MATH 220A

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Assignment: Homework 5

23.2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

Proof. Assume for contradiction, that $\bigcup A_n$ is not connected. Then there exists nonempty open subsets $U, V \subset \bigcup A_n$ such that $U \cap V = \varnothing$ and $U \cup V = \bigcup A_n$. By Lemma 23.2, for each n, A_n is entirely contained in either U or V. Moreover, as $\bigcup A_n$ is a sequence of subspaces, it follows that for some m, $A_m \subset U$, but $A_{m+1} \not\subset U$, and instead, $A_{m+1} \in V$. However, since $A_m \cap A_{m+1} \neq \varnothing$, then $A_m \cap A_{m+1} \subset U \cap V \neq \varnothing$, which is a contradiction. □

23.6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersected $\text{Bd}A$.

Proof. (Two contradiction proofs in a row, don't hate me!) Assume that C does not intersect the boundary of A. Then since $\overline{A} = \text{Int}A \cup \text{Bd}A$, and C intersects A, it follows that $C \cap \text{Int}A \neq \varnothing$. By similar reasoning, we have that $C \cap \text{Int}(X - A) \neq \varnothing$. Since $\text{Int}A \cap \text{Int}(X - A) = \varnothing$ and $\text{Int}A \cup \text{Int}(X - A) = C$, then the two previous sets are a seperation of C and therefore C is not connected. □

24.1

(a) Show that no two of the spaces $(0, 1)$, $(0, 1]$, $[0, 1]$ are homeomorphic.

Proof. By Corollary 24.2, each of the above intervals are connected in $\mathbb{R}$. However, note that if $a \in (0, 1)$ and we let $A = (0, 1)/\{a\}$, then A is not connected since we can let $U = (0, a)$ and $V = (a, 1)$ and this would constitute a serperation. On the other hand, if we let $B = (0, 1]/\{1\} = (0, 1)$, then B is connected. Further note that if $(0, 1)$ was homeomorphic to $(0, 1]$, then A would be homeomorphic to B, however this is not the case as A is not connected but B is. Therefore, $(0, 1)$ is not homeomorphic to $(0, 1]$. By the same reasoning, we can show that $(0, 1)$ is not homeomorphic to $[0, 1)$. Now note that if $a \in [0, 1]$ such that $0 < a < 1$, then $[0, 1]/\{a\}$ is not connected since we can let $U = [0, a)$ and $V = (a, 1]$ and this forms a seperation on $[0, 1]/\{a\}$. However, by removing either 0 or 1 from $[0, 1]$ we preserve its connectedness. Though doing so results in either $(0, 1]$ or $[0, 1)$, both of which we have shown not to be homeomorphic to (0, 1). □

(b) Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

Proof. By Example 4 in section 24, we have that the punctured euclidean space $\mathbb{R}^n - \{0\}$ is path connected and therefore connected, for $n > 1$. However, $\mathbb{R} - \{0\}$ is not connected since $U = (-\infty, 0)$ and $V = (0, \infty)$ forms a seperation on the set. Therefore, for $n > 1$, $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic. $\square$

24.3 Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. What happens if X equals $[0, 1)$ or $(0, 1)$.

Proof. Consider the funtion $h(x) = f(x) - id(x)$, where $id : X \to X$ is the identity map on X. Since both f and id are continuous on X, then h is continuous on X. Note that if either $f(0) = 0$ or $f(1) = 1$, then this proves the claim. On the other hand, if $f(0) \neq 0$ and $f(1) \neq 1$, then considering the range of f, this would imply that $f(0) > 0$ and $f(1) < 1$. Now given that h is continuous on X, X is connected, and X is an ordered set in the order topology, then we may apply the Intermediate Value Theorem. Namely, we have that $h(1) < 0 < h(0)$ and thus by IVP, there exists some $x \in X$ such that $h(x) = 0$. This implies that $f(x) - id(x) = f(x) - x = 0$. Hence, $f(x) = x$.

If $X = [0, 1)$ or $X = (0, 1)$, then we want to show that there does not exist a fixed point. To do this, we need a counterexample consisting of a function continuous on X, but such that there is no $x \in X$ for which $f(x) = x$. Letting $f(x) = (x + 1)/2$, we see that f is continuous as a scalar multiple of a continuous function. Now if we assume that $f(x) = x$, for some $x \in X$, then we have that $(x + 1)/2 = x$, which implies that $x = 1$. However, this cannot be as $1 \notin X$ for $X = [0, 1)$, nor for $X = (0, 1)$. $\square$

24.10 Show that if U is an open connected subspace of $\mathbb{R}^2$, then U is path connected.

Proof. Let $x_0 \in U$, and define P as the set of all points, x, for which there exists a path connecting x_0 to x. Then if $x \in P$, then $x \in U$ and since U is open, there exists a neighborhood V of x contained in U. Specifically, we may assume that $V \cap U = V$. Now we cite Example 3 in Section 24, which states that the unit ball B^n is path connected in $\mathbb{R}^n$. Thus, open balls in $\mathbb{R}^2$ are path connected. With V being such an open ball around x, we observe that if γ is a path connecting x_0 and x, then for any other $x' \in V$, we can construct a path γ' from x to x'. Thus taking $\lambda = \gamma' \circ \gamma$, we have a path from x_0 to x'. This implies that for all $x' \in V$, there exists a path connecting it to x_0. Hence, $V \subset P$. Therefore, P is open.

To show that P is also closed, we want to show that $U - P$ is open. Letting $x \in U - P$, then as U is open, there exists an open ball B such that $x \in B$ and $B \subset U$. Now assume that $x' \in B \cap P$. Then $x' \in P$ which implies that there exists a path from x_0 to x'. But we also have that $x \in B$ and so there is a path from x' to x. As before we can then construct a path from x_0 to x. This implies that $x \in P$, which is a contradiction. Therefore, $B \cap P = \varnothing$ and so $U - P$ is open. Hence, P is closed. We have shown that for every such $P \subset U$, P is open and closed which implies that P is connected. $\square$