MATH 220A

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Instructor: Dr. Martins — Assignment: Homework 3

6. Let A, B and A_α denote subsets of a space X. Prove the following:

   (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

   Proof. Let $x \in \overline{A}$. Then either $x \in A$ or $x \in A'$. If $x \in A$, then as $A \subset B$, it follows that $x \in B \subset \overline{B}$. If instead, $x \in A'$, then since x is a limit point of A, we have for every neighborhood, U, of x, there exists $a \in U$ such that $a \neq x$ and $a \in A$. Since $A \subset B$, then it follows that for every neighborhood of x, there exists $a \in U$ such that $a \neq x$ and $a \in B$. Hence, x is a limit point of B. Thus, $x \in B' \subset \overline{B}$. Therefore, $\overline{A} \subset \overline{B}$. □

   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   Proof. Let $x \in \overline{A \cup B}$. Then either $x \in A \cup B$ or $x \in (A \cup B)'$. In the former case, it follows from $A \subset \overline{A}$ and $B \subset \overline{B}$, that $x \in A \cup B \subset \overline{A} \cup \overline{B}$. If $x \in (A \cup B)'$, then for any neighborhood, U, of x, there exits $y \in U$ such that $y \neq x$ and $y \in A \cup B$. This implies that for every neighborhood, U, of x, there exists $y \in U$ such that $y \neq x$ and $y \in A$ or $y \in B$. Thus x is a limit point of either A or B. Hence, $x \in A' \cup B' \subset \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

   Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then either $x \in A$ or $x \in A'$. If $x \in A$, then $x \in A \cup B \subset \overline{A \cup B}$. If $x \in A'$, then for every neighborhood, U, of x, there exists $y \in U$ such that $y \neq x$ and $y \in A \subset A \cup B$. Hence, x is a limit point of $A \cup B$ and so $x \in (A \cup B)' \subset \overline{A \cup B}$. An identical argument can be used if we take $x \in \overline{B}$. Hence, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

   (c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A}_\alpha$; give an example where equality fails.

   Proof. Let $x \in \bigcup \overline{A}_\alpha$. Then for some particular α, we have that $x \in \overline{A}_\alpha$. This implies that either $x \in A_\alpha$ or $x \in A'_\alpha$. If $x \in A_\alpha$, then $x \in \bigcup A_\alpha \subset \overline{\bigcup A_\alpha}$. If $x \in A'_\alpha$, then for any neighborhood, U, of x, there exists $a \in U$ such that $a \neq x$ and $a \in A_\alpha \subset \bigcup A_\alpha$. This implies that x is a limit point of $\bigcup A_\alpha$. Hence, $x \in (\bigcup A_\alpha)' \subset \overline{\bigcup A_\alpha}$. Therefore $\bigcup \overline{A}_\alpha \subset \overline{\bigcup A_\alpha}$.

   For an example where equality fails, consider the following: Take $q \in \mathbb{Q}$ and let $U_q = \{q\}$. Then as any finite set is closed, it follows that $\overline{U_q} = U_q$ and thus

$$\bigcup_{q \in \mathbb{Q}} \overline{U}_q = \mathbb{Q},$$

   whereas

$$\overline{\bigcup_{q \in \mathbb{Q}} U_q} = \overline{\mathbb{Q}} = \mathbb{R}.$$

□

13. Show that X is Hausdorff if and only if the ***diagonal*** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Proof. Assume that X is Hausdorff. Then we want to show that Δ contains all of its limit points. So then let $a \times b$ be a limit point of Δ and assume that $a \neq b$. Then since X is Hausdorff, there exists neighborhoods U_1, U_2 such that $U_1 \cap U_2 = \varnothing$ and $a \in U_1$ and $b \in U_2$. As open sets, we have that $U_1 \times U_2$ is itself an open set containing $a \times b$. Moreover, since $a \times b$ is a limit point of Δ, then there exists some $x \times x \in U_1 \times U_2$ such that $x \times x \in \Delta$. However, this implies that $x \in U_1 \cap U_2$, which is a contradiction. Therefore, $a = b$ and $a \times b \in \Delta$. Thus Δ contains all of its limit points.

Assume that Δ is closed in $X \times X$. Consider $a, b \in X$ such that $a \neq b$. Since Δ is closed, then its complement is open and $a \times b$ is an element of the complement. Thus there exists U_1 and U_2 such that $a \times b \in U_1 \times U_2$ and $U_1 \times U_2 \subset \Delta^c$. Suppose that $U_1 \cap U_2 \neq \varnothing$, then there exists $u \times u \in U_1 \times U_2$. However, $u \times u \in \Delta$ and so $\Delta \cap \Delta^c \neq \varnothing$, which is a contradiction. Therefore, $U_1 \cap U_2 = \varnothing$ and thus X is Hausdorff. □

19. If $A \subset X$, we define the ***boundary*** of A by the equation

$$\mathrm{Bd}A = \overline{A} \cap (\overline{X - A}).$$

(a) Show that $\mathrm{Int}A$ and $\mathrm{Bd}A$ are disjoint, and $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$.

Proof. By definition, we may write

$$\mathrm{Int}A = \bigcup_{U \subset A} U, \quad U \text{ is open in } A.$$

For contradiction, assume that $x \in \mathrm{Int}A \cap \mathrm{Bd}A$. Then for some set, U, open in A, we have that $x \in U$ and so $x \in A$. Additionally, we have that $x \in \overline{A}$ and $x \in (\overline{X - A})$. The first implying the either $x \in A$ or $x \in A'$. However, we know that $x \in U \subset A$. Moreover, with $x \in (\overline{X - A})$, we get that either $x \in X - A$ or $x \in (X - A)'$. As $x \in X - A$ is not possible, then we conclude that $x \in (X - A)'$. In summary, we have shown that x is contained in an open set, U, of A and that x is a limit point of $X - A$. Being an element U implies that there exists a neighborhood, N of x such that $N \subset U$. Also, being a limit point of $X - A$, we have that for any neighborhood, say N, there exists a point $y \in N$ such that $x \neq y$ and $y \in X - A$. However, this would imply that $N \not\subset U$, which is a contradiction. Therefore, $\mathrm{Int}A \cap \mathrm{Bd}A = \varnothing$.

Now assume that $x \in \overline{A}$. Then either $x \in A$ or $x \in A'$. If $x \in A$, then there are two possibilities: x is an isolated point, i.e., it is an element of A but not contained in any neighborhood, or there is some neighborhood $U \subset A$ which contains x. The latter implies that x is in the interior of A and we are done. Otherwise, if x is an isolated point, then for every neighborhood U containing x, there exists $y \in U$ such that $y \neq x$ and $y \in X - A$. Thus $x \in (\overline{X - A})'$. Hence, $x \in \mathrm{Bd}A$. In short, if $x \in A$, then $x \in \mathrm{Int}A \cup \mathrm{Bd}A$.

If $x \in A'$ and there exists $U \subset A$ such that $x \in U$, then $x \in \text{Int}A$. Otherwise, if for every neighborhood, U, of x, we have that $U \not\subset A$ and $U \cap A \neq \varnothing$, t hen as before, every neighborhood of x contains points not in A and thus making x a limit point of $X - A$. Hence, if $x \in A'$, then $x \in \text{Int}A \cup \text{Bd}A$. And so $\overline{A} \subset \text{Int}A \cup \text{Bd}A$.

Assume that $x \in \text{Int}A \cup \text{Bd}A$. If $x \in \text{Int}A$, then $x \in U \subset A \subset \overline{A}$, for some open set U. If $x \in \text{Bd}A$, then by definition, $x \in \overline{A}$. Therefore, $\text{Int}A \cup \text{Bd}A \subset \overline{A}$. □

(b) Show that $\text{Bd}A = \varnothing \Leftrightarrow A$ is both open and closed.

Proof. Assume that $\text{Bd}A = \varnothing$, then every point of A is an interior point. Meaning, for every $x \in A$, there exists an open set $U \subset A$ such that $x \in U$. Thus A is open. On the other hand, if x is a limit point of A, then $x \in \overline{A}$, and for any neighborhood, U, of x we have that if there exists $y \in U$ such that $y \notin A$, then that implies x is a limit point of $X - A$, which would imply that $x \in \overline{X - A}$ and hence $x \in \overline{A} \cap (\overline{X - A})$, a contradiction. Therefore, A contains all of its limit points and A is thereby closed.

Assume that A is both open and closed. Since A is open, then for every point, x, in A, there exists a neighborhood, U, of x such that $U \subset A$. Hence $A = \text{Int}A$, and since, by (a), the interior and boundary are disjoint, then it follows that $\text{Bd}A = \varnothing$. □

(c) Show that U is open $\Leftrightarrow \text{Bd}U = \overline{U} - U$.

Proof. Assume that U is open. Then $X - U$ is closed. Thus $\overline{X - U} = X - U$ and so $\text{Bd}U = \overline{U} \cap (X - U)$. Then if $x \in \text{Bd}U$, we have that $x \in \overline{U}$ and $x \notin U$. Hence, $x \in \overline{U} - U$.

Now assume that $\text{Bd}U = \overline{U} - U$ and let $x \in U$. Then x cannot be an isolated point since that would imply that $x \in \text{Bd}U$ which implies $x \notin U$. Thus x is either an interior point or a limit point. If x is an interior point, then we are done. If x is a limit point, then $x \in U' = \overline{U} - U$ and this implies $x \notin U$. Hence, every point of U is an interior point. Therefore, U is open. □

(d) If U is open, is it true that $U = \text{Int}(\overline{U})$? Justify your answer.

Proof. This is not true. Suppose A is open. Then $A \subset \text{Int}A$, since the interior is the union of all open sets containing A. By definition of interior, we also have that $\text{Int}A \subset A$. Thus if A is open, then $A = \text{Int}A$. Now the question is, if A is open does $\text{Int}A = \text{Int}(\overline{A})$. For a counter example, consider the set $A = (-1, 0) \cup (0, 1)$. We have that A is open and that $\text{Int}A = A$. However, $\overline{A} = [-1, 1]$ and so $\text{Int}(\overline{A}) = (-1, 1)$. Thus the two sets are not equal. □

1. Find the boundary and interior of each of the following subsets of $\mathbb{R}^2$.

(a) $A = \{x \times y \mid y = 0\}$.

Proof. We claim that $\text{Int}A = \varnothing$ and that $\text{Bd}A = A$. To show this we need to compute $\overline{A}$ and $\overline{\mathbb{R}^2 - A}$. Consider $\mathbb{R}^2 - A = A^c = \{x \times y \mid y \neq 0\}$. Then for any $x \times y \in A^c$, if we let $r = |y|/2$, then the neighborhood with radius r, call it U_r contains $x \times y$ and $U_r \subset A^c$. Hence, A^c is open, which implies A is closed. Hence, $\overline{A} = A$. Moreover, for any $x \times 0 \in A$, we have that $\lim_{n\to\infty} x \times \frac{1}{n} = x \times 0$. Hence, every point of A is a limit point of $\mathbb{R}^2 - A$. Hence, for any $x \times 0 \in A$, we have that $x \times 0 \in \overline{A}$ and $x \times 0 \in \overline{\mathbb{R}^2 - A}$. Thus every point of A is a boundary point which implies $\text{Bd}A = A$ and $\text{Int}A = \varnothing$. □

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$.

Proof. Similar to the process before, we will first compute $\overline{B}$. That is we need to find the limit points of B. Suppose that $u \times v$ is a limit point of B such that $u \times v \notin \text{Int}B$. Then for every $r > 0$, the neighborhood U_r around $u \times v$, with radius r, contains a point $x \times y \in B$. Observe that if $u < 0$, then $u \times v$ could not be a limit point of B. Hence, $B' = \{x \times y \mid x \geq 0\}$ and so $\overline{B} = \{x \times y \mid x \geq 0\}$. Next, we observe that for any $x \times y \in B$, letting $r = \min\{|x|, |y|\}/2$, then $U_r(x \times y) \subset B$. Hence, B is open in $\mathbb{R}^2$ and so $\overline{\mathbb{R}^2 - B} = \mathbb{R}^2 - B$. This implies that

$$\text{Bd}B = \overline{B} \cap (\mathbb{R}^2 - B) = \overline{B} - B$$

which consists of the points formed from the positive x-axis and the positive and negative y-axis. □

(c) $C = A \cup B$.

Proof. The set C contains the entire right side of the cartesian plan, plus the points on the negative x-axis, and does not contain the points on the y-axis. This implies that $\overline{C} = \{x \times y \mid x \geq 0 \text{ or } y = 0\}$. We also have that $\mathbb{R}^2 - C = \{x \times y \mid x < 0 \text{ and } y \neq 0\}$. Thus $\overline{\mathbb{R}^2 - C} = \{x \times y \mid x \leq 0\}$. Thus the boundary of C is the set of points along the y-axis and the negative x-axis. □

(d) $D = \{x \times y \mid x \text{ is rational}\}$.

Proof. Like before, we want to first compute the closure of D. As $\overline{\mathbb{Q}} = \mathbb{R}$ then $\overline{D} = \mathbb{R}^2$. Then as for the closure of the complement, we first observe that every irrational number can be expressed as the limit of a sequence of rational numbers and vice versa, therefore $\overline{\mathbb{R}^2 - D} = \mathbb{R}^2$. Hence, $\text{Bd}D = \mathbb{R}^2 \cap \mathbb{R}^2 = \mathbb{R}^2$. □

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$.

Proof. □

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$.

Proof. □