Homework 2 - Riemann Integrals

You can use all the results discussed in class. They are collected in the notes posted on Canvas.

Problem 1
Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Determine (with proof) whether $f \in \mathcal{R}[a,b]$ and if so, compute $\int_0^1 f$.

Problem 2 Let f and g be bounded functions on $[a,b]$. Assume that $f \in \mathcal{R}[a,b]$ and $f(x) = g(x)$ for all $x \in [a,b] \setminus F$, where F is a finite subset of $[a,b]$. Prove that

$$g \in \mathcal{R}[a,b] \quad \text{and} \quad \int_a^b f = \int_a^b g \ .$$

Does the same conclusion hold if F is not finite? Give a proof or a counterexample.

Problem 3 Let $f : [a,b] \longrightarrow \mathbb{R}$ be an increasing function. Prove that $f \in \mathcal{R}[a,b]$.

Problem 4 [*Master's Exam, Fall 2017, Spring 2018*]
Let $f \in C[a,b]$ satisfy

$$\int_a^x f = \int_x^b f \quad \text{for all } x \in [a,b].$$

Prove that $f(x) = 0$ for all $x \in [a,b]$.

Problem 5 [*Master's Exam, Spring 2021*]
Let $f, g \in C[a,b]$ with $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b fg = f(c) \int_a^b g \ .$$

Does the same conclusion hold if g is not assumed to be nonnegative? Give a proof or a counterexample.

ADDITIONAL PROBLEMS

Problem A1 [*Master's Exam, Fall 2017, Spring 2019*]
Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where } m, n \in \mathbb{N} \text{ with } gcd(m,n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine (with proof) whether $f \in \mathcal{R}[a,b]$ and if so, compute $\int_0^1 f$.

Problem A2 Let $f : [1, \infty[\to [0, \infty[$ be decreasing. Prove that the sequence

$$\left\{ \sum_{k=1}^{n} f(k) - \int_1^n f \right\}_n$$

is convergent.

Problem A3 Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Prove that, if there exists a sequence of partitions $\{\sigma_n\}_n$ of $[a,b]$ such that $(\bar{S}(f,\sigma_n) - \underline{S}(f,\sigma_n)) \longrightarrow 0$, then $f \in \mathcal{R}[a,b]$ and

$$\lim_{n\to\infty} \bar{S}(f,\sigma_n) = \lim_{n\to\infty} \underline{S}(f,\sigma_n) = \int_a^b f.$$

An equivalent definition of Riemann integral

Let $f : [a,b] \to \mathbb{R}$ be bounded. If $\sigma = \{x_0, x_1, \ldots, x_N\} \in \mathcal{P}[a,b]$ we say that $\tau = \{t_1, t_2, \ldots, t_N\}$ is a selection of points within σ if $x_k \le t_{k+1} \le x_{k+1}$ for all $k \in \{0, 1, \ldots, N-1\}$. We also refer to the couple (σ, τ) as a partition pair. In this case, we define the Riemann sum of f relative to the partition pair (σ, τ) as

$$S(f, \sigma, \tau) = \sum_{k=0}^{N} f(t_{k+1})(x_{k+1} - x_k).$$

Problem A4
Let $f : [a,b] \to \mathbb{R}$ be bounded. Prove that the following are equivalent:

a) $f \in \mathcal{R}[a,b]$

b) There exists $L \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $\sigma_\epsilon \in \mathcal{P}[a,b]$ such that $|S(f,\sigma,\tau) - L| < \epsilon$ for every selection of points τ within σ_ϵ.

Moreover, when *b*) holds, we have $L = \int_a^b f$.

Note: this exercise shows that condition *b*) can be taken as an alternative definition of Riemann integral. Sometimes, this is summarized in calculus books by the informal notation

$$\lim_{N\to\infty} \sum_{k=0}^{N} f(t_{k+1})(x_{k+1} - x_k) = L$$