MATH 230B

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Assignment: Homework 02

1. Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Determine (with proof) whether $f \in \mathcal{R}[a, b]$ if so, compute $\int_0^1 f$.

Solution. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, \dots, x_n\}$ such that the mesh $||P|| < \varepsilon/n$. Since the irriationals are dense in $\mathbb{R}$, then for any Δx_i, there exists an irrational number r such that $x_{i-1} < r < x_i$ and $f(r) = 0$. This implies that $m_i = 0$ for all $i \in \{1, \dots, n\}$ and so $L(f, P) = 0$. Hence

$$\begin{aligned} U(f, P) - L(f, P) &= U(f, P) && (1) \\ &= \sum_{i=1}^{n} M_i \Delta x_i && (2) \\ &\leq \sum_{i=1}^{n} \Delta x_i && (3) \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{n} && (4) \\ &= \varepsilon. && (5) \end{aligned}$$

Note that the inequality from (2) to (3) holds since (3) represents the case where $M_i = 1$ for all i, which in general is not true (consider $x_{i-1} = 3/4$ and $x_i = 5/6$, there is no $n \in \mathbb{N}$ such that $3/4 \leq 1/n \leq 5/6$ and thus $M_i = 0$). Therefore by Theorem 6.6, $f \in \mathcal{R}[0, 1]$.

To compute $\int_0^1 f$, we first note that the integrability of f implies that

$$\sup\{L(f, P) |\ P \in \mathcal{P}([0, 1]) = \int_0^1 f = \inf\{U(f, P) |\ P \in \mathcal{P}([0, 1])\}.$$

So computing $\sup\{L(f, P) |\ P \in \mathcal{P}([0, 1])\}$ is sufficient. As was noted earlier, for any partition P, $L(f, P) = 0$ since the irrationals are dense in $\mathbb{R}$. This means the set has 0 as its only element. Thus the set is closed and bounded and so it attains its sup which is 0. Therefore

$$\int_0^1 f = 0.$$

∎

2. Let f and g be boundeds function on $[a, b]$. Assume that $f \in \mathcal{R}[a, b]$ and $f(x) = g(x)$ for all $x \in [a, b] \backslash F$, where F is a finite subset of $[a, b]$. Prove that

$$g \in \mathcal{R}[a, b] \quad \text{and} \quad \int_a^b f = \int_a^b g.$$

Does the same conclusion hold if F is not finite? Give a proof or counterexample.

Proof. Assume that $F = \{s\}$ contains only one element. Let $\varepsilon > 0$, then since $f \in \mathcal{R}[a, b]$, there exists a partition $P = \{x_0, \dots, x_n\}$ such that

$$U(f, P) - L(f, P) < \frac{\varepsilon}{2}.$$

Moreover, because $F \subset [a, b]$, then for some $i \in \{1, \dots, n\}$, we have that $x_{i-1} \leq s \leq x_i$. Now select $p, r \in [a, b]$ such that $x_{i-1} \leq p \leq s \leq r \leq x_i$ and $K|r - q| < \varepsilon/2$, where $K = \max\{|\inf f(x) - g(s)|, |\sup f(x) - g(s)|\}$ for $p \leq x \leq r$. Now define $P^* = P \cup \{p, r\}$. Then P^* is a refinement of P and thus

$$U(f, P^*) - L(f, P^*) < \frac{\varepsilon}{2}.$$

Now we consider the following 3 cases:

(i) $\inf f(x) \leq g(s) \leq \sup f(x)$ for $p \leq x \leq r$. This implies that $\inf f(x) = \inf g(x)$ and $\sup f(x) = \sup g(x)$ for $p \leq x \leq r$ and thus $L(f, P^*) = L(g, P^*)$ and $U(f, P^*) = U(g, P^*)$. Thus $U(g, P^*) - L(g, P^*) < \varepsilon/2 < \varepsilon$.

(ii) $g(s) < \inf f(x)$ for $p \leq x \leq r$. This implies that $\inf g(x) < \inf f(x)$ for $p \leq x \leq r$. Thus $L(g, P^*) < L(f, P^*)$ and $U(g, P^*) = U(f, P^*)$. Note that

$$\begin{aligned} L(f, P^*) - L(g, P^*) &= (\inf f(x) - g(s))(r - q) \\ &\leq K(r - p) \\ &< \varepsilon/2 \\ &\Rightarrow L(g, P^*) < L(f, P^*) - \varepsilon/2. \end{aligned}$$

With this we have

$$\begin{aligned} U(g, P^*) - L(g, P^*) &= U(f, P^*) - L(g, P^*) \\ &< U(f, P^*) - L(f, P^*) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

(iii) $\sup f(x) < g(s)$ for $p \leq x \leq r$. This case is shown with a similar argument to part (ii), the only difference is that we use the fact that $U(g, P^*) - U(f, P^*) < \varepsilon/2$.

This shows that for all $\varepsilon > 0$, there exists a partition P such that $U(g, P) - L(g, P) < \varepsilon$. Thus $g \in \mathcal{R}[a, b]$. Since g is integrable on $[a, b]$, then

$$\begin{aligned}\int_a^b g &= \int_a^s g + \int_s^b g \\ &= \int_a^s f + \int_s^b f \\ &= \int_a^b f.\end{aligned}$$

The above argument can then be repeated to show that it holds for $|F| = n$ for all $n \in \mathbb{N}$.

Finally, let

$$f(x) = 1 \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0, 1] \cap \mathbb{I}. \end{cases}$$

Clearly f is integrable and both f and g are bounded. Also $f(x) = g(x)$ for all $x \in [0, 1] \backslash ([0, 1] \cap \mathbb{I})$. Since g is not integrable, then the result does not hold. $\square$

3. Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Prove that $f \in \mathcal{R}[a, b]$.

Proof. Let $\varepsilon > 0$. Since f is increasing then either $f(a) = f(b)$, in which case f is constant and therefore integrable, or $f(a) < f(b)$. Supposing the latter case, let $P = \{x_0, \ldots, x_n\}$ such that for some $\delta < \varepsilon/(f(b) - f(a))$, $\Delta x_i = \delta$ all $i \in \{1, \ldots, n\}$. Then because f is increasing, it follows that $m_i = \inf f(x) = f(x_{i-1})$ and $M_i = \sup f(x) = f(x_i)$ for $x_{i-1} \leq x \leq x_i$. Hence,

$$\begin{aligned}U(f, P) - L(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i \\ &= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\delta \\ &= \delta(f(b) - f(a)) \\ &< \frac{\varepsilon}{f(b) - f(a)}(f(b) - f(a)) = \varepsilon.\end{aligned}$$

Therefore $f \in \mathcal{R}[a, b]$. $\square$

4. Let $f \in C[a, b]$ satisfy

$$\int_a^x f = \int_x^b f \quad \text{for all } x \in [a, b].$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Since f is continuous on $[a,b]$. Then by part 1 of the Fundamental Theorem of Calucus, $I_f'(x) = f(x)$ for all $x \in [a,b]$. It then follows from part 2 of FHT that $I_f(x)$ is an antiderivative of f over $[a,b]$ and thus over $[a,y]$ and $[y,b]$ for any $y \in [a,b]$. Moreover,

$$\int_a^y f = I_f(y) - I_f(a) \text{ and } \int_y^b = I_f(b) - I_f(y).$$

Combining this with our assumption of equality, we get that

$$I_f(y) - I_f(a) = I_f(b) - I_f(y) \Rightarrow I_f(y) = \frac{I_f(b) + I_f(a)}{2}.$$

The right-hand side being a constant gives

$$I_f'(y) = 0 = f(y)$$

for all $y \in [a,b]$. Therefore $f(x) = 0$ for all $x \in [a,b]$. □

5. Let $f, g \in C[a,b]$ with $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

Does the same conclusion hold if g is not assumed to be nonnegative? Give a proof or a counterexample.

Proof. Let $m = \inf f(x)$ and $M = \sup f(x)$ for $x \in [a,b]$. Then since f is continuous and $[a,b]$ is closed, then $m = \min f(x) = f(a^*)$ and $M = \max f(x) = f(b^*)$ for $x \in [a,b]$ and $a^*, b^* \in [a,b]$. Thus $m \leq f(x) \leq M$ for all $x \in [a,b]$. Moreover, since $g(x) \geq 0$ for all $x \in [a,b]$, then

$$mg(x) \leq f(x)g(x) \leq Mg(x) \text{ for all } x \in [a,b]$$

and by monotonicity we get that

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g. \tag{6}$$

Assuming $\int_a^b g > 0$, then from (6) it follows that

$$f(a^*) = m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M = f(b^*). \tag{7}$$

Now consider the integral function $I_f(x) = \int_a^x f$. Since f is continuous on $[a,b]$, then by part 1 of FHT, $I_f'(x) = f(x)$ for all $x \in [a,b]$. From 7 and Theorem 5.12 (Rudin) it follows that there exists some $c \in (a^*, b^*)$ such that

$$I_f'(c) = f(c) = \frac{\int_a^b fg}{\int_a^b g} \Rightarrow \int_a^b fg = f(c) \int_a^b g.$$

If $\int_a^b g = 0$, then since $g(x) \geq 0$ and since g is continuous on $[a, b]$, then $g(x) = 0$ for all $[a, b]$. This implies that $f(x)g(x) = 0$ for all $x \in [a, b]$. Hence

$$\int_a^b fg = \int_a^b 0 = 0 \Rightarrow \int_a^b fg = f(c) \int_a^b g = f(c) \int_a^b 0 = 0.$$

Which holds for any $c \in [a, b]$. For a counter example let $a = 0.5$, $b = 1.5$, $f(x) = x$, and $g(x) = \log x$. Then we have that

$$\int_a^b fg = \int_{0.5}^{1.5} x \log x \approx 0.0427916$$

and

$$\int_a^b g = \int_{0.5}^{1.5} \log x \approx -0.0452287.$$

Seeing as the range of f is $[0.5, 0.5]$ which is all positive values, then it is clear that there does not exist any $c \in [0.5, 1.5]$ such that

$$\int_a^b fg = f(c) \int_a^b g.$$

□