MATH 230B

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Assignment: PRACTICE

1. Let $a < b$ be real numbers and $f : (a, b) \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$ for all $x \in (a, b)$ where $M > 0$. Prove that $\lim_{x\to b^-} f(x)$.

   Proof. We need to show that for $f(t_n) \to L$ as $n \to \infty$ for all sequences $\{t_n\}$ in (a, b) such that $t_n \to b$. To start we will show that f is uniformly continuous on (a, b). Let $x, y \in (a, b)$. Then f is continuous on $[x, y]$ and differentiable on (x, y). With these two premises in hand, we can apply the MVT to obtain some $z \in (x, y)$ such that

   $$f(y) - f(x) = f'(z)(y - x). \tag{1}$$

   Since f' is bounded my M, then from (1) it follows that

   $$f(y) - f(x) \leq M(y - x). \tag{2}$$

   Furthermore, since this is true for all $x, y \in (a, b)$, then we get that

   $$|f(y) - f(x)| \leq M|y - x|. \tag{3}$$

   Hence, f is Lipschitz continuous on (a, b). Thus, in letting $\varepsilon > 0$ and choosing $\delta = \varepsilon/M$, then for any $x, y \in (a, b)$ such that $|y - x| < \delta$, then

   $$|f(y) - f(x)| \leq M|y - x| < M\delta = M\frac{\varepsilon}{M} = \varepsilon. \tag{4}$$

   Therefore f is uniformly continuous over (a, b). Now that uniform continuity has been established, we move on to showing the initial statement of this proof.

   Let c_{nn} be any sequence of (a, b) such that $c_n \to b$ as $n \to \infty$. Recall that every convergent sequence in Cauchy (since for any $\varepsilon > 0$, there exists N such that for all $n > N$, $|x_n - a| < \varepsilon/2$ and so for $m, n > N$, the triangle inequality gives us $|x_n - x_m| \leq |x_n - a| + |a - x_m| < \varepsilon$). With this we have that $\{c_n\}$ is Cauchy. Letting $\varepsilon > 0$, then since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in (a, b)$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Moreover, since $\{c_n\}$ is Cauchy, then there exists some N such that for all $m, n \geq N$ we get $|c_n - c_m| < \delta$ and hence $|f(c_n) - f(c_m)| < \varepsilon$. Therefore, $\{f(c_n)\}$ is Cauchy and therefore convergent. Hence, $\lim_{n\to\infty} f(c_n) \in \mathbb{R}$ for all sequences in (a, b) that converge to b. Therefore $\lim_{x\to b^-} f(x)$ exists. □

2. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. Prove that if $\lim_{x\to\infty} f(x) = M \in \mathbb{R}$, then there exists a sequence $\{x_n\}_n$ in $(0, \infty)$ such that $f'(x_n)$ converges to 0.

Proof. Let $\varepsilon > 0$ then there exists $\delta > 0$ such that for all $x \in (0, \infty)$ with $x > \delta$, $|f(x) - M| < \varepsilon$. By the Archimedean principle, there exists $N(\varepsilon) > \delta$ with $N \in \mathbb{Z}$ such that for any $n > N(\varepsilon)$ we have $|f(n) - M| < \varepsilon$. Hence $\{f(n)\}_n$ is convergent and therefor Cauchy. Thus, letting $m = N(\varepsilon) + 1$ and $n = m + 1$, then

$$|f(n) - f(m)| < \varepsilon. \tag{5}$$

From (5) it follows that

$$\frac{f(n) - f(m)}{n - m} \leq \frac{|f(n) - f(m)|}{|n - m|} < \varepsilon. \tag{6}$$

Then since f is differentiable over (m, n) and continuous over $[m, n]$ then by the MVT, there exists $z = z(N(\varepsilon)) \in (m, n)$ such that

$$f'(z) = \frac{f(n) - f(m)}{n - m} < \varepsilon. \tag{7}$$

Now define the following sequence $\{z_k\}_k$ where for each $k \in \mathbb{N}$, we define $\varepsilon = 1/k$, and from which we obtain $N(\varepsilon)$, and from that we obtain $m = N(\varepsilon) + 1$ and $n = m + 1$, and finally $z_k \in (m, n)$ which satisfies (7). From this we see that $\{f'(z_k)\}$ is a sequence which converges to zero. $\square$

3. Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ where } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Determine (with proof) whether $f \in \mathcal{R}[0, 1]$ and if so, compute $\int_0^1 f$.

Proof. Let $\varepsilon > 0$, then let k be the smallest integer such that $\frac{1}{k} < \varepsilon/2$. Let $P = \{x_0, \ldots, x_{2k+1}\}$ be a partition where $x_0 = 0$, $x_{2k} = 1 - \delta/2$, $x_{2k+1} = 1$, and for each $i \in \{2, \ldots, k\}$, $x_{2(k-i)+1} = \frac{1}{i} - \frac{\delta}{2}$, and $x_{2(k-i)+2} = \frac{1}{i} + \frac{\delta}{2}$, where

$$\delta < \min\{\frac{k\varepsilon - 1}{k(k-2)}, \frac{1}{2}(\frac{1}{k} - \frac{1}{k+1})\}.$$

The term in the left of the min comes from requiring that δ be such that

$$\frac{1}{k} - \frac{\delta}{2} + (k-2)\delta + \frac{\delta}{2} < \varepsilon.$$

Where the first two terms denote the distance between x_0 and x_1, the term in the middle is the sum of the rectangle's widths around $1/2, 1/3, \ldots, 1/k$, and the last term is the length between x_{2k} and x_{2k+1}. The term in the right of the min assures that δ is smaller than the length between $\frac{1}{k}$ and $\frac{1}{k+1}$. With this we have that for any $s \in \{2, \ldots, 2k+1\}$, then

$$M_s = \sup_{[x_{s-1}, x_s]} f = \begin{cases} 1 & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd.} \end{cases}$$

This is because if s is even, then there is some $i \in \{2, \ldots, k\}$ such that $s = 2(k-i)+2$ and between x_s and x_{s-1} is $\frac{1}{i}$ and so f would take on a value of 1 over that interval. Whereas, if s is odd, then there is no $i \in \mathbb{N}$ such that $x_{s-1} \leq \frac{1}{i} \leq x_s$ and so f is 0 over the interval. With this in mind we compute the upper sum

$$\begin{aligned} S(f,P) &= \sum_{s=1}^{2k+1} M_s \Delta x_s \\ &= \frac{1}{k} - \frac{\delta}{2} + \sum_{s=2}^{2k} M_s \delta + \frac{\delta}{2} \\ &= \frac{1}{k} + (k-2)\delta \\ &< \frac{1}{k} + (k-2)\frac{k\varepsilon - 1}{(k-2)k} \\ &= \frac{1}{k} + \frac{k\varepsilon - 1}{k} \\ &= \frac{k\varepsilon}{k} = \varepsilon. \end{aligned}$$

Seeing as $L(f,P) = 0$ since between any two x_{i-1} and x_i there is an irrational number, then it follows that

$$U(f,P) - L(f,P) = U(f,P) < \varepsilon.$$

Therefore $f \in \mathcal{R}[0,1]$. To compute the integral we only need to note that $L(f,P) = 0$ for all partitions (reason given above) and so the $\underline{S}(f) = 0$. Hence

$$\int_0^1 f = 0.$$

$\square$