# Homework 1 - Differentiability

MATH 230B

You can use all the results discussed in class. They are collected in the notes posted on Canvas.

**Problem 1** [*Master's Exam, Fall 2017, Fall 2018, Spring 2019*]
Let

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

How many times is $f$ differentiable on $\mathbb{R}$?
For which $n \in \mathbb{N}$ do we have $f \in C^n(\mathbb{R}) \setminus C^{n+1}(\mathbb{R})$?

**Problem 2** [*Master's Exam, Fall 2017*]
Let $f : ]0,1] \to \mathbb{R}$ be differentiable with $0 < f'(x) < 1$ for all $x \in ]0,1]$.
Prove that the sequence $\{f(1/n)\}_n$ has a limit.

**Problem 3** [*Master's Exam, Fall 2020*]
Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$ and differentiable on $]0,1[$, with $f'(x) \neq 1$ for all $x \in ]0,1[$. Prove that there exists a unique fixed point for $f$ in $[0,1]$.

**Problem 4** Let $f : [0,\infty[ \to \mathbb{R}$ be continuous on $[0,\infty[$ with $f(0) = 0$ and differentiable on $]0,\infty[$ with $f'$ increasing on $]0,\infty[$. Prove that the function $\dfrac{f(x)}{x}$ is increasing on $]0,\infty[$.

**Problem 5** Let $a < b$, $x_0 \in ]a,b[$ and $f \in C^{2n}(]a,b[)$ for some $n \in \mathbb{N}$. Suppose that $f^{(k)}(x_0) = 0$ for all $k \in \{1, 2, \ldots, 2n-1\}$ and $f^{(2n)}(x_0) > 0$. Prove that $f$ has a local minimum at $x_0$.

## ADDITIONAL PROBLEMS

**Problem A1** [*Master's Exam, Spring 2018* ]
Let $a < b$ be real numbers and $f : ]a, b[ \to \mathbb{R}$ be differentiable with $|f'(x)| \le M$ for all $x \in ]a, b[$, where $M > 0$. Prove that $\lim_{x \to b^-} f(x)$ exists.

**Problem A2** [*Master's Exam, Spring 2021* ]
Let $f : ]0, \infty[ \to \mathbb{R}$ be differentiable. Prove that if $\lim_{x \to \infty} f(x) = M \in \mathbb{R}$, then there exists a sequence $\{x_n\}_n$ in $]0, \infty[$ such that $f'(x_n)$ converges to 0.

**Problem A3** [*Master's Exam, Spring 2020* ]
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $|f'(x)| \le M$ for all $x \in \mathbb{R}$, where $0 \le M < 1$. Let $x_1 \in \mathbb{R}$ and consider the recursion

$$x_{n+1} = f(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Prove that $\{x_n\}_n$ converges to the unique fixed point of $f$.

**Problem A4** [*Master's Exam, Spring 2018* ]
Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Hint: prove that for all $x \neq 0$ and $n \in \mathbb{N}$ we have:

- $e^{-\frac{1}{x^2}} \le n! x^{2n}$
- $f^{(n)}(x) = e^{-\frac{1}{x^2}} p_n\left(\frac{1}{x}\right)$, where $p_n(x)$ is a polynomial function.