Lie Theory Notes

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Recall: An ideal I iof I of L is a subspace such that

$$[x, y] \in I, \quad \forall x \in I, \forall y \in L.$$

Definition 1.1. A **simple** Lie algebra L is one with no non-trivial ideal and L is not abelian.

- **Q**: What does it mean for L to be abelian?
- **A**: We mean $[x, y] = 0$ for any $x, y \in L$.

Given a vector space, you can always attach an abelian Lie algebra structure. Suppose $dim(L) = 1$. Then any basis for L has one basis element.

- **Q**: What are the different types of $[\cdot, \cdot]$ can we define?
- **A**: For all $a, b \in L$, $[a, b] = (ab)[e_1, e_1] = 0$. So for any 1-dimensional Lie algebra, is has to be abelian. Thus, up to isomorphism, there is only one Lie algebra on dimension 1 and it is abelian.

Lemma 1.1. *Suppose I, J are ideals of L. Then*

$$I + J = \{x + y \mid x \in I, y \in J\}$$

and

$$[I, J] = \{[x, y] \mid x \in I, y \in J\} \tag{1}$$

are ideals. ((1) is the set of all linear combinations of elements, or the span/set generated by, the elements in $[I, J]$).

Example 1.1. $[L, L]$ is an ideal. Note if $[L, L] = \{0\}$, then L is abelian. Suppose $dim(()L) = 3$. Then is could be (with $L' := [L, L]$) that

- $dim(L') = 3 \to [L, L] = L$
- $dim(L') = 2$
- $dim(L') = 1$
- $dim(L') = 0 \to L$ is abelian.

Definition 1.2. The set

$$Z(L) := \{x \in L \mid [x, y] = 0, \forall y \in L\}$$

is called the **center** of L.

Lemma 1.2. *$Z(L)$ is an ideal of L.*

- **Q**: Is $Z(L)$ abelian? i.e., what is $[Z(L), Z(L)] = Z(L)' =?$
- **A**: Yes. Note the differences between L' and $Z(L)$. So the L' is the non-abelian piece and the center is the abelian piece.

We want to start classifying simple Lie algebras. Later we'll be interested in cases where $L' = L$ since if it doesn't satisfy this it will not be simple.

1.1 Some chit-chat

For any $x \in L$, with $\dim(L) = n$, we can define a map $\mathrm{ad}_x : L \to L$ where $y \mapsto \mathrm{ad}_x(y) := [x, y]$ and this is a homomorphism, i.e., $\mathrm{ad}_x \in \mathrm{gl}(L) = \mathrm{gl}_n(\mathbb{C})$. Next level: there exists a map

$$\mathrm{ad} : L \to \mathrm{gl}; \quad x \mapsto \mathrm{ad}_x \tag{2}$$

and ad is a Lie algebra homomorphism. Is ad 1-1? In other words, does $\ker(\mathrm{ad}) = \{0\}$? In other words,

$$\ker(\mathrm{ad}) = \{x \in L \mid \mathrm{ad}_x = 0\} = \{\mathrm{ad}_x \mid \forall y \in L\} = \{[x, y] = 0 \mid \forall y \in L\} = Z(L).$$

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2.1 Structure Constants

Remark. $\mathfrak{sl}_2(\mathbb{C}) = \{A \in \mathrm{Mat}_2(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$, $[A, B] = 0$. 3 dimensional Lie algebra with basis $\mathcal{B} = \{h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$

Suppose we have

$$[x, y] = h, \quad [h, x0] = 2x, \quad [h, y] = -2y.$$

For a fixed $x \in L$, define $\mathrm{ad}_x : L \to L$, by $y \mapsto \mathrm{ad}_x y := [x, y]$. This is a linear map, i.e., $\mathrm{ad}_x \in \mathrm{End}(L) \cong \mathrm{Mat}_n(\mathbb{C}) \cong \mathfrak{gl}(L) \cong \mathfrak{gl}_n(\mathbb{C})$.

2.2 Representations

Definition 2.1. A Lie algebra homomorphism $\varphi : L \to \mathrm{gl}(V) \cong \mathrm{gl}_n(\mathbb{C})$ for some vector space V and $\dim(V) = n$ is called **representation** of L.

Definition 2.2. We call a vector space V an L-module if there is a map

$$\begin{aligned} L \times V &\to V \\ (x, v) &\mapsto x.v \end{aligned}$$

such that $\forall x, y \in L$, $u, v \in V$, and $\alpha, \beta \in F$ where

1. $(\alpha x + \beta y).u = \alpha(x.u) + \beta(y.u)$
2. $x.(\alpha u + \beta v) = \alpha(x.u) + \beta(x.v)$

x, y .u=x.(y.u)-y.(x.u)

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Recall: $\varphi : L \to \mathfrak{gl}_n(V)$ if φ is a Lie homomorphism. Often refter ro V as the representation.

A vector space V is an L-module if $L \times V \to V$ and $(x, v) \mapsto x.v$. This action is biliniear.

Example 3.1. Let $\varphi = \mathrm{ad}$, $V = L$. We've seen $\mathrm{ad} : L \to \mathfrak{gl}_n(L)$ is a representation and $x \mapsto \mathrm{ad}(x) := \mathrm{ad}_x$.