INVITATION TO LIE ALGEBRAS AND REPRESENTATIONS

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Abstract. In this paper, we outline the rudiments of the representation theory of semisimple Lie algebras. We build the necessary theory in order to analyze the representations of $\mathfrak{sl}_2$, which includes proving that representations of semisimple Lie algebras are completely reducible and preserve the Jordan decomposition. We only assume the reader has a working knowledge of linear algebra and a little familiarity with abstract algebra. Most of the proofs follow from [1], [2] and [4].

Contents

Throughout this paper, $\mathbf{F}$ denotes an algebraically closed field of characteristic zero (e.g., $\mathbb{C}$) and all vector spaces (including Lie algebras and representations) are finite-dimensional over $\mathbf{F}$.

1. Introduction

Lie algebras arise in the context of Lie groups, defined to be groups that are also smooth manifolds, with smooth product and inverse operations. The Lie algebra of a Lie group is defined to be the tangent space to the Lie group at the identity, which is a vector space that also turns out to be closed with respect to a certain bilinear operation. This makes Lie algebras a much less complicated object of study than Lie groups, and thus it is a nice fact that so much information about a Lie group is contained on its Lie algebra.

In this paper, we develop some rudiments of the representation theory of semisimple Lie algebras. Representation theory studies the homomorphisms of a group or algebra into the group or algebra of endomorphisms of a vector space. It tries to

Date: August 2014.

use the well-known ideas of linear algebra to understand more abstract algebraic structures.

One of the reasons representations of Lie algebras are studied is that they give so much information about representations of Lie groups, which were (and still are) an object of great interest to mathematicians. However, as it will become evident, Lie algebras are too vague an algebraic structure for us to readily study their representations. For instance, it is a general goal of representation theory to determine all irreducible representations of a group or an algebra, in the hope this will describe all of its representations. But we are not even sure this helps in our Lie algebra case, for we do not know whether all Lie algebra representations are direct sums of irreducible representations. It is necessary, therefore, to find certain special kinds of Lie algebras that will be well behaved enough for our purposes, which will turn out to be the **semisimple** Lie algebras.

The strategy of the paper is as follows. In Section 2, we define the basic concepts of Lie algebras and representation theory. Then, in Sections 3 and 4, we develop the necessary background for the definition of a semisimple Lie algebra, which involves discussion about solvable and nilpotent Lie algebras. In sections 5, 6 and 7 we work on results that show how the good behavior of semisimple Lie algebras is important for studying representations. In particular, we show representations of Lie algebra are completely reducible (Section 6) and preserve the Jordan decomposition (Section 7). Finally, we work in an important example in Section 8—analyzing the representations of the semisimple Lie algebra $\mathfrak{sl}_2(\mathbf{F})$.

2. Basics of Lie algebras and representations

Definition 2.1. A **Lie algebra** $\mathfrak{g}$ is a vector space $\mathfrak{g}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, which satisfies, for every $x, y, z \in \mathfrak{g}$:

(i) $[x, y] = -[y, x]$
(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

Item (ii) above is sometimes refered to as the **Jacobi identity**. An imporant observation that immediately follows from item (i), the so-called **skew-commutativity**, is that, given $x \in \mathfrak{g}$, we have $[x, x] = 0$. A Lie algebra is said to be **abelian** if $[x, y] = 0$ for every $x, y \in \mathfrak{g}$.

As an example, consider the Lie algebra $\mathfrak{gl}(V)$ of endomorphisms (i.e., linear transformations $V \to V$) of a vector space V. It is the vector space of endomorphisms of V with the Lie bracket defined as

$$[x, y] := xy - yx$$

for every endomorphisms $x, y \in \mathfrak{gl}(V)$, where the concatenation xy denotes composition of functions. The bracket (also called a **commutator** if defined as above) is bilinear and satisfies items (i) and (ii), so $\mathfrak{gl}(V)$ is indeed a Lie algebra.

Given this vector space of linear transformations $V \to V$, one can also define the **ring of endomorphisms** $\operatorname{End} V$, where the ring operation is just composition of functions. I mention this because $\operatorname{End} V$ should not be thought of as the same object as $\mathfrak{gl}(V)$, for their 'product' operations are distinct.

A **Lie algebra homomorphism** $\rho : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map that preserves the Lie bracket: $\rho([x, y]) = [\rho(x), \rho(y)]$ for every $x, y \in \mathfrak{g}$. A **Lie algebra isomorphism** is an invertible Lie algebra homomorphism.

From linear algebra, we know that $\operatorname{End} V$ is in bijection with $M_n(\mathbf{F})$, the ring of all n by n matrices with coefficients in $\mathbf{F}$, where $n = \dim V$. In this spirit, we also remark the Lie algebra $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}_n(\mathbf{F})$, which is the vector space of all n by n matrices in $\mathbf{F}$ together with a commutator. Matrix realizations will be important throughout the paper.

A (Lie) **subalgebra** $\mathfrak{h}$ of a Lie algebra is a vector space closed under the Lie bracket, i.e., such that $[x, y] \in \mathfrak{h}$ for every $x, y \in \mathfrak{h}$. An **ideal** $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a subalgebra such that $[x, y] \in \mathfrak{i}$ for every $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$. Another way of phrasing this is saying that an ideal $\mathfrak{i}$ of $\mathfrak{g}$ is a subalgebra such that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$, where $[\mathfrak{g}, \mathfrak{i}]$ is the subalgebra generated by elements of the form $[x, y]$, with $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$. An example of ideal that will be important throughout the paper is the **center** $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, defined as $\{x \in \mathfrak{g} \colon [x, z] = 0 \text{ for all } z \in \mathfrak{g}\}$. (The '$Z$' comes from the German *Zentrum.*)

Ideals allow us to talk about **quotients** of Lie algebras. Given a Lie algebra $\mathfrak{g}$ and an ideal $\mathfrak{i}$ of $\mathfrak{g}$, we define the **quotient algebra** $\mathfrak{g}/\mathfrak{i}$ as the quotient space

$$\mathfrak{g}/\mathfrak{i} = \{x + \mathfrak{i} \colon x \in \mathfrak{g}\}$$

of cosets with the Lie bracket defined as $[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}$ for every $x, y \in \mathfrak{g}$. Since $\mathfrak{i}$ is an ideal, this bracket operation is well defined.

As a consequence of the above definitions, we have the so-called **isomorphism theorems**:

Proposition 2.2. *(i). If $\rho : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then $\mathfrak{g}/\ker\rho \cong \operatorname{Im}\rho$.*

(ii). If $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of $\mathfrak{g}$, there is an isomorphism $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$.

Proof. See [8], pp. 9-10. (There, the discussion is about rings, but it is straightforward to translate it to the Lie algebra case.) □

A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We will denote $\rho(x)v$ by $x \cdot v$ when convenient, where $x \in \mathfrak{g}$ and $v \in V$.

We say that ρ gives V the structure of a **$\mathfrak{g}$-module**, which is a vector space V together with an operation $\mathfrak{g} \times V \to \mathfrak{g}$ given by $(x, v) \mapsto \rho(x)v$, called the **action** of $\mathfrak{g}$ on V. (We say $\mathfrak{g}$ **acts** on V via ρ.) We often refer to the $\mathfrak{g}$-module V itself as a representation, but that will not lead to ambiguity because, as we will see, the notions of $\mathfrak{g}$-modules and representations are equivalent.

Given two $\mathfrak{g}$-modules V and W, we say a linear map $\phi : V \to W$ is a **$\mathfrak{g}$-module homomorphism** if

$$\phi(x \cdot v) = x \cdot \phi(v)$$

for every $x \in \mathfrak{g}$ and $v \in V$. A **$\mathfrak{g}$-module isomorphism** is a $\mathfrak{g}$-module homomorphism that is also a vector space isomorphism.

We say a subspace $W \subseteq V$ of a $\mathfrak{g}$-module V is a **$\mathfrak{g}$-submodule** or a **subrepresentation** if it obeys $\rho(x)V \subseteq V$ for every $x \in V$. A $\mathfrak{g}$-module is called **irreducible** if it has no nonzero proper $\mathfrak{g}$-submodules. (In particular, $V = 0$ is not irreducible.)

Given a $\mathfrak{g}$-submodule W of V, we are able to define the **quotient $\mathfrak{g}$-module** by endowing the quotient space $V/W = \{v + W \colon v \in V\}$ with an action $\mathfrak{g} \times V/W \to \mathfrak{g}$ given by $(x, v + W) \mapsto \overline{\rho}(x)(v + W)$, where $\overline{\rho}(x)(v + W) := \rho(x)v + W$. Notice that $\overline{\rho} : \mathfrak{g} \to \mathfrak{gl}(V/W)$ is indeed a Lie algebra homomorphism. Moreover, because W is a $\mathfrak{g}$-submodule, the map $\overline{\rho}(x)$ (which we can also denote by $\overline{\rho(x)}$) is well-defined for all $x \in \mathfrak{g}$.

If $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(U)$ and $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(V)$ are representations, then we define the **direct sum** representation $\rho_1 \oplus \rho_2 : \mathfrak{g} \to \mathfrak{gl}(U \oplus V)$ by

$$(\rho_1 \oplus \rho_2)(x)(v_1, v_2) = (\rho_1(x)(v_1), \rho_2(x)(v_2)),$$

where $x \in \mathfrak{g}$ and $(v_1, v_2) \in U \oplus V$. In this case, we regard the $\mathfrak{g}$-module $U \oplus V$ as a direct sum of $\mathfrak{g}$-modules.

We say a $\mathfrak{g}$-module V is **completely reducible** if it can be written as a direct sum of irreducible $\mathfrak{g}$-submodules. This is equivalent to saying that for every $\mathfrak{g}$-submodule W, there is a $\mathfrak{g}$-submodule W' such that $V = W \oplus W'$. The proof of this last statement is non-trivial, although boring. Consult [7], p. 119 for details. There, and later in the paper, the following lemma is used:

Lemma 2.3 (Schur)**.** *Let $\phi : V \to V$ be a $\mathfrak{g}$-module homomorphism of V into itself, with V irreducible. Then, $\phi = \lambda \,\mathrm{id}$ for some $\lambda \in \mathbf{F}$*

Proof. The field $\mathbf{F}$ is algebraically closed, so ϕ has an eigenvector $v \neq 0$ associated to some eigenvalue λ. Since ϕ is a $\mathfrak{g}$-module endomorphism, the linear maps ϕ and $\rho(x)$ commute for every $x \in \mathfrak{g}$. In particular, $\phi(\rho(x)v) = \rho(x)\phi(v)$ where v is the eigenvector above, which implies $\phi(\rho(x)v) = \lambda\, \rho(x)v$. Thus, $\rho(x)v$ is also an eigenvector of ϕ to eigenvalue λ, for every $x \in \mathfrak{g}$. Since V is irreducible and v nonzero, the submodule generated by v must be all of V. Thus every vector in V is an eigenvector of ϕ to eigenvalue λ and $\phi = \lambda \,\mathrm{id}$. □

An important example of a representation is the **adjoint representation** $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}\, x(y) = [x, y]$, for any $x, y \in \mathfrak{g}$. The adjoint representation makes $\mathfrak{g}$ itself be a $\mathfrak{g}$-module, in which the $\mathfrak{g}$-submodules are the ideals of $\mathfrak{g}$.

3. Solvability and nilpotency; theorems of Lie and Engel

Before talking about semisimple Lie algebras, it is necessary to talk about **solvable** and **nilpotent** Lie algebras.

First we define the **lower central series** of $\mathfrak{g}$ recursively by

$$\mathscr{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad \mathscr{D}_k\mathfrak{g} = [\mathfrak{g}, \mathscr{D}_{k-1}\mathfrak{g}]$$

and the **derived series** of $\mathfrak{g}$ by

$$\mathscr{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad \mathscr{D}^k\mathfrak{g} = [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}],$$

where $[\mathfrak{g}, \mathfrak{g}] =: \mathscr{D}\mathfrak{g}$. The first thing to observe here is that both $\mathscr{D}_k\mathfrak{g}$ and $\mathscr{D}^k\mathfrak{g}$ are ideals in $\mathfrak{g}$ for every $k > 0$. (For the derived series, this can be shown using induction on k and the Jacobi identity.)

We now make the following

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra.

- We say $\mathfrak{g}$ is **nilpotent** if $\mathscr{D}_k\mathfrak{g} = 0$ for some k.
- We say $\mathfrak{g}$ is **solvable** if $\mathscr{D}^k\mathfrak{g} = 0$ for some k.

Three simple observations follow:

Proposition 3.2. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is also nilpotent.
(ii) If $\mathfrak{g}$ is solvable, then so are all of its subalgebras and homomorphic images.
(iii) If $\mathfrak{h}$ is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathfrak{g}$ is also solvable.

Proof. **(i).** If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then for some k we have $\mathscr{D}_k\mathfrak{g} \subseteq Z(\mathfrak{g})$. Therefore, $\mathscr{D}_{k+1}\mathfrak{g} = 0$.

(ii). Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we have $\mathscr{D}^k\mathfrak{h} \subseteq \mathscr{D}^k\mathfrak{g}$. Moreover, if ρ is a Lie algebra homomorphism, then it can be shown by induction that $\rho(\mathscr{D}^k\mathfrak{g}) = \mathscr{D}^k\rho(\mathfrak{g})$.

(iii). If $\mathfrak{g}/\mathfrak{h}$ is solvable, then $\mathscr{D}^k\mathfrak{g} \subseteq \mathfrak{h}$ for some k. Therefore, since $\mathfrak{h}$ is solvable, it follows that $\mathscr{D}^m\mathfrak{h} = 0$ for some m and hence $\mathscr{D}^{k+m}\mathfrak{g} = 0$. □

Observe that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if there exists $k \in \mathbb{N}$ such that for every $x_1, \ldots, x_n, y \in \mathfrak{g}$, we have

$$[x_1, [\ldots, [x_k, y] \ldots]] = \operatorname{ad} x_1 \cdots \operatorname{ad} x_k(y) = 0.$$

In particular, if we take $x_1 = \cdots = x_k$, we have $(\operatorname{ad} x)^k = 0$, that is, $\operatorname{ad} x$ is a nilpotent endomorphism. Under these circumstances, we say x is **ad-nilpotent** and we can readily observe that *every element of a nilpotent Lie algebra is ad-nilpotent*. The converse of this will be important for us in the future:

Theorem 3.3 (Engel)**.** *Let $\mathfrak{g}$ be a Lie algebra. If all elements of $\mathfrak{g}$ are ad-nilpotent, then $\mathfrak{g}$ is nilpotent.*

Before proving this, we need a few lemmas. The first one will be quite useful by itself:

Lemma 3.4. *Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism. Then $\operatorname{ad} x$ is also nilpotent.*

Proof. Let $y \in \operatorname{End} V$. Observe that x defines two different endomorphisms of $\operatorname{End} V$ —left translation, defined by $\ell_x(y) := xy$ and right translation, defined by $r_x(y) := yx$. Notice furthermore that $\operatorname{ad} x = \ell_x - r_x$ and that ℓ_x and r_x commute. Now, it is a fact, in any ring with unity, that the sum or difference of two nilpotent elements that commute is also nilpotent. To see this, let a and b be the commuting nilpotent elements. If N is such that $a^N = b^N = 0$, just take some $M > 2N$ and calculate

$$(a+b)^M = a^M + \cdots + \binom{M}{\lceil M/2 \rceil} a^{\lfloor \frac{M}{2} \rfloor} b^{\lceil \frac{M}{2} \rceil} + \binom{M}{\lfloor M/2 \rfloor} a^{\lceil \frac{M}{2} \rceil} b^{\lfloor \frac{M}{2} \rfloor} + \cdots + b^M.$$

Since $\lceil M/2 \rceil > N$, it is clear that the above sum is zero and hence $a+b$ is nilpotent. Thus, $\operatorname{ad} x = \ell_x - r_x$ is nilpotent. □

Lemma 3.5. *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms. Then, there exists a nonzero $v \in V$ that is an eigenvector to eigenvalue zero for every endomorphism $x \in \mathfrak{g}$.*

Proof. We will proceed by induction on $\dim \mathfrak{g}$. The case $\dim \mathfrak{g} = 1$ is easy—since every endomorphism $x \in \mathfrak{g}$ is nilpotent, some power of x is a nonzero eigenvector to eigenvalue zero. Otherwise, our goal will be to find an ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ (i.e., $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$). If such ideal exists, there is some $y \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathbf{F}y$. Moreover, since $\dim \mathfrak{h} < \dim \mathfrak{g}$, we know, by induction, that there is some eigenvector $v \in V$ to eigenvalue zero for every endomorphism in $\mathfrak{h}$; i.e., a v such that $\mathfrak{h}(v) = 0$. Denoting by W the subspace of V consisting of vectors annihilated by $\mathfrak{h}$, it suffices to find some $v \in W$ that is also annihilated by y. Now, let $x \in \mathfrak{h}$ and $w \in W$. We have

$$xy(w) = yx(w) + [x, y](w).$$

Above, the first term on the right hand side is zero because $w \in W$ and $x \in \mathfrak{h}$. The second term is also zero—and here is the importance of $\mathfrak{h}$ being an ideal—we have that $[x, y] \in \mathfrak{h}$. Thus, it follows that $y(w) \in W$ and W is therefore y-stable. Now, using the fact that y is nilpotent, we just have to pick the largest k such that $y^k \neq 0$ and some $v \in W$ such that $y^k(v) \neq 0$. We have just shown $y^k(v) \in W$, so it must be our desired common eigenvector.

Now we prove this codimension 1 ideal $\mathfrak{h}$ exists. Let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$. We will show, still using induction on $\dim \mathfrak{g}$ assuming the result of the theorem for vector spaces of dimension less than $\dim \mathfrak{g}$, that $\mathfrak{h}$ is in fact an ideal of codimension 1. Indeed, consider $\mathfrak{g}$ as a $\mathfrak{g}$-module via the adjoint representation. Since $\mathfrak{g}$ consists of nilpotent endomorphisms, it follows, as remarked above, in Lemma 3.4, that $\operatorname{ad} \mathfrak{g}$ also consists of nilpotent endomorphisms. Now, the action of $\operatorname{ad} \mathfrak{h}$ preserves $\mathfrak{h}$ since $\mathfrak{h}$ is a Lie subalgebra. Therefore $\operatorname{ad} \mathfrak{h}$ acts on the quotient $\mathfrak{g}/\mathfrak{h}$. Now, by induction, since $\dim \mathfrak{g}/\mathfrak{h} < \dim \mathfrak{g}$, we have that there is some nonzero coset $y + \mathfrak{h} \in \mathfrak{h}$ such that $\operatorname{ad} \mathfrak{h}(y + \mathfrak{h}) = \mathfrak{h}$. In other words, there exists some $y \notin \mathfrak{h}$ such that $\operatorname{ad} x(y) \in \mathfrak{h}$ for every $x \in \mathfrak{h}$. This implies that $\mathfrak{h}$ is an ideal of codimension 1 in the subalgebra $\mathfrak{g}'$ spanned by $\mathfrak{h}$ and y. But since $\mathfrak{h}$ is a maximal proper subalgebra, it follows $\mathfrak{g}' = \mathfrak{g}$, so that $\mathfrak{h}$ is indeed an ideal of codimension 1 in $\mathfrak{g}$. □

Engel's theorem now follows:

Proof of 3.3. We proceed by induction on $\dim \mathfrak{g}$. The base case $\dim \mathfrak{g} = 1$ is trivial, so suppose $\dim \mathfrak{g} > 1$. Consider the image of $\mathfrak{g}$ under the adjoint representation, $\operatorname{ad} \mathfrak{g} \subseteq \mathfrak{gl}(V)$. By hypothesis, every element of $\operatorname{ad} \mathfrak{g}$ is nilpotent. Therefore, by the above theorem, there exists some nonzero $x \in \mathfrak{g}$ killed by $\operatorname{ad} \mathfrak{g}$, that is, $[\mathfrak{g}, x] = 0$. This implies $x \in Z(\mathfrak{g})$ and therefore $Z(\mathfrak{g}) \neq 0$, so $\dim \mathfrak{g}/Z(\mathfrak{g}) < \dim \mathfrak{g}$. Thus, since $\mathfrak{g}/Z(\mathfrak{g})$ also consists of ad-nilpotent elements, we conclude by induction that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. The desired conclusion follows by using observation (i) in Proposition 3.2. □

We now develop analogous results about solvable Lie algebras.

Theorem 3.6. *Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then, there exists a nonzero $v \in V$ that is an eigenvector for every endomorphism $x \in \mathfrak{g}$.*

Proof. We adapt the proof of 3.5 to solvable Lie algebras. We will use induction on $\dim \mathfrak{g}$, with the case in which $\dim \mathfrak{g} = 0$ being evident. The first step is to find an ideal $\mathfrak{h}$ of $\mathfrak{g}$ with codimension 1. Then, we find a set W of common eigenvectors for the endomorphisms of $\mathfrak{h}$ (Step 2) and verify this set is stabilized bt $\mathfrak{g}$ (Step 3). Finally, we locate some $z \in W$ which is a common eigenvector for all of $\mathfrak{g}$ (Step 4). We now flesh out those steps:

Step 1. To find an ideal of codimension 1, consider the quotient $\mathfrak{g}/\mathscr{D}\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, we have that $\mathscr{D}\mathfrak{g}$ is *properly* contained in $\mathfrak{g}$ and thus $\mathfrak{g}/\mathscr{D}\mathfrak{g}$ is a nontrivial abelian ideal. Now, let $\mathfrak{a}$ be a subspace of codimension 1 in $\mathfrak{g}/\mathscr{D}\mathfrak{g}$. The preimage of $\mathfrak{a}$ under the projection map $\pi : x \mapsto x + \mathscr{D}\mathfrak{g}$, let us name it $\mathfrak{h}$, has also codimension 1. Moreover, since $\mathfrak{h}$ contains $\mathscr{D}\mathfrak{g}$, it follows that $\mathfrak{h}$ is an ideal.

Step 2. By the above discussion, $\dim \mathfrak{h} < \dim \mathfrak{g}$. By induction we therefore conclude that $\mathfrak{h}$ has a common eigenvector v. Let us denote the subspace consisting of such eigenvectors by

$$W = \{w \in W \colon x(v) = \lambda(x)w \text{ for all } x \in \mathfrak{h}\},$$

where λ is a linear functional in $\mathfrak{h}^*$.

Step 3. Given $x \in \mathfrak{g}$ and $w \in W$, our task is to show that $x(w) \in W$. Let $y \in \mathfrak{h}$. Since

$$\begin{aligned} yx(w) &= xy(w) - [x,y](w) \\ &= \lambda(y)x(w) - \lambda([x,y])(w), \end{aligned}$$

the last equation above being justified by the fact that $\mathfrak{h}$ is an ideal, we will be done if we can show $\lambda([x,y]) = 0$.

With x and w as above, let $n > 0$ be the smallest integer such that the sequence $(w, x(w), \ldots, x^n(w))$ is linearly dependent. Define W_i as the vector space spanned by $\underline{w_i} := (w, x(w), \ldots, x^{i-1}(w))$ and set $W_0 = 0$. Notice that, for $i \leq n$ we have $\dim W_i = i$ and for $i \geq n$ we have $\dim W_i = n$. Now, every $y \in \mathfrak{h}$ leaves W_i invariant. This is evident if $i = 0$ and given $i > 0$, we have

$$yx^i(w) = xyx^{i-1}(w) - [x,y]x^{i-1}(w),$$

where the right hand side is in W_{i+1} by induction, as desired. Finally, we claim each $y \in \mathfrak{h}$ is represented, with respect to the basis $\underline{w_n}$ of W by an upper triangular matrix with $\lambda(y)$ on its diagonal entries. The claim follows from the congruence

$$yx^i(w) \equiv \lambda(y)x^i(w) \mod W_i,$$

which can be shown by induction. This is trivial for $i = 0$. Otherwise, observe that

$$\begin{aligned} yx^i(w) &= (xy - [x,y])x^{i-1}(w) \\ &\equiv \lambda(y)x^i(w) - [x,y]x^{i-1}(w) \mod W_i \\ &\equiv \lambda(y)x^i(w) \mod W_i, \end{aligned}$$

with the first step being justified by induction and the second by the fact that $\mathfrak{h}$ stabilizes W_i.

Therefore, we can conclude that $\mathrm{Tr}(y) = n\,\lambda(y)$ for any $y \in \mathfrak{h}$. Since $\mathfrak{h}$ is an ideal, we can use the same formula for $[x,y]$ and conclude $\mathrm{Tr}([x,y]) = n\,\lambda([x,y])$. Since the trace of a commutator is always zero, we get $\lambda([x,y]) = 0$.

Step 4. Just take some $z \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathbf{F}z$. We now reap the labour of the last step—since $\mathfrak{g}$ stabilizes W, we know that z can be considered an endomorphism of W. Using the algebraic closure of $\mathbf{F}$, it follows that z has an eigenvector $v \in W$. This v is the desired common eigenvector for $\mathfrak{g}$. □

Corollary 3.7 (Lie's Theorem)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Then, $\rho(\mathfrak{g})$ stabilizes a flag of subspaces in V.*

Proof. By a **flag** of subspaces, we mean a sequence of subspaces $0 = V_0 \subset \cdots \subset V_n = V$ with $\dim V_i = i$ and we say that the flag is **stabilized** by $\mathfrak{g}$ if $x(V_i) \subseteq V_i$ for every $x \in \mathfrak{g}$.

If $\mathfrak{g}$ is a solvable subaglebra of $\mathfrak{gl}(V)$, it is an easy induction on $\dim V$ using the above theorem to show $\mathfrak{g}$ stabilizes some flag in V. The base case is trivial and in the inductive step it suffices to take the common eigenvector and use the inductive hypothesis on its complement.

Recall (point (ii) of 3.2) that homomorphic images of solvable Lie algebras are also solvable. Therefore, the endomorphisms on $\rho(\mathfrak{g})$ stabilize a flag in V. □

Corollary 3.8. *Let $\mathfrak{g}$ be a Lie algebra. Then, $\mathfrak{g}$ is solvable if and only if $\mathscr{D}\mathfrak{g}$ is nilpotent.*

Proof. Suppose $\mathfrak{g}$ is solvable. Then, by Lie's theorem applied to the adjoint representation, there exists a flag of ideals

$$0 = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim \mathfrak{g}_i = i$. That is, taking a basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$ such that $(x_1, \ldots, x_i)$ spans $\mathfrak{g}_i$, we know that the matrices of $\operatorname{ad} \mathfrak{g}$ are upper triangular. This follows from the fact that each $\mathfrak{g}_i$ is an ideal, so taking some $\operatorname{ad} x \in \operatorname{ad} \mathfrak{g}$, we have that $\operatorname{ad}(x)\, x_i \in \mathfrak{g}_i$. Thus $\operatorname{ad}(x)\, x_i$, the vector of which the coordinates are the entries of the i-th column of the matrix of $\operatorname{ad} x$, can be written as a linear combination of $x_1, \ldots, x_i$.

Therefore, the matrices of $[\operatorname{ad} \mathfrak{g}, \operatorname{ad} \mathfrak{g}] = \operatorname{ad}[\mathfrak{g}, \mathfrak{g}]$ are *strictly* upper triangular, and hence nilpotent. Thus, if $x \in \mathscr{D}\mathfrak{g}$, we have that $\operatorname{ad} x$ is nilpotent, so that by Engel's theorem $\mathscr{D}\mathfrak{g}$ is nilpotent.

Conversely, suppose $\mathscr{D}\mathfrak{g}$ is nilpotent. Then, $\mathscr{D}\mathfrak{g}$ is solvable, which implies that $\mathfrak{g}$ itself is solvable. □

4. Semisimple Lie algebras

Definition 4.1. A Lie algebra $\mathfrak{g}$ is **semisimple** if it has no nonzero solvable ideals.

In order to improve the above definition, we make the following observation:

Claim 4.2. *If $\mathfrak{i}$ and $\mathfrak{j}$ are solvable ideals of a Lie algebra $\mathfrak{g}$, then $\mathfrak{i} + \mathfrak{j}$ is solvable.*

Proof. Notice that $\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$ is solvable by Proposition 3.2 (ii), for it is the image of the ideal $\mathfrak{i}$ under the projection map $x \mapsto x + \mathfrak{i} \cap \mathfrak{j}$. By the first isomorphism theorem (2.2 (*i*)), we have $\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j}) \cong (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$, which implies that $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$ is solvable. Thus, the desired result follows from Proposition 3.2 (iii). □

Now, let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{r}$ be a maximal solvable ideal of $\mathfrak{g}$. If $\mathfrak{i}$ is any other solvable ideal, by the above claim, $\mathfrak{i} + \mathfrak{r}$ is also solvable and therefore equal to $\mathfrak{r}$, by maximality. We see therefore that this maximal solvable ideal is uniquely defined, and we call it the **radical** of $\mathfrak{g}$, denoted $\operatorname{Rad} \mathfrak{g}$. Thus, *a Lie algebra $\mathfrak{g}$ is semisimple if and only if* $\operatorname{Rad} \mathfrak{g} = 0$.

There is another useful and easy characterization of semisimplicity— *a Lie algebra $\mathfrak{g}$ is semisimple if and only if it has no nonzero abelian ideals.* One direction follows from the fact that if $\mathfrak{g}$ has a nonzero solvable ideal $\mathfrak{h}$, then for some k we have that $\mathscr{D}^k\mathfrak{h}$ is nonzero and $[\mathscr{D}^k\mathfrak{h}, \mathscr{D}^k\mathfrak{h}] = 0$. Conversely, if $\mathfrak{h}$ is a nonzero abelian ideal, then $\mathfrak{h}$ is solvable.

An application: we say a representation ρ of a Lie algebra $\mathfrak{g}$ is **faithful** if it is injective. Now, consider the adjoint representation ad. Notice that its kernel $\ker \operatorname{ad}$ is precisely the center $Z(\mathfrak{g})$, which is an abelian ideal. Thus, using the above characterization, *if $\mathfrak{g}$ is semisimple, then* ad *is faithful.*

5. Killing form and Cartan's criterion

We work on a new criterion of solvability that turns out to have enlightening consequences about semisimple Lie algebras. First, we need to recall an important theorem of linear algebra:

Theorem 5.1 (Jordan decomposition)**.** *Let V be a vector space and $x \in \operatorname{End} V$ be an endomorphism. Then,*

(i) *There exists unique* $s, n \in \operatorname{End} V$ *such that* s *is diagonalizable,* n *is nilpotent and* $sn = ns$.

(ii) *There exists polynomials* $p(X), q(X) \in \mathbf{F}[X]$ *without constant term such that* $s = p(X)$ *and* $n = q(X)$. *In particular,* s *and* n *commute with every endomorphism that commutes with* x.

Proof. See [1] pp. 17-18 or [6] pp. 27-28. □

Lemma 5.2. *Let* A *and* B *be subspaces of* $\mathfrak{gl}(V)$, *with* $A \subseteq B$ *and let* $M = \{x \in \mathfrak{gl}(V) \colon [x, B] \subseteq A\}$. *Suppose* $x \in M$ *satisfies* $\operatorname{Tr}(xy) = 0$ *for all* $y \in M$. *Then,* x *is nilpotent.*

Proof. Let $x = s + n$ be the Jordan decomposition of x. Choose a basis $\underline{v} = (v_1, \ldots, v_m)$ for V consisting eigenvectors of s, with respect to which the matrix of s can be written as $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Our goal is to show that the eigenvectors $\lambda_1, \ldots, \lambda_n$ are all zero, so x is nilpotent. We can rephrase this as follows: consider our ground field $\mathbf{F}$ as a $\mathbb{Q}$-vector space and let E be the subspace spanned by the eigenvalues of s. If we could show that $E = 0$, we would be done. We will, instead, show that the dual space $E^* = \operatorname{Hom}(E, \mathbb{Q})$ is zero, which is also valid because a finite-dimensional vector space has the same dimension as its dual.

Let e_{ij} denote the matrix with 1 in the position (i, j) and zeros elsewhere. Then, $e_{jk}e_{kl} = \delta_{il}e_{il}$. This implies that

$$\begin{aligned}
[e_{ij}, e_{kl}] &= e_{ij}e_{kl} - e_{kl}e_{ij} \\
&= \delta_{il}e_{il} - \delta_{kj}e_{kj}.
\end{aligned}$$

Now, let $f \in E^*$ be a linear functional. Consider the endomorphism $y \in \mathfrak{gl}(V)$ of which the matrix with respect to $\underline{v}$ is $\operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n))$. By bilinearity and the above calculation, we have

$$\begin{aligned}
\operatorname{ad} s(e_{ij}) = [s, e_{ij}] &= \left[\sum_k \lambda_k e_{kk}, e_{ij}\right] \\
&= \sum_k \lambda_k \big(\delta_{ki}e_{kj} - \delta_{jk}e_{ik}\big) \\
&= (\lambda_i - \lambda_j)\, e_{ij}.
\end{aligned}$$

Similarly, $\operatorname{ad} y(e_{ij}) = (f(\lambda_i) - f(\lambda_j))\, e_{ij}$. Now, there exists a polynomial $r(X) \in \mathbf{F}[X]$ without constant term such that $r(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$. For, if $\lambda_i - \lambda_j = \lambda_k - \lambda_l$, it follows by the linearity of f that $f(\lambda_i) - f(\lambda_j) = f(\lambda_k) - f(\lambda_l)$. By this reason, we can use Lagrange interpolation to deduce the existence of the polynomial. Therefore, *we are able to write* $\operatorname{ad} y$ *as a polynomial on* $\operatorname{ad} s$—we have that $r(\operatorname{ad} s) = \operatorname{ad} y$.

If $x = s + n$ is the Jordan decomposition of x, then $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ is the Jordan decomposition of $\operatorname{ad} x$. This is a particular case of a theorem from Section 7, but much easier to show. Indeed, we know that $\operatorname{ad} s$ and $\operatorname{ad} n$ are also respectively diagonalizable and nilpotent. Moreover, because ad is a Lie algebra homomorphism, they commute—if $[s, n] = 0$, then $\operatorname{ad}[s, n] = [\operatorname{ad} s, \operatorname{ad} n] = 0$. So applying the uniqueness part of the Jordan decomposition theorem, we have that $\operatorname{ad} s$ and $\operatorname{ad} n$ are indeed the diagonalizable and nilpotent parts of $\operatorname{ad} x$. This implies that $\operatorname{ad} s$ is a polynomial without constant term on $\operatorname{ad} x$. However, as seen above,

$\operatorname{ad} y$ is a polynomial without constant term on $\operatorname{ad} s$, so that $\operatorname{ad} y$ is a polynomial without constant term on $\operatorname{ad} x$. Therefore, if $x \in M$, it follows that $y \in M$.

Hence, we know by hypothesis that $\operatorname{Tr}(xy) = 0$. That is, $\sum f(\lambda_i)\lambda_i = 0$. Since the outputs of f are rational numbers, on the left side of the latter equation we have a $\mathbb{Q}$-linear combination of the basis elements of E. Thus, applying f to both sides of the equation, we get $\sum f(\lambda_i)^2 = 0$. Therefore, $f(\lambda_i) = 0$ for $1 \leq i \leq n$. Thus, since the λ_i form a basis for E, it follows that $f = 0$, as desired. □

Now we are able to prove the desired criterion for solvability, the so-called **Cartan's criterion:**

Theorem 5.3 (Cartan's criterion)**.** *If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, with $\operatorname{Tr}(xy) = 0$ for all $x \in \mathfrak{g}$ and $y \in \mathscr{D}\mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. Observe that, given $x, y, z \in \mathfrak{gl}(V)$ we have:

$$(\star) \qquad \operatorname{Tr}([x, y]z) = \operatorname{Tr}(x[y, z]).$$

To become convinced of this, just write $[x, y]z = xyz - yxz$, $x[y, z] = xyz - xzy$ use the linearity of the trace and the fact that $\operatorname{Tr}(y(xz)) = \operatorname{Tr}((xz)y)$.

By Corollary 3.8, to show $\mathfrak{g}$ is solvable, it suffices to show $\mathscr{D}\mathfrak{g}$ is nilpotent. And to show $\mathscr{D}\mathfrak{g}$ is nilpotent, by Engel's theorem and Lemma 3.4, it suffices to show that every endomorphism of $\mathscr{D}\mathfrak{g}$ is nilpotent. To do so, we will use Lemma 5.2, with $A = \mathscr{D}\mathfrak{g}$ and $B = \mathfrak{g}$, so M is defined as

$$M = \{x \in \mathfrak{gl}(V) \colon [x, \mathfrak{g}] \subset \mathfrak{g}\}.$$

Let $x \in \mathscr{D}\mathfrak{g}$. It is clear that $x \in M$. To apply the lemma, we need to show that given any $z \in M$, we have $\operatorname{Tr}(xz) = 0$. Indeed, since $x \in \mathscr{D}\mathfrak{g}$, we can write x as a linear combination $x = \sum_i \alpha_i [x_i, y_i]$, where the $[x_i, y_i]$ generate $\mathscr{D}\mathfrak{g}$. Therefore

$$\operatorname{Tr}(xz) = \operatorname{Tr}\left(\sum_i \alpha_i [x_i, y_i] z\right) = \sum_i \alpha_i \operatorname{Tr}(x_i[y_i, z])$$

by $(\star)$. However, since $z \in M$, we have that $[y_i, z] \in \mathscr{D}\mathfrak{g}$, whence follows that $\operatorname{Tr}(x_i[y_i, z]) = 0$ by hypothesis. Therefore, $\operatorname{Tr}(xz) = 0$, so x is nilpotent. □

Now, if $\mathfrak{g}$ is a Lie algebra, it is useful to define a bilinear form on $\mathfrak{g}$ by $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$, the so-called **Killing form**, due to Wilhelm Killing. It is clear that κ is symmetric and *associative* in the sense of $(\star)$: $\kappa([x, y], z) = \kappa(x, [y, z])$ for every $x, y, z \in \mathfrak{g}$. Using the Killing form, we can generalize the above criterion to any Lie algebra:

Corollary 5.4. *If $\mathfrak{g}$ is a Lie algebra such that $\kappa(x, y) = 0$ for every $x \in \mathscr{D}\mathfrak{g}$ and $y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. We apply Cartan's criterion to $\operatorname{ad} \mathfrak{g}$ to conclude $\operatorname{ad} \mathfrak{g}$ is solvable. Now, the ideal $\ker \operatorname{ad} = Z(\mathfrak{g})$ is solvable for it is abelian. Moreover, by the first isomorphism theorem, we have that $\mathfrak{g}/\ker \operatorname{ad} \cong \operatorname{ad} \mathfrak{g}$ is also solvable. Thus, by Proposition 3.2 (*iii*), $\mathfrak{g}$ itself is solvable. □

Before proceeding to the result, that links this discussion with semisimple Lie algebras, a few words on symmetric bilinear forms are necessary. We say that a symmetric bilinear form $\beta : V \times V \to \mathbf{F}$ on a vector space V is **non-degenerate** if its **radical** (or **kernel**), defined as

$$\ker \beta = \{v \in V : \beta(v, w) = 0 \text{ for all } w \in V\}$$

is zero. Non-degeneracy can also be characterized in the following manner—let $\beta^\flat : V \to V^*$ be a map defined by $\beta^\flat(v) = \beta(v, \cdot)$. Then, one can see that β is non-degenerate if and only if $\ker \beta^\flat = 0$; that is, *if and only if $\beta^\flat$ is an isomorphism.*

Now, let W be a subspace of V and consider its **orthogonal complement,** defined as

$$W^\perp = \{v \in V : \beta(v, w) = 0 \text{ for all } w \in W\},$$

where β is non-degenerate and symmetric as above. The reason why non-degeneracy is important for us can be summarized in the following claim:

Claim 5.5. *With V, W and β as above,* $\dim W + \dim W^\perp = \dim V$. *In particular, if $W \cap W^\perp = 0$, we have $V = W \oplus W^\perp$.*

Proof. Our strategy will be to use the Rank-Nullity theorem on the linear map $\beta^\sharp : V \to W^*$ defined by

$$\beta^\sharp(v) = \beta|_W(v, \cdot),$$

which is the same as $\beta^\flat$ except that it returns the linear functionals restricted to W. Observe that $W^\perp = \ker \beta^\flat$ follows readily from the definition of $W^\perp$.

We claim that $\operatorname{Im} \beta^\sharp = W^*$. Indeed, let $w^* \in W^*$ and let W' be a subspace of V such that $V = W \oplus W'$. We define $v^* \in V^*$ as being the functional w^* when it takes values on W and the zero functional when it takes values on W'. The non-degeneracy of β implies that $\beta^\flat$ is bijective, so there exists $v \in V$ such that $\beta^\flat(v) = v^*$. Now, $\beta^\flat(v)|_W = \beta^\sharp(v)$ and thus $\beta^\sharp(v) = v^*|_W = w^*$. This shows that $\operatorname{Im} \beta^\sharp = W^*$. Therefore,

$$\begin{aligned}\dim V &= \dim \operatorname{Im} \beta^\sharp + \dim \ker \beta^\sharp \\ &= \dim W^* + \dim W^\perp \\ &= \dim W + \dim W^\perp.\end{aligned}$$

The second statement of the claim follows by the definition of internal direct sum. ☐

Theorem 5.6. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form κ is non-degenerate.*

Proof. Notice that $\ker \kappa$ is an ideal of $\mathfrak{g}$, due to the associativity of κ. Now, suppose $\mathfrak{g}$ is semisimple. Then, $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ for every $x \in \ker \kappa$ and $y \in \mathfrak{g}$. Therefore, since $\mathscr{D} \ker \kappa \subseteq \mathfrak{g}$, we can apply the corollary to Cartan's criterion and the above remark to conclude $\ker \kappa$ is a solvable ideal of $\mathfrak{g}$. Thus, since $\mathfrak{g}$ is semisimple, $\ker \kappa = 0$ and κ is non-degenerate.

Conversely, suppose $\ker \kappa = 0$. To show $\mathfrak{g}$ is semisimple it is sufficient to show that every abelian ideal of $\mathfrak{g}$ is a subset of $\ker \kappa$. So let $\mathfrak{h}$ be an abelian ideal and suppose $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$. Observe that the endomorphism $(\operatorname{ad} x \operatorname{ad} y)^2$ is nilpotent:

$$\mathfrak{g} \xrightarrow{\operatorname{ad} y} \mathfrak{g} \xrightarrow{\operatorname{ad} x} \mathfrak{h} \xrightarrow{\operatorname{ad} y} \mathfrak{h} \xrightarrow{\operatorname{ad} x} [\mathfrak{h}, \mathfrak{h}] = 0,$$

the diagram above being justified by the fact that $\mathfrak{h}$ is an abelian ideal and $x \in \mathfrak{h}$. Therefore, we have that $\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$, and thus $\mathfrak{h} \subset \ker \kappa$. ☐

Before proceeding to the next theorem, which is what finally sheds light on the adjective "semisimple," we make a remark. We say a Lie algebra $\mathfrak{g}$ is a **direct sum** of ideals $\mathfrak{h}_1, \dots, \mathfrak{h}_k$ if $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ as a vector space direct sum. This condition forces $\mathfrak{h}_i \cap \mathfrak{h}_j = 0$ whenever $i \neq j$, which implies $[\mathfrak{h}_i, \mathfrak{h}_j] = 0$. Thus, we can think of the Lie algebra direct sum as the Cartesian product $\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_k$ with linear operations and the Lie bracket defined componentwise. Now, a definition: we say a Lie algebra $\mathfrak{g}$ is **simple** if it has no nonzero ideals and is not abelian.

Theorem 5.7. *A semisimple Lie algebra $\mathfrak{g}$ can be written as a direct sum of simple ideals.*

Proof. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and consider its orthogonal complement with the respect to the Killing form, $\mathfrak{h}^\perp = \{x \in \mathfrak{g} \colon \kappa(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$. We can use the corollary to Cartan's criterion on the intersection $\mathfrak{h} \cap \mathfrak{h}^\perp$ to conclude that $\mathfrak{h} \cap \mathfrak{h}^\perp$ is a solvable ideal. Thus, by the semisimplicity of $\mathfrak{g}$, we conclude $\mathfrak{h} \cap \mathfrak{h}^\perp = 0$. Because of that, we can use the non-degeneracy of κ to conclude, using Claim 5.5, that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$.

We now use induction on $\dim \mathfrak{g}$ to further refine this sum. If $\mathfrak{g}$ has no nontrivial proper ideals, then $\mathfrak{g}$ is simple and we are done. Otherwise, let $\mathfrak{h}_1$ be a minimal nonzero ideal of $\mathfrak{g}$. As remarked on the above paragraph, we have the direct sum decomposition $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_1^\perp$. Let $\mathfrak{h}$ be an ideal in $\mathfrak{h}_1$. Then,

$$\begin{aligned} [\mathfrak{g}, \mathfrak{h}] = [\mathfrak{h}_1 \oplus \mathfrak{h}_1^\perp, \mathfrak{h}] &= [\mathfrak{h}_1, \mathfrak{h}] \oplus [\mathfrak{h}_1^\perp, \mathfrak{h}] \\ &= [\mathfrak{h}_1, \mathfrak{h}] \subseteq \mathfrak{h}, \end{aligned}$$

which implies that $\mathfrak{h}_1$ is also an ideal in $\mathfrak{g}$. Similarly, ideals in $\mathfrak{h}_1^\perp$ are also ideals in $\mathfrak{g}$. This implies that $\mathfrak{h}_1$ and $\mathfrak{h}_1^\perp$ are also semisimple. By induction and minimality, we therefore conclude $\mathfrak{h}_1$ is simple. Now, by induction, we know $\mathfrak{h}_1^\perp$ can be written as a direct sum of simple ideals. The decomposition follows. □

Corollary 5.8. *The decomposition in the above theorem is unique (up to permutations). Moreover, if $\mathfrak{g} = \bigoplus_i \mathfrak{h}_i$ is the simple ideal decomposition of $\mathfrak{g}$, every ideal of $\mathfrak{g}$ is a sum of some of the $\mathfrak{h}_i$.*

Proof. By the above theorem, we know there are simple ideals $\mathfrak{h}_1, \dots, \mathfrak{h}_k$ such that $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$. Now, suppose $\mathfrak{a}$ is any ideal of $\mathfrak{g}$ and let π_i be the projection map onto $\mathfrak{h}_i$. It is clear that $\pi_i(\mathfrak{a})$ is an ideal, for, given $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$, we have

$$[x, \pi_i(y)] = \pi_i[x, y] \in \pi_i(\mathfrak{a}).$$

Since $\pi_i(\mathfrak{a}) \subseteq \mathfrak{h}_i$, the simplicity of $\mathfrak{h}_i$ implies either $\pi_i(\mathfrak{a}) = 0$ or $\pi_i(\mathfrak{a}) = \mathfrak{h}_i$. In the latter case, we have

$$\mathfrak{h}_i = [\mathfrak{h}_i, \mathfrak{h}_i] = [\mathfrak{h}_i, \pi_i(\mathfrak{h}_i)] = [\mathfrak{h}_i, \mathfrak{a}] \subseteq \mathfrak{a}.$$

Since $\mathfrak{g} = \bigoplus_i \mathfrak{h}_i$, it follows

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{g} = \bigoplus_{i=1}^{k} (a \cap \mathfrak{h}_i) = \bigoplus_{\mathfrak{h}_i \subseteq \mathfrak{a}} \mathfrak{h}_i,$$

as desired. In particular, if $\mathfrak{a}$ is simple, $\mathfrak{a} = \mathfrak{h}_i$ for some i. □

Corollary 5.9. *If $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g} = \mathscr{D}\mathfrak{g}$.*

Proof. Let $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ be the simple ideal decomposition of $\mathfrak{g}$. We have

$$\begin{aligned}[\mathfrak{g}, \mathfrak{g}] = \left[\bigoplus_{i=0}^{k} \mathfrak{h}_i, \mathfrak{g}\right] &= \bigoplus_{i=0}^{k} [\mathfrak{h}_i, \mathfrak{g}] \\ &= \bigoplus_{i=0}^{k} \mathfrak{h}_i = \mathfrak{g}.\end{aligned}$$

That $[\mathfrak{h}_i, \mathfrak{g}] = \mathfrak{h}_i$ follows from the fact that $[\mathfrak{h}_i, \mathfrak{g}]$ is a nonzero ideal in $\mathfrak{g}$ contained in the simple ideal $\mathfrak{h}_i$. (The ideal $[\mathfrak{h}_i, \mathfrak{g}]$ is nonzero because a semisimple Lie algebra has no nonzero abelian ideals.) □

6. Weyl's theorem on complete reducibility

The goal of this section is to prove the following:

Theorem 6.1 (Weyl)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Then, ρ is completely reducible.*

Before doing so, it will be useful to construct an endomorphism in $\mathfrak{gl}(V)$ which commutes with every element of $\rho(\mathfrak{g})$. Let $\mathfrak{g}$ be semisimple and ρ be *faithful.* Let β be a bilinear form on $\mathfrak{g}$ given by $\beta(x, y) = \mathrm{Tr}(\rho(x)\rho(y))$. Notice that β is associative in the same sense as the Killing form, that is, $\beta([x, y], z) = \beta(x, [y, z])$ because ρ is a Lie algebra homomorphism. (Clearly, β *is* the Killing form when $\beta = \mathrm{ad}$.) Moreover, β is nondegenerate. Indeed, the image of its radical $\mathfrak{r}$, defined by

$$\rho(\mathfrak{r}) = \{\rho(x) \in \rho(\mathfrak{g})\colon\ \mathrm{Tr}(\rho(x)\rho(y)) = 0 \text{ for all } y \in \mathfrak{g}\}$$

is a solvable ideal by Cartan's Criterion. Since ρ is faithful, we have $\rho(\mathfrak{r}) \cong \mathfrak{r}$ which implies that $\mathfrak{r}$ itself is solvable. Thus, since $\mathfrak{g}$ is semisimple, it follows that $\mathfrak{r} = 0$.

Since β is non-degenerate, given a basis $(x_1, \dots, x_n)$ of $\mathfrak{g}$, there exists a unique dual basis $(y_1, \dots, y_n)$ such that $\beta(x_i, y_j) = \delta_{ij}$. This follows for $(\beta(x_1, \cdot), \dots, \beta(x_n, \cdot))$ is a basis for the dual V^* and $(y_1, \dots, y_n)$ is the corresponding dual basis of $V^{**} = V$.

Using this pair of dual bases relative to β, we can now define the **Casimir endomorphism** of the representation ρ as

$$c_\rho = \sum_i \rho(x_i)\rho(y_i).$$

Lemma 6.2. *With ρ and $\mathfrak{g}$ given as above, the Casimir endomorphism c_ρ commutes with every endomorphism of $\rho(\mathfrak{g})$.*

Proof. Given $x, y \in \mathfrak{g}$, we can write in terms of our bases (x_i) and (y_j)

$$[x, x_i] = \sum_j a_{ij} x_j \quad \text{and} \quad [x, y_i] = \sum_i b_{ij} y_j,$$

so that using the orthonormality relations and associativity of β, we can compute

$$\begin{aligned} a_{ik} = \beta\left(\sum_j a_{ij} x_j, y_k\right) &= \beta([x_i, x], y_k) \\ &= \beta(x_i, [x, y_k]) \\ &= \beta(x_i, -\sum_j b_{kj} y_j) = -b_{ki}. \end{aligned}$$

Now, observe that for endomorphisms in $\mathfrak{gl}(V)$, it is valid that

$$[x, yz] = [x, y]z + y[x, z], \tag{$\diamond$}$$

which can be verified by expanding the commutators on the right side. Thus, given $x \in \mathfrak{g}$, we have

$$\begin{aligned}
[\rho(x), c_\rho] &= \left[\rho(x), \sum_i \rho(x_i)\rho(y_i)\right] \\
&= \sum_i \rho([x, x_i])\rho(y_i) + \sum_i \rho(x_i)\rho([x, y_i]) \text{ by } (\diamond) \\
&= \sum_{i,j} a_{ij}\rho(x_j)\rho(y_i) + \sum_{i,j} b_{ij}\rho(x_i)\rho(y_j) \\
&= \sum_{i,j} a_{ij}\rho(x_j)\rho(y_i) - \sum_{i,j} a_{ji}\rho(x_i)\rho(y_j) = 0,
\end{aligned}$$

which proves the lemma. □

We can rephrase the above result as $c_\rho : V \to V$ *is a* $\mathfrak{g}$*-module homomorphism.* If V is irreducible, it follows from Schur's lemma that c_ρ is a scalar multiplication. By what? Its trace is

$$\operatorname{Tr} \sum_i \rho(x_i)\rho(y_i) = \sum_i \beta(x_i, y_i) = \dim \mathfrak{g},$$

so c_ρ is just multiplication by $\dim \mathfrak{g}/\dim V$.

We can now prove Weyl's theorem:

Proof of 6.1. Throughout the proof, we will assume ρ is faithful. No generality will be lost, as a consequence of Corollary 5.8. For, since $\ker\rho$ is an ideal in $\mathfrak{g}$, it is a sum of certain simple ideals. Thus, if we restrict ρ to the direct sum of the other simple ideals that do not constitute $\ker\rho$, and show the restriction is completely reducible, we will be done. Indeed, if V becomes a direct sum of irreducible $\mathfrak{g}$-submodules under this restriction of ρ, this will not change if you allow ρ to take values on $\ker\rho$. Now, onto the proof:

Step 1. First we consider the case in which ρ has an irreducible $\mathfrak{g}$-submodule of codimension 1. Since ρ is faithful, we can talk about its Casimir endomorphism c_ρ. Since W is irreducible, c_ρ acts on W by multiplication by a positive scalar $\lambda = \dim \mathfrak{g}/\dim W$.

Now, consider the quotient $\mathfrak{g}$-module V/W (as in p.4). It is one-dimensional, which implies the action of $\mathfrak{g}$ on it is trivial. To see this, first notice that the actions of the elements of $\mathfrak{g}$ on V/W are just scalar multiplications, since $\dim V/W = 1$. Moreover, since $\mathfrak{g}$ is semisimple, we have that $\overline{\rho}(\mathfrak{g}) = [\overline{\rho}(\mathfrak{g}), \overline{\rho}(\mathfrak{g})]$, which implies all endomorphisms of $\overline{\rho}(\mathfrak{g})$ are generated by commutators of the form $[\overline{\rho}(x), \overline{\rho}(y)]$, with $x, y \in \mathfrak{g}$. Since $\overline{\rho}(x)$ and $\overline{\rho}(y)$ are just scalar multiples of the identity endomorphism, their commutator is zero and therefore so are all elements of $\overline{\rho}(\mathfrak{g})$.

In particular, the Casimir endomorphism $\overline{c_\rho}$ is also zero in V/W. This implies that for every coset $v + W$, we have

$$\overline{c_\rho}(v + W) = c_\rho(v) + W = W,$$

which implies $c_\rho(v) \in W$ and therefore $\operatorname{Im} c_\rho \subseteq W$. The endomorphism c_ρ is not the zero endomorphism when restricted to W (it is multiplication by $\dim \mathfrak{g}/\dim V > 0$).

Thus, $\operatorname{Im} c_\rho = W$ and $\ker c_\rho \cap W = 0$, so, by the Rank-Nullity theorem we get $V = W \oplus \ker c_\rho$. To finish this step, it suffices to show that $\ker c_\rho$ is indeed a $\mathfrak{g}$-submodule of V. To see this, suppose that $v \in \ker c_\rho$. Then, we can use the fact that the Casimir endomorphism commutes with every other endomorphism to get

$$\phi(x)c_\rho v = c_\rho \phi(x)v = 0,$$

which implies that $\phi(x)v \in \ker c_\rho$ for any $x \in \mathfrak{g}$. Thus, $\ker c_\rho$ is stable under the action of $\mathfrak{g}$.

Step 2. Consider now the second case, in which W is not irreducible, but still of codimension 1. We will show, by induction on $\dim W$, that there exists a one-dimensional complement to W. The result is evident if $\dim W = 0$, so let us assume otherwise. Suppose $Z \subset W$ is a proper nontrivial $\mathfrak{g}$-submodule of W. Now, observe that W/Z is a $\mathfrak{g}$-submodule with codimension 1 in V/Z. Then, either by Step 1 or the inductive hypothesis, there exists $\overline{y} \in V/Z$ such that $\mathbf{F}\overline{y}$ is a $\mathfrak{g}$-submodule of V/Z and $V/Z = W/Z \oplus \mathbf{F}\overline{y}$. Now, let $Y = \pi^{-1}(\mathbf{F}\overline{y})$, where π is the projection map $v \mapsto v + Z = \overline{v}$. Then, Z is a $\mathfrak{g}$-submodule of codimension 1 in Y and that $Y = Z \oplus \mathbf{F}y$. However, we do not know whether $\mathbf{F}y$ is a $\mathfrak{g}$-submodule. But $\dim Z < \dim W$, so we can use the inductive hypothesis to conclude there exists $x \in V$ such that $\mathbf{F}x$ is a $\mathfrak{g}$-submodule and $Y = Z \oplus \mathbf{F}x$. Since $y \notin W$ by construction, it follows that $x \notin W$ as well, or else we would have $Y \subseteq W$. Therefore, we can conclude $V = W \oplus \mathbf{F}x$ is a $\mathfrak{g}$-module direct sum, as desired.

Step 3. Finally, we proceed to the general case, where W is any nonzero proper $\mathfrak{g}$-submodule of V. Let $\operatorname{Hom}(V, W)$ be the space of linear maps $V \to W$. It can be made into a $\mathfrak{g}$-module via the action

$$(x \cdot f)(v) = \rho(x)f(v) - f\rho(x)(v) \tag{$\bullet$}$$

where $f \in \operatorname{Hom}(V, W)$, $x \in \mathfrak{g}$ and $v \in V$. To make sure there is no ambiguity, we remark that above, concatenation denotes composition of functions and the dot denotes the action of $\mathfrak{g}$ on $\operatorname{Hom}(V, W)$. Now, let $\mathscr{V}$ be the vector subspace of $\operatorname{Hom}(V, W)$ consisting of maps that, restricted to W, are just the multiplication by some scalar, say α. Observe that $\mathscr{V}$ is a $\mathfrak{g}$-submodule—given $x \in \mathfrak{g}$ and $w \in W$, we have

$$(x \cdot f)(w) = \rho(x)(\alpha w) - \alpha(\rho(x)v) = 0,$$

which implies $x \cdot f$ is scalar multiplication by zero when restricted to W. Define now $\mathscr{W}$ as the subspace of $\operatorname{Hom}(V, W)$ consisting of maps that are zero when restricted to W. Using the calculation above, we see $\mathscr{W}$ is also a $\mathfrak{g}$-submodule and the action of $\mathfrak{g}$ maps $\mathscr{V}$ into $\mathscr{W}$. Notice that $\mathscr{W}$ has codimension 1 in $\mathscr{V}$, which puts us back in the situation of Step 2. It follows there is some $f \in \mathscr{V}$ such that $\mathbf{F}f$ is a $\mathfrak{g}$-submodule and $\mathscr{V} = \mathscr{W} \oplus \mathbf{F}f$.

Now, by the same argument used in the second paragraph of Step 1, the fact that $\mathbf{F}f$ is one-dimensional implies it is acted upon trivially by $\mathfrak{g}$. Therefore, by ($\bullet$) we have $f\rho(x)(v) = \rho(x)f(v)$. This implies $\ker f$ is a $\mathfrak{g}$-submodule of V. Since $f \colon V \to W$ is a nonzero scalar multiple of the identity on W, it follows that $\ker f \cap W = 0$. This also implies that $\operatorname{Im} f = W$, so $\dim \ker f = \dim V - \dim W$. Thus, $V = W \oplus \ker f$ is a direct sum of $\mathfrak{g}$-modules. $\square$

7. Preservation of the Jordan decomposition

This section introduces an application of Weyl's theorem that turns out to be crucial for the study of the representation of semisimple Lie algebras. A consequence of the theorem to be proved is:

Corollary 7.1. *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(W)$ and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. If $x = s + n$ is the Jordan decomposition of $x \in L$, then $\rho(x) = \rho(s) + \rho(n)$ is the usual Jordan decomposition of $\rho(x)$.*

This nice behaviour is due to the semisimplicity of $\mathfrak{g}$. Let us see how could this fail for a non-semisimple Lie algebra. Consider the one-dimensional Lie algebra $\mathbf{F}$. It is abelian, so it is not semisimple. Every element of $\mathbf{F}$, if seen as a 1 by 1 matrix, is diagonal (hence diagonalizable), but nilpotent if and only if it is zero. However, under the representation

$$\rho : t \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

every element in $\rho(\mathbf{F})$ is nilpotent. Thus, if s and n are the diagonalizable and nilpotent parts of $t \in \mathbf{F}$, it is *not true* that $\rho(s)$ and $\rho(n)$ are the diagonalizable and nilpotent parts of $\rho(t)$ unless $t = 0$.

The following lemma is essential:

Lemma 7.2. *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then, for any $x \in \mathfrak{g}$, the diagonalizable and nilpotent parts of x are in $\mathfrak{g}$.*

Proof. Let $x \in \mathfrak{g}$ and let $x = s + n$ be its usual Jordan decomposition, with s diagonalizable and n nilpotent. Our goal is to show that s and n also lie in $\mathfrak{g}$, and to do so we will try to write $\mathfrak{g}$ in terms of subalgebras which somehow clearly contain s and n.

Since $\mathfrak{g}$ is a Lie algebra, we know $[x, \mathfrak{g}] \subseteq \mathfrak{g}$. From the Jordan decomposition theorem, we know s and n are polynomials p and q on x without a constant term, so $[s, \mathfrak{g}] \subseteq \mathfrak{g}$ and $[n, \mathfrak{g}] \subseteq \mathfrak{g}$. This shows that s and n belong to $\mathfrak{n} := \{x \in \mathfrak{gl}(V) : [x, \mathfrak{g}] \subseteq \mathfrak{g}\}$, which is a subalgebra by the Jacobi identity. We have therefore found one of our subalgebras, $\mathfrak{n}$, but we are not done because $\mathfrak{n}$ contains $\mathfrak{g}$ properly—the scalar multiples of the identity are in $\mathfrak{n}$ but not in $\mathfrak{g}$.

Now, consider V as a $\mathfrak{g}$-module under the action $(x, v) \mapsto x(v)$ and let W be a $\mathfrak{g}$-submodule. Another subalgebra that interests us is defined by

$$\mathfrak{g}_W = \{y \in \mathfrak{gl}(V) \colon y \cdot W \subseteq W \text{ and } \operatorname{Tr}(y|_W) = 0\}.$$

It is clear that $\mathfrak{g}_W$ is a subalgebra of $\mathfrak{gl}(V)$—given $x, y \in \mathfrak{g}_W$, it follows that $[x, y] \cdot W \subseteq W$ and the trace of a commutator is always zero. Moreover, $\mathfrak{g} \subseteq \mathfrak{g}_W$. This follows because W is $\mathfrak{g}$-invariant and since every $y \in \mathfrak{g}$ is a sum of commutators, $y|_W$ is a sum of restrictions of commutators, thus also being traceless.

We claim that if $x \in \mathfrak{g}_W$ has Jordan decomposition $x = s + n$, then $s \in \mathfrak{g}_W$ and $n \in \mathfrak{g}_W$. That W is invariant under s and n follows from the fact that s and n are polynomials on x, remarked above. Moreover, $n|_W$ is traceless because it is nilpotent. Thus, $s|_W = x|_W - n|_W$ is also traceless, as desired.

We claim now that the subalgebra

$$\mathfrak{g}' = \bigcap_{W \leq V} \mathfrak{g}_W \cap \mathfrak{n}$$

is $\mathfrak{g}$. If this is indeed true, then the lemma follows. For, as seen above, we will have that if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}'$, then $s, n \in \mathfrak{g}'$. To prove that $\mathfrak{g}' = \mathfrak{g}$, consider $\mathfrak{g}'$ as a $\mathfrak{g}$-module under the adjoint representation. Under these circumstances, it is clear that $\mathfrak{g}$ is a $\mathfrak{g}$-submodule of $\mathfrak{g}'$, for $\operatorname{ad}\mathfrak{g}\,(\mathfrak{g}') \subseteq \mathfrak{g}$ due to the fact that $\mathfrak{g}$ is an ideal in $\mathfrak{n}$ and hence in $\mathfrak{g}'$. Now we reap the toil of having proven Weyl's theorem. Using it, we know there is another $\mathfrak{g}$-submodule of $\mathfrak{h}$ such that $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{h}$. Since $\mathfrak{h}$ is a $\mathfrak{g}$-submodule, it is clear that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, while we have just remarked that $[\mathfrak{g}, \mathfrak{g}'] \subseteq \mathfrak{g}$, which implies $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{g}$. Thus, $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{g} \cap \mathfrak{h}$ and $[\mathfrak{g}, \mathfrak{h}] = 0$ by the directness of the sum. Now, let $y \in \mathfrak{h}$. To show that $y = 0$ (and therefore $\mathfrak{h} = 0$), it suffices to show that y is zero when restricted to every irreducible $\mathfrak{g}$-submodule W of V. This is because V is a direct sum of irreducible submodules and $\mathfrak{h}$ preserves W because $\mathfrak{h} \subseteq \mathfrak{g}_W$. Now, since y commutes with $\mathfrak{g}$ (for $[\mathfrak{g}, \mathfrak{h}] = 0$), which implies y is a $\mathfrak{g}$-module homomorphism. Thus, by Schur's lemma, we have that $y|_W$ must be the identity map multiplied by scalar. But $y \in \mathfrak{g}_W$, implies $\operatorname{Tr}(y|_W) = 0$, and therefore $y|_W = 0$. □

If $\mathfrak{g}$ is a Lie algebra, then $\operatorname{ad}\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$. Thus, the above lemma guarantees us that the diagonalizable and nilpotent parts of any element of $\operatorname{ad}\mathfrak{g}$ are in $\operatorname{ad}\mathfrak{g}$. More interestingly, if $\mathfrak{g}$ is *semisimple*, we know that the map $\mathfrak{g} \to \operatorname{ad} L$ is injective, for the kernel of ad, which is the center of $\mathfrak{g}$, is zero. Thus, taking some $\operatorname{ad} x \in \operatorname{ad}\mathfrak{g}$, it follows that there exists unique $s, n \in \mathfrak{g}$ such that $\operatorname{ad} s$ and $\operatorname{ad} n$ are respectively the diagonalizable and nilpotent parts of $\operatorname{ad} x$. We call the decomposition $x = s + n$ the **abstract Jordan decomposition** of x into its **ad-diagonalizable** and **ad-nilpotent** parts.

It is an important consequence of the above lemma that if $\mathfrak{g}$ is a semisimple subalgebra of $\mathfrak{gl}(V)$, *the abstract and usual Jordan decompositions coincide*. For, we know that each of the decompositions is unique and diagonalizable (resp. nilpotent) implies ad-diagonalizable (resp. ad-nilpotent). By construction, we know that the ad-nilpotent and ad-diagonalizable parts of the abstract Jordan decomposition are in $\mathfrak{g}$. However, prior to proving the lemma above, we did not know whether the nilpotent and diagonalizable parts of the usual Jordan decomposition were in $\mathfrak{g}$. Moreover, since ad-diagonalizable (resp. ad-nilpotent) *does not* imply diagonalizable (resp. nilpotent), we were unable to show the decompositions did indeed coincide.

We are now ready to prove the theorem from which Corollary 7.1 follows.

Theorem 7.3. *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra and let $x \in \mathfrak{g}$. Then, if $x = s + n$ is the abstract Jordan decomposition of x, $\rho(x) = \rho(s) + \rho(n)$ is the usual Jordan decomposition of $\rho(x)$.*

Proof. Since the usual and abstract Jordan decompositions of $\rho(x)$ coincide as seen above, it suffices to show that $\rho(x) = \rho(s) + \rho(n)$ is the abstract Jordan decomposition of $\rho(x)$. Since s is diagonalizable, we have that $\operatorname{ad} s$ is diagonalizable, so that $\mathfrak{g}$ is generated by eigenvectors $v_1, \ldots, v_n$ of $\operatorname{ad} s$. Moreover, since $\operatorname{ad} s(v_i) = \lambda_i v_i$ for some $\lambda_i \in \mathbf{F}$, it follows that

$$\rho(\operatorname{ad} s(v_i)) = \operatorname{ad}\rho(s)(\rho(v_i)) = \lambda_i \rho(v_i),$$

because ρ is a Lie algebra homomorphism. Thus, $\rho(\mathfrak{g})$ is spanned by the eigenvectors of $\operatorname{ad}\rho(s)$. Therefore, $\operatorname{ad}\rho(s)$ is diagonalizable. Similarly, it is not hard to see that $\operatorname{ad}\rho(n)$ is nilpotent—if there exists k such that $n^k = 0$, then $\operatorname{ad} \circ \rho(n^k) = 0$. Thus, $\rho(x) = \rho(s) + \rho(n)$ is the abstract Jordan decomposition of $\rho(x)$. □

8. $\mathfrak{sl}_2(\mathbf{F})$-MODULES

In this section, we classify the irreducible representations of the **special linear Lie algebra** $\mathfrak{sl}_2(\mathbf{F})$, defined to be the space of all 2 by 2 traceless matrices with coefficients in $\mathbf{F}$, having the commutator as its Lie bracket. We choose a basis for $\mathfrak{sl}_2(\mathbf{F})$ given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Calculating their commutators, we obtain

$$(\circ) \qquad [h,x] = 2x, \qquad [h,y] = -2y, \qquad [x,y] = h,$$

and we see $\mathfrak{sl}_2(\mathbf{F})$ is indeed a Lie algebra. In fact, it is a subalgebra of $\mathfrak{gl}_2(\mathbf{F})$. Now, the commutation relations $(\circ)$ show that $\mathfrak{sl}_2(\mathbf{F})$ is *simple*—it is not hard to check, by hand, that every one or two-dimensional subspace of $\mathfrak{sl}_2(\mathbf{F})$ is not stabilized by $\operatorname{ad}\mathfrak{sl}_2(\mathbf{F})$. This shows that $\mathfrak{sl}_2(\mathbf{F})$ is semisimple, in fact, the simplest nontrivial example of a semisimple Lie algebra. Analyzing $\mathfrak{sl}_2(\mathbf{F})$-modules will finally be an opportunity for us to apply the machinery developed throughout the paper.

Let $\rho : \mathfrak{sl}_2(\mathbf{F}) \to \mathfrak{gl}(V)$ be an irreducible representation. To simplify the notation, we will denote $X := \rho(x)$, $H := \rho(h)$ and $Y := \rho(y)$. It is now that we reap the labour of having proven the preservation of the Jordan decomposition (and Weyl's theorem). For, it is clear that h is diagonalizable, whence $\operatorname{ad} h$ is a diagonalizable endomorphism. Therefore, $h = h + 0$ is the abstract Jordan decomposition of h, and using the above Corollary 7.1 we conclude $H = H + 0$ is the usual Jordan decomposition of H. In short, *H is a diagonalizable endomorphism*—it has a basis of eigenvectors. We can thus decompose V in a direct sum of eigenspaces

$$V = \bigoplus_{\lambda \in \mathbf{F}} V_\lambda,$$

where $V_\lambda = \{v \in V : H(v) = \lambda v\}$ makes sense for all $\lambda \in \mathbf{F}$, being zero when λ is not an eigenvalue.

It is now natural to try to describe the actions of X and Y on V, and their relation to the above eigenspace decomposition. Let $v \in V_\lambda$. We can use the above $(\circ)$ relations, which are preserved since ρ is a Lie algebra homomorphism to compute

$$\begin{aligned} HY(v) &= (YH - 2Y)(v) \\ &= \lambda Y(v) - 2Y(v) = (\lambda - 2)Y(v). \end{aligned}$$

Thus, we conclude that $Y(v) \in V_{\lambda-2}$—that is, *Y maps the eigenspace for λ into the eigenspace for $\lambda - 2$*. By a similar calculation, it is possible to conclude that *X maps the eigenspace for λ into the eigenspace for $\lambda + 2$*.

The above analysis implies that all eigenvalues of H are congruent to each other mod 2. For, with the λ given above, we know that

$$\bigoplus_{j \in \mathbb{Z}} V_{\lambda+2j}$$

is invariant under $\mathfrak{sl}_2(\mathbf{F})$ and thus it is a submodule of V. By the irreducibility of V we therefore conclude the above direct sum is all of V. Now, since V is finite-dimensional (and $\dim V = \sum_{k \in \mathbb{Z}} \dim V_{\lambda+2k}$), we conclude the sequence of numbers $(\lambda + 2k)$ eventually terminates. Call α this greatest eigenvalue (which is also called the **heighest weight** of V). Then, we have $V_\alpha \neq 0$ and $V_{\alpha+2} = 0$.

Let $v_0 \in V_\alpha$ be nonzero. Hoping to find a basis for V, we are interested in knowing what happens when we repeatedly apply Y to v_0. To make things tidier, we set

$$v_{-1} = 0 \quad \text{and} \quad v_j = \frac{1}{j!} Y^j(v_0) \quad (j \geq 0)$$

and claim

Proposition 8.1. *There exists $m \in \mathbb{N}$ such that $(v_0, \ldots, v_m)$ is a basis for V.*

Proof. In order to prove this, we first show that the span of the sequence of vectors $(v_0, v_1, \ldots)$ is left invariant under the $\mathfrak{sl}_2(\mathbf{F})$ action and thus, by the irreducibility of V, spans V. First of all, observe that

$$\text{(a)} \qquad H(v_j) = (\alpha - 2j)v_j,$$

for Y maps V_α into $V_{\alpha-2}$ and therefore Y^j maps V_α into $V_{\alpha-2j}$.

Next, notice that

$$\text{(b)} \qquad Y(v_j) = Y \frac{1}{j!} Y^j(v_0) = (j+1)v_{j+1}.$$

Finally, regarding the action of X, we will use induction to show

$$\text{(c)} \qquad X(v_j) = (\alpha - j + 1)v_{j-1} \quad (j \geq 0).$$

The case in which $j = 0$ follows trivially. Now, observe that

$$\begin{aligned} jX(v_j) &= \frac{j}{j!} XY^j(v_0) \\ &= \frac{1}{(j-1)!}(YX + H)Y^{j-1}(v_0) \quad \text{by } (\circ) \\ &= (YX + H)v_{j-1} \\ &= (\alpha - j + 2)Y(v_{j+2}) + (\alpha - 2(j-1))v_{j-1} \quad \text{by (a) and induction} \\ &= (\alpha - j + 2)(j-1)v_{j-1} + (\alpha - 2j + 2)v_{j-1} \quad \text{by (b)} \\ &= j(\alpha - j + 1)v_{j-1}. \end{aligned}$$

Hence, by (a), (b), (c) and the fact that $v_{-1} = 0$, we see that the action of $\mathfrak{sl}_2(\mathbf{F})$ indeed preserves $(v_0, v_1, \ldots)$, whence we conclude the v_j span V.

Now, notice that each v_j for $j \geq 0$ is an eigenvector for a different eigenvalue. Hence, they are all independent, and therefore $(v_0, v_1, \ldots)$ is a basis for V. Since V is finite-dimensional, there is some $m \in \mathbb{N}$ such that $v_m \neq 0$ and $v_{m+1} = 0$. This implies, by (b), that $v_{m+j} = 0$ for all $j \geq 1$. $\square$

The calculations made above have some nice consequences. For instances, plugging in $j = m+1$ in the formula (c) above shows us that the maximum eigenvalue α is a positive integer m. Thus, we have discovered a surprising fact: *all eigenvalues of H are positive integers.* Next, notice that since $(v_0, \ldots, v_m)$ is a basis and each v_j spans a distinct eigenspace V_{a-2j}, it follows that each eigenspace is one-dimensional. We can now write our direct sum decomposition of V more explicitly as

$$(\triangle) \qquad V = \bigoplus_{j=0}^{m} V_{2j-m}.$$

Therefore, we have

Theorem 8.2. *For every $m \in \mathbb{N}$, there exists a unique (up to isomorphism) irreducible $\mathfrak{sl}_2(\mathbf{F})$-module, denoted $V(m)$, such that the Lie algebra actions are given by (a), (b) and (c).*

Corollary 8.3. *Let V be a $\mathfrak{sl}_2(\mathbf{F})$-module. Then, the eigenvalues of H are all integers, each occuring along with its negative an equal number of times. Moreover, the number of terms in the decomposition into irreducible submodules is precisely* $\dim V_0 + \dim V_1$.

Proof. If $V = 0$, there is nothing to prove. Otherwise, both assertions follow from the direct sum decomposition of the irreducible submodules ($\triangle$) along with Weyl's theorem. $\square$

Finally, to give some concreteness to our discussion, we can try to identify some of the most obvious $\mathfrak{sl}_2(\mathbf{F})$-modules with the $V(m)$ we have discovered. In the trivial representation given by $\rho(z) = 0$ for all $z \in \mathfrak{sl}_2(\mathbf{F})$, it is clear that the only eigenvalue of $\rho(h)$ (and thus the highest weight of V) is 0. Thus, we identify the trivial representation with $V(0)$.

On the other hand, consider the two-dimensional **standard representation** $\rho : \mathfrak{sl}_2(\mathbf{F}) \to \mathbf{F}^2$ given by $\rho(z) = z$ for all $z \in \mathfrak{sl}_2(\mathbf{F})$. (I.e., this is just the 2 by 2 matrices described in the beginning of the section multiplying column vectors in $\mathbf{F}^2$.) Define x and y to be the standard basis vectors of $\mathbf{F}^2$. To find the eigenvalues of $H =: \rho(h)$, just notice that $H(x) = x$ and $H(y) = -y$. Since the representation is two-dimensional, we can readily identify it with $V(1)$.

Now, consider the symmetric square[1] $\operatorname{Sym}^2 V$. Its basis is given by (x^2, xy, y^2), and we can compute the action of H by

$$\begin{aligned} H(x^2) &= x \cdot H(x) + H(x) \cdot x = 2x^2, \\ H(xy) &= x \cdot H(y) + H(x) \cdot y = 0, \\ H(y^2) &= y \cdot H(y) + H(y) \cdot y = -2y^2, \end{aligned}$$

whence follows that $\operatorname{Sym}^2 V = V_2 \oplus V_0 \oplus V_{-2}$ can be identified with $V(2)$.

More generally, consider $\operatorname{Sym}^m V$, which has basis $(x^m, x^{m-1}y, \ldots, y^m)$. We can fully describe the action of H by the following calculation:

$$H(x^{m-k}y^k) = x^{m-k} \cdot H(y^k) + H(x^{m-k}) \cdot y^k.$$

By induction, one shows that $H(x^j) = jx^j$. The base case is trivial and

$$\begin{aligned} H(x^j) &= x \cdot H(x^{j-1}) + H(x) \cdot x^{j-1} \\ &= (j-1)x^j + x^j = jx^j \end{aligned}$$

[1] Let V be a $\mathbf{F}$-vector space with basis $(e_1, \ldots, e_n)$. Consider the polynomial algebra $\mathbf{F}[e_1, \ldots, e_n]$ and take the subspace consisting only of polynomials of homogeneous degree k (e.g., if $k = 3$ we would only be considering the linear combinations of terms like $e_i \cdot e_j \cdot e_k$). The k-th symmetric power of V, denoted $\operatorname{Sym}^k V$ can be identified with this subspace. A representation of a Lie algebra $\mathfrak{g}$ acts on an element of the symmetric product as it would on an element of a tensor product, i.e., $\rho(x)(v \cdot w) = (\rho(x)v) \cdot w + v \cdot (\rho(x)w)$. This works because the symmetric power can be constructed as a certain quotient of the tensor power. See [3], pp. 444-446 for a detailed account.

by induction. Similarly, $H(y^j) = -jy^j$. Therefore,

$$\begin{aligned} H(x^{m-k}y^k) &= -kx^{m-k}y^k + (m-k)x^{m-k}y^k \\ &= (m-2k) \cdot x^{m-k}y^k. \end{aligned}$$

This shows that $x^{m-k}y^k \in V_{m-2k}$. Thus, $\operatorname{Sym}^m V$ decomposes itself in a direct sum of one-dimensional eigenspaces for $m, m-2, \ldots, -m$. That is, we can identify $\operatorname{Sym}^m V$ with $V(m)$. We have therefore found a concrete realization of $V(m)$ as the m-th symmetric power of the standard 2-dimensional representation $\mathbf{F}^2$.

(Perhaps now the reader will be interested in studying the representations of other semisimple Lie algebras! Further knolwedge (including more on the general theory of representations of semisimple Lie algebras) can be found in [1] and [2].)

Acknowledgments. It is a pleasure to thank my mentors Benjamin Gammage and Clark Butler for all their help and guidance. I also thank Peter May and the other involved people for organizing the REU. (Also, I apologize for the illegal length of my paper.)

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