STAT 215A Problem Set 3, Spring 2022

1. Out of 12 balls in an urn, 5 of them are green. Consider a sample of size $n = 4$ without replacement. Let A_k be the event that a sample of size 4 contains exactly k green balls for $0 \le k \le 4$. Moreover, let B_j be the event that the ball selected at the j^{th} step is green for $1 \le j \le 4$.

   (a) Determine $P(B_j)$ for each $1 \le j \le 4$.

   (b) Compute $P(A_3|B_2)$.

   (c) Compute $P(B_j|A_3)$ for each $1 \le j \le 4$.

   (d) Would your answers to any of the parts (a)-(c) above change if the sampling is done with replacement?

2. Prove the following for indicator functions $I_A(\omega)$ where $\omega \in \Omega$ and $A \in \mathcal{F}$ in a measurable space $(\Omega, \mathcal{F})$:

   (a) $I_\emptyset = 0, I_\Omega = 1, I_A + I_{A^c} = 1, I_{A\cap B} = I_A I_B, I_{A\cup B} = I_A + I_B - I_A I_B, I_{A\Delta B} = (I_A - I_B)^2$;

   (b) $I_{\bigcup_{i=1}^{n} A_i} = 1 - \prod_{i=1}^{n}(1 - I_{A_i})$; and

   (c) $I_{\bigcup_{i=1}^{n} A_i} = \sum_{i=1}^{n} I_{A_i}$, if $\{A_i\}_{i=1}^{n}$ is a disjoint collection.

3. Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, assume g is a continuous and strictly increasing function. Show that:

   (a) the function $g(X) : \Omega \to \mathbb{R}$ is a random variable;

   (b) the expressions $X + Y$ and $X \cdot Y$ are random variables (on the same space).

4. Let F be a cdf, $n \in \mathbb{N}$ and $0 < \lambda < 1$. Prove that F^n and $1 - (1 - F)^n$ are also cdf.

5. Let f and g be two probability density functions (pdf) and $0 < \lambda < 1$. Prove that:

   (a) $\lambda f + (1 - \lambda)g$ is a pdf

   (b) fg is not necessarily a pdf

6. Let $X \sim Unif(0, 1)$ with pdf $f(x) = 1$ for $0 < x < 1$. Define $Y = -\ln(X)/2$.

   (a) Show that Y is a continuous r.v. with range $(0, \infty)$.

   (b) Determine the cdf of Y.

   (c) Determine a pdf for Y.

   (d) Compute $P(Y > 2)$.

7. Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{x_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence that converges to $x \in \mathbb{R}$. If $B_n = \{\omega \in \Omega : x_n < X(\omega) \le x\}$ for $n \in \mathbb{N}$, prove that:

   (a) $B_n \in \mathcal{F}$ for each $n \in \mathbb{N}$;

   (b) $\{B_n\}_{n\in\mathbb{N}}$ is an increasing sequence of events;

   (c) $\bigcap_{n=1}^{\infty} B_n = \{\omega \in \Omega : X(\omega) = x\}$.

8. Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $P(X \ge 0) = 1$. For $x \ge 0$ and $n \in \mathbb{N}$, define a sequence of events $\{A_n(x)\}_{n\in\mathbb{N}}$ as follows: $A_n(x) = \{\omega \in \Omega : X(\omega) > x - 1/n\}$ for $n \in \mathbb{N}$. Prove that:

   (a) $\bigcap_{n=1}^{\infty} A_n(x) \in \mathcal{F}$ for all $x \ge 0$;

   (b) $\{\omega \in \Omega : X(\omega) \ge x\} \in \mathcal{F}$ for all real x.