Analysis Notes

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Theorem 1.1 (4.26)**.** *Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Then f is differentiable almost everywhere.*

Note: A monotone function can have at most countably many discontinuities "almost everywhere"=except a set of measure zero.

A monotone function is continuous almost everywhere. A monotone function can fail to be differentiable at uncountable many points.

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Note: Ex. 4.12 needs Grönwrall inequality.

2.1 The Riemann Integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Then

$$\overline{S}(f, p) =? \quad shit$$

graph goes here.

Example 2.1 (5.3)**.** $f : [a, b] \to \mathbb{R}$, $f(x) = c$, $\forall x \in [a, b]$.

$$\overline{S}(f, p) = \sum_{i=1}^{n} c \cdot \Delta x_i = x \sum_{i=1}^{n} \Delta x_i = c(b - a)$$

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then*

1. *If P_1, P_2 are two partitions such that $P_1 \subseteq P_2$ then $\underline{S}(f, P_1) \leq \underline{S}(f, P_2)$ and $\overline{S}(f, P_1) \geq \overline{S}(f, P_2)$.*
2. *If P and Q are any partitions, then*
$$\underline{S}(f, P) \leq \overline{S}(f, Q).$$
3. $\underline{(}f) \leq \overline{S}(f)$
4. *$\underline{S}(f) = \overline{S}(f)$ iff $\forall \varepsilon > 0$, there exists P_ε partition such that*
$$\overline{S}(f, P_\varepsilon) - \underline{S}(f, P_\varepsilon) < \varepsilon.$$

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is riemann integrable on $[a,b]$.*

Proof. f is continuous on a compact set, so f is uniformly continuous on $[a,b]$. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that $\forall x, y \in [a,b]$ with $|x-y| < \delta$, we have $f(x) - f(y) < \frac{\varepsilon}{2(b-a)}$. Then choose a partition $P_\varepsilon = \{x_0, \ldots, x_n\}$ such that $\|P_\varepsilon\| < \delta$, then $\forall x, y \in [x_{i-1}, x_i]$ we have
$$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$$
so $M_i - m_i \leq \frac{\varepsilon}{2(b-a)}$, $\forall 1 \leq i \leq m$
$$\overline{S}(f, P_\varepsilon) - \underline{S}(f, P_\varepsilon)$$
□

Definition 2.1. Let $f : [a,b] \to \mathbb{R}$and $x \in [a,b]$ and $h > 0$. Then
$$\operatorname{osc}(f)(x-h, x+h) = \sup\{|f(x_1) - f(x_2)| \mid x_1, x_2 \in (x-h, x+h) \cap [a,b]\}$$
If $0 < h_1 < h_2$ then $\operatorname{osc}(f)(x-h_1, x+h_1) \leq \operatorname{osc}(x-h_2, x+h_2)$

Theorem 2.3. *Let $fL[a,b] \to \mathbb{R}$ and $x \in [a,b]$. Then f is continuous at x if and only if $osc(f)(x) = 0$.*

Proof. Suppose thath f is continuous at x. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\forall y \in (x-\delta, x+\delta) \cap [a,b]$ we have $|f(x) - f(y)| < \varepsilon/2$. then $\forall x_1, x_2 \in (x-\delta, x+\delta) \cap [a,b]$ we have $|f(x_1) - f(x_2)| < \varepsilon$. Hence, $\operatorname{osc}(f)(x-\delta, x+\delta) \leq \varepsilon$. Then $0 < h < \delta$, $\operatorname{osc}(f)(x-h, x+h) \leq \varepsilon$. So $\operatorname{osc}(f)(x) \leq \varepsilon$, $\forall \varepsilon > 0$. Therefore, $\operatorname{osc}(f)(x) = 0$.

Suppose $\operatorname{osc}(f)(x) = 0$. Let $\varepsilon > 0$. Then, $\exists H > 0$ such that
$$\operatorname{osc}(f)(x-h, x+h) < \varepsilon, \quad \forall 0 < h < H$$
□

Let D be the set of discontinuities of f on $[a,b]$. Then define
$$D_k = \{x \in [a,b] \mid \operatorname{osc}(f)(x) \geq \frac{1}{k}\}$$
Then $D = \bigcup_{k \in \mathbb{N}} D_k$. And Riemann integrability $\iff$ $\mu(D_k) = 0$ for all $k \in \mathbb{N}$.