ISYE 6420: Bayesian Statistics **Spring 2020**

Homework #3

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Problem 1

a). Let's start with MLE estimate. Using Poisson pmf $\frac{\theta^x e^{-\theta}}{x!}$ we first calculate the likelihood $f(x|\theta)$ based on our experiment data:

$$f(x|\theta) = \frac{\theta^1 e^{-\theta}}{1!} \times \frac{\theta^2 e^{-\theta}}{2!} \times \frac{\theta^0 e^{-\theta}}{0!} \times \frac{\theta^1 e^{-\theta}}{1!} = e^{-4\theta}\left(\theta^1 \frac{\theta^2}{2}\theta^1\right) = \frac{e^{-4\theta}\theta^4}{2}$$

In order to find MLE estimate we need to maximize likelihood $f(x|\theta)$:

$$\hat{\theta}_{MLE} = argmax_\theta \frac{e^{-4\theta}\theta^4}{2}$$

We simplify as follows:

$$\hat{\theta}_{MLE} = argmax_\theta e^{-4\theta}\theta^4 = argmax_\theta \ln\left(e^{-4\theta}\theta^4\right) = argmax_\theta \left(4ln\theta - 4\theta\right)$$

$$d/d\theta\left(4ln\theta - 4\theta\right) = 4/\theta - 4; d/d\theta^2\left(4ln\theta - \theta\right) = -4/\theta^2$$

At $\theta = 1$ $d/d\theta = 0$ and ; $d/d\theta^2 < 0$, so we conclude that $\theta = 1$ is a maximum. So that

$$\hat{\theta}_{MLE} = 1$$

We now turn to Bayes estimator $\hat{\theta}_b$:

$$\hat{\theta}_b \propto f(x|\theta)\pi(\theta)$$

For our experiment that means

$$\hat{\theta}_b \propto \frac{e^{-4\theta}\theta^4}{2}\theta^{-1/2}$$

$$\hat{\theta}_b \propto e^{-4\theta}\theta^{3.5}$$

We can recognize Gamma distribution with parameters $\alpha = 4.5$ and $\beta = 4$:

$$\hat{\theta}_b \propto \mathcal{G}a\left(4.5, 4\right)$$

The mean of Gamma is α/β, so that our Bayes estimator becomes:

$$\hat{\theta}_b = \frac{4.5}{4} = 1.125$$

b). We start with the lower bound L:

$$\int_{-\infty}^{L} \pi(\theta|x)d\theta = \alpha/2$$

We can substitute the integral with the Gamma cdf:

$$F_X(L) = \alpha/2$$

We solve for L numerically using the Brent solver implemented via scipy *root_scalar* function.

$$F_X(L) - 0.05/2 = 0; L \approx 0.3375$$

Let's turn to the upper bound U:

$$\int_{-\infty}^{U} \pi(\theta|x)d\theta = 1 - \alpha/2$$

Alternatively

$$F_X(U) = 1 - \alpha/2$$

Solving for U gives us:

$$F_X(U) - 1 + 0.05/2 = 0; U \approx 2.3778$$

Resulting in the credible set $[0.3375, 2.3778]$ of length $l = 2.0403$.

c). In order to find HPD credible set we are essentially optimizing for the shortest credible set. Let's translate the problem into the "optimization language" and solve it with scipy *optimize* package:
We minimize the function $f(L, U)$:

$$f(L, U) = U - L$$

Subject to constraints:

$$U > L$$

$$F_X(U) - F_X(L) \geq 1 - \alpha$$

After providing an initial guess $L = 0$ and $U = 3$ and solving with scipy we obtain the HPD credible set $[0.23782339, 2.17403328]$. The length of the HPD set is $l \approx 1.9362$.

d). To find the MAP estimator we need to calculate the mode of the posterior. We calculated previously

$$\pi(\theta|x) \propto e^{-4\theta}\theta^{3.5}$$

The mode of $\pi(\theta|x)$ is a point where the posterior peaks. To find the mode we therefore need to calculate $argmax_\theta \pi(\theta|x)$, essentially solving

$$argmax_\theta e^{-4\theta}\theta^{3.5}$$

Let's run the scipy minimize procedure:

$$\hat{\theta}_{MAP} \approx 0.874999997$$

Now we check $\hat{\theta}_{MAP}$ with the analytical value – mode of the $\mathcal{G}a\,(4.5, 4)$:

$$mode = \frac{\alpha - 1}{\beta}$$

$$mode = \frac{4.5 - 1}{4} = 3.5/4 = 0.875$$

We see that, indeed,

$$\hat{\theta}_{MAP} = 0.875$$

e). Let's start with p_0 and p_1:

$$p_0 = \int_{\Theta_0} \pi(\theta|x)d\theta = \mathbb{P}^{\theta|X}(H_0)$$
$$p_1 = \int_{\Theta_1} \pi(\theta|x)d\theta = \mathbb{P}^{\theta|X}(H_1)$$

We use posterior $\mathcal{G}a\,(4.5, 4)$ cdf to calculate p_0 and p_1:

$$p_1 = F_X(1.0) \approx 0.46585$$

$$p_0 = 1 - p_1; p_0 \approx 0.53415$$

So, based on the posterior (and the fact that our Jeffreys' prior is non-informative) we would prefer hypothesis H_0.

Note. The script for solving Q1 is implemented in *hw3.py*, function *solve_q1()* (included in the zip archive). To run the code just run *'python hw3.py'*.

Problem 2

a). Let's start with posterior probabilities $p0$ and $p1$. We know that

$$p_0 = P\left(H_0|X\right) = \left[1 + \frac{\pi_1}{\pi_0} \cdot \frac{m_1(x)}{f(x|0.5)}\right]^{-1}$$

$$f(x|p) = \left(\begin{array}{c} n \\ x \end{array}\right) p^x (1-p)^{n-x}$$

$$m_1(x) = \int_{0.5}^{1} f(x|p) 2dp$$

Substituting with our values we get

$$f(x|0.5) = \left(\begin{array}{c} 16 \\ 15 \end{array}\right) 0.5^{15} 0.5^{1} = \left(\begin{array}{c} 16 \\ 15 \end{array}\right) 0.5^{16} \approx 0.00024414$$

$$m_1(x) = \int_{0.5}^{1} \left(\begin{array}{c} 16 \\ 15 \end{array}\right) p^{15}(1-p)^1 2dp = 32 \int_{0.5}^{1} p^{15}(1-p)^1 dp = 32 \int_{0.5}^{1} \left(p^{15} - p^{16}\right) dp$$

$$m_1(x) = 32 \times \frac{p^{16}}{16} - \frac{p^{17}}{17}\bigg|_{0.5}^{1} = 32 \times \left(\left(\frac{1}{16} - \frac{1}{17}\right) - \left(\frac{0.5^{16}}{16} - \frac{0.5^{17}}{17}\right)\right) \approx 0.1176309$$

$$p_0 = \left[1 + \frac{0.05}{0.95} \cdot \frac{0.1176309}{0.00024414}\right]^{-1} \approx 0.037938$$

$$p_1 = 1 - p_0$$

$$p_1 \approx 1 - 0.037938 \approx 0.962062$$

We now calculate the Bayes factors:

$$B_{01} = \frac{f(x|0.5)}{m_1(x)}$$

$$B_{01} \approx \frac{0.00024414}{0.1176309} \approx 0.0020755$$

$$B_{10} = \frac{1}{B_{01}}$$

$$B_{10} = \frac{1}{0.0020755} \approx 481.816$$

b). Let's calibrate the BF for H_1: $log_{10} B_{10} \approx 2.68288$. According to the Jeffreys' scale we have decisive evidence against H_0 ($\log_{10} B_{10}(x) > 2$).

Note. The script for solving Q2 is implemented in *hw3.py*, function *solve_q2()* (included in the zip archive). To run the code just run *'python hw3.py'*.

References

[1] Engineering Biostatistics: An Introduction using MATLAB and WinBUGS. Brani Vidakovic - Wiley Series in Probability and Statistics.