COMPREHENSIVE EXAM
ALGEBRA
[Spring/Fall] [Year]

Part I: Group Theory (Do 4 of the following 5 problems)

1. Assuming Cauchy's Theorem for all finite abelian groups, prove Cauchy's Theorem for all finite groups.

2. Let G be a finite group.

   (a) If $[G : Z(G)] = n$ where n is a positive, show that every conjugacy class has at most n elements.

   (b) Suppose that the size of each conjugacy class in G is at most 2. Show that for all $g \in G$, the centralizer $C_G(g)$ is a normal subgroup of G.

3. Let G be a finite group and suppose that H is a maximal subgroup of G. In other words, if K is any subgroup of G with $H \subseteq K \subseteq G$, then either $H = K$ or $K = G$. Assuming H is normal subgroup of G, prove that the index $[G : H]$ must be prime.

4. (a) If G is a cyclic group, prove that every subgroup of G is also cyclic.

   (b) Suppose $g \in G$ and $\mathrm{ord}(g) = n$. Given a positive integer m, if $d = \gcd(n, m)$, prove that

$$\mathrm{ord}(a^m) = \mathrm{ord}(a^d).$$

5. Let G be a group of order $182 = 2 \cdot 7 \cdot 13$.

   (a) Show that G has a normal 13-Sylow subgroup.

   (b) Show that G has a cyclic subgroup H with $|H| = 91$.

   (c) List two non-isomorphic groups of order 182. You do not have to prove that the two groups you list are not isomorphic.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let R be a commutative ring with unity, and let P be a prime ideal in R.

   (a) If I and J are ideals in R with $I \cap J \subseteq P$, show that one of I or J is a subset of P.

   (b) If R is finite, then explain why R/P is a field.

2. Let E be a field and let F be a subfield of E. Let c and d be elements of E, both algebraic over F, where the minimal polynomial of c has degree n, and the minimal polynomial of d has degree m.

   (a) Prove that the set $\{1, c, c^2, \ldots, c^{n-1}\}$ is linearly independent over F.

   (b) Suppose m and n are relatively prime. Determine, with proof, the degree of the extension $F(c, d)$ over F.

3. For each prime p and each positive integer n, show that there is a field with p^n elements by constructing a splitting field for a suitable polynomial $f(x)$ in $\mathbb{Z}_p[x]$.

4. Let ζ be a primitive 7-th root of unity.

   (a) Determine, with proof, the degree of the extension $[\mathbb{Q}(\zeta) : \mathbb{Q}]$.

   (b) Determine the number of intermediate fields L with $\mathbb{Q} \subsetneq L \subsetneq \mathbb{Q}(\zeta)$.

   (c) List all fields L from part (b) and determine their degree over $\mathbb{Q}$.

5. (a) Give the splitting field K for $f(x) = x^3 - 5$ over $\mathbb{Q}$.

   (b) Explain why $K/\mathbb{Q}$ is a Galois extension.

   (c) Determine, with proof, the Galois group of K over $\mathbb{Q}$. We will denote this group by G.

   (d) Let σ be an element of G where σ has order 3. Determine the fixed field of $\langle \sigma \rangle$.