STAT 215Ag

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Instructor: Dr. Cetin — Assignment: Homework 01

1. Let F_1 and F_2 be two sigma-algebras on a sample space Ω. We know that $F_1 \cap F_2$ is a sigma-algebra on Ω Show that $F_1 \cup F_2$ is not necessarily a sigma-algebra on Ω.

   Solution. Consider an experiment where two coins are flipped at the same time. If a coin lands on heads, we will denote that with a 1 and tails with a 0. Then the sample space is defined as
$$\Omega = \{(0,0),(0,1),(1,0),(1,1)\}.$$
   Let $A = \{(0,0)\}$ and $B = \{(1,0)\}$ be two subsets of Ω. Then consider the two following σ-algebras
$$\begin{aligned}\mathcal{F}_1 &= \{\varnothing, A, A^c, \Omega\}\\ \mathcal{F}_2 &= \{\varnothing, B, B^c, \Omega\}\end{aligned}$$
   which happen to be the smallest σ-algebras containing A and B, respectively. The union of these two σ-algebras is
$$\begin{aligned}\mathcal{F}_1 \cup \mathcal{F}_2 &= \{\varnothing, A, B, A^c, B^c, \Omega\}\\ &= \Big\{\varnothing,\ \{(0,0)\},\ \{(1,0)\},\ \{(0,1),(1,0),(1,1)\},\ \{(0,0),(0,1),(1,1)\},\ \Omega\Big\}.\end{aligned}$$
   With this we can see that $A \cup B = \{(0,0),(1,0)\}$ is not an element of $\mathcal{F}_1 \cup \mathcal{F}_2$. Thus it is not true that $\mathcal{F}_1 \cup \mathcal{F}_2$ is closed under unions. Therefore, it is not a σ-algebra. ■

2. Let $\{A_k : k = 1,2,\ldots,n\}$ be finite collection of events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following inequality for any such finite collection of n events:
$$\mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big) \geq \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{1\leq j<k\leq n} \mathbb{P}(A_j \cap A_k).$$
   Proof. We proceed by induction on n. For our base case we let $n = 2$. Then
$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1 \cup (A_2 \setminus A_1))\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus A_1)\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2 \setminus (A_1 \cap A_2))\\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).\end{aligned}$$
   With equality we satisfy the claim for $n = 2$.

Now assume the claim is true for some $n \geq 2$. Let $A_{n+1} \in \mathcal{F}$ and let $A = \bigcup_{k=1}^{n} A_k$. Then

$$\begin{aligned}
\mathbb{P}\big(\bigcup_{k=1}^{n+1} A_k\big) &= \mathbb{P}\big(\bigcup_{k=1}^{n} A_k \cup A_{n+1}\big) \\
&= \mathbb{P}(A \cup A_{n+1}) \\
&= \mathbb{P}(A) + \mathbb{P}(A_{n+1}) - \mathbb{P}(A \cap A_{n+1}) \\
&= \mathbb{P}\big(\bigcup_{k=1}^{n} A_k\big) + \mathbb{P}(A_{n+1}) - \mathbb{P}\big(\bigcup_{k=1}^{n} A_k \cap A_{n+1}\big) \\
&\geq \sum_{k=1}^{n} \mathbb{P}(A_k) + \mathbb{P}(A_{n+1}) - \sum_{1 \leq j < k \leq n} \mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_{n+1}) \\
&\geq \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq j < k \leq n+1} \mathbb{P}(A_j \cap A_k).
\end{aligned}$$

Therefore the claim is true for $n + 1$ and so it is true for all n. □

3. Let A, B, and C be three events in a probaility space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Let D be the event that exactly one event in the set A, B occurs. Show that $\mathbb{P}(D) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

   Proof. We want to determine the probability of either A or B happening but not both. Thus we are looking for $\mathbb{P}(A \triangle B)$. Since $A \triangle B = (A \setminus B) \cup (B \setminus A)$, then we have

$$\begin{aligned}
\mathbb{P}(A \triangle B) &= \mathbb{P}((A \setminus B) \cup (B \setminus A)) \\
&= \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) \\
&= \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B \setminus (A \cap B)) \\
&= \big(\mathbb{P}(A) - \mathbb{P}(A \cap B)\big) + \big(\mathbb{P}(B) - \mathbb{P}(A \cap B)\big) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).
\end{aligned}$$

□

   (b) Let E be the event that at least one event in $\{A, B, C\}$ occurs. Describe $\mathbb{P}(E)$ in terms of the probabilities of the events in $\{A, B, C\}$ and/or their intersection probabilities.

   Solution. To approach this we can ask in what ways can at least one of these events happen. At least one of these events can happen in the following ways

$$\{A, A \cap B, A \cap C, B, B \cap C, C\}.$$

   Thus

$$E = A \cup (A \cap B) \cup (A \cap C) \cup B \cup (B \cap C) \cup C = A \cup B \cup C.$$

Then by the Inclusion-Exclusion Principle, we have

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

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