MATH 230B

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Instructor: Dr. Ricciotti Assignment: Homework 04

1. Let $a_k \geq 0$, $b_k > 0$ for all $k \in \mathbb{N}$. Assume that $\lim_{k\to\infty} \frac{a_k}{b_k} = \lambda$, where $0 < \lambda < \infty$. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} b_k$ is convergent.

   Proof. Assume that $\sum_{k=1}^{\infty} b_k$ is convergent. Let $\varepsilon = 1$. Then since $\lim_{n\to\infty} a_n/b_n = \lambda$, then there exists $N \in \mathbb{N}$ such that

   $$\left|\frac{a_n}{b_n} - \lambda\right| < 1 \Leftrightarrow \left|\frac{a_n}{b_n}\right| - |\lambda| < 1 \Leftrightarrow \frac{a_n}{b_n} < 1 + \lambda \Leftrightarrow a_n < (1+\lambda)b_n$$

   for all $n \geq N$. Hence for any $M \geq N$

   $$\sum_{k=N}^{M} a_k < \sum_{k=N}^{M} (1+\lambda)b_k \leq (1+\lambda)\sum_{k=1}^{\infty} b_k < \infty.$$

   The last inquality holds by assumption. Noting that $\sum_{k=1}^{N-1} a_k$ is finite, then the above inequality implies that the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ is bounded from above. Moreover, this sequence is monotonically increasing as all of its terms are positive. Therefore the sequence of parthial sums is convergent and so

   $$\sum_{k=1}^{\infty} a_k < \infty.$$

   Now assume that $\sum_{k=1}^{\infty} a_k < \infty$. Let $\varepsilon > 0$ such that $\varepsilon < \lambda$. Then there exists $N \in \mathbb{N}$ such that

   $$\left|\frac{a_n}{b_n} - \lambda\right| < \varepsilon \Leftrightarrow (\lambda - \varepsilon)b_n < a_n < (\lambda + \varepsilon)b_n$$

   for all $n \geq N$. This implies that for any $M \geq N$

   $$\sum_{k=N}^{M} b_k < (\lambda - \varepsilon)\sum_{k=N}^{M} a_k \leq (\lambda - \varepsilon)\sum_{k=1}^{\infty} a_k < \infty.$$

   Since $\sum_{k=1}^{N-1} b_k < \infty$, then the sequence of partial sums $S_n = \sum_{k=1}^{n} b_k$ is bounded above. Since the sequence is monotincally increasing, then it converges. Therefore

   $$\sum_{k=1}^{\infty} b_k < \infty.$$

   □

2. Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} a_k^2$ is convergent.

Proof. Since $\sum_{k=1}^{\infty} a_k$ converges, then from Theorem 3.23 (Rudin), it follows that $\lim_{k\to\infty} a_k = 0$. Letting $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $|a_k| < 1$ for all $k \geq N$. Since $a_k \geq 0$ for all $k \in \mathbb{N}$, then $a_k < 1$ for all $k \geq N$. Hence, $a_k^2 \leq a_k < 1$ for all $k \geq N$. This implies that for all $M \geq N$

$$\sum_{k=N}^{M} a_k^2 \leq \sum_{k=N}^{M} a_k \leq \sum_{k=1}^{\infty} a_k < \infty.$$

By Theorem 3.25 (Rudin),

$$\sum_{k=1}^{\infty} a_k^2 < \infty.$$

□

3. Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} \sqrt{a_k a_{k+1}}$ is convergent.

Proof. Before proceeding, we note that if $a, b \geq 0$, then $(\sqrt{a} - \sqrt{b})^2 \geq 0$, which implies that

$$a + b - 2\sqrt{ab} \geq 0 \Leftrightarrow \sqrt{ab} \leq \frac{a+b}{2}. \tag{1}$$

From (1) it follows that for all $k \in \mathbb{N}$

$$\sqrt{a_k a_{k+1}} \leq \frac{a_k + a_{k+1}}{2}.$$

Thus for any $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \sqrt{a_k a_{k+1}} \leq \frac{1}{2} \sum_{k=1}^{n} a_k + a_{k+1} = \frac{a_1}{2} + \sum_{k=1}^{n-1} a_{k+1}.$$

This implies that the sequence of partial sums $S_n = \sum_{k=1}^{n} \sqrt{a_k a_{k+1}}$ is bounded above and therefore converges. □

4. Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ is convergent.

Proof. Assume that $\sum_{k=1}^{\infty} a_k$ is convergent. Then since $a_k \geq 0$ for all $k \in \mathbb{N}$, we have that

$$\frac{a_k}{1 + a_k} \leq a_k$$

for all $k \in \mathbb{N}$. By Theorem 3.25 (Rudin) it follows that

$$\sum_{k=1}^{\infty} \frac{a_k}{1 + a_k} < \infty.$$

Now assume that $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ is convergent. Then $\lim_{k\to\infty} \frac{a_k}{1+a_k} = 0$. Letting $0 < \varepsilon < 1$, then there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$\begin{aligned} \frac{a_k}{1+a_k} < \varepsilon &\Leftrightarrow a_k < \varepsilon(1+a_k) \\ &\Leftrightarrow a_k(1-\varepsilon) < \varepsilon \\ &\Leftrightarrow a_k < \frac{\varepsilon}{1-\varepsilon}. \end{aligned}$$

As $\varepsilon \to 0$, then $\frac{\varepsilon}{1-\varepsilon} \to 0$ and therefore $a_k \to 0$ as $k \to \infty$. Since $\lim_{k\to\infty} a_k = 0$, then for any $1 < B$, there exists N such that for all $k \geq N$, we have that $1 + a_k < B$. Then $\frac{B}{1+a_k} > 1$ for all $k \geq N$. Thus for all $k \geq N$

$$a_k \leq \frac{Ba_k}{1+a_k}.$$

By Theorem 3.23 (Rudin),

$$\sum_{k=1}^{\infty} a_k < \infty.$$

□

5. Prove that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then $\sum_{k=1}^{\infty} k^2 a_k$ is not convergent.

Proof. If $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then $\sum_{k=1}^{\infty} |a_k|$ is divergent. For contradiction, assume that $\sum_{k=1}^{\infty} k^2 a_k$ converges. Then $\lim_{k\to\infty} k^2 a_k = 0$. If $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$,

$$|k^2 a_k| = k^2 |a_k| < \varepsilon \Rightarrow |a_k| < \frac{1}{k^2}.$$

Given that by Theorem 3.28 (Rudin), $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, then by Theorem 3.23 (Rudin), $\sum_{k=1}^{\infty} |a_k|$ converges. This contradicts our assumption that $\sum_{k=1}^{\infty} a_k$ converges conditionally. Therefore $\sum_{k=1}^{\infty} k^2 a_k$ is divergent. □

6. Let $a_n \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} \inf ka_k = 0$. Is it true that $\lim_{k\to 0} ka_k = 0$?

Proof. For contradiction, assume that $\lim_{n\to\infty} \inf ka_k > 0$. Then no subsequences of $\{ka_k\}$ converge to 0. Hence, there exists $r > 0$ and some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $na_n > r$ which implies that $a_n > r/n$. Thus

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} a_k \geq \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{\infty} \frac{r}{k}.$$

Since $\sum_{k=N}^{\infty} r/k = \infty$, then this contradicts the assumption that $\sum_{k=1}^{\infty} a_k$ is convergent. □