4.3 EXERCISES (Part 1)

BMED6420
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Consult the class slides, hints, and cited literature for the comprehensive solutions of exercise problems.

E and Var. Find $\mathbb{E}X$ and $\mathbf{Var}(X)$ is X has a density
(a) $f(x) = 3x^2, \ 0 \le x \le 1$;
(b) $f(x) = \sin(x), \ 0 \le x \le \pi/2$.

Uniform on Unit Circle. Let bivariate random variable (X, Y) has a density

$$f(x, y) = \frac{1}{\pi}, \ x^2 + y^2 \le 1.$$

(a) Find $f_{Y|X}(y|x)$.
(b) What is the conditional density $f(y|x)$ for $x = 0$?

Fungi Spores. Fungi spores are eliptical in shape with minor and major axes used to describe their size. Let X and Y denote lengths of the minor and major axes (in micrometers):

$$\begin{aligned} &X \sim \mathcal{E}xp(1), \\ &Y|X = x \sim f(y|x) = e^{-(y-x)}, y \ge x. \end{aligned}$$

Find
(a) joint distribution $f_{(X,Y)}(x, y)$;
(b) $\mathbb{P}(Y \le 2|X = 1)$;
(c) $\mathbb{E}(Y|X = 1)$;
(d) constant c, such that $\mathbb{P}(Y \le c) = 0.5$.
(e) Are X and Y independent?

Joint, Marginals and Conditionals #1. Let random variables X and Y have joint density $f_{(X,Y)}(x, y) = \lambda^2 e^{-\lambda y}, \ 0 \le x \le y$.
Find
(a) marginal densities $f_X(x)$ and $f_Y(y)$;
(b) conditional densities of X given $Y = y$, $f_{X|Y}(x|y)$, and of Y given $X = x$, $f_{Y|X}(y|x)$.
(c) the CDF of (X, Y), $F_{(X,Y)}(x, y) = \mathbb{P}(X \le x, Y \le y)$.

Joint, Marginals and Conditionals #2. Let random variables X and Y have joint density $f_{(X,Y)}(x, y) = Cxy^2, \ x \ge 0, \ y \ge 0, \ x + y \le 1$.
(a) Show that $C = 60$, $f_X(x) = 20x(1 - x)^3, \ 0 \le x \le 1$ and $f_{Y|X}(y|x) = \frac{3y^2}{(1-x)^3}, \ 0 \le y \le 1 - x$.

(b) Find $f_Y(y)$ and $f_{X|Y}(x|y)$.

Joint, Marginals and Conditionals #3. Let random variables X and Y have joint density $f_{(X,Y)}(x,y) = \begin{cases} \frac{1+xy}{4}, & -1 \le x \le 1, \ -1 \le y \le 1 \\ 0, & \text{else} \end{cases}$

Find
(a) marginal densities $f_X(x)$ and $f_Y(y)$;
(b) conditional densities of X given $Y = y$, $f_{X|Y}(x|y)$, and of Y given $X = x$, $f_{Y|X}(y|x)$.
(c) Are X and Y independent?

Joint, Marginals and Conditionals #4. Let random variables X and Y have joint density $f_{(X,Y)}(x,y) = \begin{cases} C(x^2 - y^2)e^{-x} & 0 \le x < \infty, \ -x \le y \le x \\ 0, & \text{else} \end{cases}$

Find
(a) constant C,
(b) marginal densities $f_X(x)$ and $f_Y(y)$, and
(c) conditional distributions $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

A 2D PDF. Let

$$f(x,y) = \begin{cases} \frac{3}{8}(x^2 + 2xy), & 0 \le x \le 1, \ 0 \le y \le 2 \\ 0, & \text{else} \end{cases}$$

be a bivariate PDF of a random vector (X, Y).
(a) Show that $f(x, y)$ is a density.
(b) Show that marginal distributions are $f_X(x) = \frac{3}{2}x + \frac{3}{4}x^2, \ 0 \le x \le 1$, and $f_Y(y) = \frac{1}{8} + \frac{3}{8}y, \ 0 \le y \le 2$.
(c) Show $\mathbb{E}X = 11/16$ and $\mathbb{E}Y = 5/4$.
(d) Show that conditional distributions are

$$\begin{aligned} f(x|y) &= \frac{3x(x+2y)}{1+3y}, \ 0 \le x \le 1, \quad \text{for any fixed } y \in [0,2], \\ f(y|x) &= \frac{2y+x}{4+2x}, \ 0 \le y \le 2, \quad \text{for any fixed } x \in [0,1]. \end{aligned}$$

In a Circle. Let random variables X and Y have joint density
$f_{(X,Y)}(x,y) = \begin{cases} C\sqrt{1 - x^2 - y^2} & x^2 + y^2 \le 1 \\ 0, & \text{else} \end{cases}$

Find
(a) constant C, and

(b) marginal densities $f_X(x)$ and $f_Y(y)$.

Weibull – Gamma. Let X be distributed as Weibull $\mathcal{W}ei(r,\lambda)$, $r>0$ known, with a density

$$f(x|\lambda) = \lambda r x^{r-1} \exp\{-\lambda x^r\},\ \ \lambda > 0, x \geq 0.$$

Assume gamma $Ga(\alpha,\beta)$ prior on λ,

$$\pi(\lambda|\alpha,\beta) = \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} \exp\{-\beta\lambda\},\ \ \lambda \geq 0, \alpha, \beta > 0,$$

(a) Is the problem conjugate?
(b) Find the posterior distribution.
(c) If $X_1 = 5, X_2 = 3$, and $X_3 = 1$ are observed from $\mathcal{W}ei(2,\lambda)$ and $\pi(\lambda)$ is gamma $\mathcal{G}a(3,5)$, what is the posterior in this case? What are the mean and variance of λ according to this posterior distribution.

Normal – Inverse Gamma. Let $X \sim \mathcal{N}(\mu,\sigma^2)$, with μ known, and σ^2 is inverse gamma $\mathcal{IG}(\alpha,\beta)$, with a density

$$\pi(\sigma^2|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)(\sigma^2)^{\alpha+1}} \exp\left\{-\frac{\beta}{\sigma^2}\right\}, \sigma^2 \geq 0, \alpha, \beta > 0.$$

(a) Is the problem is conjugate?
(b) If the sample $(X_1, \dots, X_n)$

4.3430	0.5850	4.4345	6.2605	3.9778
4.4538	2.3931	3.5877	1.4254	5.0694

comes from normal $\mathcal{N}(3,\sigma^2)$ distribution, find the posterior distribution of σ^2 when the prior is $\mathcal{IG}(0.1, 0.1)$.

Uniform - Pareto. Suppose $X = (X_1, \dots, X_n)$ is a sample from $\mathcal{U}(0,\theta)$. Let θ have Pareto $\mathcal{P}a(\theta_0,\alpha)$ distribution. Show that the posterior distribution is $\mathcal{P}a(\max\{\theta_0, x_1, \dots, x_n\}$ $\alpha + n)$.

Gamma – Inverse Gamma. Let $X \sim \mathcal{G}a\left(\frac{n}{2}, \frac{1}{2\theta}\right)$, so that X/θ is χ^2_n. Let $\theta \sim \mathcal{IG}(\alpha,\beta)$. Show that the posterior is $\mathcal{IG}(n/2+\alpha, x/2+\beta)$.

Negative Binomial - Beta. If $X = (X_1, \dots, X_n)$ is a sample from $\mathcal{NB}(m,\theta)$ and $\theta \sim \mathcal{B}e(\alpha,\beta)$, show that the posterior for θ is beta $\mathcal{B}e(\alpha + mn, \beta + \sum_{i=1}^n x_i)$.

Horse-Kick Fatalities and Gamma Prior. During the latter part of the nineteenth century, Prussian officials gathered information on the hazards that horses posed to cavalry soldiers. Fatal accidents for 10 cavalry corps were collected over a period of 20 years (Preussischen Statistik). The number of fatalities due to kicks, x, was recorded for each year and each corps. The table below shows the distribution of x for these 200 "corps-years."

x = number of deaths	Observed number of corps-years in which x fatalities occurred
0	109
1	65
2	22
3	3
4	1
	200

Altogether there were 122 fatalities [109(0) +65(1) + 22(2) +3(3) + 1(4)], meaning that the observed fatality *rate* was 122/200, or 0.61 fatalities per corps-year. Von Bortkiewicz (1898) proposed a Poisson model for x with a a rate λ.

If the prior on λ is gamma $\mathcal{G}a(5, 9)$ find the posterior for λ. What are posterior mean and variance?

Counts of Alpha Particles. Rutherford and Geiger provided counts of α-particles in their experiment. The counts, given in the table below, can be well modeled by Poisson distribution.

X	0	1	2	3	4	5	6	7	8	9	10	11	≥ 12
Frequency	57	203	383	525	532	408	273	139	45	27	10	4	2

(a) Find sample size n and sample mean $\bar{X}$. In calculations for $\bar{X}$ take ≥ 12 as 12.

(b) Elicit a gamma prior for λ with rate parameter $\beta = 10$ and shape parameter α selected in such a way that the prior mean is 7.

(c) Find the Bayes estimator of λ using the prior from (b). Is the problem conjugate? Use the fact that $\sum_{i=1}^{n} X_i \sim \mathcal{P}oi(n\lambda)$.

(d) Write a WinBUGS script that simulates the Bayes estimator for λ and compare its output with the analytic solution from (c).

Estimating Chemotherapy Response Rates. An oncologist believes that 90% of cancer patients will respond to a new chemotherapy treatment and that it is unlikely that this proportion will be below 80%. Elicit a beta prior that models the oncologist's beliefs.

Hint: For elicitation of the prior use $\mu = 0.9$, $\mu - 2\sigma = 0.8$ and expressions for μ and σ for beta.

During the trial, in 30 patients treated, 22 responded.

(a) What are the likelihood and posterior distributions.

(b) Plot the prior, likelihood, and posterior in a single figure.

(c) Using WinBUGS, find the Bayes estimator of the response rate and compare it to the posterior mean.

Jeremy and Variance from Single Observation. Jeremy believes that his IQ test scores have normal distribution with mean 110 and unknown variance σ^2. He takes a test and scores $X = 98$.

(a) Show that inverse gamma prior $\mathcal{IG}(\alpha, \beta)$ is the conjugate for σ^2 if the observation X is normal $\mathcal{N}(\mu, \sigma^2)$ with μ known. What is the posterior?

(b) Find a Bayes estimator of σ^2 and its standard deviation in Jeremy's model if the prior on σ^2 is an inverse gamma $\mathcal{IG}(3, 100)$.

Hint: Random variable Y is said to have an inverse gamma $\mathcal{IG}(\alpha, \beta)$ distribution if its density is given by

$$f(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha) y^{\alpha+1}} \exp\left\{-\frac{\beta}{y}\right\}, \ \alpha, \beta > 0.$$

The mean of Y is $EY = \frac{\beta}{\alpha - 1}$, $\alpha > 1$ and the variance is $Var(Y) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$, $\alpha > 2$.

(c) Use WinBUGS to solve this problem and compare the MCMC approximations with exact values from (b).

Hint: Express the likelihood terms of precision τ with gamma $\mathcal{G}a(\alpha, \beta)$ prior, but then calculate and monitor $\sigma^2 = \frac{1}{\tau}$.

Easy 2D. Joint distribution for X and Θ is

$$f(x, \theta) = 2x, \ 0 \le x \le 1, \ 0 \le \theta \le 1.$$

Find
(a) marginal distributions $m_X(x)$ and $m_\Theta(\theta)$,
(b) conditional distributions $f(x|\theta)$ and $\pi(\theta|x)$.
(c) $P(\Theta^2 \le X \le \Theta)$, and
(d) $P(X^2 \le \Theta \le X)$

Hint. Because of factorization theorem, Θ and X are independent and you should obtain $f(x|\theta) = m_X(x)$ and $\pi(\theta|x) = m_\Theta(\theta)$.

For (c) and (d) conditioning is needed. For example, for random variables X, Y and functions g_1, g_2

$$\begin{aligned}
&P(g_1(Y) \le X \le g_2(Y)) = \\
&E[P(g_1(Y) \le X \le g_2(Y))|Y] = \\
&\int P(g_1(y) \le X \le g_2(y)) f_Y(y) dy =
\end{aligned}$$

$$\int [F_X(g_2(y)) - F_X(g_1(y))] f_Y(y) dy.$$

Bayes and Bats. By careful examination of sound and film records it is possible to measure the distance at which a bat first detects an insect. The measurements are modeled by normal distribution $N(\theta, 10^2)$, where θ is the unknown mean distance (in cm).

Experts believe that the prior suitably expressing uncertainty about θ is $\theta \sim N(50, 10^2)$.

Three measurements are obtained: 62, 52, and 68.

(a) Find the posterior distribution of θ given the observations.

[Hint: Since three measurements are taken and $\bar{X}$ is sufficient for the parameter θ, as a model (likelihood) use $\bar{X}|\theta$.]

(b) Test the hypothesis H_0 that $\theta \geq 50$ in a Bayesian fashion,

(c) What is the 95% credible set for θ.

[For parts (b) and (c) use the posterior obtained in (a).]

Hints/Results/Solutions

E and Var. (a) $\mathbb{E}X = 3/4$, $\mathbb{E}X^k = \frac{3}{3+k}$, $\mathbf{Var}(X) = 3/80$ (b) $\mathbb{E}X = 1$, $\mathbb{E}X^2 = \pi - 2$, $\mathbf{Var}(X) = \pi - 3$.

Uniform on Unit Circle. (a) *Hint.* Show $f_X(x) = \frac{2}{\pi}\sqrt{1-x^2}, \ -1 \le x \le 1$.

Fungi Spores. (a) $f_{(X,Y)}(x,y) = e^{-y}\mathbf{1}(0 \le x \le y)$. If x is integrated out, the marginal for Y is $f_Y(y) = ye^{-y}, \ y \ge 0$, which is gamma $\mathcal{G}a(2,1)$ distribution.

Joint, Marginals and Conditionals #1. It is useful to draw the support (domain) of the joint density. This is unbounded region in the first quadrant of xOy plane above the line $y = x$. There $x \in [0, y]$ for $y \ge 0$, or equivalently, $y \in [x, \infty)$, for $x \ge 0$.

(a) From $f_X(x) = \int_x^\infty f_{(X,Y)}(x,y)\,dy$ we find $X \sim \mathcal{E}xp(\lambda)$.

From $f_Y(x) = \int_0^y f_{(X,Y)}(x,y)\,dx$ we find $Y \sim \mathcal{G}a(2,\lambda)$.

(b) $X|Y = y \sim \mathcal{U}(0,y)$ $\quad$ $Y|X = x \sim \lambda e^{-\lambda(y-x)}, \ y \ge x$.

(c) Hint.

$$F(x,y) = \iint_D f_{(X,Y)}(u,v)dudv,$$

where the domain of integration D depends whether $x \le y$ or not.

If $x \le y$ then the domain of integration of $f(u,v) = \lambda^2 e^{-\lambda v}$ is

$$\{0 \le v \le u; \ 0 \le u \le x\}$$

If $x > y$ then the domain has two parts:

$$\{0 \le v \le u; \ 0 \le u \le y\} \cup \{0 \le v \le y; \ y < u \le x\}$$

After integration,

$$F(x,y) = \begin{cases} \lambda(x-y)(1-e^{-\lambda y}) + \lambda y - 1 + e^{-\lambda y}, & x > y \\ \lambda x - 1 + e^{-\lambda x}, & x \le y \end{cases}$$

Joint, Marginals and Conditionals #2.

Plot the domain of (X,Y). It has the following two equivalent descriptions: $0 \le x \le 1 - y; \ 0 \le y \le 1$ and $0 \le y \le 1 - x; \ 0 \le x \le 1$.

$f_X(x) = 60x \int_0^{1-x} y^2 \, dy = ...$

$f_Y(y) = 30y^2(1-y)^2, \ 0 \le y \le 1$.

For any y from $(0,1)$, $f_{X|Y}(x|y) = \frac{2x}{(1-y)^2}, 0 \le x \le 1-y$.

Joint, Marginals and Conditionals #3. (a) $f_X(x) = 1/2, \; -1 \le x \le 1; \quad f_Y(y) = 1/2, \; -1 \le y \le 1$,

(b) $f_{X|Y}(x|y) = \frac{1+xy}{2}, \; -1 \le x \le 1; \quad f_{Y|X}(y|x) = \frac{1+xy}{2}, \; -1 \le y \le 1$;

(c) Since $f_{X|Y}(x|y) \ne f_X(x)$, the components X and Y are not independent.

Joint, Marginals and Conditionals #4.

(a) Since

$$1 = \int_0^\infty \int_{-x}^{x} C(x^2 - y^2)e^{-x} dx dy = \frac{4C}{3}\int_0^\infty x^3 e^{-x} dx = \frac{4C}{3} \times \Gamma(4) = 8C.$$

Thus, $C = 1/8$. [Recall, $\Gamma(n) = (n-1)!$]

(b) $f_X(x) = \frac{1}{6}x^3 e^{-x} = \frac{x^{4-1}}{\Gamma(4)} e^{-1\cdot x}, \; x \ge 0$. This is Gamma $\mathcal{G}a(4,1)$ distribution.

To find $f_Y(y)$, when integrating out x, for $y \ge 0$, the integration is $\int_y^\infty$, and for $y < 0$ the integration is $\int_{-y}^\infty$. You will need to do integration by part. When the smoke clears,

$$f_Y(y) = \begin{cases} \frac{1}{4}(1+y)e^{-y}, & y \ge 0 \\ \frac{1}{4}(1-y)e^{y}, & y < 0 \end{cases}$$

or simpler, $f_Y(y) = \frac{1}{4}(1+|y|)e^{-|y|}, \; -\infty < y < \infty$.

(c) The conditional $f_{Y|X}(y|x) = \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right), \; -x \le y \le x$.

The conditional $f_{X|Y}(x|y)$ is simply $\frac{f(x,y)}{f_Y(y)}, \; |y| \le x < \infty$.

A 2D PDF.

In a Circle. (a) Use polar coordinates,

$$\int_{x^2+y^2 \le 1} C\sqrt{1-x^2-y^2} dx dy = C\int_0^{2\pi} \int_0^1 r\sqrt{1-r^2} dr d\theta.$$

After some calculation $C = \frac{3}{2\pi}$.

$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2} dy = [\text{after } a = \sqrt{1-x^2} \ge 0] = \frac{3}{2\pi}\int_{-a}^{a}\sqrt{a^2-y^2} dy = \frac{3}{2\pi} \times \frac{\pi a^2}{2} = \frac{3}{4}(1-x^2), -1 \le x \le 1$.

Because of symmetry: $f_Y(y) = \frac{3}{4}(1-y^2), -1 \le y \le 1$.

Gamma – Inverse Gamma.

Hint: The likelihood is proportional to $\frac{x^{n/2-1}}{(2\theta)^{n/2}} e^{-x/(2\theta)}$, and the prior to $\frac{\beta^\alpha}{\theta^{\alpha+1}} e^{-\beta/\theta}$. Find their product and match the distribution for θ. There is no need to find marginal and apply

Bayes theorem since the problem is conjugate.

Solution: The likelihood for $X \sim \mathcal{G}a\left(\frac{n}{2}, \frac{1}{2\theta}\right)$ and the prior $\theta \sim \mathcal{IG}(\alpha, \beta)$ are proportional to

$$\frac{1}{(2\theta)^{n/2}} \exp\left\{-\frac{x}{2\theta}\right\}, \quad \text{and} \quad \frac{1}{\theta^{\alpha+1}} \exp\left\{-\frac{\beta}{\theta}\right\},$$

respectively, if all constant terms are ignored. The product is proportional to

$$\frac{1}{\theta^{n/2+\alpha+1}} \exp\left\{-\frac{x/2+\beta}{\theta}\right\},$$

which can be recognized as un-normalized density of inverse gamma $\mathcal{IG}\left(\frac{n}{2}+\alpha, \frac{x}{2}+\beta\right)$ distribution.

Negative Binomial - Beta.

Negative binomial $\mathcal{NB}(r, p)$ distribution is given by the pmf,

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \ x = 0, 1, 2, \ldots$$

Random variable X with negative binomial distribution represents number of failures before rth success in repeated Bernoulli experiments.

Here the likelihood is proportional to

$$\theta^{mn}(1-\theta)^{\sum_{i=1}^n x_i}.$$

Prior $\pi(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ leads to posterior proportional to

$$\theta^{mn}(1-\theta)^{\sum_{i=1}^n x_i} \times \theta^{\alpha-1}(1-\theta)^{\beta-1} = \theta^{mn+\alpha-1}(1-\theta)^{\sum_{i=1}^n x_i+\beta-1},$$

which is a kernel of beta $\mathcal{B}e(mn+\alpha, \sum_{i=1}^n x_i + \beta)$ distribution,

Horse-Kick Fatalities and Gamma Prior. *Hint.* Show that the likelihood is proportional to

$$\lambda^{122} e^{-200\lambda} \propto \left(\frac{\lambda^0}{0!} e^{-\lambda}\right)^{109} \times \left(\frac{\lambda^1}{1!} e^{-\lambda}\right)^{65} \cdots$$

Use conjugate structure of the likelihood/prior pair.

Jeremy and Variance from Single Observation.

The posterior is proportional to

$$\frac{1}{(\sigma^2)^{1/2}} \exp\left\{-\frac{1/2 \cdot (X-\mu)^2}{\sigma^2}\right\} \times \frac{1}{(\sigma^2)^{\alpha-1}} \exp\left\{-\frac{\beta}{\sigma^2}\right\},$$

so we conclude that the posterior is inverse gamma

$$\mathcal{IG}\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}(X-\mu)^2\right).$$

(b) For Jeremy's data,

$$\sigma^2|X \sim \mathcal{IG}\left(\frac{7}{2}, 172\right),$$

leading to

$$\mathbb{E}\sigma^2 = 172/2.5 = 68.8, \ \mathrm{Var}(\sigma^2) = \frac{172}{(5/2)^2 \cdot (3/2)} = 3155.626666\ldots \quad std(\sigma^2) = 56.175.$$

```
(c)

model{
 X <- 98
 X ~ dnorm(110, tau)
 tau ~ dgamma(3, 100)
sigma2 <- 1/tau
}
mean sd MC_error val2.5pc median val97.5pc start sample
sigma2 68.83 56.14 0.05659 21.47 54.2 204.0 1001 1100000
```