IYSE 6420 Fall 2020 Homework2

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1. 2-D Density Tasks

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y, \lambda > 0 \\ 0, & else \end{cases}$$

(a) marginal distribution $f_X(x)$ is exponential E(λ).

Exponential: $f(x) = \lambda e^{-\lambda x}, \lambda > 0, x \ge 0$

When $\lambda > 0, x \ge 0$

$$\begin{aligned} f_X(x) &= \int_x^\infty f(x,y)dy \\ &= \int_x^\infty \lambda^2 e^{-\lambda y} dy \\ &= \left[-\lambda e^{-\lambda y}\right]_x^\infty \\ &= 0 - (-\lambda e^{-\lambda x}) \\ &= \lambda e^{-\lambda x} \end{aligned}$$

So $f_X(x)$ is exponential $\mathcal{E}(\lambda)$

(b) marginal distribution $f_Y(y)$ is Gamma Ga(2, λ).

Gamma: $f(y,\alpha,\beta) = \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}, y > 0, \alpha > 0, \beta > 0$

When $y > 0, , \lambda > 0$

$$\begin{aligned} f_Y(y) &= \int_0^y f(x,y)dx \\ &= \int_0^y \lambda^2 e^{-\lambda y}\, dx \\ &= \lambda^2 e^{-\lambda y}(y-0) \\ &= \lambda^2 e^{-\lambda y} y^{2-1} \end{aligned}$$

Let $\beta = \lambda, \alpha = 2$

$$f_Y(y) = \beta^\alpha e^{-\beta y} y^{\alpha-1}$$

So $f_Y(y)$ is Gamma $\mathcal{G}a(2,\lambda)$

(c) conditional distribution $f(y|x)$ is shifted exponential, $f(y|x) = \lambda e^{-\lambda(y-x)}, y \ge x$

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} \\ &= \lambda e^{-\lambda(y-x)} \end{aligned}$$

(d) conditional distribution $f(x|y)$ is uniform $\mathcal{U}(0, y)$

$$f(x|y) = \frac{f(x,y)}{f(y)}$$
$$= \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 e^{-\lambda y} y}$$
$$= \frac{1}{y}$$
$$= \begin{cases} \frac{1}{y} & 0 \le x \le y \\ 0 & else \end{cases}$$
$$= \mathcal{U}(0, y)$$

2. Weibull Lifetimes

$$f(x \mid \nu, \theta) = \nu\theta x^{\nu-1} e^{-\theta x^{\nu}}, \qquad x \ge 0$$
$$f(x|\theta) = 3\theta x^2 e^{-\theta x^3}, \qquad \nu = 3$$

Gamma:

$$f(x, \alpha, \beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, x > 0, \alpha > 0, \beta > 0$$

(a) For the prior suggested by the expert, find the posterior distribution of θ.

Prior

$$\pi(\theta) = 2e^{-2\theta}, \qquad \theta > 0$$

Likelihood

$$\pi(X|\theta) = \prod_{i=1}^{3} 3\theta x_i^2 e^{-\theta x_i^3}$$
$$= 3^3 2^2 3^2 2^2 \theta^3 e^{-43\theta}$$
$$= 3^5 2^4 \theta^3 e^{-43\theta}$$

Posterior

$$\pi(\theta|X) \propto \pi(X|\theta)\pi(\theta)$$
$$\propto 3^5 2^4 \theta^3 e^{-43\theta} \times 2e^{-2\theta}$$
$$\propto 6^5 \theta^3 e^{-45\theta}$$

Let $\alpha = 4, \beta = 45$

$$\pi(\theta|X) = \frac{\beta^{\alpha} \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}$$
$$= \frac{45^4 \theta^3 e^{-45\theta}}{\sqrt{4}}$$

(b) What are the posterior mean and variance? No need to integrate if you recognize to which family of distributions the posterior belongs.

The posterior is Gamma distribution, $\pi(\theta|X) \sim \mathcal{G}a(4, 45)$

Mean

$$E(X) = \frac{\alpha}{\beta} = \frac{4}{45}$$

Variance

$$Var(X) = \frac{\alpha}{\beta^2} = \frac{4}{45^2} = \frac{4}{2025}$$

3. Silver-Coated Nylon Fiber

(a) Suppose λ = 1/5, find the probabilities that

Probability density function

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Cumulative distribution function

$$F(x, \lambda) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(i) a run continues for at least 5 hours.

$$\begin{aligned} P(T \geq 5) &= 1 - P(T < 5) \\ &= 1 - \left(1 - e^{-\lambda x}\right) \\ &= e^{-\frac{5}{5}} \\ &= 0.368 \end{aligned}$$

(ii) a run lasts less than 10 hours.

$$\begin{aligned} P(T < 10) &= 1 - e^{-\lambda x} \\ &= 1 - e^{-\frac{10}{5}} \\ &= 0.865 \end{aligned}$$

(iii) a run continues for at least 10 hours, given that it has lasted 5 hours.

Since $P(T \geq 10, T \geq 5) = P(T \geq 10)$

$$\begin{aligned} P(T \geq 10 \mid T \geq 5) &= \frac{P(T \geq 10, T \geq 5)}{P(T \geq 5)} \\ &= \frac{P(T \geq 10)}{P(T \geq 5)} \\ &= \frac{1 - P(T < 10)}{1 - P(T < 5)} \\ &= \frac{e^{-2}}{e^{-1}} \\ &= e^{-1} \\ &= 0.368 \end{aligned}$$

(b) Now suppose that the rate parameter λ is unknown, but there are three measurements of interblockage times, T_1 = 2, T_2 = 4, and T_3 = 8.

(i) How would classical statistician estimate λ ?

Classical statistician takes no prior knowledge into consideration

$$\bar{T} = \frac{2 + 4 + 8}{3} = \frac{14}{3}$$

$$\lambda = \frac{1}{\bar{T}} = \frac{3}{14}$$

(ii)What is the Bayes estimator of λ if the prior is $\pi(\lambda) = \frac{1}{\sqrt{\lambda}},\ \lambda > 0$

Prior

$$\pi(\lambda) = \frac{1}{\sqrt{\lambda}}$$

Likelihood

$$\begin{aligned}\pi(T|\lambda) &= \lambda e^{-2\lambda}\lambda e^{-4\lambda}\lambda e^{-8\lambda} \\ &= \lambda^3 e^{-14\lambda}\end{aligned}$$

Posterior

$$\begin{aligned}\pi(\lambda|T) &\propto \pi(T|\lambda)\,\pi(\lambda) \\ &\propto \lambda^{\frac{5}{2}} e^{-14\lambda} \\ &= Constant \times (\lambda^{1-\frac{7}{2}} e^{-14\lambda})\end{aligned}$$

Gamma: $f(x,\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, x > 0, \alpha > 0, \beta > 0$

Let $\beta = 14, \alpha = \frac{7}{2}, x = \lambda$, we can see the posterior distribution is Gamma.

Bayes estimator

$$\hat{\lambda} = \frac{\alpha}{\beta} = \frac{7}{28}$$