MATH 230B

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Instructor: Dr. Domokos | Assignment: Homework 1

4.1 Let $f : [a, b] \to \mathbb{R}$. Prove that if f has a local maximum (or minimum) at $x_0 = a$ or $x_0 = b$ and f is differentiable at x_0, then there exists $\delta > 0$ such that

$$f'(x_0)(x - x_0) \leq 0 \quad (\text{or } f'(x_0)(x - x_0) \geq 0)$$

for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$

Proof. Since f has a local maximum at $x_0 = a$, then there exists $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ we have that

$$f(x) \leq f(x_0) \Rightarrow f(x) - f(x_0) \leq 0.$$

Moreover, we have that for all $\in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, that $x - x_0 \geq 0$. Hence,

$$\lim_{x \searrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \leq 0$$

since f is differentiable at x_0. Thus, for any $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$ it follows that $x - x_0 \geq 0$ and so $f'(x_0)(x - x_0) \leq 0$ □

4.2 Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that if f is continuous at x_0 and $|f|$ is differentiable at x_0, then f is differentiable at x_0.

Proof. We need to show that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Since f is continuous at x_0, then for any $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that $|f(x) - f(x_0)| < \varepsilon_1$ when $|x - x_0| < \delta_1$. We also have that $|f|$ is differentiable at x_0. Hence, for any $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ such that

$$\left| \frac{|f(x)| - |f(x_0)|}{x - x_0} - L_1 \right| < \varepsilon_2 \tag{1}$$

for some $L_2 \in \mathbb{R}$ and whenever $|x - x_0| < \delta_2$. Thus, we need to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L_2 \right| < \varepsilon, \tag{2}$$

for some $L_2 \in \mathbb{R}$.

Let $\delta = \min\{\delta_1, \delta_2\}$, then for $|x - x_0| < \delta$, it follows that $|f(x) - f(x_0)| < \varepsilon_1$ holds and that (1) holds. Thus multiplying (1) by $|x - x_0|$ we obtain

$$||f(x)| - |f(x_0)| - L_1(x - x_0)| < \varepsilon_2|x - x_0| < \varepsilon_1\delta.$$

By the Triangle Inequality we get that

$$\begin{aligned}||f(x)| - |f(x_0)| - L_1(x - x_0)| &\leq ||f(x)| - |f(x_0)|| + |L_1||x - x_0| \\ &\leq |f(x) - f(x_0)| + |L_1|\delta \\ &< \varepsilon_1 + |L_1|\delta.\end{aligned}$$

Now if we select $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 + |L_1|\delta \leq \varepsilon_2\delta$, then it follows that (2) holds iff $|f(x) - f(x_0) - L_2(x - x_0)| < \varepsilon\delta$. And by the Triangle Inequality we get that $|f(x) - f(x_0) - L_2(x - x_0)| < \varepsilon_1 + |L_2|\delta$. Finally, if $\varepsilon = \varepsilon_2 + |L_2| - |L_1|$, then

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - L_2\right| < \varepsilon.$$

Therefore the limit exists and f is differentiable at x_0. □

4.3 Suppose that f is differentiable on (a, b) and f' is monotone increasing on (a, b). Prove that f' is continuous on (a, b).

Proof. For contradiction, assume that f' is discontinuous on (a, b). Then by Corollary 3.19 and Theorem 3.20, f' can only have at most countably many jump discontinuities. Let $x_0 \in (a, b)$ such that x_0 is a discontinuity of the first kind. Then by Theorem 3.18

$$\lim_{x \nearrow x_0} f'(x) = L_1 \leq f'(x_0) \leq L_2 = \lim_{x \searrow x_0} f'(x).$$

Now let $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subset (a, b)$. Then since f' is monotone increasing, it follows that f' satisfies the IVP on the above interval. Finally, letting $\varepsilon = \min\{|f'(x_0) - L_1|, |f(x_0) - L_2|\}$, it follows that for any y such that $|f'(x_0) - y| < \varepsilon$, we get that there does not exist $x \in (x_0 - \delta, x_0 + \delta)$ such that $f'(x) = y$. This contradicts the IVP. □

4.4 Let

$$f(x) = \begin{cases} x^p \sin\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

(a) For what values of p is f continuous on $\mathbb{R}$?

Proof. We saw from Example 4.6 that for $p = 0$, f is not continuous at $x = 0$. For $p > 0$, it follows from the fact that $\lim_{x \to 0} f(x) = 0$, that f is continuous at 0 and thus continuous on $\mathbb{R}$. □

(b) For what values of p is f differentiable on $\mathbb{R}$?

Proof. Similar to the reasoning above, for $p = 0$, f is not continuous at $x = 0$ and thus not differentiable at $x = 0$. For $p = 1$, we refer to Example 4.7 to conclude that f is not differentiable at $x = 0$. For $p > 1$, we have that

$$\frac{f(x) - f(0)}{x} = \frac{x^p \sin \frac{1}{x} - 0}{x} = x^{p-1} \sin \frac{1}{x}$$

and since $p > 1$, then $p - 1 > 0$ and so $\lim_{x\to 0} x^{p-1} \sin \frac{1}{x} = 0$. Hence f is differentiable on $\mathbb{R}$ for $p > 1$. □

(c) For what values of p is f' continuous on $\mathbb{R}$?

Proof. For $x \neq 0$, f' is continuous as it is the product, sum, and composition of continuous functions. We have that

$$f'(x) = px^{p-1} \sin \frac{1}{x} - x^{p-2} \cos \frac{1}{x}$$

and so for $0 \leq p \leq 1$, $\lim_{x\to 0} f'(x)$ does not exist and so f' is not continuous. For $p \geq 2$, we have that $\lim_{x\to 0} f'(x) = 0$ and thus f' is continuous for $p \geq 2$. □

(d) For what values of p is f differentiable on $\mathbb{R}$?

Proof. Note that the exponents on the x term in f'' are $p-2$ and $p-3$, respectively. Thus from the difference quotient we obtain a power of $p - 4$ on some of the x terms. Hence, for $p > 4$, it follows that f is twice differentiable on $\mathbb{R}$. □

4.5 Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and assume that there exists $0 \leq M < +\infty$ such that $|f'(x)| \leq M$ for all $x \in (a, b)$.

(1) Show that f is uniformly continuous on (a, b).

Proof. We begin by trying to show that f is Lipschitz continuous and then using Theorem 3.43 to prove that f is uniformly continuous. With this in mind, we need to show that for all $x_1, x_2 \in (a, b)$, we have that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|.$$

First, let $\varepsilon > 0$ and define $a' = a + \varepsilon$ and $b' = b - \varepsilon$ such that $a' < b$ and $a < b'$. Then since $(a', b') \subset (a, b)$, it follows that f is continuous on $[a', b']$ and differentiable on (a', b'). Thus by Lagrange's Theorem, there exists $x_0 \in (a', b')$ such that

$$f(b') - f(a') = f'(x_0)(b' - a') \leq M(b' - a').$$

Since ε was arbitrary, then the above holds for all $a', b' \in (a, b)$. Hence, f is Lipschitz continuous on (a, b). By Theorem 3.43, f is uniformly continuous on (a, b). □

(2) Give an example of a function which has unbounded derivative, but still is uniformly continuous.

Proof. Let $f(x) = \sqrt{x}$. Then f is differentiable on $(0, +\infty)$. Moreover, for any $\varepsilon > 0$, if $\delta = \varepsilon^2$, then it follows that if $x, y \in (0, +\infty)$ such that $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \varepsilon$. Hence, f is uniformly continuous.

However, we have that $f'(x) = \frac{1}{2\sqrt{x}}$ and $\lim_{x\to+\infty} \frac{1}{2\sqrt{x}} = +\infty$. Thus, $f'(x)$ is not bounded. □

4.6 Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that if $f'(x) \neq 0$ for all $x \in (a, b)$, then f is strictly monotone increasing or decreasing on $[a, b]$.

Is it true that if f is strictly monotone increasing $f'(x) \neq 0$ for all $x \in (a, b)$?

Proof. Assume that $f'(x) \neq 0$ for all $x \in (a, b)$. Then by the converse of Theorem 4.9, f does not obtain a local maximum or local minimum on (a, b). Now suppose that f is not strictly monotonically increasing and not monotonically decreasing. Then there exists $x_1, x_2, x_3, x_4 \in (a, b)$ such that $x_1 < x_2$ and $f(x_1) \geq f(x_2)$, and $x_3 < x_4$ and $f(x_3) \leq f(x_4)$.

Since f is continuous on $[x_1, x_2]$, as well as on $[x_3, x_4]$ and differentiable on (x_1, x_2) and (x_3, x_4), then by Lagrange's Theorem, there exists $y_1 \in (x_1, x_2)$ and $y_2 \in (x_3, x_4)$ such that

$$f'(y_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0 \quad \text{and} \quad f'(y_2) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} \geq 0.$$

By assumtion, $f'(x) \neq 0$, for all $x \in (a, b)$, and so it has to be the case that $f'(y_1) < 0$ and $f'(y_2) > 0$.

WLOG, assume that $y_1 < y_2$. Define the function g to be the restriction of f to $[y_1, y_2]$. Then clearly g is differentiable on $[y_1, y_2]$, as f is. Thus by Theorem 4.11, g' has the Intermidiate Value Property. And since $g'(y_1) < 0 < g'(y_2)$, then there exists $y \in (y_1, y_2)$ such that $g'(y) = f'(y) = 0$. This is a contradiction. Therefore, f must either be strictly monotonically increasing or decreasing. □

4.7 Let $f : (0, +\infty) \to \mathbb{R}$ be differentiable on $(0, +\infty)$. Prove that if $\lim_{x\to+\infty} f(x) = M \in \mathbb{R}$, then for all $\varepsilon > 0$ there exists $x(\varepsilon) > 0$ such that $|f'(x(\varepsilon))| < \varepsilon$.

Proof. Let $\varepsilon_0 = \varepsilon/2 > 0$. Then by assumption, there exists $N > 0$ such that for all $x > N$, $|f(x) - M| < \varepsilon_0$. Now choose $x_1, x_2 > M$ such that $x_2 - x_1 > 1$. Then $|f(x_1) - M| < \varepsilon_0$ and $|f(x_2) - M| < \varepsilon_0$ and so by the Triangle Inequality, it follows that $|f(x_2) - f(x_1)| < 2\varepsilon_0 = \varepsilon$.

Note that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Thus by Lagrange's Theorem, there exists $x_0 \in (x_1, x_2)$ such that

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{\varepsilon}{x_2 - x_1} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$, there exists x_0 such that $|f'(x_0)| < \varepsilon$. □

4.8 Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$ and differetiable on $(0,1)$. Show that if $f'(x) \neq 1$ for all $x \in (0,1)$, then there exists a unique $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

Proof. For contradiction, assume that $f(x) \neq x$ for all $x \in [0,1]$. Then it follows that f cannot be a constant function since this would imply that $f(x) = c \in [0,1]$ and so for $x = c$, we get $f(c) = c$.

Define

$$S = \{|f(x) - x| : x \in [0,1]\}.$$

Then S is closed and bounded above by 1 and below by 0. Thus by the greastest lower bound property, $\alpha = \inf(S)$ exists, and $\alpha \in S$. Hence, there exists $x_0 \in [0,1]$ such that $f(x_0) - x_0 = \alpha$. If $\alpha = 0$, then $f(x_0) = x_0$ which is a contradiction. Thus $\alpha > 0$.

Now define $g(x) = f(x) - x$. Note that g is continuous on $[0,1]$ and differentiable on $(0,1)$. By definition, α is a local minimum of g. Note that if $x_0 \in (0,1)$, then by Theorem 4.9, $g'(x_0) = 0$, which implies that $f'(x_0) = 1$. Thus either $g(0) = \alpha$ or $g(1) = \alpha$ and $g'(x) \neq 0$ for all $x \in (0,1)$. By Exercise 4.6, this implies that g is either strictly monotone increasing or strictly monotone decreasing.

If $g(0) = \alpha$, then g has to be strictly monotone increasing otherwise, for some $0 < x$, $g(x) < \alpha$ which implies $\alpha \neq \inf(S)$. It follows that $g(1) > \alpha$, which implies that $f(1) > 1 + \alpha$ which is impossible.

If $g(1) = \alpha$, then we have that $f(1) = 1 + \alpha$ which is impossible. Therefore, there exists $x \in [0,1]$ such that $f(x) = x$.

To prove uniqueness, suppose there is $x_1, x_2 \in [0,1]$ such that $f(x_1) = x_1$ and $f(x_2) = x_2$. Then $g(x_1) = g(x_2)$. By Theorem 4.13, there exists $x_0 \in (x_1, x_2)$ such that $g'(x_0) = 0$ which implies that $f'(x_0) = 1$ contradicting our assumption. □

4.11 Suppose that f is differentiable on $[a,b]$ and f' is continuous on $[a,b]$. Prove that f is absolutley continuous on $[a,b]$.

Proof. Seeing as f' is continuous on $[a,b]$, which is closed and bounded, then by Theorem 3.26, $f'([a,b])$ is closed and bounded. Now select $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in [a,b]$. Moreover, since f is continuous on $[a,b]$ and differentiable on $[a,b]$, then for any $x, y \in [a,b]$, there exists $x_0 \in [a,b]$ such that

$$\begin{aligned} f'(x_0) = \frac{|f(y) - f(x)|}{|y - x|} &\leq L \\ \Rightarrow |f(y) - f(x)| &\leq L|y - x|. \end{aligned}$$

Hence, f is Lipschitz continuous. By Theorem 3.43, f is absolutely continuous. □

4.14 Let $f, g : [a, +\infty) \to \mathbb{R}$ be continuous on $[a, \infty)$ and differentiable on (a, ∞). Assume that $f(a) = g(a)$ and that $f'(x) \leq g'(x)$ for all $a < x$. Prove that

(1) $f(x) \leq g(x)$ for all $a \leq x$.

Proof. Assume for contradiction that there exists $a \leq x$ such that $g(x) < f(x)$. Then since both f and g are conintuous on $[a, x]$ and differentiable on (a, x), Theorem 4.15 yields some $x_0 \in (a, x)$ such that

$$g'(x_0) = f'(x_0)\frac{g(x) - g(x)}{f(x) - f(a)}$$

Seeing as $f(a) = g(a)$ and $g(x) < f(x)$, then it follows that

$$g'(x_0) = f'(x_0)\frac{g(x) - g(x)}{f(x) - f(a)} < f'(x_0).$$

Thus, for some $a < x_0$, we have $g'(x_0) < f'(x_0)$, which contradicts our assumption. □

(2) $1 + \ln x \leq x$ for all $1 \leq x$.

Proof. Letting $f(x) = 1 + \ln x$ and $g(x) = x$, then we have that $f'(x) = 1/x$ and $g'(x) = 1$. Thus for any $1 \leq x$, it follows that $f'(x) \leq g'(x)$. Moreover, since $\ln 1 = 0$, then we also have that $f(1) = g(x)$. Lastly, both f and g are continuous on $[1, +\infty)$ and differentiable on $(1, +\infty)$. Therefore, by part (1), we have that $f(x) = 1 + \ln x \leq x = g(x)$ for all $1 \leq x$. □

(3) $0 \leq \sin x \leq x$ for all $0 \leq x \leq 1$.

Proof. We will show this in two parts. To show the left inequality, we note that for all $0 \leq x \leq \pi$, we have $0 \leq \sin$ and since $1 < \pi$, then this suffices to show the left inequality. As for the right side, we note that the second derivative of $\sin x$ is $-\sin x$ and that $-\sin x \leq 0$ for all $0 \leq x \leq \pi$. By Theorem 4.16, this implies that $\cos x$ is monotonically decreasing on $[0, 1] \subset [0, \pi]$. Also note that $|\cos x| \leq 1$ for all $x \in \mathbb{R}$. Hence, $\cos x \leq 1$ for all $0 \leq x \leq 1$. Therefore, by part (a), we have that $0 \leq \sin x \leq x$ for all $0 \leq x \leq 1$. □

(4) $1 - \frac{x^2}{2} \leq \cos x \leq 1$ for all $0 \leq x \leq 1$.

Proof. From part (a) we can conclude that $\cos x \leq 1$ for all $0 \leq x \leq 1$. Now we note that if $f(x) = 1 - \frac{x^2}{2}$ and $g(x) = \cos x$, then f and g are continuous on $[0, 1]$ and differentiable on $(0, 1)$. We also have that $f(0) = 1 = \cos 0$. Finally, we have that $f'(x) = -x$ and $g'(x) = -\sin x$ and by part (3), we know $\sin x \leq x$ for all $0 \leq x \leq 1$. Thus $-x \leq -\sin x$ for all $0 \leq x \leq 1$. Hence, by (1), the desired result follows. □