Measure Theory Notes

Quin Darcy

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1 General Measure Spaces

Definition 1.1. Let M be a nonempty set. Then a collection of subsets $\sigma \subseteq \mathcal{P}(M)$ is called a σ-algebra if

1. $M \in \sigma$
2. $A \in \sigma \to M \backslash A \in \sigma$
3. $A_1, A_2, \dots \in \sigma \to \bigcup_{n=1}^{\infty} A_n \in \sigma$

The pair (M, σ) is called a measurable space.

Definition 1.2. A measure $\mu : \sigma \to \overline{R}$ on a measure space (M, σ) is a map satisfying

1. $\mu(\varnothing) = 0$
2. $A_1, A_2, \dots \in \sigma$, $A_i \cap A_j = \varnothing$ when $i \neq j$
$$\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n).$$

The tuple (M, σ, μ) is called a measure space.

2 Properties of a Meausre

Given a measure space (M, σ, μ)

1. $A_1, A_2 \in \sigma$ and $A_1 \subset A_2$, then $\mu(A_1) \leq \mu(A_2)$.
2. $A_1, A_2, \dots \in \sigma$ then
$$\mu\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \mu(A_n).$$
3. Contiuity from below. An increasing sequence of measurable sets $A_1 \subseteq A_2 \subseteq \cdots$ where $\bigcup_{n\geq 1} A_n = A$, then
$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$
4. Contiuity from above. A decreasing sequence of measurable sets $A_1 \supseteq A_2$