STAT 215A

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Instructor: Dr. Cetin — Assignment: Quiz 02

1. Let $X_1, X_2, \ldots, X_{10}$ be 10 independent random variables on a probability space $(\Omega, \mathcal{F}, P)$, each with a continuous unifrom distribution on the interval $(0, a)$ where $a > 0$.

   (a) Let $Y = \max\{X_i : i = 1, 2, \ldots, 10\}$. More explicitly, $Y(\omega) = \max\{X_i(\omega) : i = 1, 2, \ldots, 10\}$ for $\omega \in \Omega$. Show that Y is a random variable on $(\Omega, \mathcal{F}, P)$. In other words, prove that $\{\omega \in \Omega : Y(\omega) \leq y\} \in \mathcal{F}$ for all $y \in \mathbb{R}$.

   Proof. We begin by noting that if $x \in \mathbb{R}$, then $\{\omega \in \Omega : X(\omega) \leq x\} = X^{-1}((-\infty, x]) \in \mathcal{F}$ since X is a random variable and thus a measurable function. Taking that same $x \in \mathbb{R}$, we have that

$$\begin{aligned}
\{\omega \in \Omega : Y(\omega) \leq x\} &= \{\omega \in \Omega : \max\{X_1(\omega), \ldots, X_{10}(\omega)\} \leq x\} \\
&= \{\omega \in \Omega : (X_1(\omega) \leq x) \wedge \cdots \wedge (X_{10}(\omega) \leq x)\} \\
&= \{\omega \in \Omega : X_1(\omega) \leq x\} \cap \cdots \cap \{\omega \in \Omega : X_{10}(\omega) \leq x\} \\
&\in \mathcal{F},
\end{aligned}$$

   as the intersection of a sequence of elements of $\mathcal{F}$, and this follows from the properties of σ-algebras in which $\mathcal{F}$ is one on Ω. Therefore, for all $x \in \mathbb{R}$, $Y^{-1}((-\infty, x]) = \{\omega \in \Omega : Y(\omega) \leq x\} \in \mathcal{F}$ and so Y is an $\mathcal{F}$-measurable function and thus Y is a random variable on $(\Omega, \mathcal{F})$. $\square$

   (b) In fact, Y is a continuous random variable. Justify this by writing the cdf of Y, $F_Y(y)$, as an integral of an explicit nonnegative function $f_Y(\cdot)$ such that $F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt$, for all $y \in \mathbb{R}$. Recall that $f_Y(\cdot)$ is a pdf for Y.

   Solution. Since for each $i \in \{1, \ldots, 10\}$, $X_i \sim \textit{Unif}\,(0, a)$, then the pdf of X_i is

$$f_{X_i}(x) = \begin{cases} \frac{1}{a} & \text{if } 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

   for all $x \in \mathbb{R}$ and for all $i \in \{1, \ldots, 10\}$. And the cdf of X_i is

$$F_{X_i}(x) = \int_{-\infty}^{x} f_{X_i}(t)dt = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{x}{a} & \text{if } 0 < x < a \\ 1 & \text{if } a \leq x \end{cases}.$$

With this, we have that

$$\begin{aligned}
F_Y(x) &= P(\{\omega \in \Omega : Y(\omega) \leq x\}) && (1)\\
&= P(\{\omega \in \Omega : \max\{X_i(\omega), \dots, X_{10}(\omega)\} \leq x\}) && (2)\\
&= P(\{\omega \in \Omega : (X_1(\omega) \leq x) \wedge \cdots \wedge (X_{10}(\omega) \leq x)\}) && (3)\\
&= P(\{\omega \in \Omega : X_1(\omega) \leq x\} \cap \cdots \cap \{\omega \in \Omega : X_{10}(\omega) \leq x\}) && (4)\\
&= P(\{\omega \in \Omega : X_1(\omega) \leq x\}) \cdots P(\{\omega \in \Omega : X_{10}(\omega) \leq x\}) && (5)\\
&= \prod_{i=1}^{10} F_{X_i}(x) && (6)\\
&= (F_{X_1}(x))^{10} && (7)
\end{aligned}$$

Note that (5) holds since the X_i's are independent. Additionally, the last equality holds since the cdf's of each X_i are all equal so we arbitrarily selected $F_{X_1}(x)$. Finally, we can obtain $f_Y(x)$ by differentiating and getting

$$f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{d}{dx}(F_{X_1}(x))^{10} = 10(F_{X_1}(x))^9 F'_{X_1}(x) = 10(F_{X_1}(x))^9 f_{X_1}(x).$$

Therefore

$$F_Y(x) = 10 \int_{-\infty}^{x} (F_{X_1}(t))^9 f_{X_1}(t) dt.$$

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(c) Compute $P(Y > 2a/3)$.

Solution. We begin by noting

$$P(Y > 2a/3) = 1 - P(Y \leq 2a/3) = F_Y(2a/3).$$

Then using (7) from above we have that

$$F_Y(2a/3) = (F_X(2a/3))^{10}.$$

Since $0 < \frac{2a}{3} < a$, then $F_X(2a/3) = 2/3$. Thus

$$P(Y > 2a/3) = 1 - \left(\frac{2}{3}\right)^{10} \approx 0.983.$$

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2. An urn contains 5 balls numbered 1, 2, 3, 4, and 5. We randomly select two ballss, one after another and without replacement, and record the number on each ball. Let X_1 denote the number on the first ball selected, and X_2 denote the number on the second ball selected. So, clearly X_1 and X_2 are two discrete random variables on $(\Omega, \mathcal{F})$ where $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{F}$ is the power set of Ω. Moreover, define $Z = \min\{X_1, X_2\}$. One can check that Z is also a discrete random variable on $(\Omega, \mathcal{F})$.

(a) Determine the pmf of Z.

Solution. To begin, we first note that there are 5 ways to select the first ball, and 4 ways to select the second ball. This gives us a sample space $|\Omega| = 20$. For any $x \in \{1, 2, 3, 4, 5\}$, there are two ways in which the $Z = x$. However, if $x = 5$, then there is no outcome in which $Z = x$ and so $p_Z(Z \geq 5) = 0$. Thus for $x \in \{1, 2, 3, 4\}$, then either $X_1 = x < X_2$ or $X_2 = x < X_1$. With the two options mentioned, there remains $5 - x$ many numbers that are bigger than x in the set $\{1, 2, 3, 4, 5\}$. Thus

$$p_Z(x) = \begin{cases} \frac{2(5-x)}{20} & \text{if } x \in \{1, 2, 3, 4\} \\ 0 & \text{otherwise.} \end{cases}$$

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(b) Compute $E[Z]$ and $E[Z^2]$.

Solution. To compute the expexted value of Z, we have

$$E[Z] = \sum_{x=1}^{5} x p_Z(x) = \frac{8}{20} + \frac{12}{20} + \frac{12}{20} + \frac{8}{20} = 2.$$

Letting $g(x) = x^2$, then using LOTUS, we can compute

$$E[Z^2] = \sum_{x=1}^{5} g(x) p_Z(x) = \frac{100}{20} = 50.$$

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