The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and its eigenvalues

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1 Definition and remarks

The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is defined to be the set of all $n \times n$ matrices with trace 0 and whose entries are complex numbers. As the title of this paper suggests, this set is apparently a "Lie algebra". Before continuing any further, we will first define what this is, and prove that, indeed, $\mathfrak{sl}_n(\mathbb{C})$ is a Lie algebra.

Definition 1.1. A Lie algebra is a vector space L over some field F equipped with a binary operation $[\cdot, \cdot] : L \times L \to L$ such that for all $x, y, z \in L$, the following conditions hold:

(i) $[\cdot, \cdot]$ is a bilinear map.

(ii) $[x, x] = 0$

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$

With the above definition in hand, we claim that defining $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{sl}_n(\mathbb{C})$, then this will satisfy the above three conditions.