Master's Exam in Real Analysis
May 2018

Part 1: Problems 1-7 **Do six problems in Part 1**.

1. A *real number* x is called an algebraic number if x is the root of a polynomial with integer coefficients. Prove that the set of algebraic numbers is countable.

2. (a) Prove that any uncountable set $S \subset \mathbb{R}^n$ has a limit point.

   (b) Let $\mathbb{N}$ be the set of natural numbers. Prove that the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is uncountable.

3. Let (X, d_X), (Y, d_Y) be metric spaces and $f : X \to Y$ be continuous. Assume that X is connected, prove that $f(X)$ is connected.

4. Prove that compact subsets of metric spaces are closed and bounded. Is the converse true? Justify your answer.

5. Suppose that $x_n, y_n \geq 0$ for all n. Prove that

$$\limsup_{n\to\infty}(x_n y_n) \leq (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n),$$

   provided that the product on the right is not of the form $0 \cdot \infty$. Give an example for which the inequality is strict.

6. (a) Let $\{x_n\}$ be a real sequence. Suppose that there is an $a > 1$ such that

$$|x_{n+1} - x_n| \leq a^{-n},$$

   for all $n \in \mathbb{N}$. Prove that $x_n \to x$ for some $x \in \mathbb{R}$.

   (b) Let a and b be two distinct real numbers. Let $\{x_n\}$ be a sequence defined in the following:

$$x_0 = a, x_1 = b, x_n = \frac{x_{n-1} + x_{n-2}}{2} \text{ for } n \geq 2.$$

   Determine whether the sequence $\{x_n\}$ converges. Find the limit if it converges.

7. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

   (a) Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \ldots + na_n}{n} = 0.$$

   (b) Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Prove that $\sum_{n=1}^{\infty} a_n$ converges. Is the converse true. Justify your answer.

Part 2: Problems 8-14 **Do six problems in Part 2**

8. (a) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous with $f(0) = f(1)$. Prove that for each $n \in \mathbb{N}$, there exists an $x \in [0,1]$ such that
$$f(x) = f\Big(x + \frac{1}{n}\Big).$$
(b) Let $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b) with $|f'(x)| \le M$ for $x \in (a,b)$ and some $M \ge 0$. Prove that
$$\lim_{x \to b^-} f(x),$$
exists.

9. Let f be defined as follows:
$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & : x \neq 0 \\ 0 & : x = 0. \end{cases}$$
Prove that f is infinitely differentiable at 0 with $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

10. Let f be a bounded function on $[-1,1]$ and let
$$\alpha(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 2, & \text{if } 0 < x. \end{cases}$$
Prove that $f \in \mathcal{R}(\alpha)$ on $[-1,1]$ if and only if $f(0+) = f(0)$. Compute the integral $\displaystyle\int_{-1}^{1} f\, \mathrm{d}\alpha$ when $f \in \mathcal{R}(\alpha)$.

11. (a) Suppose that f is a continuous, and nonnegative function on the interval $[0,1]$. Let $M = \sup\{f(x) : x \in [0,1]\}$. Prove that
$$\lim_{n\to\infty} \left[\int_0^1 f(t)^n dt\right]^{\frac{1}{n}} = M.$$
(b) Let f be continuous on $[a,b]$ such that
$$\int_a^x f(t)\mathrm{d}t = \int_x^b f(t)\mathrm{d}t, \forall x \in [a,b].$$
Prove that $f(x) = 0$ on $[a,b]$.

12. Suppose that $\{f_k\}$ is a sequence of continuous functions on $[a,b]$. Suppose also that $f_k \to f(x)$ pointwise on $[a,b]$ where f is a continuous function and for each $x \in [a,b]$ the sequence $f_n(x) \le f_{n+1}(x)$, $n = 1, 2, 3, \ldots$. Prove **directly** that $f_n \to f$ uniformly on $[a,b]$.

13. Suppose that $f_n(x)$ is differentiable on $[a,b]$ for $n \ge 1$ with $|f_n'(x)| \le M$, $x \in [a.b]$ and some $M > 0$. If $f_n(x)$ converges to $f(x)$ pointwise on $[a,b]$, prove that $f_n(x)$ converges to $f(x)$ uniformly on $[a,b]$.

14. Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{kx}{1 + k^4x^2}$ converges uniformly on $[a,\infty)$ for $a > 0$, but does not converge uniformly on $(0,\infty)$.