# Analysis

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Settembre 2024

## 1 Introduction

### 1.1 Introductory questions

1) What is exponential growth?

2) When does one say something is like squaring the circle?

3) If the growth rate of the population is decreasing, does this mean that the population is decreasing?

4) How can one construct an equilateral triangle with ruler and compass?

5) Is it true that there are more natural numbers than even numbers?

6) If when it rains John (always) takes an umbrella, what can we say if John has not taken an umbrella?

### 1.2 Overview of the course

To understand the basic mathematics necessary for economy it is important to acquire some familiarity with real numbers. The system of real numbers completes that of rational numbers in such a way as to allow the definition of certain limiting processes which lie at the foundation of differential calculus. Mathematical notions are developed in steps, following a transparent process. In particular by clarifying the vocabulary, by giving explicit definitions, and by using conventional methods to justify the arguments employed.

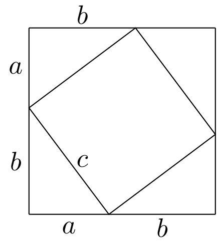
Thus, after a brief discussion of the insufficiency of rational numbers, the course will hint at some basics of the formalism used in mathematics. Then we shall dwell deeper into the notion of real number and introduce the notion of limit. Limits will then be used to define derivatives and integrals, which lay at the foundation of differential calculus.

## 1.3 Are there enough rational numbers?

Between two rational numbers $\frac{a}{b}$ e $\frac{c}{d}$ there always is a third: for instance $\frac{a}{b} < \frac{1}{2}(\frac{a}{b} + \frac{c}{d}) < \frac{c}{d}$. So there are many, many rational numbers. Does there exist a rational number whose square is 2?

# 2 Some statements and proofs

## 2.1 Pythagoras Theorem



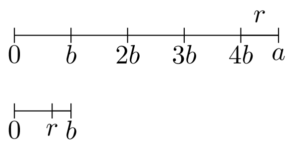

## 2.2 Commensurability and incommensurability

Two segments are commensurable if there exists a segment of which the two given segments are integral multiples. Two commensurable segments thus have a commun measure. Two segments which are not commensurable are said to be incommensurable.

Can you give an example of two incommensurable segments?

### 2.2.1 Euclid's algorithm

Given segments of length $a$ and $b$ how can we divide the first by the second?




We have

$$a = qb + r \ , \text{ with } 0 \leq r < b \ .$$

We call $q$ the *quotient* and $r$ the *remainder* (of the division of $a$ by $b$). If $r = 0$, then $a$ is an integral multiple of $b$, that is $b$ divides $a$. If instead $r > 0$, then we can start over again and find $q_1$ such that

$$b = q_1 r + r_1 \ , \text{ with } 0 \leq r_1 < r \ .$$

If $r_1 > 0$, then we find $q_2$ so that

$$r = q_2 r_1 + r_2 \ , \ \text{con } 0 \leq r_2 < r_1 \ .$$

More generally, we might find

$$\begin{array}{rcl}
a & = & qb + r \\
b & = & q_1 r + r_1 \\
r & = & q_2 r_1 + r_2 \\
. & & \\
. & & \\
r_k & = & q_{k+2} r_{k+1} + r_{k+2} \\
. & & \\
. & & \\
r_{n-1} & = & q_{n+1} r_n + r_{n+1} \ ,
\end{array}$$

with intgers $q_k$ and $r_{n+1} < r_n < r_{n-1} < \cdots < r_1 < r$. If $r_{n+2} = 0$–say–and $r_n = q_{n+2} r_{n+1}$, one can show that *$a$ et $b$ are multiples of the last non-zero remainder $r_{n+1}$*. However, in general it is not clear that the algorithm stops. When does it stop?

### 2.2.2 Decimal expansion and divisibility criteria

Using the Euclidean algorithm one can show that every integer $n$ can be written

$$n = c_k 10^k + c_{k-1} 10^{k-1} + \cdots + c_1 10 + c_0 1$$

with $c_i$ between 0 and 9 uniquely determined by $n$. The decimal representation results from the juxtaposition of the $c_i$'s

$$n = c_k c_{k-1} \cdots c_1 c_0 \ .$$

One can deduce divisibility criteria for $n$ from the expansion.

### 2.2.3 Column operations

Have you ever wondered how Romans performed additions and multiplications?

## 2.3 The binomial formula

**Theorem.** Let $x$ and $y$ be numbers [1]. The for any natural number $n$ we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

[1] It suffices that in the "number system" to which $x$ and $y$ belong there are defined operations "sum" and "multiplication" for which $ab = ba$, $a + b = b + a$, $a(b + c) = ab + ac$ and $1a = a$.

where for $k = 0$ we let $\begin{pmatrix} n \\ 0 \end{pmatrix} = 1$ and for $k \geq 1$ the *binomial coefficients* [2] are given by

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots \;\; k} \quad . \;\; (*)$$

*Proof.* The statement is true for $n = 0$. By definition

$$(x+y)^n = (x+y)^{n-1}(x+y) \; .$$

So let us suppose the statement is true for $n-1$. The right hand side becomes

$$\left( \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} x^k y^{n-1-k} \right) (x+y)$$

which we can rewrite

$$\sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ k \end{pmatrix} (x^{k+1}y^{n-1-k} + x^k y^{n-k}) \; .$$

Decomposing the sum and changing the numbering we see that to verify the formula for $n$ we have to check that for $k \geq 1$ there holds

$$\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ k \end{pmatrix} \quad . \;\; (**)$$

This follows from (*), which ends the proof.

The binomial coefficients are determined by (**), which means that one may compute them using the so-called *Pascal triangle* [3]

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

which may be expanded to an infinite number of lines. Which is the next line?

**Corollary.** The binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$ equals the number of ways to choose $k$ objects among $n$.

[2]The binomial coefficients are often denoted $C_n^k$.

[3]This triangle has been discovered a number of times: Omar Alkhaijâma (Mondo arabo, 1080), Tsu shi Kih (Cina, 1303), Stifel (Germania, 1544), Cardano (Italia, 1545), Pascal (Francia, 1654).

## 2.4 Propositional calculus

One defines operations on propositions as follows.

**Negation / NOT** Given a proposition $p$, its negation $\neg p$ is true if $p$ is false, and is false if $p$ is true. In tabular form:

| $p$ | $\neg p$ |
|---|---|
| $T$ | $F$ |
| $F$ | $T$ |

**Disjunction / OR** Given two propositions $p$ and $q$, their disjunction $p \vee q$ is the proposition which is only true when one of the two proposition $p$ or $q$ is true.

**Conjunction / AND** Given two propositions $p$ and $q$, their conjunction $p \wedge q$ is defined as $\neg(\neg p \vee \neg q)$. We see that $p \wedge q$ is true if and only if $p$ and $q$ are both true.

**Implication** Given two propositions $p$ and $q$, the implication $p \Rightarrow q$ is defined as $\neg p \vee q$. The expression $p \Rightarrow q$ is read as "if $p$, then $q$". We see that $p \Rightarrow q$ is true if $p$ false or if $q$ is true. If $p \Rightarrow q$ is an implication, then the implication $q \Rightarrow p$ is its *reciprocal.*

**Equivalence** The equivalence of two propositions $p$ and $q$, denoted $p \Leftrightarrow q$, is defined as $(p \Rightarrow q) \wedge (q \Rightarrow p)$. We thus have that $p \Leftrightarrow q$ is true if and only if $p$ and $q$ are both true or false at the same time.

In tabular form:

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \Rightarrow q$ | $p \Leftrightarrow q$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $V$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |

**Double negation** One checks easily that $p$ is equivalent to $\neg(\neg p)$.

**Contrapositive** One checks that $p \Rightarrow q$ is equivalent to the *contrapositive* implication $\neg q \Rightarrow \neg p$.

**Excluded middle** One checks that for any proposition $p$ the proposition $p \vee \neg p$ is always true.

## 2.5 Quantifiers

To do mathematics we need quantifiers, for instance to deal with statements like "for every pair of integers $x$ and $y$, there existes an integer $z$ such that $x + y = z$". We thus need to deal with "open expressions" (or predicates) such as

1) $Gx =$ "$x$ is bold"
2) $Gxy =$ "$x$ is the father of $y$"
3) $Pxy =$ "$x$ is perpendicular to $y$"
4) $Gxyz =$ "$x$ gives $y$ to $z$"
5) $Gxyz =$ "$x$ is between $y$ and $z$ on the circle $z$"
6) $Sxyz =$ "$z$ is the sum of $x$ and $y$"
7) $Rxyzw =$ "$x$ pays $y$ to $z$ on behalf of $w$"

Here $x$ ($y$, $z$, etc.) are viewed as variables and $Gx$–say–is a way to express that $G$ depends on $x$, like a function. The variable $x$ behaves like a pronoun.

The *universal quantifier* is written $\forall$ and the *existential quantifier* is written $\exists$. These are prefixes which have a bearing on the variables. The expressions

$$\forall x\ Gx \quad \textit{and} \quad \exists x\ Gx$$

are read "for every $x$ we have $Gx$" and "there exists $x$ such that $Gx$. There holds the equivalence

$$\neg(\exists x Rx) \Leftrightarrow \forall x(\neg Rx) \ .$$