STAT 215A

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Instructor: Dr. Cetin — Assignment: Homework 04

1. Let $\{X_k : k \in \mathbb{N}\}$ be a sequence of i.i.d. discrete random variables such that each member has the following pmf: $P(X_k = 1) = p$ and $P(X_k = -1) = 1 - p$, where $0 < p < 1$. For $n \in \mathbb{N}$, define $S_n = \sum_{k=1}^{n} X_k$.

   (a) Determine the pmf of $S_3 = \sum_{k=1}^{3} X_k$.

   Solution. We begin by noting that S_n is a discrete random variable as the sum of discrete random variables. Moreover, for any $n \in \mathbb{N}$, we have $-n \leq S_n \leq n$. Also, if $S_n = m$ where $-n \leq m \leq n$, then this means that there are a total of m-many occurances of a 1 and $n-m$ many occurances of -1. There are $\binom{n}{m}$ ways in which m many 1s can occur in the sum, and to each of those ways corresponds a way in which $n-m$ many -1s can occur in the sum. Lastly, m many 1s occur with probability p^m and $n-m$ many -1s occur with probability $(1-p)^{n-m}$. Therefore

$$p(x) = \mathbb{P}(S_3 = x) = \begin{cases} \binom{3}{x} p^x (1-p)^{3-x} & \text{if } 0 \leq x \leq 3, \\ \binom{3}{3+x} p^{3+x} (1-p)^{-x} & \text{if } -3 \leq x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

■

   (b) Find the mean and variance of S_n, for each n.

   Solution. Let $n \in \mathbb{N}$. Then for any $1 \leq k \leq n$, we have that

$$E[X_k] = \sum_x xp(x) = (1)p + (-1)(1-p) = 2p - 1.$$

   And since each X_k is independent

$$\begin{aligned} E[S_n] &= E[\sum_{k=1}^{n} X_k] \\ &= \sum_{k=1}^{n} E[X_k] \\ &= \sum_{k=1}^{n} 2p - 1 \\ &= n(2p-1). \end{aligned}$$

Warning: Below I obtained two different answers based on two different approached and I am not sure which one is correct so I decided to include them both.

(i) To calculate the variance, we start by computing $E[S_n^2]$. Letting $Y = g(S_n)$, where $g(X) = X^2$, then by LOTUS, we have that

$$
\begin{aligned}
E[Y] &= E[S_n^2] \\
&= \sum_{x=-n}^{n} g(x)p(x) \\
&= \sum_{x=-n}^{-1} g(x)p(x) + \sum_{x=0}^{n} g(x)p(x) \\
&= \sum_{x=-n}^{-1} x^2 \binom{n}{n+x} p^{n+x}(1-p)^{-x} + \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x^2 \binom{n}{n-x} p^{n-x}(1-p)^{x} + \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\
&= n(p-1)(n(p-1)-p) + p((n-1)np+n) \\
&= 2n^2p^2 - 2n^2p - 2np^2 + 2np + n^2.
\end{aligned}
$$

Thus the variance is

$$
\begin{aligned}
Var(S_n) &= E[(S_n-\mu)^2] \\
&= E[S_n^2] - \mu^2 \\
&= 2n^2p^2 - 2n^2p - 2np^2 + 2np + n^2 - (2np-n)^2 \\
&= 2np(1+n-p-np).
\end{aligned}
$$

(ii) Note that for any $1 \le k \le n$, we have that

$$Var(X_k) = E[X_k^2] - \mu^2 = 2(1-p).$$

Since all of the X_k are independent, then we have that

$$Var(S_n) = Var(\sum_{k=1}^{n} X_k) = \sum_{k=1}^{n} Var(X_k) = 2n(1-p).$$

■

(c) Consider the symmetric case where $p = 1/2 = 1-p$. Determine $Cov(S_n, S_m)$ for all $m, n \in \mathbb{N}$.

Solution. ■

2. Let X be a continuous r.v. with the following pdf: For fixed $C > 0$, let $f(x) = \frac{C}{x^{C+1}}$ for $x > 1$ (and it is zero otherwise).

   (a) Verify that f is a valid pdf for all $C > 0$. Then, determine the cdf of X.

   Solution. For any $x > 1$, $x^{C+1} > 1$ since $C > 0$ and thus $f(x) > 0$ for all $x > 1$. The indefinite integral of $f(x)$ is

$$\int \frac{C}{x^{C+1}} dx = C \int x^{-C-1} dx = C(-\frac{1}{C} x^{-C}) = -x^C.$$

   Thus

$$\int_1^\infty \frac{C}{x^{C+1}} = -x^C \Big|_1^\infty = \lim_{x\to\infty} -\frac{1}{x^C} + 1 = 1.$$

   This shows that $f(x)$ is a valid pdf. ■

   (b) Compute the expected value of X.

   Solution. We start by computing the following indefinite integral:

$$\int x \frac{C}{x^{C+1}} dx = C \int x^{-C} dx = \frac{C}{1-C} x^{1-C}.$$

   Thus if $C > 1$, then

$$E[X] = \int_1^\infty x \frac{C}{x^{C+1}} dx = \frac{C}{1-C} \left(\lim_{x\to\infty} \frac{x}{x^C} - 1 \right) = -\frac{C}{1-C}.$$

   Otherwise, if $0 < C < 1$, then $E[X] = \infty$. ■

   (c) Determine the median of X.

   Solution. To find the median, we need to find a value $m > 0$ such that

$$\int_1^m \frac{C}{x^{C+1}} dx = \frac{1}{2}.$$

   Using part (a), we can state this as

$$-x^{-C} \Big|_1^m = \frac{1}{2}.$$

   From this it follows that

$$-\frac{1}{m} + 1 = \frac{1}{2} \Rightarrow m = 2.$$

   ■

   (d) Determine the cdf of $Y = 1/X$.

Solution. We note that since $f(x)$ is strictly decreasing over $[1, \infty)$, then $y = g(x) = 1/x$ is strictly increasing and thus continuous and has a well defined inverse. Additionally, since $f(x)$ has a range $(0, C)$, then $g(x)$ has a range $(\frac{1}{C}, \infty)$. It is also differentiable over $(0, \infty)$, with derivative $g'(x) = -1/x^2$. For any $y \in (\frac{1}{C}, \infty)$, we have that $g^{-1}(y) = 1/y$ and so

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))} = \frac{Cy^{C+1}}{-y^2} = -Cy^{C-1}.$$

Thus the cdf of Y is

$$F_Y(y) = \int_0^y -Ct^{C-1} dt.$$

■