IYSE 6420 Fall 2020 Homework3

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1. Maxwell

Sample y1, . . . , yn, comes from Maxwell distribution with a density

$$\boldsymbol{f(y|\theta) = \sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}y^2 e^{\frac{-\theta y^2}{2}}, y \geq 0, \theta > 0}$$

Assume an exponential prior on θ

$$\boldsymbol{\pi(\theta) = \lambda e^{-\lambda\theta}, \theta > 0, \lambda > 0}$$

(a) Show that posterior belongs to gamma family and depends on data via $\sum_{i=1}^{n} y_i^2$

Using Bayes theorem:

$$\pi(\theta|y) = \frac{f(y|\theta) \times \pi(\theta)}{m(y)}$$
$$= \frac{f(y_1, \dots, y_n|\theta)\pi(\theta)}{\int_0^\infty f(y_1, \dots, y_n|\theta)\,\pi(\theta)d\theta}$$
$$= \frac{f(y_1|\theta)f(y_2|\theta)\dots f(y_n|\theta)\pi(\theta)}{\int_0^\infty f(y_1|\theta)f(y_2|\theta)\dots f(y_n|\theta)\,\pi(\theta)d\theta}$$
$$= \frac{\prod_{i=1}^{n}\sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}y_i^2 e^{\frac{-\theta y_i^2}{2}}\lambda e^{-\lambda\theta}}{\int_0^\infty \prod_{i=1}^{n}\sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}y_i^2 e^{\frac{-\theta y_i^2}{2}}\lambda e^{-\lambda\theta}d\theta}$$
$$= \frac{\prod_{i=1}^{n}\sqrt{\frac{2}{\pi}}\theta^{\frac{3}{2}}y_i^2\lambda e^{-\theta\left(\sum_{i=1}^{n}\frac{y_i^2}{2}+\lambda\right)}}{\prod_{i=1}^{n}\sqrt{\frac{2}{\pi}}y_i^2\lambda\int_0^\infty \theta^{\frac{3}{2}}e^{-\theta\left(\sum_{i=1}^{n}\frac{y_i^2}{2}+\lambda\right)}d\theta}$$

Let $\beta = \sum_{i=1}^{n}\frac{y_i^2}{2} + \lambda$

$$\pi(\theta|y) \propto \theta^{\frac{3n}{2}}e^{-\theta\beta}\beta^{\frac{3n+2}{2}}$$

The distribution belongs to gamma family: $f(\theta, \alpha, \beta)$ where $\alpha = \frac{3n+2}{2}$, $\beta = \sum_{i=1}^{n}\frac{y_i^2}{2} + \lambda$
Since λ and 2 are constant, it depends on data via $\sum_{i=1}^{n} y_i^2$

(b) For λ = 1/2 and y_1 = 1.4, y_2 = 3.1, and y_3 = 2.5, find Bayes estimator for θ. How the Bayes estimator compares to the MLE and prior mean. The MLE for θ is $\frac{3n}{\sum_{i=1}^{n} y_i^2} = \frac{3}{\overline{y^2}}$

Bayes estimator

$$E(\theta) = \frac{\alpha}{\beta}$$

$$= \frac{11}{2\left(\sum_{i=1}^{n}\frac{y_i^2}{2}+\lambda\right)}$$
$$= \frac{11}{2\left(\frac{1.4^2}{2}+\frac{3.1^2}{2}+\frac{2.5^2}{2}+\frac{1}{2}\right)}$$
$$= 0.584$$

MLE

$$\frac{3n}{\sum_{i=1}^{n} y_i^2} = \frac{3}{\bar{y}^2}$$
$$= \frac{3\times 3}{1.4^2+3.1^2+2.5^2} = \frac{50}{99} = 0.505$$

Prior mean

$$\frac{1}{\lambda} = 2$$

Prior mean > Bayes estimator > MLE, since using Bayes estimator we are taking prior belief into consideration, so the result is between MLE and prior. The result is closer to MLE because λ is small. If λ is large it will be closer to prior mean.

(c) Using MATLAB/Octave/R/Python to calculate the 95% equitailed credible set for θ.

95% equitailed credible set for θ

(0.20274964145781615, 1.16472100217966)

Python Code

```
from scipy.stats import gamma

alpha = 5.5
beta = 0.5 * (1.4**2 + 3.1**2 + 2.5**2 + 1)

gamma(alpha, scale=1/beta).interval(0.95)
# (0.20274964145781615, 1.16472100217966)
```

(d) Find a prediction for a future single observation. For this, you will need the mean of Maxwell, which is $E[Y] = 2\sqrt{\frac{2}{\pi\theta}}$

A prediction for a future single observation

2.2445

Python Code

```
import math

alpha = 5.5
beta = 0.5 * (1.4**2 + 3.1**2 + 2.5**2 + 1)

rv = gamma(alpha, scale = 1/beta)
```

```python
def mean_maxwell(theta):
    return 2 * (2/math.pi/theta) ** (0.5)

rv.expect(mean_maxwell)
# 2.2444986268225957
```

2. Jeremy Mixture

(a) Show that for likelihood $f(x|\theta)$ and mixture prior

$$\pi(\theta) = \epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta), \theta \in \Theta$$

the posterior is a mixture of

$$\pi(\theta \mid x) = \epsilon'\pi_1(\theta \mid x) + (1-\epsilon')\pi_2(\theta \mid x)$$

where

$$\pi_i(\theta \mid x) = \frac{f(x \mid \theta)\pi_i(\theta)}{m_i(x)}, m_i(x) = \int_\Theta f(x \mid \theta)\pi_i(\theta)d\theta, i = 1, 2, \text{ and}$$

$$\epsilon' = \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}$$

Posterior

$$\begin{aligned}
\pi(\theta \mid x) &= \frac{f(x|\theta) \times \pi(\theta)}{\int_\Theta f(x|\theta) \times \pi(\theta)d\theta} \\
&= \frac{f(x|\theta)\,\{\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)\}}{\int_\Theta f(x|\theta)\,\{\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)\}d\theta} \\
&= \frac{f(x|\theta)\,\epsilon\pi_1(\theta)}{\int_\Theta f(x|\theta)\,\epsilon\pi_1(\theta)d\theta + \int_\Theta f(x|\theta)\,(1-\epsilon)\pi_2(\theta)d\theta} \\
&\quad + \frac{f(x|\theta)\,(1-\epsilon)\pi_2(\theta)}{\int_\Theta f(x|\theta)\,\epsilon\pi_1(\theta)d\theta + \int_\Theta f(x|\theta)\,(1-\epsilon)\pi_2(\theta)d\theta} \\
&= \frac{f(x|\theta)\,\epsilon\pi_1(\theta)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} + \frac{f(x|\theta)\,(1-\epsilon)\pi_2(\theta)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} \\
&= \epsilon'\pi_1(\theta \mid x) + (1-\epsilon')\pi_2(\theta \mid x)
\end{aligned}$$

(b) Now we assume $X|\theta \sim \mathcal{N}(\theta, 80)$ and the prior for θ is a mixture

$$\theta \sim \pi(\theta) = \frac{2}{3}\mathcal{N}(110, 60) + \frac{1}{3}\mathcal{N}(100, 200)$$

Find the posterior and Bayes estimator for θ if X = 98.

$$f(x|\theta) = X|\theta \sim \mathcal{N}(\theta, 80) = \frac{1}{\sqrt{2\pi 80}} \exp\{-\frac{1}{2 \times 80}(x-\theta)^2\}$$

$$\theta \sim \pi(\theta) = \frac{2}{3}\mathcal{N}(110,60) + \frac{1}{3}\mathcal{N}(100,200)$$

$$= \frac{2}{3}\frac{1}{\sqrt{2\pi 60}}\exp\{-\frac{1}{2\times 60}(\theta-110)^2\} + \frac{1}{3}\frac{1}{\sqrt{2\pi 200}}\exp\{-\frac{1}{2\times 200}(\theta-100)^2\}$$

Posterior

$$\pi(\theta \mid x) = \frac{f(x|\theta)\times\pi(\theta)}{\int_\Theta f(x|\theta)\times\pi(\theta)d\theta}$$

from part(a)
If likelihood and mixture prior

$$f(x|\theta) \; and \; \pi(\theta) = \epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta), \theta \in \Theta$$

Then posterior

$$\pi(\theta \mid x) = \epsilon'\pi_1(\theta \mid x) + (1-\epsilon')\pi_2(\theta \mid x)$$

Where

$$\pi_i(\theta \mid x) = \frac{f(x \mid \theta)\pi_i(\theta)}{m_i(x)}, m_i(x) = \int_\Theta f(x \mid \theta)\pi_i(\theta)d\theta, i = 1{,}2, \text{ and}$$

$$\epsilon' = \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}$$

$$\epsilon' = \frac{1}{1 + \frac{(1-\epsilon)m_2(x)}{\epsilon m_1(x)}}$$

From the equation $\theta \sim \pi(\theta) = \frac{2}{3}\mathcal{N}(110{,}60) + \frac{1}{3}\mathcal{N}(100{,}200)$

$$\epsilon = \frac{2}{3}, 1-\epsilon = \frac{1}{3}, \pi_1(\theta) = \mathcal{N}(110{,}60), \pi_2(\theta) = \mathcal{N}(100{,}200)$$

$$\pi_1(\theta \mid x) = \frac{f(x \mid \theta)\pi_1(\theta)}{\int_\Theta f(x \mid \theta)\pi_1(\theta)d\theta}$$

$$\pi_2(\theta \mid x) = \frac{f(x \mid \theta)\pi_2(\theta)}{\int_\Theta f(x \mid \theta)\pi_2(\theta)d\theta}$$

If X = 98, follows Normal likelihood + Normal prior

$$\pi_1(\theta \mid x) \sim N\left(\frac{60}{80+60}x + \frac{80}{80+60}\times 110, \frac{60\times 80}{60+80}\right)$$

$$if\ x = 98\ then\ posterior\ \pi_1(\theta \mid x) \sim N(104.9, 34.3)$$

$$\pi_2(\theta \mid x) \sim N\left(\frac{200}{80+200}x + \frac{80}{80+200}\times 100, \frac{200\times 80}{200+80}\right)$$

$$if\ x = 98\ then\ posterior\ \pi_2(\theta \mid x) \sim N(98.6, 57.1)$$

Now we need to calculate $\frac{m_1(x)}{m_2(x)}$ to get ϵ'

$$\frac{m_1(x)}{m_2(x)} = \frac{\int_\Theta f(x \mid \theta)\pi_1(\theta)d\theta}{\int_\Theta f(x \mid \theta)\pi_2(\theta)d\theta}$$

$$= \frac{f(x \mid \theta)\pi_1(\theta)}{\pi_1(\theta \mid x)} \Bigg/ \frac{f(x \mid \theta)\pi_2(\theta)}{\pi_2(\theta \mid x)}$$

$$= \pi_1(\theta)\pi_2(\theta \mid x) \Big/ \pi_2(\theta)\pi_1(\theta \mid x)$$

$$= \frac{\mathcal{N}(110,60)N(98.6,57.1)}{\mathcal{N}(100,200)N(104.9,34.3)}$$

$$\epsilon' = \frac{1}{1 + \frac{(1-\epsilon)m_2(x)}{\epsilon m_1(x)}}$$

$$= \frac{1}{1 + \frac{1}{2} \times \frac{m_2(x)}{m_1(x)}}$$

$$= \frac{1}{1 + \frac{1}{2} \times \frac{\pi_2(\theta)\pi_1(\theta \mid x)}{\pi_1(\theta)\pi_2(\theta \mid x)}}$$

$$= \frac{2\pi_1(\theta)\pi_2(\theta \mid x)}{2\pi_1(\theta)\pi_2(\theta \mid x) + \pi_2(\theta)\pi_1(\theta \mid x)}$$

So

$$\pi(\theta \mid x) = \epsilon'\pi_1(\theta \mid x) + (1-\epsilon')\pi_2(\theta \mid x)$$

$$= \frac{2\pi_1(\theta)\pi_2(\theta \mid x)\pi_1(\theta \mid x)}{2\pi_1(\theta)\pi_2(\theta \mid x) + \pi_2(\theta)\pi_1(\theta \mid x)} + \frac{\pi_2(\theta)\pi_1(\theta \mid x)\pi_2(\theta \mid x)}{2\pi_1(\theta)\pi_2(\theta \mid x) + \pi_2(\theta)\pi_1(\theta \mid x)}$$

$$= \frac{\pi_2(\theta \mid x)\pi_1(\theta \mid x)}{2\pi_1(\theta)\pi_2(\theta \mid x) + \pi_2(\theta)\pi_1(\theta \mid x)}\{2\pi_1(\theta) + \pi_2(\theta)\}$$

$$= \frac{2\pi_1(\theta) + \pi_2(\theta)}{\frac{2\pi_1(\theta)}{\pi_1(\theta \mid x)} + \frac{\pi_2(\theta)}{\pi_2(\theta \mid x)}}$$

When $x = 98$

$$= \frac{\frac{2}{\sqrt{2\pi 60}}\exp\left\{-\frac{1}{2\times 60}(\theta-110)^2\right\} + \frac{1}{\sqrt{2\pi 200}}\exp\left\{-\frac{1}{2\times 200}(\theta-100)^2\right\}}{\frac{\frac{2}{\sqrt{2\pi 60}}\exp\left\{-\frac{1}{2\times 60}(\theta-110)^2\right\}}{\frac{1}{\sqrt{2\pi 34.3}}\exp\left\{-\frac{1}{2\times 34.3}(\theta-104.9)^2\right\}} + \frac{\frac{1}{\sqrt{2\pi 200}}\exp\left\{-\frac{1}{2\times 200}(\theta-100)^2\right\}}{\frac{1}{\sqrt{2\pi 57.1}}\exp\left\{-\frac{1}{2\times 57.1}(\theta-98.6)^2\right\}}}$$

$$\frac{\frac{e^{-1/400(\theta-100)^2}}{20\sqrt{\pi}} + \frac{e^{-1/120(\theta-110)^2}}{\sqrt{30\pi}}}{1.51217e^{0.0145773(\theta-104.9)^2-1/120(\theta-110)^2} + 0.534322e^{0.00875657(\theta-98.6)^2-1/400(\theta-100)^2}}$$

The mean of the posterior is a Bayes estimator of a parameter

3. Mendel's Experiment with Peas

For the height trait, Mendel's model suggests that 3/4 of the plants grown from a cross between tall and short height strains of pea lines will be of the tall height variety. After breeding 1064 of these plants, 787 resulted as the tall height variety. The reasonable model for the number of tall height results from n experiments is binomial Bin(n, p). Complete a Bayesian model with beta Be(12, 4) prior on the unknown proportion p.

(a) What are prior and posterior means?

Binomial Likelihood and Beta Prior

$$X \mid p \sim \text{Bin}\,(n,p); f(x \mid p) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$p \sim \text{Be}\,(\alpha,\beta); \pi(p) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$
$$\pi(p \mid x) \propto f(x \mid p)\pi(p) =$$
$$= C \times p^{x+\alpha-1}(1-p)^{n-x+\beta-1}$$
$$\Rightarrow p \mid X \sim \text{Be}(x+\alpha, n-x+\beta)$$

Prior mean

$$X \sim \text{Be}(\alpha,\beta) \Rightarrow EX = \frac{\alpha}{\alpha+\beta}$$
$$= \frac{12}{12+4} = 0.75$$

Posterior mean

$$E(p \mid X) = \frac{x+\alpha}{n+\alpha+\beta} = \frac{787+12}{1064+12+4} = 0.74$$

(b) Find posterior probability of hypothesis H0 : p ≤ 3/4?

Posterior

$$\pi(p \mid x) \Rightarrow p \mid X \sim \text{Be}(x+\alpha, n-x+\beta)$$
$$= \text{Be}(787+12, 1064-787+4)$$

CDF

$$\int_0^{\frac{3}{4}} \pi(p \mid x)dp = \int_0^{\frac{3}{4}} \text{Be}(799, 281)\, dp$$
$$= 0.776$$

Python Code

```
from scipy.stats import beta

a, b = 799, 281
rv = beta(a, b)
rv.cdf(0.75)
# 0.7758595145276612
```

**(c) Find a 95% equitailed credible set for the true proportion of tall height plants obtained
from the given cross.**

95% equitailed credible set

(0.7132478379195061, 0.7655405496526497)

Python Code

```
from scipy.stats import beta

a, b = 799, 281
rv = beta(a, b)
rv.interval(0.95)
# (0.7132478379195061, 0.7655405496526497)
```