ISYE 6420: HW3

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Question 1

Part A

The following information is given:

(1) $f(r) = \xi rexp\{\frac{-\xi r^2}{2}\}$

(2) $\pi(\xi) = \lambda e^{-\lambda\xi}$

Therefore:

(3) $f(r \mid \xi)\pi(\xi) = \xi\lambda rexp\{-\xi(\lambda + \frac{r^2}{2})\}$

Since r is observed, it is a constant and we can express this as

(4) $f(r \mid \xi)\pi(\xi)\ \alpha\ \xi exp\{-\xi(\lambda + \frac{r^2}{2})\}$

If θ followed a $Gamma(\alpha, \hat{\lambda})$ distribution, then it would have the pdf:

$f(\theta) = c(\hat{\lambda}\theta)^{\alpha-1}exp\{-\hat{\lambda}\theta\}$ for some constant, c. Note that this parameterization is using the rate parameter instead of a scale parameter. Note that (4) follows this form such that $\alpha = 2$ and $\hat{\lambda} = \lambda + \frac{r^2}{2}$

With an observed value for r, $\pi(\xi \mid r)\ \alpha\ f(r \mid \xi)\pi(\xi)$

Therefore, $pi(\xi \mid r) \sim Gamma(2, \lambda + \frac{r^2}{2})$

Part B

In part a, we demonstrated that $\pi(\xi \mid r) \sim Gamma(2, \lambda + \frac{r^2}{2})$ for a single observation r. However, in this scenario, we have multiple observations.

$f(r_1, r_2, ..., r_n \mid \xi) = \prod_{i=1}^{n} f(r_i \mid \xi) = c\xi^n exp\{\xi \sum_{i=1}^{n} \frac{r_i^2}{2}\}$ for some constant, c equal to the product of our observations.

From part A, we saw that $\pi(\xi) = \lambda e^{-\lambda\xi}$ for some lambda. So,

$f(r_1, r_2, ..., r_n \mid \xi)\pi(\xi)\ \alpha\ \xi^n exp\{\xi(\lambda + \sum_{i=1}^{n} \frac{r_i^2}{2})\}$

Therefore, $\pi(\xi \mid r_1, r_2, ...r_n) \sim Gamma(n + 1, \lambda + \sum_{i=1}^{n} \frac{r_i^2}{2})$ which is consistent with the special case of i=1 found in part A.

Given the four observations mentioned in the question statement

$\pi(\xi \mid r_1, r_2, r_3, r_4) \sim Gamma(5, \lambda + 27)$ for some lambda.

With this parameterization, $E\xi = \frac{\alpha}{\hat{\lambda}} = \frac{2}{\lambda+27}$ which is a bayes estimate.

Part C

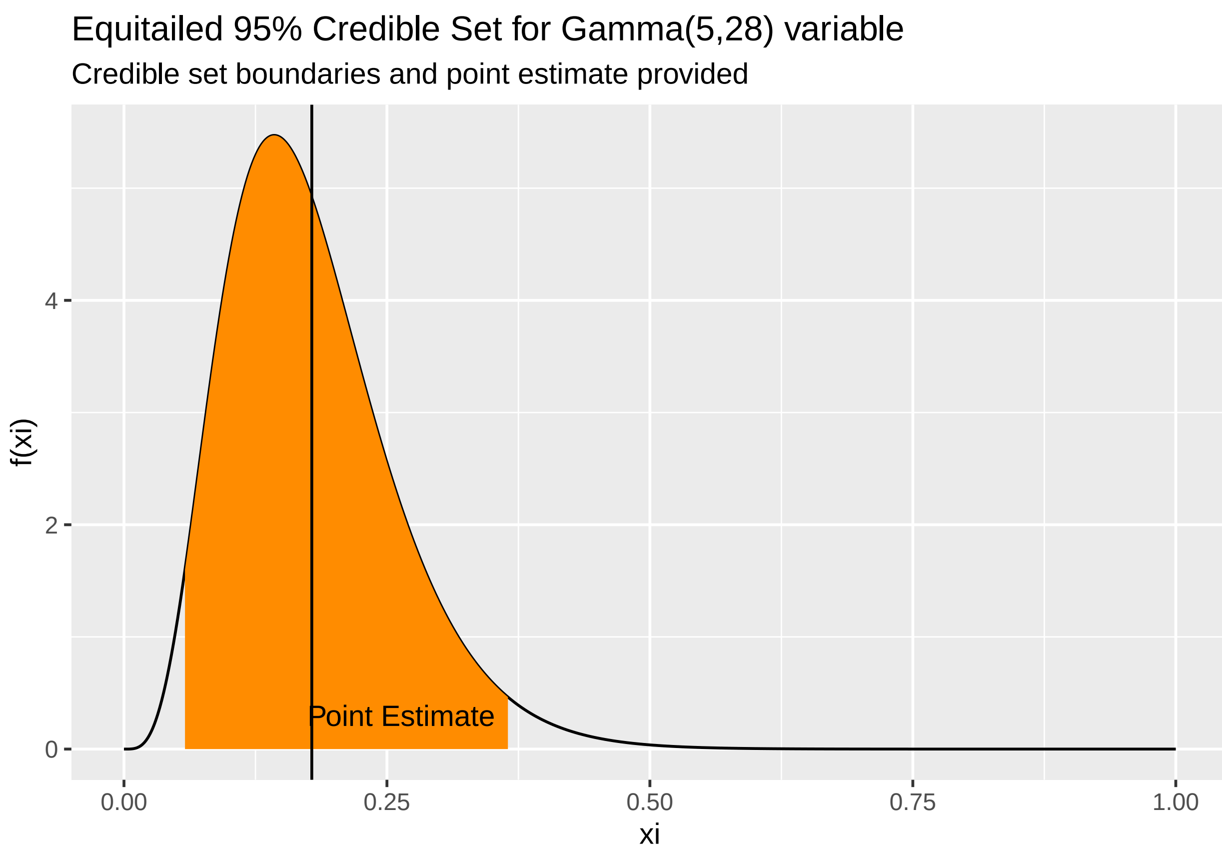
When $\lambda = 1$, the posterior follows the following distribution: $\pi(\xi \mid r_1, r_2, r_3, r_4) \sim Gamma(5, 28)$

The 95% equitailed credible set is $[0.05798, 0.36577]$

```r
alpha=5
beta=28
type_1=0.05

lb<-qgamma(type_1/2,alpha,rate=beta)
ub<-qgamma(1-(type_1/2),alpha,rate=beta)

point.estimate=alpha/beta

data<-data.frame(t=seq(0,1,0.001))%>%
  mutate(pdf=dgamma(t,alpha,rate=beta))
```

```
## Warning: package 'bindrcpp' was built under R version 3.5.1
```

```r
data%>%
  ggplot(aes(x=t,y=pdf))+
    geom_line()+
    geom_ribbon(aes(x=ifelse(t<lb | t>ub,NA,t),ymin=0,ymax=pdf),fill="darkorange")+
    geom_vline(xintercept=point.estimate)+
    annotate('text',label='Point Estimate',x=point.estimate+0.085,y=0.3)+
  labs(title='Equitailed 95% Credible Set for Gamma(5,28) variable',subtitle = 'Credible set boundaries
```



Question 2

Part A

Eliciting a prior

We are provided the following information about p:

(1) $\pi(p) \sim Gamma(\alpha, \beta)$

(2) $\mu_p = 0.9$

(3) $\mu_p - 2\sigma = 0.8$

We also know from wikipedia that for a beta distribution the following is true:

(4) $\mu = \frac{\alpha}{\alpha+\beta}$
(5) $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

From (2) and (4), we know that $\alpha = 9\beta$, which can be used to simplify σ^2 to a function of a single variable, β

(6) $\sigma^2 = \frac{9\beta^2}{(10\beta)^2(10\beta+1)}$

Using (3) and (6), we can solve for β

$\sigma = 0.05$

$0.05^2 = \frac{9\beta^2}{(10\beta)^2(10\beta+1)}$

This yields $\beta = 3.5$, which allows us to solve for α as $\alpha = 31.5$

Therefore we've elicited a prior of $\pi(p) \sim Beta(31.5, 3.5)$ which has a pdf of

(7) $\pi(p) = cp^{\alpha-1}(1-p)^{\beta-1}$ for $alpha = 31.5$ and $\beta = 3.5$ and some constant, c

Finding the likelihood distribution

Given a proportion p, the probability of experiencing k successes after n trials can be modeled as a binomial distribution. That is,

(8) $f(n, k \mid p) = \binom{n}{k} p^k(1-p)^{n-k}$ such that $n = 30$ and $k = 22$

Finding the posterior

Using (7) and (8) we get that:

$f(n, k \mid p)\pi(p) = cp^k(1-p)^{n-k}p^{\alpha-1}(1-p)^{\beta-1}$

$f(n, k \mid p)\pi(p)\ \alpha\ p^{\alpha+k-1}(1-p)^{n-k+\beta-1}$

Therefore, the posterior can be modeled as:

$\pi(p \mid n, k) \sim Beta(\alpha + k, n - k + \beta)$ using the above mentioned values of $n = 30, k = 22, \alpha = 31.5, \beta = 3.5$, we get that

$\pi(p \mid n = 30, k = 22) \sim Beta(53.5, 11.5)$ and has pdf:

$f(p \mid n = 30, k = 22) = cp^{52.5}(1-p)^{10.5}$

Finding the bayes estimator

Using equation (4) we get that

$Ep = \frac{53.5}{53.5+11.5}$ therefore our bayes estimator of p is

$Ep = 0.82308$

This intuitively makes sense. Although our experiment yielded a frequnetist estimate of $\frac{22}{30} = 0.733$, our prior distribution was fairly narrow and therefore had a lot of influence over the posterior. I want to know who this "expert" is and why they have such confidence that the variance is so low. Typical egotistical scientist.

Part B

Using the code below, we find the equitailed credible interval of [0.7222,0.9051]

```
alpha<-53.5
beta<-11.5
type_1<-0.05
lb<-qbeta(type_1/2,alpha,beta)
ub<-qbeta(1-(type_1/2),alpha,beta)

lb
```

```
## [1] 0.7222732
```

```
ub
```

```
## [1] 0.9051004
```

Part C

Using the code below, we confirm that H_0 is accepted and H_1 is rejected. However we could have also inferred this since H_1 is outside the bounds of the credible interval.

```
critical_value<-4/5
h1<-pbeta(critical_value,alpha,beta)
h0<-1-h1

h0
```

```
## [1] 0.7054277
```

```
h1
```

```
## [1] 0.2945723
```

Part D

See the included ODC file for winbugs models.