MATH 220A

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Due Date: 4/4/21
Assignment: Homework 4

19.1 Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity implies the open set definition.

Proof. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and let $V \subset \mathbb{R}$ be any open set in the range of f. Then we want to show that $f^{-1}(V)$ is open. To do this we let $a \in f^{-1}(V)$ and show that a is contained in a neighborhood which is contained in $f^{-1}(V)$. Since f is continuous, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$ for all $x \in \mathbb{R}$ such that $|x-a| < \delta$. Hence, $a \in (a-\delta, a+\delta) \subset f^{-1}(V)$. Therefore every point of $f^{-1}(V)$ is an interior point. □

19.5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ is homeomorphic with $[0, 1]$.

Proof. We begin by defining the following function $f : (a, b) \to (0, 1)$:

$$f(x) = \frac{x-a}{b-a}.$$

We want to show that f is a bijection, f is continuous, and f^{-1} is continuous. We will first let $x_1, x_2 \in (a, b)$ and assume that $f(x_1) = f(x_2)$. Then this implies that

$$\frac{x_1 - a}{b-a} = \frac{x_2 - a}{b-a} \Rightarrow x_1 - a = x_2 - a \Rightarrow x_1 = x_2.$$

Hence, f is injective. Next, if $y \in (0, 1)$, then choosing $x \in (a, b)$ such that $x = y(b-a) + a$, then we get that $f(x) = y$. Hence, f is surjective.

Having shown that f is a bjection, we now want to show that f and f^{-1} are both continuous over (a, b) and $(0, 1)$, respectively. Let $c \in (a, b)$ and let $\varepsilon > 0$. Then for $\delta = \varepsilon|b-a|$, it follows that for all $x \in (a, b)$

$$\begin{aligned}
|x-c| < \delta &\Rightarrow |x-c| < \varepsilon|b-a| \\
&\Rightarrow \frac{|x-c|}{|b-a|} < \varepsilon \\
&\Rightarrow \left|\frac{x-c}{b-a}\right| < \varepsilon \\
&\Rightarrow \left|\frac{x-a}{b-a} - \frac{c-a}{b-a}\right| < \varepsilon \\
&\Rightarrow |f(x) - f(c)| < \varepsilon.
\end{aligned}$$

For all $c \in (a,b)$. This shows that f is continuous. Now we need to show that $f^{-1}(y) = y(b-a)+a$ is continuous at every $x \in (0,1)$. Letting $\varepsilon > 0$ and selecting the same $\delta = \varepsilon/|b-a|$ as before we find that for all $y \in (0,1)$

$$\begin{aligned}
|y-c| &< \delta \\
&\Rightarrow |y-c| < \frac{\varepsilon}{|b-a|} \\
&\Rightarrow |y-c||b-a| < \varepsilon \\
&\Rightarrow |(y-c)(b-a)| < \varepsilon \\
&\Rightarrow |y(b-a) - c(b-a)| < \varepsilon \\
&\Rightarrow |y(b-a) + a - c(b-a) - a| < \varepsilon \\
&\Rightarrow |f^{-1}(y) - f^{-1}(c)| < \varepsilon.
\end{aligned}$$

Therefore (a,b) is homeomorphic to $(0,1)$. Lastly, we note that both f and f^{-1} are defined on the end points of their respective intervals and that $f(a) = 0$, $f(b) = 1$, $f^{-1}(0) = a$, and $f^{-1}(1) = b$. Hence, the given function, f, also shows that $[a,b]$ and $[0,1]$ are homeomorphic. $\square$

19.13. Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \overline{A} \to Y$, then g is uniquely determined by f.

Proof. Let $g : \overline{A} \to Y$ and $h : \overline{A} \to Y$ be continuous extensions of f. Assume there exists $x \in \overline{A}$ such that $g(x) \neq h(x)$. Then since Y is Hausdorff and $g(x), h(x) \in Y$, then there exists neighborhoods U_1 and U_2 such that $g(x) \in U_1$, $h(x) \in U_2$, and $U_1 \cap U_2 = \varnothing$. We also note that $x \in g^{-1}(U_1)$ and $x \in h^{-1}(U_2)$. Hence, $x \in g^{-1}(U_1) \cap h^{-1}(U_2)$, which, by continuity, is open as it is the intersection of open sets. With $x \in \overline{A}$, then either $x \in A$ or $x \in A'$. If $x \in A$, then $f(x) = g(x) = h(x)$, which is a contradiction. If $x \in A'$, then there exists some $v \in A$ such that $x \neq v$ and $v \in g^{-1}(U_1) \cap h^{-1}(U_2)$. This implies that $f(v) = g(v) = h(v)$. Hence, $U_1 \cap U_2 \neq \varnothing$, which is a contradiction. Therefore, there does not exist $x \in \overline{A}$ such that $g(x) \neq h(x)$ and thus $g = h$. $\square$

20.1. (a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology on $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then since for each $1 \leq i \leq n$, we have that $|x_i - y_i| \geq 0$, then $|\mathbf{x} - \mathbf{y}| \geq 0$. Similarly, if $|\mathbf{x} - \mathbf{y}| = 0$, then $|x_i - y_i| = 0$ for all i and thus, $x_i = y_i$ for all i, which implies that $\mathbf{x} = \mathbf{y}$.

Since for each $1 \leq i \leq n$, $d(x_i, y_i) = d(y_i, x_i)$, i.e., $|x_i - y_i| = |y_i - x_i|$, then if $d'(\mathbf{x}, \mathbf{y}) \neq d'(\mathbf{y}, \mathbf{x})$, then that would imply that $|x_i - y_i| \neq |y_i - x_i|$, which is a contradiction, since d is a metric on $\mathbb{R}$.

Finally, letting $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}$. Then

$$d(\mathbf{x}, \mathbf{z}) = |x_1 - z_1| + \cdots + |x_n - z_n|$$

and

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) = (|x_1 - y_1| + |y_1 - z_1|) + \cdots + (|x_n - y_n| + |y_n - z_n|)$$

Finally, we note that for each i, $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ and thus by induction it can be shown that $|x_1 - z_1| + \cdots + |x_n - z_n| \leq (|x_1 - y_1| + |y_1 - z_1|) + \cdots + (|x_n - y_n| + |y_n - z_n|)$.

To show that d' induces the usual topology on $\mathbb{R}^n$, we will show that the topology induced by d' is the same as the topology induced by ρ, which by Theorem 20.3, is the same as the standard topology on $\mathbb{R}^n$. To do this, we will let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then consider the two metrics: $d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$, and $\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} = |x_i - y_i|$, for some $1 \leq i \leq n$. This implies that $|x_1 - y_1| + \cdots + |x_n - y_n| \leq n|x_i - y_i|$. Hence, $d'(\mathbf{x}, \mathbf{y}) \leq n\rho(\mathbf{x}, \mathbf{y})$. By Lemma 20.2, any basis element $B_{d'}(\mathbf{x}, \varepsilon)$, of size ε, contains a basis element $B_\rho(\mathbf{x}, \varepsilon/n)$. Similarly, for any basis element $B_\rho(\mathbf{x}, \varepsilon)$ contains a basis element $B_{d'}(\mathbf{x}, \varepsilon)$ since $\rho(\mathbf{x}, \mathbf{y}) \leq d'(\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Thus, the topologies induced by d' and ρ are the same as the standard topology on $\mathbb{R}^n$.

Finally, the basis elements under d' appear as follows:

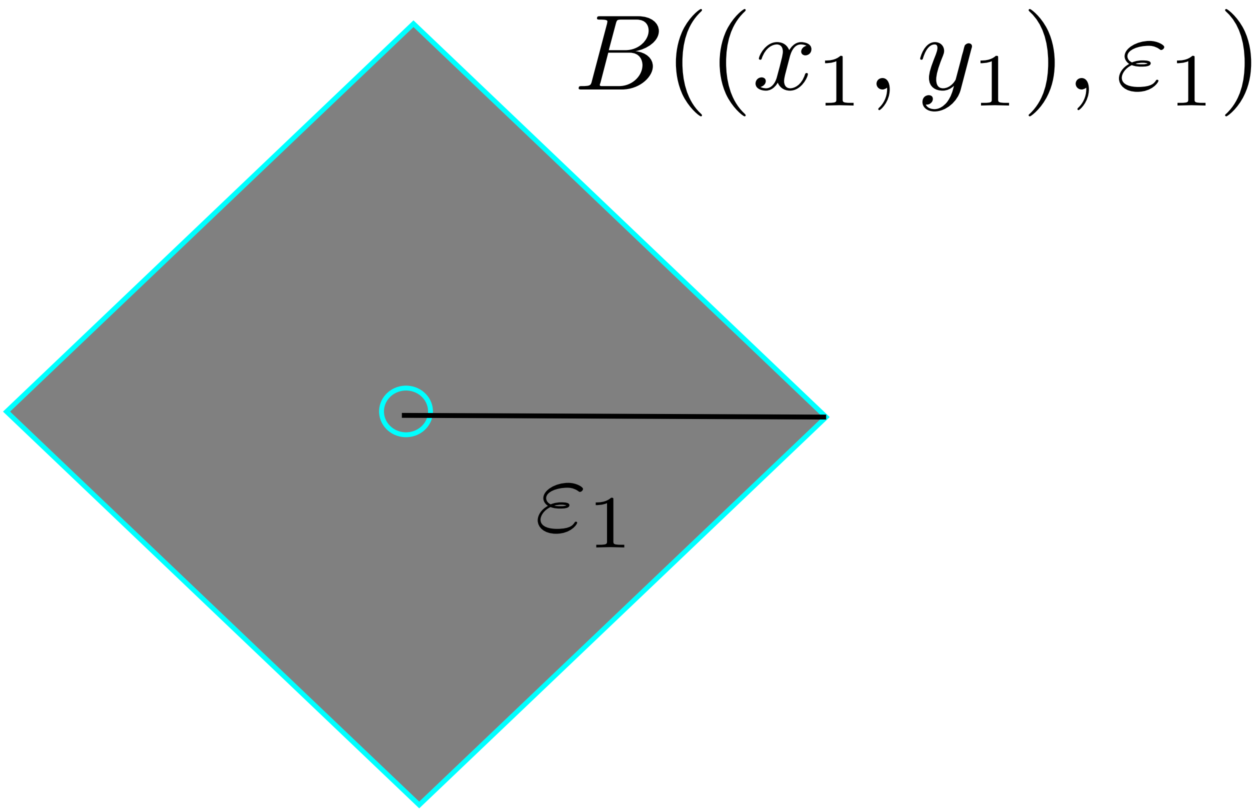

Figure 1: Basis

□

21.10 Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:

$$A = \{x \times y \mid xy = 1\}$$
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\}$$
$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}$$

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$. Then if $\varepsilon > 0$, and $a \in \mathbb{R}$, if we choose $\delta = \varepsilon$, it follows that $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. Hence, f is continuous. A similar argument can be used to show that $g(y) = y$ is continuous. By Lemma 21.4, the function fg is a continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Finally, considering the singleton set $\{1\}$, which is closed, we may then apply Theorem 18.1 which states that $(fg)^{-1}(\{1\}) = A$ is closed.

To show that S^1 is closed, we first need to show that the function $f(x) = x^2$ is continuous. Letting $\varepsilon > 0$, and $a \in \mathbb{R}$, then letting $K = \min\{1, \frac{\varepsilon}{1+2|a|}\}$. It follows that if $\delta = K/2$, then $|x - a| < 1$ and so $|x| < 1 + |a|$. Thus, $|x^2 - a^2| = |x + a||x - a| \leq |x - a||1 + 2|a|| < \delta|1 + 2|a|| = \varepsilon$. Thus, $f(x) = x^2$ is continuous. This implies that $g(y) = y^2$ is continuous and by Lemma 21.4, $f + g$ is continuous. By Theorem 18.1, $(f + g)^{-1}(\{1\})$ is closed.

Lastly, we consider the set $[-1, 1]$ which is closed in $\mathbb{R}$ and letting f and g be the same functions as above, we get that $(f + g)$ is continuous, and that $(f + g)^{-1}([-1, 1]) = B^2$, which by Theorem 18.1, is closed. □