Homework 5 - Sequences of Functions

MATH 230B

You can use all the results discussed in class. They are collected in the notes posted on Canvas.

Problem 1
Let $A_n \subseteq \mathbb{R}$ satisfy $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$ and denote $A = \bigcap\limits_{n\in\mathbb{N}} A_n$. Prove that the sequence of indicator functions $f_n = 1_{A_n}$ converges pointwise to the function $f = 1_A$ on $\mathbb{R}$.

Problem 2 [*Master's Exam, Spring 2020*]
Discuss the pointwise/uniform convergence on $[0, 1]$ of the sequence of functions

$$f_n(x) = \frac{nx}{1 + n^3x^2}$$

Problem 3 [*Master's Exam, Fall 2021*]
Let $f : [a, b] \to \mathbb{R}$ be a function and let $f_n \in C[a, b]$ for all $n \in \mathbb{N}$. Assume that $\{x_n\}_n$ is a sequence in $[a, b]$ that converges to $x \in [a, b]$.

a) Prove that, if $\{f_n\}_n$ converges uniformly to f on $[a, b]$, then the numerical sequence $\{f_n(x_n)\}_n$ converges to $f(x)$.

b) Is the same conclusion true if $\{f_n\}_n$ converges to f pointwise on $[a, b]$?

Problem 4 Let g be a continuous function on $\mathbb{R}$. Compute (with proof) the following limit

$$\lim_{n\to\infty} \int_0^1 \frac{nxg(x)}{1 + n^2x}\,\mathrm{d}x$$

Problem 5 [*Master's Exam, Fall 2020*]
Let $\{f_n\}_n$ be a sequence of functions with domain $[0, 1]$. Assume that there exists $L > 0$ such that

$$|f_n(x) - f_n(y)| \leq L|x - y| \quad \text{for all} \quad x, y \in [0, 1], n \in \mathbb{N}.$$

Prove that if $\{f_n\}_n$ converges pointwise to f on $[0, 1]$, then $\{f_n\}_n$ converges uniformly to f on $[0, 1]$.

ADDITIONAL PROBLEMS

Problem A1 Let $\{f_k\}_k$ be the sequence of functions defined as

$$f_k(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad 0 < x < 1 - \frac{1}{k} \\ 1 - \frac{1}{k} & \text{if} \quad x \geq 1 - \frac{1}{k} \end{cases}$$

Find (with proof) the pointwise limit on $[0, \infty[$ and determine if the convergence is uniform on this interval.

Problem A2 [*Master's Exam, Spring 2020*]
Let $\alpha : [a, b] \to \mathbb{R}$ be increasing. Assume that $\{f_n\}_n$ is a sequence of functions that are Riemann-Stieltjes integrable on $[a, b]$ with respect to α. Prove that, if $\{f_n\}_n$ converges uniformly to a function f on $[a, b]$, then f is Riemann-Stieltjes integrable on $[a, b]$ with respect to α and

$$\lim_{n\to\infty} \int_a^b f_n \, \mathrm{d}\alpha = \int_a^b f \, \mathrm{d}\alpha$$

Problem A3 [*Master's Exam, Spring 2018*]
Suppose that $\{f_n\}_n$ is a sequence of differentiable function on $[a, b]$. Assume that there exists $M > 0$ such that $|f_n'(x)| \leq M$ for all $x \in [a, b]$ and $n \in \mathbb{N}$. Prove that if $\{f_n\}_n$ is pointwise convergent on $[a, b]$, then it is also uniformly convergent on $[a, b]$.

Problem A4 [Dini's Theorem] [*Master's Exam, Spring 2018*]
Let $\{f_n\}_n$ be a sequence of continuous functions on $[a, b]$ that converges pointwise to a continuous function f on $[a, b]$. Prove that, if $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and for all $x \in [a, b]$, then f_n converges to f uniformly on $[a, b]$.

Problem A5 [*Master's Exam, Fall 2017*]
Let $f \in C[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n) \, \mathrm{d}x = f(0).$$