MATH 230B

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Due Date: 04/15/2022
Assignment: Homework 05

1. Let $A_n \subseteq \mathbb{R}$ satisfy $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ and denote $A = \bigcap_{n\in\mathbb{N}} A_n$. Prove that the sequence of indicator functions $f_n = 1_{A_n}$ converges pointwise to the function $f = 1_A$ on $\mathbb{R}$.

   Proof. Let $x \in \mathbb{R}$. Then either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in \bigcap_{n\in\mathbb{N}} A_n$ which implies that for all $n \in \mathbb{N}$, $x \in A_n$. Thus $f_n(x) = 1$ for all $n \in \mathbb{N}$. The final implication is that for $N = 1$ and for $n \geq N$, then $|f_n(x) - 1| = 0 < \varepsilon$, for all $\varepsilon > 0$. Therefore $\lim_{n\to\infty} f_n(x) = 1$ for all $x \in A$.

   If $x \notin A$, then $x \notin \bigcap_{n\in\mathbb{N}} A_n$ which implies that for some $m \geq 1$, $x \notin A_m$. Thus for all $n \geq m$, $x \in A_n$, and so $f_n(x) = 0$ for all $n \geq m$. Letting $\varepsilon > 0$, $N = m$, and $n \geq N$, then $|f_n(x) - 0| < \varepsilon$. Hence $\lim_{n\to\infty} f_n(x) = 0$ for all $x \notin A$. Therefore

   $$\lim_{n\to\infty} f_n(x) = f.$$

   $\square$

2. Discuss the pointwise/uniform convergence on $[0, 1]$ of the sequence of functions

   $$f_n(x) = \frac{nx}{1 + n^3x^2}.$$

   Solution. Letting $x \in [0, 1]$, then

   $$\begin{aligned}
   \lim_{n\to\infty} \frac{nx}{1 + n^3x^2} &= \lim_{n\to\infty} \frac{n}{1 + n^3x^2} \cdot x \\
   &= \lim_{n\to\infty} \frac{1/n^2}{1/n^3 + x^2} \cdot x \\
   &= \frac{0}{0 + x^2} \cdot x \\
   &= 0.
   \end{aligned}$$

   This shows that the sequence $\{f_n(x)\}$ converges for all $x \in [0, 1]$ and thus $\{f_n\}$ converges pointwise on $[0, 1]$ to $f = 0$.

   Let $x \in [0, 1]$ and $n \in \mathbb{N}$, then

   $$\begin{aligned}
   \lim_{t\to x} \frac{f_n(t) - f_n(x)}{t - x} &= \lim_{t\to x} \frac{\frac{n(t-x)}{(1+n^3x^2)(1+n^3t^2)}}{t - x} \\
   &= \lim_{t\to x} \frac{n}{(1 + n^3x^2)(1 + n^3t^2)} \\
   &= \frac{n}{(1 + n^3x^2)^2} \in \mathbb{R}.
   \end{aligned}$$

Thus $f_n(x)$ is differentiable over $[0,1]$. With this we can solve for the maximum, if it exists, of f_n

$$\begin{aligned} f_n'(x) = 0 &\Leftrightarrow \frac{n(1-n^3x^2)}{(1+n^3x^2)^2} = 0 \\ &\Leftrightarrow 1 - n^3x^2 = 0 \\ &\Leftrightarrow x = \pm\frac{1}{\sqrt{n^3}}. \end{aligned}$$

Thus

$$\sup_{x\in[0,1]} \{f_n(x) - 0\} = f(1/\sqrt{n^3}) = \frac{n(1/\sqrt{n^3})}{1+n^3(1/\sqrt{n^3})^2} = \frac{1}{2\sqrt{n}}.$$

In summary, we have that $\{f_n\}$ converges pointwise to $f = 0$ and that

$$\lim_{n\to\infty} \sup\{f_n(x) - f\} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

By Theorem 7.9 (Rudin), $\{f_n\}$ converges uniformly to $f = 0$. ■

3. Let $f : [a,b] \to \mathbb{R}$ be a function and let $f_n \in C[a,b]$ for all $n \in \mathbb{N}$. Assume that $\{x_n\}$ is a sequence that converges to $x \in [a,b]$.

   (a) Prove that, if $\{f_n\}_n$ converges uniformly to f on $[a,b]$, then the numerical sequence $\{f_n(x_n)\}_n$ converges to $f(x)$.

   Proof. Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly, then there exists N_1 such that for all $n > N_1$, we have
   $$|f_n(y) - f(y)| < \frac{\varepsilon}{2}$$
   for all $y \in [a,b]$. It also follows from the uniform convergence (and that each f_n is continuous on $[a,b]$) that f is continuous on $[a,b]$, by Theorem 7.12 (Rudin). Specifically, f is continuous at x. This means that for some $\delta > 0$ and any $y \in [a,b]$ with $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon/2$. Moreover, since $x_n \to x$, then there exists N_2 such that for all $n > N_2$, we have $|x_n - x| < \delta$. Letting $N = \max\{N_1, N_2\}$, then for any $n > N$ it follows that
   $$|x_n - x| < \delta \Rightarrow |f(x_n) - f(x)| < \frac{\varepsilon}{2}$$
   and since $n > N > N_1$, then
   $$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}.$$
   Finally, from the triangle inequality, we get that
   $$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.$$
   Therefore $\lim_{n\to\infty} f_n(x_n) = f(x)$. □

(b) Is the same conclusion true if $\{f_n\}_n$ converges to f pointwise on $[a,b]$?

Solution. Let $\{f_n\}$ be a sequence of functions defined as

$$f_n(x) = nx^n(1-x)$$

for each $n \in \mathbb{N}$, where $f_n : [0,1] \to \mathbb{R}$. For each n, f_n is continuous. We also have that for all $x \in (0,1)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx^n(1-x) = 0.$$

The last equality holds since if we let $a_n = nx^n$, where $x \in (0,1)$, then

$$\begin{aligned}\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \limsup_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{nx^n}\right| \\ &= \limsup_{n\to\infty} \left(\frac{n+1}{n}\right)|x| \\ &= \limsup_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right)|x| \\ &= |x| < 1.\end{aligned}$$

Thus $\sum_{n=1}^{\infty} a_n$ converges which implies that $a_n = nx^n \to 0$ as $n \to \infty$. Additionally, $f_n(0) = 0 = f_n(1)$. Now define $x_n = 1 - 1/n$. Then $\lim_{n\to\infty} x_n = x = 1$. As stated before $f_n(x) = f_n(1) = 0$. However,

$$\begin{aligned}\lim_{n\to\infty} f_n(x_n) &= \lim_{n\to\infty} n\left(1-\frac{1}{n}\right)^n\left(\frac{1}{n}\right) \\ &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \\ &= \frac{1}{e}.\end{aligned}$$

In summary we have a sequence of continuous functions $\{f_n\}$ which converge pointwise to $f = 0$ and we have a sequence x_n which converges to $x \in [0,1]$. However, $\lim_{n\to\infty} f_n(x_n) \neq f(x)$. ■

4. Let g be a continuous function on $\mathbb{R}$. Compute (with proof) the following limit

$$\lim_{n\to\infty}\int_0^1 \frac{nxg(x)}{1+n^2x}dx.$$

Solution. Let $x \in [0,1]$ be fixed. Then

$$\lim_{n\to\infty}\frac{nx}{1+n^2x} = \lim_{n\to\infty}\frac{x}{\frac{1}{n}+nx} = 0.$$

Thus if

$$f_n(x) = \frac{nx}{1+n^3x^2}$$

then $\{f_n\}$ converges to $f = 0$ pointwise over $[0,1]$. Let $n \in \mathbb{N}$ and $x_1, x_2 \in [0,1]$ with $x_1 < x_2$. Then

$$\begin{aligned} n^3x_1x_2 &= n^3x_1x_2 \\ &\Leftrightarrow nx_1 + n^3x_1x_2 < nx_2 + n^3x_1x_2 \\ &\Leftrightarrow nx_1(1+n^2x_2) < nx_2(1+n^2x_1) \\ &\Leftrightarrow \frac{nx_1}{1+n^2x_1} < \frac{nx_2}{1+n^2x_2}. \end{aligned}$$

This proves that

$$f_n(x) = \frac{nx}{1+n^2x}$$

is strictly increasing over $[0,1]$ for all $n \in \mathbb{N}$. It follows that for $x = 1$, $f_n(x) = \sup_{x\in[0,1]} f_n(x)$ since f_n is strictly increasing and is continuous over a compact interval for all $n \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} \sup\{f_n(x) - f\} = \lim_{n\to\infty} \frac{n}{1+n^2} = 0.$$

Therefore $\{f_n\}$ converges uniformly to $f = 0$. Now we note that since $g(x)$ is continuous over $[0,1] \subset \mathbb{R}$, then g is bounded by some $M \in \mathbb{R}$. Thus

$$\begin{aligned} \left|\lim_{n\to\infty} \int_0^1 \frac{nxg(x)}{1+n^2x} dx\right| &\leq \left|\lim_{n\to\infty} \int_0^1 \frac{nxM}{1+n^2x} dx\right| \\ &= \left|M \int_0^1 \lim_{n\to\infty} \frac{nx}{1+n^2x} dx\right| \\ &= \left|M \int_0^1 0 dx\right| \\ &= 0. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \int_0^1 \frac{nxg(x)}{1+n^2x} dx = 0.$$

∎

5. Let $\{f_n\}_n$ be a sequence of functions with domain $[0,1]$. Assume that there exists $L > 0$ such that

$$|f_n(x) - f_n(y)| \leq L|x-y| \text{ for all } x, y \in [0,1],\ n \in \mathbb{N}.$$

Prove that if $\{f_n\}_n$ converges pointwise to f on $[0,1]$, then $\{f_n\}_n$ converges uniformly to f on $[0,1]$.

Proof. □