Homotopy Type Theory

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Introduction

- Type theory is a formal deductive system
- Topology studies spaces and their deformations
- Homotopy type theory is a novel interpretation

This talk is based on the following paper:

Á. Pelayo, M.A. Warren, *Homotopy Type Theory and Voevodsky's Univalent Foundations*, Bulletin of the American Mathematical Society, 51:4 (2014), 597-648.

Rephrased with an emphasis on intuition. Some concepts are left undefined.

Type theory

Types

The most basic judgement in type theory is $x : A$, meaning x is a term of the type A.

- Type Universes
- Propositional and judgemental equalities
- Type Formers

The Curry-Howard Correspondence

- One way to give meaning to terms and types
- Types are propositions
- Terms are proofs
- Curry-Howard-Lambek correspondence

Function Types

Given two types A and B, a type of functions $A \to B$ between them can be constructed. A term of this type is a function from the domain A into the codomain B.

Such functions are often defined by λ-abstractions.

A dependent type is a type parametrised by another type. Formally, a dependent type B depending on a type A is a function $B : A \to \mathcal{U}$.

The $\prod$-type is a generalisation of the function type where the codomain can depend on the value of the variable term.

Given a type A and a dependent type $B : A \to \mathcal{U}$ this is expressed as:

$$\prod_{x:A} B(x).$$

Pairing two types A and B leads to the pair type $A \times B$. Terms of this type have the form (a, b) for $a : A$ and $b : B$.

Dependent Pair Types

Analogue to the dependent function type, the $\sum$-type generalises pair types. For elements of this pair type the second component may depend on the first.

Again for A a type and $B : A \to \mathcal{U}$ a dependent type, this is denoted

$$\sum_{x:A} B(x).$$

- Major deviation from set theory
- Identities are captured in a type $\mathsf{Id}_A(a, b)$
- Identity types hold no unique information
- In homotopy type theory they do, leading to higher types:

$$\mathsf{Id}_{\mathsf{Id}_A(a,b)}(c, d)$$

Topology

Interesting spaces contain structure. One example of such a structure is a metric, a notion of distance.

A topology captures the structure of a space by encoding how elements are spatially related to each other.

Topology

A topology on a set X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying the following three axioms:

1. The sets $\varnothing$ and X are elements of $\mathcal{T}$;
2. The intersection of two sets in $\mathcal{T}$ is an element of $\mathcal{T}$;
3. Any union of sets taken from $\mathcal{T}$ is an element of $\mathcal{T}$.

Such a pair $(X, \mathcal{T})$ forms a topological space.

Continuous maps are maps that do not behave unpredictably.

Continuity

A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ between topological spaces is called continuous if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

Paths between two points are examples of such maps.

Path

A path in a topological space X is a continuous map $[0, 1] \to X$.

Homeomorphisms

The most elementary definition of the equivalence of two topological spaces is given by a homeomorphism.

Homeomorphism

A homeomorphism is a bijection between two topological spaces such that both the map and its inverse are continuous.

Two spaces are said to be homeomorphic if there exists a homeomorphism between them.

Manifolds form possibly the most important classes of spaces in topology.

Topological Manifold

An n-dimensional topological manifold is a topological Hausdorff space that is second countable and such that every point of this space admits an open neighbourhood homeomorphic to $\mathbb{R}^n$.

Homotopies are continuous deformations between maps.

Homotopy

A homotopy is an indexed family of maps $f_t(x) : X \to Y$ between topological spaces so that the associated map $F : [0, 1] \times Y$ defined by $F(x, t) = f_t(x)$ is continuous.

The maps f_0 and f_1 are said to be homotopic, denoted $f_0 \simeq f_1$.

Homotopies can be used to define a weaker notion of equivalence between topological spaces.

Homotopy Equivalence

A continuous map $f : X \to Y$ between topological spaces is called a homotopy equivalence if there exists a continuous map $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

If these maps exist the spaces are called homotopy equivalent, written $X \simeq Y$.

Homotopy type theory

A fiber bundle is a topological space to which copies of another topological space are attached.

Fiber Bundle

A fiber bundle is a structure (E, B, π, F) where E, B and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition.

The space E is called the total space, B the base space and F the fiber.

If the projection map π has the homotopy lifting property this construction is called a fibration. Fibrations are more general in the sense that the fibers need not be the same space but are instead required to be homotopy equivalent.

Dependent Types as Fibrations

It is possible to interpret dependent types as fibrations!

Consequences

- Gives us $\prod$-types
- Construction of $\sum$-types from $\prod$-types
- Identity terms are paths
- Higher identity terms are homotopies

Context

- Usefulness of this interpretation
- This area is being developed, unfinished as of today
- The univalence axiom

Questions