MATH 210A

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Assignment: Homework 6

1. Recall (p. 7 and hw 2) that if φ is an action of G on S, $s \in S$, and $t \in G(s)$, then $G_t = gG_sg^{-1}$. Assume that $S = G$, and the action is conjugacy.

   (a) Give an example to show that G_a is not necessarily a normal subgroup of G.

   Solution. Let $G = S_3$. Let $\sigma = (12)$. Then $G_\sigma = \{(1), (12)\}$. In order for G_σ to be normal in G, it must be the case that for all $\gamma \in G_\sigma$, $\tau \circ \gamma \circ \tau^{-1} \in G_\sigma$, for all $\tau \in G$. Thus, let $\gamma = (12)$, then let $\tau = (13)$. It follows that $(13)(12)(13) = (23)$. However, $(23) \notin G_\sigma$. Thus, G_σ is not normal in G.

   (b) Prove that if $a \in Z(G)$, then $G_a \triangleleft G$.

   Proof. Assume that $a \in Z(G)$. Then for all $g \in G$, $ag = ga$. Thus, for all $g \in G$, $gag^{-1} = a$. Now consider the stabilizer of a. We have that $G_a = \{g \in G \mid gag^{-1} = a\}$. However, since $a \in Z(G)$, then $G_a = G$. Since $G \triangleleft G$, then $G_a \triangleleft G$. □

   (c) Give an example to show $G_a \triangleleft G$ does not imply that $a \in Z(G)$.

   Solution. Let $G = S_3$ and let $\sigma = (123)$. Then $G_\sigma = \{(1), (123), (132)\} \neq G$. Thus, $(123) \notin Z(G)$.

2. Assume that p is prime and $o(G) = p^n$. Using the class equation, prove that $Z(G) \neq \{e\}$.

   Proof. Consider the class equation

$$|G| = \big|Z(G)\big| + \sum_{N(s) \neq G} \frac{|G|}{\big|N(s)\big|}.$$

   Since for each $N(s)$, we are stipulating that $N(s) \neq G$ and so $[G\colon N(s)] > 1$. Moreover, since $p \mid |G|$ and $p \mid [G\colon N(s)]$, then it follows that $p \mid \big|Z(G)\big|$. Because $e \in Z(G)$, then $\big|Z(G)\big| \geq 1$. Thus, $Z(G)$ has at least p many elements. Therefore, $\big|Z(G)\big| > 1$. Hence, $Z(G) \neq \{e\}$. □

3. Assume that p is prime, and $o(G) = p^2$. Prove that G has a normal subgroup of order p.

   Proof. By the first Sylow Theorem, G contains a subgroup H of order p. By Lagrange's Theorem, $[G\colon H] = |G|/|H| = p^2/p = p$. Since p is the smallest prime factor of $o(G)$, then by the second result on Pg. 21, we have that $H \triangleleft G$. □

4. Assume that p is prime, and $o(G) = p^2$. Using the class equation, prove that G is Abelian.

Proof. By problem 3 of Homework 2, we proved that $Z(G) \subseteq_g G$. Thus, by Lagrange's Theorem, $\left|Z(G)\right| \mid p^2$. Thus, $\left|Z(G)\right| \in \{1, p, p^2\}$. Since

$$|G| = \left|Z(G)\right| + \sum_{N(s) \neq G} \frac{|G|}{\left|N(s)\right|}$$

and since $N(s) \neq G$, then $[G\colon N(s)] > 1$, for each $s \in G$. Thus, it follows that $p \mid [G\colon N(s)]$, thus p divides the sum. So since p divides $|G|$ and p divides $\sum_{N(s)\neq G} \frac{|G|}{\left|N(s)\right|}$, then $p \mid \left|Z(G)\right|$. Thus, $\left|Z(G)\right| \in \{p, p^2\}$.

Assume $\left|Z(G)\right| = p$. Let $g \in G$ and $a \in Z(G)$, then $gag^{-1} \in Z(G)$ and thus, $Z(G) \triangleleft G$. By Lagrange's Theorem, $[G\colon Z(G)] = p$. Thus, the group $G/Z(G)$ has order p. Now let $a \in G$ such that $a \neq e$. Then consider the subgroup $\langle aZ(G)\rangle$ of $G/Z(G)$. Since $a \neq e$, then $\left|\langle aZ(G)\rangle\right| > 1$. Thus, $\left|\langle aZ(G)\rangle\right|$ divides p by Langrange's Theorem. Thus, $\left|\langle aZ(G)\rangle\right| = p$. Hence, $\left|\langle aZ(G)\rangle\right| = \left|G/Z(G)\right|$ and $\langle aZ(G)\rangle \subseteq G/Z(G)$. Thus, $\langle aZ(G)\rangle = G/Z(G)$. Therefore, $G/Z(G)$ is cyclic.

Now let a denote the generator of $G/Z(G)$. Then for since the set of cosets of $Z(G)$ in G partition G, it follows that for all $g, h \in G$, there exists $n, m \in \mathbb{N}$ such that $g = a^n x$ and $h = a^m y$, for some $x, y \in Z(G)$. Thus,

$$\begin{aligned}
gh &= (a^n x)(a^m y) \\
&= a^n a^m xy \\
&= a^{n+m} xy \\
&= a^{m+n} xy \\
&= a^m a^n xy \\
&= a^m a^n yx \\
&= a^m y a^n x \\
&= (a^m y)(a^n x) \\
&= hg.
\end{aligned}$$

Hence, for all $g, h \in G$, $gh = hg$. Thus, $Z(G) = G$ and $\left|Z(G)\right| = p^2$. Therefore, G is abelian. $\square$

5. Determine (with proof) the structure(s) of a group of order 15.

Proof. We have that $|G| = 15 = 3 \times 5$. Thus, by Sylow III, it follows that $n_3 \equiv 1 (\text{mod } 3)$ and $n_3 \mid 15$. Similarly, $n_5 \equiv 1 (\text{mod } 5)$ and $n_5 \mid 15$. Since 15 has factors $1, 3, 5$, and 15, then amongst these numbers, those which satisfy $n_3 \equiv 1 (\text{mod } 3)$ and $n_3 \mid 15$ is only the number 1. And those which satisfy $n_5 \equiv 1 (\text{mod } 5)$ and $n_5 \mid 15$ is only the number 1. Thus, $P_3 \triangleleft G$ and $P_5 \triangleleft G$. Moreover, since both P_3 and P_5 have orders less than or equal to 5, then by Question 7 on Homework 1, both P_3 and P_5 are Abelian. Thus, $P_3 \cong \mathbb{Z}_3$ and $P_5 \cong \mathbb{Z}_5$. Hence, $G \cong \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_{15}$. Hence, G is cyclic. $\square$

6. Assume that $o(G) = 5^2 7^2$. Determine the possibilities for n_5 and n_7, and determine what can be concluded in each case about the Sylow 5-subgroup(s) and the Sylow 7-subgroup(s), and prove that G is Abelian.

Proof. By Sylow III, we have that $n_5 \equiv 1(\text{mod } 5)$ and $n_5 \mid 875$. Similarly, $n_7 \equiv 1(\text{mod } 7)$ and $n_7 \mid 875$. The number 875 has the following factors: 1, 5, 7, 25, 35, 49, and 875. Thus, $n_5 = 1$ or $n_5 = 49$, and $n_7 = 1$. Thus, $P_7 \lhd G$. If $n_5 = 1$, then $P_5 \lhd G$ and $P_5 P_7 \cong P_5 \times P_7 \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$. Additionally, since the orders of P_7 and P_5 are prime, then they are cyclic. Now let $P_5 = \langle a \rangle$ and let $P_7 = \langle b \rangle$. Note that $\text{Aut}(P_7) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$. Now let $\theta \colon P_5 \to \text{Aut}(P_7)$ be a homomorphsim defined by $\theta(g) = \varphi_k$. If $g = e$, then $\theta(e) = \varphi_1$. If $g = a$, then $o(\theta(a)) \mid 5$. Thus, $o(\theta(b)) = 1$ or $o(\theta(b)) = 5$. Hence, $\varphi_{k^5} = \varphi_1$. However, only φ_1 satisfies this condition. Note that since $P_7 \lhd G$, then $aba^{-1} \in \langle b \rangle$. Thus, φ_1 gives us $aba^{-1} = b$. And so $ab = ba$ and G is therefore Abelian. Additionally, $P_5 \cong \mathbb{Z}_5$ and $P_7 \cong \mathbb{Z}_7$. $\square$

7. Assume that p is prime, $p \neq 2$. Determine, with explanation, all groups of order $2p$.

Proof. By Sylow III, we have that $n_2 \equiv 1(\text{mod } 2)$ and $n_2 \mid 2p$, and $n_p \equiv 1(\text{mod } p)$ and $n_p \mid 2p$. Thus, $n_2 = 1$ or $n_2 = p$. Similarly, $n_p = 1$. Thus, $P_p \lhd G$. Since both P_2 and P_p are of prime order, then they are cyclic. Let $a, b \in G$ with $o(a) = 2$ and $o(b) = p$. Let $P_2 = \langle a \rangle$ and $P_p = \langle b \rangle$. It follows that since $P_2 \cap P_p = \{e\}$, then $G = P_2 P_p$. We have that $\text{Aut}(P_p) = \{\varphi_k \colon 1 \leq k \leq p-1\}$, where $\varphi_k(x) = x^k$. Assume that $\theta \colon P_2 \to \text{Aut}(P_p)$ is a homomorphism, where $\theta(g) = \varphi_k$.

Each φ_k corresponds to $aba^{-1} = b^k$ since $P_p = \langle b \rangle \lhd G$, and thus $aba^{-1} \in \langle b \rangle$. Note that $o(\theta(a)) \mid 2$. Thus, $o(\theta(a)) = 1$ or $o(\theta(a)) = 2$. If $o(\theta(a)) = 1$, then $\varphi_k = \varphi_1$ and thus $aba^{-1} = b$. Thus, $ab = ba$ and hence $G \cong \mathbb{Z}_{2p}$ In the case that $o(\theta(a)) = 2$, then we need $(\varphi_k)^2 = \varphi_1$, where φ_1 is the identity in $\text{Aut}(P_p)$. Thus, we need $\varphi_{k^2} = \varphi_1$. Hence, for each $1 \leq k \leq p-1$, we are looking for those k in which $k^2 \equiv 1(\text{mod } p)$. By the results on Pg. 23, there are $(2, p-1)$ solutions or no solutions. Since $2 \mid p-1$, there are 2 solutions. We can see that $k = \pm 1$. Thus, there are two groups of order $2p$, namely $\mathbb{Z}_{pq}$ and a nonabelian group of order pq whose structure is defined by $a^2 = b^p = e$ and $ab = b^{-1}a$. The latter group defined the structure of D_p. $\square$