Assignment 4

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Question 1. *Proof.* First we show that $\Pi \in \mathbf{NP}$

Let $\pi \in \Pi$.

As every Π logical expression is a CNF logical expression with exactly three literals, $\pi \in 3SAT$.

Since 3SAT is **NP**-complete, there exists a polynomial time nondeterministic algorithm N solving 3SAT.

Now,

$$\begin{aligned} &\pi \text{ is a yes-instance of } \Pi \\ \iff &\pi \text{ is a yes-instance of } 3SAT \\ \iff &\text{algorithm } N \text{ outputs a yes on input } \pi \end{aligned}$$

Therefore $\Pi \in \mathbf{NP}$.

Next, we prove that $3SAT \alpha \Pi$.

Let $C = c_1 \wedge c_2 \wedge ... \wedge c_m$ be a CNF logical expression in which each c_i contains exactly three literals, i.e. C is a problem instance of 3SAT. Let $U = \{u_1, u_2, ..., u_n\}$ be the set of variables in C.

For each c_i, $1 \leq i \leq m$ we shall construct a Π logical expression $c_i' \in \{x_i, y_i, z_i, a_i, b_i, c_i, d_i\}$ such that the Π CNF expression $C' = c_1' \wedge c_2' \wedge ... \wedge c_m'$ is satisfiable with exactly 1 literal per clause being assigned a true value if and only if C is satisfiable.

Let $c_j = x_j \vee y_j \vee z_j$. The corresponding c_j' is defined as

$$(\neg x_j \vee a_j \vee b_j) \wedge (y_j \vee b_j \vee c_j) \wedge (\neg z_j \wedge c_j \wedge d_j)$$

where a_j, b_j, c_j, d_j are newly created literals. Let $U' = U \cup (\bigcup_{j=1}^{m} \{a_j, b_j, c_j, d_j\})$.

By construction each clause of C' contains exactly three literals.

It is also easily verifiable that $|c'_j| = 3|c_j|$ since each clause in C produces exactly 3 clauses in C'. Additionally, 4 new variables are introduced into U' for each clause in C, i.e. $|U'| = 4|U|$.

Thus the transformation takes a total of $O(3|C| + 4|U|) = O(|C| + |U|)$ operations. I.e. the transformation can be done in polynomial time.

Next we shall show that,

C' is satisfiable if and only if C is is satisfiable

Let $t : U \to \{true, false\}$ be a truth assignment satisfying C.
As $C = c_1 \wedge c_2 \wedge ... \wedge c_2$ is true under t, at least one literal in each c_j must evaluate to true. 7 cases arise: (note that unassigned variables are assumed to be false)

1. $t(x_j) = true$.
   In this case we assign $a_j = true$ and $b_j = true$. We thus have a single literal in each clause of c'_j evaluated to true, and the expression is satisifed.

2. $t(y_j) = true$.
   c'_j is true, and each clause has a single true literal.

3. $t(z_j) = true$
   We assign $c_j = true$ and thus have a single positive literal in each clause, and the expression is satisifed.

4. $t(x_j, y_j) = true$
   We assign $a_j = true$ and $d_j = true$.

5. $t(x_j, z_j) = true$
   We assign $a_j = true$ and $c_j = true$.

6. $t(y_j, z_j) = true$
   We assign $d_j = true$.

7. $t(x_j, y_j, z_j) = true$
   See case 4.

In all of the above cases,

C is satisfiable $\Rightarrow$ C' is satisfiable and each clause has a single true literal

Conversely let $t' : U \to \{true, false\}$ be a truth assignment satisfying C' such that each clause contains a single true literal.

Suppose in the original 3SAT expression $\exists c_i, x_i = y_i = z_i = false$, i.e. c_i is unsatisfiable. Since t' was assumed to be a valid truth assignment for Π, one of b_i or c_i must be true (otherwise we have a contradiction). If b_i is true, we have $\neg x$ and b_i both true in the same clause, which contradicts the assumption that t' is a valid truth assignment. Similarly if c_i is true, we have $\neg z$ and c_i belonging to the same clause, a contradiction of the validity of t'. Therefore c_i must be satisifiable.

I.e.,

$$\text{C' is satisfiable and each clause has a single true literal} \Rightarrow \text{C is satisfiable}$$

Thus we have,

$$\text{C is satisfiable} \iff \text{C' is satisfiable and each clause has a single true literal}$$

Hence we have 3SAT α Π.
Thus $\Pi \in \mathbf{NP} \land 3\text{SAT}\alpha\Pi$.
Hence $\Pi \in$ **NP-**Complete. $\square$