MATH 210B

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Instructor: Dr. Shannon **Assignment:** Homework 7

10. Let $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{5}, i)$. For $0 \leq j \leq 3$, define $\varphi_j \in \mathrm{Aut}(E)$ by $\varphi_j(i) = (-1)^j i$ and $\varphi_j(\sqrt[4]{5}) = i^j \sqrt[4]{5}$, and for $4 \leq j \leq 7$, define $\varphi_j \in \mathrm{Aut}(E)$ by $\varphi_j(i) = (-1)^{j+1} i$ and $\varphi_j(\sqrt[4]{5}) = i^j \sqrt[4]{5}$.

    (a) Find $G(E/\mathbb{Q}(i))$, $G(E/\mathbb{Q}(\sqrt[3]{2}))$, $G(E/\mathbb{Q}(\sqrt[4]{5}))$, $G(E/\mathbb{Q}(\sqrt[4]{5}, i))$.

    Solution. Considering all the automorphisms which map i to i we get that $G(E/\mathbb{Q}(i)) = \{\varphi_0, \varphi_2, \varphi_5, \varphi_7\}$. Note that $x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over E and so E only contains one root of this polynomial. Thus, every automorphism must map $\sqrt[3]{2}$ to itself and so $G(E/\mathbb{Q}(\sqrt[3]{2})) = \mathrm{Aut}(E)$. Considering all the automorphisms which map $\sqrt[4]{5}$ to itself, we get that $G(E/\mathbb{Q}(\sqrt[4]{5}) = \{\varphi_0, \varphi_4\}$. Finally, there is only one automorphism which fixes both i and $\sqrt[4]{5}$ and so $G(E/\mathbb{Q}(\sqrt[4]{5}, i)) = \{\varphi_0\}$.

    (b) Find $[E\colon \mathbb{Q}(i)]$, $[E\colon \mathbb{Q}(\sqrt[3]{2})]$, $[E\colon \mathbb{Q}(\sqrt[4]{5})]$, $[E\colon \mathbb{Q}(\sqrt[4]{5}, i)]$.

    Solution. Since the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}(i)$ is $x^3 - 2$ and the minimal polynomial of $\sqrt[4]{5}$ over $\mathbb{Q}(i)$ is $x^4 - 5$, then we have that $[E\colon \mathbb{Q}(i)] = 3 \cdot 4 = 12$. Similarly, since $x^4 - 5$ is the minimal polynomial of $\sqrt[4]{5}$ over $\mathbb{Q}(\sqrt[3]{2})$ and $x^2 + 1$ is the minimal polynomial of i over $\mathbb{Q}(\sqrt[3]{2})$, then $[E\colon \mathbb{Q}(\sqrt[3]{2})] = 4 \cdot 2 = 8$. Again, we have that $x^3 - 2$ is the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}(\sqrt[4]{5})$ and $x^2 + 1$ is the minimal polynomial of i over $\mathbb{Q}(\sqrt[4]{5})$ and so $[E\colon \mathbb{Q}(\sqrt[4]{5})] = 3 \cdot 2 = 6$. Lastly, the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}(\sqrt[4]{5}, i)$ is $x^3 - 2$ and so $[E\colon \mathbb{Q}(\sqrt[4]{5}, i)] = 3$.

    (c) Determine which of these extensions are normal.

    Solution. Note that $x^3 - 2$ is irreducible over $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt[4]{5})$, and $\mathbb{Q}(\sqrt[4]{5}, i)$, and E contains a root of $x^3 - 2$, yet $x^3 - 2$ does not split over E and so none of these extensions are normal. Now note that $x^3 - 2$ is reducible over $\mathbb{Q}(\sqrt[3]{2})$, but its nonlinear irreducible factor is $x^2 + \sqrt[3]{2}x + \sqrt[3]{4}$. E does not contain a root of this polynomial and so all other irreducible polynomials over $\mathbb{Q}(\sqrt[3]{2})$ split over E. Thus, $E/\mathbb{Q}(\sqrt[3]{2})$ is a normal extension.

    (d) Determine which of these extensions are separable.

    Solution. We have that both $\sqrt[3]{2}$ and $\sqrt[4]{5}$ are algebraic over $\mathbb{Q}(i)$ and neither $x^3 - 2$ and $x^4 - 5$ have multiple roots in any extension. Thus, $E/\mathbb{Q}(i)$ is separable. By the same reasoning we can see that i, $\sqrt[3]{2}$, and $\sqrt[4]{5}$ are all algebraic over each of the base fields and all of the respective minimal polynomials do not contain multiple roots. Thus, every extension is separable.

(e) For each K, determine $F_{G(E/K)}$, and determine if $F_{G(E/K)} = K$.

Solution. Based on part (a), we have that

$$\begin{aligned}
F_{G(E/\mathbb{Q}(i))} &= \mathbb{Q}(\sqrt[3]{2}, i) \\
F_{G(E/\mathbb{Q}(\sqrt[3]{2}))} &= \mathbb{Q}(\sqrt[3]{2}) \\
F_{G(E/\mathbb{Q}(\sqrt[4]{5}))} &= \mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{5}) \\
F_{G(E/\mathbb{Q}(\sqrt[4]{5}, i))} &= \mathbb{Q}(\sqrt[3]{2}, \sqrt[4]{5}, i).
\end{aligned}$$

$E/\mathbb{Q}(\sqrt[3]{2})$ satisfies $F_{G(E/K)} = K$.