MATH 296C

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Instructor: Dr. Krauel Assignment: Midterm 1

1. Consider the Lie algebra $L = \mathfrak{o}_4(\mathbb{C})$.

   (a) Find a basis and compute its dimension.

   Solution. To find a basis we can start by considering some $A \in \mathfrak{gl}_n(\mathbb{C})$ such that $A^t = -A$. If we let
$$A = (a_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$
then we have that $A^t = (a_{j,i})$. Thus we need entries such that $(a_{j,i}) = -(a_{i,j})$. It follows that the entries down the main diagonal must satisfy $a_{i,i} = -a_{i,i}$ which implies $a_{i,j} = 0$ for all entries with $i = j$. For those entries not on the main diagonal we have that the lower triangle is equal to the negative of the upper triengle. Thus the general form of the matrix can be written as
$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$+ d\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
With this it follows that $\dim(\mathfrak{o}_4(\mathbb{C})) = 6$. ■

   (b) Describe the trivial module and its dimension.

   Solution. To obtain the trivial module we let $L = \mathfrak{o}_4(\mathbb{C})$ and let $F = \mathbb{C}$ and define a map $L \times F \to F$ such that $(x, a) \mapsto x.a := 0$, for all $x \in L$ and $a \in F$. To show that this is a module, we let $x, y \in L$, $u, v \in V$, and $\alpha, \beta \in F$. Then we have that
$$\begin{aligned} (\alpha x + \beta y).u &= 0 = \alpha(0) + \beta(0) = \alpha(x.u) + \beta(y.u) \\ x.(\alpha u + \beta v) &= 0 = \alpha(0) + \beta(0) = \alpha(x.u) + \beta(x.v) \\ [x, y].u &= 0 = x.(0) + y.(0) = x.(y.u) + y.(x.u). \end{aligned}$$
As a module, then by Lemma 4.7, it is a representation of L. Thus, with $F = \mathbb{C}$, then the trivial representation is a 1-dimensional representation and hence a 1-dimensional module. ■

(c) Describe the natural module and its dimension.

Solution. As a subalgebra of $\mathfrak{gl}(\mathbb{C}^4)$, we can define a map $\mathfrak{o}_4(\mathbb{C}) \times \mathbb{C}^4 \to \mathbb{C}^4$ such that for any $x \in \mathfrak{o}_c(\mathbb{C})$ and $(a,b,c,d) \in \mathbb{C}^4$, where we let $v = (a,b,c,d)^T$, then taking x to be a basis element we get

$$x.v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ -a \\ 0 \\ 0 \end{pmatrix}.$$

Finally, with $\mathbb{C}^4$ as our $\mathfrak{o}_4(\mathbb{C})$-module, then we conclude that the natural module is 4 dimensional. ∎

2. Consider the Lie algebra $L = \mathfrak{sl}_3(\mathbb{C})$. Let E_{ij} denote the 3×3 matrix with a 1 in the ij-th entry, and zeros elsewhere. Set $H_1 := E_{11} - E_{22}$, $H_2 := E_{22} - E_{33}$, $X_1 := E_{12}$, $X_2 := E_{23}$, $X_3 := E_{13}$, $Y_1 := E_{21}$, $Y_2 := E_{32}$, $Y_3 := E_{31}$.

(a) Compute ad_{H_2} with respect to the basis above.

Solution.

$$[H_2, H_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$$[H_2, X_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -X_1$$

$$[H_2, X_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 2X_2$$

$$[H_2, X_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = X_3$$

$$[H_2, Y_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = Y_1$$

$$[H_2, Y_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -2Y_2$$

$$[H_2, Y_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -Y_3$$

∎

(b) Consider the subset $\mathcal{H} = \text{span}\{H_1, H_2\}$ of L. Prove or disprove that $\mathcal{H}$ is an abelian Lie subalgebra.

Proof. To prove that $\mathcal{H}$ is a Lie subalgebra, we need to show that $\mathcal{H}$ is a subspace and $[x, y] \in \mathcal{H}$ for all $x, y \in \mathcal{H}$. Additionally, we need to show that $[x, y] = 0$ for all $x, y \in \mathcal{H}$.

First, we let $x, y \in \mathcal{H}$ and let $c \in \mathbb{C}$. Then we have that for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$

$$x = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta - \alpha & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$
$$y = \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \delta - \gamma & 0 \\ 0 & 0 & -\delta \end{pmatrix}$$

Now we observe that

$$\begin{aligned} x + y &= \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta - \alpha & 0 \\ 0 & 0 & -\beta \end{pmatrix} + \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \delta - \gamma & 0 \\ 0 & 0 & -\delta \end{pmatrix} \\ &= \begin{pmatrix} \alpha + \gamma & 0 & 0 \\ 0 & (\beta + \delta) - (\alpha + \gamma) & 0 \\ 0 & 0 & -(\beta + \delta) \end{pmatrix} \\ &= (\alpha + \gamma) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + (\beta + \delta) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ &= (\alpha + \gamma)H_1 + (\beta + \delta)H_2 \in \mathcal{H}. \end{aligned}$$

Thus, $\mathcal{H}$ is closed under adition. Now we look at

$$kx = k(\alpha H_1 + \beta H_2) = k\alpha H_1 + k\beta H_2 \in \mathcal{H}.$$

Therefore, $\mathcal{H}$ is a subspace. To finish showing that $\mathcal{H}$ is a subalgebra, we observe that

$$\begin{aligned} [x, y] &= [\alpha H_1 + \beta H_2, \gamma H_1 + \delta H_2] \\ &= [\alpha H_1, \gamma H_1] + [\alpha H_1, \delta H_2] + [\beta H_2, \gamma H_1] + [\beta H_2, \delta H_2] \\ &= (\alpha\gamma)[H_1, H_1] + (\alpha\delta)[H_1, H_2] + (\beta\gamma)[H_2, H_1] + (\beta\delta)[H_2, H_2] \\ &= (\alpha\gamma)(0) - (\alpha\delta)(0) + (\beta\gamma)(0) + (\beta\delta)(0) \\ &= 0 \in \mathcal{H}. \end{aligned}$$

The final line both completes the prove that $\mathcal{H}$ is a subalgebra and it also proves that $\mathcal{H}$ is abelian. $\square$

(c) Prove or disprove that $\mathcal{H}$ is an ideal.

Proof. We will show that $\mathcal{H}$ is not an ideal. Consider $H_2 \in \mathcal{H}$ and $Y_1 \in \mathfrak{sl}_3(\mathbb{C})$. Then by part 2.(a), we have that $[H_2, Y_1] = Y_1$. If $\mathcal{H}$ were an ideal, then it would follow that $Y_1 \in \mathcal{H}$. However, this would imply that

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta - \alpha & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{C}$, which is not possible. Therefore, $\mathcal{H}$ is not an ideal. □

(d) Can you find a subalgebra of L that is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$? If yes, find one and prove that are isomorphic. If not, prove why not.

Proof. Yes. Consider the set $\text{span}(H_1, X_1, Y_1)$. We claim that this is a subalgebra of L and that it is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To see that it is a subalgebra we first note that it is a subspace since for any $u, v \in \text{span}(H_1, X_1, Y_1)$, we have $\alpha_1, \alpha_2, \beta_2, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$ such that

$$\begin{aligned} u + v &= (\alpha_1 H_1 + \beta_1 X_1 + \gamma_1 Y_1) + (\alpha_2 H_1 + \beta_2 X_2 + \gamma_2 Y_2) \\ &= (\alpha_1 + \alpha_2) H_1 + (\gamma_1 + \gamma_2) X_1 + (\beta_1 + \beta_2) Y_1 \end{aligned}$$

$\in \text{span}(H_1, X_1, Y_1)$.

Similarly, for any $k \in \mathbb{C}$, it follows that $ku \in \text{span}(H_1, X_1, Y_1)$. Finally, we calculate

$$\begin{aligned} [u, v] &= [\alpha_1 H_1 + \beta_1 X_1 + \gamma_1 Y_1, \alpha_2 H_1 + \beta_2 X_1 + \gamma_2 Y_1] \\ &= (\beta_1 \gamma_2 - \gamma_1 \beta_2) H_1 + (2\alpha_1 \beta_2 - 2\beta_1 \alpha_2) X_1 + (2\gamma_1 \alpha_2 - 2\alpha_1 \gamma_2) Y_1 \\ &\in \text{span}(H_1, X_1, Y_1). \end{aligned}$$

We also require that for any $v \in \text{span}(H_1, X_1, Y_1)$, with $v =$Therefore the set is a subalgebra. Now we must show that it is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To do this consider a linear map $\phi : \text{span}(H_1, X_1, Y_1) \to \mathfrak{sl}_2(\mathbb{C})$ such that

$$H_1 \mapsto h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad X_1 \mapsto x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad Y_1 \mapsto y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

With this map we must show that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \text{span}(H_1, X_1, Y_1)$. So then let $x, y \in \text{span}(H_1, X_1, Y_1)$. Then for $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$, we have that

$$x = \alpha_1 H_1 + \beta_1 X_1 + \gamma_1 Y_1 \quad \text{and} \quad y = \alpha_2 H_1 + \beta_2 X_1 + \gamma_2 Y_1.$$

Thus by our calculation earlier we have

$$\begin{aligned} \phi([x, y]) &= \phi((\beta_1 \gamma_2 - \gamma_1 \beta_2) H_1 + (2\alpha_1 \beta_2 - 2\beta_1 \alpha_2) X_1 + (2\gamma_1 \alpha_2 - 2\alpha_1 \gamma_2) Y_1) \\ &= (\beta_1 \gamma_2 - \gamma_1 \beta_2) \phi(H_1) + (2\alpha_1 \beta_2 - 2\beta_1 \alpha_2) \phi(X_1) + (2\gamma_1 \alpha_2 - \alpha_1 \gamma_2) \phi(Y_1) \\ &= (\beta_1 \gamma_2 - \gamma_1 \beta_2) h + (2\alpha_1 \beta_2 - 2\beta_1 \alpha_2) x + (2\gamma_1 \alpha_2 - \alpha_1 \gamma_2) y. \end{aligned}$$

And we have that

$$\begin{aligned}
[\phi(x), \phi(y)] &= [\phi(\alpha_1 H_1 + \beta_1 X_1 + \gamma_1 Y_1), \phi(\alpha_2 H_1 + \beta_2 X_1 + \gamma_2 Y_1)] \\
&= [\alpha_1 \phi(H_1) + \beta_1 \phi(X_1) + \gamma_1 \phi(Y_1), \alpha_2 \phi(H_1) + \beta_2 \phi(X_1) + \gamma_2 \phi(Y_1)] \\
&= [\alpha_1 h + \beta_1 x + \gamma_1 y, \alpha_2 h + \beta_2 x + \gamma_2 y] \\
&= (2\alpha_1\beta_2)x - (2\alpha_1\gamma_2)y - (2\beta_1\alpha_2)x + (\beta_1\gamma_2)h + (2\gamma_1\alpha_2)y - (\gamma_1\beta_2)h \\
&= (\beta_1\gamma_2 - \gamma_1\beta_2)h + (2\alpha_1\beta_2 - 2\beta_1\alpha_2)x. + (2\gamma_1\alpha_2 - 2\alpha_1\gamma_2)y
\end{aligned}$$

Therefore $\phi([x, y]) = [\phi(x), \phi(y)]$ and ϕ is a Lie algebra homomorphism. Finally, we need to show that ϕ is bijective.

Let $v \in \mathfrak{sl}_2(\mathbb{C})$. Then for some $\alpha, \beta, \gamma \in \mathbb{C}$, we have that $v = \alpha h + \beta x + \gamma y$. Then letting $w = \alpha H_1 + \beta X_1 + \gamma Y_1$, it follows that $\phi(w) = v$ and so ϕ is surjective. Now let $u \in \ker \phi$. Then for some $\alpha, \beta, \gamma \in \mathbb{C}$

$$\begin{aligned}
\phi(u) &= \phi(\alpha H_1 + \beta X_1 + \gamma Y_1) \\
&= \alpha\phi(H_1) + \beta\phi(X_1) + \gamma\phi(Y_1) \\
&= \alpha h + \beta x + \gamma y \\
&= 0,
\end{aligned}$$

which implies that $\alpha = \beta = \gamma = 0$. Thus, $\ker \phi = \{0\}$. Therefore, ϕ is injective. □

(e) Prove directly that $\mathfrak{sl}_3(\mathbb{C})$ is a simple Lie algebra.

Proof. □

(f) Is the adjoint module of $\mathfrak{sl}_3(\mathbb{C})$ an irreducible module?

Solution. Suppose that U is a nonzero submodule of the adjoint module of $\mathfrak{sl}_3(\mathbb{C})$. Then $\dim(U) \geq 1$. Assume that $\dim(U) = 1$. Then it is spanned by one of the 8 basis elements. Suppose it is spanned by H_1. As a submodule, we require that $x.H_1 = [x, H_1] \in U$ for all $x \in L$. Thus, $[X_2, H_1] = X_2 \in U$, implying that $\dim(U) > 1$. Now suppose that U is instead spanned by H_2. Then $[X_1, H_2] = X_1 \in U$, implying the same contradiction. In fact we will find that for the 6 remaining basis elements, we have that

$$\begin{aligned}
&[Y_3, X_1] = Y_2, \qquad && [Y_2, X_2] = -H_2, \qquad && [Y_2, X_3] = -X_1, \\
&[X_1, Y_1] = H_1, && [X_2, Y_2] = H_2, && [X_2, Y_3] = Y_1.
\end{aligned}$$

■

The above implies that $\dim(U) > 1$. Similarly, if $\dim(U) = 2$, then there are $\binom{8}{2} = 28$ possibilities in terms of which 2 basis elements could span U. In each case we find that a third basis element can be produced. The same argument can be applied if $\dim(U) = 3, 4, 5, 6, 7$. This fact follows from the calculation in 2.(a). That $X_1, X_2, X_3, Y_1, Y_2, Y_3$ occurs, then so long as U contains any one basis element, the rest can be shown to be in U, implying that $U = \mathfrak{sl}_3(\mathbb{C})$. Hence, the adjoint module is in fact irreducible.

3. Let V be a finite-dimensional vector space. A bilinear form $b : V \times V \to \mathbb{C}$ is said to be **symmetric** if $b(x, y) = b(y, x)$ for any $x, y \in V$. A symmetric bilinear form is **nondegenerate** if $b(x, y) = 0$ for all $x \in V$ implies $y = 0$. Let L be a finite-dimensional complex Lie algebra. The **Killing form** on L is defined as the function $\kappa : L \times L \to \mathbb{C}$ given by $\kappa(x, y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$.

   (a) Prove that κ is a symmetric bilinear form.

   Proof. We will first show that κ is a bilinear form. Letting $x, u, v, w \in L$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, then since $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is a Lie algebra homomorphism (Lemma 2.12) and $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\mathbb{C})$ (Proposition 1.34), then it follows that

$$\begin{aligned}
\mathrm{ad}_{\alpha x+\beta u} \circ \mathrm{ad}_{\gamma v+\delta w} &= (\mathrm{ad}(\alpha x + \beta u)) \circ (\mathrm{ad}(\gamma v + \delta w)) \\
&= (\alpha\mathrm{ad}(x) + \beta\mathrm{ad}(u)) \circ (\gamma\mathrm{ad}(v) + \delta\mathrm{ad}(w)) \\
&= (\alpha\mathrm{ad}(x) \circ \gamma\mathrm{ad}(v)) + (\alpha\mathrm{ad}(x) \circ \delta\mathrm{ad}(w)) \\
&\quad + (\beta\mathrm{ad}(u) \circ \gamma\mathrm{ad}(v)) + (\beta\mathrm{ad}(u) \circ \delta\mathrm{ad}(w)) \\
&= (\mathrm{ad}_{\alpha x} \circ \mathrm{ad}_{\gamma v}) + (\mathrm{ad}_{\alpha x} \circ \mathrm{ad}_{\delta w}) \\
&\quad + (\mathrm{ad}_{\beta u} \circ \mathrm{ad}_{\gamma v}) + (\mathrm{ad}_{\beta u} \circ \mathrm{ad}_{\delta w}).
\end{aligned}$$

   Hence,

$$\begin{aligned}
\kappa(\alpha x + \beta u, \gamma v + \delta w) &= \mathrm{tr}(\mathrm{ad}_{\alpha x+\beta u} \circ \mathrm{ad}_{\gamma v+\delta w}) \\
&= \mathrm{tr}(\mathrm{ad}_{\alpha x} \circ \mathrm{ad}_{\gamma v}) + \mathrm{tr}(\mathrm{ad}_{\alpha x} \circ \mathrm{ad}_{\delta w}) \\
&\quad + \mathrm{tr}(\mathrm{ad}_{\beta u} \circ \mathrm{ad}_{\gamma v}) + \mathrm{tr}(\mathrm{ad}_{\beta u} \circ \mathrm{ad}_{\delta w}) \\
&= \alpha\gamma\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_v) + \alpha\delta\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_w) \\
&\quad + \beta\gamma\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_v) + \beta\delta\mathrm{tr}(\mathrm{ad}_u \circ \mathrm{ad}_w) \\
&= \alpha\gamma\kappa(x, v) + \alpha\beta\kappa(x, w) + \beta\gamma\kappa(u, v) + \beta\delta\kappa(u, w).
\end{aligned}$$

   Therefore, κ is bilinear. □