MATH 296C Portfolio

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Contents

1 Introduction

This course covered many small topics, but what this class covered, in the broadest view, was the classification of Lie algebras. We started by defining what a Lie algebra is and saw that it was no more exotic, in definition, than a vector space with a tad more machinery. After the initial definitions, we looked at some examples of Lie algebras and there was one particular Lie algebra which became the object of our attention for most of the semester. The set of $n \times n$ matrices over $\mathbb{C}$ with trace 0, or $\mathfrak{sl}_n(\mathbb{C})$. For many of our computations, we specifically looked at the case where $n = 2, 3$. This Lie algebra proved quite demonstrative of properties found in all Lie algebras.

The path to classifying Lie algebras is quite involved, but on the way there we made friends with many interesting topics including modules and representations of Lie algebras. In simple terms, we were able to peer behind the Lie curtains by observing how it acted on something else, a vector space. It turns out that a finite dimensional Lie algebra behaves as if (not sure about wording) its elements are matrices in a vector space and the algebraic properties of the Lie algebra are preserved.

With our module in hand, we can ask many linear algebra type questions, and use linear algebra tools to break it down into more fundamental pieces. The thinking here is that if we can take stock of the fundamental pieces, then we can say something about all (not quite all) Lie algebras, as they are composed of these fundamental pieces, or rather, their modules and representations are.

2 Exercise Solutions

1.1 Find the matrix A that represents f in example 1.26.

Solution. We can solve for A as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus $a = 2$ and $c = 1$. Which gives

$$\begin{pmatrix} 2 & b \\ 1 & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+b \\ 1+d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

the last system yields $b = -1$ and $d = 1$. Hence

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

$\blacksquare$

1.5 Let $\mathfrak{sl}_2(\mathbb{R}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a + d = 0, a, b, c, d \in \mathbb{R}\}$. That is, $\mathfrak{sl}_2(\mathbb{R})$ consists of 2×2 trace 0 matrices with entries in $\mathbb{R}$.

(a) Show that $\mathfrak{sl}_2(\mathbb{R})$ is a Lie algebra with bracket $[A, B] = AB - BA$.

Proof. As a subset of $\mathfrak{gl}_2(\mathbb{R})$, which is a Lie algebra, we can prove the claim by showing that $\mathfrak{sl}_2(\mathbb{R})$ is a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{R})$. Recall that a Lie subalgebra is a subspace closed under the bracket of the outer space. So do prove our claim, consider any $c \in \mathbb{R}$ and any $u, v \in \mathfrak{sl}_2(\mathbb{R})$. It follows that $\mathrm{tr}(u+v) = \mathrm{tr}(u)+\mathrm{tr}(v) = 0+0 = 0$ and so $u+v \in \mathfrak{sl}_2(\mathbb{R})$. Similarly, we have that $\mathrm{tr}(cu) = c\mathrm{tr}(u) = c0 = 0$. Thus, $cu \in \mathfrak{sl}_2(\mathbb{R})$ and it is therefore a subspace.

Taking the same $u, v \in \mathfrak{sl}_2(\mathbb{R})$, and letting

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$\begin{aligned} [u, v] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} - \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix} \\ &= \begin{pmatrix} bg-fc & af+bh-eb-fd \\ ce+dg-ga-he & cf-gb \end{pmatrix}. \end{aligned}$$

By the commutativity of the reals, it follows that $\mathrm{tr}([u, v]) = bg-fc+cf-gb = 0$. Hence, $[u, v] \in \mathfrak{sl}_2(\mathbb{R})$. □

(b) Find a basis for $\mathfrak{sl}_2(\mathbb{R})$.

Solution. Considering that the definition of this set requires that the entries be elements of $\mathbb{R}$ and that the trace must be zero, then we find that we have two "free" entries in the upper right and lower left, and that we have only one free entry along the diagonal since the other must be its additive inverse. Hence, a basis for this Lie algebra is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

■

(c) Find $\dim_{\mathbb{R}}(\mathfrak{sl}_2(\mathbb{R}))$.

Solution. Considering the basis provided in part (b), we can conclude that this Lie algebra has dimension 3. ■

1.7 Prove that every Lie algebra $(L, [\cdot, \cdot])$ over a field K is a K-algebra if one sets $x \cdot y = [x, y]$.

Proof. We first recall the definition of a K-algebra and hence need to show that L is a vector space over K such that L is a ring with under addition and $[\cdot, \cdot]$, however, associativity is not required for the second operation. Additionally, we need to show that for any $k, l \in K$, and any $x, y \in L$, we have $[kx, ly] = (kl)[x, y]$.

By the definition of a Lie algebra, L is a vector space over some field K. As such, it is an abelian group with respect to addition and a magma with respect to $[\cdot,\cdot]$. By the bilinearity of $[\cdot,\cdot]$, it follows that $[kx, ly] = k[x, ly] = l[kx, y] = (kl)[x, y]$. Thus, $(L, [\cdot,\cdot])$ is a K-algebra. □

2.9 Prove $\mathfrak{sl}_2(K)$ is simple if and only if the characteristic of K is not 2.

Proof. Assume that the characteristic of K is 2 and consider the following set

$$S = \text{span}\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}.$$

S is a subspace of $\mathfrak{sl}_2(K)$ by the definition of span. Moreover, if we notice that

$$\forall u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(K) \quad \text{and} \quad \forall v = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \in S,$$

we get that

$$\begin{aligned} [u, v] &= \begin{pmatrix} 0 & -2bk \\ 2ck & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S. \end{aligned}$$

Seeing as we just found a one-dimensional (non-trivial) ideal of $\mathfrak{sl}_2(K)$, we can conclude that $\mathfrak{sl}_2(K)$ is not simple when the characteristic of K is two.

Now assume that $\mathfrak{sl}_2(K)$ is simple and consider the following basis:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have the following commutation relations:

$$[x, y] = h, \qquad [x, h] = -2x, \qquad [y, h] = 2y.$$

Now suppose that I is a nonempty ideal of $\mathfrak{sl}_2(K)$. Then since the latter was assumed to be simple, it follows that $I = \mathfrak{sl}_2(K)$. However, if K had characteristic 2, then this would imply that $[x, h] = 0$ and $[y, h] = 0$, rendering I one-dimensional which is a contradiction. Hence, the characteristic is not two. □

2.10 Prove that $\mathfrak{sl}_n(\mathbb{C})$ is an ideal of $\mathfrak{gl}_n(\mathbb{C})$.

Proof. To show that $\mathfrak{sl}_n(\mathbb{C})$ is an ideal, we must show that it is a subspace such that $[x, y] \in \mathfrak{sl}_n(\mathbb{C})$ for all $x \in \mathfrak{gl}_n(\mathbb{C})$ and all $y \in \mathfrak{sl}_n(\mathbb{C})$. That it is a subspace is immediate since it is a subset and it is a vector space under the same operations. Now let $x \in \mathfrak{gl}_n(\mathbb{C})$ and $y \in \mathfrak{sl}_n(\mathbb{C})$. We want to show that $[x, y] \in \mathfrak{sl}_n(\mathbb{C})$. Thus we need that $\text{tr}([x, y]) = 0$. By the fact that $\text{tr}(AB) = \text{tr}(BA)$ for all $A, B \in \text{Mat}_{n\times n}(\mathbb{C})$, then we have that $\text{tr}([x, y]) = \text{tr}(xy) - \text{tr}(yx) = 0$, as desired. □

2.12 Find the structure constants for $\mathfrak{sl}_2(\mathbb{C})$.

Solution. By the above commutation relations, the structure constants are as follows: Letting a_{xy}^k denote the kth coefficient on the linear combination representing $[x, y]$, then we have

$$
\begin{array}{lll}
a_{xy}^0 = 0; & a_{xy}^1 = 0; & a_{xy}^2 = 1; \\
a_{xh}^0 = -2; & a_{xh}^1 = 0; & a_{xh}^2 = 0; \\
a_{yh}^0 = 0; & a_{yh}^1 = 2; & a_{yh}^2 = 0.
\end{array}
$$

■

5.1 Recall $\mathcal{V}_2$ of Example 4.6.

(a) Find all weights of h for $\mathcal{V}_2$.

Solution. We first recall that

$$h.(aX^2 + bXY + cY^2) := (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})(aX^2 + bXY + cY^2) = 2aX^2 - 2cY^2.$$

Moreover, using $\{X^2, XY, Y^2\}$ as a basis for $\mathcal{V}_2$, then we can apply h to each of the basis elements to obtain:

$$
\begin{aligned}
h.X^2 &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})X^2 = 2X^2 \\
h.XY &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})XY = XY - XY = 0 \\
h.Y^2 &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})Y^2 = -2Y^2.
\end{aligned}
$$

This suffices to show that the only weights of h for $\mathcal{V}_2$ are 2, 0, and -2. ■

(b) Describe (find a basis) all weight spaces.

Solution. By the above calculation, we can see that $w_0 = X^2$ is the highest weight vector with weight 2. By Lemma 5.6, it follows that

$$
\begin{aligned}
w1 &= \frac{1}{1!}y^1.w_0 = Y\frac{\partial}{\partial X}(X^2) = 2XY \\
w2 &= \frac{1}{2!}y^2.w_0 = \frac{1}{2}(Y^2\frac{\partial^2}{\partial X^2})(X^2) = Y^2.
\end{aligned}
$$

Thus $w_0 = X^2, w_1 = 2XY, w_2 = Y^2$ gives us a basis for $V(2)$. Moreover, by Theorem 5.8, $V(2) \cong \mathcal{V}_2$. Thus by part (a), we have that $V_{-2} = \text{span}(w_2)$, $V_0 = \text{span}(w_1)$, and $V_2 = \text{span}(w_0)$. ■

(c) Express $\mathcal{V}_2$ as a direct sum of its weight spaces as in Theorem 5.1.

Solution. By parts (a) and (b), we found that the weight spaces of h are V_{-2}, V_0, and V_2 and as such it follows that

$$\mathcal{V}_2 = \bigoplus_{i=0}^{2} V_{-2+2i} = \mathrm{span}(X^2) \oplus \mathrm{span}(2XY) \oplus \mathrm{span}(Y^2).$$

■

5.2 Define a vector space $\mathcal{V}_3$ similar to example 4.6.

(a) Prove that $\mathcal{V}_3$ is an $\mathfrak{sl}_2$-module.

Solution. Letting $\mathcal{V}_3 = \{aX^3 + bX^2Y + cXY^2 + dY^3 \mid a, b, c, d \in \mathbb{C}\}$, then for the same reason that $\mathcal{V}_2$ is a $\mathbb{C}$-linear space, we have that $\mathcal{V}_3$ is a $\mathbb{C}$-linear space. We will also re-use the same following relations. For all $v \in \mathcal{V}_3$, and for x, y, h as basis elements of $\mathfrak{sl}_2$, then

$$x.v = (X\frac{\partial}{\partial Y})v, \qquad y.v = (Y\frac{\partial}{\partial X})v, \qquad h.v = (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})v.$$

With this we will let $\alpha_1 x + \beta_1 y + \gamma_1 h, \alpha_2 x + \beta_2 y + \gamma_2 h \in \mathfrak{sl}_2$ and we let $u = a_1X^3 + b_1X^2Y + c_1XY^2 + d_1Y^3$ and $v = a_2X^3 + b_2X^2Y + c_2XY^2 + d_2Y^3$ be two arbitrary elements of $\mathcal{V}_3$. Then

$$\begin{aligned}
((\alpha_1 x + \beta_1 y + \gamma_1 h) + (\alpha_2 x + \beta_2 y + \gamma_2 h)).u &= ((\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)h).u \\
&= (\alpha x + \beta y + \gamma h).u \\
&= (\alpha X\frac{\partial}{\partial Y} + \beta Y\frac{\partial}{\partial X} + \gamma(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}))u \\
&= \alpha(X\frac{\partial}{\partial Y})u + \beta(Y\frac{\partial}{\partial X})u + \gamma(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})u \\
&= \alpha(x.u) + \beta(y.u) + \gamma(h.u) \\
&= (\alpha_1 + \alpha_2)(x.u) + (\beta_1 + \beta_2)(y.u) + (\gamma_1 + \gamma_2)(h.u) \\
&= \alpha_1(x.u) + \beta_1(y.u) + \gamma_1(h.u) \\
&\quad + \alpha_2(x.u) + \beta_2(y.u) + \gamma_2(h.u),
\end{aligned}$$

which proves M1. REMINDER: Finish showing M2 and M3. ■

(b) Find all the weights of h for $\mathcal{V}_3$.

Solution. Letting $\{X^3, X^2Y, XY^2, Y^3\}$ be a basis for $\mathcal{V}_2$, then we can find the weights of h by applying h to each basis element. Doing this we get

$$\begin{aligned}
h.X^3 &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})X^3 = 3X^3 \\
h.X^2Y &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})X^2Y = X^2Y \\
h.XY^2 &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})XY^2 = -XY^2 \\
h.Y^3 &= (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})Y^3 = -3Y^3.
\end{aligned}$$

Hence, the weights of h are $-3, -1, 1, 3$. ■

(c) Describe all weight spaces.

Solution. By part (b), we see that the highest weight is 3 with weight vector $w_0 = X^3$. With this we can use Lemma 5.6 to calculate the remaining weight spaces.

$$\begin{aligned} w_1 &= \frac{1}{1} y.w_0 = Y\frac{\partial}{\partial X}(X^3) = 3X^2Y \\ w_2 &= \frac{1}{2} y^2.w_0 = \frac{1}{2}(Y^2\frac{\partial^2}{\partial X^2})X^3 = 3XY^2 \\ w_3 &= \frac{1}{6}(Y^3\frac{\partial^3}{\partial X^3})X^3 = Y^3. \end{aligned}$$

Thus, we have that the weight spaces of $\mathcal{V}_3$ are $V_{-3} = \text{span}(w_3)$, $V_{-1} = \text{span}(w_2)$, $V_1 = \text{span}(w_1)$, and $V_3 = \text{span}(w_0)$. ■

(d) Express $\mathcal{V}_3$ as a direct sum of its weight spaces as in Theorem 5.1.

Solution. With $-3, -1, 1, 3$ being distinct eigenvalues for h, then by Theorem 5.1 it follows that

$$\mathcal{V}_3 \cong \bigoplus_{i=0}^{3} V_{-3+2i}$$

■

6.1 Fill in the question marks of equation (6.1).

Solution.

$\mathrm{ad}_{h_1}(h_1) = 0$	$\mathrm{ad}_{h_2}(h_1) = 0$
$\mathrm{ad}_{h1}(h_2) = 0$	$\mathrm{ad}_{h_2}(h_2) = 0$
$\mathrm{ad}_{h_1}(x_1) = 2x_1$	$\mathrm{ad}_{h_2}(x_1) = -x_1$
$\mathrm{ad}_{h_1}(x_2) = -x_2$	$\mathrm{ad}_{h_2}(x_2) = 2x_2$
$\mathrm{ad}_{h_1}(x_3) = x_3$	$\mathrm{ad}_{h_2}(x_3) = x_3$
$\mathrm{ad}_{h_1}(y_1) = -2y_1$	$\mathrm{ad}_{h_2}(y_1) = y_1$
$\mathrm{ad}_{h_1}(y_2) = y_2$	$\mathrm{ad}_{h_2}(y_2) = -2y_2$
$\mathrm{ad}_{h_1}(y_3) = -y_3$	$\mathrm{ad}_{h_2}(y_3) = -y_3$

■

6.2 Prove Lemma 6.1.

Proof. We prove this by direct calculation. To start, note that $\alpha_{ii}(h) = h_{ii} - h_{ii} = 0$, for all $1 \leq i \leq 3$. Thus,

$$\begin{aligned} L_{\alpha_{11}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h, v] = 0 \quad \text{for all } h \in H\} \\ L_{\alpha_{22}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h, v] = 0 \quad \text{for all } h \in H\} \\ L_{\alpha_{33}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h, v] = 0 \quad \text{for all } h \in H\}. \end{aligned}$$

Hence, $L_{\alpha_{11}} = L_{\alpha_{22}} = L_{\alpha_{33}}$, as desired. □

6.3 Compute the remaining $L_{\alpha_{ij}}$ for $1 \leq i, j \leq 3$ from Subsection 6.1.

Solution. We'll start by computing all the $\alpha_{ij}(h)$ for $1 \leq i, j \leq 3$ and all $h \in H$. We have that

$\alpha_{12}(h_1) = 2$	$\alpha_{12}(h_2) = -1$
$\alpha_{13}(h_1) = 1$	$\alpha_{13}(h_2) = 1$
$\alpha_{21}(h_1) = -2$	$\alpha_{21}(h_2) = 1$
$\alpha_{23}(h_1) = -1$	$\alpha_{23}(h_2) = 2$
$\alpha_{31}(h_1) = -1$	$\alpha_{31}(h_2) = -1$
$\alpha_{32}(h_1) = 1$	$\alpha_{32}(h_2) = -2$

Thus, we have

$$\begin{aligned} L_{\alpha_{11}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = 2v, [h_2, v] = -v\} \\ L_{\alpha_{13}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = v, [h_2, v] = v\} \\ L_{\alpha_{21}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = -2v, [h_2, v] = v\} \\ L_{\alpha_{23}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = -v, [h_2, v] = 2v\} \\ L_{\alpha_{31}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = -v, [h_2, v] = -v\} \\ L_{\alpha_{32}} &= \{v \in \mathfrak{sl}_3(\mathbb{C}) \mid [h_1, v] = v, [h_2, v] = -2v\}. \end{aligned}$$

■

3 Section Writing

Below is a description of a program that was written to generate the roots of $\mathfrak{sl}_n(\mathbb{C})$. Additionally, the program generates images showing a selection of eigenvalues of $\mathfrak{sl}_n(\mathbb{C})$.

4 How the roots are calculated

1. Generate basis for $\mathfrak{sl}_n(\mathbb{C})$, given some $n \in \mathbb{Z}_{\geq 1}$: This is done in 3 parts.
   (a) Calculate dimension of $\mathfrak{sl}_n(\mathbb{C})$. We use the formula
   $$\dim(\mathfrak{sl}_n(\mathbb{C})) = n - 1 + 2\sum_{i=1}^{n-1} i,$$
   where the $n-1$ term is a count of how many basis elements have entries only on the main diagonal, and the sum is a count of all the basis elements with entries off the main diagonal.

   (b) As the formula suggests, we simply fill an array with 1's at all entries a_{ij}, where $i \neq j$, i.e.,
   ```
   for i in range(n):
       for j in range(n):
           if (i != j)
               B[matnum][i][j] = 1
               matnum++.
   ```
   Similarly, for the diagonal elements, we use
   ```
   for i in range(n-1):
       B[matnum][i][i] = 1
       B[matnum][i+1][i+1] = -1
       matnum++.
   ```
   Here `n` is the number the user entered and `matnum` is a count of the matrices since `B` is an array containing as many matrices as there are dimensions in $\mathfrak{sl}_n(\mathbb{C})$, and each matrix is $n \times n$.

4.1 A Quick Visual Detour

We stop here after step 1 for a moment since there is something interesting that we can look at with what we have so far. With a basis in hand, we can, in theory, generate all of $\mathfrak{sl}_n(\mathbb{C})$. However, this is not possible on a physical computer, as it would require taking the span of the basis elements which is an uncountably infinite set.

We can instead consider a finite spanning set. Suppose for instance that we want to see how the eigenvalues of $\mathfrak{sl}_n(\mathbb{C})$ are distributed in $\mathbb{C}$ as we range over n. To do this, we can, as stated before, take a particular finite spanning set of the basis, and plot all of the complex eigenvalues of each element in this spanning set.

But how do we choose the elements which populate our finite spanning set? In other words, any finite selection would in a sense be arbitrary. I could not think of a way around this aside from making the choice as natural and uniform as possible. With this I ask you, dear reader, what is more uniform and natural that the complex unit circle? Each point is unit length, and it occupies all 4 quadrants. As you ponder the answer to that question, let us just roll with it ("roll" ... "circle"..., you get it) and see where it takes us.

Let $N \in \mathbb{N}_{\geq 0}$, then we can choose N many points from the complex unit circle by letting $\theta = 2\pi/N$ and taking $M = \{c \in \mathbb{C} \mid \cos(\theta k) + i\sin(\theta k) = c, \text{ for } 0 \leq k < N\}$. We can now thing of the set M as the set containing all the multiples which we shall use to form our linear combinations. For example, suppose $M = \{\alpha_1, \alpha_2\}$ and our basis $B = \{x, y, h\}$. Then we want to calculate the following linear combinations (which we'll denote A_i):

$$
\begin{aligned}
A_1 &= \alpha_1 x + \alpha_1 y + \alpha_1 h & \qquad A_2 &= \alpha_1 x + \alpha_1 y + \alpha_2 h \\
A_3 &= \alpha_1 x + \alpha_2 y + \alpha_1 h & \qquad A_4 &= \alpha_1 x + \alpha_2 y + \alpha_2 h \\
A_5 &= \alpha_2 x + \alpha_1 y + \alpha_1 h & \qquad A_6 &= \alpha_2 x + \alpha_1 y + \alpha_2 h \\
A_7 &= \alpha_2 x + \alpha_2 y + \alpha_1 h & \qquad A_8 &= \alpha_2 x + \alpha_2 y + \alpha_2 h
\end{aligned}
$$

If it is not already clear, given a basis with b many elements, and a set of N many multiples, then we will have a spanning set containing N^b many elements. So in our particular case our spanning set will contain

$$N^{\dim(\mathfrak{sl}_n(\mathbb{C}))}$$

many elements.

Alright, now with our linear combinations, we need only calculate the eigenvalues for each A_i. So finally, supposing all such eigenvalues are collected neatly into a set, then the last step is to plot each element of this set. Below are the results of this.

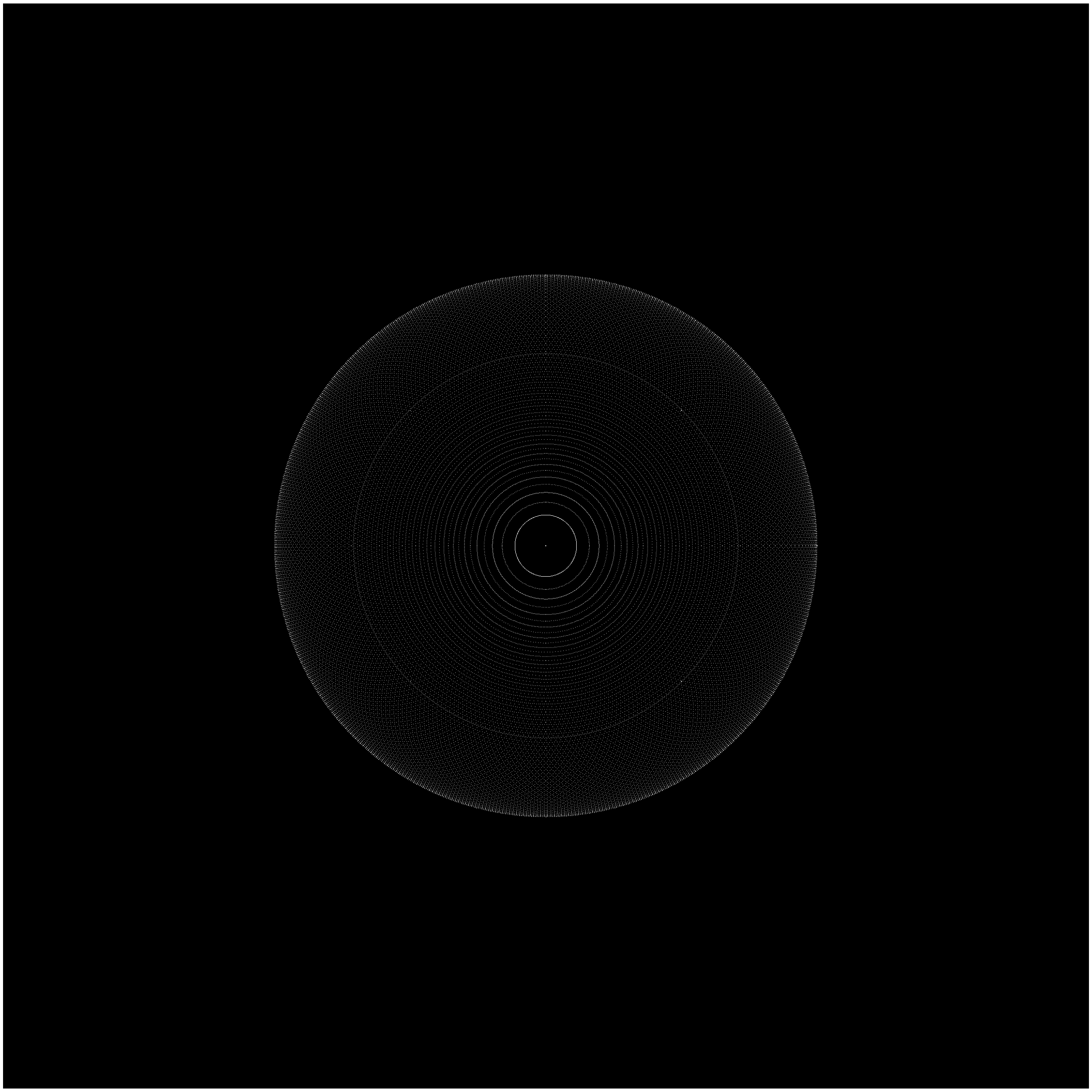

Figure 1: 14,706,125 2x2 matrices; 29,411,760 distinct eigenvalues

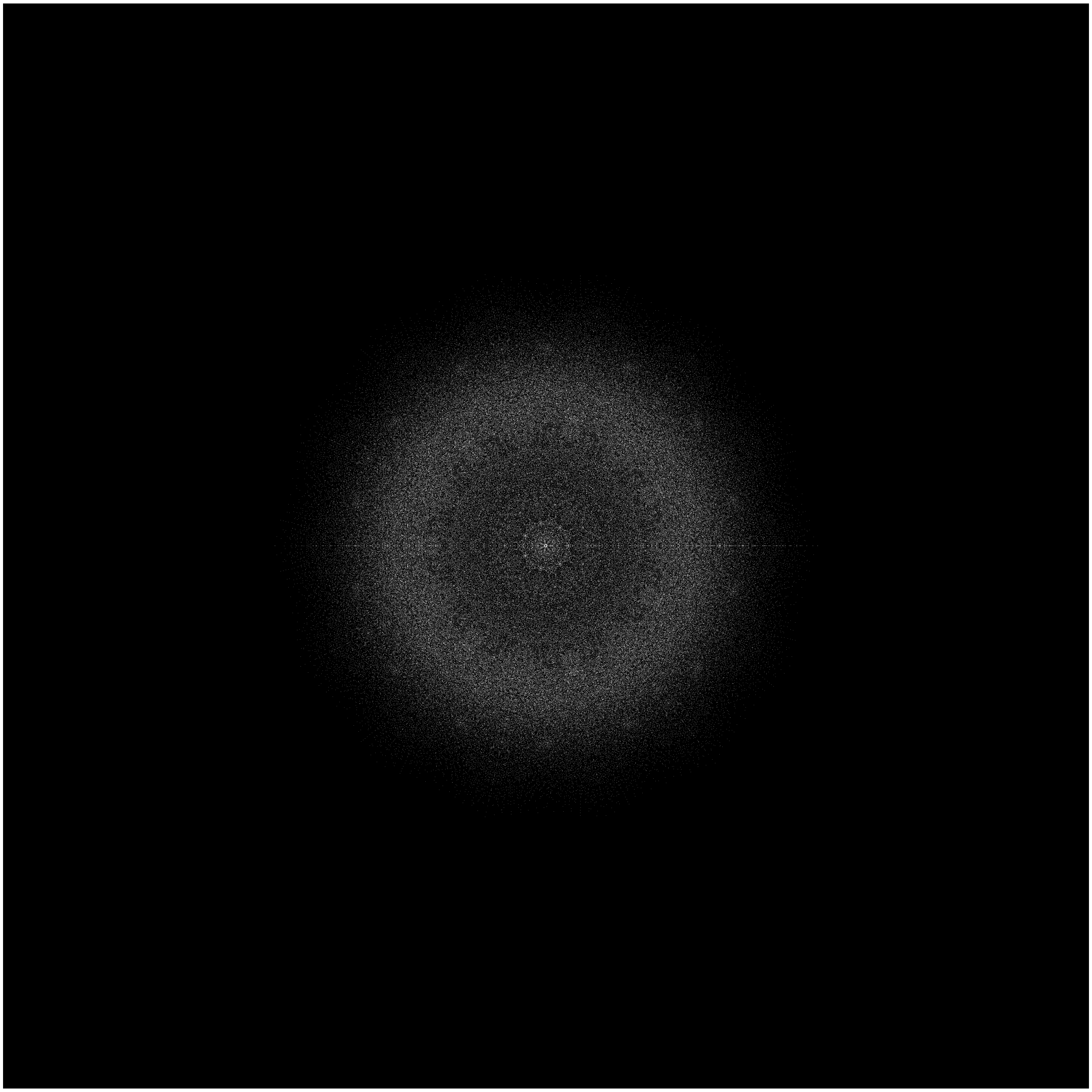

Figure 2: 16,777,216 3x3 matrices; 50,322,992 distinct eigenvalues

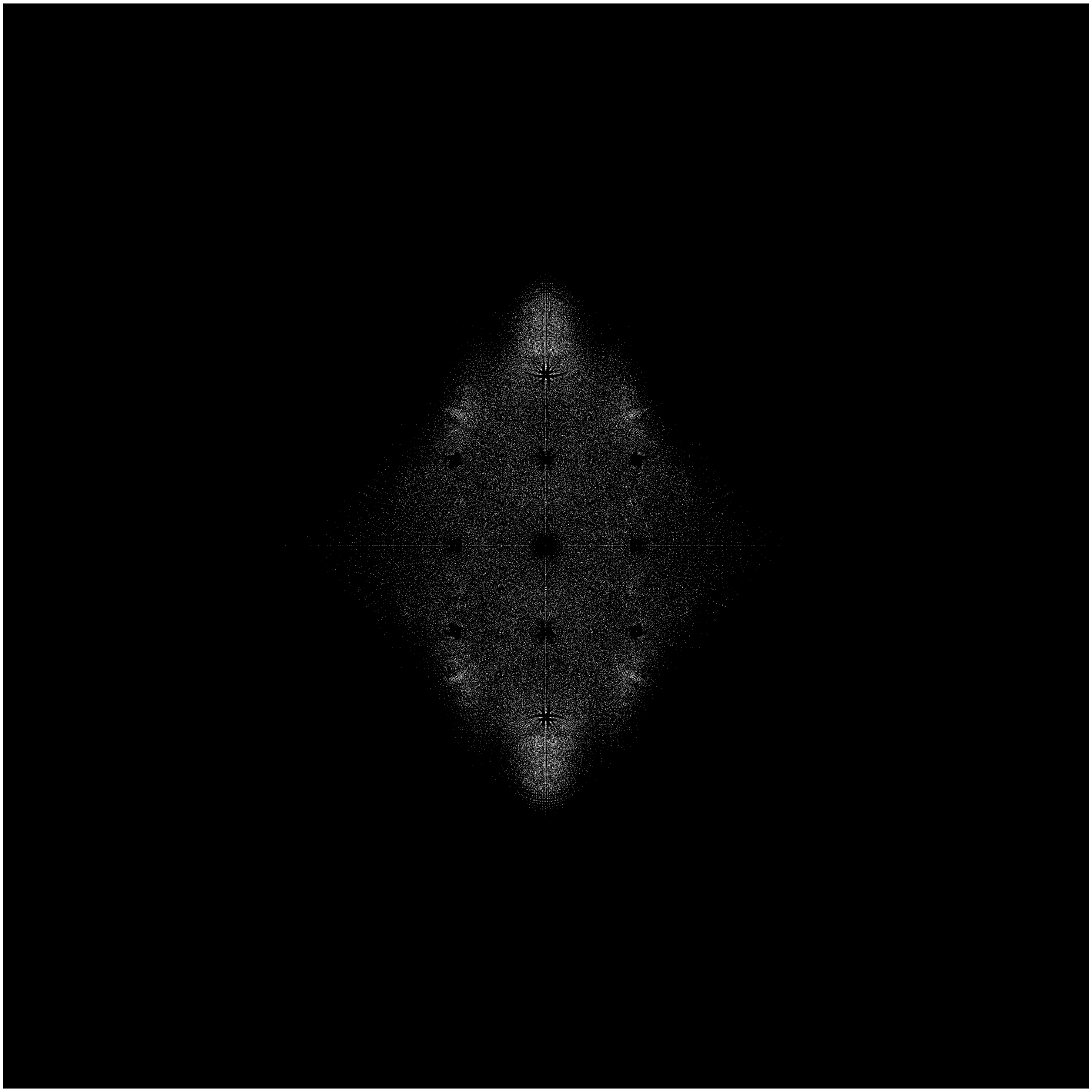

Figure 3: 14,348,907 4x4 matrices; 57,138,453 distinct eigenvalues

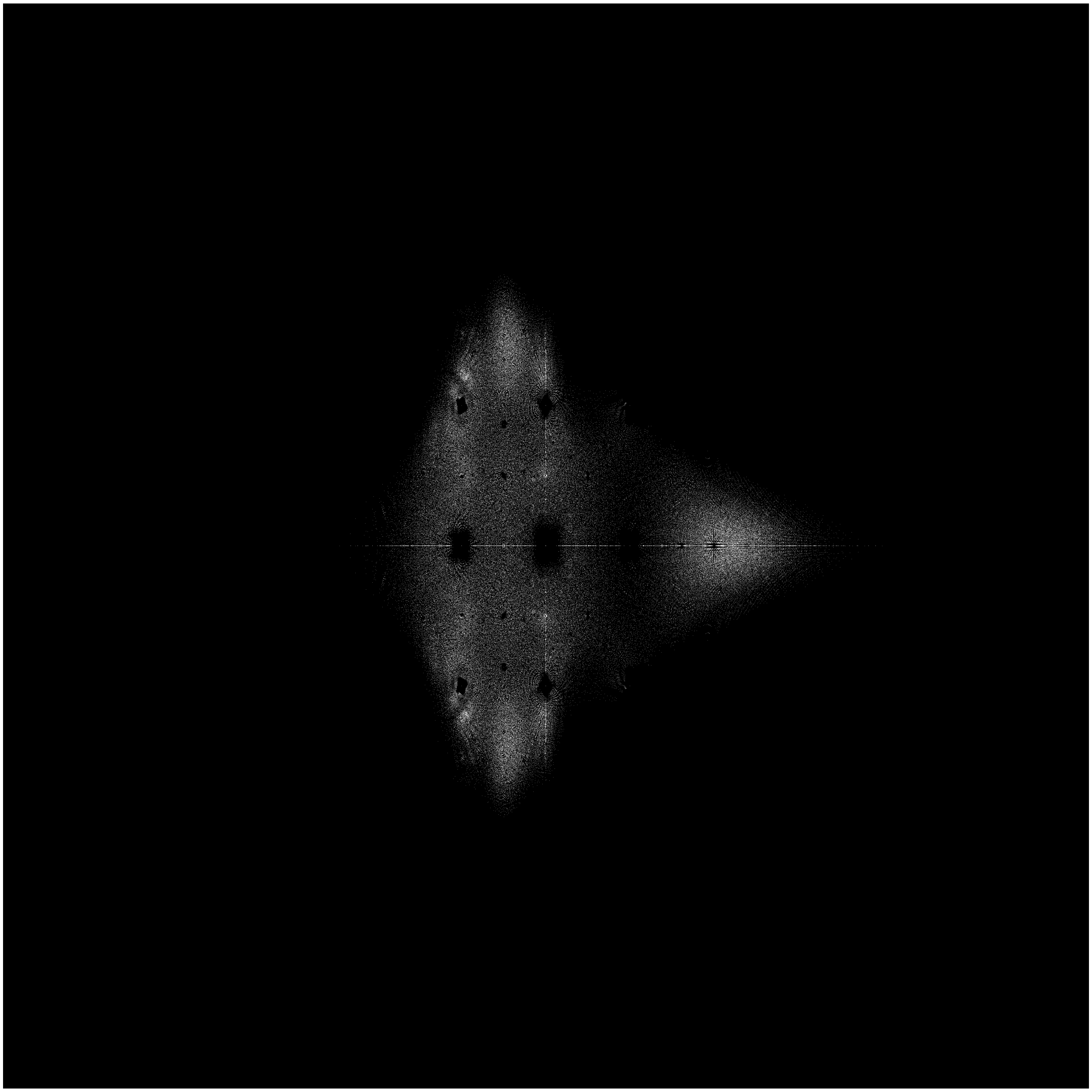

Figure 4: 16,777,216 5x5 matrices; 81,914,833 distinct eigenvalues

4.2 Continuing where we left off

3. Obtain the Cartan subalgebra. This step is actually quite easy in the case of $\mathfrak{sl}_n(\mathbb{C})$ since a basis for this subalgebra can be found by simply selecting, from the elements of the entire basis, those elements with entries on the main diagonal.

4. Calculate the adjoint representation of each basis element of the Cartan subalgebra. The reason we are doing this is because we can think of the roots of the Lie algebra as the nonzero weights of the adjoint representation. Calculating the adjoint representaion is relatively simple on paper, but quite dry in code. Essentially, we take each (Cartan) basis element, bracket it with every (entire) basis element, express the result as a linear combination of the (entire) basis elements, and use the coefficients in this linear combination as a row in the adjoint representaion matrix.

5. Calculate the eigenvectors of each adjoint representation. The weight vectors are those who are eigenvectors to every adjoint representation matrix. Hence, the weights are the eigenvalues of such eigenvectors.

6. Filter through these eigenvalues, removing those with 0 in ever component and voila you have your roots!

5 Referring

1. (Pg. 18) In A2 of an associative K-algebra, we have $a, b, c \in A$, but c is never used.

2. (Pg. 27) At the bottom of this page, it is slightly unclear at first glance that the super script is not an exponent in the summation.

3. (Pg. 31) In Example 4.1, "identity" should be replaced with with "identify".

4. (Pg. 33) In Example 4.6, there is an incomplete "mathbb".

5. (Pg. 35) In Example 4.8, "$u \in \mathfrak{sl}_2$" is never used.

6. (Pg. 39) No proof for Lemma 4.17

7. (Pg. 8) On this page the definition of a vector space was given, but throughout the rest of the notes we used K-linear space. Admittedly, I assumed an equivalence between the two structures. Further emphasis on whether or not this equivalence exists or deciding to use only one term instead of both would be helpful.

8. (Pg. 53) It is stated that $\dim(L_{\alpha_{12}}) = \dim(L_{\alpha_{21}}) = 1$. Upon first consideration, this is not clear (at least to me). One more sentence of justification might be helpful for the dummies like me.

9. (Pg. 56) It is mentioned that our definition of a Cartan subalgebra is really the definition of a maximal toral subalgebra. It would be interesting to take a little (1-2 paragraphs) of what a toral algebra is.

10. (Pg. 57) At the beginning of Section 7.2, it may be worth mentioning the dimension of L and how that relates to the number of eigenvectors of $\mathrm{ad}(H)$, as it is stated that there is a basis of L which consists of commen eigenvectors for the elements of $\mathrm{ad}(H)$.

11. (Pg. 58) In the proof at the bottom of the page, in the string of equalities it is written "$\kappa(\alpha(h)x, y) = \kappa([h, x])$", whereas it should be "$= \kappa([h, x], y)$.

12. (Pg. 60) In the third paragraph it is stated that for each $\alpha \in \Phi$, there is a $t_\alpha \in H$ such that
$$\kappa(t_\alpha, h) = \alpha(h).$$
It is not super clear to me why the killing form κ, and the linear functional α would agree at every $h \in H$, provided the first component of κ is fixed with t_α.

13. (Pg. 60) In the proof at the bottom, it is stated that for any nonzero $x_\alpha \in L_\alpha$, we take $y_\alpha \in L_{-\alpha}$ such that $\kappa(x_\alpha, y_\alpha) = \frac{2}{\kappa(t_\alpha, t_\alpha)}$. It is not clear to me why we know that there is always an x_α and y_α such that we get this output from the Killing form. In general, as the Killing form is defined in terms of the trace of the composition of adjoint representations, it is a very daunting object and a lot of results using it are tough to see initially without fully working though all the calculations.

14. (Pg. 65) I think more time spent on the reflection $\sigma_\alpha(\beta)$ would be nice. Not to suggest total handholding, but a parallel analogy to $\mathbb{R}^2$ and its inner product, as seen in Calc 3, and the one we developed in terms of the Killing form might be helpful.

15. (Pg. 69) In Example 8.5.1 the first line is "Note that we don't begin with $\theta = \pi/3$". This was confusing since prior to that being stated, I had no reason to expect to be looking at the case where $\theta = \pi/3$. It felt like I missed something.

    In this example it is also stated that the angle between α and $\alpha + \beta$ is $\pi/3$. This is clear from the picture, but a full calculation of this in terms of identity $\langle \alpha, \alpha + \beta \rangle \langle \alpha + \beta, \alpha \rangle = 4\cos^2\theta$ would have been very helpful.

6 Reflections

1. (A proof I liked) My favorite proof was that of Lemma 4.7. This is mainly because of the result itself, but the proof was short and simple. Considering how useful of a result this was throughout the semester, such a brief proof was quite great to see!

2. (A proof I struggled with) The proof of Lemma 4.8 was a little rough only because I was (am) not able to see the implications or the significance of the result. Though now I see that it is interesting that there is essentially only one L-module homomorphism if V is a irreducible L-module.

3. (A proof that I still can't quite understand) The proof of Theorem 5.4 is still sitting on the shelf in my brain. As embarassing it is to admit it, the heavy volume of calculations present in this proof has deterred me from fully reading and computing along with it. For this reason, I have only been able to take the theoreom on faith.