Master's Exam in Real Analysis
May 2020

Part 1 - Solve 6 problems from Part 1.

1. Prove that:
(a) $\mathbb{Q} \cap [0,1]$ is countable.
(b) $[0,1]$ is uncountable.
(c) $[0,1] \setminus \mathbb{Q}$ is uncountable.

2. Let X be a metric space, $n \in \mathbb{N}$ and $A_i \subset X$, for $1 \leq i \leq n$. Prove that:
(a)

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}\,.$$

(b) Is (a) true if $n = \infty$? Prove it or give a counterexample.

3. Let X be a compact metric space and $\{x_n\}$ be a sequence from X. Show that $\{x_n\}$ is convergent if and only if it is a Cauchy sequence.

4. Let $x_1 = 4$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{9}{x_n}\right)\,, \ \forall \ n \in \mathbb{N}\,.$$

Show that $\{x_n\}$ is a convergent sequence and find its limit.

5. Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}\,.$$

(a) Prove that the series is convergent.
(b) Show that the sum of the series is an irrational number.

6. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if} \quad x = \frac{m}{n} \text{ in lowest terms} \\ 0 & \text{if} \quad x = 0 \text{ or } x \notin \mathbb{Q}\,. \end{cases}$$

(a) For any $x_0 \in [0,1]$, find $\lim_{x \to x_0} f(x)$.
(b) At which points is f continuous?
(c) What type of discontinuities does f have?

7. Let f be a continuous mapping of a compact metric space X into a metric space Y. Prove that $f(X)$ is compact.

Part 2 - Solve 6 problems from Part 2.

8. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function.
(a) Show that if there exists some $L \geq 0$ such that $|f'(x)| \leq L$ for all $x \in (a, b)$, then f is uniformly continuous.
(b) Is the converse true? Prove it or give a counterexample.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that there exists some $0 \leq L < 1$ with $|f'(x)| \leq L$ for all $x \in \mathbb{R}$. Consider an arbitrary, but fixed $x_1 \in \mathbb{R}$ and the sequence $\{x_n\}$ given by $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ is convergent to an $x \in \mathbb{R}$ and $f(x) = x$.

10. Let $f : [-1, 1] \to \mathbb{R}$ be a bounded function and

$$\alpha(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad x > 0 \,. \end{cases}$$

Show that $f \in \mathcal{R}(\alpha)$ on $[-1, 1]$ if and only if f is continuous at $x = 0$.

11. Let α be monotonically increasing and bounded on $[a, b]$. Suppose that $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for all $n \in \mathbb{N}$ and that f_n converges uniformly to f on $[a, b]$. Prove that $f \in \mathcal{R}(\alpha)$ and

$$\int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha \,.$$

12. Let $f_n : [0, 1] \to \mathbb{R}$ be a continuous function for all $n \in \mathbb{N}$.
(a) Show that if the sequence $\{f_n\}$ converges uniformly to a function f on $[0, 1]$, then for all $x \in [0, 1]$ and for all sequences $\{x_n\}$ from $[0, 1]$ converging to x, we have $\{f_n(x_n)\} \to f(x)$.
(b) Is the converse true? Prove it or give a counterexample.

13. Let

$$f_n(x) = \frac{nx}{1 + n^3 x^2} \,, \; x \geq 0 \,, \; n \in \mathbb{N} \,.$$

(a) Investigate the pointwise and uniform convergences of the sequence $\{f_n\}$ on [0,1].
(b) Investigate the pointwise and uniform convergences of the series $\sum_{n=1}^{\infty} f_n$ on $[0, 1]$.

14. Let $f : [0, 1] \to \mathbb{R}$ be continuous and assume that

$$\int_0^1 f(x) x^n \, dx = 0, \quad \text{for all} \;\; n \in \mathbb{N}.$$

Prove that $f(x) = 0$ for all $x \in [0, 1]$.