Topological Quantum Field Theory

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- The goal of this presentation is to explain topological quantum field theory, rigorously for the most part.
- I will assume familiarity with quantum mechanics and quantum computing as the starting point.
- The sheets collect definitions. Some terms will be left undefined.

Introduction

A quantum field theory can be seen as a translation from the geometric and dynamical structures of spacetime to an algebraic description in terms of states and observables. A topological quantum field theory serves the same goal, but for topological structures rather than strictly geometric ones.

Mathematics

Topologies generalise metrics by encoding spatial relations in sets.

Topology

A topology on a set X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following three axioms:

1. The sets $\varnothing$ and X are elements of $\mathcal{T}$;
2. The intersection of two sets in $\mathcal{T}$ is an element of $\mathcal{T}$;
3. Any union of sets taken from $\mathcal{T}$ is an element of $\mathcal{T}$.

A pairing $(X, \mathcal{T})$ of a set and a topology on the set forms a topological space.

Examples: small metric space, bent ellipse.

Continuous maps are maps that do not show unpredictable behaviour.

Continuity

A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces is called continuous if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

Homeomorphisms fill the role of isomorphisms for topological spaces.

Homeomorphism

A homeomorphism is a bijection between two topological spaces such that both the map and its inverse are continuous.

If a homeomorphism between two spaces exists the spaces are called homeomorphic.

Example: ellipse.

Manifolds are topological spaces that locally appear to be linear.

Topological Manifold

An n-dimensional topological manifold is a topological Hausdorff space that is second countable and such that every point of the space admits an open neighbourhood homeomorphic to $\mathbb{R}^n$.

Examples: S^2, earth.

Topological manifolds can be equipped with different kinds of useful structures. The following are examples.

- Smooth manifolds
- Riemannian manifolds
- Symplectic manifolds

A boundary, as the complement of the interior, is the edge of a manifold.

If possible, an orientation determines a consistent choice of direction for the normal vectors to the surface of a manifold.

Examples: I, S^2.

Cobordism is an equivalence relation on classes of manifolds.

Cobordism

A cobordism C between two n-dimensional compact manifolds M, N is an $(n+1)$-dimensional compact manifold with boundary so that $\partial C = M \amalg N$.

Two manifolds are called cobordant if a cobordism between them exists. If M and N are oriented as well it is required that $\partial C = M \amalg N^*$.

Examples: D, pants.

Categories bundle objects and their transformations into one generalising structure.

Category

A category $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ of objects, a class $\mathrm{Mor}(\mathcal{C})$ of morphisms and a composition rule for the morphisms so that:

1. The composition rule is associative;
2. Every object admits an identity morphism that acts as an identity in compositions.

Only locally small categories are considered.

A covariant functor is a composition preserving map between categories.

Covariant Functor

A covariant functor is a map $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories that assigns an object $F(X)$ to every object X and a morphism $F(f) : F(A) \to F(B)$ to every morphism $f : A \to B$ so that the following conditions hold:

1. If f is an identity morphism, then $F(f)$ is an identity morphism;
2. If a composition $f \circ g$ is well-defined, then $F(f \circ g) = F(f) \circ F(g)$.

Topological Quantum Field Theory

A monoidal category is a category equipped with a structure that generalises the tensor product. In the same vein, the definition of an involutive category achieves the same for adjoints.

The Hilb Category

The category *Hilb* of Hilbert spaces and bounded linear operators is an example of both a monoidal and an involutive category.

The *nCob* category is defined as the category with $(n-1)$-dimensional compact oriented manifolds as its objects, and n-dimensional cobordisms as its morphisms. It is both monoidal and involutive through disjoint unions and cobordism reversal respectively.

A Functor from nCob to Hilb

An $(n+1)$-dimensional topological quantum field theory is a functor *nCob* $\to$ *Hilb* that maps manifolds to Hilbert spaces, cobordisms to bounded linear operators and preserves the monoidal and involutive structures.

Axiomatisation

Topological quantum field theory can be axiomatised through the Atiyah-Segal axioms, whereas axiomatic quantum field theory is still incomplete.

Topological Quantum Computing

Non-Abelian Anyons

Non-Abelian Anyons are particles that occur in two-dimensional systems. Exchanges alter the state of the system beyond phase shifts, meaning the configuration is important.

Braiding

Under pairwise reorderings the particles trace worldlines through spacetime, thus forming braids. These braids are elements of the braid group.

Viewing the particles as part of a 2-manifold and the braids as part of a cobordism representing time evolution, these components can be mapped to the domain of quantum mechanics through a $(2+1)$-TQFT.

The Result

The model of topological quantum computation that emerges has the following favourable properties.

- It is universal.
- The states are less susceptible to decoherence.
- It has an intrinsic resistance to error.

Questions

Literature

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