STAT 215A

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Instructor: Dr. Cetin — Assignment: PRACTICE

1. Let Ω be a sample space for a particular experiment. Moreover, let A and B be two proper subsets of Ω.

   (a) Show that $A \cup B = A \cup (B \backslash A)$.

   Proof. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \notin A$, then $x \in B \backslash A$ and thus $x \in A \cup (B \backslash A)$. If $x \in A$, then $x \in A \cup (B \backslash A)$. Thus $A \cup B \subset A \cup (B \backslash A)$. Let $x \in A \cup (B \backslash A)$. Then $x \in A$ or $x \in B$ and $x \notin A$. If $x \in A$, then $x \in A \cup B$. If $x \notin A$, then $x \in B$ and so $x \in A \cup B$. Therefore $A \cup B = A \cup (B \backslash A)$. $\square$

   (b) If A and B are not disjoint, determine the σ-algebra generated by A and B.

   Proof. Any σ-algebra containing A and B must contain A^c and B^c. It must contain $A \cup B$, $A^c \cup B$, $A \cup B^c$, $A \cap B$, $A^c \cup B^c$. Thus, the σ-algebra generated by A and B is given by

   $$\Omega = \{\varnothing, A, B, A^c, B^c, A \cup B, A \cap B, A^c \cup B, A \cup B^c, A^c \cup B^c, A^c \cap B^c, X\}.$$

   $\square$

2. Let $(\Omega, \mathcal{F})$ be a measurable space, and $B, A_1, A_2, \dots, \in \mathcal{F}$. Prove the following distributive laws:

   (a) $B \cap \left(\bigcup_{k=1}^{\infty} A_k\right) = \bigcup_{k=1}^{\infty}(B \cap A_k)$.

   Proof. Let S_1 denote the set on the left, and S_2 denote the set on the right. Let $x \in S_1$, then $x \in B$ and for some $t \in \mathbb{Z}^+$, $x \in A_t$. Thus, $x \in B \cap A_t$. Hence

   $$x \in B \cap A_t \cup \big(\bigcup_{k=1}^{\infty}(B \cap A_k)\big) = \bigcup_{k=1}^{\infty}(B \cap A_k).$$

   Thus $S_1 \subset S_2$. Now let $x \in S_2$. Then for some $t \in \mathbb{Z}^+$, $x \in B \cap A_t$. Hence

   $$x \in B \cap \big(\bigcup_{k=1}^{\infty} A_k\big).$$

   Thus $S_2 \subset S1$ and so $S_1 = S_2$. $\square$

   (b) $B \cup \left(\bigcap_{k=1}^{\infty} A_k\right) = \bigcap_{k=1}^{\infty}(B \cup A_k)$.

Proof. Let S_1 denote the set on the left and S_2 denote the set on the right. Let $x \in S_1$. Then $x \in B$ or $x \in A_k$ for every k. If $x \in B$, then $x \in B \cup A_k$ for every k. Hence

$$x \in \bigcap_{k=1}^{\infty} (B \cup A_k).$$

Hence $x \in S_2$ and so $S_1 \subset S_2$. Now let $x \in S_2$. Then $x \in B \cup A_k$ for every k. If $x \in B$, then $x \in B \cup \big(\bigcap_{k=1}^{\infty} A_k\big)$. If $x \in A_k$ for every k, then $x \in \bigcap_{k=1}^{\infty} A_k$ and thus $x \in B \cup \big(\bigcap_{k=1}^{\infty} A_k\big)$. Therefore $S_2 \subset S_1$ and so $S_1 = S_2$. □

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and B be a nonempty subset of Ω.

   (a) Let $L = \{B \cap A : A \in \mathcal{F}\}$. Show that L is a σ-algebra on B.

   Proof. (i) Since $\varnothing \in \mathcal{F}$, then $B \cap \varnothing = \varnothing \in L$. (ii) Let $E \in L$. Then for some $A' \in \mathcal{F}$, we have that $E = B \cap A'$. We want to show that $B \backslash E \in L$, that is, we need some $A \in \mathcal{F}$ such that $B \backslash E = B \cap A$. Expanding this out we have that

$$\begin{aligned}
&x \in B \backslash (B \cap A') \\
&\quad \Leftrightarrow (x \in B) \wedge (x \notin B \wedge x \notin A') \\
&\quad \Leftrightarrow (x \in B) \wedge (x \in (B \cap A')^c) \\
&\quad \Leftrightarrow (x \in B) \wedge (x \in B^c \vee x \in A'^c) \\
&\quad \Leftrightarrow (x \in B \wedge x \in B^c) \vee (x \in B \wedge x \in A'^c) \\
&\quad \Leftrightarrow (x \in B \cap B^c) \vee (x \in B \cap A'^c) \\
&\quad \Leftrightarrow x \in (B \cap B^c) \cup (B \cap A'^c) \\
&\quad \Leftrightarrow x \in B \cap A'^c.
\end{aligned}$$

   Since $A'^c \in \mathcal{F}$, then this proves that if $E \in L$, then $E^c \in L$. (iii) Finally, let $E_1, E_2, \ldots$ be a sequence of elements of L. Then for each E_i in this sequence, there is some $A_i \in \mathcal{F}$ such that $E_i = B \cap A_i$. Hence

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} (B \cap A_i) = B \cap \big(\bigcup_{k=1}^{\infty} A_k\big).$$

   The last equality following from exercise 2.(a). Since each $A_1, A_2, \ldots$ is a sequence of elements of $\mathcal{F}$, then the countable union of the sequence is also in $\mathcal{F}$. Thus

$$\bigcup_{k=1}^{\infty} E_k \in L.$$

   Therefore L is a σ-algebra on B. □

   (b) Let L be as in part (a). Is (Ω, L) a measurable space? In other words, is L a σ-algebra on Ω?

Solution. As a counter example, let $\Omega = \{0, 1\}$ and let $\mathcal{F} = \{\varnothing, \Omega\}$. If $B = \{0\}$, then $L = \{\varnothing, \{0\}\}$ which is not a σ-algebra on Ω since $\{0\}^c = \{1\} \notin L$ and so L is not closed under complementations. ∎

4. Complete the proof of the inclusion-exclusion formula for any two events A and B in a probability space $(\Omega, \mathcal{F}, P)$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof. To prove this equality, we want to appeal to the property of a measure which states that the measure of a countable union of disjoint events is the countable sum of the measure of each event. First we want to re-express $A \cup B$ as union of disjoint sets. To do this we refer to exercise 1.(a) to write $A \cup B = A \cup (B \backslash A)$. So then letting $E_1 = A$, $E_2 = B \backslash A$, and $E_i = \varnothing$ for all $i > 2$, then we have that

$$\begin{aligned} P\Big(\bigcup_{k=1}^{\infty} E_k\Big) &= \sum_{k=1}^{\infty} P(E_k) \\ &= P(A) + P(B \backslash A) + P(\varnothing) + P(\varnothing) + \cdots \\ &= P(A) + P(B \backslash (A \cap B)) + 0 + 0 + \cdots \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

This completes the proof for two events. □

5. Complete the inclusion-exclusion formula for any three events A, B, and C in a probability space $(\Omega, \mathcal{F}, P)$:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Proof. As we did in exercise 4, we want to re-express $A \cup B \cup C$ as a union of disjoint events. We have

$$\begin{aligned} A \cup B \cup C &= A \cup (B \cup C) \\ &= A \cup (B \cup (C \backslash B)) \\ &= A \cup (B \cup (C \backslash B)) \backslash A. \end{aligned}$$

By exercise 4, it follows that

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup (B \cup (C \backslash B)) \backslash A) \\ &= P(A) + P((B \cup (C \backslash B)) \backslash A) - P(A \cap (B \cup (C \backslash B)) \backslash A) \\ &= P(A) + P((B \cup (C \backslash B)) \backslash A) \\ &= P(A) + P(B \cup (C \backslash B)) - P(A \cap (B \cup (C \backslash B)) \\ &= P(A) + P(B) + P(C \backslash B) - P((A \cap B) \cup (A \cap (C \backslash B))) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap (C \backslash B)) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P((A \cap C) \backslash B) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P((A \cap C) \backslash (A \cap B \cap C)) \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

$\square$

6. The general inclusion-exclusion formula for the union of a finite number of events. Let $\{A_k : k = 1, 2, \dots, n\}$ be a collection of n events in a probability space $(\Omega, \mathcal{F}, P)$. Then for each $n \in \mathbb{N}$,

$$P\Big(\bigcup_{k=1}^{n} A_k\Big) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{n+1} P\Big(\bigcap_{n=1}^{n} A_k\Big).$$

Proof. Eh, just please do this one later. Maybe tomorrow morning. Base case is covered by exercise 4. Adding another set to the union you can break it down into a union of n sets which is covered by the inductive hypothesis and a single set. Together this is essentially the case of unioning two events together. $\square$

7. Boole's inequality. Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Prove that $P\big(\bigcup_{k=1}^{n} A_k\big) \leq \sum_{k=1}^{n} P(A_k)$, for all $n \in \mathbb{N}$.

   Proof. We proceed by induction on n. Let $n = 2$. Then if $A_1, A_2 \in \mathcal{F}$, we have that

   $$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2).$$

   This verifies our base case. Now assume the hypothesis is true for some $n \geq 2$. Let $A_1, \dots, A_n, A_{n+1} \in \mathcal{F}$. Then

   $$P\Big(\bigcup_{k=1}^{n+1} A_k\Big) = P\Big(\Big(\bigcup_{k=1}^{n} A_k\Big) \cup A_{n+1}\Big) \leq \sum_{k=1}^{n} P(A_k) + P(A_{n+1}) = \sum_{k=1}^{n+1} P(A_k).$$

   Hence, the claim is true for all $n \in \mathbb{N}$. $\square$

   (b) Prove that $P\big(\bigcup_{k=1}^{\infty} A_k\big) \leq \sum_{k=1}^{\infty} P(A_k)$.

   Proof. Let $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathcal{F}$ and let $A_{n+k} = \varnothing$ for all $k \in \mathbb{N}$. Then we have a sequence $A_1, A_2, \dots$ of events in $\mathcal{F}$. From part (a) of this exercise it follows that

   $$\begin{aligned} P\Big(\bigcup_{k=1}^{\infty} A_k\Big) &= P\Big(\Big(\bigcup_{k=1}^{n} A_k\Big) \cup \Big(\bigcup_{k=n+1}^{\infty} A_k\Big)\Big) \\ &\leq P\Big(\bigcup_{k=1}^{n} A_k\Big) + P\Big(\bigcup_{k=n+1}^{\infty} A_k\Big) \\ &\leq \sum_{k=1}^{n} P(A_k) + P(\varnothing) \\ &= \sum_{k=1}^{n} P(A_k) + \sum_{k=n+1}^{\infty} 0 \\ &= \sum_{k=1}^{\infty} P(A_k). \end{aligned}$$

□

8. We are given two urns, each containing a collection of colored balls. Urn I contians two white balls and three blue balls, and Urn II contains three white balls and four blue balls. Select one of the urns randomly and then draw a ball from that urn selected.

   (a) Describe the sample space Ω for this experiment. Is it an equiprobable space?

   Solution. Begin by letting U_1, U_2 denote the two urns. Then we can express their contents as

$$U_1 = \{w_{11}, w_{12}, b_{11}, b_{12}, b_{13}\}$$
$$U_2 = \{w_{21}, w_{22}, w_{23}, b_{21}, b_{22}, b_{23}, b_{24}\}.$$

   Since any outcome of this experiment is the selection of a single ball, then the set of all possible outcomes is $\Omega = U_1 \cup U_2$. To show that this is an equiprobable space, we have to show that it is finite and that for all $\omega_i, \omega_j \in \Omega$, with $i \neq j$, $P(\{\omega_i\}) = P(\{\omega_j\})$, for some probability measure P. First, we will let $\mathcal{F} = \mathcal{P}(\Omega)$ be the σ-algebra. Then define $P : \mathcal{F} \to [0, \infty]$ by

$$P(A) = \frac{|A|}{|\Omega|}$$

   for all $A \in \mathcal{F}$. It is clear to see P satisfies all three conditions of a probability measure.

   It also follows that for any $\{\omega_i\}, \{\omega_j\} \in \mathcal{F}$ with $i \neq j$, that

$$P(\{\omega_i\}) = \frac{|\{\omega_i\}|}{|\Omega|} = \frac{1}{|\Omega|} = \frac{|\omega_j|}{|\Omega|} = P(\{\omega_j\}).$$

   Finally, since Ω is finite, then this proves that $(\Omega, \mathcal{F}, P)$ is an equiprobability space. ■

   (b) What is the probability that the ball picked is blue?

   Solution. The event of picking a blue ball can be expressed as the set

$$B = \{b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{24}\}.$$

   Since $B \subset \Omega$, then $B \in \mathcal{F}$ and therefore measureable. And so

$$P(B) = \frac{|B|}{|\Omega|} = \frac{7}{12}.$$

   ■

   (c) Given that the ball picked is blue, what is the probability that it comes from Urn I?

Solution. Let A denote the event of selecting Urn I and let B denote the event that a blue ball was selected. Then

$$A = \{w_{11}, w_{12}, b_{11}, b_{12}, b_{13}\}$$
$$B = \{b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}, b_{24}\}.$$

Thus the probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{7}.$$

■

9. Using the same setup as in the previous exercise, now we draw a ball at random from Urn I, put it into Urn II and then pick a ball from Urn II.

   (a) What is the probability that both balls picked are blue?

   Solution. There are two different approaches to this problem. The first will use the following result: For any events A and B such that $0 < P(B) < 1$,

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c).$$

   Letting A denote the event that the last ball chosen is blue and B denote the event that the first ball chosen is blue, then $P(B) = 3/5$, $P(B^c) = 2/5$, and

$$P(A \mid B) = \frac{5}{8} \qquad\qquad P(A \mid B^c) = \frac{1}{2}.$$

   Hence

$$P(A) = \frac{5}{8} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{23}{40}.$$

   However this gives us the probability that the last ball chosen is blue. So we must subtract the probability that the first ball was white and the last ball was blue, which is $1/5$, and thus the probability that both balls were blue is $3/8$.

   The second approach requires us to redefine a new Ω based on the fact that we are making two choices, the latter being dependent on the former. Let

$$\Omega = \bigcup_{\omega \in U_1} \{\omega\} \times U_2 \cup \{\omega\}.$$

   Where the left side of the cartesian product represents our first choice from urn I and the right side represents urn II after adding our selected ball to it. Since adding a ball to urn II gives it a total of 8 elements, and there are 5 possible balls from urn I to add to urn II, then $|\Omega| = 5 \cdot 8 = 40$. So then the events in which both balls selected were blue is the set containing a blue component in the left and right. For each blue ball from urn I, of which there are 3, there correpsonds 5 ordered pairs with the blue urn I ball on the left and a blue urn II ball on the right. Hence, there are a total of $3 \cdot 5 = 15$ of these ordered pairs. Therefore, if A is the event that both the left and right components are blue, then

$$P(A) = \frac{15}{40} = \frac{3}{8}.$$

■

(b) Given that the ball drawn from urn II was blue, what is the probability that the first ball drawn was also blue?

Solution. Using the same setup as the second half of part (a), we are interested in all the times both the left and right components are blue out of all the times the right components were blue. The latter count would be $2 \cdot 4 = 8$ for the cases in which we chose white balls first, and $3 \cdot 5 = 15$ for the cases in which we chose blue balls first. Thus, there are 23 instances in which a blue ball was chosen second. Of those, 15 of them include the times in which a blue ball was chosen first. Hence, the probability that the first ball was blue given that the second ball was blue is 15/23. ■