MATH 230A

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Instructor: Dr. Domokos **Assignment:** Homework 2

2.1 Show that for all $x, y \in \mathbb{R}$ we have

$$\left||x| - |y|\right| \leq |x - y|.$$

Proof. Let $a = x - y$ and $b = y$. Then

$$|a + b| = |x| \leq |a| + |b| = |x - y| + |y| \Leftrightarrow |x| - |y| \leq |x - y|.$$

Then if $a = y - x$ and $b = x$ it follows that

$$|a + b| = |y| \leq |a| + |b| = |y - x| + |x| \Leftrightarrow |y| - |x| \leq |x - y|.$$

And since $\left||x| - |y|\right| = \max\{|x| - |y|, |y| - |x|\}$, and either one is less than or equal to $|x - y|$, then

$$\left||x| - |y|\right| \leq |x - y|.$$

□

2.3 Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for any $x, y \in \mathbb{R}^m$ the following inequality holds

$$\sum_{i=1}^{m} |x_i y_i| \leq \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{m} |y_i|^q\right)^{\frac{1}{q}}.$$

Proof. Let

$$A = \sum_{i=1}^{m} |x_i|^p, \quad B = \sum_{i=1}^{m} |y_i|^q.$$

Then Young's Inequality gives us that

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}$$

for any $x, y \in \mathbb{R}$. Then by letting $x = x_i^p / A$ and $y = y_i^q / B$, then Young's Inequality gives

$$\frac{x_i}{A^p} \frac{y_i}{B^q} \leq \frac{x_i^p}{Ap} + \frac{y_i^q}{Bq}.$$

for all i. Hence

$$\sum_{i=1}^{m} \left| \frac{x_i}{A^{\frac{1}{p}}} \frac{y_i}{B^{\frac{1}{q}}} \right| \leq \left(\sum_{i=1}^{m} \left| \frac{x_i^p}{Ap} \right| + \left| \frac{y_i^q}{Bq} \right| \right) = \left| \frac{\sum_{i=1}^{m} x_i^p}{Ap} \right| + \left| \frac{\sum_{i=1}^{m} y_i^q}{Bq} \right| = \left| \frac{A}{Ap} \right| + \left| \frac{B}{Bq} \right| = 1.$$

Thus

$$\sum_{i=1}^{m} |x_i y_i| \leq A^{\frac{1}{p}} B^{\frac{1}{q}} = \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{m} |y_i|^q\right)^{\frac{1}{q}}.$$

□

2.4 Let $p \geq 1$. Prove that for any $x, y \in \mathbb{R}^m$ the following inequality holds

$$\left(\sum_{i=1}^{m}|x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}}$$

Proof. If $p = 1$ then the inequality holds by the Triangle inequality. Let $p > 1$ and let $q = p/(q-1)$. Then

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Thus

$$\begin{aligned}
\sum_{i=1}^{m}|x_i+y_i|^p &= \sum_{i=1}^{m}|x_i+y_i||x_i+y_i|^{p-1} \\
&= \sum_{i=1}^{m}|x_i||x_i+y_i|^{p-1} + \sum_{i=1}^{m}|y_i||x_i+y_i|^{p-1} \\
&\leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}|x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}|x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}|x_i+y_i|^{p}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}|x_i+y_i|^{p}\right)^{\frac{1}{q}} \\
&= \left[\left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}}\right]\left(\sum_{i=1}^{m}|x_i+y_i|^{p}\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\sum_{i=1}^{m}|x_i+y_i|^p}{\left(\sum_{i=1}^{m}|x_i+y_i|^p\right)^{\frac{1}{q}}} \leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}} \\
\Leftrightarrow &\left(\sum_{i=1}^{m}|x_i+y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m}|y_i|^p\right)^{\frac{1}{p}}
\end{aligned}$$

□

2.6 Let X be a metric space and $A \subseteq B \subseteq X$. Show that $\overline{A} \subseteq \overline{B}$.

Proof. Let $x \in \overline{A}$. Then by Definition 2.11, for all closed sets $F \subseteq X$, if $A \subseteq F$ then $x \in F$. Thus, if B is closed then $x \subseteq B \subseteq \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$. Now assume B is open. Then since $\overline{B}$ is an intersection of closed sets, then by Theorem 2.10, $\overline{B}$ is closed. Moreover since $x \in \overline{A}$, then either x is a limit point of A or an isolated point of A. If x is an isolated point, then $x \in A \subseteq \overline{B}$ (since an isolated point would not be added by taking the closure of A). If x is a limit point of A, then for all $r > 0$, $(B_r(x)\backslash\{x\}) \cap A \neq \varnothing$. However, since $A \subseteq B$, then the previous line implies that for all $r > 0$, $(B_r(x)\backslash\{x\}) \cap B \neq \varnothing$ which implies that x is a limit point of B and thus $x \in \overline{B}$. Hence $\overline{A} \subseteq \overline{B}$. □

2.7 Let X be a metric space and $A_i \subseteq X$, for all $i \in \mathbb{N}$.

(1) Prove that

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

Proof. Let x be an element of the left-hand side. Then x is in the intersection of all closed sets containing $\bigcup_{i=1}^{n} A_n$. By Theorem 2.10, the right hand side is closed and since $A_i \subseteq \overline{A_i}$ for all i, then $\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} \overline{A_i}$. Hence x is an element of the right-hand side. Thus

$$\overline{\bigcup_{i=1}^{n} A_i} \subseteq \bigcup_{i=1}^{n} \overline{A_i}.$$

Now let x be an element of the right-hand side. Then for some $i \in \mathbb{N}$, $x \in \overline{A_i}$. Then x is a member of the intersection of all closed sets containing A_i. Since the left-hand side is closed and the left-hand side contains A_i, then it follows that x is an element of the left-hand side. Hence

$$\bigcup_{i=1}^{n} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{n} A_i}.$$

Therefore

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

$\square$

(2) Is it true that

$$\overline{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i} \quad ?$$

Solution. No. By the countability of $\mathbb{Q}$, let q_i denote the ith rational, for all $i \in \mathbb{N}$. Then define $A_i = \{q_i\}$. Then we can write $\mathbb{Q} = \cup_{i=1}^{\infty} A_i$ and thus

$$\mathbb{R} = \overline{\bigcup_{i=1}^{\infty} A_i}.$$

However, since $A_i = \overline{A_i}$ for all i, then

$$\mathbb{Q} = \bigcup_{i=1}^{\infty} \overline{A_i}.$$

Therefore

$$\overline{\bigcup_{i=1}^{n} A_i} \neq \bigcup_{i=1}^{n} \overline{A_i}.$$

2.8 Let X be a metric space and $A \subseteq X$. Prove that A' is a closed set.

Proof. Let x be a limit point of A'. Then for all $r > 0$, there exists $a \in A'$ such that $a \neq x$ and $a \in B_r(x)$. Select some $0 < r_0 < d(a, x)$. Then $B_{r_0}(a) \subset B_r(x)$. Moreover, since $a \in A'$ then a is a limit point of A which implies that there exists $b \in A$ such that $b \neq a$ and $b \in B_{r_0}(a)$. However, we notice that $b \in B_{r_0}(a) \subset B_r(x)$ and that $b \neq x$ since $r_0 < d(b, x)$. Thus, for all $r > 0$, there exists $b \in A$ such that $b \neq x$ and $b \in B_r(x)$. Therefore x is a limit point of A and so $x \in A'$. Thus, for all limit points, x, of A', $x \in A'$. By Theorem 2.14, A' is closed. $\square$

2.10 Let $\varnothing \neq A \subseteq \mathbb{R}$. Assume that all the points of A are isolated. Show that A is at most countable.

Proof. Let $x \in A$. Then since x is not a limit point of A, there exists $r > 0$ such that $(B_r(x) \backslash \{x\}) \cap A = \varnothing$. By Theorem 1.33, there exists $q \in \mathbb{Q}$ such that $x < q < x + r$. Now define a map $f \colon A \to \mathbb{Q}$ such that $f(x) = q$, where for some $r > 0$ such that $(B_r(x) \backslash \{x\}) \cap A = \varnothing$, we have that $q \in B_r(x)$ and $q \in \mathbb{Q}$. Then we see that if $f(x_1) = f(x_2)$, this implies that $q_1 = q_2$ where $x_1 < q_1 < x_1 + r_1$ and $x_2 < q_2 < x_2 + r_2$, for some $r_1, r_2 > 0$. However this implies that $B_{r_1}(x_1) \cap B_{r_2}(x_2) \neq \varnothing$. By construction it follows that $x_1 = x_2$. Thus f is injective and since $\mathbb{Q}$ is countable, then A is at most countable. $\square$

2.11 Let $A \subseteq \mathbb{R}$ be an uncountable set. Show that A has limit points. Will this result be true if we assume A is countable?

Proof. Let A' denote all the limit points of A and S denote the set of all isolated points of A. Then for any $x \in A$, it follows that x is either a limit point of A or it is not a limit point of A. Thus for any $x \in A$, we have that $x \in A' \cup S$ and so $A \subseteq A' \cup S$. By Exercise 2.10, the set S is at most countable. Since A is uncountable, then A' is uncountable and therefore $A' \neq \varnothing$ and so A has limit points.

If A is countable, then the result does not hold for every case. Consider $\mathbb{Z} \subseteq \mathbb{R}$ which is countable, but every point is an isolated point and thus no point is a limit point. $\square$

2.12 Show that $\overline{\mathbb{I}} = \mathbb{R}$.

Proof. Since both $\mathbb{Q}$ and $\mathbb{I}$ are dense in $\mathbb{R}$, then that for any $x \in \mathbb{R}$, and for any $r > 0$, there exists $s \in \mathbb{R}$ such that $x < s < x + r$. Thus $(B_r(x) \backslash \{x\}) \cap \mathbb{R} \neq \varnothing$. Hence, $\mathbb{R}$ contains all its limit points and as such $\mathbb{R}$ is closed. Additionally, we have that $\mathbb{I} \subseteq \mathbb{R}$. Thus, by Definition 2.11, $\overline{\mathbb{I}} \subseteq \mathbb{R}$. Now if $x \in \mathbb{R}$, then either $x \in \mathbb{Q}$ or $x \in \mathbb{I}$. If $x \in \mathbb{Q}$, then by the density of $\mathbb{I}$ in $\mathbb{R}$, it follows that for all $r > 0$, there exists $i \in \mathbb{I}$ such that $x < i < x + r$. Thus for all $r > 0$, $(B_r(x) \backslash \{x\}) \cap \mathbb{I} \neq \varnothing$. Hence, for any $x \in \mathbb{Q}$, x is a limit point of $\mathbb{I}$ which, by Theorem 2.14, implies that $x \in \overline{\mathbb{I}}$. Finally, if $x \in \mathbb{I}$, then since $\mathbb{I} \subseteq \overline{\mathbb{I}}$, then $x \in \overline{\mathbb{I}}$. Thus $\mathbb{R} \subseteq \overline{\mathbb{I}}$. Therefore $\overline{\mathbb{I}} = \mathbb{R}$. $\square$

2.14 Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Show that

$$\lim_{n\to\infty} x_n = x \Longrightarrow \lim_{n\to\infty} |x_n| = |x|.$$

Is the converse true?

Proof. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \varepsilon$. Thus by Exercise 2.1, for all $n \geq N$

$$\big||x_n| - |x|\big| \leq |x_n - x| < \varepsilon.$$

Therefore, $\lim_{n\to\infty}|x_n| = |x|$. The converse is not true since $\lim_{n\to\infty}\big|(-1)^n\big| = 1$, but the same sequence without the absolute value does not converge at all. $\square$

2.15 Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Show that

$$\lim_{n\to\infty} x_n = 0 \Longleftrightarrow \lim_{n\to\infty} |x_n| = 0.$$

Proof. Assume that $\lim_{n\to\infty} x_n = 0$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|x_n| < \varepsilon$. Thus by Exercise 2.1, for all $n \geq N$, $\big||x_n|\big| < |x_n| < \varepsilon$. Thus $\lim_{n\to\infty}|x_n| = 0$.

Assume that $\lim_{n\to\infty}|x_n| = 0$. Then since $\big||x_n|\big| = |x_n|$, it follows that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $\big||x_n|\big| = |x_n| < \varepsilon$. Therefore $\lim_{n\to\infty}|x|_n = 0$. $\square$

2.16 Let $\{x_n\}$ be a sequence in a metric space X. Prove that

$$\lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} d(x_n, x) = 0.$$

Proof. Assume that $\lim_{n\to\infty} x_n = x$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. Thus $\lim_{n\to\infty} d(x_n, x) = 0$. Now assume that $\lim_{n\to\infty} d(x_n, x) = 0$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) = 0$. Thus, by definition $\lim_{n\to\infty} x_n = x$. $\square$

2.17 Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space X. Show that if $\{x_n\} \to x$ and $\{d(x_n, y_n)\} \to 0$, then $\{y_n\} \to x$.

Proof. Assume that $\{x_n\} \to x$ and that $\{d(x_n, y_n)\} \to 0$. Now let $\varepsilon > 0$ and set $\varepsilon_0 = \varepsilon/2$, then there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n_1 \geq N_1$ and $n_2 \geq N_2$, we have $d(x_{n_1}, x) < \varepsilon_0$ and that $d(x_{n_2}, y_{n_2}) < \varepsilon_0$. Let $N = \max\{N_1, N_2\}$. Then by the triangle inequality, for all $n \geq N$

$$d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) < 2\varepsilon_0 = \varepsilon.$$

Therefore $\{y_n\} \to x$. $\square$

2.18 Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$. Suppose $\{x_n\} \to x \neq 0$ and that $\{x_n \cdot y_n\}$ is also convergent. Prove that $\{y_n\}$ is convergent. What can be said about the case $x = 0$?

Proof. Let $\lim_{n\to\infty} x_n \cdot y_n = L$, for some $L \in \mathbb{R}$. Then since $\{x_n\} \to x \neq 0$, by Theorem 2.24

$$\begin{aligned} &\lim_{n\to\infty} x_n \cdot y_n = L \\ &\Leftrightarrow \lim_{n\to\infty} y_n = \frac{L}{\lim_{n\to\infty} x_n} \\ &\Leftrightarrow \lim_{n\to\infty} y_n = \frac{L}{x}. \end{aligned}$$

Thus $\{y_n\} \to L/x$. In the case that $\{x_n\} \to 0$, $\{y_n\}$ diverges to infinity. □

2.19 Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences of real numbers. Suppose that

$$x_n \leq y_n \leq z_n, \quad \forall n \in \mathbb{N}$$

and there exists $a \in \mathbb{R}$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a.$$

Show that $\lim_{n\to\infty} y_n = a$.

Proof. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have that $|x_n - a| < \varepsilon$ and $|z_n - a| < \varepsilon$. Thus $-\varepsilon < x_n - a < \varepsilon$ and $-\varepsilon < z_n - a < \varepsilon$. Since $x_n - a < y_n - a < z_n - a$, then it follows that $-\varepsilon < y_n - a < \varepsilon$ and thus $|y_n - a| < \varepsilon$. Therefore $|y_n| \to a$. □

2.20 Let $\{x_n\}$ be a sequence in $\mathbb{R}$ with $x_n > 0$ for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L.$$

(a) Show that is $L < 1$, then $\{x_n\} \to 0$.

Proof. Let $\varepsilon > 0$ such that $L + \varepsilon < 1$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon \Leftrightarrow x_{n+1} < (L+\varepsilon)x_n < x_n.$$

Thus $\{x_n\}$ is monotone decreasing for $n \geq N$ and since $x_n > 0$ for all n, then it is bounded below by 0. By Theorem 2.27, $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = l$. Then if $l > 0$, it follows from Theorem 2.24 that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{l}{l} = 1.$$

This contradicts our assumption, thus $l \leq 0$. However, if $l < 0$, then for all $n \geq N$, for some N, we would have that $x_{n+1} < x_n < 0$ which is also a contradiction. Therefore $l = 0$ and thus $\{x_n\} \to 0$. □

(b) Show that if $L > 1$, then $\{x_n\} \to \infty$.

Proof. Let $\varepsilon > 0$ and let $L = 1 + 2\varepsilon$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon \Leftrightarrow (1+\varepsilon)x_n = (L-\varepsilon)x_n < x_{n+1} < (L+\varepsilon)x_n$$

Thus $(1+\varepsilon)^m x_N < x_{N+m}$ for all $m \in \mathbb{N}$. This implies $\{x_n\}$ is monotonically increasing and unbounded. Therefore $\{x_n\} \to \infty$. □

(c) What can we say if $L = 1$?

Solution. Inconclusive.

2.22 Let $k \in \mathbb{N}$ and the sequence $\{x_n\}$ defined by:

$$x_1 = k, \quad x_{k+1} = \frac{x_n}{2} + \frac{k}{2x_n}, \quad \forall n \in \mathbb{N}.$$

Prove $\{x_n\}$ is convergent and find its limit.

Proof. We begin by noting that the recursive definition given allows us to write

$$x_{k+1} = \frac{x_n}{2} + \frac{k}{2x_n} \Leftrightarrow x_n^2 - 2x_{n+1}x_n + k = 0.$$

Thus x_n is a real root of the above quadratic equation. From this and the quadratic formula it follows that $4x_{n+1}^2 - 4k \geq 0$ and thus $x_{n+1}^2 \geq k$ for all $n \geq 1$.

Now suppose that $n \geq 2$. Then the difference

$$x_n - x_{n+1} = x_n - \left(\frac{x_n}{2} + \frac{k}{2x_n}\right) = \frac{x_n}{2} - \frac{k}{2x_n} = \frac{1}{2}\left(\frac{x_n^2 - k}{x_n}\right).$$

Since $x_{n+1}^2 \geq k$ for all $n \geq 1$, then $x_n^2 \geq k$ for all $n \geq 2$. Thus

$$x_n - x_{n+1} = \frac{1}{2}\left(\frac{x_n^2 - k}{x_n}\right) \geq 0 \Leftrightarrow x_n \geq x_{n+1}$$

for all $n \geq 2$. Hence $\{x_n\}_{n=2}^{\infty}$ is a monotonically decreasing sequence. Furthermore, since $x_n^2 \geq k \geq 0$ for all $n \geq 2$, then $\{x_n\}$ is bounded below by 0. Thus by Theorem 2.27, $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = a$. Then it follows that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{k}{2x_n}\right) \Leftrightarrow a = \frac{a}{2} + \frac{k}{2a} \Leftrightarrow a^2 = k.$$

Hence $a = \pm\sqrt{k}$. However, since $x_n \geq 0$ for all $n \geq 2$, then $a \geq 0$ and thus $\{x_n\} \to \sqrt{k}$. □

2.24 Let the sequence $\{x_n\}$ be defined by:

$$x_1 = 2, \quad x_{n+1} = \sqrt{2 + \sqrt{x_n}}, \quad \forall n \in \mathbb{N}.$$

Prove that $\{x_n\}$ is convergent. What can we say about its limit.

Proof. We begin by first considering the sequence $y_{n+1} = \sqrt{y_n}$. We wish to show that this sequence is monotonic. For $y_1 = \sqrt{2}$, we have that $y_1 = \sqrt{2} < \sqrt{\sqrt{2}} = y_2$. Let this be the base case for an inductive proof that $y_{n+1} < y_n$ for all n. Then assume that for some $k > 1$, $y_{k+1} < y_k$. Then taking the square root of both sides we get that $\sqrt{y_{k+1}} = y_{k+2} < y_{k+1} = \sqrt{y_k}$. Therefore the sequence is monotonically decreasing.

Now consider the given sequence. We have that $x_1 = 2$ and $x_2 = \sqrt{2+\sqrt{2}}$ and thus $x_1 > x_2$. We claim that $\{x_n\}$ is monotone and decreasing. Having just shown the base case, assume that for $k > 1$, that $x_{k+1} < x_k$. Then

$$\begin{aligned} x_{k+1} = \sqrt{2 + \sqrt{\sqrt{2 + \sqrt{x_{k-1}}}}} &< \sqrt{2 + \sqrt{x_{k-1}}} = x_k \\ \Leftrightarrow \sqrt{x_{k+1}} = \sqrt{\sqrt{2 + \sqrt{\sqrt{2 + \sqrt{x_{n-1}}}}}} &< \sqrt{\sqrt{2 + \sqrt{x_{k-1}}}} = \sqrt{x_k} \\ \Leftrightarrow 2 + \sqrt{x_{k+1}} &< 2 + \sqrt{x_k} \\ \Leftrightarrow x_{k+2} = \sqrt{2 + \sqrt{x_{k+1}}} &< \sqrt{2 + \sqrt{x_k}} = x_{k+1} \end{aligned}$$

Therefore the result holds for all $n \in \mathbb{N}$ and the sequence is monotone and decreasing. Additionally, the sequence is bounded below by 1 and thus $\{x_n\}$ is convergent. □

2.26 Let

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \quad \forall n \in \mathbb{N}.$$

Show that $\{x_n\}$ is convergent. What can we say about its limit?

Proof. We begin by showing the sequence is monotone and increasing. Let $n \geq 2$. Then

$$\begin{aligned} x_n - x_{n-1} &= \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad - \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-2}\right) \\ &= \frac{1}{2n} + \frac{1}{2n-1} - \frac{1}{n} \\ &= \frac{4n-1}{n(4n-2)} - \frac{1}{n}. \end{aligned}$$

Moreover, since
$$\frac{4n-1}{4n-2} > 1 \Leftrightarrow \frac{4n-1}{n(4n-2)} > \frac{1}{n}$$
then $x_n - x_{n-1} > 0$ and thus $x_n > x_{n-1}$. Hence, the sequence is monotone and increasing. Next, we claim it is bounded above by 1. We proceed by induction on n and prove that $x_n \leq 1$ for all $n \in \mathbb{N}$. For the base case we let $n = 1$ and obtain $x_1 = \frac{1}{2} \leq \frac{1}{2}$. Now assume that for some $k > 1$ that
$$x_k = \frac{1}{k+1} + \cdots + \frac{1}{2k} \leq \frac{1}{2}.$$
Note that
$$x_{k+1} = x_k - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} = x_k - \frac{1}{k+1} + \frac{4k+3}{(2k+1)(2k+2)}.$$
Furthermore, we have that
$$\frac{4k+3}{(2k+1)(2k+2)} - \frac{1}{k+1} = \frac{2k+1}{2(2k+1)(k+1)} = \frac{1}{2(k+1)} \leq \frac{1}{2}.$$
So since $x_k \leq \frac{1}{2}$ and $\frac{1}{2(k+1)} \leq \frac{1}{2}$, then
$$x_{k+1} = x_k + \frac{1}{2(k+1)} \leq \frac{1}{2} + \frac{1}{2} = 1.$$
Therefore this sequence is monotonically increasing and bounded above by 2 and so by Theorem 2.27, it is convergent. $\square$