COMPREHENSIVE EXAM
ALGEBRA
Spring 2019

Part I: Group Theory (Do 4 of the following 5 problems)

1. Let $\sigma = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8) \in S_8$ and let $G = S_8$
   (a) Determine, with explanation, the size of the conjagacy class of σ in $G = S_8$
   (b) Prove $C_G(\sigma) = \langle\sigma\rangle$ (where $C_G(\sigma)$ denotes the centralizer of σ in $G = S_8$).
   (c) Determine $|C_H(\sigma)|$ for $H = A_8$.

2. Assume that H and K are normal subgroups of a group G and that $G = HK$. Further assume that each element $g \in G$ can be written uniquely in the form $g = hk$ where $h \in H$ and $k \in K$.
   (a) Prove that $H \cap K = \{e\}$
   (b) Prove that if $h \in H$ and $k \in K$, then $hk = kh$.
   (c) Define $\phi : G \to H \times K$ by $\phi(hk) = (h, k)$ for each $g = hk \in HK$ ($h \in H$ and $k \in K$). Prove that ϕ is an isomorphism.

3. Let G be a group and let $a \in G$ such that $\circ(a) = n$.
   (a) Let $i, j \in \mathbb{Z}$. Prove that if $\circ(a^i) = \circ(a^j)$, then $\gcd(i, n) = \gcd(j, n)$
   (b) Let $s \in \mathbb{Z}$ and let $d = \gcd(s, n)$. Prove $\circ(a^d) = \circ(a^s) = \dfrac{n}{d}$.
   (c) Define $U(n) = \{[k] \in \mathbb{Z}_n : \gcd(k, n) = 1\}$. You may assume that $U(n)$ is a group under the multiplicative operation in $\mathbb{Z}_n$. Assume that the group G is cyclic with $G = \langle a\rangle$ where $\circ(a) = n$. Prove that $\mathrm{Aut}(G)$, the group of automorphisms of G, is isomorphic to $U(n)$.

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4. (a) Let G be a cyclic group of order n and assume $k \mid n$. Prove that G has exactly one subgroup of order k.
   (b) Let G be a finite group such that p is a prime and p divides $|G|$. Let P be a p-Sylow subgroup of G such that P is cyclic and $P \lhd G$. Let H be a subgroup of P. Prove $H \lhd G$.

5. Let G be a group of order p^2q^2 where p and q are primes and $p < q$.
   (a) Prove that G is not simple.
   (b) Prove that if $p = 5$ and $q = 7$, then G is Abelian.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let ω be a primitive 3rd root of unity.

   (a) Describe each element of $\text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \omega)/\mathbb{Q})$ as a permutation on the subscripts of

   $$c_1 = \sqrt[3]{2}, \qquad c_2 = \omega \cdot \sqrt[3]{2}, \qquad c_3 = \omega^2 \cdot \sqrt[3]{2}.$$

   (b) For $H = \langle (1\ 2\ 3) \rangle$, find the subfield of $\mathbb{Q}(\sqrt[3]{2}, \omega)$ that corresponds to H under the Fundamental Theorem of Galois Theory.

2. Let R be a commutative ring with unity. For each ideal I in R and each $a \in R$. Define

   $$(I : a) = \{r \in R : ar \in I\}$$

   (a) Prove that $(I : a)$ is an ideal.

   (b) Prove that if I is a prime ideal, then $(I : a) = I$ or $(I : a) = R$.

   (c) Let $R = \mathbb{Q}[x]$, let $I = \langle x^4 - 1 \rangle$ and let $a = x^2 + 1$. Prove that $(I : a) = \langle x^2 - 1 \rangle$

3. Let $p = (x^2 - 4)(x^2 + 3) \in \mathbb{Q}[x]$. Let $I = \langle p \rangle$.

   (a) Determine, with proof, all ideals in $\mathbb{Q}[x]$ which contain I.

   (b) Determine all ideals in $\mathbb{Q}[x]/I$.

   (c) For which of the ideals, J in part (a) is $\mathbb{Q}[x]/J$ a field? (Explain)

4. Let E be an extension field of the field F. Let $c \in E$.

   (a) Prove that $c \in E$ is algebraic over F if and only if $[F(c) : F]$ is finite.

   (b) Suppose $[E : F] = p$ for some prime p. Prove that E is a simple extension of F.

   (c) Suppose $[F(c) : F] = 5$. Determine, with explanation, $[F(c^3) : F]$.

5. Let R be a principal ideal domain.

   (a) Let $I = \langle a \rangle$ be a nonzero ideal in R. Prove that I is a prime ideal if and only if a is irreducible.

   (b) Prove that every nonzero prime ideal in R is a maximal ideal.

   (c) Suppose $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ where each I_n is an ideal in R. Prove that there exists some natural number N such that $I_j = I_k$ for all $j, k \geq N$. (Hint: for a first step, show $\bigcup_{n \geq 0} I_n$ is an ideal in R.)