COMPREHENSIVE EXAM
ALGEBRA
Fall 2019

Part I: Group Theory (Do 4 of the following 5 problems)

1. (a) Find, with explanation, an element σ of maximal order in S_7
   (b) Prove that S_7 is not an Abelian group.
   (c) Determine, with proof, $C(\sigma)$, the centralizer of σ in S_7 (where σ is the element found in part (a))

2. Let G and J be groups and let $\phi : G \to J$ be a group epimorphism (surjective homomorphism).
   (a) Prove that if J is Abelian, then any subgroup of G that contains $\ker \phi$ is a normal subgroup of G.
   (b) Suppose $K \lhd J$. Prove there exists $H \lhd G$ such that $\ker \phi \subseteq H$ and G/H is isomorphic to J/K.

3. Give an example (or explain why none exists) of a
   (a) nontrivial homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_7$ (don't simply cite a theorem about homomorphism from $\mathbb{Z}_m$ to $\mathbb{Z}_n$).
   (b) homomorphism from S_3 onto $\mathbb{Z}_2$.
   (c) homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_4$ onto $\mathbb{Z}_8$

4. Let G be a finite group.
   (a) Suppose $H \lhd G$ and $g \in G$. Then $gH \in G/H$. Prove that $\circ(gH)$ divides $\circ(g)$.
   (b) Let p be a prime such that $p \mid |G|$. Let K be a p-Sylow subgroup of G and let $N = N(K)$ be the normalizer of K in G. Suppose $g \in N$ such that the order of g is a power of p. Prove $g \in K$.

5. Let G be a group of order $1309 = 7 \cdot 11 \cdot 17$. Prove that G is cyclic.

Part II: Ring and Field Theory (Do 4 of the following 5 problems)

1. In $\mathbb{Q}[x]$, let I be the ideal generated by x^2+x+1 in $\mathbb{Q}[x]$ and let J be the ideal generated by x^3+x^2+x in $\mathbb{Q}[x]$. That is,
$$I = (x^2+x+1)$$
$$J = (x^3+x^2+x)$$
   (a) Find a basis for $\mathbb{Q}[x]/I$ as a vector space over $\mathbb{Q}$.
   (b) Find an inverse for $x+I \in \mathbb{Q}[x]/I$
   (c) Show $x+J \in \mathbb{Q}[x]/J$ does not have an inverse.

2. Let R be a commutative ring with unity.
   (a) Prove that R is a field if and only if the only ideals of R are the trivial ideals ($\{0\}$ and R).
   (b) Let I be an ideal of R. Suppose that every nonzero coset of R/I contains a unit of R. Prove that the only ideals in R/I are the trivial ideals (I and R/I).

3. Let $F = \mathbb{Q}$ and let $E = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots)$
   (a) Prove that E is an algebraic extension of F.
   (b) Prove that $[E:F] = \infty$.

4. Let I and J be ideals of a ring R. Define
$$I + J = \{a + b : a \in I \text{ and } b \in J\}$$
   (a) Prove that $I+J$ is an ideal of R.
   (b) Prove that if $I+J = R$ and $I \cap J = \{0\}$, then R/I is isomorphic to J.

5. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so ω is a primitive cube root of unity.
   (a) Let
$$\alpha_1 = \omega^{1/3} + \omega^{-1/3}$$
$$\alpha_2 = \omega^{2/3} + \omega^{-2/3}$$
$$\alpha_3 = \omega^{4/3} + \omega^{-4/3}$$
   Show that α_1 is a root of $f(x) = x^3 - 3x + 1$.
   Note: Moving forward, you may use the fact that α_2 and α_3 are also roots of $f(x)$.
   (b) Show that the splitting field for $f(x)$ over $\mathbb{Q}$ is $E = \mathbb{Q}(\alpha_1)$. (Hint: consider $(\alpha_1)^2$)
   (c) Determine, with proof, the degree of $\mathbb{Q}(\omega^{1/3})$ over $\mathbb{Q}(\omega)$.