MATH 296C

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Due Date: 5/17/21
Assignment: Final Exam

1. Explain your work for everything done.

   a. Do **one** of the following:

   (I) Construct a rank 2 root system associated with the angle $\theta = 3\pi/4$. (Include a diagram of the roots, along with their labels.)

   Solution. Taking $\alpha, \beta \in \Phi$ such that $\beta \neq \pm\alpha$ and the angle between these roots is $3\pi/4$, then as this is an obtuse angle, we get that $\alpha + \beta \in \Phi$. Additionally, we get that $-\alpha, -\beta, -\alpha - \beta \in \Phi$. Next we want to calculate the angle between α and $\alpha + \beta$. Thus, we want to calculate the θ which satisfies

$$\langle \alpha + \beta, \alpha \rangle \langle \alpha, \alpha + \beta \rangle = 4 \cos^2 \theta.$$

   By Table 1 from the notes, we have that $\langle \alpha, \beta \rangle = -1$ and $\langle \beta, \alpha \rangle = -2$. Using this, we find that

$$\begin{aligned}
\langle \alpha + \beta, \alpha \rangle &= \frac{2(\alpha + \beta, \alpha)}{(\alpha, \alpha)} \\
&= \frac{2(\alpha, \alpha) + 2(\beta, \alpha)}{(\alpha, \alpha)} \\
&= 2 + \langle \beta, \alpha \rangle \\
&= 2 + (-2) \\
&= 0.
\end{aligned}$$

   Hence, $\cos^2 \theta = 0$. This implies that $\theta = \pi/2$. Next, we consider the reflection of $-\beta$ over α:

$$\begin{aligned}
\sigma_\alpha(-\beta) &= -\beta - \frac{2(-\beta, \alpha)}{(\alpha, \alpha)}\alpha \\
&= -\beta + \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \\
&= -\beta + \langle \beta, \alpha \rangle \alpha \\
&= -\beta - 2\alpha.
\end{aligned}$$

   From this we also get that $\pm(2\alpha + \beta) \in \Phi$. Using the same technique from above, we can determing the angle between α and $2\alpha + \beta$. We get that $\theta = \pi/4$. Thus our root system is $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\}$. This root system is isomorphic to B_2. ■

b. Find the Cartan matrix for your choice of (a).

Solution. Seeing that $\Delta = \{\alpha, \beta\}$ is a base for our choice in (a). This is true since α and β generated the root system in (a). With this we calculate the Cartan matrix as

$$C = \begin{bmatrix} \langle \alpha, \alpha \rangle & \langle \alpha, \beta \rangle \\ \langle \beta, \alpha \rangle & \langle \beta, \beta \rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

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(A) Performing the same procedure as above we get that with the two roots $\alpha, \beta \in \Phi$, it follows that $-\alpha, -\beta \in \Phi$. And since $\theta = 5\pi/6$ is obtuse, then $\pm(\alpha + \beta)$. Via reflections across the roots in Φ, we also obtain $\pm(\alpha+2\beta)$, $\pm(2\alpha+\beta)$, $\pm(3\alpha+2\beta)$, and $\pm(3\alpha + \beta)$. This gives us a root system isomorphic to G_2.

(B) In part I, we found that Φ had 8 roots, 4 of them positive. So the Weyl group would be the set of all compositions of the elements $\{\sigma_\alpha, \sigma_\beta, \sigma_{\alpha+\beta}, \sigma_{2\alpha_\beta}\}$, since $\sigma_\alpha(\beta) = \sigma_{-\alpha}(\beta)$ (at least I think so?).

(C) If $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2$, then lets assume $\langle \alpha, \beta \rangle = 1$. This forces $\langle \beta, \alpha \rangle = 2$. Hence, $2 = 4\cos^2\theta$ which implies $\cos\theta = \pm\sqrt{2}/2$. Note that we have $(\alpha, \beta) > 0$, and thus θ is acute. Hence, we get that $\cos\theta = \sqrt{2}/2$ and thus $\theta = \pi/4$. Moreover, we find that $(\beta, \beta)/(\alpha, \alpha) = 2$.

If instead, $\langle \beta, \alpha \rangle = -2$, everythig above is the same except we have that $(\alpha, \beta) < 0$ and so the angle is obtuse. Thus, $\cos\theta = -\sqrt{2}/2$, and hence $\theta = 3\pi/4$.

2. Do the following.

(A) State the roots of G_2.

Solution. As vectors in a 2-dimensional subspace of a 3-dimensional space, the roots are

$$\begin{array}{cccc} (1,-1,0) & (-1,1,0) & (2,-1,-1) & (-2,1,1) \\ (1,0,-1) & (-1,0,1) & (1,-2,1) & (-1,2,-1) \\ (0,1,-1) & (0,-1,1) & (1,1,-2) & (-1,-1,2) \end{array}$$

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(B) Provide the Cartan matrix of G_2.

Solution. The Cartan matrix of G_2 is

$$C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

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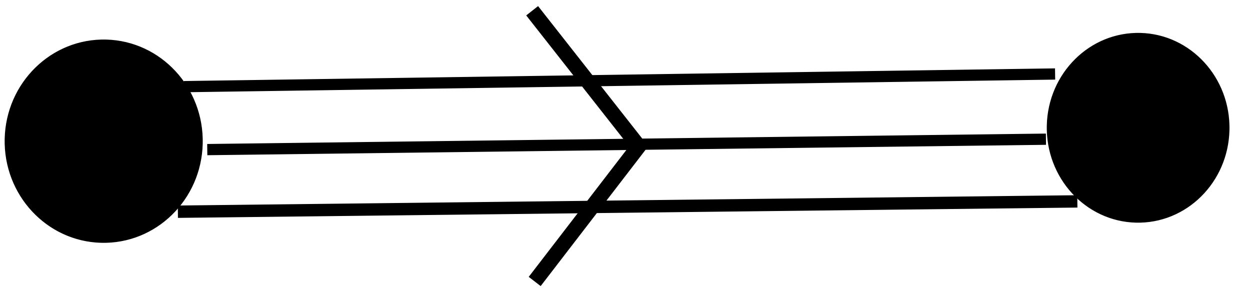
(C) Explain how one can obtain the Cartan matrix from a base of roots.

Solution. Given a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of a root system E, the Cartan matrix can be found by calculating $\langle \alpha_i, \alpha_j \rangle$ for all $1 \leq i, j \leq n$ and it is the result of each of these calculations which which will serve as the entries in the matrix. For example, the entry at the i-th row and j-th column would be $\langle \alpha_i, \alpha_j \rangle$. ∎

(D) Provide the Dynkin diagram for G_2.

Solution.



Sorry for how crazy big this turned out. I'm a lot better at LaTeX/InkScape than this, I swear! ∎

(E) Explain how one can obtain the Cartan matrix from the Dynkin diagram. In the explanation, actually construct the Cartan matrix.

Solution. Given a Dynkin diagram, we first count the number of vertices. This will tell us how many simple roots are in our base, Δ. Let's say there are n many. Between any two vertices, we count the number of lines (edges) connecting them. This number is n_{α_i,α_j} for $\alpha_i, \alpha_j \in \Delta$. Knowing this number gives us $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$. This number is always going to be an element of $\{0, 1, 2, 3\}$. Since $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$, for all i, j, then this leaves a finite number of possibilities for what $\langle \alpha_i, \alpha_j \rangle$, and $\langle \alpha_j, \alpha_i \rangle$ can be. Each possibility for one decides the value for the other. Supposing $\langle \alpha_i, \alpha_j \rangle = a$ and $\langle \alpha_j, \alpha_i \rangle = b$, where $ab = n_{\alpha_i,\alpha_j}$, we have now just calculated two entries of the Cartan matrix. Namely, the ij-th entry and the ji-th entry. Continuing in this way, looking at every consecutive pair of vertices, this will yield all entries in the matrix. ∎

(F) Explain how one can obtain the Dynkin diagram from the Cartan matrix. In the explanation, actually construct the Dynkin diagram.

Solution. Given the Cartan matrix, C, we can begin our Dynkin diagram by drawing as many vertices as there are rows or columns in the matrix. At this point, we can recall that the ij-th entry in the matrix is $\langle \alpha_i, \alpha_j \rangle$, and so we can associate each consecutive pair of vertices with a particular entry in the matrix. The entry will tell us the number of lines or edges which connect the consecutive pair of points, via the product of the entry with its transpose, i.e., $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$.

Additionally, we stipulate that if this product is greater than 1, then we draw an arrow pointing towards the vertex whose associated root has a greater magnitude betweeen the pair. ■

3. In your own words, write a 1 page overview of what we covered on Lie algebras. Attempt to weave the themes together.

Solution. To start, I must say that this is one of those classes that I would happily take again. It is a subject one joyously swan dives into if given the time. In this class we began by simply giving a definition. The definition was of a Lie algebra, and it is essentially (details omitted) a vector space equipped with a strange operator. But this humble beginning resulted in significant implications.

We were exposed to a few examples of these Lie algebras, but we set our sights on one in particular as it proved a very demonstrative choice for many of the concepts we looked at. The Lie algebra we focused on was $\mathfrak{sl}_2(\mathbb{C})$. We quickly moved to studying how this Lie algebra "acted" on other spaces. It was through this lense that we found an entry point to the classification of certain types of Lie algebras.

In my naive understanding, with an operator called the adjoint operator, this allowed us to place the elements into the setting which was where we studied the techniques of decomposing a space. This operator allowed us to see how the Lie algebra acted on itself and we were able to decompose the Lie algebra into simpler structures. Moreover, we looked at particular elements of the Lie algebra which we called semisimple. And these elements it was these elements which specifically provided us with the constituents of the decomposition.

The semisimple elements, had adjoint representations which were diagonalizable and yielded eigenvalues, which then in turn, yielded eigenspaces, whose direct sum was the Lie algebra.

In taking a closer look at the set of the adjoint representation of the semisimple (and abelian) elements of the Lie algebra, we found that these elements had common eigenvectors. For such a common eigenvector, we considered the eigenvalues associated to it with respect to the particular semisimple element for which it was an eigenvector. Said less horribly, being a simulateous eigenvector for all of these semisimple elements means that for each semisimple element, the eigenvector has a different eigenvalue. The set of these eigenvalues is what we called a root. We also considered the root to be a linear functional whose inputs were the semisimple elements and whose outputs were the eigenvalue for that given input.

Okay, we have this set of roots, big deal! But wait, we also looked at something called the Killing form which is a symmetric bilinear nondegenerate map, you know, one of those things ... This Killing form allowed us to define an inner product on the space spanned by our roots which is quite significant since this gives rise to a kind of geometry in the space! Namely, we were able to discuss lengths and angles, like one does on Thanksgiving at grandma's house.

So along with this geometry, we also looked at ways to reflect the roots across a hyperplane. This one of the tools needed in recovering all of the roots of a Lie algebra. In addition to the reflection, there were many small results about the inner product and the properties of the geometry which all together suggested that at the level of root spaces, there are only finitely many of them. And if any arbitrary one of these special types of Lie algebras corresponds to one of these root systems, then this implies that a classifiaction of the root systems might be able to pull back into a classification of the Lie algebras themselves. This turns out to be true!

Better yet, the root spaces have a few different, very compact, forms. We looked at root diagrams, which emphasized the geometry of the root space as the diagrams are like a crystallographic arrangement of arrows. Then there are the Cartan matrices whose entries give you all the information you need about the inner product between any two roots. Finally, there are the Dynkin diagrams, which, like the Cartan matrix, tells you about the inner product of any two roots, but it also conveys information regarding the lengths of the roots which is something seen in the root diagram.

All in all, and despite this (very) inaccurate summaries, this was one of, if not, the most interesting class I have ever had the pleasure of taking. Thank you very much for teaching it, Dr. Krauel. You did an incredible job! ■