# **HOMOTOPY TYPE THEORY**

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#### **INTRODUCTION**

- Type theory is a formal deductive system
- · Topology studies spaces and their deformations
- Homotopy type theory is a novel interpretation

#### Introduction

This talk is based on the following paper:

Á. Pelayo, M.A. Warren, Homotopy Type Theory and Voevodsky's Univalent Foundations, Bulletin of the American Mathematical Society, 51:4 (2014), 597-648.

Rephrased with an emphasis on intuition. Some concepts are left undefined.



#### **TYPES**

The most basic judgement in type theory is x : A, meaning x is a term of the type A.

- Type Universes
- Propositional and judgemental equalities
- Type Formers

## THE CURRY-HOWARD CORRESPONDENCE

- · One way to give meaning to terms and types
- Types are propositions
- · Terms are proofs
- Curry-Howard-Lambek correspondence

## **FUNCTION TYPES**

Given two types A and B, a type of functions  $A \rightarrow B$  between them can be constructed. A term of this type is a function from the domain A into the codomain B.

Such functions are often defined by  $\lambda$ -abstractions.

### **DEPENDENT TYPES**

A dependent type is a type parametrised by another type. Formally, a dependent type B depending on a type A is a function  $B: A \to \mathcal{U}$ .

## **DEPENDENT FUNCTION TYPES**

The  $\prod$ -type is a generalisation of the function type where the codomain can depend on the value of the variable term.

Given a type A and a dependent type  $B:A \to \mathcal{U}$  this is expressed as:

$$\prod_{x:A} B(x).$$

## **PAIR TYPES**

Pairing two types A and B leads to the pair type  $A \times B$ . Terms of this type have the form (a, b) for a : A and b : B.

### **DEPENDENT PAIR TYPES**

Analogue to the dependent function type, the  $\sum$ -type generalises pair types. For elements of this pair type the second component may depend on the first.

Again for A a type and  $B:A\to \mathcal{U}$  a dependent type, this is denoted

$$\sum_{x:A} B(x)$$
.

### **IDENTITY TYPES**

- Major deviation from set theory
- Identities are captured in a type  $Id_A(a,b)$
- · Identity types hold no unique information
- In homotopy type theory they do, leading to higher types:

$$\operatorname{Id}_{\operatorname{Id}_A(a,b)}(c,d)$$



#### INTUITION

Interesting spaces contain structure. One example of such a structure is a metric, a notion of distance.

### **TOPOLOGY**

A topology captures the structure of a space by encoding how elements are spatially related to each other.

# **Topology**

A topology on a set X is a set  $\mathcal{T} \subseteq \mathcal{P}(X)$  satisfying the following three axioms:

- 1. The sets  $\varnothing$  and X are elements of  $\mathcal{T}$ ;
- 2. The intersection of two sets in  $\mathcal{T}$  is an element of  $\mathcal{T}$ ;
- 3. Any union of sets taken from  $\mathcal{T}$  is an element of  $\mathcal{T}$ .

Such a pair (X, T) forms a topological space.

# **CONTINUITY AND PATHS**

Continuous maps are maps that do not behave unpredictably.

# **Continuity**

A map  $f:(X,\mathcal{T}_X)\to (Y,\mathcal{T}_Y)$  between topological spaces is called continuous if for all  $U\in\mathcal{T}_Y$ ,  $f^{-1}(U)\in\mathcal{T}_X$ .

Paths between two points are examples of such maps.

#### **Path**

A path in a topological space X is a continuous map  $[0,1] \rightarrow X$ .

#### **HOMEOMORPHISMS**

The most elementary definition of the equivalence of two topological spaces is given by a homeomorphism.

# **Homeomorphism**

A homeomorphism is a bijection between two topological spaces such that both the map and its inverse are continuous.

Two spaces are said to be homeomorphic if there exists a homeomorphism between them.

#### **TOPOLOGICAL MANIFOLDS**

Manifolds form possibly the most important classes of spaces in topology.

# **Topological Manifold**

An n-dimensional topological manifold is a topological Hausdorff space that is second countable and such that every point of this space admits an open neighbourhood homeomorphic to  $\mathbb{R}^n$ .

## **HOMOTOPIES**

Homotopies are continuous deformations between maps.

# **Homotopy**

A homotopy is an indexed family of maps  $f_t(x): X \to Y$  between topological spaces so that the associated map  $F: [0,1] \times Y$  defined by  $F(x,t) = f_t(x)$  is continuous.

The maps  $f_0$  and  $f_1$  are said to be homotopic, denoted  $f_0 \simeq f_1$ .

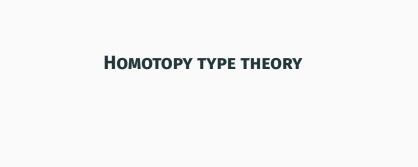
# **HOMOTOPY EQUIVALENCE**

Homotopies can be used to define a weaker notion of equivalence between topological spaces.

# **Homotopy Equivalence**

A continuous map  $f:X\to Y$  between topological spaces is called a homotopy equivalence if there exists a continuous map  $g:Y\to X$  such that  $g\circ f\simeq 1_X$  and  $f\circ g\simeq 1_Y$ .

If these maps exist the spaces are called homotopy equivalent, written  $X \simeq Y$ .



### **FIBER BUNDLES**

A fiber bundle is a topological space to which copies of another topological space are attached.

#### **Fiber Bundle**

A fiber bundle is a structure  $(E, B, \pi, F)$  where E, B and F are topological spaces and  $\pi : E \to B$  is a continuous surjection satisfying a local triviality condition.

The space *E* is called the total space, *B* the base space and *F* the fiber.

#### **FIBRATIONS**

If the projection map  $\pi$  has the homotopy lifting property this construction is called a fibration. Fibrations are more general in the sense that the fibers need not be the same space but are instead required to be homotopy equivalent.

# **DEPENDENT TYPES AS FIBRATIONS**

It is possible to interpret dependent types as fibrations!

## **CONSEQUENCES**

- Gives us ∏-types
- Construction of  $\sum$ -types from  $\prod$ -types
- · Identity terms are paths
- Higher identity terms are homotopies

#### **CONTEXT**

- Usefulness of this interpretation
- This area is being developed, unfinished as of today
- · The univalence axiom

