subseteq K$.

   (a) Prove that if $a \in D$, then $F(a) \subseteq D$.

   (b) Prove that D is a field.

5. Let F be a field with extension field E such that $a, b \in E$ are algebraic over F. Assume the minimal polynomial of a over F has degree n and the minimal polynomial for b over F has degree m where m and n are relatively prime.

   (a) Prove that $[F(a, b) : F] = mn$.

   (b) Prove that $F(a) \cap F(b) = F$.