Stat 215A, Problem Set # 1, Spring 2022

02/01/2022

This problem set consists of eight problems about sets (events), sigma-algebras and properties of probability measures. It is not for grading but for practice purposes. You can contribute to the discussions via Canvas with your questions, solution ideas or responses to other questions. A much shorter homework assignment which will be posted by this Friday is the first grading item.

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1. Let Ω be a sample space for a particular experiment. Moreover, let A and B be two proper subsets of Ω.

   (a) Show that $A \cup B = A \cup (B \backslash A)$.

   (b) If A and B are not disjoint, determine the $\sigma-$algebra generated by A and B.

2. Let $(\Omega, \mathcal{F})$ be a measurable space, and $B, A_1, A_2, ... \in \mathcal{F}$. Prove the following distributive laws:

   (a) $B \cap (\bigcup_{k=1}^{\infty} A_k) = \bigcup_{k=1}^{\infty} (B \cap A_k)$

   (b) $B \cup (\bigcap_{k=1}^{\infty} A_k) = \bigcap_{k=1}^{\infty} (B \cup A_k)$.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and B be a nonempty subset of Ω.

   (a) Let $L = \{B \cap A : A \in \mathcal{F}\}$. Show that L is a $\sigma-$algebra on B.

   (b) Let L be as in part (a). Is (Ω, L) a measurable space? In other words, this is about checking if L is a $\sigma-$algebra on Ω.

4. Complete the proof of the inclusion-exclusion formula for any two events A and B in a probability space $(\Omega, \mathcal{F}, P)$:
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
   You can also try this version for three events:
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

5. The general inclusion-exclusion formula for the union of a finite number of events. Let $\{A_k : k = 1, 2, ..., n\}$ be a collection of n events in a probability space $(\Omega, \mathcal{F}, P)$. Then, for each $n \in \mathbb{N}$,
$$P(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\bigcap_{k=1}^{n} A_k).$$
   Hint. Use induction on n and utilitize the result for two events (from problem 4).

6. Boole's inequality. Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$.

   (a) Prove that $P(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} P(A_k)$, for all $n \in \mathbb{N}$.

   (b) Prove that $P(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} P(A_k)$.

7. Bonferroni's inequality. Let $\{A_k : k = 1, 2, ..., n\}$ be a collection of n events in a probability space $(\Omega, \mathcal{F}, P)$. Prove that $P(\bigcap_{k=1}^{n} A_k) \geq \sum_{k=1}^{n} P(A_k) - (n-1)$ for each $n \geq 2$.

8. Let $\{A_k : k = 1, 2, ..., n\}$ be a collection of n events in a probability space $(\Omega, \mathcal{F}, P)$. Define a new collection of events as follows: Let $B_1 = A_1$, $B_2 = A_2/A_1, B_3 = A_3/(A_1 \cup A_2), ...,$ and $B_n = A_n/(\bigcup_{k=1}^{n-1} A_k)$.

   (a) Show that $B_k \in \mathcal{F}$ for each k.

   (b) Show that $P(\bigcup_{k=1}^{n} A_k) = P(\bigcup_{k=1}^{n} B_k)$.