Master's Exam in Real Analysis
May 2019

Part 1: Problems 1-7 **Do six problems in Part 1.**

1. (a) Prove that the set of isolated points of a subset $S \subseteq \mathbb{R}^k$ is countable.

   (b) Prove that the set of all binary sequences is uncountable.

2. Prove that compact subsets of metric spaces are closed and bounded. Is the converse true? Justify your answer.

3. Prove that any connected metric space with at least two points is uncountable.

4. Let X be a metric space.

   (a) Show that the set
   $$C_r(q) = \{x \in X \mid d(x,q) \leq r\}$$
   is closed for any $q \in X$ and $r > 0$.

   (b) Suppose $\{p_n\}$ is a sequence in X and $p_n \to p \in X$. Prove that the set
   $$\{p_n \mid n \in \mathbb{N}\} \cup \{p\}$$
   is compact.

5. Let $s, s_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Prove $s_n \to s$ implies
   $$\frac{s_1 + s_2 + \cdots + s_n}{n} \to s.$$
   Prove or disprove the converse.

6. (a) Suppose $a_n \geq 0$ for all $n \in \mathbb{N}$. Let $s_k = \sum_{n=1}^{k} a_n$. Prove $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums $\{s_k\}$ is bounded.

   (b) Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Prove that if $\alpha > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

7. Let X and Y be metric spaces where X is compact and Y is complete. Suppose $f : X \to Y$ is continuous and $\{x_n\}$ is a Cauchy sequence in X. Prove that $\{f(x_n)\}$ converges in Y.

Part 2: Problems 8-14 <u>**Do six problems in Part 2.**</u>

8. Investigate the continuity and differentiability of

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

on $\mathbb{R}$. How many derivatives does f have? How many are continuous?

9. Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \ \exists\, p, q \in \mathbb{N} \text{ coprime} \\ 1 & x = 0 \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad .$$

Prove f is Riemann integrable and find $\int_0^1 f dx$.

10. Suppose f is bounded on $[a,b]$ and $c \in (a,b)$. Let $k > 0$ and

$$\alpha(x) = \begin{cases} 0 & x \in [a,c) \\ k & x \in [c,b] \end{cases} \quad .$$

Prove $f \in \mathcal{R}(\alpha)$ if and only if $\lim\limits_{x \to c^-} f(x) = f(c)$. Determine $\int_a^b f d\alpha$ if this holds.

11. Let f be Riemann integrable and define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

Prove F is continuous. Prove that if, in addition, f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

12. Let E be a set.

    (a) Suppose $\{h_n\}$ is a sequence of bounded real-valued functions on E which converges uniformly. Prove that $\{h_n\}$ is uniformly bounded.

    (b) Suppose $\{f_n\}$ and $\{g_n\}$ are sequences of bounded real-valued functions which converge uniformly on E. Prove that $\{f_n g_n\}$ converges uniformly on E.

13. Suppose $\{f_n\}$ is a sequence of real-valued functions on a compact metric space K. Suppose $\{f_n\}$ is equicontinuous and pointwise convergent. Prove that $\{f_n\}$ is uniformly convergent.

14. (a) Suppose $\{f_n\}$ is a sequence of continuous real-valued functions on a metric space X, and $f_n \to f$ uniformly. Let $\{x_n\}$ be a sequence of points in X converging to $x \in X$. Prove that $\lim\limits_{n \to \infty} f_n(x_n) = f(x)$.

    (b) Suppose $\{f_n\}$ is a uniformly bounded sequence of Riemann integrable functions on $[a,b]$. Let $F_n : [a,b] \to \mathbb{R}$ be defined by $F_n(x) = \int_a^x f_n(t)dt$. Prove that $\{F_n\}$ contains a uniformly convergent subsequence.