MATH 230B

Name: Quin Darcy — Due Date: 02/04/2022
Instructor: Dr. Ricciotti — Assignment: Homework 01

1. Let
$$f(x) = \begin{cases} x^5 \sin(\frac{1}{x^3}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$
How many times is f differentiable on $\mathbb{R}$?
For which $n \in \mathbb{N}$ do we have $f \in C^n(\mathbb{R}) \backslash C^{n+1}(\mathbb{R})$?

Solution. For $x \in \mathbb{R} - \{0\}$, we have $f(x) = x^5 \sin(\frac{1}{x^3})$. Seeing as f is the product, composition, and quotient of differentiable functions, then by Theorem 5.3 and Theorem 5.7, we can conclude that f is differentiable for all $x \in \mathbb{R} - \{0\}$. Using the Product and Chain Rule, we obtain
$$f'(x) = 5x^4 \sin\Big(\frac{1}{x^3}\Big) - 3x \cos\Big(\frac{1}{x^3}\Big)$$
for all $x \in \mathbb{R} - \{0\}$. To obtain $f'(0)$, if it exists, we can use the definition of derivative and compute
$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^5 \sin\big(\frac{1}{h^3}\big)}{h} = \lim_{h \to 0} h^4 \sin\Big(\frac{1}{h^3}\Big) = 0.$$
The last equality holding since $0 \leq |h^4 \sin\big(\frac{1}{h^3}\big)| \leq h^4$ and since $h^4 \to 0$ as $h \to 0$. Thus
$$f'(x) = \begin{cases} 5x^4 \sin\big(\frac{1}{x^3}\big) - 3x \cos\big(\frac{1}{x^3}\big), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$
Seeing as $f'(x)$ is the product, composition, and quotient of continuous functions, provided that $x \neq 0$, then by Theorem 4.7 and Theorem 4.9, $f'(x)$ is continuous for $x \neq 0$. For $x = 0$, we note that since both the trigonometric terms are bounded as $x \to 0$ and that $5x^4, 3x \to 0$ as $x \to 0$, then $\lim_{x \to 0} f'(x) = f'(0) = 0$, meaning f' is continuous at $x = 0$. Thus $f \in C^1(\mathbb{R})$.

We now compute f''. By the same reasoning as above, for $x \neq 0$, we have that $f'(x)$ is differentiable and the Product and Chain Rules gives us
$$f''(x) = \left(20x^3 - \frac{9}{x^3}\right) \sin\Big(\frac{1}{x^3}\Big) - (15x + 3) \cos\Big(\frac{1}{x^3}\Big).$$

Using the same method as before, we let $x = 0$ and get

$$\begin{aligned}
\lim_{h\to 0} \frac{f'(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{5h^4 \sin\left(\frac{1}{h^3}\right) - 3h\cos\left(\frac{1}{h^3}\right)}{h} \\
&= \lim_{h\to 0} \left(5h^3 \sin\left(\frac{1}{h^3}\right) - 3\cos\left(\frac{1}{h^3}\right)\right) \\
&= \lim_{h\to 0} 5h^3 \sin\left(\frac{1}{h^3}\right) - \lim_{h\to 0} 3\cos\left(\frac{1}{h^3}\right) \\
&= \lim_{h\to 0} 3\cos\left(\frac{1}{h^3}\right).
\end{aligned}$$

Seeing as the limit in the last equality does not exist, then we can conclude that f' is not differentiable at $x = 0$. Therefore f is differentiable once on $\mathbb{R}$ and for $m = 1$ is it true that $f \in C^m(\mathbb{R})\backslash C^{m+1}(\mathbb{R})$. ■

2. Let $f : (0, 1] \to \mathbb{R}$ be differentiable with $0 < f'(x) < 1$ for all $x \in (0, 1]$. Prove that the sequence $\{f(1/n)\}_n$ has a limit.

Proof. Let $x, y \in (0, 1]$. Then f is continuous on $[x, y]$ and differentiable on (x, y). By the Mean Value Theorem, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(x - y) < (1)(x - y) < |x - y|,$$

and since this is true for all $x, y \in (0, 1]$, then we can conclude that f is Lipschitz on this interval and thus uniformly continuous on $(0, 1]$. With this in mind, we note that $\{1/n\}_n$ is a Cauchy sequence. Now letting $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we get

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon.$$

Combining this with the fact that f is uniformly continuous on $(0, 1]$, we have

$$f(1/n) - f(1/m) < \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon.$$

Hence, $\{f(1/n)\}_n$ is Cauchy and therefore convergent. □

3. Let $f : [0, 1] \to [0, 1]$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$, with $f'(x) \neq 1$ for all $x \in (0, 1)$. Prove that there exists a unique fixed point for f in $[0, 1]$.

Proof. For contradiction, assume that $f(x) \neq x$ for all $x \in [0, 1]$. Then define the set

$$S = \{|f(x) - x| : x \in [0, 1]\}.$$

Seeing as $f(x)$, x, and $|x|$ are all continuous on $[0, 1]$, then $g(x) = |f(x) - x|$ is continuous on $[0, 1]$ as a composition and difference of continuous functions. Thus the image of $[0, 1]$ under g, S, is closed and bounded.

By the greatest lower bound property, $\alpha = \inf(S)$ exists and $\alpha \in S$ since S is closed. Hence there exists $x_0 \in [0,1]$ such that $\alpha = |f(x_0) - x_0|$. If $\alpha = 0$ and $f(x) > x$, then $|f(x) - x| = f(x) - x = 0$ and so $f(x) = x$, which is a contradiction. Similarly, if $x > f(x)$, then $|f(x) - x| = x - f(x) = 0$ and so $x = f(x)$, another contradiction. Thus $\alpha > 0$.

Define $g(x) = f(x) - x$. Then g is continuous on $[0,1]$, differentiable on $(0,1)$, and α is a local minimum at some x_0. If $x_0 \in (0,1)$, then by Theorem 5.8 $g'(x_0) = 0$, which implies $f'(x_0) = 1$ which is not possible by assumption. Thus either $g(0) = \alpha$ or $g(1) = \alpha$. Coupled with the fact that $g(x) \neq 0$ for all $x \in (0,1)$ implies that g is strictly monotone.

If $g(0) = \alpha$, then g must be strictly monotone increasing. Thus $g(1) > \alpha$ which implies $f(1) > 1 + \alpha \notin [0,1]$. This is not possible by definition of f.

If $g(1) = \alpha$, then $f(1) = 1 + \alpha \notin [0,1]$, which is not possible by the same reasoning above. Therefore there must exist some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

To show uniqueness, let $x_1, x_2 \in [0,1]$ such that $f(x_1) = x_1$ and $f(x_2) = x_2$. Then $g(x_1) = 0 = g(x_2)$. By Rolle's Theorem, there exists $x_0 \in (x_1, x_2)$ such that $g'(x_0) = 0$ which implies $f'(x_0) = 1$, a contradiction unless $x_1 = x_2$. [I'm sorry this was so long! I got carried away and had too much fun with it.] $\square$

4. Let $f : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$ with $f(0) = 0$ and differentiable on $(0,\infty)$ with f' increasing on $(0,\infty)$. Prove that the function $\frac{f(x)}{x}$ is increasing on $(0,\infty)$.

Proof. Let $g(x) = \frac{f(x)}{x}$ and select an arbitrary $x_0 \in (0,\infty)$. Since f and x are continuous on $[0,\infty)$, then g is continuous on this interval. Similarly, since f and x are differentiable on $(0,\infty)$, then g is differentiable on this interval.

By assumption, f is continuous on $[0,x_0] \subset [0,\infty)$ and differentiable on $(0,x_0)$. Thus by Theorem 5.10, there exists $a \in (0,x_0)$ such that

$$f'(a) = \frac{f(x_0) - f(0)}{x_0 - 0} = \frac{f(x_0)}{x_0} \Leftrightarrow x_0 f'(a) = f(x_0). \tag{1}$$

We established that g is differentiable on $(0,\infty) \supset (0,x_0)$. Thus for each $x \in (0,x_0)$, it follows from Theorem 5.3(c)

$$g'(x) = \frac{xf'(x) - (1)f(x)}{x^2}. \tag{2}$$

If we want to prove that g is increasing on $(0,\infty)$, then we can use Theorem 5.11(a) to show that $g'(x) \geq 0$ for all $x \in (0,\infty)$. Using (2), we then need to show

$$\frac{xf'(x) - f(x)}{x} \geq 0$$

for all $x \in (0, \infty)$. By assumption, $f'(a) \leq f'(x_0)$ and thus $x_0 f'(a) \leq x_0 f'(x_0)$ since $a < x_0$. However, by (1) we have $x_0 f'(a) = f(x_0)$. Hence $f(x_0) \leq x_0 f'(x_0)$ which implies that $x_0 f'(x_0) - f(x_0) \geq 0$. Since $x_0 > 0$, then

$$g'(x_0) = \frac{x_0 f'(x_0) - f(x_0)}{x} \geq 0.$$

Finally, since x_0 was arbitrary, then $g'(x) \geq 0$ for all $x \in (0, \infty)$ as desired. □

5. Let $a < b$, $x_0 \in (a, b)$ and $f \in C^{2n}((a, b))$ for some $n \in \mathbb{N}$. Suppose that $f^{(k)}(x_0) = 0$ for all $k \in \{1, 2, \ldots, 2n-1\}$ and $f^{(2n)}(x_0) > 0$. Prove that f has a local minimum at x_0.

Proof. Since $f^{(2n)}$ is continuous and $f^{(2n)}(x_0) > 0$, then there exists $\delta > 0$ such that for all $x \in N_\delta(x_0) = \{x \in \mathbb{R} \mid |x - x_0| < \delta\}$, we have $f^{(2n)}(x) \geq 0$. Let $x \in N_\delta(x_0)$. Then by Taylor's theorem, there exists a c in between x_0 and x such that

$$f(x) = \sum_{k=0}^{2n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(2n)}(c)}{(2n)!}(x - x_0)^{2n} \tag{3}$$

$$= f(x_0) + \frac{f^{(2n)}(c)}{(2n)!}(x - x_0)^{2n}, \tag{4}$$

where (4) was a result of one of our assumptions. Since c is in between x_0 and x, then $c \in N_\delta(x_0)$ and thus $f^{(2n)}(c) \geq 0$. Moreover, since $2n$ is even, then $(x - x_0)^{2n} \geq 0$ for all x. Hence

$$\frac{f^{(2n)}(c)}{(2n)!}(x - x_0)^{2n} \geq 0. \tag{5}$$

Combining (4) and (5), the implication is

$$f(x) - f(x_0) \geq 0 \tag{6}$$

for all $x \in N_\delta(x_0)$. Thus $f(x_0) \leq f(x)$ for all $x \in N_\delta(x_0)$. Therefore $f(x_0)$ is a local minimum. □

6. Let $a < b$ be real numbers and $f : (a, b) \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$ for all $x \in (a, b)$, where $M > 0$. Prove that $\lim_{x \to b^-} f(x)$ exists.

Proof. To prove this claim, we will appeal to the sequential definition of limit and show that for any sequence $\{x_n\}$ in (a, b) that converges to b, we have that $\{f(x_n)\}$ is a convergent sequence.

We begin by showing that f is uniformly continuous on (a, b). Let $x, y \in (a, b)$. Then f is differentiable on (x, y) and continuous on $[x, y]$. Thus, by the MVT, there exists some $z \in (x, y)$ such that

$$f'(z) = \frac{f(y) - f(x)}{y - x} \Leftrightarrow |f(y) - f(x)| \leq M|y - x|,$$

which shows that f is Lipschitz continuous. Simply selecting $x, y \in (a, b)$ such that $|y - x| < \varepsilon / < M$, for any ε, then shows that f is uniformly continuous on (x, y) and as x and y were arbitrary, then f is uniformly continuous on (a, b).

Let $\varepsilon > 0$, then since f is uniformly continuous, there exists $\delta > 0$ such that for $x, y \in (a, b)$, $|x - y| < \delta$, then $|f(y) - f(x)| < \varepsilon$. Letting $c_n \to b$ be a sequence in (a, b), then by its convergence it follows that it is a Cauchy sequence. Hence, there exists $N \in \mathbb{N}$, such that for any $m > n \geq N$, we have that

$$|c_m - c_n| < \delta \to |f(c_m) - f(c_n)| < \varepsilon.$$

Therefore $\{f(c_n)\}$ is Cauchy and therefore convergent. □

7. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. Prove that if $\lim_{x\to\infty} f(x) = M \in \mathbb{R}$, then there exists a sequence $\{x_n\}_n$ in $(0, \infty)$ such that $f'(x_n)$ converges to 0.

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = M$, then there exists $\delta > 0$ such that for any $x > \delta$, we have that $|f(x) - M| < \varepsilon$. Moreover, by the Archemedian principle, there exists an integer $N > \delta$. Thus for any integer $n \geq N$, we have $|f(n) - M| < \varepsilon$. Therefore $\{f(n)\}_n$ is a convergent sequence and hence Cauchy. Taking the same ε as before, there exists N_ε such that for any $m, n \geq N_\varepsilon$, we have $|f(n) - f(m)| < \varepsilon$. Hence, we will let $n = N_\varepsilon$ and $m = n + 1$. Then

$$|f(m) - f(n)| < \varepsilon \Rightarrow \frac{f(m) - f(n)}{m - n} \leq \frac{|f(m) - f(n)|}{|m - n|} < \varepsilon. \tag{7}$$

Since f is differentiable on $[n, m] \subset (0, \infty)$, then there exists $z \in (n, m)$ such that

$$f'(z) = \frac{f(m) - f(n)}{m - n} < \varepsilon.$$

We now define the following sequence $\{z_k\}_k$. For each $k \in \mathbb{N}$, let $\varepsilon = 1/k$, then we obtain N_ε and let $n = N_\varepsilon$, $m = n+1$. And finally by the MVT, there exists $z_k \in (n, m)$ such that $f'(z_k) < \varepsilon = 1/k$. Since $1/k \to 0$ as $k \to \infty$, then $f'(z_k) \to 0$ as $k \to \infty$. □

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$, where $0 \leq M < 1$. Let $x_1 \in \mathbb{R}$ and consider the recursion

$$x_{n+1} = f(x_n) \text{ for all } n \in \mathbb{N}.$$

Prove that $\{x_n\}_n$ converges to the unique fixed point of f.

Proof. Let $a, b \in \mathbb{R}$. Then f is differentiable on (a, b) and continuous on $[a, b]$. By the MVT, there exists $z \in (a, b)$ such that

$$f'(z) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow |f(b) - f(a) \leq M|b - a|.$$

Thus, f is Lipschitz continuous on $\mathbb{R}$. It follows that

$$|f(x_{n+1}) - f(x_n)| = |x_{n+2} - x_{n+1}| \leq M|x_{n+1} - x_n| \leq M^{n-1}|x_2 - x_1|.$$

Since $0 \leq M < 1$, then $(x_{n+1} - x_n) \to 0$ as $n \to \infty$. Letting $m, n \in \mathbb{N}$ with $m < n$, then call $k = n - m$. We have that

$$\begin{aligned} |x_n - x_m| &= |x_{m+k} - x_m| \\ &\leq |x_{m+k} - x_{m+k-1}| + \cdots + |x_{m+1} - x_m| \\ &< (M^{k-1} + \cdots + M + 1)|x_{m+1} - x_m| \\ &< \sum_{i=0}^{\infty} M^i |x_{m+1} - x_m| \\ &= \frac{|x_{m+1} - x_m|}{1 - M}. \end{aligned}$$

Finally, since the last equality converges to 0 as $m \to \infty$, then $|x_n - x_m| \to 0$ and is therefore Cauchy. Thus $\{x_n\}$ is convergent.

Let $x = \lim_{n\to\infty} x_n$. Then

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

Thus x is a fixed point of f. Finally, assume that y is another fixed point of f and take $x < y$. Then f is differentiable on (x, y) and continuous on $[x, y]$. By the MVT, there exists $z \in (x, y)$ such that

$$f'(z) = \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1.$$

This contradicts that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Therefore x must be unique. □

9. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Proof. Letting $x \in \mathbb{R} - \{0\}$, then $f(x) = e^{-1/x^2}$ and

$$f'(x) = 2\frac{e^{-\frac{1}{x^2}}}{x^3}.$$

To obtain $f'(0)$, if it exists, we consider

$$\lim_{h\to 0} \frac{f(h+0) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

Letting $g(h) = h$, then we can write the previous statement as

$$\lim_{h\to 0} \frac{f(h)}{g(h)}$$

and since both f and g are differentiable on $(0, \infty)$ and $g'(h) = 1 \neq 0$. □