# STAT 215A Problem Set 2, Spring 2022

**For the problems 1-3, consider the following scenario (**adopted from Example 5 on page 10 of the text by Grimmett and Stirzaker): We are given two urns, each containing a collection of colored balls. Urn I contains two white and three blue balls, while urn II contains three white and four blue balls.

1. Select one of the urns randomly and then draw a ball from that urn selected.
   (a) Describe a sample space $\Omega$ for this experiment. Is it an equiprobable space?
   (b) What is the probability that the ball picked is blue?
   (c) Given that the ball picked is blue, what is the probability that it comes from urn I?

2. Draw a ball at random from urn I, put it into urn II and then pick a ball from urn II.
   (a) What is the probability that both balls picked are blue?
   (b) What is the probability that the ball picked is blue?
   (c) Given that the ball picked from urn II is blue, what is the probability that the first ball drawn was also blue?

3. Draw a ball at random from urn I, and then pick a ball from urn II. What is the probability that
   (a) the first ball is blue?
   (b) the second ball is blue?
   (c) both balls are blue?
   (d) at least one ball is blue?

4. Let A and B two events in a probability space such that $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$.
   (a) Show that $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$ and $\frac{3}{4} \leq \mathbb{P}(A \cup B) \leq 1$.
   (b) Give some numerical examples where the bounds in part (a) are attained tightly.

5. Let $\{A_k : k \in \mathbb{N}\}$ be a sequence of *certain* events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in other words, $\mathbb{P}(A_k) = 1$, for every $k \in \mathbb{N}$. Prove that $\mathbb{P}(\bigcap\limits_{k=1}^{\infty} A_k) = 1$.

6. Let $A$ and $B$ be two events in a probability space such that $\mathbb{P}(A)P(B) > 0$. Prove that
   (a) $\mathbb{P}(A|B) = \mathbb{P}(B|A)$ if and only if $\mathbb{P}(A) = \mathbb{P}(B)$.
   (b) If $\mathbb{P}(A|B) > \mathbb{P}(A)$, then $\mathbb{P}(B|A) > \mathbb{P}(B)$.
   (c) If $A \cap B$ is a null event, then $A$ and $B$ are dependent events. Note that neither $A$ nor $B$ is null since $\mathbb{P}(A)P(B) > 0$.
   (d) If $A$ and $B$ are independent events, then the events $A^C$ and $B^C$ are also independent.

7. Give an example of a collection of events $\{E_1, E_2, ..., E_n\}$ such that the events are pairwise independent but not (mutually) indepedent.

8. Give an example of a collection of events $\{E_1, E_2, E_3\}$ such that the events satisfy the multiplication rule
$$\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(E_1)\mathbb{P}(E_2)\mathbb{P}(E_3),$$
but they are not pairwise independent.

9. A potentially biased coin is tossed repeatedly. Each time, there is a fixed probability $p$ of getting a head independently of the other tosses, where $0 < p < 1$.
   (a) Let $A$ be the event that a head occurs eventually. Determine $\mathbb{P}(A)$.
   **Hint**. It may be useful to write A as a union of a disjoint sequence of events $\{A_n\}$ where $A_n$ is the event that the first head occurs at the $k^{th}$ toss.

(b) For $n \geq 0$, let $p_n$ be the probability that an even number of heads would occur after $n$ tosses. Show that $p_0 = 1$ and $p_n$ satisfies the difference equation

$$p_n = p(1 - p_{n-1}) + (1 - p)p_{n-1},$$

for $n \geq 1$.

(c) Solve the difference equation above.

**Hint**: Consider two cases: $p = 0.5$, and $p \neq 0.5$. Simplify the equation above for each case. For the non-symmetric case ($p \neq 0.5$), try a solution of the form $p_n = \alpha + \beta\gamma^n$ and solve for the parameters $\alpha, \beta$ and $\gamma$.

Some other suggested problems can be attempted from the textbook by Grimmett and Stirzaker including the exercises 1.5.7, 1.5.9, 1.8.7, 1.8.18 and 1.8.30. You may also find it useful to go over the examples of the section 1.7. In particular, can you provide an easier solution to the second part of example 1 (probability that A occurs at least one in 7 deals)?