MATH 210B

Name: Quin Darcy **Due Date:** 4/15/20
Instructor: Dr. Shannon **Assignment:** Homework 9

5. Determine, with explanation, if it is always true that if E/F is finite, and $E/L/F$, then $|G(E/F)| = |G(E/L)||G(L/F)|$.

Solution. It is not always true. Let $E = \mathbb{Q}(i, \sqrt[4]{2})$, $L = \mathbb{Q}(\sqrt[4]{2})$, and $F = \mathbb{Q}$. Then E is the splitting field for the separable polynomial $x^4 - 2 \in F[x]$ and so E/F is Galois. Thus,

$$|G(E/F)| = [E\colon F] = [E\colon L][L\colon F].$$

Now consider E/L. Since E is the splitting field of the separable polynomial $x^2 + 1$ over L, then E/L is Galois and so $|G(E/L)| = [E\colon L]$. However, L contains a root of $x^4 - 2 \in F[x]$, but $x^4 - 2$ does not split over L and thus L/F is not a normal extension. Hence, L/F is not Galois and so $|G(L/F)| < [L\colon F]$. Thus,

$$|G(E/F)| > |G(E/L)||G(L/F)|.$$

6. Recall from (1) of HW 4 that $\mathbb{Q}(i, \sqrt{2}) = \mathbb{Q}(\sqrt{i})$.

   (a) Let ζ be a primitive 8^{th} root of unity. Explain why $\zeta \in \mathbb{Q}(i, \sqrt[8]{2})$ and thus why it follows that $\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}$ is a Galois extension.

   Solution. Since ζ is an 8^{th} root of unity, then using De Moivre's theorem, we can write,

$$\zeta = e^{2\pi i/8} = e^{\pi i/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i).$$

   Now observe that $\sqrt[8]{2} \in \mathbb{Q}(i, \sqrt[8]{2})$ and $(\sqrt[8]{2})^{16} = \sqrt{2} \in \mathbb{Q}(i, \sqrt[8]{2})$. It follows that $\frac{\sqrt{2}}{2} \in \mathbb{Q}(i, \sqrt[8]{2})$. Finally, since i is also an element, then $1 + i$ is an element and so $\frac{\sqrt{2}}{2}(1+i) = \zeta \in \mathbb{Q}(i, \sqrt[8]{2})$. Because ζ is primitive, then $x^8 - 1$ splits completely in $\mathbb{Q}(i, \sqrt[8]{2})$. Finally, we can factor

$$x^8 - 2 = (x - \sqrt[8]{2})(x + \sqrt[8]{2})(x - \zeta^2\sqrt[8]{2})(x + \zeta^2\sqrt[8]{2})(x^4 + \sqrt{2}).$$

   Letting $u = x^2$, then $x^4 + \sqrt{2} = u^2 + \sqrt{2}$. Using the quadratic formula, we find that $u = \pm i\sqrt[4]{2}$ and so $x^4 - \sqrt{2} = (x^2 - i\sqrt[4]{2})(x^2 + i\sqrt[4]{2})$. Applying the quadratic formula on each of these factors we obtain

$$x^4 + \sqrt{2} = (x - \zeta\sqrt[8]{2})(x + \zeta\sqrt[8]{2})(x - \zeta^3\sqrt[8]{2})(x + \zeta^3\sqrt[8]{2}).$$

   Hence, the roots of $x^8 - 2$ are $\pm\sqrt[8]{2}, \pm\zeta\sqrt[8]{2}, \pm\zeta^2\sqrt[8]{2}, \pm\zeta^3\sqrt[3]{2}$, all of which are distinct and so $x^8 - 2$ is separable and splits in $\mathbb{Q}(i, \sqrt[8]{2})$. Now note that since $\zeta \in \mathbb{Q}(i, \sqrt[8]{2})$, then we get that $\mathbb{Q}(\zeta, \sqrt[8]{2}) \subseteq \mathbb{Q}(i, \sqrt[8]{2})$. Similarly, since $\zeta^2 = i$ it follows that $\mathbb{Q}(i, \sqrt[8]{2}) \subseteq \mathbb{Q}(\zeta, \sqrt[8]{2})$ and so $\mathbb{Q}(i, \sqrt[8]{2}) = \mathbb{Q}(\zeta, \sqrt[8]{2})$ and the former is the splitting field of $x^8 - 2$. Therefore, $\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}$ is Galois.

(b) Explain why $o\big(\text{Aut}(\mathbb{Q}(i, \sqrt[8]{2})\big) = 16$. Define $\alpha, \beta \in \text{Aut}\big(\mathbb{Q}(i, \sqrt[8]{2})\big)$ by $\alpha(\sqrt[8]{2}) = \zeta\sqrt[8]{2}$, $\alpha(i) = i$; $\beta(\sqrt[8]{2}) = \sqrt[8]{2}$, $\beta(i) = -i$. Determine $\alpha(\zeta)$ and $\beta(\zeta)$. Then determine $\alpha^2, \ldots, \alpha^7$ and $\beta\alpha, \ldots, \beta\alpha^7$. Explain why $\text{Aut}(\mathbb{Q}(i, \sqrt[8]{2}) = \{e, \alpha, \ldots, \alpha^7, \beta, \ldots, \beta\alpha^7\}$.

Solution. Given any $\sigma \in \text{Aut}(\mathbb{Q}(i, \sqrt[8]{2})$, σ must act on the roots of $x^8 - 2$ by permuting them and σ must also permute the roots of $x^2 + 1$. Thus, there are $8 \cdot 2 = 16$ possible automorphisms and so $o\big(\text{Aut}(\mathbb{Q}(i, \sqrt[8]{2})\big) = 16$.

Using the fact that $\alpha(\sqrt[8]{2}) = \zeta\sqrt[8]{2}$ and $\alpha(i) = i$, then it follows that

$$\alpha(\sqrt{2}) = \alpha\big((\sqrt[8]{2})^4\big) = \big(\alpha(\sqrt[8]{2})\big)^4 = \big(\zeta\sqrt[8]{2})^4 = \zeta^4(\sqrt[4]{2})^4 = (-1)\sqrt{2} = -\sqrt{2}.$$

Thus,

$$\alpha(\zeta) = \alpha\left(\frac{\sqrt{2}}{2}(1+i)\right) = \alpha\left(\frac{\sqrt{2}}{2}\right)\alpha(1+i) = -\frac{\sqrt{2}}{2}(1+i) = -\zeta.$$

Hence, $\alpha(\zeta) = \zeta^5$. Now we can look at

$$\beta(\sqrt{2}) = \beta\big((\sqrt[8]{2})^4\big) = \big(\beta(\sqrt[8]{2})\big)^4 = \big(\sqrt[8]{2})^4 = \sqrt{2}.$$

Combining this with the fact that $\beta(i) = -i$, then

$$\beta(\zeta) = \beta\left(\frac{\sqrt{2}}{2}(1+i)\right) = \beta\left(\frac{\sqrt{2}}{2}\right)\beta(1+i) = \frac{\sqrt{2}}{2}(1-i) = -\zeta^3.$$

To determine $\alpha^2, \ldots, \alpha^7, \beta, \ldots, \beta\alpha^7$, we look to the following table: Since both

	id	α	α^2	α^3	α^4	α^5	α^6	α^7	β	$\beta\alpha$	$\beta\alpha^2$	$\beta\alpha^3$	$\beta\alpha^4$	$\beta\alpha^5$	$\beta\alpha^6$	$\beta\alpha^7$
i	i	i	i	i	i	i	i	i	$-i$	$-i$	$-i$	$-i$	$-i$	$-i$	$-i$	$-i$
$\sqrt[8]{2}$	$\sqrt[8]{2}$	$\zeta\sqrt[8]{2}$	$-\zeta^2\sqrt[8]{2}$	$-\zeta^3\sqrt[8]{2}$	$-\sqrt[8]{2}$	$-\zeta\sqrt[8]{2}$	$\zeta^2\sqrt[8]{2}$	$\zeta^3\sqrt[8]{2}$	$\sqrt[8]{2}$	$-\zeta^3\sqrt[8]{2}$	$\zeta^2\sqrt[8]{2}$	$\zeta\sqrt[8]{2}$	$-\sqrt[8]{2}$	$\zeta^3\sqrt[8]{2}$	$-\zeta^2\sqrt[8]{2}$	$-\zeta\sqrt[8]{2}$
ζ	ζ	$-\zeta$	ζ	$-\zeta$	ζ	$-\zeta$	ζ	$-\zeta$	$-\zeta^3$	ζ^3	$-\zeta^3$	$-\zeta^3$	ζ^3	$-\zeta^3$	ζ^3	$-\zeta^3$

α and β are automorphisms, then each $\alpha^2, \ldots, \alpha^7$ and $\beta\alpha, \ldots, \beta\alpha^7$ are automorphisms. We can see by the table that each of these maps are distinct and there are 16 of them. Thus $\text{Aut}(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q})) = \{id, \alpha, \ldots, \alpha^7, \beta, \beta\alpha, \ldots, \beta\alpha^7\}$.

(c) Find the elements of $G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i))$, $G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(\sqrt{2}))$, $G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i\sqrt{2}))$, and in each case determine what well known group the Galois group is isomorphic to.

Solution. Using the table in part (b), we get

$$G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i)) = \{id, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\} \cong \mathbb{Z}_8$$
$$G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(\sqrt{2})) = \{id, \beta, \beta\alpha^2, \beta\alpha^4, \beta\alpha^6\} \cong \mathbb{Z}_5$$
$$G(\mathbb{Q}(i, \sqrt[8]{2})/\mathbb{Q}(i, \sqrt{2})) = \{id\} \cong \{e\}.$$

7. On HW8 we found $G(E/F)$ for $E = \mathbb{Q}(\sqrt[4]{5}, i)$, $F = \mathbb{Q}$. Construct the lattice of subgroups of $G(E/F)$, and the corresponding lattice of subfields of E over F, Identify all the normal extensions in the lattice of subfields.

Solution. To start, lets recall that a basis for $\mathbb{Q}(\sqrt[4]{5}, i)$ over $\mathbb{Q}$ is $\{1, \sqrt[4]{5}, \sqrt{5}, \sqrt[4]{5^3}, i, i\sqrt[4]{5}, i\sqrt{5}, i\sqrt[4]{5^3}\}$. Next, from HW 8 we found that $G(\mathbb{Q}(\sqrt[4]{5}, i)/\mathbb{Q}) = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7\}$, where

$$\varphi_0 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto \sqrt[4]{5} \\ i \mapsto i \end{cases} \quad \varphi_1 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto -\sqrt[4]{5} \\ i \mapsto i \end{cases} \quad \varphi_2 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto i\sqrt[4]{5} \\ i \mapsto i \end{cases} \quad \varphi_3 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto -i\sqrt[4]{5} \\ i \mapsto i \end{cases}$$

$$\varphi_4 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto \sqrt[4]{5} \\ i \mapsto -i \end{cases} \quad \varphi_5 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto -\sqrt[4]{5} \\ i \mapsto -i \end{cases} \quad \varphi_6 := \begin{cases} a \in \mathbb{Q} \to a \\ \sqrt[4]{5} \mapsto i\sqrt[4]{5} \\ i \mapsto -i \end{cases} \quad \varphi_7 := \begin{cases} a \in \mathbb{Q} \mapsto a \\ \sqrt[4]{5} \mapsto -i\sqrt[4]{5} \\ i \mapsto -i. \end{cases}$$

From this, we found a correspondence between the automorphisms and the following permutations:

$$\begin{array}{llll} \varphi_0 := (1) & \varphi_1 := (13)(24) & \varphi_2 := (1234) & \varphi_3 := (1432) \\ \varphi_4 := (24) & \varphi_5 := (13) & \varphi_6 := (12)(34) & \varphi_7 := (14)(23). \end{array}$$

Additionally, from the Cayley table we found that $G(\mathbb{Q}(\sqrt[4]{5}, i)/\mathbb{Q}) \cong D_8$.

Now we wish to find the fixed field of each subgroup generated by each automorphism. To do this, we begin by constructing a table which indicates where each automorphism sends each basis element.

	1	$\sqrt[4]{5}$	$\sqrt{5}$	$\sqrt[4]{5^3}$	i	$i\sqrt[4]{5}$	$i\sqrt{5}$	$i\sqrt[4]{5^3}$
φ_0	1	$\sqrt[4]{5}$	$\sqrt{5}$	$\sqrt[4]{5^3}$	i	$i\sqrt[4]{5}$	$i\sqrt{5}$	$i\sqrt[4]{5^3}$
φ_1	1	$-\sqrt[4]{5}$	$\sqrt{5}$	$-\sqrt[4]{5^3}$	i	$-i\sqrt[4]{5}$	$i\sqrt{5}$	$-i\sqrt[4]{5^3}$
φ_2	1	$i\sqrt[4]{5}$	$-\sqrt{5}$	$-i\sqrt[4]{5^3}$	i	$-\sqrt[4]{5}$	$-i\sqrt{5}$	$\sqrt[4]{5^3}$
φ_3	1	$-i\sqrt[4]{5}$	$-\sqrt{5}$	$i\sqrt[4]{5^3}$	i	$\sqrt[4]{5}$	$-i\sqrt{5}$	$-\sqrt[4]{5^3}$
φ_4	1	$\sqrt[4]{5}$	$\sqrt{5}$	$\sqrt[4]{5^3}$	$-i$	$-i\sqrt[4]{5}$	$-i\sqrt{5}$	$-i\sqrt[4]{5^3}$
φ_5	1	$-\sqrt[4]{5}$	$\sqrt{5}$	$-\sqrt[4]{5^3}$	$-i$	$i\sqrt[4]{5}$	$-i\sqrt{5}$	$i\sqrt[4]{5^3}$
φ_6	1	$i\sqrt[4]{5}$	$-\sqrt{5}$	$-i\sqrt[4]{5^3}$	$-i$	$\sqrt[4]{5}$	$i\sqrt{5}$	$-\sqrt[4]{5^3}$
φ_7	1	$-i\sqrt[4]{5}$	$-\sqrt{5}$	$i\sqrt[4]{5^3}$	$-i$	$-\sqrt[4]{5}$	$i\sqrt{5}$	$\sqrt[4]{5^3}$

Here the blue cells indicate the unchanged basis elements and the pink cells indicate a sign flip.

Now for each automorphism, φ_i, we want to determine when is $\varphi_i(x) = x$. To answer this we need to solve

$$\begin{aligned} &\varphi_i(a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}) \\ &\quad = a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3} \end{aligned}$$

for each i. This gives us the following equations:

1. $$\begin{aligned}&a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

2. $$\begin{aligned}&a_0 - a_1\sqrt[4]{5} + a_2\sqrt{5} - a_3\sqrt[4]{5^3} + a_4 i - a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} - a_7 i\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

3. $$\begin{aligned}&a_0 + a_1 i\sqrt[4]{5} - a_2\sqrt{5} - a_3 i\sqrt[4]{5^3} + a_4 i - a_5\sqrt[4]{5} - a_6 i\sqrt{5} + a_7\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

4. $$\begin{aligned}&a_0 - a_1 i\sqrt[4]{5} - a_2\sqrt{5} + a_3 i\sqrt[4]{5^3} + a_4 i + a_5\sqrt[4]{5} - a_6 i\sqrt{5} - a_7\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

5. $$\begin{aligned}&a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} - a_4 i - a_5 i\sqrt[4]{5} - a_6 i\sqrt{5} - a_7 i\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

6. $$\begin{aligned}&a_0 - a_1\sqrt[4]{5} + a_2\sqrt{5} - a_3\sqrt[4]{5^3} - a_4 i + a_5 i\sqrt[4]{5} - a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

7. $$\begin{aligned}&a_0 + a_1 i\sqrt[4]{5} - a_2\sqrt{5} - a_3 i\sqrt[4]{5^3} - a_4 i + a_5\sqrt[4]{5} + a_6 i\sqrt{5} - a_7\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

8. $$\begin{aligned}&a_0 - a_1 i\sqrt[4]{5} - a_2\sqrt{5} + a_3 i\sqrt[4]{5^3} - a_4 i - a_5\sqrt[4]{5} + a_6 i\sqrt{5} + a_7\sqrt[4]{5^3}\\ &= a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} + a_4 i + a_5 i\sqrt[4]{5} + a_6 i\sqrt{5} + a_7 i\sqrt[4]{5^3}\end{aligned}$$

Solving for these equations we get that

1. Each basis element was fixed and so
$$F_{\langle\varphi_0\rangle} = \mathbb{Q}(\sqrt[4]{5}, i).$$
2. $a_1 = 0$, $a_3 = 0$, $a_5 = 0$, $a_7 = 0$. Thus,
$$F_{\langle\varphi_1\rangle} = \{a_0 + a_2\sqrt{5} + a_4 i + a_6 i\sqrt{5} \mid a_i \in \mathbb{Q}\}.$$
3. $a_1 = a_5$, $a_2 = 0$, $a_3 = -a_7$, $a_6 = 0$. Thus,
$$F_{\langle\varphi_2\rangle} = \{a_0 + a_1(\sqrt[4]{5} + i\sqrt[4]{5}) + a_3(\sqrt[4]{5^3} - i\sqrt[4]{5^3}) + a_4 i \mid a_i \in \mathbb{Q}\}.$$
4. $a_1 = -a_5$, $a_2 = 0$, $a_3 = a_7$, $a_6 = 0$. Thus,
$$F_{\langle\varphi_3\rangle} = \{a_0 + a_1(\sqrt[4]{5} - i\sqrt[4]{5}) + a_3(\sqrt[4]{5^3} + i\sqrt[4]{5^3}) + a_4 i \mid a_i \in \mathbb{Q}\}.$$
5. $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$. Thus,
$$F_{\langle\varphi_4\rangle} = \{a_0 + a_1\sqrt[4]{5} + a_2\sqrt{5} + a_3\sqrt[4]{5^3} \mid a_i \in \mathbb{Q}\}.$$
6. $a_1 = 0$, $a_3 = 0$, $a_4 = 0$, $a_6 = 0$. Thus,
$$F_{\langle\varphi_5\rangle} = \{a_0 + a_2\sqrt{5} + a_5 i\sqrt[4]{5} + a_7 i\sqrt[4]{5^3} \mid a_i \in \mathbb{Q}\}.$$
7. $a_1 = a_5$, $a_2 = 0$, $a_3 = -a_7$, $a_4 = 0$. Thus,
$$F_{\langle\varphi_6\rangle} = \{a_0 + a_1(\sqrt[4]{5} + i\sqrt[4]{5}) + a_3(\sqrt[4]{5^3} - i\sqrt[4]{5^3}) + a_6 i\sqrt{5} \mid a_i \in \mathbb{Q}\}.$$
8. $a_1 = -a_5$, $a_2 = 0$, $a_3 = a_7$, $a_4 = 0$. Thus,
$$F_{\langle\varphi_7\rangle} = \{a_0 + a_1(\sqrt[4]{5} - i\sqrt[4]{5}) + a_3(\sqrt[4]{5^3} + i\sqrt[4]{5^3}) + a_6 i\sqrt{5} \mid a_i \in \mathbb{Q}\}.$$

We will now simplify the above fixed fields.

1. $F_{\langle\varphi_0\rangle} = \mathbb{Q}(\sqrt[4]{5}, i)$.
2. $F_{\langle\varphi_1\rangle} = \mathbb{Q}(\sqrt{5}, i)$.
3. $F_{\langle\varphi_2\rangle} = \mathbb{Q}(\sqrt[4]{5} + i\sqrt[4]{5})$
4. $F_{\langle\varphi_3\rangle} = \mathbb{Q}(\sqrt[4]{5} - i\sqrt[4]{5})$
5. $F_{\langle\varphi_4\rangle} = \mathbb{Q}(\sqrt[4]{5})$.
6. $F_{\langle\varphi_5\rangle} = \mathbb{Q}(i\sqrt[4]{5})$.
7. $F_{\langle\varphi_6\rangle} = \mathbb{Q}(\sqrt[4]{5} + i\sqrt[4]{5})$.
8. $F_{\langle\varphi_7\rangle} = \mathbb{Q}(\sqrt[4]{5} - i\sqrt[4]{5})$.

We can see that there are only 6 distinct fixed fields listed here. We know that there should be as many subfields as there are subgroups of D_8, of which there are 10. Recall that D_8 can be generated by two elements and so consider the two following automorphisms:

$$\varphi_2 := (1234) \quad \text{and} \quad \varphi_4 := (24).$$

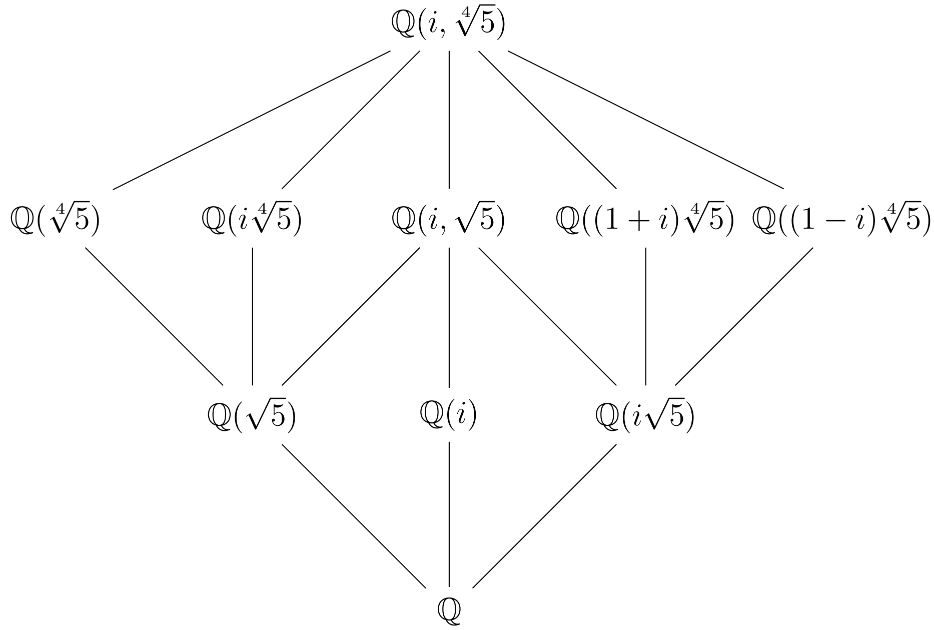
We claim that these two automorphisms generate the entire Galois group. First note,

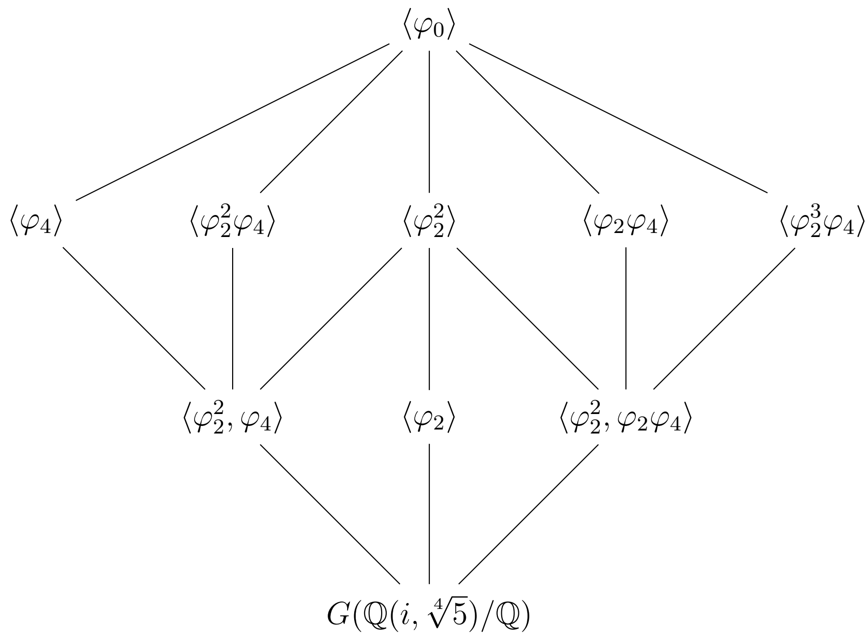
$$\begin{aligned}
\langle\varphi_2\rangle &:= \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\} \\
\langle\varphi_4\rangle &:= \{\varphi_0, \varphi_4\} \\
\langle\varphi_2^2\rangle &:= \{\varphi_0, \varphi_1\} \\
\langle\varphi_2^2, \varphi_4\rangle &:= \{\varphi_0, \varphi_1, \varphi_4, \varphi_5\} \\
\langle\varphi_2\varphi_4\rangle &:= \{\varphi_0, \varphi_6\} \\
\langle\varphi_2^2, \varphi_2\varphi_4\rangle &:= \{\varphi_0, \varphi_1, \varphi_6, \varphi_7\} \\
\langle\varphi_2^2\varphi_4\rangle &:= \{\varphi_0, \varphi_5\} \\
\langle\varphi_2^3\varphi_4\rangle &:= \{\varphi_0, \varphi_7\}
\end{aligned}$$

Performing similar calculations as above, we find that

1. $F_{\langle\varphi_2\rangle} = \mathbb{Q}(i)$.
2. $F_{\langle\varphi_4\rangle} = \mathbb{Q}(\sqrt[4]{5})$
3. $F_{\langle\varphi_2^2\rangle} = \mathbb{Q}(i, \sqrt{5})$
4. $F_{\langle\varphi_2^2,\varphi_4\rangle} = \mathbb{Q}(\sqrt{5})$
5. $F_{\langle\varphi_2\varphi_4\rangle} = \mathbb{Q}(\sqrt[4]{5} + i\sqrt[4]{5})$
6. $F_{\langle\varphi_2^2,\varphi_2\varphi_4\rangle} = \mathbb{Q}(i\sqrt{5})$
7. $F_{\langle\varphi_2^2\varphi_4\rangle} = \mathbb{Q}(i\sqrt[4]{5})$
8. $F_{\langle\varphi_2^3\varphi_4\rangle} = \mathbb{Q}(\sqrt[4]{5} - i\sqrt[4]{5})$.

Finally, given the subgroup relations that are clear from the above list, we can construct the subgroup and subfield lattice.


$\mathbb{Q}(i, \sqrt[4]{5})$
$\mathbb{Q}(\sqrt[4]{5})$
$\mathbb{Q}(i\sqrt[4]{5})$
$\mathbb{Q}(i, \sqrt{5})$
$\mathbb{Q}((1+i)\sqrt[4]{5})$
$\mathbb{Q}((1-i)\sqrt[4]{5})$
$\mathbb{Q}(\sqrt{5})$
$\mathbb{Q}(i)$
$\mathbb{Q}(i\sqrt{5})$
$\mathbb{Q}$


$\langle \varphi_0 \rangle$
$\langle \varphi_4 \rangle$
$\langle \varphi_2^2\varphi_4 \rangle$
$\langle \varphi_2^2 \rangle$
$\langle \varphi_2\varphi_4 \rangle$
$\langle \varphi_2^3\varphi_4 \rangle$
$\langle \varphi_2^2, \varphi_4 \rangle$
$\langle \varphi_2 \rangle$
$\langle \varphi_2^2, \varphi_2\varphi_4 \rangle$
$G(\mathbb{Q}(i, \sqrt[4]{5})/\mathbb{Q})$