Assignment 1

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Question 1. (i). We want to prove that when Algorithm Summation terminates its execution,

$$y = \sum_{j=0}^{n} a_j x^j$$

Proof. We shall apply induction to prove that after the mth iteration of the for loop the following invariant holds true:

$$y = \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}$$

(Induction basis) First, we note that y is initialized to 0 in Line 1. When $m = 1$ and $i = n$ in Line 3,

$$\begin{aligned} y &= a_n + x * y \\ &= a_n + x * 0 \\ &= a_n \\ &= a_n x^0 \\ &= \sum_{j=0}^{0} a_{n-j} x^{n-i-j} \end{aligned}$$

(Induction Hypothesis) Suppose for iteration $m - 1 < n$, $y = \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j-1}$
(Induction Step) When Line 3 is executed for the mth time,

$$
\begin{aligned}
y &= a_{n-i} + x \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j-1} \\
&= a_{n-i} + \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j} \\
&= a_{n-i} x^0 + \sum_{j=0}^{m-2} a_{n-j} x^{n-i-j} && \text{(Induction Hypothesis)} \\
&= \sum_{j=0}^{m-1} a_{n-j} x^{n-i-j}
\end{aligned}
$$

Therefore we can conclude that the invariant $y = \sum_{j=0}^{m-1} a_{n-j}x^{n-i-j}$ holds $\forall m > 0$.

On the n+1th iteration (where $i = 0$) we then have,

$$
\begin{aligned}
y &= \sum_{j=0}^{n} a_{n-j} x^{n-j} \\
&= a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x^1 + a_0 x^0 \\
&= \sum_{j=0}^{n} a_j x^j && \text{(By reversing the order of the terms)}
\end{aligned}
$$

□

Question 1. (ii).
Key Operation: Multiplication of real numbers.
Input size: n (The size of the input array)

The Summation algorithm must perform the key operation on each of the n elements of the input array.

1. Worst-case Time Complexity: $T(n) = n$

2. Average-case Time Complexity: $T_{ave}(n) = n$

Question 3. I assert that the claim is false, and will provide a counter example such that $f \in O(g) \wedge h(f) \notin O(h(g))$.

Proof. Let

$$f(n) = n^2,$$
$$g(n) = n^3,$$
$$h(n) = \frac{1}{n}$$

Lemma I. $f \in O(g)$, using Theorem 0.2

$$\begin{aligned}
&\lim_{n\to\infty} \frac{n^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0 (>= 0) && \text{(Theorem 0.2)}
\end{aligned}$$

Given the defintion of f and h we can see that $h(f(n)) = \dfrac{1}{n^2}$.
Given the defintion of g and h we can see that $h(g(n)) = \dfrac{1}{n^3}$.
All that remains is to prove that $\frac{1}{n^2} \notin O(\frac{1}{n^3})$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^3}} \\
&= \lim_{n\to\infty} \frac{n^3}{n^2} \\
&= \lim_{n\to\infty} n \\
&= \infty (\notin \mathbb{R}^+)
\end{aligned}$$

Hence combining Lemma I with the above proof we have created a counter example such that,

$$f \in O(g) \wedge h(f) \notin O(h(g))$$

□