Bayesian Prediction

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Recall

$$m(x) = \int f(x|\theta)\,\pi(\theta)\,d\theta \quad \text{marginal}$$

Sometimes called prior predictive distribution.

- $$f(x_{n+1}|x_1,\dots,x_n) = \int f(x_{n+1}|\theta)\,\pi(\theta|x_1,\dots,x_n)\,d\theta$$

  posterior predictive distribution

- $$\hat{X}_{n+1} = \int x_{n+1}\cdot f(x_{n+1}|x_1,\dots,x_n)\,dx_{n+1} = E^f X_{n+1}$$

  predictive mean (prediction for X_{n+1})

- $$\int (x_{n+1}-\hat{X}_{n+1})^2 f(x_{n+1}|x_1,\dots,x_n)\,dx_{n+1}$$

  predictive variance

__Example__ $X_1, \ldots, X_n \sim \mathcal{E}xp(\lambda)$, $\pi(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda}$, $\lambda \geq 0$

Posterior $\pi(\lambda | x_1, \ldots, x_n) \propto \lambda^{n+\alpha-1} \exp\{-(\sum_{i=1}^n x_i + \beta) \cdot \lambda\}$

$$\sim \mathcal{G}a(\alpha + n, \beta + \sum x_i)$$

Posterior predictive distribution is

$$f(x_{n+1} | x_1, \ldots, x_n) = \int \lambda e^{-\lambda x_{n+1}} \cdot \pi(\lambda | x_1, \ldots, x_n)\, d\lambda$$

$$= \frac{(n+\alpha)(\sum_{i=1}^n x_i + \beta)^{n+\alpha}}{(\sum_{i=1}^n x_i + \beta + x_{n+1})^{n+\alpha+1}}, \quad x_{n+1} > 0$$

$$x_{n+1} + \sum x_i + \beta \sim Pa\left(\sum_{i=1}^n x_i + \beta,\ n+\alpha\right) \quad [\text{Pareto}]$$

$X \sim Pa(c, \alpha)$

$$f(x) = \frac{\alpha}{c}\left(\frac{c}{x}\right)^{\alpha+1}, \quad x \geq c$$

$$EX = \frac{\alpha c}{\alpha - 1}, \alpha > 1; \quad \mathrm{Var}X = \frac{\alpha c^2}{(\alpha-1)^2(\alpha-2)}, \alpha > 2$$

$$X_{n+1} + \sum X_i + \beta \sim Pa\left(\sum X_i + \beta,\ n+\alpha\right)$$

$$\hat{X}_{n+1} = EX_{n+1} = \frac{\left(\sum_{i=1}^{n} X_i + \beta\right)(n+\alpha)}{n+\alpha-1} - \sum X_i - \beta = \boxed{\frac{\sum_{i=1}^{n} X_i + \beta}{n+\alpha-1}}$$

Show, $\hat{\sigma}^2_{\hat{X}_{n+1}} = \dfrac{\left(\sum_{i=1}^{n} X_i + \beta\right)^2 (n+\alpha)}{(n+\alpha-1)^2 (n+\alpha-2)}$.

For example, if $X_1 = 2.1$, $X_2 = 5.5$, $X_3 = 6.4$, $X_4 = 8.7$, $X_5 = 4.9$, $X_6 = 5.1$, and $X_7 = 2.3$ and $\lambda \sim \mathcal{G}a(2, 1)$, then $\hat{X}_8 = \frac{9}{2}$, $\hat{\sigma}^2_{\hat{X}_8} = 26.0357$

Easier if only $\hat{X}_{n+1}$ is wanted

$$\hat{X}_{n+1} = \int_{\Theta} \mu(\theta)\, \pi(\theta \mid x_1, \dots, x_n)\, d\theta$$

where $\mu(\theta) = \int x \cdot f(x \mid \theta)\, dx$ is the mean of X.