ISYE 6420: HW2

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September 14, 2019

Question 1

We are told that $P(c = k) \sim Poisson(5)$ Where c is the event the the number of cell groups in a 3d petri dish is equal to k. Therefore $P(c = k) = e^{-5}\frac{5^k}{k!}$

Part A

$P(c = 0) = e^{-5}\frac{5^0}{0!}$ which equals 0.67%

Part B

$P(c \geq 1) = 1 - P(c = 0)$ which equals 99.32%

Part C

$P(c > 8) = 1 - P(c \leq 8) = 1 - \sum_{k=0}^{8} e^{-5}\frac{5^k}{k!}$ which equals 6.81%

Part D

$P(4 \leq c \leq 6) = \sum_{k=4}^{6} e^{-5}\frac{5^k}{k!}$ which equals 49.7%

The code used for calculation is below

```r
# Define a general function for the pmf
poisson<-function(lambda,k){
  return(exp(-lambda)*(lambda^k)/factorial(k))
}

# Define a general function a range of values
poisson.cuml<-function(lambda,lb,ub){
  p=0
  for(i in lb:ub){
    p=p+poisson(lambda,i)
  }
  return(p)
}

# Part A
poisson(5,0)
```

```
## [1] 0.006737947
```

```r
#Part B
1-poisson(5,0)
```

```
## [1] 0.9932621
```

```r
# Part C
q=poisson.cuml(5,0,8)
1-q
```

```
## [1] 0.06809363
```

```r
# Part D
poisson.cuml(5,4,6)
```

```
## [1] 0.4971575
```

Question 2

We are told that $t \sim Exp(\frac{1}{10})$ where t is the time between blockages. The cumulative distribution function will be useful for the following questions and is given below:

$F(t \leq x) = 1 - e^{\frac{-x}{10}}$

Part A

$P(t \geq 10) = 1 - P(t < 10) = 36.79\%$

Part B

$P(t < 15) = 77.69\%$

Part C

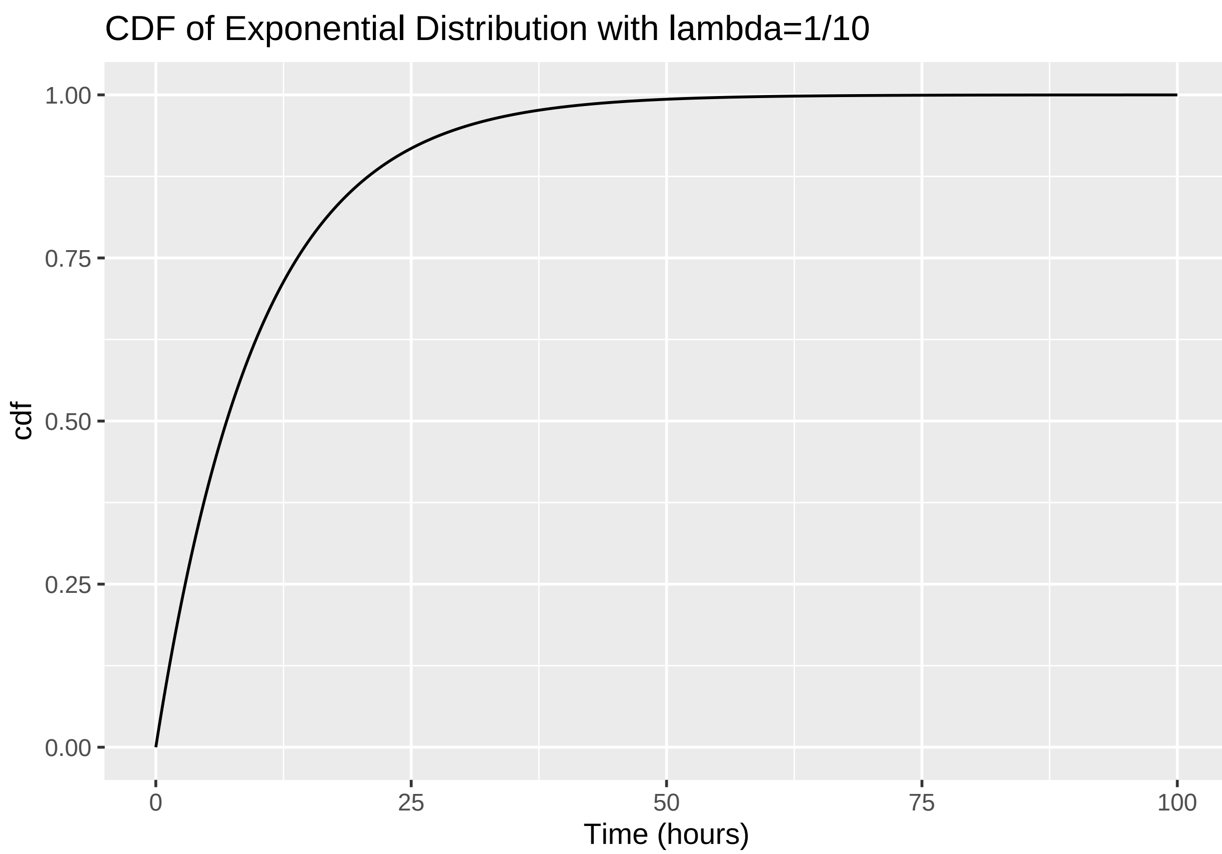
Using the memoryless property of the exponential distribution, we know that this probability is the same as Part A of this question $P(t \geq 20 \mid t \geq 10) = P(t \geq 10) = 36.79\%$

The code used for calculation is below

```r
# Define a general function for evaluting the cdf of an exponential random variable
exponential.cuml<-function(lambda,x){
  return(1-exp(-1*lambda*x))
}

# Here's a quick look at the CDF
data.frame(time=seq(0,100,0.01))%>%
  mutate(cdf=exponential.cuml(1/10,time))%>%
  ggplot(aes(x=time,y=cdf))+geom_line()+labs(title='CDF of Exponential Distribution with lambda=1/10',x=
```

```
## Warning: package 'bindrcpp' was built under R version 3.5.1
```




```
lambda<-1/10

# Part A
1-exponential.cuml(lambda,10)
```

```
## [1] 0.3678794
```

```
# Part B
exponential.cuml(lambda,15)
```

```
## [1] 0.7768698
```

```
# Part C
1-exponential.cuml(lambda,10)
```

```
## [1] 0.3678794
```

Question 3

Part A

See bottom of the document for additional integration details using software

$f_x(x) = \int_x^\infty f(x,y)dy = \int_x^\infty \lambda^2 e^{-\lambda y} dy$

$\int_x^\infty \lambda^2 e^{-\lambda y} dy = \lambda(-e^{-\lambda y})|_x^\infty$

$\lambda(-e^{-\lambda y})|_x^\infty = \lambda(e^{-\lambda y})$

Which upon inspection is the pdf of an exponential distribution

Part B

See bottom of the document for additional integration details using software

$f_y(y) = \int_0^y f(x,y)dx = \int_0^y \lambda^2 e^{-\lambda y} dy$

$\int_0^y \lambda^2 e^{-\lambda y} dy = \lambda^2 y e^{-\lambda y}$

$f_y(y) = \lambda^2 y e^{-\lambda y}$

For a $\mathsf{gamma}(2, \lambda)$ distribution, the pdf is provided below

$f(y) = \frac{\lambda^2 y^{2-1} e^{-\lambda}}{\Gamma(2)} = \lambda^2 y e^{-\lambda}$

Which is equivalent to the result for $f_y(y)$

Part C

Additional algebra details provided at the bottom of the document

$f(y \mid x) = \frac{f(x,y)}{f(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda y}}$

$f(y \mid x) = \lambda e^{-\lambda(y-x)}$

Which is an exponential distribution with $\lambda = y - x$

Part D

$f(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}}$

$f(x \mid y) = \frac{1}{y-0}$

which is equivalent to a uniform distribution of the form $\mathsf{U}(0, \mathsf{y})$

Question 4

Part A

The classical statistician would take the 3 observations and directly calculate the rate parameter without assuming any prior information. Using this method we get the average time between blockages is 8 hours.

Therefore, the estimation of the rate parameter using this method is $\frac{1}{8}$

Part B

$f(T \mid \lambda) = \prod_{i=1}^{3} \lambda e^{-\lambda t_i} = \lambda^3 e^{-24\lambda}$

$f(T \mid \lambda)\pi(\lambda) = \lambda^3 e^{-24\lambda} \lambda^{-\frac{1}{2}} = \lambda^{\frac{5}{2}} e^{-24\lambda}$

Additionally, we know that $f(T \mid \lambda)\pi(\lambda) = \pi(\lambda \mid T)m(T)$ However, $m(T)$ is a function of T and since T has been observed it is a constant. Therefore,

$\pi(\lambda \mid T) \ \alpha \ f(T \mid \lambda)\pi(\lambda)$

$\pi(\lambda \mid T) \ \alpha \ \lambda^{\frac{5}{2}} e^{-24\lambda}$

Note that if random variable, θ is $\theta \sim Gamma(\alpha, \beta)$ then $f(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}$

When α and β are known this can be re-expressed as $f(\theta) = c\theta^{\alpha-1} e^{-\beta\theta}$ for some constant, c.

Upon inspection of $\pi(\lambda \mid T)$, we see that $\pi(\lambda \mid T) \sim Gamma(\frac{7}{2}, 24)$. We know that the expectation of a Gamma distribution is $\frac{\alpha}{\beta}$ which is our bayes estimator.

Therefore, the bayes estimator is $\frac{7}{48}$