MATH 210B

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Instructor: Dr. Shannon **Assignment:** Homework 8

13. For each of the following extensions, E, of F, determine if E/F is a Galois extension, find the elements of $G(E/F)$, and determine what group $G(E/F)$ is isomorphic to.

(a) $E = \mathbb{Q}(\omega, \sqrt[3]{2})$, $F = \mathbb{Q}$.

Solution. Note that by HW 5, E is the splitting field for $x^3 - 2 \in F[x]$, which factors as $(x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(\omega^2 i^2 \sqrt[3]{2})$. Since the roots of $x^3 - 2$, are all distinct, then $x^3 - 2$ is separable over F, so that E is the splitting field of an irreducible polynomial in $F[x]$. By Theorem (??), E/F is Galois.

Let $\varphi \in G(E/F)$. Then $\varphi \in \text{Aut}(E)$ and for all $a \in F$, $\varphi(a) = a$. By HW/Thm (??), the minimal polynomials for $\sqrt[3]{2}, \omega$ over F, $p(x) = x^3 - 2$ and $x^2 + x + 1$, have roots α, β, respectively, such that $\varphi(\alpha) \in \{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\}$ and $\varphi(\beta) \in \{\omega, \omega^2\}$. This gives us $6 = [E\colon F] = \left|G(E/F)\right|$ automorphisms. These are defined as follows:

$$\varphi_1 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \omega \mapsto \omega \end{cases} \qquad \varphi_2 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \omega\sqrt[3]{2} \\ \omega \mapsto \omega \end{cases} \qquad \varphi_3 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \omega^2\sqrt[3]{2} \\ \omega \mapsto \omega \end{cases}$$

$$\varphi_4 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \omega \mapsto \omega^2 \end{cases} \qquad \varphi_5 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \omega\sqrt[3]{2} \\ \omega \mapsto \omega^2 \end{cases} \qquad \varphi_6 := \begin{cases} a \in F \mapsto a \\ \sqrt[3]{2} \mapsto \omega^2\sqrt[3]{2} \\ \omega \mapsto \omega^2. \end{cases}$$

By Thm/HW (??) $G(E/F)$ is a group. The cayley table for this group is

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6
φ_1	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6
φ_2	φ_2	φ_3	φ_1	φ	φ	φ
φ_3	φ_3	φ_1	φ_2	φ	φ	φ
φ_4	φ_4	φ_6	φ_5	φ	φ	φ
φ_5	φ_5	φ_4	φ_6	φ	φ	φ
φ_6	φ_6	φ_5	φ_4	φ	φ	φ

$\longrightarrow$

	(1)	(123)	(132)	(23)	(132)	(123)
(1)	(1)	(123)	(132)	(23)	(132)	(123)
(123)	(123)	(132)	(1)	φ	φ	φ
(132)	(13)	(1)	φ_2	φ	φ	φ
(23)	(23)	φ_6	φ_5	φ	φ	φ
(132)	(132)	φ_4	φ_6	φ	φ	φ
(123)	(123)	φ_5	φ_4	φ	φ	φ

(b) $E = \mathbb{Q}(\sqrt[4]{5}, i)$, $F = \mathbb{Q}$.

Solution. The minimal polynomial of $\sqrt[4]{5}$ over F is $x^4 - 5$. The roots of this polynomial are $\sqrt[4]{5}, -\sqrt[4]{5}, i\sqrt[4]{5}, -i\sqrt[4]{5}$ and so $x^4 - 5$ splits over E. Additionally, E is the smallest field containing F and the roots of $x^4 - 5$ and as such, E is the splitting field for x^4-5. Since $\sqrt[4]{5} \notin F(i)$, then $x^2-\sqrt[4]{25}$ is irreducible over $F(i)$ and so $[E\colon F(i)] = 2$. Similarly, since x^2+1 is irreducible over F, then $[F(i)\colon F] = 2$. Thus, $[E\colon F] = [E\colon F(i)][F(i)\colon F] = 4$. Now note that any automorphism of E fixes F by HW 1 and so $G(E/F) = \mathrm{Aut}(E)$. Moreover, for any $\varphi \in \mathrm{Aut}(E)$, we have that

$$\varphi(\sqrt[4]{5}) = \pm\sqrt[4]{5} \quad \text{and} \quad \varphi(i) = \pm i.$$

Thus, $\big|\mathrm{Aut}(E)\big| = \big|G(E/F)\big| = 4$ and so $\big|G(E/F)\big| = [E\colon F]$. Thus, by Theorem 5, E/F is Galois. Finally, let $\varphi, \tau \in G(E/F)$ such that

$$\varphi(\sqrt[4]{5}) = -\sqrt[4]{5},\ \varphi(i) = i \quad \text{and} \quad \tau(\sqrt[4]{5}) = \sqrt[4]{5},\ \tau(i) = -i.$$

Letting $\alpha_1 = \sqrt[4]{5}, \alpha_2 = -\sqrt[4]{5}, \alpha_3 = i\sqrt[4]{5}$, and $\alpha_4 = -i\sqrt[4]{5}$, then

	α_1	α_2	α_3	α_4	
1_E	α_1	α_2	α_3	α_4	(1)
φ	α_2	α_1	α_3	α_4	(12)
τ	α_1	α_2	α_4	α_3	(34)
$\varphi\tau$	α_2	α_1	α_4	α_3	(12)(34)

From this table we can see that $G(E/F) \cong \{(1), (12), (34), (12)(34)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) $E = \mathbb{Q}(\sqrt[4]{5}, i)$, $F = \mathbb{Q}(i)$.

Solution. By part (b), we know that $|\mathrm{Aut}(E)| = 4$ and two of these automorphisms map i to $-i$. Thus, $G(E/F) = \{1_E, \varphi\}$, where 1_E is the identity on E and $\varphi(\sqrt[4]{5}) = -\sqrt[4]{5}$ and $\varphi(i) = -i$. Thus, $|G(E/F)| = 2$. Additionally, the minimal polynomial of $\sqrt[4]{5}$ over F is $x^2 - \sqrt[4]{25}$ and so $[E\colon F] = 2$. Thus, $|G(E/F)| = [E\colon F]$ and by Theorem 5, E/F is Galois. Finally, by part (b), it follows that $G(E/F) = \{1_E, \varphi\} \cong \{(1), (12)\} \cong \mathbb{Z}_2$.

(d) $E = \mathbb{Q}(\sqrt[4]{5}, i)$, $F = \mathbb{Q}(\sqrt[4]{5})$.

Solution. By parts (b) and (c) it follows that $G(E/F) = \{1_E, \tau\}$. Additionally, since the minimal polynomial of i over F is $x^2 + 1$, then $[E\colon F] = 2$. Thus, $|G(E/F)| = [E\colon F]$ and by Theorem 5, E/F is Galois. Finally, $G(E/F) \cong \{(1), (34)\} \cong \mathbb{Z}_2$.

(e) $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, $F = \mathbb{Q}$.

Solution. Since E is the splitting field of the separable polynomial $(x^2-2)(x^2-3)$, then $|G(E/F)| = [E\colon F] = 4$ and so E/F is Galois. Thus, for any $\varphi \in G(E/F)$ we'll have that $\varphi(\sqrt{2}) = \pm\sqrt{2}$ and $\varphi(\sqrt{3}) = \pm\sqrt{3}$. In particular, we can find $\varphi_1, \varphi_2 \in G(E/F)$ such that

$$\begin{aligned} \varphi_1(\sqrt{2}) &= \sqrt{2}, & \varphi_1(\sqrt{3}) &= -\sqrt{3} \\ \varphi_2(\sqrt{2}) &= -\sqrt{2} & \varphi_2(\sqrt{3}) &= \sqrt{3}. \end{aligned}$$

This results in $G(E/F) = \{\varphi_0, \varphi_1, \varphi_2, \varphi_1\varphi_2\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.