MATH

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Due Date: #/#/#
Assignment: Homework #

16.1 Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Proof. Given a basis $\mathcal{B}$ of X, we will let $\mathcal{B}_1 = \{B \cap Y \mid B \in \mathcal{B}\}$. Similarly, we will let $\mathcal{B}_2 = \{B \cap A \mid B \in \mathcal{B}_1\}$. Lastly, we let $\mathcal{B}_3 = \{B \cap A \mid B \in \mathcal{B}\}$. By Lemma 16.1, $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$ are bases for the topologies $\mathcal{T}_1$ for Y, $\mathcal{T}_2$ for A, and $\mathcal{T}_3$ for A, respectively. We want to show that $\mathcal{T}_2 = \mathcal{T}_3$.

To this end, we let $B \in \mathcal{B}_2$ and choose any $x \in B$. Then we need to show that there exists $B' \in \mathcal{B}_3$ such that $x \in B' \subset B$. Since $\mathcal{B}_3$ is a basis for A and $x \in A$, then by Definition 13.1, there exits $V \in \mathcal{B}_3$ such that $x \in V$. Note that $B = B_1 \cap A$ for $B_1 \in \mathcal{B}_2$ and $V = B_2 \cap A$ for $B_2 \in \mathcal{B}_3$. Moreover, $B_1 = T \cap Y$ for $T \in \mathcal{B}$ and $B_2 = U \cap A$ for $U \in \mathcal{B}$. Hence

$$B \cap B' = (B_1 \cap A) \cap (B_2 \cap A) = ((T \cap Y) \cap A) \cap ((U \cap A) \cap A) = (T \cap U) \cap A.$$

Seeing as $x \in (T \cap U)$, then there exists $\hat{B} \in \mathcal{B}$ such that $x \in \hat{B} \subset T \cap U$. It follows that $x \in \hat{B} \cap A \subset B \cap B' \subset B$. Hence, $\mathcal{T}_2 \subset \mathcal{T}_3$. The other inclusion is shown in a similar way, and so we can conclude that $\mathcal{T}_2 = \mathcal{T}_3$. $\square$

16.4 A map $f : X \to Y$ is said to be an ***open map*** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Proof. Let $\mathcal{B}_1$ be a basis for X and $\mathcal{B}_2$ be a basis for Y. Then by Theorem 15.1, $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis for $X \times Y$. Note that if $U \subset X$ is open, then it is equal to the union of some collection of elements $B \in \mathcal{B}_1$. Seeing as for any such $(B, B') \in \mathcal{B}_1 \times \mathcal{B}_2$, we have that $\pi_1((B, B')) = B$, which is open in X, and the fact that maps preserve unions, it then follows that if $U = \bigcup_{B \in C} B$, then $\pi_1((U, V)) = U$. Hence, if W is open in $X \times Y$, then $\pi_1(W)$ is open in X. The same argument is used to show that π_2 is an open map. $\square$

17.2 Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. By assumption, $Y - A$ is open in Y and $X - Y$ is open in X. We want to show that $X - A$ is open in X. $\square$