Please show and explain your work. Justification is necessary for credit. Also sign your name here acknowledging you will not cheat (Looking up solutions online or in books, or talking with others):_________________

1. (*10 points*) Consider the Lie algebra $L = \mathfrak{o}_4(\mathbb{C})$.

a. (*5 pts*) Find a basis and compute its dimension.

b. (*2 pts*) Describe the trivial module and its dimension.

c. (*3 pts*) Describe the natural module and its dimension.

2. (*25 points*) Consider the Lie algebra $L = \mathfrak{sl}_3(\mathbb{C})$. Let E_{ij} denote the 3×3 matrix with a 1 in the ij-th entry (i is row, j is column), and zeros elsewhere. Set $H_1 := E_{11} - E_{22}$, $H_2 := E_{22} - E_{33}$, $X_1 := E_{12}$, $X_2 := E_{23}$, $X_3 := E_{13}$, $Y_1 := E_{21}$, $Y_2 := E_{32}$, $Y_3 := E_{31}$. Note

$$\mathcal{B} = \{H_1, H_2, X_1, X_2, X_3, Y_1, Y_2, Y_3\}$$

is a basis of L.

a. (*5 pts*) Compute ad_{H_2} with respect to the basis above.

b. (*5 pts*) Consider the subset $\mathcal{H} = \mathrm{span}\{H_1, H_2\}$ of L. Prove or disprove that $\mathcal{H}$ is an abelian Lie subalgebra.

c. (*2 pts*) Prove or disprove that $\mathcal{H}$ is an ideal.

d. (*5 pts*) Can you find a subalgebra of L that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$? If yes, find one and show they are isomorphic. If not, prove why not.

e. (*5 pts*) Prove directly that $\mathfrak{sl}_3(\mathbb{C})$ is a simple Lie algebra.

f. (*3 pts*) Is the adjoint module of $\mathfrak{sl}_3(\mathbb{C})$ an irreducible module? Justify your answer.

3. (*15 points*)
Let V be a finite-dimensional vector space. A bilinear form $b\colon V \times V \to \mathbb{C}$ is said to be **symmetric** if $b(x, y) = b(y, x)$ for any $x, y \in V$. A symmetric bilinear form is **nondegenerate** if $b(x, y) = 0$ for all $x \in V$ implies $y = 0$ (note that since it is symmetric this can hold in either input). Let L be a finite-dimensional complex Lie algebra. The **Killing form** on L is defined as the function $\kappa\colon L \times L \to \mathbb{C}$ given by $\kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. (Note: For for $A, B, C \in \mathrm{End}(V)$ the trace function satisfies (i) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and (ii) $\mathrm{tr}([A, B]C) = \mathrm{tr}(A[B, C])$.)

a. (*5 pts*) Prove that κ is a symmetric bilinear form.

b. (*2 pts*) Prove that κ satisfies $\kappa([x, y], z) = \kappa(x, [y, z])$ (this property is called **associativity** of the bilinear form).

c. (*8 pts*) Prove that κ is non-degenerate on $\mathfrak{sl}_2(\mathbb{C})$.