# Master's Exam in Real Analysis
December 2018

## Part 1: Problems 1-7 **Do six problems in Part 1**.

1. (a) Let $\mathbb{N}$ be the set of natural numbers. Prove that the power set $\mathcal{P}(\mathbb{N})$ is uncountable.

   (b) Prove that the collection of isolated points of any set $S \subseteq \mathbb{R}^n$ is countable.

2. Prove that the Cantor set is perfect.

3. Prove that compact subsets of metric spaces are closed and bounded. Is the converse true? Justify your answer.

4. (a) Prove that any connected subset $E$ of $\mathbb{R}$ is an interval, i.e., if $x, y \in E$ and $x < y$, then $x < z < y$ implies $z \in E$.

   (b) Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be continuous. Assume that $X$ is connected, prove that $f(X)$ is connected.

5. (a) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

   provided the sum on the right is not of the form $\infty - \infty$.

   (b) Prove that if $x_n \to x$ as $n \to \infty$, then

$$\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

6. (a) If $s_n \to s$, where $s_n, s \in \mathbb{R}$ for $n \in \mathbb{N}$, then show that $\dfrac{s_1 + s_2 + \ldots + s_n}{n} \to s$. Does the converse hold true?

   (b) Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Suppose that there is an $r > 1$ such that

$$|x_{n+1} - x_n| < r^{-n}, \text{ for } n = 1, 2, 3, \ldots, .$$

   Prove that $\{x_n\}$ converges.

7. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

   (a) Assume that $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Is the converse true? Justify your answer.

   (b) Prove that if $a_n \ge 0$ for all $n \in \mathbb{N}$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges.

## Part 2: Problems 8-14     **Do six problems in Part 2**

8. Let $f : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$ with $|f'(x)| \leq M$ for $x \in (a, b)$ and some $M \geq 0$. Prove that
$$\lim_{x \to b^-} f(x)$$
exists.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:
$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Study the continuity and differentiability of $f$ on $\mathbb{R}$. How many times is $f$ differentiable?

10. Let $f$ be a bounded function on $[-1, 1]$ and let
$$\alpha(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 2, & \text{if } 0 < x. \end{cases}$$
Prove that $f \in \mathcal{R}(\alpha)$ on $[-1, 1]$ if and only if $f(0+) = f(0)$. Compute the integral $\displaystyle\int_{-1}^{1} f \, \mathrm{d}\alpha$ when $f \in \mathcal{R}(\alpha)$.

11. Suppose that $f$ is bounded on $[a, b]$, $f$ has only finitely many points of discontinuity on $[a, b]$, and $\alpha$ is continuous at every point at which $f$ is discontinuous. Prove that $f$ is Riemann-Stieltjes integrable on $[a, b]$.

12. (a) Suppose that $f(x)$ is a continuous, nonnegative function on the interval $[0, 1]$. Let $M = \sup\{f(x) : x \in [0, 1]\}$. Prove that
$$\lim_{n \to \infty} \left[ \int_0^1 f(t)^n dt \right]^{\frac{1}{n}} = M.$$
(b) Let $f(x)$ and $g(x)$ be continuous on $[a, b]$ with $\displaystyle\int_a^b f(x) \, \mathrm{d}x = \int_a^b g(x) \, \mathrm{d}x$. Prove that there is a $c$ in $[a, b]$ with $f(c) = g(c)$.

13. Suppose that $f_n(x)$ is differentiable on $[a, b]$ for $n \geq 1$ with $|f_n'(x)| \leq M$, for $x \in [a, b]$ and some $M > 0$. If $\{f_n(x)\}$ converges to $f(x)$ pointwise on $[a, b]$, prove that $\{f_n(x)\}$ converges to $f(x)$ uniformly on $[a, b]$.

14. For what values of $x \geq 0$ does the series
$$\sum_{k=1}^{\infty} \frac{1}{k + k^2 x}$$
converge? Is the convergence uniform on the set where the series converges? Justify your answer.