Analysis Notes

Quin Darcy

Jan, 26 2021

1 Differetntiable Functions

Definition 1.1. Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. If $x \in (a, b)$ and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, then we say that f is differentiable at x and use the notation

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Note also that if $x = a$ and and $\lim_{h\searrow 0} \frac{f(a+h)-f(a)}{h}$ exists, then f is differentiable at a. Similarly for $x = b$.

Lemma 1.1. *Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. If f is differentiable at x, then there exists a fuction $\phi(h)$ defined on a small neighborhood of 0 such that*

$$f(x + h) - f(x) = \big(f'(x) + \phi(h)\big)h,$$

and

$$\lim_{h\to 0} \phi(h) = 0.$$

2 Integration

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. For any $h > 0$ we define the **oscillation** of f on the interval $(x - h, x + h)$ as

$$\operatorname{osc}(f)(x - h, x + h) = \sup\{|f(x_1) - f(x_2)| \mid x_1, x_2 \in (x - h, x + h) \cap [a, b]\}.$$

We define the oscillation of f at x as

$$\operatorname{osc}(f)(x) = \lim_{h\searrow 0} \operatorname{osc}(f)(x - h, x + h).$$

Notice that if $0 < h_1 < h_2$, then

$$\operatorname{osc}(f)(x - h_1, x + h_1) \le \operatorname{osc}(f)(x - h_2, x + h_2).$$

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ and $x \in [a,b]$. Then f is coninuous at x if and only if $osc(f)(x) = 0$.*

Proof. First, assume that f is continuous at x. Then we get that for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(y)| < \varepsilon/2$, for all $y \in (x - \delta, x + delta) \cap [a,b]$. This implies that for any $x_1, x_2 \in (x - \delta, x + \delta) \cap [a,b]$, we get that

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(x)| + |f(x) - f(x_2)| < \varepsilon.$$

Hence,

$$\operatorname{osc}(f)(x - \delta, x + \delta) \leq \varepsilon.$$

We also note that by one remark made in the definition, we have that for any $0 < h < \delta$, we have

$$\operatorname{osc}(f)(x - h, x + h) \leq \operatorname{osc}(f)(x - \delta, x + \delta).$$

In summary, we showed that for all ε, there exists δ such that for any $0 < h < \delta$, we have

$$\operatorname{osc}(f)(x - h, x + h) \leq \varepsilon$$

which implies that $\operatorname{osc}(f)(x) = 0$.

Now assume that $\operatorname{osc}(f)(x) = 0$. Then for all $\varepsilon > 0$, there exists some $H > 0$ such that for any $0 < h < H$ we have that

$$\operatorname{osc}(f)(x - h, x + h) \leq \varepsilon.$$

Now let $x_2 = x$ and fix $0 < h < H$, then by Definition 2.1, we get

$$|f(x_1) - f(x)| < \varepsilon, \quad \forall x_1 \in (x - h, x + h) \cap [a,b],$$

which shows that f is continuous at x. □

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$be a bounded function and $\gamma > 0$. Then the set*

$$D_\gamma = \Big\{x \in [a,b] \mid osc(f)(x) \geq \gamma\Big\}$$

is compact.

Proof. As D_γ is a subset of a compact set, then all we need to show it that it is closed. That it, we want to show that its complement

$$D_\gamma^c = \{x \in [a,b] \mid \operatorname{osc}(f)(x) < \gamma\}$$

is open. with this in mind, consider $x \in D_\gamma^c$. Membership of this set means that

$$\lim_{h \searrow 0} \operatorname{osc}(f)(x - h, x + h) < \gamma.$$

This means that there exists some $\varepsilon > 0$, such that

$$\operatorname{osc}(f)(x) < \gamma - \varepsilon.$$

By the properties of infimums, there exists some h_ε such that

$$\operatorname{osc}(f)(x - h_\varepsilon, x + h_\varepsilon) < \gamma.$$

Hence, for any $z \in (x - \frac{h_\varepsilon}{2}, x + \frac{h_\varepsilon}{2}) \cap [a, b]$ we have that

$$(z - \frac{h_\varepsilon}{2}, z + \frac{h_\varepsilon}{2}) \subseteq (x - h_\varepsilon, x + h_\varepsilon)$$

and thus

$$\operatorname{osc}(f)(z - \frac{h_\varepsilon}{2}, z + \frac{h_\varepsilon}{2}) < \gamma.$$

Therefore $\operatorname{osc}(f)(z) < \gamma$ and $z \in D_\gamma^c$, which imples that D_γ^c is open in $[a, b]$. $\square$