Proof Portfolio
Math 322

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Chapter 1

Reflection

Which of your proofs do you feel is the best? Why do you think this is?

I feel like my proofs for problem 1 and my proof for problem 5 are the best. I think that they are the easiest to follow, and the proofs are correct.

Which of your proofs do you feel least confident about? Why do you think this is?

I feel less confident about my proofs for problems 2 and 4. That is because they are lengthy, and can feel overwhelming when reading through them. I wanted them to be as detailed as possible, which could be why they are so lengthy.

What do you feel is your greatest strength when writing proofs for this portfolio?

I think my greatest strength was solving the proofs on paper. I found it difficult and time consuming when trying to transfer my work from my paper to my computer, the work on my paper was much more efficient.

Reflect on the entire semester and common on how you feel you have progressed in writing proofs. You should specifically address what you think has been your biggest improvement and what insights ("Aha!" moments) you have had about the process of proving results and writing up your proofs for your portfolio.

Prior to this class, I had no experience writing proofs. I discovered that the proof writing process needs to appeal to any audience. A proof should be interpretable across disciplines. Most of my writing before this class I would just assume that people could follow along with a proof, but I learned that is not the case, and you have to lead people so that they can follow along on their own.

Do you think that the proof portfolio was valuable in helping you improve your skills in writing proofs? If so, please indicate which parts were most helpful. If not, please comment on why you feel it was not valuable. In both cases, please give any suggestions you might have on how the instructor could improve the implementation of the project in the future.

I feel like the portfolio was valuable in improving my skills writing proofs. I liked that it was a semester long project, and we could continually get feedback on different parts, week after week. The feedback each week really helped me refine some of the proofs.

Chapter 2

Problems

2.1 Problem 1

Conjecture 1A: If m is an odd integer, then $\dfrac{m^4-1}{4}$ is an even integer.

Proof. Let m be an odd integer. By definition of odd integers, there exists $q \in \mathbb{Z}$ such that

$$m = 2q+1$$

By substitution,

$$\begin{aligned}
\frac{m^4-1}{4} &= \frac{(2q+1)^4-1}{4} \\
&= \frac{(2q+1)(2q+1)(2q+1)(2q+1)-1}{4} \\
&= \frac{(16q^4+32q^3+24q^2+8q+1)-1)}{4} \\
&= \frac{16q^4+32q^3+24^2+8q}{4} \\
&= \frac{8(2q^4+4q^3+3q^2+q)}{4}
\end{aligned}$$

By closure property of integers, there exists $z \in \mathbb{Z}$ such that

$$z = 2q^4+4q^3+3q^2+q$$

By substitution,

$$\frac{m^4-1}{4} = \frac{8z}{4} = 2z$$

By the definition of even integers, $\frac{m^4-1}{4}$ is an even integer.

□

Conjecture 1B: If m is an odd integer, then $\dfrac{m^4-1}{5}$ is an even integer.

Counterexample. Let m be the odd integer 5. By substitution,

$$\begin{aligned}\frac{m^4-1}{5} &= \frac{5^4-1}{5}\\ &= \frac{625-1}{5}\\ &= \frac{624}{5}\end{aligned}$$

$\frac{624}{5}$ is not an element of the integers, therefore, the conjecture is false.

□

2.2 Problem 2

Conjecture 2B: For all $a,b \in \mathbb{Z}$, $(a-3)b^2$ is even if, and only if, a is odd or b is even.

Consider the following cases, where a is odd or b is even :
1. a is odd and b is odd.
2. a is odd and b is even.
3. a is even and b is even.

By proving these cases true, we can prove the ”If a is odd or b is even, then $(a-3)b^2$ is even” to be true. Since the initial conjecture is bi-conditional, we will need to prove ”If $(a-3)b^2$ is even, then a is odd or b is even. We can prove this statement by proving the contrapositive.

Proof. **Consider the following contrapositive statement** : If a is even and b is odd, then $(a-3)b^2$ is odd. Assume a is an even integer and b is an odd integer. By definition of even and odd integers, there exists m and $n \in \mathbb{Z}$ such that

$$\begin{aligned}a &= 2m\\ b &= 2n+1\end{aligned}$$

By substitution,

$$\begin{aligned}(a-3)b^2 &= (2m-3)(2n+1)^2\\ &= (2m-3)(4n^2+4n+1)\\ &= 8n^2m+8nm+2m-12n^2-12n-3\\ &= 8n^2m+8nm+2m-12n^2-12n-4+1\\ &= 2(4n^2m+4nm+m-6n^2-6n-2)+1\end{aligned}$$

By closure property of integers, there exists $q \in \mathbb{Z}$ such that

$$q = 4n^2m+4nm+m-6n^2-6n-2$$

By substitution,

$$2(4n^2m+4nm+m-6n^2-6n-2)+1 = 2q+1$$

Therefore, by substitution again

$$(a-3)b^2 = 2q+1$$

This implies that $(a-3)b^2$ is odd, proving the contrapositive statement. Therefore, if $(a-3)b^2$ is even, then a is odd or b is even.

Consider the following cases :

Case 1. Assume a and b are odd integers. By definition of odd integers, there exists m and $n \in \mathbb{Z}$ such that

$$\begin{aligned}a &= 2m+1\\ b &= 2n+1\end{aligned}$$

By substitution,

$$\begin{aligned}(a-3)b^2 &= (2m+1-3)(2n+1)^2\\ &= (2m-2)(4n^2+4n+1)\\ &= 8n^2m+8nm+2m-8n^2-8n-2\\ &= 2(4n^2m+4nm+m-4n^2-4n-1)\end{aligned}$$

By closure property of integers, there exists $z \in \mathbb{Z}$ such that

$$z = 4n^2m + 4nm + m - 4n^2 - 4n - 1$$

By substitution,

$$2(4n^2m + 4nm + m - 4n^2 - 4n - 1) = 2z$$

By substitution, again

$$(a-3)b^2 = 2z$$

By definition of even integers, this shows that $(a-3)b^2$ is even. Therefore, $(a-3)b^2$ is even when a is odd and b is odd.

Case 2. Assume a is odd and b is even. By definition of even and odd integers, there exists m and $n \in \mathbb{Z}$ such that

$$\begin{aligned} a &= 2m + 1 \\ b &= 2n \end{aligned}$$

By substitution,

$$\begin{aligned} (a-3)b^2 &= (2m + 1 - 3)(2n)^2 \\ &= (2m - 2)(4n^2) \\ &= 8n^2m - 8n^2 \\ &= 2(4n^2m - 4n^2) \end{aligned}$$

By closure property of integers, there exists $z \in \mathbb{Z}$ such that

$$z = 4n^2m - 4n^2$$

By substitution,

$$2(4n^2m - 4n^2) = 2z$$

By substitution, again

$$(a-3)b^2 = 2z$$

By definition of even integers, $(a-3)b^2$ is even. Therefore, $(a-3)b^2$ is even when a is odd and b is even.

Case 3. Assume a and b are even. By definition of even integers, there exists m and $n \in \mathbb{Z}$ such that

$$\begin{aligned} a &= 2m \\ b &= 2n \end{aligned}$$

By substitution,

$$\begin{aligned} (a-3)b^2 &= (2m-3)(2n)^2 \\ &= (2m-3)(4n^2) \\ &= 8mn^2 - 12n^2 \\ &= 2(4mn^2 - 6n^2) \end{aligned}$$

By closure property of integers, there exists $z \in \mathbb{Z}$ such that

$$z = 4mn^2 - 6n^2$$

By substitution,

$$2(4mn^2 - 6n^2) = 2z$$

By substitution, again

$$(a-3)b^2 = 2z$$

By definition of even integers, $(a-3)b^2$ is even. Therefore, $(a-3)b^2$ is even when a is even and b is even.
We have satisfied *all* cases for this proof, and proved the contrapositive, so we can say that for all $a, b \in \mathbb{Z}$, $(a-3)b^2$ is even if, and only if, a is odd or b is even.

$\square$

2.3 Problem 3

Problem 3A: Prove that for all integers m and n, if n is an odd integer, then there does not exist an integer x such that

$$x^2 + 2mx + 2n = 0.$$

Consider the following four cases :

1. m is even and x is even
2. m is even and x is odd
3. m is odd and x is even
4. m is odd and x is odd

Proof. **Case 1.** Let m and x be even integers, and let n be an odd integer. By definition of even and odd integers, there exists some integers a, k, and b such that

$$m = 2a$$
$$n = 2k + 1$$
$$x = 2b$$

By substituting these values into the original equation, we get

$$\begin{aligned}(2b)^2 + 2(2a)(2b) + 2(2k+1) &= 0\\ 4b^2 + 8ab + 4k + 2 &= 0\\ 2(2b^2 + 4ab + 2k + 1) &= 0\\ 2(2(b^2 + 2ab + k) + 1) &= 0\end{aligned}$$

By closure property of integers, there exists some integer q such that

$$q = b^2 + 2ab + k$$

By substitution,

$$\begin{aligned}2(2(b^2 + 2ab + k) + 1) &= 0\\ 2(2q + 1) &= 0\end{aligned}$$

Divide 2 on both sides,

$$\frac{2(2q+1)}{2} = \frac{0}{2}$$
$$2q+1 = 0$$

Case 2. Let m be an even integer and x and n be odd integers. By definition of even and odd integers, there exists some integers a, k, and b such that

$$m = 2a$$
$$n = 2k+1$$
$$x = 2b+1$$

By substituting these values into the original equation, we get

$$(2b+1)^2 + 2(2a)(2b+1) + 2(2k+1) = 0$$
$$4b^2 + 4b + 1 + 8ab + 4a + 4k + 2 = 0$$
$$2(2b^2 + 2b + 4ab + 2a + 2k + 1) + 1 = 0$$

By closure property of integers, there exists some integer q such that

$$q = 2b^2 + 2b + 4ab + 2a + 2k + 1 \tag{2.1}$$

By substitution,

$$2(2b^2 + 2b + 4ab + 2a + 2k + 1) + 1 = 0$$
$$2q + 1 = 0$$

Case 3. Let m and n be odd integers, and let x be an even integer. By definition of even and odd integers, there exists some integers a, k, and b such that

$$m = 2a+1$$
$$n = 2k+1$$
$$x = 2b$$

By substituting these values into the original equation, we get

$$
\begin{aligned}
(2b)^2 + 2(2a+1)(2b) + 2(2k+1) &= 0 \\
4b^2 + 8ab + 4b + 4k + 2 &= 0 \\
2(2b^2 + 4ab + 2b + 2k + 1) &= 0 \\
2(2(b^2 + 2ab + b + k) + 1) &= 0
\end{aligned}
$$

By closure property of integers, there exists some integer q such that

$$q = b^2 + 2ab + b + k$$

By substitution,

$$
\begin{aligned}
2(2(b^2 + 2ab + b + k) + 1) &= 0 \\
2(2q+1) &= 0 \\
\frac{2(2q+1)}{2} &= \frac{0}{2} \\
2q + 1 &= 0
\end{aligned}
$$

Case 4. Let m, x, and n be odd integers. By definition of odd integers, there exists some integers a, k, and b such that

$$
\begin{aligned}
m &= 2a+1 \\
n &= 2k+1 \\
x &= 2b+1
\end{aligned}
$$

By substituting these values into the original equation, we get

$$
\begin{aligned}
(2b+1)^2 + 2(2a+1)(2b+1) + 2(2k+1) &= 0 \\
4b^2 + 4b + 1 + 8ab + 4a + 4b + 2 + 4k + 2 &= 0 \\
2(2b^2 + 2b + 4ab + 2a + 2b + 2k + 2) + 1 &= 0
\end{aligned}
$$

By closure property of integers, there exists an integer q such that

$$q = 2b^2 + 2b + 4ab + 2a + 2b + 2k + 2$$

By substitution,

$$\begin{aligned} 2(2b^2+2b+4ab+2a+2b+2k+2)+1=0 \\ 2q+1=0 \end{aligned}$$

Every case results in the equation $2q+1=0$. This implies that there is some odd integer that equals zero. Zero is not an element of the odd integers, so this means there can't be a value for x that would result in the equation being equal to zero. Since we have exhausted all values of x and m, we can say that for all integers m and n, if n is an odd integer, then there *does not* exist an integer x such that $x^2+2mx+2n=0$ □

2.4 Problem 4

Theorem 4: For all odd natural numbers n with $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^n}{n}\right)=1.$$

We will use proof by mathematical induction for this.

Proof. **Base Case:** $n=3$

$$n_3=\tfrac{3}{2}*\tfrac{2}{3}=1\ \checkmark$$

The statement is true for the base case, so we can continue on to the inductive step.

Inductive Step: Assume the proposition is true for $k=n$.

$$(1+\tfrac{1}{2})*(1-\tfrac{1}{3})*(1+\tfrac{1}{4})***(1+\tfrac{(-1)^k}{k})=1$$

If we can show that the next two values in the sequence, when multiplied together, are equal to one we can prove this statement. The next two values are $(k+1)$ and $(k+2)$. The product given by these two would be:

$$\begin{aligned} &[1+\frac{(-1)^{k+1}}{k+1}]*[1+\frac{(-1)^{k+2}}{k+2}] \\ &=1+\frac{-1^{k+1}}{k+1}+\frac{-1^{k+2}}{k+2}+\frac{(-1^{k+1})(-1^{k+2})}{(k+1)(k+2)} \\ &=1+\frac{-1^{k+1}}{k+1}+\frac{-1^{k+2}}{k+2}+\frac{-1^{2k+3}}{(k+1)(k+2)} \end{aligned}$$

We need to combine the fractions, we will multiply accordingly to get the same denominator for all values.

$$
\begin{aligned}
&= 1 + \frac{-1^{k+1}}{k+1} + \frac{-1^{k+2}}{k+2} + \frac{-1^{2k+3}}{(k+1)(k+2)} \\
&= 1 + \frac{(k+1)(-1^{k+2}) + (k+2)(-1^{k+1}) + (-1^{2(k+1)+1})}{(k+1)(k+2)} \\
&= 1 + \frac{(k+1)(-1^{k+2}) + (k+2)(-1^{k+1}) - 1}{(k+1)(k+2)}
\end{aligned}
$$

Let k be an odd number. There exists some $q \in \mathbb{Z}$ such that $k = 2q+1$.

$$
\begin{aligned}
&= 1 + \frac{(2q+2)(-1^{2q+3}) + (2q+3)(-1^{2q+2}) - 1}{(2q+2)(2q+3)} \\
&= 1 + \frac{(2q+2)(-1^{2(q+1)+1}) + (2q+3)(-1^{2(q+1)}) - 1}{(2q+2)(2q+3)} \\
&= 1 + \frac{(2q+2)(-1) + (2q+3)(1) - 1}{(2q+2)(2q+3)} \\
&= 1 + \frac{-2q-2+2q+3-1}{(2q+2)(2q+3)} \\
&= 1 + \frac{0}{(2q+2)(2q+3)} \\
&= 1
\end{aligned}
$$

We see that the product of the following two terms is equal to one. Therefore, the theorem holds by mathematical induction.

□

2.5 Problem 5

Theorem 5: Let $a_1, a_2, \ldots, a_n, \ldots$ be the sequence where $a_1 = a_2 = a_3 = 1$, and for every natural number n,

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

Then for every natural number n with $n > 1$, $a_n \leq 2^{n-2}$.

Proof. We will prove the statement using math induction. First consider the base case: Let $n = 2$

$$a_2 \leq 2^0$$
$$1 \leq 1\checkmark$$

The base case holds! Now we need to consider the inductive step.
Let $a_n \leq 2^{n-2}$, multiplying both sides by 2 allows us to show $n+1$ for 2^{n-2}

$$a_n \leq 2^{n-2}$$
$$2a_n \leq 2^{n-1}$$

We need to show $2a_n \geq a_{n+1}$

$$2a_n \geq a_{n+1}$$
$$a_n + a_n \geq a_{n+1}$$

Let $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, substitute this for one a_n

$$a_n + a_{n-1} + a_{n-2} + a_{n-3} \geq a_{n+1}$$

Now we apply the original rule , $a_{n+3} = a_{n+2} + a_{n+1} + a_n$, where

$$a_{n+1} = a_n + a_{n-1} + a_{n-2}$$

Then,

$$a_n + a_{n-1} + a_{n-2} + a_{n-3} \geq a_n + a_{n-1} + a_{n-2}$$
$$a_{n+1} + a_{n-3} \geq a_{n+1}\checkmark$$

This must be true, since all values of a_n are positive. Therefore the statement, for every natural number n with $n > 1$, $a_n \leq 2^{n-2}$, is true by math induction. □