Master's Exam in Real Analysis
December 2017

Part 1: Problems 1-7 **Do six problems in Part 1**.

1. (a) Let A be the collection of all sequences of 0s and 1s. Prove that A is uncountable.
   (b) Let B be the set that consists of all sequences of 0s and 1s for which the number of 1s is finite. Determine whether B is countable or uncountable. Give a proof of your assertion.

2. Prove that the Cantor set is perfect.

3. Prove that any connected metric space with at least two distinct points is uncountable.

4. (a) Let X and Y be metric spaces and $f : X \to Y$ be a continuous mapping. Prove that if $K \subset X$ is compact, then $f(K)$ is compact.
   (b) Prove that the intersection of an arbitrary collection of compact sets of metric space X is compact.

5. (a) Let $\{x_n\}, \{y_n\}$ be sequences of real numbers. Prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

   provided the sum on the right is not of the form $\infty - \infty$.
   (b) Prove that if $x_n \to x$ as $n \to \infty$, then

$$\limsup(x_n + y_n) = \limsup x_n + \limsup y_n$$

6. Let $\{x_n\}$ be defined inductively by $x_1 = 1$, $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \geq 1$. Prove that $\lim x_n = \frac{3}{2}$.

7. (a) Assume that $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges. Show that

$$\lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{n} = 0.$$

   (b) Prove that the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ implies the convergence of $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ where $a_n \geq 0$ for all $n \in \mathbb{N}$.

Part 2: Problems 8-14 **Do six problems in Part 2**

8. Let $f : (0,1] \to \mathbb{R}$ be differentiable with $0 < f'(x) < 1, \forall x \in (0,1]$. Define a sequence $\{a_n\}$:
$$a_n = f\Big(\frac{1}{n}\Big).$$
Prove that $\lim_{n\to\infty} a_n$ exists.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:
$$f(x) = \begin{cases} x^5 \sin\Big(\frac{1}{x^3}\Big) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Study the continuity and differentiability of f on $\mathbb{R}$. How many times is f differentiable?

10. Let f be a bounded function on $[a,b]$ and let
$$\alpha(x) = \begin{cases} 0, & \text{if } a \le x < c \\ 2, & \text{if } c \le x \le b. \end{cases}$$
Prove that $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if $f(c-) = f(c)$. Compute the integral $\displaystyle\int_a^b f\,\mathrm{d}\alpha$ when $f \in \mathcal{R}(\alpha)$.

11. (a) Let f be continuous on $[a,b]$ such that
$$\int_a^x f(t)\mathrm{d}t = \int_x^b f(t)\mathrm{d}t, \forall x \in [a,b].$$
Prove that $f(x) = 0$ on $[a,b]$.

    (b) Let $f : [0,1] \to \mathbb{R}$ be defined as follows:
$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q}, x \neq 0, , p, q \in \mathbb{N} \text{ with no common factor} \\ 0, & \text{otherwise.} \end{cases}$$
for $0 \le x \le 1$. Determine whether f is Riemann integrable. If it is, what is $\displaystyle\int_0^1 f dx$? Give your reason.

12. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that
$$\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0).$$

13. Suppose that $f_n(x)$ is differentiable on $[a,b]$ for $n \ge 1$ with $|f_n'(x)| \le M$, $x \in [a.b]$ and some $M > 0$. If $f_n(x)$ converges to $f(x)$ pointwise on $[a,b]$, prove that $f_n(x)$ converges to $f(x)$ uniformly on $[a,b]$.

14. Let
$$f_n(x) = \frac{nx}{1 + n^3x^2}, \ x \in [0,1], n \in \mathbb{N}.$$
Investigate the pointwise and uniform convergence of $\{f_n\}$ and $\displaystyle\sum_{n=1}^{\infty} f_n$ on $[0,1]$