MATH 220A

Name: Quin Darcy Due Date: 2/19/21
Instructor: Dr.Martins Assignment: Homework 2

1. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

   Proof. Let $\mathfrak{J}$ denote the topology generated by $\mathcal{A}$, and let $\mathcal{T}$ be any topology on X which contains $\mathcal{A}$. Suppose $S \in \mathfrak{J}$. Then by Lemma 13.1, $S = \bigcup B_\alpha$ for $B_\alpha \in \mathcal{A}$. Furthermore, since $\mathcal{A} \subseteq \mathcal{T}$ and $\mathcal{T}$ is closed under unions, then $S \in \mathcal{T}$. Hence, S is in the intersection of all topologies containing $\mathcal{A}$. Thus, $\mathfrak{J} \subseteq \mathcal{T}$, which implies $\mathfrak{J} \subseteq \bigcap \mathcal{T}$.

   To obtain the other inclusion, we note that $\mathfrak{J}$ is a topology which contains $\mathcal{A}$ and so $\bigcap \mathcal{T} \subseteq \mathfrak{J}$. Thus proving the equality.

   If $\mathcal{A}$ is a subbasis and $\mathfrak{J}$ is the topology generated by $\mathcal{A}$. Then if $S \in \mathfrak{J}$, we can write S as a union of finite intersections of elements of $\mathcal{A}$. Thus, as $\mathcal{A} \subseteq \mathcal{T}$, then $S \in \mathcal{T}$ since $\mathcal{T}$ is closed under finite intersections. Hence, $\mathfrak{J} \subseteq \bigcap \mathcal{T}$.

   To prove the other inclusion, we can use the same argument as above. Namely, with $\mathfrak{J}$ being a topology which contains $\mathcal{A}$, it is then in the intersection of all topologies containing $\mathcal{A}$. Thus, $\bigcap T \subseteq \mathfrak{J}$. □

7. Consider the following topologies on $\mathbb{R}$.

   $\mathfrak{J}_1 =$ the standard topology.
   $\mathfrak{J}_3 =$ the finite complement topology.
   $\mathfrak{J}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis.
   $\mathfrak{J}_5 =$ the topology having all sets $(-\infty, a) = \{x \mid x < a\}$ as basis.

   Determine, for each of these topologies, which of the others it contains.

   Proof. □

8. (a) Apply Lemma 13.2 to show that the countable collection

   $$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational } \}$$

   is a basis that generates the standard topology on $\mathbb{R}$.

   Proof. □

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ rational }\}.$$

Proof. □

1. Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. □