Solution | **Homework 1** | ISyE 6420 September 8, 2020

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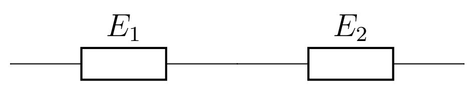
1 Circuit.

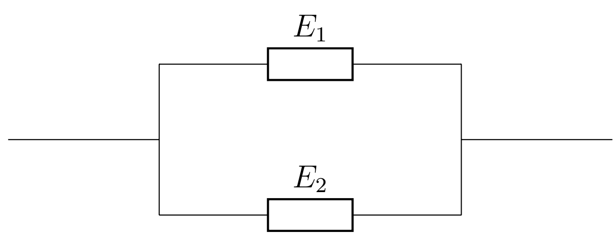
Let the probability of an element E_i of a system S being operational during time interval T as p_{E_i}. Suppose all elements work independently. We consider the following two ways of connection of elements.

- For sequential connection of elements, that is



  Then $p_S = p_{E_1} \cdot p_{E_2}$.

- For parallel connection of elements, that is



  Then $p_S = p_{E_1 \cup E_2} = p_{E_1} + p_{E_2} - p_{E_1} \cdot p_{E_2}$. This is similar to the union of events. Alternatively, We let $q_{E_i} = 1 - p_{E_i}$ be the probability that element E_i fails during time interval T, then

$$\begin{aligned} p_S &= 1 - q_S = 1 - q_{E_1} \cdot q_{E_2} \\ &= 1 - (1 - p_{E_1}) \cdot (1 - p_{E_2}) = p_{E_1} + p_{E_2} - p_{E_1} \cdot p_{E_2}. \end{aligned}$$

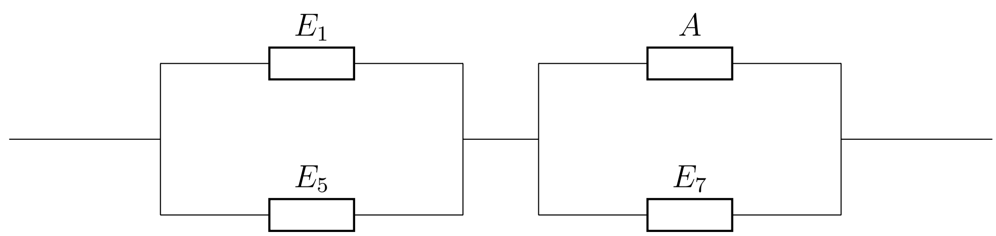
(a) We consider the following two hypotheses:

$$H_1 : E_6 \text{ works during time interval } T;$$
$$H_2 : E_6 \text{ fails during time interval } T \text{ (This is equivalent to } \overline{H_1} \text{).}$$

We know that $\mathrm{P}(H_1) = 0.5$ and $\mathrm{P}(H_2) = 0.5$.
By law of Total Probability, we have

$$\mathrm{P}(S) = \mathrm{P}(S|H_1)\mathrm{P}(H_1) + \mathrm{P}(S|H_2)\mathrm{P}(H_2).$$

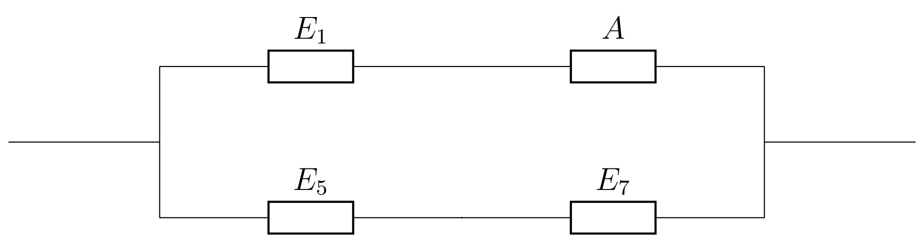
Under hypothesis H_1, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are parallel-connected.



We know that the probability that A is operational during time interval T is $p_A = 1 - q_{E_2} \cdot q_{E_3} \cdot q_{E_4} = 1 - (0.7)(0.3)(0.6) = 0.874$. Thus, we have

$$\begin{aligned} p_{S|H_1} &= (1 - q_{E_1} \cdot q_{E_5})(1 - q_A \cdot q_{E_7}) \\ &= (1 - (0.5)(0.1))(1 - (0.126)(0.3)) = 0.9141. \end{aligned}$$

Under hypothesis H_2, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are parallel-connected.



We have

$$\begin{aligned} p_{S|H_2} &= 1 - (1 - p_{E_1} \cdot p_A)(1 - p_{E_5} \cdot p_{E_7}) \\ &= 1 - (1 - (0.5)(0.874))(1 - (0.9)(0.7)) \\ &= 0.7917. \end{aligned}$$

Hence, we have

$$P(S) = (0.9141)(0.5) + (0.7917)(0.5) = 0.8529.$$

(b) We try to find $P(H_1|S)$. By Bayes' rule, we have

$$P(H_1|S) = \frac{P(S|H_1)P(H_1)}{P(S)} = \frac{(0.9141)(0.5)}{0.8529} = 0.5359.$$

2 Two Batches.

We let the first batch be the batch with all products conforming, and let the second batch be the batch that contains 10% non-conforming products. We define the following events.

$$
\begin{aligned}
&H_1 : \text{The first batch is selected;} \quad H_2 : \text{The second batch is selected;}\\
&A : \text{The first selected product from the selected batch is conforming;}\\
&B : \text{The second selected product is non-conforming.}
\end{aligned}
$$

We know that $\mathrm{P}(H_1) = \mathrm{P}(H_2) = \frac{1}{2}$ as the batch is selected randomly. We also know that $\mathrm{P}(A|H_1) = 1$ and $\mathrm{P}(A|H_2) = 0.9$ based on the problem setting. We hope to find $\mathrm{P}(B|A)$. By law of Total Probability, we have

$$
\begin{aligned}
\mathrm{P}(A) &= \mathrm{P}(A|H_1)\mathrm{P}(H_1) + \mathrm{P}(A|H_2)\mathrm{P}(H_2)\\
&= (1)(0.5) + (0.9)(0.5) = 0.95,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{P}(B|A) &= \mathrm{P}(B, H_1|A) + \mathrm{P}(B, H_2|A)\\
&= \mathrm{P}(B|H_1, A)\mathrm{P}(H_1|A) + \mathrm{P}(B|H_2, A)\mathrm{P}(H_2|A).
\end{aligned}
$$

By Bayes rule, we have

$$
\begin{aligned}
\mathrm{P}(H_1|A) &= \frac{\mathrm{P}(A|H_1)\mathrm{P}(H_1)}{\mathrm{P}(A)} = \frac{(1)(0.5)}{0.95} = \frac{10}{19},\\
\mathrm{P}(H_2|A) &= \frac{\mathrm{P}(A|H_2)\mathrm{P}(H_2)}{\mathrm{P}(A)} = \frac{(0.9)(0.5)}{0.95} = \frac{9}{19}.
\end{aligned}
$$

Hence, we have

$$
\mathrm{P}(B|A) = (0)\left(\frac{10}{19}\right) + (0.1)\left(\frac{9}{19}\right) = \frac{9}{190}.
$$

3 Machine.

We let M be the component whose failure probability is $1/2$, and consider the following events.

$$
\begin{aligned}
&A : \text{The machine fails;}\\
&B : \text{Component } M \text{ fails.}
\end{aligned}
$$

We use $\overline{(\cdot)}$ to denote the complement event of one event.

(a) Based on the law of total probability, we know that

$$\mathrm{P}(A) = P(A, B) + P(A, \overline{B}) = P(A|B)P(B) + P(A|\overline{B})P(\overline{B}).$$

We know that the machine fails when all components fail or three components fail. We condition on whether the component M fails.

If the component M fails, then the machine fails when at least two of the rest components fail. This means that

$$P(A|B) = (1-p)^3 + \binom{3}{1} p(1-p)^2.$$

If the component M does not fail, then the machine fails when all other three components fail. This means that

$$P(A|\overline{B}) = (1-p)^3.$$

Thus, we have

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|\overline{B})P(\overline{B}) \\ &= \left(\frac{1}{2}\right)\left((1-p)^3 + \binom{3}{1} p(1-p)^2\right) + \left(\frac{1}{2}\right)(1-p)^3 \\ &= (0.5)((0.6)^3 + 3(0.4)(0.6)^2)) + (0.5)(0.6)^3 = 0.432. \end{aligned}$$

Hence, the probability that the machine will fail is 0.432.

(b) We try to find $P(B|A)$ in this problem.

By Bayes's rule, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

As we already know that $P(A|B) = (1-p)^3 + 3p(1-p)^2$ and $P(A) = (0.5)(2(1-p)^3 + 3p(1-p)^2)$, we have

$$\begin{aligned} P(B|A) &= \frac{(0.5)((1-p)^3 + 3p(1-p)^2)}{(0.5)(2(1-p)^3 + 3p(1-p)^2)} \\ &= \frac{1+2p}{2+p}. \end{aligned}$$