Assignment 1

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Question 1 (i). We want to prove that when Algorithm Summation terminates its execution,

$$y = \sum_{j=0}^{n} a_j x^j$$

Proof. We shall apply induction to prove that after the mth iteration of the for loop the following invariant holds true:

$$y = \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}$$

(Induction Basis) First, we note that y is initialized to 0 in Line 1. When $m = 1$ and $i = n$ in Line 3,

$$\begin{aligned}
y &= a_n + x * y \\
&= a_n + x * 0 \\
&= a_n \\
&= a_n x^0 \\
&= \sum_{j=0}^{0} a_{n-j} x^{n-i-j}
\end{aligned}$$

(Induction Hypothesis) Suppose for iteration $m - 1 < n$, $y = \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j-1}$
(Induction Step) When Line 3 is executed for the mth time,

$$
\begin{aligned}
y &= a_{n-i} + x \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j-1} \\
&= a_{n-i} + \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j} \\
&= a_{n-i} x^0 + \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j} \\
& && \text{(Induction Hypothesis)} \\
&= \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}
\end{aligned}
$$

Therefore we can conclude that the invariant $y = \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}$ holds $\forall m > 0$.

On the n+1th iteration (where $i = 0$) we then have,

$$
\begin{aligned}
y &= \sum_{j=0}^{n} a_{n-j} x^{n-j} \\
&= a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0 x^0 \\
&= \sum_{j=0}^{n} a_j x^j && \text{(By reversing the order of the terms)}
\end{aligned}
$$

□

Question 1 (ii).
Key Operation: Multiplication of real numbers.
Input size: n (The size of the input array)

The Summation algorithm must perform the key operation on each of the n elements of the input array.

1. Worst-case Time Complexity: $T(n) = n$

2. Average-case Time Complexity: $T_{ave}(n) = n$

Question 2. $lgn \in \theta(lglg(n!))$

To prove that $lgn \in \theta(lglg(n!))$ we shall prove that $lgn \in O(lglg(n!)) \wedge lgn \in \Omega(lglg(n!))$.

Proof. First we shall prove that $lgn \in O(lglg(n!))$, that is we shall determine a $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} lgn &\leq clglg(n!) && \forall n > n_0 \\ lgn &\leq lglg^c(n!) \\ 2^{lgn} &\leq 2^{lglg^c(n!)} \\ n &\leq lg^c n! \end{aligned}$$

By letting c = 1, we obtain

$$\begin{aligned} n &\leq lg(n!) \\ 2^n &\leq n! \end{aligned}$$

Lemma 1. $2^n < n!, \forall n \geq 4$
We shall prove this by induction on n.
(Induction Basis) When n = 4, $2^4 < 4! = 16 < 24$.
(Induction Hypothesis) Assume $2^k < k!, k \geq 4$.
(Induction Step) We then have,

$$\begin{aligned} 2^k(k+1) &< k!(k+1) \\ &= (k+1)! && (I) \end{aligned}$$

Also since $k \geq 4$,

$$\begin{aligned} 2 &< k+1 \\ 2^{k+1} &< (k+1)2^k && (II) \end{aligned}$$

By transitivity of (I) and (II) we obtain $2^{k+1} < (k+1)!$.
Therefore by Induction $2^n < n!, \forall n \geq 4$.

Therefore for $c = 1$, and $n_0 = 4$ we have $lg(n) \leq clglg(n!),\ \forall n > n_0$.
Therefore $lg(n) \in O(lglg(n!))$.

Now we shall prove that $lg(n) \in \Omega(lglg(n!))$, that is we shall determine a $c \in R^+$ and $n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} lg(n) &\geq clglg(n!) && \forall n > n_0 \\ lg(n) &\geq lglg^c(n!) \\ 2^{lgn} &\geq 2^{lglg^c(n!)} \\ n &\geq lg^c n! \end{aligned}$$

By letting $c = \frac{1}{2}$ we obtain,

$$n \geq \sqrt{lg(n!)}$$
$$n^2 \geq lg(n!)$$

It can be seen that $lg(n!) \leq nlg(n)$ (III) via,

$$\begin{aligned} lg(n!) &= lg(n * (n-1) * (n-2) * ... * 1) && \text{(Definition of } n!) \\ &= lg(n) + lg(n-1) + ... + lg(1) && (lg(mn) = lg(m) + lg(n)) \\ &\leq lg(n) + lg(n) + ... + lg(n) \\ &= nlgn \end{aligned}$$

Additionally $\forall n > 1$,

$$\begin{aligned} 2^n &> n \\ n &> lg(n) \\ n^2 &> nlg(n) && (IV) \end{aligned}$$

By combining (III) and (IV) we have, $n^2 > nlg(n) \geq lg(n!),\ \forall n > 1$.
Therefore for $c = \frac{1}{2}$ and $n_0 = 1$, $lg(n) \geq clglg(n!),\ \forall n > n_0$.
Therefore $lg(n) \in O(lglg(n!)) \wedge lg(n) \in \Omega(lglg(n!))$.
Therefore $lg(n) \in \theta(lglg(n!))$.

□

Question 3. I assert that the claim is false, and will provide a counter example such that $f \in O(g) \wedge h(f) \notin O(h(g))$.

Proof. Let

$$f(n) = n^2,$$
$$g(n) = n^3,$$
$$h(n) = \frac{1}{n}$$

Lemma I. $f \in O(g)$, using Theorem 0.2

$$\begin{aligned} &\lim_{n\to\infty} \frac{n^2}{n^3} \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0 (\geq 0) && \text{(Theorem 0.2)} \end{aligned}$$

Therefore by Theorem 0.2 we have $f \in O(g)$.

Given the definition of f and h we can see that $h(f(n)) = \dfrac{1}{n^2}$.
Given the definition of g and h we can see that $h(g(n)) = \dfrac{1}{n^3}$.
All that remains is to prove that $\frac{1}{n^2} \notin O(\frac{1}{n^3})$. Using Theorem 0.2,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^3}} \\ &= \lim_{n\to\infty} \frac{n^3}{n^2} \\ &= \lim_{n\to\infty} n \\ &= \infty (\notin \mathbb{R}^+) \end{aligned}$$

Hence combining Lemma I with the above proof we have created a counter example such that,

$$f \in O(g) \wedge h(f) \notin O(h(g))$$

Therefore the claim is false. □

Question 4. I assert that the following function ordering respects the relationship $g1 = o(g2)$, $g2 = o(g3)$, ..., $g7 = o(g8)$. Functions which are asymptotically equivalent are contained within { }.

$$10^{100},\ \{\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1},\quad lg(n)\},\ 10^{lglg(n)},\ 2^{\sqrt{2lg(n)}},\ 4^{lgn},\ n^{lgn},\ 2^n$$

A series of proofs follow to confirm this ordering,

Proof. $n^{lgn} = o(2^n)$

For any $c \in R^+$ define $n_0 = \begin{cases} 17 & if\ c \geq 1 \\ \lceil \frac{16}{c} \rceil & if\ c < 1 \end{cases}$

(i) $n^{lgn} < 2^n \Rightarrow lg^2 n < n$, which holds $\forall n > 16$.
If $c \geq 1$ then $c2^n > 2^n > n^{lgn},\ \forall n \geq 17 = n_0$.

(ii) Left as an excercise for the reader. □

Proof. $4^{lgn} = o(n^{lgn})$

It can been seen that $4^{lgn} = n^{lg(4)} = n^2$. Using Theorem 0.2,

$$\lim_{n\to\infty} \frac{n^2}{n^{lgn}}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^{lgn}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{lgn-2}}$$

$$=0$$

Therefore $4^{lgn} = o(n^{lgn})$. □

Proof. $2^{\sqrt{2lgn}} \in o(4^{lgn})$

It can be seen that $4^{lgn} = (2^2)^{lgn} = 2^{2lgn}$. By letting x represent $2lgn$ and applying Theorem 0.2 we have,

$$\lim_{n\to\infty} \frac{2^{\sqrt{x}}}{2^x}$$

$$= \lim_{n\to\infty} \frac{\frac{2^{\sqrt{x}}}{2^{\sqrt{x}}}}{\frac{2^x}{2^{\sqrt{x}}}}$$

$$= \lim_{n\to\infty} \frac{1}{2^{x-\sqrt{x}}}$$

$$= 0$$

Therefore $2^{\sqrt{2lgn}} = o(4^{lgn})$. □

Proof. $10^{lglg(n)} = o(2^{\sqrt{2lgn}})$ Left as an excercise for the reader. □

Proof. $lg(n) = o(10^{lglg(n)})$

It can be seen that $10^{lglg(n)} = lg^{lg10}n$, where $lg10 > 1 = 1 + \epsilon$ for some $\epsilon > 2$. Using

Theorem 0.2 we have,

$$\lim_{n\to\infty}\frac{lg(n)}{lg^{1+\epsilon}n}$$

$$=\lim_{n\to\infty}\frac{\frac{lg(n)}{lg(n)}}{\frac{lg^{1+\epsilon}n}{lg(n)}}$$

$$=\lim_{n\to\infty}\frac{1}{lg^{\epsilon}n}$$

$$=0$$

Therefore $lg(n)=o(10^{lglg(n)})$. □

Proof. $\sum_{k=1}^{n}\frac{k^2}{3k^3+2k^2+1}=\theta(lgn)$

We shall first establish that,

$$\sum_{k=1}^{n}\frac{k^2+2}{3k^3+2k^2+1}=\theta(\sum_{k=1}^{n}\frac{1}{k})$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}=\infty^+$ and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\frac{n^2+2}{3n^3+3n^2+1}}{\frac{1}{n}}\\
=&\lim_{n\to\infty}\frac{n^3+2n}{3n^3+3n^2+1}\\
=&\lim_{n\to\infty}\frac{\frac{n^3}{n^3}+\frac{2n}{n^3}}{\frac{3n^3}{n^3}+\frac{3n^2}{n^3}+\frac{1}{n^3}}\\
=&\frac{1}{3}
\end{aligned}$$

By the The Stolz-Cesaro Theorem,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\frac{n^2+2}{3n^3+3n^2+1}}{\frac{1}{n}}\\
=&\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{k^2+2}{3k^3+2k^2+1}}{\sum_{k=1}^{n}\frac{1}{k}}\\
=&\frac{1}{3} && (>0)
\end{aligned}$$

Therefore by Theorem 0.2 $\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} = \theta(\sum_{k=1}^{n} \frac{1}{k})$ (I).

Additionally we have $\sum_{k=1}^{n} \frac{1}{k} = \theta(lgn)$ via,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{lgn} \\
=&\lim_{n\to\infty} \frac{ln(n) + \gamma + \frac{1}{2n} + o(\frac{1}{n})}{lgn} && \text{(Hint)} \\
=&\lim_{n\to\infty} \frac{ln(n) + \gamma}{lgn} \\
=&\lim_{n\to\infty} \frac{ln(n) + \gamma}{\frac{ln(n)}{ln(2)}} \\
=&\lim_{n\to\infty} \frac{\frac{ln(n)}{ln(n)} + \frac{\gamma}{ln(n)}}{\frac{ln(n)}{ln(n)ln(2)}} \\
=&ln(2) && (> 0)
\end{aligned}$$

Therefore by Theorem 0.2 $\sum_{k=1}^{n} \frac{1}{k} = \theta(lgn)$ (II).

Given (I) and (II) and the transitivity of θ we obtain $\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} = \theta(lgn)$. □

Proof. $10^{100} = o(lgn)$

Since it was shown that lgn and $\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1}$ are asymptotically equivalent, it is sufficient to show that 10^{100} is little-o of either one to confirm the ordering. To this end we apply Theorem 0.2,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{10^{100}}{lgn} \\
=&\lim_{n\to\infty} \frac{0}{\frac{lg(e)}{n}} && \text{(L'Hopital's rule)} \\
=&0
\end{aligned}$$

Therefore by Theorem 0.2 we have $10^{100} = o(lgn)$. □

Question 5 (a). $T(n) = 9T(\frac{n}{3}) + n^2 lg(n) + 2n$

Proof. Using the general formula for recurrences we note that,

$$a = 9, b = 3, f(n) = n^2 lg(n) + 2n$$

Lemma I. $f(n) \in \theta(n^2 lg(n))$, using Theorem 0.2

$$\lim_{n\to\infty} \frac{n^2 lg(n) + 2n}{n^2 lg(n)}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2 lg(n) + 2n}{n^2}}{\frac{n^2 lg(n)}{n^2}}$$

$$= \lim_{n\to\infty} \frac{lg(n) + \frac{2}{n}}{lg(n)}$$

$$= \lim_{n\to\infty} \frac{lg(n)}{lg(n)}$$

$$= 1(> 0)$$

By Lemma I we have $f(n) \in \theta(n^2 lg(n)) = \theta(n^{log_b a} log^k n)$ where $k = 1(\geq 0)$. Therefore using Case 2 of the general recurrence formula we have,

$$T(n) \in \theta(n^2 lg^2 n)$$

$\square$

Question 5 (b). $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

Proof. Using the general formula for recurrences we note that,

$$a = 3, b = 3, f(n) = \sqrt{n}$$

Therefore, $f(n) = \sqrt{n} = O(n^{1-\frac{1}{2}}) = O(n^{log_b a - \epsilon})$ where $\epsilon = \frac{1}{2} > 0$. Therefore using Case 1 of the general recurrence theorem we have,

$$T(n) \in \theta(n^{log_3 3}) = \theta(n)$$

$\square$

Question 5 (c). $T(n) = 8T(\frac{n}{4}) + n^2 lg^2 n$

Proof. Using the general formula for recurrences we note that,

$$a = 8, b = 4, f(n) = n^2 lg^2 n$$

Lemma I. $f(n) \in \Omega(n^2)$ using Theorem 0.2,

$$\lim_{x\to\infty} \frac{n^2}{n^2 lg^2 n}$$

$$= \lim_{x\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2 lg^2 n}{n^2}}$$

$$= \lim_{x\to\infty} \frac{1}{lg^2 n}$$

$$= 0$$

Hence $f(n) \in \Omega(n^2)$.

By Lemma I we have $f(n) \in \Omega(n^{log_b a+\epsilon})$ where $\epsilon = \frac{1}{2} > 0$. Moreover, for sufficiently large n,

$$\begin{aligned}
af(\frac{n}{b}) &= 8(\frac{n}{4})^2 lg^2 \frac{n}{4} \\
&= 8(\frac{n^2}{16}) lg^2 \frac{n}{4} \\
&= \frac{1}{2}(n^2 lg^2 \frac{n}{4}) \\
&= \frac{1}{2}\big(n^2(lgn-2)^2\big) \\
&= \frac{1}{2}\big(n^2(lg^2 n - 4lgn + 4)\big) \\
&= \frac{1}{2}n^2 log^2 n - 2n^2 lgn + 2n^2 \\
&\leq \frac{1}{2}n^2 log^2 n && \text{when } n \geq 2 \\
&= cn^2 log^2 n && \text{where } 0 < c = \frac{1}{2} < 1
\end{aligned}$$

By Case 3 of the general recurrence theorem we thus have $T(n) \in \theta(n^2 lg^2 n)$. $\square$

Question 6. $T(n) = T(\frac{n}{2} + 7)$

Proof. We guess that $T(n) = O(n^2)$.
(Induction Hypothesis) We first assume that $T(k) \leq ck^2,\ \forall k < n$ (I).

(Induction Step) Then, when $n > 14$

$$
\begin{aligned}
& n > 14 \Rightarrow 2n > 14 + n \Rightarrow \frac{n}{2} + 7 < n \\
& \Rightarrow T(\frac{n}{2} + 7) \leq c(\frac{n}{2} + 7)^2 \qquad \text{(by I)}
\end{aligned}
$$

Therefore, for $n > 14$,

$$
\begin{aligned}
T(n) &= T(\frac{n}{2} + 7) + n^2 \\
&\leq c(\frac{n}{2} + 7)^2 + n^2 \\
&\leq \frac{1}{4}cn^2 + 7cn + 49c + n^2
\end{aligned}
$$

For $c \geq 4$,

$$
\begin{aligned}
& c \geq 4 \\
& \frac{c}{4} \geq 1 \\
& \frac{c}{4} + \frac{c}{4} \geq 1 + \frac{c}{4} \\
& \frac{c}{2} \geq \frac{c}{4} + 1 \\
& \frac{1}{2}cn^2 \geq \frac{1}{4}cn^2 + n^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T(n) &\leq \frac{1}{4}cn^2 + 7cn + 49c + n^2 \\
&\leq \frac{1}{2}cn^2 + 7cn + 49c \qquad (c \geq 4) \\
&\leq cn^2 \qquad (n \geq 20)
\end{aligned}
$$

Hence, $T(n) \leq cn, \ \forall n \geq 20$ and any $c \geq 4$.

(Inductive Basis) Let $c' = max(\{T(n)/n^2 \mid 17 \leq n \leq 19\} \bigcup \{4\})$.
Then $T(n) \leq c'n^2, \ \forall n, \ 17 \leq n \leq 19$.
Hence $T(n) \leq c'n^2, \ \forall n \geq 17$.
i.e. $T(n) = O(n^2)$.

$\square$