ISYE 6420: Bayesian Statistics **Spring 2020**

Homework #3

Instructor: Roshan Vengazhiyil, Brani Vidakovic *Name:* Nick Korbit, *gtID:* 903263968

Problem 1

a). We first find the probability of an individual component to stay alive at time t:

$$p = e^{-0.1 \times t^{3/2}}$$

$$p = e^{-0.1 \times 3^{3/2}} \approx 0.594749$$

Then we have two options to find the probability that a k-out-of-n system is still operational when checked at time $t = 3$ (P_3^s). First, we can leverage the probability mass function and sum up probabilities that exactly 4, 5, 6, 7 and 8 components are alive at $t = 3$:

$$\Pr(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

For $n = 8$, $k = 4$, $p = 0.594749$ we have:

$$P_3^s = Pr(X = 4) + Pr(X = 5) + Pr(X = 6) + Pr(X = 7) + Pr(X = 8)$$

$$P_3^s \approx 0.818094$$

We can also utilize the cumulative distribution function. Cdf for binomial distribution is

$$F(k; n, p) = \Pr(X \leq k) = \sum_{i=0}^{\lfloor k \rfloor} \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i}$$

In order to evaluate the system staying alive at $t = 3$ we need to compute

$$P_3^s = 1 - F(3; 8, p)$$

So that

$$P_3^s \approx 0.818094$$

Confirming our previous result.

b). Let's apply the Bayes formula to find the probability that at time $t = 3$ exactly 5 components were operational (event X) given the system was found operational (event H):

$$P(X|H) = \frac{P(H|X)P(X)}{P(H)}$$

If 5 components are operational, then the system is 100% alive, so that $P(H|X)$ is a sure event, $P(H|X) = 1$. We have already calculated $P(H)$, it's basically P_3^s. And $P(X)$ can be calculated as $\Pr(X = 5)$:

$$P(X) = pmf(5, 8, 0.594749) \approx 0.277351$$

Then $P(X|H)$ is

$$P(X|H) = 0.277351/0.818094 \approx 0.339021$$

Note. The script for solving Q1 is implemented in *hw2q2.py*, function *solve_q1()* (included in the zip archive). To run the code just run *'python hw2q2.py'*.

Problem 2

a). In order to find the probability that a randomly chosen measurement can be classified as accurate we need to integrate the pdf on the selected interval $(-0.5, 0.5)$. We utilize the fact that the pdf is symmetrical wrt y-axis:

$$p = \int_{-0.5}^{0.5} \frac{3x^2}{16} dx = 2\int_{0}^{0.5} \frac{3x^2}{16} dx = 2\frac{0.5^3}{16} = 0.015625$$

So that the desired probability is about 1.6%.

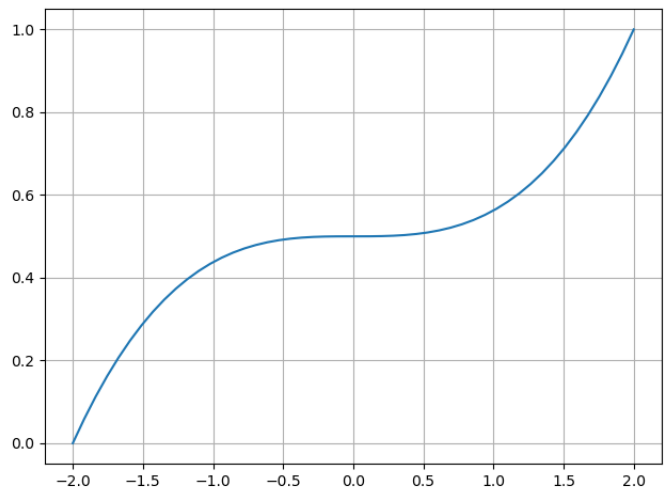
b). We now find and plot a cdf. Remember that the cdf of a continuous random variable is

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

For our function:

$$F(x) = \int_{-2}^{x} \frac{3t^2}{16} dt = \frac{x^3}{16} - \frac{(-2)^3}{16} = \frac{x^3}{16} + \frac{1}{2}$$

Let's plot the cdf on the $(-2, 2)$ interval:



Combining all intervals the cdf is:

$$F(x) = \begin{cases} 0, & x \leq -2 \\ \frac{x^3}{16} + \frac{1}{2}, & -2 < x < 2 \\ 1, & x \geq 2 \end{cases}$$

c). Finding $\mathbb{E}[Y]$ in our case is equivalent to finding a second moment:

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f(x)dx$$

For our function:

$$\mathrm{E}\left[X^2\right] = \int_{-2}^{2} x^2 \frac{3x^2}{16} dx = 2\int_{0}^{2} \frac{3x^4}{16} dx = 2(\frac{3 \cdot 2^5}{16 \cdot 5} - 0) = \frac{3 \cdot 2^2}{5} = 2.4$$

So that the expected loss is 2.4 thousands of dollars.

d). To compute the probability that the loss is less than y we utilize the cdf of X:

$$P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X < \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx$$

$$P(X^2 < 3) = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{3x^2}{16}dx = 2\int_0^{\sqrt{3}} \frac{3x^2}{16}dx = 2\frac{x^3}{16}\Big|_0^{\sqrt{3}} = \frac{2(\sqrt{3}^3)}{16} \approx 0.649519$$

So that the probability that the loss is less than 3 thousand is approximately 65%.

Note. The script for plotting Q2.B chart is implemented in *hw2q2.py*, function *solve_q2()* (included in the zip archive). To run the code just run *'python hw2q2.py'*.

Problem 3

a). We find the marginal distribution $f_X(x)$ by integrating the joint pdf wrt y:

$$f_X(x) = \int_0^1 (x+y)dy = x + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$$

Or, more formally,

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le 1 \\ 0, & else \end{cases}$$

b). Let's then find the conditional distribution $f(y|x)$:

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}}$$

Or, more formally,

$$f(y|x) = \begin{cases} \frac{x+y}{x+\frac{1}{2}}, & 0 \le x \le 1; 0 \le y \le 1 \\ 0, & else \end{cases}$$

Problem 4

a). Let's apply the Bayes theorem:

$$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{m(x)}$$

The posterior $\pi(\theta|x)$ belongs to the Pareto family:

$$\pi(\theta|x) = \frac{\alpha c^\alpha}{\theta^{\alpha+1}}\mathbf{1}(\theta > c)$$

The prior $\pi(\theta)$ is:

$$\pi(\theta) = \frac{1}{\theta}\mathbf{1}(\theta > 0)$$

The conditional $f(x|\theta)$ is:

$$f(x|\theta) = \theta^{-34}\mathbf{1}(\theta > M)$$

We now calculate the joint probability:

$$f(x,\theta) = f(x|\theta)\pi(\theta) = \frac{1}{\theta}\mathbf{1}(\theta > 0)\theta^{-34}\mathbf{1}(\theta > M) = \theta^{-35}\mathbf{1}(\theta > 0)\mathbf{1}(\theta > M) = \theta^{-35}\mathbf{1}(\theta > M)$$

The marginal $m(x)$ is:

$$m(x) = \int_\Theta f(x,\theta)d\theta = \int_M^\infty \theta^{-35}\mathbf{1}(\theta > M)d\theta = \int_M^\infty \theta^{-35}d\theta = -\frac{1}{34x^{34}}\Big|_M^\infty = 0 - \left(-\frac{1}{34M^{34}}\right) = \frac{1}{34M^{34}}$$

Getting back to the Bayes theorem:

$$\frac{\alpha c^{\alpha}}{\theta^{\alpha+1}}\mathbf{1}(\theta > c) = \frac{\theta^{-35}\mathbf{1}(\theta > M)}{\frac{1}{34M^{34}}} = \frac{34M^{34}}{\theta^{35}}\mathbf{1}(\theta > M)$$

So that we can derive that $\alpha = 34$ and $c = M = 0.54876$.

b). As we have a Pareto distribution, we can calculate $E[\theta]$:

$$E[\theta] = \frac{34 \cdot 0.54876}{34-1} \approx 0.565389$$

We now calculate lower and upper bounds, L and U respectively:

$$\int_{-\infty}^{L} \pi(\theta|x)d\theta = \alpha/2$$

We can substitute the integral with the cdf:

$$cdf(L) = \alpha/2$$

$$\left[1 - (M/L)^{34}\right]\mathbf{1}(L > M) = \frac{\alpha}{2}$$

$$L = \frac{M}{\sqrt[34]{1-\frac{\alpha}{2}}}$$

$$L = \frac{0.54876}{\sqrt[34]{1-\frac{0.05}{2}}} \approx 0.549169$$

The upper bound U is:

$$\int_{-\infty}^{U} \pi(\theta|x)d\theta = 1 - \alpha/2$$

$$\left[1 - (M/U)^{34}\right]\mathbf{1}(U > M) = 1 - \frac{\alpha}{2}$$

$$U = \frac{M}{\sqrt[34]{\frac{\alpha}{2}}}$$

$$U = \frac{0.54876}{\sqrt[34]{\frac{0.05}{2}}} \approx 0.611648$$

So that the $[U; L]$ bounds are $[0.549169; 0.611648]$ and $\theta = 0.6$ is inside the interval.

References

[1] Engineering Biostatistics: An Introduction using MATLAB and WinBUGS. Brani Vidakovic - Wiley Series in Probability and Statistics.