MATH 220A

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Instructor: Dr.Martins — Assignment: Homework 2

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational }\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Proof. Let $\mathcal{T}_1$ be the topology generated by $\mathcal{B}$ and let $\mathcal{T}_2$ be the standard topology. Then we must show that for any $U \in \mathcal{T}_2$ and $x \in U$, there exists $B \in \mathcal{T}_2$ such that $x \in B \subset U$.

Let $U \in \mathcal{T}_2$ and $x \in U$. Then by Lemma 13.1, U is the union of some collection of open intervals in $\mathbb{R}$. With $x \in U$, then we have that for some $a, b \in \mathbb{R}$, $x \in (a, b)$. If $a, b \in \mathbb{Q}$, then we are done since in this case, $(a, b) \in \mathcal{T}_2$ and $x \in (a, b) \subset U$. If a, b are not rational, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that there exists $a < a_2 < x < b_2 < b$, where $a_2, b_2 \in \mathbb{Q}$. Thus $(a_2, b_2) \in \mathcal{T}_2$ and $x \in (a_2, b_2) \subset U$. Therefore, by Lemma 13.2, $\mathcal{T}_1$ is a basis for $\mathcal{T}_2$. □

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational }\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Proof. We want to show that $\mathcal{C} \neq \mathbb{R}_l$, where $\mathbb{R}_l$ is the lower limit topology on $\mathbb{R}$. To prove this we select the set $[\sqrt{2}, 3) \in \mathbb{R}_l$ and let $x = \sqrt{2}$. From here we appeal to the negation of Lemma 13.2. Namely, to show that there does not exist a $B \in \mathcal{C}$ such that $x \in B \subset [\sqrt{2}, 3)$. This follows from $x = \sqrt{2}$, and so for all $[a, b) \subset [\sqrt{2}, 3)$ such that $x \in [a, b)$ it implies $a = \sqrt{2}$ and thus $[a, b) \not\subset \mathcal{C}$. □

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned}
A &= \{x \mid \tfrac{1}{2} < |x| < 1\},\\
B &= \{x \mid \tfrac{1}{2} < |x| \leq 1\},\\
C &= \{x \mid \tfrac{1}{2} \leq |x| < 1\},\\
D &= \{x \mid \tfrac{1}{2} \leq |x| \leq 1\},\\
E &= \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}_+ \}.
\end{aligned}$$

Proof.

(a) Considering the standard topology on $\mathbb{R}$ and letting a basis for this topology be denoted by $\mathcal{B}$, then by Lemma 16.1, $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a topology on Y. As such, we have that $A = (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 1)$ is open in $\mathbb{R}$ as it is the union of two open sets in X, and $A \subset Y$. Therefore, A is open in Y and open in $\mathbb{R}$.

(b) Considering the standard topology on $\mathbb{R}$, we have that $B = [-1, -\frac{1}{2}) \cup (-\frac{1}{2}, 1]$, which is not open in $\mathbb{R}$. However, we can rewrite

$$B = Y \cap \Big((-\frac{3}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{2}) \Big)$$

which is the union of two basis elements of the standard topology intersected with Y, and thus B is in the topology on Y. Hence, B is open in Y.

(c) We can rewrite

$$C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$$

which is not open in $\mathbb{R}$. Moreover, we cannot do as we did above and express C as the intersection of open sets of $\mathbb{R}$ with Y since for all open sets $U \in \mathbb{R}$ such that $C \subset U$, we have $U \cap Y \neq C$. To see why, suppose there is such an open set $U \in \mathbb{R}$. Then since $\frac{1}{2} \in C \subset U$, it follows that there exists an open set $U' \subset U$ such that $\frac{1}{2} \in U'$. However, this implies that there exists an open set $U'' \subset C$ such that $\frac{1}{2} \in U''$, which is not possible.

(d) By the same argument above, we can conclude that D is neither open in $\mathbb{R}$, nor is D open in Y.

(e) To show that E is open in Y, we will show that E is open in $\mathbb{R}$ and appeal to Lemma 16.1. We need to show that E is an element of the topology on $\mathbb{R}$. Thus, we need to show that E is either a basis element or the union of basis elements.

Let $x \in E$. Then if $0 < x$, by the Archmedian principle, there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} < x < \frac{1}{n}$. Thus, for some $n \in \mathbb{N}$, we have that $x \in \Big(\frac{1}{n+1}, \frac{1}{n}\Big)$. If $x < 0$, then immediately we see that $\frac{1}{x} \notin \mathbb{Z}_+$ and that $x \in (-1, 0)$, which is open in $\mathbb{R}$. Therefore,

$$E \subseteq (-1, 0) \cup \bigcup_{n \in \mathbb{N}} \Big(\frac{1}{n+1}, \frac{1}{n}\Big)$$

which is the union of basis elements of the standard topology on $\mathbb{R}$. If x is an element of the set on the right, then either $x \in (-1, 0)$ which implies $x \in E$, or for some $n \in \mathbb{N}$, $x \in (\frac{1}{n+1}, \frac{1}{n})$ which implies that $1/x \notin \mathbb{Z}_+$. In either case, we have that $0 < |x| < 1$. Therefore, $x \in E$ and

$$E = (-1, 0) \cup \bigcup_{n \in \mathbb{N}} \Big(\frac{1}{n+1}, \frac{1}{n}\Big).$$

Thus E is open in $\mathbb{R}$ and thus open in Y.

□

1. Let $[a,b] \subset \mathbb{R}$ be a closed interval and consider the following set of functions:

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f(x) \text{ is continuous on } [a,b]\ \}.$$

Using the properties of the Riemann integral, show that d_{L^1} is a metric on the space $C([a,b])$.

Proof. Let $f, g \in C([a,b])$. Then we want to show that $\|f(x) - g(x)\| \geq 0$. If $f(x) \neq g(x)$, then define $h(x) = f(x) - g(x)$. By properties of continuous functions, $h(x) \in C([a,b])$. Furthermore, we have that

$$-|h(x)| \leq h(x) \leq |h(x)|$$

and since $h(x)$ is continuous $[a,b]$, then $|h(x)|$ is continuous on $[a,b]$ and thus Riemann integrable on $[a,b]$. Thus taking the integral of the above inequality, we get that

$$0 \leq \left| \int_a^b h(x)dx \right| \leq \int_a^b |h(x)|dx.$$

And if $f(x) = g(x)$, then $f(x) - g(x) = 0$ which implies that $\|f(x) - g(x)\|_{L^1} = 0$. Therefore, $\|f(x) - g(x)\| \geq 0$, with equality when $f(x) = g(x)$.

Next we let $f(x), g(x) \in C([a,b])$. We need to show that $\|f(x) - g(x)\| = \|g(x) - f(x)\|$. This property follows immediately from the commutativity of the real numbers. Namely, that $f(x) - g(x) = g(x) - f(x)$ for all $x \in [a,b]$.

Let $f(x), g(x), h(x) \in C([a,b])$. We need to show that

$$\|f(x) - h(x)\| \leq \|f(x) - g(x)\| + \|g(x) - h(x)\|.$$

By properties of continuous functions, $f(x) - h(x)$, $f(x) - g(x)$, and $g(x) - h(x)$ are all continuous on $[a,b]$. Moreover, we have that

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|.$$

Lastly, we have that $|f(x) - h(x)|$, $|f(x) - g(x)|$, and $|g(x) - h(x)|$ are all continuous on $[a,b]$ and thus Riemann integrable on $[a,b]$. Thus taking the integral of the above inequality we get that

$$\int_a^b |f(x) - h(x)|dx \leq \int_a^b |f(x) - g(x)|dx + \int_a^b |g(x) - h(x)|dx$$

as desired. $\square$