COMPREHENSIVE EXAM
ALGEBRA
Fall 2018

Part I: Group Theory (Do 4 of the following 5 problems)

1. (a) Determine, up to isomorphism, all Abelian groups of order $180 = 2^2 \cdot 3^2 \cdot 5$
   (b) For each isomorphism class described in part (a), determine, with explanation, the number of elements of order 6.

2. (a) Let G be a group and let $x \in G$ with $\circ(x) = n$. Prove
$$x^t = e \iff n|t.$$
   (b) Prove that, in a finite Abelian group, for any $x, y \in G$, we have
$$\circ(xy) \text{ divides } lcm(\circ(x), \circ(y))$$

3. Let $n \geq 3$ be an integer and let $i \in \{1, 2, \ldots, n\}$.
   (a) Show that $H_i = \{\sigma \in S_n : \sigma(i) = i\}$ is a subgroup of S_n.
   (b) In S_6, find the number of elements in the conjugacy class that contains (14).

4. Let G be a group. For each $g \in G$, define $\phi_g : G \to G$ by
$$\phi_g(x) = gxg^{-1}$$
   (a) Prove that $\phi_g \in \text{Aut}(G)$, where $\text{Aut}(G)$ denotes the group of automorphisms on G.
   (b) Prove that ϕ_g is the identity automorphism if and only if $g \in Z(G)$, where $Z(G)$ denotes the center of G.
   (c) Let $H = \{\phi_g : g \in G\}$. Prove $H \lhd \text{Aut}(G)$. (You may assume $H \leq \text{Aut}(G)$.)

5. (a) Let G be a group of order $325 = 5^2 \cdot 13$. Prove that G is Abelian.
   (b) Let G be a group of order $992 = 2^5 \cdot 31$. Show that G is not simple.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. Let R be a commutative ring, I an ideal. For each $a \in R$ define

$$I_a = \{r \in R : ar \in I\}$$

   (a) Prove that I_a is an ideal in R, for each $a \in R$.

   (b) Let $R = \mathbb{Q}[x]$ and let $I = (x^4 - 1)$ and let $a = x^2 - 1$. Prove $I_a = (x^2 + 1)$.

2. Let F be a field and let E be an extension field of F.

   (a) Let $a \in E$. Prove $[F(a) : F]$ is finite if and only if a is algebraic over F.

   (b) Let $A = \{a \in E : a \text{ is algebraic over } F\}$. Prove that A is a subfield of E.

3. (a) Let R be a principal ideal domain (PID). Let $I = (a)$ be an ideal in R. Prove that I is a maximal ideal if and only if a is an irreducible element.

   (b) Prove that every Euclidean domain (ED) is a PID.

4. Let E be an extension field of F. Let $G =$Gal(E/F).

   (a) Suppose S and T are subsets of G. Prove $S \subseteq T \implies E^T \subseteq E^S$. (Note: E^T denotes the fixed field of T.)

   (b) Suppose H and K are subsets of G such that $E^H = E^K$. Prove that $E^{H \cup K} = E^H$.

   (c) Let $E = \mathbb{Q}(\sqrt[4]{2}, i)$, then E is the splitting field for $p = x^4 - 2 \in \mathbb{Q}[x]$. Consider $\phi \in \text{Gal}(E/\mathbb{Q})$ such that $\phi(\sqrt[4]{2}) = -\sqrt[4]{2}$ and $\phi(i) = -i$. Let $H = (\phi)$, the subgroup generated by ϕ. Determine, with explanation, E^H.

5. Determine each of the following (with explanation).

   (a) All ideals in $\mathbb{Q}\,[x]\,/\,(x^4 + 4x^2 - 5)$.

   (b) All fields which are homomorphic images of $\mathbb{Z}_{30}$.

   (c) Describe all subfields of $\mathbb{C}$ which are homomorphic images of $\mathbb{Q}\,[x]\,/\,(x^3 + 2x)$.