MATH 230B

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1. Prove that if the series of functions $\displaystyle\sum_{k=1}^{\infty} f_k$ is uniformly convergent on an interval I, then the sequence of functions $\{f_k\}_k$ converges uniformly to the zero function on I.

   Proof. Assume that $\sum_{k=1}^{\infty} f_k$ is uniformly convergent. Then by Definition 7.7 (Rudin), the sequence of partial sums $\{s_n\}_n$, where

   $$s_n(x) = \sum_{k=1}^{n} f_k(x)$$

   converges uniformly to some $s : I \to \mathbb{R}$ on I. We note that this implies that for any $x \in I$, $\lim_{n\to\infty} f_n(x) = 0$ and so the sequence $\{f_n\}_n$ converges pointwise to $f = 0$ over I. We want to show that

   $$\lim_{n\to\infty} \sup_{x\in I} |f_n(x)| = 0.$$

   Letting $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have

   $$\sup_{x\in I} |s_n(x) - s(x)| = \sup_{x\in I} \left| \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{\infty} f_k(x) \right| = \sup_{x\in I} \left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \frac{\varepsilon}{2}. \tag{1}$$

   It follows from the triangle inequality and (1) that

   $$\begin{aligned} |f_{n+1}(x)| &= \left| f_{n+1}(x) + \sum_{k=n+2}^{\infty} f_k(x) - \sum_{k=n+2}^{\infty} f_k(x) \right| \\ &\leq \left| \sum_{k=n+1}^{\infty} f_k(x) \right| + \left| \sum_{k=n+2}^{\infty} f_k(x) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

   By (1) and the above inequality we then get that

   $$\sup_{x\in I} |f_n(x)| < \varepsilon.$$

   Therefore $\{f_n\}_n$ converges uniformly to $f = 0$ on I. $\square$

2. Discuss the pointwise/uniform convergence on $[0, 1]$ of

   $$\sum_{k=1}^{\infty} \frac{kx}{1 + k^3x^2}.$$

Solution. Letting $x \in (0,1]$, then we note that for any $k \in \mathbb{N}$

$$\frac{kx}{1+k^3x^2} = \frac{x}{\frac{1}{k}+k^2x^2} \le \frac{x}{k^2x^2} = \frac{1}{k^2x}.$$

Thus for any fixed x, the series

$$\sum_{k=1}^{\infty} \frac{nx}{1+n^3x^2}$$

converges by the Comparison Test with $\frac{1}{k^2x}$. Additionally, for $x=0$, then the partial sums $s_n(x) = 0$ for all $n \in \mathbb{N}$. The series therefore converges pointwise on $[0,1]$. To show that the series does not converge uniformly we will show that there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n > m \ge N$ and $x \in [0,1]$ such that

$$\left| \sum_{k=m}^{n} f_k(x) \right| \ge \varepsilon.$$

Letting $\varepsilon = 1/9$, $m \in \mathbb{N}$, $x = 1/m \in [0,1]$, and $n = 2m$, then we have that

$$\begin{aligned}
\sum_{k=m}^{n} \frac{kx}{1+k^3x^2} &= \sum_{k=m}^{2m} \frac{k(\frac{1}{m})}{1+\frac{k^3}{m^2}} \\
&= \sum_{k=m}^{2m} \frac{mk}{m^2+k^3} \\
&\ge \sum_{k=m}^{2m} \frac{m \cdot m}{m^2+(2m)^3} \\
&= \sum_{k=m}^{2m} \frac{m^2}{m^2+8m^3} \\
&= \frac{m^2(2m-m)}{m^2+8m^3} \\
&\ge \frac{m^3}{m^3+8m^3} \\
&= \frac{m^3}{9m^3} \\
&= \frac{1}{9} = \varepsilon.
\end{aligned}$$

Therefore the series is not uniformly convergent on $[0,1]$. ■

3. Prove that the series

$$\sum_{k=1}^{\infty} \frac{kx}{1+k^4x^2}$$

is uniformly convergent on $[a,\infty)$, for all $a > 0$, but is not uniformly convergent on $[0,\infty]$.

Proof. Letting $x \in [a, \infty)$, then we have that for any $n \in \mathbb{N}$

$$\frac{nx}{1+n^4x^2} \leq \frac{nx}{n^4x^2} = \frac{1}{n^3x}.$$

This implies that

$$\sup_{x\in[a,\infty)} \left|\frac{nx}{1+n^4x^2}\right| \leq \sup_{x\in[a,\infty)} \left|\frac{1}{n^3x}\right| = \frac{1}{n^3a}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3a} < \infty$, then by the Comparison test

$$\sum_{k=1}^{\infty} \sup_{x\in[a,\infty)} \left|\frac{kx}{1+k^4x^2}\right| < \infty.$$

Therefore by the Weierstrass M-Test, the given series converges uniformly for over $[a, \infty)$ for any $a > 0$. $\square$