1 2-D Density Tasks.

(a) For $x > 0$, we have

$$f_X(x) = \int_x^\infty f(x,y)dy = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \left[\lambda^2\Big(-\frac{1}{\lambda}\Big)e^{-\lambda y}\right]\Bigg|_x^\infty = \lambda e^{-\lambda x},$$

which matches with the pdf of exponential distribution with rate parameter λ. Thus, we know that marginal distribution $f_X(x)$ is exponential $\mathcal{E}(\lambda)$.

(b) For $y > 0$, we have

$$f_Y(y) = \int_0^y f(x,y)dx = \int_0^y \lambda^2 e^{-\lambda y} dy = \left[\lambda^2 e^{-\lambda y} x\right]\Bigg|_0^y = \lambda^2 y e^{-\lambda y},$$

which matches with the pdf of gamma distribution with shape parameter 2 and rate parameter λ. Thus, we know that marginal distribution $f_Y(y)$ is Gamma $\mathcal{G}a(2,\lambda)$.

(c) For $y \geq x$, we have

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, y \geq x,$$

which shows that conditional distribution $f_{Y|X}(y|x)$ is shifted exponential.

(d) For $x \leq y$, we have

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y}, y \geq x,$$

which shows that conditional distribution $f_{X|Y}(x|y)$ is uniform $\mathcal{U}(0,y)$.

2 Weibull Lifetimes.

(i) For a given shape parameter ν, the probability density function of X is

$$f(x|\theta) = \nu\theta x^{\nu-1}\exp\{-\theta x^\nu\}, x \geq 0,$$

and thus the likelihood is proportional to $\theta^n \exp\{-\theta \sum_{i=1}^n x_i^\nu\}$. Note that we see this because ν and x are given, and thus can be treated as constants.

As $\theta \sim \exp(\lambda)$, where λ is known as 2, it is proportional to $\exp\{-\lambda\theta\}$. Thus, the posterior is proportional to

$$\theta^n \exp\{-\theta \sum_{i=1}^n x_i^\nu\}\exp\{-\lambda\theta\} = \theta^n \exp\{-\theta(\lambda + \sum_{i=1}^n x_i^\nu)\},$$

which is a kernel of Gamma distribution $\mathcal{G}amma(n+1, \lambda + \sum_{i=1}^n x_i^\nu)$.

By plugging in the value of $\nu = 3, \lambda = 2$ and $\sum_{i=1}^3 X_i^3 = 43$, we obtain the posterior as $\mathcal{G}amma(4, 45)$.

(ii) As the posterior $\theta|X \sim \mathcal{G}amma(n+1, 2 + \sum_{i=1}^n x_i^3)$, we have

$$E[\theta|X] = \frac{n+1}{2 + \sum_{i=1}^n x_i^3},$$
$$Var(\theta|X) = \frac{n+1}{(2 + \sum_{i=1}^n x_i^3)^2}.$$

As $n = 3$ and $\sum_{i=1}^3 X_i^3 = 43$, we know that

$$E[\theta|X] = \frac{n+1}{2 + \sum_{i=1}^n x_i^3} = \frac{4}{45},$$
$$Var(\theta|X) = \frac{n+1}{(2 + \sum_{i=1}^n x_i^3)^2} = \frac{4}{2025}.$$

3 Silver-Coated Nylon Fiber.

(a) (i) $P(X > 5) = 1 - p(X < 5)$. Since the cumulative distribution function (CDF) of the Exponential Distribution is

$$1 - e^{(-\lambda x)}$$

This gives $P(X > 5) = e^{-1}$.

(ii) Using the CDF again, we get $P(X < 10) = 1 - e^{-2}$

(iii) Using Bayes Rule:

$$P(X > 10|X > 5) = \frac{P(X > 5|X > 10)P(X > 10)}{/P(X > 5)} = \frac{e^{-2}}{e^{-1}} = e^{-1}$$

This demonstrates the memoryless property of the Exponential Distribution.

(b) (i) Note that the mean of the $\mathcal{E}(\lambda)$ is $1/\lambda$. Therefore, the classical estimate of λ is

$$\hat{\lambda} = \frac{1}{\bar{T}} = \frac{3}{2+4+8} = \frac{3}{14}.$$

(ii) Denote $\boldsymbol{T} = (T_1, T_2, T_3)$. The likelihood can be found as

$$p(\boldsymbol{T}|\lambda) = \prod_{i=1}^{3} \lambda \exp(-\lambda T_i) = \lambda^3 \exp(-\lambda(T_1 + T_2 + T_3)).$$

We first compute the posterior distribution of λ as

$$\begin{aligned}\pi(\lambda \big| \boldsymbol{T}) \propto p(\boldsymbol{T}|\lambda)\pi(\lambda) &= \lambda^3 \exp(-\lambda(T_1+T_2+T_3)) \cdot \frac{1}{\sqrt{\lambda}} \\ &= \lambda^{5/2} \exp(-\lambda(T_1+T_2+T_3)).\end{aligned}$$

Therefore, $[\lambda \big| \boldsymbol{T}] \sim \mathcal{G}a(7/2, T_1+T_2+T_3)$. Under the minimum mean square error, the Bayes estimator is

$$\mathbb{E}_{\lambda|\boldsymbol{T}}\lambda = \frac{7/2}{T_1+T_2+T_3} = \frac{7}{28}.$$