Topology - Homework Assignment 2

Question 1 Let $[a, b] \subset \mathbb{R}$ be a closed interval and consider the following set of functions:

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f(x) \text{ is continuous on } [a,b]\}$$

We define the L^1-norm of $f(x) \in C([a,b])$ by:

$$\|f(x)\|_{L^1} = \int_a^b |f(x)|dx$$

and the L^1-distance between two functions by:

$$d_{L^1}\big(f(x), g(x)\big) = \|f(x) - g(x)\|_{L^1} = \int_a^b |f(x) - g(x)|dx$$

Using the properties of the Riemann integral, show that d_{L^1} is a metric on the space $C([a,b])$. (The topology induced by this metric is called the L^1-topology).

Question 2 Let X be a set and d_1 and d_2 be two metrics on X. Furthermore suppose that there are constants $A_{12}, A_{21} > 0$ such that:

$$\begin{aligned} d_1(x,y) &\leq A_{12} d_2(x,y) \quad \text{for all } x, y \in X \\ d_2(x,y) &\leq A_{21} d_1(x,y) \quad \text{for all } x, y \in X \end{aligned}$$

Show that the metric topology induced by d_1 is the same as the metric topology induced by d_1.

Question 3 Consider the following norm on $\mathbb{R}^n$:

$$\|\vec{x}\|_{\max} = \max_{1 \leq i \leq n} (|x_i|)$$

Define $\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_{\max}$. Show that ρ is a metric on $\mathbb{R}^n$ and show that the metric topology induced by ρ coincides with the standard topology of $\mathbb{R}^n$.

Question 4 (Extra Credit) Let $[a, b] \subset \mathbb{R}$ be a closed interval and consider the following set of functions:

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f(x) \text{ is continuous on } [a,b]\}$$

we define the L^2-inner product $\langle \cdot, \cdot \rangle_{L^2} : C([a,b]) \times C([a,b]) \to \mathbb{R}$ by:

$$\langle f(x), g(x) \rangle_{L^2} = \int_a^b f(x)g(x)dx$$

the L^2-norm is defined by:

$$\|f(x)\|_{L^2} = \sqrt{\langle f(x), f(x) \rangle_{L^2}} = \sqrt{\int_a^b f(x)^2 dx}$$

and the L^2-distance between two functions by:

$$d_{L^2}\big(f(x), g(x)\big) = \|f(x) - g(x)\|_{L^2} = \sqrt{\int_a^b |f(x) - g(x)|^2 dx}$$

Using the properties of the Riemann integral, show that $\langle \cdot, \cdot \rangle_{L^2}$ is an **inner product** on the vector space $C([a,b])$. Then show that d_{L^2} is a metric on the space $C([a,b])$. (The topology induced by this metric is called the L^2-topology).

Relevant properties of the Riemann integral:

Continuity and integrability: If $f(x)$ is a continuous function on the interval $[a,b] \subset \mathbb{R}$, then $f(x)$ is Riemann integrable on $[a,b]$.

Vanishing integral: If $f(x)$ is a continuous function on the interval $[a,b] \subset \mathbb{R}$ such that $f(x) \geq 0$ for all x in $[a,b]$, then:

$$\int_a^b f(x)dx = 0 \Rightarrow f(x) = 0 \text{ for all } x \in [a,b]$$

Linearity: If $f(x)$ and $g(x)$ are two functions which are Riemann integrable on the interval $[a,b] \subset \mathbb{R}$ then for any numbers $\lambda, \mu \in \mathbb{R}$ the function $\lambda f(x) + \mu g(x)$ is Riemann integrable on $[a,b]$ and

$$\int_a^b \lambda f(x) + \mu g(x) dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$$

Monotonicity: If $f(x)$ and $g(x)$ are two functions which are Riemann integrable on the interval $[a,b] \subset \mathbb{R}$ such that $f(x) \leq g(x)$ for all x in $[a,b]$ then:

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Definition of an inner-product

Let V be a real vector space. A function $B : V \times V \to \mathbb{R}$ is an **inner product** on V if it satisfies the following conditions:

(i) Bilinearity For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in V and real numbers λ and μ:

$$B(\lambda\vec{u} + \mu\vec{v}, \vec{w}) = \lambda B(\vec{u}, \vec{w}) + \mu B(\vec{v}, \vec{w})$$
$$B(\vec{u}, \lambda\vec{v} + \mu\vec{w}) = \lambda B(\vec{u}, \vec{v}) + \mu B(\vec{u}, \vec{w})$$

(ii) Symmetry For any vectors $\vec{u}$ and $\vec{v}$ in V

$$B(\vec{u}, \vec{v}) = B(\vec{v}, \vec{u})$$

(iii) Positive definiteness For any vector $\vec{v}$ in V

$$B(\vec{v}, \vec{v}) \geq 0$$

furthermore $B(\vec{v}, \vec{v}) = 0 \Rightarrow \vec{v} = \vec{0}$.