MATH 220A

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Assignment: Midterm 2 Notes

1. Let A, B and A_α denote subsets of a topological space X. Prove the following:

   (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

   Proof. We have that $A \subset B$, and that $B \subset \overline{B}$. And so $A \subset \overline{B}$. This means that $\overline{B}$ is a closed set containing A. Since $\overline{A}$ is defined to be the intersection of all closed sets containing A, then it follows that $\overline{B}$ is in this intersection. Thus $\overline{A} \subset \overline{B}$. $\square$

   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   Proof. Since $A \subset A \cup B$, then by (a), $\overline{A} \subset \overline{A \cup B}$. Similarly, with $B \subset A \cup B$, it follows that $\overline{B} \subset \overline{A \cup B}$. Therefore, $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Now with $A \subset \overline{A}$, comes $A \subset \overline{A} \cup \overline{B}$. And with $B \subset \overline{B}$, comes $B \subset \overline{A} \cup \overline{B}$. Thus $A \cup B \subset \overline{A} \cup \overline{B}$. Hence, by (a), $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$, since $\overline{A} \cup \overline{B}$ is a closed subset as a finite union of closed sets. $\square$

   (c) $\bigcup_{\alpha \in J} \overline{A}_\alpha \subset \overline{\bigcup_{\alpha \in J} A_\alpha}$.

   Proof. Let $\alpha_0 \in J$ be any fixed element. Then $A_{\alpha_0} \subset \bigcup_{\alpha \in J} A_\alpha$. Hence, by (a), $\overline{A}_{\alpha_0} \subset \overline{\bigcup_{\alpha \in J} A_\alpha}$. Since this choice of α_0 was arbitrary, then the containment hold for all such $\alpha \in J$. Hence, $\bigcup_{\alpha \in J} \overline{A}_\alpha \subset \overline{\bigcup_{\alpha \in J} A_{\alpha \in J}}$. $\square$

2. Let A and B be two subsets of a set X. Then $A \cap B \neq \varnothing$ if and only if $(A \times B) \cap \Delta \neq \varnothing$, where $\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

   Proof. Suppose that $A \cap B \neq \varnothing$, then there is some $x \in X$ such that $x \in A \cap B$. With $x \in X$, we get that $(x, x) \in \Delta$ and also that $(x, x) \in A \times B$. Hence, $(x, x) \in (A \times B) \cap \Delta$ and thus $(A \times B) \cap \Delta \neq \varnothing$.

   Conversely, assume that $(A \times B) \cap \Delta \neq \varnothing$. Then there exists some $(x, y) \in (A \times B) \cap \Delta$. Since $(x, y) \in \Delta$, then we may write $x = y$ and so we have that $(x, x) \in A \times B$. Hence, $x \in A$ and $x \in B$. Therefore, $A \cap B \neq \varnothing$. $\square$

3. Show that X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \in X \times X \mid x \in X\}$$

   is closed in $X \times X$.

Proof. Assume that X is a Hausdorff topological space. We will show that Δ is closed in $X \times X$ by showing that its complement, $X - \Delta$, is open in $X \times X$.

Let $(x, y) \in X - \Delta$. Then $(x, y) \notin \Delta$ which implies that $x \neq y$. Since X is Hausdorff, then we may find open neighborhoods, $U \subset X$, of x, and $V \subset X$, of V, such that $U \cap V = \varnothing$. Hence, $U \times V$ is an open neighborhood of (x, y) in $X \times X$. Further, we recall that $U \cap V = \varnothing$, which by 2., we get that $(U \times V) \cap \Delta = \varnothing$. This implies that $U \times V \subset (X - \Delta)$. Hence, $X - \Delta$ is open.

Suppose now that Δ is closed in $X \times X$. Let x and y be any two distict points in X and consider the pair $(x, y) \in X - \Delta$. Since $X - \Delta$ is open, then there is a neighborhood $U \times V$ of (x, y) contained in $X - \Delta$, where U is a neighborhood of x and V is a neighborhood of y. With $U \times V$ being a neighborhood in $X - \Delta$, this implies that $(U \times V) \cap \Delta = \varnothing$. Hence, by 2., it follows that $U \cap V = \varnothing$. Therfore, X is Hausdorff. $\square$

4. If $A, U \subset X$, we define the boundary of A by:

$$\partial A = \overline{A} \cap \overline{X - A}.$$

   (a) Show that $\mathring{A}$ and ∂A are disjoint, and that $\overline{A} = \mathring{A} \cup \partial A$.