Homework 4 - Series

MATH 230B

You can use all the results discussed in class. They are collected in the notes posted on Canvas.

Problem 1

Let $a_k \geq 0$, $b_k > 0$ for all $k \in \mathbb{N}$. Assume that $\lim_{k\to\infty} \frac{a_k}{b_k} = \lambda$, where $0 < \lambda < \infty$. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} b_k$ is convergent.

Problem 2 [*Master's Exam, Fall 2018*]

Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} a_k^2$ is convergent.

Problem 3 [*Master's Exam, Fall 2017*]

Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} \sqrt{a_k a_{k+1}}$ is convergent.

Problem 4 [*Master's Exam, Spring 2021*]

Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ is convergent.

Problem 5 [*Master's Exam, Fall 2021*]

Prove that if $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then $\sum_{k=1}^{\infty} k^2 a_k$ is not convergent.

ADDITIONAL PROBLEMS

Problem A1

Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ is convergent, then $\liminf_{k\to\infty} ka_k = 0$. Is it true that $\lim_{k\to\infty} ka_k = 0$?

Problem A2 Let $a_k > 0$ for all $k \in \mathbb{N}$. Prove that $\sum_{k=1}^{\infty} a_k$ is convergent if and only if $\sum_{k=1}^{\infty} \frac{a_k}{S_k}$ is convergent, where $S_k = \sum_{j=1}^{k} a_j$.

Problem A3 (Kronecker's Lemma) [*Master's Exam, Fall 2017*]
Prove that if $\sum_{k=1}^{\infty} \frac{a_k}{k}$ is convergent, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0$.

Problem A4 Determine if the following series are conditionally/absolutely convergent:

a) $\sum_{k=1}^{\infty} \frac{k^\alpha}{k!}$ for $\alpha \in \mathbb{R}$

b) $\sum_{k=2}^{\infty} \frac{1}{(\log(k))^\alpha}$ for $\alpha \in \mathbb{R}$

c) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k + (-1)^k}$

Problem A5

Let $\{a_k\}_k$ be a sequence in $\mathbb{R} \setminus \{0\}$. Determine if the following statements are true or false. Give a proof or counterexample as appropriate:

a) If $\liminf_{k\to\infty} \sqrt[k]{|a_k|} > 1$ then $\sum_{k=1}^{\infty} |a_k|$ is divergent

b) If $\liminf_{k\to\infty} \sqrt[k]{|a_k|} < 1$ then $\sum_{k=1}^{\infty} a_k$ is convergent

c) If $\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 2$ then $\sum_{k=1}^{\infty} a_k$ is not convergent