STAT 215A

Name: Quin Darcy
Instructor: Dr. Cetin

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Assignment: Homework 03

1. Consider the experiment of rolling a balanced six-sided die repeatedly until a six occurs for the first time. Let Y represent the variable for the number of the rolls before the first six appears. So, the possible values of Y are $0, 1, 2, \dots,$ (Y is unbounded).

   (a) Determine the pmf of Y.

   Solution. We begin by noting that in this experiment

$$\Omega = \{6, 16, 26, 36, 46, 56, 116, \dots\}.$$

   And so Y is a discrete random variable $Y : \Omega \to \mathbb{R}$ with range$(Y) = \{1, 2, \dots\}$. Then the pmf of Y is defined as

$$p_Y(k) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = k\}).$$

   This means that for any $k \in \mathbb{N}$, $p_Y(k)$ is the probability that we roll a die $k-1$ times in a row and never get a 6, but on the kth time we do roll a 6. Observe that to roll a die once and not obtain a 6 has a probability $\frac{5}{6}$ and to roll a die and obtain a 6 has probability $\frac{1}{6}$. Since each dice roll is independent from the previous roll, then it follows that the probability of rolling a non-six value $k-1$ times in a row, followed by a 6 on the kth roll has probability

$$p_Y(k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) = \frac{5^{k-1}}{6^k}.$$

   Note that for all $k \in \{1, 2, \dots,\}$, $0 \leq p_Y(k) \leq 1$, and

$$\sum_{k=1}^{\infty} p_Y(k) = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = 1.$$

   This shows that $p_Y(k)$ satisfies both conditions of a pmf. ■

   (b) Compute $P(Y > 1 + k | Y > 1)$, where $k \in \mathbb{N}$.

   Solution. For any $k \in \mathbb{N}$, let A denote the event that $Y > 1 + k$, and let B denote the event that $Y > 1$. Seeing as k starts at 1, then clearly $A \subseteq B$. Hence $A \cap B = A$, and so

$$\mathbb{P}(Y > 1 + k \mid Y > 1) = \frac{\mathbb{A} \cap \mathbb{B}}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

To determine $\mathbb{P}(A)$, we first want to compute the probability of A^c which is the event that we get a 6 in the first $1+k$ rolls.

$$\mathbb{P}(Y \leq 1+k) = \sum_{i=1}^{1+k} p_Y(i) = \sum_{i=1}^{1+k} \frac{5^{i-1}}{6^i} = 1 - \left(\frac{6}{5}\right)^{-k-1} = 1 - \left(\frac{5}{6}\right)^{k+1}.$$

Hence

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = \left(\frac{5}{6}\right)^{k+1}.$$

Similarly, $\mathbb{P}(B) = 1 - \mathbb{P}(B^c)$. And B^c is the event that $Y \leq 1$. Seeing as Y can be 1 at the smallest, then this is equivalent to $\mathbb{P}(Y = 1) = \frac{1}{6}$. Thus $\mathbb{P}(B) = 1 - \frac{1}{6} = \frac{5}{6}$. Therefore

$$\mathbb{P}(Y > 1+k \mid Y > 1) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\left(\frac{5}{6}\right)^{k+1}}{\left(\frac{5}{6}\right)} = \left(\frac{5}{6}\right)^{k}.$$

■

2. Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $P(X \geq 0) = 1$. Moreover, suppose that $Y = 1 - e^{-2X}$.

   (a) Explain why Y is also a r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$, and then describe the cdf of Y in terms of the cdf of X.

   Solution. ■

   (b) If $X \sim Exp(2)$, then determine the pdf of Y.

   Solution. ■