MATH 296C

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Instructor: Dr. Krauel | Assignment: Homework 1

1.1 Find the matrix A that represents f in example 1.26.

Solution. We can solve for A as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus $a = 2$ and $c = 1$. Which gives

$$\begin{pmatrix} 2 & b \\ 1 & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+b \\ 1+d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

the last system yields $b = -1$ and $d = 1$. Hence

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

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1.5 Let $\mathfrak{sl}_2(\mathbb{R}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a + d = 0, a, b, c, d \in \mathbb{R}\}$. That is, $\mathfrak{sl}_2(\mathbb{R})$ consists of 2×2 trace 0 matrices with entries in $\mathbb{R}$.

(a) Show that $\mathfrak{sl}_2(\mathbb{R})$ is a Lie algebra with bracket $[A, B] = AB - BA$.

Proof. As a subset of $\mathfrak{gl}_2(\mathbb{R})$, which is a Lie algebra, we can prove the claim by showing that $\mathfrak{sl}_2(\mathbb{R})$ is a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{R})$. Recall that a Lie subalgebra is a subspace closed under the bracket of the outer space. So do prove our claim, consider any $c \in \mathbb{R}$ and any $u, v \in \mathfrak{sl}_2(\mathbb{R})$. It follows that $\text{tr}(u+v) = \text{tr}(u)+\text{tr}(v) = 0+0 = 0$ and so $u+v \in \mathfrak{sl}_2(\mathbb{R})$. Similarly, we have that $\text{tr}(cu) = c\text{tr}(u) = c0 = 0$. Thus, $cu \in \mathfrak{sl}_2(\mathbb{R})$ and it is therefore a subspace.

Taking the same $u, v \in \mathfrak{sl}_2(\mathbb{R})$, and letting

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$\begin{aligned}
[u, v] &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} - \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix} \\
&= \begin{pmatrix} bg - fc & af + bh - eb - fd \\ ce + dg - ga - he & cf - gb \end{pmatrix}.
\end{aligned}$$

By the commutativity of the reals, it follows that $\text{tr}([u, v]) = bg-fc+cf-gb = 0$. Hence, $[u, v] \in \mathfrak{sl}_2(\mathbb{R})$. □

(b) Find a basis for $\mathfrak{sl}_2(\mathbb{R})$.

Solution. Considering that the definition of this set requires that the entries be elements of $\mathbb{R}$ and that the trace must be zero, then we find that we have two "free" entries in the upper right and lower left, and that we have only one free entry along the diagonal since the other must be its additive inverse. Hence, a basis for this Lie algebra is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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(c) Find $\dim_{\mathbb{R}}(\mathfrak{sl}_2(\mathbb{R}))$.

Solution. Considering the basis provided in part (b), we can conclude that this Lie algebra has dimension 3. ■

1.7 Prove that every Lie algebra $(L, [\cdot,\cdot])$ over a field K is a K-algebra if one sets $x \cdot y = [x, y]$.

Proof. We first recall the definition of a K-algebra and hence need to show that L is a vector space over K such that L is a ring with under addition and $[\cdot,\cdot]$, however, associativity is not required for the second operation. Additionally, we need to show that for any $k, l \in K$, and any $x, y \in L$, we have $[kx, ly] = (kl)[x, y]$.

By the definition of a Lie algebra, L is a vector space over some field K. As such, it is an abelian group with respect to addition and a magma with respect to $[\cdot,\cdot]$. By the biliearity of $[\cdot,\cdot]$, it follows that $[kx, ly] = k[x, ly] = l[kx, y] = (kl)[x, y]$. Thus, $(L, [\cdot,\cdot])$ is a K-algebra. □

2.9 Prove $\mathfrak{sl}_2(K)$ is simple if and only if the characteristic of K is not 2.

Proof. Assume that the characteristic of K is 2 and consider the following set

$$S = \text{span}\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}.$$

S is a subspace of $\mathfrak{sl}_2(K)$ by the definition of span. Moreover, if we notice that

$$\forall u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(K) \quad \text{and} \quad \forall v = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \in S,$$

we get that

$$\begin{aligned} [u, v] &= \begin{pmatrix} 0 & -2bk \\ 2ck & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S. \end{aligned}$$

Seeing as we just found a one-dimensional (non-trivial) ideal of $\mathfrak{sl}_2(K)$, we can conclude that $\mathfrak{sl}_2(K)$ is not simple when the characteristic of K is two.

Now assume that $\mathfrak{sl}_2(K)$ is simple and consider the following basis:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have the following commutation relations:

$$[x,y] = h, \qquad [x,h] = -2x, \qquad [y,h] = 2y.$$

Now suppose that I is a nonempty ideal of $\mathfrak{sl}_2(K)$. Then since the latter was assumed to be simple, it follows that $I = \mathfrak{sl}_2(K)$. However, if K had characteristic 2, then this would imply that $[x,h] = 0$ and $[y,h] = 0$, rendering I one-dimensional which is a contradiction. Hence, the characteristic is not two. □

2.10 Prove that $\mathfrak{sl}_n(\mathbb{C})$ is an ideal of $\mathfrak{gl}_n(\mathbb{C})$.

Proof. To show that $\mathfrak{sl}_n(\mathbb{C})$ is an ideal, we must show that it is a subspace such that $[x,y] \in \mathfrak{sl}_n(\mathbb{C})$ for all $x \in \mathfrak{gl}_n(\mathbb{C})$ and all $y \in \mathfrak{sl}_n(\mathbb{C})$. That it is a subspace is immediate since it is a subset and it is a vector space under the same operations. Now let $x \in \mathfrak{gl}_n(\mathbb{C})$ and $y \in \mathfrak{sl}_n(\mathbb{C})$. We want to show that $[x,y] \in \mathfrak{sl}_n(\mathbb{C})$. Thus we need that $\mathrm{tr}([x,y]) = 0$. By the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, then we have that $\mathrm{tr}([x,y]) = \mathrm{tr}(xy) - \mathrm{tr}(yx) = 0$, as desired. □

2.12 Find the structure constants for $\mathfrak{sl}_2(\mathbb{C})$.

Solution. By the above commutation relations, the structure constants are as follows: Letting a_{xy}^k denote the kth coefficient on the linear combination representing $[x,y]$, then we have

$$\begin{array}{lll} a_{xy}^0 = 0; & a_{xy}^1 = 0; & a_{xy}^2 = 1; \\ a_{xh}^0 = -2; & a_{xh}^1 = 0; & a_{xh}^2 = 0; \\ a_{yh}^0 = 0; & a_{yh}^1 = 2; & a_{yh}^2 = 0. \end{array}$$

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5.1 Recall $\mathcal{V}_2$ of Example 4.6.

(a) Find all weights of h for $\mathcal{V}_2$.

Solution. We first recall that

$$h.(aX^2 + bXY + cY^2) := (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})(aX^2 + bXY + cY^2) = 2aX^2 - 2cY^2.$$

Moreover, using $\{X^2, XY, Y^2\}$ as a basis for $\mathcal{V}_2$, then we can apply h to each of the basis elements to obtain:

$$h.X^2 = (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})X^2 = 2X^2$$
$$h.XY = (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})XY = XY - XY = 0$$
$$h.Y^2 = (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})Y^2 = -2Y^2.$$

This suffices to show that the only weights of h for $\mathcal{V}_2$ are 2, 0, and -2. ■

(b) Describe (find a basis) all weight spaces.

Solution. By the above calculation, we can see that $w_0 = X^2$ is the highest weight vector with weight 2. By Lemma 5.6, it follows that

$$w1 = \frac{1}{1!}y^1.w_0 = Y\frac{\partial}{\partial X}(X^2) = 2XY$$
$$w2 = \frac{1}{2!}y^2.w_0 = \frac{1}{2}(Y^2\frac{\partial^2}{\partial X^2})(X^2) = Y^2.$$

Thus $w_0 = X^2, w_1 = 2XY, w_2 = Y^2$ gives us a basis for $V(2)$. Moreover, by Theorem 5.8, $V(2) \cong \mathcal{V}_2$. Thus by part (a), we have that $V_{-2} = \text{span}(w_2)$, $V_0 = \text{span}(w_1)$, and $V_2 = \text{span}(w_0)$. ■

(c) Express $\mathcal{V}_2$ as a direct sum of its weight spaces as in Theorem 5.1.

Solution. By parts (a) and (b), we found that the weight spaces of h are V_{-2}, V_0, and V_2 and as such it follows that

$$\mathcal{V}_2 = \bigoplus_{i=0}^{2} V_{-2+2i} = \text{span}(X^2) \oplus \text{span}(2XY) \oplus \text{span}(Y^2).$$

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5.2 Define a vector space $\mathcal{V}_3$ similar to example 4.6.

(a) Prove that $\mathcal{V}_3$ is an $\mathfrak{sl}_2$-module.

Solution. Letting $\mathcal{V}_3 = \{aX^3 + bX^2Y + cXY^2 + dY^3 \mid a, b, c, d \in \mathbb{C}\}$, then for the same reason that $\mathcal{V}_2$ is a $\mathbb{C}$-linear space, we have that $\mathcal{V}_3$ is a $\mathbb{C}$-linear space. We will also re-use the same following relations. For all $v \in \mathcal{V}_3$, and for x, y, h as basis elements of $\mathfrak{sl}_2$, then

$$x.v = (X\frac{\partial}{\partial Y})v, \qquad y.v = (Y\frac{\partial}{\partial X})v, \qquad h.v = (X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y})v.$$

With this we will let $\alpha_1 x + \beta_1 y + \gamma_1 h, \alpha_2 x + \beta_2 y + \gamma_2 h \in \mathfrak{sl}_2$ and we let $u = a_1X^3 + b_1X^2Y + c_1XY^2 + d_1Y^3$ and $v = a_2X^3 + b_2X^2Y + c_2XY^2 + d_2Y^3$ be two arbitrary elements of $\mathcal{V}_3$. Then

$$\begin{aligned}((\alpha_1 x + \beta_1 y + \gamma_1 h) + (\alpha_2 x + \beta_2 y + \gamma_2 h)).u &= ((\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y + (\gamma_1 + \gamma_2)h).u \\ &= (\alpha x + \beta y + \gamma h).u \\ &= (\alpha X \frac{\partial}{\partial Y} + \beta Y \frac{\partial}{\partial X} + \gamma(X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}))u \\ &= \alpha(X \frac{\partial}{\partial Y})u + \beta(Y \frac{\partial}{\partial X})u + \gamma(X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y})u \\ &= \alpha(x.u) + \beta(y.u) + \gamma(h.u) \\ &= (\alpha_1 + \alpha_2)(x.u) + (\beta_1 + \beta_2)(y.u) + (\gamma_1 + \gamma_2)(h.u) \\ &= \alpha_1(x.u) + \beta_1(y.u) + \gamma_1(h.u) \\ &\qquad + \alpha_2(x.u) + \beta_2(y.u) + \gamma_2(h.u),\end{aligned}$$

which proves M1. REMINDER: Finish showing M2 and M3. ■

(b) Find all the weights of h for $\mathcal{V}_3$.

Solution. Letting $\{X^3, X^2Y, XY^2, Y^3\}$ be a basis for $\mathcal{V}_2$, then we can find the weights of h by applying h to each basis element. Doing this we get

$$\begin{aligned}h.X^3 &= (X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y})X^3 = 3X^3 \\ h.X^2Y &= (X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y})X^2Y = X^2Y \\ h.XY^2 &= (X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y})XY^2 = -XY^2 \\ h.Y^3 &= (X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y})Y^3 = -3Y^3.\end{aligned}$$

Hence, the weights of h are $-3, -1, 1, 3$. ■

(c) Describe all weight spaces.

Solution. By part (b), we see that the highest weight is 3 with weight vector $w_0 = X^3$. With this we can use Lemma 5.6 to calculate the remaining weight spaces.

$$\begin{aligned}w_1 &= \frac{1}{1} y.w_0 = Y \frac{\partial}{\partial X}(X^3) = 3X^2Y \\ w_2 &= \frac{1}{2} y^2.w_0 = \frac{1}{2}(Y^2 \frac{\partial^2}{\partial X^2})X^3 = 3XY^2 \\ w_3 &= \frac{1}{6}(Y^3 \frac{\partial^3}{\partial X^3})X^3 = Y^3.\end{aligned}$$

Thus, we have that the weight spaces of $\mathcal{V}_3$ are $V_{-3} = \operatorname{span}(w_3)$, $V_{-1} = \operatorname{span}(w_2)$, $V_1 = \operatorname{span}(w_1)$, and $V_3 = \operatorname{span}(w_0)$. ■

(d) Express $\mathcal{V}_3$ as a direct sum of its weight spaces as in Theorem 5.1.

Solution. With $-3, -1, 1, 3$ being distinct eigenvalues for h, then by Theorm 5.1 it follows that

$$\mathcal{V}_3 \cong \bigoplus_{i=0}^{3} V_{-3+2i}$$

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