MATH 210B

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Instructor: Dr. Shannon **Assignment:** Homework 10

4. For each of the following, construct the lattice of subgroups, of $G(E/F)$, and the corresponding lattice of subfields of E over F. Identify all the normal extensions in the lattice of subfields, and identify which are conjugates of each other. For (b) and (c), express $G(E/F)$ as a permutation group.

   (a) $G(\mathbb{Q}(i, \sqrt[8]{2}))/\mathbb{Q}(i\sqrt{2}))$.

   Solution. To begin we note that if $\zeta = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ denotes a primitive 8th root of unity, then

$$x^8 - 2 = \prod_{k=0}^{7}(x - \zeta^k \sqrt[8]{2}).$$

   And this is the minimal polynomial of $\sqrt[8]{2}$ over $\mathbb{Q}$. Additionally, we have that for any $\sigma \in \operatorname{Aut}(\mathbb{Q}(i, \sqrt[8]{2}))$ it is defined by

$$\sigma := \begin{cases} \sqrt[8]{2} \mapsto \zeta^k \sqrt[8]{2} \\ i \mapsto (-1)^j i \end{cases}, \quad \text{for } 0 \le k \le 7,\ 0 \le j \le 1.$$

   We know from HW 9 that $[\mathbb{Q}(i, \sqrt[8]{2})\colon \mathbb{Q}] = [\mathbb{Q}(i, \sqrt[8]{2})\colon \mathbb{Q}(\sqrt[8]{2})][\mathbb{Q}(\sqrt[8]{2})\colon \mathbb{Q}] = 2 \cdot 8 = 16$. Additionally, since $\mathbb{Q}(i, \sqrt[8]{2})$ is the splitting field of $x^8 - 2$, then $\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}$ is normal. Since $\operatorname{char}(\mathbb{Q}) = 0$, then the extension is also separable and therefore the extension is Galois. Moreover, since $\mathbb{Q} \subseteq \mathbb{Q}(i\sqrt{2}) \subseteq \mathbb{Q}(i, \sqrt[8]{2})$, then $\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2})$ is Galois.

   To determine the elements of $G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2}))$, we must find those $\sigma \in \operatorname{Aut}(\mathbb{Q}(i, \sqrt[8]{2}))$ such that $\sigma(i\sqrt{2}) = i\sqrt{2}$. Hence, we need

$$\sigma(i\sqrt{2}) = \sigma(i)\sigma(\sqrt{2}) = (-1)^j i\sigma\big((\sqrt[8]{2})^4\big) = (-1)^j i\big(\zeta^k \sqrt[8]{2}\big)^4 = (-1)^j i\zeta^{4k}\sqrt{2}$$

   to equal $i\sqrt{2}$. Going through all possible values of j and k, we find that the following automorphisms fix $i\sqrt{2}$.

$$\sigma_0 := \begin{cases} \sqrt[8]{2} \mapsto \sqrt[8]{2} \\ i \mapsto i \end{cases} \qquad \sigma_1 := \begin{cases} \sqrt[8]{2} \mapsto \zeta^2 \sqrt[8]{2} \\ i \mapsto i \end{cases} \qquad \sigma_2 := \begin{cases} \sqrt[8]{2} \mapsto -\sqrt[8]{2} \\ i \mapsto i \end{cases}$$

$$\sigma_3 := \begin{cases} \sqrt[8]{2} \mapsto -\zeta^2 \sqrt[8]{2} \\ i \mapsto i \end{cases} \qquad \sigma_4 := \begin{cases} \sqrt[8]{2} \mapsto \zeta \sqrt[8]{2} \\ i \mapsto -i \end{cases} \qquad \sigma_5 := \begin{cases} \sqrt[8]{2} \mapsto \zeta^3 \sqrt[8]{2} \\ i \mapsto -i \end{cases}$$

$$\sigma_6 := \begin{cases} \sqrt[8]{2} \mapsto -\zeta \sqrt[8]{2} \\ i \mapsto -i \end{cases} \qquad \sigma_7 := \begin{cases} \sqrt[8]{2} \mapsto -\zeta^3 \sqrt[8]{2} \\ i \mapsto -i \end{cases}.$$

Another fact to note is that since $\mathbb{Q}(i, \sqrt[8]{2})$ is the splitting field of $x^8 - 2$ over $\mathbb{Q}(i\sqrt{2})$, then by (8) of HW 8, $G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2}))$ is isomorphic to a subgroup of S_8. Namely,

$$\begin{array}{lll} \sigma_0 := (1) & \sigma_1 := (1526)(3748) & \sigma_2 := (12)(34)(56)(78) \\ \sigma_3 := (1625)(3847) & \sigma_4 := (1324)(5867) & \sigma_5 := (1728)(3645) \\ \sigma_6 := (1423)(5768) & \sigma_7 := (1827)(3546). & \end{array}$$

And given that $\sigma_4\sigma_5 = \sigma_1$ and $\sigma_5\sigma_4 = \sigma_3$, then we can conclude that the Galois group is isomorphic to a nonabelian subgroup of S_8 of order 8. Furthermore, since $o(\sigma_2) = 2$ and $\sigma_1^2 = \sigma_4^2 = \sigma_5^2 = \sigma_2$, then we can conclude that the Galois group is isomorphic to the quaternion group Q. As such we can associate $\sigma_0 \to 1$, $\sigma_2 \to -1$, $\sigma_1 \to i$, $\sigma_4 \to j$, and $\sigma_5 \to k$. By relabelling, it follows that

$$G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2})) = \{\sigma_0, \sigma_2, \sigma_1, -\sigma_1, \sigma_4, -\sigma_4, \sigma_5, -\sigma_5\}.$$

And so the lattice of subgroups of the Galois group is

$$\begin{tikzcd}[arrows=dash] & G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2})) \arrow[dl] \arrow[d] \arrow[dr] & \\ \{\sigma_0, \sigma_1, \sigma_2, -\sigma_1\} \arrow[dr] & \{\sigma_0, \sigma_4, \sigma_2, -\sigma_4\} \arrow[d] & \{\sigma_0, \sigma_5, \sigma_2, -\sigma_5\} \arrow[dl] \\ & \{\sigma_0, \sigma_2\} \arrow[d] & \\ & \{\sigma_0\} & \end{tikzcd}$$

Finally, to construct the lattice of subfields we must determine the fixed fields of: $\langle\sigma_0\rangle$, $\langle\sigma_2\rangle$, $\langle\sigma_1\rangle$, $\langle\sigma_4\rangle$, and $\langle\sigma_5\rangle$. To begin, it is clear that $F_{\langle\sigma_0\rangle} = \mathbb{Q}(i\sqrt{2})$. Then taking

$$\{1, \sqrt[8]{2}, \sqrt[4]{2}, \sqrt{2}, i, i\sqrt[8]{2}, i\sqrt[4]{2}, i\sqrt{2}\}$$

as a basis for $\mathbb{Q}(i, \sqrt[8]{2})$ over $\mathbb{Q}(i\sqrt{2})$ then solving for when $\sigma_i(x) = x$ for $i = 1, 2, 4, 5$, we get that

$$F_{\langle\sigma_2\rangle} = \mathbb{Q}(i, \sqrt[4]{2}) \qquad F_{\langle\sigma_1\rangle} = \mathbb{Q}(i, \sqrt{2})$$
$$F_{\langle\sigma_4\rangle} = \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) \qquad F_{\langle\sigma_5\rangle} = \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$$

Thus, the lattice of subfields is

$$\begin{tikzcd}
 & \mathbb{Q}(i, \sqrt[8]{2}) \arrow[d, dash] & \\
 & \mathbb{Q}(i, \sqrt[4]{2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(i, \sqrt{2}) \arrow[dr, dash] & \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}) \arrow[d, dash] & \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2}) \arrow[dl, dash] \\
 & \mathbb{Q}(i\sqrt{2}) &
\end{tikzcd}$$

Now we must determine the normal extensions. Since $\langle\sigma_1\rangle$, $\langle\sigma_4\rangle$, and $\langle\sigma_5\rangle$ are all normal subgroups (because they all have index 2), then the extensions to which they correspond are normal. Lastly, we find that $\sigma_4(\langle\sigma_1\rangle) = \langle\sigma_5\rangle$, $-\sigma_5(\langle\sigma_1\rangle) = \langle\sigma_4\rangle$. And so since each of the subgroups are conjugate, then

$$\mathbb{Q}(i, \sqrt{2}), \quad \mathbb{Q}(\sqrt[4]{2} + i\sqrt[4]{2}), \quad \mathbb{Q}(\sqrt[4]{2} - i\sqrt[4]{2})$$

are all conjugate.

(b) The Galois group of $(x^2 - 3)(x^3 - 5)$ over $\mathbb{Q}$.

Solution. To begin we first note that

$$(x^2 - 3)(x^3 - 5) = (x - \sqrt{3})(x + \sqrt{3})(x - \sqrt[3]{5})(x - \omega\sqrt[3]{5})(x - \omega^2\sqrt[3]{5}).$$

And so we need the splitting field for this polynomial. It must include each of these roots. Thus, $\mathbb{Q}(\omega, \sqrt{3}, \sqrt[3]{5})$ is an extension that would admit each root. The degree of this extension over $\mathbb{Q}$ is

$$[\mathbb{Q}(\omega, \sqrt{3}, \sqrt[3]{5}) \colon \mathbb{Q}(\sqrt{3}, \sqrt[3]{5})][\mathbb{Q}(\sqrt{3}, \sqrt[3]{5}) \colon \mathbb{Q}(\sqrt[3]{5})][\mathbb{Q}(\sqrt[3]{5}) \colon \mathbb{Q}] = 2 \cdot 2 \cdot 3 = 12.$$

Moreover, this must be the splitting field since the minimal polynomials for ω, $\sqrt{3}$, and $\sqrt[4]{5}$ all have degrees $2, 2$, and 3, respectively, and none of the roots of the polynomials are common. Hence, $\mathbb{Q}(\omega, \sqrt{3}, \sqrt[3]{5})/\mathbb{Q}$ is a Galois extension.

We know that any automorphism of this extension is defined by where it sends $\omega, \sqrt{3}, \sqrt[3]{5}$. And so we have that for any automorphism σ of the extension,

$$\sigma(\omega) \in \{\omega, \omega^2\}, \quad \sigma(\sqrt{3}) \in \{\sqrt{3}, -\sqrt{3}\}, \quad \sigma(\sqrt[3]{5}) \in \{\sqrt[3]{5}, \omega\sqrt[3]{5}, \omega^2\sqrt[3]{5}\}.$$

Thus, the elements of the Galois group are

$$\sigma_0 := \begin{cases} \sqrt[3]{5} \mapsto \sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega \end{cases} \qquad \sigma_1 := \begin{cases} \sqrt[3]{5} \mapsto \omega\sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega \end{cases} \qquad \sigma_2 := \begin{cases} \sqrt[3]{5} \mapsto \omega^2\sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega \end{cases}$$

$$\sigma_3 := \begin{cases} \sqrt[3]{5} \mapsto \sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega \end{cases} \qquad \sigma_4 := \begin{cases} \sqrt[3]{5} \mapsto \omega\sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega \end{cases} \qquad \sigma_5 := \begin{cases} \sqrt[3]{5} \mapsto \omega^2\sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega \end{cases}$$

$$\sigma_6 := \begin{cases} \sqrt[3]{5} \mapsto \sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega^2 \end{cases} \qquad \sigma_7 := \begin{cases} \sqrt[3]{5} \mapsto \omega\sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega^2 \end{cases} \qquad \sigma_8 := \begin{cases} \sqrt[3]{5} \mapsto \omega^2\sqrt[3]{5} \\ \sqrt{3} \mapsto \sqrt{3} \\ \omega \mapsto \omega^2 \end{cases}$$

$$\sigma_9 := \begin{cases} \sqrt[3]{5} \mapsto \sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega^2 \end{cases} \qquad \sigma_{10} := \begin{cases} \sqrt[3]{5} \mapsto \omega\sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega^2 \end{cases} \qquad \sigma_{11} := \begin{cases} \sqrt[3]{5} \mapsto \omega^2\sqrt[3]{5} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \omega \mapsto \omega^2 \end{cases}$$

Before we associate each of these automorphisms with an element from S_5, let us note that from HW 9, we saw that $G(\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$ and so

$$S_3 \cong G(\mathbb{Q}(\omega, \sqrt[3]{5})/\mathbb{Q}) \subseteq G(\mathbb{Q}(\omega, \sqrt{3}, \sqrt[3]{5})/\mathbb{Q}).$$

What this means is that

$$G(\mathbb{Q}(\omega, \sqrt[3]{5})/\mathbb{Q}) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_6, \sigma_7, \sigma_8\}$$

Thus, the remaining 6 automorphisms in our list can be obtained by applying σ_3 to each of the 6 in the above set. Hence, we need only associate the 6 in the set with a permutation and σ_3 with a permutation and we can recover the rest.

Letting

$$c_1 = \sqrt[3]{5}, c_2 = \omega\sqrt[3]{5}, c_3 = \omega^2\sqrt[3]{5}, c_4 = \sqrt{3}, c_5 = -\sqrt{3},$$

then we have that

$$\sigma_0 := (1) \qquad \sigma_1 := (123) \qquad \sigma_2 := (132)$$
$$\sigma_6 := (23) \qquad \sigma_7 := (12) \qquad \sigma_8 := (13).$$

And we also have that

$$\sigma_3 := (45).$$

Thus, the remaining 6 permutations are

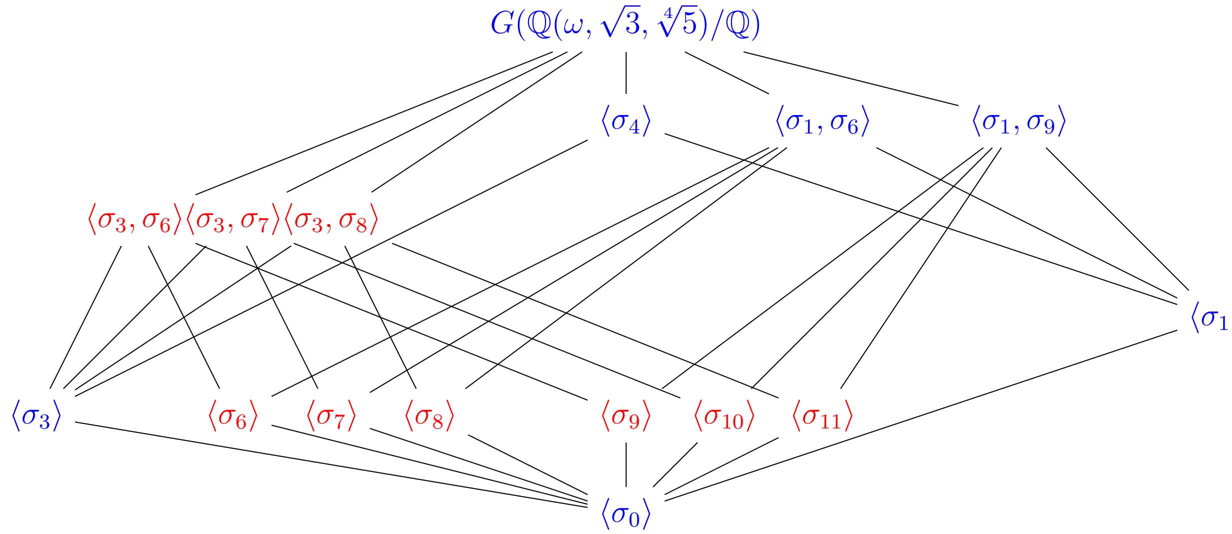
$$\sigma_0\sigma_3 = \sigma_3 := (45) \qquad \sigma_1\sigma_3 = \sigma_4 := (123)(45) \qquad \sigma_2\sigma_3 = \sigma_5 := (132)(45)$$
$$\sigma_6\sigma_3 = \sigma_9 := (23)(45) \qquad \sigma_7\sigma_3 = \sigma_{10} := (12)(45) \qquad \sigma_8\sigma_3 = \sigma_{11} := (13)(45).$$

Now we will determine all the subgroups of the Galois group. Since the order is 12, then we are looking for subgroups of order, 2,3,4,6. To help with this calculation, we will refer to the Cayley table below:

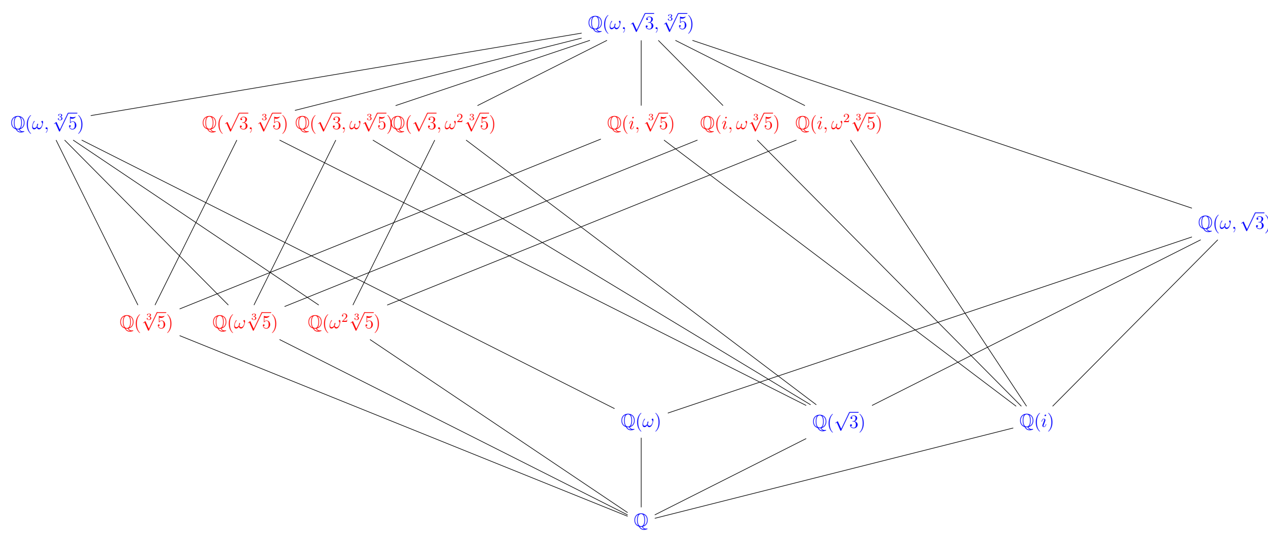
	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_0	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
σ_1	σ_1	σ_2	σ_0	σ_4	σ_5	σ_3	σ_7	σ_8	σ_6	σ_{10}	σ_{11}	σ_9
σ_2	σ_2	σ_0	σ_1	σ_5	σ_3	σ_4	σ_8	σ_6	σ_7	σ_{11}	σ_9	σ_{10}
σ_3	σ_3	σ_4	σ_5	σ_0	σ_1	σ_2	σ_9	σ_{10}	σ_{11}	σ_6	σ_7	σ_8
σ_4	σ_4	σ_5	σ_3	σ_1	σ_2	σ_0	σ_{10}	σ_{11}	σ_9	σ_7	σ_8	σ_6
σ_5	σ_5	σ_3	σ_4	σ_2	σ_0	σ_1	σ_{11}	σ_9	σ_{10}	σ_8	σ_6	σ_7
σ_6	σ_6	σ_8	σ_7	σ_9	σ_{11}	σ_{10}	σ_0	σ_2	σ_1	σ_3	σ_5	σ_4
σ_7	σ_7	σ_6	σ_8	σ_{10}	σ_9	σ_{11}	σ_1	σ_0	σ_2	σ_4	σ_3	σ_5
σ_8	σ_8	σ_7	σ_6	σ_{11}	σ_{10}	σ_9	σ_2	σ_1	σ_0	σ_5	σ_4	σ_3
σ_9	σ_9	σ_{11}	σ_{10}	σ_6	σ_8	σ_7	σ_3	σ_5	σ_4	σ_0	σ_2	σ_1
σ_{10}	σ_{10}	σ_9	σ_{11}	σ_7	σ_6	σ_8	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2
σ_{11}	σ_{11}	σ_{10}	σ_9	σ_8	σ_7	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1	σ_0

$$\langle\sigma_0\rangle = \{\sigma_0\} \qquad \langle\sigma_1\rangle = \{\sigma_0, \sigma_1, \sigma_2\} \qquad \langle\sigma_2\rangle = \langle\sigma_1\rangle$$
$$\langle\sigma_3\rangle = \{\sigma_0, \sigma_3\} \qquad \langle\sigma_4\rangle = \{\sigma_0, \sigma_4, \sigma_2, \sigma_3, \sigma_1, \sigma_5\} \qquad \langle\sigma_5\rangle = \langle\sigma_4\rangle$$
$$\langle\sigma_6\rangle = \{\sigma_0, \sigma_6\} \qquad \langle\sigma_7\rangle = \{\sigma_0, \sigma_7\} \qquad \langle\sigma_8\rangle = \{\sigma_0, \sigma_8\}$$
$$\langle\sigma_9\rangle = \{\sigma_0, \sigma_9\} \qquad \langle\sigma_{10}\rangle = \{\sigma_0, \sigma_{10}\} \qquad \langle\sigma_{11}\rangle = \{\sigma_0, \sigma_{11}\}$$
$$\langle\sigma_1, \sigma_6\rangle = \{\sigma_0, \sigma_1, \sigma_2, \sigma_6, \sigma_7, \sigma_8\} \qquad \langle\sigma_1, \sigma_9\rangle = \{\sigma_0, \sigma_1, \sigma_2, \sigma_9, \sigma_{10}, \sigma_{11}\} \qquad \langle\sigma_3, \sigma_6\rangle = \{\sigma_0, \sigma_3, \sigma_6, \sigma_9\}$$
$$\langle\sigma_3, \sigma_7\rangle = \{\sigma_0, \sigma_3, \sigma_7, \sigma_{10}\} \qquad \langle\sigma_3, \sigma_8\rangle = \{\sigma_0, \sigma_3, \sigma_8, \sigma_{11}\}$$

Based on these subgroups, we can construct the lattice of subgroups of the Galois group.



$G(\mathbb{Q}(\omega, \sqrt{3}, \sqrt[4]{5})/\mathbb{Q})$
$\langle \sigma_4 \rangle$
$\langle \sigma_1, \sigma_6 \rangle$
$\langle \sigma_1, \sigma_9 \rangle$
$\langle \sigma_3, \sigma_6 \rangle$
$\langle \sigma_3, \sigma_7 \rangle$
$\langle \sigma_3, \sigma_8 \rangle$
$\langle \sigma_1 \rangle$
$\langle \sigma_3 \rangle$
$\langle \sigma_6 \rangle$
$\langle \sigma_7 \rangle$
$\langle \sigma_8 \rangle$
$\langle \sigma_9 \rangle$
$\langle \sigma_{10} \rangle$
$\langle \sigma_{11} \rangle$
$\langle \sigma_0 \rangle$



$\mathbb{Q}(\omega, \sqrt{3}, \sqrt[3]{5})$
$\mathbb{Q}(\omega, \sqrt[3]{5})$
$\mathbb{Q}(\sqrt{3}, \sqrt[3]{5})$
$\mathbb{Q}(\sqrt{3}, \omega\sqrt[3]{5})$
$\mathbb{Q}(\sqrt{3}, \omega^2\sqrt[3]{5})$
$\mathbb{Q}(i, \sqrt[3]{5})$
$\mathbb{Q}(i, \omega\sqrt[3]{5})$
$\mathbb{Q}(i, \omega^2\sqrt[3]{5})$
$\mathbb{Q}(\omega, \sqrt{3})$
$\mathbb{Q}(\sqrt[3]{5})$
$\mathbb{Q}(\omega\sqrt[3]{5})$
$\mathbb{Q}(\omega^2\sqrt[3]{5})$
$\mathbb{Q}(\omega)$
$\mathbb{Q}(\sqrt{3})$
$\mathbb{Q}(i)$
$\mathbb{Q}$

The calculation of the first subgroup lattice went as follows:

- **Subsets:** With the given subgroups, we need to determine which are contained in the others. On the left will be the subgroup and on the right will be a list of all which it contains.

$$\begin{aligned}
\langle\sigma_0\rangle &: \langle\sigma_0\rangle & \langle\sigma_3,\sigma_6\rangle &: \langle\sigma_0\rangle, \langle\sigma_3\rangle, \langle\sigma_6\rangle, \langle\sigma_9\rangle \\
\langle\sigma_3\rangle &: \langle\sigma_0\rangle & \langle\sigma_3,\sigma_7\rangle &: \langle\sigma_0\rangle, \langle\sigma_3\rangle, \langle\sigma_7\rangle, \langle\sigma_{10}\rangle \\
\langle\sigma_6\rangle &: \langle\sigma_0\rangle & \langle\sigma_3,\sigma_8\rangle &: \langle\sigma_0\rangle, \langle\sigma_3\rangle, \langle\sigma_8\rangle, \langle\sigma_{11}\rangle \\
\langle\sigma_7\rangle &: \langle\sigma_0\rangle & \langle\sigma_1\rangle &: \langle\sigma_0\rangle \\
\langle\sigma_8\rangle &: \langle\sigma_0\rangle & \langle\sigma_4\rangle &: \langle\sigma_0\rangle, \langle\sigma_1\rangle, \langle\sigma_3\rangle, \langle\sigma_1\rangle \\
\langle\sigma_9\rangle &: \langle\sigma_0\rangle & \langle\sigma_1,\sigma_6\rangle &: \langle\sigma_0\rangle, \langle\sigma_1\rangle, \langle\sigma_6\rangle, \langle\sigma_7\rangle, \langle\sigma_8\rangle \\
\langle\sigma_{10}\rangle &: \langle\sigma_0\rangle & \langle\sigma_1,\sigma_9\rangle &: \langle\sigma_0\rangle, \langle\sigma_1\rangle, \langle\sigma_9\rangle, \langle\sigma_{10}\rangle, \langle\sigma_{11}\rangle \\
\langle\sigma_{11}\rangle &: \langle\sigma_0\rangle
\end{aligned}$$

- **Index**: On the left is the subgroup and on the right is its index in the groups which contain it, listed in increasing order.

$$\begin{aligned}
\langle\sigma_3\rangle &: 2,2,2,3,6 & \langle\sigma_1\rangle &: 2,2,2,4 \\
\langle\sigma_6\rangle &: 2,2,2,3,6 & \langle\sigma_3,\sigma_6\rangle &: 3 \\
\langle\sigma_7\rangle &: 2,2,2,3,6 & \langle\sigma_3,\sigma_7\rangle &: 3 \\
\langle\sigma_8\rangle &: 2,2,2,3,6 & \langle\sigma_3,\sigma_8\rangle &: 3 \\
\langle\sigma_9\rangle &: 2,2,2,3,6 & \langle\sigma_4\rangle &: 2 \\
\langle\sigma_{10}\rangle &: 2,2,2,3,6 & \langle\sigma_1,\sigma_6\rangle &: 2 \\
\langle\sigma_{11}\rangle &: 2,2,2,3,6 & \langle\sigma_1,\sigma_9\rangle &: 2
\end{aligned}$$

- **Normality**: In order for a subgroup H of a group G to be normal, it must be the case that for all $x \in G$, $xHx^{-1} = H$. Take for example $\sigma_1 : (123)$.

(c) The Galois group of $(x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$.

Solution. The splitting field for this polynomial is $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. The automorphisms of this extension are 8 in total and are defined by

$$\sigma := \begin{cases} \sqrt{2} \mapsto (-1)^i\sqrt{2} \\ \sqrt{3} \mapsto (-1)^j\sqrt{3} \\ \sqrt{5} \mapsto (-1)^k\sqrt{5} \end{cases}, \quad \text{for } \ 0 \le i,j,k \le 1$$

The Galois group is therefore isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and hence all of its subgroups are normal since the group is abelian. The automorphisms are

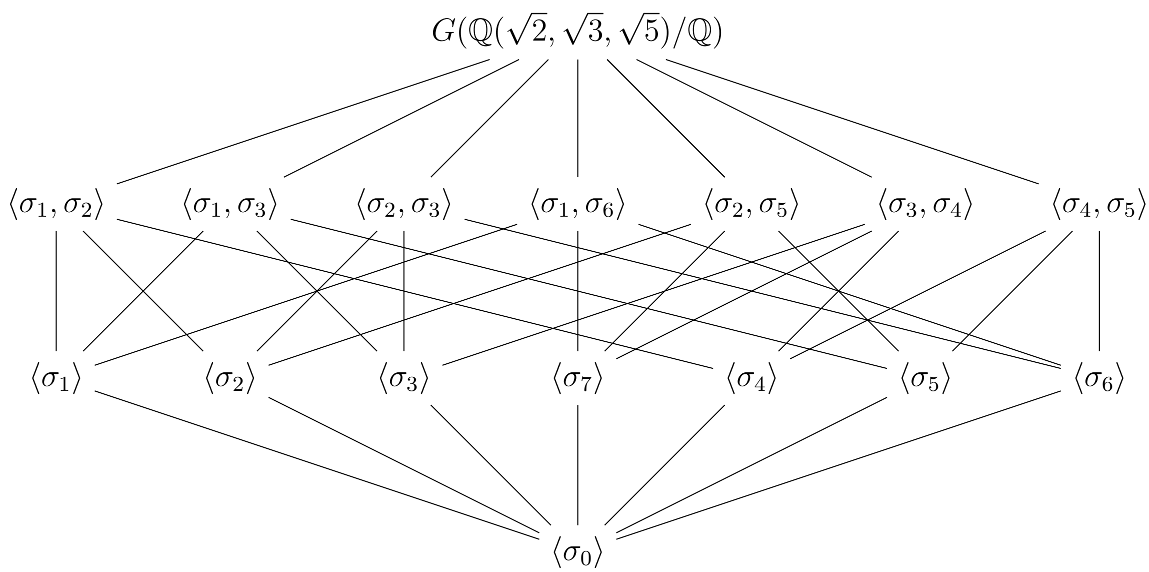
$$\sigma_0 := \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_1 := \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_2 := \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_3 := \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

$$\sigma_4 := \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_5 := \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases} \quad \sigma_6 := \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases} \quad \sigma_7 := \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

The subgroups of the Galois group are

$$\begin{array}{llll}
\langle \sigma_0 \rangle = \{\sigma_0\} & \langle \sigma_1 \rangle = \{\sigma_0, \sigma_1\} & \langle \sigma_2 \rangle = \{\sigma_0, \sigma_2\} & \langle \sigma_3 \rangle = \{\sigma_0, \sigma_3\} \\
\langle \sigma_4 \rangle = \{\sigma_0, \sigma_4\} & \langle \sigma_5 \rangle = \{\sigma_0, \sigma_5\} & \langle \sigma_6 \rangle = \{\sigma_0, \sigma_6\} & \langle \sigma_7 \rangle = \{\sigma_0, \sigma_7\} \\
\langle \sigma_1, \sigma_2 \rangle = \{\sigma_0, \sigma_1, \sigma_2, \sigma_4\} & \langle \sigma_1, \sigma_3 \rangle = \{\sigma_0, \sigma_1, \sigma_3, \sigma_5\} & \langle \sigma_1, \sigma_6 \rangle = \{\sigma_0, \sigma_1, \sigma_6, \sigma_7\} & \\
\langle \sigma_2, \sigma_3 \rangle = \{\sigma_0, \sigma_2, \sigma_3, \sigma_6\} & \langle \sigma_2, \sigma_5 \rangle = \{\sigma_0, \sigma_2, \sigma_5, \sigma_7\} & \langle \sigma_3, \sigma_4 \rangle = \{\sigma_0, \sigma_3, \sigma_4, \sigma_7\} & \\
\langle \sigma_4, \sigma_5 \rangle = \{\sigma_0, \sigma_4, \sigma_5, \sigma_6\} & & &
\end{array}$$

And the lattice of subgroups is



To determine the fixed fields of each subgroup we will look at what happens to the basis elements for a basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}, \sqrt{30}\}$. Here, the blue cells denote a sign change.

We need to find when $\sigma(x) = x$ for each of these sigma. We have

	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{10}$	$\sqrt{15}$	$\sqrt{30}$
σ_0	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{10}$	$\sqrt{15}$	$\sqrt{30}$
σ_1	1	$-\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{10}$	$\sqrt{15}$	$-\sqrt{30}$
σ_2	1	$\sqrt{2}$	$-\sqrt{3}$	$\sqrt{5}$	$-\sqrt{6}$	$\sqrt{10}$	$-\sqrt{15}$	$-\sqrt{30}$
σ_3	1	$\sqrt{2}$	$\sqrt{3}$	$-\sqrt{5}$	$\sqrt{6}$	$-\sqrt{10}$	$-\sqrt{15}$	$-\sqrt{30}$
σ_4	1	$-\sqrt{2}$	$-\sqrt{3}$	$\sqrt{5}$	$\sqrt{6}$	$-\sqrt{10}$	$-\sqrt{15}$	$\sqrt{30}$
σ_5	1	$-\sqrt{2}$	$\sqrt{3}$	$-\sqrt{5}$	$-\sqrt{6}$	$\sqrt{10}$	$-\sqrt{15}$	$\sqrt{30}$
σ_6	1	$\sqrt{2}$	$-\sqrt{3}$	$-\sqrt{5}$	$-\sqrt{6}$	$-\sqrt{10}$	$\sqrt{15}$	$\sqrt{30}$
σ_7	1	$-\sqrt{2}$	$-\sqrt{3}$	$-\sqrt{5}$	$\sqrt{6}$	$\sqrt{10}$	$\sqrt{15}$	$-\sqrt{30}$

$$\begin{aligned}\sigma_1 : &a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ =&a_0 - a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} - a_4\sqrt{6} - a_5\sqrt{10} + a_6\sqrt{15} - a_7\sqrt{30}\end{aligned}$$

This implies that $a_1 = 0, a_4 = 0, a_5 = 0, a_7 = 0$. And so

$$F_{\langle\sigma_1\rangle} = \{a_0 + a_2\sqrt{3} + a_3\sqrt{5} + a_6\sqrt{15} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{3}, \sqrt{5}).$$

Next we have

$$\begin{aligned}\sigma_2 : &a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ =&a_0 + a_1\sqrt{2} - a_2\sqrt{3} + a_3\sqrt{5} - a_4\sqrt{6} + a_5\sqrt{10} - a_6\sqrt{15} - a_7\sqrt{30}\end{aligned}$$

which gives $a_2 = 0, a_4 = 0, a_6 = 0, a_7 = 0$. Thus

$$F_{\langle\sigma_2\rangle} = \{a_0 + a_1\sqrt{2} + a_3\sqrt{5} + a_5\sqrt{10} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2}, \sqrt{5}).$$

Next we have

$$\begin{aligned}\sigma_3 : &a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ =&a_0 + a_1\sqrt{2} + a_2\sqrt{3} - a_3\sqrt{5} + a_4\sqrt{6} - a_5\sqrt{10} - a_6\sqrt{15} - a_7\sqrt{30}\end{aligned}$$

which gives $a_3 = 0, a_5 = 0, a_6 = 0, a_7 = 0$. Thus,

$$F_{\langle\sigma_3\rangle} = \{a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_4\sqrt{6} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

Next we have

$$\begin{aligned}\sigma_4 : &a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ =&a_0 - a_1\sqrt{2} - a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} - a_5\sqrt{10} - a_6\sqrt{15} + a_7\sqrt{30}\end{aligned}$$

which gives $a_1 = 0, a_2 = 0, a_5 = 0, a_6 = 0$. Thus,

$$F_{\langle\sigma_4\rangle} = \{a_0 + a_3\sqrt{5} + a_4\sqrt{6} + a_7\sqrt{30} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{5}, \sqrt{6}).$$

Next we have

$$\begin{aligned}\sigma_5 : a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ = a_0 - a_1\sqrt{2} + a_2\sqrt{3} - a_3\sqrt{5} - a_4\sqrt{6} + a_5\sqrt{10} - a_6\sqrt{15} + a_7\sqrt{30}\end{aligned}$$

which gives $a_1 = 0, a_3 = 0, a_4 = 0, a_6 = 0$. Thus

$$F_{\langle\sigma_5\rangle} = \{a_0 + a_2\sqrt{3} + a_5\sqrt{10} + a_7\sqrt{30} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{3}, \sqrt{10}).$$

Next we have

$$\begin{aligned}\sigma_6 : a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\\ = a_0 + a_1\sqrt{2} - a_2\sqrt{3} - a_3\sqrt{5} - a_4\sqrt{6} - a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30}\end{aligned}$$

which gives $a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0$. Thus

$$F_{\langle\sigma_6\rangle} = \{a_0 + a_1\sqrt{2} + a_6\sqrt{15} + a_7\sqrt{30} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{2}, \sqrt{15}).$$

Finally, we have

$$\begin{aligned}\sigma_7 : & a_0 + a_1\sqrt{2} + a_2\sqrt{3} + a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} + a_7\sqrt{30} \\ = & a_0 - a_1\sqrt{2} - a_2\sqrt{3} - a_3\sqrt{5} + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} - a_7\sqrt{30}\end{aligned}$$

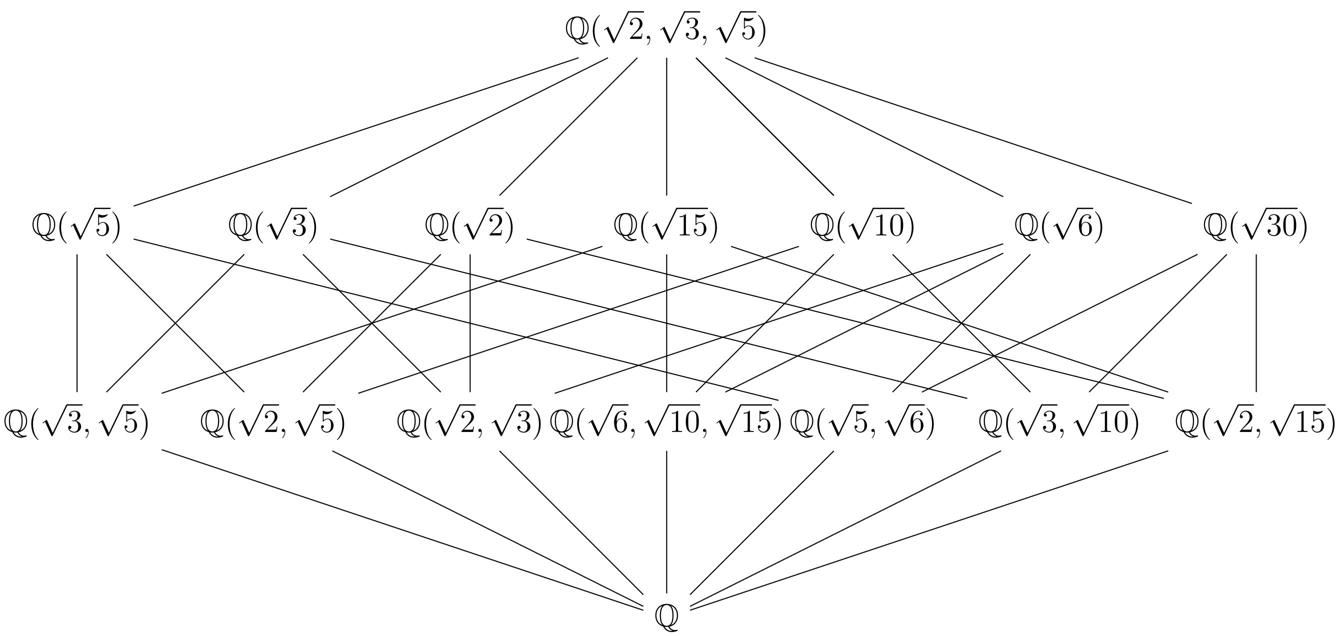
which gives $a_1 = 0, a_2 = 0, a_3 = 0, a_7 = 0$. Thus

$$F_{\langle\sigma_7\rangle} = \{a_0 + a_4\sqrt{6} + a_5\sqrt{10} + a_6\sqrt{15} \mid a_i \in \mathbb{Q}\} = \mathbb{Q}(\sqrt{6}, \sqrt{10}, \sqrt{15}).$$

To determine the fixed fields of those subgroups generated by 2 elements, we simply take the intersection of each individual fixed field corresponding to the elements in the subgroup. Hence

$$\begin{aligned}F_{\langle\sigma_1,\sigma_2\rangle} &= \mathbb{Q}(\sqrt{3}, \sqrt{5}) \cap \mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\sqrt{5}) \\ F_{\langle\sigma_1,\sigma_3\rangle} &= \mathbb{Q}(\sqrt{3}, \sqrt{5}) \cap \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{3}) \\ F_{\langle\sigma_2,\sigma_3\rangle} &= \mathbb{Q}(\sqrt{2}, \sqrt{5}) \cap \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2}) \\ F_{\langle\sigma_1,\sigma_6\rangle} &= \mathbb{Q}(\sqrt{3}, \sqrt{5}) \cap \mathbb{Q}(\sqrt{2}, \sqrt{15}) = \mathbb{Q}(\sqrt{15}) \\ F_{\langle\sigma_2,\sigma_5\rangle} &= Q(\sqrt{2}, \sqrt{5}) \cap \mathbb{Q}(\sqrt{3}, \sqrt{10}) = \mathbb{Q}(\sqrt{10}) \\ F_{\langle\sigma_3,\sigma_4\rangle} &= \mathbb{Q}(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}(\sqrt{5}, \sqrt{6}) = \mathbb{Q}(\sqrt{6}) \\ F_{\langle\sigma_4,\sigma_5\rangle} &= \mathbb{Q}(\sqrt{5}, \sqrt{6}) \cap \mathbb{Q}(\sqrt{3}, \sqrt{10}) = \mathbb{Q}(\sqrt{30})\end{aligned}$$

And so the lattice of subfields is



Finally, since all of the subgroups of the Galois group are normal, then each subgroup is equal to all of its conjugates and thus each extension is equal to each of its conjugates.