CS60-454/554
Design and Analysis of Algorithms
Winter 2017

Assignment 1

Due Date: February 7 (before lecture)

1. The following algorithm takes $x, a_i, 0 \leq i \leq n$, as input and returns $\sum_{j=0}^{n} a_j x^j$ as output.

   Algorithm `Summation`

   Input: $x, a_i, 0 \leq i \leq n$;

   Output: y;

   begin

   $y := 0$;

   for $i := n$ **step** -1 **downto** 0 **do**

   $\quad y := a_i + x * y$;

   end.

   (i) Prove the correctness of the algorithm.

   Lemma 1: At the end of the kth iteration of the **for** loop, $y = \sum_{j=1}^{k} x^{j-1} a_{n-(k-j)}$.

   Proof: (By induction on k)

   Induction basis:

   Consider the first iteration (i.e. when $k = 1$).

   Since y was initialized to 0 before execution entering the **for** loop and $i = n$,

   $$\begin{aligned} y = a_n + x * 0 = a_n = x^0 a_{n-0} &= x^{1-1} a_{n-(1-1)} \\ &= \textstyle\sum_{j=1}^{1} x^{j-1} a_{n-(1-j)} \\ &= \textstyle\sum_{j=1}^{k} x^{j-1} a_{n-(k-j)} \qquad (\because k = 1) \end{aligned}$$

   Induction hypothesis:

   Suppose at the end of the $(k-1)$th iteration, $y = \sum_{j=1}^{k-1} x^{j-1} a_{n-((k-1)-j)}$

   Induction step:

   During the kth iteration, $i = n - (k-1)$.

   Therefore, $y := a_i + x * y$

   $$\begin{aligned} \Rightarrow y &= a_{n-(k-1)} + x(\textstyle\sum_{j=1}^{k-1} x^{j-1} a_{n-((k-1)-j)}) \qquad \text{(by the induction hypothesis)} \\ &= a_{n-(k-1)} + \textstyle\sum_{j=1}^{k-1} x^{j} a_{n-((k-1)-j)} \\ &= a_{n-(k-1)} + \textstyle\sum_{1 \leq j \leq k-1} x^{j} a_{n-((k-1)-j)} \qquad \text{(equivalent notation)} \\ &= a_{n-(k-1)} + \textstyle\sum_{1+1 \leq j+1 \leq (k-1)+1} x^{(j+1)-1} a_{n-((k-1)-((j+1)-1))} \\ &= a_{n-(k-1)} + \textstyle\sum_{2 \leq j \leq k} x^{j-1} a_{n-(k-j)} \qquad \text{(Replace } j+1 \text{ with } j) \\ &= x^{1-1} a_{n-(k-1)} + \textstyle\sum_{2 \leq j \leq k} x^{j-1} a_{n-(k-j)} \\ &= \textstyle\sum_{1 \leq j \leq k} x^{j-1} a_{n-(k-j)} \end{aligned}$$

$= \sum_{j=1}^{k} x^{j-1} a_{n-(k-j)}$ $\square$

The **for** loop terminates its execution at the end of the $(n+1)$th iteration.

By Lemma 1, $y = \sum_{j=1}^{n+1} x^{j-1} a_{n-((n+1)-j)}$

$= \sum_{1 \leq j \leq n+1} x^{j-1} a_{j-1}$ (equivalent notation)

$= \sum_{0 \leq j-1 \leq n} x^{j-1} a_{j-1}$

$= \sum_{0 \leq j \leq n} x^{j} a_{j}$ (Replace $j-1$ with j)

$= \sum_{j=0}^{n} a_j x^j.$ (equivalent notation)

Hence, **Algorithm** `Summation` correctly returns $\sum_{j=0}^{n} a_j x^j$ as its output. ■

(*ii*) Analyze its time complexity.

The first assignment statement takes $O(1)$ time.

The **for** statement take $O(1)$ time.

Since $+$, $*$ and $:=$ each takes $O(1)$ time, the body of the **for** loop takes $O(1)$ time.

Since the **for** loop iterates $n+1$ times, therefore the **for** loop takes $(O(1)+O(1))*(n+1) = O(1)*(n+1) = O(n)$ time.

Algorithm `Summation` thus takes $O(1) + O(n) = O(n)$ time. ■

2. Prove that: $\lg n \in \Theta(\lg\lg(n!))$.

Solution:

$$
\begin{aligned}
&\forall n \geq 4,\ 2^n \leq n! \leq n^n \\
&\quad \Rightarrow \lg(2^n) \leq \lg(n!) \leq \lg(n^n) \\
&\quad \Rightarrow \lg\lg(2^n) \leq \lg\lg(n!) \leq \lg\lg(n^n) \\
&\quad \Rightarrow \lg n \leq \lg\lg(n!) \leq \lg(n \lg n) \qquad (\because \lg(2^n) = n; \lg(n^n) = n\lg n) \\
&\quad \Rightarrow \lg n \leq \lg\lg(n!) \leq \lg n + \lg\lg n \\
&\quad \Rightarrow \lg n \leq \lg\lg(n!) \leq \lg n + \lg n \qquad (n \geq 4) \\
&\quad \Rightarrow \lg n \leq \lg\lg(n!) \leq 2\lg n
\end{aligned}
$$

By letter $c_1 = 1, c_2 = 2, n_0 = 4$, we have:

$\exists c_1, c_2 > 0, n_0 \in \mathbf{N}, c_1 \lg n \leq \lg(\lg(n!) \leq c_2 \lg n, \forall n \geq n_0$

$\Rightarrow \lg\lg(n!) \in \Theta(\lg n)$

$\Rightarrow \lg n \in \Theta(\lg\lg(n!))$. (Theorem 0.5(b)) ■

3. Prove or disprove: Let $f, g : \mathbf{N} \to \mathbf{R}^+ \cup \{0\}$ and $h : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$.

If $f(n) \in O(g(n))$, then $h(f(n)) \in O(h(g(n)))$.

Solution: false!

Disprove with a counterexample: let $f(n) = 2n, g(n) = n, h(n) = 2^n$.

Then $2n \leq 3n, \forall n \geq 1$

$\Rightarrow f(n) \leq 3g(n), \forall n \geq 1$

$\Rightarrow \exists c \in \mathbf{R}^+, \exists n_0 \in \mathbf{N}, f(n) \leq cg(n), \forall n \geq n_0$ (note: $c = 3, n_0 = 1$)

$\Rightarrow f(n) \in O(g(n))$.

Moreover $h(f(n)) = 2^{2n} = (2^2)^n = 4^n$ and $h(g(n)) = 2^n$.

Since $\lim_{n\to\infty} \frac{2^n}{4^n} = \lim_{n\to\infty} (\frac{1}{2})^n = 0$,

by Theorem 0.2(d), $2^n \in o(4^n)$

$$\Rightarrow 2^n \in O(4^n) \wedge 2^n \notin \Theta(4^n) \quad \text{(Theorem 0.1(c))}$$
$$\Rightarrow 2^n \in O(4^n) \wedge 2^n \notin O(4^n) \cap \Omega(4^n) \quad \text{(Theorem 0.1(b))}$$
$$\Rightarrow 2^n \in O(4^n) \text{ and } \sim (2^n \in O(4^n) \wedge 2^n \in \Omega(4^n)) \quad \text{(Definition of } \cap\text{)}$$
$$\Rightarrow 2^n \in O(4^n) \text{ and } 2^n \notin O(4^n) \vee 2^n \notin \Omega(4^n) \quad \text{(E16, 60-231 Courseware)}$$
$$\Rightarrow 2^n \in O(4^n) \text{ and } 2^n \in O(4^n) \Rightarrow 2^n \notin \Omega(4^n) \quad \text{(E18, 60-231 Courseware)}$$
$$\Rightarrow 2^n \notin \Omega(4^n) \quad \text{(I3, 60-231 Courseware)}$$
$$\Rightarrow 4^n \notin O(2^n) \quad \text{(Theorem 0.1(a))}$$
$$\Rightarrow h(f(n)) \notin O(h(g(n))). \qquad \blacksquare$$

4. Rank the following functions by order of growth; that is, find an ordering $g_1, g_2, \ldots, g_8$ of the functions satisfying $g_1 = o(g_2), g_2 = o(g_3), \cdots, g_7 = o(g_8)$. Partition your lists into equivalences classes such that functions $f(n)$ and $g(n)$ are in the same class if and only if $f(n) = \Theta(g(n))$.

$$10^{100},\ \sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1},\ \lg n,\ 10^{\lg\lg n},\ 2^{\sqrt{2\lg n}},\ 4^{\lg n},\ n^{\lg n},\ 2^n$$

Solution:

The functions in ascending order of rate of growth is as follows:

$$10^{100},\ \lg n,\ \sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1},\ 10^{\lg\lg n},\ 2^{\sqrt{2\lg n}},\ 4^{\lg n},\ n^{\lg n},\ 2^n$$

(i) $\underline{10^{100} \in o(\lg n)}$:

$\lim_{n\to\infty} \frac{10^{100}}{\lg n} = 10^{100} \lim_{n\to\infty} \frac{1}{\lg n} = 10^{100} \cdot 0 = 0.$

By Theorem $0.2(d)$, $10^{100} \in o(\lg n)$. $\square$

(ii) $\underline{\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} \in \Theta(\lg n)}$:

$$\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} \geq \sum_{k=1}^{n} \frac{k^2}{3k^3+2k^3+k^3} = \sum_{k=1}^{n} \frac{k^2}{6k^3} = \frac{1}{6}\sum_{k=1}^{n} \frac{1}{k}$$
$$\Rightarrow \sum_{k=1}^{n} \frac{k^2}{3k^3+2k^3+k^3} \geq \frac{1}{6}(\ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n})) \qquad (\because \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n}))$$
$$\geq \frac{1}{6} \ln n$$
$$= \frac{1}{6}\frac{\lg n}{\lg e}$$
$$= \frac{1}{6\lg e} \lg n, \forall n \geq 1. \cdots \text{ (A)}$$

$$\sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} \leq \sum_{k=1}^{n} \frac{k^2+2k^2}{3k^3} = \sum_{k=1}^{n} \frac{3k^2}{3k^3} = \sum_{k=1}^{n} \frac{1}{k}$$

$\Rightarrow \sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} \leq \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n}). \cdots$ (I)

Since $\lim_{n\to\infty} \frac{\gamma+\frac{1}{2n}}{\ln n} = \lim_{n\to\infty} \frac{\gamma}{\ln n} + \lim_{n\to\infty} \frac{1}{2n\ln n} = 0 + 0 = 0$

$\Rightarrow \gamma + \frac{1}{2n} \in O(\ln n)$ (Theorem 0.2(a))

$\Rightarrow \exists c' > 0, \exists n' \in \mathbf{N}, \gamma + \frac{1}{2n} \leq c' \ln n, \forall n \geq n'$ (Definition of O) $\cdots$ (II)

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\ln n} = \lim_{n\to\infty} \frac{1}{n\ln n} = 0$

$\Rightarrow \frac{1}{n} \in O(\ln n)$ (Theorem 0.2(a)) $\cdots$ (III)

$\forall g(n), g(n) \in o(\frac{1}{n})$

$\Rightarrow g(n) \in O(\frac{1}{n}) \wedge g(n) \notin \Theta(\frac{1}{n})$ (Theorem 0.1(c))

$\Rightarrow g(n) \in O(\frac{1}{n})$ (I2; 60-231 Courseware) $\cdots$ (IV)

$\Rightarrow g(n) \in O(\frac{1}{n}) \wedge \frac{1}{n} \in O(\ln n)$ ((IV), (III), I6; 60-231 Courseware)

$\Rightarrow g(n) \in O(\ln n)$ (Theorem 0.3(c))

$\Rightarrow \exists c'' > 0, \exists n'' \in \mathbf{N}, g(n) \leq c'' \ln n, \forall n \geq n''$ (Definition of O) $\cdots$ (V)

Therefore, $\forall g(n) \in o(\frac{1}{n})$,

$$\begin{aligned} &\ln n + \gamma + \tfrac{1}{2n} + g(n) \\ &\leq \ln n + c' \ln n + c'' \ln n, \forall n \geq \max\{n', n''\} \quad \text{(by (II),(V))} \\ &= (1 + c' + c'') \ln n, \forall n \geq \max\{n', n''\} \end{aligned}$$

$\Rightarrow \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n}) \leq (1 + c' + c'') \ln n, \forall n \geq \max\{n', n''\} \cdots$ (VI)

Hence,
$$\begin{aligned} \sum_{k=1}^{n} \frac{k^2+2}{3k^3+2k^2+1} &\leq \ln n + \gamma + \frac{1}{2n} + o(\frac{1}{n}) \\ &\leq (1 + c' + c'') \ln n, \forall n \geq \max\{n', n''\} \quad \text{(by (VI))} \\ &= \frac{(1+c'+c'')}{\lg e} \lg n, \forall n \geq \max\{n', n''\} \cdots \text{(B)} \end{aligned}$$

From (A) and (B), we have:

$$\frac{1}{6\lg e} \lg n \leq \sum_{k=1}^{n} \frac{k^2}{3k^3+2k^3+k^3} \leq \frac{(1+c'+c'')}{\lg e} \lg n, \forall n \geq \max\{1, n', n''\}$$

$$\Rightarrow \exists c_1, c_2 > 0, \exists n_0 \in \mathbf{N}, c_1 \lg n \leq \sum_{k=1}^{n} \frac{k^2}{3k^3+2k^3+k^3} \leq c_2 \lg n, \forall n \geq n_0$$

$$\Rightarrow \sum_{k=1}^{n} \frac{k^2}{3k^3+2k^3+k^3} \in \Theta(\lg n) \quad \text{(Definition of } \Theta). \qquad \square$$

(iii) $\underline{\lg n \in o(10^{\lg\lg n})}$:

$10^{\lg\lg n} = (\lg n)^{\lg 10}$ $(a^{\log_b c} = c^{\log_b a})$

$= \lg^{3+\varepsilon} n$ ($\lg 10 = 3 + \varepsilon$, for some $\varepsilon, 0 < \varepsilon < 1$).

$\lim_{n\to\infty} \frac{\lg n}{10^{\lg\lg n}} = \lim_{n\to\infty} \frac{\lg n}{\lg^{3+\varepsilon} n} = \lim_{n\to\infty} \frac{1}{\lg^{2+\varepsilon} n} = 0$

$\Rightarrow \lg n \in o(10^{\lg\lg n})$. (Theorem 0.4($d$)) $\square$

(iv) $\underline{10^{\lg\lg n} \in o(2^{\sqrt{2\lg n}})}$:

$10^{\lg\lg n} = 2^{\lg(10^{\lg\lg n})} = 2^{(\lg 10)(\lg\lg n)}$

$$\begin{aligned} \lim_{n\to\infty} \frac{10^{\lg\lg n}}{2^{\sqrt{2\lg n}}} &= \lim_{n\to\infty} \frac{2^{(\lg 10)(\lg\lg n)}}{2^{\sqrt{2\lg n}}} \\ &= \lim_{n\to\infty} \frac{1}{2^{\sqrt{2\lg n} - (\lg 10)(\lg\lg n)}} \\ &= \lim_{n\to\infty} \frac{1}{2^{\sqrt{2\lg n}(1 - \frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})}}. \end{aligned}$$

Since $\frac{d}{dn}(\lg 10 \lg\lg n) = (\lg 10) \lg e \frac{1}{\lg n} \lg e \frac{1}{n} = (\lg 10) \lg^2 e \frac{1}{n\lg n}$

$\frac{d}{dn}(\sqrt{2\lg n}) = \frac{1}{2} \frac{1}{\sqrt{2\lg n}} 2 \lg e \frac{1}{n} = \frac{\lg e}{n\sqrt{2\lg n}}$

$\lim_{n\to\infty} \frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}} = \lim_{n\to\infty} \frac{(\lg 10)\lg^2 e \frac{1}{n\lg n}}{\frac{\lg e}{n\sqrt{2\lg n}}}$ (L'Hôpita's rule)

$= \lim_{n\to\infty} \frac{\sqrt{2}(\lg 10)\lg e}{\sqrt{\lg n}}$

$= \sqrt{2}(\lg 10)\lg e \lim_{n\to\infty} \frac{1}{\sqrt{\lg n}}$

$= 0. \cdots$ (I)

Therefore $\lim_{n\to\infty} \sqrt{2\lg n}(1 - \frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})$

$= (\lim_{n\to\infty} \sqrt{2\lg n})(1 - \lim_{n\to\infty} \frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})$ ($\lim_{n\to\infty} AB = \lim_{n\to\infty} A \cdot \lim_{n\to\infty} B$)

$= (\lim_{n\to\infty} \sqrt{2\lg n})(1-0)$ (by (I))

$= \lim_{n\to\infty} \sqrt{2\lg n}$

$= \infty \cdots$ (II)

$\Rightarrow \lim_{n\to\infty} 2^{\sqrt{2\lg n}(1-\frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})} = 2^{\lim_{n\to\infty}\sqrt{2\lg n}(1-\frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})}$ ($\lim_{n\to\infty} 2^{f(n)} = 2^{\lim_{n\to\infty} f(n)}$)

$= \infty.$ (by (II))

Hence, $\lim_{n\to\infty} \frac{10^{\lg\lg n}}{2^{\sqrt{2\lg n}}} = \lim_{n\to\infty} \frac{1}{2^{\sqrt{2\lg n}(1-\frac{(\lg 10)(\lg\lg n)}{\sqrt{2\lg n}})}} = \frac{1}{\infty} = 0.$

By Theorem 0.4(d), $10^{\lg\lg n} \in o(2^{\sqrt{2\lg n}})$. $\square$

(v) $\underline{2^{\sqrt{2\lg n}} \in o(4^{\lg n})}$:

Since $4^{\lg n} = (2^2)^{\lg n} = 2^{2\lg n}$,

$\lim_{n\to\infty} \frac{2^{\sqrt{2\lg n}}}{4^{\lg n}} = \lim_{n\to\infty} \frac{2^{\sqrt{2\lg n}}}{2^{2\lg n}} = \lim_{n\to\infty} \frac{1}{2^{2\lg n - \sqrt{2\lg n}}} = \lim_{n\to\infty} \frac{1}{2^{2\lg n(1-\frac{\sqrt{2\lg n}}{2\lg n})}}$

$= \lim_{n\to\infty} \frac{1}{2^{2\lg n(1-\frac{1}{\sqrt{2\lg n}})}}$

$\lim_{n\to\infty} 2\lg n(1 - \frac{1}{\sqrt{2\lg n}}) = (\lim_{n\to\infty} 2\lg n)(\lim_{n\to\infty}(1 - \frac{1}{\sqrt{2\lg n}}))$

$= (\lim_{n\to\infty} 2\lg n)(1 - \lim_{n\to\infty} \frac{1}{\sqrt{2\lg n}})$

$= (\lim_{n\to\infty} 2\lg n)(1-0)$

$= (\lim_{n\to\infty} 2\lg n)$

$= \infty.$

Hence, $\lim_{n\to\infty} \frac{2^{\sqrt{2\lg n}}}{4^{\lg n}} = \lim_{n\to\infty} \frac{1}{2^{2\lg n(1-\frac{1}{\sqrt{2\lg n}})}} = \frac{1}{\infty} = 0.$

By Theorem 0.4(d), $2^{\sqrt{2\lg n}} \in o(4^{\lg n})$. $\square$

(vi) $\underline{4^{\lg n} \in o(n^{\lg n})}$:

$\lim_{n\to\infty} \frac{4^{\lg n}}{n^{\lg n}} = \lim_{n\to\infty} \frac{n^{\lg 4}}{n^{\lg n}} = \lim_{n\to\infty} \frac{n^2}{n^{\lg n}} = \lim_{n\to\infty} \frac{1}{n^{\lg n - 2}} = 0.$

By Theorem 0.4(d), $4^{\lg n} \in o(n^{\lg n})$. $\square$

(vii) $\underline{n^{\lg n} \in o(2^n)}$:

$n^{\lg n} = 2^{\lg n^{\lg n}} = 2^{\lg n \lg n} = 2^{\lg^2 n}. \cdots$ (I)

$\lim_{n\to\infty} \frac{n^{\lg n}}{2^n} = \lim_{n\to\infty} \frac{2^{\lg^2 n}}{2^n}$ (by (I))

$= \lim_{n\to\infty} \frac{1}{2^{n-\lg^2 n}}$

$= \lim_{n\to\infty} \frac{1}{2^{n(1-\frac{\lg^2 n}{n})}} \cdots$ (II)

Since $\lim_{n\to\infty} n(1 - \frac{\lg^2 n}{n}) = (\lim_{n\to\infty} n)(\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{\lg^2 n}{n})$

$= (\lim_{n\to\infty} n)(1 - \lim_{n\to\infty} \frac{2\lg e \lg n}{n})$ (L'Hôpita's rule)

$$
\begin{aligned}
&= (\lim_{n\to\infty} n)(1 - \lim_{n\to\infty} \tfrac{2\lg^2 e}{n}) \quad \text{(L'Hôpita's rule)}\\
&= (\lim_{n\to\infty} n)(1-0)\\
&= \lim_{n\to\infty} n\\
&= \infty
\end{aligned}
$$

Hence, $\lim_{n\to\infty} \frac{n^{\lg n}}{2^n} = \lim_{n\to\infty} \frac{1}{2^{n(1-\frac{\lg^2 n}{n})}}$ (by (II))

$$
\begin{aligned}
&= \tfrac{1}{\infty}\\
&= 0
\end{aligned}
$$

$\Rightarrow n^{\lg n} \in o(2^n)$ (Theorem 0.4(d)) ■

5. Solve the following recurrences:

(a) $T(n) = 9T(\frac{n}{3}) + n^2 \lg n + 2n$

(b) $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

(c) $T(n) = 8T(\frac{n}{4}) + n^2 \lg^2 n$

Solution:

(a) $T(n) = 9T(\frac{n}{3}) + n^2 \lg n + 2n$

$a = 9, b = 3 \Rightarrow \log_b a = \log_3 9 = 2.$

Since $\lim_{n\to\infty} \frac{n^2 \lg n + 2n}{n^2 \lg n} = \lim_{n\to\infty} \frac{n^2 \lg n}{n^2 \lg n} + \lim_{n\to\infty} \frac{2n}{n^2 \lg n}$

$$
\begin{aligned}
&= \lim_{n\to\infty} 1 + 2\lim_{n\to\infty} \tfrac{1}{n \lg n}\\
&= 1 + 2\cdot 0\\
&= 1,
\end{aligned}
$$

$n^2 \lg n + 2n \in \Theta(n^2 \lg n)$ (Theorem 0.2(c))

$\Rightarrow f(n) = \Theta(n^2 \lg n) = \Theta(n^{\log_b a} \lg^k n)$, where $k = 1$.

By Case 2, $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n) = \Theta(n^2 \lg^2 n)$. □

(b) $T(n) = 3T(\frac{n}{3}) + \sqrt{n}$

$a = 3, b = 3 \Rightarrow \log_b a = \log_3 3 = 1.$

$\sqrt{n} = n^{\frac{1}{2}} = O(n^{\frac{1}{2}})$ (Theorem 0.3(a))

$$
\begin{aligned}
&= O(n^{1-\frac{1}{2}})\\
&= O(n^{\log_b a - \varepsilon}), \text{ where } \varepsilon = \tfrac{1}{2} > 0.
\end{aligned}
$$

By Case 1, $T(n) = \Theta(n^{\log_b a}) = \Theta(n^1) = \Theta(n)$. □

(c) $T(n) = 8T(\frac{n}{4}) + n^2 \lg^2 n$

$a = 8, b = 4 \Rightarrow \log_b a = \log_4 8.$

since $1 < \log_4 8 < 2$, $\log_4 8 + \varepsilon = 2$, for some $\varepsilon > 0$.

It follows that $\lim_{n\to\infty} \frac{n^2 \lg^2 n}{n^{\log_4 8}} = \lim_{n\to\infty} n^{\varepsilon} \lg^2 n = \infty$

$\Rightarrow n^2 \lg^2 n = \Omega(n^{\log_4 8})$. (Theorem 0.2(b))

Moreover, $af(\frac{n}{b}) = 8(\frac{n}{4})^2 \lg^2(\frac{n}{4}) = \frac{1}{2}n^2(\lg n - 2)^2$

$$
\begin{aligned}
&= \tfrac{1}{2}n^2(\lg^2 n - (4\lg n - 4)).\\
&\le \tfrac{1}{2}n^2 \lg^2 n, \text{ for } \underline{n \ge 2}
\end{aligned}
$$

$\Rightarrow af(\frac{n}{b}) \le cf(n)$, for some $c, 0 < c < 1$ and $n \ge 2$.

By Case 3, $T(n) = \Theta(f(n)) = \Theta(n^2 \lg^2 n)$. ■

6. Solve the recurrence: $T(n) = T(\frac{n}{2} + 7) + n^2$.

Solution: The given recurrence is similar to $T(n) = T(\frac{n}{2}) + n^2$.

Since $a = 1, b = 2$, and $\log_b a = \log_2 1 = 0$.

Then,

$$\begin{aligned} f(n) = n^2 &= \Omega(n^2) \quad \text{(Theorem 0.3}(d)) \\ &= \Omega(n^{0+2}) \\ &= \Omega(n^{\log_b a + \varepsilon}) \quad (\because \log_b a = 0 \text{ and } \varepsilon = 2 (> 0))) \end{aligned}$$

Moreover, $af(\frac{n}{b}) = (\frac{n}{2})^2 = \frac{1}{4}n^2 \leq cn^2 = cf(n)$, for some $0 < c < 1$ and $\forall n \geq 1$.

By Case 3, $T(n) = \Theta(f(n)) = \Theta(n^2)$.

To verify that $T(n) = O(n^2)$, we apply induction on n.

Induction hypothesis: Suppose $T(k) \leq ck^2, \forall k < n$, for some $c > 0$.

Induction Step: When $\underline{n > 14}$, $\frac{n}{2} > 7 \Rightarrow \frac{n}{2} + 7 < \frac{n}{2} + \frac{n}{2} = n$.

It follows that

$$\begin{aligned} T(n) &= T(\tfrac{n}{2} + 7) + n^2 \\ &\leq c(\tfrac{n}{2} + 7)^2 + n^2 && \text{(Induction hypothesis)} \\ &= c(\tfrac{n^2}{4} + 7n + 49) + n^2 \\ &= \tfrac{c}{4}n^2 + (7cn + 49c + n^2) \\ &= cn^2 - \tfrac{3c}{4}n^2 + (7cn + 49c + n^2) \\ &= cn^2 - (\tfrac{c}{4}n^2 + \tfrac{c}{2}n^2) + (7cn + 49c + n^2) \\ &= cn^2 - (\tfrac{c}{4}n^2 - n^2) - \tfrac{c}{2}n^2 + (7cn + 49c) \\ &\leq cn^2 - \tfrac{c}{2}n^2 + (7cn + 49c) && ((\tfrac{c}{4}n^2 - n^2) \geq 0, \text{ for } \underline{c \geq 4}) \\ &\leq cn^2 - \tfrac{c}{2}n^2 + (7cn + 7cn) && (49c \leq 7cn, \text{ when } \underline{n \geq 7}) \\ &\leq cn^2 - \tfrac{c}{2}n^2 + 14cn \\ &\leq cn^2 - (\tfrac{c}{2}n^2 - 14cn) \\ &\leq cn^2 && ((\tfrac{c}{2}n^2 - 14cn) \geq 0, \text{ when } \underline{n \geq 28}) \end{aligned}$$

Hence, for any $c \geq 4$ and $n_0 = \max\{14, 7, 28\} = 28$, $T(n) \leq cn^2, \forall n \geq n_0$.

Induction basis: Let $c' = \max\{\frac{T(n)}{n^2} | 21 \leq n \leq 27\}$.

Then, $T(n) \leq c'n^2, 21 \leq n \leq 27$.

By letting $C = \max\{c', c\}$ and $n_0 = 21$, we have: $\exists C > 0, \exists n_0 \in \mathbf{N}, T(n) \leq Cn^2, \forall n \geq n_0$.

Hence, $T(n) = O(n^2)$. ■