Taylor's Theorem Notes

Quin Darcy

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1 Needed Results

Theorem 1 (Mean value Theorem)**.** *If f is a continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

2 Two Variations of Taylor's Theorem

Theorem 2 (Rudin)**.** *Suppose f is a real function of $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for all $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k. \tag{1}$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n. \tag{2}$$

Proof. Let M be the number that satisfies

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n. \tag{3}$$

With this M in hand, define

$$g(t) = f(t) - P(t) - M(t - \alpha)^n \quad (a \le t \le b). \tag{4}$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. Now consider

$$\begin{aligned}
g(t) &= f(t) - \big(f(\alpha) + f'(\alpha)(t-\alpha) + \frac{f''(\alpha)}{2!}(t-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}(t-\alpha)^{n-1}\big) - M(t-\alpha)^n \\
&\Rightarrow g'(t) = f'(t) - \big(f'(\alpha) + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-2)!}(t-\alpha)^{n-2}\big) - nM(t-\alpha)^{n-1} \\
&\Rightarrow g''(t) = f''(t) - \big(f''(\alpha) + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-3)!}(t-\alpha)^{n-3}\big) - n(n-1)M(t-\alpha)^{n-2} \\
&\vdots \\
&\Rightarrow g^{(n-1)}(t) = f^{(n-1)}(t) - \big(f^{(n-1)}(\alpha)\big) - n(n-1)\cdots(2)M(t-\alpha) \\
&\Rightarrow g^{(n)}(t) = f^{(n)}(t) - n!M,
\end{aligned}$$

For all $a < t < b$. Hence, the proof will be complete if we can show $g^{(n)}(x) = 0$ for some x between α and β.

Note that $P(\alpha) = f(\alpha)$ since all the other terms in the sum have a $(t-\alpha)^k$ attached, which goes to zero when evaluated at α. Moreover, $P'(\alpha) = f'(\alpha)$ for the same reason, except the first $f(\alpha)$ is elimiated by the derivative. Generally, we have $P^{(k)} = f^{(k)}(\alpha)$, for each $k = 0, 1, \ldots, n-1$. Thus

$$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$$

Also note that our choice of M in (2) gives that

$$M = \frac{f(\beta) - P(\beta)}{(\beta-\alpha)^n}$$

which implies

$$\begin{aligned}
g(\beta) &= f(\beta) - P(\beta) - \left(\frac{f(\beta) - P(\beta)}{(\beta-\alpha)^n}\right)(\beta-\alpha)^n \\
&= f(\beta) - P(\beta) - f(\beta) + P(\beta) \\
&= 0.
\end{aligned}$$

Thus we have that g is a continuous function on $[\alpha, \beta]$ which is differentiable in (α, β), and so by Theorem (MVT), there exists $x_1 \in (\alpha, \beta)$ such that $g'(x_1) = (f(\beta) - f(\alpha))/(\beta - \alpha) = 0$. Since $g'(x_1) = 0$, then we can conclude that $g''(x_2) = 0$ for some $x_2 \in (\alpha, x_1)$. Continuing in this manner for n steps, we arrive at some $x_n \in (\alpha, x_{n-1})$ such that $g^{(n)}(x_n) = 0$. And since $x_n \in (\alpha, \beta)$, then we have shown what we needed. □

Theorem 3 (Bloch)**.** *Let $[a, b] \subseteq \mathbb{R}$ be a non-degenerate (i.e., $(a, b), [a, b], (a, b]$, etc.) closed bounded interval, let $c \in (a, b)$, let $f : [a, b] \to \mathbb{R}$ be a function and let $n \in \mathbb{N} \cup \{0\}$. Suppose that $f^{(k)}$ exists and is continuous on $[a, b]$ for each $k \in \{0, \ldots, n\}$, and that $f^{(n+1)}$ exists on (a, b). Let $x \in [a, b]$ then there is some p strictly between x and c (except that $p = c$ when $x = c$) such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(p)}{(n+1)!}(x-c)^{n+1}.$$

Proof. For $x = c$, the theorem holds trivially since everything else zeros out leaving $f(c) = f(c)$.

Now suppose that $x \neq c$. Then there is a unique $B \in \mathbb{R}$ such that the following equation holds (simply solve for B):

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + B(x-c)^{n+1}$$

$\square$