COMPREHENSIVE EXAM
ALGEBRA
Spring 2018

Part I: Group Theory (Do 4 of the following 5 problems)

1. (a) Let G be a group and let $H \triangleleft G$. Prove that G/H is cyclic if and only if there is an element $a \in G$ with the property that for every element $x \in G$, there is an integer n such that $xa^n \in H$.
   (b) Prove that every element of the group $\mathbb{Q}/\mathbb{Z}$ has finite order.
   (c) Prove that $\mathbb{Q}/\mathbb{Z}$ is not a cyclic group.

2. (a) Let G be a group with $|G| = 520 = 2^3 \cdot 5 \cdot 13$. Prove that G is not simple.
   (b) Let G be a group with $|G| = 36 = 2^2 \cdot 3^2$. Prove that G is not simple.

3. Note: $675 = 3^3 \cdot 5^2$.
   (a) Up to isomorphism, describe all Abelian groups of order 675.
   (b) Consider $G = \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$.
      i. Determine, with explanation, the number of elements of order 15 in G.
      ii. Determine, with explanation, the number of elements of order 45 in G

4. Let G be a finite group such that $p||G|$. Let H be a p-Sylow subgroup of G and let ϕ be an automorphism on G.
   (a) Prove that $\phi(H)$ is a p-Sylow subgroup of G.
   (b) Prove that if $H \triangleleft G$, then $\phi(H) = H$.

5. (a) In S_7, find two elements σ and τ of order 6 that are not conjugates.
   (b) Determine, with explanation, the number of conjugates of σ and the number of conjugates of τ.
   (c) Determine, with explanation the centralizer of σ and the centralizer of τ.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let R be a principal ideal domain. Let $I = (a)$ be a nonzero ideal in R. Prove that the following are equivalent.
   (a) I is a maximal ideal,
   (b) I is a prime ideal,
   (c) a is an irreducible element of R.

2. Let $f(x) = x^6 - 3 \in \mathbb{Q}[x]$ and let E be the splitting field for $f(x)$ over $\mathbb{Q}$.
   (a) Prove $E = \mathbb{Q}(\sqrt[6]{3}, i)$.
   (b) Determine, with explanation, $|\mathrm{Gal}\,(f(x)/\mathbb{Q})\,|$.
   (c) Let σ be the automorphism on E which sends $\sqrt[6]{3}$ to $\omega\sqrt[6]{3}$ (where ω is a primitive 6th root of unity) and sends i to i. Determine, with explanation, the permutation on the roots of $f(x)$ determined by σ.

3. Let p be prime.
   (a) Prove that, if F is a finite field of characteristic p, then $|F| = p^n$ for some $n \geq 1$.
   (b) Let $n \geq 1$. Prove that any field of order p^n is a splitting field for $x^{p^n} - x \in \mathbb{Z}_p[x]$.

4. (a) Find a basis for $\mathbb{Q}[x]/(x^2 + x + 1)$ as a vector space over $\mathbb{Q}$.
   (b) Find a multiplicative inverse for $x + (x^2 + x + 1) \in \mathbb{Q}[x]/(x^2 + x + 1)$.
   (c) Show that $x + (x^3 + x^2 + x) \in \mathbb{Q}[x]/(x^3 + x^2 + x)$ does not have a multiplicative inverse.

5. Let I and J be ideals in a ring R. Define $I + J = \{a + b : a \in I, b \in J\}$
   (a) Prove that $I + J$ is an ideal in R.
   (b) Prove that if $I + J = R$ and $I \cap J = 0$, then R/I is isomorphic to J.