MATH 230A

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Instructor: Dr. Domokos **Assignment:** Homework 1

1.1 Let $A, B, C \subseteq X$. Prove the following:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof .

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow (x \in A) \vee (x \in B \cap C) \\ &\Leftrightarrow (x \in A) \vee (x \in B \wedge x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \cap (x \in A \vee x \in C) \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

$\square$

(c) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

Proof .

$$\begin{aligned} x \in A \backslash (B \cup C) &\Leftrightarrow (x \notin A) \wedge (x \in B \cup C) \\ &\Leftrightarrow (x \in A) \wedge (x \notin B \wedge x \notin C) \\ &\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\Leftrightarrow (x \in A \backslash B) \wedge (x \in A \backslash C) \\ &\Leftrightarrow x \in (A \backslash B) \cap (A \backslash C). \end{aligned}$$

$\square$

(e) $(A \cup B)^c = A^c \cap B^c$.

Proof .

$$\begin{aligned} x \in (A \cup B)^c &\Leftrightarrow x \notin A \cup B \\ &\Leftrightarrow (x \notin A) \wedge (x \notin B) \\ &\Leftrightarrow (x \in A^c) \wedge (x \in B^c) \\ &\Leftrightarrow x \in A^c \cap B^c. \end{aligned}$$

$\square$

1.2 Prove that

(a)

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right]=(0,1]$$

Proof. Let x be an element of the LHS. Then by Archimedes Principle, there exists $n\in\mathbb{N}$ such that $1/n<x$. Thus, $x\in[1/n,1]\subseteq(0,1]$. Hence,

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right]\subseteq(0,1].$$

Now assume $x\in(0,1]$. Then $x>0$ and so by Archimedes Principle, there exists $n\in\mathbb{N}$ such that $1/n<x$ and thus

$$x\in\left[\frac{1}{n},1\right]\subseteq\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right]\leftrightarrow(0,1]\subseteq\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right].$$

Therefore,

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n},1\right]=(0,1].$$

□

(b)

$$\bigcap_{n=1}^{\infty}\left(0,\frac{n+1}{n}\right)=(0,1]$$

.

Proof. Note that for all n, $1\leq(n+1)/n$ and so we immediately get that

$$(0,1]\subseteq\bigcap_{n=1}^{\infty}\left(0,\frac{n+1}{n}\right).$$

Thus, if x is an element of the intersection, then $x\in(0,1]$. Hence,

$$\bigcap_{n=1}^{\infty}\left(0,\frac{n+1}{n}\right)\subseteq(0,1].$$

Therefore,

$$\bigcap_{n=1}^{\infty}\left(0,\frac{n+1}{n}\right)=(0,1].$$

□

1.3 Prove that $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume, for contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then there exists $a, b \in \mathbb{Z}$ such that $b \neq 0$, $\gcd(a, b) = 1$ and $\sqrt{2} = a/b$. Taking the square of both sides we obtain

$$2 = \frac{a^2}{b^2} \to 2b^2 = a^2. \tag{1}$$

This means that a^2 is an even number. Note that if a is odd, i.e., $a = 2k + 1$ for some $m \in \mathbb{Z}$, then $a^2 = 4k^2 + 4k + 1$ which is an odd number. Hence, if a^2 is even, then a is even. Thus we can write $a = 2m$ for some $m \in \mathbb{Z}$. Substituting into (1), we get

$$2b^2 = (2m)^2 = 4m^2 \to b^2 = 2m^2.$$

Hence, b is an even number. Thus, $2 \mid a$ and $2 \mid b$ which implies $2 \mid \gcd(a, b)$. Thus, $\gcd(a, b) \neq 1$ which is a contradiction. Therefore $\sqrt{2} \notin \mathbb{Q}$. □

1.4 Let $f \colon X \to Y$ be a function. Show that f is one-to-one if and only if for every $y \in Y$, the pre-image $f^{-1}(\{y\})$ contains at most one element.

Proof. ? □

1.5 Let $f \colon X \to Y$ be a function. Show that f is onto if and only if for every $y \in Y$, the pre-image $f^{-1}(\{y\}) \neq \varnothing$.

Proof. Assume f is onto. Then by Definition 1.5, for every $y \in Y$ there exists $x \in X$ such that $f(x) = y$. Hence, if $y \in Y$, then $f^{-1}(\{y\}) = \{x \in X \mid f(x) \in \{y\}\}$ which is necessarily nonempty by the previous line. Now assume that for every $y \in Y$, $f^{-1}(\{y\}) \neq \varnothing$. Let $y \in Y$ and let $x \in f^{-1}(\{y\})$, then $f(x) = y$. Thus for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Therefore, f is onto. □

1.6 Let $f \colon X \to Y$ be a function. Show that f is one-to-one if and only if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for every pair of sets $A_1, A_2 \subseteq X$.

Proof. Assume that f is one-to-one. Let $A_1, A_2 \subseteq X$. By Proposition 1.4.(b), we know $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Let $y \in f(A_1) \cap f(A_2)$. Then $y \in f(A_1) \wedge y \in f(A_2)$. This implies that there is some $x_1 \in A_1$ and $x_2 \in A_2$ such that $f(x_1) = y = f(x_2)$. Since f is one-to-one, then $x_1 = x_2$. Hence, $x_1 \in A_1 \wedge x \in A_2$. Thus $x_1 \in A_1 \cap A_2$. Thus, $f(x_1) = y \in f(A_1 \cap A_2)$. Therefore, $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

Assume that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all $A_1, A_2 \subseteq X$. Let $x_1, x_2 \in X$ and assume $y \in f(\{x_1\}) \cap f(\{x_2\})$. Then $f(x_1) = y = f(x_2)$. Additionally, we have that $y \in f(\{x_1\} \cap \{x_2\})$. Thus either $\{x_1\} \cap \{x_2\} = \varnothing$ or $x_1 = x_2$. The latter cannot be true since this would imply $f(\varnothing) = y$. Hence, $x_1 = x_2$ and therefore f is one-to-one. □

1.7 Let $f\colon X \to Y$ be a function. Show that f is one-to-one if and only if $f^{-1}(f(A)) = A$ for every set, $A \subseteq X$.

Proof. Assume that f is one-to-one and that $A \subseteq X$. If $a \in A$, then $f(a) \in f(A)$. The latter is true iff $a \in f^{-1}(f(A))$. Hence, $A \subseteq f^{-1}(f(A))$. Now let $x \in f^{-1}(f(A))$. This implies that $f(x) \in f(A)$. Thus there exists $a \in A$ such that $f(x) = f(a)$ and since f is one-to-one, then $x = a$. Thus $x \in A$ and so $f^{-1}(f(A)) \subseteq A$. Therefore $f^{-1}(f(A)) = A$.

Assume $f^{-1}(f(A)) = A$ for all $A \subseteq X$. Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. By our assumption we have that $f^{-1}(f(\{x_1\})) = \{x_1\}$ and $f^{-1}(f(\{x_2\})) = \{x_2\}$. Since $f(x_1) = f(x_2)$, it follows that $f(x_1), f(x_2) \in f(\{x_1\})$. Similarly, $f(x_1), f(x_2) \in f(\{x_2\})$. If there were some $y \neq f(x_1) = f(x_2)$ such that $y \in f(\{x_1\})$, then $f(x_1) = y \neq f(x_1)$ which would contradict that f is well-defined. Hence $f(\{x_1\}) = f(x_1)$ and $f(\{x_2\}) = f(x_2)$. Thus

$$f^{-1}(f(\{x_1\})) = f^{-1}(f(x_1)) = \{x_1\} = \{x_2\} = f^{-1}(f(x_2)) = f^{-1}(f(\{x_2\}))$$

Which implies that $x_1 = x_2$. Therefore f is one-to-one. □

1.8 Let $f\colon X \to Y$ be a function. Show that f is onto if and only if $f(f^{-1}(B)) = B$ for all $B \subseteq Y$.

Proof. Assume f is onto. Let $y \in B$. Then since f is onto, $f(f^{-1}(y)) = y$ and $f^{-1}(y) \in f^{-1}(B)$. Thus, $y = f(f^{-1}(y)) \in f(f^{-1}(B))$. Hence $B \subseteq f(f^{-1}(B))$. Now let $y \in f(f^{-1}(B))$. Then $f^{-1}(y) \in f^{-1}(B)$. Thus there is some $y' \in B$ such that $f^{-1}(y) = f^{-1}(y')$. Since f is onto then $f(f^{-1}(y)) = y = y' = f(f^{-1}(y'))$. Hence $y = y' \in B$ and so $f(f^{-1}(B)) \subseteq B$. Therefore $f(f^{-1}(B)) = B$.

Assume $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ and let $y \in Y$. Then $\{y\} \subseteq Y$ and $f(f^{-1}(\{y\})) = \{y\}$. Thus either there exists $x \in f^{-1}(\{y\})$ such that $f(x) \in \{y\}$, or $f^{-1}(\{y\}) = \varnothing$. If the latter were true, then for all $x \in f^{-1}(\{y\})$, $f(x) \in \{y\}$ is true vacuously. Therefore, there exists $x \in f^{-1}(\{y\}) \subseteq X$ such that $f(x) = y$. Thus f is onto. □

1.9 Let $f\colon X \to Y$ be a function. Show that f is one-to-one and onto if and only if $f(A^c) = (f(A))^c$ for all $A \subseteq X$.

Proof. Assume f is a bijection. Let $y \in (f(A))^c$. Then $y \notin f(A)$ which implies that for all x, if $f(x) = y$, then $x \in A^c$ which implies $f(x) \in f(A^c)$. Since y was arbitrary, then the previous statement holds for all $y \in (f(A))^c$. Moreover, since f is bijective, then for all $y \in (f(A))^c$, there exists $x \in X$ such that $f(x) = y$. Hence, if $y \in (f(A))^c$, then $y = f(x) \in f(A^c)$. Thus, $(f(A))^c \subseteq f(A^c)$. Now we argue the contrapositive. Assume $y \notin (f(A))^c$. Then $y \in f(A)$ which implies there exists $x \in A$ such that $f(x) = y$. This implies that there exists $x \notin A^c$ (which is unique since f is bijective) such that $f(x) = y$ and $y \notin f(A^c)$. Hence, $f(A^c) \subseteq (f(A))^c$. Therefore $f(A^c) = (f(A))^c$.

Assume $f(A^c) = (f(A))^c$ for all $A \subseteq X$. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Then $x_1 \in X \backslash \{x_2\}$ and $x_2 \in X \backslash \{x_1\}$. Thus $f(\{x_1\}) \subseteq f(X \backslash \{x_2\}) \subseteq Y \backslash \{f(x_2)\}$. Hence $f(x_1) \neq f(x_2)$ and so f is one-to-one. Finally since $f(X \backslash \varnothing) = f(X) = Y \backslash f(\varnothing) = Y \backslash \varnothing = Y$, then the codomain of f is all of Y. Hence f is onto. □

1.10 Let $f \colon X \to Y$ and $g \colon Y \to Z$ be one-to-one and onto functions. Show that $g \circ f \colon X \to Z$ is invertible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. Let $x_1, x_2 \in X$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. Then if follows that $g(f(x_1)) = g(f(x_2))$ and since g is one-to-one, this implies that $f(x_1) = f(x_2)$. Similarly, since f is one-to-one, then this implies $x_1 = x_2$. Hence, $g \circ f$ is one-to-one. Now let $z \in Z$. Then since g is onto, there exists $y \in Y$ such that $g(y) = z$. Similarly, since $y \in Y$ and f is onto, then there exists some $x \in X$ such that $f(x) = y$. Thus, $g(f(x)) = z$ and so $g \circ f$ is onto. Therefore $g \circ f$ is invertible. Finally, let $z \in Z$. Then $(g \circ f)^{-1}(z) = x$ for some $x \in X$. This means that $f(x) = y$ for $y \in Y$ and $g(y) = z$. Hence $f^{-1}(y) = x$ and $g^{-1}(z) = y$. Thus

$$(f^{-1} \circ g^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x.$$

Therefore $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ for all $z \in Z$ which implies the left and right hand side are equal. □

1.11 Prove Proposition 1.14.

(a) ***Proof.*** Let $f \colon X \to X$ be defined by $f(x) = x$. Then f is a bijection and therefore $X \sim X$. □

(b) ***Proof.*** Assume that $X \sim Y$. Then there exists a bijection $f \colon X \to Y$ and since f is a bijection, then $f^{-1} \colon X \to Y$ exists. Furthermore, we have that $f^{-1}(x) = f^{-1}(y)$ gives $f(f^{-1}(x)) = f(f^{-1}(y))$ and so $x = y$. Thus f^{-1} is one-to-one. Lastly, if $x \in X$, then $f^{-1}(f(x)) = x$. Hence, f^{-1} is onto. Therefore f^{-1} is one-to-one and onto and thus $Y \sim X$. □

(c) ***Proof.*** Assume $X \sim Y$ and $Y \sim Z$. Then $f \colon X \to Y$ and $g \colon Y \to Z$ are each bijections and by problem 1.10, $g \circ f \colon X \to Z$ is a bijection. Therefore $X \sim Z$. □

1.12 Prove that the union of two countable sets is countable.

Proof. Let A and B be countable sets and let $S = A \cup B$. Since A and B are both countable, we can denote each as $\{a_k\}$ and $\{b_k\}$, respectively, for $k = 1, 2, \cdots$. Define f by

$$f(n) = \begin{cases} a_n & \text{if} \quad 2 \mid n \\ b_n & \text{if} \quad 2 \nmid n. \end{cases}$$

Now assume $2 \mid n$ and that $n = m$. Then $f(n) = a_n$ and $f(m) = a_m$ from which it follows $a_n = a_m$. Similarly, if $2 \nmid n$, then $b_n = b_m$. Hence, $f(n) = f(m)$ for all $n, m \in \mathbb{N}$

and so $f\colon \mathbb{N} \to S$ is well-defined. Now take some $x_i \in S$. Then if $x_i \in A$, it follows that $2 \mid i$ and $f(i) = x_i$. Similarly, if $x_i \in B$, then $2 \nmid i$ and $f(i) = x_i$. Hence, f is onto.

Finally, define $g\colon S \to \mathbb{N}$ by $g(x) = n$ such that n is the smallest natural number such that $f(n) = x$. This n exists by the well-ordering principle. Assume $g(x_i) = g(x_j)$. Then $f(g(x_i)) = x_i = x_j = f(g(x_j))$. Therefore g is one-to-one. Moreover since $g(S) \subseteq \mathbb{N}$ and $S \sim g(S)$ then by Lemma 1.15, $S \sim g(S) \sim \mathbb{N}$. □

1.13 Prove that the union of countably many countable sets is countable.

Proof. Let $A_1, A_2, \ldots$ be a countable collection of countable sets. Then we want to show that $\bigcup_{n=1}^{\infty} A_n \sim \mathbb{N}$. Define $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_1 \cup A_2)$, $\ldots$. Now note that for any $i, j \in \mathbb{N}$ such that $i < j$, if $a \in B_i$, then

$$x \in A_i \backslash (A_1 \cup \cdots \cup A_{i-1})$$

which implies that $x \in A_i$. However, since

$$B_j = A_j \backslash (A_1 \cup \cdots \cup A_i \cup \cdots \cup A_{j-1})$$

then it follows that $x \notin B_j$. Hence $B_i \cap B_j = \varnothing$.

Now we note that if $b \in \bigcup_{n=1}^{\infty} B_n$, then there exists some $i \in \mathbb{N}$ such that $b \in B_i$ and by definition, this implies that $b \in A_i \subseteq \bigcup_{n=1}^{\infty} A_n$. Hence,

$$\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n.$$

Next, we let $a \in \bigcup_{n=1}^{\infty} A_n$ and define $S = \{n \in \mathbb{N} \mid a \in A_n\}$. Clearly, $S \subseteq \mathbb{N}$ and so by the Well-Ordering Principle, S has a smallest element, call it $n_0 = \min(S)$. It follows that $a \in A_{n_0}$ and $a \notin (A_1 \cup \cdots \cup A_{n_0 - 1})$. Therefore

$$a \in A_{n_0} \backslash (A_1 \cup \cdots \cup A_{n_0 - 1}) = B_{n_0} \subseteq \bigcup_{n=1}^{\infty} B_n.$$

Hence, $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$ and therefore

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Now we will let denote the prime numbers as $p_1, p_2, \ldots$ and define the following function:

$$f\colon \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$$

where, since each B_n is countable then there is a bijection $g_n\colon B_n \to \mathbb{N}$, and so for all $b \in \bigcup_{n=1}^{\infty} B_n$, $f(b) = p_n^{g_n(b)}$. Note that $g_n(b) \in \mathbb{N}$ which means $f(b) \in \mathbb{N}$. Now let $b, b' \in \bigcup_{n=1}^{\infty} B_n$ such that $f(b) = f(b')$. Since b and b' are elements of the union, then there exists $i, j \in \mathbb{N}$ such that $b \in B_i$ and $b' \in B_j$. Since we showed that $B_i \cap B_j = \varnothing$ if $i \neq j$, then either $i \neq j$ or $B_i = B_j$. If $i \neq j$, then $b \neq b'$. However, by assumption

$$f(b) = p_i^{g_i(b)} = p_j^{g_j(b')} = f(b')$$

which implies $p_i \mid p_j^{g_j(b')}$ which further implies that for some $k \leq g_j(b')$, $p_i = p_j^m$ which is a contradiction since $p_i \neq p_j$. Hence, $B_i = B_j$. Thus $f(b) = p_i^{g_i(b)} = p_i^{g_i(b')} = f(b')$. This equality only holds if $g_i(b) = g_i(b')$ and since g_i is one-to-one, then it follows that $b = b'$. Therefore, f is one-to-one.

Moreover, we have that

$$\bigcup_{n=1}^{\infty} B_n \sim f(\bigcup_{n=1}^{\infty} B_n) \subseteq \mathbb{N}.$$

Thus if $f(\bigcup_{n=1}^{\infty} B_n)$ is finite, then $\bigcup_{n=1}^{\infty} B_n$ is finite and hence countable. Otherwise, if $f(\bigcup_{n=1}^{\infty} B_n)$ is infinite, then $f(\bigcup_{n=1}^{\infty} B_n) \sim \mathbb{N}$ and by Proposition 1.14, $\bigcup_{n=1}^{\infty} B_n \sim \mathbb{N}$ and therefore countable. Finally, since $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, then the latter is countable. $\square$

1.14 Let A be the collection of all sequences of the digits 0 and 1, for which the number of digits 1 is finite. Show that A is countable.

Proof. First we need to show that $\mathbb{N}^n$ is countable for all $n \in \mathbb{N}$. Define $f_n \colon \mathbb{N}^n \to \mathbb{N}$ by $(a_1, \ldots, a_n) \mapsto 2^{a_1} 3^{a_2} \cdots p_n^{a_n}$, where p_n is the nth prime number. Now assume that $f_n((a_1, \cdots, a_n)) = f_n((b_1, \cdots, b_n))$. Then $2^{a_1} \cdots p_n^{a_n} = 2^{b_1} \cdots p_n^{b_n}$. Rewrite the left and right hand sides by only including powers greater than or equal to 1. From this we get

$$p_i^{a_i} \cdots p_j^{a_j} = p_r^{b_r} \cdots p_s^{b_s}.$$

Wlog, assume that $j - i \leq s - r$. Clearly, $p_i^{a_i}$ divides the left and right hand sides and so either $p_i^{a_i} \mid p_r^{b_r}$ or $p_i^{a_i} \nmid p_r^{b_r}$. If the former is true, then $p_i = p_r$ and $a_i \leq b_r$. Thus dividing both sides by $p_i^{a_i}$ we obtain

$$1 \cdots p_j^{p_j} = p_r^{b_r - a_i} \cdots p_s^{b_s}.$$

However, if $a_i < b_r$ then the right hand side and left hand side are both multiples of p_r, but this is a contradiction since the left hand side no longer contains any power of p_r. Hence, if $p_i^{a_i} \mid p_r^{b_r}$, then $a_i = b_r$. Additionally, if $p_i^{a_i} \nmid p_r^{b_r}$, then either $p_i < p_r$ or $p_i > p_r$. If $p_i < p_r$, then the left hand side is a multiple of p_i whereas the right is not which is a contradiction. Similarly, the same argument holds if $p_i > p_r$. Thus, $p_i = p_r$ and $a_i = b_r$. Finally, if $j - i < s - r$, then dividing the left and right hand side by the left hand side we obtain

$$1 = p_t^{b_t} \cdots p_v^{b_v}$$

which is a which contradicts our assumption that all exponents were greater than or equal to 1. Hence, $j - i = s - r$ and thus $a_i = b_i$ for all $1 \leq i \leq n$. Therefore, f_n is one-to-one. Finally, letting $x \in \mathbb{N}$, by the fundamental theorem of arithmetic, we can uniquely express x as $2^{a_1}\cdots p_n^{a_n}$ and hence $f_n((a_1,\dots,a_n))=x$. Thus f_n is onto. Thus $\mathbb{N}^n \sim \mathbb{N}$ and so $\mathbb{N}^n$ is countable.

Let $A_n \subseteq A$ be the set of all sequences in A which contain n many 1's. Then define the function $g_n\colon A_n \to \mathbb{N}^n$ in the following way: If $a \in A$ and $a = a_1, a_2, \dots,$, then

$$a_1, a_2, \dots \mapsto (\alpha_1, \alpha_2, \dots, \alpha_n)$$

if and only if $a_{\alpha_i}=1$ for all $i=1,2,\dots,n$ and 0 otherwise. Assume that $g_n(a)=g_n(b)$ for some $a,b\in A_n$. Then $(\alpha_1,\dots,\alpha_n)=(\beta_1,\dots,\beta_n)$ which implies that $\alpha_i=\beta_i$ for all i. Hence, for each a_i and b_i, we have that either $a_i=b_i=0$ or $a_{\alpha_i}=b_{\alpha_i}=1$. Therefore, $a=b$ and g_n is one-to-one. Next, if we take $(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, then if a is the sequence for which $a_{\alpha_1},\dots,a_{\alpha_n}$ are all 1 and the rest are 0, then $g_n(a)=(\alpha_1,\dots,\alpha_n)$ and g_n is onto. Thus $A_n\sim\mathbb{N}^n$ and so A_n is countable for all n.

The last part of this argument is to show that $A=\bigcup_{n=1}^\infty A_n$. If $a\in A$, then a contains m many 1's for some $m\in\mathbb{N}$ and thus $a\in A_m\subseteq\bigcup_{n=1}^\infty A_n$. Conversely, if $a\in\bigcup_{n=1}^\infty A_n$, then for some $m\in\mathbb{N}$, $a\in A_m$ and thus a is a sequence of digits 0 and 1 for which there are m many digits 1. Hence, $a\in A$ and the equality has been shown. Thus, since for each $n\in\mathbb{N}$, A_n is countable, then by problem 1.13,

$$A=\bigcup_{n=1}^\infty A_n$$

is countable. □

1.15 Let X be the set of all numbers from $[0,1]$ whose decimal expansion contains only the digits 3 or 5. Is X countable or uncountable?

Solution. Assume that X is countable. Then we can list each element of X in the following way: $x_1, x_2, \dots,$ where $x_i = 0.x_{i1}x_{i2}\dots$ for all $i\in\mathbb{N}$. In other words, x_{mn} is the nth digit of the mth decimal expansion. Now define a new decimal expansion $y=0.y_1y_2\dots$ such that

$$y_i=\begin{cases}3 & \text{if} \quad x_{ii}=5\\ 5 & \text{if} \quad x_{ii}=3\end{cases}.$$

We can see that $y\in[0,1]$ and that each of its digits are either 3 or 5. Hence, $y\in X$. However, we claim that y is not in the list provided above. For if we assume that there is some x_t such that $y=x_t$, then this implies $y_i=x_{ii}$ for all i. But this is not the case since by construction $y_t\neq x_{tt}$. Therefore y is not in the list above and the elements of X cannot be enumerated. Hence X is uncountable.

1.17 Show that if $\text{card}(X) = n$, then $\text{card}(2^X) = 2^n$.

Proof. We proceed by induction on $n \in \mathbb{N}$. Define

$$P(n) := \text{card}(X) = n \to \text{card}(2^X) = 2^n.$$

For the base case, we assume $\text{card}(X) = 0$. Then $X = \varnothing$ and thus $2^X = \{\varnothing\}$. Hence $\text{card}(2^X) = 1 = 2^0$. Thus $P(1)$ holds.

Now assume $P(k)$ holds for some $k \geq 2$. Then $\text{card}(X) = k$ implies $\text{card}(2^X) = 2^k$. Let Y be a set such that $\text{card}(Y) = k + 1$ and take $y \in Y$ and set $\hat{Y} = Y \backslash \{y\}$. Then clearly, $\text{card}(\hat{Y}) = k$ and hence $\text{card}(2^{\hat{Y}}) = 2^k$. It follows that for any subset $S \subseteq Y$, that $y \in S$ or $y \notin S$. Moreover, in collecting all the subsets $S \subseteq Y$ such that $y \notin S$, we find that there are 2^k of them since this collection is precisely $\hat{Y}$. It follows then that $Y \backslash \hat{Y}$ contains all the elements of $\hat{Y}$ but with y as a member, of which there are 2^k such elements. Hence, $\text{card}(Y) = 2 \cdot 2^k = 2^{k+1}$. Thus $P(k+1)$ holds. Therefore $P(n)$ holds for all $n \in \mathbb{N}$. □

1.18 Show that $2^{\mathbb{N}} \sim \mathbb{R}$.

Proof. First note that since $\mathbb{N} \in 2^{\mathbb{N}}$, then $2^{\mathbb{N}}$ is not finite. Next, assume for contradiction that $2^{\mathbb{N}} \sim \mathbb{N}$. Then there exists a bijection $h \colon \mathbb{N} \to 2^{\mathbb{N}}$ such that $h(n) = S_n$. Now define a function $f \colon 2^{\mathbb{N}} \to [0, 1]$ in the following way: for any $S_i \in 2^{\mathbb{N}}$

$$f(S_i) = \sum_{n=1}^{\infty} \frac{s_{in}}{10^n}; \quad s_{in} = \begin{cases} 3 & \text{if} \quad n \in S_i \\ 5 & \text{if} \quad n \notin S_i \end{cases}.$$

We want to show this function is one-to-one. Assume for $S_i, S_j \in 2^{\mathbb{N}}$ that $f(S_i) = f(S_j)$. Then $0.s_{i1}s_{i2} \cdots = 0.s_{j1}s_{j2} \ldots$ which implies $s_{ik} = s_{jk}$ for all $k \in \mathbb{N}$. Thus, if $k \in S_i$, then $s_{ik} = 3 = s_{jk}$ and hence $k \in S_j$. Thus $S_i \subseteq S_j$. Similarly, it follows by the same reasoning that $S_j \subseteq S_i$ and thus $S_i = S_j$. Therefore f is one-to-one.

Define $r = 0.g_1g_2\ldots$, where $g_i = 3$ if $s_{ii} = 5$ or $g_i = 5$ if $s_{ii} = 3$. Assume there exists some $S_i \in 2^{\mathbb{N}}$ such that $f(S_i) = r$. Then $0.g_1g_2\ldots g_i \cdots = 0.s_{i1}s_{i2}\ldots s_{ii}\ldots$, which implies $g_i = s_{ii}$, but this is not possible by construction of r and hence for all $S_i \in 2^{\mathbb{N}}$, $f(S_i) \neq r$. Finally, define $S' = \{n \in \mathbb{N} \mid g_n = 3\}$, where g_n is the nth digit in the decimal expansion r.Then it follows by construction that $f(S') = r$. However, $S' \in 2^{\mathbb{N}}$ yet $f(S') = r \neq f(S_i)$ for all $i \in \mathbb{N}$ which implies $S' \neq S_i$ for all $i \in \mathbb{N}$. Hence $S' \notin h(\mathbb{N})$ which is a contradiction.

Therefore $2^{\mathbb{N}} \not\sim \mathbb{N}$. Thus, by the Continuum Hypothesis, $\text{card}(2^{\mathbb{N}}) \geq \text{card}(\mathbb{R})$. However, since f is one-to-one, then $2^{\mathbb{N}} \sim f(2^{\mathbb{N}}) \subseteq [0, 1] \sim \mathbb{R}$ and hence
$\text{card}(2^{\mathbb{N}}) \not> \text{card}(\mathbb{R})$ which implies $\text{card}(2^{\mathbb{N}}) = \text{card}(\mathbb{R})$. Therefore $2^{\mathbb{N}} \sim \mathbb{R}$. □

1.19 Prove Proposition 1.27.

***Proof*.** Assume $\alpha^* = \inf A$. Then by Definition 1.26, α^* is a lower bound of A (part (i) of Proposition 1.27) and if α is a lower bound of A, then $\alpha \leq \alpha^*$. Let B be the set of all lower bounds of A. By definition, for any $b \in B$, $b \leq \alpha^*$. Conversely, if $b \leq \alpha^*$, then b is a lower bound of A and hence $b \in B$. Now let $\varepsilon > 0$. It follows that $\alpha^* < \alpha^* + \varepsilon$. Thus $\alpha^* + \varepsilon \notin B$, otherwise $\alpha^* + \varepsilon \leq \alpha^*$ which is not the case. Hence, $\alpha^* + \varepsilon$ is not a lower bound of A. Thus there exists some $a \in A$ such that $a < \alpha^* + \varepsilon$. Moreover, since $a \in A$ then $\alpha^* \leq a$ and thus $\alpha^* \leq a < \alpha^* + \varepsilon$ (part (ii) of Proposition 1.27).

Conversely, assume that α^* is a lower bound of A and that for all $\varepsilon > 0$, there exists $a_\varepsilon \in A$ such that $\alpha^* \leq a_\varepsilon < \alpha^* + \varepsilon$. From the second condition it follows that $\alpha^* - \varepsilon \leq \alpha^*$ for all $\varepsilon > 0$. Then if b is any lower bound of A, then there exists $\varepsilon > 0$ such that $b + \varepsilon = \alpha^*$ and hence $b = \alpha^* - \varepsilon \leq \alpha^*$. Therefore $\alpha^* = \inf A$. $\square$

1.20 Prove Proposition 1.29.

***Proof*.** Let $\beta^* = \sup A$. Then β^* is an upper bound of A (condition (i) of the proposition) and for any upper bound a of A, $\beta \leq a$. Let B be the set of all upper bounds of A. Then by definition if $b \in B$, $\beta^* \leq b$. Conversely, if $\beta^* \leq b$, then b is an upper bound of A and hence $b \in A$. Let $\varepsilon > 0$. Then $\beta^* - \varepsilon < \beta^*$ and hence $\beta^* - \varepsilon \notin B$. Thus $\beta^* - \varepsilon$ is not an upper bound of A. This implies that there exists $a \in A$ such that $\beta^* - \varepsilon < a$. But since $a \in A$, then $a \leq \beta^*$. Hence $\beta^* - \varepsilon < a \leq \beta^*$ (part (ii) of the proposition).

Conversely, assume that β^* is an upper bound of A and that for all $\varepsilon > 0$, there exists $b_\varepsilon \in A$ such that $\beta^* - \varepsilon < b_\varepsilon \leq \beta^*$. Then $\beta^* \leq \beta^* + \varepsilon$ for all $\varepsilon > 0$. Thus if b is any upper bound of A then $\beta^* \leq b$ and so there exists $\varepsilon > 0$ such that $\beta^* - \varepsilon = b$. Hence $\beta^* \leq b = \beta^* - \varepsilon$. Therefore $^* = \sup A$. $\square$

1.21 Let $A \subseteq \mathbb{R}$ be a nonempty set which is bounded from above. Show that if $\sup A \notin A$, then for all $\varepsilon > 0$ the open interval $(\sup A - \varepsilon, \sup A)$ contains infinitely many terms of A.

***Proof*.** Let $\varepsilon > 0$ and define $S = (\sup A - \varepsilon, \sup A)$. Assume for contradiction that for some $n \in \mathbb{N}$ S is finite and $S = \{s_1, s_2, \ldots, s_n\}$. Being finite implies that S has a maximum, call it $s = \max(S)$. Let $R = \{\sup A - s_i : s_i \in S\}$. Note that $R \subseteq \mathbb{R}$ and R is nonempty since otherwise $\sup A - (\sup A - \varepsilon) = 0$ which would imply $\varepsilon = 0$. Further, note that R is bounded below by 0 since if for some $1 \leq i \leq n$, $\sup A - s_i < 0$ then $\sup A < s_i$ which implies $\sup A$ is not an upper bound of A. Therefore by Axiom 5, $\inf R$ exists.

We claim that $\inf R = \sup A - s$. First, let $r \in R$, then for some $s_i \in S$, $r = \sup A - s_i$. If $\sup A - s_i < \sup A - s$ then $s < s_i$ which is a contradiction since $s = \max(S)$. Thus $\sup A - s$ is a lower bound of R. Let b be a lower bound of R. Then for all $s_i \in S$, $b \leq \sup A - s_i$ and hence $b \leq \sup A - s$. Therefore $\sup A - s = \inf R$. Finally, by Proposition 1.20 there exists $s' \in S$ such that $\sup A - s < s' \leq \sup A$. Thus $s > \sup A - s'$. But this contradicts that $s = \inf R$. Therefore S is not finite and S contains infinitely many terms if A. $\square$

1.21 Let $A, B \subseteq \mathbb{R}$ be nonempty bounded sets, and let $c \in \mathbb{R}$. Define the following sets:

$$
\begin{aligned}
A + B &= \{a + b \mid a \in A, b \in B\} \\
A - B &= \{a - b \mid a \in A, b \in B\} \\
A \cdot B &= \{ab \mid a \in A, b \in B\} \\
cA &= \{ca \mid a \in A\}.
\end{aligned}
$$

Prove that:

(1) $\inf(A + B) = \inf(A) + \inf(B)$

Proof. Let $\alpha = \inf(A+B)$, $\alpha_1 = \inf A$, and $\alpha_2 = \inf B$. Assume $\alpha < \alpha_1+\alpha_2$. Let $a \in A, b \in B$. Then $\alpha \leq a + b$, $\alpha_1 \leq a$, and $\alpha_2 \leq b$. Thus $\alpha_1 + \alpha_2 \leq a + b$. Hence $\alpha_1 + \alpha_2$ is a lower bound of $A + B$. Thus $\alpha_1 + \alpha_2 \leq \alpha$ which is a contradiction. Assume $\alpha_1 + \alpha_2 < \alpha$. Let $a \in A$, $b \in B$. Then $\alpha \leq a + b$ and $\alpha_2 \leq b$. Hence $\alpha - \alpha_2 \leq a$. Thus $\alpha - \alpha_2$ is a lower bound of A which implies $\alpha - \alpha_2 \leq \alpha_1$. Thus $\alpha \leq \alpha_1 + \alpha_2$ which is a contradiction. Therefore $\alpha = \alpha_1 + \alpha_2$, or $\inf(A + B) = \inf A + \inf B$. $\square$

(3) $\sup(-A) = -\inf(A)$

Proof. Let $\alpha = \sup(-A)$ and $\beta = -\inf(A)$. For any $a \in A$ we have that $-a \leq \alpha$ and $-\beta \leq a$ which implies $-a \leq \beta$. Thus β is an upper bound of $-A$ and so $\alpha \leq \beta$. Similarly, $-\alpha \leq a$ and so $-\alpha$ is a lower bound of A. Thus $-\alpha \leq -\beta$. Hence $\beta \leq \alpha$. Therefore $\alpha = \beta$. $\square$

(5) $\sup(A - B) = \sup A - \inf B$.

Proof. Let $C = -B$. Then by (1) and (2) it follows that

$$\sup(A - B) = \sup(A + C) = \sup A + \sup C = \sup A + \sup(-B) = \sup A - \inf B.$$

$\square$

(7) $\sup(cA) = c \sup A$ if $c > 0$.

Proof. Let $ca \in cA$. Then $ca \leq \sup(cA)$. Similarly, for any $a \in A$ we have that $a \leq \sup A$ and thus for any $c > 0$, $ca \leq c \sup A$. Thus $c \sup A$ is an upper bound of cA and hence $\sup(cA) \leq c \sup A$. Also note that since $ca \leq \sup(cA)$ for all $c > 0$ and $a \in A$, then $a \leq \frac{1}{c} \sup(cA)$ and hence $\frac{1}{c} \sup(cA)$ is an upper bound of A. As such it follows that $\sup A \leq \frac{1}{c} \sup(cA)$ and hence $c \sup A \leq \sup(cA)$. Therefore $c \sup A = \sup(cA)$. $\square$

(9) $\sup(cA) = c \inf A$ if $c < 0$.

Proof. Let $ca \in cA$. Then $ca \leq \sup(cA)$ and so $\frac{1}{c} \sup(cA) \leq a$. Hence $\frac{1}{c} \sup(cA)$ is a lower bound of A. Thus $\frac{1}{c} \sup(cA) \leq \inf A$. Additionally, if $a \in A$ then $\inf A \leq a$ and so $ca \leq c \inf A$. Thus $c \inf A$ is an upper bound of cA which implies $\sup(cA) \leq c \sup A$. Hence $\sup A \leq \frac{1}{c} \sup(cA)$. Since $\inf A \leq \sup A$ then $\inf A \leq \frac{1}{c} \sup(cA)$. Thus $\frac{1}{c} \sup(cA) = \inf A$ and hence $\sup(cA) = c \inf A$. $\square$

(11) Is it true that $\sup(A \cdot B) = \sup A \cdot \sup B$?

Solution. Let $A = [-2, -1]$ and $B = [0, 1]$ then $A \cdot B = [-2, 0]$. Thus $\sup(A \cdot B) = 0$ and $\sup A \cdot \sup B = (-1)(1) = -1$. Thus this claim is not true.

1.23 State and prove the density property of the irrational numbers.

Proof. *If $x, y \in \mathbb{R}$ and $x < y$, then there exists $i \in \mathbb{I}$ such that $x < i < y$.*

By Theorem 1.33, for any $x, y \in \mathbb{R}$ with $x < y$ there exists $p \in \mathbb{Q}$ such that $x < q < y$. Moreover, by Corollary 1.32 there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{y-p}{2}$. Thus $p + \frac{\sqrt{2}}{n} < p + \frac{2}{n} < y$ and since $x < p$ then $x < p + \frac{\sqrt{2}}{n}$. Hence $x < p + \frac{\sqrt{2}}{n} < y$. Now assume that $p + \frac{\sqrt{2}}{n} \in \mathbb{Q}$. Then there exists $a, b \in \mathbb{Z}$ such that $b \neq 0$, $\gcd(a, b) = 1$, and $p + \frac{\sqrt{2}}{n} = \frac{a}{b}$. Hence $\sqrt{2} = \frac{(a-bp)n}{b} \in \mathbb{Q}$. However this contradicts problem 1.3. Therefore $p + \frac{\sqrt{2}}{n} \in \mathbb{I}$. $\square$