MATH 210A

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Instructor: Dr. Shannon **Assignment:** Homework 3

1.

(a) Assume that $N \triangleleft G$, and define $\varphi\colon G \times N \to N$ by $\varphi((g,n)) = g \star n \star g^{-1}$. Prove that φ is an action of G on N.

Proof. We want to show that for all $h, g \in G$ and all $n \in N$ that

$$\varphi(h \star g, n) = \varphi(h, \varphi(g,n)) \quad \text{and} \quad \varphi(e,n) = n.$$

Let $h, g \in G$ and let $n \in N$. Then

$$\begin{aligned}
\varphi(h \star g, n) &= (h \star g) \star n \star (h \star g)^{-1} \\
&= (h \star g) \star n \star (g^{-1} \star h^{-1}) \\
&= h \star (g \star n \star g^{-1}) \star h^{-1} \\
&= \varphi(h, g \star n \star g^{-1}) \\
&= \varphi(h, \varphi(g,n)).
\end{aligned}$$

Note that the fourth equality holds since N is normal which means that for all $g \in G$ and $n \in N$, $g \star n \star g^{-1} \in N$ and so $(h, g \star n \star g^{-1}) \in G \times N$. We have satisfied the first of the two equalities. Now consider

$$\varphi(e,n) = e \star n \star e^{-1} = e \star n \star e = n.$$

Therefore, φ is an action of G on N. $\square$

(b) Assume that $H \subseteq_g G$. Prove that $H \triangleleft G$ iff H is a union of conjugacy classes.

Proof. Assume $H \triangleleft G$, and let

$$A = \bigcup_{h \in H} C(h),$$

where $C(h)$ denote the conjugacy class of $h \in H$. We want to show that $H \triangleleft G$ implies $H = A$. Let $m \in H$. Then for some $g \in G$, $m \in gHg^{-1}$ by Exercise 4, part (a). Thus, for some $h \in H$, $m = g \star h \star g^{-1}$. Thus, $m \in C(h)$. Since $C(h) \subseteq A$, then $m \in A$. Thus, $H \subseteq A$. Now let $a \in A$. Then for some $h \in H$, $a \in C(h)$. Thus, for some $g \in G$, $a = g \star h \star g^{-1}$. However, by Exercise 4, part (a), $g \star h \star g^{-1} \in H$ for all $g \in G$. Thus, $a \in H$. Hence, $A \subseteq H$. Therefore, $H = A$.

Assume H is equal to a union of conjugacy classes. We want to show that for all $g \in G$, $gHg^{-1} = H$. Let $g \in G$ and let $m \in gHg^{-1}$. Then for some $h \in H$, $m = g \star h \star g^{-1}$. Thus, $m \in C(h) \subseteq H$ and hence, $m \in H$. Thus, $gHg^{-1} \subseteq H$. Now let $m \in H$. Then, for some $h \in H$, $m \in C(h)$. Thus, $m = g \star h \star g^{-1}$, for some $g \in G$. Thus, $m \in gHg^{-1}$. Thus, $H \subseteq gHg^{-1}$. Therefore, for all $g \in G$, $gHg^{-1} = H$. Thus, $H \triangleleft G$. $\square$

2. Assume that H is a subgroup of G. Define $N(H) = \{g \in G \colon gHg^{-1} = H\}$.

   a) Prove that $N(H)$ is a subgroup of G.

      Proof. To prove that $N(H) \subseteq_g G$, we will first show that $N(H) \neq \varnothing$. Since $eHe^{-1} = H$, then it follows that $e \in N(H)$ and thus $N(H) \neq \varnothing$. Next, we must show that for all $h, g \in G$, if $h, g \in N(H)$, then $h \star g^{-1} \in N(H)$. Let $h, g \in G$ and assume $h, g \in N(H)$. Then we have that $hHh^{-1} = H$ and $gHg^{-1} = H$. We want to show that $h \star g^{-1} \in N(H)$. Thus, we want to show that

      $$(h \star g^{-1})H(h \star g^{-1})^{-1} = h \star (g^{-1}Hg) \star h^{-1} = H.$$

      Note that since $gHg^{-1} = H$ and, by Homework 2, $gHg^{-1} = g^{-1}Hg$ (this holds since $g^{-1} \star h \star g = g \star h' \star g^{-1}$, provided $h = g \star (g \star h' \star g^{-1}) \star g^{-1} \in H$). Thus, with $g^{-1}Hg = H$, then

      $$h \star (g^{-1}Hg) \star h^{-1} = hHh^{-1} = H.$$

      Thus, $h \star g^{-1} \in N(H)$. Therefore, $N(H)$ is a subgroup of G. □

   b) Prove that $H \lhd N(H)$.

      Proof. To prove that $H \lhd N(H)$, we will show that for all $g \in N(H)$, $gHg^{-1} = H$. Let $g \in N(H)$. Then $gHg^{-1} = H$. Therefore, $H \lhd N(H)$. □

   c) Prove that if M is a subgroup of G, and $H \lhd M$, then $M \subseteq N(H)$.

      Proof. Assume $M \subseteq_g G$ and $H \lhd M$. Let $m \in M$, then since $H \lhd M$ it follows that $mHm^{-1} = H$. Since $M \subseteq G$, then we have that $m \in G$ and $mHm^{-1} = H$. Thus, $m \in N(H)$. Therefore, $M \subseteq N(H)$. □

3. Prove, by induction, that for $n \geq 2$, $(a_1a_2 \ldots a_n)^{-1} = (a_na_{n-1} \ldots a_2a_1)$.

   Proof. Let $P(n) := (a_1a_2 \ldots a_n)^{-1} = (a_na_{n-1} \ldots a_1)$. Then we want to show that for all $n \in \mathbb{N}$, $P(n)$ holds.

   <u>Base case:</u> Let $n = 2$. Then we can write the 2-cycle as (a_1a_2). Since this is a permutation, then we can also write it as a bijective map $\sigma \colon \{a_1, a_2\} \to \{a_1, a_2\}$, where $\sigma(a_1) = a_2$ and $\sigma(a_2) = a_1$. Since for each 2-cycle permutation, it is its own inverse, it follows that $\sigma(\sigma(a_1)) = \sigma(a_2) = a_1$ and $\sigma(\sigma(a_2)) = \sigma(a_1) = a_2$. Thus, denoting this in cycle notation, we have that $(a_1a_2) = (a_1a_2)^{-1} = (a_2a_1)$. Thus, $P(2)$ holds.

   <u>Inductive step:</u> Assume that for some $k \in \mathbb{N}$, where $k > 2$, $P(k)$ holds. Then

   $$(a_1a_2 \ldots a_k)^{-1} = (a_ka_{k-1} \ldots a_1). \tag{1}$$

   By the result on pg. 9, any permutation can be written as the product of transpositions. Thus,

   $$\begin{aligned}(a_1a_2 \ldots a_{k+1})^{-1} &= \big((a_1a_{k+1}) \cdots (a_1a_2)\big)^{-1} \\ &= (a_1a_2)^{-1} \cdots (a_1a_{k+1})^{-1} \\ &= (a_2a_1) \cdots (a_{k+1}a_1) \\ &= (a_{k+1}a_k \ldots a_1).\end{aligned}$$

Thus, $P(k+1)$ holds. Therefore, for all $n \in \mathbb{N}$, $P(n)$ holds. $\square$

5.

a) Let D be a subset of S_n consisting of all of the odd permutations in S_n. Define $\theta\colon A_n \to D$ by $\theta(\sigma) = \sigma(12)$ (that is $\sigma \circ (12)$). Prove that θ is 1-1 and onto D.

Proof. Assume that for some $\sigma_1, \sigma_2 \in A_n$, that $\theta(\sigma_1) = \theta(\sigma_2)$. Then it follows that $\sigma_1(12) = \sigma_2(12)$. Composing both sides with (12), we get $\sigma_1 = \sigma_2$. Thus, θ is 1-1.

Let $\delta \in D$. Then if we take $\sigma \in A_n$ such that $\sigma = \delta(12)$, we get that $\theta(\delta(12)) = \delta$. Therefore, θ is onto. $\square$

b) Prove that $A_n \subseteq_g S_n$, and that $A_n \lhd S_n$.

Proof. Since the identity $(1) = (a_1a_2)(a_1a_2)$, for any $a_1, a_2 \in \{1, \ldots, n\}$ then it follows that (1) is an even permutation and thus $(1) \in A_n$. Thus, $A_n \neq \varnothing$. By definition, A_n contains all even permutations on n letters, and thus $A_n \subseteq S_n$. Now we wish to show that for all $\sigma \circ \gamma^{-1} \in A_n$ whenever $\sigma, \gamma \in A_n$. Let $\sigma, \gamma \in A_n$. Then both σ and γ can be written as the product of an even number of transpositions. Thus, we may write

$$\sigma = (a_1a_{2m+1}) \cdots (a_1a_2) \quad \text{and} \quad \gamma = (b_1b_{2k+1}) \cdots (b_1b_2),$$

for $a_1, \ldots, a_{2m+1}, b_1, \ldots b_{2k+1} \in \{1, \ldots n\}$. So we have that σ is the product of $2m$ transpositions and is the product of $2k$ transpositions. By the result proven in Exercise 3, we have that

$$\gamma^{-1} = (b_2b_1) \cdots (b_{2k+1}b_1).$$

Note that the number of transpositions remains unchanged. Thus,

$$\sigma \circ \gamma^{-1} = (a_1a_{2m+1}) \cdots (a_1a_2)(b_2b_1) \cdots (b_{2k+1}b_1)$$

is the product of $2m + 2k = 2(m+k)$ transpositions. Thus, $\sigma \circ \gamma^{-1} \in A_n$. Therefore, $A_n \subseteq_g S_n$.

To prove that $A_n \lhd S_n$, we will show that $[S_n\colon A_n] = 2$ and appeal to Exercise 5 on Homework 2. By part (a) of Exercise 5, we showed that $|A_n| = |D|$, where D is the set of all odd permutations. Since S_n consists of all odd and even permutations, we have that $A_n \cap D = \varnothing$ and $A_n \cup D = S_n$. Thus, $\{A_n, D\}$ is a partition on S_n. Now let $\sigma \in S_n/A_n$, then either $\sigma \in A_n$ or $\sigma \in S_n - A_n = D$. Thus, $S_n/A_n = \{A_n, D\}$ which implies that $[S_n\colon A_n] = 2$. Thus, $A_n \lhd S_n$. $\square$

6. In S_5, find $\Big|c\big((123)(45)\big)\Big|$, and find $N\big((123)(45)\big)$.

Solution. On pg. 7 we proved that

$$\big|c(s)\big| = \frac{|G|}{\big|N(s)\big|},$$

where G was a group and $s \in S$, for a set S. Thus, replacing these terms with the terms in our question, we get

$$\Big|c\big((123)(45)\big)\Big| = \frac{|S_5|}{\Big|N\big((123)(45)\big)\Big|}.$$

We know that $|S_5| = 5! = 120$, so finding either $\Big|c\big((123)(45)\big)\Big|$ or $\Big|N\big((123)(45)\big)\Big|$ will give us the other.

Recall that

$$N\big((123)(45)\big) = \{\sigma \in S_5 \colon \sigma \circ (123)(45) = (123)(45) \circ \sigma\}.$$

Thus, we are looking for a σ such that

$$\sigma \circ (123)(45) \circ \sigma^{-1} = (123)(45),$$

of which there are 6 since $o\big((123)(45)\big) = 6$. Thus,

$$\Big|c\big((123)(45)\big)\Big| = 120/6 = 20 \quad \text{and} \quad \Big|N\big((123)(45)\big)\Big| = 6.$$

4. Assume H and K are subgroups of $(G, \star)$, and that $N \triangleleft G$.

   a) Prove that HK is a subgroup of G iff $HK = KH$.

   Proof. Assume that $HK \subseteq_g G$. Let $x \in HK$. Then since HK is a subgroup, $x^{-1} \in HK$. Thus, there exists $h \in H$ and $k \in K$ such that $x^{-1} = h \star k$. Thus, $x = k^{-1} \star h^{-1}$, which is an element of KH. Thus, $x \in KH$. Hence, $HK \subseteq KH$. Let $x \in KH$. Then by the same reasoning as before, $x^{-1} \in KH$. Thus, there exists $k \in K$ and $h \in H$ such that $x^{-1} = k \star h$. Thus, $x = h^{-1} \star k^{-1}$, which is an element of HK. Thus, $x \in HK$. Hence, $KH \subseteq HK$. Therefore, $HK = KH$.

   Assume that $HK = KH$. Let $x, y \in HK$. Then there exists $g, h \in H$ and $j, k \in K$ such that $x = g \star j$ and $y = h \star k$. Thus,

   $$x \star y^{-1} = (g \star j) \star (k^{-1} \star h^{-1}) = g \star (j \star k^{-1} \star h^{-1}).$$

   Since $j \star k^{-1} \star h^{-1} \in KH$ and $KH = HK$, then there exists some $h' \in H$ and $k' \in K$ such that $j \star k^{-1} \star h^{-1} \star = h' \star k'$. Thus,

   $$x \star y^{-1} = g \star h' \star k'.$$

   We see that $g \star h' \in H$ and $k' \in K$. Thus, $g \star h' \star k' \in HK$. Thus, $x \star y^{-1} \in HK$. Therefore, $HK \subseteq_g G$. □

   b) Prove that NH is a subgroup of G.

   Proof. Let $x \in NH$. Then there exists $n \in N$ and $h \in H$ such that $x = n \star h$. Note that $n \star h \in Nh$, and since N is normal, then $Nh = hN$. Thus, $n \star h \in hN$ which implies that there exists some $n' \in N$ such that $n \star h = h \star n'$, and this is an element of HN. Thus, $x \in HN$ and hence $NH \subseteq HN$. By the same argument, we can show $HN \subseteq NH$. Thus, $NH = HN$, and by (a), NH is therefore a subgroup of G. $\square$