MATH 210A

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Instructor: Dr. Shannon **Assignment:** Homework 7

1. Assume that $\varphi\colon G \to H$ is a homomorphism, and that $M \subseteq_g G$. Let $K = \ker(\varphi)$. Let γ be the function φ restricted to M (that is $\gamma(x) = \varphi(x)$ for all $x \in M$, and $\mathrm{dom}(\gamma) = M$). Then $\gamma\colon M \to \varphi(M)$ is a homomorphism onto $\varphi(M)$. Prove that $\ker(\gamma) = K \cap M$. Using this, and the FHT, prove that $o(\varphi(M)) \mid o(M)$.

   Proof. Let $x \in \ker(\gamma)$. Then $\gamma(x) = e_H$. Since $\ker(\gamma) \subseteq M \subseteq_g G$ and thus $x \in \ker(\varphi)$. Thus, $x \in K$. Additionally, since $\ker(\gamma) \subseteq M$, then $x \in M$. Consequently, $x \in K \cap M$. Thus, $\ker(\gamma) \subseteq K \cap M$. Now let $x \in K \cap M$. Then $x \in K$ and $x \in M$. Since $x \in K$ then $\varphi(x) = e_H$. Moreover, since $x \in M$, then $\gamma(x) = \varphi(x) = e_H$. Thus, $x \in \ker(\gamma)$. Hence, $K \cap M \subseteq \ker(\gamma)$. Thus, $\ker(\gamma) = K \cap M$.

   Since $\ker(\gamma) \lhd M$, then by FHT, $M/(K \cap M) \cong \varphi(M)$. Moreover, since $\ker(\gamma) = K \cap M$ and $\ker(\gamma) \lhd M$, then by Lagrange's Theorem, $o(K \cap M) \mid o(M)$. Thus, $o(M/(K \cap M))$ divides $o(M)$. Thus, since $M/(K \cap M) \cong \varphi(M)$, then $o(M/(K \cap M)) = o(\varphi(M))$. Thus, $o(\varphi(M)) \mid o(M)$. $\square$

3. Without simply citing the results that we have proved for groups of order pq, determine the structure of all groups of order 21.

   Proof. Let G be a group of order 21. To start, we note that the prime factorization of 21 is 3×7 and so by Sylow I, G contains a 3-Sylow subgroup and a 7-Sylow subgroup. By Sylow III, we know that $n_3 \equiv 1(\text{mod } 3)$, $n_3 \mid 7$, $n_7 \equiv 1(\text{mod } 7)$, and $n_7 \mid 3$. From these statements it follows that $n_3 = 1$ or $n_3 = 7$ and $n_7 = 1$. Thus, $P_7 \lhd G$.

   Now let $\langle a \rangle$ be a 3-Sylow subgroup and let $\langle b \rangle$ be the 7-Sylow subgroup. Assume $\theta\colon \langle a \rangle \to \mathrm{Aut}(\langle b \rangle)$ is a homomorphism where for each $h \in \langle a \rangle$, we have $\theta(h) = \varphi_k$. Since $\langle b \rangle \lhd G$, then it follows that $aba^{-1} \in \langle b \rangle$. Thus, $aba^{-1} = b^k$ and to each $\varphi_k \in \mathrm{Aut}(\langle b \rangle)$ there corresponds an instance of $aba^{-1} = b^k$.

   Since θ is a homomorphism, then it follows that $o(\theta(a)) \mid 3$. Given that $\theta(a) = \varphi_k$, then it follows that $o(\varphi_k) = 1$ or $o(\varphi_k) = 3$. In the former case where $o(\varphi_k) = 1$ we have that θ is a trivial homomorphism which maps each element of $\langle a \rangle$ to the identity map φ_1. It follows from this that $aba^{-1} = b$ and so $ab = ba$ which implies that G is abelian. Moreover, $G \cong \mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{21}$. In the latter case where $o(\varphi_k) = 3$ it follows that $\varphi_{k^3} = \varphi_1$. Thus, in this case we are looking for all values k such that $k^3 \equiv 1(\text{mod } 7)$.

   We can see that since $2^3 = 8 \equiv 1(\text{mod } 7)$, then 2 is a solution. By the result from number theory referenced on page 23, we have that there are 3 solutions and they are of the form 1, d, and d^2, where d is a solution. Thus, we have $1, 2,$ and 4. Furthermore, as noted on page 23, each of the solutions to $k^3 \equiv 1(\text{mod } 7)$ are equivalent. Thus, any group G of order 21 is either isomorphic to the abelian group $\mathbb{Z}_{21}$ or a nonabelian group of order 21 where $G = \langle a \rangle \langle b \rangle$, $a^3 = 1 = b^7$, and $ab = b^2 a$. $\square$

4. Assume that p is prime, and that $o(G) = p^n$. Prove that G is solvable.

Proof. Assume that G is Abelian. By Sylow I, G contains subgroups of order p^k for all $1 \leq k \leq n-1$. Since G is Abelian, then all of these subgroups are normal in G. Now let $H_{k+1} \subseteq_g G$ such that $o(H) = p^{k+1}$, for some $0 \leq k+1 \leq n$. Then by Sylow I, H_{k+1} contains a subgroup H_k of order p^{k-1} and since $H_k \lhd G$, then $H_k \lhd H_{k+1}$. Continuing in this way, we can construct a normal series for G

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G,$$

where for each H_i, we have that $o(H_i) = p^i$. Now consider the factor H_{i+1}/H_i. It follows that $o(H_{i+1}/H_i) = p$ and thus H_{i+1}/H_i is Abelian. Thus, the normal series above has factors which are all abelian. Hence, ig G is Abelien, then G is solvable.

Now assume that G is not Abelian and that $n \geq 3$. Continuing by complete induction on n, assume that if $o(H) = p^t$, where $t < n$, then H is solvable. Now consider the subgroup $Z(G)$. Since G is not Abelian, then $Z(G) \neq G$ and thus, $o(Z(G)) = p^t$ for some $t < n$. Thus, by assumption, $Z(G)$ is solvable. Since $Z(G)$ is Abelian, then $Z(G) \lhd G$. Additionally, since $o(Z(G)) = p^t$, from which it follows that $o(G/Z(G)) = p^{n-t}$, then since $n - t < n$, then by assumption $G/Z(G)$ is solvable. Thus, by the result on page 24, it follows that G is solvable. $\square$

5. Let G be a group of order 8.

   (a) Prove that if $g^2 = e$ for all $g \in G$, then G is Abelian.

   Proof. Let $a, b \in G$. Then by assumption, both $a^2 = e$ and $b^2 = e$. Thus, $a = a^{-1}$ and $b = b^{-1}$. It follows then that $ab = a^{-1}b^{-1} = (ba)^{-1}$. However, since $ba \in G$, then $(ba)^2 = e$ and so $(ba)^{-1} = ba$. Thus, $ab = (ba)^{-1} = ba$. Hence, G is Abelian. Additionally, if we let $a, b \in G$ be two non-equal elements of order 2 and let $c \in G$ such that c is distinct from e, a, b, ab, then it follows that the set $\{e, a, b, c, ab, ac, bc, abc\}$ is a subgroup of G of order 8. Thus, $G = \{e, a, b, c, ab, ac, bc, abc\}$. Hence, $G = \langle a \rangle \langle b \rangle \langle c \rangle$. Furthermore, since each of these subgroups are normal in G and $\langle a \rangle \cap \langle b \rangle = \{e\}$, $\langle a \rangle \cap \langle c \rangle = \{e\}$, and $\langle b \rangle \cap \langle c \rangle = \{e\}$, then it follows that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. $\square$

   (b) Assume that there exists $a \in G$ such that $o(a) = 4$, and that no element of G has order 8. Explain why $\langle a \rangle \lhd G$. Assume that $b \notin \langle a \rangle$. Prove that $b^2 \in \langle a \rangle$.

      (i) If $b^2 = e$ explain why $G = \langle a \rangle \langle b \rangle$, and prove that either $bab^{-1} = a$ or $bab^{-1} = a^{-1}$, and determine the structure of the groups in these two cases.

      (ii) If $o(b) = 4$, then prove $b^2 = a^2$, and again prove that either $bab^{-1} = a$ or $bab^{-1} = a^{-1}$, and determine the structure of the groups in these two cases (if $bab^{-1} = a^{-1}$ prove that $Z(G) = \{e, a^2\}$)

   Proof. By assumption, $o(a) = 4$ and if it is the only element of G with order 4, then every other element of G must have order $1, 2$, or 8. If G has an element of order 8, then G is cyclic and thus Abelian, from which it would follow that $\langle a \rangle \lhd G$.

Now assume G does not contain an element of order 8, and assume $H \subseteq_g G$ such that $o(H) = 4$ and $H \neq \langle a \rangle$. Clearly, $a \notin H$ and $H \cap \langle a \rangle = \{e\}$. Note that every element of H except for the e has order 2. Now let $x \in G$ such that $x \neq e$, then $x \in \langle a \rangle$ or $x \in H$.

If $x \in \langle a \rangle$, then $x\langle a \rangle = \langle a \rangle = \langle a \rangle x$. Now assume $x \in H$. Then $x \notin \langle a \rangle$. Thus, $x\langle a \rangle \subseteq H$. This implies that $xa \in H$ and since $x \in H$, then $x(xa) = x^2a = a \in H$. This is a contradiction. Thus, for all $x \in G$ we have that $x \in \langle a \rangle$. This would imply that $G = \langle a \rangle$ which is not possible by assumption. Thus, there exists no other subgroup H of G of order 4. Hence, $\langle a \rangle$ is the only subgroup of G of order 4. By 4.(c) of Homework 2, it follows that $\langle a \rangle \triangleleft G$.

Now assume that $b \notin \langle a \rangle$. By virtue of being an element of G, it follows that $o(b) \mid 8$. Thus, $o(b) = 1, 2, 4, 8$. We know that $o(b) \neq 1, 4$ since if $o(b) = 1$, then $b = e \in \langle a \rangle$, and $o(b) \neq 4$ since we have assumed that a is the only element of order 4. Thus, $o(b) = 2$ or $o(b) = 8$.

If $o(b) = 2$, then $b^2 = e \in \langle a \rangle$. If $o(b) = 8$, then $G = \langle b \rangle$. However, if this were the case then we would have that $o(b^2) = 4$ and $o(b^6) = 4$. This is a contradiction since a is the only element of order 4. Thus, $o(b) \neq 8$ and so $o(b) = 2$. Thus, $b^2 = e \in \langle a \rangle$.

(i) Assume that $b^2 = e$. Let $x \in \langle a \rangle\langle b \rangle$. Then for some $i, j \in \mathbb{Z}^+$, we have that $x = a^ib^j$ and since $a^i, b^j \in G$, then $a^ib^j \in G$. Thus, $x \in G$. Hence, $\langle a \rangle\langle b \rangle \subseteq G$. Note that $o(\langle a \rangle\langle b \rangle) = 8$. Thus, since $\langle a \rangle\langle b \rangle$ is a subset of G with the same cardinality of G, then it follows that $G = \langle a \rangle\langle b \rangle$.

Since $\langle a \rangle \triangleleft G$, then it follows that $bab^{-1} \in \langle a \rangle$. Thus, for some $k \in \mathbb{Z}$, we have that $bab^{-1} = a^k$. Assume that $bab^{-1} = e$. Then $ba = b$ and thus, $a = e$ which is a contradiction. Now assume that $bab^{-1} = a^2$. Then this would imply that $a = b^{-1}a^2b$. Thus, $a^2 = (b^{-1}a^2b)(b^{-1}a^2b) = b^{-1}a^4b = b^{-1}eb = b^{-1}b = e$. This is also a contradiction. Thus, it follows that $bab^{-1} = a$ or $bab^{-1} = a^3 = a^{-1}$. If $aba^{-1} = a$, then it follows that $ab = ba$ and thus $G = \langle a \rangle\langle b \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. If $bab^{-1} = a^{-1}$, then $ba = a^{-1}b$. In this case we have a group of order 4×2, where $o(a) = 4$, $o(b) = 2$, $G = \langle a \rangle\langle b \rangle$, and $ba = a^{-1}b$. Thus, in this case it follows that $G \cong D_4$.

(ii) Let $H = \langle a \rangle$ and assume that every element of $G - H$ has order 4 and let b be such an element. Then $o(b) = 4$ and it follows that $o(b^2) = 2$, thus $b^2 \in H$ and since the only element of H with order 2 is a^2, then we have that $a^2 = b^2$. Now assume that $bab^{-1} = a^2$, then it follows that $bab^{-1} = b^2$ and thus $ba = b^3$, hence $a = b^2 = a^2$. Thus, this cannot be the case. Now assume that $bab^{-1} = e$. Then $a = e$ which is a contradiction. Thus, $bab^{-1} = a$ or $bab^{-1} = a^3 = a^{-1}$.

Now let $bab^{-1} = a^{-1}$, then it follows that $ba = a^3b$. Thus, the set $H \cup Hb = \{e, a, b, ab, a^2, a^3, a^2b, a^3b\}$ is a subgroup of order 8 such that $a^4 = b^4 = e$, $a^2 = b^2$, and $ba = a^3b$. Thus, $G \cong Q_8$. □

(c) Determine all groups of order 8.

Solution. $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8, Q_8.

6. Assume that p is prime, $o(G) = p^t$, $b \in G$, $b^{p^k} \neq e$, and $o(b^{p^k}) = p^m$. Prove that $o(b) = p^{k+m}$.

Proof. Let $o(b) = p^y$. Since $o(b^{p^k}) = p^m$, then $(b^{p^k})^{p^m} = b^{p^{k+m}} = e$. Thus, $p^y \mid p^{k+m}$. It follows from this that p^{k+m} is a multiple of p^y. In particular, the multiple is some power of p. Thus, $p^{k+m} = p^x p^y$. $\square$