Assignment 1

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Question 1 (i). We want to prove that when Algorithm Summation terminates its execution,

$$y = \sum_{j=0}^{n} a_j x^j$$

Proof. We shall apply induction to prove that after the mth iteration of the for loop the following invariant holds true:

$$y = \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}$$

(Induction basis) First, we note that y is initialized to 0 in Line 1. When $m = 1$ and $i = n$ in Line 3,

$$\begin{aligned} y &= a_n + x * y \\ &= a_n + x * 0 \\ &= a_n \\ &= a_n x^0 \\ &= \sum_{j=0}^{0} a_{n-j} x^{n-i-j} \end{aligned}$$

(Induction Hypothesis) Suppose for iteration $m - 1 < n$, $y = \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j-1}$
(Induction Step) When Line 3 is executed for the mth time,

$$
\begin{aligned}
y &= a_{n-i} + x\sum_{j=0}^{m-2} a_{n-j}x^{n-i-j-1} \\
&= a_{n-i} + \sum_{j=0}^{m-2} a_{n-j}x^{n-i-j} \\
&= a_{n-i}x^0 + \sum_{j=0}^{m-2} a_{n-j}x^{n-i-j} \\
& \qquad\qquad\qquad\qquad \text{(Induction Hypothesis)} \\
&= \sum_{j=0}^{m-1} a_{n-j}x^{n-i-j}
\end{aligned}
$$

Therefore we can conclude that the invariant $y = \sum_{j=0}^{m-1} a_{n-j}x^{n-i-j}$ holds $\forall m > 0$.

On the n+1th iteration (where $i = 0$) we then have,

$$
\begin{aligned}
y &= \sum_{j=0}^{n} a_{n-j}x^{n-j} \\
&= a_nx^n + a_{n-1}x^{n-1} + ... + a_1x^1 + a_0x^0 \\
&= \sum_{j=0}^{n} a_jx^j && \text{(By reversing the order of the terms)}
\end{aligned}
$$

□

Question 1 (ii).
Key Operation: Multiplication of real numbers.
Input size: n (The size of the input array)

The Summation algorithm must perform the key operation on each of the n elements of the input array.

1. Worst-case Time Complexity: $T(n) = n$
2. Average-case Time Complexity: $T_{ave}(n) = n$

Question 2. $lgn \in \theta(lglg(n!))$

To prove that $lgn \in \theta(lglg(n!))$ we shall prove that $lgn \in O(lglg(n!)) \wedge lgn \in \Omega(lglg(n!))$.

Proof. First we shall prove that $lgn \in O(lglg(n!))$, that is we shall determine a $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} lgn &\leq clglg(n!) && \forall n > n_0 \\ lgn &\leq lglg^c(n!) \\ 2^{lgn} &\leq 2^{lglg^c(n!)} \\ n &\leq lg^c n! \end{aligned}$$

By letting c = 1, we obtain

$$\begin{aligned} n &\leq lg(n!) \\ 2^n &\leq n! \end{aligned}$$

Lemma 1. $2^n < n!, \forall n \geq 4$
We shall prove this by induction on n.
(Induction Basis) When n = 4, $2^4 < 4! = 16 < 24$.
(Induction Hypothesis) Assume $2^k < k!, k \geq 4$.
(Induction Step) We then have,

$$\begin{aligned} 2^k(k+1) &< k!(k+1) \\ &= (k+1)! && (I) \end{aligned}$$

Also since $k \geq 4$,

$$\begin{aligned} 2 &< k+1 \\ 2^{k+1} &< (k+1)2^k && (II) \end{aligned}$$

By transitivity of (I) and (II) we obtain $2^{k+1} < (k+1)!$.
Therefore by Induction $2^n < n!, \forall n \geq 4$.

Therefore for $c = 1$, and $n_0 = 4$ we have $lg(n) \leq clglg(n!),\ \forall n > n_0$.
Therefore $lg(n) \in O(lglg(n!))$.

Now we shall prove that $lg(n) \in \Omega(lglg(n!))$, that is we shall determine a $c \in R^+$ and $n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} lg(n) &\geq clglg(n!) && \forall n > n_0 \\ lg(n) &\geq lglg^c(n!) \\ 2^{lgn} &\geq 2^{lglg^c(n!)} \\ n &\geq lg^c n! \end{aligned}$$

By letting $c = \frac{1}{2}$ we obtain,

$$n \geq \sqrt{lg(n!)}$$
$$n^2 \geq lg(n!)$$

It can be seen that $lg(n!) \leq nlg(n)$ (III) via,

$$\begin{aligned}
lg(n!) &= lg(n*(n-1)*(n-2)*...*1) && \text{(Definition of } n!) \\
&= lg(n) + lg(n-1) + ... + lg(1) && (lg(mn) = lg(m) + lg(n)) \\
&\leq lg(n) + lg(n) + ... + lg(n) \\
&= nlgn
\end{aligned}$$

Additionally $\forall n > 1$,

$$\begin{aligned}
2^n &> n \\
n &> lg(n) \\
n^2 &> nlg(n) && (IV)
\end{aligned}$$

By combining (III) and (IV) we have, $n^2 > nlg(n) \geq lg(n!),\ \forall n > 1$.
Therefore for $c = \frac{1}{2}$ and $n_0 = 1$, $lg(n) \geq clglg(n!),\ \forall n > n_0$.
Therefore $lg(n) \in O(lglg(n!)) \wedge lg(n) \in \Omega(lglg(n!))$.
Therefore $lg(n) \in \theta(lglg(n!))$.

□

Question 3. I assert that the claim is false, and will provide a counter example such that $f \in O(g) \wedge h(f) \notin O(h(g))$.

Proof. Let

$$f(n) = n^2,$$
$$g(n) = n^3,$$
$$h(n) = \frac{1}{n}$$

Lemma I. $f \in O(g)$, using Theorem 0.2

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0 (\geq 0) && \text{(Theorem 0.2)}
\end{aligned}$$

Given the defintion of f and h we can see that $h(f(n)) = \dfrac{1}{n^2}$.

Given the defintion of g and h we can see that $h(g(n)) = \dfrac{1}{n^3}$.

All that remains is to prove that $\frac{1}{n^2} \notin O(\frac{1}{n^3})$. Using Theorem 0.2,

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^3}}$$
$$= \lim_{n\to\infty} \frac{n^3}{n^2}$$
$$= \lim_{n\to\infty} n$$
$$= \infty (\notin \mathbb{R}^+)$$

Hence combining Lemma I with the above proof we have created a counter example such that,

$$f \in O(g) \wedge h(f) \notin O(h(g))$$

□

Question 4. I assert that the following function ordering respects the relationship $g1 = o(g2)$, $g2 = o(g3)$, ..., $g7 = o(g8)$.

$$10^{100},\ weirdsum,\ lg(n),\ 10^{lglg(n)},\ 2^{\sqrt{2lg(n)}},\ 4^{lgn},\ n^{lgn},\ 2^n$$

A series of proofs follow to confirm this ordering,

Proof. $n^{lgn} = o(2^n)$

Given that $n^k = o(2^n), \forall k > 0$ was proven on Page 48 of Chapter 0, and $lgn > 0, \forall n > 1$ we have $n^{lgn} = o(2^n)$. □

Proof. $4^{lgn} = o(n^{lgn})$

It can been seen that $4^{lgn} = n^{lg(4)} = n^2$. Using Theorem 0.2,

$$\lim_{n\to\infty} \frac{n^2}{n^{lgn}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^{lgn}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{lgn-2}}$$

$$=0$$

Therefore $4^{lgn} = o(n^{lgn})$. $\square$

Proof. $2^{\sqrt{2lgn}} \in o(4^{lgn})$

It can be seen that $4^{lgn} = (2^2)^{lgn} = 2^{2lgn}$. By letting x represent $2lgn$ and applying Theorem 0.2 we have,

$$\lim_{n\to\infty} \frac{2^{\sqrt{x}}}{2^x}$$

$$= \lim_{n\to\infty} \frac{\frac{2^{\sqrt{x}}}{2^{\sqrt{x}}}}{\frac{2^x}{2^{\sqrt{x}}}}$$

$$= \lim_{n\to\infty} \frac{1}{2^{x-\sqrt{x}}}$$

$$= 0$$

Therefore $2^{\sqrt{2lgn}} = o(4^{lgn})$. $\square$

Proof. $lg(n) = o(10^{lglg(n)})$

It can be seen that $10^{lglg(n)} = lg^{lg10}n$, where $lg10 > 1 = 1 + \epsilon$ for some $\epsilon > 2$. Using Theorem 0.2 we have,

$$\lim_{n\to\infty} \frac{lg(n)}{lg^{1+\epsilon}n}$$

$$= \lim_{n\to\infty} \frac{\frac{lg(n)}{lg(n)}}{\frac{lg^{1+\epsilon}n}{lg(n)}}$$

$$= \lim_{n\to\infty} \frac{1}{lg^{\epsilon}n}$$

$$=0$$

Therefore $lg(n) = o(10^{lglg(n)})$. $\square$

Question 5 (a). $T(n) = 9T(\frac{n}{3}) + n^2lg(n) + 2n$

Proof. Using the general formula for recurrences we note that,

$$a = 9, b = 3, f(n) = n^2lg(n) + 2n$$

Lemma I. $f(n) \in \theta(n^2lg(n))$, using Theorem 0.2

$$\lim_{n\to\infty} \frac{n^2lg(n) + 2n}{n^2lg(n)}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2lg(n)+2n}{n^2}}{\frac{n^2lg(n)}{n^2}}$$

$$= \lim_{n\to\infty} \frac{lg(n) + \frac{2}{n}}{lg(n)}$$

$$= \lim_{n\to\infty} \frac{lg(n)}{lg(n)}$$

$$= 1 (> 0)$$

By Lemma I we have $f(n) \in \theta(n^2lg(n)) = \theta(n^{log_b a} log^k n)$ where $k = 1(\geq 0)$. Therefore using Case 2 of the general recurrence forumla we have,

$$T(n) \in \theta(n^2lg^2n)$$

□

Question 5 (b). $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

Proof. Using the general formula for recurrences we note that,

$$a = 3, b = 3, f(n) = \sqrt{n}$$

Lemma 1. $f(n) \in O(n^{log_3 2})$

$$\frac{1}{2} \leq log_3 2$$
$$\sqrt{n} \leq n^{log_3 2} \qquad (n \geq 0)$$

Therefore for c = 1, and $n_0 = 0$ we have,

$$\sqrt{n} <= cn^{log_3 2}, \forall n_0 > 0$$

Hence $f(n) \in O(n^{log_3 2})$.

By Lemma I we have $f(n) \in O(n^{log_b a - \epsilon})$ where $\epsilon = 1 (> 0)$. Therefore using Case 1 of the general recurrence forumla we have,

$$T(n) \in \theta(n^{log_3 3}) = \theta(n)$$

$\square$

Question 5 (c). $T(n) = 8T(\frac{n}{4}) + n^2 lg^2 n$

Proof. Using the general formula for recurrences we note that,

$$a = 8, b = 4, f(n) = n^2 lg^2 n$$

Lemma I. $f(n) \in \Omega(n^{log_4 9})$

$$\begin{aligned} log_4 9 &\leq 2 \\ n^{log_4 9} &\leq n^2 \\ &\leq n^2 lg^2 n && (n \geq 2) \end{aligned}$$

Therefore for $c = 1$ and $n_0 = 2$ we have,

$$n^2 lg^2 n \geq c n^{log_4 9}$$

Hence $f(n) \in \Omega(n^{log_4 9})$.

By Lemma I we have $f(n) \in \Omega(n^{log_b a + \epsilon})$ where $\epsilon = 1$. Moreover, for sufficiently large n,

$$\begin{aligned} af(\frac{n}{b}) &= 8(\frac{n}{4})^2 lg^2 \frac{n}{4} \\ &= 8(\frac{n^2}{16}) lg^2 \frac{n}{4} \\ &= \frac{1}{2}(n^2 lg^2 \frac{n}{4}) \\ &= \frac{1}{2}\big(n^2 (lgn - 2)^2\big) \\ &= \frac{1}{2}\big(n^2 (lg^2 n - 4lgn + 4)\big) \\ &= \frac{1}{2} n^2 log^2 n - 2n^2 lgn + 2n^2 \\ &\leq \frac{1}{2} n^2 log^2 n && \text{when } n \geq 2 \\ &= cn^2 log^2 n && \text{where } 0 < c = \frac{1}{2} < 1 \end{aligned}$$

By Case 3 of the general recurrence forumla we thus have $T(n) \in \theta(n^2 lg^2 n)$. $\square$