1 Maxwell.

(a) We let $\boldsymbol{y} = (y_1, \ldots, y_n)$ and find the likelihood as

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \theta^{3/2} y_i^2 \exp\left\{-\theta y_i^2/2\right\} \\ &= \left(\sqrt{\frac{2}{\pi}}\right)^n \theta^{\frac{3}{2}n} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\} \\ &\propto \theta^{3n/2} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\}. \end{aligned}$$

Then the posterior distribution is

$$\begin{aligned} \pi(\theta|\boldsymbol{y}) &\propto \theta^{\frac{3}{2}n} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\} \cdot \exp\{-\lambda\theta\} = \theta^{\frac{3}{2}n} \exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda\right)\theta\right\} \\ &\sim \mathcal{G}amma\left(\frac{3}{2}n + 1, \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda\right). \end{aligned}$$

Hence, the posterior is $\mathcal{G}amma(\alpha, \beta)$ where $\alpha = \frac{3}{2}n + 1$ and $\beta = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda$, which means that it depends on data via $\sum_{i=1}^{n} y_i^2$.

(b) Since $\sum_{i=1}^{3} y_i^2 = (1.4)^2 + (3.1)^2 + (2.5)^2 = 17.82$ and the mean of $\mathcal{G}amma(\alpha, \beta)$ is $\frac{\alpha}{\beta}$, we calculate the Bayes estimator as

$$\widehat{\theta}_B = E[\theta|\boldsymbol{y}] = \frac{\frac{3}{2}n + 1}{\frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda} = \frac{\frac{3}{2}(3) + 1}{\frac{1}{2}(17.82) + \frac{1}{2}} = 0.5845.$$

The MLE for θ is $\widehat{\theta}_{MLE} = \frac{3n}{\sum_{i=1}^{n} y_i^2} = \frac{(3)(3)}{17.82} = 0.5051$, and the prior mean is $E[\theta] = 1/\lambda = 2$. Based on our results, the relationship among Bayes estimator, MLE and prior mean is

$$\text{MLE}(0.5051) < \text{Bayes estimator}(0.5845) < \text{prior mean}(2).$$

(c) We use the following Matlab code.

```
gaminv(0.025, 4.5+1, 1/(17.82/2+0.5))  % 0.2027
gaminv(0.975, 4.5+1, 1/(17.82/2+0.5))  % 1.1647
```

The 95% equitailed credible set for θ to be $[0.2027, 1.1647]$.

(d) As $EY = 2\sqrt{\frac{2}{\pi\theta}}$ and $\pi(\theta|x) \sim \mathcal{G}amma(5.5, 9.41)$, we find the predictive value for a single future observation as

$$\begin{aligned}\widehat{y}_{n+1} &= \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta = \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{9.41^{5.5}}{\Gamma(5.5)}\theta^{4.5}e^{-9.41\theta}d\theta \\ &\approx 2.2445.\end{aligned}$$

2 Jeremy Mixture.

(a) The derivation is shown as follows.

$$\begin{aligned}\pi(\theta|x) &= \frac{f(x|\theta)\pi(\theta)}{m(x)} = \frac{f(x|\theta)(\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta))}{\int_\Theta f(x|\theta)(\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta))d\theta} \\ &= \frac{\epsilon f(x|\theta)\pi_1(\theta) + (1-\epsilon)f(x|\theta)\pi_2(\theta)}{\epsilon\int_\Theta f(x|\theta)\pi_1(\theta)d\theta + (1-\epsilon)\int_\Theta f(x|\theta)\pi_2(\theta)d\theta} \\ &= \frac{\epsilon m_1(x)\pi_1(\theta|x) + (1-\epsilon)m_2(x)\pi_2(\theta|x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} \\ &= \frac{\epsilon m_1(x)\pi_1(\theta|x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} + \frac{(1-\epsilon)m_2(x)\pi_2(\theta|x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} \\ &= \epsilon'\pi_1(\theta|x) + (1-\epsilon')\pi_2(\theta|x),\end{aligned}$$

where $\epsilon' = \dfrac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}$.

(b) We first find the posterior distribution given the likelihood $X|\theta \sim \mathcal{N}(\theta, \sigma^2)$ when σ^2 is fixed, and the prior $\theta \sim \mathcal{N}(\theta_0, \sigma_0^2)$. We have

$$\begin{aligned}\pi(\theta|x) \propto f(x|\theta)\pi(\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}} \\ &= \frac{1}{2\pi\sigma\sigma_0}\exp\left\{-\frac{(x-\theta)^2\sigma_0^2 + (\theta-\theta_0)^2\sigma^2}{2\sigma^2\sigma_0^2}\right\} \\ &= \frac{1}{2\pi\sigma\sigma_0}\exp\left\{-\frac{\theta^2 - \frac{2(\sigma_0^2 x + \sigma^2\theta_0)}{\sigma_0^2+\sigma^2}\theta + \frac{\sigma_0^2x^2+\sigma^2\theta_0^2}{\sigma_0^2+\sigma^2}}{\frac{2\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}}\right\} \\ &\propto \mathcal{N}\left(\frac{\sigma_0^2 x + \sigma^2\theta_0}{\sigma_0^2+\sigma^2}, \frac{\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}\right).\end{aligned}$$

Based on the above result and $\pi_1(\theta) \sim \mathcal{N}(110, 60), \pi_2(\theta) \sim \mathcal{N}(100, 200)$ and $\sigma_0^2 = 80$, we have

$$\pi_1(\theta|x) \sim \mathcal{N}\left(\frac{(60)(98)+(80)(110)}{60+80}, \frac{(80)(60)}{60+80}\right) = \mathcal{N}\left(\frac{734}{7}, \frac{240}{7}\right),$$
$$\pi_2(\theta|x) \sim \mathcal{N}\left(\frac{(200)(98)+(80)(100)}{200+80}, \frac{(80)(200)}{200+80}\right) = \mathcal{N}\left(\frac{690}{7}, \frac{400}{7}\right).$$

We then calculate the marginal distribution of X as following.

$$\begin{aligned}
m(x) &= \int_{-\infty}^{\infty} f(x|\theta)\pi(\theta)d\theta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}\right\} d\theta \\
&= \frac{1}{2\pi\sigma\sigma_0} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\sigma_0^2+\sigma^2)\theta^2 - 2(\sigma_0^2 x + \sigma^2\theta_0)\theta + \sigma_0^2 x^2 + \sigma^2\theta_0^2}{2\sigma^2\sigma_0^2}\right\} d\theta \\
&= \frac{1}{2\pi\sigma\sigma_0} \int_{-\infty}^{\infty} \exp\left\{-\frac{\left(\theta - \frac{\sigma_0^2 x+\sigma^2\theta_0}{\sigma^2+\sigma_0^2}\right)^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\} \exp\left\{\frac{\left(\frac{\sigma_0^2 x+\sigma^2\theta_0}{\sigma^2+\sigma_0^2}\right)^2 - \frac{\sigma_0^2 x^2+\sigma^2\theta_0^2}{\sigma_0^2+\sigma^2}}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\} d\theta \\
&= \frac{1}{2\pi\sigma\sigma_0} \exp\left\{\frac{(\frac{\sigma_0^2 x+\sigma^2\theta_0}{\sigma^2+\sigma_0^2})^2 - \frac{\sigma_0^2 x^2+\sigma^2\theta_0^2}{\sigma_0^2+\sigma^2}}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\} \\
&\quad \cdot \left(\sqrt{2\pi\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}} \exp\left\{-\frac{(\theta - \frac{\sigma_0^2 x+\sigma^2\theta_0}{\sigma^2+\sigma_0^2})^2}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\} d\theta \\
&= \frac{\sqrt{2\pi\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}}{2\pi\sigma\sigma_0} \exp\left\{\frac{(\frac{\sigma_0^2 x+\sigma^2\theta_0}{\sigma^2+\sigma_0^2})^2 - \frac{\sigma_0^2 x^2+\sigma^2\theta_0^2}{\sigma_0^2+\sigma^2}}{2\frac{\sigma_0^2\sigma^2}{\sigma_0^2+\sigma^2}}\right\}.
\end{aligned}$$

Thus for $\pi_1(\theta) \sim \mathcal{N}(110, 60)$, we have $\sigma^2 = 80, \theta_0 = 110$ and $\sigma_0^2 = 60$. Given that $X = 98$, we have

$$m_1(98) = 0.03372\exp\{-0.5143\} = 0.0202.$$

Similarly, for $\pi_2(\theta) \sim \mathcal{N}(100, 200)$, we have $\sigma^2 = 80, \theta_0 = 100$, and $\sigma_0^2 = 200$. Given that $X = 98$, we have

$$m_2(98) = 0.02384\exp\{-0.0071\} = 0.0237.$$

Thus, we compute the ϵ' as

$$\epsilon' = \frac{\frac{2}{3}m_1(x)}{\frac{2}{3}m_1(x) + \frac{1}{3}m_2(x)} = \frac{\frac{2}{3}(0.0202)}{\frac{2}{3}(0.0202) + \frac{1}{3}(0.0237)} = 0.6303.$$

Then the posterior is given as

$$\pi(\theta|x) = 0.6303 \cdot \mathcal{N}\left(\frac{734}{7}, \frac{240}{7}\right) + 0.3697 \cdot \mathcal{N}\left(\frac{690}{7}, \frac{400}{7}\right),$$

and the Bayes estimator for θ is

$$\delta_B(98) = E[\theta|x] = (0.6303)\left(\frac{734}{7}\right) + (0.3697)\left(\frac{690}{7}\right) = 102.5333.$$

3 Mendel's Experiment with Peas.

(a) To calculate the prior mean, use the mean formula for $\mathcal{B}$eta distribution. This is just $\frac{\alpha}{\alpha+\beta}$. The posterior distribution of p is $\mathcal{B}eta(\alpha + 787, \beta + 1064 - 787) = \mathcal{B}eta(799, 281)$. Using the same formula to calculate the posterior mean yields $\frac{799}{799+281} \approx 0.7398$

(b) Taking the CDF of the above posterior $\mathcal{B}$eta distribution, we get: $P(p \leq 0.75) = 0.7759$. (Done in R)

```
> pbeta(0.75,799,281)
```

(c) Using the "bayestestR" library in R:

```
> dist = rbeta(10000, 799, 281)
> ci(dist, method = "ETI", 0.95)
```

Result: [0.71, 0.77]