GE 2318

Lecture 9: Fractals

G Ron Chen

Acknowledgement: Un-copyrighted photos and pictures were taken from the Internet

From Chaos 混沌 to Fractals 分形

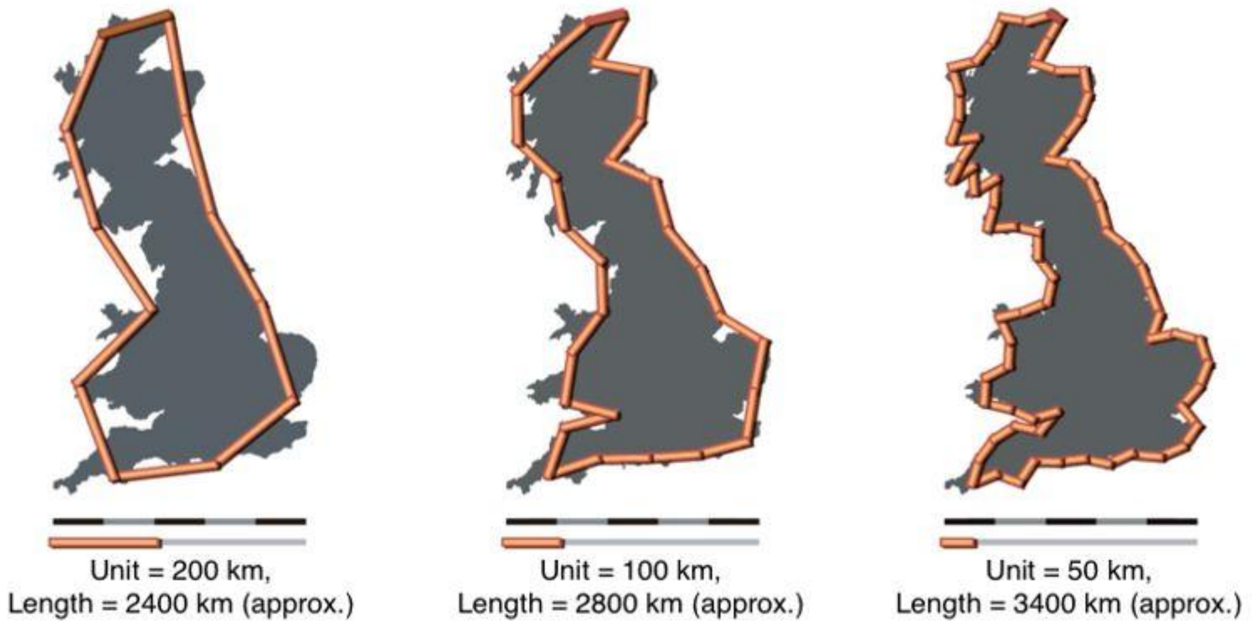
Fractals … from the beginning

What is this?

In Bible, Middle Ages (1220-1230)

source

Q: How Long is the Coast of Great Britain?



Benoit Mandelbrot
(1924 – 2010)

A: It depends on the length of your ruler

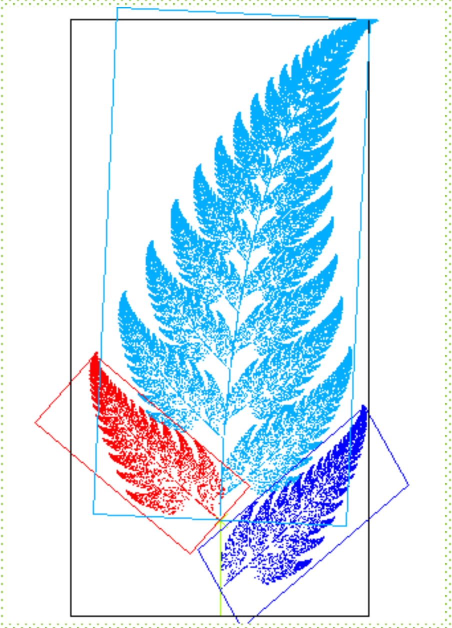
Benoit Mandelbrot

Professor of Mathematical Sciences
Yale University

Coined the word "fractal" in 1975

Fractals

- “Fractus” – Latin adjective, meaning “to break”, so as to create irregular fragments
- Fractal – a never ending pattern that repeats itself at different scales
  - Pattern within pattern
  - Self-similarity
- Each part is like the whole, but smaller (or larger)



Benoit Mandelbrot: [Fractals and the art of roughness]

Self-Similarity in Nature

Romanesco Broccoli

Fern

Sea Shell

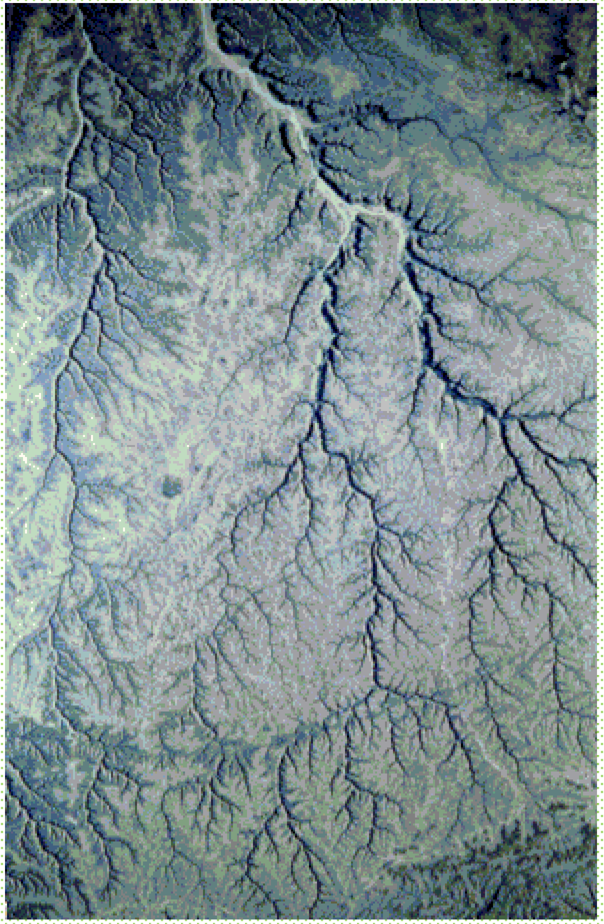
Lightening

Fractals in Nature

Self-Similarity – Drainage Patterns

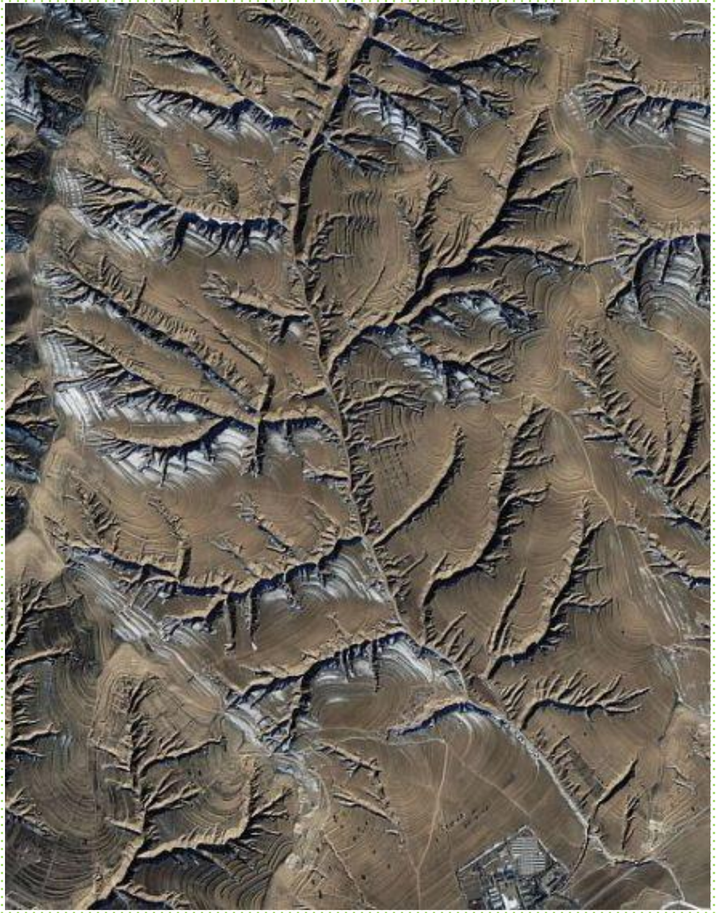
patterns, within patterns, within patterns, …

大自然的藝術 -- 特德·格拉姆比奧拍攝，展示了澳大利亞西北部金伯利地區的幹涸三角洲



Self-Similarity – Drainage Patterns

Egypt



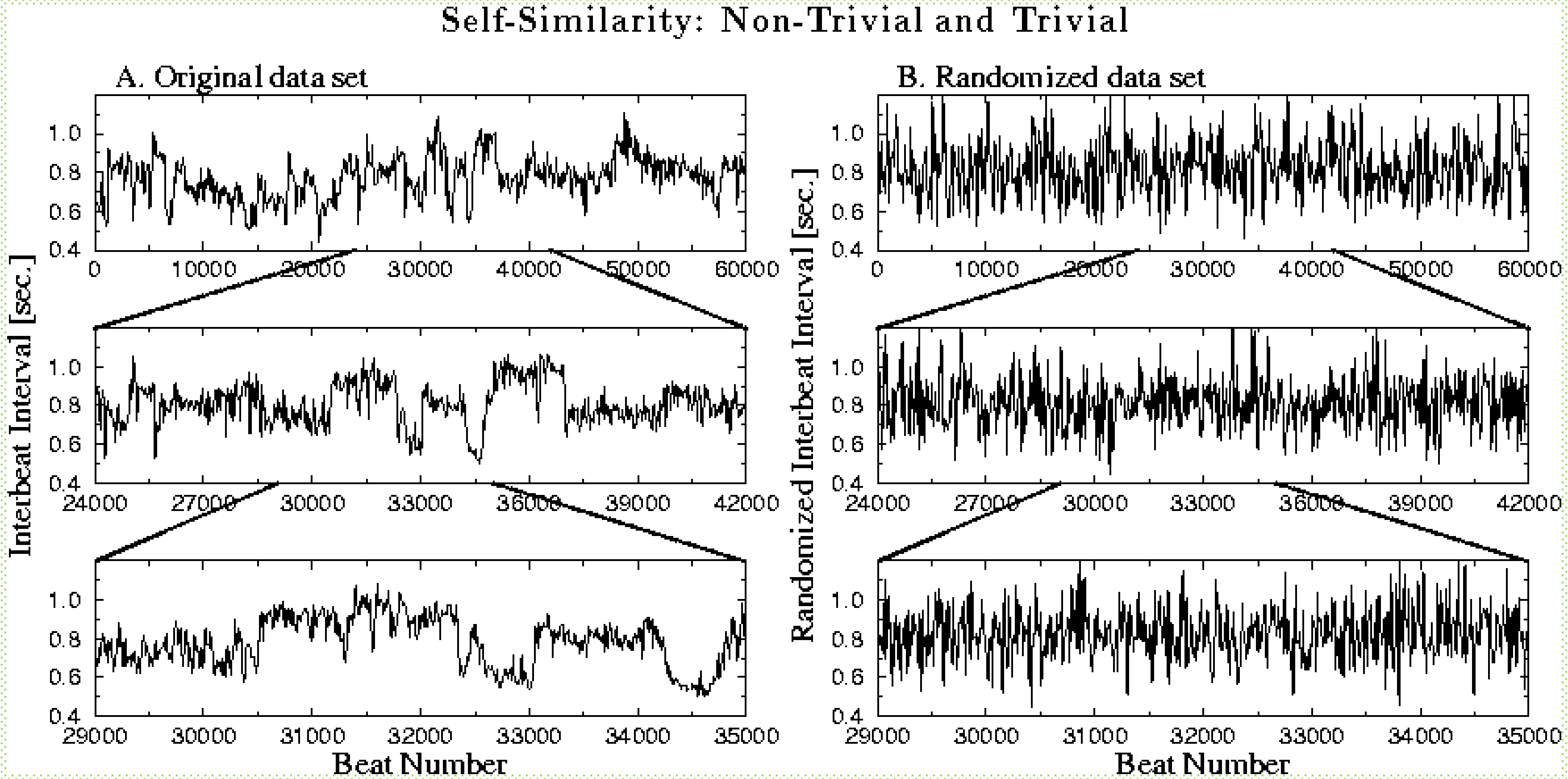
Ke Feng, China

Self-Similarity – Drainage Patterns

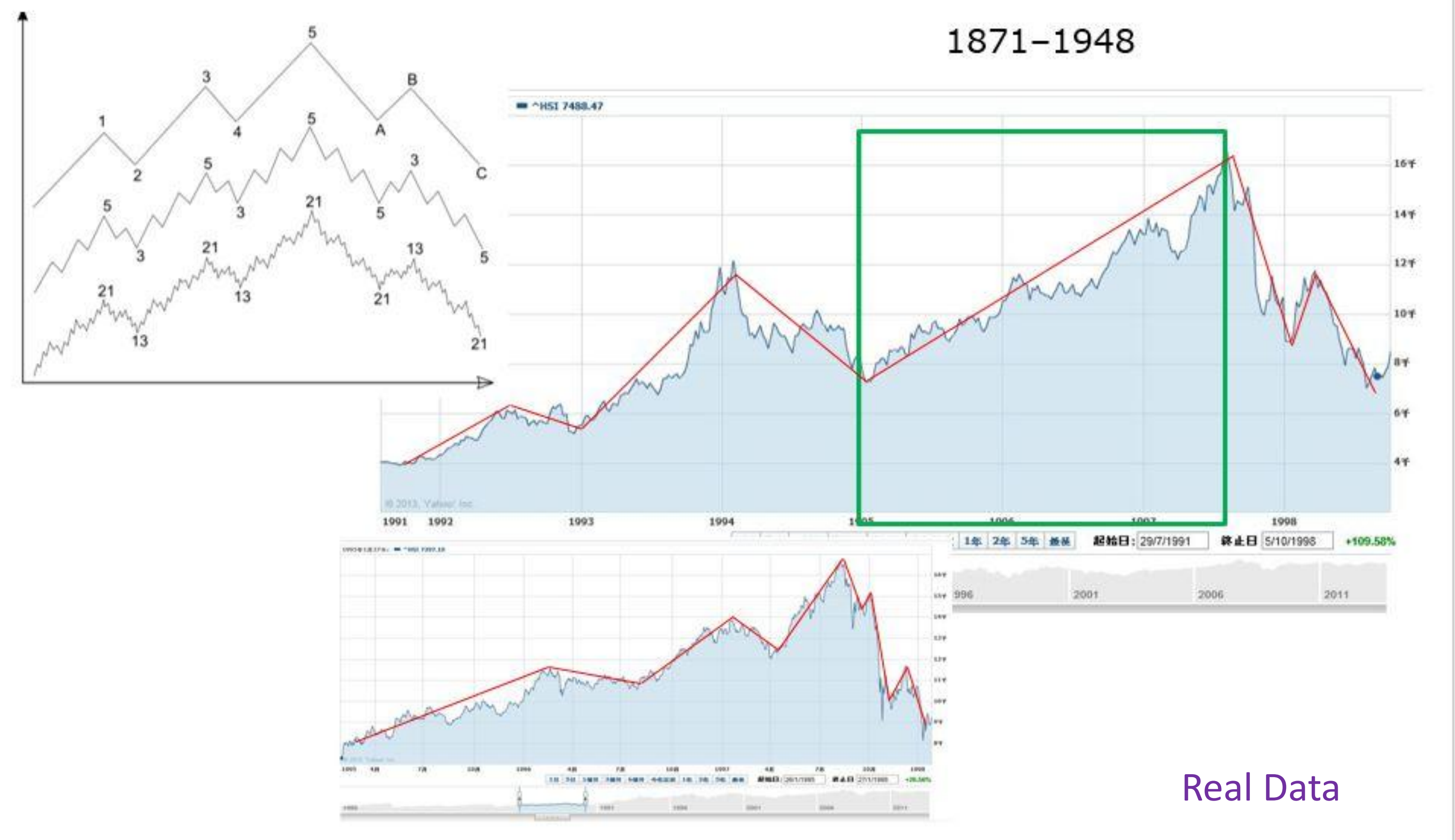
钱塘江 China

Self-Similarity in Heart Beating

Heart Beat Intervals

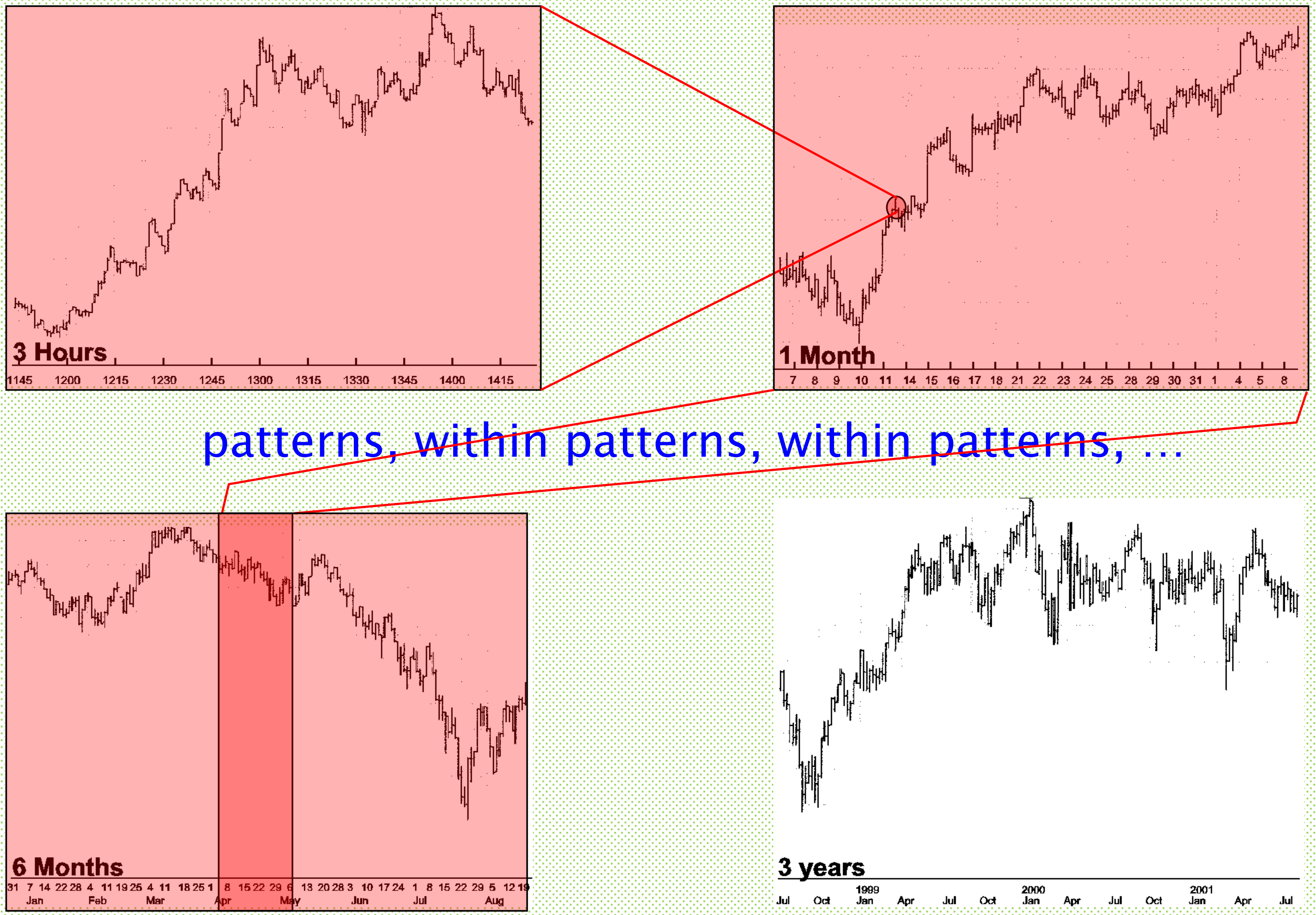


Self-Similarity in Stock Market



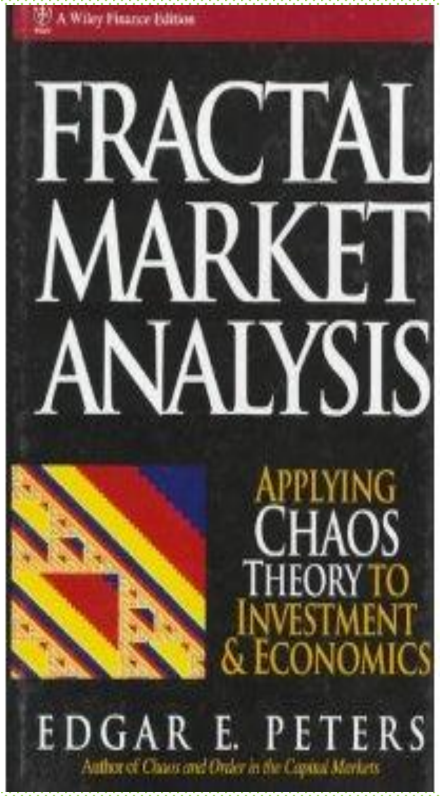
Ralph Nelson Elliott: Wave Principle (1938)

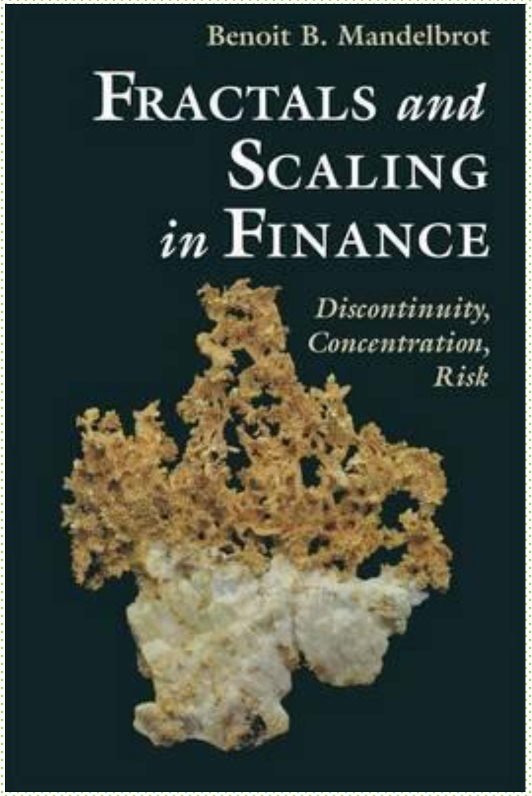
Self-Similarity – Dow Jones Average

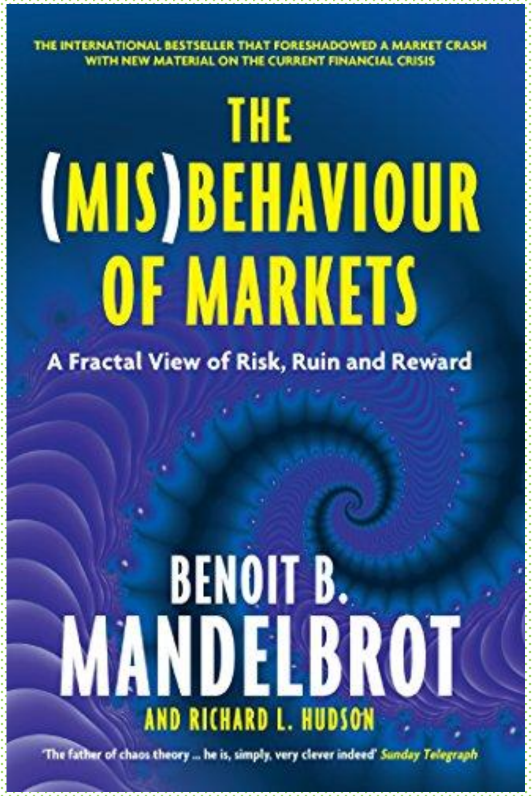


patterns, within patterns, within patterns, …

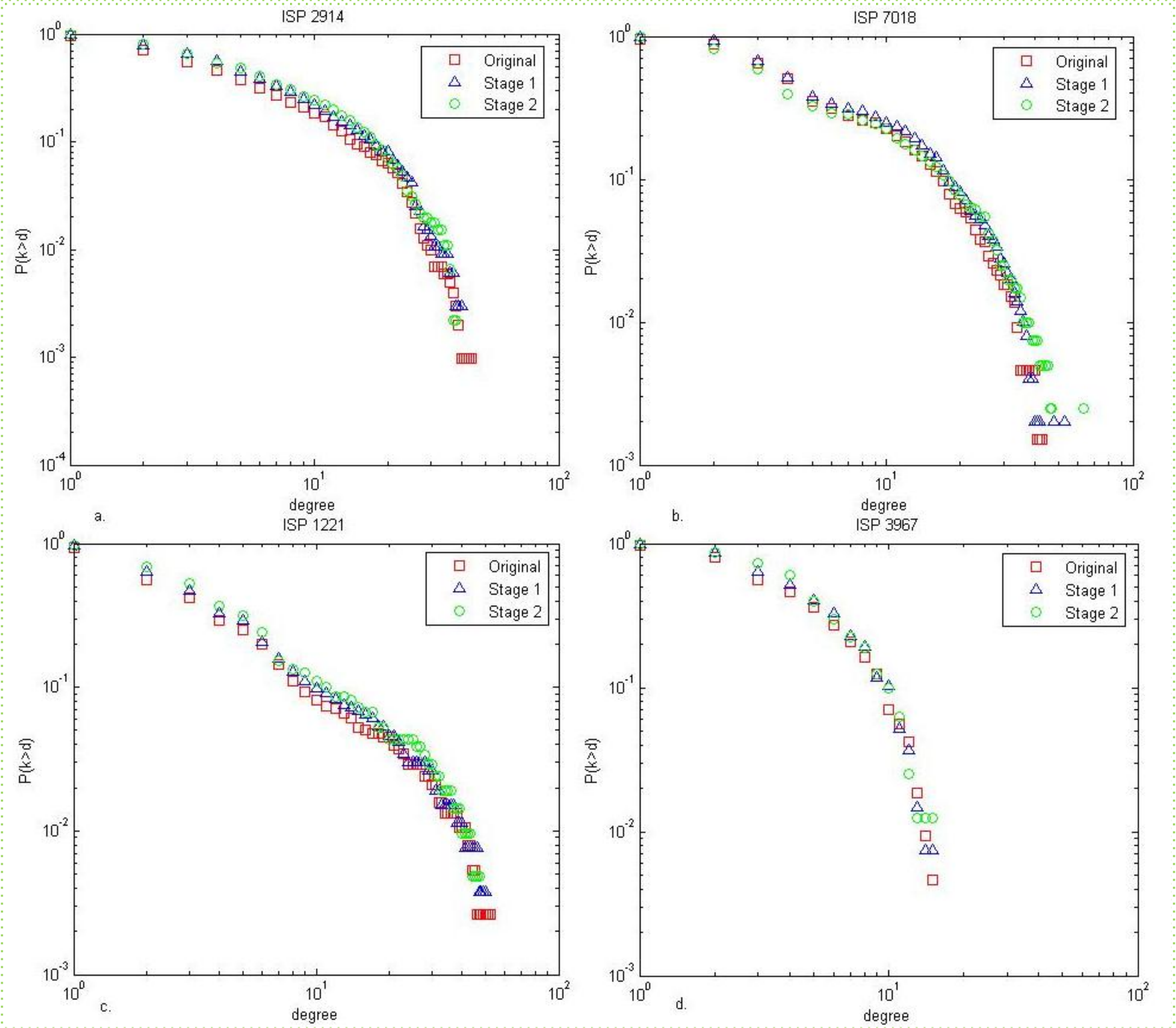
More Reading …







Self-Similarity in Internet



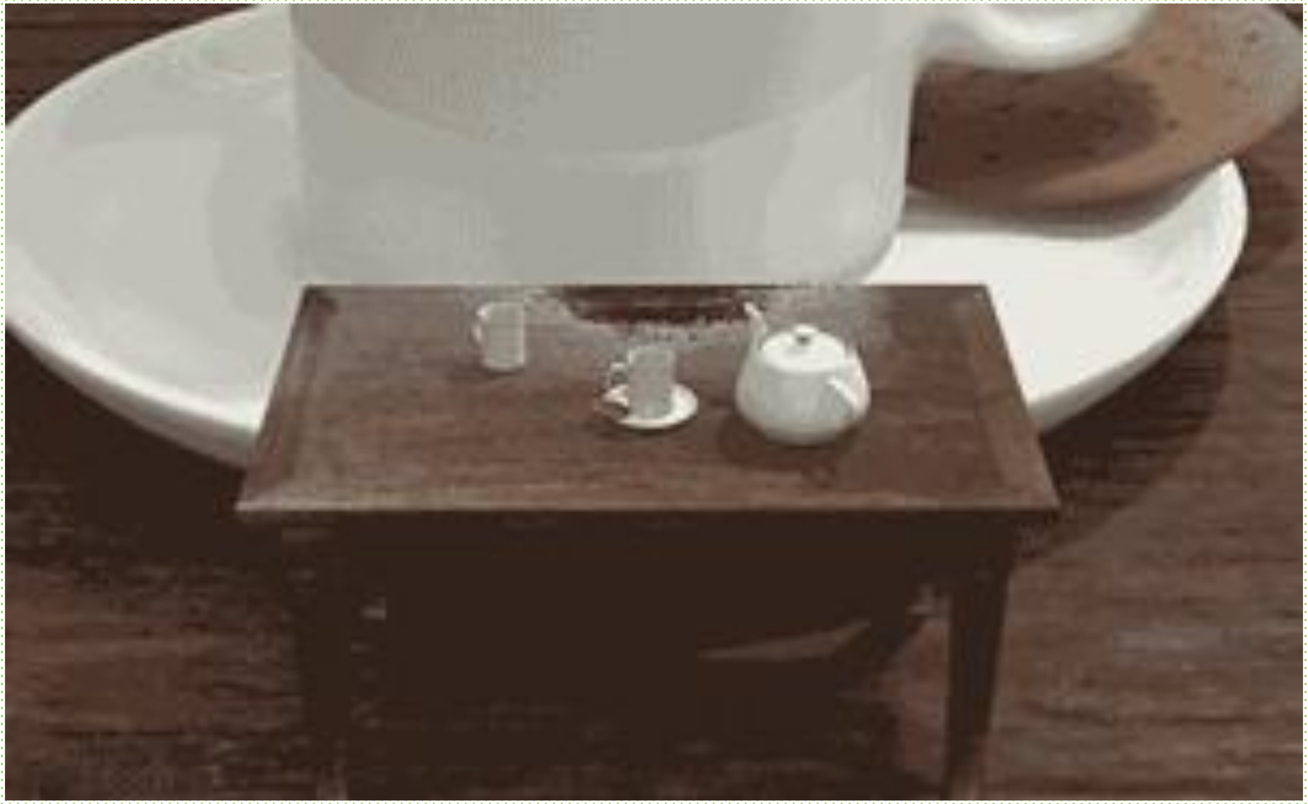
Routers Level

Distributions at
different scales:

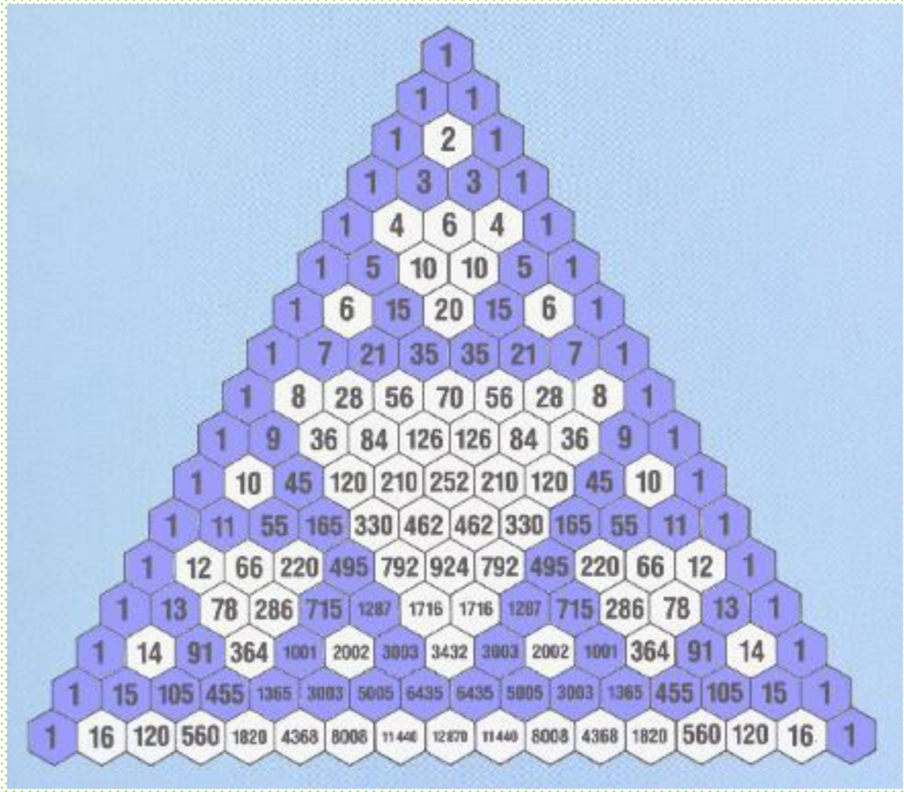
All are power-laws with
similar slopes

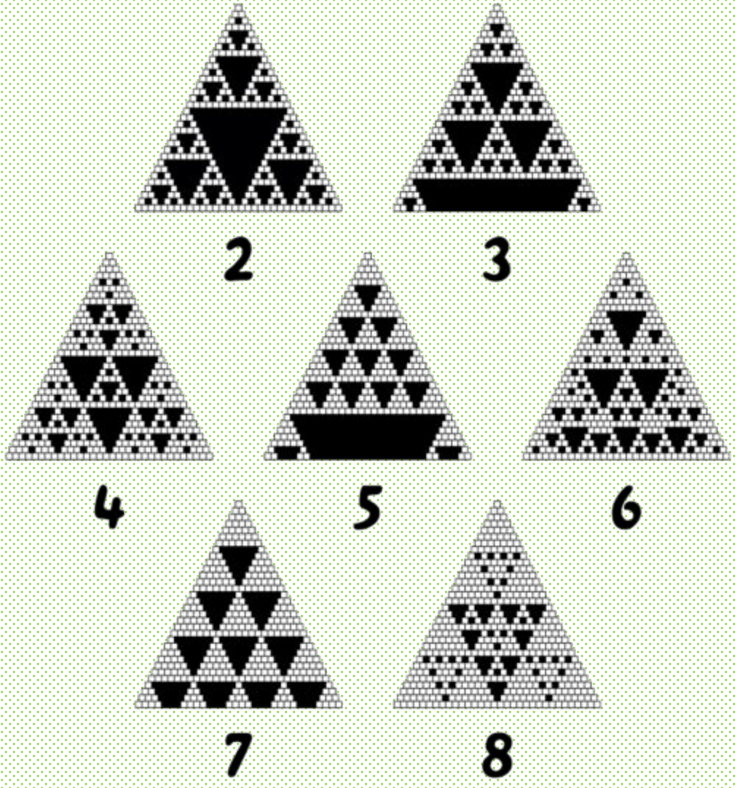
Y. Tang, K.T. Ko (2007): Scale-invariant
in terms of node-degree distributions

Self-Similarity



Pascal Triangle





Even numbers
are highlighted

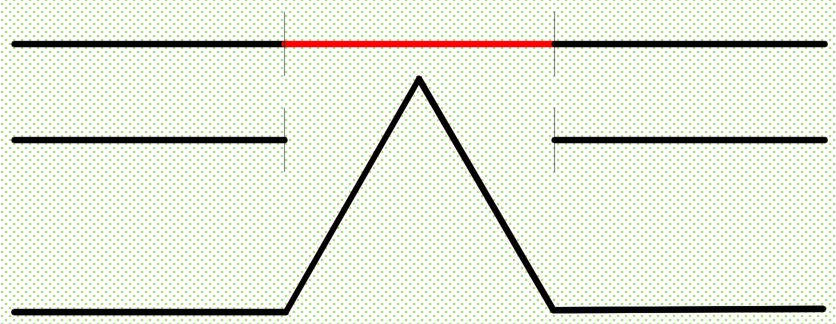
Pascal Triangle

Fractal Geometry

Fractals can be generated by iterations of a function (map)

Helge von Koch
(1870 - 1924)

Koch Curve

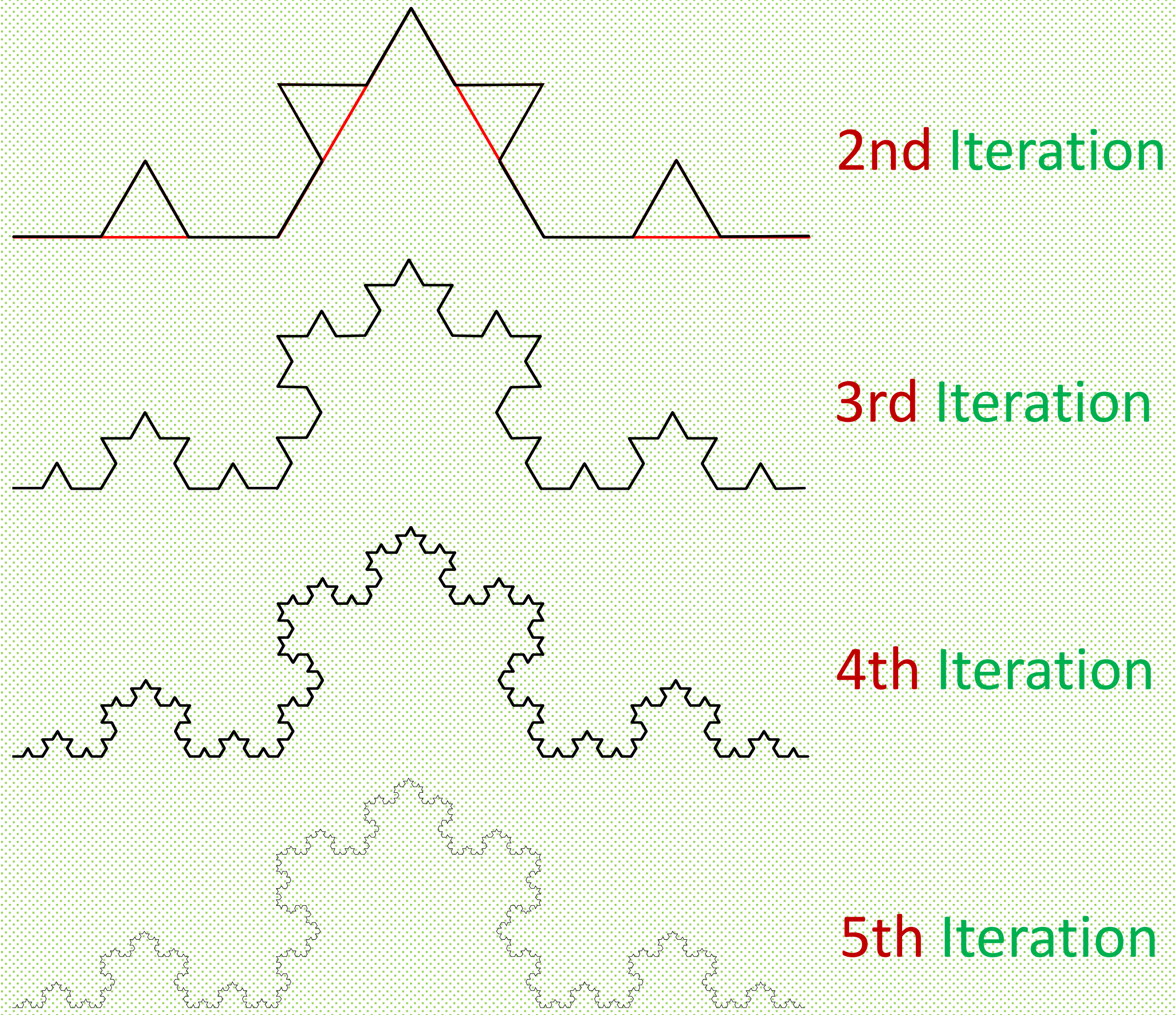


1st Iteration

1. Begin with a line
2. Divide the line into thirds
3. Remove the middle part
4. Add two lines to form a triangle in the middle third of the original line

Repeat Steps 1-4 for many times

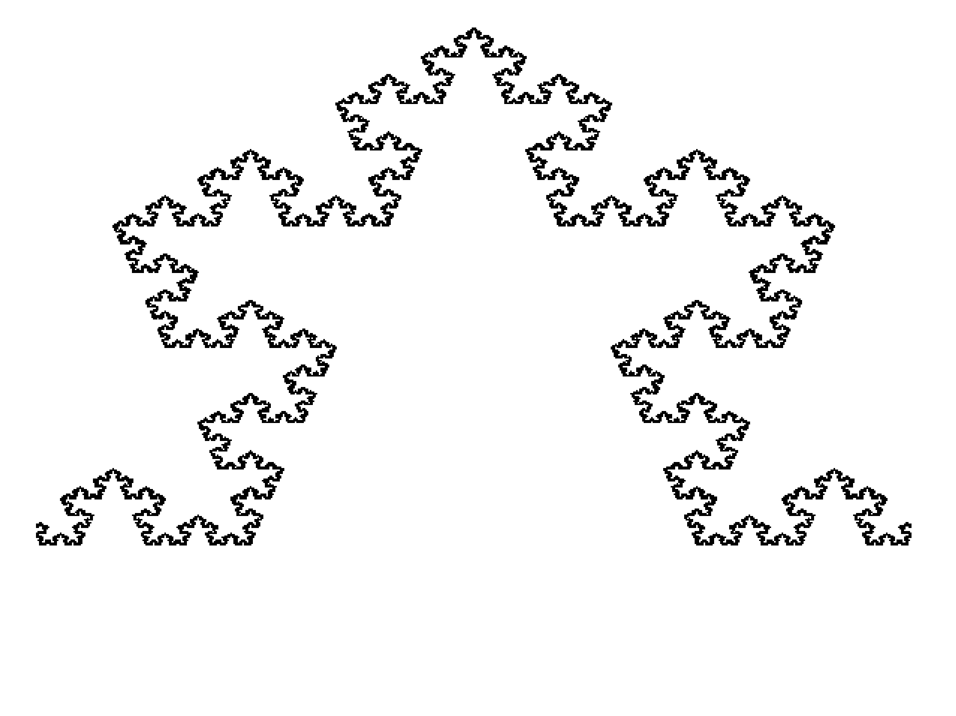
Fractal Geometry - Koch Curve

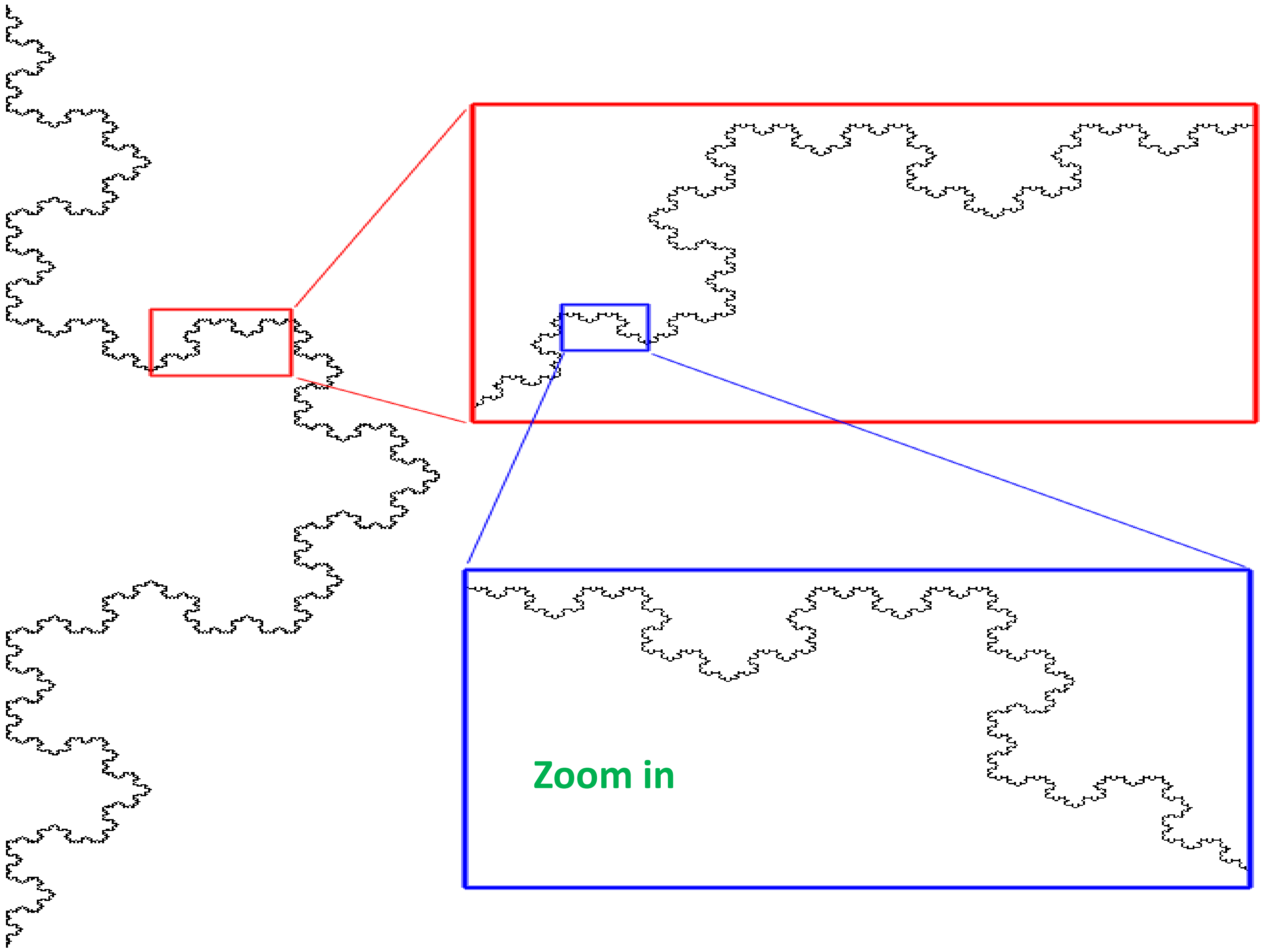


Koch Curve

Iterations

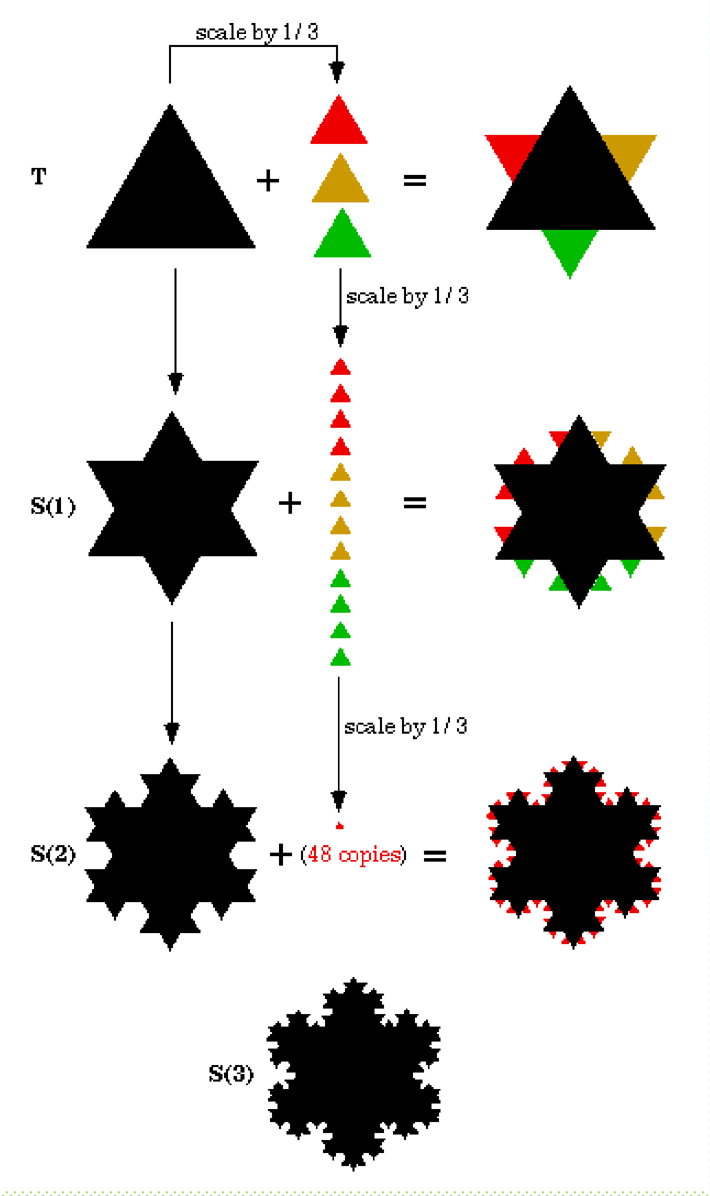
Self-Similarity



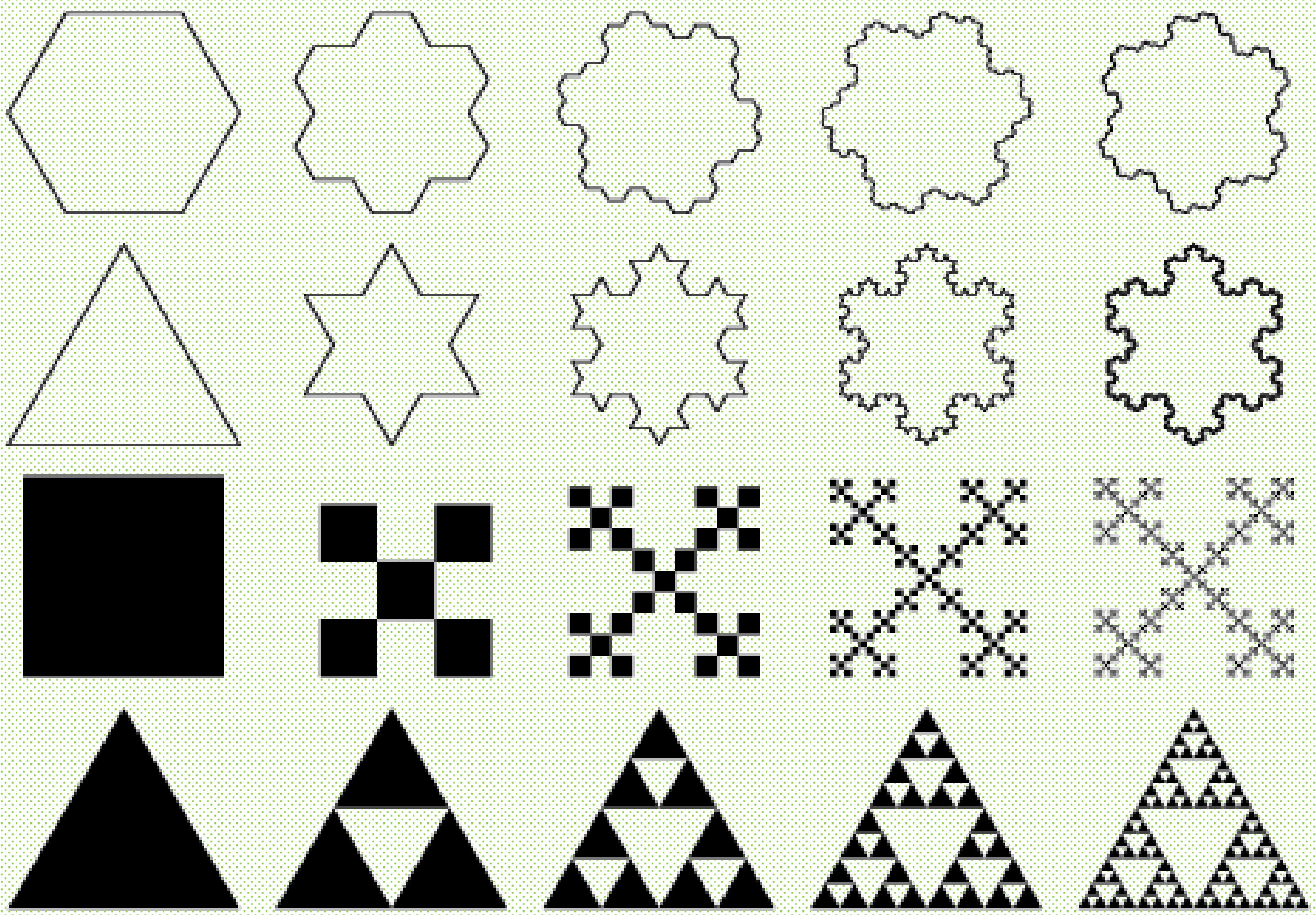

Zoom in

Koch Snowflake

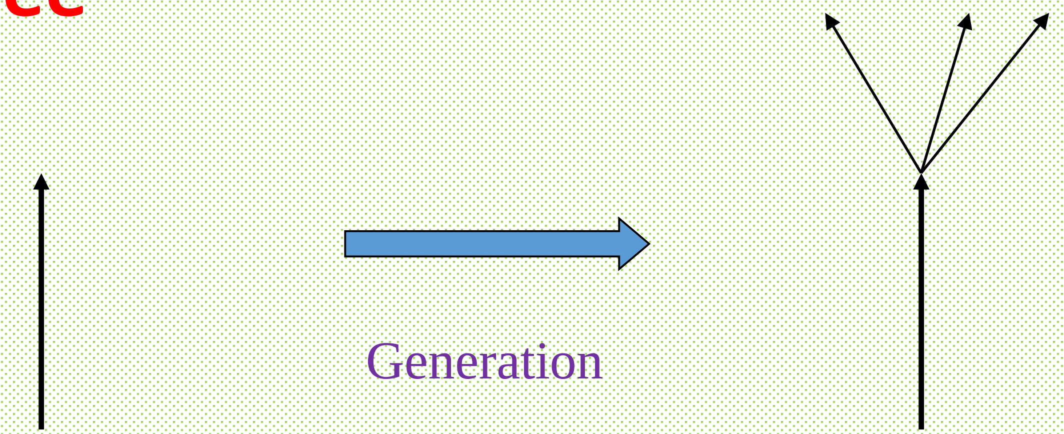
1. Start with an equilateral triangle
2. Divide each side of the triangle into three segments, and remove the middle one
3. At each side, draw two lines with the same length as the removed segment and form a small equilateral triangle
4. Repeat Steps 2-3 infinitely

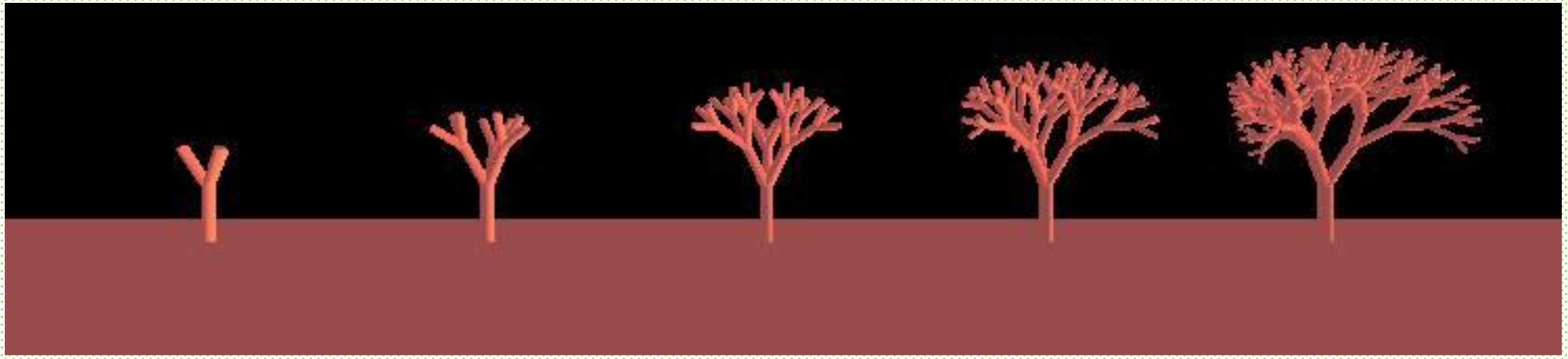


Many Ways to Generate **Fractals**



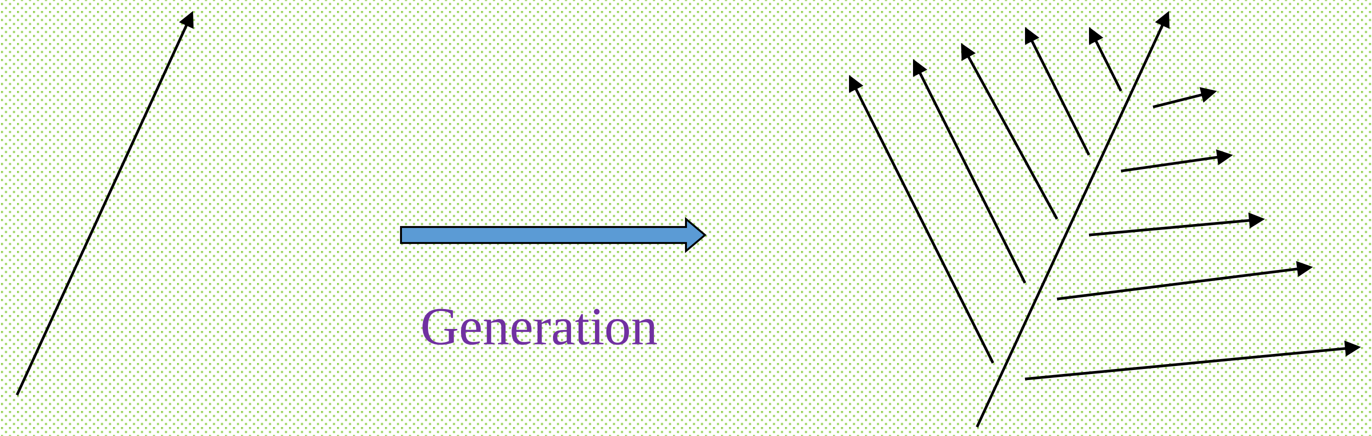
Fractal Tree

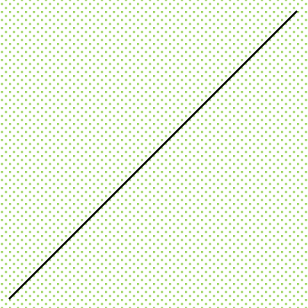




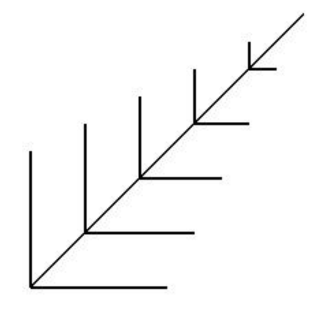
Iteration 1 Iteration 2 Iteration 3 Iteration 4 Iteration 5

Fractal Fern

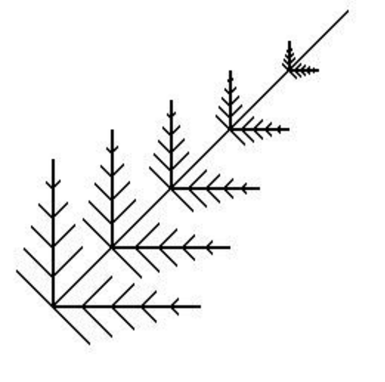




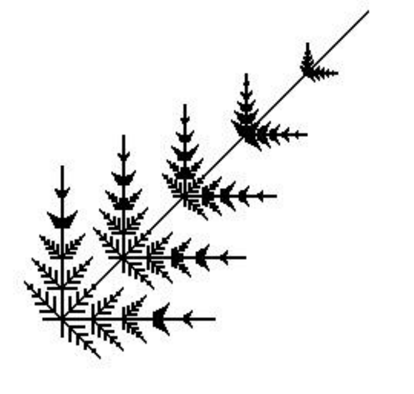
Iteration 0



Iteration 1

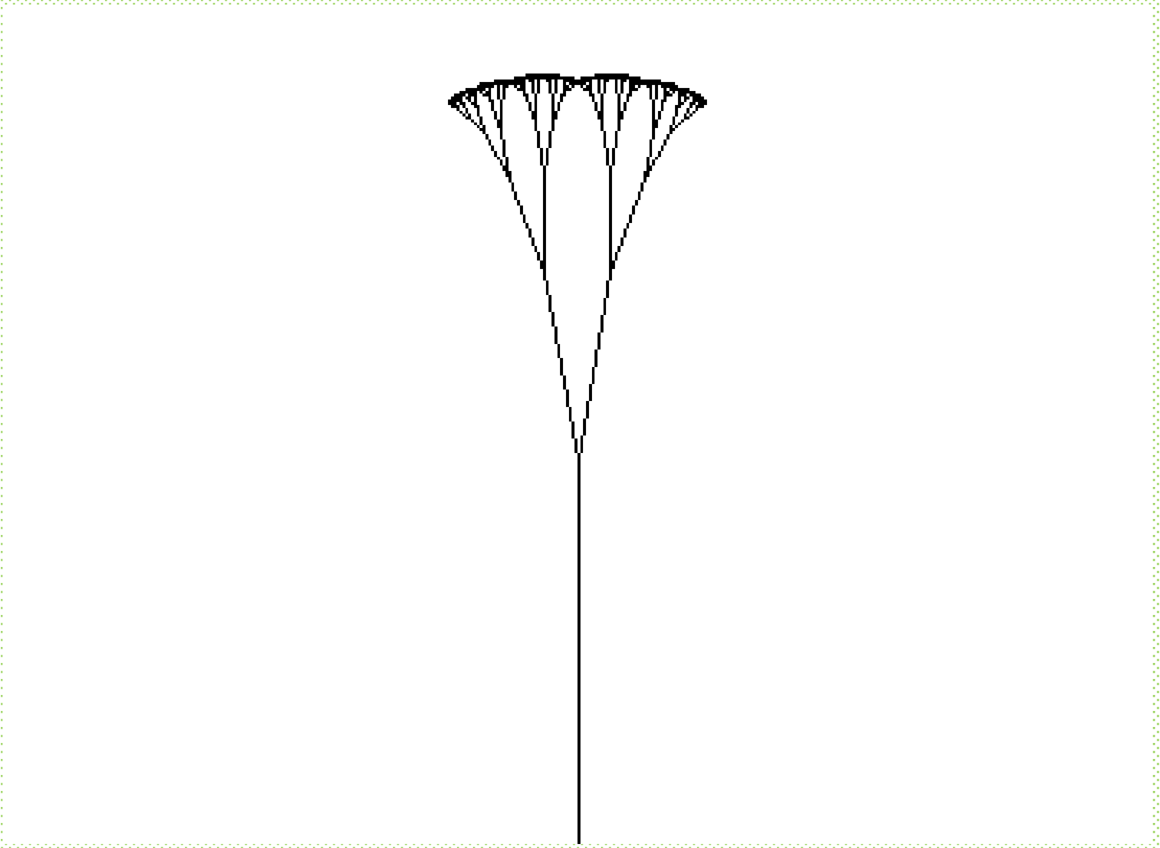


Iteration 2



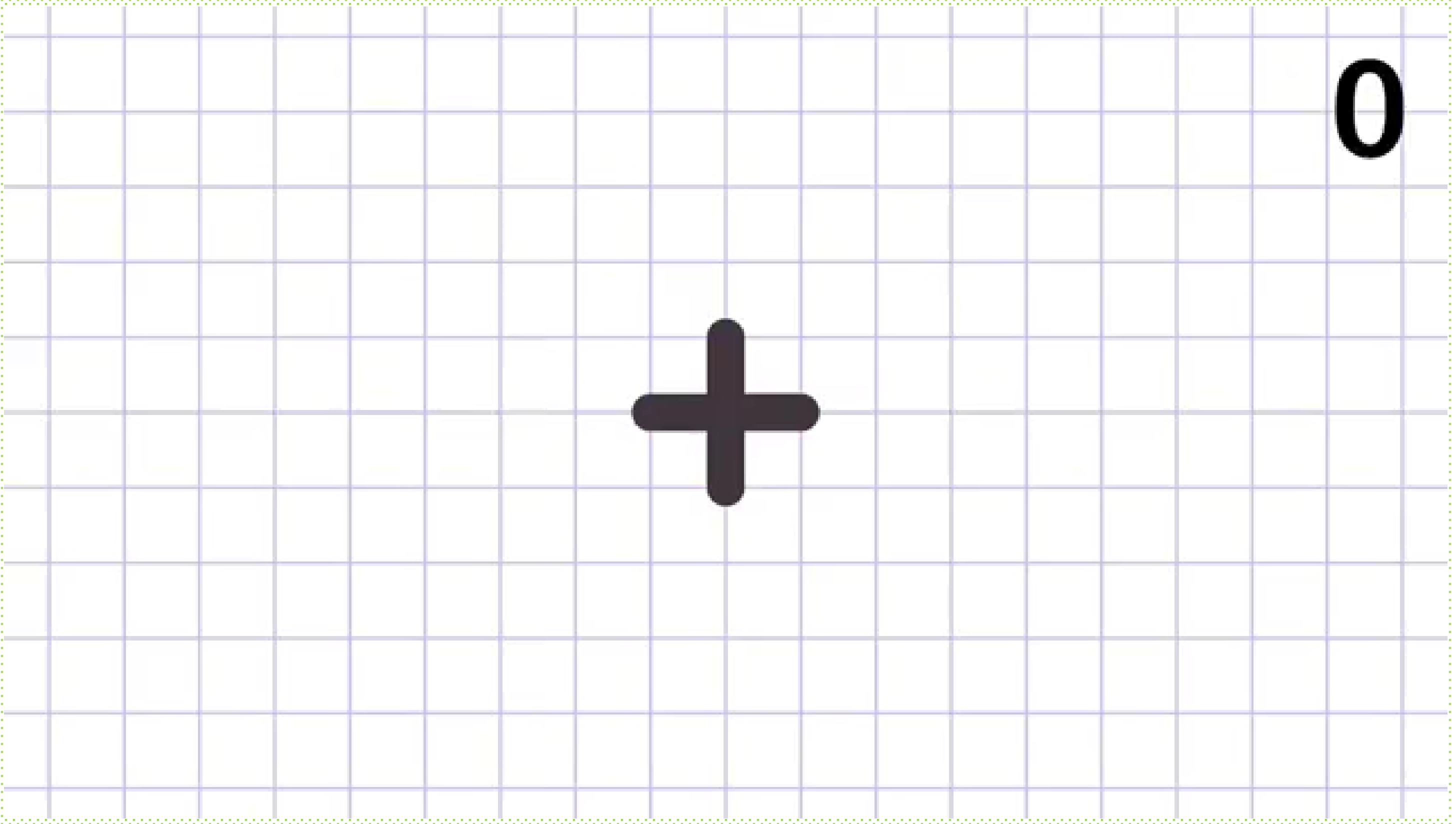
Iteration 3

Fractals

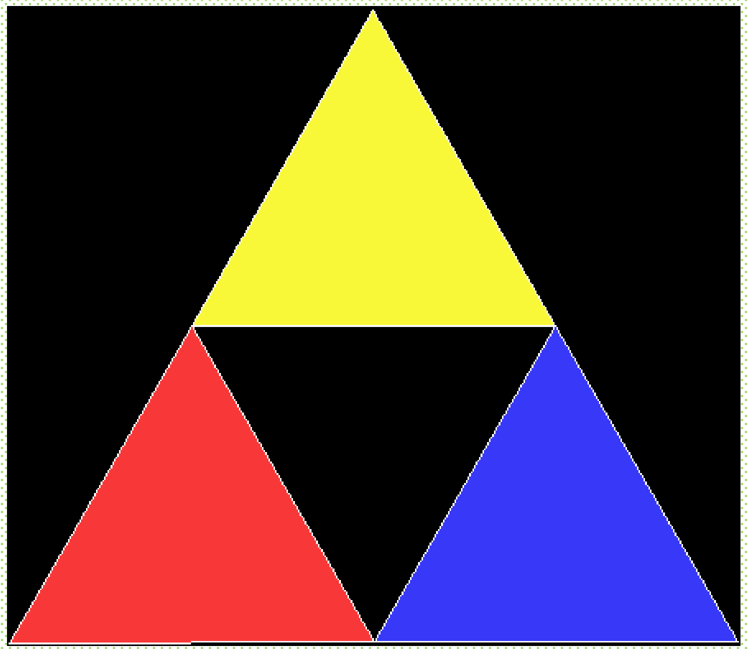


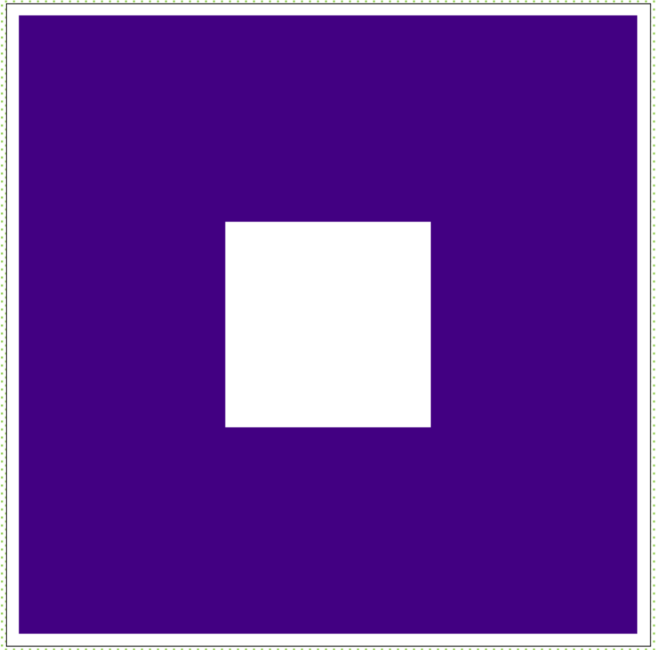
Growing Fractal Three

Growing Fractals

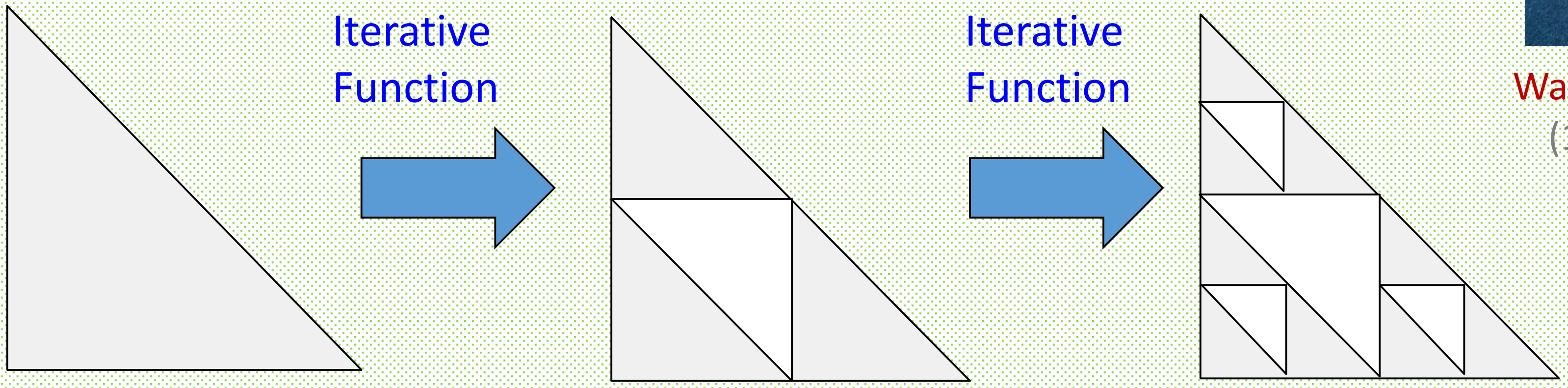


Generating Fractals

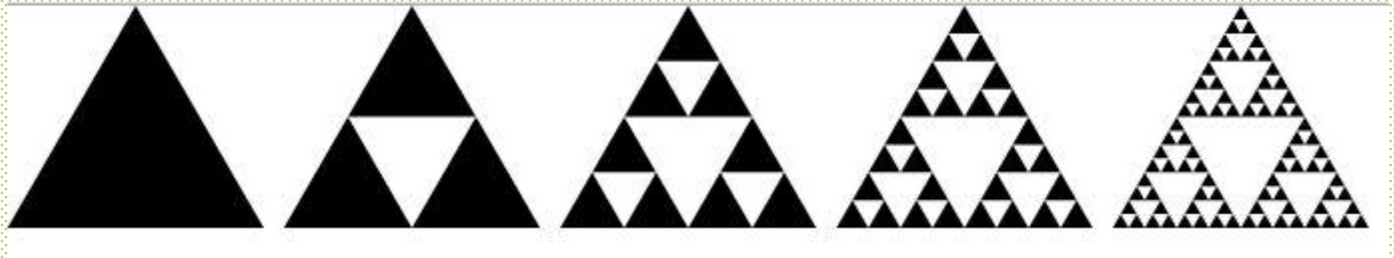




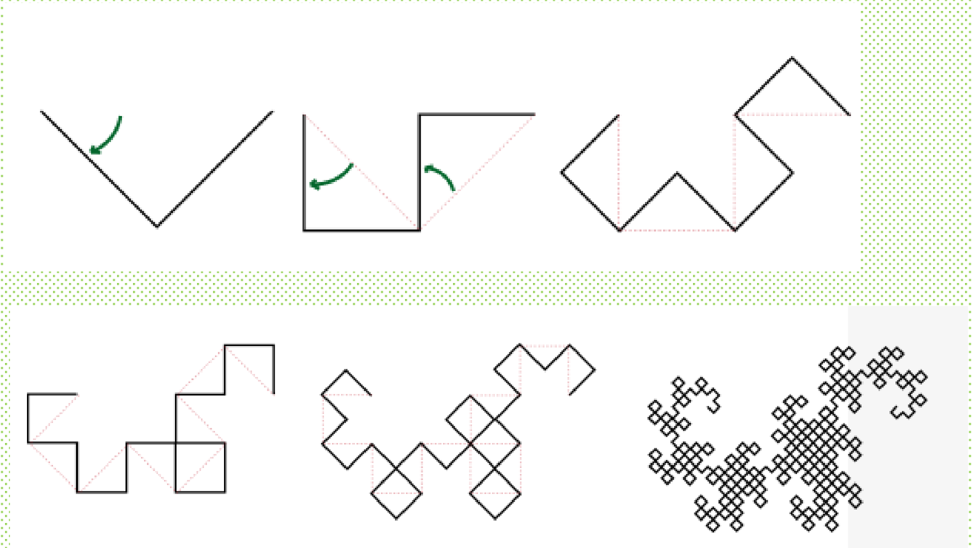
Sierpinski Triangle

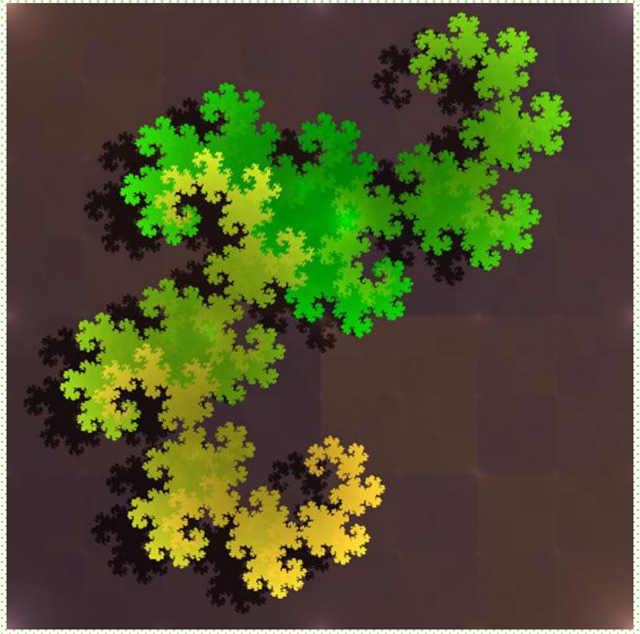


Wacław Sierpiński
(1882 – 1969)



Dragon Curve



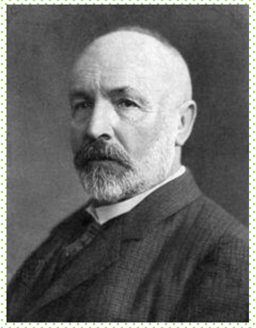


Dragon curve from paper-folding: Reference

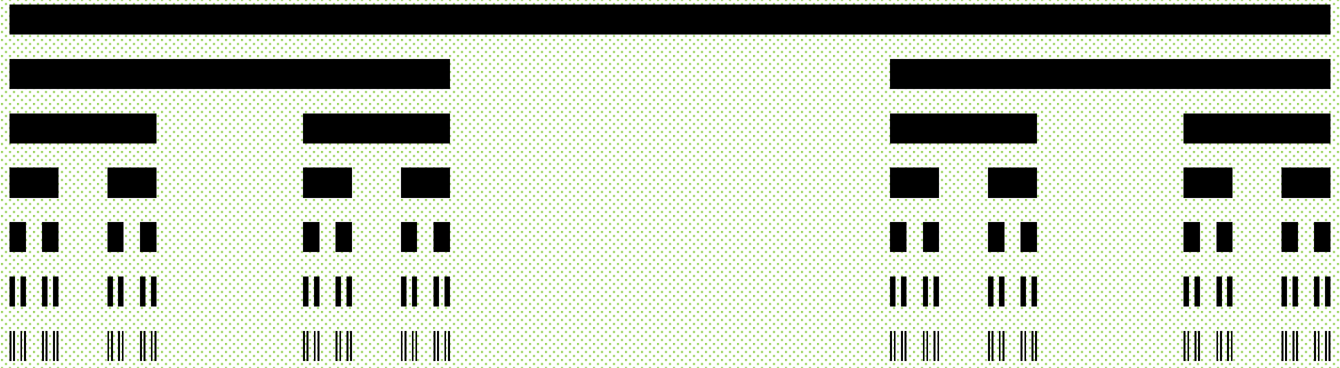
Cantor Set

Generation of a Cantor Set:

- Start with a line of unit length
- Divide it into three segments and remove the middle one
- Repeat the process infinitely



Georg F.L.P. Cantor
(1845-1918)



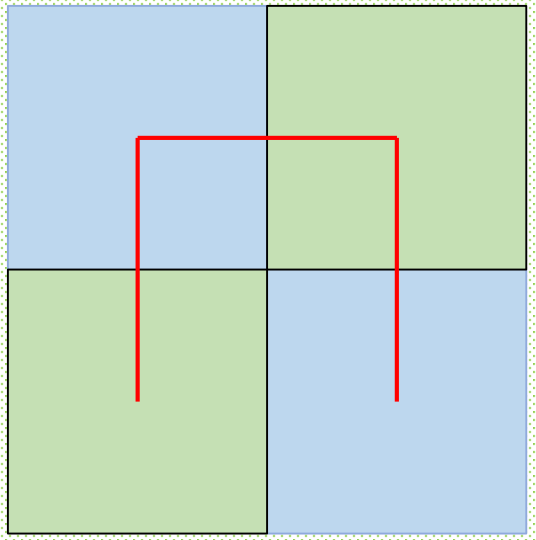
Self-similarity

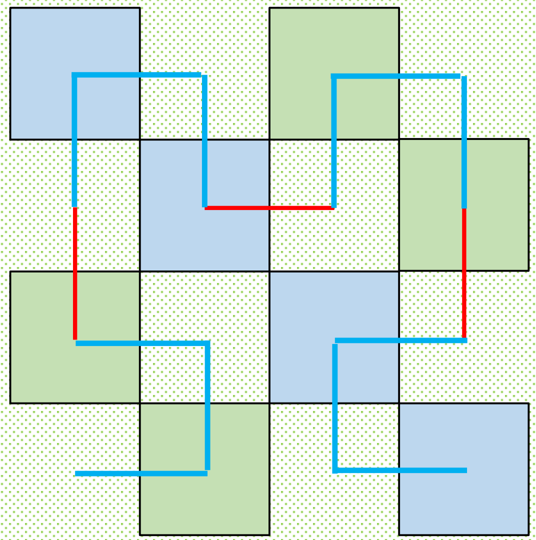
Curve Filling Curves

Question: Can a curve fill up a unit square?

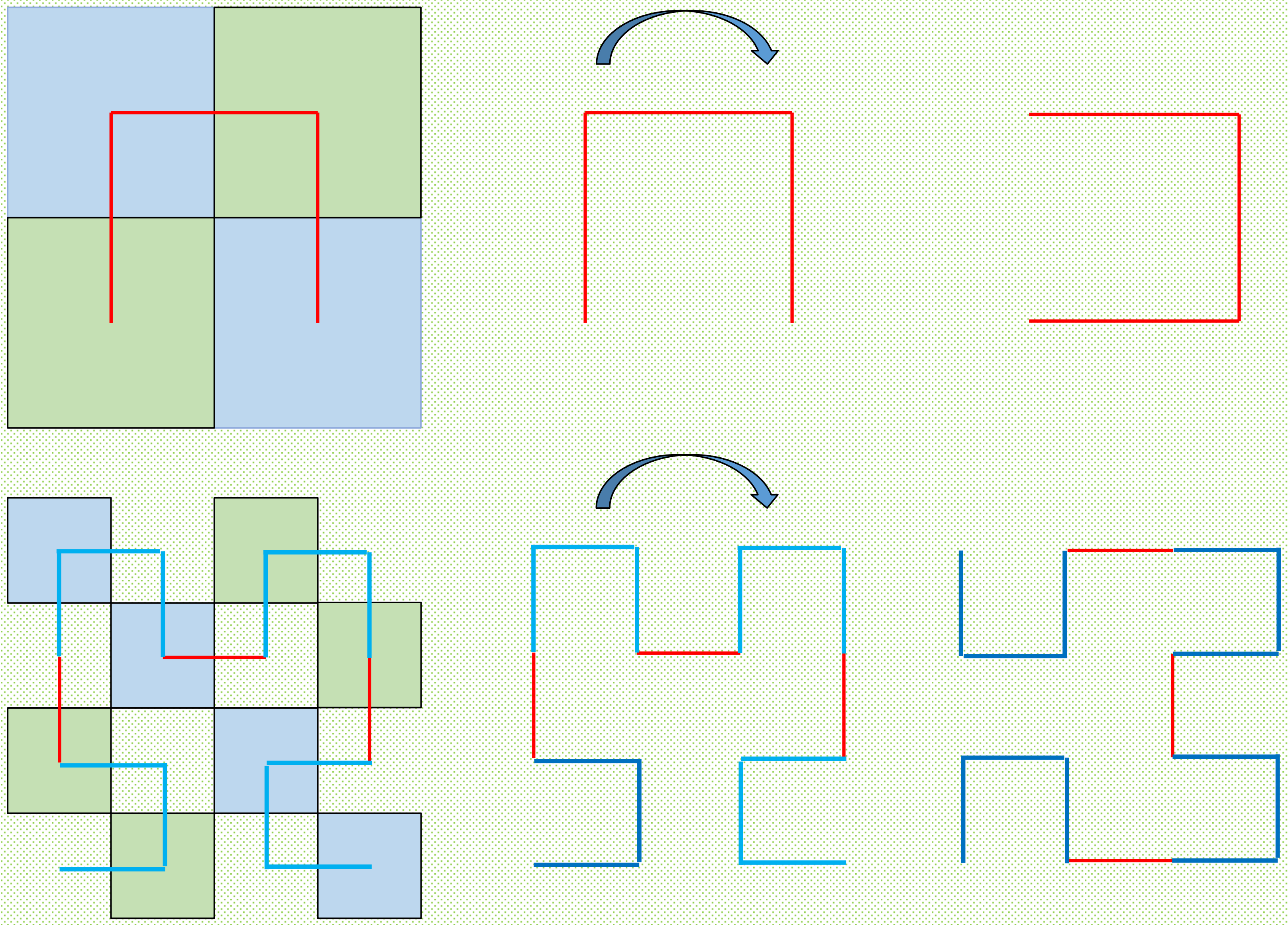
Hilbert Curve

- Created by David Hilbert (1891)
- Start with a string and lay it over a grid of squares (the string should pass through each square once and once only, without crossing)
- Repeat this infinitely many times

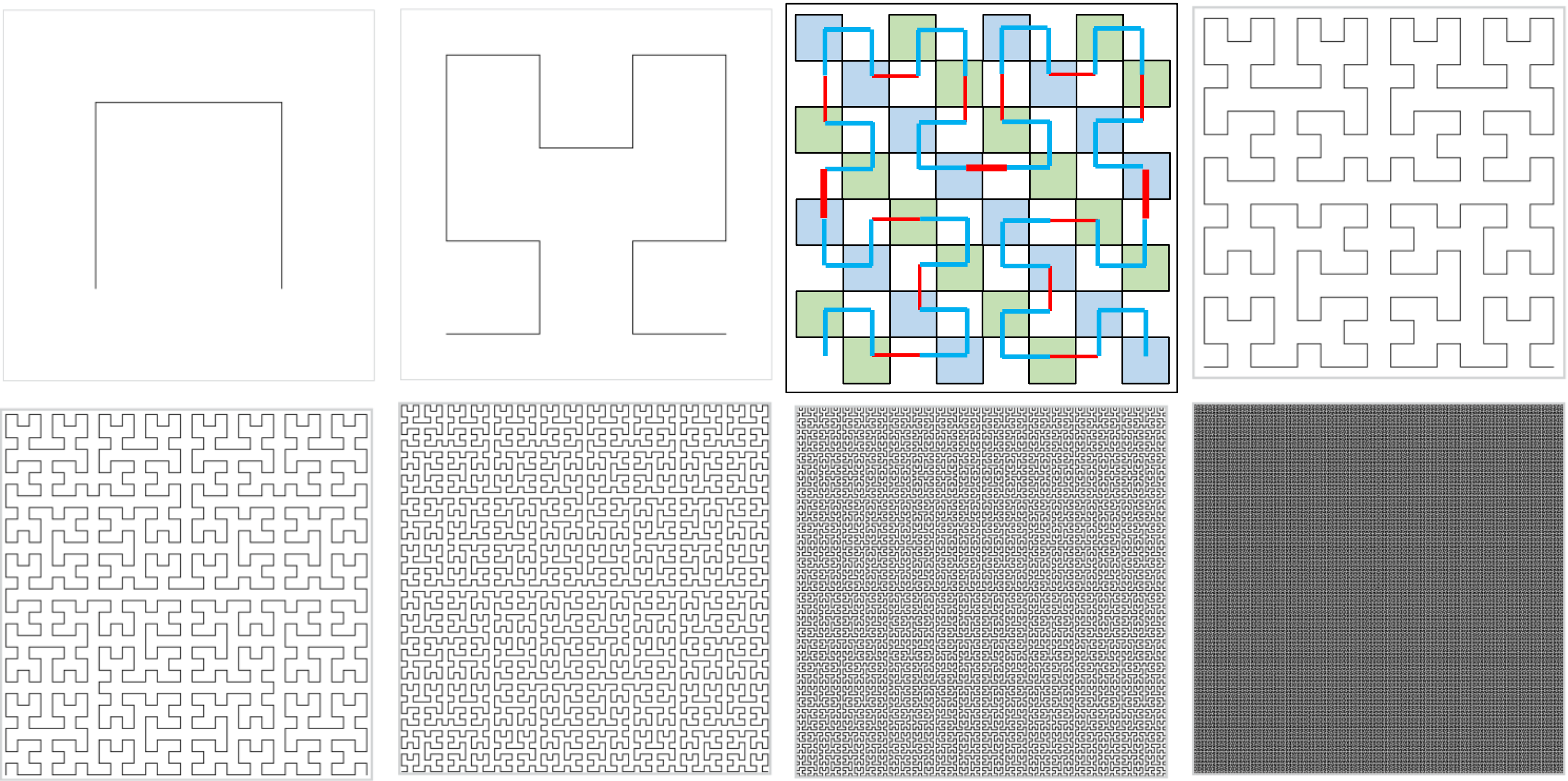




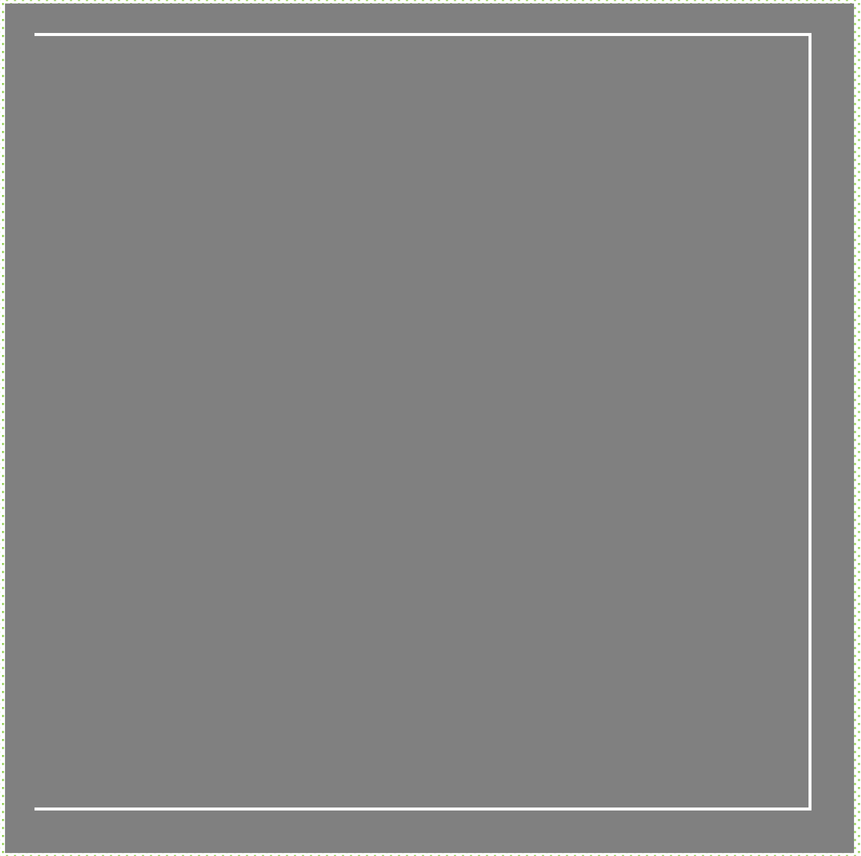
Hilbert Curve



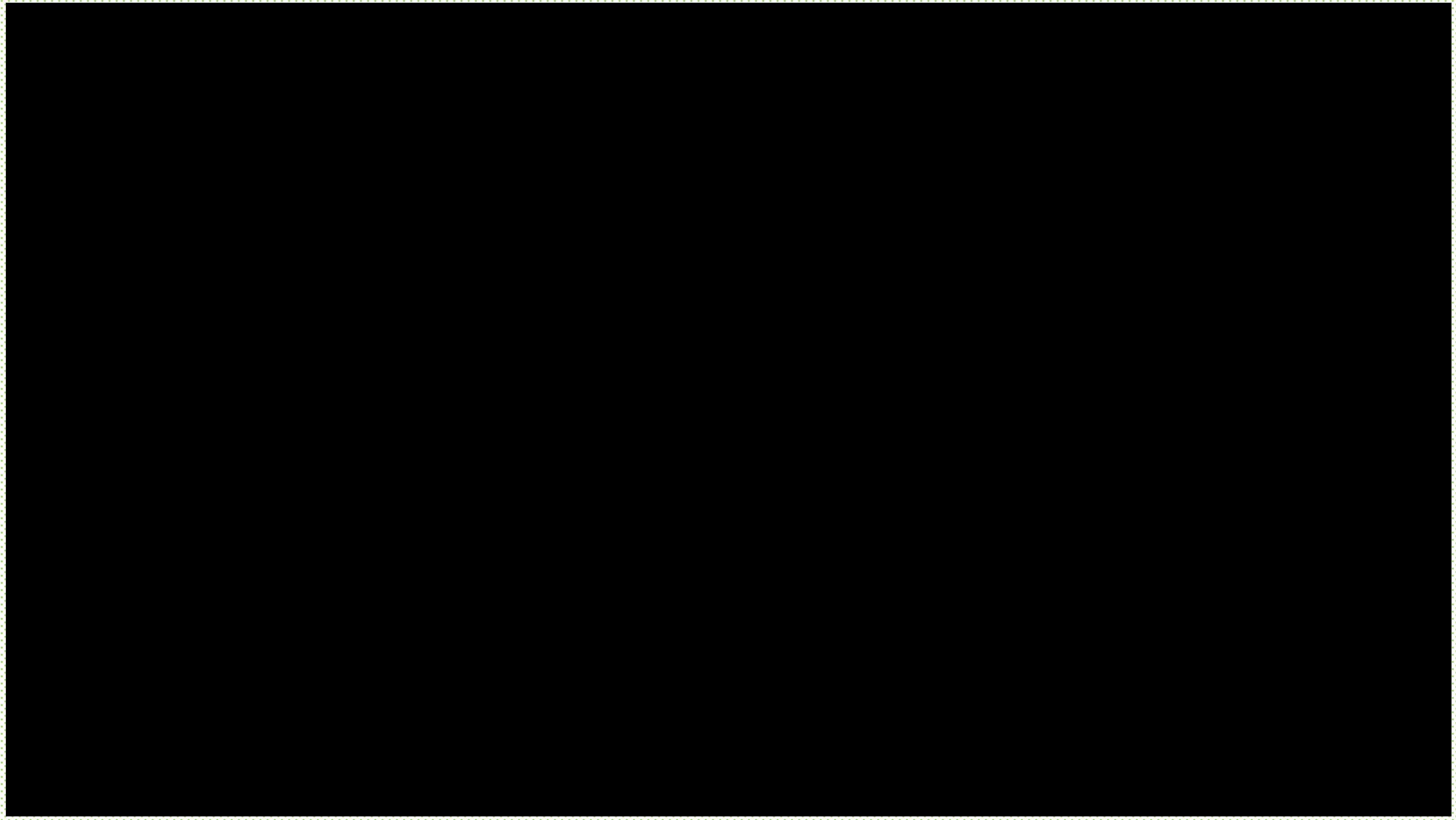
First 8 orders of the Hilbert curve



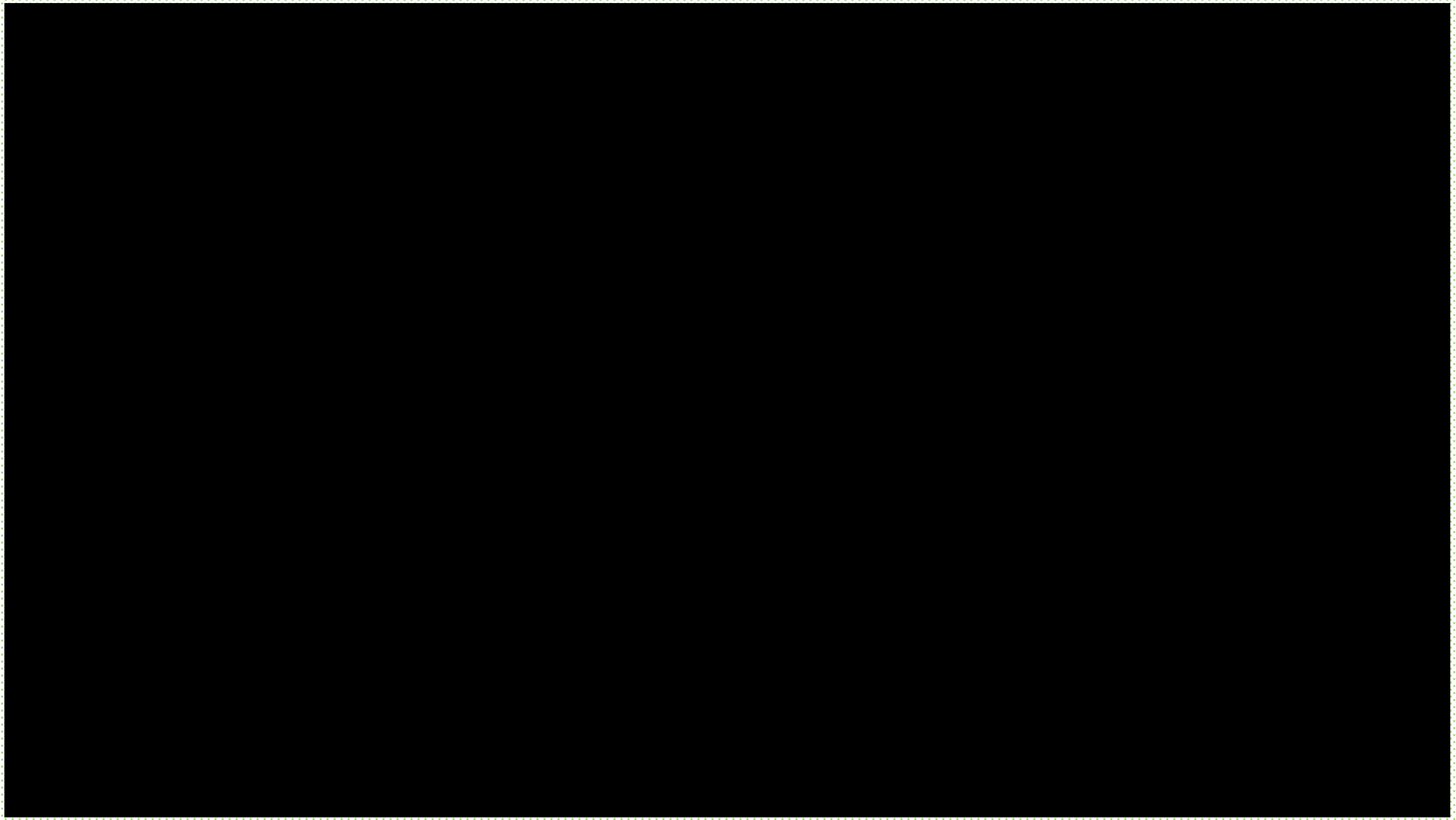
Hilbert Curve



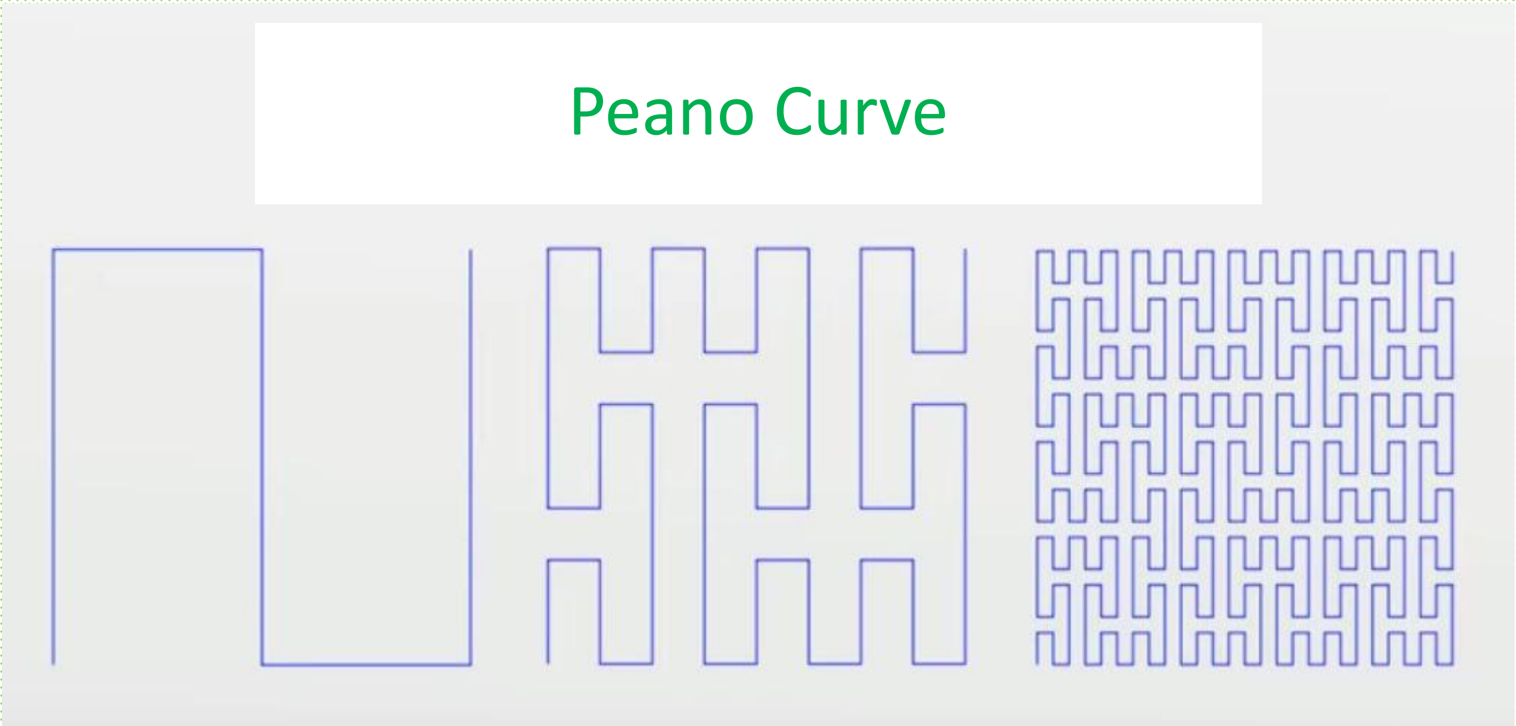
2D Hilbert Curve



3D Hilbert Curve



More Space Filling Curves

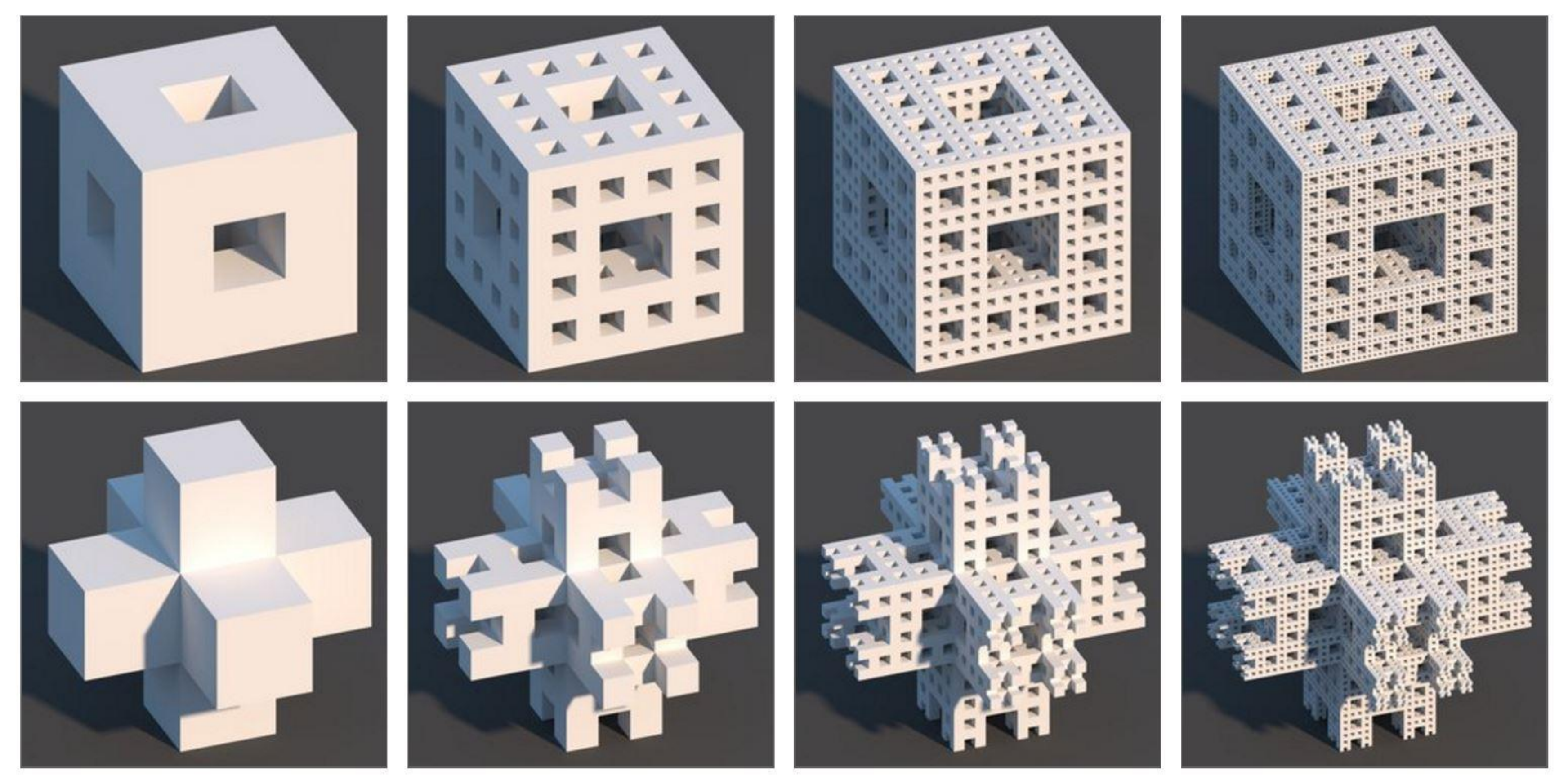


Peano Curves

Space Filling Curves

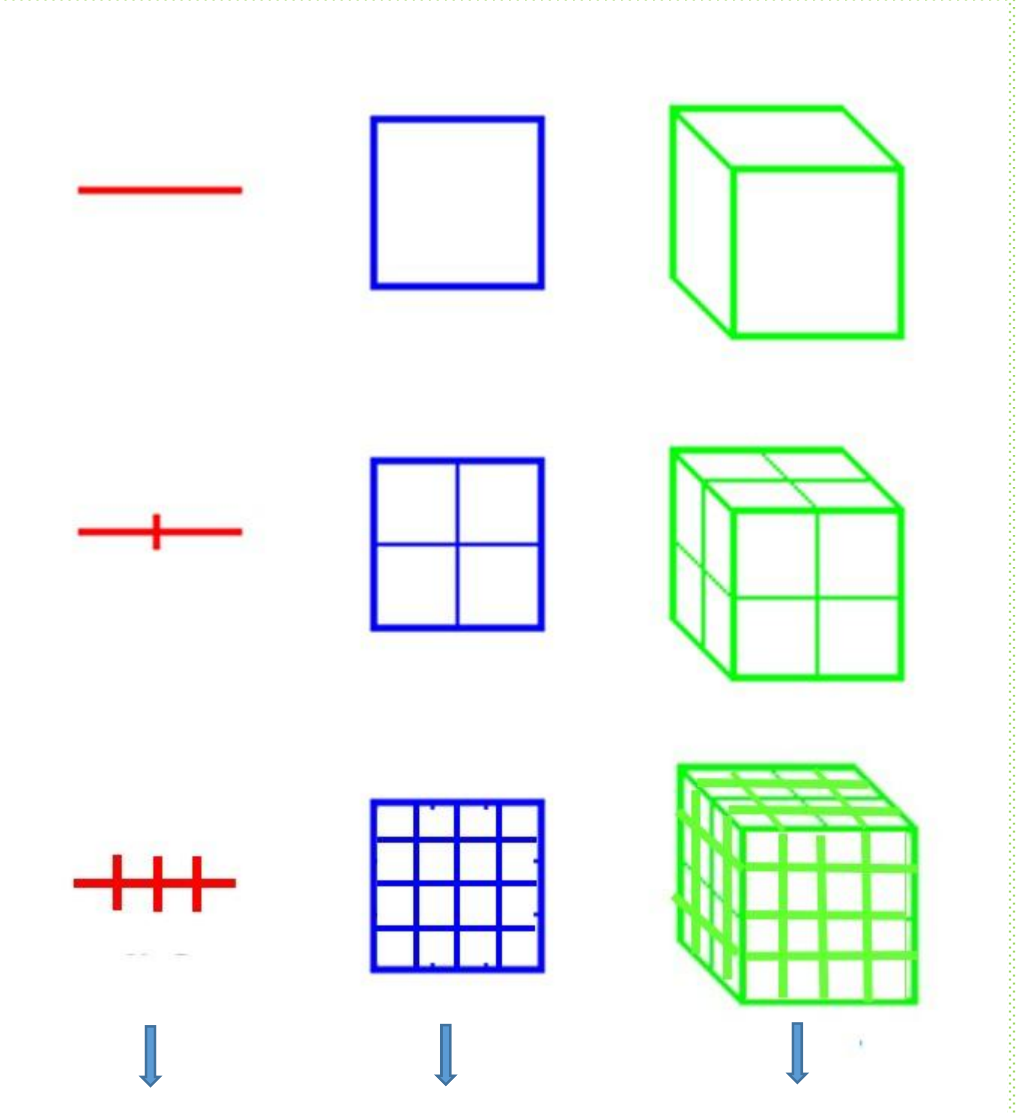
More … …

Generating Fractals



Fractal Dimension

D – Dimension
N – Number of self-similar pieces that each piece produces
L – Length of each new side (Let $R = 1/L$)



Red line

$D = 1$

$N = 2, \quad L = 1/2$

$R = 1/L = 2$

Blue square

$D = 2$

$N = 4, \quad L = 1/2$

$R = 1/L = 2$

Green cube

$D = 3$

$N = 8, \quad L = 1/2$

$R = 1/L = 2$

Dimension formula:

$D = \log(N)/\log(R)$

Verifying:

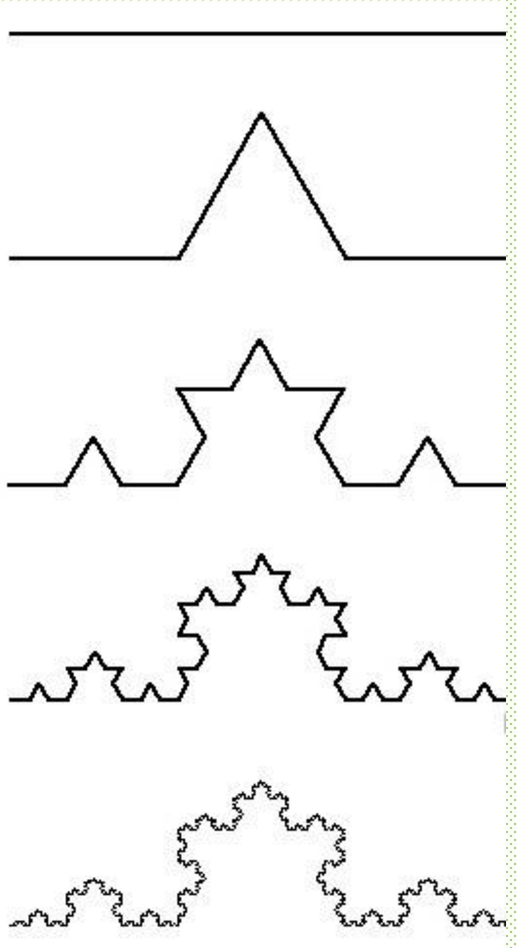
$D = \dfrac{\log(2)}{\log(2)} = 1$ Red

$D = \dfrac{\log(4)}{\log(2)} = 2$ Blue

$D = \dfrac{\log(8)}{\log(2)} = 3$ Green

$$\boxed{D = \log(N)/\log(R)}$$

Example: Koch Curve



Number of self-similar pieces:

$N = 4,\ L = 1/3$

$R = 1/L = 3$

$D = \log(N)/\log(R)$

$= \log(4)/\log(3)$

$= 1.26$

$\boxed{D = \log(N)/\log(R)}$

Example: Sierpinski triangles

Number of self-similar pieces:

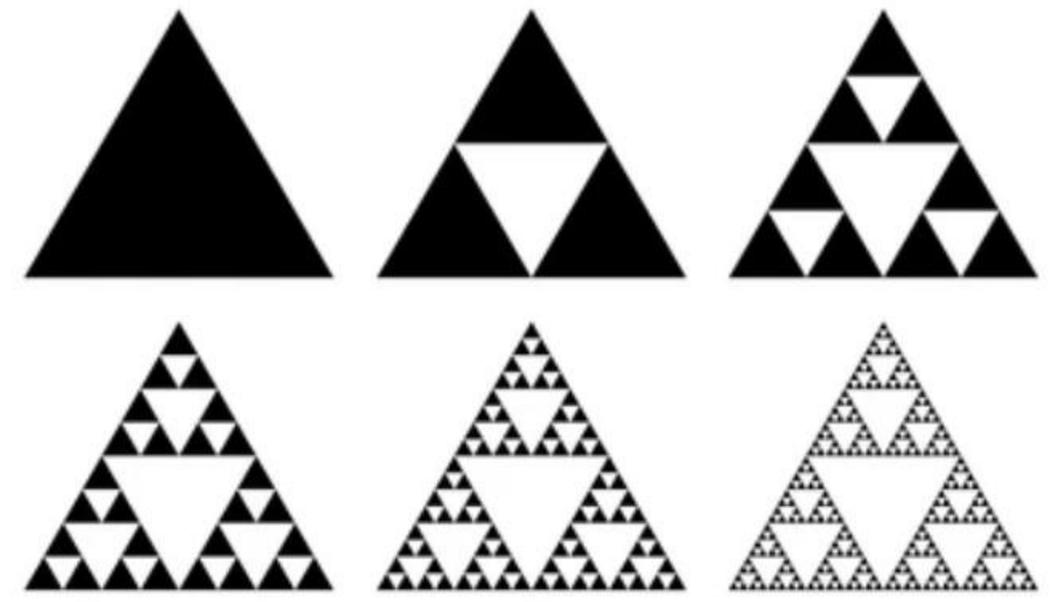
$N = 3,\ L = 1/2$

$R = 1/L = 2$

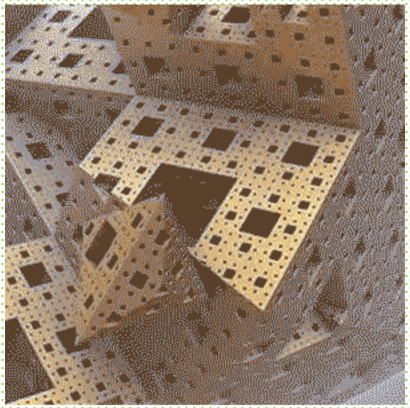
$D = \log(N)/\log(R)$

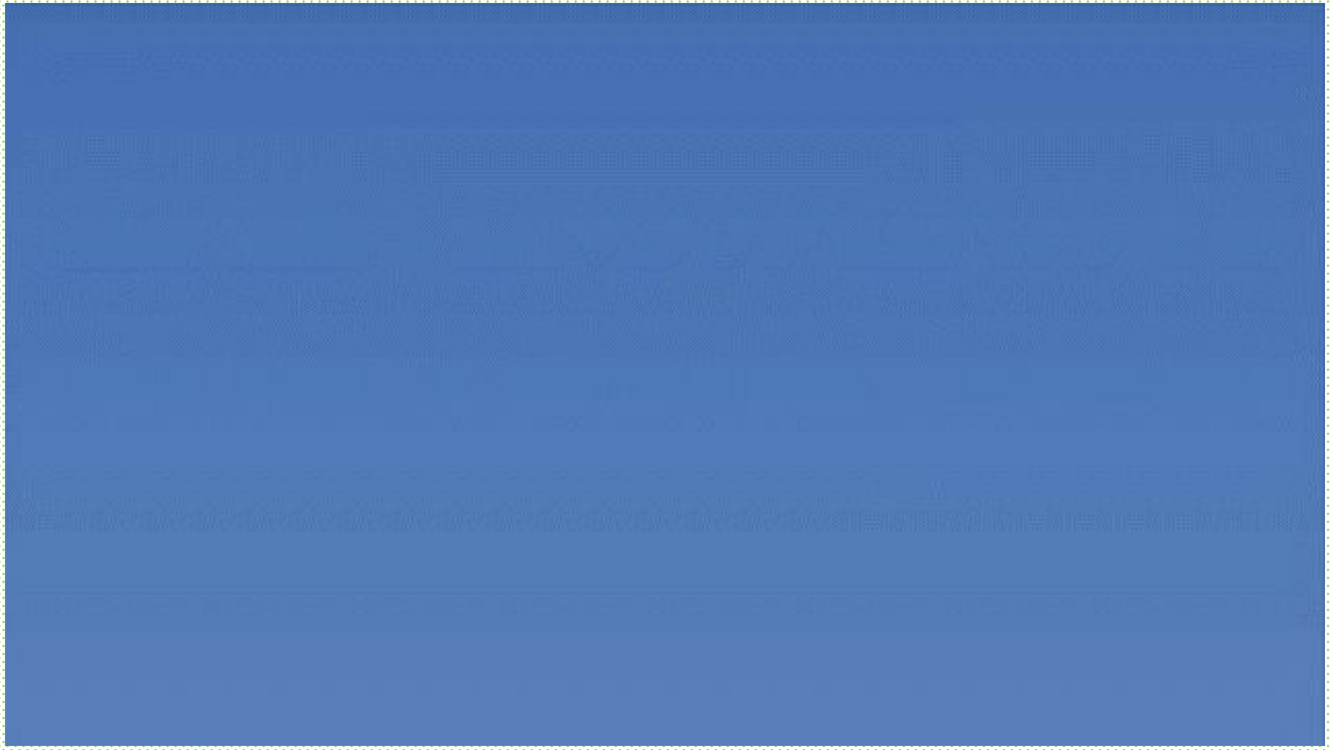
$= \log(3)/\log(2)$

$= 1.585$



Generating Fractals





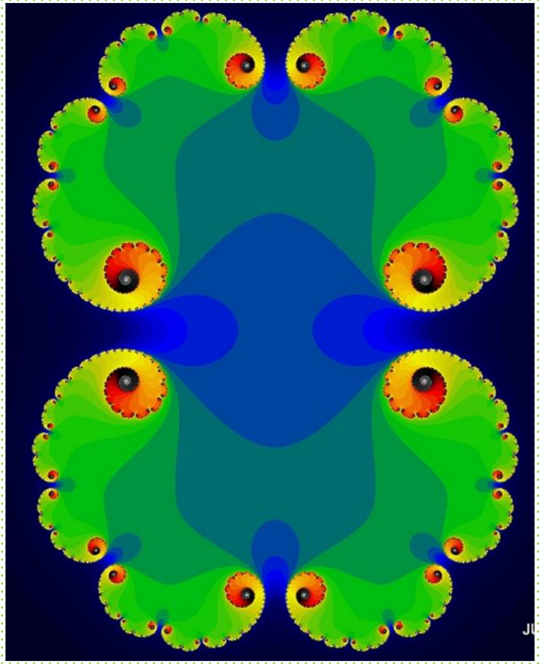
Julia Set

- Gaston Julia studied the iterations of polynomials and rational functions in the early 20th century
- Consider: $z(k+1) = z^2(k) - C$

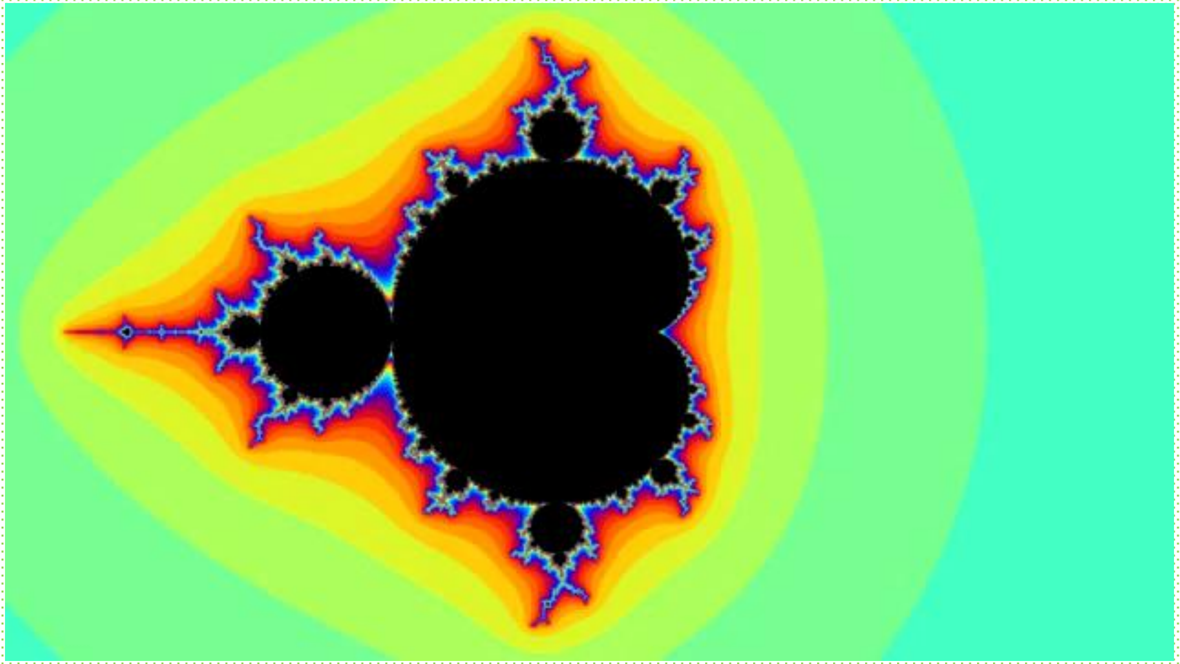
  where C is a real or complex constant
- **Julia Set:** The set of initial points $z(0)$ that do not approach infinity as $k = 1,2,\cdots \to \infty$

Gaston Julia
(1893-1978)

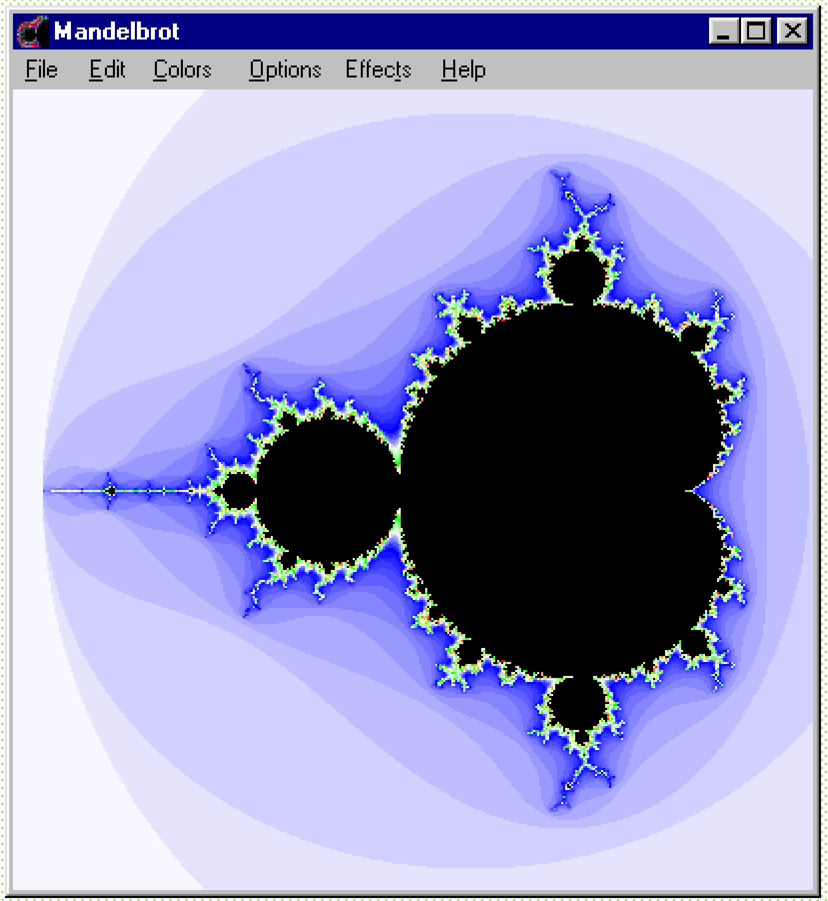
Julia Set



$C = 0.279$



Fractal Geometry in Mandelbrot Set



Geometrical Self-Similarity

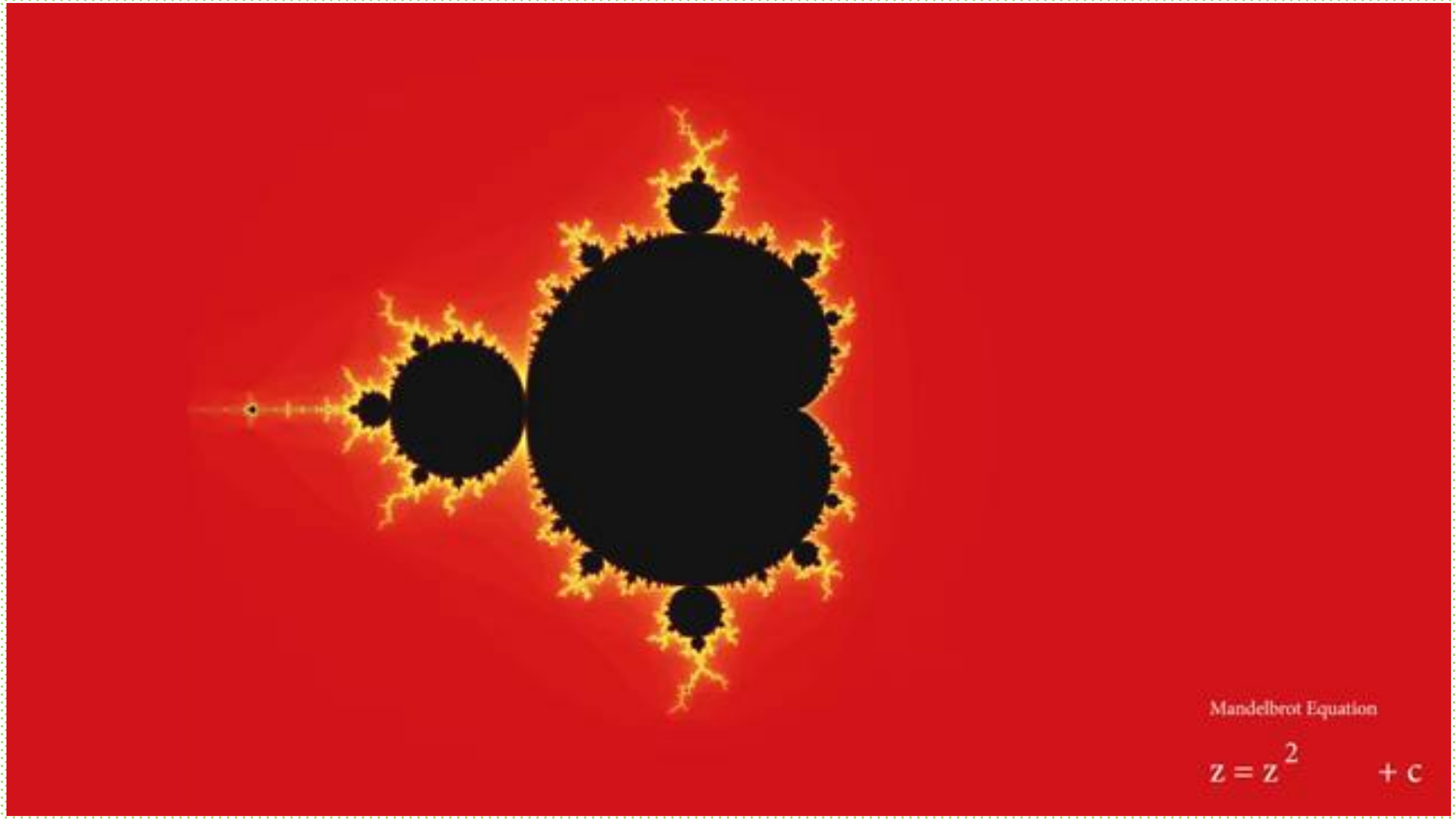
Take a point on the complex number plane, place its value into the Mandelbrot equation, and iterate it for 1000 times.

If the numbers converge to one value, color the pixel black

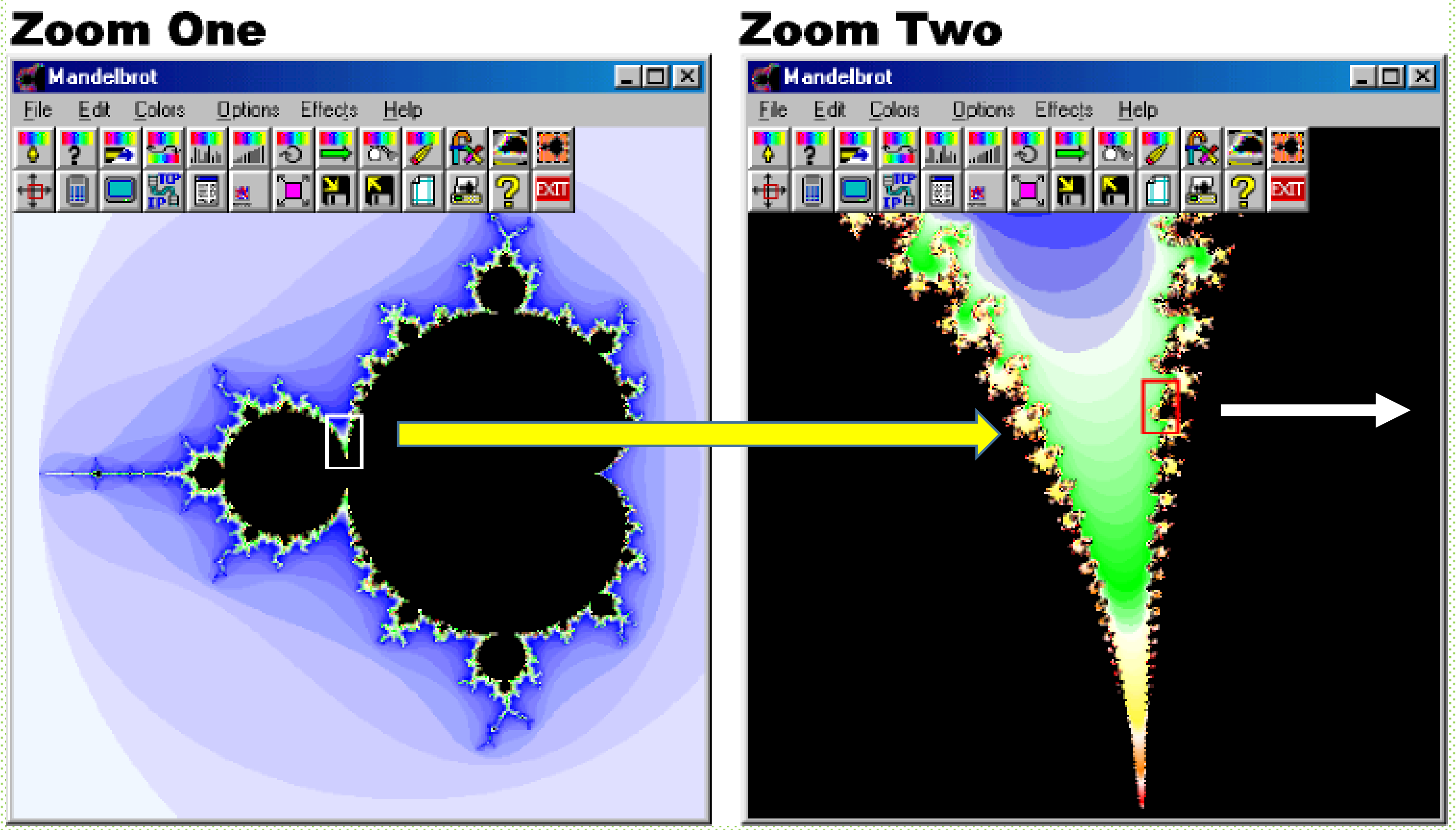
If the numbers diverge to infinity, then color it differently, say green, blue, light blue, ...

Finally, you have the results of the calculating Mandelbrot set

Mandelbrot Set -- Fractals



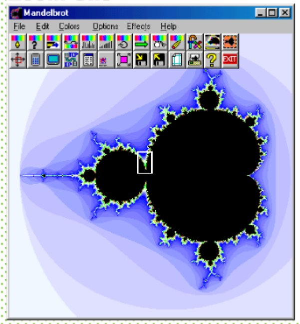
Zoom into the Mandelbrot Set



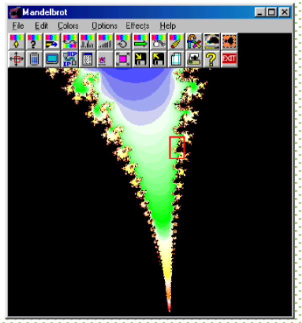
→ Smaller Scale

Zoom into the Mandelbrot Set

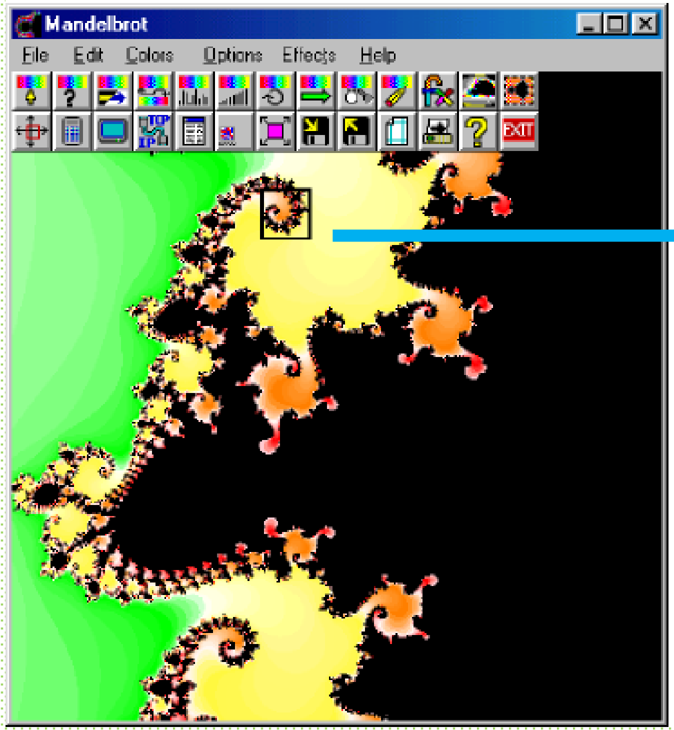
Zoom One



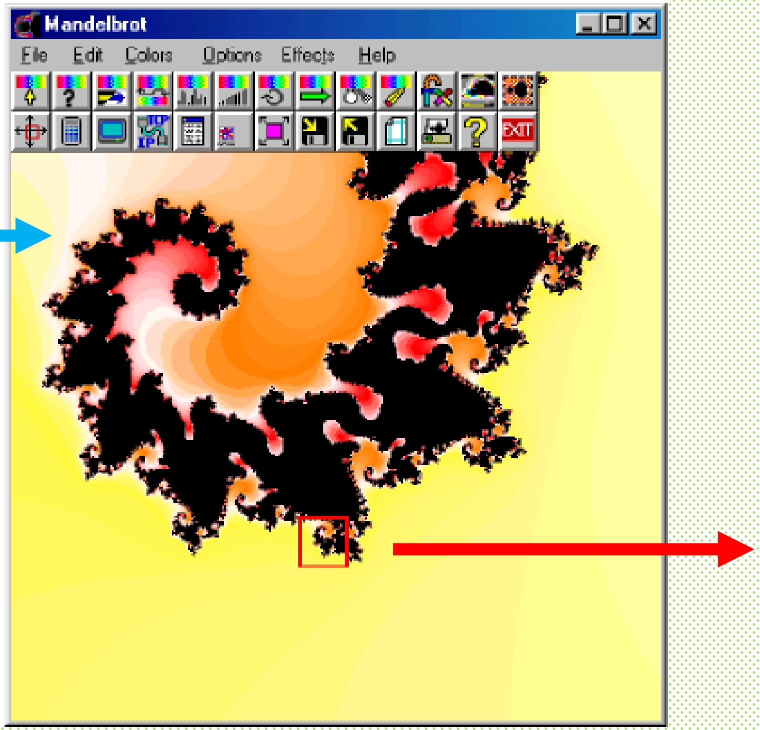
Zoom Two



Zoom Three

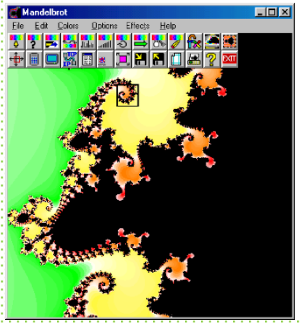


Zoom Four

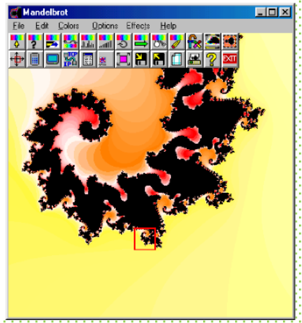


Zoom into the Mandelbrot Set

Zoom Three

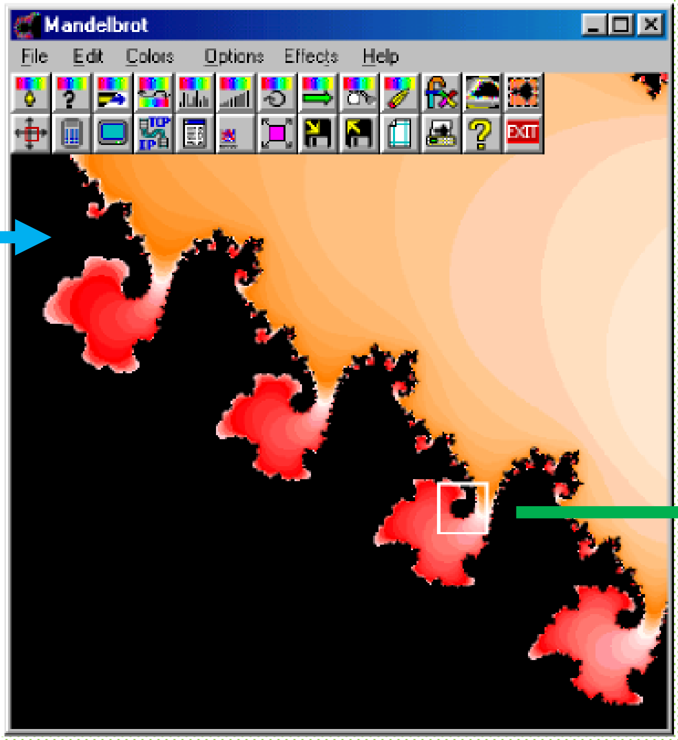


Zoom Four



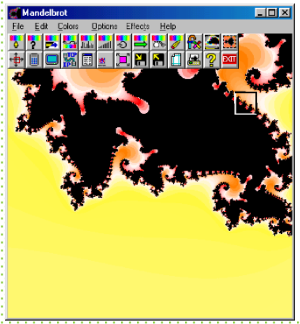
Zoom Five

Zoom Six

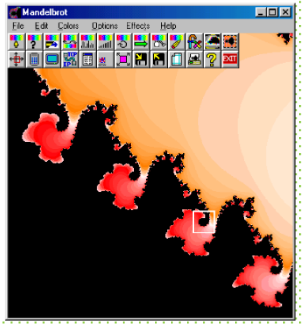


Zoom into the Mandelbrot Set

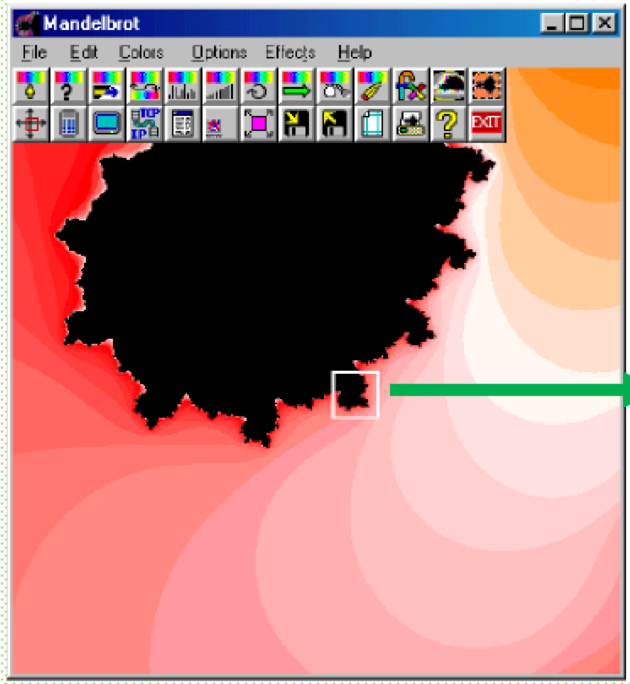
Zoom Five



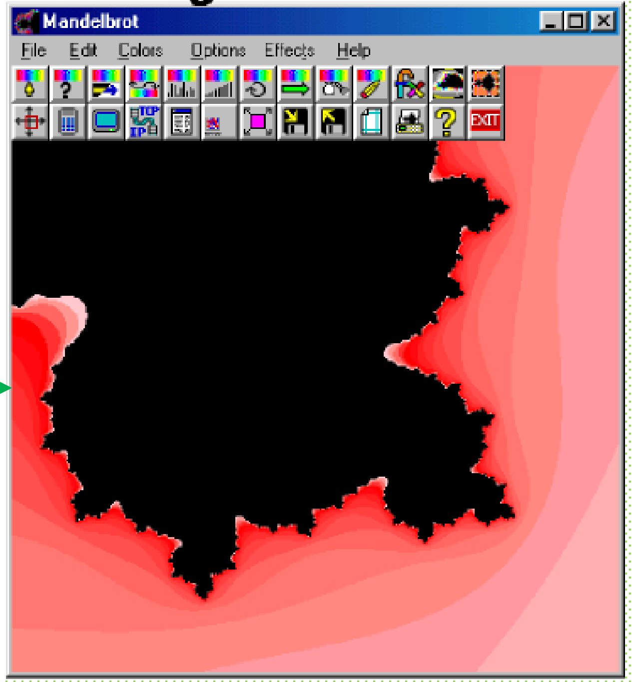
Zoom Six

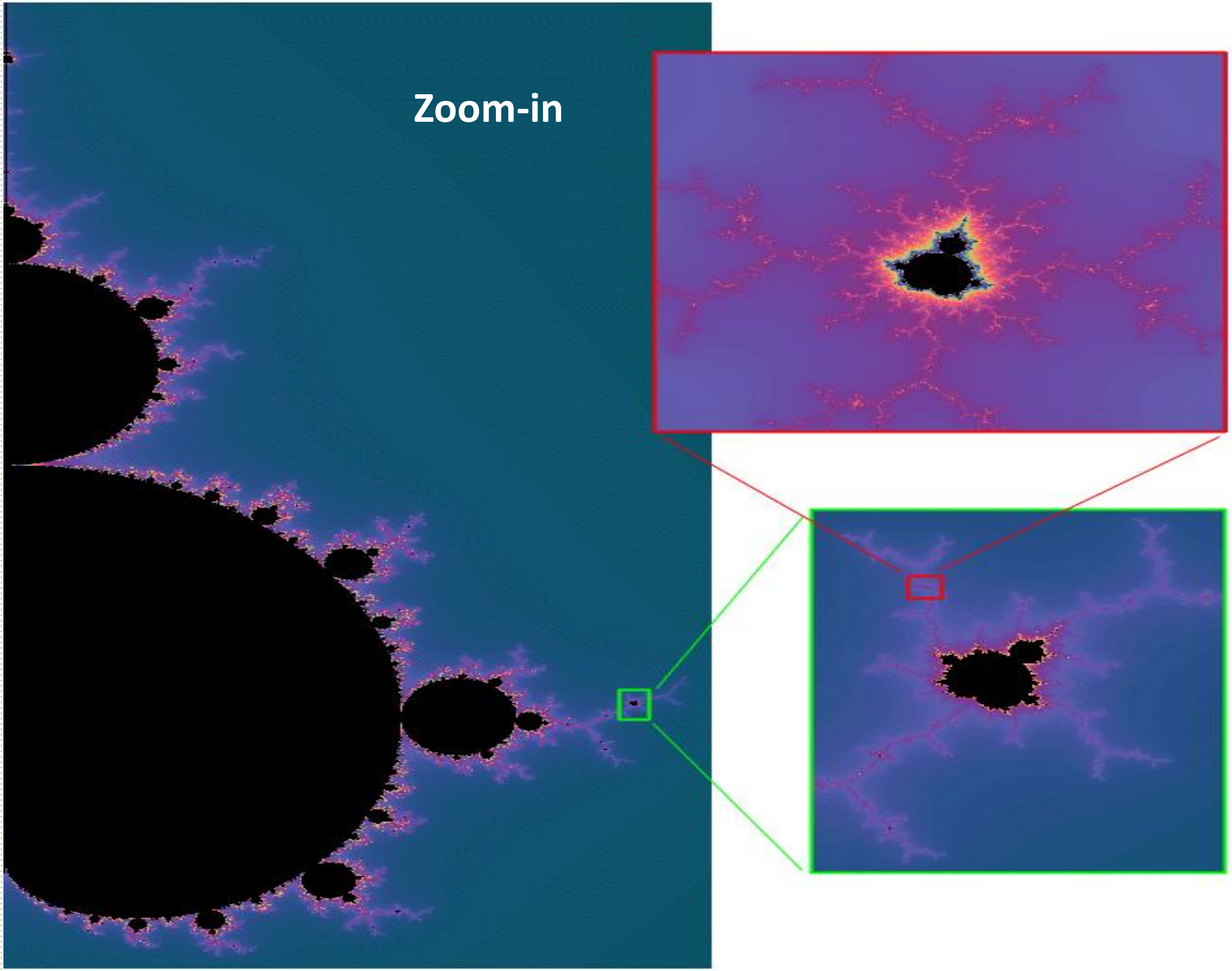


Zoom Seven



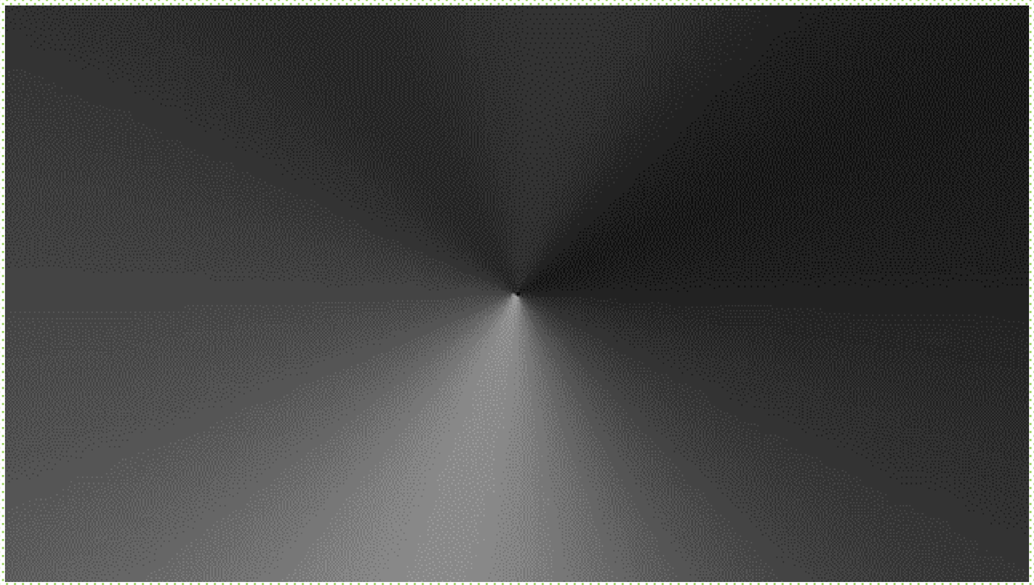
Zoom Eight

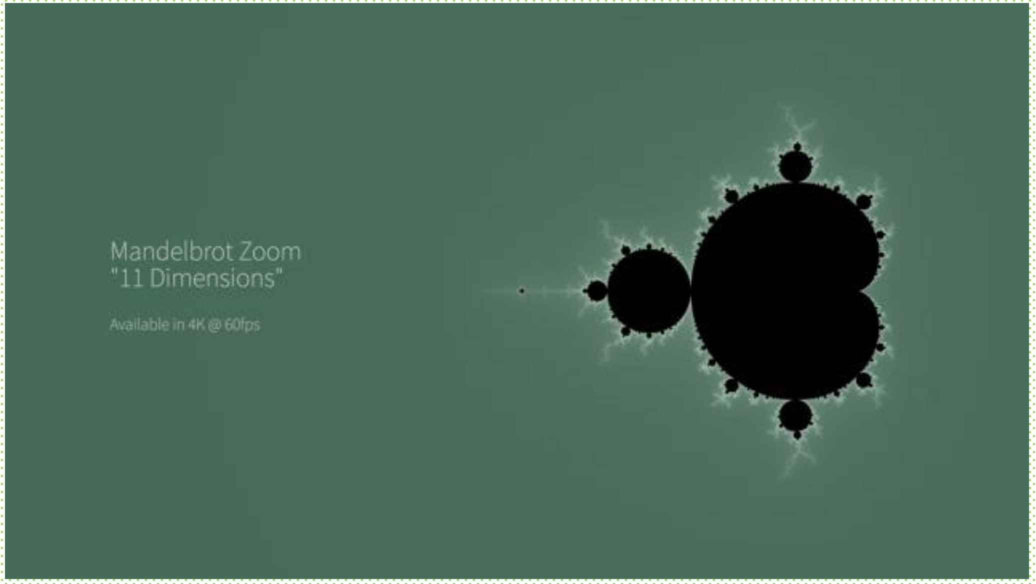



Zoom-in

Mandelbrot Fractals

Mandelbrot Fractals



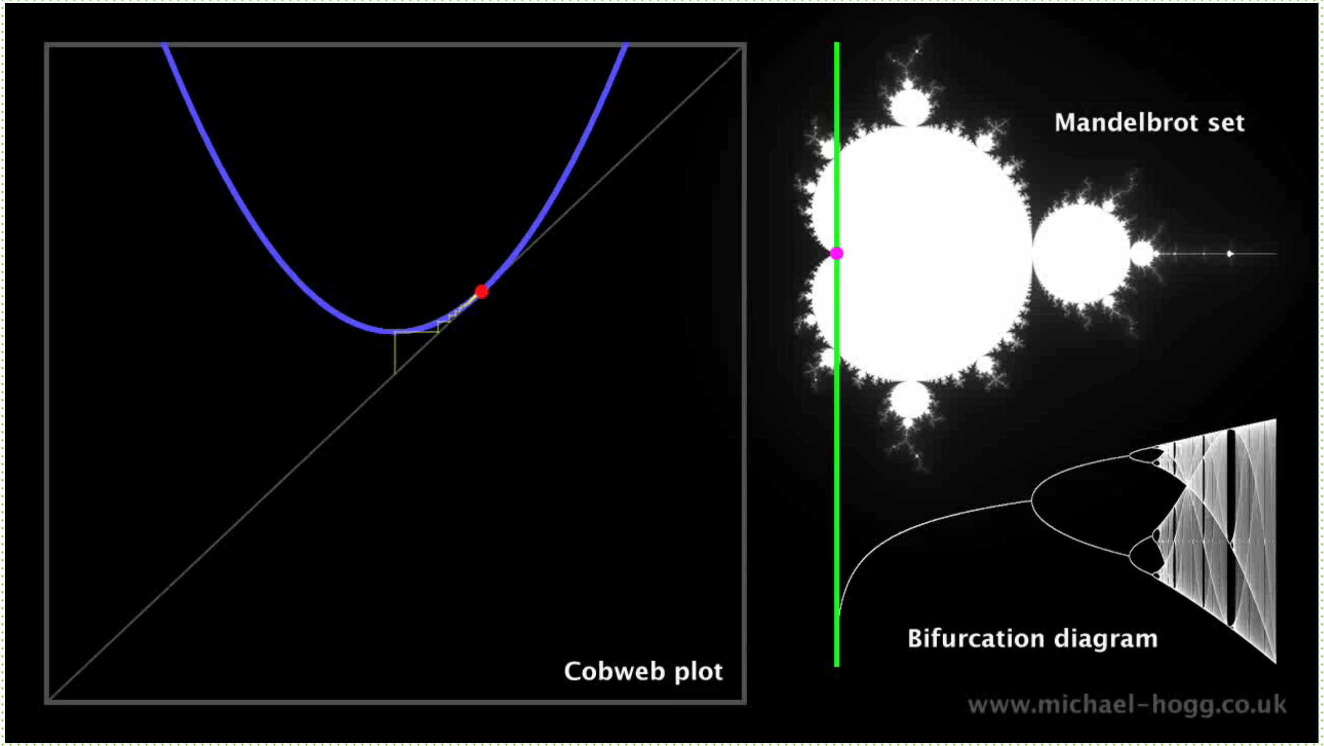


Self-Similarity in CHAOS

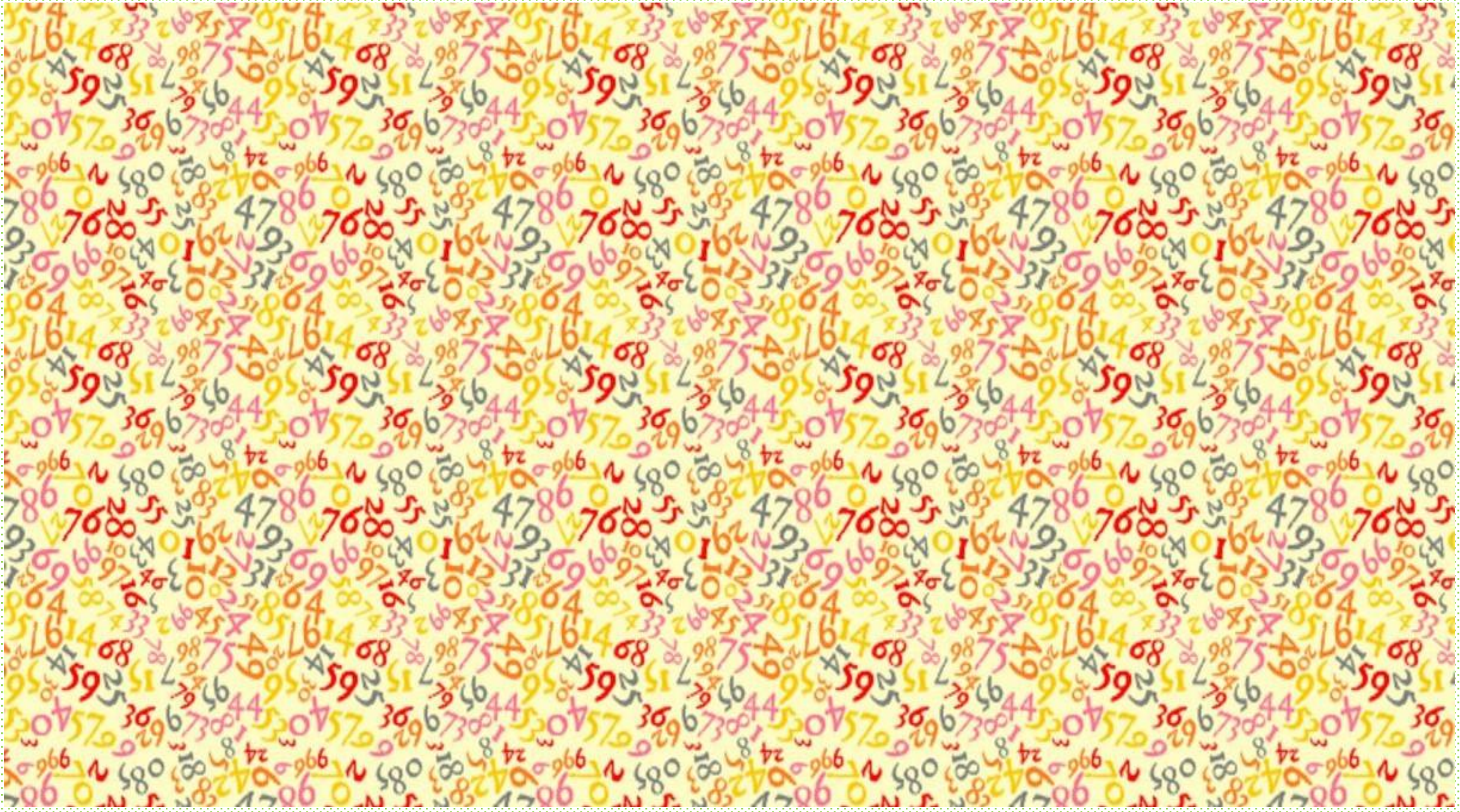
Logistic Map and Mandelbrot Set

Logistic Map has self-similarity – Fractal structure

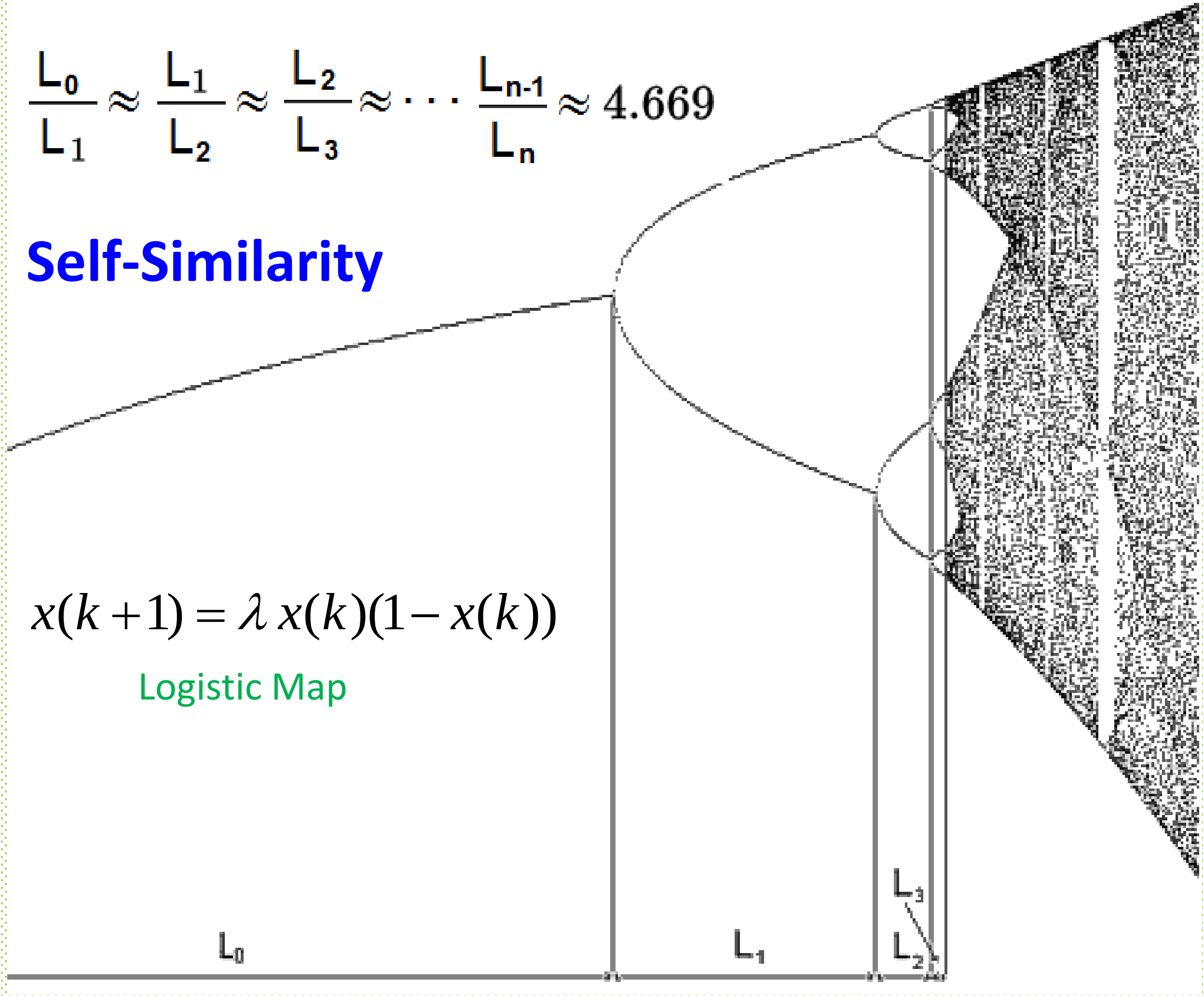
Mandelbrot Set has self-similarity – Fractal structure

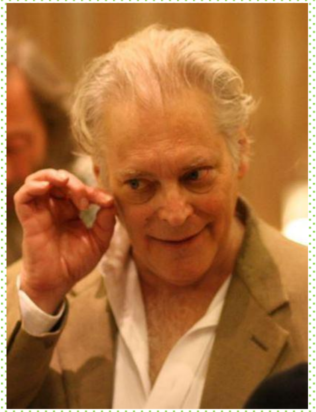


Some Interesting Numbers



A Magic Ratio = 4.669… Feigenbaum Constant



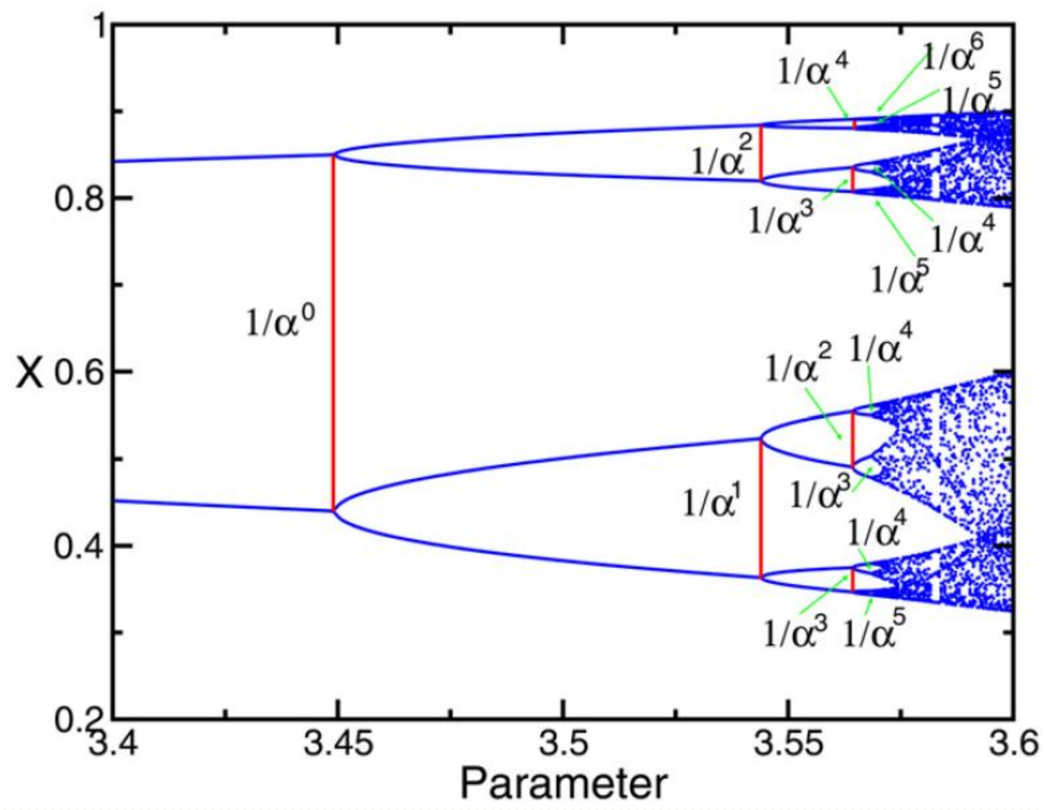


Mitchell Feigenbaum
(1944 -2019)

Fibonacci Numbers

- Numbers of branches corresponding to the various powers of $1/\alpha$ follow the sequence:

$$1/\alpha^0,\ 1/\alpha^1,\ 2/\alpha^2,\ 3/\alpha^3,\ 5/\alpha^4,\ 8/\alpha^5,\ 13/\alpha^6,\ 21/\alpha^7,\ 34/\alpha^8,\ 55/\alpha^9,\ \cdots,$$



Sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

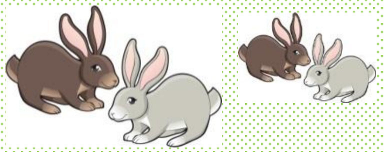
Fibonacci Numbers

Leonardo Fibonacci
(1170 – 1250)

Fibonacci Story

- Leonardo Fibonacci published a book, **Liber abaci**, in 1202
- One question found in the book:

1. John has a pair of child rabbits
2. Every month, each pair of adult rabbits begets a new pair
3. It takes a month for a child rabbit to become an adult rabbit
4. How many pairs of rabbits are there after one year?

Time	No of pairs
At start	1 pair (child) 
After 1 month	1 pair 
After 2 months	2 pairs 
After 3 months	3 pairs 
After 4 months	5 pairs 
After 5 months	8 pairs 
:	:

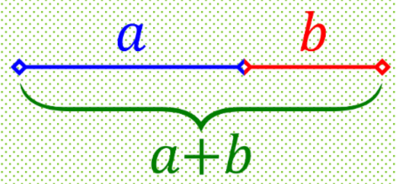
Fibonacci Sequence

- Fibonacci sequence:
    - 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .
- i-th number + (i+1)-th number = (i+2)-th number

1+1=2, 1+2=3, 2+3=5, 3+5=9, 5+8=13, 8+13=21, 13+21=34, 21+34=55, 34+55=89, …

- (i-th number) / [(i+1)-th number]
    - 1, 0.5, 0.666, .., 0.6, 0.615, 0.619, 0.6176, 0.618181818…, 0.617977528, …
    - Final value is: 0.6180339887….
- Golden Ratio [(i+1)-th number] / (i-th number) 1/0.6180339887… = 1.618 …

黃金比例 Golden Ratio



$$\frac{a+b}{a} = \frac{a}{b}$$

At each step, roundoff it and keep 3 dismal numbers

每步運算四捨五入，然後保留3位小數：

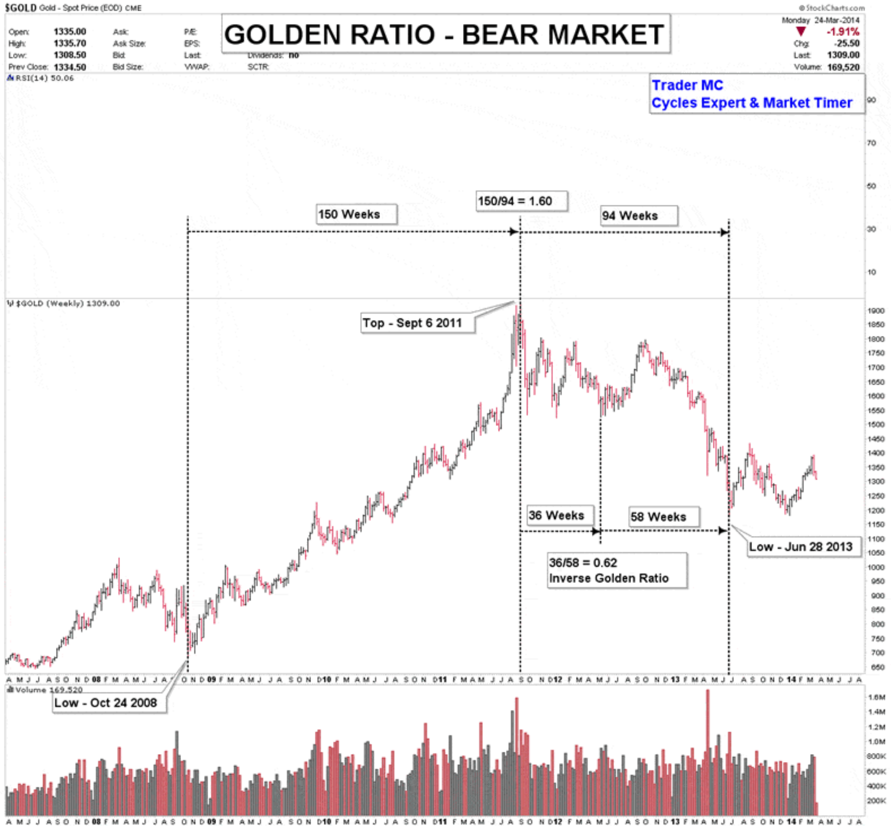
If $a = 1,\ b = 0.618$ **then**

$$\frac{a+b}{a} = \frac{a}{b} \rightarrow \frac{1.618}{1} = \frac{1}{0.618} = 1.618$$

If $a = 1.618,\ b = 1$ **then**

$$\frac{a+b}{a} = \frac{a}{b} \rightarrow \frac{2.618}{1.618} = \frac{1.618}{1} = 1.618$$

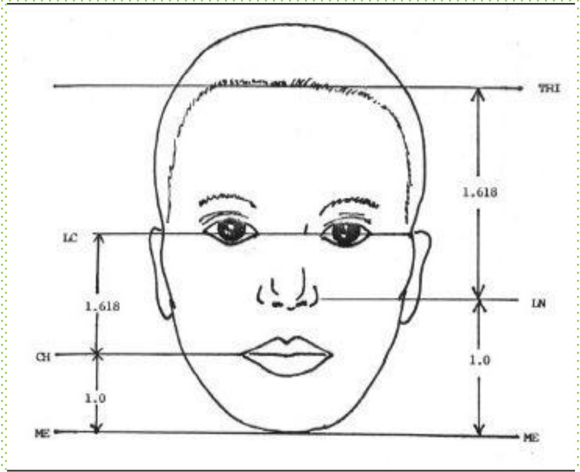
In the Great Pyramid of Giza, the length of each side of the base is 756 feet with a height of 481 feet. The ratio of the base to the height is roughly 1.5717, which is close to the Golden ratio 1.618



http://www.marketoracle.co.uk/Article44964.html

$$\phi = 1.618033........ = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}$$

$$\phi = 1.618033.......... = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}}$$



Fibonacci Music

EBREAK

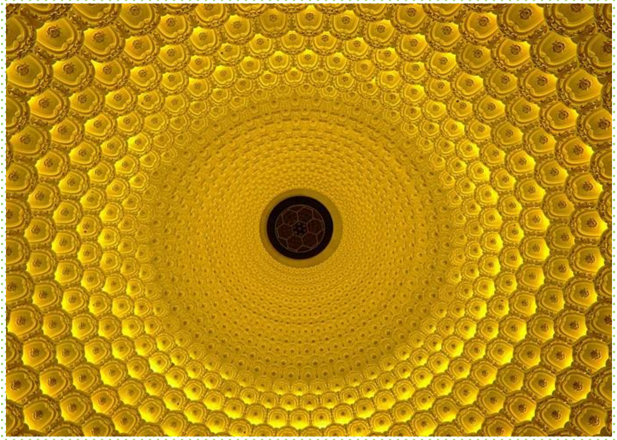
10 Minutes

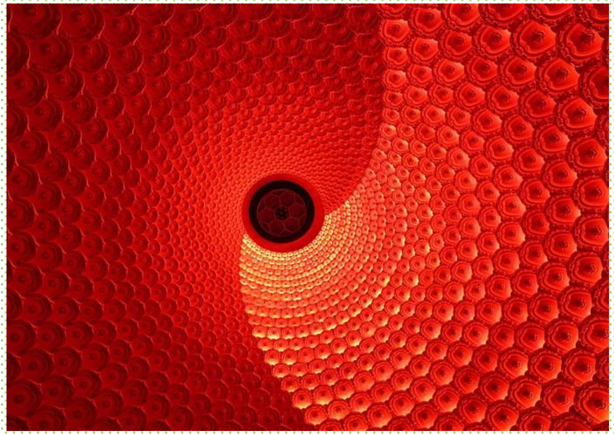
Fractals

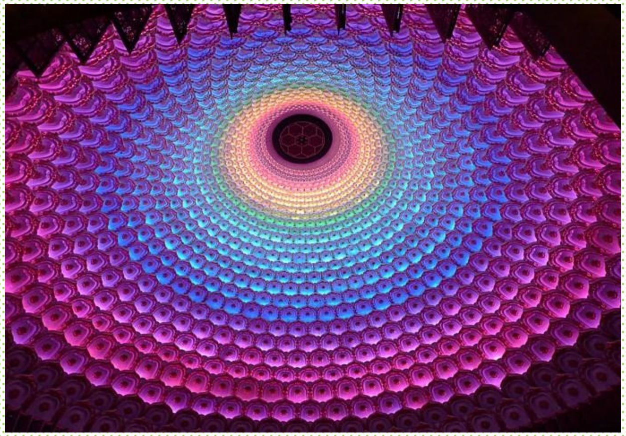
Features of Fractals

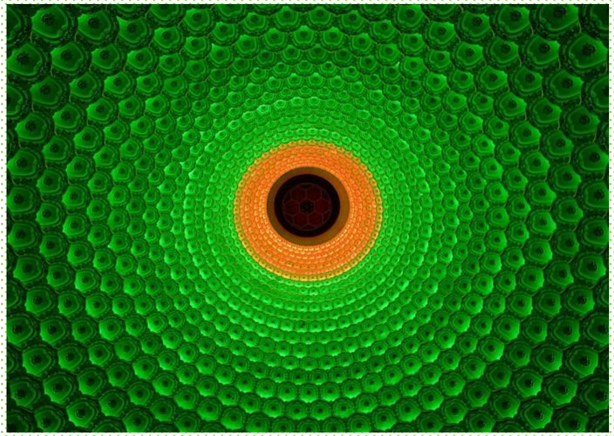
- Self-similarity in different scales
- Patterns in patterns
- A fine structure at any arbitrarily-small scale
- Generated by recursive iterations

Applications of Fractals

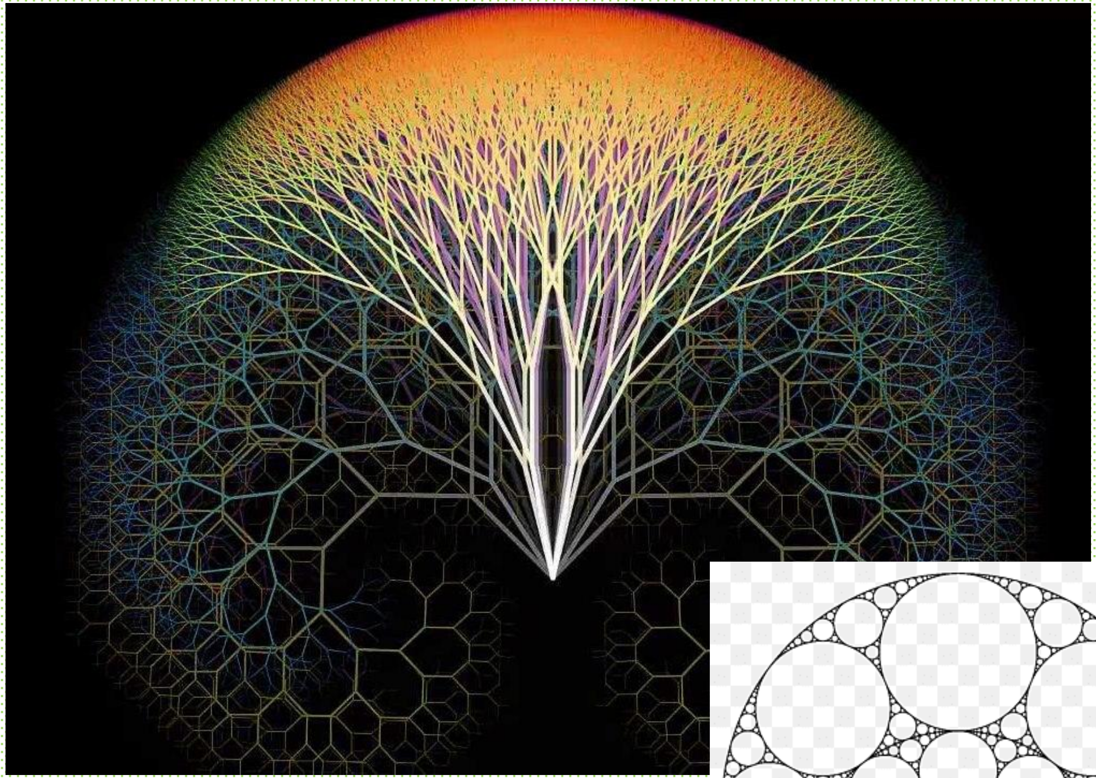


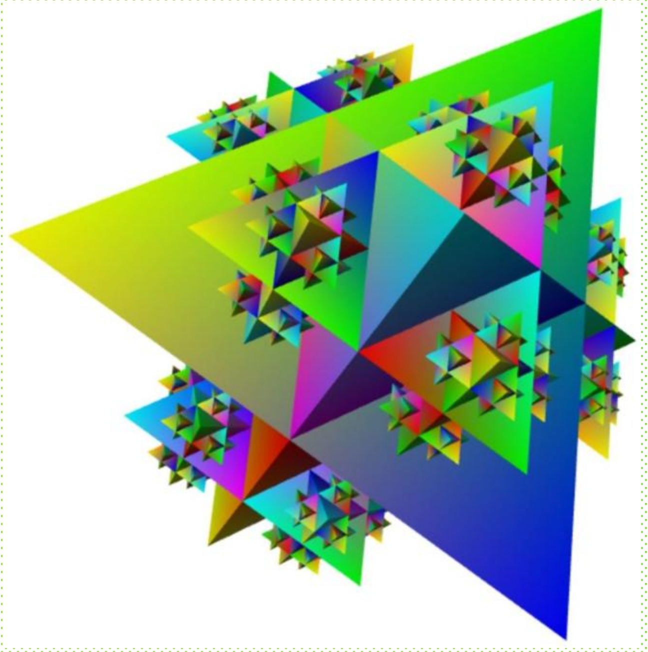


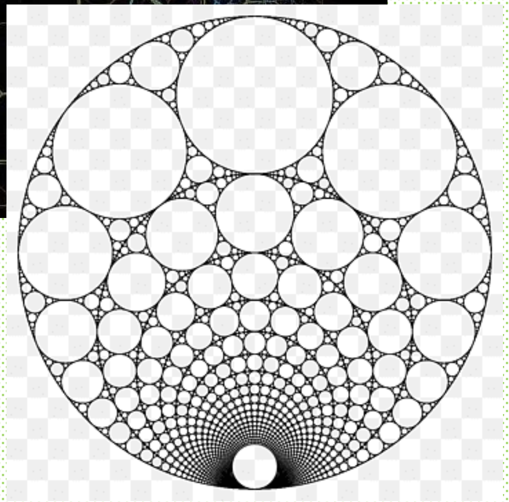




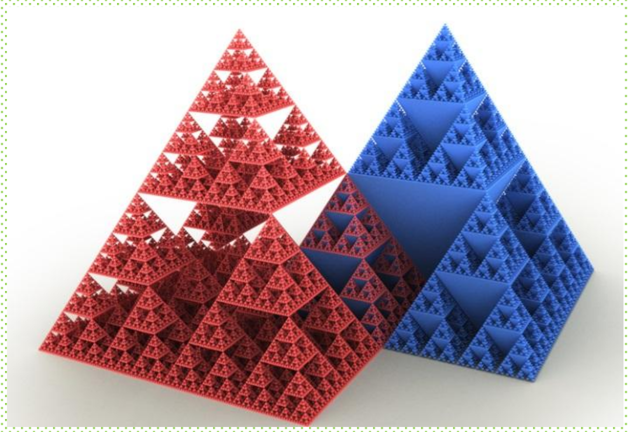
Fractal Arts



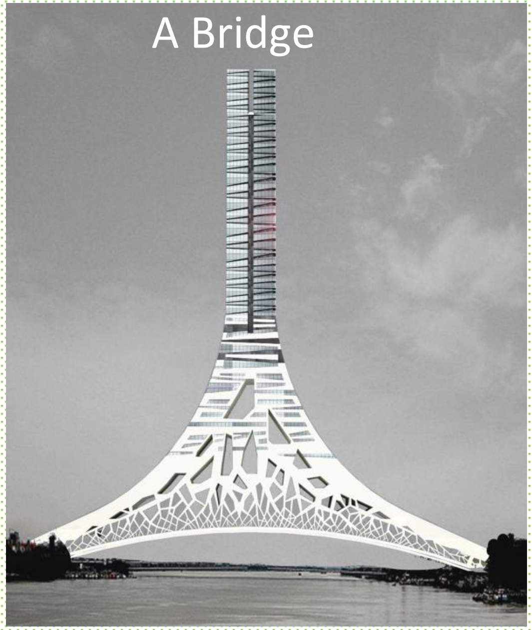




Applications in Architecture



Geometry to build lighter structures



Mosaics (Church Santa Maria in Trastevere, Rome, Italy)

Van Gogh: Starry Night

Applications in Art Design

- Accessory:
  - Ear-rings, necklace, …



- Drawing:



Fashion Show

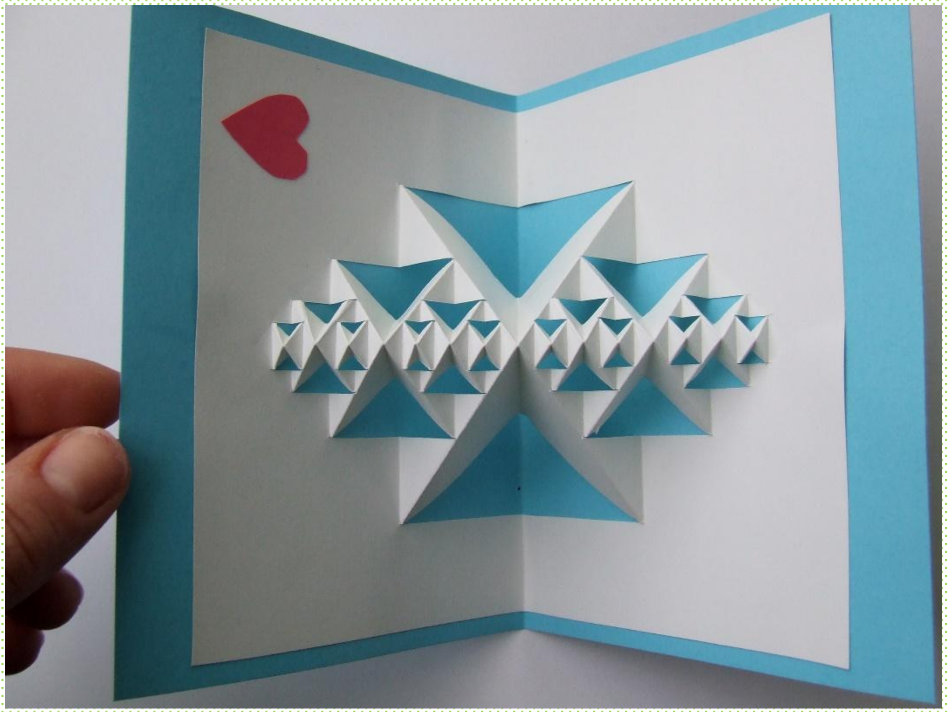
2010 Tokyo

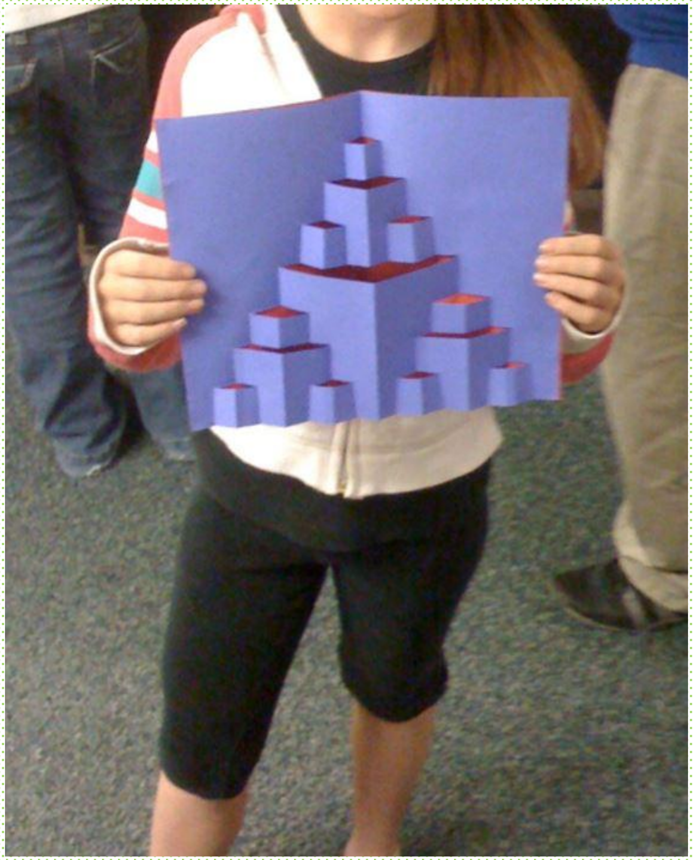
Courtesy of Kazu Aihara

Applications in Dress Design

- Fashion:

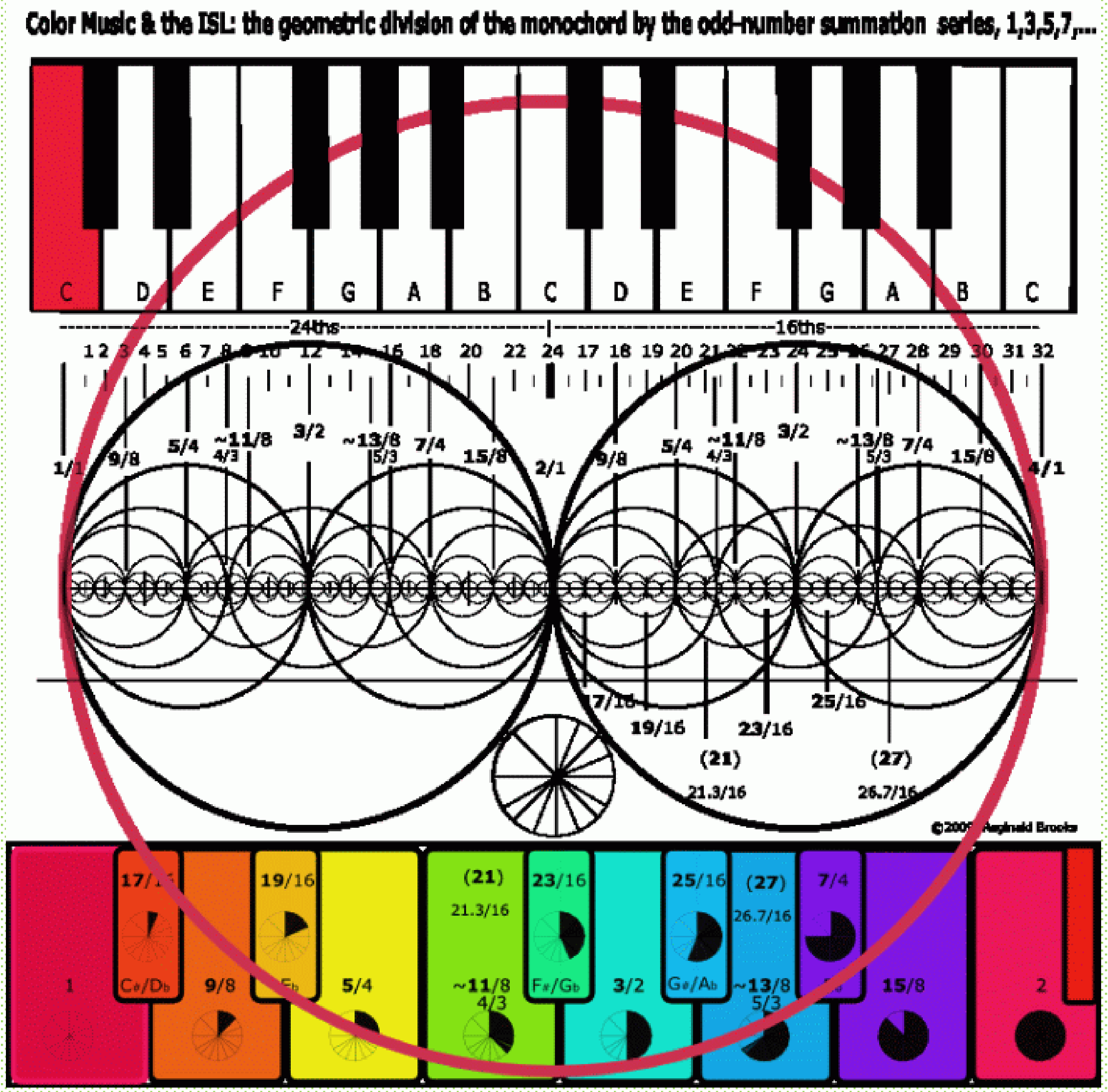

Fractal Paper Cuts



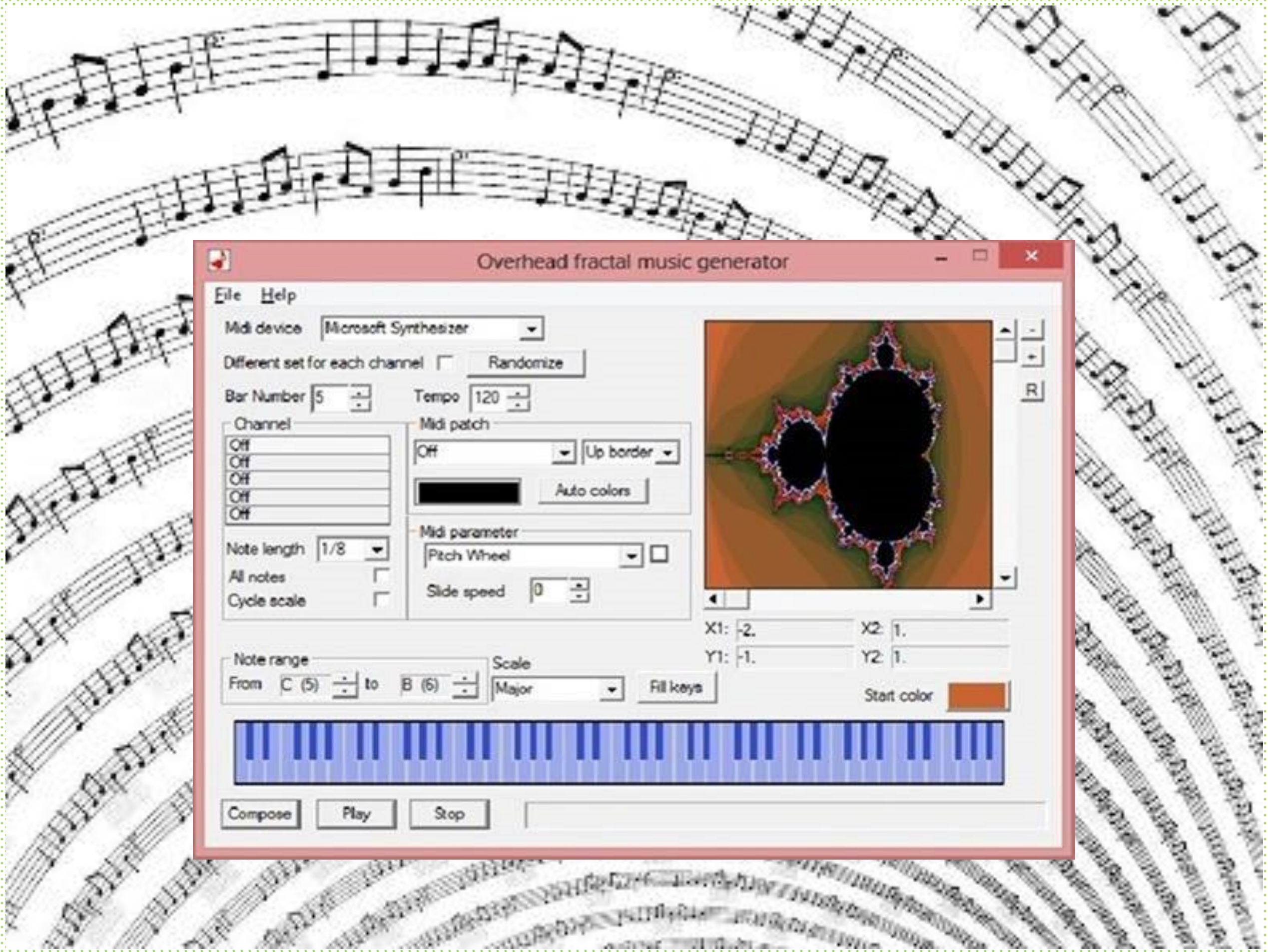


Fractal Music

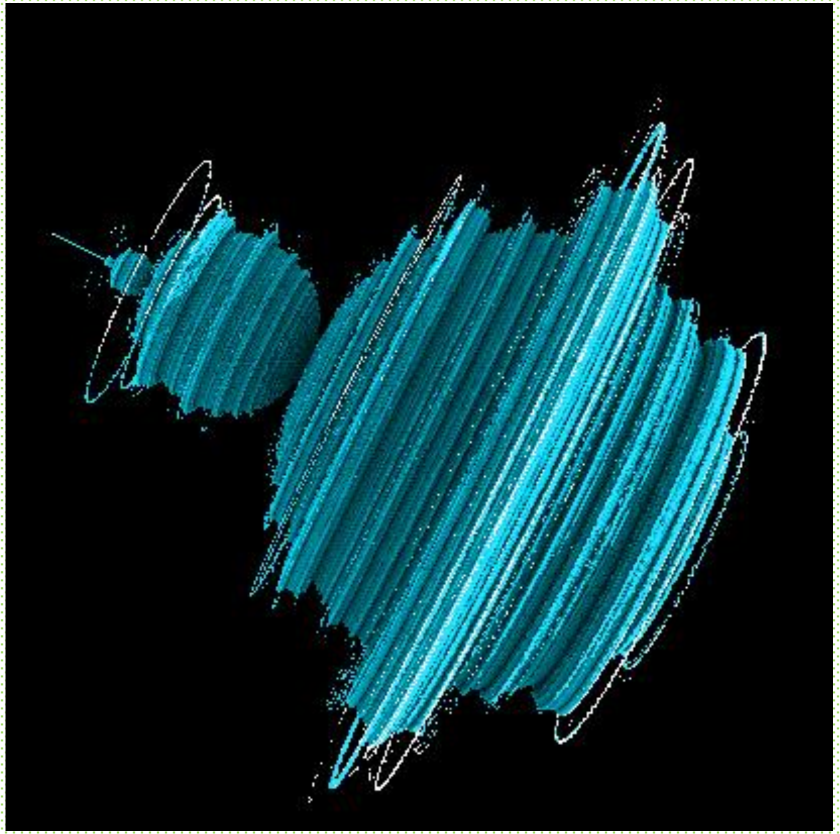
Fractal Music



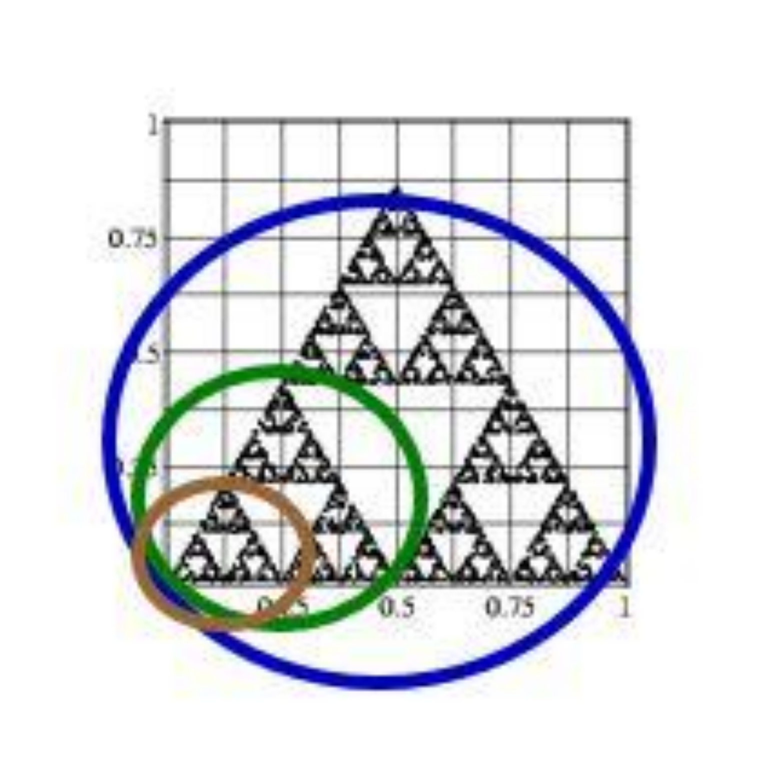
Fractal Music



Fractal Vibration



Fractals in Technology



FRACTAL ANTENNA

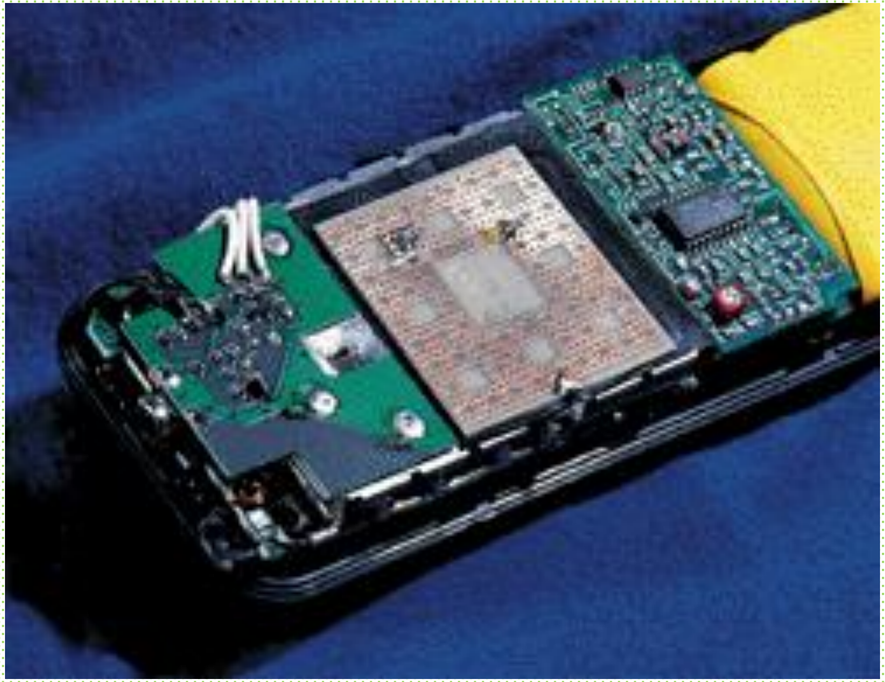
Fractal Antenna

- Fractal antenna design for radio communication

- Many length scales

→ Broadband

Broadening frequency bands



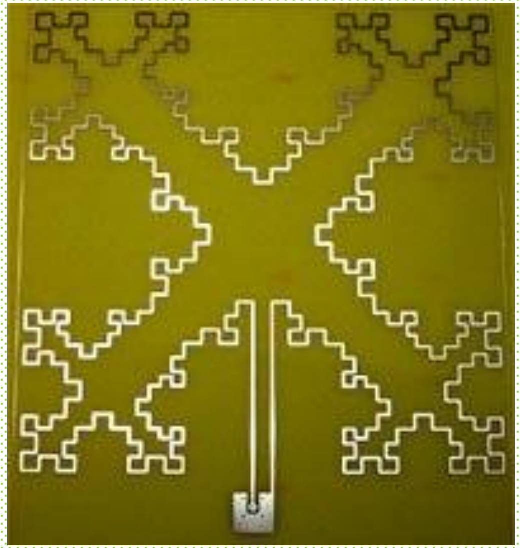
Fractal Antenna

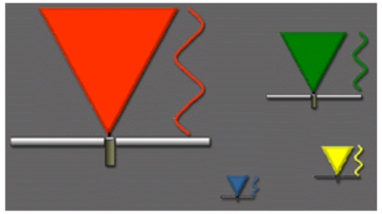
- **Fractal Property:**

  Finite area but infinite perimeter

- [Useful for antenna]()

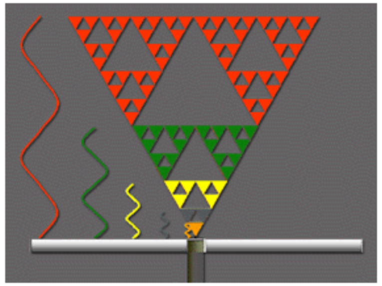
- For antenna, the frequency depends on the length and the shape of line





Four antennas (with a wave cartoon) intended to be used on four discrete frequency bands [www.fractus.com, 2000].

Figure 1 Four separate antennas.

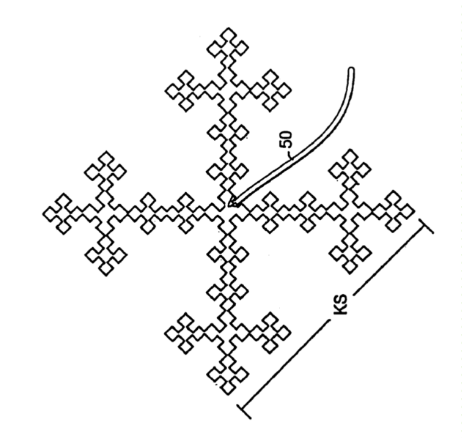


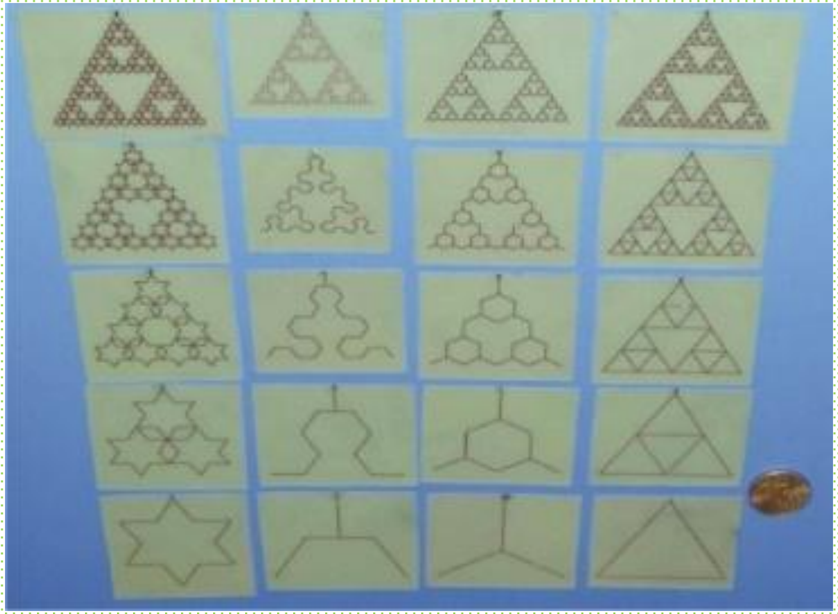
One antenna intended to be use for four discrete frequency bands [www.fractus.com, 2000].

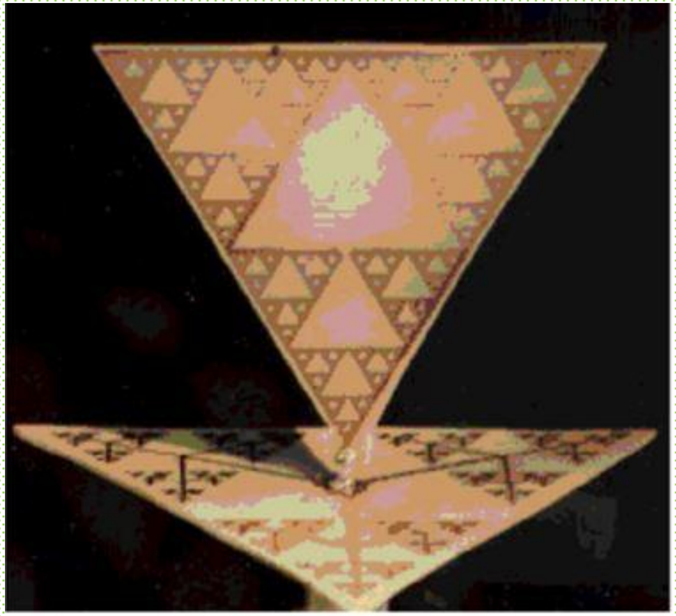
Figure 2 One antenna for four bands.

Fractal Antenna

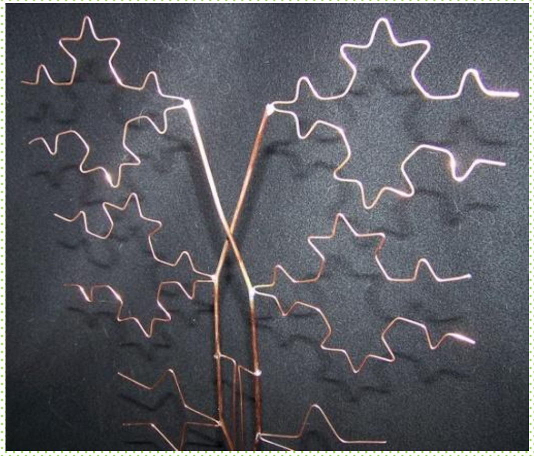
- Antennas built with only a small number of iterations of a fractal process can exhibit sensitivity at several frequencies
- Smaller size with multiple frequencies







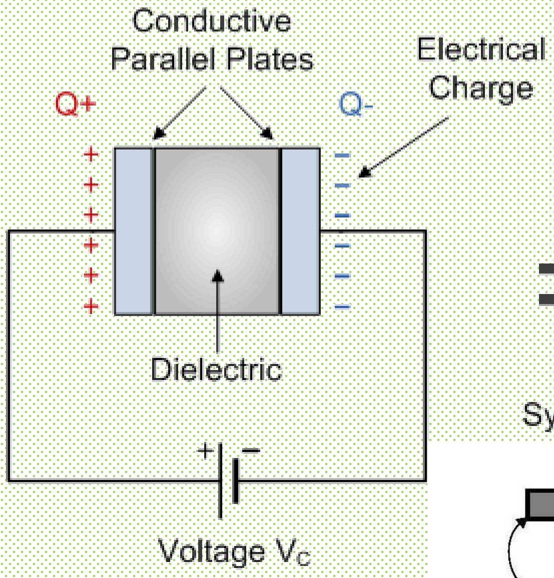
Fractal Antenna

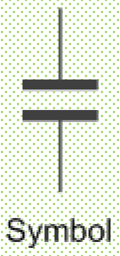


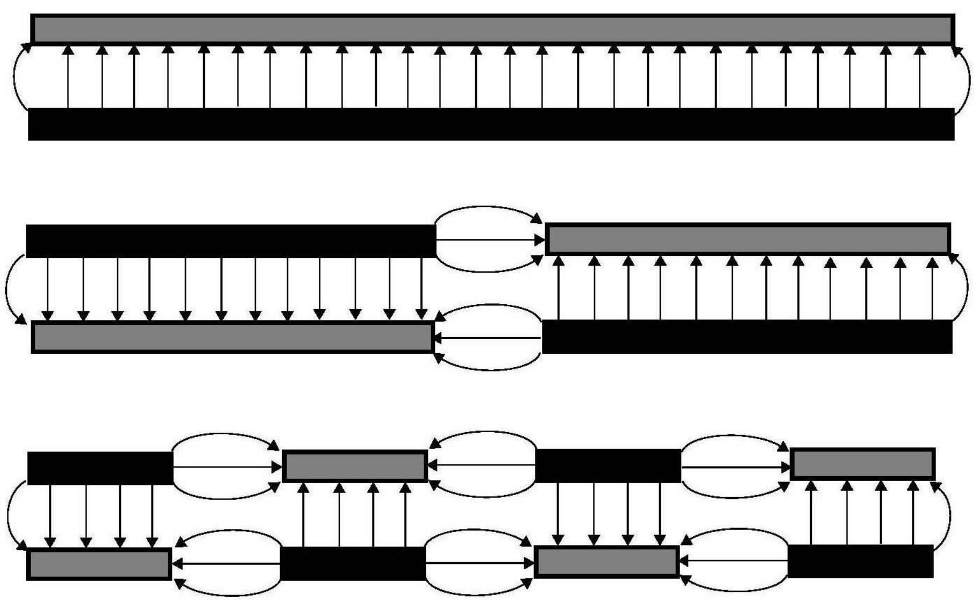
Shortcoming:

Interference

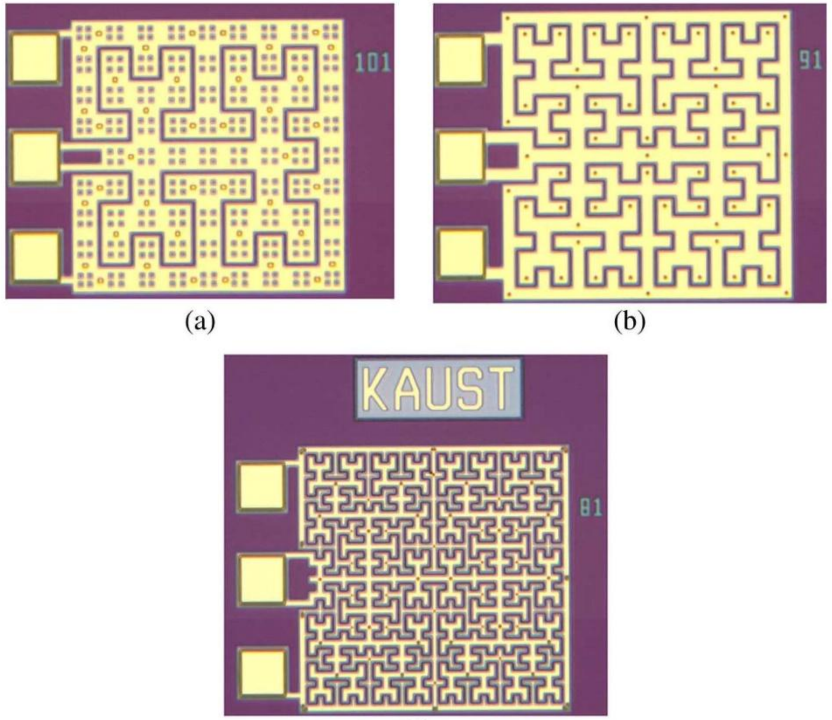
Fractal Capacitor





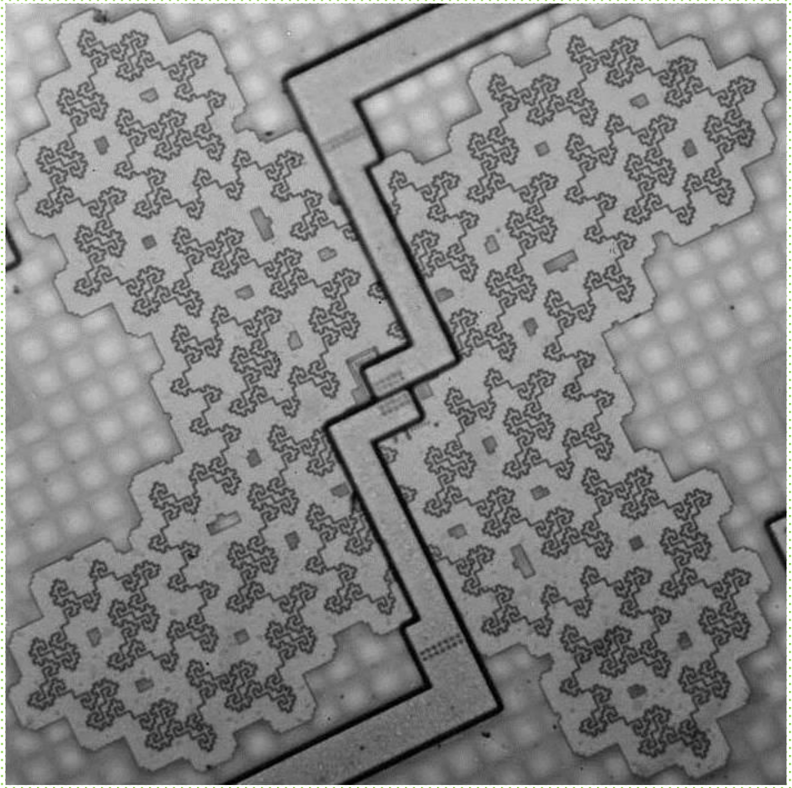


Small size but
Large capacity



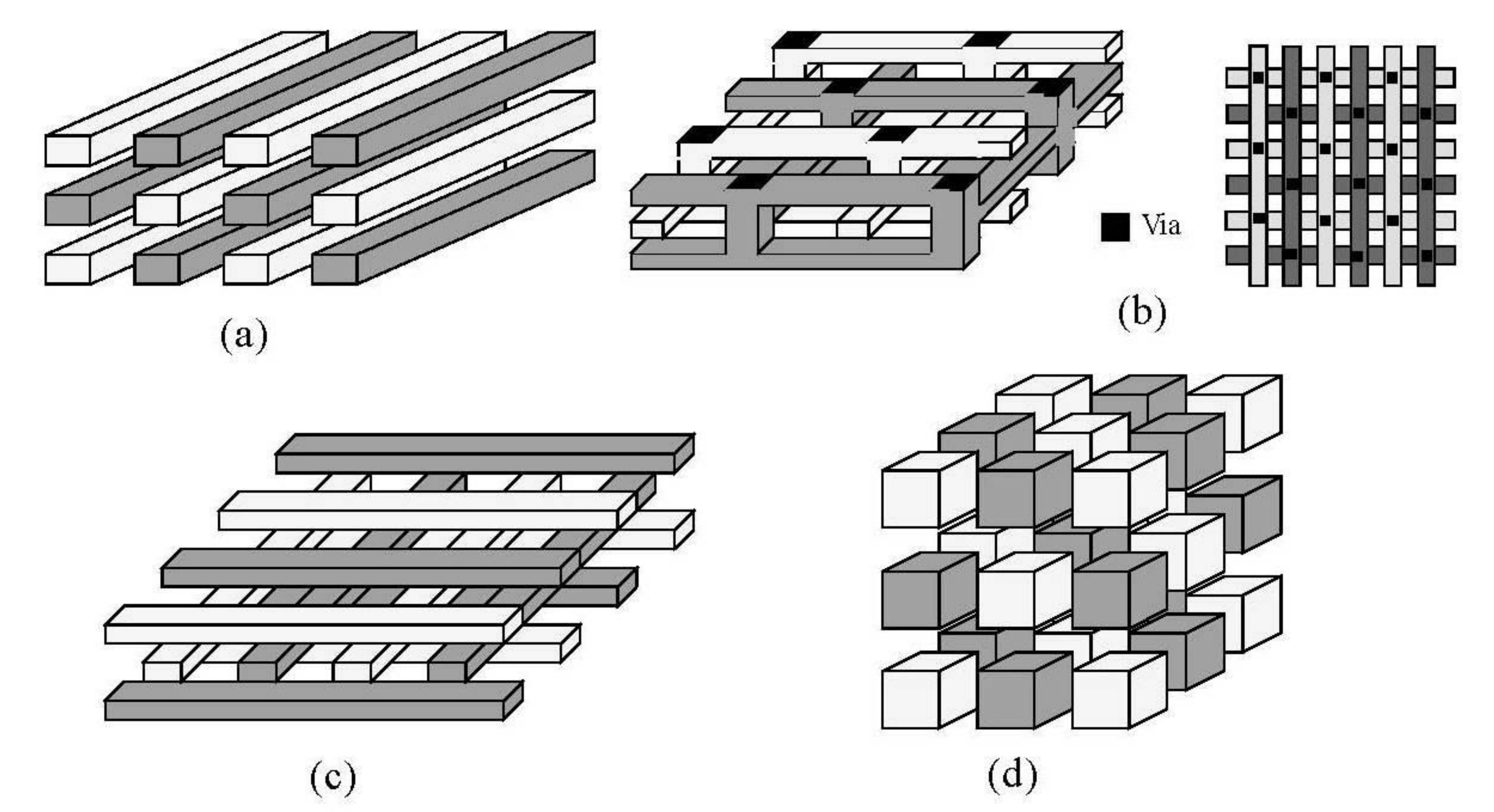
(a) (b) (c)

Fractal Capacitor



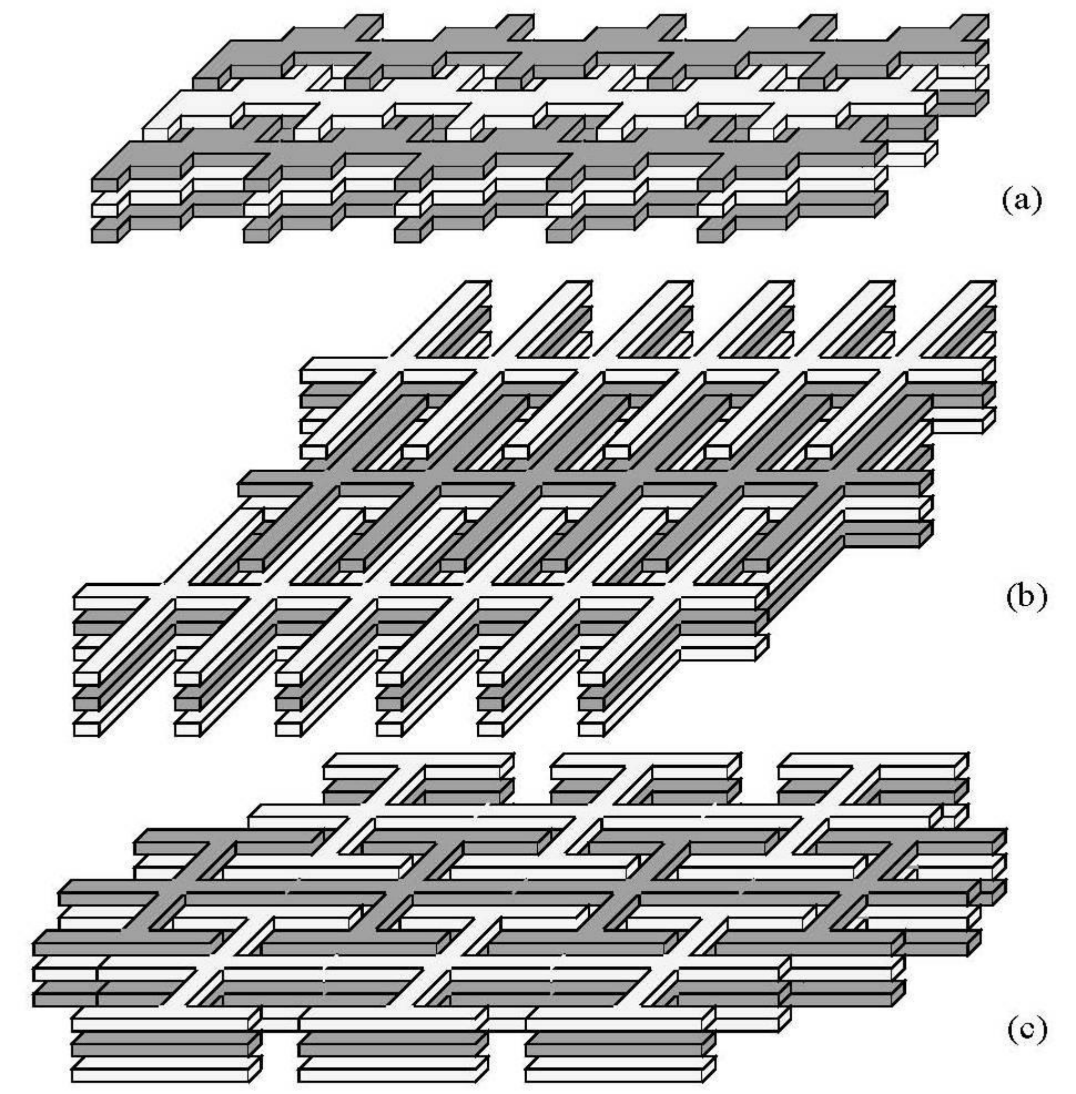
Shortcoming: Interference

Fractal Capacitor

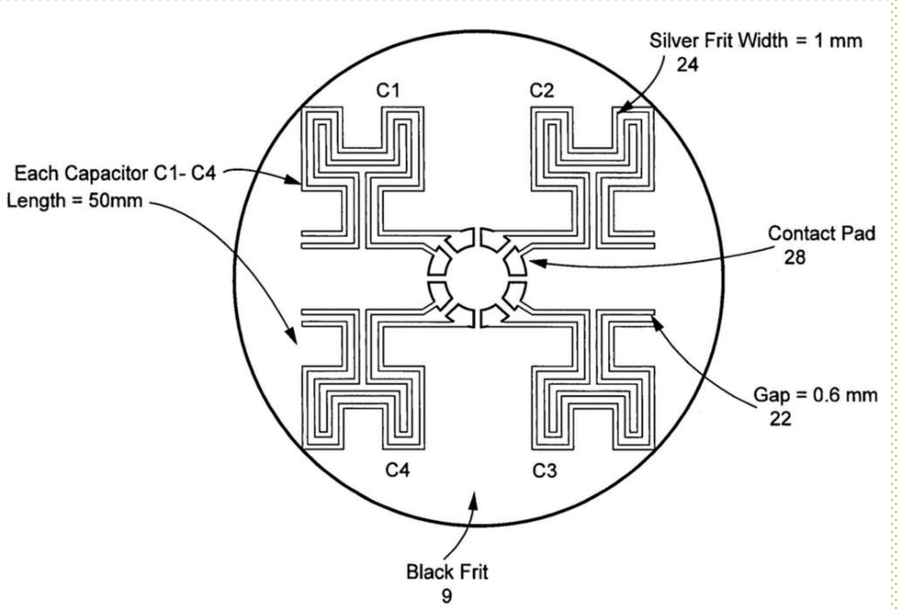


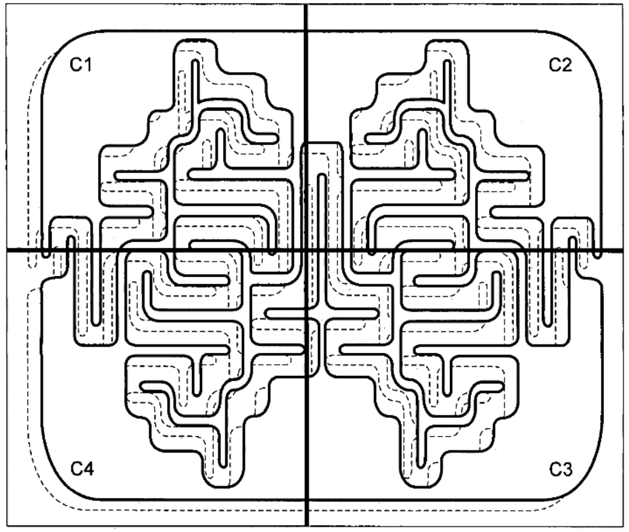
(a) (b) (c) (d)

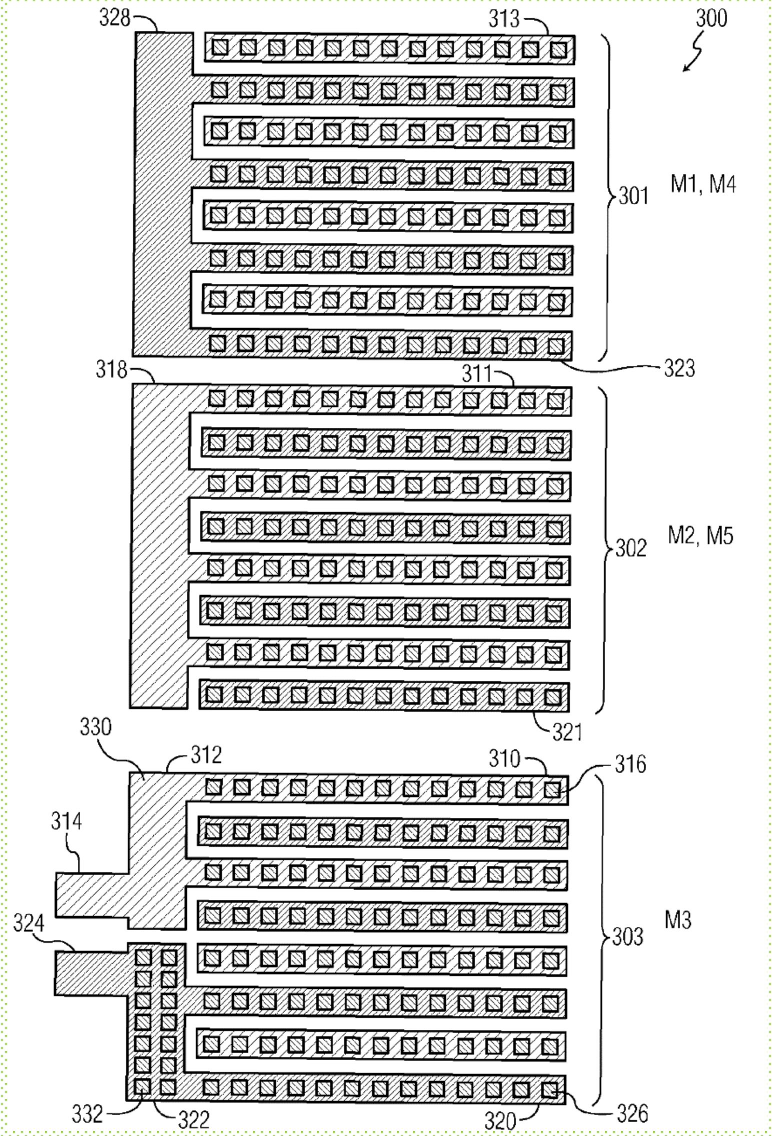
Fractal Capacitor



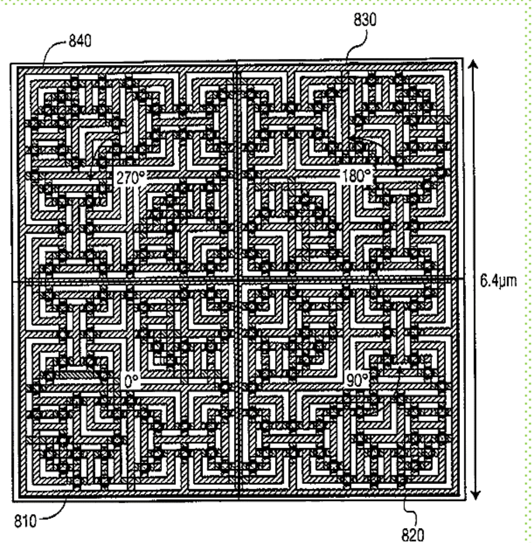
Fractal Capacitor

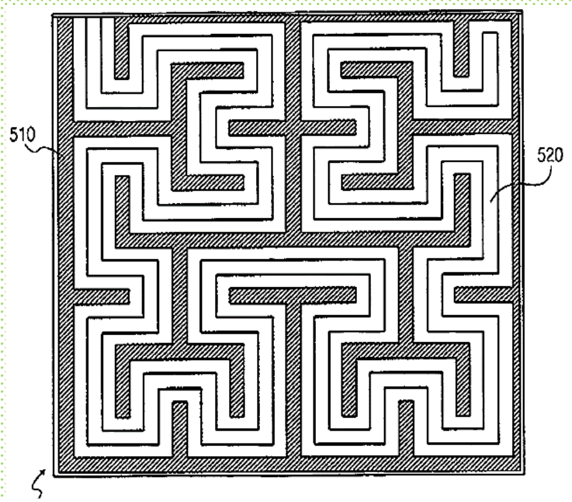


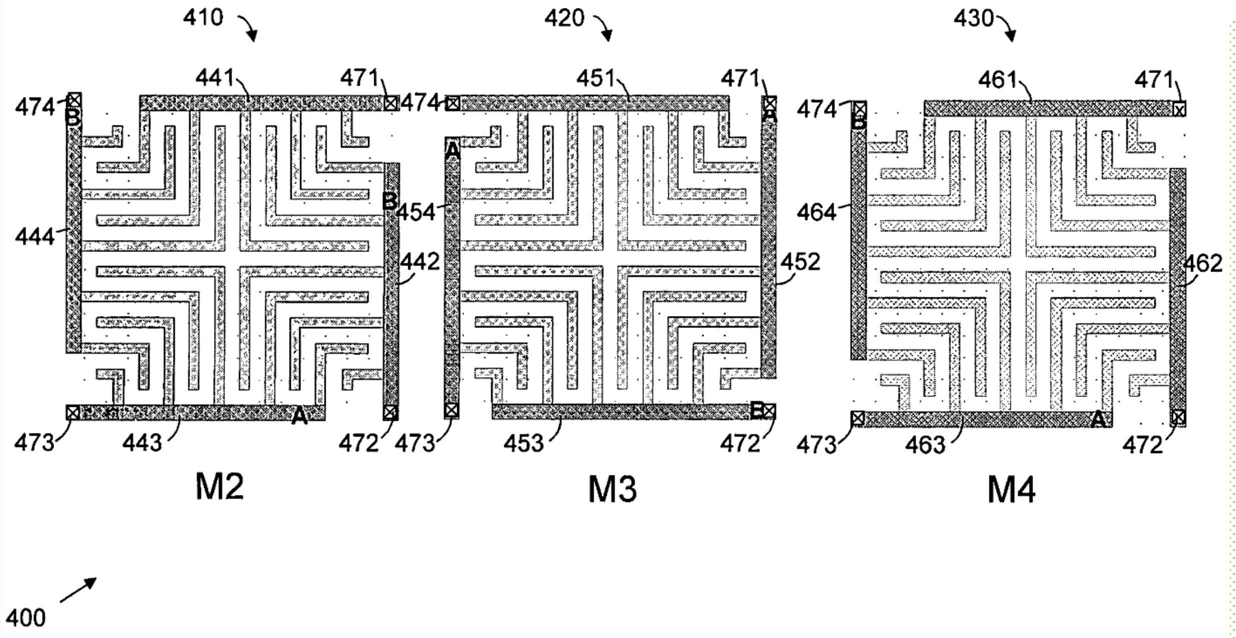




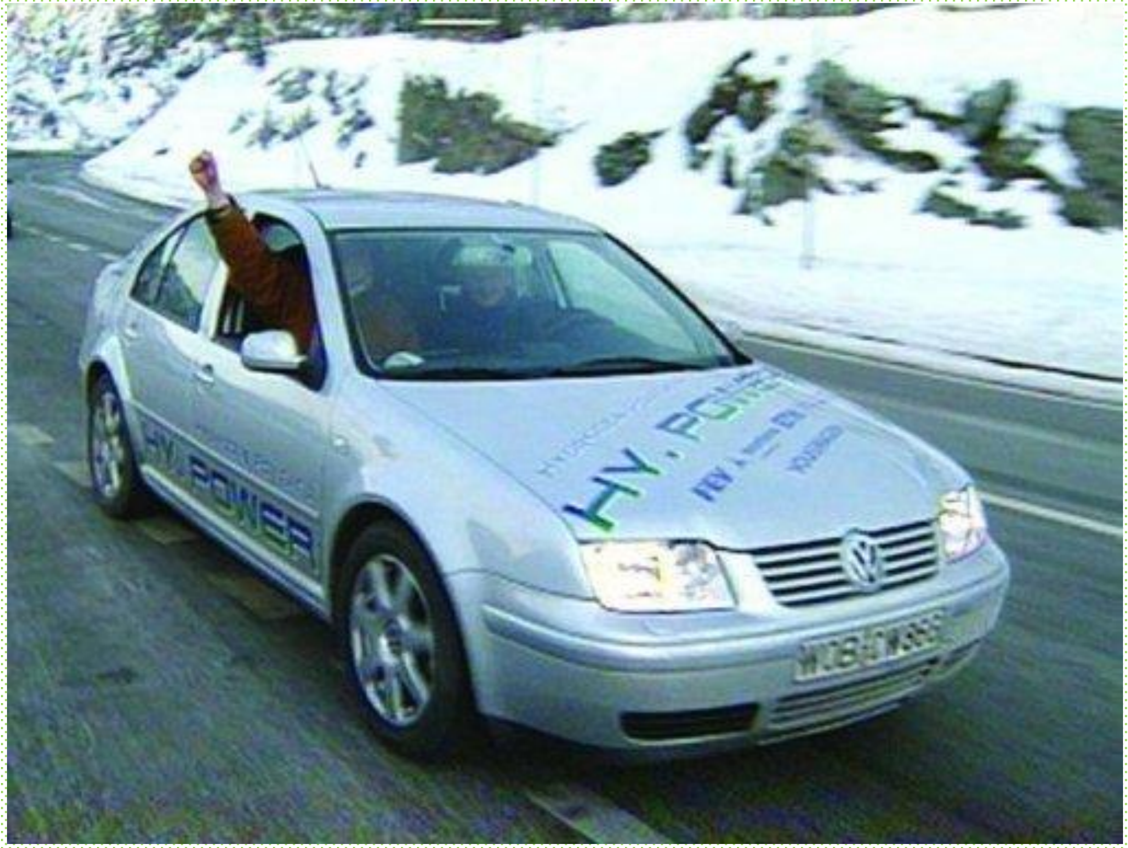
Fractal Capacitor







Fractal Capacitor



Field Test

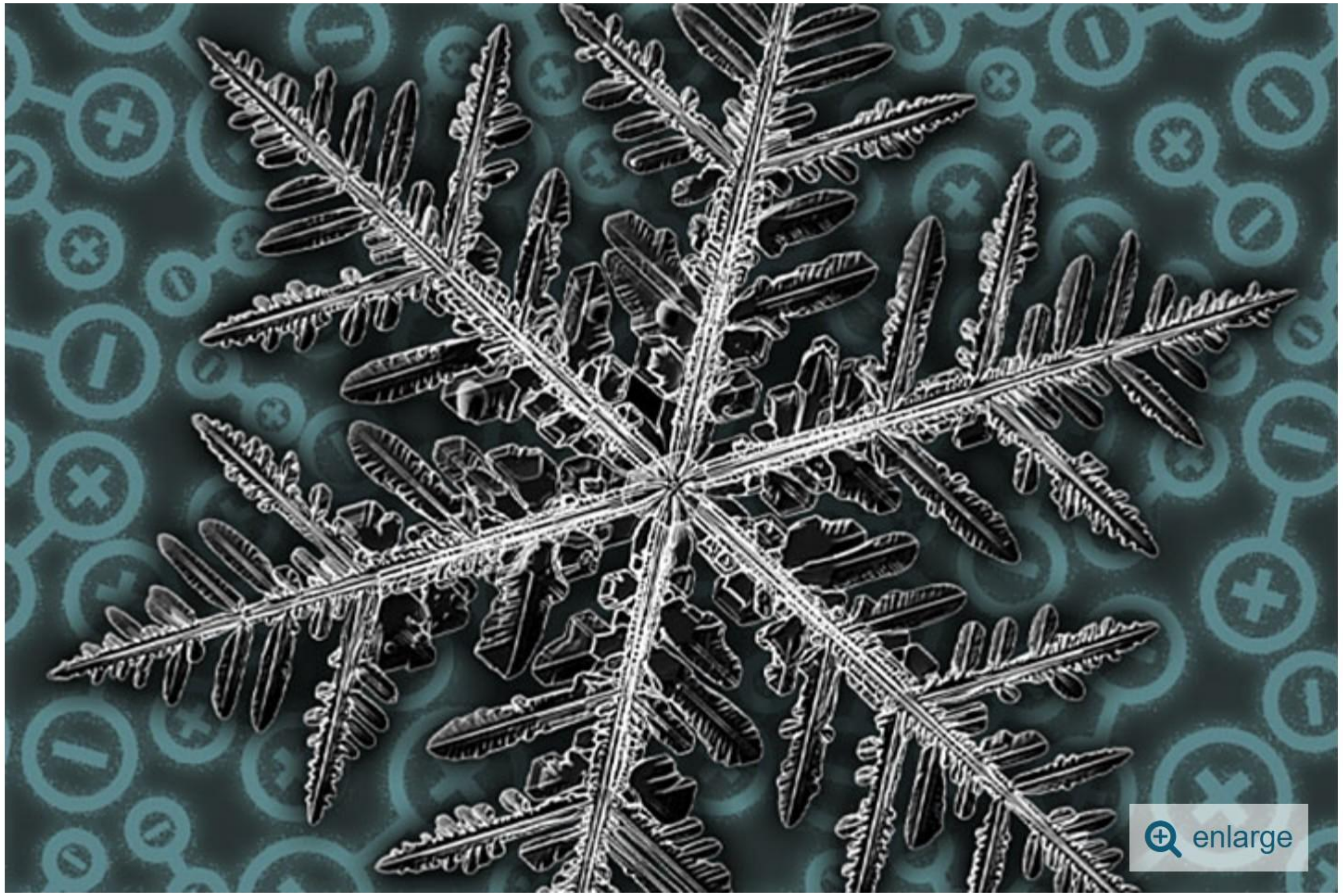
Switzerland 2002

分形電容器應用的一個成功試驗性例子是由瑞士 Paul Scherrer Institute 公司研製的分形超級電容器（supercapacitor 或 ultracapacitor）[6] 安裝在一輛名為 *Hy.Power* 的燃料驅動小汽車裏，用作汽車爆發加速時的拖動功率補給。2002 年 1 月 16 日，*Hy.Power* 成功地爬上了位於瑞士 Brig 與義大利 Domodossola 之間海拔兩千多米高的 Simplon 山口（圖 13）。這段山路極為陡峭，而且當時山頂氣候條件惡劣，同類型的小汽車只能望山興歎 [7]

Scientists discover fractal patterns in a quantum material

The X-ray-focusing lens used in the experiment is based on a design used in lighthouses for centuries.

October 18, 2019



The repeating patterns in a snowflake are a classic example of beautiful, geometric fractals. Now MIT scientists have discovered fractal-like patterns in the magnetic configurations of a quantum material for the first time.

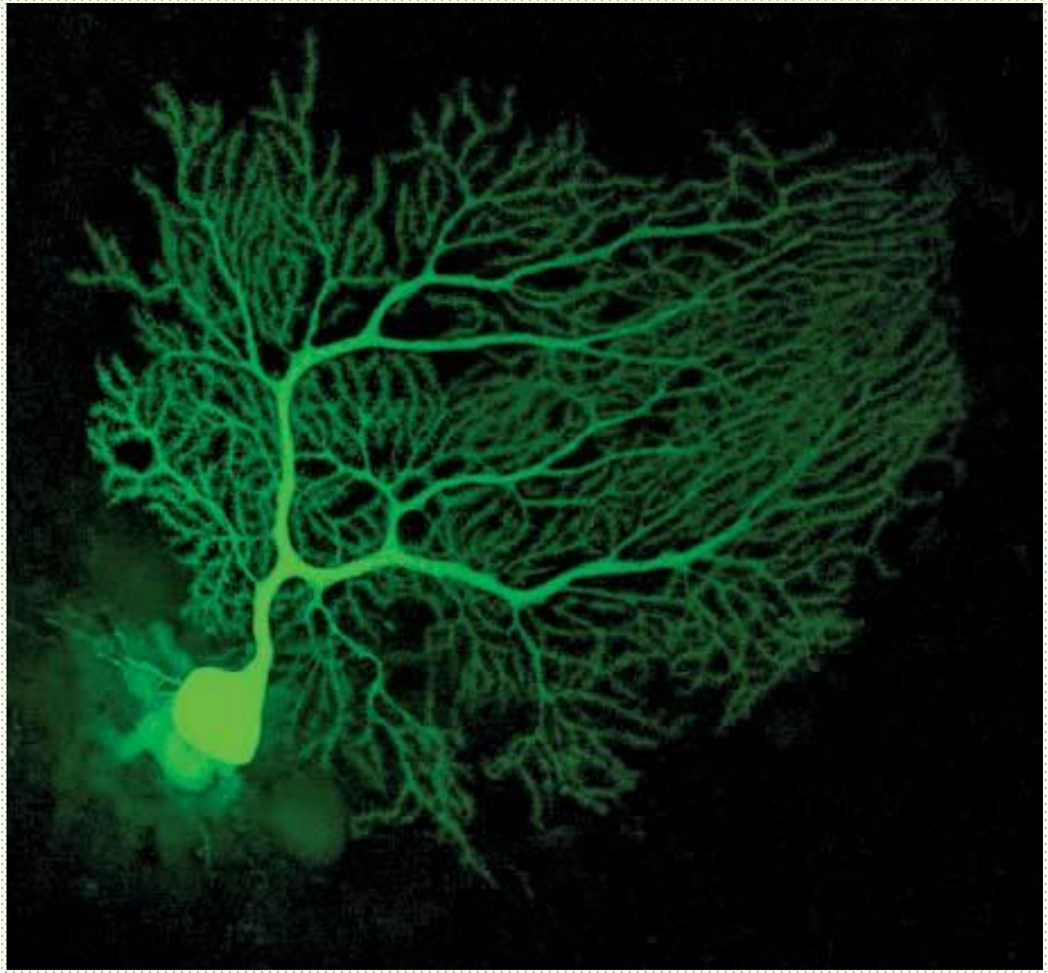
Fractals in Materials

Invisible cloak made by fractal materials

Invisible cloak

Fractals in Biology

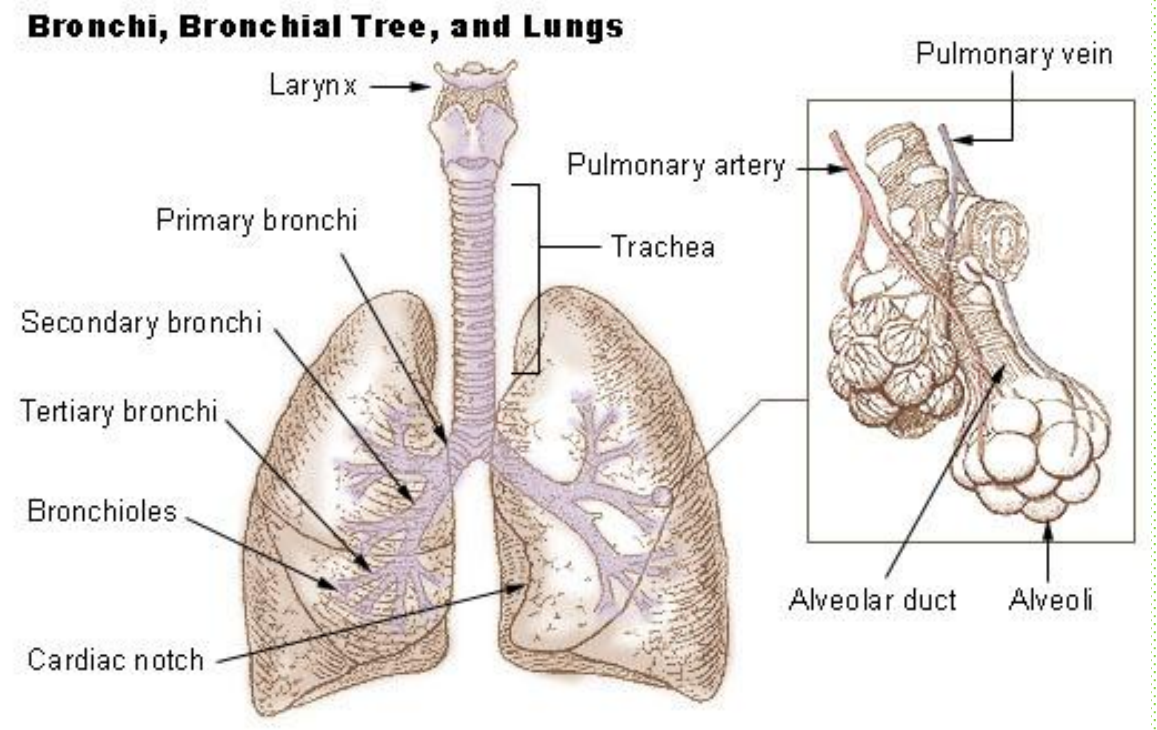
Fractals in Human Heart

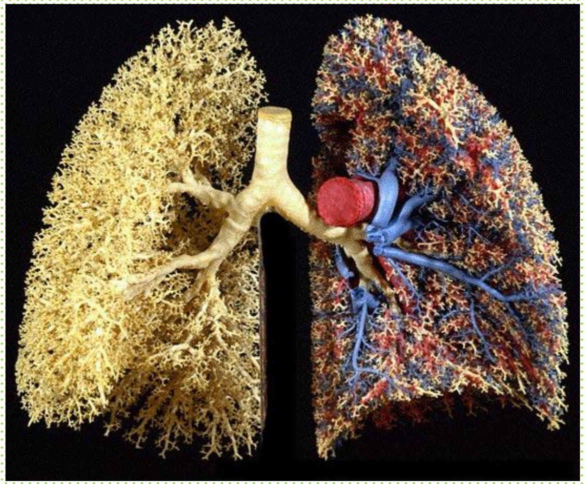


Purkinje Cell in Human Heart

Fractals in Human Body

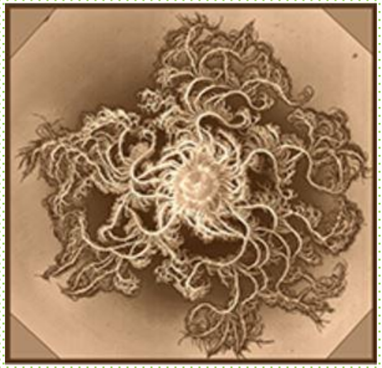
- Human respiratory, circulatory, and nervous systems have fractal architectures
- Branches → Branches → ……

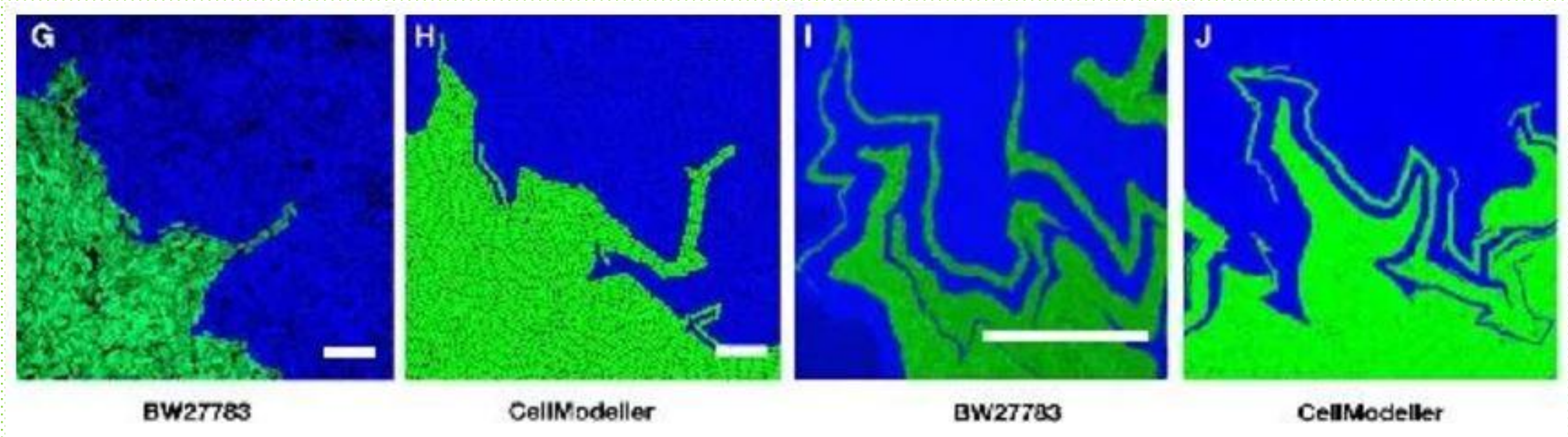




Fractals in Bacteria Colony

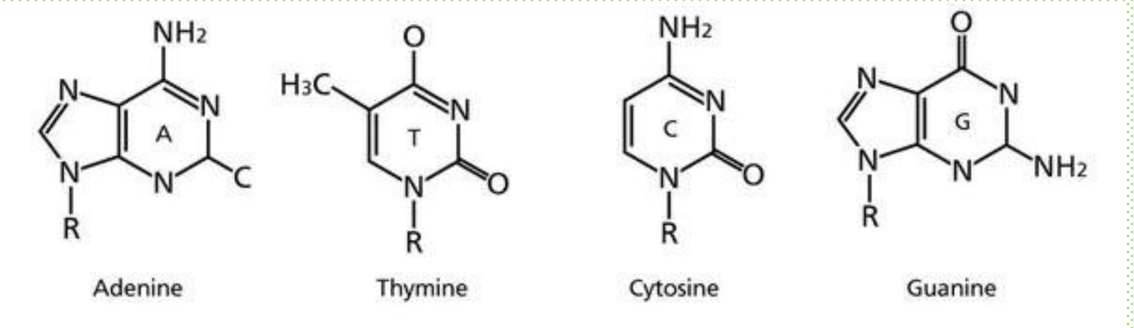
- Fractals have been applied to analyzing the growth of bacteria



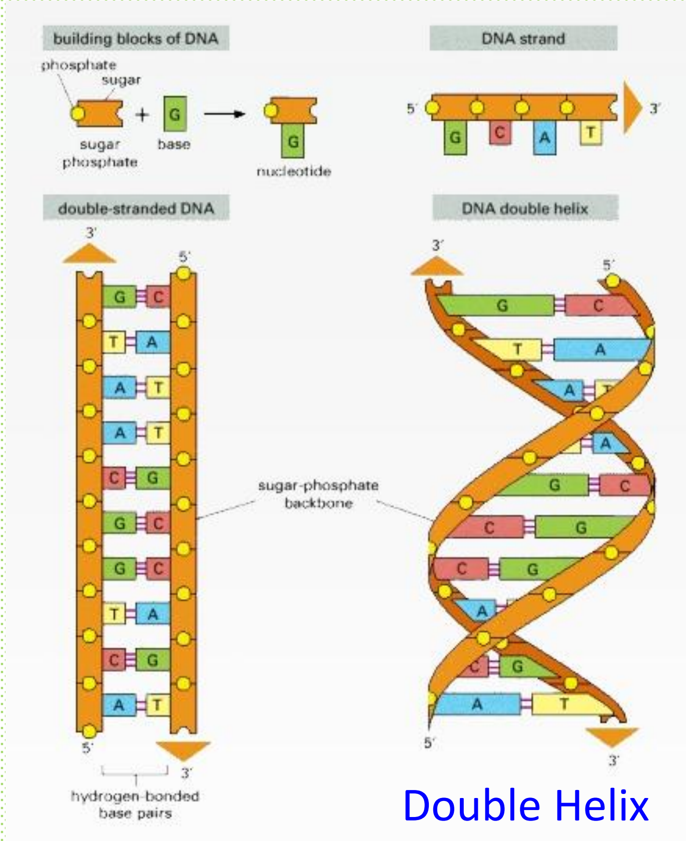


Fractal Representation of DNA Sequences

4 Basic elements:

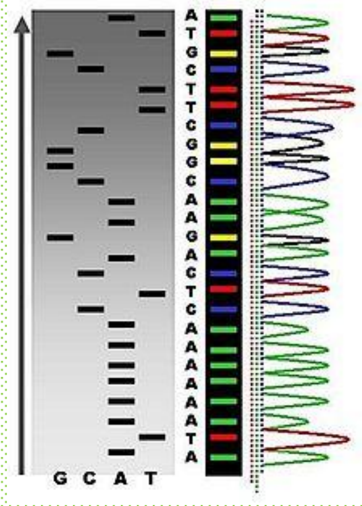


A **T** **C** **G**



Double Helix

- C and G are paired together
- A and T (U) are paired together



GTAACGGTCA ….

Fractal Representation of DNA Sequences

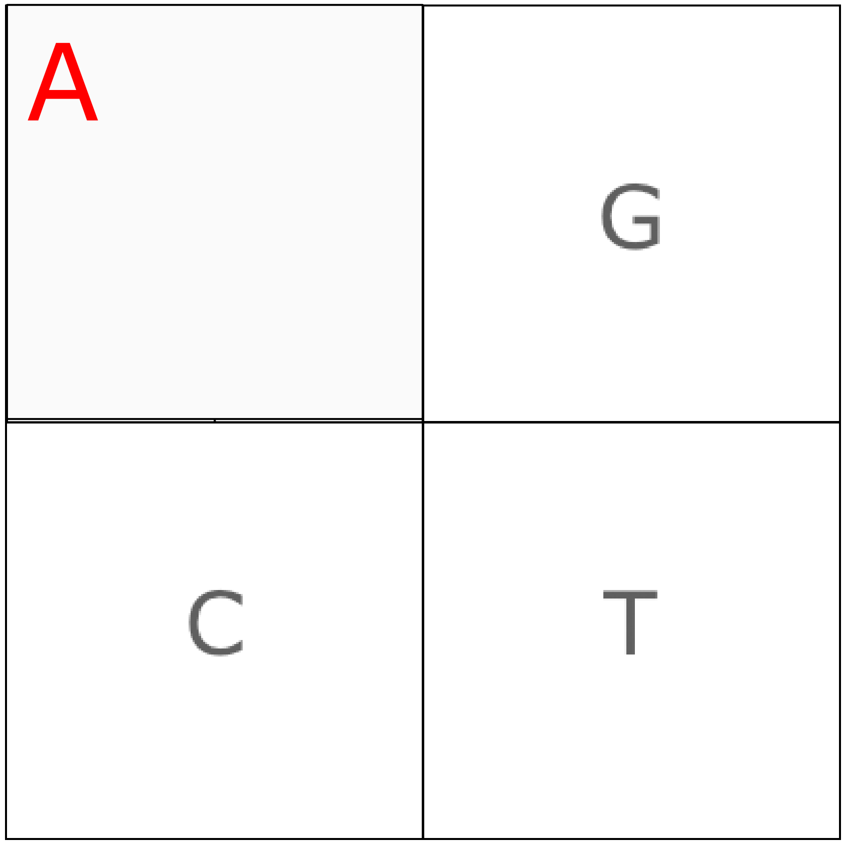
- A handy approach to dealing with large amount of data in DNA sequence

1. Divide a square into four parts, representing A,C,T,G

2. Each part is further divided into 4, and so on

3. Each pixel is associated with a specific word and the word frequencies are displayed by the intensity of each pixel

Fractal



Example:

GCTATATATATATATAGCGCAGTAAGTAAGTAAGGC

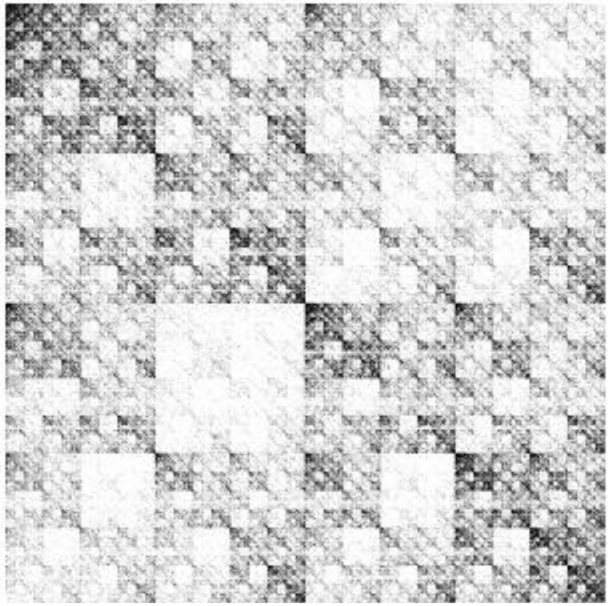
AA	AG	GA	GG
AC	AT	GC	GT
CA	CG	TA	TG
CC	CT	TC	TT

Pattern	Freq	Pattern	Freq
AA	2	AG	4
AC		AT	6
GA		GG	1
GC	4	GT	2
CA	1	CG	1
CC		CT	1
TA	9	TG	
TC		TT	

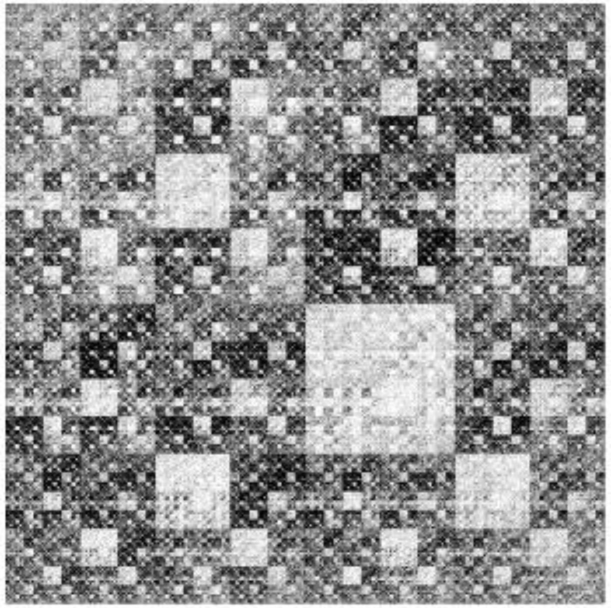
Self-Similarity

In DNA sequences

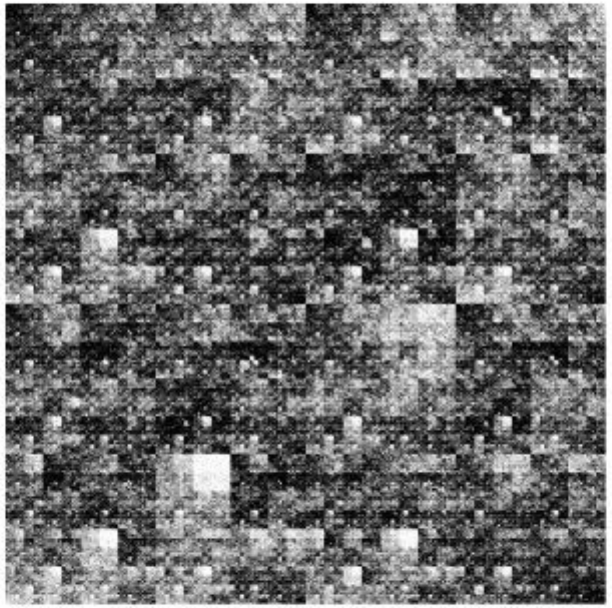
Mycoplasma genitalium



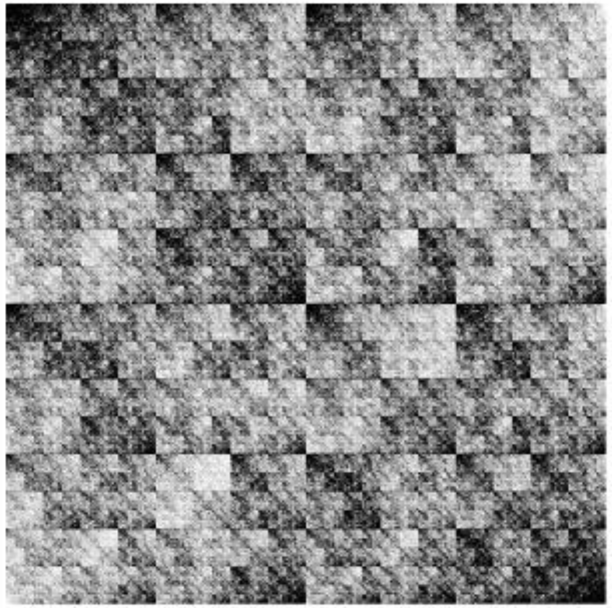
Synechococcus sp.



Escherichia coli



Bacillus subtilis



The grey color depends on the occurrence frequency

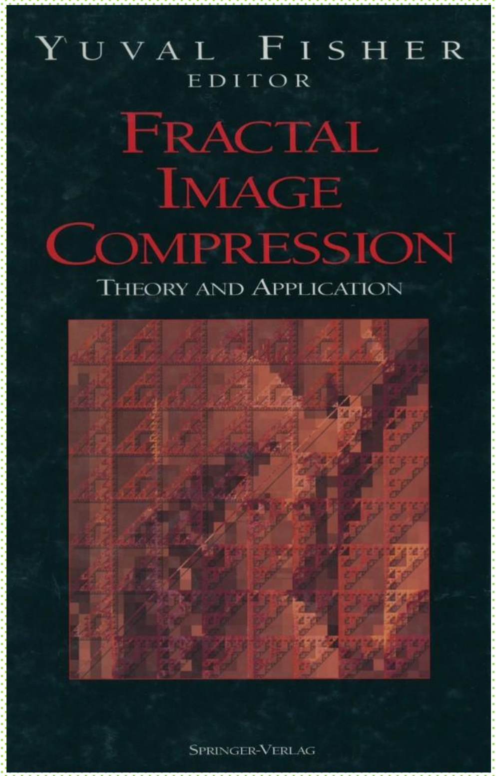
Fractal Data Compression

- Image compression means to use an alternative way to store/represent an image so that the file size is smaller (easier to store and process)
- Use fractals to compress images
- Idea: Search for self-similarities

  in an image and then represent it

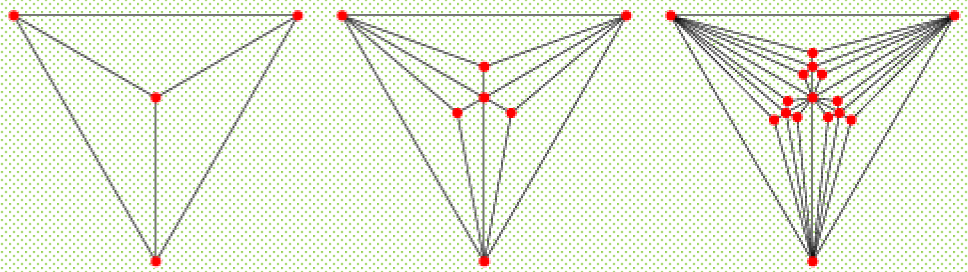
  with an iterated function system

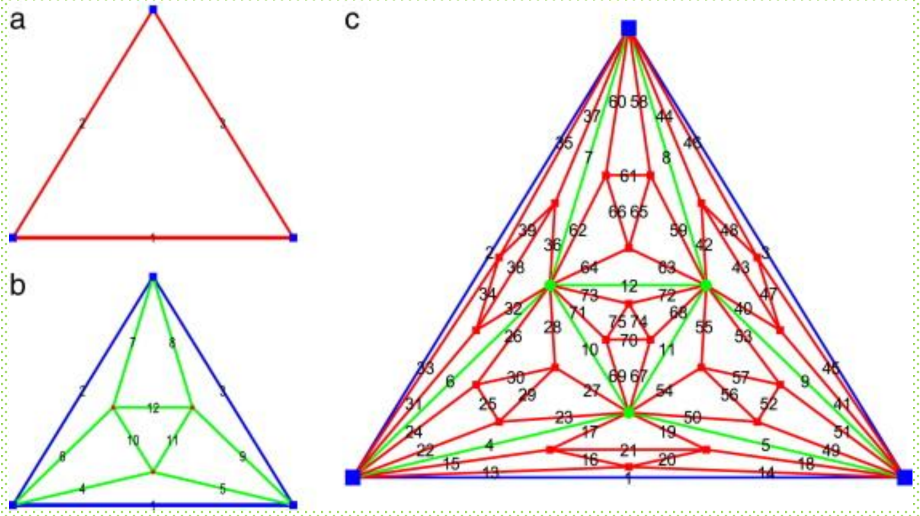
[Fractal Compression]

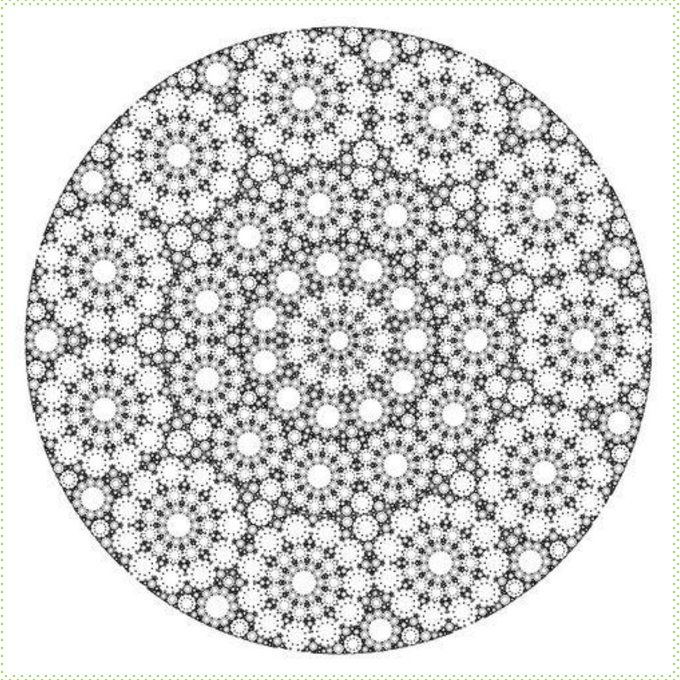


Fractal Networks

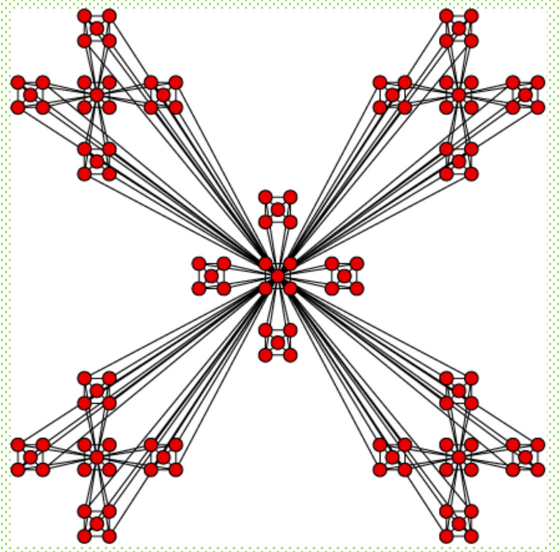
Apollonian networks

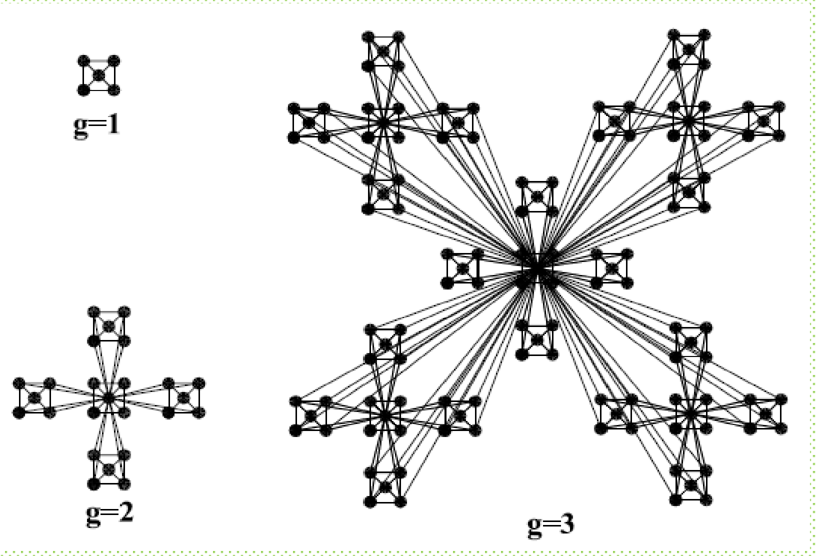




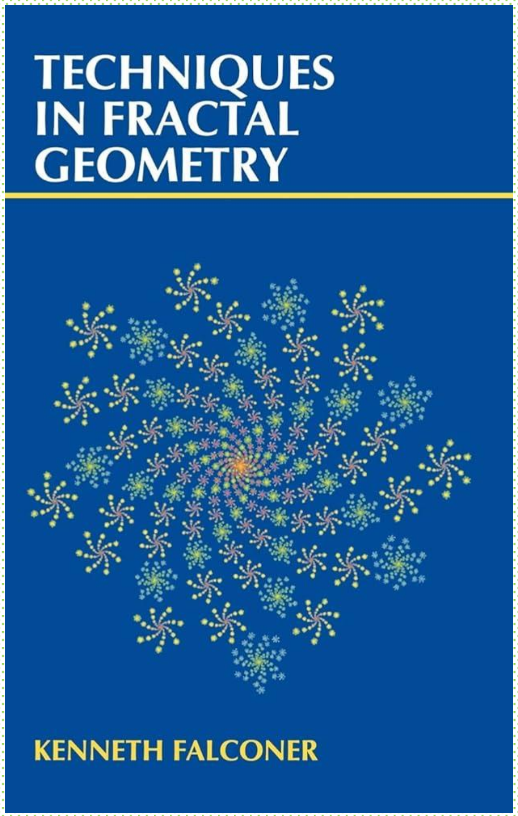


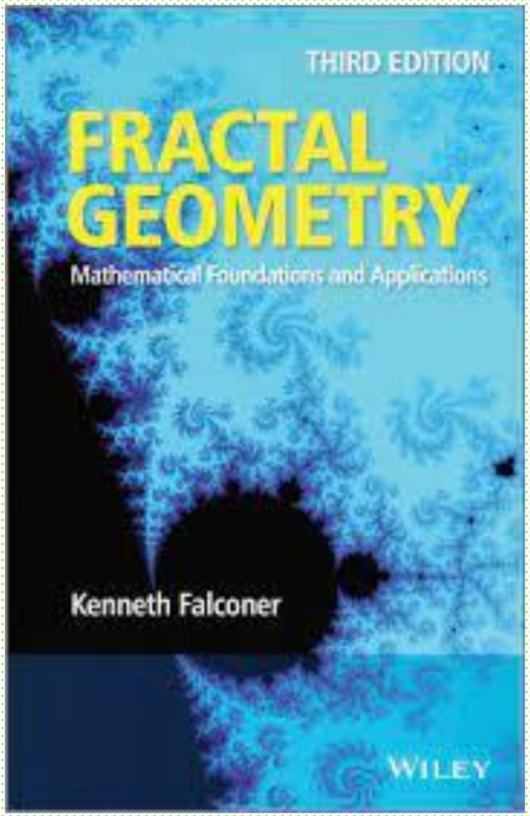
Fractal Networks

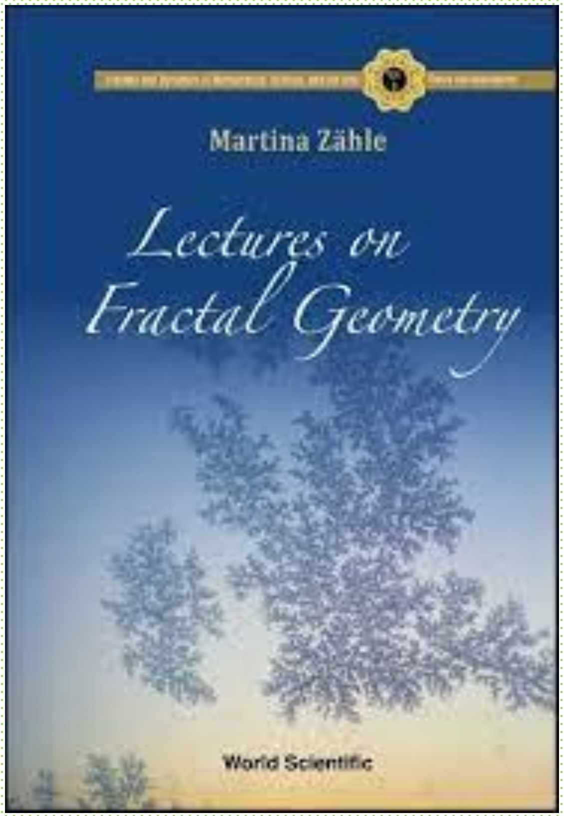


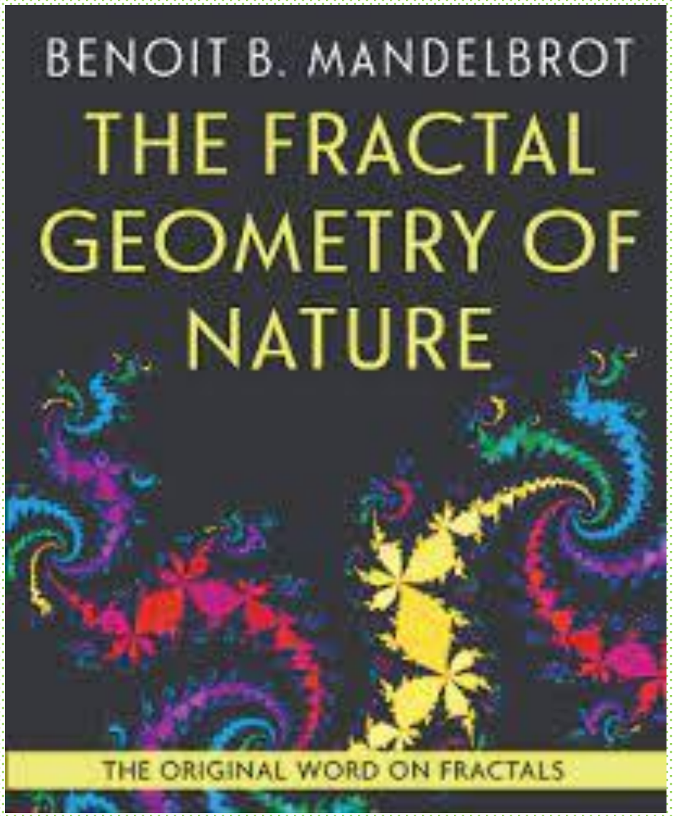


Step g =	1	2	3	N
Maximum degree $d_{\max}$	4	$4+4\times 4$ $=20$	$4+4\times 4+4\times 4\times 4$ $=84$	$4+4\times 4+\cdots+4\times 4\times 4+\cdots+4^N$ $=\frac{4}{3}\left(4^N-1\right)$
Minimum degree $d_{\min}$	3	3	3	3
How many nodes with $d_{\min}$	4	4	$4+4\times 4$ $=20$	$4+4\times 4+\cdots+4\times 4\times 4+\cdots+4^{N-1}$ $=\frac{4}{3}\left(4^{N-1}-1\right) \quad (N\geq 2)$









More Applications

End