Quinton Crum Homework #3 MATH 450

Problem #12:

A.

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ ? & & 1 & 0 \\ & & & 1 \end{bmatrix} \quad U = \begin{bmatrix} & & & \\ 0 & & ? & \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix}$$

$k=1$

$u_{11} = a_{11} = 1$, $u_{12} = a_{12} = 1/2$, $u_{13} = a_{13} = 1/3$, $u_{14} = a_{14} = 1/4$

$l_{21} = \frac{a_{21}}{a_{11}} = \frac{1/2}{1} = \frac{1}{2}$, $l_{31} = \frac{a_{31}}{a_{11}} = \frac{1/3}{1} = 1/3$, $l_{41} = \frac{a_{41}}{a_{11}} = \frac{1/4}{1} = \frac{1}{4}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & & 1 & 0 \\ 1/4 & & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix}$$

$k=2$

$$\begin{aligned} u_{22} &= a_{22} - l_{21}u_{12} \\ &= 1/3 - 1/2 \cdot 1/2 \\ &= 1/12 \end{aligned} \qquad \begin{aligned} u_{23} &= a_{23} - l_{21}u_{13} \\ &= 1/4 - 1/2 \cdot 1/3 \\ &= 1/12 \end{aligned}$$

$$\begin{aligned} u_{24} &= a_{24} - l_{21}u_{14} \\ &= 1/5 - 1/2 \cdot 1/4 \\ &= 3/40 \end{aligned} \qquad \begin{aligned} l_{32} &= \frac{1}{u_{22}}\left[a_{32} - l_{31} \cdot u_{12}\right] \\ &= 12\left[1/4 - 1/3 \cdot 1/2\right] \\ &= 12\left[1/12\right] = 1 \end{aligned}$$

$$\begin{aligned} l_{42} &= \frac{1}{u_{22}}\left[a_{42} - l_{41}u_{12}\right] \\ &= 12\left[1/5 - 1/4 \cdot 1/2\right] \\ &= 12\left[3/40\right] = 9/10 \end{aligned}$$

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$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 1 & 1 & 0 \\ 1/4 & 9/10 & & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & & \\ 0 & 0 & 0 & \end{bmatrix}$$

$k=3$

$$\begin{aligned} u_{33} &= a_{33} - (L_{31} u_{13} + L_{32} u_{23}) \\ &= 1/5 - (1/3 \cdot 1/3 + 1 \cdot 1/12) \\ &= 1/5 - (1/9 + 1/12) \\ &= 1/5 - 7/36 = \frac{1}{180} \end{aligned}$$

$$\begin{aligned} u_{34} &= a_{34} - (L_{31} u_{14} + L_{32} u_{24}) \\ &= 1/6 - (1/3 \cdot 1/4 + 1 \cdot 3/40) \\ &= 1/6 - (1/12 + 3/40) \\ &= 1/6 - 19/120 = 1/120 \end{aligned}$$

$$\begin{aligned} L_{43} &= \frac{1}{u_{33}} [a_{43} - (L_{41} u_{13} + L_{42} u_{23} \\ &= 180 \left[1/6 - \left(\frac{1}{4} \cdot \frac{1}{3} + \frac{9}{10} \cdot \frac{1}{12}\right) \right] \\ &= 180 \left[1/6 - \left(\frac{1}{12} + \frac{9}{120}\right) \right] \\ &= 180 \left[1/6 - \frac{19}{120} \right] \\ &= 180 \left[1/120 \right] \\ &= 3/2 \end{aligned}$$

$k=4$

$$\begin{aligned} u_{44} &= a_{44} - (L_{41} u_{14} + L_{42} u_{24} + L_{43} u_{34}) \\ &= 1/7 - \left(1/4 \cdot \frac{1}{4} + \frac{9}{10} \cdot \frac{3}{40} + \left(\frac{3}{2}\right) \cdot \frac{1}{120}\right) \\ &= 1/7 - \left(1/16 + 27/400 + (3/240)\right) \\ &= 1/7 - \left(\frac{13}{100} + \frac{3}{240}\right) = \frac{1}{7} - \frac{57}{400} = 1/2800 \end{aligned}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/3 & 1 & 1 & 0 \\ 1/4 & 9/10 & 3/2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/12 & 1/12 & 3/40 \\ 0 & 0 & \frac{1}{180} & 1/120 \\ 0 & 0 & 0 & 1/2800 \end{bmatrix}$$

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B)

$Ly = b$ Forward Substitution

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & 1 & 1 & 0 \\ \frac{1}{4} & \frac{9}{10} & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$1 \cdot y_1 + 0 \cdot y_2 + 0 \cdot y_3 + 0 \cdot y_4 = y_1 = 1$

$\frac{1}{2} \cdot y_1 + 1 \cdot y_2 + 0 \cdot y_3 + 0 \cdot y_4 = \frac{1}{2} \cdot 1 + y_2 = 2 \Rightarrow y_2 = \frac{3}{2}$

$\frac{1}{3} \cdot y_1 + 1 \cdot y_2 + 1 \cdot y_3 + 0 \cdot y_4 = \frac{1}{3} \cdot 1 + 1 \cdot \frac{3}{2} + y_3 = 3 \Rightarrow y_3 = \frac{7}{6}$

$\frac{1}{4} \cdot y_1 + \frac{9}{10} \cdot y_2 + \frac{3}{2} \cdot y_3 + 1 \cdot y_4 = \frac{1}{4} \cdot 1 + \frac{9}{10} \cdot \frac{3}{2} + \frac{3}{2} \cdot \frac{7}{6} + y_4 = 4$

$\Rightarrow y_4 = 4 - \frac{1}{4} - \frac{27}{20} - \frac{21}{12} = \frac{15}{4} - \frac{27}{20} - \frac{21}{12} = \frac{48}{20} - \frac{21}{12} = \frac{-13}{20} = y_4$

$Ux = y$ Backwards Sub

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & 0 & \frac{1}{2800} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{7}{6} \\ \frac{13}{20} \end{bmatrix}$$

$x_4 = \frac{13}{20} \cdot 2800 = 182$

$x_3 = \left(\frac{7}{6} - \frac{1}{120} x_4\right) 180 = \left(\frac{7}{6} - \frac{91}{60}\right) 180 = \frac{-21}{60} \cdot 180 = -63$

$x_2 = \left(\frac{3}{2} - \frac{1}{12} x_3 - \frac{3}{40} x_4\right) 12 = \left(\frac{3}{2} + \frac{63}{12} - \frac{546}{40}\right) 12 = \left(\frac{81}{12} - \frac{546}{40}\right) 12 = \frac{-69}{10} \cdot 12 = \frac{-414}{5}$

$x_1 = \left(1 - \frac{1}{2} x_2 - \frac{1}{3} x_3 - \frac{1}{4} x_4\right) = \left(1 + \frac{414}{10} + \frac{63}{3} - \frac{182}{4}\right) = \left(\frac{424}{10} + \frac{63}{3} - \frac{182}{4}\right)$

$= \frac{317}{5} - \frac{182}{4} = \frac{179}{10}$

Problem 13

A)

In [ ]:
```
import numpy as np
import math
import matplotlib.pyplot as plt
```

In [ ]:
```
def create_L_matrix(n):
    L = np.identity(n=n)
    return L

def create_U_matrix(n):
    U = np.zeros((n,n))
    return U

def hilbert(n):
    x = np.arange(1, n+1) + np.arange(0, n)[:, np.newaxis]
    return 1.0/x

def lu_decomp(A):
    n = len(A)

    L = create_L_matrix(n)
    U = create_U_matrix(n)


    for k in range(0,n):
        # Ukk
        temp_sum_ukk = 0
        for s in range(0,k):
            temp_sum_ukk += L[k,s]*U[s,k]

        U[k,k] = A[k,k] - temp_sum_ukk

        #Ukj
        for j in range(k+1,n):
            temp_sum_ukj = 0
            for s in range(0,k):
                temp_sum_ukj += L[k,s]*U[s,j]

            U[k,j] = A[k,j] - temp_sum_ukj

        #Lik
        for i in range(k+1,n):
            temp_sum_lik = 0
            for s in range(0,k):
                temp_sum_lik += L[i,s]*U[s,k]

            L[i,k] = (1/U[k,k])*(A[i,k] - temp_sum_lik)

    return L,U
```

In [ ]:
```
A = hilbert(4)

results = lu_decomp(A)
```

```
print("L: ")
print(results[0])
print()
print()
print('U:')
print(results[1])
np.set_printoptions(precision=55)
```

```
L:
[[1.                 0.                 0.
  0.                ]
 [0.5                1.                 0.
  0.                ]
 [0.3333333333333333 1.0000000000000004 1.
  0.                ]
 [0.25               0.9000000000000004 1.4999999999999951
  1.                ]]


U:
[[1.0000000000000000e+00 5.0000000000000000e-01 3.3333333333333333e-01
  2.5000000000000000e-01]
 [0.0000000000000000e+00 8.3333333333333331e-02 8.3333333333333334e-02
  7.5000000000000001e-02]
 [0.0000000000000000e+00 0.0000000000000000e+00 5.5555555555555536e-03
  8.3333333333333276e-03]
 [0.0000000000000000e+00 0.0000000000000000e+00 0.0000000000000000e+00
  3.5714285714295447e-04]]
```

B)

In [ ]:

```
from audioop import reverse


def forwardSub(L,b):
    L_inv = np.linalg.inv(L)
    y = np.matmul(L_inv,b)
    return y

def backwardsSub(U,y):
    #x = np.zeros((len(U),1))
    #print(x)
    #for i in reversed(range(0,len(U))):
        #temp_sum = 0
        #for j in range(i+1,len(U)):
            #print('i: '+str(i)+'  j: '+str(j))
            #temp_sum += x[j]/U[i,j]
        #x[i][0] = (y[i]-temp_sum)/U[i,i]



    U_inv = np.linalg.inv(U)
    x = np.matmul(U_inv,y)
    return x
```

In [ ]:

```
A = hilbert(4)
b = np.matrix([[1],[2],[3],[4]])

results = lu_decomp(A)
```

```
y = forwardSub(results[0],b)
print('Forward substitution results   y: ')
print(y)
print()

x = backwardsSub(results[1],y)
print('Backwards substitution results  x: ')
print(x)
```

```
Forward substitution results   y:
[[1.                ]
 [1.5               ]
 [1.1666666666666666 ]
 [0.6500000000000048]]

Backwards substitution results  x:
[[  -63.999999999977085]
 [   899.9999999997318  ]
 [-2519.999999999342   ]
 [ 1819.9999999995673  ]]
```

C)

In [ ]:

```
A = hilbert(4)

checking = np.matmul(A,x)
print('Ab = x where x equals:')
print(checking)
```

```
Ab = x where x equals:
[[1.0000000000001137]
 [2.                ]
 [3.                ]
 [3.999999999999943 ]]
```

As we can see the outcome of the matrix multiplication of matrix A with the solution vector x gives us with in a reasonable range for the true value of b.

Problem # 14

A.
$$\|X_1\|_2 = \left(3^2 + (-4)^2 + 0^2 + 1.5^2\right)^{1/2}$$
$$= (27.25)^{1/2}$$
$$\approx 5.22$$

$$\|X_1\|_\infty = \max_{1 \le i \le n} |X_i| = 4$$

$$\|X_2\|_2 = \left(2^2 + 1^2 + (-3)^2 + 4^2\right)^{1/2}$$
$$= (30)^{1/2}$$
$$\approx 5.477$$

$$\|X_2\|_\infty = \max_{1 \le i \le n} |X_i| = 4$$

$$\|X_3\|_2 = \left(\sin^2 k + \cos^2 k + (2k)^2\right)^{1/2}$$
$$= \left(1 + 4k^2\right)^{1/2}$$

$$\|X_3\|_\infty = \max_{1 \le i \le n} |X_i| = 2k \qquad k \in \mathbb{N}$$

$$\|X_4\|_2 = \left(\frac{16}{(k+1)^2} + \frac{4}{k^4} + k^4 e^{-2k}\right)^{1/2}$$
$$= \left(\frac{16}{k^2+2k+1} + \frac{4}{k^4} + \frac{k^4}{e^{+2k}}\right)^{1/2}$$

$$\|X_4\|_\infty = \max_{1 \le i \le n} |X_i| = \begin{cases} \dfrac{4}{k+1} & -\infty < k \le -2.5 \\ & -1.5 \le k \le -0.5 \\ & 1 \le k < +\infty \\ 2/k^2 & -0.5 \le k \le 1 \\ k^2/e^k & -2.5 < k < -1.5 \end{cases}$$

$$B.\quad \|A\|_\infty = \max_{1\le i\le m}\left(\sum_{j=1}^{n} |a_{ij}|\right)$$

$$= \max(10+15,\ 0+1)$$

$$= 25$$

$$\|B\|_\infty = \max(10+0,\ 15+1)$$

$$= 16$$

$$\|C\|_\infty = \max(2+1+0,\ 1+2+1,\ 0+1+2)$$

$$= 4$$

$$\|D\|_\infty = \max(4+1+7,\ 1+4+0,\ 7+0+4)$$

$$= 12$$

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Problem #15

A.

let $a = \max_{1 \le i \le n} |v_i| = \|v\|_\infty$

$$\cancel{\text{thus}}\ (\|v\|_1)^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i^2 \le \sum_{i=1}^{n} a^2$$

$$\le n a^2$$

$$\le n (\|v\|_\infty)^2$$

$C = \sqrt{n}$ $$\le \sqrt{n}\ \|v\|_\infty$$

$$\sum_{i=1}^{n} v_i^2 \ge \cancel{\sum} a^2$$

$$\ge 1 \cdot a^2 = 1 \cdot (\|v\|_\infty)^2$$

$c = 1$

B. The result of part a) does not imply that:

let $v_1 = \begin{pmatrix} 0 \\ -8 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

$\|v_1\|_1 = 8 \le \|v_2\|_1 = 9$ this holds, however,

$\|v_1\|_\infty = 8 \le \|v_2\|_\infty = 7$ Does not hold true!

Problem 16

A)

In [ ]:
```
import numpy as np
import math
import matplotlib.pyplot as plt
```

In [ ]:
```
def create_A_b(n):
    A = np.zeros((n,n))
    b = np.empty((n,1))

    for i in range(len(A)):
        A[i,i] = 2.01    # i=j

        for j in range(len(A)):
            if(i == j+1):
                A[i,j] = -1
            if(i == j-1):
                A[i,j] = -1

    # b creation
        b[i][0] = (1/100)*(math.sin(2*i*math.pi/50))
    return A,b


def jacobi_randomXnot(A,b,stepNumber):
    x = np.random.rand(len(b),1)
    u = np.zeros((len(b),1))
    A_nodaig = np.copy(A)
    temp = np.arange(0,len(A))
    A_nodaig[temp,temp] = 0

    for k in range(0,stepNumber):
        for i in range(len(A)):
            u[i,0] = (1/A[i,i])*(b[i,0] - np.sum(np.matmul(A_nodaig[i],x))) #(1/
        x = np.copy(u)

    return x
```

In [ ]:
```
Ab = create_A_b(5)
jacobi_randomXnot(Ab[0],Ab[1],15)
```

Out[ ]:
```
array([[0.03560611],
       [0.08552933],
       [0.07401939],
       [0.08872267],
       [0.03957281]])
```

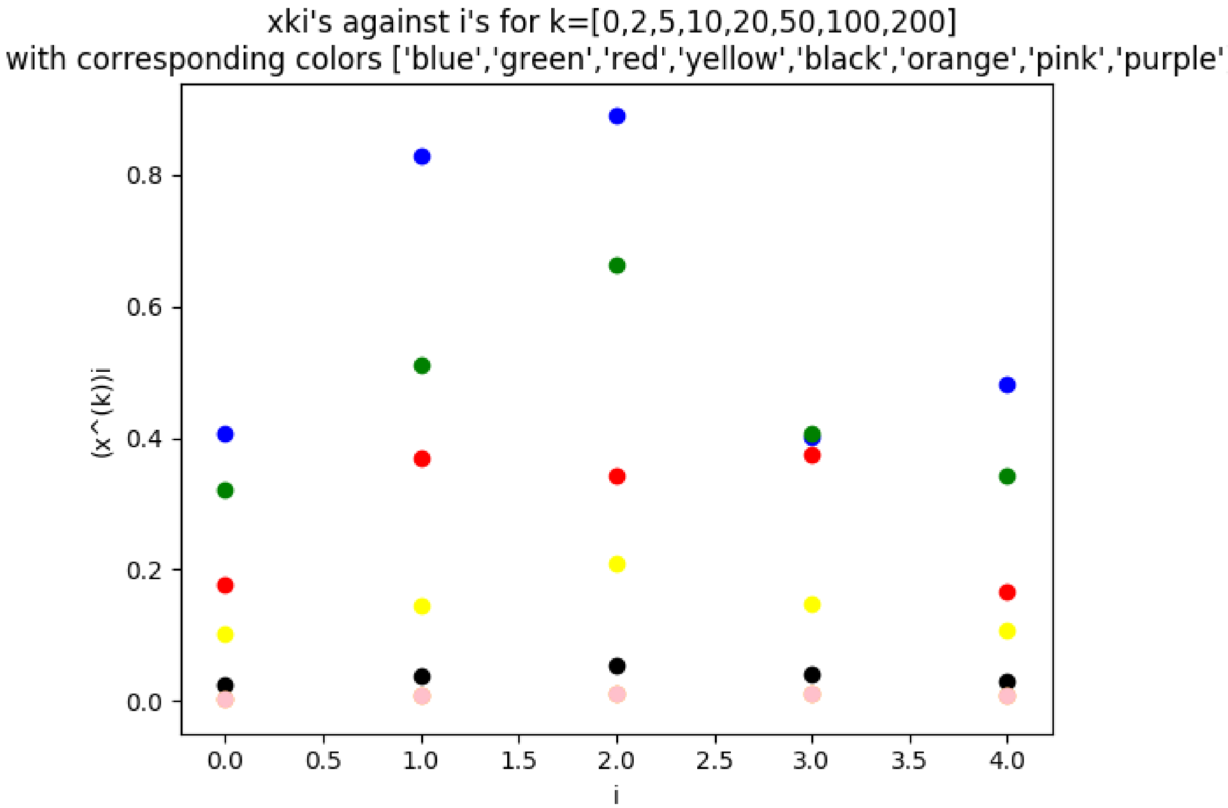
B)

In [ ]:
```
from cProfile import label
from turtle import color


def jacobi_plot(A,b,stepNumber):
```

```python
    plotMe = np.array([0,2,5,10,20,50,100,200])
    colorMe = np.array(['blue','green','red','yellow','black','orange','pink','p
    x = np.random.rand(len(b),1)
    u = np.zeros((len(b),1))
    A_nodaig = np.copy(A)
    temp = np.arange(0,len(A))
    A_nodaig[temp,temp] = 0

    fig, ax = plt.subplots(1,1)

    for k in range(0,stepNumber):
        for i in range(len(A)):
            u[i,0] = (1/A[i,i])*(b[i,0] - np.sum(np.matmul(A_nodaig[i],x))) #(1/
            if(len(np.where(plotMe == k)[0]) > 0):
                #plt.plot(i,u[i,0],'o',c=colorMe[np.where(plotMe == k)[0][0]],la
                ax.scatter(i,u[i,0],color = colorMe[np.where(plotMe == k)[0][0]]

        x = np.copy(u)

    plt.xlabel('i')
    plt.ylabel('(x^(k))i')
    plt.title('xki\'s against i\'s for k=[0,2,5,10,20,50,100,200] \n with corres





    plt.show()
```

In [ ]:

```python
Ab = create_A_b(5)
jacobi_plot(Ab[0],Ab[1],200)
```



From the plot above we can observe the values of $(x^{(k)})_i$ gets closer and closer to zero as the number of iterations increases for all the is. This even occurs fairly evenly when the starting values are different. In other words the $(x^{(k)})_i$s for some value k seem to get closer and closer to equal as k increases.