Analysis Notes

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1 Proofs

Definition 1.1. Let X, Y be metric spaces and $f : X \to Y$.

(i) We say that f is **uniformly continuous** on X if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $d(f(x_1), f(x_1)) < \varepsilon$ for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$.

(ii) We say that f is **Lipschitz continuous** on X if there exists $L \geq 0$ such that

$$d(f(x_1), f(x_2)) < Ld(x_1, x_2), \quad \forall x_1, x_2 \in X.$$

The number L is called the *Lipschitz constant* of f.

(iii) If f is Lipschitz continuous and $0 \leq L < 1$, then we say that f is a **contraction**.

Theorem 1.1. *Let X, Y be metric spaces and $f : X \to Y$. Regarding the continuity of f on X, the following implications hold:*

$$\textit{Lipschitz continuous} \Rightarrow \textit{uniformly continuous} \Rightarrow \textit{continuous.}$$

Proof. If f is Lipschitz continuous, then for any $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) = \frac{\varepsilon}{L+1}$. By doing this we get that for all $x, y \in X$

$$|x - y| < \delta \Rightarrow |x - y| < \frac{\varepsilon}{L + 1} \Rightarrow |x - y|(L + 1) < \varepsilon$$

and so

$$|f(x) - f(y)| < L|x - y| < (L + 1)|x - y| < \varepsilon.$$

Hence, f is uniformly continuous. □

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in (a, b)$. If f has a local maximum (or minimum) at x_0 and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. □

Theorem 1.3 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. If f is constant on $[a, b]$, then $f'(x) = 0$ for all $x \in (a, b)$.

If f is not constant on $[a, b]$, then it attains its maximum and minimum of $[a, b]$. From $f(a) = f(b)$ it follows taht one of them must occur inside (a, b). To see why this is, assume the opposite; that f attains both its minimum and maximum not in (a, b). Then $f(a)$ is either a maximum or minimum (let's assume its a max) which means $f(b)$ is also the max. So where is the minimum?? That's right, it's in (a, b). Anyways, there is some local minimum or maximum point $x_0 \in (a, b)$ and by Theorem 1.2, $f'(x_0) = 0$. □

Theorem 1.4 (Lagrange's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that*

$$f(b) - f(a) = f'(x_0)(b - a).$$

Proof. Consider

$$h(x) = x\Big(f(b) - f(a)\Big) - f(x)\Big(b - a\Big).$$

Seeing as $(f(b) - f(a))$ and $(b - a)$ are constants, and x and $f(x)$ are both differentiable on (a, b), then so is $h(x)$. □

2 The Riemann Integral

Definition 2.1. Consider a closed and bounded interval $[a, b]$. A partition P of $[a, b]$ is a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$. If we have two partitions P and Q of the same interval $[a, b]$, we say that Q is a refinement of P, if $P \subseteq Q$.

Definition 2.2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We define the following quantities:

$$\begin{aligned} m_i(f) &= \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\}, \\ M_i(f) &= \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\} \\ \Delta x_i &= x_i - x_{i-1}. \end{aligned}$$

The norm of the partition P is defined as

$$\|P\| = \max\{\Delta x_i \mid 1 \leq i \leq n\}$$

The lower Riemann sum of f is associated to the partition P is

$$\underline{S}(f, P) = \sum_{i=1}^{n} m_i(f)\Delta x_i.$$

The upper sum associated with the partition P is

$$\overline{S}(f, P) = \sum_{i=1}^{n} M_i(f)\Delta x_i.$$

The lower Riemann sum of f is

$$\underline{S}(f) = \sup\{\underline{S}(f, P) \mid \text{for all partitions } P\}.$$

The upper Riemann sum of f is

$$\overline{S}(f) = \inf\{\overline{S}(f, P) \mid \text{for all partitions } P\}.$$

We say that f is Riemann integrable over $[a, b]$ if

$$\underline{S}(f) = \overline{S}(f).$$

If f is Riemann integrable, then the common value of the upper and lower Riemann sum is denoted by

$$\int_a^b f(x)dx.$$

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then the following statements hold:*

(1) If P_1, P_2 are partitions of $[a, b]$, $P_1 \subset P_2$, then

$$\underline{S}(f, P_1) \leq \underline{S}(f, P_2) \quad \text{and} \quad \overline{S}(f, P_1) \geq \overline{S}(f, P_2).$$

(2) If P and Q are any two partitions of $[a, b]$, then

$$\underline{S}(f, P) \leq \overline{S}(f, Q).$$

(3)

$$\underline{S}(f) \leq \overline{S}(f)$$

(4) $\underline{S}(f) = \overline{S}(f)$ if and only if for all $\varepsilon > 0$ there exists a partition P_ε such that

$$\overline{S}(f, P_\varepsilon) - \underline{S}(f, P_\varepsilon) < \varepsilon.$$

Theorem 2.2.