Assignment 4

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Question 1. *Proof.* First we show that $\Pi \in \mathbf{NP}$

Let $\pi \in \Pi$.

As every Π logical expression is a CNF logical expression with exactly three literals, $\pi \in 3SAT$.

Since 3SAT is **NP**-complete, there exists a polynomial time nondeterministic algorithm N solving 3SAT.

Now,

$$\begin{aligned} &\pi \text{ is a yes-instance of } \Pi \\ \iff &\pi \text{ is a yes-instance of } 3SAT \\ \iff &\text{algorithm } N \text{ outputs a yes on input } \pi \end{aligned}$$

Therefore $\Pi \in \mathbf{NP}$.

Next, we prove that $3SAT \alpha \Pi$.

Let $C = c_1 \wedge c_2 \wedge ... \wedge c_m$ be a CNF logical expression in which each c_i contains exactly three literals, i.e. C is a problem instance of 3SAT. Let $U = \{u_1, u_2, ..., u_n\}$ be the set of variables in C.

For each c_i, $1 \leq i \leq m$ we shall construct a Π logical expression $c'_i \in \{x_i, y_i, z_i, a_i, b_i, c_i, d_i\}$ such that the Π CNF expression $C' = c'_1 \wedge c'_2 \wedge ... \wedge c'_m$ is satisfiable with exactly 1 literal per clause being assigned a true value if and only if C is satisfiable.

Let $c_j = x_j \vee y_j \vee z_j$. The corresponding c'_j is defined as

$$(\neg x_j \vee a_j \vee b_j) \wedge (y_j \vee b_j \vee c_j) \wedge (\neg z_j \wedge c_j \wedge d_j)$$

where a_j, b_j, c_j, d_j are newly created literals. Let $U' = U \cup (\bigcup_{j=1}^{m} \{a_j, b_j, c_j, d_j\})$.

By construction each clause of C' contains exactly three literals.

It is also easily verifiable that $|c'_j| = 3|c_j|$ since each clause in C produces exactly 3 clauses in C'. Additionally, 4 new variables are introduced into U' for each clause in C, i.e. $|U'| = 4|U|$.

Thus the transformation takes a total of $O(3m+4m) = O(m) = O(|C|)$ operations. I.e. the transformation can be done in polynomial time.

Next we shall show that,

C' is satisfiable if and only if C is is satisfiable

Let $t : U \to \{true, false\}$ be a truth assignment satisfying C.
As $C = c_1 \wedge c_2 \wedge ... \wedge c_2$ is true under t, at least one literal in each c_j must evaluate to true. 7 cases arise: (note that unassigned variables are assumed to be false)

1. $t(x_i) = true$.
   In this case we assign $a_i = true$ and $b_i = true$. We thus have a single literal in each clause of c'_j evaluated to true, and the expression is satisifed.
2. $t(y_i) = true$.

□